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Multi-amalgamation of rules with application conditions in $\mathcal{M}$-adhesive categories

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Amalgamation is a well-known concept for graph transformations that is used to model synchronised parallelism of rules with shared subrules and corresponding transformations. This concept is especially important for an adequate formalisation of the operational semantics of statecharts and other visual modelling languages, where typed attributed graphs are used for multiple rules with nested application conditions. However, the theory of amalgamation for the double-pushout approach has so far only been developed on a set-theoretical basis for pairs of standard graph rules without any application conditions.

For this reason, in the current paper we present the theory of amalgamation for $\mathcal{M}$-adhesive categories, which form a slightly more general framework than (weak) adhesive HLR categories, for a bundle of rules with (nested) application conditions. The two main results are the Complement Rule Theorem, which shows how to construct a minimal complement rule for each subrule, and the Multi-Amalgamation Theorem, which generalises the well-known Parallelism and Amalgamation Theorems to the case of multiple synchronised parallelism. In order to apply the largest amalgamated rule, we use maximal matchings, which are computed according to the actual instance graph. The constructions are illustrated by a small but meaningful running example, while a more complex case study concerning the firing semantics of Petri nets is presented as an introductory example and to provide motivation.

1. Introduction and related work

1.1. *Historical background for amalgamation*

The concepts of adhesive (Lack and Sobociński 2005) and weak adhesive high-level replacement (HLR) (Ehrig *et al.* 2006) categories were a breakthrough for the double-pushout approach of algebraic graph transformations (Rozenberg 1997). Almost all the main results for graph transformation systems could be formulated and proved in these categorical frameworks and instantiated to a large variety of HLR systems, including various kinds of graph and Petri net transformation systems (Ehrig *et al.* 2006). These

results included the Local Church–Rosser, Parallelism and Concurrency Theorems, the Embedding and Extension Theorem, the completeness of critical pairs, and the Local Confluence Theorem (Ehrig *et al.* 2014). Ehrig *et al.* (2010) showed that $\mathcal{M}$-adhesive categories, which are a slightly weaker version, are also sufficient for formulating graph transformations in such a general categorical setting.

While most graph transformation models for distributed systems concentrate on the topological aspects of the system (Castellani and Montanari 1983; Degano and Montanari 1987), the application of the main theorems for the analysis of such systems is also of interest. One example is the Parallelism Theorem (Ehrig and Kreowski 1976), which states that two parallel independent transformations can be combined and are equivalent to a single transformation using the corresponding parallel rule. However, a weaker form of parallel independence is often required for distributed systems: two transformations do not have to be completely parallel independent, but may overlap dependently on certain well-defined elements. This generalisation of the Parallelism Theorem is called the Amalgamation Theorem, where the assumption of parallel independence is dropped and some synchronisation takes place. It was developed in Böhm *et al.* (1987) on a set-theoretical basis for a pair of standard graph rules without application conditions.

The synchronisation of two rules p_1 and p_2 is expressed by a common subrule p_0, which we call the *kernel rule* in the current paper. The subrule concept is formalised by a so-called *kernel morphism*, which is a rule morphism from p_0 to p_i. Given two such kernel morphisms, the rules p_1 and p_2 can be glued along p_0 to give an amalgamated rule $\tilde{p}$ representing the synchronised effects of p_1 and p_2. Now, two transformations *via* p_1 and p_2 are *amalgamable* if they are parallel independent except for the elements matched by the kernel rule. In this case, and in a similar way to the Parallelism Theorem, the two transformations can be combined and are equivalent to a single transformation using the amalgamated rule. This is the main statement of the Amalgamation Theorem: each amalgamable pair of transformations $G \Rightarrow G_i$ $(i = 1, 2)$ *via* p_1 and p_2 leads to an amalgamated transformation $G \Rightarrow H$ *via* $\tilde{p}$.

Moreover, the Complement Rule Theorem in Böhm *et al.* (1987) allows us to construct a complement rule $\overline{p}$ out of a kernel morphism from p_0 to p. Using the kernel rule p_0 and the complement rule $\overline{p}$, we can construct a concurrent rule $p_0 *_E \overline{p}$ equal to p. The Concurrency Theorem then allows us to decompose each transformation $G \Rightarrow H$ *via* p into sequences $G \Rightarrow G_i \Rightarrow H$ *via* p_0 and $\overline{p}$. Moreover, an amalgamated transformation can also be sequentialised in this way.

1.2. *Other parallel models of computation in graph transformation*

Parallel rewriting was first studied at the level of strings. Motivated by examples from biology, 'Lindenmayer Systems', or L-systems for short, were developed as a mathematical theory of parallel languages in the 1970s. The main idea of L-systems is to simultaneously replace all letters of a string according to a given set of rules. This idea was generalised to graphs, which led to various kinds of parallel graph grammars and graph-L-systems (Rozenberg and Lindenmayer 1976).

There are several other graph transformation based approaches and tools that realise the transformation of multi-object structures. PROGRES (Schürr *et al.* 1999) and Fujaba (Fischer *et al.* 2000) feature so-called set-valued nodes, which can be duplicated as often as necessary. Both approaches handle multi-objects pragmatically. Object nodes are identified to be, optionally, matched once, arbitrarily often or at least once, and adjacent arcs are treated accordingly. This concept focuses on multiple instances of single nodes instead of graph parts.

Other approaches that realise amalgamated graph transformation are AToM3, GReAT and GROOVE. Of these, AToM3 supports the explicit definition of interaction schemes in different rule editors (de Lara *et al.* 2004), while GROOVE implements rule amalgamation based on nested graph predicates (Rensink and Kuperus 2009). Although nesting extends the expressiveness of these transformations, writing and understanding these predicates is a fairly complicated task, and it seems to be difficult to relate them to or integrate them in the theoretical results for graph transformation. By contrast, the GReAT tool can use a group operator to apply delete, move or copy operations to each match of a rule (Balasubramanian *et al.* 2007).

Grønmo *et al.* (2009) adopted a related conceptual approach, which aimed at the transformation of collections of similar subgraphs. In that work, all the collection operators (multi-objects) in a rule are replaced by the mapped number of collection match copies. Similarly, Hoffmann *et al.* (2006) defined a cloning operator, where cloned nodes roughly correspond to multi-objects.

However, none of these approaches investigated the formal analysis of amalgamated graph transformation.

1.3. *Applications of amalgamation*

The concepts of amalgamation were applied to communication based systems in Taentzer and Beyer (1994), Taentzer (1996) and Ermel (2006), and transferred to the single-pushout approach of graph transformation in Löwe (1993). Amalgamation was used in Biermann *et al.* (2010a) to define a model transformation that translates simple business process models written in the Business Process Modelling Notation (BPMN) to executable processes formulated in the Business Process Execution Language for Web Services (BPEL). Amalgamation also plays a key role in the modelling of the operational semantics for visual languages (Ermel 2006). Golas *et al.* (2011) and Golas (2011) presented a complex case study for the operational semantics of statecharts based on typed attributed graphs and multi-amalgamation. An advantage of amalgamation is that we do not need helper structures or a complex external control structure to cover complex semantical steps in our approach. The result is a model-independent definition that is not only visual and intuitive, but also allows us to show termination and forms a solid basis for applying further graph transformation based analysis techniques.

The theory of amalgamation presented in the current paper has been implemented in AGG (Taentzer 2004) and in our EMF transformation tool EMF Henshin (Biermann *et al.* 2010b), which has been extended by visual editors for amalgamated rules and application conditions (Biermann *et al.* 2010c).

1.4. *The aim of the current paper*

In most applications, we need amalgamation for n rules (called multi-amalgamation), which are based not only on standard graph rules, but on various kinds of typed and attributed graph rules, including (nested) application conditions. While some of the tools provide an *ad hoc* implementation of multi-amalgamation, the underlying theory is not elaborated. The main idea of the current paper is to fill this gap between theory and applications. To this end, we have developed the theory of multi-amalgamation for $\mathcal{M}$-adhesive systems based on rules with nested application conditions. Ehrig *et al.* (2014) gives a brief description of the amalgamation of exactly two rules in this framework. Our work in the current paper allows us to instantiate the theory to a large variety of graphs and corresponding graph transformation systems, and, using weak adhesive HLR categories, to typed attributed graph transformation systems (Ehrig *et al.* 2006) as well.

The work in the current paper extends Golas *et al.* (2010) in several ways. First, we consider amalgamated transformations in any $\mathcal{M}$-adhesive category, while Golas *et al.* (2010) only used adhesive categories. Second, we present the firing semantics of Petri nets as a new case study. This semantics is much smaller and easier to survey than the semantics of statecharts in Golas *et al.* (2011), but still shows the importance of multi-amalgamation, including the use of application conditions. Moreover, we give the full proofs for the results and extend the theory by maximal matchings, which allows us to compute the maximal amalgamated rule applicable at a specific kernel match.

1.5. *Organisation of the paper*

In Section 2, we discuss how to define the semantics of Petri nets using graph transformation and show that amalgamation makes it easier to define rules without the need for any additional control structure. In Section 3, we review basic notions related to $\mathcal{M}$-adhesive categories, transformations and application conditions. In Section 4, we introduce kernel rules, multi-rules and kernel morphisms, which lead to the Complement Rule Theorem as our first main result. In Section 5, we construct multi-amalgamated rules and transformations, and then show the Multi-Amalgamation Theorem as our second main result. Maximal matchings, which are used to compute the maximal amalgamated rule, are constructed in Section 6. Finally, we present a summary of our results and discuss future work in Section 7. All the more complex proofs are collected together in Appendix A, while some technical lemmas underlying these proofs are relegated to Appendix B.

2. Firing semantics of Petri nets using amalgamation

A Petri net, or place/transition net (Reisig and Rozenberg 1998), consists of places (circles) and transitions (rectangles) with arcs between them. A place with a connecting arc to or from a transition is called its pre-place or post-place, respectively. Note that for simplicity we do not allow a place to be both a pre-arc and a post-arc of the same transition. A number of tokens is put on each place, and there is no limit on the number

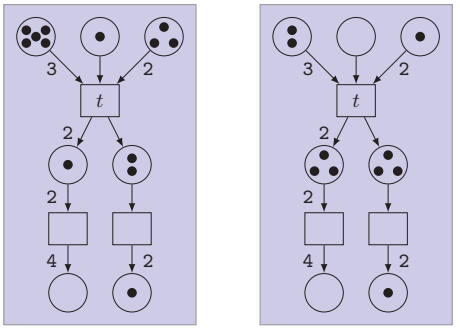
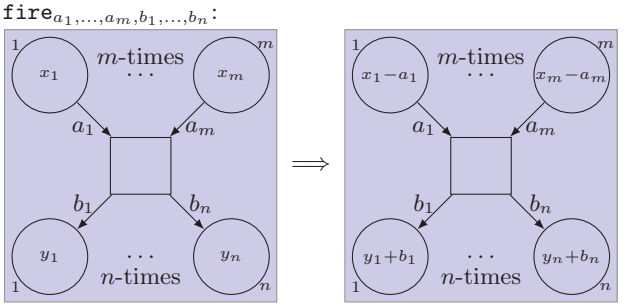


Fig. 1. (Colour online) The firing of the transition t



+ application conditions:
no other pre- and post-places
$a_i \leq x_i$ for $i = 1, \ldots, m$

Fig. 2. (Colour online) The rule scheme for firing an arbitrary transition in place/transition nets

of tokens allowed. Natural numbers at the arcs mark how many tokens are moved when the transition fires. Note that the absence of a number at an arc is an abbreviation for 1. A transition is enabled if all its pre-places hold at least as many tokens as required by the arc inscription. Firing this transition leads to the deletion of this number of tokens on the pre-places and the respective number of tokens is then added to each post-place (see Figure 1). For the modelling of the nets, we use typed attributed graphs (Ehrig *et al.* 2006), which we will not describe in detail here. For each place, there is an attribute `token` of type integer representing the number of tokens at this place. In the figures, we simply show this number inside the place.

Generally speaking, there are two main approaches in the literature for defining a rule-based semantics for models:

— In the first approach, the rules can be dependent on the actual instance of the model (Kuske *et al.* 2002), so there are some rule schemes or instructions that have to be applied to describe how to obtain the semantical rule for a concrete semantical step dependent on what the model instance looks like. In a place/transition net, for a transition with m pre-places and n post-places, we have variables $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ denoting the number of tokens for the rule.
Given the arc weights $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ for the pre-arcs and post-arcs, this leads to a rule $\texttt{fire}_{a_1,\ldots,a_m,b_1,\ldots,b_n}$ (see Figure 2) describing the token handling. We then need application conditions to ensure all the pre-places and post-places are matched. In

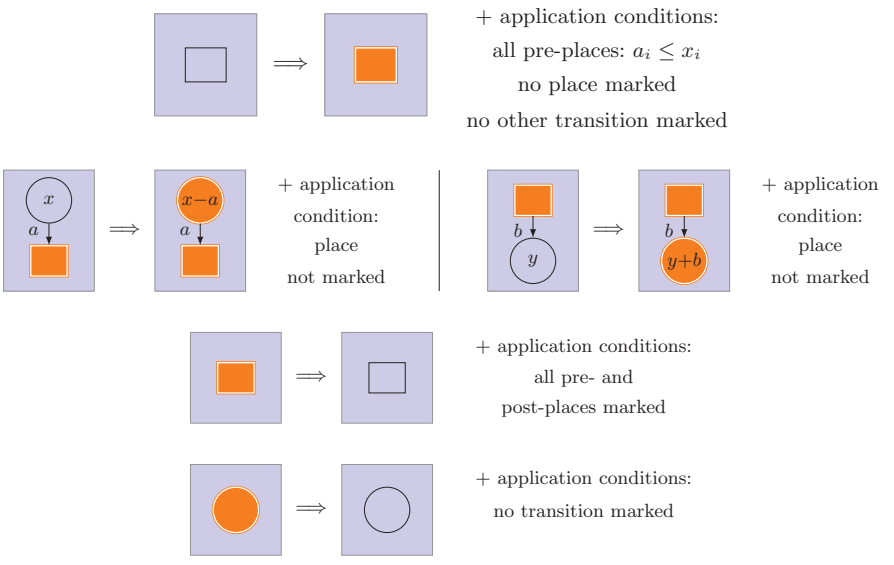


Fig. 3. (Colour online) The general rules for firing a transition in place/transition nets

addition, we have to check that the number of tokens at a pre-place is no smaller than the corresponding arc weight. This rule scheme can be interpreted for each transition that occurs, thus defining the semantics of a concrete place/transition net. Note that to obtain all the firing rules of the place/transition nets, we have to consider all combinations of values for m, n, a_i and b_j. This approach is easy to use once the rules are constructed, but when a model is changed, the semantical rules also have to be adapted. For arbitrary instances not known in advance, infinitely many rules appear, which are then difficult to analyse.

— For the second approach, general rules are applied according to some complex control structure (Varró 2002). For place/transition nets, we first have to mark an active transition to declare its firing (the top rule in Figure 3). Since we do not know in advance how many pre-places and post-places will need to be handled, we require one rule to delete a token from one pre-place and one rule to add a token in one post-place of a transition (the middle rules in Figure 3). Since we have to know which places have already been processed, we also have to mark these places. In the end, when all of the relevant places have been handled, first the transition and then the places can be unmarked (the bottom rules in Figure 3). Applying the first rule m-times and the second one n-times with the corresponding matches leads to a firing step in the Petri net. In this way, all model instances are handled using the same rules. However, even for this simple example, a lot of marking is needed to ensure the correct matches. Although the single rules are relatively easy to understand, the additional helper structures, often combined with complex control structures for more difficult examples, makes it hard to understand the complete semantics.

Even for Petri nets, whose semantics can be described relatively easily on a set-theoretical basis (Reisig and Rozenberg 1998), both graph transformation approaches

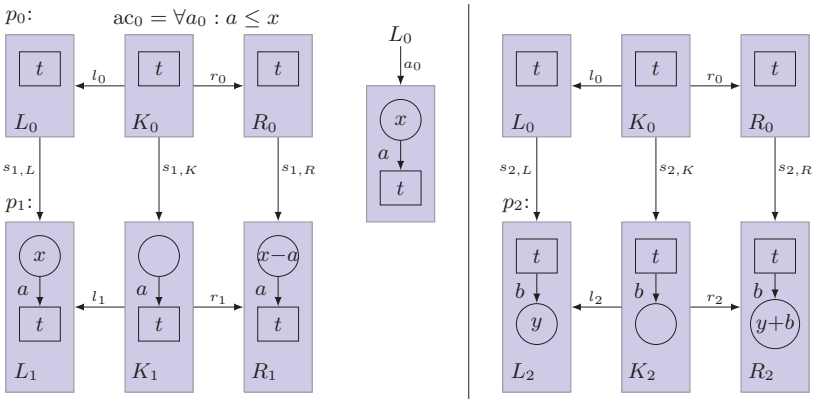


Fig. 4. (Colour online) The amalgamation semantics for firing place/transition nets

discussed above have their drawbacks. When we analyse the second approach, it is obvious that the marking of the transition represents a kind of synchronisation: instead of arbitrary matches for the transition, the handling of the pre-places and post-places has to happen at the marked transition. The marking of the pre-places and post-places is required to avoid multiple processing of the same place. Neither of these markings are required for the first approach, since in that case all places are handled at the same time. Our goal is to combine both approaches to give a universal rule application for all model instances with less additional structure so that analysis becomes easier. To do this, we use amalgamation to define an interaction scheme that provides the necessary rules. The semantical step for each model instance can be computed using maximal matchings. As shown below, for place/transition nets, we only need one kernel rule and two multi-rules to describe the complete firing semantics for all well-defined nets. When we use amalgamation, there is no need for infinitely many rules, which are difficult to analyse, or any control or helper structure. This makes the modelling of the semantics easier and prevents errors.

The semantics for place/transition nets using amalgamation is shown in Figure 4. The kernel rule p_0 appears twice in the top row. Note that we use rules in the double-pushout approach with a left-hand side L describing what must be found to apply the rule, an interface K describing what is preserved and a right-hand side R showing the resulting graph part. This means that the elements $L \backslash K$ are deleted and the elements $R \backslash K$ are created by the rule. The kernel rule selects an activated transition (but does not change or mark it), and controls the synchronisation. Note that we use an application condition ac_0, shown in the middle of Figure 4, saying that for all morphisms a_0, the attribute value of the arc a has to be smaller than that of the node x. We have two multi-rules, which define the handling of the tokens:

— p_1 on the left for the pre-places selecting a place and decreasing the number of tokens; and
— p_2 on the right for the post-places selecting a place and increasing the number of tokens.

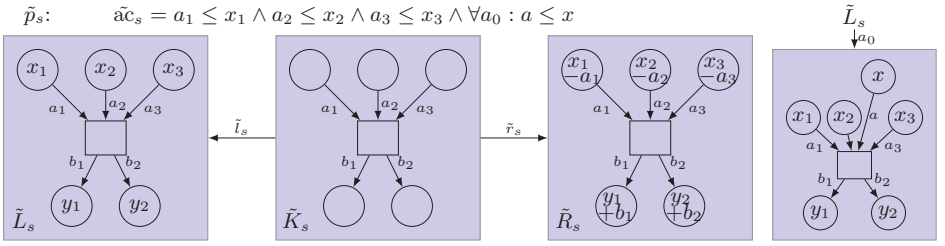


Fig. 5. (Colour online) The amalgamated rule for firing the transition t

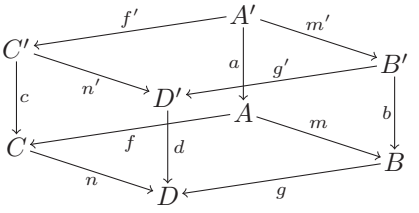
We define morphisms s_1 and s_2 from the kernel rule to the multi-rules, which form an interaction scheme. Whenever a firing step is performed, we compute a maximal weakly disjoint matching, meaning that we look for matches for the multi-rules that overlap on the kernel rule, but are disjoint outside. Such a matching is relatively easy and inexpensive to compute, and ensures that all pre-places and post-places of a chosen transition are mapped.

For example, the maximal weakly disjoint matching for the firing of the transition t in Figure 1 with kernel match t includes three matches for the multi-rule p_1 and two matches for the multi-rule p_2: one for each pre-place and post-place, respectively. Amalgamation of this maximal weakly disjoint matching leads to the amalgamated rule $\tilde{p}_s$ shown in Figure 5, which describes the complete firing of t. Note that this rule looks similar to an instantiation of the rule scheme in Figure 2, but is obtained by a very different construction mechanism, *viz.* amalgamation.

3. Review of basic notions

The basic idea of adhesive categories (Lack and Sobociński 2005) is to have a category with pushouts and pullbacks along monomorphisms satisfying the van Kampen property. Intuitively, this means that pushouts along monomorphisms and pullbacks are compatible with each other. This holds for sets and various kinds of graphs (Lack and Sobociński 2005; Ehrig *et al.* 2006), including the standard category of graphs, which we will use as a running example. $\mathcal{M}$-adhesive categories include a distinguished morphism class $\mathcal{M}$ of monomorphisms and extend adhesive categories with suitable properties: a major difference is that they only require pushouts along $\mathcal{M}$-morphisms to be *vertical weak* van Kampen squares.

Definition 3.1 (van Kampen square). A pushout, as at the bottom of the cube



with $m \in \mathcal{M}$, is a *vertical weak van Kampen square*, or $\mathcal{M}$-van Kampen square for short, if it satisfies the *vertical weak van Kampen property*, that is, for any commutative cube where the back faces are pullbacks and the vertical morphisms $b, c, d \in \mathcal{M}$, the top face is a pushout if and only if the front faces are pullbacks.

By contrast, the horizontal weak van Kampen property assumes that $f \in \mathcal{M}$ instead of $b, c, d \in \mathcal{M}$, while the (standard) van Kampen property does not require any additional $\mathcal{M}$-morphisms.

Definition 3.2 ($\mathcal{M}$-adhesive category). An *$\mathcal{M}$-adhesive category* $(\mathbf{C}, \mathcal{M})$ consists of a category $\mathbf{C}$ and a class $\mathcal{M}$ of monomorphisms in $\mathbf{C}$ that is closed under isomorphisms, composition and decomposition ($g \circ f \in \mathcal{M}$ and $g \in \mathcal{M}$ implies $f \in \mathcal{M}$) such that $\mathbf{C}$ has pushouts and pullbacks along $\mathcal{M}$-morphisms, $\mathcal{M}$-morphisms are closed under pushouts and pullbacks, and pushouts along $\mathcal{M}$-morphisms are $\mathcal{M}$-van Kampen squares.

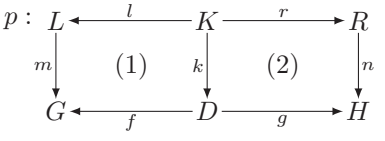
Well-known examples of $\mathcal{M}$-adhesive categories are the categories $(\mathbf{Sets}, \mathcal{M})$ of sets, $(\mathbf{Graphs}, \mathcal{M})$ of graphs, $(\mathbf{Graphs_{TG}}, \mathcal{M})$ of typed graphs, $(\mathbf{ElemNets}, \mathcal{M})$ of elementary Petri nets and $(\mathbf{PTNets}, \mathcal{M})$ of place/transition nets, where $\mathcal{M}$ is the class of all monomorphisms for all these examples, and $(\mathbf{AGraphs_{ATG}}, \mathcal{M})$ of typed attributed graphs, where $\mathcal{M}$ is the class of all injective typed attributed graph morphisms with isomorphic data type component (Ehrig *et al.* 2006).

In the double-pushout approach to transformations, rules give a general description of how to transform objects. The application of a rule to an object is called a transformation and is based on two gluing constructions, which are pushouts in the corresponding category.

Definition 3.3 (rule and transformation). A *rule* is given by a span

$$p = \left(L \xleftarrow{l} K \xrightarrow{r} R\right)$$

with objects L, K and R, called the left-hand side, interface and right-hand side, respectively, and $\mathcal{M}$-morphisms l and r. An application of such a rule to an object G *via* a match $m : L \to G$ is constructed as two pushouts (1) and (2) leading to a *direct transformation* $G \overset{p,m}{\Longrightarrow} H$:



Example 3.4. An example for a rule can be found in the top row of Figure 10. The application of the rule to the graph G leads to the transformation $G \overset{\tilde{p}_s,\tilde{m}}{\Longrightarrow} H$ shown, where both squares are pushouts.

An important extension is the use of rules with suitable application conditions. These include positive application conditions of the form $\exists a$ for a morphism $a : L \to C$, which demand a certain structure in addition to L, and negative application conditions $\neg\exists a$, forbidding such a structure. A match $m : L \to G$ satisfies $\exists a$ (respectively, $\neg\exists a$) if there is

a (respectively, no) $\mathcal{M}$-morphism $q : C \to G$ satisfying $q \circ a = m$. More precisely, we use nested application conditions (Habel and Pennemann 2009), or just application conditions for short.

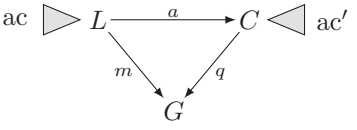
Definition 3.5 (application condition and satisfaction). An *application condition* ac over an object L is of the form

$$\mathrm{ac} = \mathrm{true}$$

or

$$\mathrm{ac} = \exists(a, \mathrm{ac}'),$$

where $a : L \to C$ is a morphism and ac' is an application condition over C. Given a condition ac over L, a morphism $m : L \to G$ *satisfies* ac, written $m \models \mathrm{ac}$, if $\mathrm{ac} = \mathrm{true}$ or $\mathrm{ac} = \exists(a, \mathrm{ac}')$ and there exists a morphism $q \in \mathcal{M}$ with $q \circ a = m$ and $q \models \mathrm{ac}'$:



Moreover, application conditions are closed under Boolean formulas (with finite or infinite index set) and satisfaction is extended in the usual way. To simplify the presentation, we will write false to abbreviate $\neg\mathrm{true}$, $\exists a$ to abbreviate $\exists(a, \mathrm{true})$ and $\forall(a, \mathrm{ac})$ to abbreviate $\neg\exists(a, \neg\mathrm{ac})$. We will also write $\mathrm{ac}_C \cong \mathrm{ac}'_C$ to denote the semantical equivalence of ac_C and ac'_C on C.

Example 3.6. In Figure 10, the application condition $\tilde{\mathrm{ac}}_s$ of the rule $\tilde{p}_s$ is stated above the rule, while the relevant morphisms are shown on the right. This condition forbids various edges coming from or going to node 1. The match morphism $\tilde{m}$ satisfies this application condition.

In the current paper, we consider rules of the form

$$p = \left(L \xleftarrow{l} K \xrightarrow{r} R, \mathrm{ac}\right),$$

where

$$\left(L \xleftarrow{l} K \xrightarrow{r} R\right)$$

is a (plain) rule and ac is an application condition on L. There are two important concepts we need in order to handle rules with application conditions, namely, the shifts of application conditions over morphisms and rules (Habel and Pennemann 2009; Ehrig *et al.* 2014).

For the shift construction over morphisms, we use a distinguished class $\mathcal{E}'$ of morphism pairs with the same codomain such that for any pair of morphisms with common codomain, a unique $\mathcal{E}'$-$\mathcal{M}$ pair factorisation exists.

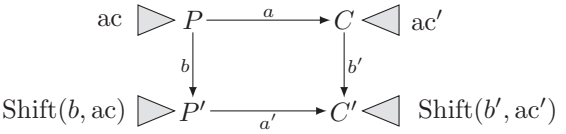
Definition 3.7 (shift over morphism). Given an application condition $\mathrm{ac} = \exists(a, \mathrm{ac}')$ over P and a morphism $b : P \to P'$, we define $\mathrm{Shift}(b, \mathrm{ac})$ to be an application condition over

P' such that

$$\text{Shift}(b, \text{ac}) = \vee_{(a',b')\in\mathcal{F}} \exists(a', \text{Shift}(b', \text{ac}'))$$

with

$$\mathcal{F} = \{(a', b') \mid (a', b') \in \mathcal{E}', b' \in \mathcal{M},\ b' \circ a = a' \circ b\}.$$



Moreover,

$$\text{Shift}(b, \text{true}) = \text{true}$$

and the construction is extended for Boolean formulas in the usual way.

Remark 3.8. $\mathcal{F}$ is finite if $\mathcal{E}'$ consists of jointly surjective pairs of morphisms, which is the case in our example categories.

Example 3.9. An example for shifting an application condition over a morphism is given on the left-hand side of Figure 7. We have that

$$\text{Shift}(v_1, \neg\exists a_1') = \neg\exists a_{11},$$

because square $(*)$ is the only possible commuting square leading to a_{11} and b_{11} jointly surjective and b_{11} injective.

Fact 3.10. Given an application condition ac over P and morphisms

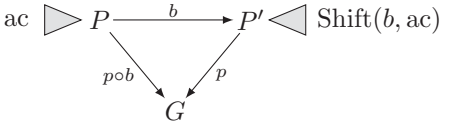
$$b : P \to P'$$
$$p : P' \to G,$$

we have

$$p \models \text{Shift}(b, \text{ac})$$

if and only if

$$p \circ b \models \text{ac}.$$



Proof. See Habel and Pennemann (2009) and Ehrig *et al.* (2014). □

By analogy with the application condition over L, which is a pre-application condition, it is also possible to define post-application conditions over the right-hand side R of a rule. Since these application conditions over R can be translated to equivalent application conditions over L, and *vice versa* (Habel and Pennemann 2009), we can restrict our rules to application conditions over L.

Definition 3.11 (shift over rule). Given a rule

$$p = \left(L \overset{l}{\leftarrow} K \overset{r}{\rightarrow} R, \mathrm{ac}\right)$$

and an application condition

$$\mathrm{ac}_R = \exists(a, \mathrm{ac}'_R)$$

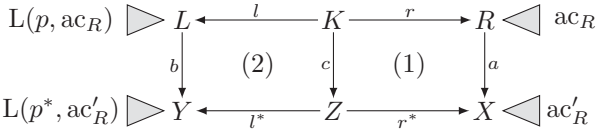
over R, we define $\mathrm{L}(p, \mathrm{ac}_R)$ to be an application condition over L with

$$\mathrm{L}(p, \mathrm{ac}_R) = \exists(b, \mathrm{L}(p^*, \mathrm{ac}'_R))$$

if $a \circ r$ has a pushout complement (1) and

$$p^* = \left(Y \overset{l^*}{\leftarrow} Z \overset{r^*}{\rightarrow} X\right)$$

is the derived rule by constructing pushout (2):



Otherwise

$$\mathrm{L}(p, \mathrm{ac}_R) = \mathrm{false}.$$

Moreover,

$$\mathrm{L}(p, \mathrm{true}) = \mathrm{true},$$

and the construction is extended to Boolean formulas in the usual way.

Example 3.12. Figure 7 gives an example of shifting an application condition over a rule shown by the two pushout squares (PO_1) and (PO_2) where

$$\mathrm{L}(p_1^*, \neg\exists a_{11}) = \neg\exists a_1.$$

Fact 3.13. Given a transformation $G \overset{p,m}{\Longrightarrow} H$ *via* a rule

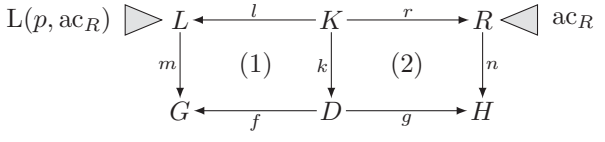
$$p = \left(L \overset{l}{\leftarrow} K \overset{r}{\rightarrow} R, \mathrm{ac}\right)$$

and an application condition ac_R over R, we have

$$m \models \mathrm{L}(p, \mathrm{ac}_R)$$

if and only if

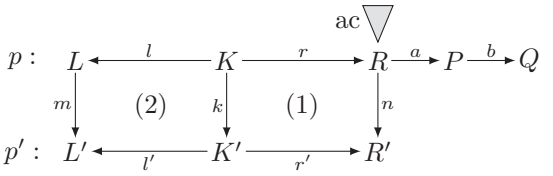
$$n \models \mathrm{ac}_R.$$



Proof. See Habel and Pennemann (2009). □

Shifts over morphisms are compositional and shifts over morphisms and rules are compatible *via* double pushouts.

Fact 3.14. Given an application condition ac on R, the double pushouts (1) and (2) and morphisms a, b



we have

$$\text{Shift}(b, \text{Shift}(a, \text{ac})) \cong \text{Shift}(b \circ a, \text{ac})$$

and

$$\text{Shift}(m, \text{L}(p, \text{ac})) \cong \text{L}(p', \text{Shift}(n, \text{ac})).$$

Proof. See Habel and Pennemann (2009) and Ehrig *et al.* (2014). □

3.1. *General assumptions*

In the rest of the paper, we assume we have an $\mathcal{M}$-adhesive category with binary coproducts, initial pushouts, $\mathcal{E}'$-$\mathcal{M}$-pair factorisation and effective pushouts (Ehrig *et al.* 2006; Golas 2011). We consider rules with (nested) application conditions (Habel and Pennemann 2009) as explained above, and in the presentation we will assume familiarity with parallelism and concurrency in the sense of Ehrig *et al.* (2006). Moreover, we use the corresponding constructions and results for the case with application conditions given in Ehrig *et al.* (2014). In the following, a *bundle* represents a family of morphisms or transformation steps with the same domain, which means that a bundle always starts at the same object.

4. Decomposition of direct transformations

In this section, we show how to decompose a direct transformation in $\mathcal{M}$-adhesive categories into transformations *via* a kernel and a complement rule, which will lead us to the Complement Rule Theorem.

A kernel morphism describes how a smaller rule, the kernel rule, is embedded into a larger rule, the multi-rule, which gets its name from the fact that it can be applied multiple times for a given kernel rule match, as described in Section 5. We will need some more technical preconditions to ensure that the embeddings of the L-, K- and R-components and the application conditions are consistent and will allow us to construct a complement rule.

Definition 4.1 (kernel morphism). Given rules

$$p_0 = \left(L_0 \xleftarrow{l_0} K_0 \xrightarrow{r_0} R_0, \mathrm{ac}_0\right)$$
$$p_1 = \left(L_1 \xleftarrow{l_1} K_1 \xrightarrow{r_1} R_1, \mathrm{ac}_1\right),$$

a *kernel morphism* $s_1 : p_0 \to p_1$, with

$$s_1 = (s_{1,L}, s_{1,K}, s_{1,R})$$

consists of $\mathcal{M}$-morphisms

$$s_{1,L} : L_0 \to L_1$$
$$s_{1,K} : K_0 \to K_1$$
$$s_{1,R} : R_0 \to R_1$$

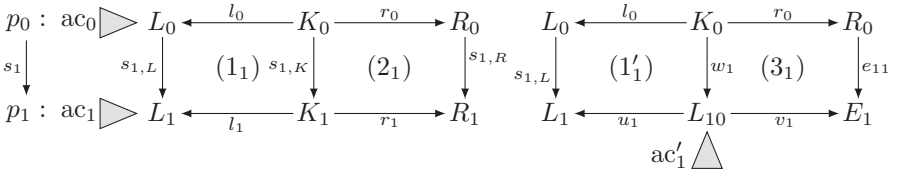
such that (1_1) and (2_1) in the following diagram are pullbacks, (1_1) has a pushout complement $(1'_1)$ for $s_{1,L} \circ l_0$, and ac_0 and ac_1 are *complement-compatible with respect to* s_1, that is, given pushout (3_1), we have

$$\mathrm{ac}_1 \cong \mathrm{Shift}(s_{1,L}, \mathrm{ac}_0) \wedge \mathrm{L}(p_1^*, \mathrm{Shift}(v_1, \mathrm{ac}'_1))$$

for some ac'_1 on L_{10} and

$$p_1^* = \left(L_1 \xleftarrow{u_1} L_{10} \xrightarrow{v_1} E_1\right).$$

In this case, p_0 is called a *kernel rule* and p_1 a *multi-rule*.



Remark 4.2. The complement compatibility of the application conditions makes sure that there is a decomposition of ac_1 into parts on L_0 and L_{10}, where we will use the latter for the application conditions of the complement rule later in the paper.

Example 4.3. To explain the concept of amalgamation, we will model a small transformation system for switching the direction of edges in labelled graphs, where we only have different labels for edges – black and dotted edges. The kernel rule p_0 is shown at the top of Figure 6. It selects a node with a black loop, deletes this loop and adds a dotted loop, all of this provided no dotted loop is already present. The matches are defined by the numbers at the nodes and can be induced for the edges by their position.

The middle and bottom rows of Figure 6 show two multi-rules p_1 and p_2, which extend the rule p_0 and also reverse an edge if no backward edge is present. They also inherit the application condition of p_0 forbidding a dotted loop at the selected node. There is a

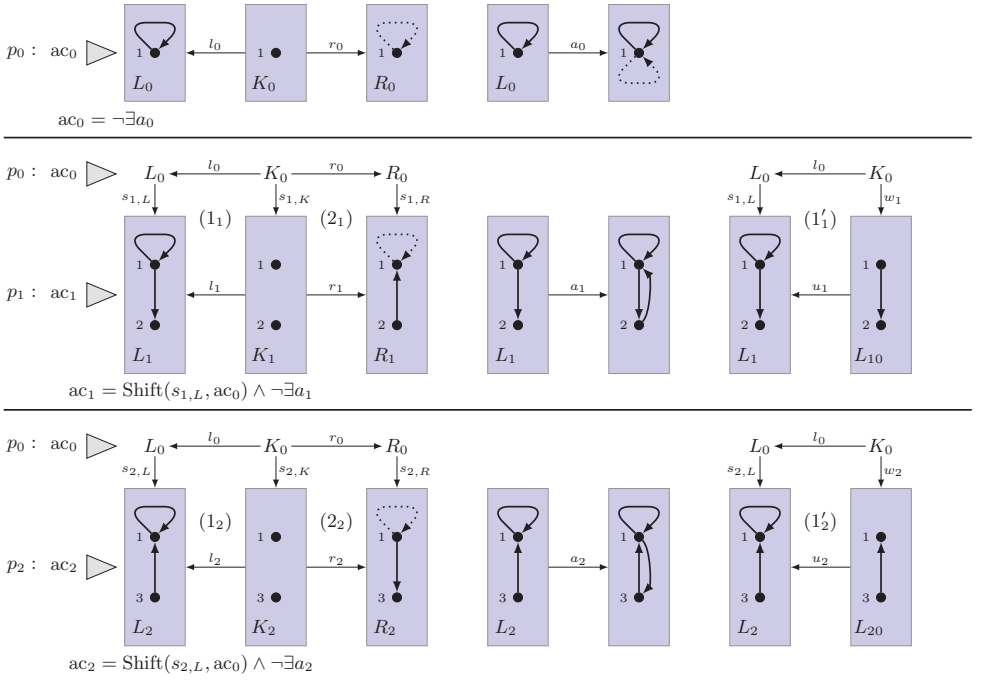


Fig. 6. (Colour online) The kernel rule p_0 and the multi-rules p_1 and p_2

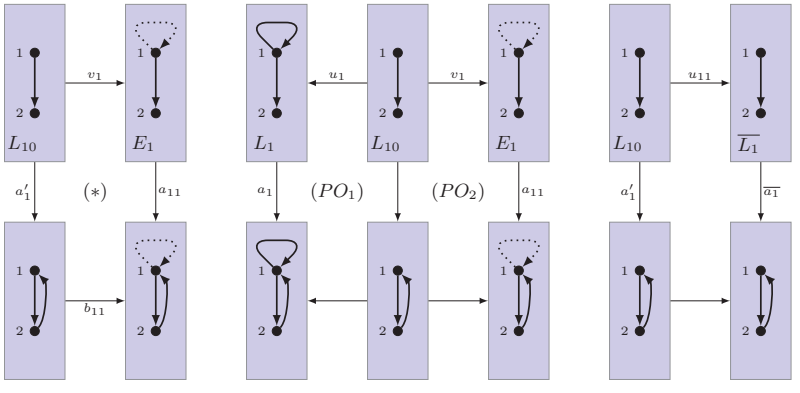


Fig. 7. (Colour online) Constructions for the application conditions

kernel morphism $s_1 : p_0 \to p_1$ as shown at the top of Figure 6 with pullbacks (1_1) and (2_1), and pushout complement $(1'_1)$. For the application conditions, we have

$$\begin{aligned} \text{ac}_1 &= \text{Shift}(s_{1,L}, \text{ac}_0) \wedge \neg\exists a_1 \\ &\cong \text{Shift}(s_{1,L}, \text{ac}_0) \wedge \text{L}(p_1^*, \text{Shift}(v_1, \neg\exists a'_1)), \end{aligned}$$

as shown on the left-hand side of Figure 7. Thus $\text{ac}'_1 = \neg\exists a'_1$, and ac$_0$ and ac$_1$ are complement compatible.

Similarly, there is a kernel morphism $s_2 : p_0 \to p_2$, as shown in the bottom row of Figure 6, with pullbacks (1_2) and (2_2), pushout complement $(1'_2)$, and ac_0 and ac_2 being complement compatible.

For a given kernel morphism, the complement rule is the remainder of the multi-rule after the application of the kernel rule, that is, it describes what the multi-rule does in addition to the kernel rule.

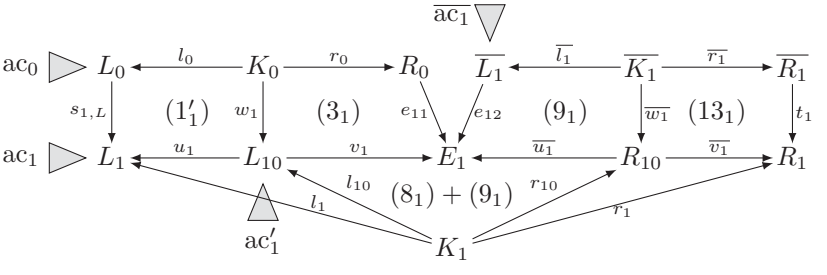
Theorem 4.4 (existence of complement rule). Given rules

$$p_0 = \left(L_0 \xleftarrow{l_0} K_0 \xrightarrow{r_0} R_0, \mathrm{ac}_0 \right)$$
$$p_1 = \left(L_1 \xleftarrow{l_1} K_1 \xrightarrow{r_1} R_1, \mathrm{ac}_1 \right),$$

and a kernel morphism $s_1 : p_0 \to p_1$, there is a canonical way to construct a rule

$$\overline{p_1} = \left(\overline{L_1} \xleftarrow{\overline{l_1}} \overline{K_1} \xrightarrow{\overline{r_1}} \overline{R_1}, \overline{\mathrm{ac}_1} \right)$$

and a jointly epimorphic cospan $R_0 \xrightarrow{e_{11}} E_1 \xleftarrow{e_{12}} \overline{L_1}$ such that the E_1-concurrent rule $p_0 *_{E_1} \overline{p_1}$ exists and $p_1 = p_0 *_{E_1} \overline{p_1}$:



See Ehrig *et al.* (2014) for the definition of E-concurrent rules for rules with application conditions.

Proof. See Section A.1 in the appendix. □

Remark 4.5. Note that when we use the construction in the appendix, the interface K_0 of the kernel rule has to be preserved in the complement rule. This canonical construction of $\overline{p_1}$ is not unique with respect to the property $p_1 = p_0 *_{E_1} \overline{p_1}$ since other choices for S_1 with $\mathcal{M}$-morphisms s_{11} and s_{13} also lead to a well-defined construction. In particular, we could choose $S_1 = R_0$ leading to

$$\overline{p_1} = E_1 \xleftarrow{\overline{u_1}} R_{10} \xrightarrow{\overline{v_1}} R_1.$$

Our choice represents a smallest possible complement, which should be preferred in most application areas.

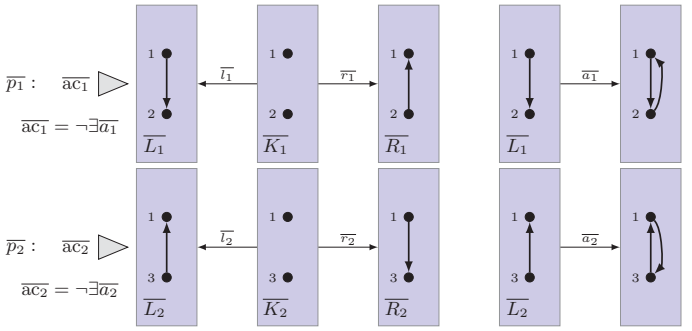


Fig. 8. (Colour online) The complement rules for the kernel morphisms

Definition 4.6 (complement rule). Given rules

$$p_0 = \left(L_0 \xleftarrow{l_0} K_0 \xrightarrow{r_0} R_0, \mathrm{ac}_0\right)$$
$$p_1 = \left(L_1 \xleftarrow{l_1} K_1 \xrightarrow{r_1} R_1, \mathrm{ac}_1\right),$$

and a kernel morphism $s_1 : p_0 \to p_1$, the canonical rule

$$\overline{p_1} = \left(\overline{L_1} \xleftarrow{\overline{l_1}} \overline{K_1} \xrightarrow{\overline{r_1}} \overline{R_1}, \overline{\mathrm{ac}_1}\right)$$

identified by Theorem 4.4 is called the *complement rule* (of s_1).

Example 4.7. Consider the kernel morphism s_1 in Figure 6. Using the construction in Theorem 4.4, we obtain the complement rule in the top row of Figure 8 with the application condition $\overline{\mathrm{ac}_1} = \neg\exists\overline{a_1}$ constructed in the right-hand side of Figure 7. Figure 9 shows the diagrams of the construction. In a similar way, we can obtain a complement rule for the kernel morphism $s_2 : p_0 \to p_2$ in Figure 6 – see the bottom row of Figure 8.

Each direct transformation *via* a multi-rule can be decomposed into a direct transformation *via* the kernel rule followed by a direct transformation *via* the complement rule.

Fact 4.8 (decomposition of transformations). Given rules

$$p_0 = \left(L_0 \xleftarrow{l_0} K_0 \xrightarrow{r_0} R_0, \mathrm{ac}_0\right)$$
$$p_1 = \left(L_1 \xleftarrow{l_1} K_1 \xrightarrow{r_1} R_1, \mathrm{ac}_1\right),$$

a kernel morphism $s_1 : p_0 \to p_1$, and a direct transformation $t_1 : G \overset{p_1,m_1}{\Longrightarrow} G_1$, we have that t_1 can be decomposed into the transformation

$$G \overset{p_0,m_0}{\Longrightarrow} G_0 \overset{\overline{p_1},\overline{m_1}}{\Longrightarrow} G_1$$

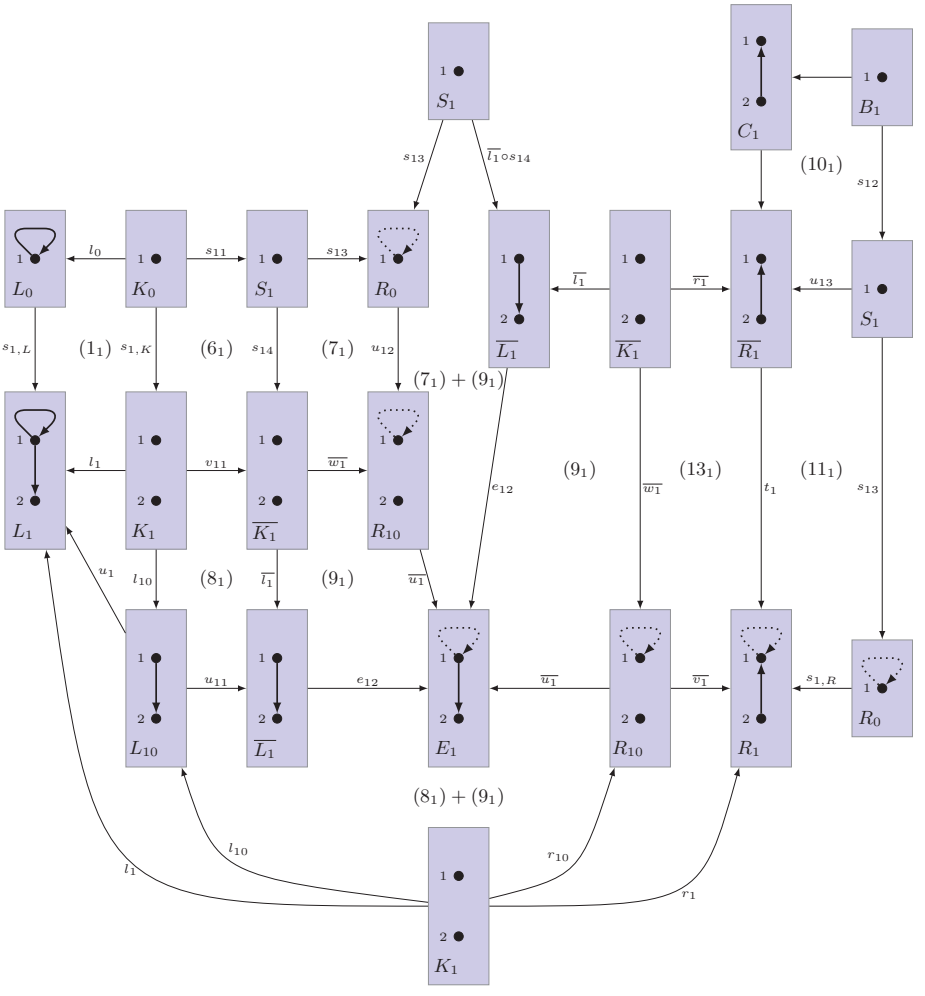
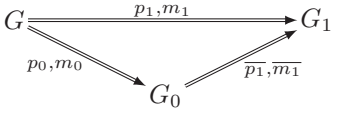


Fig. 9. (Colour online) The construction of the complement rule for the kernel morphism s_1

with

$$m_0 = m_1 \circ s_{1,L}$$

where $\overline{p_1}$ is the complement rule of s_1:



Proof. We have

$$p_1 \cong p_0 *_{E_1} \overline{p_1}.$$

The analysis part of the Concurrency Theorem (Ehrig *et al.* 2014) then implies the decomposition into

$$G \overset{p_0,m_0}{\Longrightarrow} G_0 \overset{\overline{p_1},\overline{m_1}}{\Longrightarrow} G_1$$

with $m_0 = m_1 \circ s_{1,L}$. □

5. Multi-amalgamation

Böhm *et al.* (1987) developed an Amalgamation Theorem for a pair of graph rules without application conditions, which can be seen as a generalisation of the Parallelism Theorem (Ehrig and Kreowski 1976) in which the assumption of parallel independence is dropped and pure parallelism is generalised to synchronised parallelism. In this section, we present the Multi-Amalgamation Theorem as an Amalgamation Theorem for a bundle of rules with application conditions over objects in an $\mathcal{M}$-adhesive category.

We consider not only single kernel morphisms, but bundles of morphisms over a fixed kernel rule. We can then combine the multi-rules of such a bundle to give an amalgamated rule by gluing them along the common kernel rule.

Definition 5.1 (amalgamated rule). Given rules

$$p_i = \left(L_i \xleftarrow{l_i} K_i \xrightarrow{r_i} R_i, \mathrm{ac}_i\right)$$

for $i = 0, \ldots, n$ and a bundle of kernel morphisms

$$s = (s_i : p_0 \to p_i)_{i=1,\ldots,n},$$

the *amalgamated rule*

$$\tilde{p}_s = \left(\tilde{L}_s \xleftarrow{\tilde{l}_s} \tilde{K}_s \xrightarrow{\tilde{r}_s} \tilde{R}_s, \tilde{\mathrm{ac}}_s\right)$$

is constructed using:

— the componentwise colimits of the kernel morphisms

$$\begin{aligned}\tilde{L}_s &= \mathrm{Col}\left((s_{i,L})_{i=1,\ldots,n}\right)\\ \tilde{K}_s &= \mathrm{Col}\left((s_{i,K})_{i=1,\ldots,n}\right)\\ \tilde{R}_s &= \mathrm{Col}\left((s_{i,R})_{i=1,\ldots,n}\right);\end{aligned}$$

— $\tilde{l}_s$ and $\tilde{r}_s$ are induced by $(t_{i,L} \circ l_i)_{i=0,\ldots,n}$ and $(t_{i,R} \circ r_i)_{0=1,\ldots,n}$, respectively; and
— $\tilde{\mathrm{ac}}_s$ given by

$$\tilde{\mathrm{ac}}_s = \bigwedge_{i=1,\ldots,n} \mathrm{Shift}(t_{i,L}, \mathrm{ac}_i).$$

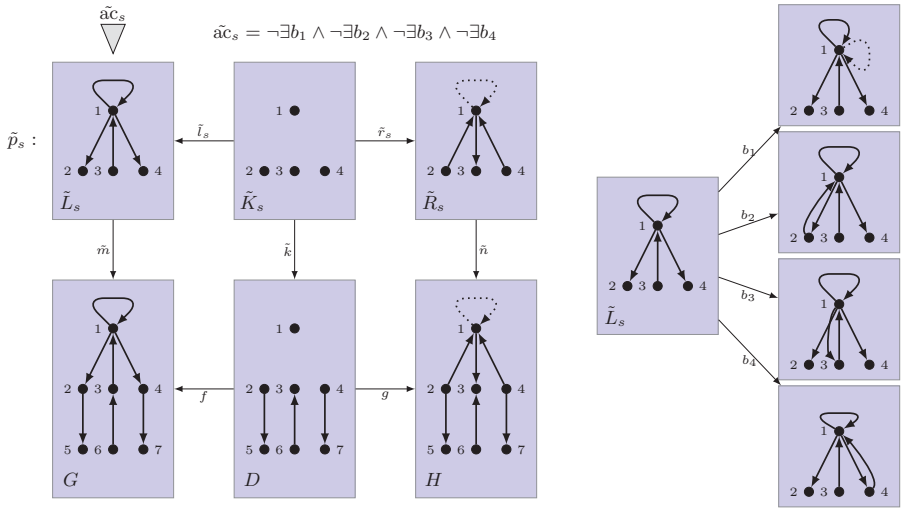
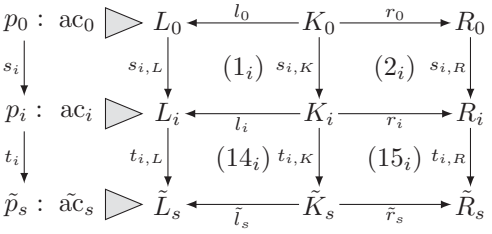


Fig. 10. (Colour online) An amalgamated transformation



Fact 5.2. The amalgamated rule is well defined and we have kernel morphisms

$$t_i = (t_{i,L}, t_{i,K}, t_{i,R}) : p_i \to \tilde{p}_s$$

for $i = 0, \ldots, n$.

Proof. See Section A.2 in the appendix. □

The application of an amalgamated rule yields an amalgamated transformation.

Definition 5.3 (amalgamated transformation). The application of an amalgamated rule to a graph G is called an *amalgamated transformation.*

Example 5.4. Consider the bundle $s = (s_1, s_2, s_3 = s_1)$ of kernel morphisms shown in Figure 6. The corresponding amalgamated rule $\tilde{p}_s$ is shown in the top row of Figure 10. This amalgamated rule can be applied to the graph G leading to the amalgamated transformation shown in Figure 10, where the application condition $\tilde{ac}_s$ is obviously fulfilled by the match $\tilde{m}$.

If we have a bundle of direct transformations of a graph G, where for each transformation one of the multi-rules is applied, we want to determine if the amalgamated rule is

applicable to G in combining all of the single transformation steps. These transformations are compatible, that is, multi-amalgamable, if the matches agree on the kernel rules and are independent outside.

Definition 5.5 (s-amalgamable). Given a bundle of kernel morphisms

$$s = (s_i : p_0 \to p_i)_{i=1,\dots,n},$$

we say a bundle of direct transformations steps

$$(G \overset{p_i,m_i}{\Longrightarrow} G_i)_{i=1,\dots n}$$

is *s-amalgamable* if:

— it has *consistent matches*, that is,

$$m_i \circ s_{i,L} = m_j \circ s_{j,L} =: m_0$$

for all $i, j = 1, \dots, n$ and

— it has *weakly independent matches*, that is, for all $i \neq j$, if we consider the pushout complements $(1'_i)$ and $(1'_j)$, there exist morphisms

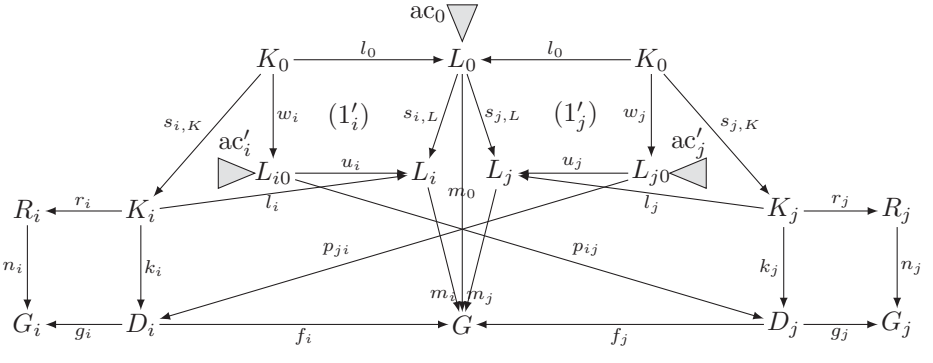
$$p_{ij} : L_{i0} \to D_j$$
$$p_{ji} : L_{j0} \to D_i$$

such that

$$f_j \circ p_{ij} = m_i \circ u_i$$
$$f_i \circ p_{ji} = m_j \circ u_j$$

and

$$g_j \circ p_{ij} \models \mathrm{ac}'_i$$
$$g_i \circ p_{ji} \models \mathrm{ac}'_j.$$

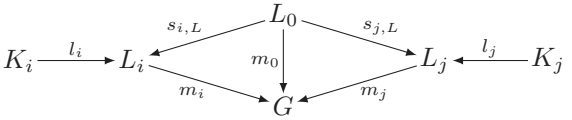


We can give a set-theoretical characterisation of weak independence without application conditions in a similar way to the characterisation of parallel independence in Ehrig *et al.* (2006).

Fact 5.6. For graphs and other set-based structures, weakly independent matching, without considering the application conditions, means that

$$m_i(L_i) \cap m_j(L_j) \subseteq m_0(L_0) \cup (m_i(l_i(K_i)) \cap m_j(l_j(K_j)))$$

for all $i \neq j$, that is, the elements in the intersection of the matches m_i and m_j are either preserved by both transformations or are also matched by m_0.



Proof. We have to prove the equivalence of

$$m_i(L_i) \cap m_j(L_j) \subseteq m_0(L_0) \cup (m_i(l_i(K_i)) \cap m_j(l_j(K_j)))$$

for all $i \neq j = 1, \ldots, n$ to the definition of weakly independent matches.

($\Leftarrow$) Let

$$x = m_i(y_i) = m_j(y_j),$$

and suppose $x \notin m_0(L_0)$. Since $(1_i')$ is a pushout, we have

$$y_i = u_i(z_i) \in u_i(L_{i0} \backslash w_i(K_0)),$$

and

$$x = m_i(u_i(z_i)) = f_j(p_{ij}(z_i)) = m_j(y_j),$$

and by pushout properties, $y_j \in l_j(K_j)$ and $x \in m_j(l_j(K_j))$. Similarly, $x \in m_i(l_i(K_i))$.

($\Rightarrow$) For $x \in L_{i0}$ and $x = w_i(k)$, we define

$$p_{ij}(x) = k_j(s_{j,K}(k)).$$

Then

$$\begin{aligned} f_j(p_{ij}(x)) &= f_j(k_j(s_{j,K}(k))) \\ &= m_j(l_j(s_{j,K}(k))) \\ &= m_j(s_{j,L}(l_0(k))) \\ &= m_i(s_{i,L}(l_0(k))) \\ &= m_i(u_i(w_i(k))) \\ &= m_i(u_i(x)). \end{aligned}$$

Otherwise, we have $x \notin w_i(K_0)$, that is, $u_i(x) \notin s_{i,L}(L_0)$, and define

$$p_{ij}(x) = y$$

with

$$f_j(y) = m_i(u_i(x)).$$

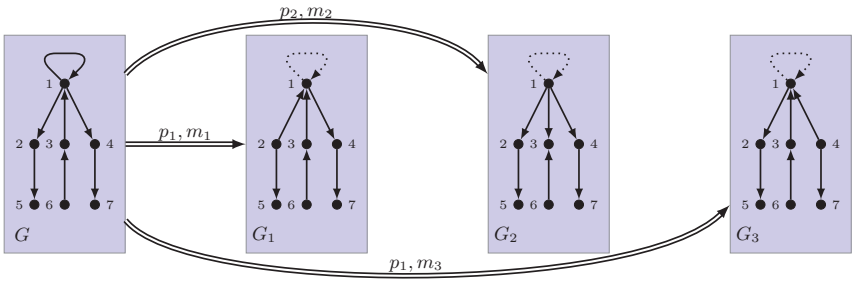


Fig. 11. (Colour online) An s-amalgamable bundle of direct transformations

This y exists because either

$$m_i(u_i(x)) \notin m_j(L_j)$$

or

$$m_i(u_i(x)) \in m_j(L_j)$$

and thus

$$m_i(u_i(x)) \in m_j(l_j(K_j)),$$

and in both cases

$$m_i(u_i(x)) \in f_j(D_j).$$

We can also define p_{ji} with the required property in a similar way. □

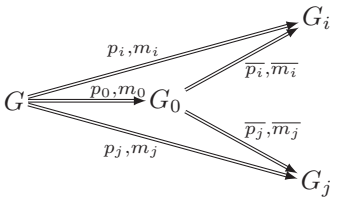
Example 5.7. Consider the bundle $s = (s_1, s_2, s_3 = s_1)$ of kernel morphisms we considered in Example 5.4. For the graph G given in Figure 10, we find matches

$$m_0 : L_0 \to G$$
$$m_1 : L_1 \to G$$
$$m_2 : L_2 \to G$$
$$m_3 : L_1 \to G$$

mapping all nodes from the left-hand side to their corresponding nodes in G, except for m_3, which maps node 2 in L_1 to node 4 in G. For all these matches, the corresponding application conditions are fulfilled, and we can apply the rules p_1, p_2, p_1, respectively, to give the bundle of direct transformations shown in Figure 11. This bundle is s-amalgamable because the matches m_1, m_2 and m_3 agree on the match m_0, and are weakly independent because they only overlap in m_0.

For an s-amalgamable bundle of direct transformations, each single transformation step can be decomposed into an application of the kernel rule followed by an application of the complement rule. Moreover, all kernel rule applications lead to the same object, and

the following applications of the complement rules are parallel independent.



Fact 5.8. Given a bundle of kernel morphisms

$$s = (s_i : p_0 \to p_i)_{i=1,\dots,n}$$

and an s-amalgamable bundle of direct transformations

$$(G \overset{p_i,m_i}{\Longrightarrow} G_i)_{i=1,\dots,n},$$

each direct transformation $G \overset{p_i,m_i}{\Longrightarrow} G_i$ can be decomposed into a transformation

$$G \overset{p_0,m_0}{\Longrightarrow} G_0 \overset{\overline{p_i},\overline{m_i}}{\Longrightarrow} G_i.$$

Moreover, the transformations

$$G_0 \overset{\overline{p_i},\overline{m_i}}{\Longrightarrow} G_i$$

are pairwise parallel independent.

Proof. See Section A.3 in the appendix. □

If a bundle of direct transformations of a graph G is s-amalgamable, we can apply the amalgamated rule directly to G to give a parallel execution of all the changes made by the single transformation steps.

Theorem 5.9 (multi-amalgamation). Consider a bundle of kernel morphisms

$$s = (s_i : p_0 \to p_i)_{i=1,\dots,n}.$$

Then:

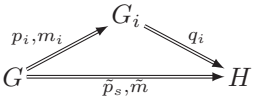
(1) *Synthesis*:
Given an s-amalgamable bundle

$$\left(G \overset{p_i,m_i}{\Longrightarrow} G_i\right)_{i=1,\dots,n}$$

of direct transformations, there is an amalgamated transformation $G \overset{\tilde{p}_s,\tilde{m}}{\Longrightarrow} H$ and transformations $G_i \overset{q_i}{\Longrightarrow} H$ over the complement rules q_i of the kernel morphisms $t_i : p_i \to \tilde{p}_s$ such that

$$G \overset{p_i,m_i}{\Longrightarrow} G_i \overset{q_i}{\Longrightarrow} H$$

is a decomposition of $G \overset{\tilde{p}_s,\tilde{m}}{\Longrightarrow} H$:



(2) *Analysis*:
Given an amalgamated transformation $G \overset{\tilde{p}_s,\tilde{m}}{\Longrightarrow} H$, there are s_i-related transformations

$$G \overset{p_i,m_i}{\Longrightarrow} G_i \overset{q_i}{\Longrightarrow} H$$

for $i = 1,\ldots,n$ such that $G \overset{p_i,m_i}{\Longrightarrow} G_i$ is s-amalgamable.

(3) *Bijective correspondence*:
The synthesis and analysis constructions are inverse to each other up to isomorphism.

Proof. See Section A.4 in the appendix. □

Remark 5.10. Note that q_i can be constructed as the amalgamated rule of the kernel morphisms

$$(p_{K_0} \to \overline{p_j})_{j \neq i},$$

where

$$p_{K_0} = \left(K_0 \overset{id_{K_0}}{\longleftarrow} K_0 \overset{id_{K_0}}{\longrightarrow} K_0, \text{true}\right)$$

and $\overline{p_j}$ is the complement rule of p_j.

For $n = 2$ and rules without application conditions, the Multi-Amalgamation Theorem specialises to the Amalgamation Theorem in Böhm *et al.* (1987). Moreover, if p_0 is the empty rule, it is just the Parallelism Theorem in Ehrig *et al.* (2014) since the transformations are parallel independent for an empty kernel match.

Example 5.11. As stated in Example 5.7, the transformations

$$G \overset{p_1,m_1}{\Longrightarrow} G_1$$
$$G \overset{p_2,m_2}{\Longrightarrow} G_2$$
$$G \overset{p_1,m_3}{\Longrightarrow} G_3$$

shown in Figure 11 are s-amalgamable for the bundle

$$s = (s_1, s_2, s_3 = s_1)$$

of kernel morphisms. Applying Fact 5.8, we can decompose these transformations into a transformation $G \overset{p_0,m_0}{\Longrightarrow} G_0$ followed by transformations

$$G_0 \overset{\overline{p_1},\overline{m_1}}{\Longrightarrow} G_1$$
$$G_0 \overset{\overline{p_2},\overline{m_2}}{\Longrightarrow} G_2$$
$$G_0 \overset{\overline{p_1},\overline{m_3}}{\Longrightarrow} G_3$$

via the complement rules, which are pairwise parallel independent. These transformations are shown in Figure 12. Moreover, Theorem 5.9 implies that we obtain an amalgamated transformation $G \overset{\tilde{p}_s,\tilde{m}}{\Longrightarrow} H$ for this bundle of direct transformations – this is the transformation already shown in Figure 10. Conversely, the analysis of this amalgamated

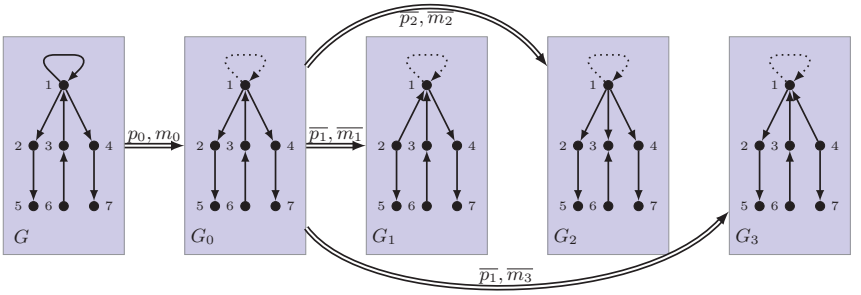


Fig. 12. (Colour online) The decomposition of the *s*-amalgamable bundle

transformation leads to the *s*-amalgamable bundle of transformations

$$G \overset{p_1,m_1}{\Longrightarrow} G_1$$
$$G \overset{p_2,m_2}{\Longrightarrow} G_2$$
$$G \overset{p_1,m_3}{\Longrightarrow} G_3$$

from Figure 11.

6. Multi-amalgamation with maximal matchings

An important extension of the theory presented so far is the introduction of interaction schemes and maximal matchings. For many interesting application areas, including the operational semantics for Petri nets and statecharts, we do not want to define the matches for the multi-rules explicitly, but obtain them dependent on the object to be transformed. For example, for the firing semantics of statcharts (Golas *et al.* 2011), an unknown number of state transitions triggered by the same event, which is highly dependent on the actual system state, can be handled in parallel. Similarly, for our Petri net semantics introduced in Section 2, the pre-places and post-places of a transition should be computed during runtime, and they are dependent on the current Petri net model.

An interaction scheme defines a bundle of kernel morphisms. In contrast to a concrete bundle, in order to apply such an interaction scheme, all possible matches for the multi-rules that agree on a given kernel match are computed and lead to an amalgamable bundle of transformations.

Definition 6.1 (interaction scheme). A kernel rule p_0 and a set of multi-rules

$$\{p_1, \ldots, p_k\}$$

with kernel morphisms $s_i : p_0 \to p_i$ form an *interaction scheme*

$$is = \{s_1, \ldots, s_k\}.$$

When given an interaction scheme, we want to apply as many rules occurring in the interaction scheme as often as possible over a certain kernel rule match. For maximal weakly independent matchings, we require the matchings of the multi-rules to be weakly

independent to ensure that the resulting bundle of transformations is amalgamable. This is the minimal requirement to meet the definition.

Definition 6.2 (maximal weakly independent matching). Given an interaction scheme

$$is = \{s_1, \ldots, s_k\}$$

with $s_j : p_0 \to p_j$ for $j = 1, \ldots, k$ and a family of matchings

$$m = (m_i : L_i' \to G),$$

where each p_i' corresponds to some p_j for $j \leqslant k$, with transformations $G \overset{p_i', m_i}{\Longrightarrow} G_i$, we say m forms a *maximal weakly independent matching* if the bundle $G \overset{p_i', m_i}{\Longrightarrow} G_i$ is multi-amalgamable and, for any rule p_j, no other match $m' : L_j \to G$ can be found such that $((m_i), m')$ fulfils this property.

This definition leads directly to the following algorithm to compute maximal weakly independent matchings for graphs and graph-like structures.

Algorithm 6.3 (maximal weakly independent matching). Given a graph G and an interaction scheme

$$is = \{s_1, \ldots, s_k\},$$

a maximal weakly disjoint matching

$$m = (m_0, m_1, \ldots, m_n)$$

can be computed as follows:

(1) Set $i = 0$ and choose a kernel matching $m_0 : L_0 \to G$ such that

$$G \overset{p_0, m_0}{\Longrightarrow} G_0$$

is a valid transformation.

(2) For as long as possible, increase i, choose a multi-rule $\hat{p}_i = p_j$ with $j \in \{1, \ldots, k\}$, and find a match $m_i : L_j \to G$ such that:

— $m_i \circ s_{j,L} = m_0$;

— $G \overset{p_j, m_i}{\Longrightarrow} G_i$ is a valid transformation;

— the matches $m_1, \ldots, m_i$ are weakly independent; and

— $m_i \neq m_\ell$ for all $\ell = 1, \ldots, i-1$.

(3) If no more valid matches for any rule in the interaction scheme can be found, return

$$m = (m_0, m_1, \ldots, m_n).$$

The maximal weakly independent matching leads to a bundle of kernel morphisms

$$s = (s_i : p_0 \to \hat{p}_i)$$

and an s-amalgamable bundle of direct transformations $G \overset{\hat{p}_i, m_i}{\Longrightarrow} G_i$.

For applications, the computation of maximal weakly independent matchings requires a lot of backtracking because a match in Step (2) is often not weakly independent from an already chosen one, which has to be checked pairwise for this new match compared to all others. While the application conditions always have to be analysed since they may state global properties of the resulting graph, at least for the elements available for the new match, some restrictions may help to enhance the computation. In many cases, it is enough to require the matches to be disjoint outside the kernel match. A typical example is the semantics of Petri nets described in Section 2, where all maximal weakly independent matchings are also weakly disjoint. This disjointness property is described formally by a certain pullback requirement. Using maximal weakly disjoint matchings for the implementation, we can rule out model parts that have already been matched.

Definition 6.4 (maximal weakly disjoint matching). Given an interaction scheme

$$is = \{s_1, \ldots, s_k\}$$

and a maximal weakly independent matching

$$m = (m_i : L'_i \to G),$$

we say m forms a *maximal weakly disjoint matching* if the square $(P_{i\ell})$ is a pullback for all $i \neq \ell$:

$$\begin{array}{ccc} L_0 & \xrightarrow{s_{i,L}} & L'_i \\ {\scriptstyle s_{\ell,L}}\downarrow & (P_{i\ell}) & \downarrow{\scriptstyle m_i} \\ L'_\ell & \xrightarrow[m_\ell]{} & G \end{array}$$

Note that for maximal weakly disjoint matchings, the pullback requirement already implies the existence of the morphisms for the weakly independent matches, and only the property for the application conditions has to be checked in addition.

Fact 6.5. Given an object G, a bundle of kernel morphisms $s = (s_1, \ldots, s_n)$ and matches $m_1, \ldots, m_n$ leading to a bundle of direct transformations $G \overset{p_i, m_i}{\Longrightarrow} G_i$ such that

$$m_i \circ s_{i,L} = m_0$$

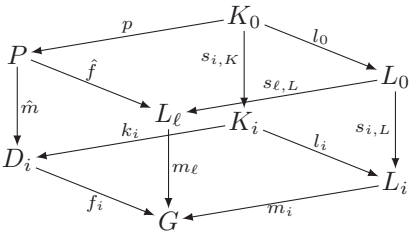
and square $(P_{i\ell})$ is a pullback for all $i \neq \ell$, the bundle $G \overset{p_i, m_i}{\Longrightarrow} G_i$ is s-amalgamable for transformations without application conditions.

Proof. By construction, the matches m_i agree on the match m_0 of the kernel rule, so it just remains to show that they are weakly independent.

Consider the transformations $G \overset{p_i, m_i}{\Longrightarrow} G_i$ with pushouts (20_i) and (21_i) in the following diagram:

$$\begin{array}{ccccc} L_i & \xleftarrow{l_i} & K_i & \xrightarrow{r_i} & R_i \\ {\scriptstyle m_i}\downarrow & (20_i) & {\scriptstyle k_i}\downarrow & (21_i) & \downarrow{\scriptstyle n_i} \\ G & \xleftarrow[f_i]{} & D_i & \xrightarrow[g_i]{} & G_i \end{array}$$

For the cube



the bottom face is pushout (20_i), the back right face is pullback (1_i) and the front right face is pullback $(P_{i\ell})$. Now construct the pullback of f_i and m_ℓ as the front left face, and since

$$\begin{aligned} m_\ell \circ s_{\ell,L} \circ l_0 &= m_i \circ s_{i,L} \circ l_0 \\ &= m_i \circ l_i \circ s_{i,K} \\ &= f_i \circ k_i \circ s_{i,K}, \end{aligned}$$

we obtain a morphism p with

$$\hat{f} \circ p = s_{\ell,L} \circ l_0$$

and

$$\hat{m} \circ p = k_i \circ s_{i,K}.$$

From pullback composition and decomposition of the right and left faces, it follows that the back left face is a pullback too. The $\mathcal{M}$-van Kampen property can now be applied to give a pushout in the top face. Since pushout complements are unique up to isomorphism, we can substitute the top face by pushout $(1'_i)$ from Definition 5.5 with $P \cong L_{\ell 0}$. Thus we have found the morphism $p_{\ell i} := \hat{m}$ with

$$f_i \circ p_{\ell i} = m_\ell \circ u_i.$$

This construction can be applied for all pairs i, ℓ leading to weakly independent matches without application conditions. □

This fact leads to a set-theoretical characterisation of maximal weakly disjoint matchings similar to the result in Fact 5.6.

Fact 6.6. For graphs and graph-based structures, valid matches $m_0, m_1, \ldots, m_n$ with

$$m_i \circ s_{i,L} = m_0$$

for all $i = 1, \ldots, n$ form a maximal weakly disjoint matching without application conditions if and only if

$$m_i(L_i) \cap m_\ell(L_\ell) = m_0(L_0)$$

for all $i \neq \ell$.

Proof. The fact that we have valid matches means that the transformations

$$G \overset{p_i, m_i}{\Longrightarrow} G_i$$

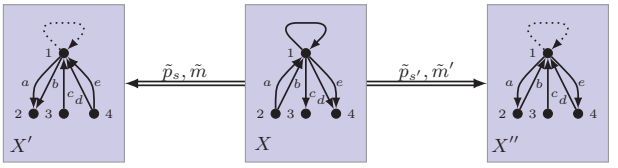


Fig. 13. (Colour online) Application of an amalgamated rule *via* maximal matchings

are well defined. In graphs and graph-like structures, $(P_{i\ell})$ is a pullback if and only if

$$m_i(L_i) \cap m_\ell(L_\ell) = m_0(L_0).$$

Fact 6.5 then implies that the matches form a maximal weakly disjoint matching without application conditions. □

Example 6.7. Consider the interaction scheme $is = (s_1, s_2)$ defined by the kernel morphisms s_1 and s_2 in Figure 6, the graph X shown in the middle of Figure 13 and the kernel rule match m_0 mapping the node 1 in L_0 to the node 1 in X.

If we choose maximal weakly independent matchings, the construction works as follows to define the following matches, where f is the edge from 1 to 2 in L_1 and g is the reverse edge in L_2:

$$\begin{aligned}
i = 1 &: \hat{p}_1 = p_1,\ m_1 : 2 \mapsto 3, f \mapsto c,\\
i = 2 &: \hat{p}_2 = p_1,\ m_2 : 2 \mapsto 4, f \mapsto d,\\
i = 3 &: \hat{p}_3 = p_2,\ m_3 : 3 \mapsto 2, g \mapsto a,\\
i = 4 &: \hat{p}_4 = p_1,\ m_4 : 2 \mapsto 4, f \mapsto e,\\
i = 5 &: \hat{p}_5 = p_2,\ m_5 : 3 \mapsto 2, g \mapsto b.
\end{aligned}$$

Thus, we find five different matches: three for the multi-rule p_1 and two for the multi-rule p_2. Note that in addition to the overlapping m_0, the matches m_3 and m_5 overlap in the node 2, while m_2 and m_4 overlap in the node 4, but since these matches are still weakly independent because the nodes 2 and 4 are not deleted by the rule applications, this is a valid maximal weakly independent matching. This leads to the bundle

$$s = (s_1, s_1, s_1, s_2, s_2)$$

and the amalgamated rule $\tilde{p}_s$, which can be applied to X to give the amalgamated transformation $X \overset{\tilde{p}_s, \tilde{m}}{\Longrightarrow} X'$, as shown on the left of Figure 13.

If we choose maximal weakly disjoint matchings instead, the matches m_4 and m_5 are no longer valid because they overlap with m_2 and m_3, respectively, in more than the match m_0. Thus, we obtain the maximal weakly disjoint matching (m_0, m_1, m_2, m_3), the corresponding bundle $s' = (s_1, s_1, s_2)$ giving the amalgamated rule $\tilde{p}_{s'}$ and the amalgamated transformation $X \overset{\tilde{p}_{s'}, \tilde{m}'}{\Longrightarrow} X''$ shown on the right of Figure 13. Note that this matching is not unique, and (m_0, m_1, m_2, m_4) could also have been chosen as a maximal weakly disjoint matching.

7. Conclusions

In the current paper, we have generalised the theory of amalgamation in Böhm *et al.* (1987) to multi-amalgamation in $\mathcal{M}$-adhesive categories, and introduced interaction schemes and maximal matchings. More precisely, the Complement Rule and Amalgamation Theorems in Böhm *et al.* (1987) are presented on a set-theoretical basis for pairs of plain graph rules without any application conditions. The Complement Rule and Multi-Amalgamation Theorems in the current paper are valid in adhesive and $\mathcal{M}$-adhesive categories for n rules with application conditions (Habel and Pennemann 2009). These generalisations are non-trivial, and are important for applications of parallel graph transformations to communication-based systems (Taentzer 1996) and to model transformations from BPMN to BPEL (Biermann *et al.* 2010a), and for modelling the operational semantics of visual languages (Ermel 2006), where interaction schemes are used to generate multi-amalgamated rules and transformations based on suitable maximal matchings.

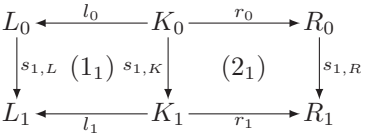
The theory of multi-amalgamation is a solid mathematical basis for analysing interesting properties of the operational semantics, such as termination, local confluence and functional behaviour. However, generalising the corresponding results in Ehrig *et al.* (2006), such as the Local Church–Rosser, Parallelism and Local Confluence Theorems, to the case of multi-amalgamated rules, and, in particular, to the operational semantics of statecharts based on amalgamated graph transformation with maximal matchings in Golas *et al.* (2011), is left for future work.

Appendix A. Proofs of facts and theorems

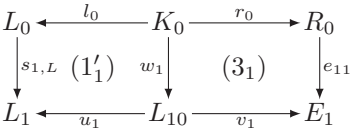
In this appendix, we prove the facts and theorems used within the main part. They rely on the technical lemmas proved in Appendix B.

A.1. *Proof of Theorem 4.4*

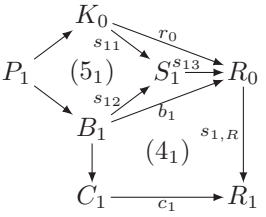
Proof. We begin by considering the construction without application conditions. Since s_1 is a kernel morphism, the diagrams (1_1) and (2_1) in



are pullbacks and (1_1) has a pushout complement $(1'_1)$ for $s_{1,L} \circ l_0$ (see Definition 4.1). We construct the pushout (3_1):



We now construct the initial pushout (4_1) over $s_{1,R}$ with $b_1, c_1 \in \mathcal{M}$, P_1 as the pullback object of r_0 and b_1, and the pushout (5_1) in



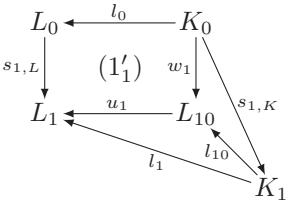
We obtain an induced morphism $s_{13} : S_1 \to R_0$ with

$$s_{13} \circ s_{12} = b_1$$
$$s_{13} \circ s_{11} = r_0$$

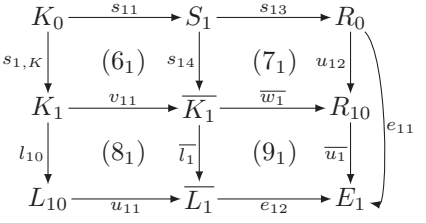
and $s_{13} \in \mathcal{M}$ by effective pushouts. Since (1_1) is a pullback, Lemma B.1 implies that there is a unique morphism $l_{10} : K_1 \to L_{10}$ with

$$l_{10} \circ s_{1,K} = w_1$$
$$u_1 \circ l_{10} = l_1,$$

and $l_{10} \in \mathcal{M}$ as in

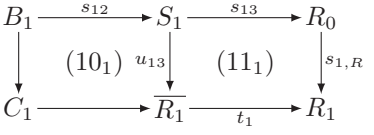


We can then construct pushouts (6_1)–(9_1)

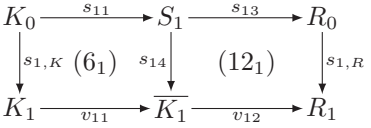


as a decomposition of pushout (3_1) above, which leads to $\overline{L_1}$ and $\overline{K_1}$ of the complement rule, so e_{11} and e_{12} are jointly epimorphic because $(7_1) + (9_1)$ is a pushout.

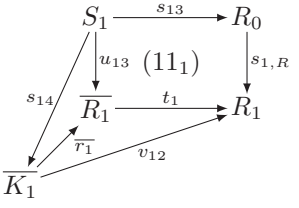
The pushout (4_1) can be decomposed into pushouts (10_1) and (11_1) as in



to give the right-hand side $\overline{R_1}$ of the complement rule. The pullback (2_1) can be decomposed into pushout (6_1) and square (12_1), which is a pullback by Lemma B.2 as shown in



Lemma B.1 now implies that there is a unique morphism $\overline{r_1} : \overline{K_1} \to \overline{R_1}$ in

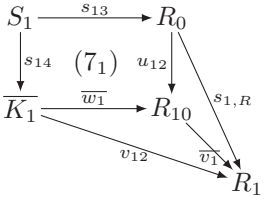


with

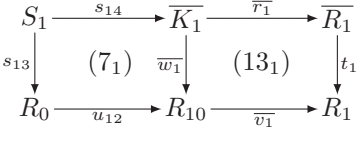
$$\overline{r_1} \circ s_{14} = u_{13}$$
$$t_1 \circ \overline{r_1} = v_{12},$$

and $\overline{r_1} \in \mathcal{M}$.

The pushout (7_1) implies that there is a unique morphism $\overline{v_1} : R_{10} \to R_1$ as shown in



and, by pushout decomposition of $(11_1) = (7_1) + (13_1)$, square (13_1) is a pushout:



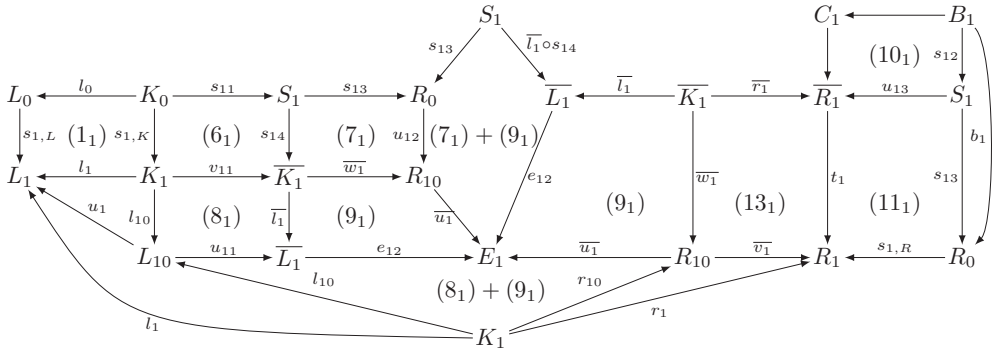
Moreover, $(8_1) + (9_1)$, as a pushout over $\mathcal{M}$-morphisms, is also a pullback, which completes the construction shown below, leading to the required rule

$$\overline{p_1} = \left(\overline{L_1} \xleftarrow{\overline{l_1}} \overline{K_1} \xrightarrow{\overline{r_1}} \overline{R_1} \right)$$

and

$$p_1 = p_0 *_{E_1} \overline{p_1}$$

for rules without application conditions.



For the application conditions, suppose

$$\mathrm{ac}_1 \cong \mathrm{Shift}(s_{1,L}, \mathrm{ac}_0) \wedge \mathrm{L}\left(p_1^*, \mathrm{Shift}(v_1, \mathrm{ac}_1')\right)$$

for

$$p_1^* = (L_1 \xleftarrow{u_1} L_{10} \xrightarrow{v_1} E_1)$$

with

$$v_1 = e_{12} \circ u_{11}$$

and ac_1' on L_{10}. We now define

$$\overline{\mathrm{ac}_1} = \mathrm{Shift}(u_{11}, \mathrm{ac}_1'),$$

which is an application condition on $\overline{L_1}$. We have to show that

$$(p_1, \mathrm{ac}_{p_0 *_{E_1} \overline{p_1}}) \cong (p_1, \mathrm{ac}_1).$$

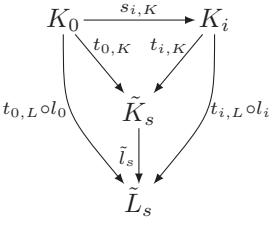
By construction of the E_1-concurrent rule,

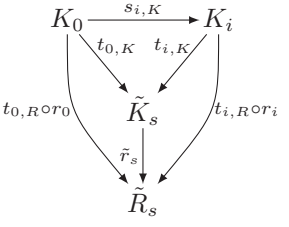
$$\begin{aligned}
\mathrm{ac}_{p_0 *_{E_1} \overline{p_1}} &\cong \mathrm{Shift}(s_{1,L}, \mathrm{ac}_0) \wedge \mathrm{L}(p_1^*, \mathrm{Shift}(e_{12}, \overline{\mathrm{ac}_1})) \\
&\cong \mathrm{Shift}(s_{1,L}, \mathrm{ac}_0) \wedge \mathrm{L}(p_1^*, \mathrm{Shift}(e_{12}, \mathrm{Shift}(u_{11}, \mathrm{ac}_1'))) \\
&\cong \mathrm{Shift}(s_{1,L}, \mathrm{ac}_0) \wedge \mathrm{L}(p_1^*, \mathrm{Shift}(e_{12} \circ u_{11}, \mathrm{ac}_1')) \\
&\cong \mathrm{Shift}(s_{1,L}, \mathrm{ac}_0) \wedge \mathrm{L}(p_1^*, \mathrm{Shift}(v_1, \mathrm{ac}_1')) \\
&\cong \mathrm{ac}_1.
\end{aligned}$$

□

A.2. *Proof of Fact 5.2*

Proof. We will begin by showing the well definedness of the morphisms $\tilde{l}_s$ and $\tilde{r}_s$:





Consider the colimits

$$\begin{array}{lll}(\tilde{L}_s, (t_{i,L})_{i=0,\ldots,n}) & \text{of} & (s_{i,L})_{i=1,\ldots,n} \\ (\tilde{K}_s, (t_{i,K})_{i=0,\ldots,n}) & \text{of} & (s_{i,K})_{i=1,\ldots,n} \\ (\tilde{R}_s, (t_{i,R})_{i=0,\ldots,n}) & \text{of} & (s_{i,R})_{i=1,\ldots,n},\end{array}$$

with

$$t_{0,*} = t_{i,*} \circ s_{i,*}$$

for $* \in \{L, K, R\}$ in the right-hand digram above. Since

$$t_{i,L} \circ l_i \circ s_{i,K} = t_{i,L} \circ s_{i,L} \circ l_0 = t_{0,L} \circ l_0,$$

we get an induced morphism $\tilde{l}_s : \tilde{K}_s \to \tilde{L}_s$ with

$$\tilde{l}_s \circ t_{i,K} = t_{i,L} \circ l_i$$

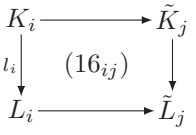
for $i = 0, \ldots, n$. Similarly, we obtain $\tilde{r}_s : \tilde{K}_s \to \tilde{R}_s$ with

$$\tilde{r}_s \circ t_{i,K} = t_{i,R} \circ r_i$$

for $i = 0, \ldots, n$. The colimit of a bundle of n morphisms can be constructed by iterated pushout constructions, which means that we only have to require pushouts over $\mathcal{M}$-morphisms. Since pushouts are closed under $\mathcal{M}$-morphisms, the iterated pushout construction leads to $t_i \in \mathcal{M}$.

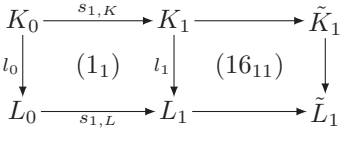
It remains to show that (14_i) and $(14_i) + (1_i)$, and (15_i) and $(15_i) + (2_i)$ in Definition 5.1 are pullbacks, and (14_i) and $(14_i) + (1_i)$ have a pushout complement for $t_{i,L} \circ l_i$. We will prove this by induction over j for the (14_i) and $(14_i)+(1_i)$ case only; the pullback property for (15_i) follows analogously.

Let $\tilde{L}_j$ and $\tilde{K}_j$ be the colimits of $(s_{i,L})_{i=1,\ldots,j}$ and $(s_{i,K})_{i=1,\ldots,j}$, respectively. We need to prove that (16_{ij}) in



is a pullback with the pushout complement property for all $i = 0, \ldots, j$.

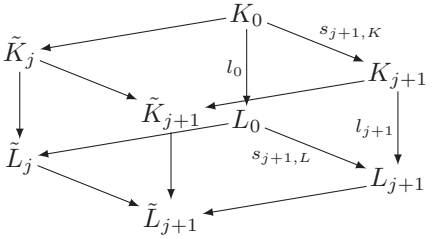
— *Base case* $(j = 1)$:
The colimits of $s_{1,L}$ and $s_{1,K}$ are L_1 and K_1, respectively, which means that $(16_{01}) = (1)_1 + (16_{11})$ and (16_{11}) are both pushouts and pullbacks:



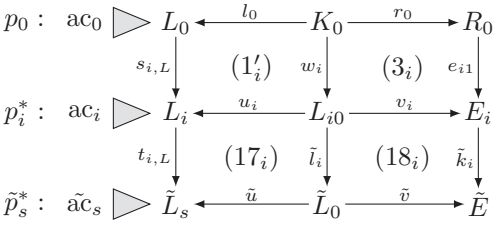
— *Induction step* $(j \to j+1)$:
We construct

$$\tilde{L}_{j+1} = \tilde{L}_j +_{L_0} L_{j+1}$$
$$\tilde{K}_{j+1} = \tilde{K}_j +_{K_0} K_{j+1}$$

as pushouts in the cube



The top and bottom faces are pushouts, the back faces are pullbacks, and, by the van Kampen property, the front faces are also pullbacks. Moreover, by Lemma B.3, the front faces have the pushout complement property, and, by Lemma B.4, this also holds for (16_{0j}) and (16_{ij}) as compositions.
Thus, for a given n, (16_{in}) is the required pullback (14_i) and $(14_i)+(1_i)$ with the pushout complement property using $\tilde{K}_n = \tilde{K}_s$ and $\tilde{L}_n = \tilde{L}_s$.
Moreover, we have pushout complements (17_i) and $(17_i) + (1'_i)$ for $t_{i,L} \circ l_i$ as in



Since ac_0 and ac_i are complement-compatible for all i, we have

$$\mathrm{ac}_i \cong \mathrm{Shift}(s_{i,L}, \mathrm{ac}_0) \wedge \mathrm{L}\left(p_i^*, \mathrm{Shift}\left(v_i, \mathrm{ac}'_i\right)\right).$$

For any ac'_i, we have

$$\begin{aligned}\mathrm{Shift}(t_{i,L}, \mathrm{L}(p_i^*, \mathrm{Shift}(v_i, \mathrm{ac}'_i))) &\cong \mathrm{L}\left(\tilde{p}_s^*, \mathrm{Shift}\left(\tilde{k}_i \circ v_i, \mathrm{ac}'_i\right)\right)\\ &\cong \mathrm{L}\left(\tilde{p}_s^*, \mathrm{Shift}\left(\tilde{v}, \mathrm{Shift}\left(\tilde{l}_i, \mathrm{ac}'_i\right)\right)\right)\end{aligned}$$

since all squares are pushouts by pushout–pullback decomposition and the uniqueness of pushout complements. We define

$$\mathrm{ac}_i^* := \mathrm{Shift}\left(\tilde{l}_i, \mathrm{ac}'_i\right)$$

as an application condition on $\tilde{L}_0$. It then follows that

$$\begin{aligned}\tilde{\text{ac}}_s &= \bigwedge_{i=1,\dots,n} \text{Shift}(t_{i,L}, \text{ac}_i)\\ &\cong \bigwedge_{i=1,\dots,n} (\text{Shift}(t_{i,L} \circ s_{i,L}, \text{ac}_0) \wedge \text{Shift}(t_{i,L}, \text{L}(p_i^*, \text{Shift}(v_i, \text{ac}_i'))))\\ &\cong \text{Shift}(t_{0,L}, \text{ac}_0) \wedge \bigwedge_{i=1,\dots,n} \text{L}(\tilde{p}_s^*, \text{Shift}(\tilde{v}, \text{ac}_i^*)).\end{aligned}$$

For $i = 0$, we define

$$\text{ac}_{s0}' = \bigwedge_{j=1,\dots,n} ac_j^*,$$

so

$$\tilde{\text{ac}}_s = \text{Shift}(t_{0,L}, \text{ac}_0) \wedge \text{L}(\tilde{p}_s^*, \text{Shift}\left(\tilde{v}, \text{ac}_{s0}'\right))$$

implies the complement-compatibility of ac_0 and $\tilde{\text{ac}}_s$.

For $i > 0$, we have

$$\text{Shift}(t_{0,L}, \text{ac}_0) \wedge \text{L}(\tilde{p}_s^*, \text{Shift}(\tilde{v}, \text{ac}_i^*)) \cong \text{Shift}(t_{i,L}, \text{ac}_i).$$

We define

$$\text{ac}_{si}' = \bigwedge_{j=1,\dots,n\setminus i} ac_j^*,$$

so

$$\tilde{\text{ac}}_s = \text{Shift}(t_{i,L}, \text{ac}_i) \wedge \text{L}(\tilde{p}_s^*, \text{Shift}(\tilde{v}, \text{ac}_{si}'))$$

implies the complement-compatibility of ac_i and $\tilde{\text{ac}}_s$. □

A.3. *Proof of Fact 5.8*

Proof. From Fact 4.8, each single direct transformation $G \overset{p_i,m_i}{\Longrightarrow} G_i$ can be decomposed into a transformation

$$G \overset{p_0,m_0^i}{\Longrightarrow} G_0^i \overset{\overline{p_i},\overline{m_i}}{\Longrightarrow} G_i$$

with

$$m_0^i = m_i \circ s_{i,L},$$

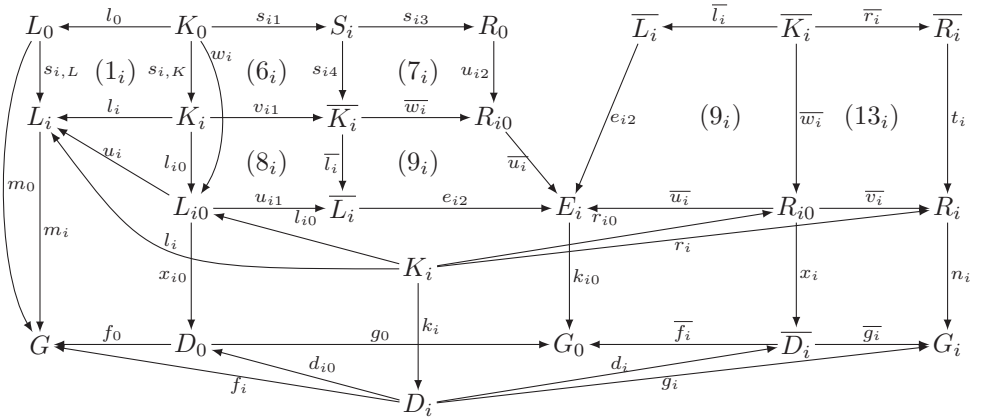
and since the bundle is s-amalgamable,

$$m_0 = m_i \circ s_{i,L} = m_0^i$$

and $G_0 := G_0^i$ for all $i = 1, \dots, n$.

We now have to show the pairwise parallel independence.

From the constructions of the complement rule and the Concurrency Theorem, we obtain the following diagram for all $i = 1, \ldots, n$:



For $i \neq j$, the weakly independent matches mean we have a morphism $p_{ij} : L_{i0} \to D_j$ with

$$f_j \circ p_{ij} = m_i \circ u_i.$$

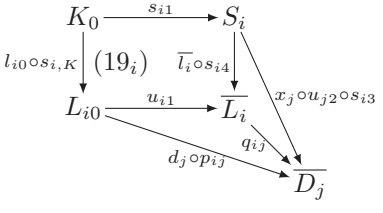
It follows that

$$\begin{aligned} f_j \circ p_{ij} \circ w_i &= m_i \circ u_i \circ w_i \\ &= m_i \circ s_{i,L} \circ l_0 \\ &= m_0 \circ l_0 \\ &= m_j \circ s_{j,L} \circ l_0 \\ &= m_j \circ u_j \circ w_j \\ &= m_j \circ u_j \circ l_{j0} \circ s_{j,K} \\ &= m_j \circ l_j \circ s_{j,K} \\ &= f_j \circ k_j \circ s_{j,K}, \end{aligned}$$

and with $f_j \in \mathcal{M}$, we have

$$p_{ij} \circ w_i = k_j \circ s_{jk}. \qquad (*)$$

Now consider the pushout $(19_i) = (6_i) + (8_i)$ in comparison with object $\overline{D_j}$ and morphisms $d_j \circ p_{ij}$ and $x_j \circ u_{j2} \circ s_{i3}$ as shown below:



We have

$$\begin{aligned} d_j \circ p_{ij} \circ l_{i0} \circ s_{i,K} &= d_j \circ p_{ij} \circ w_i \\ &= d_j \circ k_j \circ s_{j,K} && \text{(by } *\text{)} \\ &= x_j \circ r_{j0} \circ s_{j,K} \\ &= x_j \circ \overline{w_j} \circ v_{j1} \circ s_{j,K} \\ &= x_j \circ u_{j2} \circ s_{j3} \circ s_{j1} \\ &= x_j \circ u_{j2} \circ r_0 \\ &= x_j \circ u_{j2} \circ s_{i3} \circ s_{i1}. \end{aligned}$$

Now, pushout (19_i) induces a unique morphism q_{ij} with

$$\begin{aligned} q_{ij} \circ u_{i1} &= d_j \circ p_{ij} \\ q_{ij} \circ \overline{l_i} \circ s_{i4} &= x_j \circ u_{j2} \circ s_{i3}. \end{aligned}$$

For the parallel independence of

$$\begin{aligned} G_0 &\overset{\overline{p_i},\overline{m_i}}{\Longrightarrow} G_i \\ G_0 &\overset{\overline{p_j},\overline{m_j}}{\Longrightarrow} G_j, \end{aligned}$$

we have to show that $q_{ij} : \overline{L_i} \to \overline{D_j}$ satisfies

$$\overline{f_j} \circ q_{ij} = k_{i0} \circ e_{i2} =: \overline{m_i}.$$

With $f_0 \in \mathcal{M}$ and

$$\begin{aligned} f_0 \circ d_{j0} \circ p_{ij} &= f_j \circ p_{ij} \\ &= m_i \circ u_i \\ &= f_0 \circ x_{i0} \end{aligned}$$

it follows that

$$d_{j0} \circ p_{ij} = x_{i0}. \qquad (**)$$

This means that

$$\begin{aligned} \overline{f_j} \circ q_{ij} \circ u_{i1} &= \overline{f_j} \circ d_j \circ p_{ij} \\ &= g_0 \circ d_0 \circ p_{ij} && \text{(by } **\text{)} \\ &= g_0 \circ x_{i0} \\ &= k_{i0} \circ e_{i2} \circ u_{i1}. \end{aligned}$$

We also have

$$\begin{aligned} \overline{f_j} \circ q_{ij} \circ \overline{l_i} \circ s_{i4} &= \overline{f_j} \circ x_j \circ u_{j2} \circ s_{i3} \\ &= k_{j0} \circ \overline{u_j} \circ u_{j2} \circ s_{i3} \\ &= k_{i0} \circ \overline{u_i} \circ u_{i2} \circ s_{i3} \\ &= k_{i0} \circ e_{i2} \circ \overline{l_i} \circ s_{i4}. \end{aligned}$$

Since (19_i) is a pushout, u_{i1} and $\overline{l_i} \circ s_{i4}$ are jointly epimorphic, so

$$\overline{f_j} \circ q_{ij} = k_{i0} \circ e_{i2}.$$

If ac_0 and ac_i are not complement-compatible, then $\overline{\mathrm{ac}_i} = \mathrm{true}$ and, trivially,

$$\overline{g_j} \circ q_{ij} \models \overline{\mathrm{ac}_i}$$

for all $j \neq i$. Otherwise, we have

$$g_j \circ p_{ij} \models \mathrm{ac}'_i,$$

and with

$$\begin{aligned} g_j \circ p_{ij} &= \overline{g_j} \circ d_j \circ p_{ij} \\ &= \overline{g_j} \circ q_{ij} \circ u_{i1} \end{aligned}$$

it follows that

$$\overline{g_j} \circ q_{ij} \circ u_{i1} \models \mathrm{ac}'_i,$$

which is equivalent to

$$\overline{g_j} \circ q_{ij} \models \mathrm{Shift}(u_{i1}, \mathrm{ac}'_1) = \overline{ac_i}.$$ □

A.4. *Proof of Theorem 5.9*

Proof.

(1) *Synthesis*:

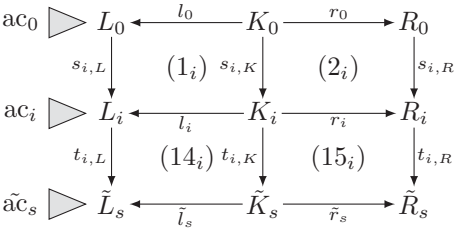
We have to show that $\tilde{p}_s$ is applicable to G leading to an amalgamated transformation $G \overset{\tilde{p}_s, \tilde{m}}{\Longrightarrow} H$ with $m_i = \tilde{m} \circ t_{i,L}$, where $t_i : p_i \to \tilde{p}_i$ is the kernel morphism constructed in Fact 5.2.

Then we can apply Fact 4.8, which implies the decomposition of $G \overset{\tilde{p}_s, \tilde{m}}{\Longrightarrow} H$ into

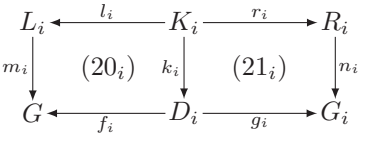
$$G \overset{p_i, m_i}{\Longrightarrow} G_i \overset{q_i}{\Longrightarrow} H,$$

where q_i is the (weak) complement rule of the kernel morphism t_i.

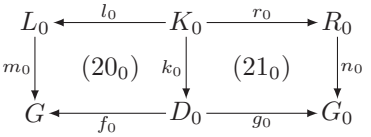
Given the kernel morphisms, the amalgamated rule and the bundle of direct transformations, we have the pullbacks (1_i), (2_i), (14_i), (15_i) (see Definition 5.1)



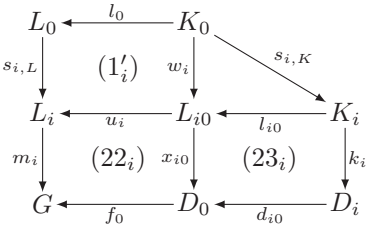
and the pushouts (20_i), (21_i) (see proof of Fact 6.5) on the right



Using Fact 5.8, we know that we can apply p_0 *via* m_0 to give a direct transformation $G \overset{p_0,m_0}{\Longrightarrow} G_0$ given by the pushouts (20_0) and (21_0):



Moreover, we can find decompositions of pushouts (20_0) and (20_i) into pushouts $(1'_i)$ and (22_i), and (22_i) and (23_i), respectively, by $\mathcal{M}$-pushout–pullback decomposition and the uniqueness of pushout complements as in

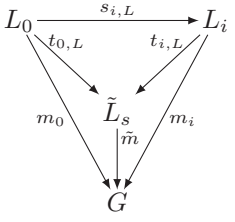


Since we have consistent matches,

$$m_i \circ s_{i,L} = m_0$$

for all $i = 1, \ldots, n$. The colimit $\tilde{L}_s$ then implies that there is a unique morphism $\tilde{m} : \tilde{L}_s \to G$ with

$$\tilde{m} \circ t_{i,L} = m_i$$
$$\tilde{m} \circ t_{0,L} = m_0.$$



Moreover,

$$\begin{aligned} m_i \models \mathrm{ac}_i &\Rightarrow \tilde{m} \circ t_{i,L} \models \mathrm{ac}_i \\ &\Rightarrow \tilde{m} \models \mathrm{Shift}(t_{i,L}, \mathrm{ac}_i) \end{aligned}$$

for all $i = 1, \ldots, n$, so

$$\tilde{m} \models \tilde{\mathrm{ac}}_s = \bigwedge_{i=1,\ldots,n} \mathrm{Shift}(t_{i,L}, \mathrm{ac}_i).$$

The fact that we have weakly independent matches means that there exist morphisms p_{ij} with

$$f_j \circ p_{ij} = m_i \circ u_i$$

for $i \neq j$. We now construct D as the limit of $(d_{i0})_{i=1,\ldots,n}$ with morphisms d_i. Now f_0 being a monomorphism with

$$\begin{aligned} f_0 \circ d_{i0} \circ p_{ji} &= f_i \circ p_{ji} \\ &= m_j \circ u_j \\ &= f_0 \circ x_{j0} \end{aligned}$$

implies that

$$d_{i0} \circ p_{ji} = x_{j0},$$

so

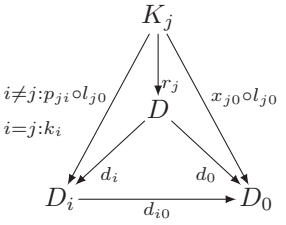
$$d_{i0} \circ p_{ji} \circ l_{j0} = x_{j0} \circ l_{j0},$$

and, together with

$$d_{i0} \circ k_i = x_{i0} \circ l_{i0},$$

limit D then implies that there exists a unique morphism r_j with

$$\begin{aligned} d_i \circ r_j &= p_{ji} \circ l_{j0} \\ d_i \circ r_i &= k_i \\ d_0 \circ r_j &= x_{j0} \circ l_{j0}. \end{aligned}$$



Similarly, f_j being a monomorphism with

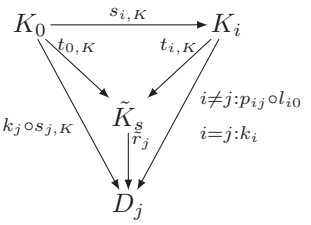
$$\begin{aligned} f_j \circ p_{ij} \circ l_{i0} \circ s_{i,K} &= m_i \circ u_i \circ w_i \\ &= m_i \circ s_{i,L} \circ l_0 \\ &= m_0 \circ l_0 \\ &= m_j \circ s_{j,L} \circ l_0 \\ &= m_j \circ l_j \circ s_{j,K} \\ &= f_j \circ k_j \circ s_{j,K} \end{aligned}$$

implies that

$$p_{ij} \circ l_{i0} \circ s_{i,K} = k_j \circ s_{j,K}.$$

Now, colimit $\tilde{K}_s$ implies that there is a unique morphisms $\tilde{r}_j$ with

$$\begin{aligned} \tilde{r}_j \circ t_{i,K} &= p_{ij} \circ l_{i0} \\ \tilde{r}_j \circ t_{j,K} &= k_j \\ \tilde{r}_j \circ t_{0,K} &= k_j \circ s_{j,K}. \end{aligned}$$



Since

$$\begin{aligned} d_{i0} \circ \tilde{r}_i \circ t_{i,K} &= d_{i0} \circ k_i \\ &= q_i \circ l_{i0} \\ &= d_{j0} \circ p_{ij} \circ l_{i0} \\ &= d_{j0} \circ \tilde{r}_j \circ t_{i,K} \end{aligned}$$

and

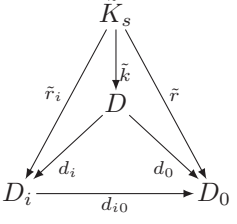
$$\begin{aligned} d_{i0} \circ \tilde{r}_i \circ t_{0,K} &= d_{i0} \circ k_i \circ s_{i,K} \\ &= k_0 \\ &= d_{j0} \circ \tilde{r}_j \circ t_{0,K}, \end{aligned}$$

colimit $\tilde{K}_s$ implies that for all i, j we have

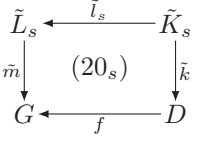
$$d_{i0} \circ \tilde{r}_i = d_{j0} \circ \tilde{r}_j =: \tilde{r}.$$

From limit D, it now follows that there exists a unique morphism $\tilde{k}$ with

$$\begin{aligned} d_i \circ \tilde{k} &= \tilde{r}_i \\ d_0 \circ \tilde{k} &= \tilde{r}. \end{aligned}$$



We now have to show that (20_s) in



with $f = f_0 \circ d_0$ is a pushout.

With

$$\begin{aligned} f \circ \tilde{k} \circ t_{i,K} &= f_0 \circ d_0 \circ \tilde{k} \circ t_{i,K} \\ &= f_0 \circ \tilde{r} \circ t_{i,K} \\ &= f_0 \circ d_{i0} \circ \tilde{r}_i \circ t_{i,K} \\ &= f_0 \circ d_{i0} \circ k_i \\ &= f_i \circ k_i \\ &= m_i \circ l_i \\ &= \tilde{m} \circ t_{i,L} \circ l_i \\ &= \tilde{m} \circ \tilde{l}_s \circ t_{i,K} \end{aligned}$$

and

$$\begin{aligned} f \circ \tilde{k} \circ t_{0,K} &= f_0 \circ d_0 \circ \tilde{k} \circ t_{0,K} \\ &= f_0 \circ \tilde{r} \circ t_{0,K} \\ &= f_0 \circ d_{i0} \circ \tilde{r}_i \circ t_{0,K} \\ &= f_0 \circ d_{i0} \circ k_i \circ s_{i,K} \\ &= f_0 \circ k_0 = m_0 \circ l_0 \\ &= \tilde{m} \circ t_{0,L} \circ l_0 \\ &= \tilde{m} \circ \tilde{l}_s \circ t_{0,K} \end{aligned}$$

and $\tilde{K}_s$ being a colimit, it follows that

$$f \circ \tilde{k} = \tilde{m} \circ \tilde{l}_s,$$

so the square commutes.

Pushout (23_i) can be decomposed into pushouts (24_i) and (25_i) in

$$\begin{array}{ccccc} K_i & \xrightarrow{r_i} & D & \xrightarrow{d_i} & D_i \\ {\scriptstyle l_{i0}}\downarrow & (24_i) & {\scriptstyle x_i}\downarrow & (25_i) & \downarrow{\scriptstyle d_{i0}} \\ L_{i0} & \xrightarrow[x_{i0}]{} & P_i & \xrightarrow[y_{i0}]{} & D_0 \end{array}$$

Using Lemma B.5, it follows that D_0 is the colimit of $(x_i)_{i=1,\ldots,n}$, because (23_i) is a pushout, D is the limit of $(d_{i0})_{i=1,\ldots,n}$, and we have morphisms p_{ij} with $d_{j0} \circ p_{ij} = q_i$. Lemma B.6 then implies that (25) in

$$\begin{array}{ccc} +K_i & \xrightarrow{+l_{i0}} & +L_{i0} \\ {\scriptstyle r}\downarrow & (25) & \downarrow{\scriptstyle \overline{d}} \\ D & \xrightarrow[d_0]{} & D_0 \end{array}$$

is also a pushout, where $+$ represents the coproduct construction with index $i = 1, \ldots, n$ with injections ι_{K_i} and $\iota_{L_{i0}}$, respectively.

Consider the n-ary coequalisers:

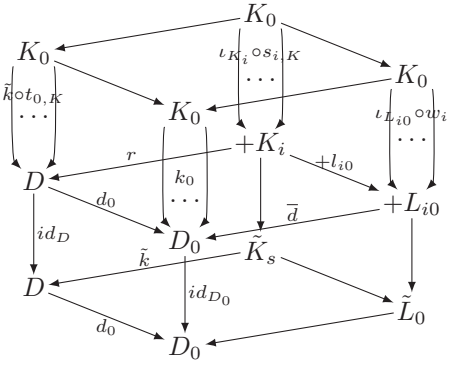
— $\tilde{K}_s$ of $(\iota_{K_i} \circ s_{i,K} : K_0 \to +K_i)_{i=1,\dots,n}$
(which is actually $\tilde{K}_s$ by construction of colimits);

— $\tilde{L}_0$ of $(iota_{L_{i0}} \circ w_i : K_0 \to +L_{i0})_{i=1,\dots,n}$
(as already constructed in Fact 5.2);

— D of $(\tilde{k} \circ t_{0,K} : K_0 \to D)_{i=1,\dots,n}$;

— D_0 of $(k_0 : K_0 \to D_0)_{i=1,\dots,n}$.

In the cube

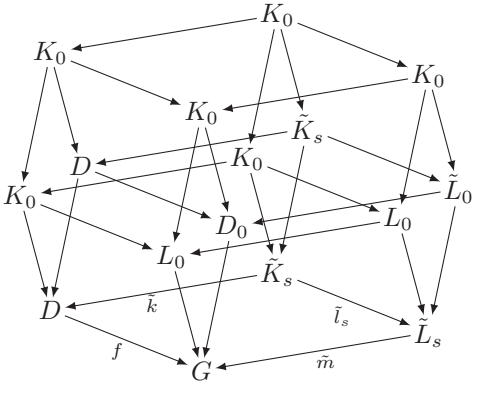


the top square with identical morphisms is a pushout, the top cube commutes and the middle square is pushout (25) from above. Using Lemma B.7, it follows that the bottom square

$$\begin{array}{ccc} \tilde{K}_s & \longrightarrow & \tilde{L}_0 \\ {\scriptstyle \tilde{k}}\downarrow & (26) & \downarrow \\ D & \xrightarrow[d_0]{} & D_0 \end{array}$$

constructed of the four coequalisers is a pushout too.

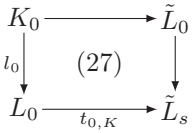
Now consider the cube



where the top and middle squares are pushouts and the two top cubes commute. Using Lemma B.7 again, it follows that (20_s) in the bottom is actually a pushout, where

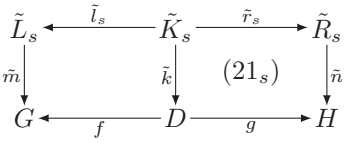
$$(27) = (1'_i) + (17_i)$$

is a pushout by composition:



We can now construct pushout (21_s), which completes the direct transformation

$$G \overset{\tilde{p}_s,\tilde{m}}{\Longrightarrow} H.$$



(2) *Analysis*:
Using the kernel morphisms t_i, we obtain transformations

$$G \overset{p_i,m_i}{\Longrightarrow} G_i \overset{q_i}{\Longrightarrow} H$$

from Fact 4.8 with

$$m_i = \tilde{m} \circ t_{i,L}.$$

We have to show that this bundle of transformation is s-amalgamable. Applying Fact 4.8 again, we obtain transformations

$$G \overset{p_0,m_0^i}{\Longrightarrow} G_0^i \overset{\overline{p_i}}{\Longrightarrow} G_i$$

with

$$m_0^i = m_i \circ s_{i,L}.$$

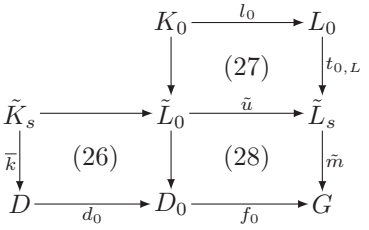
It follows that

$$\begin{aligned} m_0^i &= m_i \circ s_{i,L} \\ &= \tilde{m} \circ t_{i,L} \circ s_{i,L} \\ &= \tilde{m} \circ t_{0,L} \\ &= \tilde{m} \circ t_{j,L} \circ s_{j,L} \\ &= m_j \circ s_{j,L}, \end{aligned}$$

sos we have consistent matches with $m_0 := m_0^i$ well defined and $G_0 = G_0^i$.

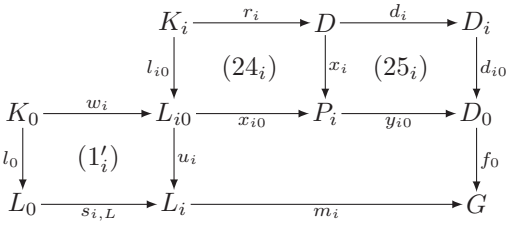
We still need to show the weakly independent matches. Given the above transformations, we have pushouts (20_0), (20_i) and (20_s) as above. We can find decompositions of (20_0)

and (20_s) into pushouts (27) + (28) and (26) + (28), respectively:

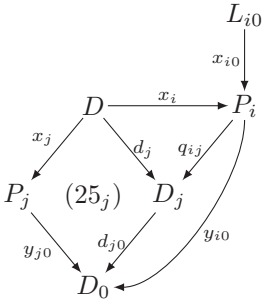


Using pushout (26) and Lemma B.8, it follows that (25) as above is a pushout since $\tilde{K}_s$ is the colimit of $(s_{i,L})_{i=1,\dots,n}$ and $\tilde{L}_0$ is the colimit of $(w_i)_{i=1,\dots,n}$, and id_{K_0} is obviously an epimorphism.

Lemma B.6 now implies that there is a decomposition into pushouts (24_i) with colimit D_0 of $(x_i)_{i=1,\dots,n}$ and pushout (25_i) by the $\mathcal{M}$-pushout–pullback decomposition



Since D_0 is the colimit of $(x_i)_{i=1,\dots,n}$ and (25_j) is a pushout, it follows that D_j is the colimit of $(x_i)_{i=1,\dots,j-1,j+1,\dots,n}$ with morphisms $q_{ij} : P_i \to D_j$ and $d_{j0} \circ q_{ij} = y_{i0}$:



Hence, we obtain for all $i \neq j$, a morphism

$$p_{ij} = q_{ij} \circ x_{i0}$$

and

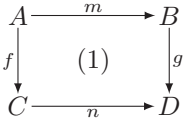
$$\begin{aligned} f_j \circ p_{ij} &= f_0 \circ d_{j0} \circ q_{ij} \circ x_{i0} \\ &= f_0 \circ y_{i0} \circ x_{i0} \\ &= m_i \circ u_i. \end{aligned}$$

(3) *Bijective correspondence*:
Because of the uniqueness of the constructions used, the above constructions are inverse to each other up to isomorphism. □

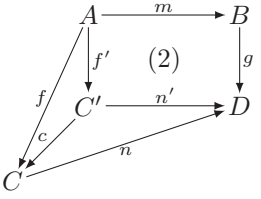
Appendix B. Additional lemmas

The following lemmas are valid in all adhesive and $\mathcal{M}$-adhesive categories, and are used in the proofs of the main theorems: Lemmas B.1 and B.2 are used in the proof of Theorem 4.4; Lemmas B.3 and B.4 are used in the proof of Fact 5.2; and Lemmas B.5, B.6, B.7 and B.8 are used in the proof of Theorem 5.9.

Lemma B.1 ($\mathcal{M}$ complement property). If

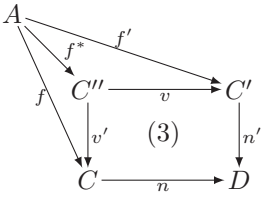


is a pushout and



is a pullback, and $n' \in \mathcal{M}$, then there exists a unique morphism $c : C' \to C$ such that $c \circ f' = f$, $n \circ c = n'$ and $c \in \mathcal{M}$.

Proof. Since (2) is a pullback, $n' \in \mathcal{M}$ implies that $m \in \mathcal{M}$, and then $n \in \mathcal{M}$ also because (1) is a pushout. We construct the pullback



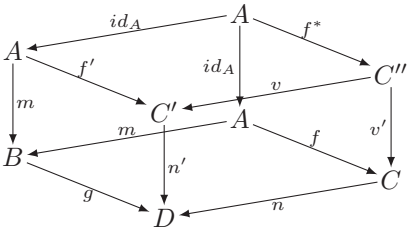
with $v, v' \in \mathcal{M}$, and since

$$n' \circ f = g \circ m = n \circ f,$$

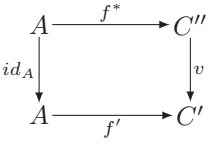
there is a unique morphism $f^* : A \to C''$ with

$$v \circ f^* = f'$$
$$v' \circ f^* = f.$$

Now consider the cube



where the bottom face is pushout (1), the back left face is a pullback because $m \in \mathcal{M}$, the front left face is pullback (2) and the front right face is pullback (3). Now, by pullback composition and decomposition, the back right face is a pullback too, and the VK property then implies that the top face is a pushout. Since



is a pushout, and pushout objects are unique up to isomorphism, this implies that v is an isomorphism and $C'' \cong C'$. We now define $c := v' \circ v^{-1}$ and have

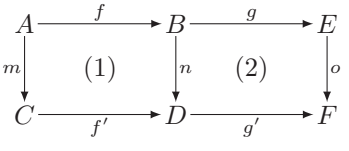
$$\begin{aligned} c \circ f' &= v' \circ v^{-1} \circ f' \\ &= v' \circ f^* \\ &= f \end{aligned}$$

and

$$\begin{aligned} n \circ c &= n \circ v' \circ v^{-1} \\ &= n', \end{aligned}$$

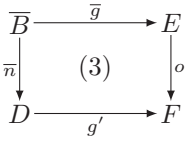
and $c \in \mathcal{M}$ by decomposition of $\mathcal{M}$-morphisms. □

Lemma B.2 ($\mathcal{M}$ pullback-pushout decomposition). Consider



If (1) + (2) is a pullback, (1) is a pushout, (2) commutes and $o \in \mathcal{M}$, then (2) is a pullback too.

Proof. With $o \in \mathcal{M}$ and the fact that (1) + (2) is a pullback and (1) is a pushout, we have that $m, n \in \mathcal{M}$. We construct the pullback

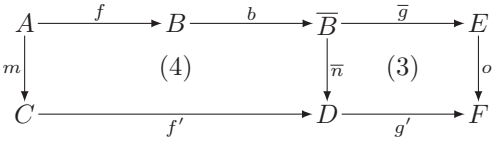


of o and g'. It then follows that $\overline{n} \in \mathcal{M}$ and we get an induced morphism $b : B \to \overline{B}$ with

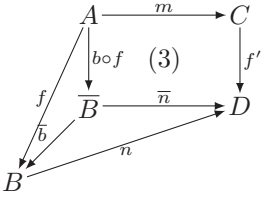
$$\overline{g} \circ b = g$$
$$\overline{n} \circ b = n,$$

and, by decomposition of $\mathcal{M}$-morphisms, $b \in \mathcal{M}$.

By pullback decomposition, (4) is a pullback too:



So we can apply Lemma B.1 with pushout (1) and $\overline{n} \in \mathcal{M}$ to obtain a unique morphism $\overline{b} \in \mathcal{M}$ with $n \circ \overline{b} = \overline{n}$ and $\overline{b} \circ b \circ f = f$:



Now $n \in \mathcal{M}$ and

$$n \circ \overline{b} \circ b = \overline{n} \circ b = n$$

implies that

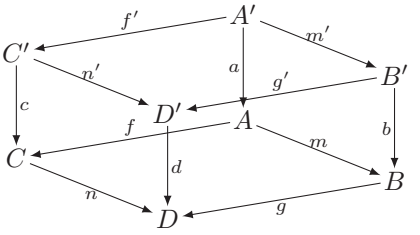
$$\overline{b} \circ b = id_B,$$

and, similarly, $\overline{n} \in \mathcal{M}$ and

$$\overline{n} \circ b \circ \overline{b} = n \circ \overline{b} = \overline{n}$$

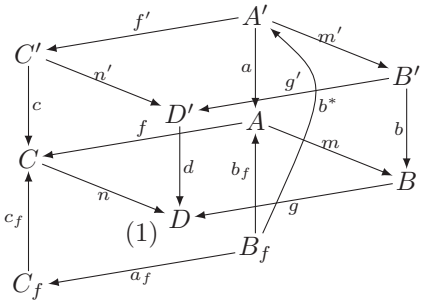
implies that $b \circ \overline{b} = id_{\overline{B}}$, which means that B and $\overline{B}$ are isomorphic, so (2) is a pullback too. □

Lemma B.3. Given the commutative cube

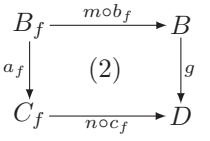


with the bottom face a pushout, the front right face has a pushout complement over $g \circ b$ if the back left face has a pushout complement over $f \circ a$.

Proof. We construct the initial pushout (1) over f:



Since the back left face has a pushout complement, there is a morphism $b^* : B_f \to A'$ such that $a \circ b^* = b_f$. Since the bottom face is a pushout, the square

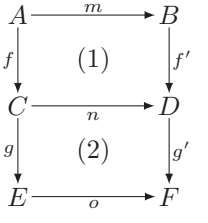


as the composition, is the initial pushout over g. Now

$$\begin{aligned} b \circ m' \circ b^* &= m \circ a \circ b^* \\ &= m \circ b_f, \end{aligned}$$

so the pushout complement of $g \circ b$ exists. □

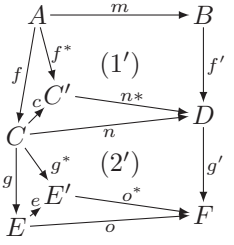
Lemma B.4. If we are given pullbacks (1) and (2)



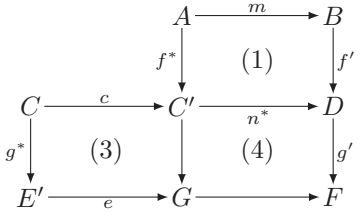
with pushout complements over $f' \circ m$ and $g' \circ n$, respectively, then (1) + (2) also has a pushout complement over $(g' \circ f') \circ m$.

Proof. Let C' and E' be the pushout complements of (1) and (2), respectively. By Lemma B.1, there are morphisms c and e such that

$$c \circ f = f^*$$
$$n^* \circ c = n$$
$$e \circ g = g*$$
$$o^* \circ e = o.$$



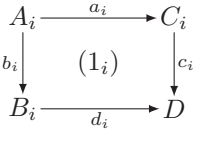
Now (2′) can be decomposed into pushouts (3) and (4):



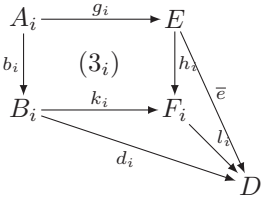
and (1′) + (4) is also a pushout and the pushout complement of $(g' \circ f') \circ m$. □

Lemma B.5. If we are given:

— the pushouts

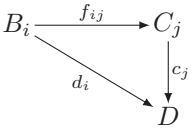


and

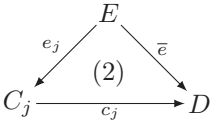


with $b_i \in \mathcal{M}$ for $i = 1, \ldots, n$;

— morphisms $f_{ij} : B_i \to C_j$ with $c_j \circ f_{ij} = d_i$ for all $i \neq j$:



— the limit

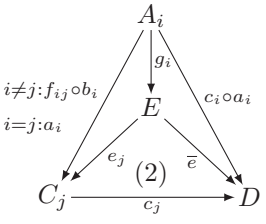


of $(c_j)_{j=1,\ldots,n}$ such that g_i is the induced morphism into E with

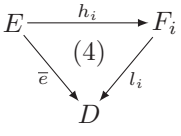
$$e_i \circ g_i = a_i$$
$$e_j \circ g_i = f_{ij} \circ b_i$$

using

$$c_j \circ f_{ij} \circ b_i = d_i \circ b_i$$
$$= c_i \circ a_i,$$



then we have



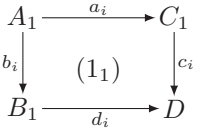
is the colimit of $(h_i)_{i=1,\ldots,n}$, where l_i is the induced morphism from pushout (3_i) compared with

$$\overline{e} \circ g_i = c_i \circ e_i \circ g_i = c_i \circ a_i = d_i \circ b_i.$$

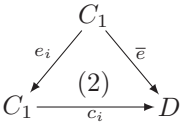
Proof. We use induction over n:

— *Base case* $(n = 1)$:

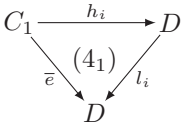
For $n = 1$, we have that C_1 is the limit of c_1, that is, $E = C_1$. It follows that $F_1 = C_1$ for the pushout $(3_1) = (1_1)$, so we have



and



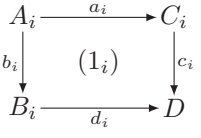
and it is obvious that



is a colimit.

— *Induction step* $(n \to n+1)$:
Consider:

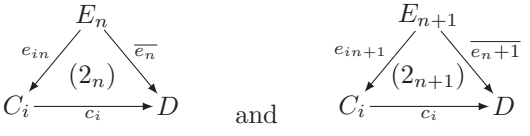
– the pushouts



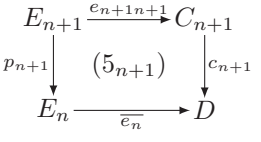
with $b_i \in \mathcal{M}$ for $i = 1, \ldots, n+1$;

– the morphisms $f_{ij} : B_i \to C_j$ with $c_j \circ f_{ij} = d_i$ for all $i \neq j$; and

– the limits

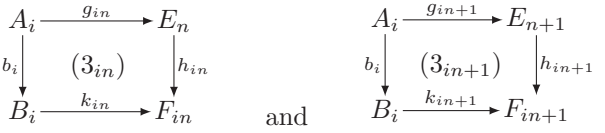


of $(c_i)_{i=1,\ldots,n}$ and $(c_i)_{i=1,\ldots,n+1}$, respectively, leading to the pullback

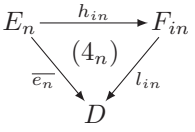


by the construction of limits.

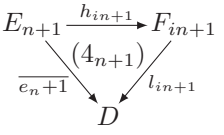
Moreover, g_{in} and g_{in+1} are the induced morphisms into E_n and E_{n+1}, respectively, leading to the pushouts



By the induction hypothesis,



is the colimit of $(h_{in})_{i=1,\ldots,n}$, and we have to show that

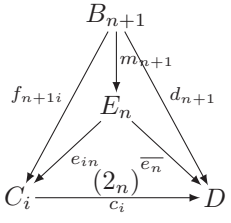


is the colimit of $(h_{in+1})_{i=1,\ldots,n+1}$.
Since (2_n) is a limit and

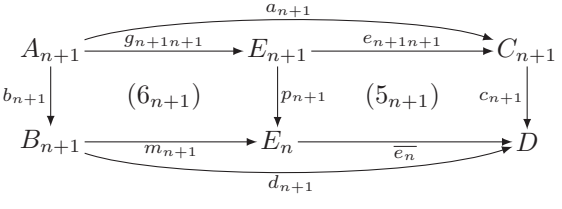
$$c_i \circ f_{n+1i} = d_{n+1}$$

for all $i = 1, \ldots, n$, we obtain a unique morphism m_{n+1} with

$$e_{in} \circ m_{n+1} = f_{n+1i}$$
$$\overline{e_n} \circ m_{n+1} = d_{n+1}.$$



Since (1_{n+1}) is a pushout and (5_{n+1}) is a pullback, by $\mathcal{M}$-pushout–pullback decomposition, (5_{n+1}) and (6_{n+1}) are pushouts too:



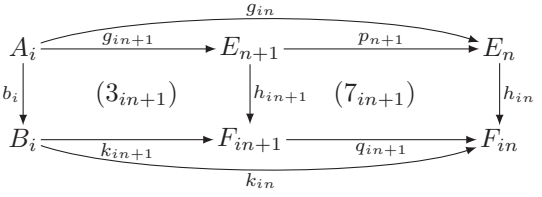
So $F_{n+1n+1} = E_n$. From pushout (3_{in+1}) and

$$h_{in} \circ p_{n+1} \circ g_{in+1} = h_{in} \circ g_{in} = k_{in} \circ b_i$$

we get an induced morphism q_{in+1} with

$$q_{in+1} \circ h_{in+1} = h_{in} \circ p_{n+1}$$
$$q_{in+1} \circ k_{in+1} = k_{in},$$

and from pushout decomposition, (7_{in+1}) is a pushout too:

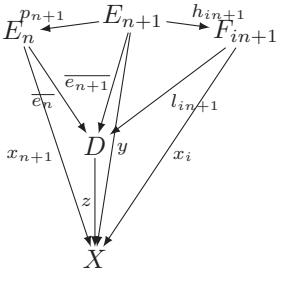


To show that (4_{n+1}) is a colimit, consider an object X and morphisms (x_i) and y with

$$x_i \circ h_{in+1} = y$$

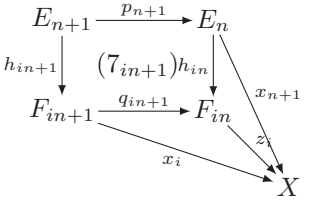
for $i = 1, \ldots, n$ and

$$x_{n+1} \circ p_{n+1} = y.$$



From pushout (7_{in+1}), we obtain a unique morphism z_i with

$$z_i \circ q_{in+1} = x_i$$
$$z_i \circ h_{in} = x_{n+1}.$$



Now, colimit (4_n) induces a unique morphism z with

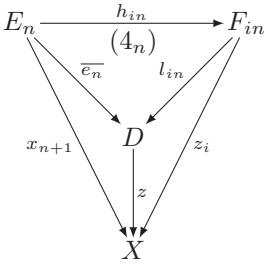
$$z \circ \overline{e_n} = x_{n+1}$$
$$z \circ l_{in} = z_i.$$

It then follows directly that

$$\begin{aligned} z \circ l_{in+1} &= z \circ l_{in} \circ q_{in+1} \\ &= z_i \circ q_{in+1} \\ &= x_i \end{aligned}$$

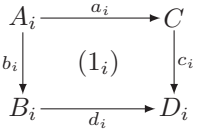
and

$$\begin{aligned} z \circ \overline{e_{n+1}} &= z \circ \overline{e_n} \circ p_{n+1} \\ &= x_{n+1} \circ p_{n+1} \\ &= y. \end{aligned}$$

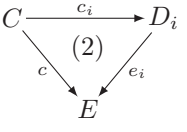


The uniqueness of z then follows directly from the construction, so (4_{n+1}) is the required colimit. □

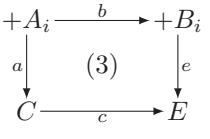
Lemma B.6. Given the diagrams



for $i = 1, \ldots, n$,



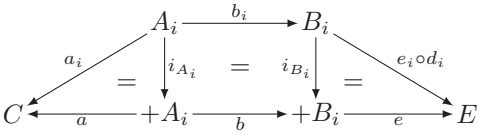
and



with $b = +b_i$, and a and e induced by the coproducts $+A_i$ and $+B_i$, respectively, we have:

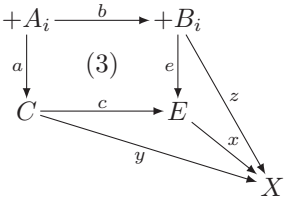
(a) If (1_i) is a pushout and (2) a colimit, then (3) is also a pushout.

(b) If (3) is a pushout, then there is a decomposition into pushout (1_i) and colimit (2) with $e_i \circ d_i = e \circ i_{B_i}$:



Proof.

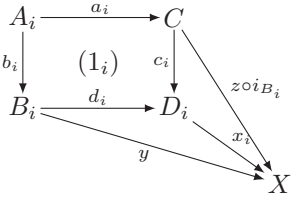
(a) We assume we are given an object X and morphisms y, z with $y \circ a = z \circ b$:



From pushout (1_i), we obtain with

$$\begin{aligned} z \circ i_{B_i} \circ b_i &= z \circ b \circ i_{A_i} \\ &= y \circ a \circ i_{A_i} \\ &= y \circ a_i, \end{aligned}$$

a unique morphism x_i

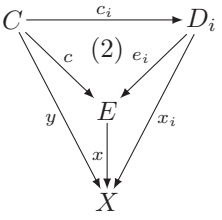


with

$$\begin{aligned} x_i \circ c_i &= y \\ x_i \circ d_i &= z \circ i_{B_i}. \end{aligned}$$

Now colimit (2) implies a unique morphism x with

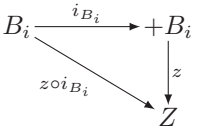
$$\begin{aligned} x \circ c &= y \\ x \circ e_i &= x_i. \end{aligned}$$



It then follows that

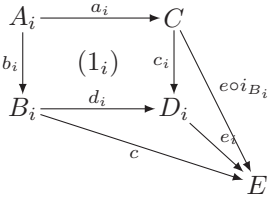
$$\begin{aligned} x \circ e \circ i_{B_i} &= x \circ e_i \circ d_i \\ &= x_i \circ d_i \\ &= z \circ i_{B_i}, \end{aligned}$$

and since z is unique with respect to $z \circ i_{B_i}$, it follows from the coproduct that $z = x \circ e$.



The uiqueness of x follows from the uniqueness of x and x_i, so (3) is a pushout.

(b) We define $a_i := a \circ i_{A_i}$ and construct the pushout



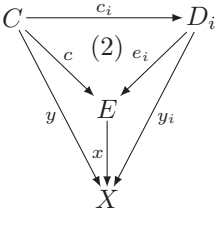
With

$$e \circ i_{B_i} \circ b_i = e \circ b \circ i_{A_i} = c \circ a_i,$$

pushout (1_i) induces a unique morphism e_i with

$$\begin{aligned} e_i \circ d_i &= e \circ i_{B_i} \\ e_i \circ c_i &= c. \end{aligned}$$

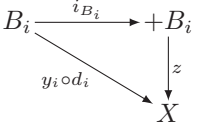
Given an object X and morphisms y and y_i with $y_i \circ c_i = y$,



we obtain a morphism z with

$$z \circ i_{B_i} = y_i \circ d_i$$

from coproduct $+B_i$



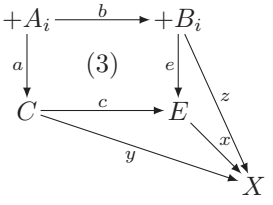
So we have

$$\begin{aligned} y \circ a \circ i_{A_i} &= y_i \circ c_i \circ a_i \\ &= y_i \circ d_i \circ b_i \\ &= z \circ i_{B_i} \circ b_i \\ &= z \circ b \circ i_{A_i}, \end{aligned}$$

and from coproduct $+A_i$, it follows that

$$y \circ a = z \circ b.$$

Now pushout (3) implies a unique morphism x with $x \circ c = y$ and $x \circ e = z$:



From pushout (1_i) and using

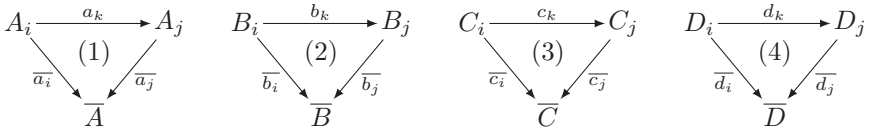
$$\begin{aligned} x \circ e_i \circ d_i &= x \circ e \circ i_{B_i} \\ &= z \circ i_{B_i} \\ &= y_i \circ d_i \end{aligned}$$

and

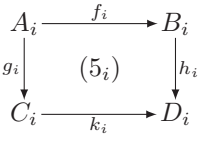
$$\begin{aligned} x \circ e_i \circ c_i &= x \circ c \\ &= y \\ &= y_i \circ c_i, \end{aligned}$$

it then follows that $x \circ e_i = y_i$, so (2) is a colimit. □

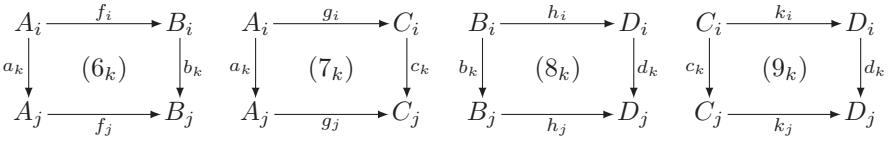
Lemma B.7. Consider the colimits



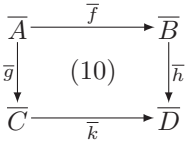
such that



is a pushout for all $i = 1,\ldots,n$ and



commute for all $k = 1,\ldots,m$. Then



is a pushout too.

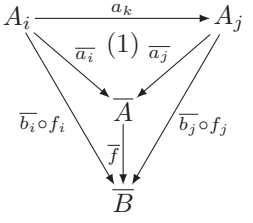
Proof. The morphisms $\overline{f}$, $\overline{g}$, $\overline{h}$ and $\overline{k}$ are uniquely induced by the colimits. We will just show the case for the morphism $\overline{f}$ as an example.

From colimit (1), with

$$\overline{b_j} \circ f_j \circ a_k = \overline{b}_j \circ b_k \circ f_i = \overline{b_i} \circ f_i,$$

we obtain a unique morphism $\overline{f}$ with

$$\overline{f} \circ \overline{a_i} = \overline{b_i} \circ f_i.$$

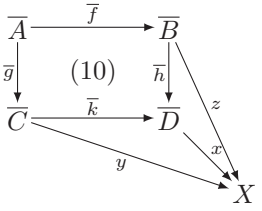


It then follows directly that

$$\overline{k} \circ \overline{h} = \overline{h} \circ \overline{f}.$$

Now consider an object X and morphisms y and z with

$$y \circ \overline{g} = z \circ \overline{f}.$$

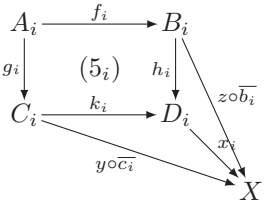


From pushout (5_i) with

$$\begin{aligned} y \circ \overline{c_i} \circ g_i &= y \circ \overline{g} \circ \overline{a_i} \\ &= z \circ \overline{f} \circ \overline{a_i} \\ &= z \circ \overline{b_i} \circ f_i, \end{aligned}$$

we obtain a unique morphism x_i with

$$\begin{aligned} x_i \circ k_i &= y \circ \overline{c_i} \\ x_i \circ h_i &= z \circ \overline{b_i}. \end{aligned}$$



For all $k = 1, \ldots, m$, we have

$$\begin{aligned} x_j \circ d_k \circ k_i &= x_j \circ k_j \circ c_k \\ &= y \circ \overline{c_j} \circ c_k \\ &= y \circ \overline{c_i} \end{aligned}$$

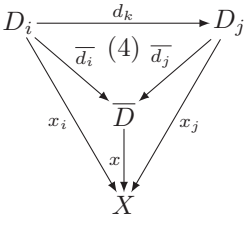
and

$$\begin{aligned} x_j \circ d_k \circ h_i &= x_j \circ h_j \circ b_k \\ &= z \circ \overline{b_j} \circ b_k \\ &= z \circ \overline{b_i}, \end{aligned}$$

and pushout (5_i) implies that

$$x_i = x_j \circ d_k.$$

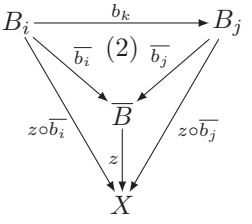
This means that colimit (4) implies a unique x with $x \circ \overline{d_i} = x_i$:



Now consider colimit (2).

$$\begin{aligned} x \circ \overline{h} \circ \overline{b_i} &= x \circ \overline{d_i} \circ h_i \\ &= x_i \circ h_i \\ &= z \circ \overline{b_i} \end{aligned}$$

implies that $x \circ \overline{h} = z$:



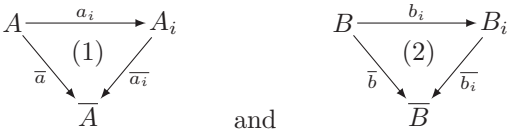
Similarly,

$$x \circ \overline{k} = y,$$

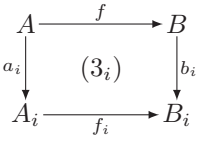
and the uniqueness follows from the uniqueness of x with respect to (4), so (10) is indeed a pushout. □

Lemma B.8. We assume:

— colimits



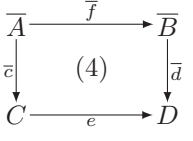
such that



commutes for all $i = 1, \ldots, n$;

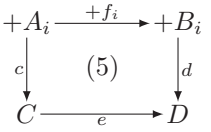
— f is an epimorphism; and

— the square



is a pushout with $\overline{f}$ induced by colimit (1).

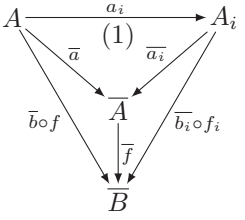
Then



is a pushout also, where c and d are induced from the coproducts.

Proof. Since (1) is a colimit and

$$\begin{aligned}\overline{b_i} \circ f_i \circ a_i &= \overline{b_i} \circ b_i \circ f \\ &= \overline{b} \circ f,\end{aligned}$$

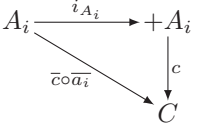


we actually get an induced $\overline{f}$ with

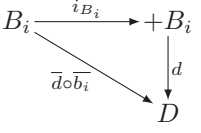
$$\begin{aligned}\overline{f} \circ \overline{a_i} &= \overline{b_i} \circ f_i \\ \overline{f} \circ \overline{a} &= \overline{b} \circ f.\end{aligned}$$

From the coproducts, we obtain induced morphisms:

— c with $c \circ i_{A_i} = \overline{c} \circ \overline{a_i}$



— d with $d \circ i_{B_i} = \overline{d} \circ \overline{b_i}$



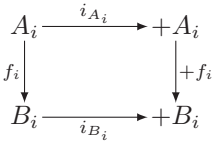
Moreover, for all $i = 1, \ldots, n$, we have

$$\begin{aligned}d \circ (+f_i) \circ i_{A_i} &= d \circ i_{B_i} \circ f_i \\ &= \overline{d} \circ \overline{b_i} \circ f_i \\ &= \overline{d} \circ \overline{f} \circ \overline{a_i} \\ &= e \circ \overline{c} \circ \overline{a_i} \\ &= e \circ c \circ i_{A_i}.\end{aligned}$$

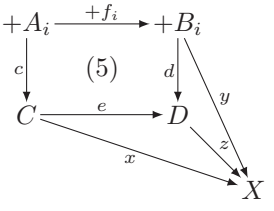
Uniqueness of the induced coproduct morphisms leads to

$$d \circ (+f_i) = e \circ c,$$

that is, (5) commutes.



We now have to show that (5) is a pushout:



Given morphisms x and y with

$$x \circ c = y \circ (+f_i),$$

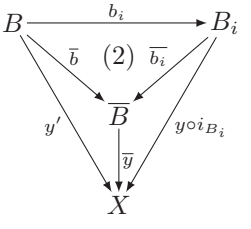
we have

$$\begin{aligned} y \circ i_{B_i} \circ b_i \circ f &= y \circ i_{B_i} \circ f_i \circ a_i \\ &= y \circ (+f_i) \circ i_{A_i} \circ a_i \\ &= x \circ c \circ i_{A_i} \circ a_i \\ &= x \circ \overline{c} \circ \overline{a_i} \circ a_i \\ &= x \circ \overline{c} \circ \overline{a} \end{aligned}$$

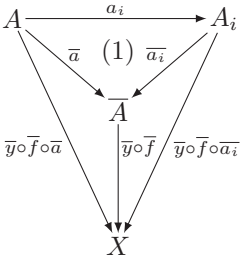
for all $i = 1, \ldots, n$. The fact that f is an epimorphism implies that

$$y \circ i_{B_i} \circ b_i = y \circ i_{B_j} \circ b_j$$

for all i, j. We now define $y' := y \circ i_{B_i} \circ b_i$ and from colimit (2)



we obtain a unique morphism $\overline{y}$ with $\overline{y} \circ \overline{b_i} = y \circ i_{B_i}$ and $\overline{y} \circ \overline{b} = y'$:



Now

$$\begin{aligned} x \circ \overline{c} \circ \overline{a_i} &= x \circ c \circ i_{A_i} \\ &= y \circ (+f_i) \circ i_{A_i} \\ &= y \circ i_{B_i} \circ f_i \\ &= \overline{y} \circ \overline{b_i} \circ f_i \\ &= \overline{y} \circ \overline{f} \circ \overline{a_i} \end{aligned}$$

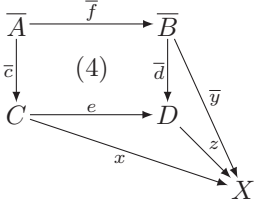
and

$$\begin{aligned} x \circ \overline{c} \circ \overline{a} &= x \circ \overline{c} \circ \overline{a_i} \circ a_i \\ &= \overline{y} \circ \overline{f} \circ \overline{i} \circ a_i \\ &= \overline{y} \circ \overline{f} \circ \overline{a}, \end{aligned}$$

and the uniqueness of the induced colimit morphism implies that

$$\overline{y} \circ \overline{f} = x \circ \overline{c}.$$

This means that X can be compared to pushout (4), and we obtain a unique morphism z with $z \circ \overline{d} = \overline{y}$ and $z \circ e = x$:



Now

$$\begin{aligned} z \circ d \circ i_{B_i} &= z \circ \overline{d} \circ \overline{b_i} \\ &= \overline{y} \circ \overline{b_i} \\ &= y \circ i_{B_i}, \end{aligned}$$

so $z \circ d = y$. Similarly, the uniqueness of z with respect to the pushout property of (5) also follows, so (5) is a pushout. □

References

Balasubramanian, D., Narayanan, A., Neema, S., Shi, F., Thibodeaux, R. and Karsai, G. (2007) A Subgraph Operator for Graph Transformation Languages. *Electronic Communications of the EASST* **6** 1–12.

Biermann, E., Ehrig, H., Ermel, C., Golas, U. and Taentzer, G. (2010a) Parallel Independence of Amalgamated Graph Transformations Applied to Model Transformation. In: Graph Transformations and Model-Driven Engineering. *Springer-Verlag Lecture Notes in Computer Science* **5765** 121–140.

Biermann, E., Ermel, C. and Taentzer, G. (2010b) Lifting Parallel Graph Transformation Concepts to Model Transformation Based on the Eclipse Modeling Framework. *Electronic Communications of the EASST* **26** 1–19.

Biermann, E., Ermel, C., Schmidt, J. and Warning, A. (2010c) Visual Modeling of Controlled EMF Model Transformation using HENSHIN. In: Proceedings of the Fourth International Workshop on Graph-Based Tools (GraBaTs 2010). (Available at `http://journal.ub.tu-berlin.de/index.php/eceasst/article/view/528`.)

Böhm, P., Fonio, H.-R. and Habel, A. (1987) Amalgamation of Graph Transformations: A Synchronization Mechanism. *Journal of Computer and System Sciences* **34** (2-3) 377–408.

Castellani, I. and Montanari, U. (1983) Graph Grammars for Distributed Systems. In: Ehrig, H., Nagl, M. and Rozenberg, G. (eds.) Graph Grammars and Their Application to Computer Science. *Springer-Verlag Lecture Notes in Computer Science* **153** 20–38.

de Lara, J., Ermel, C., Taentzer, G. and Ehrig, K. (2004) Parallel Graph Transformation for Model Simulation Applied to Timed Transition Petri Nets. *Electronic Notes in Theoretical Computer Science* **109** 17–29.

Degano, P. and Montanari, U. (1987) A Model of Distributed Systems Based on Graph Rewriting. *Journal of the ACM* **34** (2) 411–449.

Ehrig, H. and Kreowski, H.-J. (1976) Parallelism of Manipulations in Multidimensional Information Structures. In: Proceedings of MFCS 1976. *Springer-Verlag Lecture Notes in Computer Science* **45** 285–293.

Ehrig, H., Ehrig, K., Prange, U. and Taentzer, G. (2006) *Fundamentals of Algebraic Graph Transformation*, EATCS Monographs, Springer-Verlag.

Ehrig, H., Golas, U. and Hermann, F. (2010) Categorical Frameworks for Graph Transformation and HLR Systems based on the DPO Approach. *Bulletin of the EATCS* **102** 111–121.

Ehrig, H., Golas, U., Habel, A., Lambers, L. and Orejas, F. (2014) M-Adhesive Transformation Systems with Nested Application Conditions. Part 1: Parallelism, Concurrency and Amalgamation. *Mathematical Structures in Computer Science* (this volume).

Ermel, C. (2006) *Simulation and Animation of Visual Languages based on Typed Algebraic Graph Transformation*, Ph.D. thesis, Technische Universität Berlin.

Fischer, T., Niere, J., Torunski, L. and Zündorf, A. (2000) A New Graph Rewrite Language Based on the Unified Modeling Language. In: Proceedings of TAGT 1998. *Springer-Verlag Lecture Notes in Computer Science* **1764** 296–309.

Golas, U. (2011) *Analysis and Correctness of Algebraic Graph and Model Transformations*, Ph.D. thesis, Technische Universität Berlin, Vieweg and Teubner.

Golas, U., Biermann, E., Ehrig, H. and Ermel, C. (2011) A Visual Interpreter Semantics for Statecharts Based on Amalgamated Graph Transformation. *Electronic Communications of the EASST* **39** 1–24.

Golas, U., Ehrig, H. and Habel, A. (2010) Multi-Amalgamation in Adhesive Categories. In: Graph Transformations. Proceedings of ICGT 2010. *Springer-Verlag Lecture Notes in Computer Science* **6372** 346–361.

Grønmo, R., Krogdahl, S. and Møller-Pedersen, B. (2009) A Collection Operator for Graph Transformation. In: Proceedings of ICMT 2009. *Springer-Verlag Lecture Notes in Computer Science* **5563** 67–82.

Habel, A. and Pennemann, K.-H. (2009) Correctness of High-Level Transformation Systems Relative to Nested Conditions. *Mathematical Structures in Computer Science* **19** (2) 245–296.

Hoffmann, B., Janssens, D. and van Eetvelde, N. (2006) Cloning and Expanding Graph Transformation Rules for Refactoring. *Electronic Notes in Theoretical Computer Science* **152** 53–67.

Kuske, S., Gogolla, M., Kollmann, R. and Kreowski, H.-J. (2002) An Integrated Semantics for UML Class, Object and State Diagrams Based on Graph Transformation. In: Proceedings of IFM 2002. *Springer-Verlag Lecture Notes in Computer Science* **2335** 11–28.

Lack, S. and Sobociński, P. (2005) Adhesive and Quasiadhesive Categories. *Theoretical Informatics and Applications* **39** (3) 511–545.

Löwe, M. (1993) Algebraic Approach to Single-Pushout Graph Transformation. *Theoretical Computer Science* **109** 181–224.

Reisig, W. and Rozenberg, G. (1998) Lectures on Petri Nets I: Basic Models. *Springer-Verlag Lecture Notes in Computer Science* **1491**.

Rensink, A. and Kuperus, J.-H. (2009) Repotting the Geraniums: On Nested Graph Transformation Rules. *Electronic Communications of the EASST* **18** 1–15.

Rozenberg, G. (ed.) (1997) *Handbook of Graph Grammars and Computing by Graph Transformation 1: Foundations*, World Scientific.

Rozenberg, G. and Lindenmayer, A. (1976) *Automata, Languages, and Development*, North Holland.

Schürr, A., Winter, A. and Zündorf, A. (1999) The PROGRES-Approach: Language and Environment. In: *Handbook of Graph Grammars and Computing by Graph Transformation 2: Applications, Languages and Tools*, World Scientific 487–550.

Taentzer, G. (1996) *Parallel and Distributed Graph Transformation: Formal Description and Application to Communication Based Systems*, Ph.D. thesis, Technische Universität Berlin.

Taentzer, G. (2004) AGG: A Graph Transformation Environment for Modeling and Validation of Software. In: Proceedings of AGTIVE 2003. *Springer-Verlag Lecture Notes in Computer Science* **3062** 446–456.

Taentzer, G. and Beyer, M. (1994) Amalgamated Graph Transformations and Their Use for Specifying AGG – an Algebraic Graph Grammar System. In: Graph Transformations in Computer Science. *Springer-Verlag Lecture Notes in Computer Science* **776** 380–394.

Varró, D. (2002) A Formal Semantics of UML Statecharts by Model Transition Systems. In: Proceedings of ICGT 2002. *Springer-Verlag Lecture Notes in Computer Science* **2505** 378–392.