Discrete Yamabe problem for polyhedral surfaces

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October 14, 2019

Abstract

We introduce a new discretization of the Gaussian curvature on piecewise flat surfaces. As the prime new feature the curvature is scaled by the factor $1/r^2$ upon scaling the metric globally with the factor r. We develop a variational principle to tackle the corresponding discrete uniformisation theorem – we show that each piecewise flat surface is discrete conformally equivalent to one with constant discrete Gaussian curvature. This surface is in general not unique. We demonstrate uniqueness for particular cases and disprove it in general by providing explicit counterexamples. We deploy hyperbolic geometry to deal with the change of combinatorics.

1 Introduction

The Yamabe problem asks for the existence of Riemannian metrics with constant scalar curvature within confromal classes. For two-dimensional manifolds the scalar and the Gaussian curvature are equivalent, and thus the Yamabe problem is answered by the celebrated *Poincaré-Koebe uniformisation theorem*, which states that any closed oriented Riemannian surface is conformally equivalent to one with constant Gaussian curvature $K \in \{-1, 0, 1\}$. This surface is unique if $K = -1$, unique up to global scale factor if $K = 0$, and unique up to Möbius transformations if $K = 1$.

We discretize the uniformisation theorem in the realm of piecewise flat surfaces.

A *marked surface* is a closed oriented topological surface S, together with a nonempty finite set $V \subseteq S$ of *marked points.* If $\chi(S) = 0$, we assume that $|V| \geq 3$. A *PL-metric* on (S, V) is a metric d on S such that a neigbourhood of every point $i \in S$ is isometric to the Euclidean plane if $i \notin V$, and isometric to the tip of a Euclidean cone, with cone angle $\alpha_i > 0$, if $i \in V$. A *piecewise flat surface* is a surface equipped with a PL-metric.

Definition 1.1. *Let (S, V, d) be a piecewise flat surface. The* ***discrete Gaussian curvature*** *at vertex $i \in V$ is the quotient of the angle defect W_i and the area A_i of the Voronoi cell V_i. That is,*

$$K : V \to \mathbb{R}, \qquad i \mapsto K_i := \frac{W_i}{A_i},$$

where $W_i := 2\pi - \alpha_i$.

With the definition of *discrete conformal equivalence* pioneered by Feng Luo [5], and developed by Alexander Bobenko, Ulrich Pinkall and Boris Springborn [2], we prove the following theorem.

Theorem 1.2 (Discrete uniformisation theorem)**.** *For every PL-metric d on a marked surface (S, V), there exists a conformally equivalent PL-metric $\tilde{d}$ such that the surface $(S, V, \tilde{d})$ has constant discrete Gaussian curvature.*

Corollary 1.3. *The PL-metric $\tilde{d}$ is, in general, not unique.*

This paper is structured as follows. The necessary mathematical background and notation are introduced in Section 2. In Section 3 we present counterexamples for uniqueness. The proof of Theorem 1.2 is based on two variational principles, which we introduce in Section 4. Finally, we prove Theorem 1.2 in Section 5.

2 Preliminaries

In this section we review the mathematical background of this paper and fix notation. In particular, we explain the link between PL-metrics and ideal decorated polyhedra. Since the results in this section are well-known, we omit the proofs. A thorough summary of these notions can be found in [8, Sections 2-6].

We denote a *triangulation* by Δ and the *set of edges* and *faces* of Δ by E_Δ and F_Δ, respectively. A *triangulation of a marked surface* (S, V) is a triangulation of S with the vertex set equal to V. Let d be a metric on V or on $S \backslash V$. A *geodesic triangulation of* (S, V, d) is a triangulation of (S, V) where the edges are geodesics with respect to the metric d.

2.1 Voronoi and Delaunay tessellations

Voronoi tessellation Every piecewise flat surface (S, V, d) posesses a unique *Voronoi tessellation.* Let $p \in S$ and let $d(p, V)$ denote the distance of p to the set V, $d(p, V) := \min_{i \in V} d(p, i)$. Consider the set $\Gamma_V(p)$ of all geodesics realizing $d(p, V)$. The open 2-cells of the Voronoi tessellation of (S, V, d) are the connected components of

$$\mathcal{C}_2 = \{p \in S \mid |\Gamma_V(p)| = 1\}.$$

$$\mathcal{C}_1 = \{p \in S \mid |\Gamma_V(p)| = 2\} \quad \text{and} \quad \mathcal{C}_0 = \{p \in S \mid |\Gamma_V(p)| \geq 3\},$$

respectively.

For a point $i \in V$, the *Voronoi cell* V_i is the closure of the 2-cell of the Voronoi tessellation that contains i.

Delaunay tessellation and triangulation *Delaunay tessellation* of a piecewise flat surface (S, V, d) is the dual of the Voronoi tessellation. A *Delaunay triangulation* arises from the Delaunay tessellation by adding edges to triangulate the non-triangular faces.

Let d be a PL-metric on (S, V), and let Δ be a geodesic triangulation of (S, V, d). Let ijk, ijl be two triangles in F_Δ. The edge $ij \in E_\Delta$ is called a *Delaunay edge* if it satisfies the following property: the vertex l is not contained in the interior of the circumcircle of the triangle ijk.

Fact 2.1. *Let α_k, α_l be the angles opposite of the edge ij in the triangles ijk and ijl, respectively. The edge ij is Delaunay if one of the following equivalent* ***Delaunay conditions*** *holds:*
a) $\cot \alpha_k + \cot \alpha_l \geq 0$,
b) $\alpha_k + \alpha_l \leq \pi$,
c) $\cos \alpha_k + \cos \alpha_l \geq 0$.

Proposition 2.2. *A geodesic triangulation of a piecewise flat surface is Delaunay if and only if each of its edges is Delaunay.*

Ideal hyperbolic polyhedra The Delaunay tessellation of (S, V, d) possesses the *empty disc property:* Let $C \subseteq S$ be a closed 2-cell of the Delaunay tessellation. Then there exists an open disc $D_C \subseteq \mathbb{R}^2$ and a local isometry $\varphi_C : \bar{D}_C \to S$ such that $\varphi_C^{-1}(C)$ is a cyclic polygon with circumcircle ∂D_C and vertices $\varphi_C^{-1}(C \cap V)$.

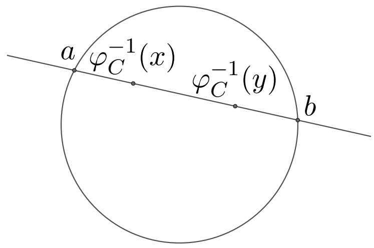


Figure 2.1

The PL-metric d on (S, V) induces a hyperbolic metric d_{hyp} on the set $S \backslash V$ by interpreting the disc D_C as the Klein-Beltrami model of hyperbolic geometry. More precisely, for $x, y \in C$, let $a, b \in \partial D_C$ be the intersection points of the line through $\varphi_C^{-1}(x)$ and $\varphi_C^{-1}(y)$ and the circumcircle ∂D_C, as illustrated in Figure 2.1. The formula

$$d_{hyp}(x, y) = \frac{1}{2} \log \left(\frac{\|\varphi_C^{-1}(x) - b\| \|\varphi_C^{-1}(y) - a\|}{\|a - \varphi_C^{-1}(x)\| \|b - \varphi_C^{-1}(y)\|} \right) \tag{1}$$

induces a hyperbolic metric with cusps on each Delaunay cell C.

The induced hyperbolic metrics on the Delaunay cells fit along the edges. The triple (S, V, d_{hyp}) is a complete finite area hyperbolic surface with $|V|$ punctures. We call it an *ideal hyperbolic polyhedron.* A *decorated ideal hyperbolic polyhedron* is an ideal hyperbolic polyhedron (S, V, d_{hyp}) with a decoration with a horocycle $\mathcal{H}_i$ at each cusp $i \in V$. Each horocycle is small enough such that, altogether, the horocycles bound disjoint cusp neighbourhoods.

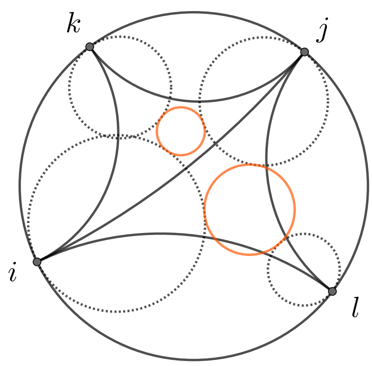
Ideal Delaunay triangulation Consider a decorated ideal hyperbolic polyhedron $(S, V, d_{hyp}, \mathcal{H})$.

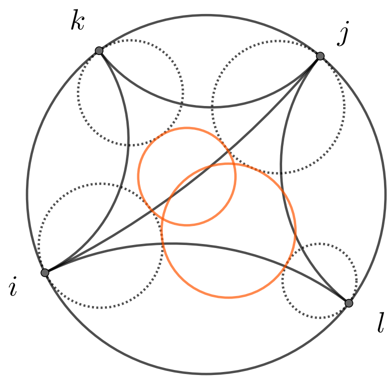
Definition 2.3. *An **ideal Delaunay decomposition** of $(S, V, d_{hyp}, \mathcal{H})$ is an ideal cell decomposition of (S, V, d_{hyp}), such that for each face f of the lift of (S, V, d_{hyp}) to the hyperbolic plane H^2 via an isometry of the universal cover, the following condition is satisfied: there exists a circle that touches all lifted horocycles centred at the vertices of f externally and does not meet any other lifted horocycles.*

*An **ideal Delaunay triangulation** is any refinement of an ideal Delaunay decomposition by decomposing the non-triangular faces into ideal triangles by adding geodesic edges.*

Theorem 2.4. *For each decorated ideal hyperbolic polyhedron with at least one cusp, there exists a unique ideal Delaunay decomposition.*

Definition 2.5. *Let $(S, V, d_{hyp}, \mathcal{H})$ be a decorated ideal hyperbolic polyhedron with a geodesic triangulation Δ. Consider an edge $ij \in E_\Delta$ and its neigbouring two triangles $ijk, ijl \in F_\Delta$. The edge ij is called **Delaunay** if the two circles touching the horocycles at vertices i, j, k and i, j, l are externally disjoint or externally tangent (see Figure 2.2).*





(a) The two orange circles are disjoint, the edge ij is Delaunay.

(b) The two orange circles intersect, the edge ij is not Delaunay.

Figure 2.2: Delaunay and non-Delaunay edge, in the Poincaré disc model of hyperbolic geometry.

Proposition 2.6. *An ideal geodesic triangulation of a decorated ideal hyperbolic polyhedron is Delaunay if and only if each of its edges is Delaunay.*

2.2 Discrete metric and Penner coordinates

Let Δ be a triangulation of a marked surface (S, V).

Discrete metric

Definition 2.7. *A* ***discrete metric*** *on* (S, V, Δ) *is a function*

$$\ell : E_\Delta \to \mathbb{R}_{>0}, \qquad \ell(ij) = \ell_{ij},$$

such that for every triangle $ijk \in F_\Delta$*, the (sharp) triangle inequalities are satisfied. That is,*

$$\ell_{ij} + \ell_{jk} > \ell_{ki}, \quad \ell_{jk} + \ell_{ki} > \ell_{ij}, \quad \ell_{ki} + \ell_{ij} > \ell_{jk}.$$

The logarithm of this function,

$$\lambda_{ij} = 2 \log \ell_{ij}, \tag{2}$$

is called the ***logarithmic lengths****.*

Fact 2.8. *Let* d *be a PL-metric on a marked surface* (S, V) *and let* Δ *be a geodesic triangulation of* (S, V, d)*. Then* d *induces a discrete metric* ℓ_d

on (S, V, Δ) by measuring the lengths of the edges in E_Δ.

Vice versa, each discrete metric ℓ on a marked triangulated surface (S, V, Δ) induces a PL-metric on (S, V), which we denote by d_ℓ.

Indeed, ℓ imposes a Euclidean metric on each triangle $ijk \in F_\Delta$ by transforming it into a Euclidean triangle with edge lengths $\ell_{ij}, \ell_{jk}, \ell_{ki}$. The metrics fit isometrically along the edges of the triangulation. Thus, by patching up the triangles along the edges we equip the marked surface with a PL-metric.

Penner coordinates Penner coordinates were introduced by Robert Penner in [7] to study the Decorated Teichmüller space.

Definition 2.9. *Let i and j be two ideal points of the hyperbolic plane. Let $\mathcal{H}_i$ and $\mathcal{H}_j$ be two horocycles, anchored at the points i and j, respectively. Let g be the hyperbolic line connecting i and j, and let $p_i = g \cap \mathcal{H}_i$, and $p_j = g \cap \mathcal{H}_j$. The* ***signed horocycle distance*** *between $\mathcal{H}_i$ and $\mathcal{H}_j$ is the hyperbolic distance between the points p_i and p_j, taken negative if $\mathcal{H}_i$ and $\mathcal{H}_j$ intersect. We denote it by λ_{ij}.*

The signed distances between edges of an ideal triangle are illustrated in Figure 2.3. The distance λ_{ij} is negative, whereas the distances λ_{jk} and λ_{ki} are positive.

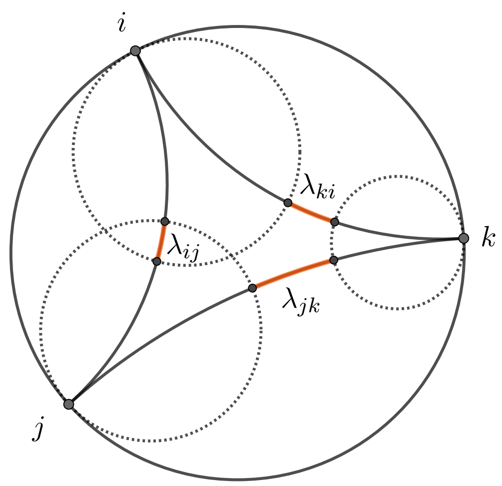


Figure 2.3: Penner coordinates of a decorated ideal hyperbolic triangle ijk, in the Poincaré disc model.

Definition 2.10. ***Penner coordinates*** *is a pair consisting of a triangulation* Δ *of* (S, V) *and a map*

$$\lambda : E_\Delta \to \mathbb{R}, \qquad ij \mapsto \lambda_{ij}.$$

Fact 2.11. *Penner coordinates* (Δ, λ) *define a decorated ideal hyperbolic polyhedron* $(S, V, d_\lambda^{hyp}, \mathcal{H})$*, such that the signed distance between the horocycles* $\mathcal{H}_i$ *and* $\mathcal{H}_j$*, with* $ij \in E_\Delta$*, is* λ_{ij}*.*

Vice versa, let Δ *be a geodesic triangulation of a decorated ideal hyperbolic polyhedron* $(S, V, d_{hyp}, \mathcal{H})$*. Then* $(S, V, d_{hyp}, \mathcal{H})$ *induces Penner coordinates* (Δ, λ) *by measuring the signed horocycle distance between horocycles* $\mathcal{H}_i$ *and* $\mathcal{H}_j$*, with* $ij \in E_\Delta$*.*

The following proposition creates an essential link between PL-metrics and ideal hyperbolic polyhedra.

Proposition 2.12. *Let* (S, V, d) *be a piecewise flat surface, let* Δ *be a Delaunay triangulation of* (S, V, d) *and let* ℓ_d *be the discrete metric induced by* d*. Let* $\lambda : E_\Delta \to \mathbb{R}$ *be defined via Equation (2). Then the hyperbolic metric* d_λ^{hyp}*, induced by the Penner coordinates* (Δ, λ)*, is isometric to the hyperbolic metric* d_{hyp} *induced by the PL-metric* d *via the formula (1).*

The following theorem links the Euclidean Delaunay and ideal Delaunay triangulations.

Theorem 2.13. *Let* (S, V) *be a marked surface with a triangulation* Δ*.*

a) Let $\ell : E_\Delta \to \mathbb{R}_{\geq 0}$ *be a discrete metric, such that* Δ *is a Delaunay triangulation of the piecewise flat surface* (S, V, d_ℓ)*. Let* λ *be the logarithmic lengths of* ℓ *(see Equation (2)). Then* Δ *is an ideal Delaunay triangulation of the decorated ideal hyperbolic polyhedron defined by Penner coordinates* (Δ, λ)*.*

b) Vice versa, let (Δ, λ) *be Penner coordinates on* (S, V)*, such that* Δ *is an ideal Delaunay triangulation of the decorated ideal hyperbolic polyhedron defined by* (Δ, λ)*. Then the map* $\ell : E_\Delta \to \mathbb{R}_{\geq 0}$*, defined by Equation (2), is a discrete metric on* (S, V, Δ)*, and* Δ *is a Delaunay triangulation of the piecewise flat surface* (S, V, d_ℓ)*.*

2.3 Discrete conformal classes

Definition 2.14. *Two PL-metrics* $d, \tilde{d}$ *on a marked surface* (S, V) *are* ***discrete conformally equivalent*** *if the induced hyperbolic metrics* d_{hyp} *and* $\tilde{d}_{hyp}$

*are isometric. This notion induces an equivalence relation, splitting the space of all PL-metrics on (S, V) into equivalence classes called **conformal classes**.*

The notion of discrete conformal equivalence for PL-metrics with prescribed fixed combinatorics was introduced by Feng Luo in [5]. The equivalence of Luo's and our definition for surfaces with prescribed fixed combinatorics has been demonstrated by Boris Springborn, Ulrich Pinkall and Alexander Bobenko in [2, Proposition 5.1.2]. Luo's definition has been generalized by Xianfeng Gu, Feng Luo, Jian Sun and Tianqui Wu in [3, 4], with the aim of comparing metrics with varying combinatorics. Springborn proved that this generalization is also equivalent to Definition 2.14, see [8, Chapter 10].

Definition 2.15. *Let d and $\tilde{d}$ be two conformally equivalent PL-metrics on (S, V). Let Δ and $\tilde{\Delta}$ be Delaunay triangulations of the piecewise flat surfaces (S, V, d) and $(S, V, \tilde{d})$, and let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be the decorations of (S, V, d_{hyp}) with respect to Δ and $\tilde{\Delta}$, respectively. The map*

$$u : V \to \mathbb{R}, \qquad i \mapsto u_i,$$

*where u_i is the signed distance from the horocycle $\mathcal{H}_i$ to the horocycle $\tilde{\mathcal{H}}_i$, is called a **conformal change**, or a **conformal factor**.*

The PL-metric $\tilde{d}$, the decoration $\tilde{\mathcal{H}}$ and the induced Penner coordinates $(\tilde{\Delta}, \tilde{\lambda})$, in dependence of the PL-metric d and the conformal change u, are denoted by $d(u), \mathcal{H}(u)$ and $(\Delta(u), \lambda(u))$, respectively.

The signed distance from the horocycle $\mathcal{H}_i$ to the horocycle $\tilde{\mathcal{H}}_i$ is illustrated in Figure 2.4. The following proposition provides a direct link between the conformal change and the discrete metrics of two discrete conformally equivalent PL-metrics.

Proposition 2.16. *Let d and $\tilde{d}$ be two conformally equivalent PL-metrics on a marked surface (S, V), related by the conformal factor $u : V \to \mathbb{R}$, and let Δ be a geodesic triangulation of the surface (S, V, d), as well as the surface $(S, V, \tilde{d})$. Then the induced discrete metrics ℓ_d and $\ell_{\tilde{d}}$ satisfy*

$$\ell_{\tilde{d}}(jk) = \ell_d(jk) e^{\frac{u_j + u_k}{2}}$$

for every edge $jk \in E_\Delta$.

Remark. *Proposition 2.16 is the original definition of discrete conformal equivalence, due to Luo [5].*

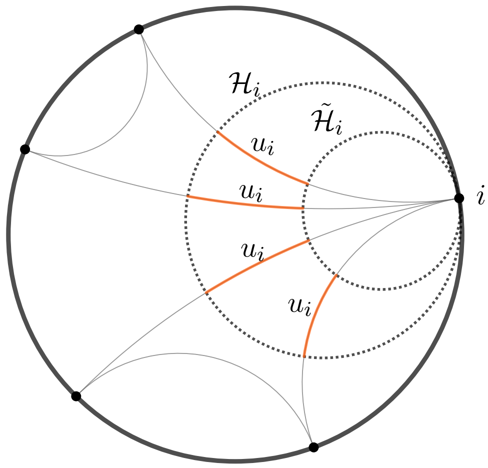


Figure 2.4: The distance between two horocycles.

Proposition 2.17. *Let (S, V, d) be a piecewise flat surface. The conformal class of the PL-metric d is parametrised by the vector space*

$$\mathbb{R}^V = \{u : V \to \mathbb{R}\}.$$

The vector space $\mathbb{R}^V$ is isomorphic to $\mathbb{R}^{|V|}$.

Furthermore, the vector space $\mathbb{R}^V$ can be partitioned as follows.

Definition 2.18. *Let Δ be a triangulation of a piecewise flat surface (S, V, d). The* ***Penner cell*** *of (S, V, d) with respect to Δ is the set*

$$\mathcal{A}_\Delta = \{u \in \mathbb{R}^V \mid \Delta \textit{ is a Delaunay triangulation of } (S, V, d(u))\}.$$

The set of all triangulations of (S, V, d) with non-empty Penner cells is denoted by $\mathfrak{D}(S, V, d)$.

Theorem 2.19. *[Hirotaka Akiyoshi [1]] The set $\mathfrak{D}(S, V, d)$ is finite.*

In particular,

$$\mathbb{R}^V = \bigcup_{\Delta \in \mathfrak{D}(S,V,d)} \mathcal{A}_\Delta,$$

and the Penner cells $\mathcal{A}_\Delta$ are closed top-dimensional cells in $\mathbb{R}^V$.

2.4 Ptolemy flip

Definition 2.20. *Let the Penner coordinates* (Δ, λ) *on* (S, V) *define a decorated ideal hyperbolic polyhedron. Let* $ij \in E_\Delta$*, and consider the quadrilateral built by the neighbouring ideal triangles* $ijk, ijl \in F_\Delta$*. The* ***Ptolemy flip*** *of* ij *consists of replacing the diagonal* ij *of the quadrilateral by the hyperbolic line connecting* k *to* l*.*

The Ptolemy flip has the following properties:

- Any edge can be Ptolemy-flipped.
- ([8], Proposition 3.6) Let $\lambda_a, \ldots, \lambda_d, \lambda_e, \lambda_f$ be the Penner coordinates of the four edges and two diagonals, respectively, of an ideal hyperbolic quadrilateral. If $\ell_x := e^{\frac{\lambda_x}{2}}$, then the lengths ℓ_x satisfy the *Ptolemy relation*

$$\ell_e \ell_f = \ell_a \ell_c + \ell_b \ell_d. \tag{3}$$

- If the original edge is not Delaunay, the flipped edge is.

The following theorem suggests that the Ptolemy flip can be used to compute ideal Delaunay triangulations.

Theorem 2.21. *[Jeffrey Weeks [9]] Let* $(S, V, d_{hyp}, \mathcal{H})$ *be a decorated ideal hyperbolic polyhedron, and let* Δ *be a geodesic triangulation on* $(S, V, d_{hyp}, \mathcal{H})$*. Iteratively flip non-Delaunay edges in* E_Δ *using the Ptolemy flip and update the triangulation until all edges are Delaunay. This algorithm terminates after finitely many flips.*

Fact 2.22. *If a pair of Penner cells has non-empty intersection,* $u \in \mathcal{A}_\Delta \cap \mathcal{A}_{\tilde{\Delta}}$*, there exists a sequence* $\Delta_0, \ldots, \Delta_m$ *of ideal triangulations such that*
- $u \in \mathcal{A}_{\Delta_0} \cap \cdots \cap \mathcal{A}_{\Delta_m}$,
- $\Delta_0 = \Delta, \Delta_m = \tilde{\Delta}$, *and*
- Δ_i *differs from* Δ_{i+1} *by a Ptolemy flip of one edge.*

3 Counterexamples

There exist piecewise flat surfaces whose conformal classes contain more than one metric with constant discrete Gaussian curvature. To demonstrate this, we first construct a so-called *symmetric* piecewise flat surface – a surface where all faces are congruent triangles and all vertices have the same degree. Due to the symmetry, this surface has constant discrete Gaussian curvature. Then we investigate a particular one-parameter family of PL-metrics in the conformal class of this surface to find another PL-metric with constant discrete Gaussian curvature.

A *tetrahedron* is a sphere with four marked points (S, V), a triangulation Δ of (S, V) that has the combinatorics of a standard tetrahedron, and a discrete metric ℓ on (S, V, Δ). The pairs of edges that are not adjacent one to another are called the *opposite edges*. We call a surface of genus two a *double torus*.

Definition 3.1. *An* ***almost symmetric tetrahedron*** *is a tetrahedron with two pairs of opposite edges of equal length. Its faces are two copies of a triangle with edge lengths a, b, c, and two copies of a triangle with edge lengths $\bar{a}, b, c$.*

Let S be a double torus, and let $V \subseteq S$ consist of two points, one white (w) and one black (b). An ***almost symmetric double torus*** *is a piecewise flat surface (S, V, d) with a triangulation Δ, that consists of four copies of a triangle with edge lengths a, b, c and four copies of a triangle with edge lengths $\bar{a}, b, c$, respectively. The edges are glued together according to the scheme represented in Figure 3.1.*

In both cases, we use the notation introduced in Figure 3.1 to label the angles.

Proposition 3.2. *The discrete Gaussian curvature of an almost symmetric tetrahedron with Delaunay edges satisfies*

$$K_1 = K_2, \qquad \text{and} \qquad K_3 = K_4.$$

Proof. Follows from the fact that $W_1 = W_2, W_3 = W_4, A_1 = A_2$ and $A_3 = A_4$. □

Definition 3.3. *An almost symmetric tetrahedron or double torus is called* ***symmetric*** *if all its faces are congruent. We call this face* ***the defining triangle*** *of the tetrahedron or double torus.*

Fact 3.4. *A symmetric tetrahedron or double torus*

- *is defined by the edge lengths a, b, c of the defining triangle. This definition is unique up to the relabeling of the edges.*
- *has constant discrete Gaussian curvature.*
- *has Delaunay edges if and only if the defining triangle is acute.*

An almost symmetric tetrahedron or double torus with edges $a, b, c, \bar{a}, b, c$ is symmetric if and only if $a = \bar{a}$.

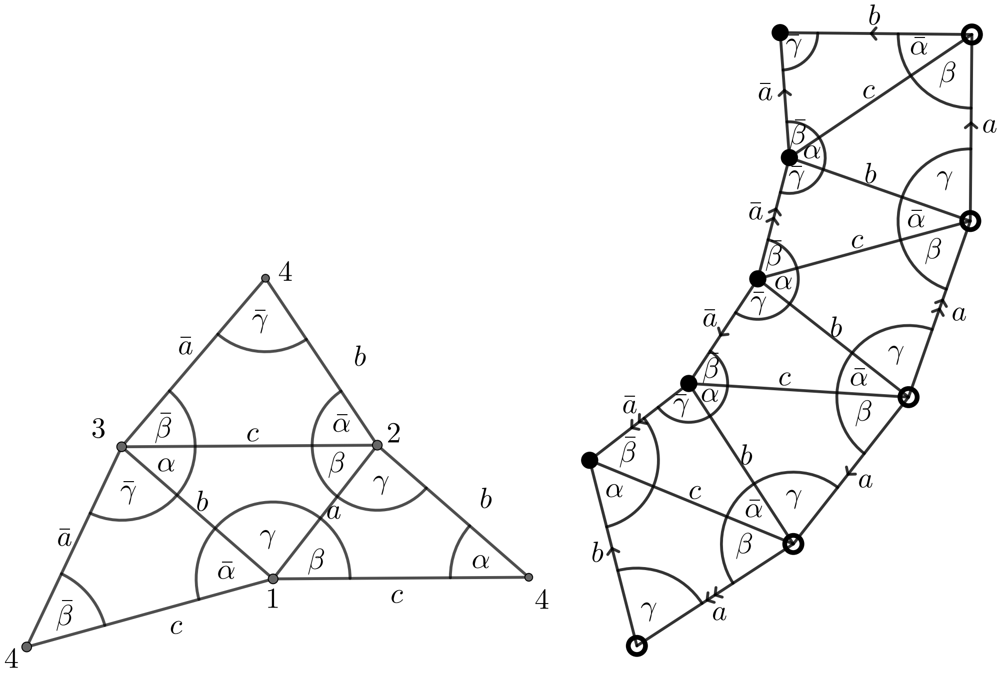


Figure 3.1: An almost symmetric tetrahedron (left) and an almost symmetric double torus (right).

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Consider the symmetric tetrahedron or double torus $\mathfrak{t}$ with defining triangle with edges a_0, b_0, c_0, satisfying the triangle inequalities, such that the edge lengths satisfy

$$a_0 = 1, \qquad a_0 \leq b_0 \leq c_0.$$

Due to the Delaunay properties (Fact 2.1) and Fact 3.4 $\mathfrak{t}$ has Delaunay edges if and only if

$$c_0^2 \leq a_0^2 + b_0^2 = 1 + b_0^2. \tag{4}$$

We apply the following conformal change to the discrete metric ℓ of $\mathfrak{t}$.

Lemma 3.5. *Let*

$$\mathcal{S}_{(b_0,c_0)} := \left[-\log(b_0^2 + c_0^2), \log(b_0^2 + c_0^2)\right].$$

Let further

$$u(v) = (u_1, u_2, u_3, u_4)(v) := (0, 0, v, v), \qquad v \in \mathbb{R}.$$

The tetrahedron given by the discrete metric $\ell(u(v))$ is almost symmetric. It has Delaunay edges if $v \in \mathcal{S}_{(b_0,c_0)}$.

Let

$$u(v) = (u_w, u_b)(v) := (0, v), \qquad v \in \mathbb{R}.$$

The double torus given by the discrete metric $\ell(u(v))$ is almost symmetric. It has Delaunay edges if $v \in \mathcal{S}_{(b_0,c_0)}$.

Proof. For each $v \in \mathbb{R}$, the tetrahedron or double torus with discrete metric $\ell(u(v))$ has edge lengths

$$a = 1, \qquad b = b_0 e^{v/2}, \qquad c = c_0 e^{v/2}, \qquad \bar{a} = e^v,$$

and is thus almost symmetric. The minimal and maximal value of the parameter v follow from Equation (4). $\square$

Corollary 3.6. *The symmetric tetrahedron or double torus with the defining triangle with edge lengths a_0, b_0, c_0 is given by the discrete metric $\ell(u(0))$.*

Fact 3.7. *The map u, defined in Lemma 3.5, parametrises the conformal class of the symmetric double torus given by the discrete metric $\ell(u(0))$ completely up to global scaling. The conformal classes of symmetric double torus thus contain only almost symmetr*

Let A and $\bar{A}$ denote the area of the triangles with side lengths a, b, c and $\bar{a}, b, c$, respectively, and let $F_a, \ldots, F_{\bar{c}}$ denote the area components as in Figure 3.2.

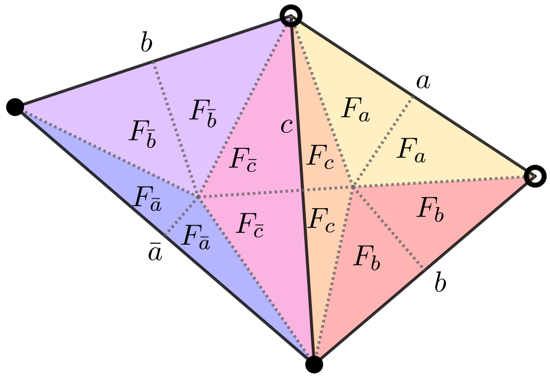


Figure 3.2

Lemma 3.8. *Let u be defined as in Lemma 3.5. A tetrahedron given by the discrete metric $\ell(u(v))$, with $v \in \mathcal{S}_{(b_0,c_0)}$, has constant discrete Gaussian curvature if and only if v is a zero of the map*

$$g_{(b_0,c_0)} : \mathcal{S}_{(b_0,c_0)} \to \mathbb{R}, \qquad v \mapsto 2\pi(F_{\bar{a}} - F_a) + (\alpha - \bar{\alpha})(A + \bar{A}).$$

A double torus with discrete metric $\ell(u(v))$, with $v \in \mathcal{S}_{(b_0,c_0)}$ has constant discrete Gaussian curvature if and only if v is a zero of the map

$$h_{(b_0,c_0)} : \mathcal{S}_{(b_0,c_0)} \to \mathbb{R}, \qquad v \mapsto \pi(F_{\bar{a}} - F_a) + (\bar{\alpha} - \alpha)(A + \bar{A}).$$

Proof. From Lemma 3.5 and Proposition 3.2 we know that the discrete Gaussian curvature of any tetrahedron given by $\ell(u(v))$ satisfies $K_1 = K_2$ and $K_3 = K_4$. The equality of the values of the discrete Gaussian curvature at vertices 1 and 3 is equivalent to the expression:

$$W_1 A_3 = W_3 A_1 \iff 2\pi(F_{\bar{a}} - F_a) = (\bar{\alpha} - \alpha)(A + \bar{A}).$$

The discrete Gaussian curvatures at vertices b and w are equal if and only if

$$W_w A_b = W_b A_w \iff \pi(F_{\bar{a}} - F_a) = (\alpha - \bar{\alpha})(A + \bar{A}).$$

$\square$

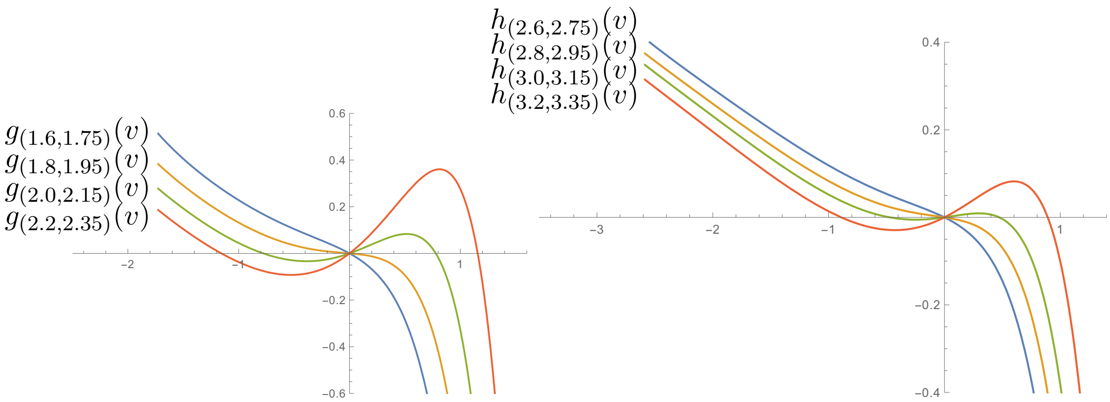


Figure 3.3: Graphs of the functions g (left) and h (right) for various values of b_0 and c_0.

The number of critical points of g and h varies depending on the choice of (b_0, c_0). Figure 3.3 illustrates the graphs of g and h for various values of (b_0, c_0). The red and green curves correspond to conformal classes with more than one metric with constant discrete Gaussian curvature.

Corollary 3.9. *Let $N(b_0, c_0)$ be the number of PL-metrics with constant discrete Gaussian curvature in the conformal class of the symmetric double torus with the defining triangle defined by (b_0, c_0). Then N varies depending on the values of (b_0, c_0). In other words, the number of PL-metrics with constant discrete Gaussian curvature in conformal class is not constant even after fixing the genus and the number of marked points.*

Concluding remarks on the uniqueness of PL-metrics with constant discrete Gaussian curvature

Uniqueness of PL-metrics with constant discrete Gaussian curvature up to global scaling in discrete conformal classes holds in three special cases:

- *S is of genus zero and $|V| = 3$.*
  This follows from the positive semi-definiteness of the second derivative of the function $\mathbb{F}$, see Fact 4.12 for the definition.

- *S is of genus one.*
  In this case the Yamabe problem is equivalent to the discrete uniformisation problem. The uniqueness follows from the positive semi-definiteness of the second derivative of function $\mathbb{E}$ (Definition 4.3) and was proved in [2].

- *S is of genus larger than one and $|V| = 1$.*
  This case is trivial, since every discrete conformal class consists of one PL-metric up to a global scaling.

4 Variational principles

4.1 Two essential building blocks

The function $\mathbb{E}$, which we will introduce shortly, was defined by Bobenko, Pinkall and Springborn in [2]. Its building block is a peculiar function f.

Definition 4.1. *Consider a Euclidean triangle with edge lengths a, b, c and angles α, β, γ, opposite to edges a, b, c, respectively. Let*

$$x = \log a, \qquad y = \log b, \qquad z = \log c,$$

as illustrated in Figure 4.1a. Let $\mathfrak{A}$ be the set of all triples $(x, y, z) \in \mathbb{R}^3$, such that (a, b, c) satisfy the triangle inequalities:

$$\mathfrak{A} = \{(x, y, z) \in \mathbb{R}^3 \mid a + b - c > 0, a - b + c > 0, -a + b + c > 0\}.$$

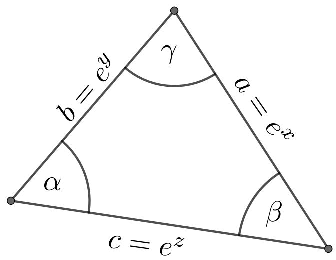
The function f is defined as follows:

$$f : \mathfrak{A} \to \mathbb{R}, \quad f(x, y, z) = \alpha x + \beta y + \gamma z + \mathbb{L}(\alpha) + \mathbb{L}(\beta) + \mathbb{L}(\gamma),$$

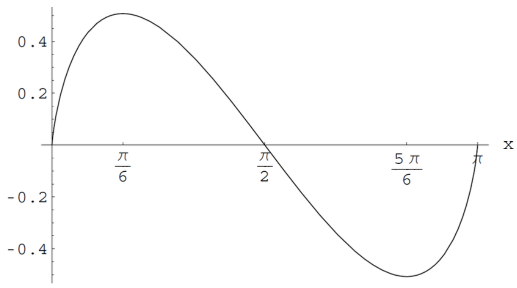
where

$$\mathbb{L}(\alpha) = -\int_0^\alpha \log|2\sin(t)|\, dt$$

is Milnor's Lobachevsky function, introduced by Milnor in [6].



(a) Logarithmic edge lengths of a triangle.



(b) Graph of Milnor's Lobachevsky function, $y = \mathbb{L}(x)$.

Figure 4.1

Fact 4.2. *Milnor's Lobachevsky function* $\mathbb{L}(x)$ *is odd,* 2π*-periodic, and smooth except at* $x \in \pi\mathbb{Z}$*.*

We first define the function $\mathbb{E}$ on the Penner cells.

Definition 4.3. *Let* (S, V, d) *be a piecewise flat surface, and let* $\Delta \in \mathfrak{D}(S, V)$*. On the Penner cell* $\mathcal{A}_\Delta$*, the function* $\mathbb{E}_\Delta$ *is defined as follows:*

$$\mathbb{E}_\Delta : \mathcal{A}_\Delta \to \mathbb{R},$$

$$\mathbb{E}_\Delta(u) = \sum_{ijk \in F_\Delta} \left(2f\left(\frac{\tilde{\lambda}_{ij}}{2}, \frac{\tilde{\lambda}_{jk}}{2}, \frac{\tilde{\lambda}_{ki}}{2}\right) - \frac{\pi}{2}(\tilde{\lambda}_{ij} + \tilde{\lambda}_{jk} + \tilde{\lambda}_{ki}) \right) + 2\pi \sum_{i \in V} u_i,$$

where $\tilde{\lambda}_{ij}$ *are the logarithmic lengths of the discrete metric induced by the PL-metric* $d(u)$*.*

Lemma 4.4. *The partial derivatives of the function* $\mathbb{E}_\Delta$ *satisfy the equation*

$$\frac{\partial \mathbb{E}_\Delta}{\partial u_i} = W_i, \tag{5}$$

where W_i *is the angle defect at vertex* i *of the piecewise flat surface* $(S, V, d(u))$*.*

Proof. Follows from [2, Proposition 4.1.2]. □

The functions f and $\mathbb{E}_\Delta$ have the following properties:

Proposition 4.5 (Properties of f and $\mathbb{E}_\Delta$)**.** *The functions* f *and* $\mathbb{E}_\Delta$ *are analytic and locally convex on* $\mathfrak{A}$ *and* $\mathcal{A}_\Delta$*, respectively. Their second derivatives are positive semidefinite quadratic forms with one-dimensional kernels, spanned by* $(1,1,1) \in \mathfrak{A}$*,* $(1,\dots,1) \in \mathcal{A}_\Delta$*, respectively. Further,*

$$f(x+t, y+t, z+t) = f(x,y,z) + \pi t \qquad \text{for all } (x,y,z) \in \mathfrak{A},$$
$$\mathbb{E}(u + t(1,\dots,1)) = \mathbb{E}(u) + 2\pi\chi(S)t \qquad \text{for all } u \in \mathcal{A}_\Delta,$$

where $\chi(S)$ *denotes the Euler characteristic of the surface* S*.*

Proof. See [2, Equation (4-5)] or [8, Propositions 7.2 and 7.7]. □

Theorem 4.6 (Extension)**.** *For a conformal factor $u \in \mathbb{R}^V$, let $\Delta(u)$ be a Delaunay triangulation of the surface $(S, V, d(u))$. The map*

$$\mathbb{E} : \mathbb{R}^V \to \mathbb{R}, \qquad u \mapsto \mathbb{E}_{\Delta(u)}(u),$$

is well-defined and twice continuously differentiable. Its second derivative is a positive semidefinite quadratic form with one-dimensional kernel, spanned by $(1, \dots, 1) \in \mathbb{R}^V$. Explicitly,

$$d^2\mathbb{E} = \frac{1}{4} \sum_{ij \in E} (\cot \alpha_k^{ij} + \cot \alpha_l^{ij})(du_i - du_j)^2.$$

Proof. Follows from [2, Proposition 4.1.6] and [8, Section 7 and 8]. □

***

Definition 4.7. *Let (S, V, d) be a piecewise flat surface, and let $\Delta \in \mathfrak{D}(S, V)$. On the Penner cell $\mathcal{A}_\Delta$, the function A_{tot}^Δ is defined as follows:*

$$A_{tot}^\Delta : \mathcal{A}_\Delta \to \mathbb{R}, \qquad A_{tot}^\Delta(u) = \sum_{ijk \in F_\Delta} A_{ijk}(u),$$

where $A_{ijk}(u)$ is the area of the triangle with vertices $i, j, k \in V$ on the piecewise flat surface $(S, V, d(u))$.

Notation 4.8. *Consider a triangle ijk, with edges and angles labeled as in Figure 4.2. Let A_{jk}^i denote half of the area of the isosceles triangle with base length ℓ_{jk} and legs of length R_{ijk}. Then*

$$A_{jk}^i = \frac{\ell_{jk}^2}{8} \cot \alpha_i^{jk}.$$

The area of the Voronoi cell V_i of a piecewise flat surface (S, V, d) satisfies the equation

$$A_i = \sum_{jk | ijk \in F_\Delta} A_{ki}^j + A_{ij}^k.$$

Lemma 4.9. *The function A_{tot}^Δ is analytic. Its partial derivatives satisfy the equation*

$$\frac{\partial A_{tot}^\Delta}{\partial u_i} = 2A_i. \tag{6}$$

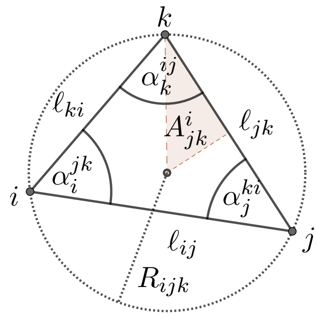


Figure 4.2

Its second derivative is

$$d^2 A_{tot}^{\Delta} = \sum_{ij \in E_\Delta} 2A_{ij}(du_i + du_j)^2 - \frac{1}{2}\sum_{ij \in E_\Delta} (R_{ijk}^2 \cot \alpha_k^{ij} + R_{ijl}^2 \cot \alpha_l^{ij})(du_i - du_j)^2,$$

where $A_{ij} = A_{ij}^k + A_{ij}^l$.

Proof. The function A_{tot}^{Δ} is analytic since the area A_{ijk} of each triangle $ijk \in F_\Delta$ is an analytic function with respect to the conformal factors. Deploying Notation 4.8,

$$\frac{\partial A_{ijk}}{\partial u_i} = 2A_{ki}^j + 2A_{ij}^k - R_{ijk}^2 \frac{\partial}{\partial u_i}(\underbrace{\alpha_i^{jk} + \alpha_j^{ki} + \alpha_k^{ij}}_{=\pi}) = 2A_{ki}^j + 2A_{ij}^k.$$

Due to the linearity of the area function,

$$\frac{\partial A_{tot}^{\Delta}}{\partial u_i} = \sum_{jk | ijk \in F_\Delta} 2A_{ki}^j + 2A_{ij}^k = 2A_i.$$

In the upcoming calculations we use the following formula, which was proved in [2, Equation (4-8)].

Lemma. *Let* a, b, c *be edge lengths of a triangle,* α, β, γ *angles opposite of* a, b, c*, respectively, and let* $\lambda_a, \lambda_b, \lambda_c$ *be the logarithmic lengths. Then*

$$2d\alpha = (\cot\beta + \cot\gamma)d\lambda_a - \cot\gamma d\lambda_b - \cot\beta d\lambda_c.$$

Since

$$\frac{\partial A_{ki}^j}{\partial u_i} = A_{ki}^j - \frac{1}{2}R_{ijk}^2 \frac{\partial \alpha_j^{ki}}{\partial u_i} = A_{ki}^j - \frac{1}{4}R_{ijk}^2 \cot \alpha_k^{ij},$$

we obtain

$$\frac{\partial^2 A_{tot}}{\partial u_i^2} = 2A_i - \frac{1}{2} \sum_{jk|ijk \in F_\Delta} R_{ijk}^2 (\cot \alpha_k^{ij} + \cot \alpha_j^{ki}).$$

Let $i, j \in V$ be two vertices. If j is not adjacent to i,

$$\frac{\partial^2 A_{tot}}{\partial u_i \partial u_j} = 0.$$

If j is adjacent to i, let $k, l \in V$ be the two vertices such that $ijk, ijl \in F_\Delta$. Since

$$\frac{\partial A_{jk}^i}{\partial u_i} = -\frac{1}{2} R_{ijk}^2 \frac{\partial \alpha_i^{jk}}{\partial u_i} = \frac{1}{4} R_{ijk}^2 (\cot \alpha_j^{ki} + \cot \alpha_k^{ij}),$$

the mixed partial derivative equals

$$\frac{\partial^2 A_{tot}}{\partial u_i \partial u_j} = \underbrace{2A_{ij}^k + 2A_{ij}^l}_{=2A_{ij}} + \frac{1}{2}(R_{ijk}^2 \cot \alpha_k^{ij} + R_{ijl}^2 \cot \alpha_l^{ij}).$$

Thus,

$$d^2 A_{tot} = \sum_{ij \in E_\Delta} 2A_{ij}(du_i + du_j)^2 - \frac{1}{2} \sum_{ij \in E_\Delta} (R_{ijk}^2 \cot \alpha_k^{ij} + R_{ijl}^2 \cot \alpha_l^{ij})(du_i - du_j)^2.$$

$\square$

Theorem 4.10 (Extension)**.** *For a conformal factor $u \in \mathbb{R}^V$, let $\Delta(u)$ be a Delaunay triangulation of the surface $(S, V, d(u))$. The map*

$$A_{tot} : \mathbb{R}^V \to \mathbb{R}, \qquad u \mapsto A_{tot}^{\Delta(u)}(u),$$

is well-defined and once continuously differentiable.

Proof. Due to Lemma 4.9 the function A_{tot} is once continuously differentiable in the interior of every Penner cell. Due to Theorem 2.21 and Fact 2.22 we may assume that at the points at the boundary of two Penner cells $u \in \mathcal{A}_{\Delta(u)} \cap \mathcal{A}_{\tilde{\Delta}(u)}$ the triangulations $\Delta(u)$ and $\tilde{\Delta}(u)$ differ by one Ptolemy flip of a diagonal of a cyclic polygon. The area of a Voronoi cell remains invariant under such Ptolemy flips due to Equation (3) and Theorem 2.13. $\square$

Remark. *The function A_{tot} is, in fact, twice continuously differentiable. This can be proved by a long and unilluminating calculation.*

4.2 The variational principles

Theorem 4.11 (Variational principle with equality constraints)**.** *Let (S, V, d) be a piecewise flat surface. The PL-metrics with constant discrete Gaussian curvature in the conformal class of the metric d are the critical points of the function*

$$\mathbb{E} : \mathbb{R}^V \to \mathbb{R}, \qquad u \mapsto \mathbb{E}(u),$$

under the constraint

$$A_{tot}(u) = 1.$$

Proof. We use the method of Lagrange multipliers. A conformal factor $u \in \mathbb{R}^V$ is a critical point of the function $\mathbb{E}$ under the constraint $A_{tot} = 1$ if and only if there exists a Lagrange multiplier $\lambda \in \mathbb{R}$, such that

$$0 = \frac{\partial(\mathbb{E} - \lambda A_{tot})}{\partial u_i} \overset{(5),(6)}{=} W_i - 2\lambda A_i.$$

This holds if and only if

$$\frac{W_i}{A_i} = 2\lambda = const.$$

□

Remark. *The parameter λ can be determined by the so-called discrete Gauss-Bonnet theorem. Indeed, let (S, V, d) be a piecewise flat surface with constant discrete Gaussian curvature K_{av} at every vertex. Denote the total area of the surface by A_{tot}, and the Euler characteristics of S by $\chi(S)$. Then,*

$$K_{av} = \frac{2\pi\chi(S)}{A_{tot}}.$$

This follows from the Euler characteristic. Let Δ be a geodesic triangulation of the surface (S, V, d). Then:

$$2\pi\chi(S) = 2\pi|V| - \pi|F_\Delta| = \sum_{i \in V}\left(2\pi - \sum_{jk|ijk \in F_\Delta} \alpha_i^{jk}\right) = \sum_{i \in V} W_i = \sum_{i \in V} A_i K_i.$$

Fact 4.12 (Alternative variational principle to Theorem 4.11)**.** *The PL-metrics with constant discrete Gaussian curvature in the conformal class of the metric d are also the critical points of the function*

$$\mathbb{F} : \mathbb{R}^V \to \mathbb{R}, \qquad u \mapsto \mathbb{F}(u) = \mathbb{E}(u) - \pi\chi(S)\log(A_{tot}).$$

Indeed,

$$0 = \frac{\partial \mathbb{F}}{\partial u_i} \overset{(5),(6)}{=} W_i - \frac{2\pi\chi(S)}{A_{tot}} A_i.$$

This holds if and only if

$$\frac{W_i}{A_i} = \frac{2\pi\chi(S)}{A_{tot}}.$$

Theorem 4.13 (Variational principle with inequality constraints)**.** *Let (S, V, d) be a piecewise flat surface. The existence of PL-metrics with constant discrete Gaussian curvature in the conformal class of the metric d follows from the existence of minima of the function $\mathbb{E}$ under the following inequality constraints:*

- *if the Euler characteristic of S satisfies $\chi(S) = 2$, the inequality constraint is*
$$A_{tot} \geq 1,$$
- *if the Euler characteristic of S satisfies $\chi(S) < 0$, the inequality constraint is*
$$A_{tot} \leq 1.$$

Proof. Proposition 4.15 shows that, if $u \in \mathbb{R}^V$ is a minimum of the function $\mathbb{E}$ under one of these constraints, then $A_{tot}(u) = 1$. Since a minimum is a critical point, the claim follows from Theorem 4.11. □

Proposition 4.14. *Let*

$$\mathcal{A}_+ = \{u \in \mathbb{R}^V \mid A_{tot}(u) \geq 1\}, \qquad \mathcal{A}_- = \{u \in \mathbb{R}^V \mid A_{tot}(u) \leq 1\},$$

be two sets in the vector space $\mathbb{R}^V$. The sets $\mathcal{A}_+$ and $\mathcal{A}_-$ have the following properties:

a) The sets $\mathcal{A}_+$ and $\mathcal{A}_-$ are closed subsets of $\mathbb{R}^V$.

b) Let $\mathbb{I} = (1, \ldots, 1) \in \mathbb{R}^V$, and let $u \in \mathbb{R}^V$ be a conformal factor. Then the rays

$$\mathcal{R}_u^+ = \left\{ u + c\,\mathbb{I} \mid c \geq -\frac{1}{2}\log A_{tot}(u) \right\}, \ \mathcal{R}_u^- = \left\{ u + c\,\mathbb{I} \mid c \leq -\frac{1}{2}\log A_{tot}(u) \right\}$$

are completely contained in the sets $\mathcal{A}_+$ and $\mathcal{A}_-$, respectively. The sets $\mathcal{A}_+$ and $\mathcal{A}_-$ are thus unbounded.

Proof. a) The proof follows from the fact that the sets $\mathcal{A}_+$ and $\mathcal{A}_-$ satisfy the equation

$$\mathcal{A}_+ = A_{tot}^{-1}([1, \infty)), \qquad \mathcal{A}_- = A_{tot}^{-1}([0, 1]).$$

b) The statement follows from the fact that

$$A_{tot}(u + c\,\mathbb{I})) = A_{tot}(u)\exp(2c).$$

□

Proposition 4.15. *Let (S, V, d) be a piecewise flat surface. If*
- *the Euler characteristic of S satisfies $\chi(S) = 2$ and the function $\mathbb{E}$ attains a minimum in the set $\mathcal{A}_+$, or*
- *the Euler characteristic of S satisfies $\chi(S) < 0$ and the function $\mathbb{E}$ attains a minimum in the set $\mathcal{A}_-$,*

the minimum lies at the boundary of the sets,

$$\partial\mathcal{A}_+ = \partial\mathcal{A}_- = \left\{ u \in \mathbb{R}^V \mid A_{tot}(u) = 1 \right\}.$$

Proof. Let $\chi(S) = 2$, let $u \in \mathcal{A}_+$ be a minimum of the function $\mathbb{E}$ in $\mathcal{A}_+$ and let $c \geq -\frac{1}{2}\log A_{tot}(u)$. Then $u + c\,\mathbb{I} \in \mathcal{A}_+$ due to Proposition 4.14 and

$$\mathbb{E}(u) \leq \mathbb{E}(u + c\,\mathbb{I}) = \mathbb{E}(u) + 2\chi(S)\pi c,$$

with equality if and only if $-\frac{1}{2}\log A_{tot}(u) = c = 0$. This implies that u lies at the boundary of $\mathcal{A}_+$.

For surfaces with $\chi(S) < 0$ the proof is analogous. □

5 Existence

In this section we prove Theorem 1.2. The proof is based on several key observations of the behaviour of sequences of conformal factors $(u_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^V$. The reason for the necessity of these observations is the deployment of Theorem 5.1.

Theorem 1.2 is proved in Section 5.1. In Section 5.2 we study the behaviour of sequences in $\mathbb{R}^V$. Finally, in Section 5.3 we prove Theorem 5.2 and Theorem 5.3, which are the main two ingredients of the proof of Theorem 1.2.

5.1 Proof of Theorem 1.2

Recall the statement of Theorem 1.2:

Theorem (Discrete uniformisation theorem)**.** *For every PL-metric d on a marked surface (S, V), there exists a conformally equivalent PL-metric $\tilde{d}$ such that the surface $(S, V, \tilde{d})$ has constant discrete Gaussian curvature.*

For surfaces of genus one ($\chi(S) = 0$) the Yamabe problem is equivalent to the discrete uniformisation problem, which has been proved in [4, Theorem 1.2] and [8, Theorem 11.1]. We prove the claim for surfaces with $\chi(S) = 2$ and $\chi(S) < 0$.

Due to Theorem 4.13 it suffices to find the minima of the function $\mathbb{E}$ in the sets $\mathcal{A}_+$ if $\chi(S) = 2$ and $\mathcal{A}_-$ if $\chi(S) < 0$. Proposition 4.14 ensures that the sets $\mathcal{A}_+$ and $\mathcal{A}_-$ are closed and unbounded. We deploy the following classical theorem from calculus. We omit the proof.

Theorem 5.1. *Let $A \subseteq \mathbb{R}^m$ be a closed set and let $f : A \to \mathbb{R}$ a continuous function. If every unbounded sequence $(x_n)_{n\in\mathbb{N}}$ in A has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that*

$$\lim_{k\to\infty} f(x_{n_k}) = +\infty,$$

then f attains a minimum in A.

To obtain minima in the sets $\mathcal{A}_+$ and $\mathcal{A}_-$, it suffices to apply the aforementioned Theorem 5.1 and to prove the following two theorems:

Theorem 5.2. *Let $\chi(S) < 0$ and let $(u_n)_{n\in\mathbb{N}}$ be an unbounded sequence in $\mathcal{A}_-$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, such that*

$$\lim_{k\to\infty} \mathbb{E}(u_{n_k}) = +\infty.$$

Theorem 5.3. *Let $\chi(S) = 2$ and let $(u_n)_{n\in\mathbb{N}}$ be an unbounded sequence in $\mathcal{A}_+$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, such that*

$$\lim_{k\to\infty} \mathbb{E}(u_{n_k}) = +\infty.$$

In the following section we show how to extract these subsequences from any given unbounded sequence in the sets $\mathcal{A}_+$ and $\mathcal{A}_-$.

5.2 Behaviour of sequences of conformal factors

We use the following standard definition from calculus.

Definition. *A sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ is called* ***properly divergent to*** $+\infty$ *if, for each $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$, such that $x_n > M$ for all $n > N$.*

A sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ is called ***properly divergent to*** $-\infty$ *if, for each $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$, such that $x_n < M$ for all $n > N$.*

Throughout this section we consider an unbounded sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^V$. We use the following convention.

Convention 5.4. *The sequence $(u_n)_{n\in\mathbb{N}}$ possesses the following properties:*

- *It lies in a Penner cell $\mathcal{A}_\Delta$ of $\mathbb{R}^V$.*
- *There exists a vertex $i^* \in V$ such that, for all $j \in V$ and $n \in \mathbb{N}$, $u_{i^*,n} \leq u_{j,n}$.*
- *Each coordinate sequence $(u_{j,n})_{n\in\mathbb{N}}$ either converges, diverges properly to $+\infty$ or diverges properly to $-\infty$.*
- *For all $j \in V$ the sequences $(u_{j,n} - u_{i^*,n})_{n\in\mathbb{N}}$ either converge or diverge properly to $+\infty$.*

Every sequence in $\mathbb{R}^V$ possesses a subsequence that satisfies these properties. The first property follows from Akiyoshi's Theorem 2.19.

In addition, we deploy the following notation:

$$\ell_{ij}^n := \ell_{ij} \exp\left(\frac{u_{i,n} + u_{j,n}}{2}\right). \tag{7}$$

Recall that due to Proposition 2.16 the map ℓ^n, defined by Equation (7), is a discrete metric on (S, V, Δ), and Δ is a Delaunay triangulation of $(S, V, d(u))$.

5.2.1 Behaviour of $(\mathbf{u_n})_{n\in\mathbb{N}}$ in one triangle

Consider a triangle in F_Δ, with vertices labeled by $1, 2, 3 \in V$ and an initial discrete metric $\ell_{12}, \ell_{23}, \ell_{31}$, uniquely determined by d. Define

$$\mathcal{A}_{123} := \{(u_1, u_2, u_3) \mid u \in \mathcal{A}_\Delta\}.$$

Let $(u_{1,n}, u_{2,n}, u_{3,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}_{123}$. This implies that, in particular, the edge lengths $\ell_{12}^n, \ell_{23}^n, \ell_{31}^n$ satisfy the triangle inequalities for all $n \in \mathbb{N}$.

Lemma 5.5. *If the sequences $(u_{1,n})_{n\in\mathbb{N}}$ and $(u_{2,n})_{n\in\mathbb{N}}$ diverge properly to $+\infty$ and the sequence $(u_{3,n})_{n\in\mathbb{N}}$ is bounded from above, there exists an $n \in \mathbb{N}$ such that*

$$\ell_{12}^n > \ell_{23}^n + \ell_{31}^n.$$

In other words, there exists no sequence in $\mathcal{A}_{123}$ where two of the coordinate sequences would diverge properly to $+\infty$ and the third would be bounded from above.

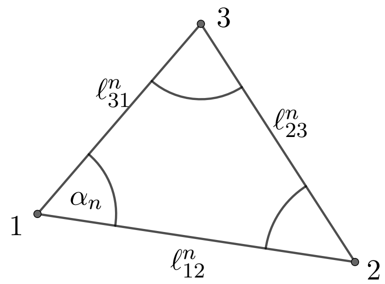


Figure 5.1

Proof. Without loss of generality we may assume that $u_{1,n} \leq u_{2,n}$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}
0 < \ell_{12} &= \exp\left(\frac{-u_{1,n} - u_{2,n}}{2}\right) \ell_{12}^n \\
&\leq \exp\left(\frac{-u_{1,n} - u_{2,n}}{2}\right) (\ell_{23}^n + \ell_{31}^n) \\
&= \exp\left(\frac{u_{3,n} - u_{1,n}}{2}\right) \left(\ell_{23} + \ell_{31} \underbrace{\exp\left(\frac{u_{1,n} - u_{2,n}}{2}\right)}_{\leq 1}\right) \\
&\leq \exp\left(\frac{u_{3,n} - u_{1,n}}{2}\right) (\ell_{23} + \ell_{31}) \xrightarrow{n\to\infty} 0.
\end{aligned}$$

This contradicts the triangle inequality

$$\ell_{12}^n \leq \ell_{23}^n + \ell_{31}^n.$$

□

Lemma 5.6. *Assume that the sequence $(u_{1,n})_{n\in\mathbb{N}}$ diverges properly to $+\infty$ and the sequences $(u_{2,n})_{n\in\mathbb{N}}$ and $(u_{3,n})_{n\in\mathbb{N}}$ converge. Then*

$$\frac{\ell_{12}^n}{\ell_{31}^n} \xrightarrow{n\to\infty} 1,$$

and the sequence of angles α_n, opposite to the edge 23 in the triangle with edge lengths $\ell_{12}^n, \ell_{23}^n, \ell_{31}^n$, satisfies

$$\alpha_n \xrightarrow{n\to\infty} 0.$$

Proof. Dividing both sides of the triangle inequality $\ell_{31}^n \leq \ell_{23}^n + \ell_{12}^n$ by ℓ_{31}^n yields the inequality

$$1 \leq \frac{\ell_{23}^n}{\ell_{31}^n} + \frac{\ell_{12}^n}{\ell_{31}^n} = \frac{\ell_{23}}{\ell_{31}} \exp\left(\frac{1}{2}(u_{2,n} - u_{1,n})\right) + \frac{\ell_{12}^n}{\ell_{31}^n}.$$

Dividing both sides of the triangle inequality $\ell_{12}^n \leq \ell_{23}^n + \ell_{31}^n$ by ℓ_{12} yields the inequality

$$1 \leq \frac{\ell_{23}}{\ell_{12}} \exp\left(\frac{1}{2}(u_{3,n} - u_{1,n})\right) + \frac{\ell_{31}^n}{\ell_{12}^n}.$$

Since, for $i = 2, 3, \exp\left(\frac{1}{2}(u_{i,n} - u_{1,n})\right) \xrightarrow{n\to\infty} 0$, we obtain

$$\frac{\ell_{23}^n}{\ell_{31}^n} \xrightarrow{n\to\infty} 0, \qquad \frac{\ell_{23}^n}{\ell_{12}^n} \xrightarrow{n\to\infty} 0.$$

The convergence of the fraction $\frac{\ell_{12}^n}{\ell_{31}^n}$ follows from the inequalities

$$1 \leq \lim_{n\to\infty} \frac{\ell_{12}^n}{\ell_{31}^n} \leq 1.$$

From the cosine rule we obtain the convergence

$$2\cos\alpha_n = \frac{\ell_{12}^n}{\ell_{31}^n} + \frac{\ell_{31}^n}{\ell_{12}^n} - \frac{(\ell_{23}^n)^2}{\ell_{31}^n \ell_{12}^n} \xrightarrow{n\to\infty} 2,$$

and thus $\alpha_n \xrightarrow{n\to\infty} 0$. □

5.2.2 Behaviour of $(\mathbf{u_n})_{\mathbf{n}\in\mathbb{N}}$ around a vertex star and influence on area

Lemma 5.5 yields the following key observation:

Corollary 5.7. *At every triangle $t = ijl \in F_\Delta$, at least two of the three sequences $(u_{i,n} - u_{i^*,n})_{n\in\mathbb{N}}$, $(u_{j,n} - u_{i^*,n})_{n\in\mathbb{N}}$, $(u_{l,n} - u_{i^*,n})_{n\in\mathbb{N}}$ converge.*

We observe the following: if, at a vertex $i \in V$, the sequence $(u_{i,n} - u_{i^*,n})_{n\in\mathbb{N}}$ diverges, then the sequence $(u_{j,n} - u_{i^*,n})_{n\in\mathbb{N}}$ at any neighbour $j \in V$ converges. We investigate the behaviour of edge lengths and angles in triangles with vertex i.

Definition and Notation 5.8. *Let $i \in V$ be a vertex. A* ***vertex star around vertex*** *i is the subset of the triangles $F_\Delta^i \subseteq F_\Delta$ that contain the vertex i. At a vertex star we use the following labeling: Let $s = \deg i$. We label the vertex i by 0 and the vertices adjacent to i by $1, \ldots, s$, such that, for each $j \in \{1, \ldots, s\}$, the vertices $0, j$ and $j+1$ belong to a triangle. Whenever necessary, we use the convention $1 - 1 = s$.*

In the following we drop the index n labeling the sequences when we talk about angles. Figure 5.2 illustrates the notation used at a vertex star.

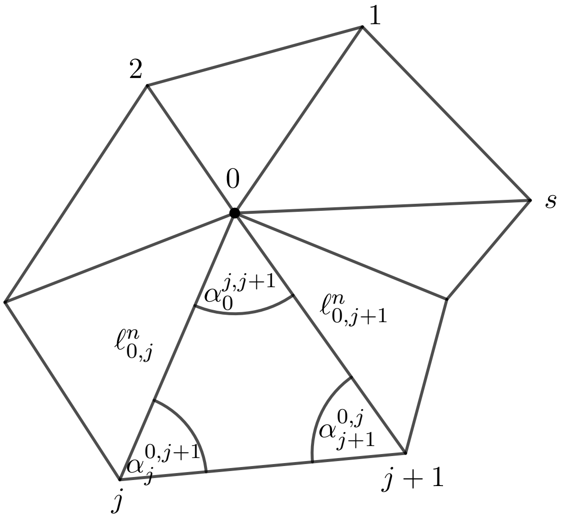


Figure 5.2

Proposition 5.9. *Let $i \in V$ be a vertex and let F_Δ^i be a vertex star around i. Assume that the sequence $(u_{0,n})_{n\in\mathbb{N}}$ diverges properly to $+\infty$. Then the sequences of angles in the triangles in F_Δ^i satisfy*

$$\lim_{n\to\infty} \alpha_0^{j,j+1} = 0, \qquad \lim_{n\to\infty} \alpha_{j+1}^{j,0} = \lim_{n\to\infty} \alpha_j^{j+1,0} = \pi/2, \qquad j \in \{1,\dots,s\}.$$

Proof. Denote the limit of a sequence of angles $\alpha_k^{i,j}$ along $(u_n)_{n\in\mathbb{N}}$ by $\bar{\alpha}_k^{i,j}$. Due to Lemma 5.6,

$$\bar{\alpha}_0^{j,j+1} = 0,$$

and thus, for all $j = 1,\dots,s$,

$$\bar{\alpha}_j^{0,j+1} + \bar{\alpha}_{j+1}^{0,j} = \pi.$$

Since the edges $0j$ are Delaunay, the Delaunay inequality

$$\bar{\alpha}_{j-1}^{0,j} + \bar{\alpha}_{j+1}^{0,j} \le \pi$$

is satisfied for each $j \in \{1,\dots,s\}$. Summing up the Delaunay inequalities for edges $01,\dots,0s$, we obtain

$$\pi s \ge \sum_{j=1}^{s} (\bar{\alpha}_{j-1}^{0,j} + \bar{\alpha}_{j+1}^{0,j}) = \sum_{j=1}^{s} (\bar{\alpha}_j^{0,j+1} + \bar{\alpha}_{j+1}^{0,j}) = \pi s.$$

In other words, each Delaunay inequality becomes an equality in the limit. Let $\varphi := \bar{\alpha}_1^{0,2}$. Then

$$\bar{\alpha}_{j-1}^{0,j} = \varphi, \qquad \bar{\alpha}_j^{0,j-1} = \pi - \varphi,$$

for all $j \in \{1, \dots, s\}$.

To show that $\varphi = \pi/2$, we apply the following equation:

Consider a triangle with sides a, b, c, and opposite angles α, β, γ. Then

$$b - a = c \frac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\gamma}{2}\right)}. \tag{8}$$

Denote the limit of the lengths of edges $\ell_{j,j+1}^n$ by $\lim_{n\to\infty} \ell_{j,j+1}^n = \bar{\ell}_{j,j+1}$. Since, for all $n \in \mathbb{N}$, holds

$$\sum_{j=1}^{s} \left(\ell_{0,j+1}^n - \ell_{0,j}^n\right) = 0,$$

in the limit

$$0 = \lim_{n\to\infty} \sum_{j=1}^{s} \left(\ell_{0,j+1}^n - \ell_{0,j}^n\right) \overset{(8)}{=} \sin\left(\frac{\pi - 2\varphi}{2}\right) \sum_{j=1}^{s} \bar{\ell}_{j,j+1}.$$

Since, for all $j = 1, \dots, s$, the sequences of conformal factors $(u_{j,n})_{n\in\mathbb{N}}$ converge,

$$\sum_{j=1}^{s} \bar{\ell}_{j,j+1} > 0.$$

We deduce that

$$\sin\left(\frac{\pi - 2\varphi}{2}\right) = 0,$$

and thus $\varphi = \pi/2$. □

The last property we need to explore is the behaviour of the area of the triangles under sequences of conformal factors.

Lemma 5.10. *Let $ijl \in F_\Delta$ be a triangle, such that the sequences $(u_{j,n} - u_{i^*,n})_{n\in\mathbb{N}}$ and $(u_{l,n} - u_{i^*,n})_{n\in\mathbb{N}}$ converge. Denote by A_{ijl}^n the area of the triangle with edge lengths $\ell_{ij}^n, \ell_{jk}^n, \ell_{ki}^n$.*

a) If the sequence $(u_{i,n} - u_{i^*,n})_{n\in\mathbb{N}}$ converges, there exists a convergent sequence of real numbers $(C_n)_{n\in\mathbb{N}}$, such that the area of the triangle ijl satisfies

$$\log A_{ijl}^n = C_n + 2u_{i^*,n}.$$

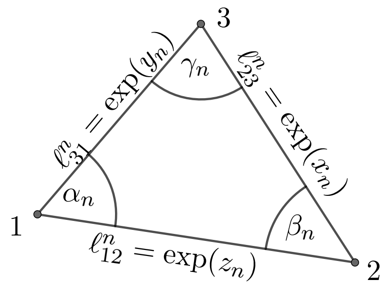


Figure 5.3

b) If the sequence $(u_{i,n} - u_{i^*,n})_{n\in\mathbb{N}}$ diverges to $+\infty$, there exists a convergent sequence of real numbers $(C_n)_{n\in\mathbb{N}}$, such that the area of the triangle ijl satisfies

$$\log A^n_{ijl} = C_n + \frac{1}{2}(u_{i,n} + 3u_{i^*,n}).$$

Proof. The proof follows from the continuity of the area function and from Convention 5.4. □

5.2.3 Behaviour of the function $\mathbb{E}$ along $(\mathbf{u_n})_{\mathbf{n}\in\mathbb{N}}$

Recall the function f from Definition 4.1. Let

$$h : \mathcal{A}_{123} \to \mathbb{R}, \qquad h(u_1, u_2, u_3) := 2f\left(\frac{\tilde\lambda_{12}}{2}, \frac{\tilde\lambda_{23}}{2}, \frac{\tilde\lambda_{31}}{2}\right) - \frac{\pi}{2}(\tilde\lambda_{12} + \tilde\lambda_{23} + \tilde\lambda_{31}).$$

Lemma 5.11. *For any real number $v \in \mathbb{R}$, the function h satisfies the equation*

$$h((u_1, u_2, u_3) + v(1, 1, 1)) = h(u_1, u_2, u_3) - \pi v.$$

Proof. Follows from the property of the function f from Proposition 4.5. □

Proposition 5.12. *Let $(u_{1,n}, u_{2,n}, u_{3,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}_{123}$. Suppose that the sequence $(u_{1,n})_{n\in\mathbb{N}}$ diverges properly to $+\infty$, and the sequences $(u_{2,n})_{n\in\mathbb{N}}$ and $(u_{3,n})_{n\in\mathbb{N}}$ converge to $\overline{u_2}, \overline{u_3}$, respectively. Then*

$$\lim_{n\to\infty} h(u_{1,n}, u_{2,n}, u_{3,n}) = -\pi\left(\log \ell_{23} + \frac{1}{2}(\overline{u_2} + \overline{u_3})\right).$$

Proof. Consider the notation as in Figure 5.3. Then,

$$\frac{1}{2}h(u_{1,n}, u_{2,n}, u_{3,n}) = \alpha_n x_n + \beta_n y_n + \gamma_n z_n + \mathbb{L}(\alpha_n) + \mathbb{L}(\beta_n) + \mathbb{L}(\gamma_n) - \frac{\pi}{2}(x_n + y_n + z_n).$$

In the limit, the sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ satisfy

$$\lim_{n\to\infty} x_n = \log \ell_{23} + \frac{1}{2}(\overline{u_2} + \overline{u_3}) =: \overline{x}, \qquad \lim_{n\to\infty} y_n = +\infty, \qquad \lim_{n\to\infty} z_n = +\infty,$$

and, due to Proposition 5.9,

$$\lim_{n\to\infty} (\alpha_n, \beta_n, \gamma_n) = \left(0, \frac{\pi}{2}, \frac{\pi}{2}\right).$$

Thus,

$$\lim_{n\to\infty} \alpha_n x_n = 0,$$

and, since the Lobachevsky function is continuous and satisfies the equations $\mathbb{L}(0) = \mathbb{L}\left(\frac{\pi}{2}\right) = 0$, in the limit we obtain

$$\lim_{n\to\infty} (\mathbb{L}(\alpha_n) + \mathbb{L}(\beta_n) + \mathbb{L}(\gamma_n)) = 0.$$

In summary,

$$\lim_{n\to\infty} h(u_{1,n}, u_{2,n}, u_{3,n}) = 2 \lim_{n\to\infty} \left[\left(\beta_n - \frac{\pi}{2}\right) y_n + \left(\gamma_n - \frac{\pi}{2}\right) z_n\right] - \pi\overline{x}.$$

We rearrange the expression $\left(\beta_n - \frac{\pi}{2}\right) y_n + \left(\gamma_n - \frac{\pi}{2}\right) z_n$ to obtain

$$\left(\beta_n - \frac{\pi}{2}\right) y_n + \left(\gamma_n - \frac{\pi}{2}\right) z_n = -\frac{1}{2}\alpha_n(y_n + z_n) + \frac{1}{2}(\beta_n - \gamma_n)(y_n - z_n).$$

In the limit, $\lim_{n\to\infty}(\beta_n - \gamma_n) = 0$, and

$$\lim_{n\to\infty} (y_n - z_n) = \log \ell_{31} - \log \ell_{12} + \frac{1}{2}(\overline{u_3} - \overline{u_2}).$$

Thus,

$$\lim_{n\to\infty} \frac{1}{2}(\beta_n - \gamma_n)(y_n - z_n) = 0.$$

It is left to determine the limit

$$\lim_{n\to\infty} \alpha_n(y_n + z_n) = \lim_{n\to\infty} \alpha_n \log \ell_{31}^n + \lim_{n\to\infty} \alpha_n \log \ell_{12}^n.$$

We apply the sine rule and the L'Hospital's rule to obtain the expression

$$\begin{aligned}\lim_{n\to\infty} \alpha_n \log \ell_{31}^n &= \lim_{n\to\infty} (\alpha_n \log \ell_{23}^n + \alpha_n \log \sin \beta_n - \alpha_n \log \sin \alpha_n) \\ &= -\lim_{n\to\infty} \alpha_n \log \sin \alpha_n = 0.\end{aligned}$$

Similarly, $\lim_{n\to\infty} \alpha_n \log \ell_{12}^n = 0$.

Altogether, we see that

$$\lim_{n\to\infty} h(u_{1,n}, u_{2,n}, u_{3,n}) = -\pi\overline{x}.$$

$\square$

Lemma 5.13. *There exists a convergent sequence $(C_n)_{n\in\mathbb{N}}$ such that the function $\mathbb{E}$ satisfies*

$$\mathbb{E}(u_n) = C_n + 2\pi \left(u_{i^*,n}\chi(S) + \sum_{j\in V} (u_{j,n} - u_{i^*,n}) \right).$$

Proof. Due to the Euler formula, $2|V| - |F_\Delta| = 2\chi(S)$. Applying Lemma 5.11 we obtain the equality

$$\begin{aligned}
\mathbb{E}(u_n) &= \sum_{ijl\in F_\Delta} h(u_{i,n}, u_{j,n}, u_{l,n}) + 2\pi \sum_{j\in V} u_{j,n} \\
&= \underbrace{\sum_{ijl\in F_\Delta} h((u_{i,n}, u_{j,n}, u_{l,n}) - u_{i^*,n}(1,1,1))}_{=:C_n} - \pi|F_\Delta| u_{i^*,n} + 2\pi \sum_{j\in V} u_{j,n} \\
&= C_n + 2\pi \left(u_{i^*,n}\chi(S) + \sum_{j\in V} (u_{j,n} - u_{i^*,n}) \right).
\end{aligned}$$

The sequence $(C_n)_{n\in\mathbb{N}}$ converges due to Corollary 5.7 and Proposition 5.12. $\square$

5.3 Proofs of Theorem 5.2 and Theorem 5.3

The two essential ingredients for the proofs of Theorem 5.2 and Theorem 5.3 are Lemma 5.10 and Lemma 5.13.

Theorem (Theorem 5.2)**.** *Let $\chi(S) < 0$ and let $(u_n)_{n\in\mathbb{N}}$ be an unbounded sequence in $\mathcal{A}_-$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, such that*

$$\lim_{k\to\infty} \mathbb{E}(u_{n_k}) = +\infty.$$

Proof. We assume that $(u_n)_{n\in\mathbb{N}}$ satisfies Convention 5.4. Due to Lemma 5.13 there exists a convergent sequence $(C_n)_{n\in\mathbb{N}}$ such that

$$\mathbb{E}(u_n) = C_n + 2\pi \left(u_{i^*,n}\chi(S) + \sum_{j\in V} (u_{j,n} - u_{i^*,n}) \right).$$

The sequence

$$\left(\sum_{j\in V}(u_{j,n}-u_{i^*,n})\right)_{n\in\mathbb{N}}$$

is bounded from below by zero due to Convention 5.4. The sequence $(u_{i^*,n})_{n\in\mathbb{N}}$ diverges to $-\infty$ due to Lemma 5.10 and the fact that $(u_n)_{n\in\mathbb{N}}$ lies in $\mathcal{A}_-$ and is unbounded. Thus,

$$\lim_{n\to\infty}\mathbb{E}(u_n)=+\infty.$$

□

Theorem (Theorem 5.3)**.** *Let $\chi(S)=2$ and let $(u_n)_{n\in\mathbb{N}}$ be an unbounded sequence in $\mathcal{A}_+$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, such that*

$$\lim_{k\to\infty}\mathbb{E}(u_{n_k})=+\infty.$$

Proof. We assume that $(u_n)_{n\in\mathbb{N}}$ satisfies Convention 5.4. Due to Lemma 5.13 there exists a convergent sequence $(C_n)_{n\in\mathbb{N}}$ such that

$$\mathbb{E}(u_n)=C_n+2\pi\left(2u_{i^*,n}+\sum_{j\in V}(u_{j,n}-u_{i^*,n})\right).$$

The sequence

$$\left(\sum_{j\in V}(u_{j,n}-u_{i^*,n})\right)_{n\in\mathbb{N}}$$

is bounded from below by zero due to Convention 5.4. We distinguish three cases.

Case 1: The sequence $(\mathbf{u}_{\mathbf{i}^*,\mathbf{n}})_{\mathbf{n}\in\mathbb{N}}$ diverges properly to $+\infty$.
It follows immediately that

$$\lim_{n\to\infty}\mathbb{E}(u_n)=+\infty.$$

Case 2: The sequence $(\mathbf{u}_{\mathbf{i}^*,\mathbf{n}})_{\mathbf{n}\in\mathbb{N}}$ converges.
Since the sequence $(u_n)_{n\in\mathbb{N}}$ is unbounded, there exists a vertex $j\in V$ such that the sequence $(u_{j,n}-u_{i^*,n})_{n\in\mathbb{N}}$ diverges properly to $+\infty$. Thus,

$$\lim_{n\to\infty}\mathbb{E}(u_n)=+\infty.$$

Case 3: The sequence $(\mathbf{u}_{\mathbf{i}^*,\mathbf{n}})_{\mathbf{n}\in\mathbb{N}}$ diverges properly to $-\infty$.
There exists a vertex $i\in V$, such that the sequence $(u_{i,n}+3u_{i^*,n})_{n\in\mathbb{N}}$ is

bounded from below. This is due to Lemma 5.10 and the fact that the sequence $(u_n)_{n\in\mathbb{N}}$ lies in $\mathcal{A}_+$. We obtain

$$2u_{i^*,n} + \sum_{j\in V}(u_{j,n} - u_{i^*,n}) = -2u_{i^*,n} + (u_{i,n} + 3u_{i^*,n}) + \sum_{j\in V, j\neq i}(u_{j,n} - u_{i^*,n}).$$

Since both sequences

$$\left(\sum_{j\in V, j\neq i}(u_{j,n} - u_{i^*,n})\right)_{n\in\mathbb{N}} \qquad \text{and} \qquad (u_{i,n} + 3u_{i^*,n})_{n\in\mathbb{N}}$$

are bounded from below, and the sequence $(-2u_{i^*,n})_{n\in\mathbb{N}}$ diverges properly to $+\infty$,

$$\lim_{n\to\infty} \mathbb{E}(u_n) = +\infty.$$

□

Acknowledgement. This research was supported by DFG SFB/Transregio 109 "Discretization in Geometry and Dynamics".

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