STRUCTURED EIGENVALUE BACKWARD ERRORS OF MATRIX PENCILS AND POLYNOMIALS WITH HERMITIAN AND RELATED STRUCTURES*

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Abstract. We derive a formula for the backward error of a complex number λ when considered as an approximate eigenvalue of a Hermitian matrix pencil or polynomial with respect to Hermitian perturbations. The same are also obtained for approximate eigenvalues of matrix pencils and polynomials with related structures like skew-Hermitian, *-even, and *-odd. Numerical experiments suggest that in many cases there is a significant difference between the backward errors with respect to perturbations that preserve structure and those with respect to arbitrary perturbations.

1. Introduction. We study the perturbation theory of the polynomial eigenvalue problem $\lambda^k A_k x + \cdots + \lambda A_1 x + A_0 x = 0$, where $A_0, \ldots, A_k$ are complex $n \times n$ matrices that carry a symmetry structure. In particular, we are interested in solving the following problem.

PROBLEM 1.1. *Let $P(z) = z^k A_k + \cdots + z A_1 + A_0$ be a regular structured matrix polynomial with $A_0, \ldots, A_k \in \mathbb{C}^{n \times n}$. Given a value $\lambda \in \mathbb{C}$, what is the smallest perturbation $(\Delta_0, \ldots, \Delta_k)$ from some perturbation set $\mathbb{S} \subseteq (\mathbb{C}^{n \times n})^{k+1}$ so that λ becomes an eigenvalue of $\widetilde{P}(z) := z^k(A_k - \Delta_k) + \cdots + z(A_1 - \Delta_1) + (A_0 - \Delta_0)$?*

The notion *structured* refers to a symmetry structure in the coefficients of the matrix polynomial as it can be found in Hermitian, alternating, or palindromic matrix polynomials. Typically, the *perturbation set* $\mathbb{S} \subseteq (\mathbb{C}^{n \times n})^{k+1}$ is then chosen in such a way that the perturbed polynomial has the same structure as the original polynomial $P(z)$. The term *smallest* is understood with respect to some weighted norm on $(\mathbb{C}^{n \times n})^{k+1}$ that is related to the spectral norm on $\mathbb{C}^{n \times n}$. The norm of the smallest perturbation $(\Delta_0, \ldots, \Delta_k)$ in Problem 1.1 can then be interpreted as the backward error of the value λ as an approximate eigenvalue of the polynomial $P(z)$.

The matrix polynomial $P(z)$ is called *Hermitian* if $P(z)^* := \sum_{j=0}^k z_j A_j^* = P(z)$, where A^* denotes the complex conjugate transpose of a matrix A. Equivalently, if all coefficient matrices are Hermitian, then $P(z)$ is a Hermitian matrix polynomial. Such polynomials occur in many applications, like structural mechanics, fluid mechanics, and signal processing; see [35] and the references therein. A structure-preserving linearization of Hermitian matrix polynomials leads to Hermitian pencils [14], thus

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making the case $k = 1$ an important special case. Other important classes of structured matrix polynomials are ∗-even or ∗-odd polynomials which satisfy $P(z)^* = P(-z)$ or $P(z)^* = -P(-z)$, respectively. Since coefficient matrices of these matrix polynomials alternate between Hermitian and skew-Hermitian matrices, the hypernym ∗-*alternating matrix polynomials* has been introduced in [25]. Important applications for ∗-even matrix polynomials are linear-quadratic optimal control theory [24, 28] and gyroscopic systems [23]. The discrete optimal control problem or the computation of the Crawford number of Hermitian pencils leads to ∗-*palindromic matrix polynomials* (see [15, 26]), which are characterized by the identities $A_j^* = A_{k-j}$ for $j = 0, \ldots, k$.

As noted in [1], "backward perturbation analysis and condition numbers play an important role in the accuracy assessment of computed solutions of eigenvalue problems." If eigenvalue problems with additional symmetry structures are considered, then the use of structure-preserving algorithms is advisable, because in this way existing symmetries in the spectrum are preserved even under roundoff errors. On the other hand, the use of general methods that do not consider the special structure of the problem may produce physically meaningless results [35]. Finally, it is well known that the perturbation theory may be fundamentally different when general versus structured methods are compared. For example, there exist systems with Hamiltonian matrices that are unstable when general perturbations are applied but stable under Hamiltonian perturbations; see [30, Example 3.5].

Therefore, there has been strong interest in the sensitivity analysis of eigenvalues and eigenpairs of structured eigenvalue problems; see, e.g., [1, 2, 3, 13, 21, 22, 34]. In particular, formulas for structured backward errors for eigenpairs of structured matrix pencils and polynomials have been developed in [1, 2]. However, there is also need for structured backward errors for *eigenvalues* of structured matrix polynomials. Indeed, if one is interested in computing the eigenvalues of a matrix polynomial but not in the eigenvectors or invariant subspaces, then the corresponding error analysis should take this into account. On the other hand, structured backward errors of eigenvalues play an important role in the solution of distance problems. For example, a formula for the structured backward error for eigenvalues of Hamiltonian matrices was developed and used for the solution of the problem of distance to bounded realness of Hamiltonian matrices in [4]. This distance has applications in the passivation of linear time-invariant control systems.

While the unstructured backward errors for eigenvalues of matrix pencils and polynomials can be easily obtained from the formulas for backward errors of eigenpairs developed in [3] by minimization over all nonzero vectors, this approach seems not to be as easy in the case of structured backward errors. Therefore, we will instead follow the strategy suggested in [20], which uses an approach via minimization problems of the maximal eigenvalues of a parameter-depending Hermitian matrix.

The main focus of this paper is on Hermitian matrix polynomials, because many other cases of structured matrix polynomials and pencils can be reduced to the Hermitian case. Indeed, if $P(z)$ is a skew-Hermitian or ∗-alternating matrix polynomial, then one may instead consider the Hermitian polynomials $iP(z)$, or $P(iz)$ and $iP(iz)$, respectively. Also, some matrix polynomials with coefficient matrices from Lie and Jordan algebras associated with an indefinite scalar product can be reduced to the Hermitian case. For example, eigenvalue problems with an underlying matrix pencil that is *skew-Hamiltonian/Hamiltonian* (see [7, 29]) satisfy $(JA_1)^* = JA_1$ and $(JA_2)^* = -JA_2$, where $n = 2m$ is even and

$$J = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}.$$

It immediately follows from the definition that the skew-Hamiltonian/Hamiltonian pencil $L(z) = A_1 + zA_2$ is equivalent to the $*$-even pencil $\widetilde{L}(z) = JA_1 + zJA_2$. Since the matrix J is unitary, the backward errors for the pencils $L(z)$ and $\widetilde{L}(z)$ will be identical if unitarily invariant norms like the spectral norm are considered. These observations do not hold for the $*$-palindromic case, though. Here, additional arguments are needed, and therefore the investigation of backward errors of palindromic polynomials is addressed in as subsequent paper.

This paper is organized a follows. In section 2, we introduce definitions and establish preliminary results that provide a setting for the main results of the paper. An outline of the technique for deriving the formulas for the structured backward error is also provided in this section. The minimization of the largest eigenvalue of an affine combination of Hermitian matrices over several real parameters plays a key role in finding the formulas for the structured backward error. This problem is discussed in detail in section 3. The formulas for the structured backward error of a complex number λ are given in section 4 for Hermitian matrix polynomials.

These formulas are extended to the case of skew-Hermitian, $*$-even, and $*$-odd matrix polynomials in section 5. In section 6, the techniques for deriving the backward error formulas are further extended to the case of perturbations that do not affect all the coefficients of the original polynomial. Finally, in section 7, we present some numerical experiments that illustrate the main results of the paper and highlight the different effects of structure preserving and arbitrary perturbations on the eigenvalues of the structured matrix polynomials under consideration.

Notation. The notation $\mathrm{Herm}(n)$ and $\mathrm{SHerm}(n)$ denote the sets of Hermitian and skew-Hermitian matrices of size $n \times n$, respectively. Given a Hermitian matrix H, $\lambda_{\max}(H)$ denotes the largest eigenvalue of H.

2. Preliminaries. In order to measure perturbations of matrix polynomials in a flexible way, we introduce a norm on $(\mathbb{C}^{n \times n})^{k+1}$ associated with a weight vector $w \in \mathbb{R}^{k+1}$.

DEFINITION 2.1. *Let $\|\cdot\|$ be the spectral norm and let $w = (w_0, \ldots, w_k) \in \mathbb{R}^{k+1}$, where $w_0, \ldots, w_k > 0$.*
  (1) *w is called a* weight vector *and its entries w_j are called* weights.
  (2) *The* reciprocal weight vector *of w is defined as $w^{-1} := (w_0^{-1}, \ldots, w_k^{-1})$.*
  (3) *For a tuple of matrices $\Delta_0, \ldots, \Delta_k \in \mathbb{C}^{n \times n}$, we define*

$$(2.1) \qquad \|(\Delta_0, \ldots, \Delta_k)\|_w := \sqrt{w_0^2 \|\Delta_0\|^2 + \cdots + w_k^2 \|\Delta_k\|^2}.$$

DEFINITION 2.2. *Let $P(z) = z^k A_k + \cdots + zA_1 + A_0$ be a matrix polynomial, where $A_0, \ldots, A_k \in \mathbb{C}^{n \times n}$, and let $\lambda \in \mathbb{C}$. Furthermore, let $w = (w_0, \ldots, w_k) \in \mathbb{R}^{k+1}$ be a weight vector and let $\mathbb{S} \subseteq (\mathbb{C}^{n \times n})^{k+1}$. Then we call*

$$\eta_w^{\mathbb{S}}(P, \lambda) := \inf \left\{ \|(\Delta_0, \ldots, \Delta_k)\|_w \;\middle|\; \det \left(\sum_{j=0}^{k} \lambda^j (A_j - \Delta_j) \right) = 0, \, (\Delta_0, \ldots, \Delta_k) \in \mathbb{S} \right\}$$

the structured backward error of λ with respect to P, $\mathbb{S}$, and w.

Thus, $\eta_w^{\mathbb{S}}(P, \lambda)$ is the norm of the smallest perturbation from $\mathbb{S}$ so that λ becomes an eigenvalue of the perturbed matrix polynomial $\widetilde{P}(z) := \sum_{j=0}^{k} z^j (A_j - \Delta_j)$. Clearly, we have $\eta_w^{\mathbb{S}}(P, \lambda) = 0$ if the matrix $P(\lambda) \in \mathbb{C}^{n \times n}$ is singular, i.e., if λ is already an

eigenvalue of $P(z)$ (including the case that the matrix polynomial $P(z)$ is singular). So, in the following we may assume that $P(z)$ is regular and that $P(\lambda)$ is nonsingular.

Remark 2.3. If $(A_0, \ldots, A_k) \in \mathbb{S}$, then we have

$$\eta_w^{\mathbb{S}}(P, \lambda) \leq \|(A_0, \ldots, A_k)\|_w < \infty,$$

because the perturbation with the tuple $(A_0, \ldots, A_k)$ results in the zero polynomial.

Observe that $\| \cdot \|_w$ is a norm on $(\mathbb{C}^{n \times n})^{k+1}$. The weights can be used to balance the importance of perturbations of individual coefficients. Sometimes zero weights are allowed in the literature with the convention that only those perturbations that change coefficients associated with nonzero weights are considered. We will treat this case differently in section 6 by restricting our perturbation class $\mathbb{S}$ accordingly.

Following the strategy used in [20] for computing structured backward errors of structured matrices, we will first reformulate the determinant equation in the definition of $\eta_w^{\mathbb{S}}(P, \lambda)$ in terms of a mapping problem. This is done in the following lemma.

LEMMA 2.4. *Let $P(z) = z^k A_k + \cdots + z A_1 + A_0$ be a matrix polynomial, where $A_0, \ldots, A_k \in \mathbb{C}^{n \times n}$, and let $\Delta_0, \ldots, \Delta_k \in \mathbb{C}^{n \times n}$ and $\lambda \in \mathbb{C}$ such that $M := P(\lambda)^{-1}$ exists. Then the following statements are equivalent:*

(a) $\det(\sum_{j=0}^{k} \lambda^j (A_j - \Delta_j)) = 0$.

(b) *There exist vectors $v_0, \ldots, v_k \in \mathbb{C}^n$ satisfying $\sum_{j=0}^{k} \lambda^j v_j \neq 0$ such that*

$$v_j = \Delta_j M (\lambda^k v_k + \cdots + \lambda v_1 + v_0) \quad for \; j = 0, \ldots, k.$$

Proof. Denote $\widetilde{P}(\lambda) := \sum_{j=0}^{k} \lambda^j (A_j - \Delta_j)$.

(a) $\Rightarrow$ (b) If (a) holds, then there exists $x \neq 0$ such that $\widetilde{P}(\lambda)x = 0$. Let $v_j := \Delta_j x$ for $j = 0, \ldots, k$. Then we have

$$(2.2) \qquad P(\lambda)x = P(\lambda)x - \widetilde{P}(\lambda)x = \sum_{j=0}^{k} \lambda^j \Delta_j \, x = \sum_{j=0}^{k} \lambda^j v_j =: v_\lambda.$$

We have $v_\lambda \neq 0$ because $P(\lambda) = M^{-1}$ is nonsingular by assumption. On multiplying (2.2) from the left with $\Delta_j M$ we obtain the identities $v_j = \Delta_j M v_\lambda$ for $j = 0, \ldots, k$.

(b) $\Rightarrow$ (a) Suppose that (b) holds and set $v_\lambda := \sum_{j=0}^{k} \lambda^j v_j$. Then

$$\widetilde{P}(\lambda)M v_\lambda = \left(P(\lambda) - \sum_{j=0}^{k} \lambda^j \Delta_j \right) M v_\lambda = v_\lambda - \sum_{j=0}^{k} \lambda^j \Delta_j \, M v_\lambda = 0,$$

because $\Delta_j \, M v_\lambda = v_j$ for $j = 0, \ldots, k$. Since $M v_\lambda \neq 0$, this implies (a). $\quad\square$

COROLLARY 2.5. *Let $P(z) = z^k A_k + \cdots + z A_1 + A_0$ be a matrix polynomial, where $A_0, \ldots, A_k \in \mathbb{C}^{n \times n}$, and let $\lambda \in \mathbb{C}$ such that $M := P(\lambda)^{-1}$ exists. Furthermore, let $\mathbb{S} \subseteq (\mathbb{C}^{n \times n})^{k+1}$. Then*

$$\eta_w^{\mathbb{S}}(P, \lambda) = \inf \left\{ \|(\Delta_0, \ldots, \Delta_k)\|_w \, \Big| \, (\Delta_0, \ldots, \Delta_k) \in \mathbb{S}, \; \exists v_0, \ldots, v_k \in \mathbb{C}^n : \right.$$

$$\left. v_\lambda := \sum_{j=0}^{k} \lambda^j v_j \neq 0, \; v_j = \Delta_j M v_\lambda, \; j = 0, \ldots, k \right\}.$$

Since the matrices $\Delta_0, \ldots, \Delta_k$ are Hermitian in our particular problem, we are lead to the following Hermitian mapping problems: *Under which conditions on $v_0, \ldots, v_k$ do there exist Hermitian matrices $\Delta_j \in \mathrm{Herm}(n)$ such that the identities*

$$(2.3) \qquad v_j = \Delta_j M v_\lambda, \quad j = 0, \ldots, k,$$

are satisfied?

These mapping problems can be condensed into the following general Hermitian mapping problem: *Under which conditions on vectors $x, y \in \mathbb{C}^n$ does there exist a Hermitian matrix $H \in \mathbb{C}^{n \times n}$ satisfying $Hx = y$?*

The answer to this problem is well known; see, e.g., [27] where solutions that are minimal with respect to the spectral or Frobenius norm are also characterized. We also refer to [19] and [32] for the more general problem of the existence of a Hermitian $H \in \mathbb{C}^{n \times n}$ such that $HX = Y$ for two matrices $X, Y \in \mathbb{C}^{n \times m}$. For convenience, we state the answer to our Hermitian mapping problem in terms that allow a direct application in this paper, and for the sake of completeness, we also provide a proof.

THEOREM 2.6. *Let $x, y \in \mathbb{C}^n$, $x \neq 0$. Then there exists a Hermitian matrix $H \in \mathrm{Herm}(n)$ such that $Hx = y$ if and only if $\mathrm{Im}\,(x^* y) = 0$. If the latter condition is satisfied, then*

$$\min \left\{ \|H\| \mid H \in \mathrm{Herm}(n),\ Hx = y \right\} = \frac{\|y\|}{\|x\|}$$

and the minimum is attained for

$$(2.4) \qquad H_0 := \frac{\|y\|}{\|x\|} \begin{bmatrix} \frac{y}{\|y\|} & \frac{x}{\|x\|} \end{bmatrix} \begin{bmatrix} \frac{y^* x}{\|x\|\,\|y\|} & 1 \\ 1 & \frac{x^* y}{\|x\|\,\|y\|} \end{bmatrix}^{-1} \begin{bmatrix} \frac{y}{\|y\|} & \frac{x}{\|x\|} \end{bmatrix}^*$$

if x and y are linearly independent and for $H_0 := \frac{yx^*}{x^* x}$ otherwise.

Proof. The identity $Hx = y$ immediately implies $\mathrm{Im}\,(x^* y) = \mathrm{Im}\,(x^* H x) = 0$, because H is Hermitian, and

$$\|H\| \geq \|y\| / \|x\| =: c.$$

In particular, this proves the "only if" part of the statement of the theorem. Conversely, let $\mathrm{Im}\,(x^* y) = 0$. Suppose first that x and y are linearly independent. Then H_0 given as in (2.4) is well defined and Hermitian, and we immediately obtain

$$H_0 \begin{bmatrix} \frac{x}{\|x\|} & \frac{y}{\|y\|} \end{bmatrix} = \frac{\|y\|}{\|x\|} \begin{bmatrix} \frac{y}{\|y\|} & \frac{x}{\|x\|} \end{bmatrix},$$

which implies $H_0 x = y$ and $H_0 y = c^2 x$. Thus, $y \pm cx$ are eigenvectors of H_0 associated with the eigenvalues $\pm c$, respectively, which implies $\|H_0\| = c$.

On the other hand, if x and y are linearly dependent, then $y = \alpha x$ with $\alpha \in \mathbb{R}$, and the matrix $H_0 = \alpha x x^* / \|x\|^2$ is Hermitian and satisfies $H_0 x = y$, and since H_0 has rank 1, $\|H_0\| = c$. $\square$

Before we derive formulas for the structured backward error of eigenvalues of Hermitian matrix polynomials of arbitrary degree, let us consider the pencil case $k = 1$ in order to illustrate the main ideas. Thus, for the moment, we assume $P(z) = zA_1 + A_0$ and for simplicity let us consider the norm (2.1) with weight vector $w = (1, 1)$.

In view of Corollary 2.5, we need to find vectors $v_0, v_1 \in \mathbb{C}^n$ with $v_\lambda := \lambda v_1 + v_0 \neq 0$ and matrices $\Delta_0, \Delta_1 \in \mathrm{Herm}(n)$ of minimal norm such that

$$(2.5) \qquad\qquad v_0 = \Delta_0 M v_\lambda \quad \text{and} \quad v_1 = \Delta_1 M v_\lambda,$$

where $M := P(\lambda)^{-1}$. By Theorem 2.6 the minimal norm $\|(\Delta_0, \Delta_1)\|_w$ for a fixed pair (v_0, v_1) is then given by

$$\|(\Delta_0, \Delta_1)\|_w^2 = \|\Delta_0\|^2 + \|\Delta_1\|^2 = \frac{\|v_0\|^2}{\|Mv_\lambda\|^2} + \frac{\|v_1\|^2}{\|Mv_\lambda\|^2} = \frac{\|v_0\|^2 + \|v_1\|^2}{\|M(\lambda v_1 + v_0)\|^2}.$$

Setting

$$v := \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \quad \text{and} \quad G := \begin{bmatrix} M^*M & \lambda\, M^*M \\ \bar\lambda\, M^*M & |\lambda|^2\, M^*M \end{bmatrix},$$

we obtain using $\|M(\lambda v_1 + v_0)\|^2 = (\bar\lambda v_1^* + v_0^*)M^*M(\lambda v_1 + v_0)$ that

$$(2.6) \qquad \|(\Delta_0, \Delta_1)\|_w^2 = \frac{\|v_0\|^2 + \|v_1\|^2}{\|M(\lambda v_1 + v_0)\|^2} = \frac{v^*v}{v^* \begin{bmatrix} M^*M & \lambda\, M^*M \\ \bar\lambda\, M^*M & |\lambda|^2\, M^*M \end{bmatrix} v} = \frac{v^*v}{v^*Gv},$$

which is just the reciprocal of the Rayleigh quotient of v with respect to the Hermitian matrix G. Since this quantity is just minimal in norm for a fixed pair (v_0, v_1), we now have to minimize (2.6) over all admissible pairs (v_0, v_1), i.e., all pairs for which there exists $\Delta_j \in \mathrm{Herm}(n)$, $j = 0, 1$, such that (2.5) is satisfied. By Theorem 2.6 those are exactly the pairs (v_0, v_1) satisfying $\mathrm{Im}\left(v_0^* M(v_0 + \lambda v_1)\right) = 0 = \mathrm{Im}\left(v_1^* M(v_0 + \lambda v_1)\right)$ and $\lambda v_1 + v_0 \neq 0$. Setting

$$H_0 := i \begin{bmatrix} M - M^* & \lambda M \\ -\bar\lambda M^* & 0 \end{bmatrix} \quad \text{and} \quad H_1 := i \begin{bmatrix} 0 & -M^* \\ M & \lambda M - \bar\lambda M^* \end{bmatrix}$$

these identities can be reformulated as

$$(2.7) \qquad 0 = -2\,\mathrm{Im}\left(v_0^* M(v_0 + \lambda v_1)\right) = i\left(v_0^* M(v_0 + \lambda v_1) - (M(v_0 + \lambda v_1))^* v_0\right)$$

$$= i\left(\begin{bmatrix} v_0 \\ v_1 \end{bmatrix}^* \begin{bmatrix} M & \lambda M \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} - \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}^* \begin{bmatrix} M^* & 0 \\ \bar\lambda M^* & 0 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right)$$

$$= v^* H_0 v,$$

and

$$(2.8) \qquad 0 = -2\,\mathrm{Im}\left(v_1^* M(v_0 + \lambda v_1)\right) = i\left(v_1^* M(v_0 + \lambda v_1) - (M(v_0 + \lambda v_1))^* v_1\right)$$

$$= i\left(\begin{bmatrix} v_0 \\ v_1 \end{bmatrix}^* \begin{bmatrix} 0 & 0 \\ M & \lambda M \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} - \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}^* \begin{bmatrix} 0 & M^* \\ 0 & \bar\lambda M^* \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right)$$

$$= v^* H_1 v.$$

Observe that $v^*Gv = \|M(\lambda v_1 + v_0)\|^2 \neq 0$ if and only if $\lambda v_1 + v_0 \neq 0$. Thus, we obtain from Corollary 2.5 that for $\mathbb{S} = \mathrm{Herm(n)}^2$ we have

$$(2.9) \quad \eta_w^{\mathbb{S}}(P, \lambda)^2 = \inf \left\{ \|(\Delta_0, \Delta_1)\|_w^2 \, \middle| \, \Delta_j \in \mathrm{Herm}(n), \, \exists v_0, v_1 \in \mathbb{C}^n \, : \, \lambda v_1 + v_0 \neq 0, \right.$$

$$\left. v_j = \Delta_j M(\lambda v_1 + v_0), j = 0, 1 \right\}$$

$$= \inf \left\{ \frac{v^* v}{v^* G v} \, \middle| \, v \in \mathbb{C}^{2n}, \, v^* G v \neq 0, \, v^* H_0 v = 0, \, v^* H_1 v = 0 \right\}$$

$$= \left(\sup \left\{ \frac{v^* G v}{v^* v} \, \middle| \, v \in \mathbb{C}^{2n} \backslash \{0\}, \, v^* H_0 v = 0, \, v^* H_1 v = 0 \right\} \right)^{-1}.$$

Note that in the latter identity the condition $v^* G v \neq 0$ could be dropped, because $\eta_w^{\mathbb{S}}(P, \lambda)$ is finite, which implies that the supremum in (2.9) will be positive. Therefore including vectors v satisfying $v^* G v = 0$ will not change the supremum of the considered set.

From these observations, we see that the structured backward error $\eta_w^{\mathbb{S}}(P, \lambda)$ can be computed by maximizing a Rayleigh quotient under two constraints. Since for Hermitian matrices the maximum of the Rayleigh quotient is equal to the maximal eigenvalue, the idea is to introduce Lagrange parameters t_0 and t_1 and minimize the function

$$(2.10) \qquad L : \mathbb{R}^2 \to \mathbb{R}, \quad (t_0, t_1) \mapsto \lambda_{\max}(G + t_0 H_0 + t_1 H_1).$$

In the next section, we will show in a more general setting that under adequate conditions on G, H_0, and H_1 the supremum in (2.9) coincides with the global minimum of L.

3. Minimizing the maximal eigenvalue of a Hermitian matrix function. As seen in the previous section for the example of Hermitian pencils, we will find out in section 4 that the computation of the structured backward error of eigenvalues of Hermitian matrix polynomials of degree k will lead to a minimization problem of a function of the form

$$L : \mathbb{R}^{k+1} \to \mathbb{R}, \quad (t_0, \ldots, t_k) \mapsto \lambda_{\max}(G + t_0 H_0 + \cdots + t_k H_k),$$

for some Hermitian matrices $G, H_0, \ldots, H_k \in \mathbb{C}^{n \times n}$. In order to analyze the extrema of L, we first need information on the partial differentiability of these kinds of functions. To this end, the following theorem provides useful information.

THEOREM 3.1. *Let $G, H \in \mathbb{C}^{n \times n}$ be Hermitian and let the map $L : \mathbb{R} \to \mathbb{R}$ be given by $L(t) := \lambda_{\max}(G + tH)$. Let the columns of the isometric matrix $U \in \mathbb{C}^{n,m}$ form an (orthonormal) basis of the eigenspace of the eigenvalue $\lambda_{\max}(G)$ of G. Then the left and right directional derivatives of L at $t = 0$ exist and we have*

$$\frac{d}{dt} L(0)_+ := \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{\lambda_{\max}(G + \varepsilon H) - \lambda_{\max}(G)}{\varepsilon} = \lambda_{\max}(U^* H U),$$

$$\frac{d}{dt} L(0)_- := \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{\lambda_{\max}(G - \varepsilon H) - \lambda_{\max}(G)}{-\varepsilon} = \lambda_{\min}(U^* H U).$$

If, in particular, $m = 1$, then L is differentiable at $t = 0$, $u := U \in \mathbb{C}^n \backslash \{0\}$, and

$$\frac{d}{dt} L(0) = \lambda_{\max}(U^* H U) = u^* H u.$$

For a proof of the above result, see [6, p. 149] or [31]. With these preparations, we are able to state and prove the main result of this section. We highlight that in the

next theorem and in the following, the term "indefinite" is understood in the sense of "strictly not semidefinite" as opposed to "not necessarily definite" as it is used in [9].

THEOREM 3.2. *Let $G, H_0, \ldots, H_k \in \mathbb{C}^{n \times n}$ be Hermitian matrices. Assume that any nonzero linear combination $\alpha_0 H_0 + \cdots + \alpha_k H_k$, $(\alpha_0, \ldots, \alpha_k) \in \mathbb{R}^{k+1} \backslash \{0\}$ is indefinite (i.e., strictly not semidefinite). Then the following statements hold:*

(1) *The function $L : \mathbb{R}^{k+1} \to \mathbb{R}$, $(t_0, \ldots, t_k) \mapsto \lambda_{\max}(G + t_0 H_0 + \cdots + t_k H_k)$ is convex and has a global minimum*

$$\lambda_{\max}^* := \min_{t_0, \ldots, t_k \in \mathbb{R}} L(t_0, \ldots, t_k).$$

(2) *If the minimum $\lambda_{\max}^*$ of L is attained at $(t_0^*, \ldots, t_k^*) \in \mathbb{R}^{k+1}$ and is a simple eigenvalue of $H_* := G + t_0^* H_0 + \cdots + t_k^* H_k$, then there exists an eigenvector $u \in \mathbb{C}^n \backslash \{0\}$ of H_* associated with $\lambda_{\max}^*$ satisfying*

$$(3.1) \qquad u^* H_j u = 0 \quad \text{for } j = 0, \ldots, k.$$

(3) *Under the assumptions of (2) we have*

$$(3.2) \qquad \sup \left\{ \frac{u^* G u}{u^* u} \,\middle|\, u \in \mathbb{C}^n \backslash \{0\},\, u^* H_j u = 0,\, j = 0, \ldots, k \right\} = \lambda_{\max}^*.$$

In particular, the supremum of the left-hand side of (3.2) *is a maximum and attained for the eigenvector u from* (2).

Proof. (1) The convexity of L is straightforward to check. Concerning the proof that L has a global minimum, we will show that there exists a constant $\varrho > 0$ such that for all $(t_0, \ldots, t_k)$ with $t_0^2 + \cdots + t_k^2 > \varrho^2$ we have $L(t_0, \ldots, t_k) \geq L(0, \ldots, 0)$. Since the closed ball

$$\mathcal{B}_\varrho := \{(t_0, \ldots, t_k) \in \mathbb{R}^{k+1} \,|\, t_0^2 + \cdots + t_k^2 \leq \varrho^2\}$$

with center at the origin and radius ϱ is compact and since L is continuous as eigenvalues depend continuously on the entries of a matrix, L has a global minimum $\lambda_{\max}^* \leq L(0, \ldots, 0)$ on $\mathcal{B}_\varrho$. By construction we then have $\lambda_{\max}^* \leq L(t_0, \ldots, t_k)$ for all $(t_0, \ldots, t_k) \in \mathbb{R}^{k+1}$, i.e., $\lambda_{\max}^*$ is the global minimum of L. Thus, define

$$c := \inf \left\{ \lambda_{\max}(\alpha_0 H_0 + \cdots + \alpha_k H_k) \,\middle|\, (\alpha_0, \ldots, \alpha_k) \in \mathbb{R}^{k+1},\, \alpha_0^2 + \cdots + \alpha_k^2 = 1 \right\}.$$

Then $c \geq 0$, because by hypothesis the matrix $\alpha_0 H_0 + \cdots + \alpha_k H_k$ is indefinite for all $(\alpha_0, \ldots, \alpha_k) \in \mathbb{R}^{k+1}$ with $\alpha_0^2 + \cdots + \alpha_k^2 = 1$, i.e., it always has positive eigenvalues. Since the function $f : (\alpha_0, \ldots, \alpha_k) \mapsto \lambda_{\max}(\alpha_0 H_0 + \cdots + \alpha_k H_k)$ is continuous (again using the well-known fact that eigenvalues depend continuously on the entries of a matrix), the infimum c is attained, because of the compactness of the unit sphere in $\mathbb{R}^{k+1}$. This implies $c > 0$, because the function f only takes positive values on the unit sphere. Next, define

$$\varrho := \frac{\lambda_{\max}(G) - \lambda_{\min}(G)}{c} \geq 0.$$

Let $(t_0, \ldots, t_k) \in \mathbb{R}^{k+1}$ and $r \geq \varrho$ so that $t_0^2 + \cdots + t_k^2 = r^2 \geq \varrho^2$. Using the fact that for two Hermitian matrices $A, B \in \mathbb{C}^{n \times n}$ we have $\lambda_{\max}(A + B) \geq \lambda_{\max}(A) + \lambda_{\min}(B)$ (see [17]), we obtain

$$L(t_0, \ldots, t_k) = \lambda_{\max}(G + t_0 H_0 + \cdots + t_k H_k) \geq \lambda_{\max}(t_0 H_0 + \cdots + t_k H_k) + \lambda_{\min}(G)$$
$$= r \cdot \lambda_{\max}\left(\frac{t_0}{r} H_0 + \cdots + \frac{t_k}{r} H_k\right) + \lambda_{\min}(G)$$
$$\geq \varrho \cdot c + \lambda_{\min}(G) = \lambda_{\max}(G) = L(0, \ldots, 0).$$

This finishes the proof of (1).

(2) By step (1), the minimum $\lambda_{\max}^*$ of L exists and by assumption it is attained at some point $(t_0^*, \ldots, t_k^*) \in \mathbb{R}^{k+1}$ and is a simple eigenvalue of the corresponding matrix $G + t_0^* H_0 + \cdots + t_k^* H_k$. Then, it follows from Theorem 3.1 that L is partially differentiable at $(t_0^*, \ldots, t_k^*)$ and

$$\frac{\partial L}{\partial t_j}(t_0^*, \ldots, t_k^*) = u^* H_j\, u, \quad j = 0, \ldots, k,$$

where u is an eigenvector of $G + t_0^* H_0 + \cdots + t_k^* H_k$ associated with $\lambda_{\max}^*$ satisfying $\|u\| = 1$. Since $\lambda_{\max}^*$ is the global minimum of L, this immediately implies $u^* H_j u = 0$ for $j = 0, \ldots, k$.

(3) Let s^* denote the left-hand side of (3.2). We show that $s^* = \lambda_{\max}^*$.

"$\geq$" By (2), there exists an eigenvector $u \in \mathbb{C}^n \backslash \{0\}$ of $G + t_0^* H_0 + \cdots + t_k^* H_k$ associated with $\lambda_{\max}^*$ satisfying $u^* H_j u = 0$ for $j = 0, \ldots, k$. Thus, we obtain

$$\lambda_{\max}^* = \frac{u^*(G + t_0^* H_0 + \cdots + t_k^* H_k)u}{u^* u} = \frac{u^* G u}{u^* u},$$

which implies that $s^* \geq \lambda_{\max}^*$.

"$\leq$" Let $u \in \mathbb{C}^n \backslash \{0\}$ be an arbitrary vector satisfying $u^* H_j\, u = 0$ for $j = 0, \ldots, k$. (By "$\geq$" there do exists such vectors.) Then we obtain

$$\frac{u^* G u}{u^* u} = \frac{u^*(G + t_0^* H_0 + \cdots + t_k^* H_k)u}{u^* u} \leq \lambda_{\max}(G + t_0^* H_0 + \cdots + t_k^* H_k) = \lambda_{\max}^*.$$

Since u was arbitrary, this implies $s^* \leq \lambda_{\max}^*$. This completes the proof. $\quad\square$

Remark 3.3. We highlight that the applicability of Theorem 3.2 relies heavily on the fact that the eigenvalue $\lambda_{\max}^*$ is a simple eigenvalue. This need not be the case, as the following example shows.

Example 3.4. Consider the Hermitian 2×2 matrices $G = 0$ and

$$H_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad H_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then for $t_0, t_1 \in \mathbb{R}$, the matrix

$$H(t_0, t_1) = G + t_0 H_0 + t_1 H_1 = \begin{bmatrix} t_0 & t_1 \\ t_1 & -t_0 \end{bmatrix}$$

has the eigenvalues $\pm\sqrt{t_0^2 + t_1^2}$, which implies in particular that any nonzero linear combination $\alpha_0 H_0 + \alpha_1 H_1$ is indefinite. Moreover, the function $L : \mathbb{R}^2 \to \mathbb{R}$ given by $L(t_0, t_1) := \lambda_{\max}(H(t_0, t_1))$ has its minimum at $(t_0^*, t_1^*) = (0, 0)$ with value $\lambda_{\max}^* = 0$, which happens to be a double eigenvalue of the zero matrix $H(0, 0)$. Nevertheless, the vector $u = \begin{bmatrix} 1 & i \end{bmatrix}^\top$ is an eigenvector of $H(0, 0)$ associated with $\lambda_{\max}^* = 0$ satisfying $u^* H_0 u = 0 = u^* H_1 u$.

Example 3.4 suggests that statement (2) of Theorem 3.2 may still be true even without the hypothesis of $\lambda_{\max}^*$ being a simple eigenvalue. The next theorem shows that in the case of the pencils where $k = 1$, this is indeed always the case.

THEOREM 3.5. *Let $G, H_0, H_1 \in \mathbb{C}^{n \times n}$ be Hermitian matrices. Assume that any linear combination $\alpha_0 H_0 + \alpha_1 H_1$, $(\alpha_0, \alpha_1) \in \mathbb{R}^2 \backslash \{0\}$, is indefinite (i.e., strictly not semidefinite). Then the following statements hold:*

(1) *The function $L : \mathbb{R}^2 \to \mathbb{R}$ given by $L(t_0, t_1) := \lambda_{\max}(G + t_0 H_0 + t_1 H_1)$ is convex and has a global minimum $\lambda_{\max}^*$.*

(2) *If the minimum $\lambda_{\max}^*$ of L is attained at $(t_0^*, t_1^*) \in \mathbb{R}^2$, then there exists an eigenvector $u \in \mathbb{C}^n \backslash \{0\}$ of $G + t_0^* H_0 + t_1^* H_1$ associated with $\lambda_{\max}^*$ satisfying*

$$(3.3) \qquad u^* H_0 u = 0 = u^* H_1 u.$$

(3) *We have*

$$(3.4)$$
$$\sup \left\{ \frac{u^* G u}{u^* u} \ \middle| \ u \in \mathbb{C}^n \backslash \{0\}, \, u^* H_0 u = 0, \, u^* H_1 u = 0 \right\} = \min_{t_0, t_1 \in \mathbb{R}} L(t_0, t_1) = \lambda_{\max}^*.$$

In particular, the supremum of the left-hand side of (3.4) *is a maximum and attained for the eigenvector u from* (2).

Proof. In view of Theorem 3.2, it remains to prove (2) for the case that $\lambda_{\max}^*$ is a multiple eigenvalue of $G + t_0^* H_0 + t_1^* H_1$. Let the columns of $U \in \mathbb{C}^{n,m}$ form an orthonormal basis of the eigenspace of $G + t_0^* H_0 + t_1^* H_1$ associated with $\lambda_{\max}^*$. Moreover, let $\alpha_0, \alpha_1 \in \mathbb{R}$ such that $\alpha_0^2 + \alpha_1^2 = 1$. By Theorem 3.1, we obtain the existence of the limit of the one-sided derivatives at $t = 0$ of the function

$$t \mapsto L(t_0^* + \alpha_0 t, t_1^* + \alpha_1 t) = \lambda_{\max}\big((G + t_0^* H_0 + t_1^* H_1) + t(\alpha_0 H_0 + \alpha_1 H_1)\big),$$

and this limit must be nonnegative, because there is a global minimum at (t_0^*, t_1^*). More precisely, we obtain from Theorem 3.1 that

$$\lambda_{\max}\big(U^*(\alpha_0 H_0 + \alpha_1 H_1)U\big) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{L(t_0^* + \alpha_0 \varepsilon, t_1^* + \alpha_1 \varepsilon) - L(t_0^*, t_1^*)}{\varepsilon} \geq 0$$

for all $\alpha_0, \alpha_1 \in \mathbb{R}$ with $\alpha_0^2 + \alpha_1^2 = 1$. Thus, for all such $\alpha = (\alpha_0, \alpha_1)$ there exists an eigenvector $x_\alpha \in \mathbb{C}^m$, $\|x_\alpha\| = 1$, associated with $\lambda_{\max}\big(U^*(\alpha_0 H_0 + \alpha_1 H_1)U\big)$ such that

$$(3.5) \qquad x_\alpha^* U^*(\alpha_0 H_0 + \alpha_1 H_1)U x_\alpha = \lambda_{\max}\big(U^*(\alpha_0 H_0 + \alpha_1 H_1)U\big) \geq 0.$$

We now show the existence of a vector $x \in \mathbb{C}^m$ with $\|x\| = 1$ such that

$$(3.6) \qquad x^* U^* H_0 U x = 0 = x^* U^* H_1 U x.$$

Then $u = Ux$ is the desired eigenvector of $G + t_0^* H_0 + t_1^* H_1$ satisfying (3.1). Recall that the joint numerical range of two Hermitian matrices $F_1, F_2 \in \mathbb{C}^{n \times n}$ is the set

$$\mathcal{W}_0(F_1, F_2) := \{(x^* F_1 x, x^* F_2 x) \in \mathbb{R}^2 \mid x \in \mathbb{C}^n, \|x\| = 1\}.$$

Thus the existence of a vector x with $\|x\| = 1$ satisfying (3.6) is equivalent to the fact that zero is in the joint numerical range $\mathcal{W}_0 := \mathcal{W}_0(U^* H_0 U, U^* H_1 U)$ of the matrices $U^* H_0 U$ and $U^* H_1 U$. Thus, let us assume that zero is *not* in $\mathcal{W}_0$. Since $\mathcal{W}_0$ is a closed convex set [18], by [16, Theorem 4.11, p. 51] this implies the existence of $\tilde{\alpha} = [\tilde{\alpha}_0, \tilde{\alpha}_1]^\top \in \mathbb{R}^2 \backslash \{0\}$ (without loss of generality we may assume $\tilde{\alpha}_0^2 + \tilde{\alpha}_1^2 = 1$) with

$$0 > \left\langle \widetilde{\alpha}, \begin{bmatrix} x^*U^*H_0Ux \\ x^*U^*H_1Ux \end{bmatrix} \right\rangle = x^*U^*(\widetilde{\alpha}_0 H_0 + \widetilde{\alpha}_1 H_1)Ux$$

for all $x \in \mathbb{C}^m$ with $\|x\| = 1$, contradicting (3.5). Hence, zero is in the joint numerical range of U^*H_0U and U^*H_1U, which finishes the proof of (2) and thus of the theorem. $\square$

Remark 3.6. If $m > 1$ in the above result, then since 0 is in the joint numerical range of the $m \times m$ Hermitian matrices U^*H_0U and U^*H_1U, the Hermitian pencil $zU^*H_0U + U^*H_1U$ is not a definite pencil (see [33] for details). Therefore its eigenvalues do not satisfy the conditions that characterize definite pencils as specified in Theorem 3.2 of [5]. These facts may be used in the numerical computation of the eigenvector x corresponding to $\lambda^*_{\max}$ such that $x^*U^*H_0Ux = x^*U^*H_1Ux = 0$ when $\lambda^*_{\max}$ is a multiple eigenvalue of $G + t_0^*H_0 + t_1^*H_1$.

Remark 3.7. Unfortunately, the argument in the proof of Theorem 3.5 cannot be generalized to the case $k > 1$, because the joint numerical range of three or more Hermitian matrices need not be convex.

Example 3.8. Consider the Hermitian 3×3 matrices $G = \text{diag}(\alpha, \alpha, \beta)$, where $\alpha > \beta \geq 0$, and

$$H_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad H_2 = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then for $t_0, t_1, t_2 \in \mathbb{R}$, the matrix

$$H(t_0, t_1, t_2) = G + t_0 H_0 + t_1 H_1 + t_2 H_2 = \begin{bmatrix} \alpha + t_0 & t_1 + it_2 & 0 \\ t_1 - it_2 & \alpha - t_0 & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

has the eigenvalues β and $\alpha \pm \sqrt{t_0^2 + t_1^2 + t_2^2}$. Again, any nonzero linear combination $\alpha_0 H_0 + \alpha_1 H_1 + \alpha_2 H_2$ is indefinite. Similar to Example 3.4, the function $L : \mathbb{R}^3 \to \mathbb{R}$ given by $L(t_0, t_1, t_2) = \lambda_{\max}\big(H(t_0, t_1, t_2)\big)$ has its minimum at $(0, 0, 0)$ with value $\lambda^*_{\max} = \alpha$, which happens to be a double eigenvalue of the matrix $H(0, 0, 0) = G$. In this case, a matrix whose columns form an orthonormal bases of the eigenspace of $H(0, 0, 0)$ associated with α is the 3×2 matrix $U = [e_1 \ e_2]$, where e_1 and e_2 denote the first two standard basis vectors. One easily checks that zero is not in the joint numerical range of

$$(3.7) \qquad U^*H_0U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ U^*H_1U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \text{and } U^*H_2U = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

and hence no eigenvector u of $H(0, 0)$ associated with $\lambda^*_{\max} = \alpha$ satisfies $u^*H_ju = 0$ for $j = 0, 1, 2$.

Note that in this example, scalar multiples of the third standard basis vector e_3 are the only vectors u satisfying $u^*H_ju = 0$ for $j = 0, 1, 2$, which shows that the left-hand side of (3.2) in Theorem 3.2 equals β, which is strictly less than $\alpha = \lambda^*_{\max}$.

The three Hermitian matrices in (3.7) are a classical example for Hermitian matrices whose joint numerical range is not convex [8, 12].

4. Backward errors of approximate eigenvalues of Hermitian polynomials. In this section, we consider Problem 1.1 for the case that $A_0, \ldots, A_k$ are Hermitian, i.e., $P(z) = \sum_{j=0}^{k} z^j A_j$ is a Hermitian matrix polynomial. The unstructured backward error $\eta_w(P, \lambda) := \eta_w^{\mathbb{S}}(P, \lambda)$ with $\mathbb{S} = (\mathbb{C}^{n \times n})^{k+1}$ is well-known and given in a notation slightly different from ours in [3, Proposition 4.6]. We restate the result here and include a proof for the sake of completeness.

THEOREM 4.1. *Let $P(z) = \sum_{j=0}^{k} z^j A_j$, where $A_0, \ldots, A_k \in \mathbb{C}^{n \times n}$ are Hermitian, and let $\lambda \in \mathbb{C}$. Then*

$$\eta_w(P, \lambda) = \frac{\sigma_{\min}\big(P(\lambda)\big)}{\|(1, \lambda, \ldots, \lambda^k)\|_{w^{-1}}},$$

where $\sigma_{\min}(M)$ stands for the smallest singular value of a matrix M.

Proof. Let $x \in \mathbb{C}^n \backslash \{0\}$. Then the backward error $\eta_w(P, \lambda, x)$ of the eigenpair (λ, x) is given by

$$(4.1) \qquad \eta_w(P, \lambda, x) := \frac{\|(P(\lambda)x\|}{\|x\| \cdot \|(1, \lambda, \ldots, \lambda^k)\|_{w^{-1}}}.$$

Indeed, if $\Delta_0, \ldots, \Delta_k \in \mathbb{C}^{n \times n}$ are perturbation matrices such that

$$\Delta P(\lambda)x := \sum_{j=0}^{k} \lambda^j \Delta_j x = P(\lambda)x,$$

that is, (λ, x) is an eigenpair of $\sum_{j=0}^{k} z^j (A_j - \Delta_j)$, then

$$\|P(\lambda)x\| = \|\Delta P(\lambda)x\| \leq \left\| \sum_{j=0}^{k} \lambda^j \Delta_j \right\| \cdot \|x\| = \left\| \sum_{j=0}^{k} \frac{\lambda^j}{w_j} (w_j \Delta_j) \right\| \cdot \|x\|$$

$$\leq \sum_{j=0}^{k} \frac{|\lambda^j|}{w_j} w_j \|\Delta_j\| \cdot \|x\| \leq \|(1, \lambda, \ldots, \lambda^k)\|_{w^{-1}} \|(\Delta_0, \ldots, \Delta_k)\|_w \|x\|$$

using the Cauchy–Schwarz inequality. This implies the "$\geq$" inequality in (4.1). On the other hand, setting

$$(4.2) \qquad \Delta_j := \frac{\bar{\lambda}^j P(\lambda)xx^*}{w_j^2 \, x^*x \|(1, \lambda, \ldots, \lambda^k)\|_{w^{-1}}^2}$$

we easily obtain $\Delta P(\lambda)x = \sum_{j=0}^{k} \lambda^j \Delta_j x = P(\lambda)x$ and equality in (4.1). Clearly we have $\eta_w(P, \lambda) = \min\{\eta_w(P, \lambda, x) \,|\, x \neq 0\}$, so the assertion immediately follows from (4.1). ☐

In the following, we require our perturbation matrices to be Hermitian as well, i.e., we want to compute the structured backward error $\eta_w^{\mathrm{Herm}}(P, \lambda) := \eta_w^{\mathbb{S}}(P, \lambda)$, where $\mathbb{S} := \mathrm{Herm}(n)^{k+1}$. If $\lambda \in \mathbb{R}$ is real, then there is no difference between the structured and the unstructured case. This fact was shown in [2] for the weight vector $w = (1, \ldots, 1)$ and easily generalizes to arbitrary weight vectors.

THEOREM 4.2. *Let $P(z) = \sum_{j=0}^{k} z^j A_j$, where $A_0, \ldots, A_k \in \mathbb{C}^{n \times n}$ are Hermitian, and let $\lambda \in \mathbb{R}$. Then*

$$\eta_w^{\mathrm{Herm}}(P, \lambda) = \eta_w(P, \lambda) = \frac{\sigma_{\min}\big(P(\lambda)\big)}{\|(1, \lambda, \ldots, \lambda^k)\|_{w^{-1}}}.$$

Proof. If λ is real, then the perturbation matrices Δ_j in (4.2) are Hermitian, which implies the desired result. $\square$

The situation is completely different if $\lambda \notin \mathbb{R}$. In this case, we obtain the structured backward error in terms of a minimization problem of the maximal eigenvalue of a parameter-depending Hermitian matrix. Here, the pencil case $k = 1$ differs from the polynomial case $k > 1$, where we require an additional hypothesis that the maximal eigenvalue of the Hermitian matrix that solves the minimization problem be a simple eigenvalue. For future reference, we state the pencil case separately.

THEOREM 4.3. *Let $P(z) = zA_1 + A_0$, where $A_0, A_1 \in \mathbb{C}^{n \times n}$ are Hermitian, let $\lambda \in \mathbb{C} \backslash \mathbb{R}$, and let $w = (w_0, w_1)$ be a weight vector. Suppose $\det P(\lambda) \neq 0$ so that $M := P(\lambda)^{-1}$ exists. Then*

$$\eta_w^{\mathrm{Herm}}(P, \lambda) = \left(\min_{t_0, t_1 \in \mathbb{R}} \lambda_{\max}(G + t_0 \, H_0 + t_1 \, H_1) \right)^{-1/2},$$

where

$$G := W^{-1} \begin{bmatrix} M^*M & \lambda M^*M \\ \bar{\lambda}M^*M & |\lambda|^2 M^*M \end{bmatrix} W^{-1}, \qquad H_0 := i \, W^{-1} \begin{bmatrix} M - M^* & \lambda M \\ -\bar{\lambda}M^* & 0 \end{bmatrix} W^{-1},$$

$$H_1 := i \, W^{-1} \begin{bmatrix} 0 & -M^* \\ M & \lambda M - \bar{\lambda}M^* \end{bmatrix} W^{-1}, \qquad W := \mathrm{diag}(w_0 I_n, w_1 I_n).$$

Theorem 4.3 is not proved separately as it is a special case of the following theorem, which we will prove in detail.

THEOREM 4.4. *Let $P(z) = \sum_{j=0}^{k} z^j A_j$, where $A_0, \ldots, A_k \in \mathbb{C}^{n \times n}$ are Hermitian, and let $w = (w_0, \ldots, w_k)$ be a weight vector. Let $\lambda \in \mathbb{C} \backslash \mathbb{R}$ be such that $\det P(\lambda) \neq 0$ so that $M := P(\lambda)^{-1}$ exists. Let $\Lambda_k := [1, \lambda, \ldots, \lambda^k]^* = [1, \bar{\lambda}, \ldots, \bar{\lambda}^k]^T$ and set*

$$\widetilde{G} := (\Lambda_k \Lambda_k^*) \otimes (M^*M) = \begin{bmatrix} M^*M & \lambda M^*M & \ldots & \lambda^k M^*M \\ \bar{\lambda}M^*M & |\lambda|^2 M^*M & \ldots & \bar{\lambda}\lambda^k M^*M \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\lambda}^k M^*M & \lambda\bar{\lambda}^k M^*M & \ldots & |\lambda|^{2k} M^*M \end{bmatrix},$$

$$\widetilde{H}_j := i\big((e_{j+1}\Lambda_k^*) \otimes M - (\Lambda_k e_{j+1}^*) \otimes M^*\big) = i \begin{bmatrix} & & -\bar{\lambda}^0 M^* & & \\ & & \vdots & & \\ \lambda^0 M \cdots & \lambda^j M - \bar{\lambda}^j M^* & \cdots \lambda^k M \\ & & \vdots & & \\ & & -\bar{\lambda}^k M^* & & \end{bmatrix}$$

for $j = 0, \ldots, k$, where e_{j+1} denotes the $(j+1)$st standard basis vector of $\mathbb{R}^{k+1}$ and

$$W := \mathrm{diag}(w_0, \ldots, w_k) \otimes I_n, \quad G = W^{-1}\widetilde{G}W^{-1}, \quad H_j = W^{-1}\widetilde{H}_j W^{-1},$$

for $j = 0, \ldots, k$. Then

$$\lambda_{\max}^* := \min_{t_0, \ldots, t_k \in \mathbb{R}} \lambda_{\max}(G + t_0 \, H_0 + \cdots + t_k \, H_k)$$

is attained for some $(t_0^*, \ldots, t_k^*) \in \mathbb{R}^{k+1}$. If $k = 1$ or $\lambda_{\max}^*$ is a simple eigenvalue of $G + t_0^* \, H_0 + \cdots + t_k^* \, H_k$, then

$$\eta_w^{\mathrm{Herm}}(P, \lambda) = \frac{1}{\sqrt{\lambda_{\max}^*}} = \left(\min_{t_0, \ldots, t_k \in \mathbb{R}} \lambda_{\max}(G + t_0 \, H_0 + \cdots + t_k \, H_k) \right)^{-1/2}.$$

Proof. Let $v_0, \ldots, v_k \in \mathbb{C}^n$ with $v_\lambda := \sum_{j=0}^k \lambda^j v_j \neq 0$ and set $v := [v_0^\top, \ldots, v_k^\top]^\top$. Then using Lemma 2.6, we find that there exist $\Delta_j \in \mathrm{Herm}(n)$ satisfying

$$(4.3) \qquad v_j = \Delta_j M v_\lambda, \quad j = 0, \ldots, k,$$

if and only if $v_j^* M v_\lambda \in \mathbb{R}$ for $j = 0, \ldots, k$. As in (2.7) and (2.8) these conditions can be reformulated as $k+1$ Hermitian constraints $v^* \widetilde{H}_j v = 0$. If these conditions are fulfilled, then according to Lemma 2.6 the minimal norms of $\Delta_j \in \mathrm{Herm}(n)$ satisfying (4.3) are given by $\|\Delta_j\| = \|v_j\|/\|M v_\lambda\|$, $j = 0, \ldots, k$. Setting $u := W v$, by reasons identical to those used to establish (2.6), the minimal norm of a tuple $(\Delta_0, \ldots, \Delta_k) \in \mathrm{Herm}(n)^{k+1}$ satisfying (4.3) is given by

$$\|(\Delta_0, \ldots, \Delta_k)\|_w^2 = \frac{w_0^2 \|v_0\|^2 + \cdots + w_k^2 \|v_k\|^2}{\|M v_\lambda\|^2} = \frac{v^* W^2 v}{v^* \widetilde{G} v} = \frac{u^* u}{u^* G u}.$$

Observe that for any vector $v = [v_0^\top, \ldots, v_k^\top]^\top$ we have $v^* \widetilde{H}_j v = u^* H_j u$ and that $0 \neq u^* G u = \|M v_\lambda\|^2$ if and only if $v_\lambda = \lambda^k v_k + \cdots + \lambda v_1 + v_0 \neq 0$. Thus, we have

$$(4.4) \quad \eta_w^{\mathrm{Herm}}(P, \lambda)^2 = \inf \left\{ \frac{u^* u}{u^* G u} \,\middle|\, u \in \mathbb{C}^{2n}, \, u^* G u \neq 0, \, u^* H_j u = 0, \, j = 0, \ldots, k \right\}$$

$$= \sup \left\{ \frac{u^* G u}{u^* u} \,\middle|\, u \in \mathbb{C}^{2n} \setminus \{0\}, \, u^* H_j u = 0, \, j = 0, \ldots, k \right\}^{-1}.$$

Note that since $\eta_w^{\mathrm{Herm}}(P, \lambda)$ is finite and positive, the supremum in the latter equality of (4.4) will not be attained by vectors u satisfying $u^* G u = 0$ and therefore the condition $u^* G u \neq 0$ is superfluous for it.

Since our aim is to apply Theorem 3.5 or Theorem 3.2 for the case of the pencils and polynomials, respectively, we need to check whether each nontrivial linear combination of $H_0, \ldots, H_k$, or, equivalently, of $\widetilde{H}_0, \ldots, \widetilde{H}_k$, is indefinite. Thus, assume that $\alpha := [\alpha_0, \ldots, \alpha_k]^\top \in \mathbb{R}^{k+1}$ is such that $H := \sum_{j=0}^k \alpha_j \widetilde{H}_j$ is semidefinite. Then

$$H = i \sum_{j=0}^k \alpha_j \big((e_{j+1} \Lambda_k^*) \otimes M - (\Lambda_k e_{j+1}^*) \otimes M^* \big) = i \big((\alpha \Lambda_k^*) \otimes M - (\Lambda_k \alpha^\top) \otimes M^* \big)$$

and we have to show that $\alpha = 0$. Setting

$$Q := \begin{bmatrix} 1 & -\lambda & 0 & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & -\lambda \\ 0 & \cdots & 0 & 1 \end{bmatrix} \quad \text{and} \quad a := \begin{bmatrix} a_0 \\ \vdots \\ a_k \end{bmatrix} := Q^* \alpha,$$

we obtain $\Lambda_k^* Q = e_1^\top$ and hence

$$(Q \otimes I_n)^* H (Q \otimes I_n)$$

$$= i \big((a e_1^\top) \otimes M - (e_1 a^*) \otimes M^* \big) = i \begin{bmatrix} a_0 M - \bar{a}_0 M^* & -\bar{a}_1 M^* & \cdots & -\bar{a}_k M^* \\ a_1 M & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_k M & 0 & \cdots & 0 \end{bmatrix}.$$

Since H is semidefinite and M is invertible, it follows that $a_1 = \cdots = a_k = 0$, i.e., $a = a_0 e_1$. In particular, $a_1 = 0$ implies that $\alpha_1 - \bar{\lambda}\alpha_0 = 0$. But this implies that $\alpha_0 = \alpha_1 = 0$, because α_0, α_1 are real and λ is nonreal. Finally, the first entry of $Q^*\alpha$ yields the identity $a_0 = \alpha_0 = 0$ and thus $a = 0$, which implies $\alpha = 0$.

If $k = 1$, then the assertion follows immediately from (4.4) and Theorem 3.5. On the other hand, if $k > 1$, then with the additional assumption on $\lambda^*_{\max}$, the assertion follows similarly from (4.4) and Theorem 3.2. $\quad\square$

Remark 4.5. In view of Example 3.8 it is crucial that the eigenvalue $\lambda^*_{\max}$ in Theorem 4.4 is a simple eigenvalue. Numerical experiments suggest that generically this is indeed the case.

Remark 4.6. Once $\lambda^*_{\max}$ and the corresponding eigenvector $u \in \mathbb{C}^{(k+1)n}$ satisfying $u^*H_j u = 0$, $j = 0, \ldots, k$, have been computed, the optimal perturbation matrices can be easily constructed using Theorem 2.6: writing $v := W^{-1}u = [v_0^\top, \ldots, v_k^\top]^\top$ with $v_j \in \mathbb{C}^n$ and $v_\lambda := \lambda^k v_k + \cdots + \lambda v_1 + v_0$, we find that the required coefficients Δ_j for $j = 0, \ldots, k$ of the minimal Hermitian perturbation are given by

$$\Delta_j := \frac{\|v_j\|}{\|Mv_\lambda\|} \begin{bmatrix} \frac{v_j}{\|v_j\|} & \frac{Mv_\lambda}{\|Mv_\lambda\|} \end{bmatrix} \begin{bmatrix} \frac{v_j^* Mv_\lambda}{\|Mv_\lambda\|\,\|v_j\|} & 1 \\ 1 & \frac{(Mv_\lambda)^* v_j}{\|Mv_\lambda\|\,\|v_j\|} \end{bmatrix}^{-1} \begin{bmatrix} \frac{v_j}{\|v_j\|} & \frac{Mv_\lambda}{\|Mv_\lambda\|} \end{bmatrix}^*$$

if v_j and Mv_λ are linearly independent and by

$$\Delta_j := \frac{v_j v_\lambda^* M^*}{v_\lambda^* M^* Mv_\lambda}$$

otherwise.

Remark 4.7. We highlight that there are situations when the perturbed matrix pencils or matrix polynomials turn out to be singular. For example, this is the case if $A_0 = [\,a\,]$ and $A_1 = [\,b\,]$ are real 1×1 matrices. As nonreal eigenvalues of Hermitian matrix pencils always occur in pairs $(\lambda, \overline{\lambda})$, the only Hermitian perturbation that makes $\lambda \in \mathbb{C}\backslash\mathbb{R}$ an eigenvalue of $zA_1 + A_0$ is $(\Delta_0, \Delta_1) = ([-a], [-b])$ resulting in the zero pencil which is singular. Similar examples can be constructed for larger dimensions n. However, numerical examples suggest that these cases are actually exceptional.

5. Matrix polynomials with related structures. As mentioned in the introduction, the problem of computing structured backward errors for eigenvalues of skew-Hermitian or $*$-alternating polynomials can be reduced to the case of Hermitian polynomials. These lead to formulas for the structured backward error of approximate eigenvalues for such polynomials.

THEOREM 5.1. *Let* $P(z) = \sum_{j=0}^k z^j A_j$ *be a skew-Hermitian matrix polynomial with* $A_0, \ldots, A_k \in \mathbb{C}^{n \times n}$, *let* $\mathbb{S} := (\mathrm{SHerm(n)})^{k+1}$, *and let* $w \in \mathbb{R}^{k+1}\backslash\{0, 0, \ldots, 0\}$ *be a weight vector. Then*

$$\eta_w^{\mathbb{S}}(P, \lambda) = \eta_w^{\mathrm{Herm}}(iP, \lambda).$$

Proof. This follows immediately from the fact that P is skew-Hermitian if and only if iP is Hermitian. $\quad\square$

For the next result let

$$\mathbb{S}_e := \left\{(\Delta_0, \ldots, \Delta_k)\big|\Delta_{2j} \in \mathrm{Herm(n)}, \Delta_{2j+1} \in \mathrm{SHerm(n)}, j = 0, \ldots, \lfloor\tfrac{k}{2}\rfloor\right\}$$

$$\text{and} \quad \mathbb{S}_o := \left\{(\Delta_0, \ldots, \Delta_k)\big|\Delta_{2j+1} \in \mathrm{Herm(n)}, \Delta_{2j} \in \mathrm{SHerm(n)}, j = 0, \ldots, \lfloor\tfrac{k}{2}\rfloor\right\},$$

i.e., $\mathbb{S}_e$ is the set of $*$-even matrix polynomials and $\mathbb{S}_o$ is the set of $*$-odd matrix polynomials.

THEOREM 5.2. *Let* $P(z) = \sum_{j=0}^{k} z^j A_j$ *be an* $*$-*alternating matrix polynomial with* $A_0, \ldots, A_k \in \mathbb{C}^{n \times n}$, *let* $Q(z) := P(iz) = \sum_{j=0}^{k} z^j (i^j A_j)$, *and let* $w \in \mathbb{R}^{k+1} \backslash \{0, \ldots, 0\}$ *be a weight vector. Then*

$$\eta_w^{\mathbb{S}_e}(P, \lambda) = \eta_w^{\mathrm{Herm}}(Q, \lambda/i)$$

if P *is* $*$-*even and*

$$\eta_w^{\mathbb{S}_o}(P, \lambda) = \eta_w^{\mathrm{Herm}}(iQ, \lambda/i)$$

if P *is* $*$-*odd.*

Proof. This follows immediately from the fact that $Q(z/i) = P(z)$ and that Q is Hermitian if P is $*$-even or skew-Hermitian if P is $*$-odd. □

6. Further restriction of perturbation sets. In some cases it may be of interest to further restrict the perturbation set $\mathbb{S} = \mathrm{Herm}(n)^{k+1}$. In particular, it may be useful to perturb only some of the coefficients of the matrix polynomial. For example, a Hermitian pencil $P(z) = zA_1 + A_0$ can be canonically identified with the A_1-self-adjoint matrix $\mathcal{H} := A_1^{-1} A_0$ if A_1 is invertible. (Recall that a matrix $\mathcal{H}$ is called A_1-self-adjoint if $\mathcal{H}^* A_1 = A_1 \mathcal{H}$; see, e.g., [9].) In this case A_1 can be interpreted as a matrix that induces a (possibly indefinite) scalar product on $\mathbb{C}^n$. If perturbations of the pencil P that allow changes only to A_0 are considered, then this results in the effect that the matrix defining the scalar product remains constant. Therefore, we briefly explain in this section how our main results can be applied to those cases as well.

To be more precise, let $I := \{i_0, \ldots, i_m\} \subseteq \{0, \ldots, k\}$ with $i_0 < \cdots < i_m$ be an index set and define

$$(6.1) \qquad\qquad \mathbb{S} := \mathbb{S}(I) := \mathcal{S}_0 \times \cdots \times \mathcal{S}_k \subseteq \mathrm{Herm}(n)^{k+1},$$

where $\mathcal{S}_j = \mathrm{Herm}(n)$ if $j \in I$ and $\mathcal{S}_j = \{0\}$ if $j \notin I$. For example, if $k = 4$ and $I = \{1, 2\}$, then $(\Delta_0, \ldots, \Delta_4) \in \mathbb{S}(I)$ if and only if $\Delta_0 = \Delta_3 = \Delta_4 = 0$ and $\Delta_1, \Delta_2 \in \mathrm{Herm}(n)$, i.e., perturbations from $\mathbb{S}$ will only change the coefficients A_1 and A_2 of a matrix polynomial $\sum_{j=0}^{4} z^j A_j$. Thus, each $(\Delta_0, \ldots, \Delta_k) \in \mathbb{S}$ can be canonically identified with a tuple $(\Delta_{i_0}, \ldots, \Delta_{i_m}) \in \mathrm{Herm}(n)^{m+1}$ and we consider

$$\|(\Delta_{i_0}, \ldots, \Delta_{i_m})\|_{\widehat{w}} = \|(\Delta_0, \ldots, \Delta_k)\|_w = \sqrt{w_{i_0}^2 \|\Delta_{i_0}\|^2 + \cdots + w_{i_m}^2 \|\Delta_{i_m}\|^2},$$

which is a norm on $\mathrm{Herm}(n)^{m+1}$ and the corresponding backward error

$$\eta_{\widehat{w}}^{\mathbb{S}}(P, \lambda) := \inf \left\{ \|(\Delta_0, \ldots, \Delta_k)\|_w \,\bigg|\, \det \left(\sum_{j=0}^{k} \lambda^j (A_j - \Delta_j) \right) = 0, \, (\Delta_0, \ldots, \Delta_k) \in \mathbb{S} \right\}.$$

Thus, the new weight vector $\widehat{w} := [w_{i_0}, \ldots, w_{i_m}]^T \in \mathbb{R}^{m+1}$ is obtained from the old weight vector $w \in \mathbb{R}^{k+1}$ by deleting the entries w_j with $j \notin I$.

Note that the computation of $\eta_{\widehat{w}}^{\mathbb{S}}(P, \lambda)$ when $\lambda \in \mathbb{R}$ has already been considered in [1] and [2]. Therefore, we only consider $\lambda \in \mathbb{C} \backslash \mathbb{R}$ and obtain the following analogue of Theorem 4.4.

THEOREM 6.1. *Let* $P(z) = \sum_{j=0}^{k} z^j A_j$, *where* $A_0, \ldots, A_k \in \mathbb{C}^{n \times n}$ *are Hermitian and* $\lambda \in \mathbb{C} \backslash \mathbb{R}$ *be such that* $M := P(\lambda)^{-1}$ *exists. Let* $I := \{i_0, \ldots, i_m\} \subseteq \{0, \ldots, k\}$,

$\mathbb{S}$ be given by (6.1), and $\widehat{w} := (w_{i_0}, w_{i_1}, \ldots, w_{i_m}) \in \mathbb{R}^{m+1}\backslash\{0, \ldots, 0\}$ be a weight vector. Let $\Lambda_m := [\lambda^{i_0}, \ldots, \lambda^{i_m}]^*$ and set

$$\widehat{G} := (\Lambda_m \Lambda_m^*) \otimes (M^* M) \quad and \quad \widehat{H}_j := i\big((e_{j+1}\Lambda_m^*) \otimes M - (\Lambda_m e_{j+1}^*) \otimes M^*\big)$$

for $j = 0, \ldots, m$, where e_{j+1} denotes the $(j+1)$st standard basis vector of $\mathbb{R}^{m+1}$. Also let

$$W := \mathrm{diag}(w_{i_0}, \ldots, w_{i_m}) \otimes I_n, \quad G := W^{-1}\widehat{G}W^{-1}, \quad H_j := W^{-1}\widehat{H}_jW^{-1}$$

for $j = 0, \ldots, m$. Then

$$\lambda_{\max}^* := \min_{t_0, \ldots, t_m \in \mathbb{R}} \lambda_{\max}(G + t_0\,H_0 + \cdots + t_m\,H_m)$$

is attained for some $(t_0^*, \ldots, t_m^*) \in \mathbb{R}^{m+1}$. If $\eta_{\widehat{w}}^{\mathbb{S}}(P, \lambda)$ is finite and $m \leq 1$ or $\lambda_{\max}^*$ is a simple eigenvalue of $G + t_0^*\,H_0 + \cdots + t_m^*\,H_m$, then

$$\eta_{\widehat{w}}^{\mathbb{S}}(P, \lambda) = \frac{1}{\sqrt{\lambda_{\max}^*}} = \Big(\min_{t_0, \ldots, t_m \in \mathbb{R}} \lambda_{\max}(G + t_0\,H_0 + \cdots + t_m\,H_m)\Big)^{-1/2}.$$

Observe that $\widehat{G}$ and $\widehat{H}_j$ are obtained from the corresponding matrices $\widetilde{G}$ and $\widetilde{H}_{i_j}$ in Theorem 4.4 by deleting the block rows and columns with indices not in I.

Remark 6.2. The proof of Theorem 6.1 proceeds in exactly the same way as the proof of Theorem 4.4. It is based on a modified version of Lemma 2.4 with setting $\Delta_j = 0$ for $j \notin I$ and requiring $v_j = 0$ for $j \notin I$ in (b). (In the case $m = 0$ [20, Theorem 4.5] is applied in place of Theorem 3.5.)

The condition $\eta_{\widehat{w}}^{\mathbb{S}}(P, \lambda) < \infty$ is indeed necessary, as there are a number of instances when this is not the case. For example, if $k = 1$ and $I = \{0\}$, then $\eta_{\widehat{w}}^{\mathbb{S}}(P, \lambda) = \infty$ for any nonreal λ if A_1 is either positive or negative definite. Also, if A_0 is nonsingular and $0 \notin I$, then $\eta_{\widehat{w}}^{\mathbb{S}}(P, 0) = \infty$ for any degree k. The latter situation is also reflected by the fact that the matrix

$$\widehat{Q} := \begin{bmatrix} 1 & -\lambda^{i_1-i_0} & 0 & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & -\lambda^{i_m-i_{m-1}} \\ 0 & \ldots & 0 & 1 \end{bmatrix}$$

replaces the matrix Q in the proof of Theorem 4.4 when establishing the indefiniteness of any nontrivial linear combination of $H_0, \ldots, H_m$ and the argument needs the fact that the vector $\Lambda_m^*\widehat{Q} = \lambda^{i_0}e_1^\top$ is nonzero. Observe that this is true if and only if $i_0 = 0$ or $\lambda \neq 0$.

Thus, we see that restricting the perturbation set in such a way that only m of k coefficient matrices are perturbed, the corresponding structured backward error can be computed by solving an $(m + 1)$-parameter optimization problem rather than a $(k + 1)$-parameter problem.

7. Numerical examples. In this section we present some numerical examples to illustrate the proposed method for computing the structured backward error $\eta_w^{\mathbb{S}}(P, \lambda)$ of some $\lambda \in \mathbb{C}$ for the case $\mathbb{S} := (\mathrm{Herm(n)})^{k+1}$ and $w := (1, 1, \ldots, 1)$. In all cases we have used the software package CVX [11, 10] in MATLAB to solve the associated optimization problem of finding

$$\lambda_{\max}^* := \min_{t_0, t_1, \ldots, t_k \in \mathbb{R}} \lambda_{\max}(G + t_0 H_0 + \cdots + t_k H_k)$$

and the points $t_0^*, t_1^*, \ldots, t_k^* \in \mathbb{R}$ that attain it as described in Theorem 4.4.
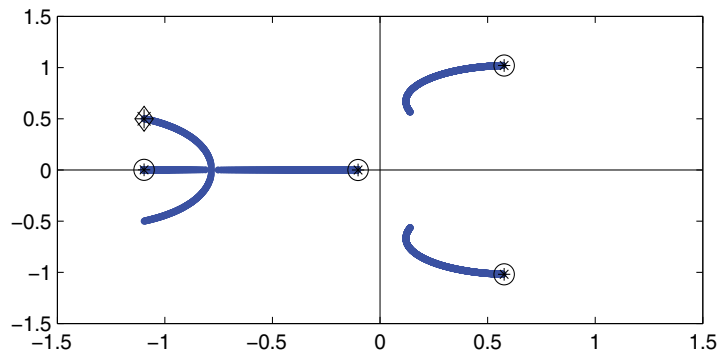
FIG. 1. *Eigenvalue perturbation curves for the Hermitian pencil in Example* 7.1 *with respect to Hermitian perturbation.*
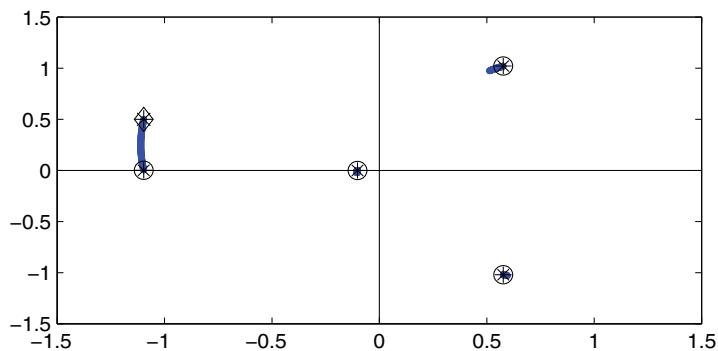

FIG. 2. *Eigenvalue perturbation curves for the Hermitian pencil in Example* 7.1 *with respect to unstructured perturbation.*

Example 7.1. $L(z) := zA_1 + A_0$ is a randomly generated Hermitian pencil of size 4×4 with eigenvalues $0.57661 \pm 1.0199i$, -1.0966, and -0.10193. The Hermitian backward error for the point $\lambda = -1.0966 + 0.5i$ which is close to the eigenvalue -1.0966 is 1.3058, while the unstructured backward error 0.47045 is much smaller, as expected.

Figure 1 illustrates the movement of the eigenvalues of the pencil $L(z)$ (marked with stars surrounded by circles) under the homotopic perturbation $L(z) + t\Delta L(z)$ as t varies from 0 to 1. Observe that the target point $-1.0966 + 0.5i$ (marked with a star surrounded by a diamond) as well as its complex conjugate become eigenvalues of $(L + \Delta L)(z)$ and this is produced by the splitting of a real eigenvalue of multiplicity 2 of $(L + t_0\Delta L)(z)$ for some $0 < t_0 < 1$. Here $\Delta L(z) := z\Delta_1 + \Delta_0$ is the optimal Hermitian perturbation satisfying $\|(\Delta_1, \Delta_0)\| = 1.3058$ such that $-1.0966 + 0.5i$ is an eigenvalue of $(L + \Delta L)(z)$.

Figure 2 illustrates the same effect with respect to unstructured homotopic perturbations $L(z) + t\widehat{\Delta L}(z)$ as t varies from 0 to 1. In this case $\widehat{\Delta L}(z) := z\widehat{\Delta}_1 + \widehat{\Delta}_0$ is a minimal non-Hermitian perturbation such that $-1.0966 + 0.5i$ is an eigenvalue of $(L + \widehat{\Delta L})(z)$. Observe that in this case the complex conjugate of $-1.0966 + 0.5i$ is not an eigenvalue of $(L + \widehat{\Delta L})(z)$ as it is not a Hermitian pencil.

Example 7.2. $L(z) := zA_1 + A_0$ is a diagonal Hermitian pencil of size 3×3 with real eigenvalues $21.393, 4.2464,$ and -3.5385. The Hermitian backward error of the point $-0.1241 + 1.4897i$ is 0.5608, while its unstructured backward error is 0.4246. This is an example for which $\lambda^*_{\max}$ is a multiple eigenvalue of $G + t_0^* H_0 + t_1^* H_1$, where $t_0^* = -0.3819$ and $t_1^* = 0.6266$.

Figure 3 traces the movement of the eigenvalues of $L(z)$ with respect to perturbations $L(z) + t\Delta L(z)$ as t varies from 0 to 1 where $\Delta L(z) := z\Delta_1 + \Delta_0$ is the optimal Hermitian perturbation satisfying $\|(\Delta_1, \Delta_0)\| = 0.5608$ such that $-1.0966 + 0.5i$ is an eigenvalue of $(L + \Delta L)(z)$.

The point $-0.1241 + 1.4897i$ (marked with a star surrounded by a diamond) and its complex conjugate become eigenvalues of $(L + \Delta L)(z)$ after the splitting of a real eigenvalue of multiplicity 2 that arises from the meeting of eigenvalue curves that originated from the unperturbed eigenvalues 21.393 and -3.5385 of $L(z)$. It is interesting to note that the eigenvalue curve originating from 21.393 moves over ∞ before it meets the curve originating from -3.5385. Figure 4 illustrates the same effect with respect to unstructured homotopic perturbations $L(z) + t\widehat{\Delta L}(z)$ as t varies from 0 to 1. In this case $\widehat{\Delta L}(z) := z\widehat{\Delta}_1 + \widehat{\Delta}_0$ is a minimal non-Hermitian perturbation such that $-1.0966 + 0.5i$ is an eigenvalue of $(L + \widehat{\Delta L})(z)$. The complex conjugate of
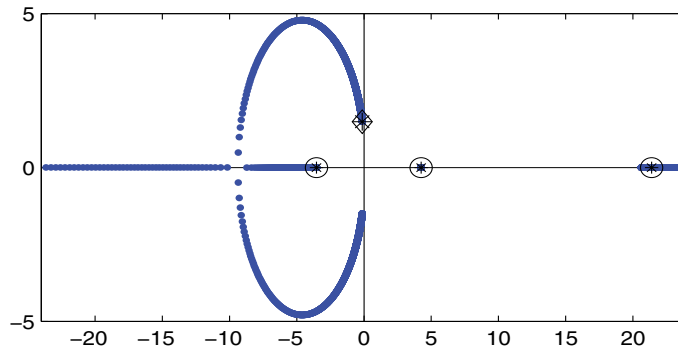

FIG. 3. *Eigenvalue perturbation curves for the Hermitian pencil in Example* 7.2 *with respect to Hermitian perturbation.*
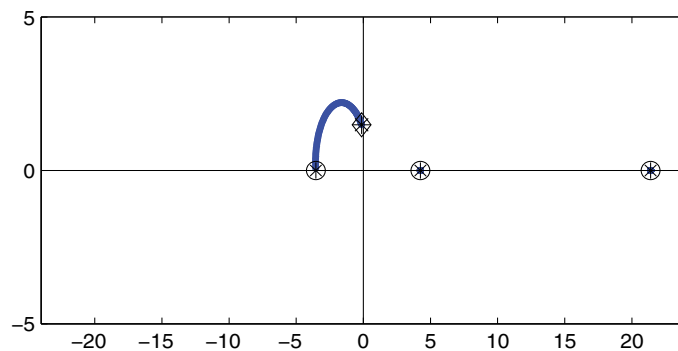

FIG. 4. *Eigenvalue perturbation curves for the Hermitian pencil in Example* 7.2 *with respect to unstructured perturbation.*
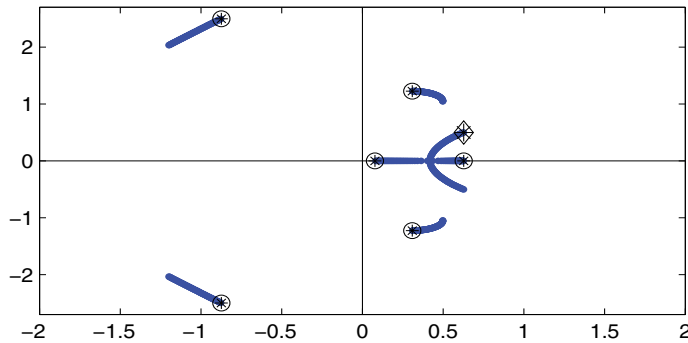
FIG. 5. *Eigenvalue perturbation curves for the Hermitian polynomial $Q(z)$ of Example 7.3 with respect to Hermitian perturbation.*
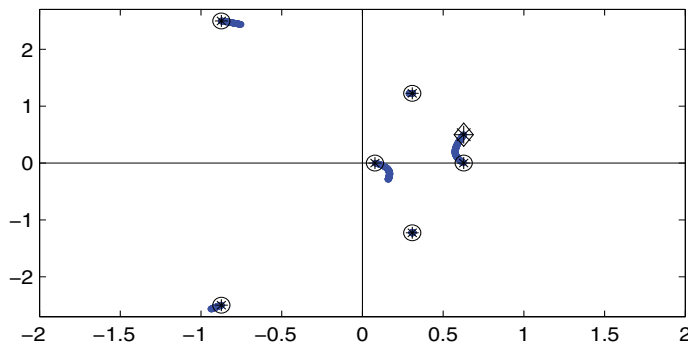

FIG. 6. *Eigenvalue perturbation curves for the Hermitian polynomial $Q(z)$ of Example 7.3 with respect to non-Hermitian perturbation.*

$-0.1241 + 1.4897i$ is not an eigenvalue of $(L + \widehat{\Delta L})(z)$ as it is not a Hermitian pencil and therefore only a single eigenvalue curve originating from -3.5385 reaches this point for $t = 1$.

Example 7.3. $Q(z) := z^2 A_2 + z A_1 + A_0$ is a Hermitian matrix polynomial of size 3×3 with eigenvalues $-0.8738 \pm 2.4984i, 0.3091 \pm 1.226i, 0.62802,$ and 0.07796. The Hermitian backward error for the point $0.62802 + 0.5i$ which is close to the real eigenvalue 0.62802 is 1.9177, whereas the backward error with respect to arbitrary perturbations is 1.3279.

Figure 5 traces the movement of the eigenvalues of $Q(z)$ with respect to perturbations $Q(z) + t\Delta Q(z)$ as t moves from 0 to 1, $\Delta Q(z)$ being the minimal Hermitian perturbation that produces an eigenvalue at $0.62802 + 0.5i$. As expected, since $(Q + \Delta Q)(z)$ is Hermitian, it has a pair of eigenvalues at $0.62802 \pm 0.5i$ which are produced by the meeting (on the real line) and splitting of eigenvalue curves originating from the two real eigenvalues of $Q(z)$.

On the other hand, Figure 6 traces the movement of the eigenvalues of $Q(z)$ with respect to perturbations $Q(z) + t\widehat{\Delta Q}(z)$ as t moves from 0 to 1. Here $\widehat{\Delta Q}(z)$ is the minimal non-structure-preserving perturbation to $Q(z)$ such that $0.62802 + 0.5i$ is an eigenvalue of $(Q + \widehat{\Delta Q})(z)$.

In further numerical experiments we have observed that for diagonal Hermitian polynomials, $\lambda^*_{\max}$ is a multiple eigenvalue of $G + t^*_0 H_0 + \cdots t^*_k H_k$. Despite this fact, it has been observed that in each of these cases it is possible to find an eigenvector x corresponding to $\lambda^*_{\max}$ satisfying $x^* H_j x = 0$ for $j = 0, 1, \ldots, k$. This aspect of such problems is still under investigation. However, we have not yet encountered a case where $\lambda^*_{\max}$ is multiple for Hermitian matrix polynomials whose coefficients are randomly generated.

We also computed the structured and unstructured backward errors of a nonreal λ whose real part is a simple eigenvalue of the Hermitian matrix polynomial. We observed that as expected, the unstructured backward error approached zero as the imaginary part of λ was reduced. However, this did not decrease the structured backward error as significantly, leading to large differences between the two backward error values. These are recorded for the Hermitian pencil considered in Example 7.1 and the Hermitian quadratic polynomial considered in Example 7.3 in Tables 1 and 2, respectively.

The situation is different if the selected complex values λ are chosen in such a way that they converge to a nonreal eigenvalue instead of a real one. In that case both the structured and unstructured backward errors tend to zero as expected. The values are recorded in Table 3 for the Hermitian pencil considered in Example 7.1 for λ converging to the eigenvalue $0.57661 + 1.0199i$ together with the backward errors for some nonreal values λ not necessarily close to eigenvalues. The latter values show

TABLE 1
Structured and unstructured eigenvalue backward errors for Hermitian pencils.

λ	t^*_0	t^*_1	$\lambda^*_{\max}$	$\eta_w(L, \lambda)$	$\eta_w^{\text{Herm}}(L, \lambda)$
-1.0966 + i	0.47	-0.73	0.5017	0.8450	1.4118
-1.0966 + 0.5i	1.12	-1.39	0.5865	0.4704	1.3058
-1.0966 + 0.1i	6.12	-6.75	0.6533	0.0978	1.2372
-1.0966 + 0.05i	12.27	-13.48	0.6561	0.0490	1.2345
-1.0966 + 0.01i	61.43	-67.37	0.6571	0.0098	1.2337
-1.0966 + 0.005i	122.87	-134.74	0.6571	0.0049	1.2337

TABLE 2
Structured and unstructured eigenvalue backward errors for quadratic Hermitian polynomials.

λ	t^*_0	t^*_1	t^*_2	$\lambda^*_{\max}$	$\eta_w(Q, \lambda)$	$\eta_w^{\text{Herm}}(Q, \lambda)$
0.62802 + i	0.06	0.09	0.03	0.81	1.0965	1.1099
0.62802 + 0.5i	-0.28	-0.54	-0.62	0.2719	1.3279	1.9177
0.62802 + 0.1i	-3.38	-5.33	-8.21	0.3852	0.2411	1.6113
0.62802 + 0.05i	-6.85	-10.88	-17.17	0.3882	0.1198	1.6051
0.62802 + 0.01i	-34.38	-54.73	-87.12	0.3891	0.0239	1.6032
0.62802 + 0.005i	-68.76	-109.48	-174.32	0.3891	0.0120	1.6031

TABLE 3
Structured and unstructured eigenvalue backward errors for Hermitian pencils.

λ	η_w	η_w^{Herm}	λ	η_w	η_w^{Herm}
-0.3200 + 0.8000i	0.9073	1.0139	-0.1364 + 0.1139i	0.4247	0.8762
0.3000 + 0.8800i	0.4733	0.4851	-1.1465 + 1.1909i	0.9531	1.4643
0.4500 + 0.9000i	0.2717	0.2790	-1.2173 + 0.0412i	0.1213	1.3267
0.5200 + 0.9500i	0.1390	0.1426	-1.4410 + 0.5711i	0.5717	1.4981
0.5600 + 0.9800i	0.0663	0.0678	-1.6041 + 0.2573i	0.4800	1.5440
0.5760 + 1.0005i	0.0227	0.0231	-2.1707 − 0.0592i	0.7687	1.7158

that the difference between the structured and unstructured backward errors may be quite significant even if the value is not close to the real line.

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