Isothermic constrained Willmore tori

vorgelegt von
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an der Fakultät II - Mathematik und Naturwissenschaften
der Technischen Universität Berlin
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften
Dr. rer. nat.

genehmigte Dissertation

Promotionsausschuss:

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Tag der wissenschaftlichen Aussprache: 30. September 2019

Berlin 2020

Zusammenfassung

Diese Dissertation beschäftigt sich mit der Klassifikation von isothermen constrained Willmore Tori. Jörg Richter hat in seiner Dissertation gezeigt, dass für jeden in $\mathbb{R}^3$ immersierten, isothermen, constrained Willmore Torus durch eine konforme Änderung der euklidischen Metrik eine Raumform konstruiert werden kann, in der die Immersion konstante mittlere Krümmung (CMC) hat. Wir modifizieren seinen Beweis, sodass er seine Gültigkeit auch dann behält, wenn die Fläche Nabelpunkte besitzt. Die Nabelpunkte isothermer constrained Willmore Tori sind entweder isoliert (Bryant Flächen mit glatten Enden) oder liegen in einer Ebene (Babich-Bobenko Tori). In beiden Fällen haben die Flächen konstante mittlere Krümmung bezüglich einer hyperbolischen Metrik. Ben Andrews and Haizhong Li haben unter der Verwendung von Sphären-Kongruenzen bewiesen, dass jeder in $\mathbb{S}^3$ eingebettete CMC Torus rotationssymmetrisch ist [2]. Wir haben ihre Konstruktion der maximalen Sphären-Kongruenz für geschlossene Flächen in einem Möbius geometrischen Setup reproduziert. Auf diese Weise können wir ihren Beweis für Tori in $\mathbb{S}^3$ auf beliebige Raumformen erweitern. Im Besonderen zeigen wir, dass Babich-Bobenko Tori nicht eingebettet werden können. Dem entsprechend ist jeder eingebettete, isotherme, constrained Willmore Torus $f : M \to \mathbb{R}^3$ entweder eine Bryant Fläche, oder hat nach einer stereographischen Projektion konstante mittlere Krümmung in $\mathbb{S}^3$. Im zweiten Fall ist der Torus eine Kanalfläche. Die Methode lässt sich auch auf periodische Zylinder in $\mathbb{R}^3$ anwenden. Dies führt zu einem neuen Beweis für die bekannte Tatsache [21], dass Delaunay Zylinder die einzigen eigentlich eingebetteten, periodischen, CMC Zylinder in $\mathbb{R}^3$ sind. Es ist bekannt, dass isotherme Kanalflächen in $\mathbb{S}^3$ Möbius äquivalent zu Rotationsflächen sind. Wir geben einen neuen Möbius geometrischen Beweis für diese Tatsache und erhalten damit folgende Klassifizierung: Jeder eingebettete, isotherme, constrained Willmore Torus $f : M \to \mathbb{R}^3$ ist entweder eine isotherme Bryant Fläche mit glatten Enden oder nach einer stereographischen Projektion Möbius äquivalent zu einer Drehfläche in $\mathbb{S}^3$.

Abstract

This dissertation treats the classification of isothermic constrained Willmore tori. Jörg Richter proved in his dissertation that for every immersion $f : M \to \mathbb{R}^3$ of an isothermic constrained Willmore torus, there exists a conformal change of the euclidean metric of $\mathbb{R}^3$ such that the surface has constant mean curvature (CMC) in a space form. We extended his proof such a way that it remains true if the surface has umbilical points. The umbilical points of isothermic constrained Willmore tori are either isolated (Bryant surfaces with smooth ends) or lie in a plane (Babich-Bobenko tori). In both cases the surface has constant mean curvature with respect to a hyperbolic metric.

Using sphere congruences, Ben Andrews and Haizhong Li proved in [2] that every embedded CMC torus $f : M \to \mathbb{S}^3$ has a rotational symmetry. We reproduced their construction of a maximal interior sphere congruence for closed surfaces in a Möbius-geometric setup. This enables us to extend their proof for CMC tori in $\mathbb{S}^3$ to arbitrary spaceforms. By this means, we can prove that Babich-Bobenko tori cannot be embedded. Further, we obtain that after a stereographic projection, every embedded isothermic constrained Willmore torus $f : M \to \mathbb{R}^3$ that is not a Bryant surface has constant mean curvature in the unit 3-sphere in $\mathbb{R}^4$ and is hence a canal surface. We also apply this method to periodic CMC cylinders in $\mathbb{R}^3$, which leads to a new proof of the well-known fact that Delaunay cylinders are the only properly embedded, periodic, CMC cylinders in $\mathbb{R}^3$. It is well-known that isothermic canal surfaces in $\mathbb{S}^3$ are Möbius equivalent to a surface of revolution. We give a new Möbius-geometric proof for this fact and obtain the following classification: After a stereographic projection, every embedded isothermic constraint Willmore torus $f : M \to \mathbb{R}^3$ is either an isothermic Bryant surface with smooth ends or Möbius equivalent to a surface of revolution in $\mathbb{S}^3$.

Acknowledgements

I would like to thank my supervisor Ulrich Pinkall for giving me this interesting subject and all his great ideas. His mathematical intuition and experience are just exceptional. He suggested most of the results of this work long time before there was the faintest idea of a proof. I am very glad about the opportunity to learn from him. I would like to thank my parents for their emotional and financial support of my studies. You always encouraged me to go on and gave me the confidence that I needed. I would like to thank my beloved girlfriend Henrike and my friends Stephan, Anno and Andy for just being who their are for me and their support on so many levels. I would like to thank Felix, Albert, Christoph Lara and Jan for helpful discussions and Lara, Christoph, Alex, Kai, Jan and Johannes for proof reading (parts of) the dissertation. Especially Lara and Johannes helped me a lot. Your support at the end of the dissertation was crucial for me; Thank you so much! This work was funded by the Berlin Mathematical School. Their scholarship gave me the time to write this thesis.

Contents

1. Introduction

For a given abstract closed surface M there exist many different immersions into the euclidean space $\mathbb{R}^3$. As a natural consequence, the question arises which immersions $f : M \to \mathbb{R}^3$ of a given surface M are the most 'round' or symmetric ones. An answer to this question was given in the year 1965 by Thomas James Willmore [32] by defining an energy for immersions of surfaces using their mean curvature H

$$W(f) := \int_M H^2 d\sigma.$$

The critical points of the *Willmore functional* W are called *Willmore surfaces*. Based on the early investigations of elastic surfaces by Sophie Germain [13], where she suggested that the mean curvature is proportional to the elastic force of thin plates, the Willmore energy plays an important role in the elasticity theory of surfaces and is still the object of modern research in this area (see for example [12]).

The theory of Willmore surfaces is also used in computer graphics when looking for the 'smoothest' realization of an abstract surface that may have some additional structure (see [10]). This leads to the notion of constrained Willmore surfaces, which are critical points of the Willmore energy under all compact variations that preserve a particular structure. The most investigated constraint, and the only one that we will consider in this work, is the conformal structure of the surfaces. Critical points of the Willmore functional in a fixed conformal class will be called *constrained Willmore surfaces*. They can be seen as the most 'round' resp. symmetric realizations of an abstract Riemann surface and are hence of great interest in conformal geometry and computer graphics. Important work to understand constrained Willmore surfaces has already been done. Ulrich Pinkall et al. gave a basic theory of constrained Willmore surfaces, including a derivation of the Euler-Lagrange equation from a variational problem [6]. Ernst Kuwert and Rainer Schätzle furthermore proved the existence of minimizers of the Willmore functional under preservation of a fixed conformal class if there exists an immersion of the surface with Willmore energy smaller than 8π [22]. A substantial survey of the previous research on constrained Willmore surfaces can be found in the article "*Towards a constrained Willmore conjecture*" [16] by Lynn Heller and Franz Pedit, where they also state several conjectures for the classification of constrained Willmore tori. For Willmore tori without constraints Fernando C. Marques and André Neves [26] proved the famous Willmore conjecture:

Conjecture 1 (Willmore 1965).
For every smooth immersed torus $f : M \to \mathbb{R}^3$ the Willmore energy satisfies: $W(f) \geq 2\pi^2$.

The Clifford torus has a Willmore energy of $2\pi^2$ and realizes the absolute minimum among all immersions of tori into $\mathbb{S}^3$ and hence in its conformal class. To find the explicit minimizers of the Willmore energy among all tori in an arbitrary prescribed conformal class is still an open problem. For the rectangular conformal class Lynn Heller and Franz Pedit conjectured the following.

Conjecture 2 (Heller, Pedit [16]).
The constrained minimizers of the Willmore energy for tori in $\mathbb{R}^3$ of rectangular types $\mathbb{R}^2/\mathbb{Z} \oplus ib\mathbb{Z}$ are (stereographic projections of) the homogeneous tori $\mathbb{S}^1 \times \mathbb{S}^1_b$ in the 3-sphere for $1 \leq b \leq \sqrt{3}$ and the 2-lobed Delaunay tori in a 3-sphere for $b > \sqrt{3}$ limiting to a twice covered equatorial 2-sphere as $b \to \infty$.

In [25] Peter Li and Shing-Tung Yau proved that compact surfaces with a Willmore energy smaller than 8π are embedded. Therefore, embedded surfaces are the best candidates for minimizing the Willmore energy and it is an interesting question which constrained Willmore tori are embedded.

Many known examples of constrained Willmore surfaces (for instance, constant mean curvature surfaces like the Delaunay tori and the homogeneous tori from Conjecture 2) are isothermic, i.e., locally they admit a conformal coordinate line parametrization. Since embedded isothermic constrained Willmore tori are good candidates to minimize the Willmore energy in their conformal class, we are interested in a classification of these tori. Being isothermic or constraint Willmore are both Möbius invariant properties, thus the classification should reflect this invariance and a theory of isothermic constraint Willmore tori should be a Möbius-geometric one. Our main result is the following theorem:

Theorem (Main theorem)**.**
Every embedded isothermic constraint Willmore torus is either an isothermic Bryant surface with smooth ends or Möbius equivalent to a surface of revolution in $\mathbb{S}^3$.

The starting point of our research is given by a theorem from Jörg Richter, which states that every isothermic constrained Willmore torus $f : M \to \mathbb{R}^3$ has constant mean curvature in a space form [30]. We extended his proof in a way that it remains true if the surface has umbilical points. In this case there exists a hyperplane E that separates $\mathbb{R}^3$ in two hyperbolic spaces $\mathbf{H}^3_\pm$ whose ideal boundaries are given by $\partial\mathbf{H}^3_\pm = E$. The umbilical points of the surface all lie on E and the restriction of f to $\mathbf{H}^3_\pm$ has constant mean curvature. Surfaces that intersect E are called *Babich-Bobenko tori* because they were first constructed by Mikhail V. Babich and Alexander I. Bobenko using theta functions and eliptic integrals [3]. Their constant mean curvature w.r.t. the hyperbolic metric will be denoted with $\tilde{H}$ and satisfies $|\tilde{H}| < 1$. If the surface lies in one of the hyperbolic spaces $\mathbf{H}^3_\pm$ and is tangential to E, it has hyperbolic mean curvature $\tilde{H} = 1$ and is called a Bryant tori with smooth ends [9]. We prove that the Babich-Bobenko tori cannot be embedded, which leads to the following theorem:

Theorem.
An embedded, isothermic constrained Willmore torus $f : M \to \mathbb{R}^3$ is either a Bryant surface with smooth ends or there exists a stereographic projection that maps it into a unit 3-sphere in $\mathbb{R}^4$, where it has constant mean curvature.

Shortly before finishing this work Birahim Ndiaye Cheikh, Lynn Heller and Sebastian Heller published the preprint of an article [15] in which they proof that isothermic constrained Willmore tori in the conformal 3-sphere with Willmore energy below 8π are CMC surfaces in the round 3-sphere. Using the estimate of Peter Li and Shing-Tung Yau [25], and the quantization for the Willmore energy of Bryant surfaces that Christoph Bohle and G. Paul Peters proved in [5], our theorem confirms their result.

Ben Andrews and Haizhong Li proved in [2] that every embedded CMC torus $f : M \to \mathbb{S}^3$ has a rotational symmetry. In their proof they consider the maximal interior sphere congruence (MISC) of these tori, which is the map that assigns to every point $p \in M$ the biggest sphere in $\mathbb{S}^3$ that is tangent to the surface at $f(p)$ and contained in the compact region bounded by $f(M)$. In particular, they show that the curvature of theses spheres is given by the bigger principle curvature of f at the touching points.

We reproduce their construction in a projective model of Möbius geometry, whose representation in homogeneous coordinates corresponds to the five dimensional Minkowski space $\mathbb{R}^{4,1}$ on which the group of Möbius transformations acts linearly. The different space forms $\mathbb{S}^3, \mathbf{H}^3, \mathbb{R}^3$ are obtained by intersections of the light cone $\mathcal{L} := \{P \in \mathbb{R}^{4,1} | \langle P, P\rangle_{Mink} = 0\}$ with affine hyperplanes and the set of their spheres can be identified with the standard hyperboloid of one sheet $\mathcal{H}^4_+ \subset \mathbb{R}^5$. In this model a sphere congruence is given by a continuous map $\mathcal{S} : M \to \mathcal{H}^4_+$. The property of being an isothermic constrained Willmore immersion is invariant under Möbius transformations. To exploit this invariance, which also applies to sphere congruences, the projective model of Möbius geometry is the natural setup to study these surfaces and their maximal interior sphere congruence.

Another advantage of this model is that it makes it possible to generalize the method that Ben Andrews and Haizhong Li developed for CMC surfaces in $\mathbb{S}^3$ in a way that it can be used for surfaces in $\mathbb{R}^3$ and $\mathbf{H}^3$, too.

In particular, for CMC tori in $\mathbf{H}^3_\pm$ that intersect the ideal boundary $\partial\mathbf{H}^3_\pm = E$ (Babich-Bobenko tori and Bryant tori with smooth ends), the Möbius-geometric setup is very useful to study their sphere congruences. This is because given as a map $\mathcal{S} : M \to \mathcal{H}^4_+$, a sphere congruence is well defined even if the hyperbolic radius of the spheres goes towards infinity, as it is the case if the surface touches E. Using their maximal interior sphere congruence, we are able to prove that Babich-Bobenko tori can be embedded. We conjecture that isothermic Bryant tori with smooth ends cannot be embedded either. So far we don not have a proof for the conjecture, but are confident that it is possible to use the MISC to show that these surfaces cannot be embedded.

Complete and properly embedded surfaces in $\mathbb{R}^3$ can also be investigated in the setup given above. In 1989 Nicholas J. Korevaar, Rob Kusner and Bruce Solomon proved that every complete and properly embedded CMC surface in $\mathbb{R}^3$ with two ends is a Delaunay surface [21]. In their proof they use Alexandrows reflection method. We reproduced their result for embedded periodic CMC cylinders with annular ends by adapting Ben Andrews' and Haizhong Li's strategy for tori in $\mathbb{S}^3$ to periodic cylinders in $\mathbb{R}^3$.

Surfaces whose bigger principal curvatures are constant along the corresponding curvature lines are *canal surfaces*, i.e., the envelope of a 1-parameter family of spheres. After we have shown that this is the case for embedded isothermic constrained Willmore tori, that are no Bryant surfaces, and periodic CMC cylinders, we continue our investigation of these surfaces by considering them as canal surfaces in the projective model of Möbius geometry. In particular, we give a new geometric proof for the well-known fact (see for example [17]) that isothermic canal surfaces are Möbius equivalent to a surface of revolution in $\mathbb{S}^3$, and that there are no embedded CMC tori in $\mathbb{R}^3$ and $\mathbf{H}^3$.

Before we start with the derivation of the new results in Section 7, we will give a deeper introduction into the subject in the sections 2 - 6. In this introduction we will fix the notation and establish the definitions and concepts that are necessary for a basic understanding of isothermic constrained Willmore surfaces.

The first section starts with abstract Riemann surfaces that we define as 2-dimensional Riemannian manifolds equipped with an almost complex structure. Using the Levi-Civita connection of the Riemannian metric, we define holomorphic structures on several vector bundles, like the one of $\mathbb{C}$-valued functions or of quadratic differentials. Further, we show

how the connection and its induced curvature behave under a conformal change of the metric.
For the investigation of immersions of Riemann surfaces into $\mathbb{R}^3$, we make use of the quaternions by identifying the euclidean space $\mathbb{R}^3$ with the imaginary quaternions. This gives us a very simple representation for conformal transformations of $\mathbb{R}^3$. In the presence of an immersion, the vector bundle of $\mathbb{H}$-valued 1-forms splits into two complex line bundles, the normal and the tangential one. Using these line bundles, we obtain a coordinate-free description of many important quantities of the immersion and a straightforward derivation of the structure equations (Gauß and Gauß-Codazzi equation). Holomorphic quadratic differentials can also be identified with sections of these line bundles and used for the definition of isothermic surfaces and their Christoffel dual.
In Section 4 we compute how a conformal change of the euclidean metric of the ambient space $\mathbb{R}^3$,

$$\langle \cdot, \cdot \rangle_{euc} \to \tilde{\langle \cdot, \cdot \rangle} = e^{2u} \langle \cdot, \cdot \rangle_{euc}, \qquad u \in C^\infty(\mathbb{H}, \mathbb{R}),$$

changes important quantities of the immersion $f : M \to \mathbb{R}^3$, such as the Hopf differential and the mean curvature. Further, we will show how different space forms such as $\mathbf{H}^3$ and $\mathbb{S}^3$ can be obtained from $\mathbb{R}^3$ by a conformal change of the euclidean metric. This construction of different space forms for a fixed immersion in $\mathbb{R}^3$ has its analogue in the projective model of Möbius geometry and is crucial for the proof of Jörg Richters theorem.
In Section 5 we introduce the Willmore functional and prove its invariance under reparametrizations of the surface and conformal transformations of the ambient space. Further, we use the quaternionic analysis of surfaces to obtain variation formulas for immersions into $\mathbb{R}^3$ and their induced metric and conformal structure. This supports a coordinate-free derivation of the Euler-Lagrange equation for the Willmore functional.
In Section 6 we finally consider constrained Willmore surfaces, i.e., critical points of the Willmore functional in a fixed conformal class. Since the Willmore functional is invariant under reparametrizations, Teichmüller space is an important tool for the understanding of constrained Willmore surfaces. We recap the construction of the Teichmüller space from a Riemannian perspective as it was developed by Anthony J. Tromba in [18]. Then we use the strategy that Christoph Bohle, Paul Peters, and Ulrich Pinkall establish in [6] to derive the Euler-Lagrange equations of constrained Willmore surfaces. The subspace of conformal immersions of a Riemann surface inside the smooth manifold of all immersions is singular at the isothermic immersions. Therefore, it is not self-evident that every infinitesimal conformal variation of an isothermic immersion extends to a real conformal variation of the immersion. Since we are particularly interested in isothermic surfaces and because the existence of conformal variations is crucial for the derivation of the Euler-Lagrange equations, we recap some of the ideas that Ernst Kuwert and Rainer Schätzle use in [22] to prove the existence of minimizers of the Willmore functional in a fixed conformal class.

We hope that this longer introduction makes the work self-contained and accessible for those readers that are not as familiar with conformal geometry, quaternionic analysis of surfaces, or variations of immersions. The investigation of immersions into different space forms via fixing the map $f : M \to \mathbb{R}^3$ and considering a conformal change of the euclidean metric of $\mathbb{R}^3$ could be interesting for experienced readers too. Furthermore, we view the well-known results from a Riemannian perspective and present them in a common coordinate-free language. We hope this makes some of the already established proofs better understandable.

2. Riemann surfaces

The fundamental object of this work are immersions of Riemann surfaces into 3-dimensional space forms. However, before we consider immersions in Section 3, we will start with a purely intrinsic investigation of the surfaces. This section serves mainly as an introduction, where we give important definitions and fix the notation. Our approach is a Riemannian one, in the sense that, we consider the almost complex structure and the corresponding Riemannian metrics as the main objects to describe Riemann surfaces. From the Levi-Civita connection of these metrics we will derive the important notion of holomorphicity. Therefore, we start with describing the relation between the spaces of Riemannian metrics, conformal structures and complex structures. This will also be important for the construction of the Teichmüller space in section 6.

Section 2 and 3 are highly inspired by two lectures that Ulrich Pinkall gave at TU Berlin about conformal geometry (WS 2013/2014) and Riemann surfaces (WS 2018/2019). We assume that the reader is familiar with the notion of smooth manifolds and vector bundles.

Definition 1.
Let M be a smooth manifold, then $J \in \Gamma(\mathrm{End}(TM))$ with $J^2 = -\mathbb{1}$ is called an ***almost complex structure*** *on M. The set of all almost complex structures on M will be denoted by $\mathcal{A}(M)$. If M is 2-dimensional, the pair (M, J) is called a* ***Riemann surface****.*

On $\mathbb{C}$ the almost complex structure is given by the complex multiplication with i. If we identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, that is, $\mathbb{R}^2 \ni (x, y) \leftrightarrow x + iy = z \in \mathbb{C}$, the almost complex structure on $\mathbb{R}^2$ is given by

$$\hat{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Vice versa, an almost complex structure J turns the tangent spaces T_pM into complex vector spaces if we define the complex multiplication on T_pM by

$$(x + iy)X := xX + yJ_pX \qquad \forall X \in T_pM \text{ and } x + iy \in \mathbb{C}.$$

Definition 2.
Let M be a $2n$-dimensional smooth manifold. A ***complex structure*** *$c := \{U_i, \varphi_i\}_{i\in I}$ on M is a maximal atlas of M such that the transition functions $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ are biholomorphic.*
The ***set of all complex structures*** *on M will be denoted by $\mathcal{C}(M)$. A pair (M, c) is called a* ***complex manifold****.*

Proposition 1.
Every complex manifold has a canonical almost complex structure, i.e., there is a well-defined map

$$\Phi : \mathcal{C}(M) \to \mathcal{A}(M). \tag{2.1}$$

Proof. For a complex manifold (M, c) we define an almost complex structure J on M by locally pulling back $\hat{J}$ using the charts of the complex structure

$$J_p := d\phi^{-1}_{\phi(p)} \hat{J} d\phi_p \tag{2.2}$$

for some chart $(\varphi, U) \in c$ with $p \in U$. We have to show that this definition is independent of the choice of the chart. To this end, let $(\phi, V) \in c$ be another chart with $p \in V$. Now, $\varphi \circ \phi^{-1}$ is an biholomorphic function and its differential commutes with $\hat{J}$. This give rise to

$$\begin{aligned} & \hat{J} = d(\varphi \circ \phi^{-1})_{\phi(p)} \hat{J} d(\phi \circ \varphi^{-1})_{\varphi(p)} \\ \Leftrightarrow \qquad & \hat{J} = d\varphi_p d\phi^{-1}_{\phi(p)} \hat{J} d\phi_p d\varphi^{-1}_{\varphi(p)} \\ \Leftrightarrow \qquad & d\varphi^{-1}_{\varphi(p)} \hat{J} d\varphi_p = d\phi^{-1}_{\phi(p)} \hat{J} d\phi_p. \end{aligned}$$

□

Definition 3.
For $A \in \Gamma(\mathrm{End}(TM))$ the Nijenhuis-tensor $N_A \in \Omega^2(M, TM)$ is defined as:

$$N_A(X, Y) := -A^2[X, Y] - [AX, AY] + A([AX, Y] + [X, AY]),$$

where $X, Y \in \mathfrak{X}(M)$ and $[\cdot, \cdot]$ denotes the Lie bracket of vector fields on TM.

The question which almost complex structures are integrable and hence give rise to a complex structure was first answered by A. Newlander and L. Nirenberg in [28].

Theorem 1 (A. Newlander and L. Nirenberg)**.**
An almost complex manifold (M, J) is complex, if and only if the Nijenhuis-tensor of the almost complex structure vanishes, i.e., $N_J = 0$.

Corollary 1.
Every Riemann surface (M, J) is a 1-dimensional complex manifold. In particular, the map

$$\Phi : \mathcal{C}(M) \to \mathcal{A}(M)/_{\{\pm 1\}},$$

defined in Proposition 1 is bijective.

Proof.

$$N_J(X, JX) := -J^2[X, JX] - [JX, JJX] + J([JX, JX] + [X, JJX]) = 0.$$

□

2.1. Riemannian metrics, conformal structures and almost complex structures on Riemann surfaces.

Definition 4.
Let M be a smooth manifold. We define $\mathrm{Sym}(M)$ to be the vector bundle over M whose fiber at $p \in M$ consists of all symmetric bilinear maps $h_p : T_pM \times T_pM \to \mathbb{R}$. Further, let

$$\mathcal{M}(M) := \{g \in \Gamma(\mathrm{Sym}(M)) |\, g_p : T_pM \times T_pM \to \mathbb{R} \text{ is positive definite } \forall p \in M\}$$

denote the set of all ***Riemannian metrics*** *on M. A pair (M, g) is called a* ***Riemannian manifold****.*

For a Riemannian manifold (M, g), the space of endomorphism fields of the tangent bundle splits into the symmetric and skew-symmetric ones with respect to g,

$$\Gamma(\mathrm{End}(TM)) = \Gamma(\mathrm{End}_{g+}(TM)) \oplus \Gamma(\mathrm{End}_{g-}(TM)),$$

where

$$\begin{aligned}\Gamma(\mathrm{End}_{g+}(TM)) &:= \{\Gamma(\mathrm{End}(TM)) \mid A = A^*\},\\ \Gamma(\mathrm{End}_{g-}(TM)) &:= \{\Gamma(\mathrm{End}(TM)) \mid A = -A^*\}.\end{aligned}$$

The projection maps are given by

$$\begin{aligned}\pi_\pm : \Gamma(\mathrm{End}(TM)) &\to \Gamma(\mathrm{End}_{g\pm}(TM))\\ A &\mapsto \frac{1}{2}(A \pm A^*).\end{aligned}$$

Fixing $g \in \mathcal{M}(M)$, we obtain the following identification:

$$\begin{aligned}\Gamma(\mathrm{Sym}(M)) &\leftrightarrow \Gamma(\mathrm{End}_{g+}(TM))\\ g(A\cdot, \cdot) &\leftrightarrow A.\end{aligned} \tag{2.3}$$

In the same way, we can identify $\mathcal{M}(M)$ with the positive definite sections of $\mathrm{End}_{g+}(TM)$.

On a Riemannian surface (M, g), we can choose a basis on T_pM such that $g_p(\cdot,\cdot) = \langle\cdot,\cdot\rangle_{\mathbb{R}^2}$. This gives us for $A \in \Gamma(\mathrm{End}(TM))$ and $A_\pm \in \Gamma(\mathrm{End}_{g\pm}(TM))$:

$$A(p) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A_+(p) = \begin{pmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{pmatrix} \text{ and } A_-(p) = \begin{pmatrix} 0 & \frac{b-c}{2} \\ \frac{c-b}{2} & 0 \end{pmatrix},$$

for some $a, b, c, d \in \mathbb{R}$. Therefore, the rank of the vector bundles is given by: $\mathrm{rank}(\mathrm{End}(TM)) = 4$, $\mathrm{rank}(\mathrm{End}_{g-}(TM)) = 1$ and $\mathrm{rank}(\mathrm{End}_{g+}(TM)) = 3$.

Definition 5.
The trace of $A \in \Gamma(\mathrm{End}(TM))$ with respect to a Riemannian metric g is defined as

$$\begin{aligned}\mathrm{tr} : \Gamma(\mathrm{End}(TM)) &\to C^\infty(M)\\ \mathrm{tr}(A)(p) &:= g_p(AX, X) + g_p(AY, Y),\end{aligned}$$

for some orthonormal basis (ONB), X, Y of T_pM.

Recall from linear algebra:

(1) The definition of the trace is independent of the particular choice of the ONB X, Y.
(2) $\mathrm{tr}(A^*) = \mathrm{tr}(A)$.
(3) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Definition 6.
*A **conformal structure** on a smooth manifold M is an equivalence class of Riemannian metrics, where two metrics g and $\tilde{g}$ are conformally equivalent if there exists $u \in C^\infty(M)$ such that*

$$\tilde{g} = e^{2u} g. \tag{2.4}$$

*The **set of conformal structures** on M is given by $\mathcal{M}(M)/\mathcal{P}(M)$, where $\mathcal{P}(\mathcal{M}) := C^\infty(M, \mathbb{R}^+)$.*

The name conformal comes from the fact that angles are the same for every representative g of a conformal structure. For $X, Y \in T_pM \setminus \{0\}$ their enclosed angle α is given by

$$\cos(\alpha) = \frac{g_p(X,Y)}{\sqrt{g_p(X,X) g_p(Y,Y)}},$$

and hence the same for all metrics in one conformal class. If $\tilde{g}$ is conformally equivalent to g, the endomorphism field associated to $\tilde{g}$ is given by: $A = e^{2u}\mathbb{1}$.

Lemma 1.
Both the trace and the adjoint are invariant under a conformal change of the metric.

Proof. Let g and $\tilde{g}$ be conformally equivalent metrics, i.e., $\tilde{g} = ge^{2u}$. If $X, Y \in T_pM$ have length one w.r.t. g, the vectors $\tilde{X} := e^{-u}X$ and $\tilde{Y} := e^{-u}Y$ are normalized w.r.t $\tilde{g}$ and we obtain for the trace of $A \in \Gamma(\mathrm{End}(TM))$ that

$$\begin{aligned}
\mathrm{tr}_{\tilde{g}}(A)(p) &= \tilde{g}_p(A\tilde{X}, \tilde{X}) + \tilde{g}_p(A\tilde{Y}, \tilde{Y}) \\
&= e^{2u} g_p(Ae^{-u}X, e^{-u}X) + e^{2u} g_p(Ae^{-u}Y, e^{-u}Y) \\
&= g_p(AX, X) + g_p(AY, Y) \\
&= \mathrm{tr}_g(A)(p).
\end{aligned}$$

For the adjoint we get

$$g(AX,Y) = g(X, A^*Y) \Leftrightarrow e^{2u} g(AX,Y) = e^{2u} g(X, A^*Y) \Leftrightarrow \tilde{g}(AX,Y) = \tilde{g}(X, A^*Y).$$

□

Definition 7.
On a Riemann surface (M, J) we define:

$$\begin{aligned}
\Gamma(\mathrm{End}_-(TM)) &:= \{A \in \Gamma(\mathrm{End}(TM)) | AJ = -JA\}, \\
\Gamma(\mathrm{End}_+(TM)) &:= \{A \in \Gamma(\mathrm{End}(TM)) | AJ = JA\}.
\end{aligned}$$

Note that $\Gamma(\mathrm{End}(TM)) = \Gamma(\mathrm{End}_+(TM)) \oplus \Gamma(\mathrm{End}_-(TM))$ and that the projection maps are given by

$$\begin{aligned}
\pi_\pm : \Gamma(\mathrm{End}(TM)) &\to \Gamma(\mathrm{End}_\pm(TM)) \\
A &\mapsto \frac{1}{2}(A \mp JAJ).
\end{aligned}$$

In local coordinates with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $A_\pm \in \Gamma(\mathrm{End}_\pm(TM))$ have the form

$$A_+ = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ and } A_- = \begin{pmatrix} c & d \\ d & -c \end{pmatrix}.$$

A straightforward calculation shows:

- $\Gamma(\mathrm{End}_+(TM)) = \mathrm{span}_{C^\infty(M)}\{\mathbb{1}, J\}$.
- $\Gamma(\mathrm{End}_-(TM)) = \{A \in \Gamma(\mathrm{End}(TM)) |\ \mathrm{tr}(A) = 0 \text{ and } A = A^*\}$.

Definition 8.
*Let g be a Riemannian metric on a Riemann surface (M, J), then g and J are called **compatible** if J is skew adjoint with respect to g, i.e.,*

$$g(JX, Y) = -g(X, JY), \qquad \forall X, Y \in \mathfrak{X}(M). \tag{2.5}$$

Given a compatible metric g, we define a Hermitian inner product on the tangent spaces by

$$\langle \cdot, \cdot \rangle_{\mathbb{C}} : \Gamma(TM) \times \Gamma(TM) \to \mathbb{C}$$
$$\langle X, Y \rangle_{\mathbb{C}} := g(X, Y) + ig(JX, Y).$$

Note that J and g are compatible if and only if J is orthogonal w.r.t. g, i.e.,

$$g(X, Y) = g(JX, JY).$$

In the following we will write (M, J, g) for a Riemann surface with compatible metric g.

Proposition 2.
Let g be a Riemannian metric on a Riemann surface (M, J) compatible with J. Any other Riemannian metric $\tilde{g}$ is conformally equivalent to g if and only if $\tilde{g}$ and J are compatible, too.

Proof. " $\Rightarrow$ " Let g and $\tilde{g}$ be conformally equivalent. With the last Lemma J is skew adjoint w.r.t. g if and only J is skew adjoint w.r.t. $\tilde{g}$.

" $\Leftarrow$ " Let g and $\tilde{g}$ be compatible with J. By (2.3) there exists a positive definite $A \in \Gamma(\mathrm{End}_{g+}(TM))$ such that $g(A\cdot, \cdot) = \tilde{g}(\cdot, \cdot)$. The compatibility gives us for all $X, Y \in \mathfrak{X}(M)$

$$\begin{aligned} & \tilde{g}(JX, Y) = -\tilde{g}(X, JY) \\ \Leftrightarrow \quad & g(AJX, Y) = -g(AX, JY) \\ \Leftrightarrow \quad & AJ = JA \\ \Rightarrow \quad & A \in \Gamma(\mathrm{End}_{g+}(TM)) \cap \Gamma(\mathrm{End}_+(TM)). \end{aligned}$$

Since A is positive definite, there exists $u \in \mathcal{P}(M)$ such that $A = e^{2u}\mathbb{1}$ and the metrics are conformally equivalent. $\square$

Definition 9.
*A n-dimensional smooth manifold M is called **orientable** if there exists a nowhere vanishing volume form $\sigma \in \Omega^n(M)$. An **orientation** of M is an equivalence class of nowhere vanishing volume forms, where two volume forms are equivalent if there exists $u \in C^\infty(M)$ such that*

$$\tilde{\sigma} = e^{2u}\sigma. \tag{2.6}$$

An orientation of a Riemann surface (M, J) is called compatible with the almost complex structure if the basis X, JX is positively oriented, i.e., $\sigma(X, JX) > 0$ for all $X \neq 0 \in \mathfrak{X}(M)$.

Note that for an orientable manifold there exist exactly two choices of orientation.

Proposition 3.
Every Riemann surface (M, J) has a compatible conformal structure and orientation. Vice versa, every oriented Riemannian surface has a unique almost complex structure J compatible with the orientation and the conformal structure.

Proof. " $\Rightarrow$ " Let (M, J) be a Riemann surface and $\tilde{g}$ any Riemannian metric on M, then

$$g(\cdot,\cdot) := \tilde{g}(\cdot,\cdot) + \tilde{g}(J\cdot, J\cdot)$$

defines a J compatible metric because

$$g(JX, Y) = \tilde{g}(JX, Y) + \tilde{g}(J^2X, JY) = -\tilde{g}(JX, J^2Y) - \tilde{g}(X, JY) = -g(X, JY).$$

The orientation is given by the never vanishing volume form

$$\sigma(\cdot,\cdot) := g(J\cdot,\cdot).$$

" $\Leftarrow$ " We define J_p on T_pM to be the 90 degree rotation in the positive sense. Let $X, Y \in T_pM$ with $g(X, X) = g(Y, Y)$, $g(X, Y) = 0$ and $\sigma(X, Y) > 0$, then J_p is defined by

$$JX = Y \quad \text{and} \quad JY = -X.$$

This definition is independent of the choice of metric g in the conformal class and the basis X, Y. If $\tilde{X}, \tilde{Y}$ is another positively oriented orthogonal basis of T_pM with equal length, there exists $\alpha \in [0, 2\pi]$ and $\lambda \in \mathbb{R} \setminus \{0\}$ such that

$$\tilde{X} = \lambda(\cos(\alpha)X + \sin(\alpha)Y); \quad \text{and} \quad \tilde{Y} = \lambda(\cos(\alpha)Y - \sin(\alpha)X).$$

Hence,

$$J\tilde{X} = \tilde{Y} \quad \text{and} \quad J\tilde{Y} = -\tilde{X}.$$

□

Note that $-J$ has the same conformal structure than J but a different orientation. This observation leads to the following definition.

Definition 10.
On an oriented surface M, we define the set of compatible almost complex structures

$$\mathcal{A}^+(M) := \{J \in \mathcal{A}(M) |\, X, JX \text{ is positively oriented } \forall X \in \mathfrak{X}(M)\} \cong \mathcal{A}(M)_{/\{\pm\}}.$$

Corollary 2.
On an oriented surface M, the set of compatible complex structures is the same as the set of conformal structures

$$\mathcal{C}(M) \cong \mathcal{A}^+(M) \cong \mathcal{M}(M)/\mathcal{P}(M).$$

The set of Riemannian metrics on M

$$\mathcal{M}(M) := \{g \in \Gamma(\mathrm{Sym}(M)) |\, g_p : T_p \times T_pM \to \mathbb{R} \text{ is positive definite } \forall p \in M\},$$

is an open subset of $\Gamma(\mathrm{Sym}(M))$ and hence an infinite dimensional (Hilbert-)manifold. The tangent space of $\mathcal{M}(M)$ at $g \in \mathcal{M}(M)$ is given by $T_g\mathcal{M}(M) = \Gamma(\mathrm{Sym}(M))$.

Definition 11.
We define a metric on $T_g\mathcal{M}(M) \cong \Gamma(\mathrm{End}_{g+}(TM))$

$$\begin{aligned}\langle\!\langle \cdot, \cdot \rangle\!\rangle_g : \Gamma(\mathrm{End}(TM)) \times \Gamma(\mathrm{End}(TM)) &\to \mathbb{R} \\ (H_1, H_2) &\mapsto \frac{1}{2} \int_M \mathrm{tr}(H_1^* H_2) d\sigma_g,\end{aligned} \tag{2.7}$$

where $d\sigma_g := g(J\cdot, \cdot)$ denotes the volume form associated to the Riemannian metric g.

Note that:

(1) The splitting $\Gamma(\mathrm{End}(TM)) = \Gamma(\mathrm{End}_-(TM)) \oplus \Gamma(\mathrm{End}_+(TM))$ is orthogonal with respect to the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, because $\mathrm{tr}(A) = 0$ for $A \in \Gamma(\mathrm{End}_-(TM))$ and $A_-^* A_+ \in \Gamma(\mathrm{End}_-(TM))$ for $A_\pm \in \Gamma(\mathrm{End}_\pm(TM))$.
(2) The inner product is compatible with the almost complex structure J on $\Gamma(\mathrm{End}(TM))$ because
$$\mathrm{tr}((JA)^* B) = \mathrm{tr}(A^* J^* B) = \mathrm{tr}(B^* JA) = -\,\mathrm{tr}((JB)^* A).$$

.

Definition 12.
Let $g \in \mathcal{M}(M)$ be a Riemannian metric of M. A ***variation*** *of g is given by a smooth curve in $\mathcal{M}(M)$*

$$g_t : (\epsilon, \epsilon) \to \mathcal{M}(M)$$

with $g_0 = g$. The derivative

$$\dot{g} := \frac{\partial}{\partial t}\bigg|_{t=0} g_t$$

is called the ***variation field*** *of g. A variation is* ***conformal****, if the metrics g_t are conformally equivalent for all t. For a conformal variation the variation field has the form $\dot{g} = u\mathbb{1}$ for some $u \in C^\infty(M)$.*

Lemma 2.
There is an orthogonal splitting of $\Gamma(\mathrm{End}_{g+}(M)) = T_g\mathcal{M}(M)$ *into trace-free endomorphism fields and those that correspond to conformal variations of* g

$$\Gamma(\mathrm{End}_{g+}(M)) = S^c_g(M) \oplus S^t_g(M),$$

where

$$\begin{aligned} S^t_g(M) &:= \{A \in \Gamma(\mathrm{End}_{g+}(M)) |\ \mathrm{tr}(A) = 0\} = \Gamma(\mathrm{End}_-(TM)),\\ S^c_g(M) &:= \{A \in \Gamma(\mathrm{End}_{g+}(M)) |\, A = u\mathbb{1},\, u \in C^\infty(M)\}. \end{aligned}$$

Proof. For $A \in \Gamma(\mathrm{End}_{g+}(M))$ the splitting is given by

$$A = \underbrace{\frac{1}{2}\,\mathrm{tr}(A)\mathbb{1}}_{\in S^c_g(M)} + \underbrace{A - \frac{1}{2}\,\mathrm{tr}(A)\mathbb{1}}_{\in S^t_g(M)}.$$

The splitting is orthogonal because

$$\langle\!\langle \frac{1}{2}\,\mathrm{tr}(A)\mathbb{1}, A - \frac{1}{2}\,\mathrm{tr}(A)\mathbb{1} \rangle\!\rangle_g = \frac{1}{2}\int_M \mathrm{tr}\left(\frac{1}{2}\,\mathrm{tr}(A)\mathbb{1}\left(A - \frac{1}{2}\,\mathrm{tr}(A)\mathbb{1}\right)\right) d\sigma_g = 0.$$

Consider now a conformal variation of g, i.e., $g_t := e^{2u_t}g$ for some $u : (-\epsilon, \epsilon) \times M \to \mathbb{R}$ with $u_0 = 0$. The variation field is given by $\dot{g} = 2\dot{u}g$, corresponding to $2\dot{u}\mathbb{1} \in S^c_g(M)$. Vice versa, any $u\mathbb{1} \in S^c_g(M)$ defines a conformal variation of g by: $g_t := e^{tu}g$. □

Due to Lemma 1, the splitting is the same for all metrics in one conformal class.

Proposition 4.
$\mathcal{M}(M)/\mathcal{P}(M)$ *is a smooth quotient manifold of* $\mathcal{M}(M)$*. Its tangent space at* $c \in \mathcal{C}(M)$ *is given by* $T_c\mathcal{C}(M) = S^t_g(M)$ *for some* g *compatible with* c*.*

Sketch of proof: $\mathcal{M}(M)/\mathcal{P}(M)$ is a quotient manifold of $\mathcal{M}(M)$ if the the group action of $\mathcal{P}(M)$ on $\mathcal{M}(M)$ is smooth, proper and free. This is proven in [18]. If $\mathcal{O}_g$ denotes the orbit of the group action containing $g \in \mathcal{M}(M)$, we obtain with the last lemma that its tangent space is given by $T_g\mathcal{O}_g = S^c_g(M)$ and thus $T_{[g]}(\mathcal{M}(M)/\mathcal{P}(M)) = S^t_g(M)$.

Proposition 5 (Tromba)**.**
The space $\mathcal{A}^+(M)$ *is a smooth submanifold of* $\Gamma(\mathrm{End}(TM))$ *whose tangent space at* $J \in \mathcal{A}^+(M)$ *is given by*

$$T_J\mathcal{A}^+(M) = \Gamma(\mathrm{End}_-(TM)).$$

The proposition is proven by using the implicit function Theorem twice and can be found in [18].

2.2. **Quadratic differentials and Riemannian metrics.** Quadratic differentials will play a crucial role for the notion of both isothermic and constrained Willmore surfaces, which we will investigate in section 3.3 and 6, respectively . We start with the definition of real and complex quadratic differentials and consider their relation to Riemannian metrics.

Definition 13.
*A **real quadratic differential** on a smooth manifold M is a map $q : \mathcal{X}(M) \to C^\infty(M)$ that satisfies:*

(1) $q(\lambda X) = \lambda^2 q(X)$,
(2) $q(X+Y)+q(X-Y) = 2q(X)+2q(Y)$, $\quad\forall X, Y \in \mathcal{X}(M)$ *and* $\lambda \in C^\infty(M)$.

The set of all real quadratic differentials on a manifold M will be denoted with $\mathcal{RQ}(M)$.

On a Riemann surface (M, J) there is a one-to-one correspondence between real quadratic differentials and symmetric $C^\infty(M)$ bilinear maps

$$\mathcal{RQ}(M) \cong \Gamma(\mathrm{Sym}(M)).$$

For $h \in \Gamma(\mathrm{Sym}(M))$ we obtain $q \in \mathcal{RQ}(M)$ by

$$q(X) := h(X, X) \qquad \forall X \in \mathcal{X}(M).$$

Vice versa, $q \in \mathcal{RQ}(M)$ defines $h \in \Gamma(\mathrm{Sym}(M))$ by

$$h(X, Y) := \frac{1}{2}(q(X+Y) - q(X) - q(Y)).$$

Because any $h \in \Gamma(\mathrm{Sym}(M))$ defines a real valued two form $\omega \in \Omega^2(M)$

$$\omega(X, Y) := h(JX, Y),$$

we can identify a real quadratic differential with the corresponding two form

$$q(X) = \omega(X, JX).$$

Definition 14.
*On a Riemann surface (M, J) a map $q : \mathcal{X}(M) \to C^\infty(M, \mathbb{C})$ is called a **(complex) quadratic differential** if*

$$q(zX) = z^2 q(X), \qquad \forall X \in \mathcal{X}(M) \text{ and } z \in C^\infty(M, \mathbb{C}).$$

*The **set of complex quadratic differentials** on a Riemann surface (M, J) will be denoted with $\mathcal{Q}(M)$.*

In local coordinates $z : U \subset M \to \mathbb{C}$, a complex quadratic differential q has the form

$$q = f dz^2,$$

for some $f \in C^\infty(U, \mathbb{C})$.

Proposition 6.
Let $q \in \mathcal{Q}(M)$ with $q_p \neq 0$ $\forall p \in M$, then

$$g : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$$
$$g(X, Y) := \frac{1}{2}(|q(X+Y)| - |q(X)| - |q(Y)|),$$

defines a metric compatible with J.

Proof. The absolute value of a complex quadratic differential defines a real quadratic differential. Therefore, $g(X,Y) := \frac{1}{2}(|q(X+Y)| - |q(X)| - |q(Y)|)$ defines a symmetric bilinear map. This map g is positive definite, because q has no zeros. From $q(JX) = -q(X)$, we get $g(X,Y) = g(JX,JY)$, i.e., g is compatible with J. $\square$

2.3. **Holomorphic structures on vector bundles, curvature and some important differential operators.** Starting from the Levi-Civita connection, we will define holomorphic structures on several vector bundles, like the one of quadratic differentials. Further, we will show how the connection and its induced curvature behave under a conformal change of metric, which will be important for section 4. We introduce the Laplace operator and show how it can be computed using the exterior derivative and the Hodge star operator. Finally, we will consider the divergence of endomorphism fields, which will play a crucial role for holomorphic quadratic differentials and the construction of the Teichmüller space.

Theorem 2 (Fundamental Theorem of Riemannian geometry)**.**
On a Riemannian manifold (M,g) there is a unique connection ∇ on TM, called the ***Levi-Civita connection****, satisfying the following properties for all $X,Y,Z \in \mathfrak{X}(M)$:*

(1) $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$, meaning ∇ is ***metric****.*
(2) $[X,Y] = \nabla_X Y - \nabla_Y X$, that is, ∇ is ***torsion free****.*

For the proof see for example [24].

Definition 15.
Let E be a vector bundle with an almost complex structure J, i.e., $J \in \Gamma(\mathrm{End}(E))$ with $J^2 = -\mathbb{1}$. A connection ∇ on E is called a ***complex connection*** *if $\nabla J = 0$.*

Proposition 7.
If (M,J,g) is a Riemann surface, then the Levi-Civita connection w.r.t. g is complex.

Proof. Using the Levi-Civita connection and $J^* = -J$, we obtain from $g(X,JY) = -g(JX,Y)$ that

$$\begin{aligned} 0 &= g(\nabla X, JY) + g(X, \nabla(JY)) + g(\nabla(JX), Y) + g(JX, \nabla Y) \\ &= g(\nabla X, JY) + g(X, (\nabla J)Y + J\nabla Y) + g((\nabla J)X + J\nabla X, Y) + g(JX, \nabla Y) \\ &= g(X, (\nabla J)Y) + g(Y, (\nabla J)X), \end{aligned}$$

i.e., $(\nabla J) \in \Omega(M, \mathrm{End}_{g-}(TM))$. From $J^2 = -\mathbb{1}$ we have $(\nabla J)J + J\nabla J = 0$, i.e., $\nabla J \in \Omega(M, \mathrm{End}_-(TM))$. Since $\mathrm{End}_-(TM) \cap \mathrm{End}_{g-}(TM) = 0$, we finally have $\nabla J = 0$. $\square$

Having two vector bundles E_1, E_2 with connections ∇^1, ∇^2, we can define a connection on $\mathrm{Hom}(E_1, E_2)$ using the product rule:

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \Gamma(\mathrm{Hom}(E_1,E_2)) &\to \Gamma(\mathrm{Hom}(E_1,E_2)) \\ (\nabla_X A)(\psi) &:= \nabla^2_X(A\psi) - A(\nabla^1_X \psi). \end{aligned}$$

Lemma 3.
Let (M, J, g) be a Riemann surface. Using the Levi-Civita connection, we define a connection on $\mathrm{End}(TM)$*:*

$$(\nabla_X A)(Y) := \nabla_X(AY) - A(\nabla_X Y).$$

The restriction of this connection onto $\mathrm{End}_\pm(TM)$ *is a connection too.*

Proof. We have to show that $A \in \Gamma(\mathrm{End}_\pm(TM)) \Rightarrow \nabla A \in \Omega(M, \mathrm{End}_\pm(TM))$:

$$AJ \pm JA = 0 \Rightarrow 0 = (\nabla A)J + A\nabla J \pm (J\nabla A + (\nabla J)A) = (\nabla A)J \pm J\nabla A.$$

□

Note, that if we equip $\mathrm{End}(TM)$ with the almost complex structure

$$\begin{aligned} J : \Gamma(\mathrm{End}(TM)) &\to \Gamma(\mathrm{End}(TM)) \\ A &\mapsto J \circ A, \end{aligned}$$

the connection defined in Lemma 3 is complex.

Definition 16.
On a Riemann surface (M, J) the ***Hodge star operator*** *is defined by its action on 1-forms:*

$$*\omega := \omega \circ J.$$

Using the Hodge star operator, we can split the bundle of $\mathbb{C}$-valued 1-forms into the canonical and anti-canonical bundle: $\Omega(M, \mathbb{C}) = K \oplus \overline{K}$, where

$$K := \{\omega \in \Omega^1(M, \mathbb{C}) | *\omega = i\omega\}, \tag{2.8}$$
$$\overline{K} := \{\omega \in \Omega^1(M, \mathbb{C}) | *\omega = -i\omega\}. \tag{2.9}$$

Note:

- K and $\overline{K}$ are complex line bundles, where the almost complex structure is given by $J\omega := i\omega$.
- In local coordinates $z : U \subset M \to \mathbb{C}$, $\omega \in K$ and $\eta \in \overline{K}$ have the form

  $$\omega = f dz \quad \text{and} \quad \eta = g d\bar{z}$$

  for some $f, g \in C^\infty(U, \mathbb{C})$.
- With the Hodge star operator we can rewrite the wedge product of 1-forms

  $$(\omega \wedge \eta)(X, JX) = \omega(X)\eta(JX) - \omega(JX)\eta(X) = (\omega * \eta - *\omega\eta)(X).$$

In particular, we obtain for $\omega, \tilde{\omega} \in K$ and $\eta, \tilde{\eta} \in \overline{K}$

$$\begin{aligned}\omega \wedge \tilde{\omega} &= \omega * \tilde{\omega} - *\omega\tilde{\omega} = \omega i\tilde{\omega} - i\omega\tilde{\omega} = 0,\\ \eta \wedge \tilde{\eta} &= \eta * \tilde{\eta} - *\eta\tilde{\eta} = -\eta i\tilde{\eta} + i\eta\tilde{\eta} = 0,\\ \omega \wedge \eta &= -\omega i\eta - i\omega\eta = -2i\omega\eta.\end{aligned}$$

Definition 17.
*A **holomorphic structure** on a complex vector bundle (E, J) over a manifold M is a linear map*

$$\bar{\partial} : \Gamma(E) \to \Gamma(\bar{K}E)$$

satisfying for all $f \in C^\infty(M, \mathbb{C})$, $X \in \Gamma(TM)$ and $\psi \in \Gamma(E)$

$$\bar{\partial}_X(f\psi) = (\bar{\partial}_X f)\psi + f\bar{\partial}_X\psi,$$

where $\bar{\partial}_X f := \frac{1}{2}(df(X) + idf(JX))$.
*In the same way, an **anti-holomorphic structure** is a linear map*

$$\partial : \Gamma(E) \to \Gamma(KE)$$

satisfying for all $f \in C^\infty(M, \mathbb{C})$, $X \in \Gamma(TM)$ and $\psi \in \Gamma(E)$

$$\partial_X(f\psi) = (\partial_X f)\psi + f\partial_X\psi,$$

where $\partial_X f := \frac{1}{2}(df(X) - idf(JX))$.

Given a complex connection ∇ on a complex vector bundle (E, J) over a Riemann surface (M, J), we can uniquely decompose ∇ into a holomorphic and an anti-holomorphic structure:

$$\nabla = \partial + \bar{\partial}, \tag{2.10}$$

where $\partial := \frac{1}{2}(\nabla - J * \nabla)$ and $\bar{\partial} := \frac{1}{2}(\nabla + J * \nabla)$.
Example 1:
After identifying $K\overline{K}$ with $\Omega(M, \mathbb{C})$ via $\omega\eta \leftrightarrow \omega \wedge \eta$, the exterior derivative defines an holomorphic structure on K. For $f \in C^\infty(M, \mathbb{C})$ and $\omega \in K$ we have:

$$\begin{aligned}\bar{\partial}(f\omega) &= d(f\omega) = df \wedge \omega + fd\omega\\ &= (\partial f + \bar{\partial} f) \wedge \omega + fd\omega\\ &= \bar{\partial} f \wedge \omega + fd\omega\\ &= \bar{\partial} f \wedge \omega + f\bar{\partial}\omega,\end{aligned}$$

where we used that $\partial f \wedge \omega = 0$, because $\partial f, \omega \in K$.

For a complex line bundle $(L, J), (\tilde{L}, \tilde{J})$ we define

$$\begin{aligned}L^* &:= \{\omega : L \to \mathbb{C} |\, \omega \text{ is } \mathbb{C}\text{-linear}\},\\ L \otimes \tilde{L} &:= \{\eta : L^* \times \tilde{L}^* \to \mathbb{C} |\, \eta \text{ is } \mathbb{C}\text{-linear}\}.\end{aligned}$$

In particular, we obtain that $K^* = TM$ and $K^2 = K \otimes K = \mathcal{Q}(M)$, i.e., for every complex quadratic differential $q \in \mathcal{Q}(M)$ there exists $\omega, \eta \in K$ such that

$$q(zX) = \omega(zX)\eta(zX) = z\omega(X)z\eta(X) = z^2 q(X), \qquad \forall z \in C^\infty(M, \mathbb{C}) \text{ and } X \in \mathfrak{X}(M).$$

Example 2:
With the holomorphic structure on K and the identification $K^2 \cong \mathcal{Q}(M)$ we can define a holomorphic structure on $\mathcal{Q}(M)$. Let $q \in \mathcal{Q}(M)$ and $\omega, \eta \in K$ with $\omega\eta = q$. In local coordinates we have $\omega = f dz, \eta = g dz$ and hence $q = fg dz^2$. Now define

$$\bar{\partial} q := \bar{\partial}(f dz) g dz + f dz \bar{\partial}(g dz) = (\bar{\partial}(f) g + f \bar{\partial}(g)) dz^2 = \bar{\partial}(fg) dz^2.$$

Note that the holomorphic structure is defined independently of the choice of $\omega, \eta \in K$ that represent q. The ***set of holomorphic quadratic differentials*** *on M is a vector space and will be denoted with $H^0(\mathcal{Q}(M))$.*

With the famous Riemann–Roch Theorem it is possible to compute the dimension of $H^0(\mathcal{Q}(M))$.

Proposition 8.
On a compact Riemann surface M of genus $\mathfrak{g}$ the dimension of the complex vector space $H^0(\mathcal{Q}(M))$ is given by

$$\begin{aligned}
&\dim_{\mathbb{C}}(H^0(\mathcal{Q}(M))) = 0 && \text{for } \mathfrak{g} = 0,\\
&\dim_{\mathbb{C}}(H^0(\mathcal{Q}(M))) = 1 && \text{for } \mathfrak{g} = 1,\\
&\dim_{\mathbb{C}}(H^0(\mathcal{Q}(M))) = 3\mathfrak{g} - 3 && \text{for } \mathfrak{g} > 1.
\end{aligned}$$

For the proof and more information about the Riemann-Roch Theorem see for example [20].

In this work we are particularly interested in tori, i.e., genus one surfaces. For these surfaces the complex dimension of $H^0(\mathcal{Q}(M))$ is one, and because for every $c \in \mathbb{C}$ there exists a holomorphic quadratic differential $q = c\,dz^2$, these are the only ones with that property. In particular, we obtain the following corollary that will play an important role in this work.

Corollary 3.
On a torus, holomorphic quadratic differentials do not have zeros.

We will now consider how the structures obtained from the Levi-Civita connection behave under a conformal change of the metric.

Proposition 9.
Let (M, g) be a Riemannian surface with Levi-Civita connection ∇ and $\tilde{g} = e^{2u} g$ a conformally equivalent metric, then the Levi-Civita connection w.r.t. $\tilde{g}$ is given by:

$$\tilde{\nabla}_X Y = \nabla_X Y + \langle X, G \rangle Y + \langle Y, G \rangle X - \langle X, Y \rangle G,$$

where $G = \operatorname{grad} u$ and $g = \langle \cdot, \cdot \rangle$. With the anti-holomorphic structure ∂ we further obtain:

$$\tilde{\nabla} = \nabla + 2\partial u. \tag{2.11}$$

Proof. The connection $\tilde{\nabla}$ is torsion free, because

$$\tilde{\nabla}_X Y - \tilde{\nabla}_Y X = \nabla_X Y - \nabla_Y X = [X, Y].$$

We have to show that $\tilde{\nabla}$ is metric w.r.t. $\tilde{g}$. Therefore, consider

$$X\left(\tilde{g}(Y, Z)\right) = e^{2u}\left(2\langle G, X\rangle\langle Y, Z\rangle + \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle\right).$$

Further, we obtain

$$\begin{aligned}\tilde{g}(\tilde{\nabla}_X Y, Z) + \tilde{g}(\tilde{\nabla}_X Z, Y) = e^{2u}(&\langle \nabla_X Y + \langle X, G\rangle Y + \langle Y, G\rangle X - \langle X, Y\rangle G, Z\rangle \\ &+ \langle \nabla_X Z + \langle X, G\rangle Z + \langle Z, G\rangle X - \langle X, Z\rangle G, Y\rangle),\end{aligned}$$

which gives rise to

$$\left(\tilde{\nabla}_X \tilde{g}\right)(Y, Z) = X\left(\tilde{g}(Y, Z)\right) - \tilde{g}(\tilde{\nabla}_X Y, Z) - \tilde{g}(\tilde{\nabla}_X Z, Y) = 0.$$

To prove the second identity, first note that $\tilde{\nabla}_X Y$ is tensorial in the first entry and we may assume $|X| = 1$. This gives rise to

$$G = \langle X, G\rangle X + \langle JX, G\rangle JX,$$

and we obtain

$$\begin{aligned}\langle Y, G\rangle X - \langle X, Y\rangle G &= X(\langle Y, X\rangle\langle X, G\rangle + \langle JX, Y\rangle\langle JX, G\rangle) \\ &\quad - \langle X, Y\rangle(\langle X, G\rangle X + \langle JX, G\rangle JX) \\ &= \langle JX, G\rangle(\langle Y, JX\rangle X - \langle X, Y\rangle JX) \\ &= -\langle JX, G\rangle JY.\end{aligned}$$

Now, we can finish the proof:

$$\begin{aligned}\tilde{\nabla}_X Y &= \nabla_X Y + \langle X, G\rangle Y + \langle Y, G\rangle X - \langle X, Y\rangle G \\ &= \nabla_X Y + \langle X, G\rangle Y - \langle JX, G\rangle JY \\ &= \nabla_X Y + du(X)Y - du(JX)JY \\ &= \nabla_X Y + 2(\partial_X u)Y.\end{aligned}$$

□

Corollary 4.
The holomorphic structure induced by the Levi-Civita connection does not depend on the particular choice of metric but on the conformal class.

Proposition 10.
*On a Riemannian surface (M, g) the **Laplace operator** is defined pointwise by*

$$\begin{aligned}&\triangle : C^\infty(M) \to C^\infty(M) \\ &\qquad (\triangle u)_p := \operatorname{div}\operatorname{grad} u = g(\nabla_X(\operatorname{grad} u), X) + g(\nabla_Y(\operatorname{grad} u), Y),\end{aligned}$$

for some ONB X, Y of T_pM. Note that the definition is independent of the choice of X, Y. If M has an almost complex structure J compatible with g, we obtain a new expression for the Laplace operator using the Hodge star operator and the exterior derivative:

$$-\triangle u \, d\sigma = d * du,$$

where $d\sigma$ denotes the volume form associated to g.

Proof. For $X \in T_pM$ with $g(X,X) = \langle X, X\rangle = 1$ we get:

$$\begin{aligned}(d * du)(X, JX) &= X(*du(JX)) - JX(*du(X)) - *du([X, JX])\\
&= -X(\langle \operatorname{grad} u, X\rangle) - JX(\langle \operatorname{grad} u, JX\rangle) - \langle \operatorname{grad} u, J[X, JX]\rangle\\
&= -\langle \nabla_X \operatorname{grad} u, X\rangle - \langle \operatorname{grad} u, \nabla_X X\rangle - \langle \nabla_{JX} \operatorname{grad} u, JX\rangle\\
&\quad - \langle \operatorname{grad} u, \nabla_{JX} JX\rangle - \langle \operatorname{grad} u, J(\nabla_X JX - \nabla_{JX} X)\rangle\\
&= -\langle \nabla_X(\operatorname{grad} u), X\rangle - \langle \nabla_{JX}(\operatorname{grad} u), JX\rangle\\
&= - \triangle\, u\, \sigma(X, JX).\end{aligned}$$

In the calculation we used that J is parallel w.r.t. the Levi-Civita connection of g. □

On a closed Riemannian surface the Laplace operator is self-adjoint w.r.t. the scalar product

$$\langle\!\langle f, g\rangle\!\rangle := \int_M f g\, \sigma,$$

and its kernel is given by the constant functions. This is a consequence of Stokes Theorem.

Definition 18.
*Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. The **curvature tensor** w.r.t. ∇ is defined as*

$$\begin{aligned}R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M)\\
R(X, Y)Z &:= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.\end{aligned}$$

*For a plane $E \subset T_pM$ with ONB X, Y we define the **sectional curvature** of E by*

$$K := g(R(X, Y)Y, X).$$

*A complete Riemannian manifold with constant sectional curvature is called a **space form**. If (M, g) is a surface, K is called the **Gaussian curvature** of M.*

Note that R is indeed a tensor, i.e., $C^\infty(M)-$linear in each entry. Therefore, the definition of the sectional curvature is independent of the choice of basis X, Y.

Example 3:
Let (M, J) be a Riemann surface and $q \in H^0(\mathcal{Q}(M))$ a holomorphic quadratic differential without zeros. In local coordinates $z : U \subset M \to \mathbb{C}$ it has the form $q = f dz^2$, for some holomorphic function $f : U \to \mathbb{C}$ (see example 2). Because f has no zeros, there exists $\tilde{z} : U \subset M \to \mathbb{C}$ such that $d\tilde{z}^2 = f dz^2 = q$. With Proposition 6, $g := |q| = |d\tilde{z}^2|$ defines a J compatible metric on M and $\tilde{z} : (U, g) \to (\mathbb{C}, \langle\cdot,\cdot\rangle)$ is an isometry. Therefore, the Gaussian curvature of g is zero.

For more examples of space forms see Section 4.

Proposition 11.
Let (M,g) be a Riemannian manifold with $\dim(M)\geq 2$ and $E\subset T_pM$ a plane. Under the conformal change of metric $g\mapsto e^{2u}g=\tilde g$, the sectional curvature of E changes according to

$$\tilde K = e^{-2u}(K-\langle\nabla_X G,X\rangle-\langle\nabla_Y G,Y\rangle-|G^\perp|^2),$$

where $G=\operatorname{grad} u$ and $G^\perp$ denotes the projection from G onto the orthogonal complement of E. In particular, if (M,g) is a surface we obtain

$$\tilde K = e^{-2u}(K-\triangle u).$$

Proof. Let X,Y be some ONB of T_pM with respect to $g=\langle\cdot,\cdot\rangle$, then $\tilde X = e^{-u}X,\tilde Y = e^{-u}Y$ are an ONB w.r.t. $\tilde g$. We extend X,Y to vector fields on a neighborhood of p such that $(\nabla_X Y)_p=(\nabla_Y X)_p=0$. Then

$$\begin{aligned}
\tilde K &= \tilde g(\tilde R(\tilde X,\tilde Y)\tilde Y,\tilde X)\\
&= e^{-2u}\langle\tilde R(X,Y)Y,X\rangle\\
&= e^{-2u}\langle\tilde\nabla_X\tilde\nabla_Y Y-\tilde\nabla_Y\tilde\nabla_X Y-\tilde\nabla_{\underbrace{[X,Y]}_{=0}}Y,X\rangle\\
&= e^{-2u}\langle\nabla_X\tilde\nabla_Y Y+\langle\tilde\nabla_Y Y,G\rangle X+\langle X,G\rangle\tilde\nabla_Y Y-\langle X,\tilde\nabla_Y Y\rangle G\\
&\quad-\nabla_Y\tilde\nabla_X Y-\langle\tilde\nabla_X Y,G\rangle Y-\langle Y,G\rangle\tilde\nabla_X Y+\langle\tilde\nabla_X Y,Y\rangle G,X\rangle\\
&= e^{-2u}(\langle\nabla_X(\nabla_Y Y+2\langle Y,G\rangle Y-G),X\rangle+\langle\nabla_Y Y+2\langle Y,G\rangle Y-G,G\rangle\\
&\quad-\langle\nabla_Y(\nabla_X Y+\langle X,G\rangle Y+\langle Y,G\rangle X),X\rangle-\langle Y,G\rangle\langle\nabla_X Y+\langle X,G\rangle Y+\langle Y,G\rangle X,X\rangle\\
&\quad+\langle\nabla_X Y+\langle X,G\rangle Y+\langle Y,G\rangle X,Y\rangle\langle G,X\rangle)\\
&= e^{-2u}(K-\langle\nabla_X G,X\rangle+\langle\nabla_Y Y,G\rangle+2\langle Y,G\rangle^2-|G|^2\\
&\quad-\langle X,G\rangle\underbrace{\langle\nabla_Y Y,X\rangle}_{=0}-\langle\nabla_Y Y,G\rangle-\langle Y,\nabla_Y G\rangle-\langle Y,G\rangle^2+\langle X,G\rangle^2)\\
&= e^{-2u}(K-\langle\nabla_X G,X\rangle-\langle\nabla_Y G,Y\rangle-|G^\perp|^2).
\end{aligned}$$

If M is a surface, we have $E=T_pM$. Hence $G^\perp=0$ and $\triangle u=\langle\nabla_X G,X\rangle+\langle\nabla_Y G,Y\rangle$. $\square$

Definition 19.
On a Riemann surface (M,J,g) we define the divergence operator for endomorphism fields

$$\begin{gathered}
\operatorname{div}:\Gamma(\operatorname{End}(TM))\to\mathfrak{X}(M)\\
(\operatorname{div} A)_p := (\nabla_X A)_p X+(\nabla_Y A)_p Y,
\end{gathered}$$

where X,Y is some ONB of T_pM and the connection on $\operatorname{End}(TM)$ is obtained from the Levi-Civita connection w.r.t. g (compare Lemma 3).

Note that the definition is independent of the choice of the ONB but depends on the metric g.

Proposition 12.
After identifying the bilinear maps with the corresponding 2*-forms, the holomorphic structure on* $\Gamma(\mathrm{End}_+(TM))$ *defines a map*

$$\begin{aligned}\overline{D} : \Gamma(\mathrm{End}_+(TM)) &\to \Omega^2(M,TM)\\ \overline{D}A(X,Y) &:= \overline{\partial}_X AY - \overline{\partial}_Y AX = J \operatorname{div} A \, d\sigma(X,Y).\end{aligned}$$

Analogously we obtain for the anti-holomorphic structure on $\in \Gamma(\mathrm{End}_-(TM))$

$$\begin{aligned}D : \Gamma(\mathrm{End}_-(TM)) &\to \Omega^2(M,TM)\\ DA(X,Y) &:= \partial_X AY - \partial_Y AX = -J \operatorname{div} A \, d\sigma(X,Y).\end{aligned}$$

In particular, we have $A \in \Gamma(\mathrm{End}_\pm(TM))$ *is holomorphic resp. anti-holomorphic if and only if* $\operatorname{div} A = 0$.

Proof. Locally choose a normalized vector field X. For $A \in \Gamma(\mathrm{End}_-(TM))$ we have $\nabla_X A \in \Gamma(\mathrm{End}_-(TM))$ and hence $\partial_X AJX = -\partial_{JX} AX$. This gives rise to

$$\begin{aligned}DA(X,JX) &= \partial_X A(JX) - \partial_J XA(X)\\ &= ((\nabla_X - J\nabla_{JX})A)(JX)\\ &= -J(\nabla_X AX + \nabla_{JX} AJX)\\ &= -J \operatorname{div} A \, d\sigma(X,JX).\end{aligned}$$

The case of $A \in \Gamma(\mathrm{End}_+(TM))$ is proven analogously. □

3. Conformal immersions of Riemann surfaces into $\mathbb{R}^3$

In this section we will consider immersions of Riemann surfaces into $\mathbb{R}^3$ using quaternions. The quaternions are a very useful tool for the investigation because they admit a very simple representation for conformal transformations of $\mathbb{R}^3$ and allow a beautiful coordinate-free description of surfaces. We start with a brief introduction of the quaternions and their connection to conformal geometry of $\mathbb{R}^3$.

3.1. **Quaternions.** The quaternion numbers $\mathbb{H}$ are nothing but $\mathbb{R}^4$ together with a multiplication law. The identification of $\mathbb{H}$ and $\mathbb{R}^4$ is given by:

$$\mathbb{H} = \{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \,|\, x_0, x_1, x_2, x_3 \in \mathbb{R}\}.$$

With the usual addition, scalar multiplication and metric inherited from $\mathbb{R}^4$, the set of quaternions $\mathbb{H}$ becomes a four dimensional euclidean vector space with canonical basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Extending the usual multiplication of real numbers, the quaternion multiplication is defined by

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

Note that the multiplication is associative but not commutative. Similar to the complex numbers, we can split every quaternion into a real and an imaginary part

$$\mathbb{H} = \mathrm{Re}(\mathbb{H}) \oplus \mathrm{Im}(\mathbb{H}),$$

where $\mathrm{Re}(\mathbb{H}) := \mathrm{span}\{1\} = \mathbb{R}$ and $\mathrm{Im}(\mathbb{H}) := \mathrm{span}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\} = \mathbb{R}^3$. Writing $q \in \mathbb{H}$ as

$$q = \alpha + v, \text{ for some } \alpha \in \mathbb{R} \text{ and } v \in \mathbb{R}^3,$$

we obtain a new expression of the quaternion multiplication using the scalar and cross product of $\mathbb{R}^3$

$$pq = (\alpha + v)(\beta + w) = \alpha\beta - \langle v, w\rangle + \alpha w + \beta v + v \times w.$$

Vice versa, we obtain for $v, w \in \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$

$$\begin{aligned} \langle v, w\rangle &= -\frac{1}{2}(vw + wv) \\ v \times w &= \frac{1}{2}(vw - wv). \end{aligned}$$

Analogously to the complex numbers, the conjugation is defined as

$$\bar{q} = \mathrm{Re}(q) - \mathrm{Im}(q).$$

A short calculation shows:

(1) $\overline{qp} = \bar{p}\bar{q}$.
(2) $|q|^2 = q\bar{q} = \bar{q}q$.
(3) $|qp| = |q||p|$.
(4) For $q \in \mathbb{H}$ the inverse element with respect to the quaternion multiplication is given by

$$q^{-1} = \frac{\bar{q}}{|q|^2}.$$

(5) Similar to the complex numbers we can write quaternions in polar coordinates. For $q \in \mathbb{H}$ there exists $\alpha \in [0, 2\pi], v \in \mathbb{S}^2 \subset \mathbb{R}^3$ and $r \in [0, \infty)$ such that

$$q = r\left(\cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) v\right).$$

A very useful application of the quaternions is the description of rotations in $\mathbb{R}^3$.

Theorem 3.
Let $v \in \mathbb{R}^3$ with $|v| = 1$, $\alpha \in \mathbb{R}$ and $q = \cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) v$. Then we have for all $y \in \mathbb{R}^3$:

(1) $qy\bar{q} \in \mathrm{Im}(\mathbb{H}) = \mathbb{R}^3$.
(2) The map $F : \mathbb{R}^3 \to \mathbb{R}^3$, $y \mapsto qy\bar{q}$ is a rotation around v by the angle α.
(3) $\mathbb{S}^3 = \{q \in \mathbb{H} |\, |q| = 1\}$ is a double cover of $SO(3)$.

Proof. (1) A quaternion q is purely imaginary if and only if $\bar{q} = -q$.

$$\overline{qy\bar{q}} = q\bar{y}\bar{q} = -qy\bar{q}.$$

(2) F is a linear map and hence completely determined by its action on a basis. We extend v to a positive oriented orthonormal basis $\{v, b, c\}$ of $\mathbb{R}^3$. The action of F on v is given by

$$\begin{aligned} F(v) = qv\bar{q} &= \left(\cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) v\right) v \left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right) v\right) \\ &= \left(\cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) v\right) \left(\cos\left(\frac{\alpha}{2}\right) v + \sin\left(\frac{\alpha}{2}\right)\right) \\ &= \left(\cos^2\left(\frac{\alpha}{2}\right) + \sin^2\left(\frac{\alpha}{2}\right)\right) v + \cos\left(\frac{\alpha}{2}\right) \sin\left(\frac{\alpha}{2}\right) (1 + v^2) \\ &= v. \end{aligned}$$

In the last step we used that $v^2 = -1$. This gives us that the v-axis is invariant under F. Now, we consider the action of F on the plane orthogonal to v, i.e., the plane spanned by b and c. Since v and b are orthogonal, they anti-commute w.r.t. quaternionic multiplication.

$$vb = v \times b = -b \times v = -bv.$$

Using this and the addition formulas for cosine and sine we obtain

$$\begin{aligned} F(b) &= \left(\cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) v\right) b \left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right) v\right) \\ &= \left(\cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) v\right) \left(\cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) v\right) b \\ &= (\cos(\alpha) + \sin(\alpha) v)\, b = \cos(\alpha) b + \sin(\alpha)\, (v \times b) \\ &= \cos(\alpha) b + \sin(\alpha) c. \end{aligned}$$

Analogously one shows $F(c) = \cos(\alpha)c - \sin(\alpha)b$, and the matrix representation of F with respect to the basis $\{v, b, c\}$ is given by:

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Now it is easy to see that F describes a rotation around v by the angle α.
(3) Every $q \in \mathbb{S}^3$ is of the form $q = \cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) v$. Further q and $\tilde{q}$ define the same rotation if and only if $\tilde{q} = -q$. □

On $\mathbb{S}^n$ the Möbius group $\text{Möb}(n)$ is defined as the subgroup of the automorphism group of $\mathbb{S}^n$ that preserves hyper-spheres $\mathbb{S}^{n-1} \subset \mathbb{S}^n$. The Möbius group is of special interest when studying conformal geometry because any Möbius transformation is a conformal diffeomorphism. Moreover in dimensions $n \geq 3$, they are the only conformal diffeomorphisms.

Theorem 4 (Liouville).
Let $f : U \to V$ be a conformal diffeomorphism between connected open subsets $U, V \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}$, $n \geq 3$. Then there is a unique Möbius transformation $\mu \in \text{Möb}(n)$, such that $f = \mu|_U$.

For the proof see for example [17]. Using a stereographic projection, we can identify the compactification of $\mathbb{R}^n$, $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ with $\mathbb{S}^n$

$$\begin{aligned} \pi : \overline{\mathbb{R}^n} &\to \mathbb{S}^n \subset \mathbb{R}^{n+1} \\ (x_1, \dots, x_n) &\mapsto \frac{1}{|x|^2 + 1}(2x_1, \dots 2x_n, |x|^2 - 1). \end{aligned}$$

The stereographic projection is a conformal map because

$$\pi^*\langle\cdot,\cdot\rangle_{\mathbb{S}^n} = \frac{4}{(1+|x|^2)^2}\langle\cdot,\cdot\rangle_{\mathbb{R}^n}.$$

The action of Möb(n) on $\overline{\mathbb{R}^n}$ is generated by the inversion in the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, stretch rotations and translations. Considering $\mathbb{R}^3$ as $\mathrm{Im}(\mathbb{H})$, we can express the generators of Möb(3) using quaternions. For $\lambda \in \mathbb{H} \setminus \{0\}$ and $x \in \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$ we have:

- Stretch rotations: $x \mapsto \lambda x \bar{\lambda}$.
- Inversion at the unit sphere: $x \mapsto -x^{-1}$.
- Translations: $x \mapsto x + \lambda$.

3.2. **Immersions into $\mathbb{R}^3$.** After an intrinsic investigation of Riemann surfaces in the first section and studying the quaternions, we will now focus on immersions $f : M \to \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$. We start by a giving a sufficient and necessary criterion for these immersions to be conformal. Then we consider the set of $\mathbb{H}$-valued 1-forms and introduce some important quantities of immersions, like the Gaußian and mean curvatures. After deriving the well-known structure equations of surfaces in our coordinate free setup, we will finish with some computational lemmas that will be needed in the coming sections.

Let M be a two dimensional manifold and $f : M \to \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$ an immersion. If $\langle\cdot,\cdot\rangle$ denotes the euclidean scalar product on $\mathbb{R}^3$, the pullback metric $g(\cdot,\cdot) := f^*\langle\cdot,\cdot\rangle$ turns M into a Riemannian manifold. The **set of all immersions of a surface** M into $\mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$ will be denoted by $\mathcal{I}(M)$.

Proposition 13.
Let (M, J) be a Riemann surface and $f : M \to \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$ an immersion, then f is conformal if and only if there exists $N : M \to \mathbb{S}^2 \subset \mathbb{R}^3$ with

$$* \, df = N df. \tag{3.1}$$

The map N is called the unit ***normal vector field*** *of f.*

Proof. " $\Rightarrow$ " Let f be conformal, then $df(X)$ and $*df(X) = df(JX)$ are orthogonal w.r.t. the euclidean scalar product of $\mathbb{R}^3$ and we can define $N : M \to \mathbb{S}^2 \subset \mathbb{R}^3$ pointwise by

$$N_p := df(X)df(JX) = \underbrace{-\langle df(JX), df(X)\rangle}_{=0} + df(X) \times df(JX),$$

for some $X \in T_pM$ with $|df(X)| = 1$. This definition is independent of the choice of X.

" $\Leftarrow$ " If there exists a $N : M \to \mathbb{S}^2 \subset \mathbb{R}^3$ with

$$*df = N df,$$

we obtain $\langle N, df(X)\rangle = 0$ for all $X \in T_pM$ and hence

$$\begin{aligned} g(JX,Y) &= \langle df(JX), df(Y)\rangle \\ &= \langle \underbrace{-\langle N, df(X)\rangle}_{=0} + N \times df(X), df(Y)\rangle \\ &= -\langle N \times df(Y), df(X)\rangle \\ &= -g(X, JY). \end{aligned}$$

□

In the presence of a conformal immersion $f : M \to \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$, the trivial quaternionic line bundle $M \times \mathbb{H}$ splits into two complex line bundles: the tangent bundle f_*TM and the trivial normal bundle $M \times \mathrm{span}\{1, N\} \cong M \times \mathbb{C}$, where we use the identification $(1, \mathbf{i}) \leftrightarrow (1, N)$.

This splitting is reflected on the level of $\mathbb{H}$ valued 1-forms as follows: $\omega \in \Omega^1(M, \mathbb{H})$ is **tangential**(-valued) if

$$\omega N = -N\omega, \tag{3.2}$$

and **normal**(-valued) if

$$\omega N = N\omega.$$

The corresponding projection maps are given by

$$\begin{aligned} \pi_T : \Omega^1(M, \mathbb{H}) &\to \Omega^1(M, f_*TM) \\ \omega &\mapsto \frac{1}{2}(\omega + N\omega N) \\ \pi_N : \Omega^1(M, \mathbb{H}) &\to \Omega^1(M, \mathbb{C}) \\ \omega &\mapsto \frac{1}{2}(\omega - N\omega N). \end{aligned}$$

In Section 2.3 we have seen that $\Omega(M, \mathbb{C})$ splits into the canonical and anti-canonical bundle: $\Omega(M, \mathbb{C}) = K(M) \oplus \overline{K}(M)$. Also the tangential value 1-forms split into two complex line bundles

$$K_L(M) := \{\omega \in \Omega^1(M, \mathbb{H}) | *\omega = N\omega = -\omega N\}, \tag{3.3}$$
$$K_R(M) := \{\omega \in \Omega^1(M, \mathbb{H}) | *\omega = \omega N = -N\omega\}. \tag{3.4}$$

Note:

- A map $f : M \to \mathrm{Im}(\mathbb{H})$ is conformal w.r.t. N if $df \in K_L(M)$ and conformal w.r.t. $-N$ if $df \in K_{-N}(M) = K_R(M)$.
- Because the differential of an immersion never vanishes, $K_L(M)$ is spanned by df. Moreover, we can identify $\omega \in K_L(M)$ with $A \in \Gamma(\mathrm{End}_+(TM))$ via $\omega = df \circ A$.
- Similarly, every $\eta \in K_R(M)$ corresponds to $B \in \Gamma(\mathrm{End}_-(TM))$ via $\eta = df \circ B$.

For the wedge product of tangential 1-forms $\omega, \tilde{\omega} \in K_L(M)$ and $\eta, \tilde{\eta} \in K_R(M)$ we obtain

$$\begin{aligned} \omega \wedge \tilde{\omega} &= \omega * \tilde{\omega} - *\omega\tilde{\omega} = \omega N \tilde{\omega} - N\omega\tilde{\omega} = -2N\omega\tilde{\omega}, \\ \eta \wedge \tilde{\eta} &= \eta * \tilde{\eta} - *\eta\tilde{\eta} = -\eta N \tilde{\eta} + N\eta\tilde{\eta} = 2N\eta\tilde{\eta}, \\ \omega \wedge \eta &= -\omega N \eta - N\omega\eta = 0. \end{aligned} \tag{3.5}$$

Definition 20.
*Let $f : M \to \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$ be a conformal immersion with unit normal vector field N, then the **Weingarten operator** $A \in \Gamma(\mathrm{End}_{g+}(TM))$ is defined by*

$$dN(X) = df(AX), \qquad \forall X \in \mathfrak{X}(M). \tag{3.6}$$

*Its eigenvectors are called the principal curvature directions. The corresponding eigenvalues will be denoted by λ_1, λ_2 and called the **principal curvatures** of f. Their product $K := \lambda_1\lambda_2 = \det A$, is the **Gaussian curvature**. The decomposition of A into its J commuting and anti-commuting part gives us*

$$A = H\mathbb{1} + Q, \tag{3.7}$$

*where $H := \frac{1}{2}\operatorname{tr} A$ is called the **mean curvature** of the immersion f. This leads to the decomposition of dN into its $K_L(M)$ and $K_R(M)$ part*

$$dN = H df + \omega, \tag{3.8}$$

*where $\omega := df(Q) \in K_R(M)$ is called the **Hopf differential** of f.*

Evaluating (3.8) at the principal curvature directions gives us the **Gauss equation**

$$|\omega|^2 = \mu^2 |df|^2 = (H^2 - K)|df|^2, \qquad \text{where } \mu := \frac{|\lambda_1 - \lambda_2|}{2}. \tag{3.9}$$

Since f is an immersion, we obtain

$$\omega = 0 \Leftrightarrow \mu = 0 \Leftrightarrow \lambda_1 = \lambda_2. \tag{3.10}$$

These points are called **umbilical**.
Taking the derivative of (3.8), we obtain the **Gauss–Codazzi equation**:

$$d\omega = -dH \wedge df. \tag{3.11}$$

Hence, the immersion f has constant mean curvature (CMC) if and only if the Hopf differential ω is closed.

We prove now some important identities of the wedge product and the Laplace operator.

Lemma 4.
For a conformal immersion $f : M \to \mathbb{R}^3$ with Hopf differential ω and volume form $d\sigma = |df|^2$, we have:

(1) $df \wedge df = 2N d\sigma$.
(2) $\omega \wedge \omega = -N2(H^2 - K)d\sigma = -2N\mu^2 d\sigma$.
(3) $dN \wedge *dN = 2(H^2 + \mu^2)d\sigma$.

Proof. Let $X \in T_pM$ with $g(X, X) = 1$, using (3.5) and (3.8) we obtain

(1)
$$\begin{aligned}(df\wedge df)(X,JX) &= df(X)df(JX)-df(JX)df(X)\\ &= 2N|df(X)|^2\\ &= 2Nd\sigma(X,JX).\end{aligned}$$

(2)
$$\begin{aligned}(\omega\wedge\omega)(X,JX) &= \omega(X)\omega(JX)-\omega(JX)\omega(X)\\ &= -2N|\omega(X)|^2\\ &= -2N\mu^2|df(X)|^2\\ &= -2N\mu^2 d\sigma(X,JX).\end{aligned}$$

(3)
$$\begin{aligned}dN\wedge *dN &= (Hdf+\omega)\wedge(HNdf-N\omega)\\ &= H^2df\wedge Ndf-\omega\wedge N\omega\\ &= 2H^2d\sigma+2\mu^2d\sigma\\ &= 2(H^2+\mu^2)d\sigma.\end{aligned}$$

□

The wedge product defined via the quaternionic multiplication is not skew-symmetric because the quaternionic multiplication is not commutative. In order to obtain a skew symmetric wedge product, we use the euclidean scalar product of $\mathbb{H}$ and define

$$\begin{aligned}\langle\cdot\wedge\cdot\rangle:\Omega(M,\mathbb{H})\times\Omega(M,\mathbb{H})&\to\Omega^2(M,\mathbb{R})\\ \langle\eta\wedge\omega\rangle&:=\langle\eta,*\omega\rangle-\langle*\eta,\omega\rangle.\end{aligned}$$

For $h\in C^\infty(M,\mathbb{H})$, the real valued wedge product defined above satisfies the following product rule for the exterior derivative:

$$d\langle h,\eta\rangle=\langle dh\wedge\eta\rangle+\langle h,d\eta\rangle. \tag{3.12}$$

Further, $\langle *\eta\wedge\omega\rangle=\langle*\eta,*\omega\rangle+\langle\eta,\omega\rangle$ defines a positive definite symmetric bilinear map at every point in M and we can introduce a pointwise defined norm for $\eta\in\Omega^1(M,\mathbb{H})$ by

$$\|\eta\|^2:=\frac{\langle*\eta\wedge\eta\rangle}{2d\sigma}. \tag{3.13}$$

Lemma 5.
For a conformal immersion $f:M\to\mathbb{R}^3=\mathrm{Im}(\mathbb{H})$ with unit normal vector field N the Laplacians of f and N are given by

(1) $\triangle f=-\frac{1}{dA}d*df=-2HN.$

(2) $\triangle N=-2N((H^2+\mu^2)+\frac{dH\wedge df}{|df|^2}).$

If f has additionally constant mean curvature, we further get

$$\triangle N=-2N(H^2+\mu^2). \tag{3.14}$$

Proof. By Proposition 10 we have $\triangle f = -\frac{1}{dA} d * df$. With (3.5) and (3.8) we further obtain:

$$\begin{aligned} d * df &= d(Ndf) = dN \wedge df = Hdf \wedge df = H(df Ndf - Ndf df) \\ &= -2HNdf df = 2HN|df|^2. \end{aligned}$$

$$\begin{aligned} d * dN &= d(N(Hdf - \omega)) = dN \wedge Hdf + NdH \wedge df - dN \wedge \omega - Nd\omega \\ &= H^2 df \wedge df + 2NdH \wedge df - \omega \wedge \omega \\ &= 2N(H^2|df|^2 + dH \wedge df + |\omega|^2) \\ &= 2N\left((H^2 + \mu^2)|df|^2 + dH \wedge df\right). \end{aligned}$$

In the case that f has CMC, we further get $dH = d\omega = 0$ and hence:

$$\triangle N = -2N(H^2 + \mu^2). \tag{3.15}$$

□

3.3. **isothermic immersions and quadratic differentials.** So far, we used the canonical bundle K to construct quadratic differentials (see Example 2). If we have an immersion, there is a second construction not using the normal valued differentials contained in K but tangential ones. In this section we will show the correspondence between quadratic differentials, $-N$ conformal 1-forms and J anti-commuting endomorphism fields of the tangent bundle. Using this correspondence, we establish criteria when quadratic differentials are holomorphic and finally consider isothermic immersions.

Proposition 14.
In the presence of a conformal immersion $f : M \to \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$ there is an one-to-one correspondence between complex quadratic differentials and $-N$ conformal tangential 1-forms:

$$K_R(M) \ni \tau \leftrightarrow df\tau = q \in \mathfrak{Q}(M). \tag{3.16}$$

If $\tau = df \circ A$ for $A \in \Gamma(\mathrm{End}_-(TM))$, we further have

$$q(X) = -\langle AX, X\rangle_{\mathbb{C}}. \tag{3.17}$$

Proof. For $\tau \in K_R(M)$ we define

$$\begin{aligned} q &: \mathfrak{X}(M) \to C^\infty(M, \mathbb{C}) \\ &q(X) := df(X)\tau(X). \end{aligned}$$

First, we show that q takes values in the normal bundle that we identify with $M \times \mathbb{C}$

$$\begin{aligned} df(X)\tau(X) &= -\langle df(X), \tau(X)\rangle + df(X) \times \tau(X) \\ &= -\langle df(X), \tau(X)\rangle + \frac{1}{2}(df(X)\tau(X) - \tau(X)df(X)) \\ &= -\langle df(X), \tau(X)\rangle + \frac{1}{2}(-Ndf(JX)\tau(X) - N\tau(X)df(JX)) \\ &= -\langle df(X), \tau(X)\rangle + N\langle df(JX), \tau(X)\rangle. \end{aligned}$$

For $\tau = df \circ A$ we further get

$$\begin{aligned} df(X)\tau(X) &= -\langle df(X), \tau(X)\rangle + N\langle df(JX), \tau(X)\rangle \\ &= -g(AX, X) + Ng(JX, AX) \\ &= -\langle AX, X\rangle_{\mathbb{C}}. \end{aligned}$$

The differential q is quadratic, because for $z = x + \mathbf{i}y$ we have

$$q(zX) = df(zX)\tau(zX) = (x + Ny)df(X)(x - Ny)\tau(X) = (x + Ny)^2 df(X)\tau(X) = z^2 q(X).$$

Vice versa, given $q \in \mathcal{Q}(M)$ we define

$$\tau(X) := q(X)(df(X))^{-1} = q(X)\frac{\overline{df(X)}}{|df(X)|^2}.$$

Since q is normal and df^{-1} tangential, τ is also tangential. Moreover, τ is $-N$ conformal because for $\alpha + J\beta \in C^\infty(M, \mathbb{C})$ we have

$$\begin{aligned} \tau((\alpha + J\beta)X) &= q((\alpha + J\beta)X)(df((\alpha + J\beta)X))^{-1} \\ &= (\alpha + N\beta)^2 q(X)\frac{(\alpha - N\beta)}{|\alpha + N\beta|^2}\frac{\overline{df(X)}}{|df(X)|^2} \\ &= (\alpha - N\beta)\tau(X). \end{aligned}$$

□

Lemma 6.
For $\tau = df(A\cdot) \in K_R(M)$ the tangential and normal part of the exterior derivative are given by

$$\pi_T(d\tau)(X, JX) = df((\partial_X A)JX), \tag{3.18}$$
$$\pi_N(d\tau)(X, JX) = -\operatorname{tr}(JQA)Nd\sigma(X, JX) = 2\langle \tau(X), \omega(JX)\rangle N, \tag{3.19}$$

where ω denotes the Hopf differential of f. In particular,

$$\begin{aligned} &\pi_T(d\tau) = 0 \Leftrightarrow \partial A = 0, \\ &\pi_N(d\tau) = 0 \Leftrightarrow \langle \tau(X), \omega(JX)\rangle = 0 \quad \forall X \in \mathfrak{X}(M). \end{aligned}$$

Proof. The exterior derivative of τ is defined as

$$d\tau(X, JX) = \nabla_X(\tau(JX)) - \nabla_{JX}(\tau(X)) - \tau([X, JX]).$$

With the usual splitting of the directional derivative into its normal and tangential part,

$$\nabla_X(\tau(Y)) = df(\nabla_X(AY)) - \langle df(AY), dN(X)\rangle N,$$

we have

$$\begin{aligned}\pi_T(d\tau)(X,JX) &= df(\nabla_X(AJX) - \nabla_{JX}(AX) - A(\nabla_X JX - \nabla_{JX}X))\\ &= df((\nabla_X A)JX - (\nabla_{JX}A)X)\\ &= df(-J((\nabla_X A)X - J(\nabla_{JX}A)X))\\ &= df(-J(\partial_X A)X)\\ &= df((\partial_X A)JX),\end{aligned}$$

i.e., $\pi_T(d\tau) = 0 \Leftrightarrow \partial A = 0$. For the normal part we obtain with (3.8)

$$\begin{aligned}\pi_N(d\tau)(X,JX) &= \langle df(AX), dN(JX)\rangle N - \langle df(AJX), dN(X)\rangle N\\ &= g(AX, QJX + HJX)N - g(AJX, QX + HX)N\\ &= -g(JQAX, X)N - g(JQAJX, JX)N\\ &= \operatorname{tr}(-JQA)N d\sigma(X,JX).\end{aligned}$$

The second identity of (3.19) is proven by

$$\begin{aligned}\operatorname{tr}(JQA)d\sigma(X,JX) &= g(JQAX,X)N + g(JQAJX,JX)N\\ &= -2g(AX,QJX)N\\ &= -2\langle \tau(X), \omega(JX)\rangle N.\end{aligned}$$

□

Proposition 15.
Let $f : M \to \mathbb{R}^3$ be a conformal immersion of the Riemann surface (M, J) and $q = df\tau = \langle -A\cdot, \cdot\rangle_{\mathbb{C}} \in \mathcal{Q}(M)$ be a complex quadratic differential, then

$$(\bar{\partial}_X q)(Y) = \langle (-\partial_X A)Y, Y\rangle_{\mathbb{C}}.$$

In particular, q is holomorphic if one and hence all of the following equivalent conditions are satisfied:

$$\begin{aligned}\pi_T(d\tau) &= 0,\\ \partial A &= 0,\\ \operatorname{div} A &= 0.\end{aligned}$$

Proof. q is holomorphic if $\bar{\partial} q = \bar{\partial}(-g(A\cdot,\cdot) + \mathbf{i}g(A\cdot,\cdot)) = 0$, where the holomorphic structure $\bar{\partial}$ is obtained from the Levi-Civita connection of g using the product rule

$$\begin{aligned}2(\bar{\partial}_X q)(Y) &= (\nabla_X q + \mathbf{i}\nabla_{JX} q)(Y)\\ &= -g((\nabla_X A)Y, Y) - \mathbf{i}g((\nabla_{JX}A)Y, Y) + \mathbf{i}g((\nabla_X A)Y, JY) - g((\nabla_{JX}A)Y, JY)\\ &= -g((\nabla_X A)Y - J(\nabla_{JX}A)Y, Y) + \mathbf{i}g((\nabla_X A)Y - J(\nabla_{JX}A)Y, JY)\\ &= -2g((\partial_X A)Y, Y) + 2\mathbf{i}g((\partial_X A), JX)\\ &= 2\langle(-\partial_X A)Y, Y\rangle_{\mathbb{C}}.\end{aligned}$$

Therefore, we have $\bar{\partial} q = 0 \Leftrightarrow \partial A = 0$. With Proposition 12 this is equivalent to $\operatorname{div} A = 0$ and with the last lemma we finally have $\pi_T(d\tau) = 0$. □

With the usual identification of bilinear maps and 2-forms and Proposition 12, we define

$$\begin{aligned}\overline{D} : \mathcal{Q}(M) &\to \Omega^2(M, T^*M)\\ \overline{D}q &:= g(-DA, \cdot) = g(J \operatorname{div} A, \cdot)d\sigma.\end{aligned} \tag{3.20}$$

Note that $\ker(\overline{D}) = \ker(\overline{\partial}) = H^0(\mathcal{Q}(M))$. Using Proposition 14 and Corollary 3, we obtain the following corollary.

Corollary 5.
On a torus, a differential $\tau \in K_R(M)$ with $\pi_T(d\tau) = 0$ is either identically zero or has no zeros at all.

Lemma 7.
Let $f : M \to \mathbb{R}^3$ be a conformal immersion and $\tau \in K_R(M)$ with $d\tau = 0$. Then there exists $\rho : \tilde{M} \to \mathbb{R}$ such that $\omega|_{\tilde{M}} = \rho\tau$, where ω denotes the Hopf differential of f and $\tilde{M} := \{p \in M | \, \tau_p \neq 0\}$.

Proof. Since $\tau, \omega \in K_R(M)$ there exist $\alpha, \rho \in C^\infty(\tilde{M}, \mathbb{R})$ such that $\omega = (\rho + \alpha N)\tau$. The Gauss–Codazzi equation gives rise to

$$\underbrace{-dH \wedge df}_{tangential} = d\omega = d((\rho + \alpha N)\tau) = \underbrace{(d\rho + d\alpha N) \wedge \tau}_{tangential} + \alpha \underbrace{H df \wedge \tau}_{=0} + \alpha \underbrace{\omega \wedge \tau}_{normal}.$$

Because the left hand side of the equation is tangential, the normal part of the right hand side has to vanish, and we obtain $\alpha = 0$. □

Definition 21.
An immersion $f : M \to \mathbb{R}^3$ of Riemann surface M is called ***(globally-) isothermic*** *if there exists a holomorphic quadratic differential $q = df\tau$ with $d\tau = 0$.*

Note that locally the closed $\mathbb{R}^3$ valued one form τ is exact and hence defines a conformal immersion $f^* : U \subset M \to \mathbb{R}^3$ with normal $-N$. The immersion f^* is called the **Christoffel dual** of f.

Our global definition of isothermic immersions is stricter than the usual local one, where an immersion is called isothermic if it locally admits a conformal coordinate line parametrization. In the next proposition we show that these immersions are contained in our definition.

Proposition 16.
Let $f : M \to \mathbb{R}^3$ be an isothermic immersion such that away from the isolated zeros of τ, there exist local coordinates $z = x + \boldsymbol{i}y : U \subset M \to \mathbb{C}$ such that $\tilde{f} := f \circ z^{-1}$ is a conformal coordinate line parametrization, i.e.,

$$\begin{aligned}dN\left(\frac{\partial}{\partial x}\right) &= \lambda_1 d\tilde{f}\left(\frac{\partial}{\partial x}\right)\\ dN\left(\frac{\partial}{\partial y}\right) &= \lambda_2 d\tilde{f}\left(\frac{\partial}{\partial y}\right).\end{aligned}$$

Vice versa, any conformal coordinate line parametrization locally defines a holomorphic quadratic differential $q = \tau df$ with $d\tau = 0$.

Proof. " $\Rightarrow$ " Let $q = df\tau \in H^0(\mathfrak{Q}(M))$ with $d\tau = 0$ and $U \subset M$ with $\tau|_U \neq 0$. With Lemma 7 there exists $\rho \in C^\infty(U)$ such that $\rho\tau = \omega$ on U. Further, there are conformal coordinates (see example 3) $z = x + \mathbf{i}\, y : U \to \mathbb{C}$ with

$$df\tau = -\underbrace{\langle df, \tau\rangle}_{:=\alpha} + N \underbrace{\langle df\tau, N\rangle}_{:=\beta} = dz^2 = dx^2 - dy^2 + 2\mathbf{i}dxdy.$$

From $dx\left(\frac{\partial}{\partial y}\right) = dy\left(\frac{\partial}{\partial x}\right) = 0$ we get

$$\beta\left(\frac{\partial}{\partial x}\right) = 2dx\left(\frac{\partial}{\partial x}\right) dy\left(\frac{\partial}{\partial x}\right) = 0 = \beta\left(\frac{\partial}{\partial y}\right).$$

Since $dN\left(\frac{\partial}{\partial x}\right)$ and $df\left(\frac{\partial}{\partial x}\right)$ are both tangential, they are parallel if and only if

$$\langle dN\left(\frac{\partial}{\partial x}\right) \times df\left(\frac{\partial}{\partial x}\right), N\rangle = 0.$$

The Gauss equation gives rise to

$$\begin{aligned}
\langle dN\left(\frac{\partial}{\partial x}\right) \times df\left(\frac{\partial}{\partial x}\right), N\rangle &= \langle\left(Hdf\left(\frac{\partial}{\partial x}\right) + \omega\left(\frac{\partial}{\partial x}\right)\right) \times df\left(\frac{\partial}{\partial x}\right), N\rangle \\
&= \langle \rho\tau\left(\frac{\partial}{\partial x}\right) \times df\left(\frac{\partial}{\partial x}\right), N\rangle \\
&= -\rho\langle df\left(\frac{\partial}{\partial x}\right) \tau\left(\frac{\partial}{\partial x}\right), N\rangle \\
&= -\rho\beta\left(\frac{\partial}{\partial x}\right) = 0.
\end{aligned}$$

Analogously one shows that $dN\left(\frac{\partial}{\partial y}\right)$ and $d\tilde{f}\left(\frac{\partial}{\partial y}\right)$ are parallel, hence z gives us curvature line coordinates.

" $\Leftarrow$ " Let $f : U \subset \mathbb{C} \to \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$ be a conformal coordinate line parametrization with normal N, i.e.,

$$\begin{aligned}
N_x &= dN\left(\frac{\partial}{\partial x}\right) = \lambda_1 df\left(\frac{\partial}{\partial x}\right) = \lambda_1 f_x \\
N_y &= dN\left(\frac{\partial}{\partial y}\right) = \lambda_2 df\left(\frac{\partial}{\partial y}\right) = \lambda_2 f_y \\
f_y &= Nf_x = -f_x N.
\end{aligned}$$

From $\langle N, f_x\rangle = 0$ we get

$$\langle N, f_{yx}\rangle = -\langle N_y, f_x\rangle = -\lambda_2\langle f_y, f_x\rangle = 0.$$

Since f_{yx} has no real part either, it is tangential, i.e.,

$$Nf_{yx} = -f_{yx}N. \tag{3.21}$$

We define $\tau \in \Omega(U, \mathbb{H})$ by

$$\tau := f_x^{-1}dx - f_y^{-1}dy. \tag{3.22}$$

From $*dx = -dy$ and $Nf_x^{-1} = -f_x^{-1}N$ we obtain

$$*\tau = -f_x^{-1}dy - f_y^{-1}dx = Nf_x^{-1}dx - f_y^{-1}dy = -N\tau = \tau N.$$

It is left to show that τ is closed

$$\begin{aligned} d\tau &= df_x^{-1} \wedge dx - df_y^{-1} \wedge dy \\ &= \frac{\partial}{\partial y} f_x^{-1} dy \wedge dx - \frac{\partial}{\partial x} f_y^{-1} dx \wedge dy \\ &= \left(f_x^{-1} f_{yx} f_x^{-1} + f_y^{-1} f_{xy} f_y^{-1}\right) dx \wedge dy \\ &= \left(f_x^{-1} f_{yx} f_x^{-1} - f_x^{-1} N f_{xy} N f_x^{-1}\right) dx \wedge dy \\ &= \left(f_x^{-1} f_{yx} f_x^{-1} - f_x^{-1} f_{yx} f_x^{-1}\right) dx \wedge dy \\ &= 0, \end{aligned}$$

where we used (3.21) in the last step. □

Remark 1. Due to Proposition 8 there are no holomorphic quadratic differentials on a genus zero surface. Therefore, our global definition of isothermic immersions does not work in the case of spheres or ellipsoids.
In the case of genus one surfaces, holomorphic quadratic differentials do not have zeros. Their umbilical points are given by the zero set of the function ρ defined in Lemma 7. These umbilical points are special because the conformal curvature lines can be extended through them and do not collapse as in the case of ellipsoids.

4. Conformal change of metric of the ambient space

Let M be a two dimensional manifold and $f : M \to \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$ a conformal immersion with unit normal vector field $N : M \to \mathbb{S}^2$. If $\langle\cdot,\cdot\rangle$ denotes the euclidean scalar product on $\mathbb{H}$, then $g := f^*\langle\cdot,\cdot\rangle$ defines an Riemannian metric on M. Let ∇ denote the Levi-Civita connection with respect to g. In this section we want to compute how a conformal change of the euclidean metric of the ambient space

$$\langle\cdot,\cdot\rangle \to \widetilde{\langle\cdot,\cdot\rangle} = e^{2u}\langle\cdot,\cdot\rangle, \qquad u \in C^\infty(\mathbb{H}, \mathbb{R}), \tag{4.1}$$

changes important quantities of the immersion f.

Proposition 17.
Under a conformal change of the metric of the ambient space $\langle\cdot,\cdot\rangle \to \widetilde{\langle\cdot,\cdot\rangle} = e^{2u}\langle\cdot,\cdot\rangle$, the Hopf-differential ω and the mean curvature H transform as follows:

$$\tilde{\omega} = e^{-u\circ f}\omega, \tag{4.2}$$
$$\tilde{H} = e^{-u\circ f}(H + \langle N, \mathrm{grad}_f u\rangle), \tag{4.3}$$
$$\tilde{A} = e^{-u\circ f}(A + \langle N, \mathrm{grad}_f u\rangle \mathbb{1}). \tag{4.4}$$

Proof. Due to Proposition 9 the Levi-Civita connection of $\tilde{g} = e^{2u\circ f} g$ is given by:

$$\tilde{\nabla}_X Y = \nabla_X Y + g(X,G)Y + g(Y,G)X - g(X,Y)G, \qquad \text{where } G := \mathrm{grad}(u \circ f).$$

Note that $df_p(G)$ is the tangential part of $\mathrm{grad}_{f(p)} u$. Firstly, we derive an explicit expression of the new shape operator $\tilde{A}$, which is defined by

$$df(\tilde{A}X) = \tilde{\nabla}_X \tilde{N}, \qquad \forall X \in \mathfrak{X}(M).$$

With $\tilde{N} = e^{-u\circ f} N$ we get

$$\begin{aligned}
\tilde{\nabla}_X \tilde{N} &= \tilde{\nabla}_X (e^{-u\circ f} N) \\
&= -\langle df(X), \operatorname{grad}_f u\rangle \tilde{N} + e^{-u\circ f} \tilde{\nabla}_X N \\
&= -\langle df(X), \operatorname{grad}_f u\rangle \tilde{N} + e^{-u\circ f} \Big(\nabla_X N + \langle df(X), \operatorname{grad}_f u\rangle N + \langle N, \operatorname{grad}_f u\rangle df(X) \\
&\quad + \langle df(X), N\rangle \operatorname{grad}_f u \Big) \\
&= e^{-u\circ f}((\nabla_X N + \langle N, \operatorname{grad}_f u\rangle df(X)) \\
&= df(e^{-u\circ f} AX + e^{-u\circ f} \langle N, \operatorname{grad}_f u\rangle X).
\end{aligned}$$

This gives us $\tilde{A} = e^{-u\circ f}(A + \langle N, \operatorname{grad}_f u\rangle \mathbb{1})$ and we finally obtain

$$\begin{aligned}
\tilde{H} &= \frac{1}{2}\operatorname{tr}(\tilde{A}) = e^{-u\circ f}(H + \langle N, \operatorname{grad}_f u\rangle), \\
\tilde{\omega} &= e^{-u\circ f}\omega.
\end{aligned}$$

□

Proposition 18.
Under a conformal change of metric of the ambient space $\langle\cdot,\cdot\rangle \to \widetilde{\langle\cdot,\cdot\rangle} = e^{2u}\langle\cdot,\cdot\rangle$, the Gaussian curvature with respect to the new metric is given by

$$\tilde{K} = \hat{K}\circ f + \det(\tilde{A}), \tag{4.5}$$

where $\hat{K}(f(p))$ denotes the sectional curvature of $f^* T_p M \subset \mathbb{R}^3$ with respect to the metric $e^{2u}\langle\cdot,\cdot\rangle$.

Proof. By Proposition 11 we have

$$\tilde{K} = e^{-2u\circ f}(K - \triangle(u\circ f)).$$

We want to compute $\triangle(u\circ f)$. Therefore, we choose $X \in \mathfrak{X}(M)$, such that X_p and JX_p are principal directions of f at $p \in M$ and $|X_p| = 1$. For the gradient of u at p we obtain

$$\begin{aligned}
\operatorname{grad}_f u &= \langle \operatorname{grad}_f u, df(X)\rangle df(X) + \langle \operatorname{grad}_f u, df(JX)\rangle df(JX) + \langle \operatorname{grad}_f u, N\rangle N \\
&= df(\operatorname{grad}(u\circ f)) + \langle \operatorname{grad}_f u, N\rangle N.
\end{aligned} \tag{4.6}$$

Taking the derivative leads to

$$\begin{aligned}
\nabla_X(df(\operatorname{grad}(u\circ f))) &\overset{(4.6)}{=} \nabla_X \operatorname{grad}_f u - \nabla_X(\langle \operatorname{grad}_f u, N\rangle N) \\
&= \nabla_X \operatorname{grad}_f u - \langle \nabla_X \operatorname{grad}_f u, N\rangle N - \langle \operatorname{grad}_f u, dN(X)\rangle N \\
&\quad - \langle \operatorname{grad}_f u, N\rangle dN(X).
\end{aligned} \tag{4.7}$$

The decomposition of $\nabla_X(df(\operatorname{grad}(u\circ f)))$ into its normal and tangential part is given by

$$df(\nabla_X \operatorname{grad}(u\circ f)) = \nabla_X(df(\operatorname{grad}(u\circ f))) - \langle df(\operatorname{grad}(u\circ f)), dN(X)\rangle N. \tag{4.8}$$

Now we have all the ingredients together to compute the Laplacian of $u \circ f$.

$$\begin{aligned}
\triangle(u \circ f) &= \operatorname{div}\operatorname{grad}(u \circ f) \\
&= g(\nabla_X \operatorname{grad}(u \circ f), X) + g(\nabla_{JX} \operatorname{grad}(u \circ f), JX) \\
&= \langle df((\nabla_X \operatorname{grad}(u \circ f)), df(X)\rangle + \langle df((\nabla_{JX} \operatorname{grad}(u \circ f)), df(JX)\rangle \\
&\overset{(4.8)}{=} \langle \nabla_X(df(\operatorname{grad}(u \circ f))) - \langle df(\operatorname{grad}(u \circ f)), dN(X)\rangle N, df(X)\rangle \\
&\quad + \langle \nabla_{JX}(df(\operatorname{grad}(u \circ f))) - \langle df(\operatorname{grad}(u \circ f)), dN(JX)\rangle N, df(JX)\rangle \\
&\overset{(4.7)}{=} \langle \nabla_X \operatorname{grad}_f u - \langle \operatorname{grad}_f u, N\rangle dN(X), df(X)\rangle \\
&\quad + \langle \nabla_{JX} \operatorname{grad}_f u - \langle \operatorname{grad}_f u, N\rangle dN(JX), df(JX)\rangle \\
&= \langle \nabla_X \operatorname{grad}_f u, df(X)\rangle + \langle \nabla_{JX} \operatorname{grad}_f u, df(JX)\rangle - 2H\langle \operatorname{grad}_f u, N\rangle. \qquad (4.9)
\end{aligned}$$

The sectional curvature of $\mathbb{R}^3$ equipped with the euclidean scalar product is zero everywhere. With Proposition 11 we obtain for the sectional curvature of f_*TM with respect to the metric $e^{2u}\langle\cdot,\cdot\rangle$

$$\hat{K} \circ f = -e^{-2u\circ f}(\langle \nabla_X \operatorname{grad}_f u, df(X)\rangle + \langle \nabla_{JX} \operatorname{grad}_f u, df(JX)\rangle + \langle \operatorname{grad}_f u, N\rangle^2). \quad (4.10)$$

Proposition 17 and the usual calculation rules for the determinate give rise to

$$\begin{aligned}
\det(\tilde{A}) &= \det(e^{-u\circ f}(A + \langle N, \operatorname{grad}_f u\rangle \mathbb{1})) \\
&= e^{-2u\circ f}(\det(A) + \operatorname{tr}(A)\langle N, \operatorname{grad}_f u\rangle + \langle N, \operatorname{grad}_f u\rangle^2) \\
&= e^{-2u\circ f}(K + 2H\langle N, \operatorname{grad}_f u\rangle + \langle N, \operatorname{grad}_f u\rangle^2). \qquad (4.11)
\end{aligned}$$

Now we can write everything together to proof the proposition

$$\begin{aligned}
\tilde{K} &\overset{(4.5)}{=} e^{-2u\circ f}(K - \triangle(u \circ f)) \\
&\overset{(4.9)}{=} e^{-2u\circ f}(K - \langle \nabla_X \operatorname{grad}_f u, df(X)\rangle - \langle \nabla_{JX} \operatorname{grad}_f u, df(JX)\rangle + 2H\langle \operatorname{grad}_f u, N\rangle) \\
&\overset{(4.10)}{=} \hat{K} \circ f + e^{-2u\circ f}(K + 2H\langle N, \operatorname{grad}_f u\rangle + \langle N, \operatorname{grad}_f u\rangle^2) \\
&\overset{(4.11)}{=} \hat{K} \circ f + \det(\tilde{A}).
\end{aligned}$$

□

Corollary 6.
Under the conformal change of metric of the ambient space $\langle\cdot,\cdot\rangle \to \widetilde{\langle\cdot,\cdot\rangle} = e^{2u}\langle\cdot,\cdot\rangle$, *the* Gauss equation *has the form:*

$$|\tilde{\omega}|^2 = \tilde{\mu}^2|df|^2 = (\tilde{H}^2 - \tilde{K} + \hat{K} \circ f)|df|^2.$$

Example 4:
Space forms are of particular interest for us (see section 7). The simplest example of a space form is $\mathbb{R}^3$ *equipped with the euclidean metric, where the sectional curvature is zero everywhere. The hyperbolic space is an example of a space form with constant negative sectional curvature and can be obtained form* $\mathbb{R}^3$ *using a conformal change of metric.*

(1) *We consider the Poincaré ball model of the hyperbolic space:*

$$\mathbf{H}^3_k := \{x \in \mathbb{R}^3 | |x| < k\}, \qquad \langle \cdot, \cdot \rangle_{hyp} = \frac{4}{(k^2 - |x|^2)^2} \langle \cdot, \cdot \rangle.$$

The conformal factor is given by

$$u(x) = \log \left(\frac{2}{k^2 - |x|^2} \right),$$

which leads to

$$f^* du = \frac{2\langle f, df \rangle}{k^2 - |f|^2} \qquad \text{and} \qquad \mathrm{grad}_f u = \frac{2f}{k^2 - |f|^2}.$$

Therefore, we finally have

$$\tilde{H} = \frac{k^2 - |f|^2}{2} H + \langle N, f \rangle, \tag{4.12}$$

$$\tilde{\omega} = \frac{k^2 - |f|^2}{2} \omega. \tag{4.13}$$

For the sectional curvature of $(\mathbf{H}^3_k, \langle \cdot, \cdot \rangle_{hyp})$, *we have* $\hat{K} = -k^2$.

(2) *A second model of the hyperbolic space that can be obtained form* $\mathbb{R}^3$ *using a conformal change of metric is the the Poincaré half-space model. For* $b \in \mathbb{R}^3$ *we define:*

$$\mathbf{H}^3_b := \{x \in \mathbb{R}^3 | \langle x, b \rangle > 0\}, \qquad \langle \cdot, \cdot \rangle_{hyp} = \frac{1}{\langle x, b \rangle^2} \langle \cdot, \cdot \rangle.$$

Here, the conformal factor is

$$u(x) = \log \left(\frac{1}{\langle b, f \rangle} \right),$$

this gives rise to

$$f^* du = \frac{-\langle b, df \rangle}{\langle b, f \rangle} \qquad \text{and} \qquad \mathrm{grad}_f u = \frac{-b}{\langle b, f \rangle}.$$

Therefore, we finally have

$$\tilde{H} = \langle b, f \rangle H - \langle N, b \rangle = \langle b, fH - N \rangle, \tag{4.14}$$

$$\tilde{\omega} = \langle b, f \rangle \, \omega. \tag{4.15}$$

$(\mathbf{H}^3_b, \langle \cdot, \cdot \rangle_{hyp})$ *is a space form too, with sectional curvature* $\hat{K} = -|b|^2$.

(3) *An similar result is obtained for the spherical case. After a stereographic projection, we have*

$$\mathbb{S}^3_k \setminus \{\infty\} \cong \mathbb{R}^3, \qquad \langle \cdot, \cdot \rangle_{sph} = \frac{4}{(k^2 + |x|^2)^2} \langle \cdot, \cdot \rangle.$$

$(\mathbb{S}^3_k, \langle\cdot,\cdot\rangle_{sph})$ is a space form with positive sectional curvature $\hat{K} = k^2$. For the Hopf differential and the mean curvature we obtain:

$$\tilde{H} = \frac{k^2 + |f|^2}{2} H - \langle N, f\rangle, \tag{4.16}$$

$$\tilde{\omega} = \frac{k^2 + |f|^2}{2}\omega. \tag{4.17}$$

5. Variation of immersions and the Euler-Lagrange equations for the Willmore functional

For a given surface M there exist lots of possible immersions $f : M \to \mathbb{R}^3$. As a natural consequence, the question arises which immersions $f : M \to \mathbb{R}^3$ of a given surface M are the most 'round' or symmetric ones. An answer to this question was given by Thomas James Willmore [32].

Definition 22 (Willmore surfaces)**.**
*Let M be a surface and $f : M \to \mathbb{R}^3$ an immersion. The **Willmore energy** of f is defined as*

$$W(f) := \int_M H^2 d\sigma.$$

*The critical points of the Willmore functional W are called **Willmore surfaces**.*

In some papers the Willmore energy of closed immersed surfaces is defined as

$$W(f) := \int_M H^2 - K d\sigma.$$

Because the Gauss-Bonnet Theorem gives us

$$\int_M K d\sigma = 2\pi\chi(M) = 4\pi(\mathfrak{g} - 1),$$

the definitions only differ by a topological constant and their critical points are the same.

In order to obtain a coordinate-free derivation of the Euler-Lagrange equation of the Willmore functional, we combine the strategy that Ernst Kuvert and Rainer Schätzle use in [22] with the coordinate free variation formulas of the metric and its almost complex structure that Christoph Bohle, Paul Peters and Ulrich Pinkall introduce in [6].

First, note that the Willmore energy depends highly on the metric of the target space of the immersion. If we consider a conformal change of metric of $\mathbb{R}^3$, $\langle\cdot,\cdot\rangle \mapsto e^{2u}\langle\cdot,\cdot\rangle$, we obtain with Proposition 17 and 18

$$\begin{aligned} d\tilde{\sigma} &= e^{2u\circ f} d\sigma, \\ \tilde{H} &= e^{-u\circ f}(H + \langle N, \mathrm{grad}_f u\rangle), \\ \tilde{K} &= \hat{K}\circ f + \det(\tilde{A}), \end{aligned}$$

where $\hat{K}(f(p))$ denotes the sectional curvature of $f^*T_pM \subset \mathbb{R}^3$ with respect to the metric $e^{2u}\langle\cdot,\cdot\rangle$. This gives us for the Willmore energy of a closed immersed surface

$$\begin{aligned}\tilde{W}(f) &= \int_M \tilde{H}^2 - \tilde{K} d\tilde{\sigma} + 4\pi(\mathfrak{g}-1)\\ &= \int_M H^2 + \langle 2HN, \operatorname{grad}_f u\rangle + \langle N, \operatorname{grad}_f u\rangle^2 - \hat{K}\circ f - \det(\tilde{A})\, d\sigma + 4\pi(\mathfrak{g}-1)\\ &= \int_M H^2 - \hat{K}\circ f - K d\sigma + 4\pi(\mathfrak{g}-1)\\ &= \int_M H^2 - \hat{K}\circ f\, d\sigma.\end{aligned}$$

This leads to a conformal invariant definition of the Willmore energy of immersions into arbitrary 3-dimensional Riemannian manifolds that is consistent with the one for immersions into $\mathbb{R}^3$, as given above.

Definition 23.
*Let $f : M \to \hat{M}$ be a conformal immersion of a closed Riemann surface into a 3-dimensional Riemannian manifold $(\hat{M}, \hat{g})$ with sectional curvature $\hat{K}$. The **Willmore energy** of f is defined as*

$$W(f) := \int_M H^2 + \hat{K}\circ f\, d\sigma.$$

This definition is invariant under a conformal change of metric $\hat{g} \mapsto e^{2u}\hat{g}$.

Further, we obtain that the Willmore energy is invariant under isometries and scaling of the target space. In order to find the critical points of the Willmore functional we have to consider variations of immersions.

Definition 24.
*Let $f : M \to \mathbb{R}^3 \subset \operatorname{Im}(\mathbb{H})$ be an immersion. A **variation** of f is given by a one parameter family of smooth immersions*

$$\begin{aligned} f_t : (\epsilon, \epsilon) \times M &\to \mathbb{R}^3\\ (t,p) &\mapsto f_t(p)\end{aligned}$$

such that $f_0 = f$. The derivative at $t = 0$

$$\dot{f} := \left.\frac{\partial}{\partial t}\right|_{t=0} f_t$$

is called the variation vector field of f.

Note that for a given variation vector field $\dot{f}$, there exist $X \in \mathfrak{X}(M)$ and $u \in C^\infty(M)$ such that $\dot{f} = df(X) + uN$, where $df(X)$ is the tangential and uN the normal part of the variation. The tangential part of an variation corresponds to a reparametrisation of $f(M)$. For the flow of X

$$\begin{aligned}\Phi(\epsilon,\epsilon)\times M &\to M\\ (t,p) &\mapsto \Phi_t(p),\end{aligned}$$

we define $f_t := f \circ \Phi_t$, and obtain

$$\dot{f} = \frac{\partial}{\partial t}\Big|_{t=0} f \circ \Phi_t = df(X).$$

For a normal variation vector field $\dot{f} = uN$, the corresponding variation is given by

$$f_t = f + tuN.$$

Lemma 8 (Bohle, Peters Pinkall [6]).
The variations of the induced metric $g := f^*\langle \cdot, \cdot \rangle$ and of the compatible almost complex structure $J \in \Gamma(\mathrm{End}(TM))$ are given by

$$\begin{aligned} \dot{g} &= \mathcal{L}_X g = g\left((\nabla X + (\nabla X)^*)\,\cdot, \cdot\right) \\ \dot{J} &= \mathcal{L}_X J = 2J\bar{\partial}X \end{aligned}$$

for a tangential variation $\dot{f} = df(X)$ and by

$$\begin{aligned} \dot{g} &= 2u\, g\,(A\,\cdot, \cdot) \\ \dot{J} &= 2u\, JQ \end{aligned}$$

for a normal variation $\dot{f} = uN$, where A denotes the shape operator of f and Q its trace-free part, the Hopf differential. A variation is conformal if $\dot{J} = -\mathcal{L}_X J + 2u\, JQ = 0$.

Note that A is symmetric with respect to g and hence

$$Q = A - \frac{1}{2}\,\mathrm{tr}(A)\,\mathbb{1} = A - A_+ = A_-,$$

i.e., Q anti-commutes with J.

Proof. We start with the tangential variation $\dot{f} = df(X)$:

$$\dot{g} = \frac{\partial}{\partial t}\Big|_{t=0} g_t = \frac{\partial}{\partial t}\Big|_{t=0} (f \circ \Phi_t)^* \langle \cdot, \cdot \rangle = \frac{\partial}{\partial t}\Big|_{t=0} \Phi_t^* g = \mathcal{L}_X g.$$

For the second identity, let $Y, Z \in \mathfrak{X}(M)$ and consider:

$$\begin{aligned} Xg(Y,Z) &= \frac{\partial}{\partial t}\Big|_{t=0} g_t(Y,Z) = \dot{g}(Y,Z) + g(\mathcal{L}_X Y, Z) + g(\mathcal{L}_X Z, Y) \\ \Rightarrow \qquad \dot{g}(Y,Z) &= Xg(Y,Z) - g(\mathcal{L}_X Y, Z) - g(\mathcal{L}_X Z, Y) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g(\nabla_X Y - \nabla_Y X, Z) - g(Y, \nabla_X Z - \nabla_Z X) \\ &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) \\ &= g\left((\nabla X + (\nabla X)^*) Y, Z\right). \end{aligned}$$

For the almost complex structure we have $J_t = \Phi_t^* J$ and therefore,

$$\dot{J} = \frac{\partial}{\partial t}\Big|_{t=0} \Phi_t^* J = \mathcal{L}_X J.$$

The almost complex structure J is compatible with g and hence parallel w.r.t. the Levi-Civita connection of g. This gives us for $Y \in \mathfrak{X}(M)$

$$\begin{aligned}\mathcal{L}_X J(Y) &= \mathcal{L}_X(JY) - J\mathcal{L}_X Y = [X, JY] - J[X,Y] \\ &= \nabla_X(JY) - \nabla_{JY}X - J\nabla_X Y + J\nabla_Y X = J\nabla_Y X - \nabla_{JY}X \\ &= J(\nabla_Y X + J\nabla_{JY}X) = 2J\bar{\partial}_Y X.\end{aligned}$$

Now we consider the normal variation $\dot{f} = uN$. For fixed $X, Y \in \mathfrak{X}(M)$ we have

$$\begin{aligned}\dot{g}(X,Y) &= \frac{\partial}{\partial t}\Big|_{t=0} \langle df_t(X), df_t(Y)\rangle \\ &= \langle df(X), d\dot{f}(Y)\rangle + \langle df(Y), d\dot{f}(X)\rangle \\ &= \langle df(X), du(Y)N + u\, dN(Y)\rangle + \langle df(Y), du(X)N + u\, dN(X)\rangle \\ &= 2\langle uN(X), df(Y)\rangle \\ &= 2u\, g(AX, Y).\end{aligned}$$

$g_t(J_tY, Y) = 0\ \forall Y \in \mathfrak{X}(M)$ gives us

$$\begin{aligned}0 &= \dot{g}(JY, Y) + g(\dot{J}Y, Y) \\ &= 2u\, g(AJY, Y) + g(\dot{J}Y, Y) \\ &= 2u\, g((Q + \frac{1}{2}\operatorname{tr}(A)\,\mathbb{1})JY, Y) + g(\dot{J}Y, Y) \\ &= 2u\, g(QJY, Y) + g(\dot{J}Y, Y) \\ \Rightarrow \qquad g(\dot{J}Y, Y) &= g(-2uQJY, Y)\end{aligned} \tag{5.1}$$

From $g_t(Y,Y) = g_t(J_tY, J_tY)$ we obtain

$$\begin{aligned}0 &= \dot{g}(JY, JY) + 2g(\dot{J}Y, JY) - \dot{g}(Y,Y) \\ &= g(2uAJY, JY) + 2g(\dot{J}Y, JY) - g(2uAY, Y) \\ &= g(2u(JAJ + A)JY, JY) + 2g(\dot{J}Y, JY) \\ &= g(4uQJY, JY) + 2g(\dot{J}Y, JY) \\ \Rightarrow \qquad g(\dot{J}Y, JY) &= g(-2uQJY, JY).\end{aligned} \tag{5.2}$$

Combining (5.1) and (5.2) gives us

$$\dot{J} = -2uQJ = 2u\, JQ.$$

□

Corollary 7.
For a normal variation $\dot{f} = uN$, the Hodge-star operator changes according to

$$\dot{*}\omega = \omega(\dot{J}\cdot) = 2u\omega(JQ\cdot), \tag{5.3}$$

for all $\omega \in \Omega^1(M, \mathbb{H})$.

Corollary 8.
For a normal variation $\dot{f} = uN$, the volume-form $d\sigma := g(J\cdot, \cdot)$ changes according to

$$\dot{d\sigma} = 2uH d\sigma. \tag{5.4}$$

Proof.

$$\dot{d\sigma} = \frac{\partial}{\partial t}\Big|_{t=0} d\sigma_t = \frac{\partial}{\partial t}\Big|_{t=0} g_t(J_t\cdot,\cdot) = \dot{g}(J\cdot,\cdot) + g(\dot{J}\cdot,\cdot) = 2u\, g((A-Q)J\cdot,\cdot) = 2uH\, d\sigma.$$

□

We want to compute the Euler-Lagrange equations for the Willmore functional. At first we only consider tangential variations:

Lemma 9.
The Willmore functional $W(f)$ is invariant under reparametrisations, i.e., tangential variations of f.

Proof. For a tangential variation $f_t := f \circ \Phi_t = \Phi_t^* f$, we get

$$d\sigma_t = g_t(J_t\cdot,\cdot) = \Phi_t^*(g(J\cdot,\cdot)) = \Phi_t^* d\sigma.$$

From $N_t = \Phi_t^* N$, we further have

$$g_t(A_t\cdot,\cdot) = \langle df_t, dN_t\rangle = (\Phi_t^* g)(\Phi_t^* A\cdot,\cdot) = g_t(\Phi_t^* A\cdot,\cdot).$$

Therefore, $H_t = \Phi_t^* H$ and finally:

$$W(f_t) = \int_M H_t^2 d\sigma_t = \int_M \Phi_t^*(H^2 d\sigma) = \int_M H^2 d\sigma.$$

□

Remark 2. The action of the group

$$\mathrm{Diff}_0(M) := \{\Phi : M \to M \,|\, \Phi \text{ is a diffeomorphism isotopic to the identity}\}$$

on the space of immersions into $\mathbb{R}^3$ $\mathcal{I}(M)$ is given by

$$\begin{aligned} \mu : \mathrm{Diff}_0(M) \times \mathcal{I}(M) &\to \mathcal{I}(M) \\ (\Phi, f) &\mapsto f \circ \Phi. \end{aligned}$$

With Lemma 9, the Willmore functional is invariant under the group action μ and hence well defined on the quotient space $\mathcal{I}(M)/\mathrm{Diff}_0(M)$.

Lemma 10.
For a normal variation $\dot{f} = uN$, the variation of the $\mathbb{H}$ valued 2-form $\mathcal{H}d\sigma$ is given by

$$(2\mathcal{H}d\sigma)^{\cdot} = 2uNKd\sigma + Nd * du + 2Hdu \wedge *df.$$

Proof. With Lemma 5 we have $d *_t df_t = 2H_t N_t d\mu_t = 2\mathcal{H}_t d\mu_t$ and hence

$$\begin{aligned}
2(\mathcal{H}d\sigma)^{\cdot} &= \frac{\partial}{\partial t}\Big|_{t=0} d *_t df_t \\
&= (d *_t d(f + tuN))^{\cdot} \\
&= d\dot{*}df + d * d(uN) \\
&= d(df(2uJQ)) + d(u * dN + N * du) \\
&= d(-2u * \omega) + Nd * du + u\, d * dN + 2du \wedge *dN \\
&= -2du \wedge *\omega + 2udN \wedge \omega + 2uNd\omega + Nd * du + u\, d * dN + 2du \wedge *dN \\
&= 2u\omega \wedge \omega + 2uNd\omega + 2Hdu \wedge *df + Nd * du + u\, d * dN.
\end{aligned}$$

Using Lemma 4 and 5 again, we finally obtain

$$\begin{aligned}
2(\mathcal{H}d\sigma)^{\cdot} &= -4uN\mu^2 d\sigma + 2uNd\omega + 2Hdu \wedge *df + Nd * du + 2uN\left((2H^2 - K)d\sigma + dH \wedge df\right) \\
&= 2uNKd\sigma + Nd * du + 2Hdu \wedge *df.
\end{aligned}$$

□

Theorem 5.
An immersion $f : M \to \mathbb{R}^3$ of a closed Riemann surface M is a critical point of the Willmore functional

$$W(f) := \int_M H^2 d\sigma,$$

if and only if the mean curvature H of the immersion satisfies

$$\triangle H + 2H(H^2 - K) = 0. \tag{5.5}$$

Proof. From Lemma 9 we know that we only have to consider normal variations $\dot{f} = uN$. This gives us for the variation of the Willmore energy of f:

$$\begin{aligned}
\dot{W}(f) &= \int_M (H^2 d\sigma)^{\cdot} \\
&= \int_M (\langle \mathcal{H}, \mathcal{H}\rangle d\sigma)^{\cdot} = \int_M 2\langle \dot{\mathcal{H}}, \mathcal{H}\rangle d\sigma + \langle \mathcal{H}, \mathcal{H}\rangle \dot{d\mu} \\
&= \int_M 2\langle (\mathcal{H}d\sigma)^{\cdot}, \mathcal{H}\rangle - \langle \mathcal{H}, \mathcal{H}\rangle \dot{d\mu} \\
&= \int_M \langle 2uNKd\sigma + Nd * du + 2Hdu \wedge *df, HN\rangle - H^2 2uHd\sigma \\
&= \int_M u(2HK - H^3)d\sigma + Hd * du \\
&= \int_M -u\left(\triangle H + 2H(H^2 - K)\right) d\sigma
\end{aligned}$$

In the last step we used that the Laplace-operator is self-adjoint on closed manifolds and $\triangle H d\sigma = -d * dH$.

The fundamental Lemma of calculus of variations finally gives us $\dot{W}(f) = 0$ for all variations if and only if $\triangle H + 2H(H^2 - K) = 0$.

$\square$

6. The Teichmüller space and constrained Willmore immersions

Definition 25.
*An immersion $f : M \to \mathbb{R}^3$ of a Riemann surface (M, J) is called **constrained Willmore**, if f is a critical point of the Willmore functional $W(f)$ with respect to all variations that preserve the complex structure, i.e., $\dot{J} = 2uJQ + 2J\bar{\partial}X = 0$.*

Note that in contrast to the more general case of Willmore surfaces, it is not obvious that for every conformal variation vector field $\dot{f} = uN + df(X)$ with $\dot{J} = 0$ there exists a corresponding 1-parameter family of conformally equivalent immersions. We will deal with this problem later in this section.

For closed surfaces with with genus $\mathfrak{g} = 0$, there exists only one complex structure and constrained Willmore surfaces are Willmore surfaces. In this section we want to compute the Euler-Lagrange equations for constrained Willmore immersions of closed orientable 2-dimensional manifolds M with genus $\mathfrak{g} \geq 1$. We use the strategy that Christoph Bohle, Paul Peters and Ulrich Pinkall introduce in [6]. With Lemma 9 we know that the Willmore functional is invariant under reparametrisations. Therefore, the Teichmüller space is an important tool to understand constrained Willmore surfaces. We will recap the construction of the Teichmüller space from a Riemannian perspective as it was developped by Anthony J. Tromba in [18].

Definition 26.
*Let M be a smooth orientable 2 dimensional manifold with genus $\mathfrak{g} \geq 1$ and $\mathcal{C}(M)$ the set of complex structures on M, then the **Teichmüller space of** M is defined as*

$$\mathcal{T}(M) := \mathcal{C}(M)/\mathrm{Diff}_0(M),$$

where $\mathrm{Diff}_0(M) := \{\Phi : M \to M \,|\, \Phi \text{ is a diffeomorphism isotopic to the identity}\}$.

For a better understanding of the Teichmüller space we investigate the group action of $\mathrm{Diff}_0(M)$ on $\mathcal{M}(M)$,

$$\begin{aligned} \mu : \mathrm{Diff}_0(M) \times \mathcal{M}(M) &\to \mathcal{M}(M) \\ (\Phi, g) &\mapsto \Phi^* g. \end{aligned}$$

For every $\Phi \in \mathrm{Diff}_0(M)$ there exist a 1-parameter family of diffeomorphisms $\Phi_t : [-1, 1] \to \mathrm{Diff}_0(M)$ satisfying:

- $\Phi_0 = id_M$,
- $\Phi_1 = \Phi$,
- $\frac{\partial}{\partial t}|_{t=0}\Phi_t = X \in \mathfrak{X}(M)$,

i.e., Φ_t is the flow of X and any variation of $g \in \mathcal{M}(M)$ that stays in the same orbit of the group action has the form

$$g_t := \Phi_t^* g.$$

The corresponding variation vector field is hence given by

$$\frac{\partial}{\partial t}\Big|_{t=0} \Phi_t^* g = -\mathcal{L}_X g = g((\nabla X + (\nabla X)^*)\cdot, \cdot).$$

This observation leads to the following definition. Fixing a metric $g \in \mathcal{M}(M)$ we can map every vector field on M to the corresponding variation on $\mathcal{M}(M)$

$$\begin{aligned} \mathcal{L}g : \mathfrak{X}(M) &\to T_g\mathcal{M}(M) = \Gamma(\mathrm{End}_{g+}(TM)) \\ X &\mapsto \mathcal{L}_X g = g((\nabla X + (\nabla X)^*)\cdot, \cdot). \end{aligned}$$

Proposition 19.
On a closed Riemannian manifold (M, g), the map $-\mathcal{L}g$ is the adjoint of div*, i.e.,*

$$\langle\!\langle \mathcal{L}_X g, H \rangle\!\rangle = -\langle\!\langle X, \operatorname{div} H \rangle\!\rangle, \qquad \forall H \in \Gamma(\mathrm{End}_{g+}(TM)) \text{ and } X \in \mathfrak{X}(M), \tag{6.1}$$

where the scalar product on $\mathfrak{X}(M)$ is given by

$$\langle\!\langle X, Y \rangle\!\rangle := \int_M g(X, Y) d\sigma_g,$$

and the one on $\Gamma(\mathrm{End}_{g+}(TM))$ is defined as in (2.7). For $A \in \Gamma(\mathrm{End}_-(TM))$ we further have

$$2\langle\!\langle \overline{\partial} X, A \rangle\!\rangle = -\langle\!\langle X, \operatorname{div} A \rangle\!\rangle,$$

Proof. We start with the left hand side of the equation

$$\langle\!\langle \mathcal{L}_X g, H \rangle\!\rangle = \frac{1}{2}\int_M \mathrm{tr}((\nabla X + (\nabla X)^*)^* H) = \int_M \mathrm{tr}(H\nabla X)$$

because H is self-adjoint and $\mathrm{tr}(A^*) = \mathrm{tr}(A)$. Now we locally choose a vector field Y with $|Y| = 1$ and consider the integrand of the right hand side

$$\begin{aligned} g(X, \operatorname{div} H) &= g(X, (\nabla_Y H)Y + (\nabla_{JY} H)JY) \\ &= g(X, \nabla_Y(HY) - H(\nabla_Y Y) + \nabla_{JX}(HJX) - H(\nabla_{JY} JY) \\ &= Y(g(X, HY)) - g(\nabla_Y X, HY) + (JY)(g(X, HJY)) - g(\nabla_{JY} X, HJY) \\ &\quad - g(X, H(\nabla_Y Y)) - g(X, H(\nabla_{JY} JY)) \\ &= g(\nabla_Y(HX), Y) + g(\nabla_{JY}(HX), JY) - g((\nabla X)(Y), HY) - g((\nabla X)(JY), HJY) \\ &= \operatorname{div}(HX) - \mathrm{tr}(H\nabla X) \end{aligned}$$

Using Stokes Theorem, we can show that both sides of the equation are equal.

$$\langle\!\langle X, \operatorname{div} H \rangle\!\rangle = \int_M g(X, \operatorname{div} H) = \int_M \operatorname{div}(HX) - \mathrm{tr}(H\nabla X) = \int_M -\mathrm{tr}(H\nabla X).$$

For $A \in \Gamma(\mathrm{End}_-(TM))$ we further have

$$\begin{aligned}\langle\!\langle X, \operatorname{div} A\rangle\!\rangle &= \int_M -\operatorname{tr}(A\nabla X)\\ &= \int_M -\operatorname{tr}(A(\partial X+\bar\partial X))\\ &= \int_M -\operatorname{tr}(A\bar\partial X)\\ &= -2\langle\!\langle \bar\partial X, A\rangle\!\rangle.\end{aligned}$$

□

Corollary 9.
Let $f : M \to \mathbb{R}^3$ be a conformal immersion of a closed Riemann surface (M, J) and

$$\begin{aligned}\overline{D} : \mathfrak{X}(M) &\to \Gamma(\mathrm{End}_-(TM)) = T_{[f]}\mathcal{C}(M)\\ X &\mapsto \dot J = 2J\bar\partial X,\end{aligned}$$

the map that assigns to every tangential variation $\dot f := df(X)$ the corresponding change of the conformal structure (compare Lemma 8). After the usual identification of $\Gamma(\mathrm{End}_-(TM))$ with $\mathcal{Q}(M)$ (see (3.16)), the adjoint operator of $\overline{D}$ is given by the negative of the $\overline{D}$ operator on $\mathcal{Q}(M)$ that we defined in (3.20)

$$\begin{aligned}\overline{D}^* : \mathcal{Q}(M) &\to \Omega^2(M, T^*M)\\ q &\mapsto -Dq = \langle -J\operatorname{div}(A), \cdot\rangle\, d\sigma\end{aligned}$$

Proof.

$$\langle\!\langle 2J\bar\partial X, q\rangle\!\rangle = \langle\!\langle 2J\bar\partial X, -A\rangle\!\rangle = 2\langle\!\langle \bar\partial X, JA\rangle\!\rangle = -\langle\!\langle X, \operatorname{div}(JA)\rangle\!\rangle = -\langle\!\langle X, \overline{D}(q)\rangle\!\rangle.$$

□

The following diagram visualizes the spaces involved in the definition of D^*:

$$\begin{tikzcd} \mathfrak{X}(M) \arrow[r, "\overline{D}"] \arrow[d, leftrightarrow] & \Gamma(\mathrm{End}_-(TM)) \arrow[d, leftrightarrow] \\ \Omega^2(M, T^*M) & \mathcal{Q}(M) \arrow[l, "\overline{D}^*"'] \end{tikzcd}$$

Definition 27.
On a Riemann surface (M, J, g), we define the second order differential operator

$$\begin{aligned}\diamond : \mathfrak{X}(M) &\to \mathfrak{X}(M)\\ X &\mapsto \operatorname{div}\mathcal{L}_X g.\end{aligned}$$

Lemma 11.
On a closed Riemann surface (M, J, g), the $\diamond$ operator is self-adjoint and its kernel is given by the set of Killing vector fields, i.e., $\ker(\diamond) = \{X \in \mathfrak{X}(M) \,|\, \mathcal{L}_X g = 0\}$.

Proof. $\diamond$ is self-adjoint because

$$\langle\!\langle \diamond X, Y\rangle\!\rangle = \langle\!\langle \operatorname{div} \mathcal{L}_X g, Y\rangle\!\rangle = -\langle\!\langle \mathcal{L}_X g, \mathcal{L}_Y g\rangle\!\rangle = \langle\!\langle X, \operatorname{div} \mathcal{L}_Y g\rangle\!\rangle = \langle\!\langle X, \diamond Y\rangle\!\rangle.$$

Obviously $\{X \in \mathfrak{X}(M) | \, \mathcal{L}_X g = 0\} \subset \ker(\diamond)$. Vice versa, we obtain for $X \in \ker(\diamond)$

$$0 = \langle\!\langle \diamond X, X\rangle\!\rangle = -\langle\!\langle \mathcal{L}_X g, \mathcal{L}_X g\rangle\!\rangle \Rightarrow \mathcal{L}_X g = 0.$$

□

Proposition 20 (M. Berger D.Ebin 1969)**.**
There is an orthogonal splitting of $T_g\mathcal{M}(M) = \Gamma(\mathrm{End}_{g+}(TM))$ into divergence-free endomorphism fields and those that come from variations of g that corresponds to the action of $\mathrm{Diff}_0(M)$ on $\mathcal{M}(M)$. In particular

$$\Gamma(\mathrm{End}_{g+}(TM)) = S^d_g(M) \oplus S^L_g(M), \tag{6.2}$$

where $S^d_g(M) := \{A \in \Gamma(\mathrm{End}_{g+}(TM)) | \operatorname{div} A = 0\}$
and $S^L_g(M) := \{A \in \Gamma(\mathrm{End}_{g+}(TM)) | \, A = \mathcal{L}_X g, \; X \in \mathfrak{X}(M)\}$.

Proof. With the last Lemma the splitting is orthogonal if it exists. For $H \in \Gamma(\mathrm{End}_{g+}(TM))$ we want to find $X \in \mathfrak{X}(M)$ with

$$\operatorname{div} H = \operatorname{div} \mathcal{L}_X g. \tag{6.3}$$

Then we can define the divergence-free endomorphism field $H_0 := H - \mathcal{L}_X g$ and obtain the splitting

$$H = \underbrace{H_0}_{\in S^d_g(M)} + \underbrace{\mathcal{L}_X g}_{\in S^L_g(M)} .$$

A necessary condition for the existence of such an X is that $\operatorname{div} H \in \{\ker(\operatorname{div} \circ \mathcal{L})\}^\perp$. For $X \in \ker(\operatorname{div} \circ \mathcal{L})$ we obtain

$$\begin{aligned} & 0 = \langle\!\langle X, \operatorname{div} \mathcal{L}_X g\rangle\!\rangle = -\langle\!\langle \mathcal{L}_X g, \mathcal{L}_X g\rangle\!\rangle, \\ \Rightarrow \quad & 0 = \mathcal{L}_X g, \\ \Rightarrow \quad & \langle\!\langle \operatorname{div} H, X\rangle\!\rangle = -\langle\!\langle H, \mathcal{L}_X g\rangle\!\rangle = 0. \end{aligned}$$

That this condition is not only necessary but also sufficient, is a consequence of the fact that $\operatorname{div} \circ \mathcal{L}$ is an elliptic operator. This is proven in [4]. □

The Teichmüller space of M is obtained as a quotient space of $\mathcal{M}(M)$ under the action of the groups $\mathcal{P} = C^\infty(M, \mathbb{R}^+)$ and $\mathrm{Diff}_0(M)$ (see Figure 1). The group actions commute in the sense that for $\Phi \in \mathrm{Diff}_0(M)$ and $e^{2u} \in \mathcal{P}$ we obtain

$$\Phi^*(e^{2u} g)(p) = e^{2(u\circ\Phi)(p)} g_{\Phi(p)}(d\Phi\cdot, d\Phi\cdot) = e^{2(u\circ\Phi)(p)}(\Phi^* g)(p).$$

The orbits of the actions intersect transversallly (see [18]) and $\mathcal{T}(M)$ has the structure of a smooth manifold. The tangent space of of $\mathcal{T}(M)$ is hence given as the intersection of the tangent spaces of the quotient spaces $\mathcal{M}(M)/\mathcal{P}$ and $\mathcal{M}(M)/\mathrm{Diff}_0(M)$.

We summarize the results in the following theorem:

$$\begin{tikzcd} \mathcal{M}(M) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & \mathcal{M}(M)/\mathcal{P} \cong \mathcal{A}^+ \arrow[d, "\pi_2"] \\ \mathcal{M}(M)/\mathrm{Diff}_0(M) \arrow[r, "\pi_1"] & \mathcal{C}(M)/\mathrm{Diff}_0(M) = \mathcal{T}(M) \end{tikzcd}$$

FIG. 1: The construction of the Teichmüller space as quotient manifold of $\mathcal{M}(M)$

$$\begin{tikzcd} \Gamma(\mathrm{End}_{g+}(TM)) \arrow[r, "d\pi_1"] \arrow[d, "d\pi_2"] & S_g^t(M) = \Gamma(\mathrm{End}_-(TM)) \arrow[d, "d\pi_2"] \\ S_g^d(M) \arrow[r, "d\pi_1"] & S_g^d(M) \cap S_g^t(M) \cong H^0(\mathcal{Q}(M)) \end{tikzcd}$$

FIG. 2: The tangent spaces of the quotient spaces involved in the construction above.

THEOREM 6.
Let M be a smooth orientable 2 dimensional manifold with genus $\mathfrak{g} \geq 1$. The Teichmüller space $\mathcal{T}(M) := \mathcal{C}(M)/\mathrm{Diff}_0(M)$ of M is a smooth manifold of dimension $6\mathfrak{g} - 6$ for $\mathfrak{g} > 1$ and dimension 2 for $\mathfrak{g} = 1$. The tangent space at $[g] \in \mathcal{T}(M)$ is isomorphic to $H^0(\mathcal{Q}(M))$.

For the proof and more informations see again [18].

We want to understand the relation between the space of immersions $\mathcal{I}(M)$ and the Teichmüller space $\mathcal{T}(M)$. Therefore, we consider the projection that maps an immersion f to its induced complex structure

$$\begin{aligned} \Phi : \mathcal{I}(M) &\to \mathcal{M}(M)/\mathcal{P} \cong \mathcal{A}^+ \\ f &\mapsto [f^*\langle \cdot, \cdot \rangle] \cong J. \end{aligned}$$

At every point $f \in \mathcal{I}(M)$ the differential Φ defines a map

$$\begin{aligned} d\Phi_f : C^\infty(M, \mathbb{R}^3) &\to \Gamma(\mathrm{End}_-(TM)) \\ \dot{f} &\mapsto \dot{J}. \end{aligned}$$

Since the Willmore functional is invariant under reparametrisations, we are interested in unparametrized immersions, $\mathcal{I}(M)/\mathrm{Diff}_0(M)$. Therefore, we restricted the differential of Φ to normal variations and define

$$\begin{aligned} \delta : C^\infty(M) &\to \Gamma(\mathrm{End}_-(TM)) \\ u &\mapsto \dot{J} = 2uJQ. \end{aligned}$$

With the usual non-degenerate pairing between smooth functions and real valued two forms,

$$(\cdot,\cdot): C^\infty(M)\times\Omega^2(M)$$
$$(u,\omega) := \int_M u\,\omega,$$

and the identification of $\mathcal{Q}(M)$ with $\Gamma(\mathrm{End}_-(TM))$ (see Proposition 14), we obtain the adjoint operator of δ

$$\delta^*: \mathcal{Q}(M)\to\Omega^2(M).$$

The following diagram visualizes the spaces involved in the definition of δ^*:

$$\begin{tikzcd} C^\infty(M) \arrow[r, "\delta"] \arrow[d, leftrightarrow] & \Gamma(\mathrm{End}_-(TM)) \arrow[d, leftrightarrow] \\ \Omega^2(M) & \mathcal{Q}(M) \arrow[l, "\delta^*"'] \end{tikzcd}$$

Lemma 12.
The adjoint operator of δ with respect to the inner product $\langle\cdot,\cdot\rangle$ on $\Gamma(\mathrm{End}_{g+}(TM))$, defined in (2.7), is given by

$$\delta^*(q) = \mathrm{tr}(JQA)d\sigma = \pi_N(d\tau)N = -2\langle\tau, *\omega\rangle,$$

where $\tau = df(A)\in K_R(M)$ is defined by $q = df\tau$ and $\omega = df(Q)$ denotes the Hopf differential of f.

Proof.
$$\langle\!\langle \delta u, q\rangle\!\rangle = \langle\!\langle 2uJQ, q\rangle\!\rangle = \int_M \mathrm{tr}(uJQA)d\sigma = (u, \mathrm{tr}(JQA)d\sigma) = (u, \delta^*(q)).$$

This gives us $\delta^*(q) = \mathrm{tr}(JQA)d\sigma$. Lemma 6 proves the remaining identities. $\square$

Now we can compute the Euler-Lagrange equations for constrained Willmore surfaces.

Theorem 7 (Bohle, Peters, Pinkall [6])**.**
Let $f: M\to\mathbb{R}^3$ be a conformal immersion of a compact Riemann surface. Then f is constrained Willmore if and only if there exists $q = df\tau\in H^0(\mathcal{Q}(\mathcal{M}))$ such that

$$-d\tau N = \delta^*(q) = \mathrm{grad}\,W = -\left(\triangle H + 2H(H^2-K)\right)d\sigma.$$

Proof. By Definition 25 f is constrained Willmore if

$$\dot W(f) = \langle\!\langle \mathrm{grad}\,W, u\rangle\!\rangle = 0,$$

for all variations $\dot f = uN + df(X)$ with

$$\dot J = 2uJQ + 2J\bar\partial X = 0, \text{ i.e., } \delta(u) = -2J\bar\partial X = -\overline{D}(X).$$

Suppose there exists $q = \langle -A\cdot,\cdot\rangle_{\mathbb{C}}\in H^0(\mathcal{Q}(M))$ with $\mathrm{grad}(W) = \delta^*(q)$. Using Corollary 9, we obtain

$$\dot W(f) = \langle\!\langle \mathrm{grad}\,W, u\rangle\!\rangle = \langle\!\langle \delta^*(q), u\rangle\!\rangle = \langle\!\langle q, \delta(u)\rangle\!\rangle = \langle\!\langle q, -\overline{D}(X)\rangle\!\rangle = \langle\!\langle \overline{D}(q), X\rangle\!\rangle = 0.$$

Vice versa, suppose $\langle\!\langle \operatorname{grad} W, u \rangle\!\rangle = 0$ for all $u \in C^\infty(M)$ with $\delta(u) \in \operatorname{Im}(\overline{D}) = \ker(\overline{D}^*)^\perp$. Then $u \in (\delta^* \ker(\overline{D}^*))^\perp$ and hence $\operatorname{grad} W \in (\delta^* \ker(\overline{D}^*))^{\perp\perp}$. Because $\ker(\overline{D}^*) = H^0(\mathcal{Q}(M))$ is finite dimensional, we have $(\delta^* \ker(\overline{D}^*))^{\perp\perp} = \delta^* \ker(\overline{D}^*)$ and the theorem is proven. □

Remark 3. The proof of Theorem 7 does not use the specific form of the Willmore functional but only its invariance under reparametrizations. Therefore, it holds true for other functionals with the same property, like the area functional

$$A(f) := \int_M d\sigma.$$

Remark 4. Immersions of non-compact Riemann surfaces are called constrained Willmore if $\dot{W}(f) = 0$ for all conformal variations with compact support. In this case the existence of $q \in H^0(\mathcal{Q}(\mathcal{M}))$ with $\delta^*(q) = \operatorname{grad} W$ is still a sufficient condition for f being constrained Willmore, but no longer necessary because $H^0(\mathcal{Q}(M))$ is no longer finite-dimensional.

Corollary 10 (Bohle, Peters, Pinkall [6]).
If $f : M \to \hat{M}$ is a conformal immersion with constant mean curvature into a 3-dimensional space form $(\hat{M}, \hat{g})$, then the immersion f is constrained Willmore.

Proof. The immersion f has constant mean curvature H if and only if the Hopf differential is closed $d\omega = 0$, see (3.11). Hence $q := df H * \omega$ defines a holomorphic quadratic differential, and with Lemma 6 and 4 we obtain

$$\begin{aligned} \delta^*(q) &= \pi_N(d(-HN\omega))N \\ &= \pi_N(-HdN \wedge \omega)N \\ &= \pi_N(-H\omega \wedge \omega)N \\ &= \pi_N(2NH(H^2 - K)d\sigma)N \\ &= -2H(H^2 - K)d\sigma \\ &= \operatorname{grad} W_f. \end{aligned}$$

In the last step we used that for CMC surfaces the Laplacian of the mean curvature is zero, i.e., $\triangle H = 0$. □

In [30] Jörg Richter gives a different characterization of constrained Willmore surfaces. We show that it is equivalent to the one of Theorem 7. In the following we will need two $-N$ conformal differentials: τ and $\chi \in K_R(M)$. The letter τ will be used for a closed differential that makes the immersion f isothermic and χ denotes the one that guarantees f to be constrained Willmore.

Proposition 21.
A conformal immersion of a compact Riemann surface $f : M \to \mathbb{R}^3$ is constrained Willmore if and only if there exists $\eta \in \Omega^1(M, \mathbb{H})$ with:

(1) $d\eta = 0$.
(2) $*\eta = -N\eta$.
(3) $\operatorname{Re}(\eta) = dH$.

The 1-form η is unique up to the addition of some $\tau \in K_R(M)$ with $d\tau = 0$.

Proof. Due to Theorem 7, f is constrained Willmore if and only if there exists $\chi \in K_R(M)$ with

$$d\chi = \big(\triangle H + 2H(H^2 - K)\big)\, N d\sigma.$$

We define $\eta \in \Omega^1(M, \mathbb{H})$ by

$$\eta := \chi - H * \omega + dH + N * dH,$$

and obtain:

$$\begin{aligned} *\eta &= *\chi + NH * \omega + *dH - NdH \\ &= -N\,(\chi - H * \omega + dH + N * dH) \\ &= -N\eta. \end{aligned}$$

Using Lemma 4 we can compute the exterior derivative

$$\begin{aligned} d\eta &= d\chi - dH \wedge *\omega + HdN \wedge \omega + HNd\omega + dN \wedge *dH + Nd * dH \\ &= \big(\triangle H + 2H(H^2 - K)\big)\, Nd\sigma - \omega \wedge *dH + H\omega \wedge \omega - HNdH \wedge df \\ &\quad + Hdf \wedge *dH + \omega \wedge *dH - \triangle HNd\sigma \\ &= 0. \end{aligned}$$

Vice versa, given η with the properties above, we can define $\chi := \eta^{tan} + H * \omega$ and obtain

(1) $\chi \in K_R(M)$.
(2) $d\chi = \big(\triangle H + 2H(H^2 - K)\big)\, Nd\sigma$.

□

6.1. **Existence of conformal variations of immersions.** As mentioned after the definition of constrained Willmore immersions, it is not clear that for every variation vector field $\dot{f} = uN + df(X)$ with $\dot{J} = 0$ there exists a corresponding 1-parameter family of conformally equivalent immersions. In [23] Ernst Kuwert and Reiner Schätzle prove the existence of conformal variations. In this subsection we will recap their results. Therefore, let $\pi : \mathcal{A}^+(M) \to \mathcal{T}(M) = \mathcal{A}^+(M)/\mathrm{Diff}_0(M)$ denote the projection that maps an almost complex structure onto the corresponding point in the Teichmüller space and let

$$\begin{aligned} \tilde{\Phi} := \pi \circ \Phi : \mathcal{I}(M) &\to \mathcal{T}(M) \\ f &\mapsto \pi(J). \end{aligned}$$

The map $\tilde{\Phi}$ is smooth (see [23]) and the differential of $\tilde{\Phi}$ defines a map:

$$\begin{aligned} d\tilde{\Phi}_f : T_f\mathcal{I}(M) \cong C^\infty(M, \mathbb{R}^3) &\to T_{\tilde{\Phi}(f)}\mathcal{T}(M) = H^0(\mathcal{Q}(M)) \\ \dot{f} = uN + df(X) &\mapsto d\pi(2uJQ). \end{aligned}$$

Remember that a variation f_t is conformal if $d\tilde{\Phi}_{f_t}(\dot{f}_t) = 0$ for all t because $\ker d\pi = \bar{D}(\Gamma(TM))$. Let $n := \dim(H^0(\mathcal{Q}(M)))$ and $V_f := \mathrm{Im}(d\tilde{\Phi}_f)$. Clearly we have $\dim V_f < n$.

Proposition 22.
For a conformal immersion $f : M \to \mathbb{R}^3$ of a Riemann surface with $\mathfrak{g} \geq 1$ we have:

$$\dim V_f \geq n-1,$$

and $\dim V_f = n-1$ *if and only if f is isothermic.*

Proof. Suppose $\dim V_f = n-1$, then there exists $q = df\tau = \langle -A\cdot, \cdot\rangle_{\mathbb{C}} \in H^0(\mathcal{Q}(M))$ such that:

$$0 = \langle\langle q, 2uJQ\rangle\rangle = \int_M u\,\mathrm{tr}(-AJQ)d\sigma = \int_M u d\tau \qquad \forall u \in C^\infty(M, \mathbb{R}^3).$$

From the fundamental Theorem of variation we get $d\tau = 0$ and f is isothermic (see definition 21).

If $\dim V_f \leq n-1$, there exist two linearly independent holomorphic quadratic differentials $q_1 = df\tau_1, q_2 = df\tau_2 \in H^0(\mathcal{Q}(M))$ with $d\tau_1 = d\tau_2 = 0$. The quotient of two holomorphic quadratic differentials defines a meromorphic function $\rho = \alpha + N\beta : M \to \mathrm{span}\{1, N\} \cong \mathbb{C}$, such that $\tau_2 = (\alpha + N\beta)\tau_1$. The exterior derivative of τ_2 gives us :

$$\begin{aligned} 0 &= d\tau_2 = d((\alpha + N\beta)\tau_1) \\ &= (d\alpha + dN\beta + N\beta) \wedge \tau_1 \\ &= \underbrace{(d\alpha + Nd\beta) \wedge \tau_1}_{\text{tangential}} + \underbrace{\beta\omega \wedge \tau_1}_{\text{normal}}. \end{aligned}$$

The normal part is zero if either $\beta = 0$ or $\omega = 0$. If $\beta = 0$ at more than finitely many points, then α is constant. This would imply that τ_1, τ_2 are linearly dependent. So by smoothness ω has to be zero everywhere and M is totally umbilical, and hence a sphere. This contradicts $\mathfrak{g} \geq 1$. ☐

Proposition 23.
For a conformal immersion $f : M \to \mathbb{R}^3$ of a Riemann surface with $\dim V_f = n$ *and every variation vector field $\dot{f} = uN + df(X)$ with $d\tilde{\Phi}_f(\dot{f}) = 0$, there is a 1-parameter family of conformally equivalent immersions f_t with $f_0 = f$ and $\frac{\partial}{\partial t} f_t\Big|_{t=0} = \dot{f}$.*

Proof. Let $q_1, \dots q_n$ be an ONB of $T_{\tilde{\Phi}(f)}\mathcal{T}(M)$. Because $\dim(V_f) = n$, the map $d\tilde{\Phi}_f$ is surjective and there exist n linearly independent vector fields $v_1, \dots, v_n \in T_f\mathcal{I}(M) = C^\infty(M, \mathbb{R}^3)$ such that $d\tilde{\Phi}_f(v_i) = q_i$. We define:

$$\begin{aligned} F &: \mathbb{R}^{n+1} \to \mathcal{T}(M) \\ (s_1, \dots, s_n, t) &\mapsto \tilde{\Phi}(f + t\dot{f} + \sum_1^n s_i v_i). \end{aligned}$$

Since $\frac{\partial F}{\partial s_i}\Big|_{(s,t)=(0,0)} = d\tilde{\Phi}_f(v_i) = q_i$, the differential dF is surjective. The implicit function theorem gives us the existence of

$$\begin{aligned} h &: (-\epsilon, \epsilon) \to \mathbb{R}^n \\ h(t) &= (s_1(t), \dots s_n(t)), \end{aligned}$$

with $F(h(t),t) = F(0,0) = \tilde{\Phi}(f), \quad \forall t \in (-\epsilon,\epsilon)$. Now we can define the conformal variation of f with variation vector field $\dot{f}$:

$$f_t : (-\epsilon,\epsilon) \to \mathcal{I}(M)$$
$$f_t := f + t\dot{f} + \sum_1^n s_i(t)v_i.$$

□

For isothermic immersions f, the construction of conformal variations is more complicated because the map $d\tilde{\phi}_f$ is not longer surjective and we loose one dimension in the Teichmüller space to do the necessary corrections. Ernst Kuwert and Reiner Schätzle solved this problem by considering the second derivative of $\tilde{\Phi}$. Let $V \subset \mathcal{T}(M)$ be a neighborhood of $\tilde{\Phi}(f)$, $\psi : V \to \mathbb{R}^n$ local coordinates of $\mathcal{T}(M)$, $U := \tilde{\Phi}^{-1}(V)$,and

$$\hat{\Phi} : U \to \mathbb{R}^n$$
$$\hat{\Phi} := \psi \circ \tilde{\Phi}.$$

Now the second variation of f in Teichmüller space with respect to the variation vector field $\dot{f}$ and the chart ψ is given by:

$$d^2\hat{\pi}_f(\dot{f}) := \frac{\partial^2}{\partial t^2}\Big|_{t=0} \hat{\Phi}(f + t\dot{f}).$$

Proposition 24 (Kuwert, Schätzle).
For an isothermic immersion $f : M \to \mathbb{R}^3$, there exist $v_1, \ldots v_{n-1}, v_\pm \in C^\infty(M,\mathbb{R}^3)$ and $q \in V_f^\perp$ with $|q| = 1$ such that:

$$V_f = \operatorname{span}\{\tilde{\Phi}_f(v_1), \ldots, \tilde{\Phi}_f(v_{n-1})\},$$
$$\pm \langle\!\langle d^2\hat{\pi}_f(v_\pm), q\rangle\!\rangle > 0 \text{ and}$$
$$\tilde{\Phi}_f(v_\pm) = 0.$$

For the proof see [23]. These vector fields can now be used to do the corrections in the Teichmüller space and we finally obtain the following theorem.

Theorem 8 (Kuwert, Schätzle).
For a conformal immersion $f : M \to \mathbb{R}^3$, there exists a conformal variation for every smooth variation vector field $\dot{f} = uN + df(X)$ with $d\tilde{\Phi}_f(\dot{f}) = 0$.

7. Isothermic constrained Willmore immersions

In this section we will investigate isothermic constrained Willmore immersions. With Corollary 10 we know that every conformal immersion that has constant mean curvature in a 3-dimensional space form is constrained Willmore. These immersions are also isothermic because their Hopf differential is holomorphic.

The opposite does not hold in general. Fran Burstall constructed a cylinder over a plane curve that is isothermic and constrained Willmore but does not have constant mean curvature in some space form (see [6]). For every isothermic constrained Willmore tori without umbilical points Jörg Richter proved in [30] that there exist a 3-dimensional space form in which the immersion has CMC. We extend his proof so that it stays true if the surface has umbilical points. In this case, the surface is either Möbius equivalent to a minimal surface in $\mathbb{R}^3$ with planar ends or there exists a hyperplane E that separates $\mathbb{R}^3$ in two hyperbolic spaces $\mathbf{H}^3_\pm$ whose ideal boundaries are given by $\partial\mathbf{H}^3_\pm = E$ and the restriction of the surface onto $\mathbf{H}^3_\pm$ has CMC. We start our investigation with the observation that being an isothermic constrained Willmore immersion is a Möbius invariant property.

Proposition 25 (Richter [30]).
The notion of isothermic constrained Willmore immersions is invariant under Möbius transformations of the ambient space.

Proof. An immersion $f : M \to \mathbb{R}^3$ is isothermic if there exist $\tau \in K_R(M)$ with $d\tau = 0$ (see Definition 21), and constrained Willmore if there exists $\eta \in \Omega^1(M, \mathbb{H})$ with $d\eta = 0$, $*\eta = -N\eta$ and $\mathrm{Re}(\eta) = dH$ (see Proposition 21). The Möbius group is generated by translations, scalings, rotations and the inversion at the unit sphere (compare Section 3.1). Under translations neither df nor H are changed and the two differentials τ, η of an isothermic constrained Willmore immersion stay the same, too. In the quaternionic setup a stretch-rotation is by

$$f \mapsto \tilde{f} = \bar{\lambda} f \lambda, \qquad \lambda \in \mathbb{H} \setminus \{0\}. \tag{7.1}$$

With $d\tilde{f} = \bar{\lambda} df \lambda$, we get $\tilde{g} = |\lambda|^4 g$. Due to Proposition 17, the mean curvature H and the normal N change according to

$$\begin{aligned} N \mapsto& \tilde{N} = \lambda^{-1} N \lambda, \\ H \mapsto& \tilde{H} = |\lambda|^{-2} H. \end{aligned}$$

With the following definitions, $\tilde{f}$ is isothermic and constrained Willmore.

$$\tilde{\tau} := \bar{\lambda} \tau \lambda, \tag{7.2}$$
$$\tilde{\eta} := \lambda^{-1} \eta \bar{\lambda}^{-1}. \tag{7.3}$$

To finish the proof, we have to show the invariance of the properties under the inversion at the unit sphere

$$f \mapsto \tilde{f} = f^{-1}. \tag{7.4}$$

With $d\tilde{f} = -f^{-1} df f^{-1}$ we have $\tilde{g} = |f|^4 g$ and hence

$$\begin{aligned} N &\mapsto \tilde{N} = f N f^{-1}, \\ H &\mapsto \tilde{H} = |f|^2 H - 2\langle f, N \rangle, \end{aligned}$$

$$\begin{aligned} \Rightarrow \qquad d\tilde{H} &= 2\langle f, df \rangle H + |f|^2 dH - 2\langle f, dN \rangle \\ &= |f|^2 dH - 2\langle f, \omega \rangle. \end{aligned}$$

With $\tilde{\tau} = f\tau f$ we obtain

$$\begin{aligned}
d\tilde{\tau} &= df \wedge \tau f + f d\tau f + f\tau \wedge df = 0,\\
*\tilde{\tau} &= -fN\tau f = -fNf^{-1} f\tau f = -\tilde{N}\tilde{\tau},\\
*\tilde{\tau} &= f\tau N f = f\tau f f^{-1} N f = \tilde{\tau}\tilde{N},
\end{aligned}$$

and hence f^{-1} is isothermic. In order to show that f^{-1} is constrained Willmore, we define $\tilde{\eta} := 2f\omega - f\eta f$ and check the necessary properties formulated in Proposition 21.

$$\begin{aligned}
d\tilde{\eta} &= \underbrace{df \wedge (2\omega - \eta f)}_{=0} + f(2d\omega + \eta \wedge df)\\
&= f(-2dH \wedge df + \eta \wedge df)\\
&= f((\eta - 2dH) \wedge df)\\
&= -f\bar{\eta} \wedge df = 0.
\end{aligned}$$

$$\begin{aligned}
*\tilde{\eta} &= f(-2N\omega + N\eta f)\\
&= -fNf^{-1}(2f\omega - f\eta f)\\
&= -\tilde{N}\tilde{\eta}.
\end{aligned}$$

$$\begin{aligned}
\operatorname{Re}(\tilde{\eta}) &= -2\langle f, \omega\rangle - \operatorname{Re}(f\eta f) = -2\langle f, \omega\rangle - |f|^2 \operatorname{Re}(f^{-1}\eta f)\\
&= -2\langle f, \omega\rangle - |f|^2 \operatorname{Re}(\eta) = d\tilde{H}.
\end{aligned}$$

This proves that $\tilde{f}$ is constrained Willmore. □

From now on, let $f : M \to \mathbb{R}^3$ be an isothermic constrained Willmore immersion of a torus. On a closed genus one surface holomorphic quadratic differentials have no zeros (see Corollary 3), hence we obtain for the differential of the Christoffel dual of f: $\tau_p \neq 0$ for all $p \in M$. From Lemma 7 we get the existence of $\rho \in C^\infty(M, \mathbb{R})$ with $\rho\tau = \omega$. Further, there exists another function $h \in C^\infty(M, \mathbb{H})$ with $\eta = \tau h$. In order to construct a space form in which f has constant mean curvature, we will investigated the relation between τ, η and ω.

Lemma 13.
The functions $h \in C^\infty(M, \mathbb{H})$ with $\eta = \tau h$ and $\rho \in C^\infty(M, \mathbb{R})$ with $\rho\tau = \omega$ satisfy

$$d\rho = -\langle h, df\rangle. \tag{7.5}$$

After a proper choice of η (adding a suitable real multiple of τ), we further obtain

$$\chi = \beta * \tau,$$

for some $\beta \in \mathbb{R}$, where $\chi \in K_R(M)$ with $(d\chi)^{tan} = 0$ is given by $\eta^{tan} = \chi - H * \omega$ (see Proposition 21). Further, there exist $c \in \mathbb{R}$ and $b \in \mathbb{R}^3$ such that

$$h = cf + b \tag{7.6}$$

and the following real valued function is constant

$$\tilde{H} := \langle N, h\rangle + H\rho = \text{ constant.}$$

Proof. We start with the proof of the first identity by considering the differential of the mean curvature

$$dH = \mathrm{Re}(\eta) = \mathrm{Re}(\tau h) = -\langle \tau, h\rangle = \frac{1}{2}(\tau h + h\tau). \tag{7.7}$$

If we insert this in the Gauß–Codazzi equation, we get

$$\begin{aligned} & d\rho \wedge \tau = d\omega = -dH \wedge df = -\frac{1}{2}(\tau h + h\tau) \wedge df = -\frac{1}{2}\tau \wedge hdf \\ \Rightarrow \qquad & 0 = \tau \wedge (\frac{1}{2}hdf - d\rho) \\ \Rightarrow \qquad & 0 = (N - *)(\frac{1}{2}hdf - d\rho). \end{aligned}$$

Considering the real part of the last equation, this gives rise to

$$\begin{aligned} & *d\rho = \frac{1}{2}\mathrm{Re}(hNdf - Nhdf) = \mathrm{Re}(hNdf) = -\langle h, *df\rangle \\ \Rightarrow \qquad & grad(\rho) = -h. \end{aligned}$$

Since $\tau, \chi \in K_R(M)$ and $\tau_p \neq 0$, there exist functions $\alpha, \beta \in C^\infty(M, \mathbb{R})$ such that $\chi = (\alpha + \beta N)\tau$. The tangential part of $d(\chi)^{\mathrm{tan}}$ is zero, which leads to

$$\begin{aligned} & d\chi = \underbrace{(d\alpha + Nd\beta) \wedge \tau}_{tangential} + \beta \underbrace{dN \wedge \tau}_{normal} \\ \Rightarrow \qquad & (N - *)(d\alpha + d\beta N) = 0, \end{aligned}$$

i.e., $\alpha + i\beta$ is a holomorphic map. Since M is compact, α and β are constant. By a proper choice of η, $(\eta \mapsto \eta - \alpha\tau)$ we have

$$\chi = \beta N\tau = -\beta * \tau.$$

The tangential parts of τh resp. η are given by

$$\begin{aligned} (\tau h)^{tan} &= \mathrm{Re}(h)\tau + \langle h, N\rangle \tau N, \\ \eta^{tan} &= \chi - H * \omega. \end{aligned}$$

Since h was defined by the equation $\eta = \tau h$, we obtain for the tangential parts

$$\begin{aligned} 0 &= (\tau h)^{tan} - \eta^{tan} \\ &= -\chi + H * \omega + \mathrm{Re}(h)\tau + \langle h, N\rangle N\tau \\ &= \tau(\mathrm{Re}(h) + \langle N, h\rangle N) + \beta * \tau + H\rho * \tau \\ &= \tau(\mathrm{Re}(h) + N(\langle N, h\rangle + \beta + H\rho)) \\ &= \mathrm{Re}(h)\tau - (\langle N, h\rangle + \beta + H\rho) * \tau. \end{aligned}$$

Due to the fact that the 1-forms τ and $*\tau$ are linearly independent at every point $p \in M$, we get

$$\mathrm{Re}(h) = 0 \text{ and } \tilde{H} := \langle N, h\rangle + H\rho = -\beta = \text{ constant.} \tag{7.8}$$

Because η is closed, we have $0 = d\eta = d(\tau h) = -\tau \wedge dh$, which leads to $(N - *)dh = 0$. Using $\mathrm{Re}(h) = 0$, we further obtain

$$\langle N, dh\rangle = -\operatorname{Re}(N dh) = -\operatorname{Re}(*dh) = 0,$$

that is, $dh \in K_L(M)$. Since df and dh are both exact, there exists $c \in \mathbb{R}$ such that $c\, dh = df$ and $b \in \mathbb{R}^3$ with

$$ch + b = f. \tag{7.9}$$

□

Now, we can proof an extended version of Jörg Richters Theorem, that classifies isothermic contrained Willmore tori as CMC surfaces. Our modification of the proof (we never divide through the function ρ) allows us to consider tori with umbilical points. This gives us three new classes of isothermic constrained Willmore tori that were not considered in [30]. As we will see in the proof of the theorem and the discussion of the results afterwards, isothermic constrained Willmore tori with umbilical points are either Bryant surfaces with smooth ends, Babich-Bobenko tori, or minimal surfaces with planar ends in $\mathbb{R}^3$.

Theorem 9.
Let $f : M \to \mathbb{R}^3$ be an isothermic constrained Willmore immersion of a torus. Then there exists $\tilde\rho \in C^\infty(\mathbb{R}^3, \mathbb{R})$ such that

$$\widetilde{\langle \cdot, \cdot\rangle} := \tilde\rho^{-2}\langle \cdot, \cdot\rangle$$

defines a metric with constant curvature on $\mathbb{R}^3 \setminus U_0$, where $U_0 := \{x \in \mathbb{R}^3 | \tilde\rho(x) = 0\}$.
If the sectional curvature of the new metric is negative, the set U_0 is a plane or sphere and $\mathbb{R}^3$ splits into two hyperbolic 3-spaces. If the sectional curvature is zero, the set U_0 contains at most one point and for positive sectional curvature the set U_0 is empty. In all the cases, the restriction of $f(M)$ to $\mathbb{R}^3 \setminus U_0$ has constant mean curvature

$$\tilde H = \rho H - \langle \operatorname{grad}\tilde\rho, N\rangle,$$

with respect to the metric $\tilde g := f^*(\widetilde{\langle \cdot, \cdot\rangle})$.

Proof. Using (7.6), we can integrate (7.5) and compute ρ in terms of f

$$\begin{aligned} d\rho &= -\langle h, df\rangle = -\langle cf + b, df\rangle, \\ \Rightarrow \quad \rho &= \frac{-c}{2}|f|^2 - \langle b, f\rangle + k, \end{aligned} \tag{7.10}$$

for some $k \in \mathbb{R}$.

The function $\rho : M \to \mathbb{R}$ naturally extends to a function $\tilde\rho : \mathbb{R}^3 \to \mathbb{R}$ with $f^*\tilde\rho = \rho$. We use $\tilde\rho$ to define a bilinear map on $\mathbb{R}^3$

$$\hat g := \tilde\rho^{-2}\langle \cdot, \cdot\rangle. \tag{7.11}$$

Depending on c and b, given by $cf + b = h$, $\hat g$ defines a metric with constant sectional curvature that is conformally equivalent to the euclidean metric on $\mathbb{R}^3$:

- For $c, b = 0$, we have $h = 0$ and $\tilde{\rho} =$ constant, i.e., the bilinear map $\hat{g}$ defines an euclidean metric on $\mathbb{R}^3$.
- For $c = 0$ and $b \neq 0$, we get:
$$\tilde{\rho} = -\langle b, x\rangle + k = -\langle b, x - a\rangle,$$
for some $a \in \mathbb{R}^3$ and the bilinear map $\hat{g}$ defines a metric on the half-spaces
$$H^3_{\pm b,a} := \{x \in \mathbb{R}^3 | \langle b, x - a\rangle \gtrless 0\}.$$
The metric spaces $(H^3_{\pm b,a}, \tilde{g})$ are the Poincaré half space model of the hyperbolic 3-space and we obtain that $\mathbb{R}^3$ splits into two hyperbolic spaces that touch at the plane $U_0 := \{x \in \mathbb{R}^3 | \, \langle x - a, b\rangle = 0\}$, which is their common boundary at infinity.
- For $c \neq 0$, we get:
$$\tilde{\rho} = \frac{-c}{2}|x|^2 - \langle b, x\rangle + k = \frac{R^2 - |x - a|^2}{2},$$
for some $R \in \mathbb{R} \cup i\mathbb{R}$ and $a \in \mathbb{R}^3$. Depending on the value of R, we have to distinguish 3 different cases:
  - For $R \in \mathbb{R} \setminus \{0\}$, the bilinear map $\hat{g}$ defines a metric on
$$H^3_{\pm R,a} := \{x \in \mathbb{R}^3 | |x - a| \gtrless R\}.$$
The metric spaces $(H^3_{\pm R^2,a}, \tilde{g})$ are the Poincaré ball model of the hyperbolic 3-space and we obtain that $\mathbb{R}^3$ splits into two hyperbolic spaces that touch at the sphere $U_0 := \{x \in \mathbb{R}^3 | \, |x - a|^2 = |R|^2\}$, which is their common boundary at infinity.
  - For $R = 0$, the bilinear map $\hat{g}$ defines a euclidean metric on $\mathbb{R}^3 \setminus \{a\}$. This metric is obtained from the usual euclidean metric of $\mathbb{R}^3$ by an inversion in a 2-sphere with center a. The set U_0 contains only the point a, which gets mapped to infinity after the sphere inversion.
  - For $R \in i\mathbb{R} \setminus \{0\}$, the bilinear map $\hat{g}$ defines spherical metric. Using a stereographic projection, we can map $(\mathbb{R}^3, \tilde{g})$ isometrically to $\mathbb{S}^3$ such that the center a becomes the south pole. In this case, the set U_0 is empty.

With Proposition 17 the mean curvature of f with respect to $\tilde{g} := f^*\hat{g}$ is given by
$$\tilde{H} := \rho H - \langle \operatorname{grad} \tilde{\rho}, N\rangle.$$
Due to Lemma 13, the mean curvature $\tilde{H}$ is constant. □

Remark 5. The hyperbolic half-space model $\mathbf{H}^3_{\pm}$ and the hyperbolic ball model $\mathbf{H}^3_{\pm R}$ are Möbius equivalent because we can map one to the other using a stereographic projection. After another Möbius transformation, we may assume $|R| = |b| = 1$ and $a = 0$ and obtain the models described in Example 4. The properties of isothermic constrained Willmore tori, that we are interested in, are invariant under Möbius transformations. In order to simplify the formulas, we will use these standard models and switch between the half-space and ball model depending on the occasion. In all the cases - euclidean, spherical, and hyperbolic - we have
$$\langle d(\mathrm{grad}\rho), N\rangle = c\langle df, N\rangle = 0, \tag{7.12}$$
for some $c \in \mathbb{R}$, and hence $d(\mathrm{grad}\rho) \in K_L(M)$.

Corollary 11.
Let $f : M \to \mathbb{R}^3$ be a conformal, isothermic constrained Willmore immersion of a torus. If f has constant mean curvature with respect to a spherical metric, then there are no umbilical points. In the hyperbolic case, all umbilical points lie on the sphere/plane at infinity $\partial\mathbf{H}^3_\pm = \{x \in \mathbb{R}^3 |\, \rho(\tilde{x}) = 0\}$. In the euclidean case, all umbilical points get mapped to one point.

Proof. With the Gauss equation

$$|\mu||df| = |\omega|, \tag{7.13}$$

the immersion f has an umbilical point if and only if $\omega = 0$. With $\rho\tau = \omega$ and the fact that τ has no zeros, the set of umbilical points is given by the zero set of ρ. □

Remark 6. Let $f : M \to \mathbb{R}^3$ be an isothermic constrained Willmore immersion of a torus that has constant mean curvature $\tilde{H}$ with respect to the euclidean metric $\tilde{g} = \frac{4}{|f|^4}g$. If there exits at least one umbilical point, i.e., the discrete set $M_0 := \{p \in M | f(p) = 0\}$ is non empty, the constant mean curvature $\tilde{H} = \frac{|f|^2}{2}H - \langle f, N\rangle$ is zero everywhere. After an inversion in the unit sphere, $f^{-1} : M \setminus M_0 \to \mathbb{R}^3$ is a minimal surface in $\mathbb{R}^3$ with smooth ends. Those ends are given by the umbilical points of f.

Remark 7. If ρ defines a hyperbolic metric, three different cases can occur depending on the value of $\tilde{H}$:

(1) For $|\tilde{H}| > 1$, the immersion f lies entirely in one of the spaces $\mathbf{H}^3_\pm$.

(2) For $|\tilde{H}| = 1$, the immersion f lies in one of the $\mathbf{H}^3_\pm$ and touches the sphere at infinity $\partial\mathbf{H}^3_\pm$ at isolated points, see Proposition 26. These surfaces were first investigated by Robert Leamon Bryant [9] and are called **Bryant surfaces with smooth ends**.

(3) If $|\tilde{H}| < 1$, one part of the surface lies in $\mathbf{H}^3_+$ and the other one in $\mathbf{H}^3_-$. Then the normal's with respect to $\tilde{g}$,

$$\tilde{N} = \rho N,$$

have different signs in $\mathbf{H}^3_\pm$ (see Figure 3). These tori have lines of umbilical points and were first constructed by Mikhail V. Babich and Alexander I. Bobenko using theta functions and elliptic integrals [3]. We will call them **Babich-Bobenko tori**.

We continue by giving a geometric interpretation of f having constant mean curvature $|\tilde{H}| \leq 1$ in $\mathbf{H}^3_{\pm 1}$.

Proposition 26.
If $f : M \to \mathbb{R}^3$ is an immersion whose restriction to $\mathbf{H}^3_{\pm 1}$ has constant mean curvature $|\tilde{H}| = |(\rho H - d\rho(N)| \leq 1$, then the mean curvature spheres of f, $S_f(p) := \{x \in \mathbb{R}^3 | \|f(p) - \frac{N(p)}{H(p)} - x\|^2 = \frac{1}{H^2}\}$, intersect the ideal boundary $\partial\mathbf{H}^3_{\pm 1} = \{x \in \mathbb{R}^3 | \|x\| = 1\}$ at constant angle $\alpha \in [0, \pi]$.

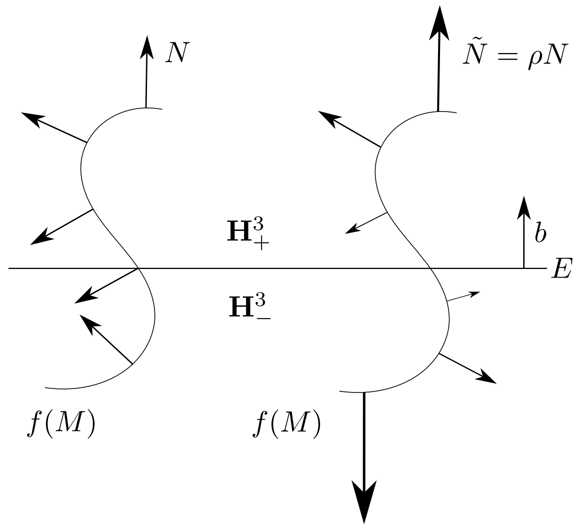


FIG. 3: The intersection of a Babich-Bobenko torus with the plane $E = \partial\, \mathbf{H}^3_\pm$ with euclidean normal's on the left and hyperbolic normal's on the right.

Proof. First, note that if $|\tilde{H}| < 1$, all the mean curvature spheres $S_f(p)$ intersect $\partial \mathbf{H}^3_{\pm 1}$. For $p \in M$ and $q \in \partial \mathbf{H}^3_{\pm 1} \cap S_f(p)$, we consider the triangle with vertices $(q, 0, c)$, where $c = f(p) - \frac{N(p)}{H(p)}$ is the center of $S_f(p)$ (see Figure 4). If α denotes the angle in the vertex q, the cosine-theorem gives us

$$\begin{aligned} & \left\| f(p) - \frac{N(p)}{H(p)} \right\|^2 = 1 + \frac{1}{H^2(p)} - \frac{2}{H(p)} \cos(\alpha) \\ \Rightarrow \quad & \cos(\alpha) = \frac{H(p)}{2} \left(1 + \frac{1}{H^2(p)} - \|f(p)\|^2 + \frac{2\langle f(p), N(p) \rangle}{H(p)} - \frac{1}{H^2(p)} \right) \\ & = H(p) \frac{1 - \|f(p)\|^2}{2} + \langle f(p), N(p) \rangle \\ & = \rho(p) H(p) - d\rho(N)(p) = \tilde{H}(p) = \text{const.} \end{aligned}$$

$\square$

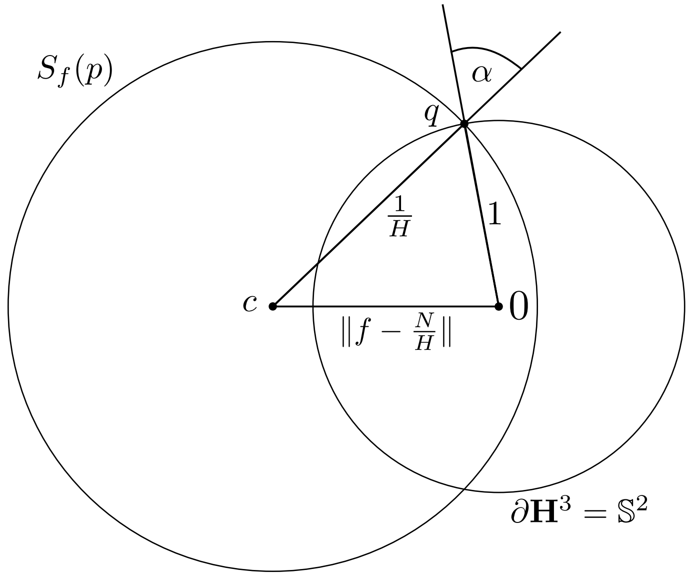


FIG. 4: The mean curvature sphere of f, $S_f(p)$ cuts the ideal boundary of the hyperbolic space $\partial\mathbf{H}^3_{\pm 1}$ in the angle α.

We summaries the previous results in a theorem.

THEOREM 10 (Classification of isothermic constrained Willmore tori as CMC surfaces)**.** *Let $f : M \to \mathbb{R}^3$ be an immersion of an isothermic constrained Willmore torus. Then f is Möbius equivalent to one of the following surfaces:*

- *A CMC surface in $\mathbf{H}^3, \mathbb{R}^3$ or $\mathbb{S}^3$*
- *A Bryant surface with smooth ends*
- *A Babich-Bobenko torus*
- *A minimal surface with smooth ends in $\mathbb{R}^3$*

We finish this section with a lemma that we will need for further computations.

LEMMA 14.
The mean curvature H of an isothermic constrained Willmore immersion $f : M \to \mathbb{R}^3$ satisfies

$$\begin{aligned} dH &= \langle \operatorname{grad}\rho, \tau\rangle, \\ \triangle H &= \frac{-2\mu^2\langle \operatorname{grad}\rho, N\rangle}{\rho}, \\ \|dH\|^2 &= \frac{\mu^2}{\rho^2}\|d\rho\|^2. \end{aligned}$$

Proof. With $H = \frac{\tilde{H}+d\rho(N)}{\rho}$ and $\langle d(\operatorname{grad}\rho), N\rangle = 0$ we get

$$\begin{aligned} dH &= \frac{\langle dN, \operatorname{grad}\rho\rangle}{\rho} - \frac{d\rho(\tilde{H} + \langle N, \operatorname{grad}\rho\rangle)}{\rho^2} \\ &= \frac{\langle \operatorname{grad}\rho, dN - Hdf\rangle}{\rho} \\ &= \frac{\langle \operatorname{grad}\rho, \omega\rangle}{\rho} \\ &= \langle \operatorname{grad}\rho, \tau\rangle. \end{aligned}$$

Let $X \in T_pM$ be a normalized principal curvature direction of f, i.e., $g(X,X) = 1$ and $dN(X) = \lambda df(X)$. We compute the norm of the differential dH that was defined in (3.13).

$$\begin{aligned} \|dH\|^2 &= *dH \wedge dH(X, JX) \\ &= \langle \operatorname{grad}\rho, *\tau\rangle \wedge \langle \operatorname{grad}\rho, *\tau\rangle(X, JX) \\ &= \frac{\mu^2}{\rho^2}\langle \operatorname{grad}\rho, *df\rangle \wedge \langle \operatorname{grad}\rho, *df\rangle(X, JX) \\ &= \frac{\mu^2}{\rho^2}\|d\rho\|^2. \end{aligned}$$

Due to the fact that $d(\operatorname{grad}\rho) \in K_L(M)$ and Lemma 4, we further have

$$d * dH = d\langle \operatorname{grad}\rho, *\tau\rangle = \langle \operatorname{grad}\rho, -dN \wedge \tau\rangle = \frac{\langle \operatorname{grad}\rho, -\omega \wedge \omega\rangle}{\rho} = \frac{\langle \operatorname{grad}\rho, N\rangle \mu^2 d\sigma}{\rho}.$$

□

8. Isothermic constrained Willmore tori with minimal dual surfaces in $\mathbb{R}^3$

Robert L. Bryant showed in [8] that for Willmore surfaces $f : M \to \mathbb{R}^3$, there exists away from the set of umbilical points M_0 a (possibly branched) dual immersion $\hat{f} : M \setminus M_0 \to \mathbb{R}^3$, whose mean curvature vanishes up to second order. The immersions f and $\hat{f}$ are conformally equivalent and $\hat{f}$ is called the **conformal transform** of f. In this section we will investigate isothermic constrained Willmore tori that, after an inversion in a 2-sphere, become minimal surfaces in $\mathbb{R}^3$ with smooth ends. After a possible translation and scaling, these surfaces have zero CMC with respect to the metric $\tilde{g} = \frac{4}{|f|^4} f^*\langle\cdot,\cdot\rangle$, and their inverse $f^{-1} := \frac{-f}{|f|^2} : M \setminus M_0 \to \mathbb{R}^3$ is the conformal transform of f. Our aim is to show that these surfaces cannot be embedded.

Lemma 15.
Let $f : M \to \mathbb{R}^3$ be an immersion of an isothermic constrained Willmore tori such that its inversion in the unit sphere $f^{-1} : M \to \mathbb{R}^3 \cup \{\infty\}$ is a minimal surface with smooth ends, then the total curvature of the minimal surface f^{-1} is finite. In particular, the end points of f^{-1} are flat.

Proof. Since the Gaussian curvature of a surface is determined by the metric, we can use the immersion f together with the Riemannian metric $\tilde{g} = \frac{4}{|f|^4} f^*\langle\cdot,\cdot\rangle$ to compute the

Gaussian curvature $\tilde{K}$ of f^{-1}, (see Proposition 11).

$$\tilde{K} = \det(\tilde{A}) = \frac{|f|^4}{4}(\det(A + \langle N, f\rangle)\,\mathrm{Id}) = \frac{|f|^4}{4}(K + 2H\langle N, f\rangle + \langle N, f\rangle^2).$$

The set of end points of f^{-1} is given by the set of umbilical points of f

$$M_0 := \{p \in M |\, f(p) = 0\}.$$

Therefore, the Gaussian curvature of f^{-1} vanishes at the end points. Further, those end points are flat, because they are umbilical. The total curvature of f^{-1} is given by

$$\begin{aligned}
\mathrm{TC}(f^{-1}) &= \int_M |\tilde{K}| d\tilde{\sigma} \\
&= \int_M \left| \frac{|f|^4}{4}(K + 2H\langle N, f\rangle + \langle N, f\rangle^2) \right| \frac{4}{|f|^4} d\sigma \\
&= \int_M \left| K + 2H\langle N, f\rangle + \langle N, f\rangle^2 \right| d\sigma.
\end{aligned}$$

This integral is finite because M is compact and the integrand is smooth. □

Now assume that $f : M \to \mathbb{R}^3$ is an embedding. Since sphere inversions are injective, and there are no compact minimal surfaces in $\mathbb{R}^3$ (see [1]), f^{-1} is an embedded complete minimal surface of finite total curvature in $\mathbb{R}^3$ with one smooth end. Hence, the surface f^{-1} satisfies all assumptions of the following theorem of Jorge and Meeks, which we can use to show that $f^{-1}(M)$ has to be a plane. But because we assumed that M is a torus, this is not possible and f cannot be embedded.

Theorem 11 (Luquesio P. Jorge, William H. Meeks III [19]).
An embedded complete minimal surface of finite total curvature in $\mathbb{R}^3$ is either a plane or has at least two ends.

Theorem 12.
isothermic constrained Willmore tori that are Möbius equivalent to a minimal surface with smooth ends in $\mathbb{R}^3$ cannot be embedded.

9. The projective model of Möbius geometry and its subgeometries

We want to continue our investigation of isothermic constrained Willmore tori by studying their sphere congruences. In Section 7, we proved that the property of being an isothermic constrained Willmore immersion is invariant under Möbius transformations. To exploit this invariance, which also applies to sphere congruences, we will consider these surfaces in a Möbius-geometric setup. The main objects of 3-dimensional Möbius geometry are the space $\mathbb{S}^3$, the set of 2-spheres in $\mathbb{S}^3$ and the set of Möbius transformations Möb(3), which map spheres to spheres. We will start by introducing these objects in a projective model. This projective model is conformal in the sense that we can measure angles, but we do not have a notion of length. To obtain Riemannian metrics, that we need for our investigation of surfaces, we will introduce the space forms described in Example 4 as subgeometries of Möbius geometry. An advantage of the projective model of Möbius geometry is that its representation in homogeneous coordinates corresponds to the five dimensional Minkowski space on which the group of Möbius transformations acts linearly. Furthermore, the different space forms and the set of their spheres can be obtained as subsets of this Minkowski space. The following short introduction to Möbius geometry is inspired by the detailed exposition in Udo Hertrich-Jeromins book [17].

The **Minkowski space** $\mathbb{R}^{4,1}$ is defined as $\mathbb{R}^5$ equipped with the bilinear form

$$\begin{aligned} \langle \cdot, \cdot \rangle &: \mathbb{R}^5 \times \mathbb{R}^5 \to \mathbb{R} \\ \langle P, Q \rangle &:= P_1Q_1 + P_2Q_2 + P_3Q_3 + P_4Q_4 - P_5Q_5, \end{aligned} \tag{9.1}$$

called the **Minkowski product**. A point $P \in \mathbb{R}^5$ is called **space-like** if $\langle P, P \rangle > 0$, **time-like** if $\langle P, P \rangle < 0$, and **light-like** if $\langle P, P \rangle = 0$. The set

$$\mathcal{L} := \{P \in \mathbb{R}^{4,1} | \langle P, P \rangle = 0\} \tag{9.2}$$

is called the **light cone**. The unit sphere with respect to the Minkowski metric is a hyperboloid of one sheet in $\mathbb{R}^5$ and will be denoted with

$$\mathcal{H}^4_+ := \{P \in \mathbb{R}^5 | \langle P, P \rangle = 1\}. \tag{9.3}$$

Analogously, consider the two sheeted hyperboloid that is given by:

$$\mathcal{H}^4_- := \{P \in \mathbb{R}^5 | \langle P, P \rangle = -1\}. \tag{9.4}$$

With the equivalence relation on $\mathbb{R}^{4,1}$

$$P \sim Q \Leftrightarrow P = \lambda Q, \qquad \text{for some } \lambda \in \mathbb{R} \setminus \{0\}, \tag{9.5}$$

we get the corresponding four dimensional projective space $\mathbf{P}^4 := \mathbb{P}(\mathbb{R}^{4,1}) = \left(\mathbb{R}^{4,1} \setminus \{0\}\right)/_\sim$, where $\mathbb{R}^{4,1}$ is the corresponding space of **homogeneous coordinates**, in which the coordinates of points will be denoted with capital letters: P_j, $j = 1, \ldots, 5$. For the corresponding **affine coordinates** $p_j = P_j/P_5, j = 1, \ldots, 4$ we will use small letters. The property of being time, space or light-like is still well defined after taking the quotient and we can think of $\mathbb{S}^3 \subset \mathbf{P}^4$ as the projectivized light cone:

$$\mathbb{P}(\mathcal{L}) = \{[P] \in \mathbf{P}^4 | \langle P, P \rangle = 0\} = \{[p, 1] \in \mathbf{P}^4 |\, p_1^2 + p_2^2 + p_3^2 + p_4^2 = 1\} \cong \mathbb{S}^3. \tag{9.6}$$

The projectivized light cone $\mathbb{P}(\mathcal{L})$ inherits a conformal structure from the Minkowski product on $\mathbb{R}^{4,1}$ but so far we do not have a metric. Later in this section we will equip $\mathbb{P}(\mathcal{L})$ with different Riemannian metrics that are compatible with this conformal structure and correspond to the space forms described in Example 4. In particular, we can interpret the affine coordinates $(p_1, \ldots, p_4)$ as Cartesian coordinates of $\mathbb{R}^4$ and equip $\mathbb{P}(\mathcal{L})$ with the usual metric of $\mathbb{S}^3 \subset \mathbb{R}^4$ (see Lemma 17).

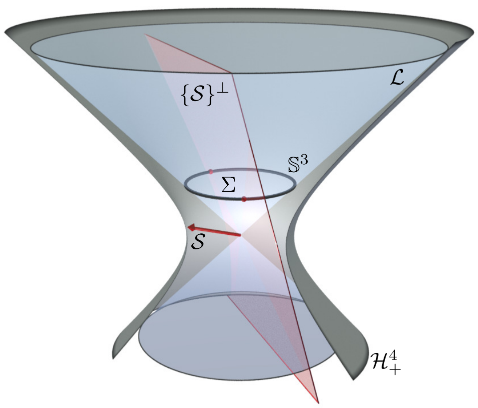
Identifying opposite points in $\mathcal{H}^4_+$ and $\mathcal{H}^4_-$, we obtain the set of points outside resp. inside $\mathbb{P}(\mathcal{L})$:

$$\begin{aligned} \mathbf{P}^4_+ &:= \mathcal{H}^4_+/_{\pm 1} = \{[P] \in \mathbf{P}^4 |\, \langle P, P \rangle > 0\}, \\ \mathbf{P}^4_- &:= \mathcal{H}^4_-/_{\pm 1} = \{[P] \in \mathbf{P}^4 |\, \langle P, P \rangle < 0\}. \end{aligned}$$

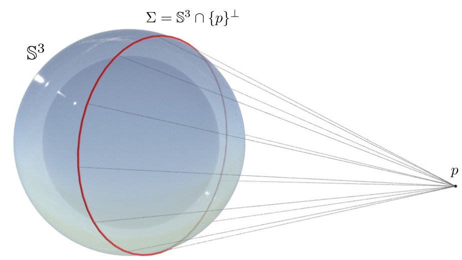
A 2-sphere in $\mathbb{P}(\mathcal{L}) \cong \mathbb{S}^3$ is given by a transversal intersection of the light cone with a hyperplane through the origin. A hyperplane intersects $\mathbb{P}(\mathcal{L})$ transversely if the normal of the plane is space-like and is tangent to $\mathbb{P}(\mathcal{L})$ if the normal is light-like. Hyperplanes that have a time-like normal do not intersect $\mathbb{P}(\mathcal{L})$. Hence we can identify the set of 2-spheres in $\mathbb{P}(\mathcal{L})$ with $\mathbf{P}^4_+$. To a given 2-sphere $\Sigma \subset \mathbb{P}(\mathcal{L}) \cong \mathbb{S}^3$ with center $[M] = [m, 1] \in \mathbb{S}^3$ and spherical radius $r \in (0, \pi)$, we assign the space-like point

$$[S] = [m, \cos(r)]. \tag{9.7}$$

Because of $\langle S, S\rangle = 1 - \cos(r)^2 = \sin(r)^2 > 0$, we have $[S] \in \mathbf{P}^4_+$ and a straight forward calculation shows that $\Sigma = S^\perp \cap \mathbb{S}^3$. If we change the orientation of $\Sigma \subset \mathbb{S}^3$, the center and radius are given by $[\tilde{M}] = [-m, 1]$ and $\tilde{r} = \pi - r$, but the point $[S] \in \mathbf{P}^4_+$ stays the same. Hence $\mathbf{P}^4_+$ is the set of unoriented 2-spheres in $\mathbb{S}^3$.



(A) A point S on the hyperboloid of one sheet $\mathcal{H}^4_+$ in $\mathbb{R}^{4,1}$ defines a hyperplane $S^\perp$, that corresponds to a 2-sphere in $\mathbb{S}^3$.



(B) In the projective picture the intersection of $\mathbb{S}^3$ with the rays, that are tangent to $\mathbb{S}^3$ and pass through the point P outside $\mathbb{S}^3$, defines a 2-sphere Σ.

One of the main reasons to choose the projective model to describe Möbius geometry is the fact that the Möbius transformations of $\mathbb{S}^3 \subset \mathbf{P}^4$ arise from the linear group $O(4, 1)$. Here

$$O(4, 1) := \{A \in \mathbb{R}^{4,1\times 4,1} |\, A^{-1} = A^t\},$$

denotes the set of linear maps that preserve the Minkowski product $\langle\cdot, \cdot\rangle$ on $\mathbb{R}^{4,1}$ and hence the light cone $\mathcal{L}$. In particular, for any $F \in \text{Möb}(3)$ there exists an $A \in O(4, 1)$ such that in homogeneous coordinates we have $F([P]) = [A(P)]$ for all $[P] \in \mathbb{S}^3$. The choice of A is unique up to a sign. For a more detailed explanation and proofs see for example [17].

$O(4, 1)$ is a Lie group and the corresponding Lie algebra is the linear space of skew symmetric matrices, which is given by

$$\mathfrak{so}(4, 1) := \{X \in \mathbb{R}^{4,1\times 4,1} |\, -X = X^t\}, \tag{9.8}$$

where the adjoint is defined w.r.t. the Minkowski product. Now we want to realize the space forms described in Example 4 as subgeometries of Möbius geometry. To this end, we change the basis of $\mathbb{R}^{4,1}$. Let $\{E_1, E_2, E_3, E_4, E_5\}$ denote the basis that we used for the definition of the Minkowski product (see (9.1)), then $E_0 := \frac{E_5 - E_4}{2}$ and $E_\infty := \frac{E_5 + E_4}{2}$ are light-like vectors and $\{E_1, E_2, E_3, E_\infty, E_0\}$ is another basis of $\mathbb{R}^{4,1}$. We will write coordinates of points relative to this basis in column vectors of the form $\begin{pmatrix} \mathbf{x} \\ \alpha \\ \beta \end{pmatrix}$, where $\mathbf{x} \in \mathbb{R}^3$ contains the coordinates w.r.t. E_1, E_2, E_3 and $\alpha, \beta \in \mathbb{R}$ denote the ones corresponding to E_∞ resp. E_0. In this new basis the Minkowski product has the form

$$\langle \cdot, \cdot \rangle : \mathbb{R}^{3+1,1} \times \mathbb{R}^{3+1,1} \to \mathbb{R}$$
$$\left\langle \begin{pmatrix} \mathbf{x_1} \\ \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \mathbf{x_2} \\ \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle := \langle \mathbf{x_1}, \mathbf{x_2} \rangle - \frac{1}{2}(\alpha_1 \beta_2 + \alpha_2 \beta_1). \tag{9.9}$$

This notation that we will use throughout the next sections is motivated by the following lemma.

Lemma 16.
We define the light-like vector $\mathcal{A}_{euc} := \begin{pmatrix} \mathbf{0} \\ 2 \\ 0 \end{pmatrix}$ *and consider the affine quadric*

$$Q_{\mathcal{A}_{euc}} := \{P \in \mathcal{L} | \, \langle P, \mathcal{A}_{euc} \rangle = -1\}.$$

If we equip $Q_{\mathcal{A}_{euc}}$ *with the metric inherited from* $\mathbb{R}^{4,1}$*, the map*

$$\Phi : \mathbb{R}^3 \to Q_{\mathcal{A}_{euc}}$$
$$\mathbf{x} \mapsto \begin{pmatrix} \mathbf{x} \\ |\mathbf{x}|^2 \\ 1 \end{pmatrix},$$

is an isometric parametrization.

Proof. From $\langle \Phi(x), A_{euc} \rangle = -1$ and $\langle \Phi, \Phi \rangle = \langle \mathbf{x}, \mathbf{x} \rangle - \frac{1}{2}(|\mathbf{x}|^2 + |\mathbf{x}|^2) = 0$, we find that Φ maps $\mathbb{R}^3$ onto $Q_{A_{euc}}$, and it is easy to see that the map is bijective. Furthermore, the parametrization is isometric because

$$\langle d\Psi, d\Psi \rangle = \left\langle \begin{pmatrix} d\mathbf{x} \\ 2\langle \mathbf{x}, d\mathbf{x} \rangle \\ 0 \end{pmatrix}, \begin{pmatrix} d\mathbf{x} \\ 2\langle \mathbf{x}, d\mathbf{x} \rangle \\ 0 \end{pmatrix} \right\rangle = \langle d\mathbf{x}, d\mathbf{x} \rangle.$$

□

After we established an isometric lift of $\mathbb{R}^3$ to the light cone, the next lemma gives us a generalization for arbitrary 3-dimensional space forms.

Lemma 17 (Hertrich-Jeromin [17]).
Let $\mathcal{A} \in \mathbb{R}^{4,1} \setminus \{0\}$ *and* $k := -|\mathcal{A}|^2$*, then the affine quadric*

$$Q_{\mathcal{A}} := \{P \in \mathcal{L} | \, \langle P, \mathcal{A} \rangle = -1\},$$

equipped with the metric inherited from $\mathbb{R}^{4,1}$*, is a space with constant sectional curvature* k*.*

Proof. If $\mathcal{A}$ is light-like, we can use an analogous construction as in lemma 16 to show that $Q_{\mathcal{A}}$ is isometric equivalent to $\mathbb{R}^3$. If $\mathcal{A}$ is space-like, the restriction of the Minkowski product to the affine hyperplane $E_{\mathcal{A}} := \{P \in \mathbb{R}^{4,1} | \, \langle P, \mathcal{A} \rangle = -1\}$ is a Minkowski product too. Hence after identifying $k^{-1}\mathcal{A}$ with the origin, we have $E_{\mathcal{A}} \cong \mathbb{R}^{3,1}$. A point

$$P \;=\; k^{-1}\mathcal{A} \;+\; P^{\perp} \in E_{\mathcal{A}} \subset \mathbb{R}^{4,1}$$

lies in the light cone $\mathcal{L}$ if and only if

$$\langle P, P\rangle_{\mathbb{R}^{4,1}} = 0 \Leftrightarrow \langle P^{\perp}, P^{\perp}\rangle_{\mathbb{R}^{3,1}} = k^{-1}.$$

Therefore, the hyperplane $E_{\mathcal{A}} \cong \mathbb{R}^{3,1}$ contains $Q_{\mathcal{A}}$ as the standard two sheeted hyperboloid with radius $\frac{1}{\sqrt{|k|}}$. If $\mathcal{A}$ is time-like, the restriction of the Minkowski product to $E_{\mathcal{A}}$ is euclidean. If we identify the origin of $E_{\mathcal{A}}$ with $k^{-1}\mathcal{A}$, $Q_{\mathcal{A}}$ becomes a round sphere of radius $\frac{1}{\sqrt{|k|}}$ in $E_{\mathcal{A}} \cong \mathbb{R}^4$. □

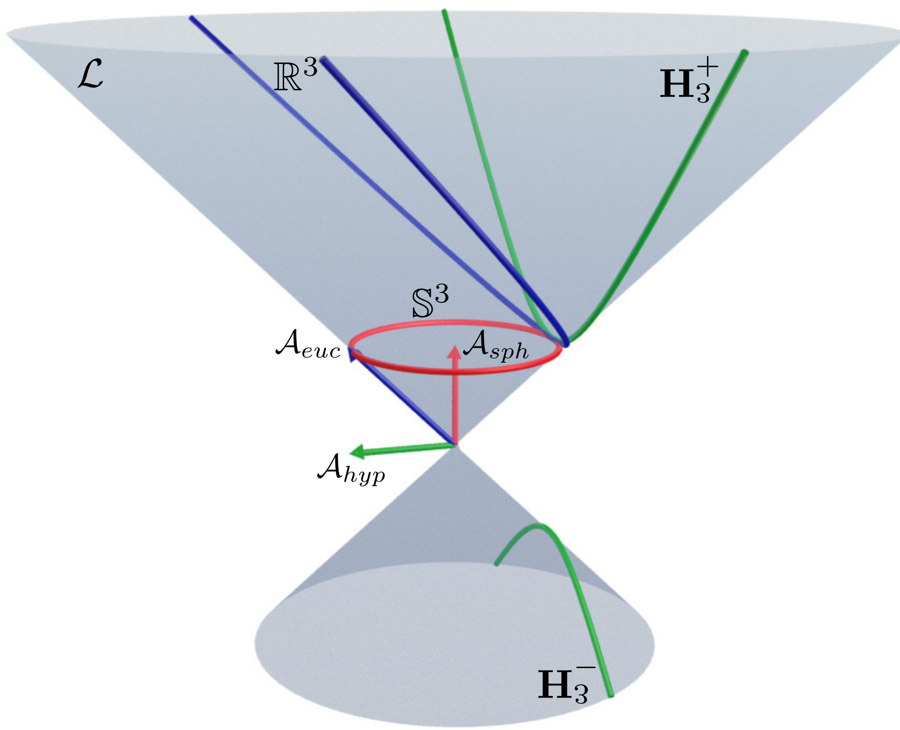


FIG. 6: The intersection of the light cone $\mathcal{L}$ with the affine hyperplane $E_{\mathcal{A}} := \{P \in \mathbb{R}^{4,1} | \langle P, \mathcal{A}\rangle = -1\}$ gives us a space with constant sectional curvature. A light like vector $\mathcal{A}_{euc}$, gives us a euclidean space form. For a time like vector $\mathcal{A}_{sph}$ the space form is spherical and for a space like vector $\mathcal{A}_{hyp}$, we obtain two hyperbolic spaces that touch at their ideal boundary.

The totally geodesic hyperplanes in $Q_{\mathcal{A}}$, that is, planes in $\mathbb{R}^3$, great 2-spheres in $\mathbb{S}^3$ and hyperbolic planes in $\mathbf{H}^3$ are given by the intersections of $Q_{\mathcal{A}}$ with hyperplanes in $\mathbb{R}^{4,1}$ whose normal's are orthogonal to $\mathcal{A}$. The set of all totally geodesic hyperplanes in $Q_{\mathcal{A}}$ is hence given by

$$\{[P] \in \mathbf{P}^4_+ | \langle P, \mathcal{A}\rangle = 0\}. \tag{9.10}$$

In particular, if $\langle \nu, x\rangle = d$ is a hyperplane in the euclidean space $\mathbb{R}^3$ with normal $\nu \in \mathbb{S}^2 \subset \mathbb{R}^3$ and offset $d \in \mathbb{R}$, the corresponding point in $\mathbf{P}^4_+$ is given by

$$[K] = \begin{bmatrix} \nu \\ 2d \\ 0 \end{bmatrix}.$$

For a sphere in $\mathbb{R}^3$ with center $c \in \mathbb{R}^3$ and radius r the corresponding point $[S] \in \mathbf{P}^4_+$ is given in homogeneous coordinates by:

$$S = \begin{pmatrix} c \\ |c|^2 - r^2 \\ 1 \end{pmatrix}.$$

10. Surfaces in the projectivized light cone

In this section, we will consider immersions $f : M \to \mathbb{P}(\mathcal{L}) \cong \mathbb{S}^3$ of a Riemannian surface M into the projectivized light cone. Here the immersion f is represented in Cartesian coordinates and hence denoted with a lower case letter. Any choice of homogeneous coordinates for the immersion f gives us a map $\Psi : M \to \mathcal{L} \subset \mathbb{R}^{4,1}$, that we will call a **lift of f to the light cone**. Because f is an immersion, the restriction of the Minkowski product to the tangent spaces of an arbitrary lift Ψ is euclidean and $\Psi^*\langle\cdot,\cdot\rangle$ defines a Riemannian metric on M. Vice versa, we can consider any immersion $\Psi : M \to \mathcal{L}$, whose pullback of the Minkowski product defines a Riemannian metric as a the lift of an immersion $f : M \to \mathbb{P}(\mathcal{L})$. At first we will show that all lifts of an immersion $f : M \to \mathbb{P}(\mathcal{L})$ are conformally equivalent and then consider special lifts corresponding to immersions into $\mathbb{R}^3$, $\mathbf{H}^3$, or $\mathbb{S}^3$, and recall some of their properties that we described in previous sections.

Lemma 18.
Let M be a Riemannian surface and $f : M \to \mathbb{P}(\mathcal{L}) \cong \mathbb{S}^3$ an immersion. If $\Psi, \tilde{\Psi} : M \to \mathcal{L}$ are two lifts of f to the light cone, they are conformally equivalent.

Proof. Let Ψ and $\tilde{\Psi}$ be lifts of f to the light cone. From $|\Psi(p)| = 0$ we obtain:

$$\langle d\Psi(X), \Psi(p)\rangle = 0, \qquad \forall X \in T_pM.$$

Further, there exists a smooth map $\rho : M \to \mathbb{R} \setminus \{0\}$ such that $\tilde{\Psi} = \rho\Psi$. The Riemannian metric induced by $\tilde{\Psi}$ is given by:

$$\langle d\tilde{\Psi}, d\tilde{\Psi}\rangle = \langle d\rho\Psi + \rho d\Psi, d\rho\Psi + \rho d\Psi\rangle = \rho^2\langle d\Psi, d\Psi\rangle + d\rho^2\langle\Psi, \Psi\rangle = \rho^2\langle d\Psi, d\Psi\rangle.$$

Therefore, the metric induced by $\tilde{\Psi}$ is conformally equivalent to the one induced by Ψ. □

Using the map Φ defined in Lemma 16, we can lift any immersion into the euclidean space isometrically to the light cone. In particular, if $f : M \to \mathbb{R}^3$ is an immersion, we call

$$\begin{aligned} \Psi : M &\to Q_{\mathcal{A}_{euc}} \subset \mathcal{L} \\ p &\mapsto \Psi(p) := \Phi(f(p)) = \begin{pmatrix} f(p) \\ |f(p)|^2 \\ 1 \end{pmatrix}, \end{aligned} \tag{10.1}$$

the **isometric lift of f to the light cone**.

Planes in $Q_{\mathcal{A}_{euc}}$ correspond to points in $\mathcal{H}^4_+$ that are orthogonal to $\mathcal{A}_{euc}$ (see (9.10)). If $N : M \to \mathbb{S}^2$ denotes the unit normal field of the immersion $f : M \to \mathbb{R}^3$, the tangent planes of the isometric lift Ψ are given by:

$$\mathcal{N} : M \to \mathcal{H}^4_+$$
$$\mathcal{N}(p) := \begin{pmatrix} N(p) \\ 2\langle N(p), f(p)\rangle \\ 0 \end{pmatrix}. \tag{10.2}$$

And indeed, we have

$$\langle \mathcal{N}(p), d\Psi_p\rangle = \langle \mathcal{N}(p), \mathcal{A}_{euc}\rangle = \langle d\Psi_p, \mathcal{A}_{euc}\rangle = 0.$$

Also, the Gauß equation can be formulated for the isometric lift. If A denotes the Weingarten operator of the immersion f, we obtain

$$d\mathcal{N}(X) = d\Psi(AX) = Hd\Psi(X) + \Omega(X) \qquad \forall X \in \mathfrak{X}(M),$$

where $\Omega(X) := \begin{pmatrix} \omega(X) \\ 2\langle \omega(X), f\rangle \\ 0 \end{pmatrix}$ is called the **lift of the Hopf differential** ω.

Choosing $X \in T_pM$ with $|X| = 1$, we can construct a frame field attached to the immersion Ψ, i.e.,

$$\{\mathcal{A}_{euc}, \Psi(p), d\Psi(X), d\Psi(JX), \mathcal{N}(p)\}$$

is a basis of $\mathbb{R}^{4,1}$ such that the Minkowski product has the form (9.9). Note that the representation of an arbitrary vector $V \in \mathbb{R}^{4,1}$ with respect to this basis is given by

$$\begin{aligned} V = &\langle V, d\Phi(X)\rangle d\Phi(X) + \langle V, d\Phi(JX)\rangle d\Phi(JX) + \langle V, \mathcal{N}\rangle\mathcal{N} - \langle V, \mathcal{A}_{euc}\rangle\Psi \\ &- \langle V, \Psi\rangle\mathcal{A}_{euc}. \end{aligned} \tag{10.3}$$

In Section 4, we started with an immersion into $\mathbb{R}^3$ and used a conformal change of metric of the ambient space $\mathbb{R}^3$ to obtain immersions into different space forms. This construction can be adapted to the new setup. If $f : M \to \mathbb{R}^3$ is an immersion and $\Psi : M \to Q_{\mathcal{A}_{euc}}$ its isometric lift to the light cone, we can change the lift to obtain immersions into different space forms $Q_{\mathcal{A}}$. Therefore, we define

$$\begin{aligned} \rho &:= -\langle \mathcal{A}, \Psi\rangle, \\ \tilde{\Psi} &:= \rho^{-1}\Psi, \\ \tilde{\mathcal{N}} &:= \mathcal{N} - \frac{\langle \mathcal{A}, \mathcal{N}\rangle}{\langle \mathcal{A}, \Psi\rangle}\Psi = \mathcal{N} + \langle \mathcal{A}, \mathcal{N}\rangle\tilde{\Psi}. \end{aligned}$$

From $|\tilde{\Psi}| = 0$ and $\langle \tilde{\Psi}, \mathcal{A}\rangle = -1$ we obtain that $\tilde{\Psi} : M \to Q_{\mathcal{A}}$ defines a lift of f to $Q_{\mathcal{A}}$. Further $\tilde{\mathcal{N}}$ denotes the tangent bundle of $\tilde{\Psi}$, because $\langle \tilde{\mathcal{N}}, \mathcal{A}\rangle = 0$ and $\langle \tilde{\mathcal{N}}, d\tilde{\Psi}\rangle = 0$. Due to Lemma 18, the metrics on M induced by Ψ and $\tilde{\Psi}$ are conformally equivalent. The conformal change is given by

$$\langle d\tilde{\Psi}, d\tilde{\Psi}\rangle = \rho^{-2}\langle d\Psi, d\Psi\rangle.$$

Lemma 19.
Let $f : M \to \mathbb{R}^3$ be an immersion and $\Psi : M \to Q_{\mathcal{A}_{euc}}$ its isometric lift. If $\tilde{\Psi} : M \to Q_{\mathcal{A}}$ defines another lift to a space form $Q_{\mathcal{A}}$ with sectional curvature k, the conformal factor ρ satisfies

$$\rho = -\frac{1}{2\langle \mathcal{A}, \mathcal{A}_{euc}\rangle}\left(2\|d\rho\|^2 + \langle \mathcal{A}, \mathcal{N}\rangle^2 + k\right). \tag{10.4}$$

Proof. The conformal factor is defined as $\rho := -\langle \mathcal{A}, \Psi\rangle$. Let $X \in T_pM$ with $|X| = 1$, then we can use (10.3) to represent $\mathcal{A}$ with respect to the frame field adapted to Ψ

$$\begin{aligned} \mathcal{A} = &\langle \mathcal{A}, d\Psi(X)\rangle d\Psi(X) + \langle \mathcal{A}, d\Psi(JX)\rangle d\Psi(JX) + \langle \mathcal{A}, \mathcal{N}\rangle \mathcal{N} \\ &- \langle \mathcal{A}, \mathcal{A}_{euc}\rangle \Psi - \langle \mathcal{A}, \Psi\rangle \mathcal{A}_{euc} \end{aligned}$$

Due to Lemma 17, the sectional curvature of $Q_{\mathcal{A}}$ is given by $k = -|A|^2$. This gives rise to

$$\begin{aligned} -k = \|\mathcal{A}\|^2 &= \langle \mathcal{A}, d\Psi(X)\rangle^2 + \langle \mathcal{A}, d\Psi(JX)\rangle^2 + \langle \mathcal{A}, \mathcal{N}\rangle^2 - 2\langle \mathcal{A}, \Psi\rangle\langle \mathcal{A}, \mathcal{A}_{euc}\rangle \\ \Rightarrow \qquad \rho = -\langle \mathcal{A}, \Psi\rangle &= \frac{-1}{2\langle \mathcal{A}, \mathcal{A}_{euc}\rangle}(2\|d\rho\|^2 + \langle \mathcal{A}, \mathcal{N}\rangle^2 + k). \end{aligned}$$

□

Example 5:
In the light cone model the spaceforms described in Example 4 are realized by the following choice of $\mathcal{A}$:

- *For the Poincare ball model we define:*
$$\mathcal{A}_{hyp} := \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \qquad \Rightarrow \rho_{hyp} = \frac{1 - |x|^2}{2}.$$
- *For the Poincare halfspace model with normal $b \in \mathbb{S}^2 \subset \mathbb{R}^3$ consider:*
$$\mathcal{A}_{hyp,b} := \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \rho_{hyp,b} = \langle b, x\rangle.$$
- *The spherical case is obtained by:*
$$\mathcal{A}_{sph} := \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \Rightarrow \rho_{sph} = \frac{1 + |x|^2}{2}.$$

In the last lemma of the section, we consider again an isometric lift Ψ of an immersion $f : M \to \mathbb{R}^3$ and compute the Laplacian of Ψ and the one of the space-like field $\mathcal{N}$ that represents the tangent bundle of Ψ.

Lemma 20.
The Laplacians of Ψ and $\mathcal{N}$ are given by:

$$\begin{aligned} \triangle\Psi &= -2H\mathcal{N} + 2\mathcal{A}_{euc}, \\ \triangle\mathcal{N} &= -2(H^2 + \mu^2)\mathcal{N} + \frac{2 * dH \wedge d\Psi}{|d\Psi|^2} + 2H\mathcal{A}_{euc}. \end{aligned}$$

Proof. With the properties of the wedge product described Equation (3.12) we have

$$d * d|f|^2 = 2d\langle f, *df\rangle = 2\langle df \wedge *df\rangle + 2\langle f, d * df\rangle, .$$

Now we can use Lemma 5 to obtain

$$\triangle |f|^2 = -4\langle HN, f\rangle + 4.$$

Therefore, the Laplacian of Ψ is given by

$$\triangle\Psi(x) = \triangle \begin{pmatrix} f \\ |f|^2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2HN \\ -4\langle HN, f\rangle + 4 \\ 0 \end{pmatrix} = -2H\mathcal{N} + 2\mathcal{A}_{euc}.$$

The computation for $\mathcal{N}$ is analogous. From

$$d * d\langle N, f\rangle = \langle d * dN, f\rangle + \langle df \wedge *dN\rangle,$$

we get with Lemma 5

$$\begin{aligned} \triangle\mathcal{N} &= \begin{pmatrix} \triangle N \\ 2\,\triangle\,(\langle N, f\rangle) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -2N(H^2+\mu^2) + 2\frac{*dH\wedge df}{|d\Psi|^2} \\ -4(H^2+\mu^2)\langle N, f\rangle + 4\frac{*dH\wedge\langle df, f\rangle}{|d\Psi|^2} + 2\frac{\langle *dN\wedge df\rangle}{|d\Psi|^2} \\ 0 \end{pmatrix} \\ &= -2(H^2+\mu^2)\mathcal{N} + 2\frac{*dH\wedge d\Psi}{|d\Psi|^2} + 2H\mathcal{A}_{euc}. \end{aligned}$$

□

11. The maximal interior sphere congruence of surfaces

Definition 28.
A ***sphere congruence*** *is a continuous map $\mathcal{S} : M \to \mathbf{P}_+ = \mathcal{H}^4_+/_{\pm 1}$, that assigns a sphere to each point in M. An immersion $\Psi : M \to \mathcal{L}$* ***envelops a sphere congruence*** *$\mathcal{S}$, if the spheres $S(p)$ are tangent to the surface at $\Psi(p)$, i.e.,*

$$\langle \mathcal{S}(p), \Psi(p)\rangle = 0 \quad \text{and} \quad \langle \mathcal{S}(p), d\Psi(X)\rangle = 0, \qquad \forall p \in M \text{ and } \forall X \in T_pM.$$

Note that if $\Psi : M \to \mathcal{L}$ is the lift of an immersion $f : M \to \mathbb{P}(\mathcal{L})$, all other lifts of f to the light cone envelop $\mathcal{S}$, too.

For a given immersion $f : M \to \mathbb{R}^3$ and its isometric lift $\Psi : M \to \mathcal{L}$, the set of spheres that are tangent to the surface at the point $\Psi(p)$ is given by:

$$\mathcal{S}(p) := \mathcal{N}(p) - \Phi(p)\Psi(p), \qquad \Phi(p) \in \mathbb{R},$$

where $\Phi(p)$ denotes the euclidean curvature of the sphere $\mathcal{S}(p)$ whose center in $\mathbb{R}^3$ is given by $c(p) := f(p) - \frac{N(p)}{\Phi(p)}$.

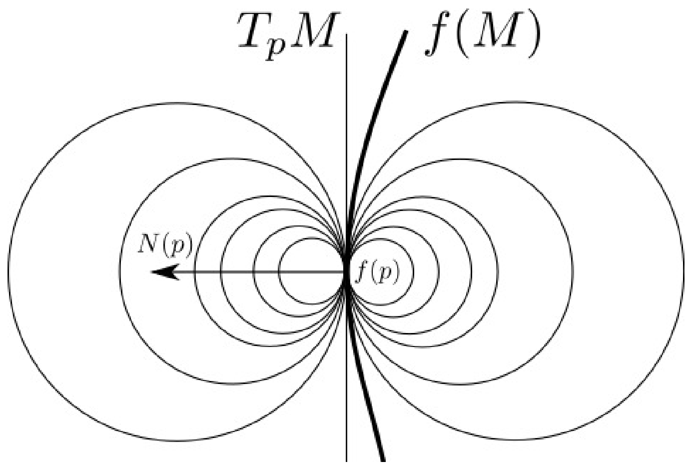


FIG. 7: The pencil of spheres tangent to the surface $f(M)$ at the point p.

If $\tilde{\Psi}$ denotes the lift of f to the space form $Q_{\mathcal{A}}$, the sphere congruence $\mathcal{S}(p)$ is still enveloped by $\tilde{\Psi}$, but the radii and centers change.

$$\begin{aligned}\mathcal{S}(p) :&= \mathcal{N}(p) - \Phi(p)\Psi(p)\\ &= \tilde{\mathcal{N}}(p) - \langle \mathcal{A}, \mathcal{N}(p)\rangle \tilde{\Psi}(p) + \Phi(p)\langle \mathcal{A}, \Psi(p)\rangle \tilde{\Psi}(p)\\ &= \tilde{\mathcal{N}}(p) - \underbrace{(-\Phi(p)\langle \mathcal{A}, \Psi(p)\rangle + \langle \mathcal{A}, \mathcal{N}(p)\rangle)}_{=:\tilde{\Phi}(p)} \tilde{\Psi}(p).\end{aligned} \tag{11.1}$$

With Proposition 17, the change of the curvature of the spheres is the same as the one of the mean curvature and the principal curvatures of f under the conformal change of metric $g \mapsto \rho^{-2}g$. Therefore, the mean curvature sphere congruence and the principal curvature sphere congruences are the same in every space form:

$$\begin{aligned}\mathcal{H}(p) &:= \mathcal{N}(p) - H(p)\Psi(p) = \tilde{\mathcal{N}}(p) - \tilde{H}(p)\tilde{\Psi}(p),\\ \mathcal{S}_i(p) &:= \mathcal{N}(p) - \lambda_i(p)\Psi(p) = \tilde{\mathcal{N}}(p) - \tilde{\lambda}_i(p)\tilde{\Psi}(p).\end{aligned} \tag{11.2}$$

DEFINITION 29.
*Let $f : M \to \mathbb{R}^3$ be a conformal embedding of a compact surface M, $\overline{M} \subset \mathbb{R}^3$ the compact region bounded by $f(M)$, and N the outpointing unit normal field, then the **maximal interior sphere congruence** of f is defined as:*

$$\begin{aligned}\mathcal{K} &: M \to \mathbf{P}_+\\ \mathcal{K}(p) &:= \mathcal{N}(p) - \Phi(p)\Psi(p),\end{aligned}$$

where $\Phi(p)$ is the smallest positive real number such that $\mathcal{K}(p)$ is entirely contained in $\overline{M}$.

Note that if we reverse the orientation of M, we can do the same construction to obtain the **maximal exterior sphere congruence**, but since $\overline{M}$ is no longer compact, there will exist points $x \in M$ with $\Phi(x) = 0$, i.e., $\mathcal{K}(x)$ is no longer a sphere but a supporting plane.

Lemma 21.
Let $f : M \to \mathbb{R}^3$ be a conformal embedding of a compact surface M and $\overline{M} \subset \mathbb{R}^3$ the compact region bounded by $f(M)$. A sphere congruence enveloped by f

$$\begin{aligned} &\mathcal{S} : M \to \mathbf{P}_+ \\ &\mathcal{S}(p) = \mathcal{N}(p) - \Phi(p)\Psi(p) \end{aligned}$$

lies entirely in $\overline{M}$ if and only if the function

$$\begin{aligned} &Z : M \times M \to \mathbb{R} \\ &(x, y) \mapsto \langle \mathcal{S}(x), \Psi(y) \rangle, \end{aligned}$$

is nowhere negative.

Proof. A point $f(y)$ lies inside the compact region bounded by the sphere $\mathcal{S}(x)$ if

$$\begin{aligned} & \left| f(x) - f(y) - \frac{N(p)}{\Phi(p)} \right|^2 < \Phi(x)^{-2} \\ \Leftrightarrow \qquad & \frac{\Phi(x)}{2} |f(x) - f(y)|^2 - \langle N(x), f(x) - f(y) \rangle < 0. \end{aligned}$$

With (10.1) and (10.2) we get for Z

$$\begin{aligned} \langle \mathcal{S}(x), \Psi(y) \rangle &= \langle \mathcal{N}(p), \Psi(y) \rangle - \Phi(p) \langle \Psi(p), \Psi(y) \rangle \\ &= \langle N(p), f(y) - f(x) \rangle - \Phi(p) \big(\langle f(x), f(y) \rangle - \frac{1}{2} (|f(x)|^2 |f(y)|^2) \big) \\ &= \frac{\Phi(x)}{2} |f(x) - f(y)|^2 - \langle N(x), f(x) - f(y) \rangle. \end{aligned}$$

□

The construction of the the maximal interior sphere congruence for a given embedding $f : M \to \mathbb{R}^3$ works pointwise. For $x_0 \in M$ we start with a large number $\Phi \in \mathbb{R}$ and shrink it till the function

$$\begin{aligned} &Z : M \times \mathbb{R} \to \mathbb{R} \\ &(\Phi, y) \mapsto \langle \mathcal{N}(x_0) - \Phi \Psi(x_0), \Psi(y) \rangle, \end{aligned}$$

is negative for some $y \in M$. Because Z is continuous and $[0, \infty)$ is closed, there exists a minimal $\Phi(x_0) \in \mathbb{R}^+$ such that $Z(\Phi(x_0), y) \geq 0$ for all $y \in M$. Therefore, we can define the maximal sphere congruence as

$$\mathcal{K} := \mathcal{N} - \Phi \Psi.$$

This construction is continuous and conformally invariant. If $\tilde{\Psi} : M \to Q_{\mathcal{A}}$ denotes the lift of f to another space form, we obtain with (11.1) that the maximal sphere congruence stays the same. We summarize our considerations in the next proposition.

Proposition 27.
If $\Psi : M \to Q_{\mathcal{A}}$ is the lift of an embedding $f : M \to \mathbb{R}^3$ to a space form $Q_{\mathcal{A}} \subset \mathcal{L}$, then there exists a continuous function $\Phi : M \to \mathbb{R}^+$ such that

$$\mathcal{K} := \mathcal{N} - \Phi\Psi,$$

is the maximal sphere congruence of Ψ.

We start our investigation of the maximal sphere congruence with a first estimate for curvature function Φ.

Lemma 22.
Let $\mathcal{K} := \mathcal{N} - \Phi\Psi$ be the maximal interior sphere congruence of an embedding $f : M \to \mathbb{R}^3$, then the curvature function Φ satisfies

$$\Phi(x) \geq \lambda_1(x) \qquad \forall x \in M,$$

where $\lambda_1(x)$ denotes the bigger principal curvature of f.

Proof. We fix $x_0 \in M$ and consider the function $Z(y) := \langle \mathcal{N}(x_0) - \Phi(x_0)\Psi(x_0), \Psi(y)\rangle$ Because $Z(y) \geq 0$ for all $y \in M$ and $Z(x_0) = 0$, Z has a global minimum at x_0.

Let $\gamma : (-\epsilon, \epsilon) \to M$ be an arclength parametrized geodesic in M with $\gamma(0) = x_0$, then $Z(t) := Z(\gamma(t))$ has a global minimum at $t = 0$. Because f envelops $\mathcal{K}$, we have

$$\begin{aligned} & dZ(X) = \langle \mathcal{N}(x_0) - \Phi(x_0)\Psi(x_0), d\Psi(X)\rangle = 0 \qquad \forall X \in T_{x_0}M \\ \Rightarrow \qquad & Z'(0) = 0 \qquad \text{for all geodesics } \gamma. \end{aligned}$$

Since γ is a geodesic, the second derivative is normal valued

$$f(\gamma(t))'' = -\langle df(\gamma'(t)), dN(\gamma'(t))\rangle N(\gamma(t)).$$

This gives us for the second derivative of $\Psi(t) := \Psi(\gamma(t))$:

$$\Psi''(t) = \begin{pmatrix} f''(t) \\ 2\langle f''(t), f(t)\rangle + 2 \\ 0 \end{pmatrix} = -\langle df(\gamma'(t)), dN(\gamma'(t))\rangle \mathcal{N}(\gamma(t)) + \mathcal{A}_{euc}.$$

Now we can compute the second derivative of Z:

$$\begin{aligned} Z''(t) &= \langle \mathcal{N}(x_0) - \Phi(x_0)\Psi(x_0), \Psi''(t))\rangle \\ &= \langle \mathcal{N}(x_0) - \Phi(x_0)\Psi(x_0), -\langle df(\gamma'(t)), dN(\gamma'(t))\rangle \mathcal{N}(\gamma(t)) + \mathcal{A}_{euc}\rangle \qquad (11.3) \\ \Rightarrow \qquad Z''(0) &= \Phi(x_0) - \underbrace{\langle df(\gamma'(0)), dN(\gamma'(0))\rangle}_{\in[\lambda_2(x_0), \lambda_1(x_0)]}. \end{aligned}$$

Since Z has a minimum at x_0, the function $Z''(0)$ cannot be negative and we obtain $\Phi(x_0) \geq \lambda_1(x_0)$. $\square$

Definition 30.
Let $\mathcal{K} := \mathcal{N} - \Phi\Psi$ be the maximal interior sphere congruence of an embedding $f : M \to \mathbb{R}^3$, the points $x \in M$ with $\Phi(x) = \lambda_1(x)$ will be called ***maximal points*** *of the maximal interior sphere congruence.*

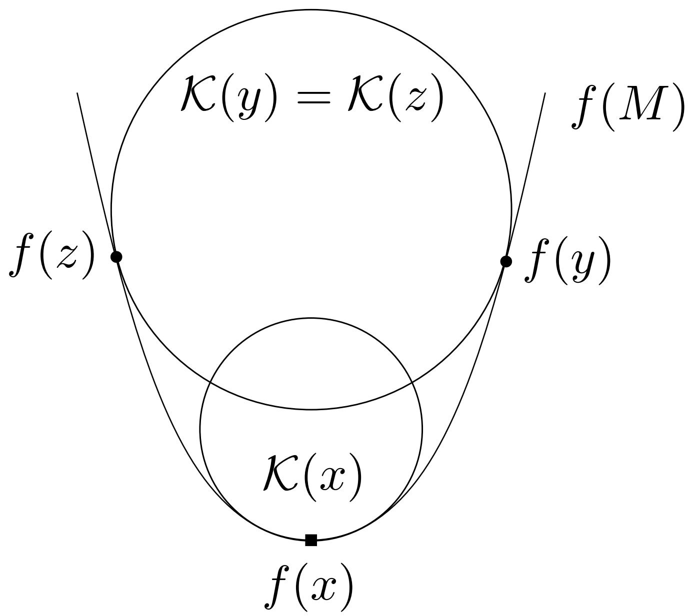


FIG. 8: Two spheres of the maximal interior sphere congruence $\mathcal{K}$. The point x is a maximal point of $\mathcal{K}$ because the sphere $\mathcal{K}(x)$ touches the surface only in the point $f(x)$. The points y, z are no maximal points, because the curvature of the sphere $\mathcal{K}(y) = \mathcal{K}(y)$ is bigger as the curvature of the surface at the touching points.

If x_0 is not a maximal point of the maximal interior sphere congruence, there exist $y_0 \neq x_0 \in M$ such that the sphere $\mathcal{K}(x_0)$ is tangent to the surface at $\psi(x_0)$ and $\Psi(y_0)$. In particular we have $\mathcal{K}(x_0) = \mathcal{K}(y_0)$, i.e.,

$$\Phi(x_0) = \Phi(y_0), \tag{11.4}$$

$$\mathcal{N}(x_0) = \mathcal{N}(y_0) - \Phi(x_0)(\Psi(x_0) - \Psi(y_0), \tag{11.5}$$

and the function $Z(x, y) := \langle \mathcal{K}(x), \Psi(y) \rangle$ has a global minimum at (x_0, y_0). If for $x_0 \in M$ there does not exist another point $y_0 \neq x_0 \in M$ with $\mathcal{K}(x_0) = \mathcal{K}(y_0)$, then x_0 is a maximal point of the maximal sphere congruence. The existence of a maximal point gives us further information about the curvature of the surface at this point.

PROPOSITION 28.
Let $f : M \to \mathbb{R}^3$ be an embedding and $x_0 \in M$ a maximal point of the maximal interior sphere congruence $\mathcal{K}$ of f. If f admits a curvature line parametrisation in a neighborhood of x_0, we have $d\lambda_1(X_1) = 0$, where $X_1 \in T_{x_0}M$ is the principal curvature direction corresponding to the bigger principle curvature λ_1. If additionally x_0 is an umbilical point, we further have $d\mu_{x_0} = 0 = dH_{x_0}$.

Proof. Let $\gamma : (-\epsilon, \epsilon) \to M$ be an arclength parametrized geodesic in M with $\gamma(0) = x_0$. If f admits a curvature line parametrisation, there exist normalized principal curvature vector fields X_1, X_2 a long γ and $\beta \in C^\infty((-\epsilon, \epsilon), [0, 2\pi])$ such that:

$$\gamma'(t) = \cos(\beta(t))X_1(t) + \sin(\beta(t))X_2(t).$$

As in the proof of Lemma 22, we consider the function $Z(t) := \langle \mathcal{K}(x_0), \Psi(\gamma(t))\rangle$, that has a global minimum at $t = 0$ for all geodesics $\gamma(t)$. Due to (11.3), the second derivative of Z is given by

$$\begin{aligned} & Z''(t) = \langle \mathcal{N}(x_0) - \Phi(x_0)\Psi(x_0), -\langle df(\gamma'(t)), dN(\gamma'(t))\rangle \mathcal{N}(\gamma(t)) + \mathcal{A}_{euc}\rangle \\ \Rightarrow \qquad & Z''(0) = \Phi(x_0) - \langle df(\gamma'(0)), dN(\gamma'(0))\rangle. \end{aligned}$$

For the geodesic with $\gamma'(0) = X_1$, we get $Z''(0) = 0$. Since Z has a global minimum at $t = 0$, the third derivative has to be zero, too. This is the case if and only if

$$\begin{aligned} 0 &= \langle df(\gamma'(t)), dN(\gamma'(t))\rangle'|_{t=0} \\ &= \Big(\cos^2(\beta(t))\lambda_1(\gamma(t)) + \sin^2(\beta(t))\lambda_2(\gamma(t)\rangle\Big)'\Big|_{t=0} \\ &= \cos^2(\beta(0))d\lambda_1(\gamma'(0)) + \sin^2(\beta(0))d\lambda_2(\gamma'(0)) \\ &\quad + 2\cos(\beta(0))\sin(\beta(0))d\beta(\gamma'(0))(\lambda_2(p_0) - \lambda_1(p_0)) \\ &= d\lambda_1(\gamma'(0)) \\ &= d\lambda_1(X_1). \end{aligned} \tag{11.6}$$

Assume now that x_0 is an umbilical point, i.e., $\lambda_1(x_0) = \lambda_2(x_0)$. Then Equation 11.6 has to be satisfied for all choices of $\beta(0) \in [0, 2\pi]$.

$$\begin{aligned} 0 &= \Big(\cos^2(\beta(t))\lambda_1(\gamma(t)) + \sin^2(\beta(t))\lambda_2(\gamma(t)\rangle\Big)'\Big|_{t=0} \\ &= \cos^2(\beta(0))d\lambda_1(\gamma'(0)) + \sin^2(\beta(0))d\lambda_2(\gamma'(0)) \\ &\quad + 2\cos(\beta(0))\sin(\beta(0))d\beta(\gamma'(0))\underbrace{(\lambda_2(p_0) - \lambda_1(p_0))}_{=0} \\ &= \cos^3(\beta(0))d\lambda_1(X_1) + \sin(\beta(0))\cos^2(\beta(0))d\lambda_1(X_2) \\ &\quad + \sin^2(\beta(0))\cos(\beta(0))d\lambda_2(X_1) + \sin^3(\beta(0))d\lambda_2(X_2). \end{aligned}$$

For $\beta(0) = 0$ we get $d\lambda_1(X_1) = 0$ and for $\beta(0) = \frac{\pi}{2}$ we have $d\lambda_2(X_2) = 0$ and hence

$$\begin{aligned} & 0 = \sin(\beta(0))\cos(\beta(0))\left(\sin(\beta(0))d\lambda_2(X_1) + \cos(\beta(0))d\lambda_1(X_2)\right) \\ \Leftrightarrow \quad & \sin(\beta(0))d\lambda_2(X_1) = -\cos(\beta(0))d\lambda_1(X_2) \end{aligned} \tag{11.7}$$

Because (11.7) has to be satisfied for all $\beta(0) \in (0, \frac{\pi}{2})$, we obtain $d\lambda_2 = d\lambda_1 = 0$ and hence

$$dH_{p_0} = \frac{1}{2}(d\lambda_1 + d\lambda_2) = 0 = \frac{1}{2}(d\lambda_1 - d\lambda_2) = d\mu_{x_0}.$$

□

12. Babich-Bobenko tori and Bryant tori with smooth ends

Let $f : M \to \mathbb{R}^3$ be an embedding of an isothermic constrained Willmore torus. With Theorem 9, we know that f has constant mean curvature $\tilde{H}$ in a space form $Q_{\mathcal{A}}$. For the following investigation we distinguish tree different cases, depending on whether the surface intersects the ideal boundary of $Q_{\mathcal{A}}$ (Babich-Bobenko tori), touches it (Bryant surfaces), or lies entirely in $Q_{\mathcal{A}}$. In this section, we will show that the Babich-Bobenko tori cannot be embedded. To this end, we assume that $f : M \to \mathbb{R}^3$ is an embedding and $\tau = \rho^{-1}\omega$ the differential of the Christoffel dual of f. Because M is a torus, the holomorphic differential τ has no zeros (see Corollary 5). We will show that in the case of Babich-Bobenko tori, these assumptions lead to a contradiction. Let $\Psi : M \to \mathcal{L}$ denote the euclidean lift of f to the light cone, and $\tilde{\Psi} : M \to Q_{\mathcal{A}}$ the lift to the space form $Q_{\mathcal{A}}$ in which f has constant mean curvature $\tilde{H}$. The maximal interior sphere congruence of f is given by:

$$\mathcal{K} = \mathcal{N} - \Phi\Psi = \tilde{\mathcal{N}} - \tilde{\Phi}\tilde{\Psi}.$$

We start with a lemma for all embedded isothermic constrained Willmore tori.

Lemma 23.
Let $f : M \to \mathbb{R}^3$ be an embedding of an isothermic constrained Willmore torus, $\Psi : M \to \mathcal{L}$ its isometric lift to the light cone and $\mathcal{K}$ its maximal interior sphere congruence. Then for every point pair $(x_0, y_0) \in M \times M$ with $Z(x_0, y_0) := \langle \mathcal{K}(x_0), \Psi(y_0)\rangle = \langle \mathcal{K}(y_0), \Psi(x_0)\rangle = Z(y_0, x_0) = 0$, the following equation for the euclidean mean curvature holds

$$\rho(x_0)H(x_0) - \rho(y_0)H(y_0) = \Phi\big(\rho(x_0) - \rho(y_0)\big).$$

If $\tilde{g} = \rho^{-2}g$ defines a hyperbolic metric, $f(x_0)$ and $f(y_0)$ lie in the same space $\mathbf{H}^3_{\pm}$. Furthermore, x_0 is an umbilical point if and only if y_0 is.

Proof. Because $\tilde{H} = -\rho H + \langle \mathcal{N}, \mathcal{A}\rangle$ is constant, we obtain with (11.5)

$$\begin{aligned}\rho(x_0)H(x_0) - \rho(y_0)H(y_0) &= \langle \mathcal{N}(x_0), \mathcal{A}\rangle - \langle \mathcal{N}(y_0), \mathcal{A}\rangle \\ &= \langle -\Phi(x_0)(\Psi(x_0) - \Psi(y_0), \mathcal{A}\rangle \\ &= \Phi\big(\rho(x_0) - \rho(y_0)\big).\end{aligned}$$

Rearranging the terms gives us

$$\rho(x_0)(\Phi - H(x_0)) = \rho(y_0)(\Phi - H(y_0)). \tag{12.1}$$

The relation between the curvature of the spheres $\mathcal{K}(x)$, $\mathcal{K}(y)$ and the bigger principal curvatures of the surface at the touching points was described in Proposition 22:

$$\Phi(x_0) = \Phi(y_0) \geq \lambda_1(y_0) \geq H(y_0) \geq \lambda_2(y_0).$$

Therefore, $\Phi(x_0) - H(x_0)$ and $\Phi(x_0) - H(y_0)$ are non negative, and both sides of Equation 12.1 are zero if and only if x_0 and y_0 are umbilical points. Finally, we obtain from (12.1) that $\rho(x_0)$ and $\rho(y_0)$ have the same sign and hence $f(x_0), f(y_0)$ lie in the same space $\mathbf{H}^3_{\pm}$ or are both umbilical. □

Note, that the Lemma 23 stays true if we reverse the orientation of M and consider the maximal exterior sphere congruence of M.

12.1. **Babich-Bobenko tori.** For the following investigations of the Babich-Bobenko tori, we choose the Poincaré half-space model. Let $f : M \to \mathbb{R}^3$ be an immersion with outpointing euclidean normal field N that has CMC with respect to the hyperbolic metric $\tilde{g} := \rho^{-2} f^*\langle\cdot,\cdot\rangle$, where $\tilde{\rho}(x) := \langle x, b\rangle$ for some $b \in \mathbb{S}^2$ and $\rho := f^*\tilde{\rho}$. In other terms, the restriction of f to

$$M_{\pm} := \{p \in M | \langle b, f(p)\rangle \gtrless 0\} \tag{12.2}$$

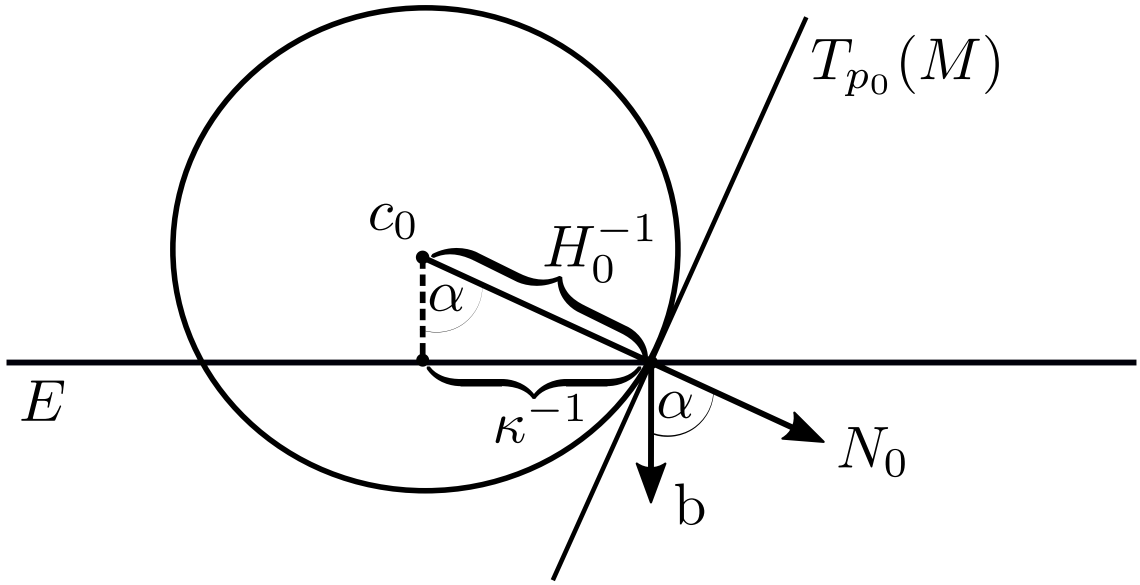


FIG. 9: The mean curvature sphere $S_f(p)$ intersects the plane E in a circle with radius $\kappa^{-1} = \frac{\sin(\alpha)}{H_0}$.

has constant mean curvature

$$\tilde{H} = \pm(\rho H - \langle b, N \rangle)), \tag{12.3}$$

with respect to the metric $\tilde{g}$. Let

$$M_0 := \{p \in M | \langle b, f(p) \rangle = 0\}, \tag{12.4}$$

denote the set of umbilical points of $f(M)$. The fact that the restriction of the immersion f onto $M_\pm$ has constant mean curvature $\pm\tilde{H}$ implies that $\tau := \rho^{-1}\omega$ is closed and hence holomorphic on $M_\pm$. Since τ is continuous on the whole of the genus one surface M, it is holomorphic everywhere (for more details see [3]). It follows from Proposition 26 that $f(M)$ cuts the plane $E := \{x \in \mathbb{R}^3 | \rho(x) = 0\}$ with constant angle $\alpha := \arccos(\tilde{H})$, and from Lemma 14 we obtain for the differential of the euclidean mean curvature H

$$dH = \langle \tau, b \rangle. \tag{12.5}$$

We now assume that $f : M \to \mathbb{R}^3$ is an embedding and consider the intersection of $f(M)$ with E, i.e, $f(M_0)$. Due to the fact that $f(M)$ is embedded, there exist regular simply closed curves $\gamma_i : \mathbb{S}^1 \to M$ such that $f(M_0) = \bigcup_i f(\gamma_i)(\mathbb{S}^1)$. The curves do not intersect each other and at least one of them, here denoted by $f(\gamma)$, bounds a topological disk in $E \setminus f(M_0)$.

LEMMA 24.
The curvature $\kappa(t)$ of $f(\gamma(t))$ as plane curve in E is given by

$$\sin(\alpha)\kappa(t) = \pm H(\gamma(t)), \tag{12.6}$$

depending on whether the normal of γ points out of or into the compact region bounded by $f(M)$. Here $H(\gamma(t))$ is the mean curvature of $f(M)$ at $\gamma(t)$ and α denotes the constant angle of intersection between $f(M)$ and E.

Proof. We assume that $f(\gamma)$ is an arclength parametrization and $n : \mathbb{S}^1 \to \mathbb{S}^1 \subset E$ denotes the outpointing normal field of $f(\gamma)$. Since $f(M)$ intersects E in a constant angle α, the normal field is given by

$$n(t) = \pm \sin(\alpha) N(\gamma(t)) + \cos(\alpha) N df(\gamma'(t)).$$

The curvature $\kappa(t)$ of the planar curve $f(\gamma)$ is defined by

$$-\kappa(t)n(t) = f(\gamma(t))'' = df(\gamma'(t))' = df(\nabla_{\gamma'(t)}\gamma'(t)) - \langle df(\gamma'(t)), dN(\gamma'(t))\rangle N(t).$$

Because $f(\gamma(t))$ lies in the plane E, all the points of the curve $f(\gamma(t))$ are umbilical points of the immersion f. With $|df(\gamma'(t))| = 1$ we obtain for the second fundamental form of f at $\gamma(t)$

$$\langle df(\gamma'(t)), dN(\gamma'(t))\rangle = H(\gamma(t)).$$

Taking the scalar product with $N(t)$ gives us the desired result

$$-H(\gamma(t)) = \langle N(t), f(\gamma(t))''\rangle = \langle N(t), -\kappa(t)n(t)\rangle = \mp \sin(\alpha)\kappa(t).$$

□

Because $f(\gamma)$ is a regular, simply closed and planar curve, we can apply the following theorem from Bernd Wegner.

Theorem 13 (**Wegner** [31])**.**
Every simply closed C^3-curve a in the plane has at least two points where the osculating circles are contained in the closure of its interior and at least two points where the osculating circles are contained in the closure of its exterior. (For the exterior case also supporting lines at points with vanishing curvature will be called osculating circles.)

Now we have all the necessary components to prove the following theorem.

Theorem 14.
There are no embedded Babich-Bobenko tori.

Proof. Assume that $f : M \to \mathbb{R}^3$ is an embedding of a Babich-Bobenko torus and let $f(\gamma)(\mathbb{S}^1)$ be a component of $f(M) \cap E$ that bounds a topological disk D in $E \setminus f(M_0)$. By Theorem 13, there exist two points x_1, x_2 on $f(\gamma)$ such that the osculating circles C_i at these points are contained in the closure of D. Further, let $S_f(x_i)$ denote the sphere that is tangent to $f(M)$ at x_i and intersects the plane E in the C_i. With Lemma 24, the radius of the sphere $S_f(x_i)$ is given by $\pm H(x_i)$, i.e., $S_f(x_i) = \mathcal{N}(x_i) - H(x_i)\Psi(x_i)$ is the mean curvature sphere of f at x_i. If the normal of $f(\gamma)$ points outside the compact region bounded by $f(M)$, we consider the maximal interior sphere congruence of f, else the exterior one. In both cases we will denote the sphere congruence by $\mathcal{K}$ and show that x_i is a maximal point of $\mathcal{K}$.

If x_i is not a maximal point, there exists another point $y_i \in M$ such that $\Psi(x_i)$ and $\Psi(y_i)$ lie in the same sphere of the maximal interior sphere congruence, i.e., $\mathcal{K}(x_i) = \mathcal{K}(y_i)$. We can now use Lemma 23 and obtain that y_i is an umbilical point because $x_i \in M_0$ is umbilical. Due to the construction of the maximal sphere congruence, the point $f(y_i)$ lies on the circle C_i if it is contained in the plane E. This gives us that the sphere $\mathcal{K}(x_i)$

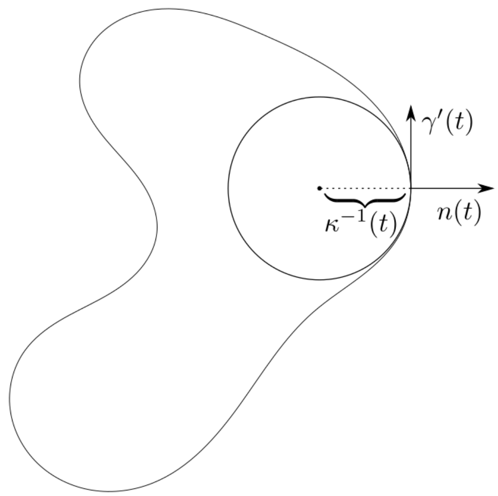


FIG. 10: The osculating circle of γ at the point $\gamma(t)$ is contained in the compact region bounded by $\gamma(\mathbb{S}^1)$.

intersects the plane E in the circle C_i and is hence identical with the mean curvature sphere of f at x_i.

At umbilical points the principal curvatures and the mean curvature are equal and x_i is a maximal point of maximal sphere congruence $\mathcal{K}$. Furthermore, there exist a conformal coordinate line parametrization in a neighborhood of x_i because Babich-Bobenko tori are isothermic (see Proposition 16). Therefore, the point x_i satisfies all assumptions of Proposition 28 and the differential of the mean curvature dH is zero at x_i. Using Equation 12.5, this leads to

$$0 = dH_{x_i} = \langle \tau_{x_i}, b \rangle.$$

Since the immersion f intersects the plane $E = \{b\}^{\perp}$ transversely and the differential $\tau \in K_R(M)$ is conformal, we finally have $\tau_{x_i} = 0$. On a torus holomorphic differentials do not have zeros and hence f cannot be embedded. $\square$

In Theorem 10 we gave a first classification of isothermic constrained Willmore tori as CMC surfaces. Now we want to figure out which of them can be embedded. Due to Alexandrov's theorem [1], totally umbilical spheres are the only closed CMC surfaces in $\mathbb{R}^3$ and $\mathbf{H}^3$. Further we have shown in Proposition 12 that isothermic constrained Willmore tori that are Möbius equivalent to a minimal surface with smooth ends in $\mathbb{R}^3$ cannot be embedded. After we have now proved that the Babich-Bobenko tori can be embedded, we obtain the following classification of embedded isothermic constrained Willmore tori:

Theorem 15 (First classification of embedded, isothermic constrained Willmore tori)**.** *An embedded, isothermic constrained Willmore torus $f : M \to \mathbb{R}^3$ is either a Bryant surface with smooth ends or there exists a stereographic projection that maps it into a unit 3-sphere in $\mathbb{R}^4$, where it has constant mean curvature.*

12.2. **Bryant tori with smooth ends.** For the investigation of the Bryant surfaces, we choose the Poincaré ball model, which we obtain by a conformal change of the euclidean metric of $\mathbb{R}^3$:

$$\langle \cdot, \cdot \rangle \mapsto \widetilde{\langle \cdot, \cdot \rangle} := \tilde{\rho}^{-2} \langle \cdot, \cdot \rangle, \quad \text{where} \quad \tilde{\rho} = \frac{1 - |x|^2}{2}.$$

An immersion $f : M \to \mathbb{R}^3$ is a Bryant surface with smooth ends if it has constant mean curvature $\tilde{H} = 1$ with respect to the metric $\tilde{g} := \rho^{-2} g = (f^* \tilde{\rho})^{-2} f^* \langle \cdot, \cdot \rangle$. These surfaces lie inside $\overline{\mathbf{H}^3} = \{x \in \mathbb{R}^3 | \, |x|^2 \leq 1\}$ and are tangential to the ideal boundary $\partial \mathbf{H}^3 = \mathbb{S}^2$ (see Proposition 26). Points in

$$M_0 := \{p \in M | f(p) \in \mathbb{S}^2\}$$

are called **end-points** of the Bryant surface. Due to Corollary 11, a point $p \in M$ is an umbilical point of f if and only if it is an end-point. If $\Psi : M \to \mathcal{L}$ denotes the euclidean lift of f to the light cone, the map $\tilde{\Psi} := \rho^{-1} \Psi$ lifts the immersion f to the space form $Q_{\mathcal{A}_{hyp}}$ in which it has constant mean curvature $\tilde{H} = 1$ (see Example 5).

Proposition 29.
Let $f : M \to \mathbb{R}^3$ be an embedding of a Bryant torus with smooth ends, $\mathcal{K} : M \to \mathbf{P}^+$ its maximal interior sphere congruence, and $x \in M_0$ an end-point of the Bryant surface, then x is a maximal point of the sphere congruence $\mathcal{K}$.

Proof. Since the Bryant surface $f(M)$ is tangent to the ideal boundary $\mathbb{S}^2$ at x, the same applies for the sphere $\mathcal{K}(x)$. We assume that x is not a maximal point of the MISC $\mathcal{K}$. Then there exists another point $y \in M$ such that $\Psi(x)$ and $\Psi(y)$ lie in the same sphere of the maximal interior sphere congruence, i.e., $\mathcal{K}(x) = \mathcal{K}(y)$. Using Lemma 23 and the fact that x is an umbilical point, we obtain that y is also an umbilical point and hence an end-point of the Bryant surface. This means the spheres $\mathcal{K}(x)$ and $\mathbb{S}^2$ are identical because they share two points $f(x), f(y)$ and the tangent space at $f(x)$. Since we assumed f to be a Bryant torus, the surface $f(M)$ is contained in $\overline{\mathbf{H}^3} = \{x \in \mathbb{R}^3 | \, |x|^2 \leq 1\}$. If one sphere $\mathcal{K}(x)$ of the maximal interior sphere congruence is given by the ideal boundary $\mathbb{S}^2$, this implies that the surface $f(M)$ lies on the 2-sphere $\mathbb{S}^2$. This contradicts the assumption that f is an embedding of a torus, and is hence impossible. Therefore, x has to be a maximal point of the MISC. □

If a Bryant torus is isothermic, there exists a conformal coordinate line parametrization in a neighborhood of the end-points (see Propostion 16), and we can use Proposition 22 to obtain the following corollary.

Corollary 12.
Let $f : M \to \mathbb{R}^3$ be an embedding of an isothermic Bryant torus with smooth ends and $x \in M_0$ an end-point. Then both the differential of the euclidean mean curvature H and the differential of the difference of the principal curvatures $\mu = \frac{\lambda_1 - \lambda_2}{2}$ have a zero at x.

Remark 8. For Babich-Bobenko tori, the zero of dH_x leads to a zero of the differential of the Christoffel dual τ, which we utilized to show that these surfaces cannot be embedded. The same is not true for isothermic Bryant tori because they are tangent to the ideal boundary. This gives us that at a touching point the zero of the function $\rho(p) := \frac{1-|f(p)|^2}{2}$ has at least order two. If we now choose a principal curvature direction X at an end-point $x \in M_0$, we obtain with Lemma 7 that

$$\rho(x)\tau(X) = \omega(X) = \mu(x)df(X).$$

The fact that μ has a zero of order two does no longer imply a zero of the holomorphic differential τ because ρ also has a zero of order at least two. However, if $f : M \to \mathbb{R}^3$ is an embedding of a Bryant torus with smooth ends, we obtain from Proposition 29 that the mean curvature sphere at the end-points is contained in the compact region bounded by $f(M)$. Furthermore, we know that at these points the euclidean mean curvature H is bigger than one and that its differential vanishes if the embedding is isothermic. Since the end-points are umbilical and the mean curvature is bigger than one, they have to be isolated. Further, we conjecture that isothermic Bryant tori with smooth ends cannot be embedded, but do not have a proof so far.

13. Embedded CMC tori

In the last section, we found that every embedded isothermic constrained Willmore torus $f : M \to \mathbb{R}^3$ is either a Bryant surface with smooth ends or there exists a stereographic projection that maps it onto a round 3-sphere where it has constant mean curvature. In both cases there exists a space form $Q_{\mathcal{A}}$ in which the surface has constant mean curvature $\tilde{H}$. In this section, we will show that the curvature of the maximal interior sphere congruence of CMC tori that are entirely contained in $Q_{\mathcal{A}}$ (this excludes Bryant tori with smooth ends) is given by the bigger principal curvature λ_1 of f, i.e.,

$$\mathcal{K} = \mathcal{N} - \lambda_1 \Psi,$$

and that λ_1 is constant along its curvature line. Later in Section 15, we will see that surfaces with this property are canal surfaces.

In 2013 Simon Brendle proofed the Lawson conjecture considering the maximal interior sphere congruence of minimal tori in $\mathbb{S}^3$ [7]. Two years later Ben Andrews and Haizhong Li [2] generalized Brendles result and showed that any CMC torus in $\mathbb{S}^3$ is axially symmetric. We adapt their strategy such that it works in our Möbius-geometric coordinate free setup and deals with all possible space forms at the same time. We think that the Möbius-geometric setup gives a better geometric understanding of the method and makes it possible to apply it in other situations, too. In particular, we use it in the next section to reproduce the well-known result that the only periodic CMC cylinders with annular ends in $\mathbb{R}^3$ are Delaunay surfaces. The famous theorem of Alexander D. Alexandrow [1] proves that there are no embedded CMC tori in $\mathbb{R}^3$ or $\mathbf{H}^3$, in Section 16 we can show this result for tori using the MISC. Furthermore, we plane to apply this method in future work to prove that Bryant surfaces with smooth ends cannot be embedded.

In this section we will always use the following notation and assumptions:
Let $f : M \to \mathbb{R}^3$ be an embedding of a torus, that has constant mean curvature $\tilde{H}$ with respect to the metric $\tilde{g}$ which we obtained by a conformal change of the euclidean metric of $\mathbb{R}^3$, i.e., $\tilde{g} := \rho^{-2}g = \rho^{-2}f^*\langle\cdot,\cdot\rangle$, with $\rho(p) \neq 0$ for all $p \in M$. Further, let $\Psi : M \to \mathcal{L}$ denote the isometric lift of f to the light cone and $\tilde{\Psi} := \rho^{-1}\Psi$ the one to the space form $Q_{\mathcal{A}}$ in which the surface is entirely contained and has constant mean curvature $\tilde{H}$.

In general it is not guaranteed that the maximal interior sphere congruence of an embedded surface is smooth. Therefore, we will first construct a smooth sphere congruence $\mathcal{K} = \mathcal{N} - \Phi\Psi$ that is entirely contained in the compact region bounded by $f(M)$ and then show that the curvature Φ has to be equal to the bigger principal curvature of f.

Let $\hat{M}$ be the compact region bounded by $f(M)$. Because M is compact, we can choose the orientation of f such that the normal's N point outside $\hat{M}$ and the mean curvature $\tilde{H}$ of f in $Q_{\mathcal{A}}$ is positive. With Lemma 21, the sphere congruence is entirely contained in $\hat{M}$ if the function

$$\begin{aligned} Z : M \times M &\to \mathbb{R} \\ (x,y) &\mapsto \langle \mathcal{K}(x), \Psi(y)\rangle, \end{aligned}$$

is nowhere negative. With Corollary 11, f has no umbilical points, i.e., $\mu = \frac{\lambda_1-\lambda_2}{2} > 0$, and we can choose the following ansatz for Φ:

$$\Phi := H + \kappa\mu \qquad \kappa \in \mathbb{R}. \tag{13.1}$$

For a sufficiently large choice of κ, the radii of the spheres become arbitrarily small and Z is a positive function. Now we shrink κ till there exist at least one pair of points $(x_0, y_0) \in M \times M$ with $Z(x_0, y_0) = 0$ and $Z(x, y) \geq 0$ for all $(x, y) \in M \times M$. Because $\mathcal{K}(x)$ contains $\Psi(x_0)$ and $\Psi(y_0)$, we get

$$\mathcal{K}(x_0) = \mathcal{N}(x_0) - \Phi(x_0)\Psi(x_0) = \mathcal{N}(y_0) - \Phi(y_0)\Psi(y_0) = \mathcal{K}(y_0).$$

And hence

$$\begin{aligned} \Phi(x_0) &= \Phi(y_0) \\ \mathcal{N}(x_0) &= \mathcal{N}(y_0) + \Phi(x_0)\underbrace{(\Psi(x_0) - \Psi(y_0))}_{:=R(x_0,y_0)}. \end{aligned} \tag{13.2}$$

The function Z attains its global minimum zero at (x_0, y_0), which is equivalent to

$$\langle \mathcal{N}(x_0), \Psi(y_0)\rangle = \Phi(x_0)\langle \Psi(x_0), \Psi(y_0)\rangle = -\Phi(x_0)\frac{|R(x_0, y_0)|^2}{2}. \tag{13.3}$$

Because Z has a global minimum at (x_0, y_0), the derivatives of Z disappear at this point. For $(0, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M)$ we get

$$0 = \nabla_{(0,Y)} Z|_{(x_0,y_0)} = \langle \mathcal{K}(x_0), d\Psi(Y)\rangle = \langle \mathcal{K}(y_0), d\Psi(Y)\rangle. \tag{13.4}$$

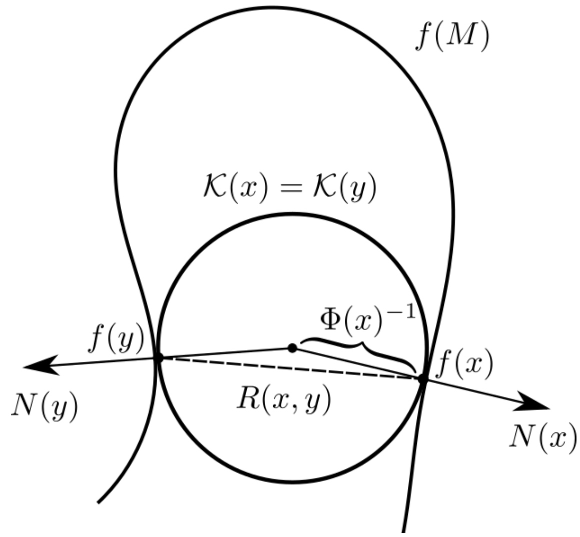


FIG. 11: The maximal interior sphere $\mathcal{K}$ is tangent to the surface $f(M)$ at the points $f(x)$ and $f(y)$.

This means, that $f(M)$ is tangent to $\mathcal{K}(x_0)$ at y_0. Let A denote the shape operator of the immersion f. Then we get for $(X, 0) \in \mathfrak{X}(M) \times \mathfrak{X}(M)$

$$\begin{aligned} 0 = \nabla_{(X,0)} Z|_{(x_0,y_0)} &= \langle d\mathcal{K}(x_0), \Psi(y_0)\rangle \\ &= \langle d\mathcal{N}(X) - \Phi(x_0) d\Psi(X) - d\Phi(X)\Psi(x_0), \Psi(y_0)\rangle \\ &= \langle d\Psi(AX - \Phi(x_0)X) - d\Phi(X)\Psi(x_0), \Psi(y_0)\rangle \\ \Rightarrow \quad & \langle d\Psi(AX - \Phi(x_0)X), \Psi(y_0)\rangle = d\Phi(X)\langle \Psi(x_0), \Psi(y_0)\rangle. \end{aligned} \tag{13.5}$$

If the vector $d\mathcal{K}(X)(x_0, y_0)$ is not zero, it is space-like and defines a sphere it self,

$$d\mathcal{K}(X) = -d\Phi(X) \begin{pmatrix} f(x_0) - \frac{df(AX - \Phi(x_0)X)}{d\Phi(X)} \\ |f(x_0)|^2 - 2\frac{\langle f(x_0), df(AX - \Phi(x_0)X)\rangle}{d\Phi(X)} \\ 1 \end{pmatrix}. \tag{13.6}$$

This sphere contains $\Psi(x_0)$ and $\Psi(y_0)$ and intersects $\mathcal{K}(x_0)$ orthogonal because

$$0 = \langle \mathcal{K}(x_0), d\mathcal{K}(X)\rangle = \langle \Psi(x_0), d\mathcal{K}(X)\rangle = \langle \Psi(y_0), d\mathcal{K}(X)\rangle.$$

In order to compute second derivatives of Z, we need to know the Laplacian of Φ. To this end, we consider a third metric $\hat{g}$ on M, which we define by the real quadratic form

$$\hat{q}(X, X) := |df(X)\tau(X)|, \tag{13.7}$$

where τ denotes the differential of the Christoffel dual of f. The metrics g and $\hat{g}$ are conformally equivalent. Considering the corresponding quadratic forms q and $\hat{q}$, we can compute the conformal factor

$$\hat{q} = |df||\tau| = |df|\frac{|\omega|}{|\rho|} = \frac{|\mu|}{|\rho|}|df|^2 = \frac{\mu}{\rho}q. \tag{13.8}$$

Since the quadratic differential $\hat{q} = df\tau$ is holomorhic, the metric $\hat{g}$ is flat, i.e., $\hat{K} = 0$ (see Example 3). The conformal change between $\hat{g}$ and $\tilde{g}$ is given by

$$\hat{g} = e^{2u}\tilde{g} = \mu\rho\tilde{g} = \tilde{\mu}\tilde{g}. \tag{13.9}$$

Lemma 25 (Gordon equation)**.**
The function $u = \frac{\ln(\mu\rho)}{2}$ *satisfies the following differential equation:*

$$\begin{aligned} d * du &= -\left(\left(\tilde{H}^2 + k\right) e^{-2u} - e^{2u}\right) d\hat{\sigma} \\ &= -\left(\left(\tilde{H}^2 + k\right) \rho^{-2} - \mu^2\right) d\sigma, \end{aligned}$$

where k *denotes the sectional curvature of the space form with metric* $\tilde{g}$*.*

Note that for $\tilde{H} = 0$ and $k = \pm 1$, this is the cosh- resp. sinh-Gordon equation for minimal surfaces in $\mathbf{H}^3$ resp. $\mathbb{S}^3$.

Proof. We consider the conformal change of metric $\tilde{g} \mapsto e^{2u}g = \hat{g}$. Due to Proposition 11, the Gauss curvature of the immersion f changes according to

$$\hat{K} = e^{-2u}(\tilde{K} - \tilde{\triangle}u),$$

where the Laplace operator is defined with respect to the old metric $\tilde{g}$

$$\tilde{\triangle}u d\tilde{\sigma} = -d * du.$$

Using the Gauss equation for an immersion into the space form with metric $\tilde{g}$ and curvature k (see Corollary 6), we further have

$$\tilde{K} d\tilde{\sigma} = \left(\tilde{H}^2 + k - \tilde{\mu}^2\right) d\tilde{\sigma}.$$

We have now all ingredients together to proof the lemma

$$\begin{aligned} -d * du = \tilde{K} d\tilde{\sigma} &= \left(\tilde{H}^2 + k - \tilde{\mu}^2\right) d\tilde{\sigma} \\ &= \left(\left(\tilde{H}^2 + k\right) \tilde{\mu}^{-1} - \tilde{\mu}\right) d\hat{\sigma} \\ &= \left(\left(\tilde{H}^2 + k\right) e^{-2u} - e^{2u}\right) d\hat{\sigma} \\ &= \left(\left(\tilde{H}^2 + k\right) \rho^{-2} - \mu^2\right) d\sigma. \end{aligned}$$

$\square$

Proposition 30.
The Laplacian of $\Phi := H + \kappa\mu$ *is given by*

$$\triangle\Phi = 2\kappa\mu\Big(\frac{\|d\mu\|^2}{\mu^2} - \frac{\|d\rho\|^2}{\rho^2} + H^2 - \mu^2 + \frac{H\langle\mathcal{A},\mathcal{N}\rangle}{\rho} - \frac{\langle\mathcal{A},\mathcal{A}_{euc}\rangle}{\rho}\Big) + \frac{2\mu^2}{\rho}\langle\mathcal{A},\mathcal{N}\rangle.$$

Proof. With $u = \frac{\ln(\mu\rho)}{2}$ we have

$$\begin{aligned} & d * du = \frac{1}{2} d\left(\frac{*d\rho}{\rho} + \frac{*d\mu}{\mu}\right) = \frac{1}{2}\left(\frac{d * d\mu}{\mu} + \frac{*d\mu \wedge d\mu}{\mu^2} + \frac{d * d\rho}{\rho} + \frac{*d\rho \wedge d\rho}{\rho^2}\right) \\ \Rightarrow \quad & d * d\mu = 2\mu\, d * du - \frac{2\|d\mu\|^2}{\mu} - \frac{\mu\, d * d\rho}{\rho} - \frac{2\mu\|d\rho\|^2}{\rho^2}. \end{aligned}$$

In Lemma 14 we computed the Laplacian of the mean curvature H,

$$d * dH = -\frac{2\mu^2}{\rho}\langle \mathcal{A}, \mathcal{N}\rangle d\sigma.$$

The formula for the Laplacian of Ψ that we derived in Lemma 20, gives us

$$d * d\rho = -\langle \mathcal{A}, d * d\Psi\rangle = -2H\langle \mathcal{A}, \mathcal{N}\rangle d\sigma + 2\langle \mathcal{A}, \mathcal{A}_{euc}\rangle d\sigma.$$

This gives rise to

$$\begin{aligned} d * d\Phi &= d * dH + \kappa d * d\mu \\ &= \frac{-2\mu^2}{\rho}\langle \mathcal{A}, \mathcal{N}\rangle d\sigma + \kappa\left(2\mu\, d * du - \frac{2\|d\mu\|^2 d\sigma}{\mu} - \frac{\mu\, d * d\rho}{\rho} - \frac{2\mu\|d\rho\|^2 d\sigma}{\rho^2}\right) \\ &= -2\kappa\mu\left(\frac{\|d\mu\|^2}{\mu^2} + \frac{\|d\rho\|^2}{\rho^2} + \frac{\tilde{H}^2 + k}{\rho^2} - \mu^2 - \frac{H\langle \mathcal{A}, \mathcal{N}\rangle}{\rho} + \frac{\langle \mathcal{A}, \mathcal{A}_{euc}\rangle}{\rho}\right) d\sigma \\ &\quad - \frac{2\mu^2}{\rho}\langle \mathcal{A}, \mathcal{N}\rangle d\sigma. \end{aligned}$$

With Proposition 17 we have $\tilde{H} = H\rho + \langle \mathcal{A}, \mathcal{N}\rangle$, which leads to

$$\frac{\tilde{H}^2}{\rho^2} = H^2 + \frac{2H\langle \mathcal{A}, \mathcal{N}\rangle}{\rho} + \frac{\langle \mathcal{A}, \mathcal{N}\rangle^2}{\rho^2}.$$

Using Equation 10.4, we finally get

$$\begin{aligned} \triangle\Phi &= 2\kappa\mu\Big(\frac{\|d\mu\|^2}{\mu^2} + \frac{\|d\rho\|^2}{\rho^2} + H^2 + \frac{2H\langle \mathcal{A}, \mathcal{N}\rangle}{\rho} + \frac{\langle \mathcal{A}, \mathcal{N}\rangle^2}{\rho^2} + \frac{k}{\rho^2} - \mu^2 - \frac{H\langle \mathcal{A}, \mathcal{N}\rangle}{\rho} \\ &\quad + \frac{\langle \mathcal{A}, \mathcal{A}_{euc}\rangle}{\rho}\Big) + \frac{2\mu^2}{\rho}\langle \mathcal{A}, \mathcal{N}\rangle \\ &= 2\kappa\mu\Big(\frac{\|d\mu\|^2}{\mu^2} - \frac{\|d\rho\|^2}{\rho^2} + H^2 - \mu^2 + \frac{1}{\rho^2}(\|d\rho\|^2 + \mathcal{A}, \mathcal{N}\rangle^2 + k) + \frac{H\langle \mathcal{A}, \mathcal{N}\rangle}{\rho} \\ &\quad + \frac{\langle \mathcal{A}, \mathcal{A}_{euc}\rangle}{\rho}\Big) + \frac{2\mu^2}{\rho}\langle \mathcal{A}, \mathcal{N}\rangle \\ &= 2\kappa\mu\Big(\frac{\|d\mu\|^2}{\mu^2} - \frac{\|d\rho\|^2}{\rho^2} + H^2 - \mu^2 + \frac{H\langle \mathcal{A}, \mathcal{N}\rangle}{\rho} - \frac{\langle \mathcal{A}, \mathcal{A}_{euc}\rangle}{\rho}\Big) + \frac{2\mu^2}{\rho}\langle \mathcal{A}, \mathcal{N}\rangle. \end{aligned}$$

□

For the computation of the second derivative of Z, we choose vector fields $X_1, X_2, Y_1, Y_2 \in \mathcal{X}(M)$ with the following properties: Firstly, X_1, X_2 are orthonormal at x_0 with respect to $\tilde{g}$ and satisfy $(\nabla X_i)|_{x_0} = 0$ and $dN(X_i) = \lambda_i df(X_i)$. Secondly, Y_1, Y_2 are orthonormal and have vanishing derivatives at y_0, i.e., $(\nabla Y_i)|_{y_0} = 0$. We further demand that $d\Psi(Y_i)(y_0)$ is given by $T(d\Psi(X_i))(x_0)$, where T denotes the reflection at the affine hyperplane that contains the center of $\mathcal{K}(x_0)$ and is perpendicular to $R(x_0, y_0)$, i.e :

$$T(d\Psi(X)) = \frac{\rho(y_0)}{\rho(x_0)}\left((d\Psi(X) - 2\langle d\Psi(X), R(x_0, y_0)\rangle \frac{R(x_0, y_0)}{|R(x_0, y_0)|^2}\right) = d\Psi(Y).$$

Using Lemma 20, we can now compute the second derivatives of Z.

Lemma 26.
At the point $(x_0, y_0) \in M \times M$ the second derivatives of Z with respect to X_i, Y_i are given by:

$$
\begin{aligned}
\sum_{i=1}^{2} \nabla_{(0,Y_i)} dZ(0,Y_i)|_{(x_0,y_0)} &= 2\rho^2(y_0)(\Phi(x_0) - H(y_0)), \\
\sum_{i=1}^{2} \nabla_{(0,X_i)} dZ(0,Y_i)|_{(x_0,y_0)} &= 2\rho(x_0)\rho(y_0)(H(x_0) - \Phi(x_0)), \\
\sum_{i=1}^{2} \nabla_{(0,X_i)} dZ(0,X_i)|_{(x_0,y_0)} &= 2\rho(x_0)^2(\Phi(x_0) - H(x_0)) - 2\kappa * d\mu \wedge \langle d\Psi, \Psi(y_0)\rangle(X_1, X_2) \\
&\quad + \Bigg(\frac{-2\kappa\rho(x_0)^2\|d\mu\|^2}{\mu} + 2\kappa\mu\|d\rho\|^2 - 2\kappa\rho(x_0)\mu H(x_0)\langle \mathcal{A}, \mathcal{N}\rangle \\
&\quad + 2\kappa\rho(x_0)\mu\langle \mathcal{A}, \mathcal{A}_{euc}\rangle + 2(\kappa^2 - 1)\rho(x_0)^2 H(x_0)\mu^2(x_0) \\
&\quad - 2\mu^2\rho(x_0)\langle \mathcal{A}, \mathcal{N}\rangle \Bigg) \langle \Psi(x_0), \Psi(y_0)\rangle.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
\sum_{i=1}^{2} \nabla_{(0,Y_i)} dZ(0,Y_i)|_{(x_0,y_0)} &= \langle \mathcal{K}(x_0), \tilde{\triangle}\Psi(y_0)\rangle \\
&= \rho^2(y_0)\langle \mathcal{K}(x_0), \triangle\Psi(y_0)\rangle \\
&= \rho^2(y_0)\langle \mathcal{K}(y_0), -2H(y_0)\mathcal{N}(y_0) + 2\mathcal{A}_{euc}\rangle \\
&= \rho^2(y_0)\langle \mathcal{N}(y_0) - \Phi(y_0)\Psi(y_0), -2H(y_0)\mathcal{N}(y_0) + 2\mathcal{A}_{euc}\rangle \\
&= 2\rho^2(y_0)(\Phi(x_0) - H(y_0)).
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=1}^{2} \nabla_{(0,X_i)} dZ(0,Y_i)|_{(x_0,y_0)} &= \sum_{i=1}^{2} \langle d\mathcal{K}(X_i), d\Psi(Y_i)\rangle \\
&= \frac{\rho(y_0)}{\rho(x_0)} \sum_{i=1}^{2} \langle d\mathcal{K}(X_i), d\Psi(X_i) - 2\langle d\Psi(X_i), R(x_0,y_0)\rangle \frac{R(x_0,y_0)}{|R(x_0,y_0)|^2}\rangle \\
&= \frac{\rho(y_0)}{\rho(x_0)} \sum_{i=1}^{2} \langle d\mathcal{N}(X_i) - \Phi(x_0)d\Psi(X_i) - d\Phi(X_i)\Psi(x_0), d\Psi(X_i)\rangle \\
&= 2\rho(x_0)\rho(y_0)(H(x_0) - \Phi(x_0)).
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=1}^{2} \nabla_{(0,X_i)} dZ(0,X_i)|_{(x_0,y_0)} &= \langle \tilde{\triangle}\mathcal{K}(x_0), \Psi(y_0)\rangle \quad = \rho(x_0)^2\langle \triangle\mathcal{K}(x_0), \Psi(y_0)\rangle \\
&= \rho(x_0)^2\langle \triangle\mathcal{N}(x_0) - \triangle\Phi(x_0)\Psi(y_0) - \Phi(x_0) \triangle \Psi(x_0), \Psi(y_0)\rangle + 2\langle d\Phi \wedge *d\Psi(X_1,X_2), \Psi(y_0)\rangle \\
&= \rho(x_0)^2\langle -2(H^2(x_0) + \mu^2(x_0))\mathcal{N}(x_0) + 2H(x_0)\mathcal{A}_{euc}, \Psi(y_0)\rangle + 2\langle *dH \wedge d\Psi(X_1,X_2), \Psi(y_0)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\rho(x_0)^2\langle 2H(x_0)\Phi(x_0)\mathcal{N}(x_0)-2\rho(x_0)^2\Phi(x_0)\mathcal{A}_{euc},\Psi(y_0)\rangle-\rho(x_0)^2\langle\triangle\Phi(x_0)\,\Psi(x_0),\Psi(y_0)\rangle\\
&-2*d\Phi\wedge\langle d\Psi,\Psi(y_0)\rangle(X_1,X_2)\\
=\;&2\rho(x_0)^2(\Phi(x_0)H(x_0)-H^2(x_0)-\mu^2(x_0))\langle\mathcal{N}(x_0),\Psi(y_0)\rangle-2\kappa*d\mu\wedge\langle d\Psi,\Psi(y_0)\rangle(X_1,X_2)\\
&-2\rho(x_0)^2(\Phi(x_0)-H(x_0))\langle\mathcal{A}_{euc},\Psi(y_0)\rangle-\rho(x_0)^2\triangle\Phi(x_0)\langle\Psi(x_0),\Psi(y_0)\rangle\\
=\;&\rho(x_0)^2\left(-\triangle\Phi(x_0)+2\Phi(x_0)(\Phi(x_0)H(x_0)-H^2(x_0)-\mu^2(x_0))\right)\langle\Psi(x_0),\Psi(y_0)\rangle\\
&-2\kappa*d\mu\wedge\langle d\Psi,\Psi(y_0)\rangle(X_1,X_2)+2\rho(x_0)^2(\Phi(x_0)-H(x_0))\\
=\;&\rho(x_0)^2\left(-\triangle\Phi(x_0)+2\left(\kappa\mu(x_0)H^2(x_0)-\kappa\mu^3(x_0)+(\kappa^2-1)H(x_0)\mu^2(x_0)\right)\right)\langle\Psi(x_0),\Psi(y_0)\rangle\\
&+2\rho(x_0)^2(\Phi(x_0)-H(x_0))-2\kappa*d\mu\wedge\langle d\Psi,\Psi(y_0)\rangle(X_1,X_2)\\
\overset{Prop.30}{=}\;&\rho(x_0)^2\Bigg(-2\kappa\mu\Big(\frac{\|d\mu\|^2}{\mu^2}-\frac{\|d\rho\|^2}{\rho^2}+H^2-\mu^2+\frac{H\langle\mathcal{A},\mathcal{N}\rangle}{\rho}-\frac{\langle\mathcal{A},\mathcal{A}_{euc}\rangle}{\rho}\Big)-\frac{2\mu^2}{\rho}\langle\mathcal{A},\mathcal{N}\rangle\\
&+2\left(\kappa\mu(x_0)H^2(x_0)-\kappa\mu^3(x_0)+(\kappa^2-1)H(x_0)\mu^2(x_0)\right)\Bigg)\langle\Psi(x_0),\Psi(y_0)\rangle\\
&+2\rho(x_0)^2(\Phi(x_0)-H(x_0))-2\kappa*d\mu\wedge\langle d\Psi,\Psi(y_0)\rangle(X_1,X_2)\\
=\;&2\rho(x_0)^2(\Phi(x_0)-H(x_0))+\Bigg(\frac{-2\kappa\rho(x_0)^2\|d\mu\|^2}{\mu}+2\kappa\mu\|d\rho\|^2-2\kappa\rho(x_0)\mu H(x_0)\langle\mathcal{A},\mathcal{N}\rangle\\
&+2\kappa\rho(x_0)\mu\langle\mathcal{A},\mathcal{A}_{euc}\rangle-2\mu^2\rho(x_0)\langle\mathcal{A},\mathcal{N}\rangle+2(\kappa^2-1)\rho(x_0)^2H(x_0)\mu^2(x_0)\Bigg)\langle\Psi(x_0),\Psi(y_0)\rangle\\
&-2\kappa*d\mu\wedge\langle d\Psi,\Psi(y_0)\rangle(X_1,X_2).
\end{aligned}
$$

□

Proposition 31.
With the assumptions above we obtain at $(x_0,y_0)\in M\times M$:

$$
\begin{aligned}
&\sum_{i=1}^{2}\left(d_{(X_i,0)}+d_{(0,Y_i)}\right)^2Z|_{(x_0,y_0)}\\
&=(1-\kappa^2)\Bigg(|R(x_0,y_0)|^2\mu^2\rho(x_0)\tilde{H}+\frac{2\mu}{\kappa|R(x_0,y_0)|^2}\sum_{i=1}^{2}\Big(\langle d\Psi(X_i),\Psi(y_0)+\frac{|R(x_0,y_0)|^2\mathcal{A}}{2\rho(x_0)}\rangle\Big)^2\Bigg).
\end{aligned}
$$

In order to prove the proposition, we will need two lemmas.

Lemma 27.
With the assumptions above we obtain at $(x_0,y_0)\in M\times M$:

$$
\begin{aligned}
&-2\kappa*d\mu\wedge\langle d\Psi,\Psi(y_0)\rangle(X_1,X_2)\\
&=\frac{2\mu(1-\kappa^2)}{|R(x_0,y_0)|^2\kappa}\sum_{i=1}^{2}\langle d\Psi(X_i),\Psi(y_0)\rangle^2+\frac{|R(x_0,y_0)|^2\mu\|d\rho\|^2}{\kappa}\\
&\quad+\frac{2\mu}{\rho(x_0)\kappa}\sum_{i=1}^{2}\langle d\Psi(X_i),\Psi(y_0)\rangle d\rho(X_i)-\frac{|R(x_0,y_0)|^2\kappa\rho(x_0)^2\|d\mu\|^2}{\mu}
\end{aligned}
$$

Proof. Completing the square and using Equation 13.5 and Lemma 14 we get

$$
\begin{aligned}
&-2\kappa * d\mu \wedge \langle d\Psi, \Psi(y_0)\rangle(X_1, X_2)\\
&= \frac{-1}{2|R(x_0,y_0)|^2\kappa\mu}\sum_{i=1}^{2} 2|R(x_0,y_0)|^2\kappa d\mu(X_i)2\kappa\mu\langle d\Psi(X_i),\Psi(y_0)\rangle\\
&= \frac{1}{2|R(x_0,y_0)|^2\kappa\mu}\sum_{i=1}^{2}\big(|R(x_0,y_0)|^2\kappa d\mu(X_i) - 2\kappa\mu\langle d\Psi(X_i),\Psi(y_0)\rangle\big)^2\\
&\quad - \frac{|R(x_0,y_0)|^2}{2\kappa\mu}\sum_{i=1}^{2}\kappa^2 d\mu(X_i)^2 - \frac{2\kappa\mu}{|R(x_0,y_0)|^2}\sum_{i=1}^{2}\langle d\Psi(X_i),\Psi(y_0)\rangle^2\\
&= \frac{1}{2|R(x_0,y_0)|^2\kappa\mu}\sum_{i=1}^{2}\big(|R(x_0,y_0)|^2 d\Phi(X_i) - |R(x_0,y_0)|^2 dH(X_i) - 2\kappa\mu\langle d\Psi(X_i),\Psi(y_0)\rangle\big)^2\\
&\quad - \frac{|R(x_0,y_0)|^2\kappa\rho(x_0)^2\|d\mu\|^2}{\mu} - \frac{2\kappa\mu}{|R(x_0,y_0)|^2}\sum_{i=1}^{2}\langle d\Psi(X_i),\Psi(y_0)\rangle^2\\
&= \frac{1}{2|R(x_0,y_0)|^2\kappa\mu}\sum_{i=1}^{2}\Big(-2\langle d\mathcal{N}(X_i) - \Phi(x_0)d\Psi(X_i),\Psi(y_0)\rangle - |R(x_0,y_0)|^2 dH(X_i)\\
&\quad - 2\kappa\mu\langle d\Psi(X_i),\Psi(y_0)\rangle\Big)^2 - \frac{|R(x_0,y_0)|^2\kappa\rho(x_0)^2\|d\mu\|^2}{\mu}\\
&\quad - \frac{2\kappa\mu}{|R(x_0,y_0)|^2}\sum_{i=1}^{2}\langle d\Psi(X_i),\Psi(y_0)\rangle^2\\
&= \frac{1}{2|R(x_0,y_0)|^2\kappa\mu}\sum_{i=1}^{2}\Big(-2\langle\Omega(X_i),\Psi(y_0)\rangle - |R(x_0,y_0)|^2 dH(X_i)\Big)^2\\
&\quad - \frac{|R(x_0,y_0)|^2\kappa\rho(x_0)^2\|d\mu\|^2}{\mu} - \frac{2\kappa\mu}{|R(x_0,y_0)|^2}\sum_{i=1}^{2}\langle d\Psi(X_i),\Psi(y_0)\rangle^2\\
&= \frac{2\mu}{|R(x_0,y_0)|^2\kappa}\sum_{i=1}^{2}\langle d\Psi(X_i),\Psi(y_0)\rangle^2 + \frac{|R(x_0,y_0)|^2\rho(x_0)^2\|dH\|^2}{\mu\kappa}\\
&\quad + \frac{2}{\kappa\mu}\sum_{i=1}^{2}\langle\Omega(X_i),\Psi(y_0)\rangle dH(X_i) - \frac{|R(x_0,y_0)|^2\kappa\rho(x_0)^2\|d\mu\|^2}{\mu}\\
&\quad - \frac{2\kappa\mu}{|R(x_0,y_0)|^2}\sum_{i=1}^{2}\langle d\Psi(X_i),\Psi(y_0)\rangle^2\\
&= \frac{2\mu(1-\kappa^2)}{|R(x_0,y_0)|^2\kappa}\sum_{i=1}^{2}\langle d\Psi(X_i),\Psi(y_0)\rangle^2 + \frac{|R(x_0,y_0)|^2\mu\|d\rho\|^2}{\kappa}\\
&\quad + \frac{2\mu}{\rho(x_0)\kappa}\sum_{i=1}^{2}\langle d\Psi(X_i),\Psi(y_0)\rangle d\rho(X_i) - \frac{|R(x_0,y_0)|^2\kappa\rho(x_0)^2\|d\mu\|^2}{\mu}
\end{aligned}
$$

□

Lemma 28.
With the assumptions above we obtain at $(x_0, y_0) \in M \times M$*:*

$$\begin{aligned}
&2\rho^2(x_0)(\Phi(x_0) - H(x_0)) + 2\rho^2(y_0)(\Phi(x_0) - H(y_0)) + 4\rho(x_0)\rho(y_0)(H(x_0) - \Phi(x_0)) \\
&= 2\kappa\mu\rho(x_0)\left(\rho(x_0) - \rho(y_0)\right) \\
&= \frac{-2\kappa\mu}{\rho(x_0)} \sum_{i=1}^{2} \langle \mathcal{A}, d\Psi(X_i)\rangle\langle d\Psi(X_i), \Psi(y_0)\rangle + 2\kappa\mu\rho(x)\Phi(x_0)\langle \mathcal{A}, \mathcal{N}(x_0)\rangle\langle\Psi(x_0), \Psi(y_0)\rangle \\
&\quad - 2\kappa\rho(x_0)\mu\langle \mathcal{A}, \mathcal{A}_{euc}\rangle\langle\Psi(x_0), \Psi(y_0)\rangle.
\end{aligned}$$

Proof. With Lemma 23 we get:

$$\begin{aligned}
&2\rho^2(x_0)(\Phi(x_0) - H(x_0)) + 2\rho^2(y_0)(\Phi(x_0) - H(y_0)) + 4\rho(x_0)\rho(y_0)(H(x_0) - \Phi(x_0)) \\
&= 2\Phi(x_0)\left(\rho(x_0) - \rho(y_0)\right)^2 + 2\rho(x_0)H(x_0)\left(\rho(y_0) - \rho(x_0)\right) \\
&\quad + 2\rho(y_0)\left(\rho(x_0)H(x_0) - \rho(y_0)H(y_0)\right) \\
&= 2\Phi(x_0)\left(\rho(x_0) - \rho(y_0)\right)^2 + 2\rho(x_0)H(x_0)\left(\rho(y_0) - \rho(x_0)\right) \\
&\quad - 2\rho(y_0)\Phi(x_0)\left(\rho(y_0) - \rho(x_0)\right) \\
&= 2\Phi(x_0)\big(\rho(x_0)^2 - \rho(x_0)\rho(y_0)\big) + 2\rho(x_0)H(x_0)\left(\rho(y_0) - \rho(x_0)\right) \\
&= 2\kappa\mu\rho(x_0)\left(\rho(x_0) - \rho(y_0)\right)
\end{aligned} \tag{13.10}$$

Since $\tilde{g}_{x_0}(X_i, X_i) = 1$, the vectorfield $\tilde{X}_i := \rho(x)^{-1}X_i$ has length one w.r.t. g at x_0 and we can use Equation 10.4 and 13.2 to obtain a new expression for $\rho(y_0)$.

$$\begin{aligned}
\rho(y_0) &= -\langle \mathcal{A}, \Psi(y_0)\rangle \\
&= -\langle \mathcal{A}, d\Psi(\tilde{X}_1)\rangle\langle d\Psi(\tilde{X}_1), \Psi(y_0)\rangle - \langle \mathcal{A}, d\Psi(\tilde{X}_2)\rangle\langle d\Psi(\tilde{X}_2), \Psi(y_0)\rangle \\
&\quad - \langle \mathcal{A}, \mathcal{N}(x_0)\rangle\langle \mathcal{N}(x_0), \Psi(y_0)\rangle + \langle \mathcal{A}, \Psi(x_0)\rangle\langle \mathcal{A}_{euc}, \Psi(y_0)\rangle + \langle \mathcal{A}, \mathcal{A}_{euc}\rangle\langle\Psi(x_0), \Psi(y_0)\rangle \\
&= -\rho(x_0)^{-2} \sum_{i=1}^{2} \langle \mathcal{A}, d\Psi(X_i)\rangle\langle d\Psi(X_i), \Psi(y_0)\rangle - \Phi(x_o)\langle \mathcal{A}, \mathcal{N}(x_0)\rangle\langle\Psi(x_0), \Psi(y_0)\rangle + \rho(x) \\
&\quad + \langle \mathcal{A}, \mathcal{A}_{euc}\rangle\langle\Psi(x_0), \Psi(y_0)\rangle.
\end{aligned} \tag{13.11}$$

Now we insert (13.11) into (13.10) to finish the proof of the lemma.

$$\begin{aligned}
2\kappa\mu\rho(x_0)\left(\rho(x_0) - \rho(y_0)\right) &= 2\kappa\mu\rho(x_0)\Big(\rho(x_0) - \rho(x_0)^{-2} \sum_{i=1}^{2} \langle \mathcal{A}, d\Psi(X_i)\rangle\langle d\Psi(X_i), \Psi(y_0)\rangle \\
&\quad + \Phi(x_o)\langle \mathcal{A}, \mathcal{N}(x_0)\rangle\langle\Psi(x_0), \Psi(y_0)\rangle - \rho(x) \\
&\quad - \langle \mathcal{A}, \mathcal{A}_{euc}\rangle\langle\Psi(x_0), \Psi(y_0)\rangle\Big) \\
&= \frac{-2\kappa\mu}{\rho(x_0)} \sum_{i=1}^{2} \langle \mathcal{A}, d\Psi(X_i)\rangle\langle d\Psi(X_i), \Psi(y_0)\rangle \\
&\quad + 2\kappa\mu\rho(x)\big(\Phi(x_0)\langle \mathcal{A}, \mathcal{N}(x_0)\rangle - \langle \mathcal{A}, \mathcal{A}_{euc}\rangle\big)\langle\Psi(x_0), \Psi(y_0)\rangle.
\end{aligned}$$

□

With the two Lemmas 27 and 28 we have all the ingredients together for the proof of Proposition 31:

Proof.

$$
\begin{aligned}
&\sum_{i=1}^{2}\left(d_{(X_i,0)}+d_{(0,Y_i)}\right)^2 Z|_{(x_0,y_0)}\\
&=\sum_{i=1}^{2}\nabla_{(0,Y_i)}dZ(0,Y_i)|_{(x_0,y_0)}+2\sum_{i=1}^{2}\nabla_{(0,X_i)}dZ(0,Y_i)|_{(x_0,y_0)}+\sum_{i=1}^{2}\nabla_{(0,X_i)}dZ(0,X_i)|_{(x_0,y_0)}\\
&\overset{Lem.26}{=} 2\rho^2(y_0)(\Phi(x_0)-H(y_0))+4\rho(x_0)\rho(y_0)(H(x_0)-\Phi(x_0))+2\rho(x_0)^2(\Phi(x_0)-H(x_0))\\
&\quad+\Bigg(\frac{-2\kappa\rho(x_0)^2\|d\mu\|^2}{\mu}+2\kappa\mu\|d\rho\|^2-2\kappa\rho(x_0)\mu H(x_0)\langle\mathcal{A},\mathcal{N}\rangle+2\kappa\rho(x_0)\mu\langle\mathcal{A},\mathcal{A}_{euc}\rangle\\
&\quad+2(\kappa^2-1)\rho(x_0)^2H(x_0)\mu^2(x_0)-2\mu^2\rho(x_0)\langle\mathcal{A},\mathcal{N}\rangle\Bigg)\langle\Psi(x_0),\Psi(y_0)\rangle\\
&\quad-2\kappa*d\mu\wedge\langle d\Psi,\Psi(y_0)\rangle(X_1,X_2)\\
&\overset{Lem.27,28}{=}-\frac{2\kappa\mu}{\rho(x_0)}\sum_{i=1}^{2}\langle\mathcal{A},d\Psi(X_i)\rangle\langle d\Psi(X_i),\Psi(y_0)\rangle+2\kappa\mu\rho(x)\Phi(x_0)\langle\mathcal{A},\mathcal{N}(x_0)\rangle\langle\Psi(x_0),\Psi(y_0)\rangle\\
&\quad-2\kappa\rho(x_0)\mu\langle\mathcal{A},\mathcal{A}_{euc}\rangle\langle\Psi(x_0),\Psi(y_0)+\Bigg(\frac{-2\kappa\rho(x_0)^2\|d\mu\|^2}{\mu}+2\kappa\mu\|d\rho\|^2-2\kappa\rho(x_0)\mu H(x_0)\langle\mathcal{A},\mathcal{N}\rangle\\
&\quad+2\kappa\rho(x_0)\mu\langle\mathcal{A},\mathcal{A}_{euc}\rangle+2(\kappa^2-1)\rho(x_0)^2H(x_0)\mu^2(x_0)-2\mu^2\rho(x_0)\langle\mathcal{A},\mathcal{N}\rangle\Bigg)\langle\Psi(x_0),\Psi(y_0)\rangle\\
&\quad+\frac{2\mu(1-\kappa^2)}{|R(x_0,y_0)|^2\kappa}\sum_{i=1}^{2}\langle d\Psi(X_i),\Psi(y_0)\rangle^2+\frac{2\mu}{\rho(x_0)\kappa}\sum_{i=1}^{2}\langle d\Psi(X_i),\Psi(y_0)\rangle d\rho(X_i)\\
&\quad+\frac{|R(x_0,y_0)|^2\mu\|d\rho\|^2}{\kappa}-\frac{|R(x_0,y_0)|^2\kappa\rho(x_0)^2\|d\mu\|^2}{\mu}\\
&=(1-\kappa^2)\Bigg(|R(x_0,y_0)|^2\mu^2\rho(x_0)(H(x_0)\rho(x_0)+\langle\mathcal{A},\mathcal{N}\rangle)+\frac{|R(x_0,y_0)|^2\mu\|d\rho\|^2}{\kappa}\\
&\quad+\frac{2\mu}{\kappa|R(x_0,y_0)|^2}\sum_{i=1}^{2}\langle d\Psi(X_i),\Psi(y_0)\rangle^2+\frac{|R(x_0,y_0)|^2}{\rho(x_0)}\langle d\Psi(X_i),\Psi(y_0)\rangle d\rho(X_i)\Bigg)\\
&=(1-\kappa^2)\Bigg(|R(x_0,y_0)|^2\mu^2\rho(x_0)\tilde{H}+\frac{2\mu}{\kappa|R(x_0,y_0)|^2}\sum_{i=1}^{2}\langle d\Psi(X_i),\Psi(y_0)\rangle^2\\
&\quad+\frac{|R(x_0,y_0)|^2}{\rho(x_0)}\langle d\Psi(X_i),\Psi(y_0)\rangle\langle\mathcal{A},d\Phi(X_i)\rangle+\langle\mathcal{A},d\Phi(X_i)\rangle^2\frac{|R(x_0,y_0)|^4}{4\rho(x_0)^2}\Bigg)\\
&=(1-\kappa^2)\Bigg(\frac{2\mu}{\kappa|R(x_0,y_0)|^2}\sum_{i=1}^{2}\left(\left\langle d\Psi(X_i),\Psi(y_0)+\frac{|R(x_0,y_0)|^2\mathcal{A}}{2\rho(x_0)}\right\rangle\right)^2\\
&\quad+|R(x_0,y_0)|^2\mu^2\rho(x_0)\tilde{H}\Bigg).
\end{aligned}
$$

□

Theorem 16.
Let $f : M \to \mathbb{R}^3$ be an embedding of a torus and $Q_{\mathcal{A}} \subset \mathcal{L}$ a space form, such that the lift of f to $Q_{\mathcal{A}} \subset \mathcal{L}$ has constant mean curvature $\tilde{H}$. Then the sphere congruence

$$\mathcal{K} := \mathcal{N} - \lambda_1 \Psi,$$

where λ_1 denotes the bigger principal curvature of f, lies entirely in the compact region bounded by $f(M)$, i.e., the function

$$\begin{aligned} Z : M \times M &\to \mathbb{R} \\ (x, y) &\mapsto \langle \mathcal{K}(x), \Psi(y) \rangle \end{aligned}$$

is nowhere negative. Furthermore, λ_1 is constant along its curvature line.

Proof. We define the sphere congruence $\mathcal{K}_\kappa := \mathcal{N} - (H + \kappa\mu)\Psi$. Because f has no umbilical points, we have $\mu > 0$ and for big enough κ the radii of $\mathcal{K}$ get arbitrarily small. Due to the fact that M is compact, there exists a minimal κ such that $\mathcal{K}_\kappa$ lies entirely in the compact region bounded by $f(M)$, and there is at least one point pair $(x_0, y_0) \in M \times M$ with $\Psi(y_0) \in \mathcal{K}_\kappa(x_0)$, i.e., Z is nowhere negative and obtains its global minimum zero at (x_0, y_0). For $\kappa = 1$ the curvature of the sphere congruence is given by the bigger principal curvature λ_1 of f, hence we obtain with Lemma 22 that κ cannot be smaller than one.

Now consider the geodesic $\gamma : (-\epsilon, \epsilon) \to M \times M$ with

$$\begin{aligned} \gamma(0) &= (x_0, y_0), \\ \gamma'(0) &= \sum_{i=1}^{2} (X_i, Y_i). \end{aligned}$$

The function $Z(t) = Z(\gamma(t))$ has a global minimum at $t = 0$ and indeed Equation 13.5 and 13.4 guarantee that $Z'(t)$ has a zero at $t = 0$.

With Proposition 31, the second derivative of Z at $t = 0$ is given by

$$\begin{aligned} Z(\gamma(t))''\big|_{t=0} &= \sum_{i=1}^{2} \left(d_{(X_i,0)} + d_{(0,Y_i)} \right)^2 Z|_{(x_0,y_0)} \\ &= (1 - \kappa^2) \Bigg(\frac{2\mu}{\kappa |R(x_0, y_0)|^2} \sum_{i=1}^{2} \left(\left\langle d\Psi(X_i), \Psi(y_0) + \frac{|R(x_0, y_0)|^2 \mathcal{A}}{2\rho(x_0)} \right\rangle \right)^2 \\ &\quad + |R(x_0, y_0)|^2 \mu^2 \rho(x_0) \tilde{H} \Bigg). \end{aligned}$$

For $\kappa > 1$, $Z''(0)$ is negative and cannot have a minimum at $t = 0$, hence we obtain $\kappa = 1$ and the sphere congruence $\mathcal{K} := \mathcal{N} - \lambda_1 \Psi$ lies entirely in the compact region bounded by $f(M)$. Due to Lemma 22, this guarantees that $\mathcal{K}$ is the maximal interior sphere congruence of f. Since every point $p \in M$ is a maximal point of $\mathcal{K}$, we can use Proposition 28 to prove that λ_1 is constant along its curvature line. □

14. Periodic CMC cylinders with annular ends

Already in 1841, Ch. Delaunay [11] constructed a class of periodic surfaces of revolution that have constant mean curvature in $\mathbb{R}^3$. In his honor these surfaces are nowadays called **Delaunay surfaces**. In 1989, Nicholas J. Korevaar, Rob Kusner and Bruce Solomon proved in their famous paper [21] that every complete and properly embedded CMC surface in $\mathbb{R}^3$ with two ends is a Delaunay surface. In their proof they use Alexandrows reflection method. We will reproduce their result for a class of cylinders, called **periodic CMC cylinders with annular ends**, by using the strategy that we developed for embedded isothermic constrained Willmore tori. To this end, we will recap which properties of a torus were necessary for the proof of Theorem 16, and then consider a class of cylinders that satisfy the same assumptions.

For the construction of the maximal interior sphere congruence it was crucial that the surface separates the ambient space into two connected components. One class of surfaces with this property is given by properly embedded surfaces with two annular ends.

Definition 31.
*Let $M, \hat{M}$ be manifolds with boundary. An embedding $f : M \to \hat{M}$ is **proper** if*

(1) $f(\partial M) = f(M) \cap \partial(\hat{M})$.
(2) *$f(M)$ is transverse to $\partial(\hat{M})$ for all points in $f(\partial M)$.*

*A properly embedded surface $f : M \to \mathbb{R}^3$ is called of **finite type**, if it is homeomorphic to a closed surface with a finite number of closed submanifolds removed. The neighborhoods of the removed submanifolds are called the ends of the surface. An **annular end** $A \subset f(M)$ is a properly embedded subset homeomorphic to the punctured unit disk $D^\circ := \{x \in \mathbb{R}^2 |\, 0 < |x| \leq 1\}$. For an annular end A of $f(M)$ the homeomorphism $\phi : A \to D^\circ$ satisfies $\lim\limits_{x\to 0} \phi(x) \to \infty$.*

We are especially interested in **properly embedded cylinders**. They are homeomorphic to a sphere with two points removed and hence have two annular ends. These surfaces separate the ambient space into two connected components, but they are not compact. Therefore, it is not clear that we can construct the maximal interior sphere congruence (MISC) in the same way as we did it for closed surfaces. By a theorem of William Hamilton Meeks [27], the annular ends of a properly embedded surface of finite type are cylindrically bounded if the surface has constant mean curvature in $\mathbb{R}^3$. This should guarantee the existence of the MISC for properly embedded CMC cylinders. However, we will consider periodic cylinders for which it is much simpler to define the MISC.

Definition 32.
*An immersion $f : M \to \mathbb{R}^3$ of a surface is called **periodic** if there exists $v \in \mathbb{R}^3$ such that $f(M)$ is invariant under translations of the lattice $\Lambda := \{av \in \mathbb{R}^3 |\, a \in \mathbb{Z}\}$.*

Let E denote the hyperplane with normal $\tilde{v} := \frac{v}{|v|}$, $T := |v|$ and

$$E_t := \{x \in \mathbb{R}^3 |\ (x - t\tilde{v}) \in E\},$$

the corresponding 1-parameter family of parallel affine hyperplanes. An immersion f is periodic with respect to the lattice $\Lambda := \{av \in \mathbb{R}^3 | \, a \in \mathbb{Z}\}$ if and only if

$$f(M) \cap E_t = (f(M) \cap E_{t+T}) - Tv \qquad \forall t \in \mathbb{R},$$

where the sum of sets $U, V \subset \mathbb{R}^3$ is defined as

$$U + V := \{u + v \in \mathbb{R}^3 | \, u \in U, v \in V\}.$$

For properly embedded periodic surfaces $f : M \to \mathbb{R}^3$, the quotient $f(M)/\Lambda$ is a closed surface in $\mathbb{R}^3/\Lambda$. After we have identified $\mathbb{R}^3/\Lambda$ with the homeomorphic equivalent space $\mathbb{S}^1 \times \mathbb{R}^2$, there exists a closed manifold $\tilde{M}$ and an embedding $\tilde{f} : \tilde{M} \to \mathbb{S}^1 \times \mathbb{R}^2$, such that $f(M)/\Lambda \cong \tilde{f}(\tilde{M})$. In particular, any properly embedded periodic cylinder with annular ends $f : M \to \mathbb{R}^3$ corresponds to an embedding of a torus $\tilde{f} : \tilde{M} \to \mathbb{S}^1 \times \mathbb{R}^2$.

For the closed surface $\tilde{f}(M)$, the maximal interior sphere congruence is well defined and continuous (see Proposition 27). If we equip $\mathbb{S}^1 \times \mathbb{R}^2$ with the metric that it inherits as quotient space from $\mathbb{R}^3$, we can use the MISC of $\tilde{f}$ and construct one for f by periodic continuation. This proves the following lemma.

Lemma 29.
Properly embedded periodic cylinders $f : M \to \mathbb{R}^3$ admit a continuous and periodic maximal sphere congruence.

Another property of tori that we used for the proof of Theorem 16 is that holomorphic differentials of a torus have no zeros. Due to this fact, the differential of the Christoffel dual $\tau \in K_L(M)$ of an isothermic torus never vanishes. This was crucial for the derivation of the Gordon equation (see Proposition 30) and further implies that CMC tori in $\mathbb{R}^3, \mathbb{S}^3$, or $\mathbf{H}^3$ have no umbilical points (see Corollary 11). Since properly embedded periodic cylinders correspond to embedded tori in $\mathbb{S}^1 \times \mathbb{R}^2$, their holomorphic differentials have no zeros either and we can use the same argumentation for these surfaces. Therefore, we can extend Theorem 16 such that it stays true for properly embedded periodic CMC cylinders in $\mathbb{R}^3$.

Theorem 17.
The bigger principal curvature of a properly embedded periodic CMC cylinder is constant along its corresponding curvature line.

15. Canal surfaces

In this section we will investigate canal surfaces. We start with a local criterion for surfaces in $\mathbb{R}^3$ and show that they are canal surfaces if and only if their bigger principal curvature is constant along its curvature line. With Theorem 16, this ensures that all embedded constrained Willmore tori are canal surfaces or Bryant surfaces with smooth ends. Then, we will come back to the projective model of Möbius geometry and present a new proof for the known fact that isothermic canal surfaces in $\mathbb{S}^3$ are Möbius equivalent to a surface of revolution.

Definition 33.
The envelope of a smooth 1*-parameter family of spheres,*

$$S_t := \{x \in \mathbb{R}^3 |\, \|x - m(t)\|^2 - r(t)^2 = 0\}, \tag{15.1}$$

is called a ***canal surface*** *if there exists a an immersion* $f : (a, b) \times \mathbb{S}^1 \to \mathbb{R}^3$ *such that*

$$f(t, \mathbb{S}^1) = C_t := S_t \cap S_t' \qquad \forall t \in (a, b),$$

where S_t' *denotes the derivative of* S_t *w.r.t. the parameter* t*, i.e.,*

$$S_t' := \{x \in \mathbb{R}^3 |\, 0 = -2\langle x - m(t), m'(t)\rangle - 2r'(t)r(t)\}. \tag{15.2}$$

The curve of centers $m : (a, b) \to \mathbb{R}^3$ *is called the* ***directrix*** *and the* C_t*'s are the* ***characteristic circles*** *of the canal surface.*

Some authors give a more general definition of canal surfaces (compare for example [17]). They do not demand that the sphere intersects its envelope in a whole circle and call surfaces with this property **elliptic canal surfaces**. Given a curve $m : (a, b) \to \mathbb{R}^3$ and a positive function $r : (a, b) \to \mathbb{R}^+$, the envelope of the 1-parameter family of spheres $S_t := \{x \in \mathbb{R}^3 |\, \|x - m(t)\|^2 - r(t)^2 = 0\}$ is not always a canal surface. A necessary condition is that S_t' defines a family of planes which is the case if and only if the curve m is regular. Further, these planes have to intersect the spheres S_t transversely. A short calculation shows that this condition is fulfilled if and only if $|r'(t)| < \|m'(t)\|$.

The canal surface can be parameterized as follows (see [14]):

$$f(s, t) := m(t) - \frac{r(t)r'(t)}{\|m'(t)\|^2} m'(t) + r(t)\frac{\sqrt{\|m'(t)\|^2 - r'(t)^2}}{\|m'(t)\|} \left(e_1(t)\cos(s) + e_2(t)\sin(s)\right), \tag{15.3}$$

where $e_1(t), e_2(t)$ and $\frac{m'(t)}{\|m'(t)\|}$ are an orthonormal basis of $\mathbb{R}^3$ for all t.

Proposition 32.
An immersion $f : (a, b) \times (c, d) \to \mathbb{R}^3$ *is part of a canal surface if and only if one of the principal curvatures is constant and non zero along the corresponding curvature line. These curvature lines are the characteristic circles of the canal surface.*

Proof. " $\Rightarrow$ " Let $f : (a, b) \times (c, d) \to \mathbb{R}^3$ be a part of a canal surface, $m : (a, b) \to \mathbb{R}^3$ its directrix and $r : (a, b) \to \mathbb{R}^+$ the function of radii. Since the immersion $f(t, s)$ is contained in the intersection of the sphere $S_t := \{x \in \mathbb{R}^3 |\, \|x - m(t)\|^2 - r(t)^2 = 0\}$ with the plane $S_t' := \{x \in \mathbb{R}^3 |\, 0 = -2\langle x - m(t), m'(t)\rangle - 2r'(t)r(t)\}$, it has to satisfy two equations:

$$0 = \|f(t, s) - m(t)\|^2 - r(t)^2, \tag{15.4}$$

$$0 = 2\langle f(t, s) - m(t), m'(t)\rangle + 2r'(t)r(t). \tag{15.5}$$

Taking the derivative of Equation 15.4 w.r.t. both parameters s and t, and using (15.5), we obtain

$$0 = \langle f_s(t, s), f(t, s) - m(t)\rangle,$$
$$0 = 2\langle f_t(t, s) - m'(t), f(t, s) - m(t)\rangle - 2r'(t)r(t) = \langle f_t(t, s), f(t, s) - m(t)\rangle.$$

This means, $f(t,s) - m(t)$ is orthogonal to $f_t(t,s)$ as well as $f_s(t,s)$ and hence a scalar multiple of the unit normal field $N(t,s)$. Equation 15.4 gives us

$$f(t,s) - m(t) = r(t)N(t,s).$$

Taking the derivative with respect to s leads to

$$f_s(s,t) = r(t)N_s(t,s),$$

i.e., $\lambda(t,s) = \frac{1}{r(t)}$ is a principal curvature of f with $\lambda_s(t,s) = 0$.

" $\Leftarrow$ " Let $f : (a,b) \times (c,d) \to \mathbb{R}^3$ be an immersion with unit normal vector field N and

$$\lambda(t)f_s(s,t) = N_s(s,t) \neq 0, \tag{15.6}$$

i.e., λ is a never vanishing principal curvature that is constant along the corresponding curvature line. We define

$$m(t,s) := f(t,s) - \frac{N(t,s)}{\lambda(t)} \tag{15.7}$$

and obtain

$$m_s(t,s) = f_s - \frac{N_s(t,s)}{\lambda(t)} = 0. \tag{15.8}$$

Now we consider the distance of f to m for a given t

$$\|f(t,s) - m(t)\| = \| - \frac{N(t,s)}{\lambda(t)}\| = \frac{1}{|\lambda(t)|} =: r(t). \tag{15.9}$$

This gives us the first equation for f being a canal surface

$$0 = \|f(t,s) - m(t)\|^2 - r(t)^2. \tag{15.10}$$

The second one we obtain by taking the derivative with respect to t:

$$\begin{aligned} 0 &= 2\langle f_t(t,s) - m'(t), f(t,s) - m(t)\rangle - 2r'(t)r(t) \\ &= -2\langle m'(t), f(t,s) - m(t)\rangle - 2r'(t)r(t) + \underbrace{2\langle f_t(t,s), f(t,s) - m(t)\rangle}_{=0}. \end{aligned}$$

□

The Theorems 16 and 17 give rise to the following corollary.

Corollary 13.
Every embedding $f : M \to \mathbb{R}^3$ of a torus that can be lifted to a space form $Q_{\mathcal{A}}$ in which it has constant mean curvature, and every properly embedded periodic CMC cylinder in $\mathbb{R}^3$ are canal surfaces.

15.1. **Canal surfaces in the projective model of Möbius geometry.** For the following investigation of canal surfaces, we use again the projective model of Möbius geometry described in Section 9 and homogeneous coordinates with respect to the basis $\{E_1, \ldots, E_5\}$ such that the Minkowski product has the form:

$$\langle P, Q\rangle = P_1Q_1 + P_2Q_2 + P_3Q_3 + P_4Q_4 - P_5Q_5.$$

Remember that the set of spheres in $\mathbb{P}(\mathcal{L}) \cong \mathbb{S}^3$ is given by:

$$\mathbf{P}^4_+ := \mathcal{H}^4_{+\backslash\pm 1} = \{[P] \in \mathbf{P}^4 |\, \langle P, P\rangle > 0\}.$$

We adapt the Definition 33 of canal surfaces to the projective setup.

Definition 34.
*Let $S : (a,b) \to \mathcal{H}^4_+$ be a regular curve. The envelope of the corresponding family of spheres $\Sigma_t = \mathbb{P}(\mathcal{L}) \cap \{S(t)\}^\perp$ is called a **canal surface** if there exists an immersion $\Psi : (a,b) \times \mathbb{S}^1 \to \mathcal{L}$ that satisfies for all t:*

$$[\Psi(t, \mathbb{S}^1)] = C_t := \mathcal{L} \cap \{S(t), S'(t)\}^\perp. \tag{15.11}$$

*The C_t are called the **characteristic circles** of the canal surface.*

Note that this definition of canal surfaces is consistent with the one that we gave for canal surfaces in $\mathbb{R}^3$. In particular, we can change the lift Ψ to obtain a canal surface in an arbitrary space form $Q_\mathcal{A} \subset \mathcal{L}$ that envelopes the family of spheres given by S.

Lemma 30 (Bohle and Pinkall [29])**.**
The envelope Ψ of a a regular curve $S : (a,b) \to \mathcal{H}^4_+$ is a canal surface if and only if the following two conditions are satisfied for all $t \in (a,b)$:

(1) $G_t := \operatorname{span}\{S(t), S'(t)\}$ defines a projective line that does not intersect $\mathbb{S}^3$
(2) $E_t := \operatorname{span}\{S(t), S'(t), S''(t)\}$ defines a projective plane that intersects $\mathbb{S}^3$ transversely.

*The planes E_t are called the **osculating planes** of the curve $S(t)$.*

Proof. At first, we prove that the conditions are necessary. Therefore, we assume that the envelope of $S(t)$ is a canal surface. Since $S(t)$ is a regular curve, $S(t)$ and $S'(t)$ are linearly independent for all t, and hence $G_t := \operatorname{span}\{S(t), S'(t)\}$ defines a projective line. If that line intersects $\mathbb{S}^3$, the circle $C_t = \mathbb{S}^3 \cap \{S(t), S'(t)\}^\perp$ is either empty or contains only two points. In both cases Ψ cannot be an immersion.

If there exist $t \in (a,b)$ such that $E(t)$ is not a projective plane that intersects $\mathbb{S}^3$ transversely, then there exists $s_0 \in \mathbb{S}^1$ such that $\Psi(t, s_0) \in \{S''(t)\}^\perp$. This gives rise to

$$0 = \frac{\partial}{\partial t}\langle \Psi(t,s_0), S(t)\rangle = \langle \Psi_t(t,s_0), S(t)\rangle + \underbrace{\langle \Psi(t,s_0), S'(t)\rangle}_{=0},$$

$$0 = \frac{\partial}{\partial t}\langle \Psi(t,s_0), S'(t)\rangle = \langle \Psi_t(t,s_0), S'(t)\rangle + \underbrace{\langle \Psi(t,s_0), S''(t)\rangle}_{=0},$$

i.e., $\Psi_t(t,s_0) \in \{S(t), S'(t)\}^\perp$. Therefore, $\Psi_t(t,s_0)$ would be tangential to the generating circle C_t and Ψ cannot be an immersion.

We prove now that the conditions are sufficient. The first condition implies that $S(t)$ is a space-like curve in $\mathcal{H}^4_+$, i.e., $|S'(t)| > 0$ for all $t \in (a,b)$. From the second condition we obtain that the restriction of the Minkowski product to the projective planes E_t has signature $(1,1,-1)$. We construct a frame for the curve $S : (a,b) \to \mathcal{H}^4_+$, representing the

family of spheres. Using the Gram Schmidt method, we transform $\{S(t), S'(t), S''(t)\}$ into an ONB $\{S(t), T(t), N(t)\}$ of E_t, with $|S(t)|^2 = |T(t)|^2 = 1$ and $|N(t)|^2 = -1$.
The frame field (S, T, N) satisfies the following system of ODE's:

$$\begin{aligned} S' &= \kappa\mu T, \\ T' &= -\kappa\mu S + \kappa N, \\ N' &= -\kappa T + Q, \end{aligned}$$

for some smooth functions $\kappa, \mu : (a, b) \to \mathbb{R}^+$ and $Q : (a, b) \to \mathbf{P}^4$ with $Q(t) \in E_t^\perp$. Note that the restriction of the Minkowski product to the projective line $L_t := E_t^\perp$ is euclidean for all t. We extend the frame field (S, T, N) to an ONB of $\mathbb{R}^{4,1}$ by choosing $M_1, M_2 : (a, b) \to L_t$ such that $M_1(t), M_2(t)$ is an ONB of L_t and $M_1'(t), M_2'(t) \in E_t$. The derivatives of (S, N, T, M_1, M_2) are now given by

$$\begin{aligned} S' &= \kappa\mu T, \\ T' &= -\kappa\mu S + \kappa N, \\ N' &= -\kappa T + aM_1 + bM_2, \\ M_1' &= -aN, \\ M_2' &= -bN. \end{aligned} \tag{15.12}$$

Since $Q(t)$ is contained in the projective line L_t, we obtain for the functions a and b

$$Q(t) = \langle Q, M_1\rangle M_1(t) + \langle Q, M_2\rangle M_2(t) = a(t)M_1(t) + b(t)M_2(t).$$

Using M_1 and M_2, we can parameterize the projective lines L_t by

$$\begin{aligned} P : (a, b) \times \mathbb{R}^2 &\to \mathbf{P}^4 \\ P(t, (x, y)) &:= xM_1(t) + yM_2(t). \end{aligned}$$

For every $t_0 \in (a, b)$, P_{t_0} defines an isometry between L_{t_0} and $\mathbb{R}^2$, and there exists functions $\alpha, \beta : \mathbb{S}^1 \to \mathbb{R}$ such that the restriction of P onto $(a, b) \times \mathbb{S}^1$ has the form:

$$P(t, s) = \alpha(s)M_1(t) + \beta(s)M_2(t).$$

Now we can define the immersion Ψ that envelops S and satisfies (15.11) as follows:

$$\Psi(t, s) := N(t) + \alpha(s)M_1(t) + \beta(s)M_2(t). \tag{15.13}$$

□

Remark 9. With the frame field defined in (15.12), the generating circles of the canal surface Ψ are now given by

$$C_t = \mathbb{S}^3 \cap \{S(t), T(t)\}^\perp.$$

For a fixed $t \in (a, b)$, any choice of lift $\Psi : (a, b) \times \mathbb{S}^1 \to \mathcal{L}$ of the canal surface is contained in the projective plane $\operatorname{span}\{N(t), M_1(t), M_2(t)\}$. In particular, there exist functions $\alpha, \beta, \gamma : (a, b) \times \mathbb{S}^1 \to \mathbb{R}$ such that

$$\Psi(t, s) = \gamma(t, s)N(t) + \alpha(t, s)M_1(t) + \beta(t, s)M_2(t). \tag{15.14}$$

Definition 35.
Let $\Psi : (a,b) \times \mathbb{S}^1 \to \mathcal{L}$ be a canal surface and $S : (a,b) \to \mathcal{H}^4_+$ the corresponding one parameter family of spheres. The frame field (S, T, N, M_1, M_2) defined in (15.12) is called the ***canonical frame field*** *adapted to S. The representation of Ψ with respect to (N, M_1, M_2) that we defined in (15.14),*

$$\Psi(t,s) = \gamma(t,s)N(t) + \alpha(t,s)M_1(t) + \beta(t,s)M_2(t),$$

will be called ***canonical representation*** *of the immersion Ψ.*

In the next lemma we will see that the envelope of $S : (a,b) \to \mathcal{H}^4_+$ has a special form if $S(t)$ defines a planar curve. This is the case if and only E_t defines the same projective plane for all t, i.e., $Q(t) \equiv 0$.

Lemma 31.
Let $\Psi : (a,b) \times \mathbb{S}^1 \to \mathcal{L}$ be a canal surface and $S : (a,b) \to \mathcal{H}^4_+$ the corresponding one parameter family of spheres. If S is a planar curve in $\mathcal{H}^4_+$, i.e., E_t defines the same projective plane for all $t \in (a,b)$, then Ψ is Möbius equivalent to a surface of revolution in $\mathbb{S}^3$.

Proof. If E_t is a constant plane, the intersection $C := E_t \cap \mathbb{S}^3$ is a fixed circle in $\mathbb{S}^3$. After applying a suitable Möbius transformation, we may assume that E_t contains the center of $\mathbb{S}^3$ and C is a great circle. The centers of the spheres Σ_t now all lie on C and hence Σ_t and C intersect orthogonally. The envelope Ψ of the one parameter family of spheres Σ_t becomes a surface of revolution. $\square$

Lemma 32.
Let $\Psi : (a,b) \times \mathbb{S}^1 \to \mathcal{L}$ be a canal surface, $S : (a,b) \to \mathcal{H}^4_+$ the corresponding one parameter family of spheres and $t_0 \in (a,b)$. Then there exists a smooth family of Möbius transformations $A : (a,b) \to O(4,1)$ such that we can roll the canal surface Ψ isometrically onto the sphere $\Sigma_{t_0} := \{S(t_0)\}^\perp \cap \mathcal{L}$:

(1) $\tilde{\Psi}(t,s) := A(t)^{-1}\Psi(t,s) \in \Sigma_{t_0}, \qquad \forall (t,s) \in (a,b) \times \mathbb{S}^1.$

(2) $\langle d\Psi, d\Psi \rangle = \langle d\tilde{\Psi}, d\tilde{\Psi} \rangle.$

Proof. We define $X : (a,b) \to \mathfrak{so}(4,1)$ by

$$X(t)P := \langle P, S(t) \rangle S'(t) - \langle P, S'(t) \rangle S(t), \qquad \forall t \in (a,b) \text{ and } P \in \mathbb{R}^{4,1} \tag{15.15}$$

and $A : (a,b) \to O(4,1)$ as the unique solution of the initial value problem

$$\begin{aligned} A'(t) &= X(t)A(t), \\ A(t_0) &= \mathbb{1}. \end{aligned} \tag{15.16}$$

We will show that $A(t)$ maps $S(t_0)$ onto $S(t)$. To this end, we will consider the function $P(t) := A(t)S(t_0)$ and show that $P(t)$ and $S(t)$ are equal because they solve the same initial value problem. In particular, $P(t)$ is the unique solution of the initial value problem:

$$\begin{aligned} P'(t) &= A'(t)S(t_0) = X(t)P(t), \\ P(t_0) &= S(t_0). \end{aligned}$$

The curve S solves the same initial value problem because

$$X(t)S(t) = \langle S(t), S(t)\rangle S'(t) - \langle S(t), S'(t)\rangle S(t) = S'(t).$$

The uniqueness of the solution gives us: $A(t)S(t_0) = S(t)$. For a fixed $t \in (a, b)$, we have $\Psi(t, s) \in \Sigma_t = \mathcal{L} \cap S(t)^\perp$. The Möbius transformations $A^{-1}(t)$ preserve the Minkowski product. This gives rise to

$$\begin{aligned} &\langle \Psi(t,s), S(t)\rangle = 0 = \langle A^{-1}(t)\Psi(t,s), A^{-1}(t)S(t)\rangle = \langle A^{-1}(t)\Psi(t,s), S(t_0)\rangle \\ \Rightarrow \qquad &A^{-1}\Psi(t,s) \in \mathcal{L} \cap S(t_0)^\perp = \Sigma_{t_0}. \end{aligned}$$

Now we compute the derivatives of $\Psi(s,t) = A(t)\tilde{\Psi}(t,s)$ using (15.15) and (15.11):

$$\begin{aligned} \Psi_s(t,s) &= A(t)\tilde{\Psi}_s(t,s), \\ \Psi_t(t,s) &= X(t)A(t)\tilde{\Psi}(t,s) + A(t)\tilde{\Psi}_t(t,s) \\ &= \underbrace{\langle \Psi(s,t), S(t)\rangle}_{=0} S'(t) - \underbrace{\langle \Psi(s,t), S'(t)\rangle}_{=0} S(t) + A(t)\tilde{\Psi}_t(t,s) \\ &= A(t)\tilde{\Psi}_t(t,s). \end{aligned}$$

Since $A(t) \in O(4,1)$, we obtain $\langle d\Psi, d\Psi\rangle = \langle d\tilde{\Psi}, d\tilde{\Psi}\rangle$. $\square$

Note that the Möbius transformation $A^{-1}(t)$ maps the sphere Σ_t onto Σ_{t_0} and the generating circles C_t onto circles $\tilde{C}_t \subset \Sigma_{t_0}$. With $R(t) := A^{-1}(t)T(t)$ the new circles are given by

$$\tilde{C}_t := A^{-1}(t)C_t = \mathcal{L} \cap \{S(t_0), R(t)\}^\perp. \tag{15.17}$$

Using the frame equations for the curve S (see 15.12), the derivative of R is given by:

$$\begin{aligned} R'(t) &= (A^{-1})'T(t) + A^{-1}(t)T'(t) \\ &= -A^{-1}(t)A'(t)A^{-1}(t)T(t) + A^{-1}(t)\left(-\mu\kappa S(t) + \kappa N(t)\right) \\ &= -A^{-1}(t)X(t)T(t) + A^{-1}(t)\left(-\kappa\mu S(t) + \kappa N(t)\right) \\ &= A^{-1}(t)\left(\kappa\mu S(t) - \kappa\mu S(t) + \kappa N(t)\right) \\ &= \kappa A^{-1}(t)N(t). \end{aligned} \tag{15.18}$$

In particular, we have $\langle R(t), R'(t)\rangle = 0$, which we will need in the proof of the next lemma.

Lemma 33.
Let $\Psi : (a,b) \times \mathbb{S}^1 \to \mathcal{L}$ be an isothermic canal surface, $S : (a,b) \to \mathcal{H}^4_+$ the corresponding one parameter family of spheres and $t_0 \in (a,b)$. Then there exist $D : (a,b) \to O(4,1)$ and locally $L : V \subset \mathbb{S}^1 \to C_0 := \mathcal{L} \cap \{S(t_0), S'(t_0)\}^\perp$ such that

$$\begin{aligned} \Psi : U \subset (a,b) \times \mathbb{S}^1 &\to \mathcal{L} \\ \Psi(t,s) &:= D(t)L(s), \end{aligned} \tag{15.19}$$

defines a conformal coordinate line parametrisation of the canal surface. Furthermore, the derivatives of Ψ are given by

$$\Psi_t(t,s) = -\kappa(t)\langle \Psi(s,t), N(t)\rangle T(t), \tag{15.20}$$

$$\Psi_s(t,s) = D(t)L'(s). \tag{15.21}$$

Proof. In Lemma 32, we defined a family of Möbius transformations $A : (a, b) \to O(4, 1)$ such that $A(t)$ maps the sphere Σ_0 onto Σ_t. Now, we are looking for a family of Möbius transformations that preserves the sphere Σ_0 and maps the circle $C_0 = \mathcal{L} \cap \{S(t_0), S'(t_0)\}^\perp$ onto $\tilde{C}_t = \mathcal{L} \cap \{S(t_0), R(t)\}^\perp$. Therefore, we define $Y : (a, b) \to \mathfrak{so}(4, 1)$ by

$$Y(t)P := \langle P, R(t)\rangle R'(t) - \langle P, R'(t)\rangle R(t), \qquad \forall t \in (a, b) \text{ and } P \in \mathbb{R}^{4,1}, \tag{15.22}$$

and obtain $B : (a, b) \to O(4, 1)$ as the unique solution of the initial value problem

$$\begin{aligned} B'(t) &= Y(t)B(t), \\ B(t_0) &= \mathbb{1}. \end{aligned} \tag{15.23}$$

With the same argumentation as in the proof of Lemma 32,

$$\begin{aligned} Y(t)R(t) &= R'(t), \\ B(t_0)R(t_0) &= R(t_0), \end{aligned}$$

give rise to

$$B(t)R(0) = R(t). \tag{15.24}$$

Further, we can conclude from:

$$\begin{aligned} Y(t)S(0) &= \langle S(0), R(t)\rangle R'(t) - \langle S(0), R'(t)\rangle R(t) \\ &= \langle S(t), T(t)\rangle R'(t) - \langle S(t), \kappa N(t)\rangle R(t) \\ &= 0, \end{aligned}$$

that $B(t)S(0) = S(0)$. In particular, $B(t)$ preserves the sphere Σ_{t_0} and maps C_0 onto $\tilde{C}_t$.

Let $\Psi : U \subset (a, b)\times \to \mathcal{L}$ be a conformal coordinate line parametrization of the isothermic canal surface. Due to Proposition 32, one family of coordinate lines is given by the generating circles C_t. For the derivative of $\tilde{\Psi} := A^{-1}(t)\Psi(t, s)$, defined in Lemma 32, we obtain

$$|\tilde{\Psi}_t(t, s)| = |\tilde{\Psi}_s(t, s)|, \tag{15.25}$$

$$\langle \tilde{\Psi}_t(t, s), \tilde{\Psi}_s(t, s)\rangle = 0 \qquad \forall\, (t, s) \in U. \tag{15.26}$$

Further, we use the Möbius transformations $A(t), B(t)$, introduced in (15.16) and (15.23), to define

$$\begin{aligned} L : U \subset (a, b) \times \mathbb{S}^1 &\to C_0 \subset \mathbb{S}^3 \\ (t, s) &\mapsto B^{-1}(t)A^{-1}(t)\Psi(t, s). \end{aligned}$$

The derivatives $L_s(s, t)$ and $L_t(s, t)$ are both tangent to the circle C_0 and hence space-like and linearly dependent. We will show that $L_t(s, t)$ is zero everywhere. Then the conformal coordinate line parametrisation of an isothermic canal surface has the form $\Psi(s, t) = D(t)L(s)$, where $D(t) := A(t)B(t)$. We start by computing the derivatives of L, by using the one of $\tilde{\Psi}$ and (15.18):

$$
\begin{aligned}
\tilde{\Psi}_s(t,s) &= B(t)L_s(t,s)\\
\tilde{\Psi}_t(t,s) &= Y(t)B(t)L(t,s) + B(t)L_t(t,s)\\
&= \langle B(t)L(t,s), R(t)\rangle R'(t) - \langle B(t)L(t,s), R'(t)\rangle R(t) + B(t)L_t(t,s)\\
&= \underbrace{\langle A(t)B(t)L(t,s), T(t)\rangle}_{=0} R'(t) - \langle A(t)B(t)L(t,s), \kappa N(t)\rangle R(t) + B(t)L_t(t,s)\\
&= -\kappa\langle \Psi(s,t), N(t)\rangle R(t) + B(t)L_t(t,s)
\end{aligned}
$$

With (15.26) we have

$$
\begin{aligned}
0 &= \langle \tilde{\Psi}_t(t,s), \tilde{\Psi}_s(t,s)\rangle\\
&= \langle B(t)L_t(t,s), B(t)L_s(t,s)\rangle - \kappa\langle \Psi(s,t), N(t)\rangle \underbrace{\langle R(t), B(t)L_s(t,s)\rangle}_{=0}\\
&= \langle L_t(t,s), L_s(t,s)\rangle.
\end{aligned}
$$

Because $\Psi(s,t)$ is an immersion, the space-like vector $L_s(t,s)$ cannot vanish and we obtain $L_t(t,s) = 0$. From $A(t)d\tilde{\Psi}(t,s) = d\Psi(t,s)$, we finally get the desired form for the derivatives of Ψ. □

Theorem 18.
An isothermic canal surface $\Psi : (a,b) \times \mathbb{S}^1 \to \mathcal{L}$ is Möbius equivalent to a surface of revolution.

Proof. Let $S : (a,b) \to \mathcal{H}^4_+$ be the 1-parameter family of spheres corresponding to the canal surface and (S,T,N,M_1,M_2) the canonical frame adapted to S. Due to Lemma 33, locally there exists a conformal coordinate line parametrization, $\Psi(t,s) = D(t)L(s)$, of the canal surface whose derivatives are given by

$$
\begin{aligned}
\Psi_t(t,s) &= -\kappa(t)\langle \Psi(s,t), N(t)\rangle T(t),\\
\Psi_s(t,s) &= D(t)L'(s).
\end{aligned}
$$

We change the lift Ψ such that the partial derivatives of Ψ have constant length one, i.e.,

$$\tilde{\Psi}(t,s) := \frac{1}{|L'(s)|}\Psi(t,s).$$

From $d\tilde{\Psi} = \frac{1}{|L'|}d\Psi + d\left(\frac{1}{|L'|}\right)\Psi$, we obtain

$$
\begin{aligned}
\tilde{\Psi}_s(t,s) &= D(t)\frac{1}{|L'(s)|}L'(s) + \left(\frac{1}{|L'(s)|}\right)_s \Psi(t,s),\\
\tilde{\Psi}_t(t,s) &= -\kappa(t)\langle \tilde{\Psi}(s,t), N(t)\rangle T(t).
\end{aligned}
$$

Due to the fact that $\Psi(s,t)$ is a conformal coordinate line parametrization and any two lifts are conformally equivalent (see Lemma 18), the derivatives of $\tilde{\Psi}$ satisfy

$$|\tilde{\Psi}_t(t,s)|^2 = |\tilde{\Psi}_s(t,s)|^2 = 1 \qquad \text{and} \qquad \langle \tilde{\Psi}_t(t,s), \tilde{\Psi}_s(t,s)\rangle = 0. \tag{15.27}$$

This gives rise to

$$\begin{aligned} & 0=\frac{\partial}{\partial s}|\tilde{\Psi}_t(t,s)|^2=2\kappa(t)^2\langle\tilde{\Psi}(t,s),N(t)\rangle\langle\tilde{\Psi}_s(t,s),N(t)\rangle \\ \Rightarrow\qquad & 0=\frac{\partial}{\partial s}\langle\tilde{\Psi}(t,s),N(t)\rangle \\ \Rightarrow\qquad & 0=\tilde{\Psi}_{ts}(t,s). \end{aligned} \tag{15.28}$$

Since $\langle\tilde{\Psi}(t,s),N(t)\rangle$ is constant with respect to the parameter s, the canonical representation of $\tilde{\Psi}$ (see Definition 35) has the form

$$\tilde{\Psi}(t,s)=\gamma(t)N(t)+\alpha(t,s)M_1(t)+\beta(t,s)M_2(t). \tag{15.29}$$

Using the frame equations (15.12), we obtain

$$\begin{aligned} 0&=\tilde{\Psi}_{st}(s,t) \\ &=\frac{\partial}{\partial t}\left(\alpha_s(t,s)M_1(t)+\beta_s(t,s)M_2(t)\right) \\ &=\alpha_{st}(t,s)M_1(t)+\beta_{st}(t,s)M_2(t)+\alpha_s(t,s)M_1'(t)+\beta_s(t,s)M_2'(t) \\ &=\underbrace{\alpha_{st}(t,s)M_1(t)+\beta_{st}(t,s)M_2(t)}_{\in N(t)^\perp}-\left(\alpha_s(t,s)a(t)+\beta_s(t,s)b(t)\right)N(t), \end{aligned}$$

$$\Rightarrow\qquad \begin{cases}\alpha_{st}(t,s)M_1(t)+\beta_{st}(t,s)M_2(t)=0,\\ \alpha_s(t,s)a(t)+\beta_s(t,s)b(t)=0.\end{cases}$$

Since $M_1(t)$ and $M_2(t)$ are linearly independent for all t, the first equation gives us $\alpha_s(t,s)=\alpha_s(s)$ and $\beta_s(t,s)=\beta_s(s)$. Due to (15.27) we further have

$$\alpha_s(s)^2+\beta_s(s)^2=1,\qquad \forall s\in V\subset\mathbb{S}^1. \tag{15.30}$$

Therefore, the second equation

$$\alpha_s(s)a(t)+\beta_s(s)b(t)=0,$$

can only be satisfied for all $s\in V$ if $a(t)=b(t)=0$. Considering the frame equations (15.12), this implies that $S(t)$ defines a planar curve in $\mathcal{H}^4_+$, and by Lemma 31 the canal surface is Möbius equivalent to a surface of revolution. □

Remark 10. The centers of the spheres enveloped by an isothermic canal surface in $\mathbb{S}^3$ lie on an circle, m. With a suitable stereographic projection we can map the circle into the x,y plane in $\mathbb{R}^3$ and its center to the origin. If we insert

$$m(t):=R\begin{pmatrix}\cos(t)\\ \sin(t)\\ 0\end{pmatrix},\quad e_1(t):=\begin{pmatrix}\cos(t)\\ \sin(t)\\ 0\end{pmatrix},\quad e_2(t):=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

in (15.3), we obtain the following conformal curvature line parametrisation for an isothermic canal torus:

$$
\begin{aligned}
f(s,t) :&= m(t) - \frac{r(t)r'(t)}{R^2}m'(t) + r(t)\frac{\sqrt{R^2 - r'(t)^2}}{R}\left(e_1(t)\cos(s) + e_2(t)\sin(s)\right) \\
&= \left(R + r(t)\frac{\sqrt{R^2 - r'(t)^2}\cos(s)}{R}\right)\begin{pmatrix}\cos(t)\\ \sin(t)\\ 0\end{pmatrix} - \frac{r(t)r'(t)}{R}\begin{pmatrix}-\sin(t)\\ \cos(t)\\ 0\end{pmatrix} \\
&\quad + r(t)\frac{\sqrt{R^2 - r'(t)^2}\sin(s)}{R}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}. \qquad (15.31)
\end{aligned}
$$

16. Classification of embedded isothermic constrained Willmore tori

In this last section, we will collect our previous results and prove the main theorem. We will further give a new proof for the well-known fact, that tori in $\mathbb{R}^3$ and $\mathbb{H}^3$ cannot be embedded.

To this end, let $f : M \to \mathbb{R}^3$ be an isothermic constrained Willmore embedding of a torus and $\Phi : M \to \mathcal{L}$ its isometric lift to the light cone $\mathcal{L} \subset \mathbb{R}^{4,1}$. Due to Theorem 10, there exist a space form $Q_{\mathcal{A}} \subset \mathcal{L}$ and a conformal change of the lift Φ, such that the restriction of the new lift $\tilde{\Phi} : M \to \mathcal{L}$ to the space form $Q_{\mathcal{A}} \subset \mathcal{L}$ has constant mean curvature $\tilde{H}$.

If the torus is not a Bryant surface, the embedding $\tilde{\Phi}(M)$ is contained in $Q_{\mathcal{A}}$ and has no umbilical points (see Theorem 10, 14 and 12). This makes it possible to apply Theorem 16 and Corollary 13, and to show that the $\Phi(M)$ is a canal surface. Since CMC surfaces are isothermic, we obtain from Theorem 18, that the embedding is Möbius equivalent to a surface of revolution. Since there are no embedded CMC tori in $\mathbb{R}^3$ and $\mathbf{H}^3$, we have proven the following classification of embedded, isothermic, constrained Willmore tori.

Theorem 19.
An embedded, isothermic constrained Willmore torus $f : M \to \mathbb{R}^3$ is either a Bryant surface with smooth ends or there exists a stereographic projection that maps it into a unit 3-sphere in $\mathbb{R}^4$, where it has constant mean curvature and is Möbius equivalent to a surface of revolution. Further, every properly embedded periodic CMC cylinder in $\mathbb{R}^3$ is a surface of revolution.

We finish this work with a new proof of the following well-known theorem.

Theorem 20.
There are no embedded CMC tori in $\mathbb{R}^3$ and $\mathbf{H}^3$.

Proof. Let $f : M \to \mathbb{R}^3$ is an embedding of a torus that has constant mean curvature $\tilde{H} = \rho H - \langle N, \operatorname{grad}_\rho \rangle$, with respect to the metric $\tilde{g} = \rho^{-2} g$. For $\rho = 1$, we obtain the euclidean case and for $\rho = \frac{1-|f|^2}{2}$, the hyperbolic one. Here, we again use the Poincaré ball model, and assume that f is contained in the unit ball of $\mathbb{R}^3$. Due to Theorem 16, the surface f is a canal torus and the bigger principle curvature λ_1 is constant along the corresponding curvature lines which are the characteristic circles of the canal surface. The centers of the spheres enveloped by the isothermic canal torus lie on a circle, that we will

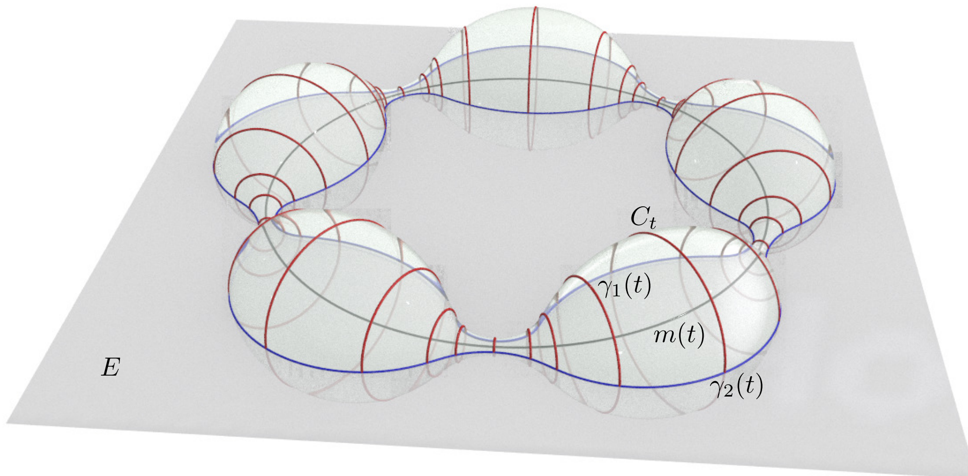


FIG. 12: The centers of the spheres, which are enveloped by a canal torus that is Möbius equivalent to a surface of revolution, lie on a circle m. The intersection of the canal torus with the plane E, containing the circle m, consists of two simple closed curves γ_1, γ_2 that are orthogonal to the characteristic circles C_t of the canal torus.

denote by m (see Theorem 18 and Lemma 31). The intersection of the canal torus with the plane E, that contains the circle m, consists of two simple closed curves γ_1, γ_2 (see Figure 12). The characteristic circles C_t intersect the plane E and hence the curves γ_1, γ_2 orthogonally. Therefore, γ_1 and γ_2 are curvature lines of the surface with respect to the principle curvature λ_2.
If we use the canal torus parametrization 15.31 from Remark 10, the planar curvature lines γ_1, γ_2 are parameterized by

$$\begin{aligned}
\gamma_1(t) &= m(t) - \frac{r(t)r'(t)}{R^2}m'(t) - r(t)\frac{\sqrt{R^2 - r'(t)^2}}{R^2}m(t) \\
&= \left(R - r(t)\frac{\sqrt{R^2 - r'(t)^2}}{R}\right)e_1(t) - \frac{r(t)r'(t)}{R}e_2(t), \\
\gamma_2(t) &= \left(R + r(t)\frac{\sqrt{R^2 - r'(t)^2}}{R}\right)e_1(t) - \frac{r(t)r'(t)}{R}e_2(t), \qquad (16.1)
\end{aligned}$$

where R denotes the radius of m, $r(t) := \lambda_1(t)^{-1}$ the radius of the enveloped sphere Σ_t, and $e_1(t) := \frac{m(t)}{R}, e_2(t) := \frac{m'(t)}{R}$ are an orthonormal frame adapted to the circle $m(t)$ (see Figure 13b). For embedded canal tori, the radius of the circle m has to be bigger than the one of the characteristic circles, i.e.

$$R > r(t) \qquad \forall t.$$

The normal $N_{1/2}(t)$ of the canal surface at the point $\gamma_{1/2}(t)$ is given by

$$N_{1/2}(t) = (\gamma_{1/2}(t) - m(t))\lambda_1(t) = \mp\frac{\sqrt{R^2 - r'(t)^2}}{R}e_1(t) - \frac{r'(t)}{R}e_2(t).$$

For the derivative of the curves and their normal's we obtain:

$$\begin{aligned}\gamma'_{1/2}(t) &= \left(\frac{r(t)r'(t)}{R} \mp r'(t)\frac{\sqrt{R^2 - r'(t)^2}}{R} \pm \frac{r(t)r'(t)r''(t)}{R\sqrt{R^2 - r'(t)^2}}\right) e_1(t)\\ &\quad + \left(R - \frac{r(t)r''(t) + r'(t)^2}{R} \mp r(t)\frac{\sqrt{R^2 - r'(t)^2}}{R}\right) e_2(t),\\ N'_{1/2}(t) &= \left(\frac{r'(t)}{R} \pm \frac{r'(t)r''(t)}{R\sqrt{R^2 - r'(t)^2}}\right) e_1(t) + \left(\frac{-r''(t)}{R} \mp \frac{\sqrt{R^2 - r'(t)^2}}{R}\right) e_2(t).\end{aligned}$$

The continuous function r takes its maximum on the compact set $\mathbb{S}^1$. Therefore, there exists $t_0 \in \mathbb{S}^1$ with: $r'(t_0) = 0$ and $r''(t_0) \leq 0$. At this point the tangents and the derivative of the normal's of the curves are given by:

$$\begin{aligned}\gamma'_{1/2}(t_0) &= \left(R - \frac{r(t_0)r''(t_0)}{R} \mp r(t_0)\right) e_2(t_0),\\ N'_{1/2}(t_0) &= \left(\frac{-r''(t_0)}{R} \mp 1\right) e_2(t_0).\end{aligned}$$

The curvature of the curves γ_1, γ_2 at this point is given by

$$\kappa_{1/2}(t_0) = \frac{\langle N'_{1/2}, \gamma'_{1/2}\rangle}{\|\gamma'_{1/2}\|^2}(t_0) = \left(\frac{-r''(t_0)}{R} \mp 1\right)\left(R - \frac{r(t_0)r''(t_0)}{R} \mp r(t_0)\right)^{-1}. \tag{16.2}$$

Due to the fact that $\gamma_{1/2}$ are principle curvature lines, their curvature is the same as the one of the canal surface a long the curves. Let $x = (0, t_0), y = (\pi, t_0) \in M = \mathbb{S}^1 \times \mathbb{S}^1$, the difference of the smaller principle curvature λ_2 at these points is given by

$$\begin{aligned}&\lambda_2(x) - \lambda_2(y)\\ &= \kappa_2(t_0) - \kappa_1(t_0)\\ &= \frac{\left(\frac{-r''(t_0)}{R} + 1\right)\left(R - \frac{r(t_0)r''(t_0)}{R} - r(t_0)\right) - \left(\frac{-r''(t_0)}{R} - 1\right)\left(R - \frac{r(t_0)r''(t_0)}{R} + r(t_0)\right)}{\left(R - \frac{r(t_0)r''(t_0)}{R} - r(t_0)\right)\left(R - \frac{r(t_0)r''(t_0)}{R} + r(t_0)\right)}\\ &= \frac{2\frac{r''(t_0)r(t_0)}{R} + 2(R - \frac{r(t_0)r''(t_0)}{R})}{\left(R - \frac{r(t_0)r''(t_0)}{R}\right)^2 - r(t_0)^2}\\ &= \frac{2R^2}{\left(R - \frac{r(t_0)r''(t_0)}{R}\right)^2 - r(t_0)^2} > 0. \end{aligned} \tag{16.3}$$

For canal surfaces, we can use Lemma 23, to obtain a different way to compute the difference of the smaller principal curvatures on a characteristic circle. To this end, remember that the difference of the principal curvatures of an CMC torus is never zero, because CMC tori have no umbilical points

$$\mu := \frac{\lambda_1 - \lambda_2}{2} = \lambda_1 - H > 0,$$

and that the bigger principle curvature of the canal torus is constant along the characteristic circles

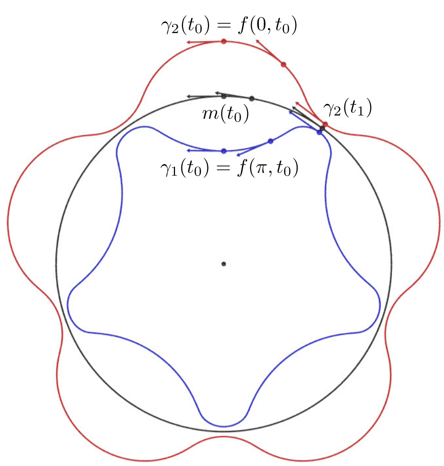
$$\lambda_1(x) = \lambda_1(y) = r(t_0)^{-1} =: \lambda_1(t_0).$$

Now, Equation 12.1 gives us

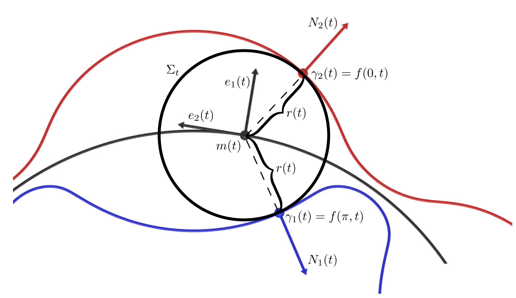
$$\begin{aligned}
& \rho(x)(\lambda_1(t_0) - H(x)) = \rho(y)(\lambda_1(t_0) - H(y)) \\
\Leftrightarrow \quad & \rho(x)\left(\frac{\lambda_1(t_0) - \lambda_2(x)}{2}\right) = \rho(y)\left(\frac{\lambda_1(t_0) - \lambda_2(y)}{2}\right) \\
\Leftrightarrow \quad & \lambda_2(x) = -\frac{2\rho(y)}{\rho(x)}\mu(y) + \lambda_1(t_0) \\
\Leftrightarrow \quad & \lambda_2(x) = -2\left(\frac{\rho(y)}{\rho(x)}\right)\mu(y) + 2\mu(y) + \lambda_2(y) \\
\Leftrightarrow \quad & \lambda_2(x) - \lambda_2(y) = 2\left(1 - \frac{\rho(y)}{\rho(x)}\right)\mu(y). \qquad (16.4)
\end{aligned}$$

Hence, the difference of the smaller principle curvatures is bigger than zero if and only if $\rho(y) < \rho(x)$. In the euclidean case, we have $1 = \rho(y) = \rho(x)$. For a hyperbolic metric, the Poincare ball model gives us $\rho(x) = \frac{1-|f(x)|^2}{2} < \frac{1-|f(y)|^2}{2} = \rho(y)$. In both cases, the Equations 16.3 and 16.4 contradict each other, and hence CMC tori cannot be embedded in the euclidean nor hyperbolic 3-space. $\square$

Note, that there exist CMC tori in the 3-sphere (see for example [15]), and that our argumentation does not contradict this, since in the spherical case the relation of the conformal factors is given by $\rho(x) = \frac{1+|f(x)|^2}{2} > \frac{1+|f(y)|^2}{2} = \rho(y)$.



(A) The centers of the 1-parameter family of spheres enveloped by the canal surface lie on a sphere m. The two curvature lines γ_1, γ_2 of the surface that lie in the same plane as the circle m are both passed counter-clockwise. The radius $r(t)$ of the enveloped spheres has a maximum at t_0 and a minimum at t_1.



(B) The orthonormal frame $(e_1(t), e_2(t))$ adapted to the curve $m(t)$ is obtained by a normalization of $(m(t), m'(t))$. The normal's of the surface, and hence of the plane curves γ_1, γ_2, are given by $N_{1/2}(t) = (\gamma_{1/2}(t) - m(t))r^{-1}(t)$.

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