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R. Shorten, P. Curran, K. Wulff, C. King and E. Zeheb

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On Spectral Conditions for Positive Realness of Transfer Function Matrices*

Robert Shorten[†], Paul Curran[‡], Kai Wulff[§], Chris King[¶], Ezra Zeheb[||]

Abstract

Necessary and sufficient conditions for strict positive realness and positive realness of general transfer function matrices are derived. The conditions are expressed in terms of eigenvalues of matrix functions of the state matrices representation of the LTI system. Illustrative numerical examples are provided.

Keywords: Positive Real (PR) Conditions; Passivity; Spectral methods.

*The material of Section 4 was completed while RS and CK were visiting the research group of Prof. Joerg Raisch at the TU Berlin.**

[†]Professor Shorten is with the Hamilton Institute, National University of Ireland, Maynooth, Co. Kildare, Ireland, and is supported by Science Foundation Ireland Grant 04/IN3/I460. E-mail: robert.shorten@nuim.ie.

[‡]Dr. Curran is with the Department of Electrical Engineering, University College Dublin, Ireland

[§]Dr. Wulff is with the Department of Electrical Engineering, TU Berlin, Germany

[¶]Professor King is with the Department of Mathematics, Northeastern University, Boston, USA

[||]Professor Zeheb is with the Department of Electrical Engineering, Technion-Israel Institute of Technology, Haifa, Israel and Jerusalem College of Engineering, Jerusalem, Israel

1 Introduction

The concept of Positive Realness (PR) and Strict Positive Realness (SPR) of a rational function appears frequently in various aspects of engineering. Roughly speaking, checking whether a dynamic system is positive real amounts to testing whether a certain matrix valued function of a frequency variable is positive definite for all frequencies. Exhaustive numerical checking of such matrices for all frequencies is expensive for large dimensional systems. Consequently, several authors over the past two decades have sought to derive easily verifiable conditions for checking whether a given transfer function is PR: see [1, 2, 3] and the references therein for a review of some of this work. Efficient techniques for checking whether a given transfer function matrix is positive real has also been the subject of much interest in the numerical Linear Algebra and VLSI communities. For the case of proper transfer function matrices, $G(s) = D + C^T(sI - A)^{-1}B$, with $D + D^T > 0$, robust numerical methods exist and are commonly used. An unfortunate fact is that all these methods fail when $D + D^T$ is singular (and consequently for strictly proper transfer function matrices). Our objective here is to summarise our results in [4] that resolve this latter question and to present compact spectral conditions to check strict positive realness, and positive realness, for strictly proper, and related, transfer function matrices.

2 Background

In the remainder of this note we use the following common definitions for Positive Realness (PR) and Strict Positive Realness (SPR).

Definition 1: A rational function in a complex variable $H(s)$ is PR if, and only if, $H(s)$ is real for real values of s, and $H(s)$ satisfies

$$\text{Real}\,[H(s)] \geq 0 \quad \text{for} \quad \text{Real}\,[s] \geq 0\,. \tag{1}$$

Definition 2: A rational function in a complex variable $H(s)$ is SPR if, and only if,

$$\exists\, \epsilon > 0 \quad \text{such that} \quad H(s - \epsilon) \text{ is PR}\,. \tag{2}$$

The following equivalent definition of positive realness of a rational function is also well known [5].

Definition 1(a): A rational function of a complex variable $H(s)$ is PR if and only if $H(s)$ is real for real values of s, and all poles of $H(s)$ are in the closed left half plane of s. If there are imaginary-axis poles, they are simple with real positive residues, and

$$\text{Real } [H(j\omega)] \geq 0 \qquad \forall\, \omega \in \mathbb{R} \tag{3}$$

where $s = \sigma + j\omega$.

Definition 3: A matrix $H(s)$ is a positive real matrix, termed a PR matrix, if and only if the rational function defined by

$$F(s) = x^T H(s) x \tag{4}$$

is a positive real function for every real m-dimensional vector x.

Definition 4: A matrix $H(s)$ is a strictly positive real matrix, termed a SPR matrix, if and only if there exists an $\epsilon > 0$ such that $F(s-\epsilon)$ is a positive real function for every real m-dimensional vector x where $F(s)$ is defined in (4). Equivalently, it is shown in [6] $H(s)$ is SPR if and only if: **(i)** all eigenvalues of $H(s)$ have negative real parts; **(ii)** $G(j\omega) + G(j\omega)^* > 0$ for all $\omega \in R$; **(iii)** either $G(j\infty) + G(j\infty)^*$ is positive definite, or it is positive semi-definite and $\lim_{\omega\to\infty} \omega^2 M^T \Big[G(j\omega) + G(j\omega)\Big]$ is positive definite where M is any $n \times p$ matrix of rank p that satisfies $M^T \Big(G(j\infty) + G(j\infty)^*\Big) M = 0$ and p is the dimension of the kernel of $G(j\infty) + G(j\infty)^*$.

3 Main results

Frequently, in practical situations, one exploits the Hamiltonian based methods to test for SPR of a given transfer function matrix [7, 8] by making use of the following theorem.

Theorem 1 *[9] Let A be a stable $[n \times n]$ real matrix. Let $B \in R^{n\times m}$, $C \in R^{n\times m}$, and let $D \in R^{m\times m}$, with $D + D^T > 0$. Then, the transfer function matrix*

$$G(j\omega) = D + C^T (j\omega I - A)^{-1} B \tag{5}$$

is SPR if and only if the matrix

$$N = \begin{bmatrix} -A + BQ^{-1}C^T & BQ^{-1}B^T \\ -CQ^{-1}C^T & A^T - CQ^{-1}B^T \end{bmatrix}, \tag{6}$$

with $Q = D + D^T$, has no eigenvalues on the imaginary axis. The matrix N is called the Hamiltonian matrix.

The Hamiltonian matrix provides a convenient method for checking whether a given transfer function matrix is SPR. The proof relies crucially on the fact that $D + D^T$ is invertible, and the theorem ceases to be valid when $D + D^T$ is singular. Much of the previous work on this topic is concerned with extending the Hamiltonian, and other matrix methods, to the singular case [10]. This mirrors the case of scalar transfer functions that we considered in earlier publications. In [11, 12], necessary and sufficient conditions for PR and SPR are derived in terms of eigenvalue-locations of matrices related to the state space realisation (A, b, c, d) (here d is a scalar). We now present a general solution for transfer function matrices that is based on the following simple observation.

Observation: *Let $\mathcal{G}$ denote the locus of eigenvalues of the matrix $G(j\omega)+G(j\omega)^*$ for all $\omega \in [-\infty, \infty]$. Let $\mathcal{G}^r$ denote the locus of eigenvalues of the matrix $G(\frac{1}{j\omega})+G(\frac{1}{j\omega})^*$ for all $\omega \in [-\infty, \infty]$. Then, $\mathcal{G}$ and $\mathcal{G}^r$* coincide.

Thus, if we can establish $G(\frac{1}{j\omega})+G(\frac{1}{j\omega})^*$ is positive definite for all $\frac{1}{\omega}$, then it automatically follows that $G(j\omega) + G(j\omega)^*$ is such as well. To this end we note the following easily established result [13].

Theorem 2 *Let $G(j\omega) = D + C^T(j\omega I - A)^{-1}B$ be a strictly proper SPR transfer function matrix. Then, $G(\frac{1}{j\omega}) = \bar{D} + \bar{C}^T(j\omega I - \bar{A})^{-1}\bar{B}$, with $\bar{A} = A^{-1}$, $\bar{B} = -A^{-1}B$, $\bar{C}^T = C^T A^{-1}$, $\bar{D} = D - C^T A^{-1}B$, and $\bar{D} + \bar{D}^T > 0$.*

Proof : From the definitions in the theorem: $\bar{D} + \bar{C}^T(j\omega I - \bar{A})^{-1}\bar{B}$

$$\begin{aligned}
&= D - C^T A^{-1}B - C^T A^{-1}(j\omega I - A^{-1})^{-1}A^{-1}B \\
&= D - C^T A^{-1}\Big\{I + (j\omega I - A^{-1})^{-1}A^{-1}\Big\}B \\
&= D - C^T A^{-1}(j\omega A - I)^{-1}j\omega AB \\
&= D + C^T(\frac{1}{j\omega}I - A)^{-1}B,
\end{aligned}$$

which proves the first part of the assertion of the theorem. The fact that $\bar{D}+\bar{D}^T > 0$ follows from the fact that $G(j\omega)$ is SPR. Note that $\bar{D} = G(0)$ and $G(0) + G(0)^* > 0$ since $G(j\omega)$ is SPR.

A. Strict positive realness

Theorem 3 *Let A be a stable $[n \times n]$ real matrix. Let $B \in R^{n\times m}$, $C \in R^{n\times m}$, and let $D \in R^{m\times m}$, with $D + D^T$ singular. Suppose that conditions (i) and (iii) are satisfied in Definition 4. Then the transfer function matrix*

$$G(j\omega) = D + C^T(j\omega I - A)^{-1}B \tag{7}$$

is SPR if and only if $G(0) + G(0)^* > 0$ and the matrix

$$\bar{N} = \begin{bmatrix} -\bar{A} + \bar{B}\bar{Q}^{-1}\bar{C}^T & \bar{B}\bar{Q}^{-1}\bar{B}^T \\ -\bar{C}\bar{Q}^{-1}\bar{C}^T & \bar{A}^T - \bar{C}\bar{Q}^{-1}\bar{B}^T \end{bmatrix}, \tag{8}$$

with $\bar{Q} = \bar{D} + \bar{D}^T$, has no eigenvalues on the imaginary axis, except at the origin, with $\bar{A} = A^{-1}$, $\bar{B} = -A^{-1}B$, $\bar{C}^T = C^TA^{-1}$ and $\bar{D} = D - C^TA^{-1}B$.

Proof : No eigenvalues on the imaginary axis except at the origin is enough to prove that $G(j\omega) + G(j\omega)^* > 0$ for all finite ω. Since $G(j\omega) + G(j\omega)^* \geq 0$ in the limit as $\omega \to \infty$ does not contradict our definition of strict positive realness we must account for this possibility. Elementary arguments reveals that this may correspond to the $\bar{N}$ being singular.

The previous discussion takes care of strict positive realness (SPR). Now we ask if there are efficient methods to check positive realness. To this end the following Lemma is useful.

B. Positive realness

Lemma 1 *[4] Let A be a stable matrix. Let $G(j\omega) = D + C^T(j\omega I - A)^{-1}B$, with $D + D^T > 0$. Then,*

$$det\Big[G(j\omega) + G(j\omega)^*\Big] = S(\omega)det\Big[j\omega I + N\Big],$$

where $S(\omega)$ is a scalar function of ω such that $S(\omega) < 0$ for all $\omega \in (-\infty, \infty)$.

Comment 1: From continuity it follows that $G(j\omega)$ is SPR if and only if N has no eigenvalues on the imaginary axis.

Comment 2: The previous assertion follows even if A is a complex matrix provided A is stable, and $D + D^T > 0$.

Armed with this result we now proceed to consider the case of positive realness. Here, three special cases can be discerned.

(i) *Hamiltonian methods apply directly :* This is the case when $D + D^T$ and/or $\bar{D} + \bar{D}^T$ are invertible.

(ii) $G(0) + G(0)^*$ *and* $D + D^T$ *are both singular.*

(iii) *Singular for all frequencies :* This is the case when $G(j\omega) + G(j\omega)^*$ is singular for all $\omega \in (-\infty, \infty)$.

Case (i) : (Hamiltonian methods apply) Let us assume without any loss of generality that $D + D^T > 0$.

Theorem 4 *Let Ω be the distinct set of frequencies for which det$\Big[G(j\omega)+G(j\omega)^*\Big] = 0$, with the elements of $\Omega = \Big\{\omega_1, \omega_2, ..., \omega_s\Big\}$, $s < n$ listed in strictly increasing order. These frequencies are the eigenvalues of N that are on the imaginary axis. $G(j\omega)$ is PR if and only if: (i) N has no eigenvalues on the imaginary axis of odd multiplicity; (ii) $G(j\Delta_i) + G(j\Delta_i)^*$ has only positive real eigenvalues for all $\Delta_i = \frac{\omega_i+\omega_{i+1}}{2}$, $i \in \Big\{1, s-1\Big\}$.*

Proof : With reference to Lemma 1, it is immediate that if N has an imaginary eigenvalue of odd multiplicity, say ω_0, then *det*$\Big[j\omega I + N\Big]$ changes sign as the frequency variable ω passes through $\omega = \omega_0$. Hence, $G(j\omega) + G(j\omega)^*$ must have at least one negative eigenvalue for some ω in the vicinity of ω_0.

Suppose now that N has only eigenvalues that are of even multiplicity on the imaginary axis. Let these eigenvalues be ordered and denoted $\Big\{\omega_1, \omega_2, ..., \omega_s\Big\}$, $s < n$. Then,

$det\Big[j\omega I + N\Big]$ never changes sign for all $\omega \in \Omega$. In this case we cannot deduce whether the eigenvalues of $G(j\omega) + G(j\omega)^*$ remain positive. However, we do note that the eigenvalues of $G(j\omega)+G(j\omega)^*$ are continuous functions of ω. Thus, if some of them become negative, they cannot become positive again without $det\Big[G(j\omega) + G(j\omega)^*\Big] = 0$ for some other ω. Hence, by selecting $\Delta_i = \frac{\omega_i+\omega_{i+1}}{2}$, for all $i \in \{1, .., s-1\}$, one can deduce the signs of the eigenvalues of the transfer function matrix in each of the open intervals (ω_i, ω_{i+1}).

<u>**Case (ii) : ($G(0) + G(0)^*$ singular)**</u> Suppose now that both $D + D^T$ and $G(0) + G(0)^*$ are both singular. Then, the Hamiltonian cannot be used to test for positive realness. However, if we assume that $det\Big[G(j\omega) + G(j\omega)^*\Big] \neq 0$ for some $\omega \in (-\infty, \infty)$, then there must exist a ω_0, such that $G(j\omega_0) + G(j\omega_0)^* > 0$ (a necessary condition for positive realness). We can then make use of the following observation.

Observation: *Let $\mathcal{G}$ denote the locus of eigenvalues of the matrix $G(j\omega)+G(j\omega)^*$ for all $\omega \in [-\infty, \infty]$. Let $\mathcal{G}^s$ denote the locus of eigenvalues of the matrix $\tilde{G}(j\delta) + \tilde{G}(j\delta)^*$ for all $\delta \in [-\infty, \infty]$, with $\delta = \omega - \omega_0$, and $\tilde{G}(j\delta) = G\Big(j(\delta + \omega_0)\Big)$. Then, $\mathcal{G}$ and $\mathcal{G}^s$* coincide.

Note that $\tilde{G}(j\delta) = D + C^T(j\delta I - \tilde{A})^{-1}B$ where $\tilde{A} = A - j\omega_0 I$. By definition, $\tilde{G}(0) + \tilde{G}(0) > 0$, and Hamiltonian methods can now be applied. Note that $\tilde{A}$ is complex but the Hamiltonian matrix can be applied from Comment 2.

<u>**Case (iii) : (Singular for all frequencies)**</u> Suppose that $G(j\omega) + G^*(j\omega)$ is singular for all $\omega \in (-\infty, \infty)$. Then, since $det[G(j\omega) + G(j\omega)^*]$ is a rational polynomial in ω, it must be identically zero for all ω. So this case can be easily identified. The question is what happens to the other eigenvalues of $G(j\omega) + G(j\omega)^*$ as ω varies.

To this end we use the following well known Theorem which states that an Hermitian matrix is positive semi-definite if and only if all its principal minors are non-negative.

This theorem allows us to test for positive realness. Since all minors of $G(j\omega)+G(j\omega)^*$ are themselves Hermitian, then these can be tested for positive realness using the methods described above.

4 A Hamiltonian equivalence class

The test for positive realness used a simple observation on the eigenvalue locus of a family of matrices. In the main results of this report we used the fact that $j\omega$ can be replaced with its reciprocal, but in the section on positive realness, we noted that other transformations can be used as well. In this section we formalise this latter fact and identify entire classes of linear systems that are equivalent from the spectral locus perspective. Checking whether $H(j\omega) + H(j\omega)^* > 0$ for any of these systems immediately implies this statement for any of the others.

Consider four complex matrices A, B, C, D, where $j\omega I - A$ is invertible for all real ω. For $z \in \mathbb{C}$ and $zI - A$ invertible, define

$$\begin{aligned} \sigma(A, B, C, D; z) = \mathrm{Spec}\Big[& D + C^*(zI - A)^{-1}B \\ & +(D + C^*(zI - A)^{-1}B)^* \Big] \end{aligned} \tag{9}$$

where C^* is the Hermitian conjugate of C, and where Spec is the spectrum, that is the set of eigenvalues.

Define the *spectral locus* corresponding to A, B, C, D to be

$$\rho(A, B, C, D) = \overline{\cup_{\omega \in [-\infty, \infty]} \sigma(A, B, C, D; j\omega)} \tag{10}$$

where $\overline{S}$ denotes the closure of S.

Now let a, b, c, d be real numbers, where we assume that b, d are not simultaneously zero. Define the following matrices:

$$\overline{A} = (cA - jaI)(bI - jdA)^{-1} \tag{11}$$
$$\overline{B} = (bI - jdA)^{-1}B \tag{12}$$
$$\overline{C} = (ad + bc)(bI + jdA^*)^{-1}C \tag{13}$$
$$\overline{D} = D + jdC^*(bI - jdA)^{-1}B \tag{14}$$

Theorem 5

$$\rho(A, B, C, D) = \rho(\overline{A}, \overline{B}, \overline{C}, \overline{D}) \tag{15}$$

Proof: Define the complex variable u by the following fractional linear transformation:

$$z = \frac{ja + bu}{c + jdu} \tag{16}$$

It follows that z is pure imaginary if and only if u is pure imaginary. Direct substitution shows that

$$D + C^*(zI - A)^{-1}B = \overline{D} + \overline{C}^*(uI - \overline{A})^{-1}\overline{B} \tag{17}$$

and hence $\sigma(A, B, C, D; z) =$

$$\sigma(\overline{A}, \overline{B}, \overline{C}, \overline{D}; u) \tag{18}$$

The mapping $z \mapsto u$ is one-to-one on the extended imaginary axis (where the point at infinity is included) and hence $\cup_{\mathrm{Re}(z)=0\cup\{\infty\}}\, \sigma(A, B, C, D; z)$

$$= \cup_{\mathrm{Re}(u)=0\cup\{\infty\}}\ \sigma(A, B, C, D; u)$$

The proof is completed by noting that $\cup_{\mathrm{Re}(z)=0\cup\{\infty\}}\, \sigma(A, BC, D; z) =$

$$\overline{\cup_{\mathrm{Re}(z)=0}\, \sigma(A, B, C, D; z)} \tag{19}$$

With these observations we can refine Theorem 1 as follows.

Theorem 6 *Let A be a stable $[n \times n]$ real matrix. Let $B \in R^{n\times m}$, $C \in R^{n\times m}$, and let $D \in R^{m\times m}$, with $D + D^T \geq 0$. Then, the transfer function matrix*

$$G(j\omega) = D + C^T(j\omega I - A)^{-1}B \tag{20}$$

is SPR if and only if the matrix

$$N = \begin{bmatrix} -\bar{A} + \bar{B}Q^{-1}C^T & \bar{B}Q^{-1}\bar{B}^T \\ -\bar{C}Q^{-1}\bar{C}^T & \bar{A}^T - \bar{C}Q^{-1}\bar{B}^T \end{bmatrix}, \tag{21}$$

with $Q = \bar{D} + \bar{D}^T$, has no eigenvalues on the imaginary axis for any a, b, c, d with $Q > 0$, except at the mapping of $\omega = \infty$.

Comment 3: Theorem 6 admits further generalisations. For example, entire classes of LTI systems can be identified that are equivalent from a passivity viewpoint. The implications of this observation are currently being investigated.

5 Example

In this section we present an example to illustrate our results.

Example : Let $A \in R^{n \times n}$ be a Hurwitz stable matrix with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

and where both $A_{11} \in R^{m \times m}$ and $A_{22} \in R^{n-m \times n-m}$ are assumed to be Hurwitz stable. A basic question that arises is whether one can find a block diagonal $P = P^T > 0$, such that

$$\begin{aligned} A_{11}^T P_{11} + P_{11} A_{11} &< 0, & (22) \\ A_{22}^T P_{22} + P_{22} A_{22} &< 0, & (23) \\ A^T P + PA &< 0. & (24) \end{aligned}$$

with

$$P = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}. \tag{25}$$

In other words, can one stabilise each of the component systems, and the entire interconnected system, simultaneously? Such problems arise in power networks, in control over wireless links, and in a variety of other decentralised control problems. A strategy to identify A matrices that admit such a P matrix in the Lyapunov equation may be proposed as follows. Let $B \in R^{n \times m}$ be the matrix of all zeros except for the last m rows and m columns which are set to $I_{m \times m}$

$$B = \begin{bmatrix} 0_{n-m \times m} \\ I_{m \times m} \end{bmatrix}.$$

Suppose now that one can find a $\hat{P} \in R^{m \times m}$ such that

$$\hat{P} B^T (j\omega I - A)^{-1} B + \left(\hat{P} B^T (j\omega I - A)^{-1} B\right)^* > 0, \ \forall \ \omega \ \in \ (-\infty, \infty), \tag{26}$$

i.e. such that $\hat{P} B^T (j\omega I - A)^{-1} B$ is strictly positive real. Then, it follows from the KYP lemma, with some additional minor assumptions, that this frequency domain condition is both necessary and sufficient for the existence of a $P = P^T > 0$ that satisfies:

$$\begin{aligned} A^T P + PA &< 0 \\ PB &= B\hat{P} \end{aligned}$$

This in turns guarantees the existence of a block diagonal P matrix with $P_{22} = \hat{P}$. To be more specific.

Let

$$A = \begin{bmatrix} -16 & -50 & 19 & 36 \\ 14 & -24 & 37 & 26 \\ 47 & -12 & -36 & -43 \\ -29 & 37 & 10 & -52 \end{bmatrix}$$

and

$$B = C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We shall now determine whether one can find a block diagonal P of the form (25) such that (22)-(24) with $P_{11}, P_{22} \in R^{2\times 2}$. We first select P_{22}. With

$$P_{22} = \begin{bmatrix} 45 & 8 \\ 8 & 23 \end{bmatrix}$$

we have that $A_{22}^T P_{22} + P_{22} A_{22} < 0$. Note A is *block diagonally* stable if and only if $P_{22} B^T (j\omega I - A)^{-1} B$ is SPR. This condition can be checked via a Hamiltonian using the reciprocal representation (Theorem 3). Substituting $\bar{A} = A^{-1}$, $\bar{B} = -A^{-1}B$, $\bar{C}^T = P_{22} B^T A^{-1}$, $\bar{D} = -C^T A^{-1} B$, yields the Hamiltonian matrix

$$\bar{N} = \begin{bmatrix} -\bar{A} + \bar{B} Q^{-1} \bar{C}^T & \bar{B} Q^{-1} \bar{B}^T \\ -\bar{C} Q^{-1} \bar{C}^T & \bar{A}^T - \bar{C} Q^{-1} \bar{B}^T \end{bmatrix},$$

with $Q = \bar{D} + \bar{D}^T$. We can verify that the eigenvalues of $\bar{N}$ are not on the imaginary axis and thus, $P_{22} B^T (j\omega I - A)^{-1} B$ is SPR. It follows that A is block diagonally stable. Using LMI's, with P_{22} given, we find that P_{11} may be taken to be

$$P_{11} = \begin{bmatrix} 84.72 & -13.89 \\ -13.89 & 85.87 \end{bmatrix}.$$

6 Conclusion

Necessary and sufficient conditions for strict positive realness and positive realness of general transfer function matrices are derived. The conditions are expressed in terms of eigenvalues of matrix functions of the state matrices representation of the LTI system.

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