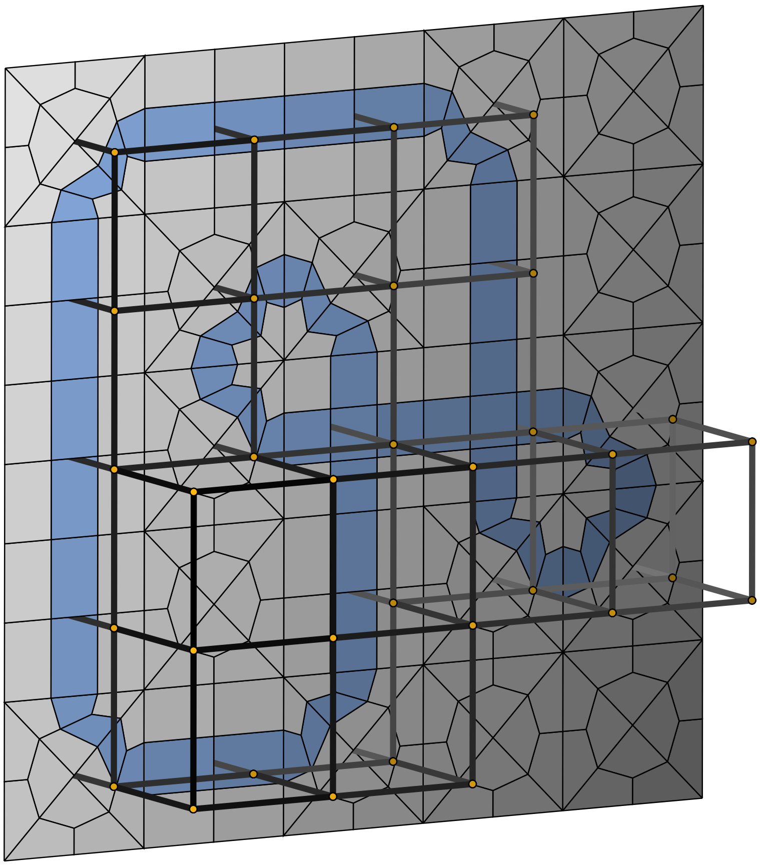
Constructions of Cubical Polytopes

Alexander Schwartz



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Constructions of Cubical Polytopes

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Abstract

In this thesis we consider cubical d-polytopes, convex bounded d-dimensional polyhedra all of whose facets are combinatorially isomorphic to the $(d-1)$-dimensional standard cube.
It is known that every cubical d-polytope P determines a PL immersion of an abstract closed cubical $(d-2)$-manifold into the polytope boundary $\partial P \cong S^{d-1}$. The immersed manifold is orientable if and only if the 2-skeleton of the cubical d-polytope ($d \geq 3$) is "edge orientable" in the sense of Hetyei. He conjectured that there are cubical 4-polytopes that are not edge-orientable.
In the more general setting of cubical PL $(d-1)$-spheres, Babson and Chan have observed that *every* type of normal crossing PL immersion of a closed PL $(d-2)$-manifold into an $(d-1)$-sphere appears among the dual manifolds of some cubical PL $(d-1)$-sphere.
No similar general result was available for cubical polytopes. The reason for this may be blamed to a lack of flexible construction techniques for cubical polytopes, and for more general cubical complexes (such as the "hexahedral meshes" that are of great interest in CAD and in Numerical Analysis).
In this thesis, we develop a number of new and improved construction techniques for cubical polytopes. We try to demonstrate that it always pays off to carry along convex lifting functions of high symmetry.
The most complicated and subtle one of our constructions generalizes the "Hexhoop template" which is a well-known technique in the domain of hexahedral meshes.

Using the constructions developed here, we achieve the following results:

- A rather simple construction yields a cubical 4-polytope (with 72 vertices and 62 facets) for which the immersed dual 2-manifold is not orientable: One of its components is a Klein bottle. Apparently this is the first example of a cubical polytope with a non-orientable dual manifold. Its existence confirms the conjecture of Hetyei mentioned above.

- More generally, all PL-types of normal crossing immersions of closed 2-manifolds appear as dual manifolds in the boundary complexes of cubical 4-polytopes. In the case of non-orientable 2-manifolds of odd genus, this yields cubical 4-polytopes with an odd number of facets. (This result is a polytopal analog to the Babson-Chan construction restricted to 2-manifolds.)

- In particular, we construct an explicit example with 19 520 vertices and 18 333 facets of a cubical 4-polytope that has a cubation of Boy's surface (an immersion of the projective plane with exactly one triple point) as a dual manifold immersion.

- Via Schlegel diagrams, this implies that for every cubification of a 3-dimensional domain there is also a cubification of opposite parity. Thus the flip graph for hexahedral meshes has at least two connected components. This answers questions by Eppstein and Thurston.

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Alexander Schwartz

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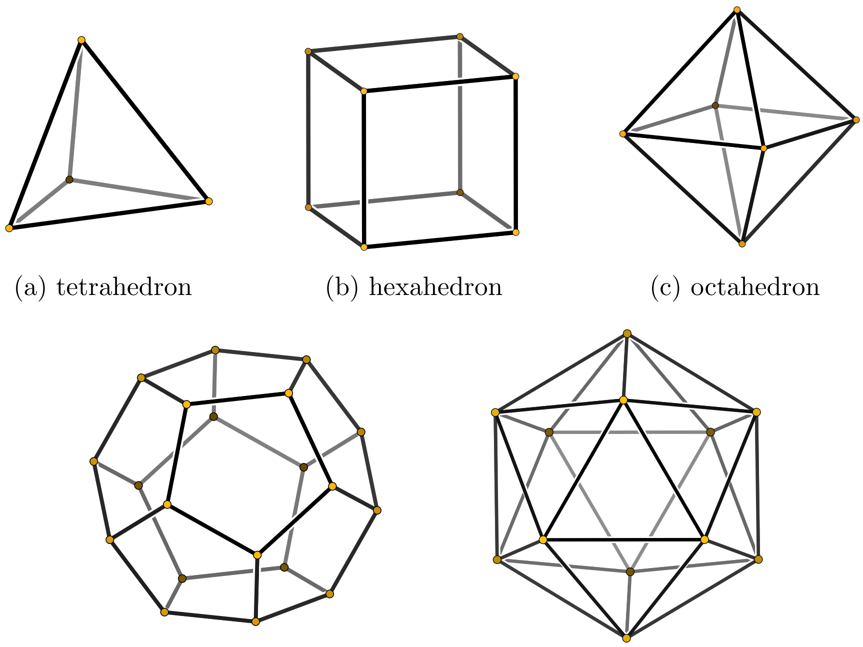
Introduction

Convex polytopes (convex bounded d-dimensional polyhedra) are classical objects of study in combinatorial geometry. Classical convex polytopes are the *Platonic solids* which were known to the ancient Greeks, and were described by Plato in his *Timaeus* 350 BC.



(a) tetrahedron (b) hexahedron (c) octahedron

(d) dodecahedron (e) icosahedron

FIGURE 1: The five *Platonic solids*.

The Platonic solids are *regular* 3-dimensional convex polytopes. A d-polytope is *regular* provided for every k with $0 \leq k \leq d-1$ and for every $(k+1)$-face F^{k+1} and $(k-1)$-face F^{k-1} incident with F^{k+1}, there is a symmetry of P (i.e. an orthogonal transformation of $\mathbb{R}^d$ mapping P onto itself) such that the two k-faces of P incident to both F^{k+1} and F^{k-1} are mapped onto each other.

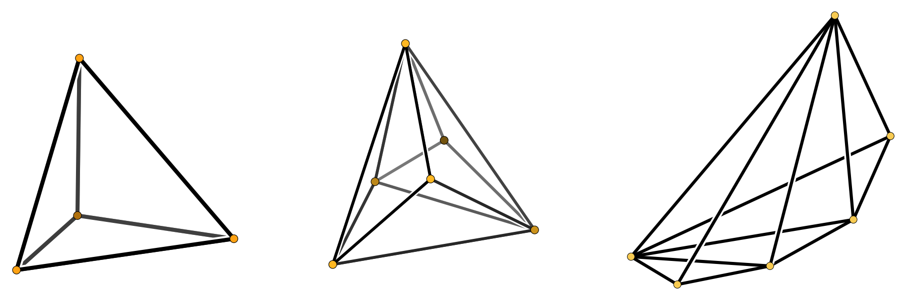
Compare Coxeter [18] or Grünbaum [28, p. 412]. Up to only five Platonic solids exist:

- The *tetrahedron* with 4 triangular faces.
- The *hexahedron*, also known as *cube*, with 6 square faces.
- The *octahedron*, alias *cross-polytope*, with 8 triangular faces.
- The *dodecahedron* with 12 pentagonal faces.
- The *icosahedron* with 20 triangular faces.

In dimension four there are six regular polytopes [28, p. 414]. It turns out that in higher dimensions $d \geq 5$ there are only three regular polytopes:

- The *regular d-simplex* whose $(d+1)$ facets are all $(n-1)$-simplices.
- The *d-cube* whose $2d$ faces are $(n-1)$-cubes.
- The *d-dimensional cross-polytope* whose 2^d facets are all $(d-1)$-simplices.

The d-dimensional *(affine) simplices*, that is, d-polytopes affinely isomorphic to the regular d-simplex, are the 'building-blocks' of the most famous class of convex polytopes: the so-called *simplicial* polytopes. A *simplicial* polytope is convex polytope whose facets are affine simplices. This implies that all proper k-faces are affine k-simplices.



(a) 3-simplex (b) stacked (c) neighborly

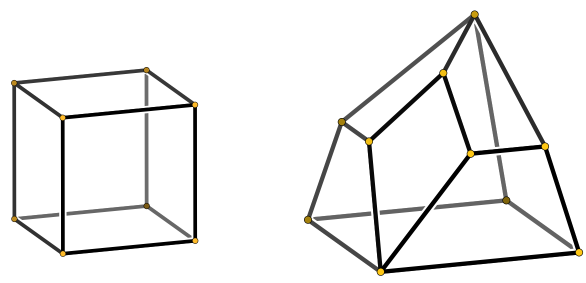
FIGURE 2: Simplicial 3-polytopes.

Simplicial polytopes and their duals, the *simple polytopes* (that is, d-polytopes whose vertices are d-valent), belong to the best understood classes of polytopes.

One interesting aspect is that a simplicial polytope encodes points in general position in the following sence: A polytope is simplicial if and only if the vertices of every facet are in general position. A random polytope that is constructed as the convex hull of a number of points with random coordinates is with high probability simplicial.

Furthermore, two classes of simplicial polytopes are of interest for the study of extremal properties of polytopes, namely the *stacked* and the *neighborly* (simplicial) polytopes.
The *lower bound theorem* (Barnette [7][8]) tells us that the minimum number of facets for a simplicial polytope with a prescribed number of vertices is attained by a *stacked* polytope. An *stacked* d-polytope with n vertices [39] is either a d-simplex, or the outcome of the *stacking operation* over a facet F of a stacked d-polytope P with $n-1$ vertices, that is, the convex hull of P with an additional point that is beyond the facet F (and beneath all others). Clearly a stacked polytope is simplicial.
A d-polytope P is *neighborly* if any subset of $\lfloor \frac{d}{2} \rfloor$ or fewer vertices is the vertex set of a face of P. This is equivalent to the condition that P has the $(\lfloor \frac{d}{2} \rfloor - 1)$-skeleton of the d-simplex. By the *upper bound theorem* (McMullen [43]) every d-polytope with n vertices has at most as many k-faces, $0 \leq k \leq d$, as a neighborly d-polytope with the same number of vertices.

In this thesis, we consider *cubical* d-polytopes, d-polytopes all of whose facets are combinatorially isomorphic to the $(d-1)$-dimensional standard cube. This class of polytopes behaves (somehow) similar to simplicial polytopes.

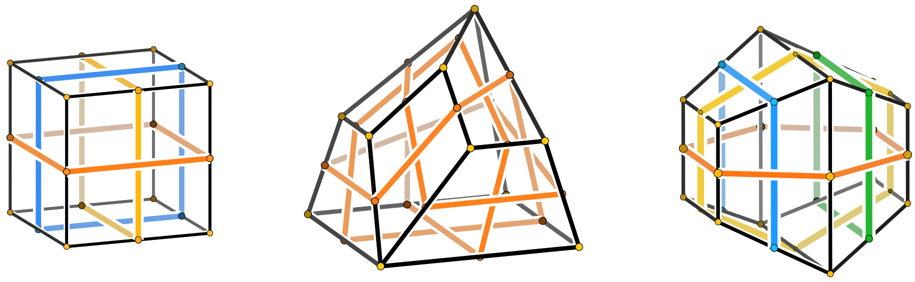


(a) 3-cube (b) cubical octahedron (c) bicuboid

FIGURE 3: Some cubical 3-polytopes.

For instance, for $d \geq 3$, every simple and simplicial d-polytope is a d-simplex. Similar, for $d \geq 3$, every simple and cubical d-polytope is combinatorially isomorphic to a d-cube.

It has been observed independently by Stanley and by MacPherson [32] that there is a correspondence between cubical polytopes or complexes and immersed manifolds. The following figure illustrates the dual manifolds of some 3-dimensional cubical polytopes.



(a) 3-cube (b) cubical octahedron (c) bicuboid

FIGURE 4: Cubical 3-polytopes and their dual manifolds. The cube has three dual manifolds, the cubical octahedron one and the bicuboid four. For the cube and the bicuboid the dual manifolds are embedded. The dual manifold of the cubical octahedron is immersed with 8 double-intersection points. In general, an immersed dual manifold of a d-polytope can have multiple-intersection points up to degree d.

A *dual manifold* of a cubical 3-polytope P is a connected component of the 1-dimensional complex whose facets are given as follows: For every facet Q (which is a quadrangle) of the polytope the 1-dimensional complex has two edges, where each of these edges join the midpoints of two opposite edges of Q. This gives an an immersion of a 1-dimensional manifold into the boundary of the 3-polytope, which is a 2-dimensional sphere. Hence, we have *codimension one* PL (=piece-wise linear) immersions. The immersion can be embedding but in general there can be *multiple-intersection points*.
Similarly, the dual manifolds can be defined for higher-dimensional cubical d-polytopes (or more generally: cubical complexes). Using the *barycentric subdivision* one obtains a codimension one PL immersion of a closed cubical $(d-2)$-manifold into a $(d-1)$-dimensional sphere. Furthermore, these immersions are *normal crossing*, that is, each *multiple-intersection point* is of degree $k \leq d$ and there is a neighborhood of each multiple intersection point that is PL isomorphic to (a neighborhood of) a point which is contained in k pair-wise perpendicular hyperplanes.
In the case of 4-polytopes, the dual manifolds are surfaces (compact 2-manifolds without boundary). Our main results show that all PL-types of normal crossing immersions of closed surfaces appear as dual manifolds in the boundary complexes of cubical 4-polytopes (Theorem 10.3).

For a lot of results that are proven for simplicial polytopes, there is only a similar conjecture for cubical polytopes. For instance the *upper* and the *lower bound conjecture* for cubical polytopes are still open. (An overview about the results concerning cubical polytopes is given in Chapter 2.)

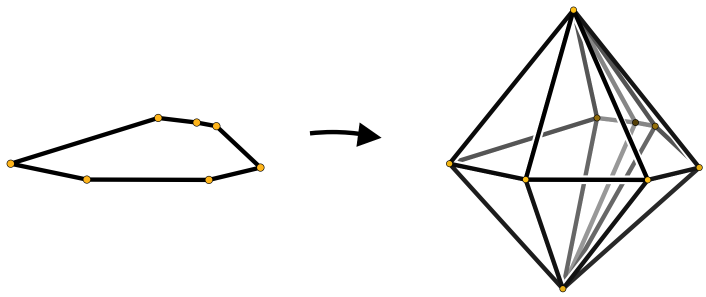


FIGURE 5: The *bipyramid* construction.

The reason for the 'absence' of several results may be traced/blamed to a lack of flexible construction techniques for cubical polytopes, and for more general cubical complexes (such as the "hexahedral meshes" that are of great interest in CAD and in Numerical Analysis).
For instance, there is no cubical analog to the *bipyramid* construction which produces a simplicial $(d+1)$-polytope from any given simplicial d-polytope, as illustrated in Figure 5.
This thesis concentrates on construction techniques for cubical polytopes and cubical balls. However, we do not try to present a complete list of all constructions. Instead we take challenging (open) problems, mainly existence problems for cubical polytopes, and try to give some appropriate constructions. In particular, we tackle the following questions:

- ▷ Is there a natural construction that produces a cubical $(d+1)$-polytope from any given cubical d-polytope?
- ▷ Are there cubical 4-polytopes that are *not edge-orientable* in the sense of Hetyei (that is, there is no orientation of the edges such that in each 2-face of P opposite edges are parallel)?
- ▷ Is there a cubical 4-polytope with a non-orientable dual manifold?

The primary objects we deal with are *regular cubical balls*, that is, cubical d-balls $\mathcal{B}$ subdividing a convex d-polytope $P = |\mathcal{B}|$ that are "regular" (also known as "coherent" or "projective"), that is, they admit concave lifting functions $f : P \to \mathbb{R}$ whose domains of linearity are given by the facets of the complex. Here our point of view is that it always pays off to carry along convex lifting functions of high symmetry. Therefore we introduce the notion of a *lifted ball*, that is a pair of a regular polytopal (in most cases cubical) subdivision and a convex lifting function.

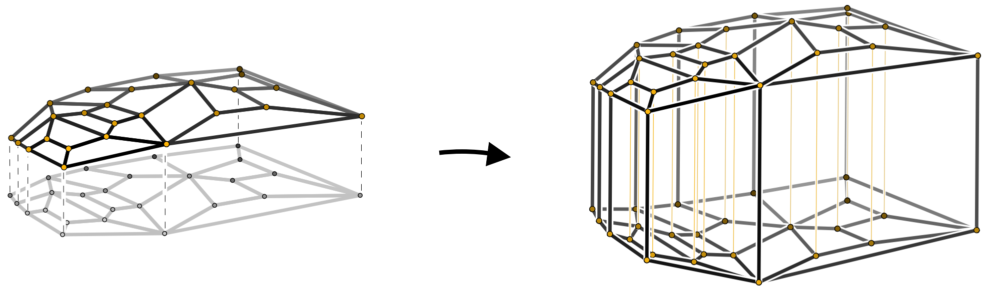


FIGURE 6: The lifted prism construction. The input is a *lifted* cubical subdivision of a d-polytope P, that is, a pair consisting of a regular cubical subdivision of P and a convex lifting function (of this subdivision). The outcome is a cubical $(d + 1)$-dimensional ball whose convex hull gives a cubical $(d + 1)$-polytope (provided the polytopal subdivision does not subdivide the boundary of P).

One of our 'main ingredients' for the construction of cubical polytopes is the *lifted prism construction* which yields a cubical $(d + 1)$-polytope from any given lifted cubical ball; compare Figure 6.
Although some of our descriptions and all our pictures are necessarily low-dimensional, many of our results do generalize to higher dimensions.

All of our constructions are implemented, and our results can be verified by examining the instances available from

`http://www.math.tu-berlin.de/~schwartz/c4p`

The production of the computer models of our instances was done using the polymake system by GAWRILOW & JOSWIG [26], a system for the construction and analysis of convex polytopes. Furthermore, a lot of tools that I wrote myself for handling cubical complexes are involved. They cover creation, verification, and visualization of cubical complexes (for $d \in \{2, 3\}$). Nevertheless, the instances can be verified by using only standard polymake tools which have been used (and thereby verified) by various users over the past years.

New Results in this thesis

- We present a simple construction which yields a rather small cubical 4-polytope (with 72 vertices and 62 facets) for which the immersed dual 2-manifold is not orientable: One of its components is a Klein bottle (Theorem 7.1). Apparently this is the first example of a cubical polytope with a non-orientable dual manifold. Thus its existence also confirms a conjecture of Hetyei.

- Our *generalized regular Hexhoop* construction (Construction 9.5) gives a generalization of the *Hexhoop template* [58] which is known from the hexa meshing community. Our approach yields a cubification (a cubical polytopal subdivision with prescribed boundary) of an input polytope with a boundary subdivision that is symmetric with respect to a given hyperplane.
  Furthermore, the construction yields a regular cubification with “prescribed heights on the boundary” (with a symmetry requirement); compare Theorem 9.4.

- Using the generalized regular Hexhoop construction, we prove that all PL-types of normal crossing immersions of closed 2-manifolds appear as dual manifolds in the boundary complexes of cubical 4-polytopes; compare Theorem 10.3. (This result is a polytopal analog to the Babson-Chan construction restricted to 2-manifolds.) In the case of non-orientable 2-manifolds of odd genus, our construction yields cubical 4-polytopes with an odd number of facets.

- In particular, we construct an explicit example with 19 520 vertices and 18 333 facets of a cubical 4-polytope with a cubation of *Boy’s surface* (that is, an immersion of the real projective plane with exactly one triple point and three double-intersection loops) as a dual manifold immersion (Theorem 11.1).

- From this, we also obtain a complete characterization of the lattice of f-vectors of cubical 4-polytopes (Corollary 12.1).

- Via Schlegel diagrams, this implies that for every cubification of a 3-dimensional domain there is also a cubification of the opposite parity. Thus the flip graph for hexahedral meshes has at least two connected components (Theorem 13.4). This answers questions by Eppstein and Thurston.

Organization of this thesis

This thesis consists of four parts.

A Basics. In Chapter 1 we recall terminology concerning convex polytopes, polytopal complexes and manifolds. A brief introduction to cubical polytopes, including a survey of known results, is given in Chapter 2. Chapter 3 is devoted to *hexa meshing* and cubical balls. The *dual manifolds* of cubical polytopes, the most important concept of this thesis, are defined in Chapter 4.

B Elementary constructions. The second part covers rather elementary construction techniques, as well as some results that we achieved using these constructions. In Chapter 6 we present a number of new construction techniques for cubical polytopes and cubical balls. The construction of a small cubical 4-polytope with a dual Klein bottle is described in Chapter 7. Further consequences are presented in Chapter 8. This includes

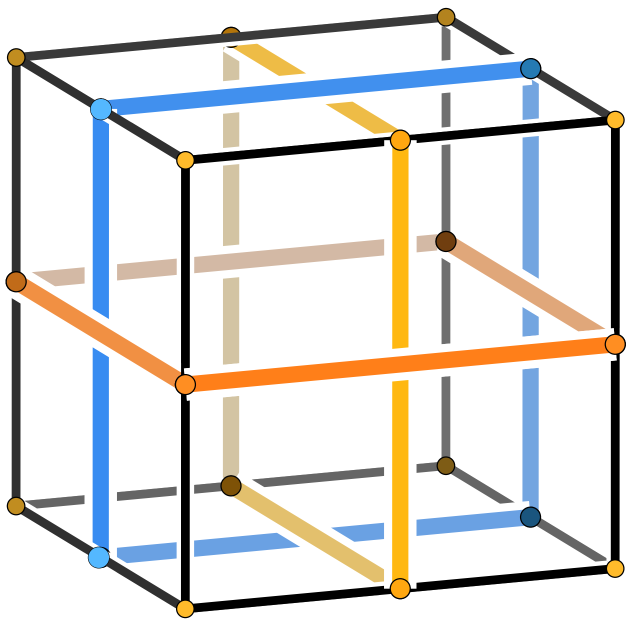
- the existence of cubical 4-polytopes with oriented dual surfaces of prescribed genus, and
- a discussion of cubical polytopes with few dual manifolds.

C Main results. This part starts with the construction of "cubifications" (Chapter 9). Our main result is given in Chapter 10: All PL-types of normal crossing immersions of closed surfaces appear as dual manifolds in the boundary complexes of cubical 4-polytopes (Theorem 10.3). In Chapter 11 we discuss the existence of cubical polytopes and spheres with an odd number of facets. In particular, we describe the construction of an instance of cubical 4-polytope that has a dual Boy's surface and an odd number of facets.
Further consequences are presented in Chapter 12, and applications to hexa meshing are discussed in Chapter 13.

D Towards higher dimensions. In Chapter 14 we sketch two further constructions, namely the *encapsulated Schlegel cap* and the *mirrored encapsulated Schlegel cap*, and give some remarks on the construction of cubical 5-polytope with prescribed dual 3-manifolds.

Part I

Basics



Chapter 1

Definitions

We assume that the readers are familiar with the basic combinatorics and geometry of convex polytopes. In particular, we will be dealing with cubical polytopes (see Grünbaum [28, Sect. 4.6]), polytopal (e. g. cubical) complexes, regular subdivisions (see Ziegler [59, Sect. 5.1]), and Schlegel diagrams [28, Sect. 3.3] [59, Sect. 5.2]. For regular cell complexes, barycentric subdivision and related notions we refer to Munkres [46] and Bayer [9]. Suitable references for the basic concepts about PL manifolds, embeddings and (normal crossing) immersions include Hudson [31] and Rourke & Sanderson [52].

Nevertheless we give a brief overview of the used notation, based on [59].

Basics. We use bold symbols for vectors, points, row vectors, etc. By $\boldsymbol{e}_k$ we denote the k-th *unit vector* (in $\mathbb{R}^d$, where the dimension d is known from the context) as a column vector, and by $\mathbb{1} = (1,\ldots,1)^T = \sum_k \boldsymbol{e}_k$ and $\mathbf{0} = (0,\ldots,0)^T$ the column vectors of all ones, respectively, of all zeros.
For an arbitrary set $X \subset \mathbb{R}^d$ we denote by ∂X its *boundary*, by $\mathrm{int}(X)$ its *interior*, and by $\mathrm{relint}(X)$ its *relative interior.*
The *affine hull*, *convex hull* and the *conical hull* (or *cone* for short) of X are known to be

$$\begin{aligned}
\mathrm{aff}(S) &= \left\{ \sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i \;:\; \{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\} \in X,\; \sum_{i=1}^{n} \lambda_i = 1 \right\}\\
\mathrm{conv}(S) &= \left\{ \sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i \;:\; \{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\} \in X,\; \sum_{i=1}^{n} \lambda_i = 1, \lambda_1,\ldots,\lambda_n \geq 0 \right\},\\
\mathrm{cone}(X) &= \left\{ \sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i \;:\; \{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\} \in X,\; \lambda_1,\ldots,\lambda_n \geq 0 \right\},
\end{aligned}$$

respectively. Clearly $\mathbf{0}$ is contained in cone(X). For a point $\boldsymbol{p} \in \mathbb{R}^d$, the *cone with apex point $\boldsymbol{p}$ spanned by* X is defined as

$$\begin{aligned}\operatorname{cone}(\boldsymbol{p}, X) &:= \left\{\boldsymbol{p} + \sum_{i=1}^{k} \lambda_i(\boldsymbol{x}_i - \boldsymbol{p}) \;:\; \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \in X,\ \lambda_1, \ldots, \lambda_k \geq 0\right\} \\ &= \boldsymbol{p} + \operatorname{cone}(X - \boldsymbol{p}).\end{aligned}$$

The *(closed) halfspace* H_+ determined by a hyperplane

$$H = \left\{\boldsymbol{x} \in \mathbb{R}^\ell \;:\; \boldsymbol{a}^T\boldsymbol{x} = a_0\right\},$$

where $a_0 \in \mathbb{R}$ and $\boldsymbol{a}^T \in (\mathbb{R}^\ell)^*$ denotes a row vector of length ℓ, is defined as

$$H_+ = \left\{\boldsymbol{x} \in \mathbb{R}^\ell \;:\; \boldsymbol{a}^T\boldsymbol{x} \geq a_0\right\}.$$

The *open halfspace* determined by H is $H_+ \setminus H$.

Polytopes. A *polytope* $P \subset \mathbb{R}^\ell$ is the convex hull of a finite number of points in $\mathbb{R}^\ell$. Equivalently, a polytope can be described as the bounded intersection of finitely many closed halfspaces $H_+ = \left\{\boldsymbol{x} \in \mathbb{R}^\ell \;:\; \boldsymbol{a}^T\boldsymbol{x} \geq a_0\right\}$. The *dimension* $\dim(P)$ of a polytope is the dimension of its affine hull. We refer to d-dimensional polytopes as *d-polytopes*.
A linear inequality $\boldsymbol{a}^T\boldsymbol{x} \leq a_0$ is *valid* for a polytope P if it is satisfied by all points of P. A *face* of a polytope P is a subset $F \subseteq P$ of the form

$$F = P \cap \left\{\boldsymbol{x} \in \mathbb{R}^\ell \;:\; \boldsymbol{a}^T\boldsymbol{x} = a_0\right\},$$

where $\boldsymbol{a}^T\boldsymbol{x} \geq a_0$ is a valid inequality for P. Again, the *dimension* of a face is the dimension of its affine hull. A face of dimension k is called a *k-face* of P. A *facet* is face of dimension $\dim(P) - 1$.
Note that the empty set and P are both faces of P, where the valid inequalities are $\mathbf{0}^T\boldsymbol{x} \leq 1$ and $\mathbf{0}^T\boldsymbol{x} \leq 0$. All other faces are called *proper faces* of P. By $\mathcal{F}(P)$ we denote the set all faces of P.

A hyperplane H in $\mathbb{R}^d$ *separates* two sets $A, B \subset \mathbb{R}^d$ if A and B are contained in different open halfspace determined by H. Let P be a d-polytope, F a facet of P, and H a hyperplane such that $F = H \cap P$. We say a point $\boldsymbol{p}$ is *beneath* a facet F of P if both P and $\boldsymbol{p}$ in the same open halfspace determined by H, and *beyond* if they are in different open halfspaces determined by H.

Two polytopes are *combinatorially isomorphic* (also known as *combinatorially equivalent*) if their face lattices are isomorphic. A *combinatorial d-cube* is a d-polytope which is combinatorially isomorphic to the standard cube. Two polytopes $P \subset \mathbb{R}^{d'}$, $Q \subset \mathbb{R}^{d''}$ are called *affinely* or *projectively isomorphic* if there is an affine or projective, respectively, map from $\mathbb{R}^{d'}$ to $\mathbb{R}^{d''}$ inducing a bijection between P to Q. Two polytopes are *congruent* if there is an isomorphism between them that is induced by an orthogonal matrix (plus a translation). The chain of implications "congruent $\Rightarrow$ affinely isomorphic $\Rightarrow$ projectively isomorphic $\Rightarrow$ combinatorially isomorphic" is well-known.

A d-polytope is *simple* if all vertices have valence d, where the *valence* of a vertex is the number of adjacent vertices in the graph. A d-polytope is *simplicial* if its dual polytope is simple, or equivalently, if each facet is a $(d-1)$-simplex.

Polytopal complexes. A *polytopal complex* $\mathcal{C}$ is a finite set of polytopes (called *faces* or *cells*) in some $\mathbb{R}^n$ that contains all the faces of its cells and that satisfies the *intersection property*, that is, the intersection of any two cells $P, Q \in C$ is a face of both. This implies that the empty set is contained in $\mathcal{C}$, and each face F of a cell $P \in \mathcal{C}$ is contained in $\mathcal{C}$. The *support* or *underlying set* of $\mathcal{C}$ is $|\mathcal{C}| := \bigcup\{F : F \in \mathrm{fac}(\mathcal{C})\}$.
A complex is *pure* if all *facets*, i.e. the maximal cells with respect to inclusion, have the same dimension. The *dimension* of a pure polytopal complex is the dimension of the facets. In the following, all polytopal complexes will be pure.
The cells of dimension k of a d-dimensional complex $\mathcal{C}$ are called *k-cells* or *k-faces* of $\mathcal{C}$. The faces of dimension 0, 1, $d-1$, and d are the *vertices*, *edges*, *ridges* and *facets*, respectively. By $\mathcal{F}_k(\mathcal{C})$, for $k \in \{-1, \dots, d\}$, we denote the set of k-cells of a d-dimensional complex, and by $\mathrm{vert}(\mathcal{C})$, $\mathrm{edges}(\mathcal{C})$, $\mathrm{ridges}(\mathcal{C})$, and $\mathrm{fac}(\mathcal{C})$ the sets of vertices, edges, ridges and facets of $\mathcal{C}$, respectively. The *face poset* of a complex is the set of its faces ordered by inclusion.
A d-dimensional complex $\mathcal{C}$ is a *subcomplex* of a d'-dimensional complex $\mathcal{C}'$ if $C \in \mathcal{C}'$ for every $C \in \mathcal{C}$. For a set of k-faces $\mathcal{S} \subset \mathcal{F}_k(\mathcal{C})$, the k-dimensional subcomplex *formed* (or *induced*) by $\mathcal{S}$ is the subcomplex of $\mathcal{C}$ given by all k-faces $F \in \mathcal{S}$ and all their faces. The *k-skeleton* $\mathcal{F}_{\leq k}(\mathcal{C})$ is the k-dimensional subcomplex of $\mathcal{C}$ formed of by all faces of $\mathcal{C}$ of dimension at most k. The 1-skeleton is the *graph* $\mathcal{G}(\mathcal{C})$ of $\mathcal{C}$.
A complex $\mathcal{S}$ is a *subdivision* (or *refinement*) of a complex $\mathcal{T}$ if every facet F of $\mathcal{S}$ is contained in a facet of $\mathcal{T}$, and $|\mathcal{S}| = |\mathcal{T}|$. A *polytopal subdivision of a d-polytope* P is a subdivision of complex with P as a single facet. Hence a polytopal subdivision of P is a polytopal complex $\mathcal{B}$ with $P = |\mathcal{B}|$.
The *barycentric subdivision* $\mathrm{sd}\,(\mathcal{C})$ of a complex $\mathcal{C}$ is a *simplicial complex*,

that is, a complex whose facets are all simplicies, with one vertex for each (barycenter of a) proper cell of $\mathcal{C}$, and a proper face for each chain of proper elements of $\mathcal{C}$.
A complex homeomorphic to the standard unit of some dimension ball is called a *ball*, and a complex homeomorphic to the standard unit sphere is called a *sphere*.
An *abstract cell* complex is a regular cell complexes; compare Munkres [46].

Face and flag numbers. The *f-vector* of a d-dimensional complex $\mathcal{C}$ is the vector

$$f(\mathcal{C}) = (f_0, f_1, \ldots, f_d),$$

where $f_k = f_k(\mathcal{C})$ denotes the number of k-faces of P. In order to simplify f-vector calculations we use the additional values $f_k(\mathcal{C}) = 0$ for $k \notin \{0, \ldots, d\}$.
The *flag vector* (or *extended f-vector*) of a d-dimensional complex $\mathcal{C}$ is the set of all *flag numbers* $f_S = f_S(\mathcal{C})$, $S = \{s_1, \ldots, s_\ell\} \subseteq \{0, \ldots, d\}$, where $f_S(\mathcal{C})$ denotes the number of chains $F_1 \subset F_2 \subset \ldots \subset F_\ell$ of faces of $\mathcal{C}$ with $\dim F_k = s_k$. We use abbreviations like $f_{01} = f_{\{0,1\}}$.

Boundary complexes of polytopes. By $\mathcal{C}(P)$ we denote the polytopal complex given by a d-polytope P and all its faces. By $\mathcal{C}(\partial P)$ we denote the *boundary complex* of P, consisting of all proper faces of P. Clearly $\mathcal{C}(\partial P)$ is a $(d-1)$-sphere. The *f-vector* of a d-polytope P is the f-vector of its boundary complex:

$$f(P) \;:=\; f(\mathcal{C}(\partial P)) \;=\; (f_0, f_1, \ldots, f_{d-1}).$$

The *Schlegel complex* of a pair (P, F) consisting of a d-polytope P and a facet F of P is the complex induced by all facets except F, that is, $\mathcal{C}(\partial P)\backslash\{F\}$.

Manifolds. We use *PL* as an abbreviation for *piecewise linear*. A d-dimensional abstract or polytopal complex $\mathcal{M}$ is a *manifold of dimension d* (or simply *d-manifold*) if each point $\boldsymbol{x} \in \mathcal{M}$ has a neighborhood in $\mathcal{M}$ that is PL-homeomorphic to a d-polytope. A manifold is *closed* if it is compact and has no boundary. A *surface* is a closed 2-manifold.
An *immersion* $j : \mathcal{M} \looparrowright \mathbb{R}^d$ of a manifold $\mathcal{M}$ is a map such that the image of each k face is PL-homeomorphic to a k-polytope. The *degree* of a point $\boldsymbol{p} \in \mathbb{R}^d$ is defined as $\deg(\boldsymbol{p}) := \#\{\boldsymbol{x} \in \mathcal{M} \,:\, j(\boldsymbol{x}) = \boldsymbol{p}\}$. A point $\boldsymbol{p} \in \mathbb{R}^d$ with $\deg(\boldsymbol{p}) \geq 2$ is a *multiple-intersection point* of j. An immersion $j : \mathcal{M} \looparrowright \mathbb{R}^d$ is *normal crossing*, if each multiple-intersection point is of degree $k \leq d$

and there is a neighborhood of each multiple-intersection point that is PL-homeomorphic to (a neighborhood of) a point which is contained in k pairwise perpendicular hyperplanes. Two normal-crossing immersion are *PL-equivalent* if their images are PL-homeomorphic.
An *embedding* $j : \mathcal{M} \hookrightarrow \mathbb{R}^d$ is an immersion such that $\deg(\boldsymbol{p}) \leq 1$ for each $\boldsymbol{p} \in \mathbb{R}^d$.

Chapter 2

Introduction to cubical polytopes

2.1 Cubical polytopes

The main objects we study in this thesis are cubical polytope like the ones illustrated in the following figure.

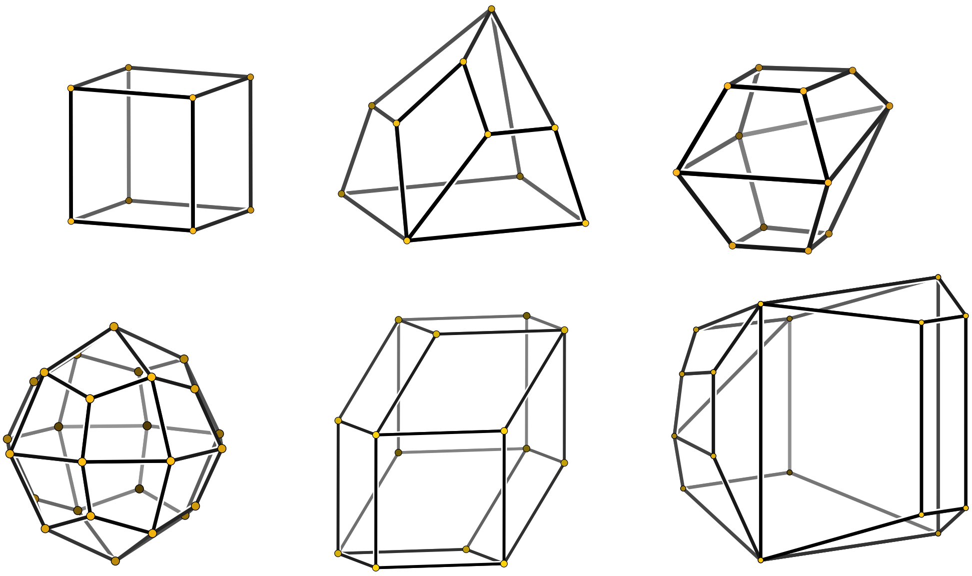


FIGURE 2.1: Some three-dimensional cubical polytopes.

Combinatorial cubes

A *combinatorial d-cube* is a d-polytope combinatorially to the *standard d-cube* (or *regular d-cube*)

$$Q^d := \operatorname{conv}((\pm 1, \pm 1, \ldots, \pm 1)) = [-1, 1]^d.$$

The face numbers of (standard) cubes can be calculated recursively according to

$$f_k(Q^{d+1}) = 2f_k(Q^d) + f_{k-1}(Q^d), \qquad 0 \le k \le d,\ 0 < d.$$

Here (and in the sequel), to simplify f-vector calculations we use the notion $f_k = 0$ if $k < 0$, and, if P is a d-polytope, $f_d(P) = 1$ and $f_k(P) = 0$ for $k > d$. Resolving the recursion yields that the f-vector of a combinatorial cube P can be calculated as

$$f_k(P) = 2^{d-k}\binom{d}{k}, \qquad 0 \le k < d.$$

The following properties are trivial but very essential:

- Every facet F of a combinatorial cube has a unique disjoint facet, the so-called *opposite facet*.
- The set of edges of a (combinatorial) d-cube has a partition into d parallel classes of edges.
- For every pair F, F' of opposite facets corresponds to a parallel class $[e]$ of edges: The set of edges $[e]$ consists of all edges which are not contained in F or in F'.

Two d-polytopes $P \subset \mathbb{R}^{d'}$, $Q \subset \mathbb{R}^{d''}$ are *projectively isomorphic* if there is a projective transformation from $\mathbb{R}^{d'}$ to $\mathbb{R}^{d''}$ inducing a bijection between P to Q.

Any two combinatorial 2-cubes $P \subset \mathbb{R}^{d'}$, $Q \subset \mathbb{R}^{d''}$ are projectively isomorphic. However, this does not extend to higher-dimensional cubes. Even in dimension three, there are distinct isomorphism classes of cubes with respect to projective isomorphisms.

To see this, take for instance a regular 3-cube, and a truncated pyramid over a non-regular quadrangle such that the base and the top facet are parallel. These two combinatorial 3-cubes are not projectively isomorphic, since for the second one, there is a pair of opposite edges of the base facet such that the lines containing the edges do not intersect in the same point as the lines containing the corresponding edges in the top facet. (This relies on the property that intersections of lines considered in $\mathbb{RP}^3$ are invariant under projective transformations.)

Polytopes without triangular faces

A lot of results about cubical polytopes are due to Gerd and Roswita Blind (see Section 2.6). In [11] they examined a class of polytopes which is closely related to cubical polytopes, the class of d-polytopes without triangular 2-faces, or equivalently, without a face which is a pyramid. Using the fact that every facet F of a d-polytope P without triangular 2-faces there is disjoint facet F' (which is not necessarily unique), they derive lower bounds for the face numbers.

Lemma 2.1 (Blind & Blind [11])
Let P be a d-polytope without triangular 2-faces. Then

$$f_k(P) \geq f_k(Q^k) = 2^{d-k}\binom{d}{k}, \qquad 0 \leq k < d.$$

Furthermore, if equality is attained for some k, then P is a combinatorial d-cube.

Later in this thesis this lemma is used to verify that some of the 4-polytopes we construct are indeed cubical (without calculating the whole face lattices).

Cubical polytopes and k-cubical polytopes

A d-dimensional polytope is *cubical* if all facets are combinatorial cubes, or equivalently, if all proper k-faces are combinatorial k-cubes. Every 2-polytope is simple and cubical.
Every simple and simplicial 2-polytope is simplex or an n-gon [59, Exercise 0.1, p. 23]. A similar result holds for simple cubical polytopes.

Lemma 2.2 (Simple cubical polytopes; c.f. [59, Exercise 0.1, p. 23])
For $d \geq 3$, every simple cubical d-polytope is a d-cube.

A d-dimensional polytope is k-*cubical* for some $0 \leq k \leq d-1$ if all j-faces, for $0 < j \leq k$, are combinatorial j-cubes. Hence, every d-polytope is 1-cubical, and every cubical polytope is $(d-1)$-cubical. A d-dimensional polytope is k-*cocubical* if its dual polytope is k-cubical.

Remark 2.3. (Prisms over cubical polytopes)
The *prism* over a cubical d-polytope P, that is, the product of P and an interval, yields a $(d+1)$-polytope Q whose facets are the facets of P, plus two copies of P. If P is a combinatorial cube, then Q is a combinatorial cube, too. Otherwise, Q is $(d-1)$-cubical, but not cubical.
(Similarly, the prism over a k-cubical d-polytope yields a k-cubical $(d+1)$-polytope.)

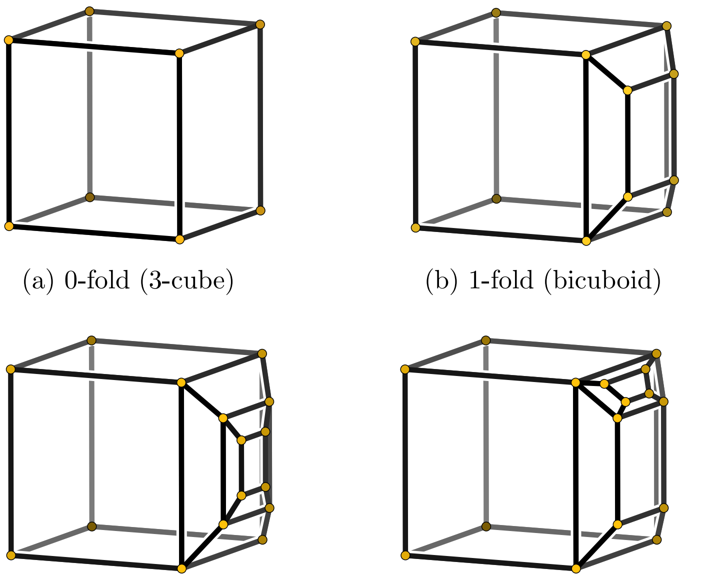
This yields the following question.

Question 2.4 (Joswig/Ziegler 2001)
Is there a natural construction that produces a cubical $(d + 1)$*-polytope from any given cubical* d*-polytope?*

We give (partial) answers to this question in Section 5.8.1.

2.2 Capping and capped cubical polytopes

A well-known class of cubical polytopes are the *capped cubical polytopes* (also known as *stacked cubical polytopes*) which we define and examine in this section (compare Figure 2.2).



(a) 0-fold (3-cube) (b) 1-fold (bicuboid)

(c) 2-fold linearly capped (d) 2-fold non-linearly capped

FIGURE 2.2: Some capped cubical 3-polytopes: The 0-fold capped cubical 3-polytope is a (combinatorial) 3-cube. There is one combinatorial type of a 1-fold capped cubical 3-polytope, known as the *bicuboid*. There are exactly two combinatorial types of 2-fold capped cubical 3-polytopes: the *lineary* and the *nonlineary* one. The lineary one is *almost simple*, that is every vertex is 3-valent or 4-valent.

This class of polytopes is an cubical analogon to the class of *stacked polytopes*, one of the classical classes of simplicial polytopes. An *stacked* d-polytope with n vertices [39] is either a d-simplex, or the outcome of the *stack operation* over a facet F of a stacked d-polytope P with $n-1$ vertices, that is, the convex hull of P with an additional point that is beyond the facet F (and beneath all others). These polytopes have the minimum number of facets for simplicial polytope for a given number of vertices.

The cap construction

The cubical analog to the (simplicial) *stack operation* is the *cap operation*. They are various equivalently descriptions for this operations which yield different geometric realizations of the same combinatorial polytope.
Blind & Blind [13] define the *cap operation* operation as follows: A polytope Q is called *capped* over a given cubical polytope P if there is a combinatorial cube C such that $Q = P \cup C$ and $P \cap C$ is a facet of P. The facet of C opposite to $P \cap C$ is called the *cap facet* of Q.
This can be interpreted as a special case of the well-known *connected sum* of two polytopes (of the same dimension); c.f. [59, Sect. 8.6] [51, Sect. 3.2].

Connected sums

Assume P_L and P_R are two d-polytopes that have two projectively isomorphic facets F_L, F_R, respectively. Let us denote the geometric type of F_L and F_R up to projective transformations by F. Then, by projective transformations, one can construct the *connected sum* $P_L \#_F P_R =: Q$ of P_L and P_R at F, which is a d-polytope which has all facets of P_L and P_R, except F_L and F_R. However, the boundary complex of F is still contained in Q. We also say, that Q is obtained by *gluing together* P_L and P_R at F. Compare Figure 2.3. If P_L and P_R are both cubical polytopes, then the polytope $P_L \# P_R$ is cubical, too.

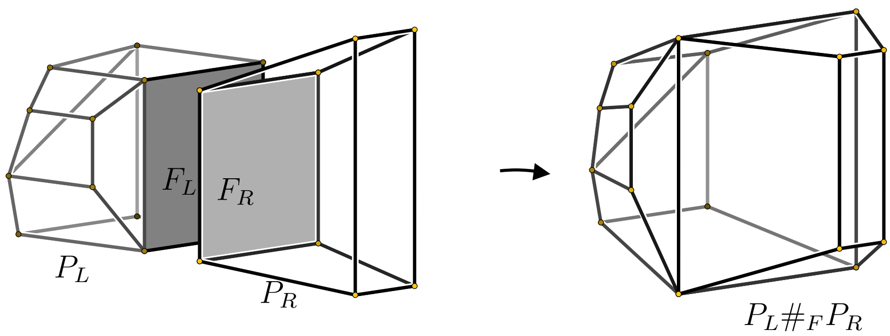


Figure 2.3: The connected sum of two cubical polytopes. In this special case the outcome is a *cap* over the facet F_L of the polytope P_L.

The *capped polytope* over a (not necessarily cubical) d-polytope P and a facet F of P is the connected sum of P and a prism over F based on F.

The truncated stack operation

Alternatively, a cap operation can be described by a stack operation followed by a truncation operation: Let P be a (not necessarily cubical) d-polytope and F a facet of F. Furthermore, let $\boldsymbol{p}$ be a point beyond F (and beneach all other facets of P), and H a hyperplane that separated $\boldsymbol{p}$ from P. Fix a positive halfspace H_+ of H such that $P \subset H_+$.
Then the capped polytope $\operatorname{capped}(P, F)$ over P is given as

$$\operatorname{capped}(P, F) := H_+ \cap \operatorname{conv}(P \cup \{\boldsymbol{p}\}).$$

Capped cubical polytopes

A cubical polytope is called *ℓ-fold capped cubical*, for some $\ell \geq 0$, if it can be obtained from a combinatorial cube by ℓ capping operations; see Blind and Blind [13]. Some 3-dimensional capped cubical polytopes are illustrated in Figure 2.2.
Every 2-polytope with $2n$ vertices is $(n-2)$-capped cubical.
In dimension $d = 3$ there are cubical 3-polytopes that are not capped, but combinatorially isomorphic to capped ones; compare Figure 2.4.

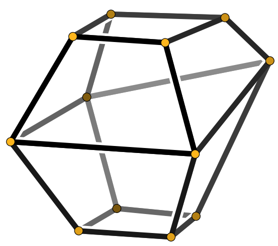


FIGURE 2.4: A cubical 3-polytope that is not capped, but combinatorially isomorphic to the 1-fold capped one.

This does not extend to higher dimensions:

Remark 2.5. (Blind & Blind [13])
For $d \geq 4$, every cubical d-polytope isomorphic to an ℓ-fold capped one is itself an ℓ-fold capped one.

This is an immediate consequence of the "*flatness*" of combinatorial d-cubes, for $d \geq 4$: A combinatorial d-polytope P is called *necessarily flat* if every polyhedral embedding of its $(d-1)$-skeleton in $\mathbb{R}^n$, $d < n$, has affine dimension at most d; compare [51]. (A *polyhedral embedding* of a k-dimensional cell complex $\mathcal{C}$ into $\mathbb{R}^n$ is an mapping of the vertices of $\mathcal{C}$ into $\mathbb{R}^n$ such that the image of every ℓ-face, $2 \leq \ell \leq k$, is an ℓ-dimensional convex polyhedron, and for no two faces the images intersect in their relative interiors.) In [51] Richter-Gebert proved that 3-dimensional prisms are necessarily flat. We adapt his proof and extend the result to simple d-polytopes, $d \geq 3$.

Lemma 2.6 *For $d \geq 3$, every simple d-polytope is necessarily flat.*

Proof. Let Q be a simple d-polytope, $d \geq 3$, and P a realization of the $(d-1)$-skeleton of Q in $\mathbb{R}^n$, $d < n$. We denote by $\mathrm{N}_P(\boldsymbol{x})$ the set of neighbors of the node $\boldsymbol{x}$ in $\mathcal{G}(P)$ and define $\overline{\mathrm{N}}_P(\boldsymbol{x}) := \mathrm{N}_P(\boldsymbol{x}) \cup \{\boldsymbol{x}\}$. In the following we show that for every vertex $\boldsymbol{u}$ we have $\mathrm{aff}(P) = \mathrm{aff}(\overline{\mathrm{N}}_P(\boldsymbol{u}))$.
We use two observations:

(a) Since P is simple, every vertex $\boldsymbol{v}$ of P is in d facets, and has d neighbor vertices. Furthermore, for every vertex $\boldsymbol{w} \in \mathrm{N}_P(\boldsymbol{v})$ there is a unique facet $F_{\boldsymbol{v},\boldsymbol{w}}$ that contains $\overline{\mathrm{N}}_P(\boldsymbol{v}) \setminus \{\boldsymbol{w}\}$ (and does not contain $\boldsymbol{w}$).

(b) Since P is a realization of the $(d-1)$-skeleton of Q, every facet F of P is a simple convex polytope. Hence $\mathrm{aff}(F) = \mathrm{aff}(\overline{\mathrm{N}}_F(\boldsymbol{u}))$ for every vertex $\boldsymbol{u}$ of F.

Let $\boldsymbol{u}$ be a vertex of P. Observation (b) implies that every facet F that contains $\boldsymbol{u}$ is contained in $\mathrm{aff}(\overline{\mathrm{N}}_F(\boldsymbol{u})) \subset \mathrm{aff}(\overline{\mathrm{N}}_P(\boldsymbol{u}))$. Hence each vertex $\boldsymbol{w}$ that is contained in facet F that contains $\boldsymbol{u}$ is contained in $\mathrm{aff}(\overline{\mathrm{N}}_P(\boldsymbol{u}))$.
Let $\boldsymbol{v}$ be any neighbor of $\boldsymbol{u}$. By Observation (a) there is a unique facet $F_{\boldsymbol{v},\boldsymbol{u}}$ that contains all vertices $\overline{\mathrm{N}}_P(\boldsymbol{v}) \setminus \{\boldsymbol{u}\}$ but not $\boldsymbol{u}$. By (b) the facet $F_{\boldsymbol{v},\boldsymbol{u}}$ is spanned by the vertices $\overline{\mathrm{N}}_F(\boldsymbol{v})$ which all are known to be in the affine hull $\mathrm{aff}(\overline{\mathrm{N}}_P(\boldsymbol{u}))$. This yields $F_{\boldsymbol{v},\boldsymbol{u}} \subset \mathrm{aff}(\overline{\mathrm{N}}_P(\boldsymbol{u}))$.
Hence a traversal of the graph of P starting at the vertex $\boldsymbol{u}$ yields $\mathrm{aff}(P) = \mathrm{aff}(\overline{\mathrm{N}}_P(\boldsymbol{u}))$. Hence, for $d \geq 4$, every simple d-polytope is necessarily flat. ☐

Lemma 2.7 (f-vectors of capped cubical polytopes)

(i) Let Q be a capped d-polytope over a cubical d-polytope P. Then the f-vector of Q satisfies

$$\begin{aligned} f_k(Q) &= f_k(P) + f_k(Q^d) - f_k(Q^{d-1}) \\ &= f_k(P) + 2^{d-k}\binom{d}{k} - 2^{d-k-1}\binom{d-1}{k}, \end{aligned}$$

for $k \in \{0, \ldots, d-1\}$.

(ii) Let P be a ℓ-fold capped cubical d-polytope. Then

$$\begin{aligned} f_k(P) &= \ell(f_k(Q^d) - f_k(Q^{d-1})) + f_k(Q^d) \\ &= (\ell+1)2^{d-k}\binom{d}{k} - \ell 2^{d-k-1}\binom{d-1}{k} \end{aligned}$$

A well-known theorem of Blind and Mani [14] states that the combinatorial structure of every simple polytope P is uniquely determined by its graph, that is, the vertex-facet incidences of P can be reconstructed (up to isomorphisms) from the graph of G. In [38] Kalai gave a very short and elegant proof of this result using the concept of acyclic orientations. Since the dual polytope of a simplicial polytope P is simple (and the vertex-facet incidence matrix of P is the transposed vertex-facet incidence matrix of the dual polytope of P), the combinatorial structure of every simplicial polytope is uniquely determined by its dual graph. It is natural to ask whether this extends to cubical polytopes.

Conjecture 2.8 (Joswig [34])
Every cubical polytope is determined by its dual graph.

The conjecture is proven for the special case of capped cubical polytopes.

Theorem 2.9 (Joswig [34])
Every ℓ-fold capped cubical polytope is determined by its dual graph.

We remark that the combinatorial structure of every capped cubical polytope can be reconstructed from its graph (rather from its dual graph).

Lemma 2.10 *Every ℓ-fold capped cubical polytope is determined by its graph.*

Proof. If Q is a cubical d-polytope over a cubical polytope P, then the 'cap' C (such that $Q = P \cup C$) can be recognized using the following observation: If W is a subset of d-valent vertices of Q such that the induced subgraph is connected, then P is a capped polytope over the convex hull $P := \text{conv}(\text{vert}(Q) \setminus W)$.
Hence we can reconstruct a capped cubical d-polytope P from its graph by iteratively recognizing the caps. □

2.3 Three-dimensional cubical polytopes

All entries of the f-vector and the flag-vector of a cubical 3-polytope are given in terms of f_0 according to

$$\begin{array}{rcl} f_1 & = & 2\,(f_0 - 2) \\ f_2 & = & f_0 - 2 \\ f_{01} & = & 4\,(f_0 - 2) \\ f_{02} & = & 4\,(f_0 - 2) \\ f_{12} & = & 4\,(f_0 - 2) \\ f_{012} & = & 8\,(f_0 - 2). \end{array}$$

This follows from Lemma 2.13. (Recall that we use abbreviations like $f_{12} := f_{\{1,2\}}$.)

Every planar, 3-connected, 4-valent and simple graph is the dual graph of a cubical 3-polytope. Hence, for small numbers of vertices the combinatorial types of cubical 3-polytopes can be enumerated; compare Table 2.1.

f_0	f_1	f_2	#
8	12	6	1
9	14	7	0
10	16	8	1
11	18	9	1
12	20	10	3
13	22	11	3
14	24	12	11
15	26	13	18
16	28	14	58
17	30	15	139
18	32	16	451

f_0	f_1	f_2	#
19	34	17	1 326
20	36	18	4 461
21	38	19	14 554
22	40	20	49 957
23	42	21	171 159
24	44	22	598 102
25	46	23	2 098 675
26	48	24	7 437 910
27	50	25	26 490 072
28	52	26	94 944 685
29	54	27	341 867 921
30	56	28	1 236 864 842

TABLE 2.1: The numbers of combinatorial types of cubical 3-polytopes with up to 30 vertices, calculated by plantri due to Brinkmann and McKay [16].

The set of f-vectors of cubical 3-polytopes is characterized by the following well-known result.

Lemma 2.11 (well known)
An integer vector (f_0, f_1, f_2) is the f-vector of a cubical 3-polytope if and only if it satisfies

$$\begin{array}{rclcrcl} f_0 & \geq & 8, & \qquad & f_1 & = & 2\,(f_0 - 2), \\ f_0 & \neq & 9, & & f_2 & = & f_0 - 2. \end{array}$$

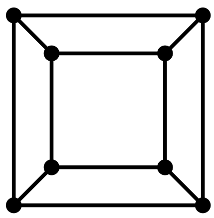


FIGURE 2.5: The graphs of the cubical 3-polytopes with $8, 10$ and 11 vertices.

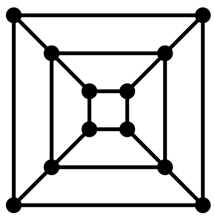


FIGURE 2.6: The graphs of the three combinatorial types of cubical 3-polytopes with 12 vertices.

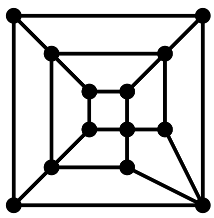


FIGURE 2.7: The graphs of the three combinatorial types of cubical 3-polytopes with 13 vertices.

Proof. By Lemma 2.1 every cubical 3-polytope has at least 8 vertices. All combinatorial types of cubical 3-polytopes with $f_0 \in \{8, 10, 11, 12, 13\}$ vertices are shown in the Figures 2.5, 2.6 and 2.7. For $n \geq 14$, a cubical 3-polytope with n vertices is given by a capped polytope over a cubical 3-polytope with $n-4$ vertices. Hence, for every $n \leq 10$ there is a cubical 3-polyope with n vertices.

It remains to show that there is no cubical 3-polytope with 9 vertices. Assume P is a cubical 3-polytope with 9 vertices. The valence of a vertex of a 3-polytope is at least 3. If there is a vertex of valence at least 5, or at least two vertices of valence 4, that P has more than 9 vertices.

Hence, assume P has one vertex $\boldsymbol{v}_1$ of valence 4. Let F be an arbitrary facet of P with vertices $\boldsymbol{v}_1, \dots, \boldsymbol{v}_4$. Then each of the vertices $\boldsymbol{v}_2, \dots, \boldsymbol{v}_4$ has a neighbor $\boldsymbol{v}_5, \dots, \boldsymbol{v}_7$, and $\boldsymbol{v}_1$ has two neighbors $\boldsymbol{v}_8$ and $\boldsymbol{v}_9$. All these vertices are different. On the other hand, there is a facet G which contains the vertices $\boldsymbol{v}_1, \boldsymbol{v}_8$ and $\boldsymbol{v}_9$. Hence, there is another vertex $\boldsymbol{v}_{10}$ which is contained in G but not belongs to $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_9\}$. This implies that there is no cubical 3-polytope with 9 vertices. $\square$

2.4 Face and flag numbers

For every d-polytope P the famous *Euler-Poincaré* formula [59, Proposition 8.17] holds:

$$\sum_{k=0}^{d-1}(-1)^k f_k(P) = 1-(-1)^d.$$

Furthermore, the f-vector of a cubical polytope is subject to restrictions that are similar to the Dehn-Sommerville equations for simplicial polytopes [59, Theorem 8.21].

Proposition 2.12
(Cubical Dehn-Sommerville equations; Grünbaum [28, Sect. 9.4.1])
Let $(f_0, \ldots, f_{d-1})$ be the f-vector of a cubical d-polytope. Then, for $0 \leq k \leq d-2$,

$$\sum_{i=k}^{d-1}(-1)^i 2^{i-k}\binom{i}{k} f_i \quad = \quad (-1)^{d-1} f_k.$$

The flag vector of a cubical polytope is determined by its f-vector.

Lemma 2.13 (Flag numbers of cubical polytopes)
Let P be a cubical d-polytope. Then the flag vector of P is determined by its f-vector. In particular, we have

$$f_{0k}(P) = f_0(Q^k) f_k(P) = 2^k f_k(P), \qquad 0 < k < d, \tag{2.1}$$

and for every $S = \{i_1, \ldots, i_\ell\} \subset \{0, \ldots, d-1\}$ we have

$$f_{\{i_1,\ldots,i_\ell\}}(P) = f_{i_\ell}(P) \prod_{k=1}^{\ell-1} f_{i_k}(Q^{i_{k+1}}). \tag{2.2}$$

Proof. Let $k \in \{1, \ldots, d-1\}$. Then f_{0k} denotes the number of incidences between vertices and k-faces of P. Since every k-face of P is a combinatorial k-cube the number f_{0k} equals the product of the number of k-faces of P and the number of vertices of the k-cube. Hence (2.1) is satisfied.
Similarly, for $S = \{i_1, \ldots, i_\ell\} \subset \{0, \ldots, d-1\}$ we obtain the equation

$$f_{\{i_1,\ldots,i_\ell\}}(P) = f_{\{i_1,\ldots,i_{\ell-1}\}}(Q^{i_\ell}) f_{i_\ell}(P)$$

Since each cube is cubical, this gives a recursion which yields (2.2). $\square$

Hence, whenever we discuss constructions of cubical polytopes (as well as cubical balls) in the sequel, we can omit the calculation of the flag vector.

Corollary 2.14 (Face and flag numbers of cubical 4-polytopes)
Let P be a cubical 4-polytope. Then its face numbers $f_k = f_k(P)$, $k \in \{0,\dots,d-1\}$, and flag numbers $f_S = f_S(P)$, $S \subseteq \{0,\dots,d-1\}$, are determined by the number of vertices f_0 and the number of facets f_3 according to

$$\begin{array}{rclcrclcrcl}
f_1 &=& 2f_3+f_0 & \qquad & f_{01} &=& 4f_3+2f_0 & \qquad & f_{012} &=& 24f_3\\
f_2 &=& 3f_3 & & f_{02} &=& 12f_3 & & f_{013} &=& 24f_3\\
 & & & & f_{03} &=& 8f_3 & & f_{123} &=& 24f_3\\
 & & & & f_{12} &=& 12f_3 & & & & \\
 & & & & f_{13} &=& 12f_3 & & f_{0123} &=& 48f_3.\\
 & & & & f_{23} &=& 6f_3 & & & &
\end{array}$$

2.5 Parity constraints

In dimension three, there is gap in the numbers of vertices of cubical polytopes: There is no cubical 3-polytope with 9 vertices, but for every $n \geq 8$, $n \neq 9$ there is a cubical 3-polytope with n vertices. This yields the question whether there are gaps in number of vertices of cubical polytopes of higher dimensions. It is known that there is a parity restriction for even-dimensional polytopes (unless the dimension is two).

Theorem 2.15 (Blind & Blind [12])
For even $d \geq 4$, every even-dimensional cubical d-polytope has an even number of vertices.

In [5] Babson & Chan generalized this result to odd-dimensional Eulerian manifolds (which includes odd-dimensional cubical spheres). However, it is unkown whether this extends to even-dimensional cubical spheres.

Open Question 2.16 (Babson & Chan [5])
Are there odd-dimensional cubical d-polytopes with odd numbers of facets (for $d \geq 5$)? Are there even-dimensional cubical spheres with odd numbers of facets?

Whereas the f-vectors of cubical 3-polytopes have been characterized, this is not the case for higher-dimension cubical polytopes:

Open Question 2.17 *What are the f-vectors of cubical 4-polytopes?*

This is a challenging open problem – and might be very difficult [60]. The f-vectors of simplicial 4-polytopes are characterized.
As a first approach one can consider the following question.

Question 2.18 *What is the $\mathbb{Z}$-affine span of f-vectors of cubical 4-polytopes?*

The lattice of f-vectors of cubical 3-spheres is known: Babson & Chan [5] gave an existences proof for cubical PL 3-spheres with odd numbers of facets. (The proof involves Construction 4.3 which is described in Section 4.3).

Corollary 2.19 (Babson & Chan [5])
The $\mathbb{Z}$-affine span of the f-vectors (f_0, f_1, f_2, f_3) of the cubical 4-spheres is characterized by

(i) *integrality ($f_i \in \mathbb{Z}$ for all i),*
(ii) *the cubical Dehn-Sommerville equations $f_0 - f_1 + f_2 - f_3 = 0$ and $f_2 = 3f_3$, and*
(iii) *the extra condition $f_0 \equiv 0 \bmod 2$.*

Note that this includes modular conditions such as $f_2 \equiv 0 \bmod 3$, which are not "modulo 2." The main result of Babson & Chan [5] says that for cubical d-spheres and $(d+1)$-polytopes, $d \geq 2$, "all congruence conditions are modulo 2." However, this refers only to the modular conditions *which are not implied by integrality and the cubical Dehn-Sommerville equations*. The first example of such a condition is, for $d = 4$, the congruence (iii) due to Blind & Blind [12].

To prove a similar result for cubical 4-polytopes we have to answer the following question.

Question 2.20 (Eppstein 2002)
Is there a cubical 4-polytope with an odd number of facets?

We give a positive answer to this question in Chapter 11. In particular, we give a constructive existence proof (Theorem 11.1).

2.6 Cubical polytopes with few vertices

There is a characterization of cubical d-polytopes with up to 2^{d+1} vertices due to Blind & Blind. A d-polytope is called *almost simple* if every vertex is d-valent or $(d+1)$-valent. A cubical d-polytope is *liftable* if its boundary can be embedded into the boundary of a $(d+1)$-dimensional cube.
They give a complete enumeration of a almost simple cubical d-polytopes, which is even valid for almost simple cubical $(d-1)$-spheres. This yields an

enumeration of a cubical d-polytopes with up to 2^{d+1} vertices. It turns out that for $d \geq 4$, with the single exception of the linearily 2-fold capped cubical d-polytope these are precisely the d-polytope which are *liftable*, that is, the d-polytopes which can be embedded into a $(d+1)$-cube.

2.7 Other known families of cubical polytopes

2.7.1 Cuboids

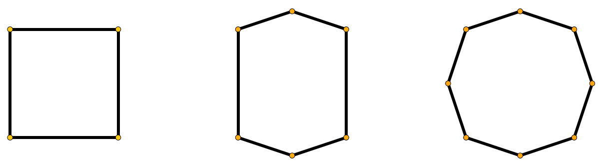
Grünbaum [28, pp. 59] describes a family of cubical polytopes, the so-called *cuboids* Q_k^d, for $0 \leq k \leq d$. The cuboid Q_0^d is a combinatorial d-dimensional cube. For $1 \leq k \leq d$, the cuboid Q_k^d is constructed from two cuboids Q_{k-1}^d which are "pasted together together at a common Q_{k-1}^{d-1}." (This requires that the two Q_{k-1}^d's are deformed beforehand.)
For $j + k \leq d - 1$ we have the recursion

$$f_j(Q_k^d) = 2f_j(Q_{k-1}^d) - 2f_j(Q_{k-1}^{d-1}),$$

whereas the number of facets satisfies the recursion relation

$$f_{d-1}(Q_k^d) = 2f_{d-1}(Q_{k-1}^d) - 2^k.$$

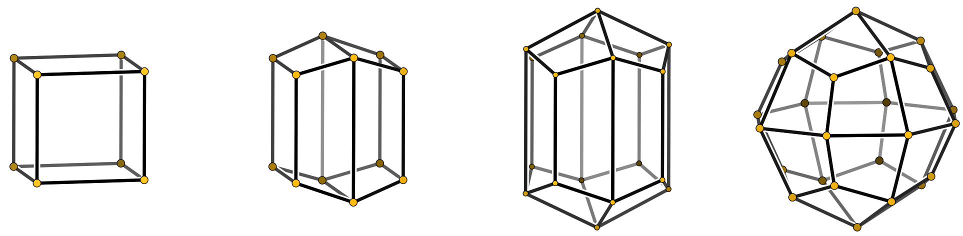
The 2- and 3-dimensional cuboids are depicted in Figure 2.8 and Figure 2.9.



(a) Q_0^2 (b) Q_1^2 (c) Q_2^2

FIGURE 2.8: The 2-dimensional cuboids Q_0^2, Q_1^2 and Q_2^2.

The boundary complex of a cuboid Q_k^d is combinatoprially isomorphic to the cubical PL $(d-1)$-sphere obtain from splitting the boundary complex of the d-dimensional standard cube $Q^d = [-1, +1]^d$ along the first k coordinate hyperplanes. (The k-th *coordinate hyperplane* of $\mathbb{R}^d$ is the hyperplane given by $x_k = 0$.)
In Section 3.3 we show that every cuboid Q_d^d, $d > 2$, is the so-called *cubical barycentric subdivision* of a d-dimensional cross-polytope. Furthermore we there obtain a coordinate representation of Q_d^d, $d > 2$, such that each Q_{d-1}^{d-1} that is contained in the boundary complex of Q_d^d is contained in a coordinate hyperplane.



(a) Q_0^3 (b) Q_1^3 (c) Q_2^3 (d) Q_3^3

Figure 2.9: The 3-dimensional cuboids Q_0^3, Q_1^3, Q_2^3 and Q_3^3.

A possible choice of vertex coordinates for a cuboid Q_k^d, $k < d$, is the set of all vertices $\boldsymbol{v}$ of Q_d^d such that $v_\ell \neq 0$ for all $\ell \geq k$. (Compare again Figure 2.8 and 2.9.)

2.7.2 Cubical zonotopes

Zonotopes are a special class of polytopes that has several descriptions: as Minkowski sums of line segments, as projections of higher-dimensional cubes, or as hyperplane arrangements. We refer to Ziegler [59, Sect. 7.3] for a detailed discussion of zonotopes. In Figure 2.10 we depict several cubical zonotopes. A cubical zonotope corresponds to a hyperplane arrangement in general position.

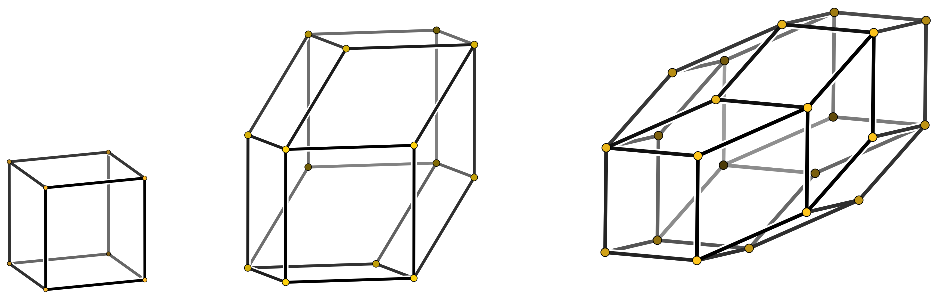


Figure 2.10: Cubical 3-dimensional zonotopes with $3, 4$ and 5 zones.

Theorem 2.21 (Babson, Finschi & Fukuda [3])
Every cubical zonotope is determined by its dual graph.

2.7.3 Neighborly cubical polytopes

Similar to simplicial neighborly polytopes, a *neighborly cubical polytope* C_d^n is a cubical d-polytope (with 2^n vertices for some $n \geq d$) which has the $(\lfloor \frac{d}{2} \rfloor - 1)$-skeleton of an n-cube. This notion was introduced by Babson, Billera & Chan [4], where neighborly cubical spheres were constructed.

Theorem 2.22
(Existence of neighborly cubical polytopes; Joswig & Ziegler [37])
For any $n, d, r \in \mathbb{N}$ with $n \geq d \geq 2r + 2$ there is a cubical d-polytope whose r-skeleton is isomorphic to the r-skeleton of the n-dimensional cube.

As an example, we here display a Schlegel diagram of a neighborly cubical 4-polytope (with the graph of the 5-cube), with f-vector $(32, 80, 96, 48)$. According to Joswig & Ziegler [37] this may be constructed as

$$C_4^5 \quad := \quad \operatorname{conv}((Q \times 2Q) \ \cup \ (2Q \times Q)), \qquad \text{where } Q = [-1, +1]^2.$$

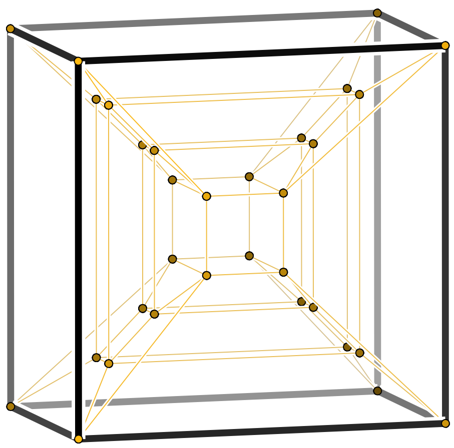


FIGURE 2.11: A Schlegel diagram of the neighborly cubical 4-polytope C_4^5 with the graph of the 5-cube.

2.8 Lower and upper bound conjectures

For simplicial polytopes there is the famous *upper bound theorem*, which was proved in 1970 by McMullen [43]. The following *cubical upper bound conjecture* is due to Kalai [4, Conj. 4.2].

Conjecture 2.23 (Cubical upper bound conjecture; Kalai [4, Conj. 4.2])
Let P be a cubical $(d-1)$-sphere with $f_0(P) = 2^n$ vertices. Then its number of facets is bounded from above by that of a neighborly cubical d-polytope C_d^n with the same number of vertices. Moreover,

$$f_i(P) \leq f_i(C_d^n) \qquad \text{for } 1 \leq i \leq d-1.$$

Furthermore, if equality is attained, then P is a neighborly cubical d-polytope.

This conjecture is proven by Joswig & Ziegler for the special case $n = d+1$ if restricted to cubical polytopes [37]. Furthermore, they present a counter-example which is a cubical sphere with $d = 4$ and $n = 6$.

Jokusch [33] stated a cubical analog to the simplicial *lower bound theorem* (Barnette [7][8]).

Conjecture 2.24 (Cubical lower bound conjecture; Jokusch [33])
Let P be a cubical d-polytope with $f_0(P) = 2^d + \ell 2^{d-1}$. Then its number of k-faces, $0 \leq k \leq d-1$, is bounded from below by the number of k-faces of a capped cubical polytope with the same number of vertices. Moreover, for any cubical d-polytope P‘ (with an arbitrary number of vertices) we have

$$f_k(P') \geq \left\lceil \frac{f_0(P)}{2^{d-1}} - 2 \right\rceil \left(2^{d-k}\binom{d}{k} - 2^{d-k-1}\binom{d-1}{k} \right) + 2^{d-k}\binom{d}{k}.$$

In the simplicial case, for $d \geq 4$, equality is attained only for stacked simplicial d-polytopes, whereas there are non-simplicial 3-polytopes for which equality holds.
Similarily, in the cubical case on assumes that $d \geq 4$, equality is attained only for capped cubical d-polytopes. As in the simplicial case, not every cubical 3-polytope for which equality holds is combinatorially isomorphic to a capped cubical polytope. (For instance, take a cubical 3-polytope with $8+4k$ vertices that is not capped. In particular, the second and the third instance displayed in Figure 2.6 are cubical 3-polytopes with 12 vertices that are not capped.)

2.9 Almost cubical polytopes

All proper faces of a cubical d-polytope have to be combinatorial cubes. We introduce an *almost cubical* d-polytope as a pair (P, F), where F is a specified facet of P such that all facets of P other than F are required to be combinatorial cubes. Thus, F need not be a cube, but it will be cubical.

Hence, P is $(d-2)$-cubical. (Of course, F can be a combinatorial cube and P a cubical polytope.)
Recall that by $\mathcal{C}(P)$ we denote the polytopal complex given by a polytope P and all its faces. By $\mathcal{C}(\partial P)$ we denote the *boundary complex* of P, consisting of all proper faces of P. If P is a cubical polytope, then $\mathcal{C}(\partial P)$ is a cubical complex. If (P, F) is almost cubical, then the *Schlegel complex* $\mathcal{C}(\partial P)\setminus\{F\}$ is a cubical complex that is combinatorially isomorphic to the Schlegel diagram of SCHLEGEL(P, F).

Chapter 3

Hexa meshing and cubical balls

A *d-dimenensional cubical ball* (*cubical d-ball* for short) is a d-dimensional polytopal PL-homeomorphic to simplex. In Chapter 5 we will see that cubical polytopes can be constructed from cubical balls. An important application for cubical balls is *hexa meshing* which is described in the next section. Furthermore, hexa meshing is important for this thesis for two reasons: From a known hexa meshing technique we learn how to construct some special cubical balls (Chapter 9). On the other hand, our results have interesting consequences for hexa meshing questions; compare Chapter 13.

3.1 Hexa meshing

In the context of computer aided design (CAD) the surface of a workpiece (for instance a part of a car, ship or plane) is often modeled by a *surface mesh*. In order to analyze physical and technical properties of the mesh (and of the workpiece), finite element methods (FEM) are widely used.

Such a surface mesh is either a *topological mesh*, that is, in our terminology a 2-dimensional regular CW complex, or a *geometric mesh*, that is, a (pure) 2-dimensional complex with polytopal cells. Common cell types of a surface mesh are triangles (2-simplices) and quadrangles. (A geometric *quadrilateral mesh*, *quad mesh* for short, is a 2-dimensional cubical complex, and a topological one is a cubical 2-dimensional regular CW complex.)

In recent years there has been growing interest in volume meshing. Tetrahedral volume meshes (simplicial 3-complexes) are well-understood, whereas there are interesting and challenging open questions both in theory and practice of hexahedral volume meshes, *hexa meshes* for short. (A geometric hexa mesh is a 3-dimensional cubical complex, and a topological one is a cubical 3-dimensional regular CW complex.)

3.2 Cubifications

A (polytopal) *cubification* of a cubical PL $(d-1)$-sphere $\mathcal{S}^{d-1}$ is a cubical (polytopal) d-ball $\mathcal{B}^d$ with boundary $\mathcal{S}^{d-1}$. Similarily, a *CW cubification* of a cubical regular CW $(d-1)$-sphere $\mathcal{S}^{d-1}$ is a cubical regular CW d-ball $\mathcal{B}^d$ with boundary $\mathcal{S}^{d-1}$. In the terminology of the meshing community, a CW/polytopal cubification of a CW/polytopal 2-sphere is known as a "topological/geometrical hexa mesh compatible with a topological/geometrical quad meshes."

A double counting argument shows that every cubical (CW/PL) $(d-1)$-sphere that admits a cubification has an even number of facets.

Lemma 3.1 *Every (pure) cubical d-complex has an even number of boundary facets.*

Proof. Let $\mathcal{C}$ be a pure cubical d-dimensional complex. We count the flag number $f_{\{d-1,d\}}$ which is the number of incidences between ridges (i.e. $d-1$-faces) and facets of $\mathcal{C}$. Every facet of $\mathcal{C}$ is incident to $2d$ ridges. Hence

$$f_{\{d-1,d\}} = 2d \cdot f_d(\mathcal{C}). \tag{3.1}$$

On the other hand, each ridge of $\mathcal{C}$ that belongs to the boundary of $\mathcal{C}$ is incident to one facet of $\mathcal{C}$, and each ridge of $\mathcal{C}$ that does not belong to the boundary of $\mathcal{C}$ is incident to two facets of $\mathcal{C}$. Hence

$$\begin{aligned} f_{\{d-1,d\}} &= f_{d-1}(\partial\mathcal{C}) + 2(f_{d-1}(\mathcal{C}) - f_{d-1}(\partial\mathcal{C})) \\ &= 2f_{d-1}(\mathcal{C}) - f_{d-1}(\partial\mathcal{C}). \end{aligned}$$

By (3.1) the number $f_{d-1}(\partial\mathcal{C})$ of facets of the boundary of $\mathcal{C}$ is even. □

Whether this condition is sufficient is a challenging open problem, even for $d = 3$ (compare [10], [21]).

Open Problem 3.2 (Cubification problem)
Has every cubical PL 2-sphere with an even number of facets a cubification?

A lot of construction techniques for cubifications are available in the CW category. In particular, every cubical CW $(d-1)$-sphere $\mathcal{S}^{d-1}$ with an even number of facets admits a CW cubification.

Theorem 3.3 (Thurston [57], Mitchell [44])
Every cubical CW $(d-1)$-sphere $\mathcal{S}^{d-1}$ with an even number of facets admits a CW cubification.

Eppstein showed in [21] that a linear number of topological cubes suffices, and Bern, Eppstein & Erickson proved the existence of a (pseudo-)shellable topological hexa mesh [10].

3.3 Cubical barycentric subdivision

The *cubical barycentric subdivision* a cubical analog to the well-known (simplicial) barycentric subdivision of a complex. Furthermore, it is a standard tool for passage from a complex to a PL-homeomorphic cubical complex. In the meshing community the special case of the cubical barycentric subdivision for 2- or 3-dimensional simplicial complexes is known as a *diced triangulation*; compare for instance Mitchell [45].

Given a simplicial complex $\mathcal{C}$, we define the *cubical barycentric subdivision* $\mathrm{cbs}(\mathcal{C})$ as the complex which has a vertex $\boldsymbol{w}_S = \mathrm{bary}(S)$ for every face $S \in \mathcal{C}$, where $\mathrm{bary}(S) := \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{s}_i$ denotes the barycentre of a simplex $S = \mathrm{conv}\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\}$, and a facet

$$\mathrm{conv}\,\{\boldsymbol{w}_S \,:\, S \subseteq F \text{ with } \boldsymbol{v} \subseteq S\}$$

for every vertex-facet incidence F facet of $\mathcal{C}$ and $\boldsymbol{v} \in \mathrm{vert}(F)$. Hence, every k-simplex of $\mathcal{C}$ is subdivided into $k+1$ different k-cubes (of $\mathrm{cbs}(\mathcal{C})$).
Observe that the set of all elements of the face poset of $\mathrm{cbs}(\mathcal{C})$ equals the set of all order relations of the face poset of $\mathcal{C}$.

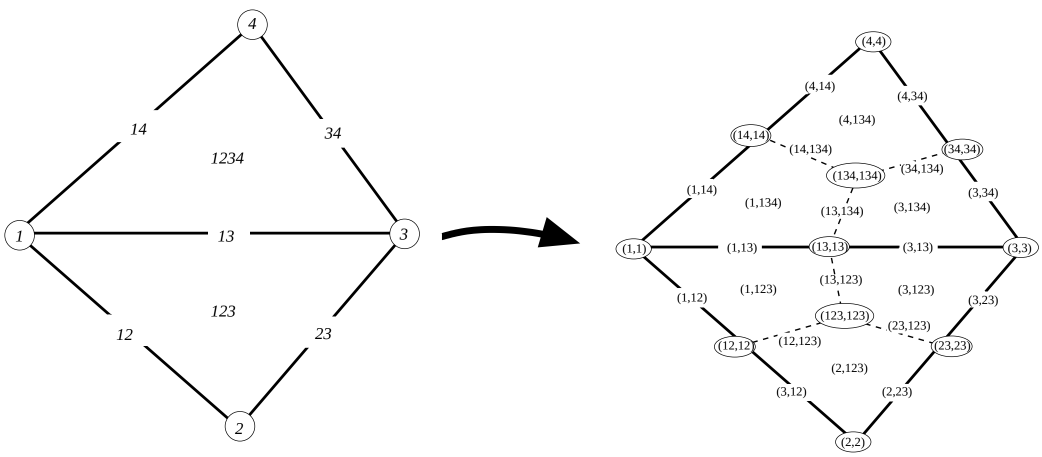
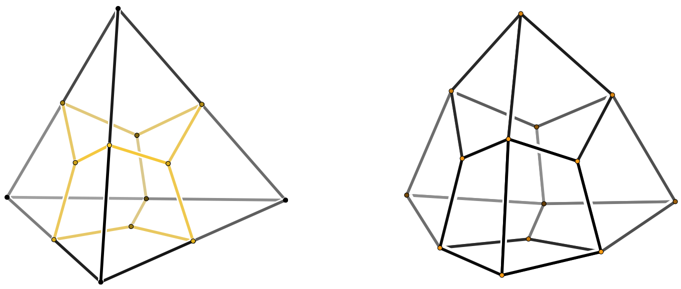


FIGURE 3.1: The cubical barycentric subdivision of a 2-dimensional simplicial complex $\mathcal{C}$ with 2 maximal cells. Each cell of $\mathrm{cbs}(\mathcal{C})$ is labeled with its corresponding chain of length two in the face poset of $\mathcal{C}$.

A generalization of the cubical barycentric subdivision (of a simplicial complex) is the *cubical barycentric cover* or short *cubical barycover* of an abstract (not necessarily simplicial) complex or poset introduced by Billera et al. [4].



(a) (b)

FIGURE 3.2: (a) The cubical barycentric subdivision of the boundary of a regular 3-dimensional simplex, and (b) the polytopal realization obtained by the Shephard construction.

3.4 The Shephard construction

In 1966 Shephard [54] presented a construction which has several interesting consequences on realization questions concerning cubical barycentric subdivisions.

His construction is based on the following observation. Assume we are given a d-dimensional cube Q such that $\mathbf{0}$ is a vertex of Q and $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ are the neighbors of $\mathbf{0}$. Then there is a projective transformation τ_ε, depending on a parameter $\varepsilon > 0$, that has the following properties:

- τ_ε fixes 0
- τ_ε fixes the hyperplane $H = \left\{\boldsymbol{x} \in R^d \, : \, \mathbb{1}^T\boldsymbol{x} = 1\right\}$ pointwise.
- Every point $\boldsymbol{x}$ with $\mathbb{1}^T\boldsymbol{x} > 1$ converges to H as ε tends to zero.

Compare Figure 3.3.

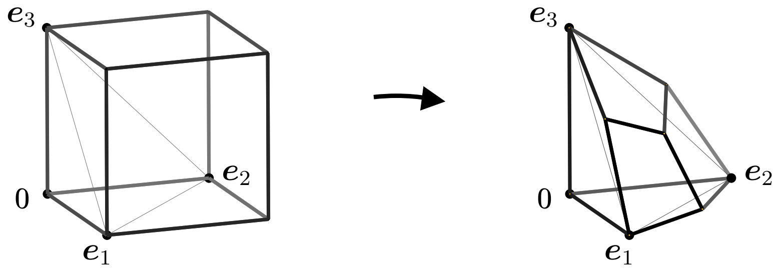


FIGURE 3.3: A projective transformation used by the Shephard construction.

By this idea we can construct a polytopal realization of the cubical barycentric subdivision of a simplicial polytope as sketched in Figure 3.4.

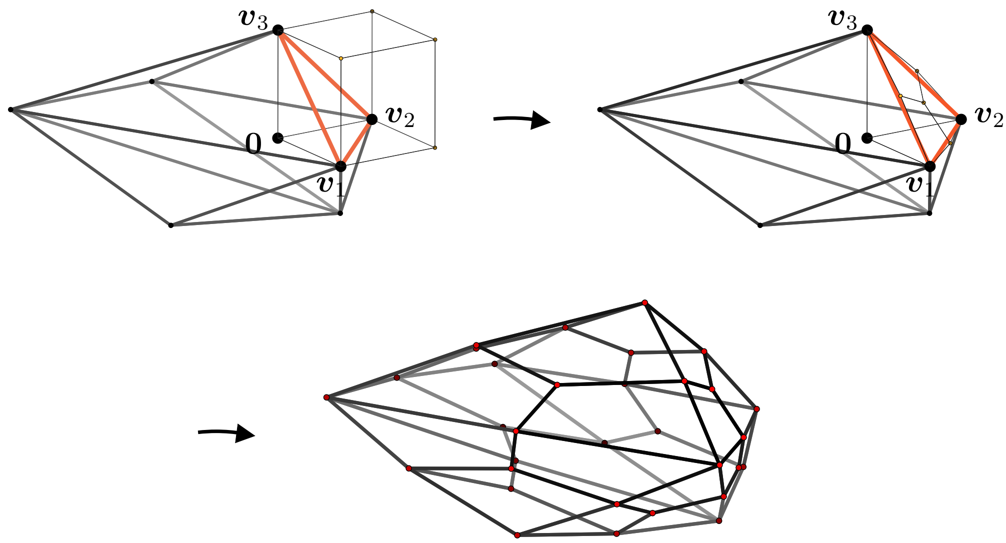


Figure 3.4: Sketch of the Shephard construction.

Consequences.

- The cubical barycentric subdivision of the boundary complex of a simplicial polytope is polytopal.
- Every d-polytope can be approximated (in the Hausdorff sense) with a d-polytope with projectively regular facets. (Recal that two d-polytopes $P \subset \mathbb{R}^{d'}$, $Q \subset \mathbb{R}^{d''}$ are *projectively isomorphic* if there is a projective transformation from $\mathbb{R}^{d'}$ to $\mathbb{R}^{d''}$ inducing a bijection between P to Q, and a combinatorial cube d-polytopes Q is *projectively regular* if it is projectively isomorphic to the d-dimensional standard cube.)
- Furthermore: Every d-polytope can be approximated in Hausdorff sense with a cubical d-polytope which admits a cubification (with a single interior point).
- The cubical barycentric subdivision of a regular simlicial complex is polytopal.

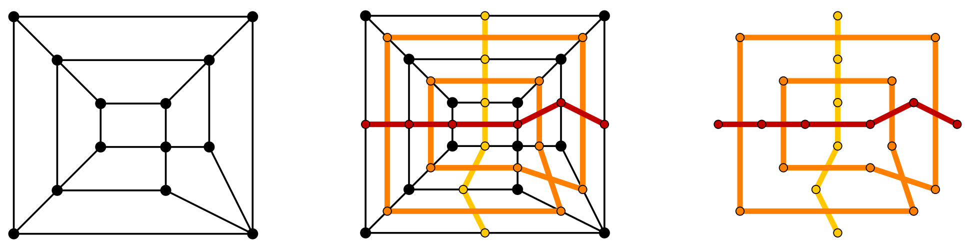
Furthermore, all these objects can be constructed with rational coordinates of small encoding length.
(In Section 5.8.3 we give some remarks on a generalization of the Shepard construction.)

Chapter 4

Dual manifolds and edge orientations

4.1 Dual manifolds

Derivative complex. For every (pure) cubical d-dimensional complex $\mathcal{C}$, $d > 1$, the *derivative complex* is an abstract cubical $(d-1)$-dimensional cell complex $\mathcal{D}(\mathcal{C})$ whose vertices may be identified with the edge midpoints of the complex, while the facets "separate the opposite facets of a facet of $\mathcal{C}$," that is, they correspond to pairs $(F, [e])$, where F is a facet of $\mathcal{C}$ and $[e]$ denotes a "parallel class" of edges of F. Thus this is a cell complex with $f_1(\mathcal{C})$ vertices and $(d-1)f_{d-1}(\mathcal{C})$ cubical facets of dimension $d-1$.



(a) $\mathcal{C}$ (b) $\mathcal{C}$ and $\mathcal{D}(\mathcal{C})$ (c) $\mathcal{D}(\mathcal{C})$

FIGURE 4.1: The derivative complex of a cubical 2-complex $\mathcal{C}$. The second figure shows on class of edges that must be oriented consistently.

For the f-vector we obtain $f_k(\mathcal{D}(\mathcal{C})) = (k+1)f_{k+1}(\mathcal{C})$, for $0 \le k \le d-1$. One can assign a polynomial $f(\mathcal{C}', t)$ to the f-vector of a d-complex $\mathcal{C}'$ by

$$f(\mathcal{C}', t) = \sum_{k=0}^{d} f_k(\mathcal{C}')t^k.$$

The f-vector polynomial of the derivative complex satisfies

$$\begin{aligned} f(\mathcal{D}(\mathcal{C}), t) &= f_1(\mathcal{C}) + 2 \cdot f_2(\mathcal{C})t + \ldots + d \cdot f_d(\mathcal{C})t^d \\ &= \sum_{k=0}^{d-1}(k+1)f_{k+1}(\mathcal{C})t^k = \frac{d}{dt}f(\mathcal{C}, t), \end{aligned}$$

which gives a motivation for the term *derivative complex.* Furthermore, the assignment $\mathcal{D}$, which maps a cubical complex onto its derivate complex, acts as a derivation with respect to products and disjoint unions:

$$\mathcal{D}(\mathcal{C}_1 \times \mathcal{C}_2) = (\mathcal{D}(\mathcal{C}_1) \times \mathcal{C}_2) \cup (\mathcal{C}_1 \times \mathcal{D}(\mathcal{C}_2)).$$

Dual manifolds. Most cubical complexes we consider in this thesis are either cubical PL spheres (for instance boundary complexes of cubical polytopes), or cubical PL balls. In these cases, the derivate complex is a (not necessarily connected) manifold, and we call each connected component of the derivative complex $\mathcal{D}(P)$ of a cubical complex $\mathcal{C}$ a *dual manifold* of $\mathcal{C}$. If the cubical complex $\mathcal{C}$ is a PL sphere then the dual manifolds of $\mathcal{C}$ are manifolds without boundary. If $\mathcal{C}$ is a cubical d-ball, then some (possibly all) dual manifolds have non-empty boundary components, namely the dual manifolds of $\partial\mathcal{C}$.

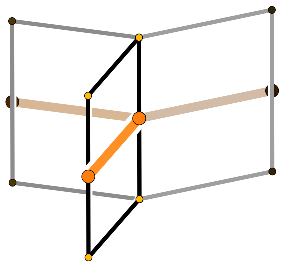


FIGURE 4.2: A 2-dimensional cubical complex (with 3 facets) and a component of its derivative complex that is not a manifold.

If the cubical complex $\mathcal{C}$ is not a sphere or a ball, then the derivative complex is not necessarily a manifold: compare Figure 4.2.

Immersions. The derivative complex, and thus each dual manifold, comes with a canonical immersion into the boundary of P. More precisely, the (simplicial) barycentric subdivision of $\mathcal{D}(P)$ has a simplicial map to the (simplicial) barycentric subdivision of the boundary complex ∂P, which is a codimension one immersion into the simplicial sphere $\mathrm{sd}\,(\mathcal{C}(\partial P))$. Restricted to a dual manifold, this immersion may be an embedding or not. Furthermore, all these immersions that are not embeddings are *normal crossing*, that is, each *multiple-intersection point* is of degree $k \leq d$ and there is a neighborhood of each multiple intersection point that is PL isomorphic to (a neighborhood of) a point which is contained in k pair-wise perpendicular hyperplanes.

In the case of cubical 3-polytopes, the derivative complex may consist of one or many 1-spheres. For example, for the 3-cube it consists of three 1-spheres, while for the "cubical octahedron" O_8 displayed in Figure 4.3 the dual manifold is a single immersed S^1 (with 8 double points).

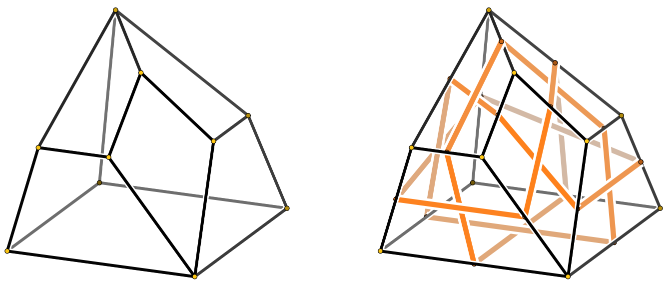


Figure 4.3: The cubical octahedron O_8 (the only combinatorial type of a cubical 3-polytope with 8 facets), and its single immersed dual manifold.

Example 4.1. In the case of cubical 4-polytopes, the dual manifolds are surfaces (compact 2-manifolds without boundary). The dual manifolds of the neighborly cubical 4-polytope C_4^5 with the graph of the 5-cube are four embedded cubical 2-spheres S^2 with f-vector $(16, 28, 14)$ — of two different combinatorial types — and one embedded torus T with f-vector $(16, 32, 16)$. Compare Figure 4.1.

Generalizations. It is possible to define the derivative complex and dual manifolds for any class of d-dimensional complexes with facets that satisfy the *unique opposite facet property*: For every facet P of the complex and for every facet F of P there is a unique opposite facet F' (which is disjoint from F).

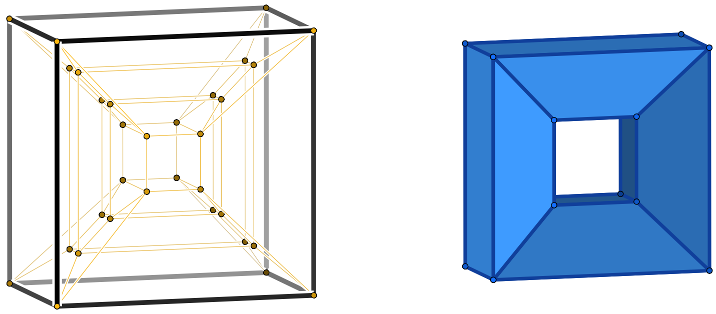


FIGURE 4.4: A Schlegel diagram of the neighborly cubical 4-polytope C_4^5 with the graph of the 5-cube, and its dual torus. All others are embedded 2-spheres.

One class of polytopes with that property are *zonotopes* (cf. [59, Sect. 7.3]). In the case of non-cubical d-dimensional zonotopes we obtain codimension one immersions of zonotopal $(d-1)$-manifolds, but these immersions are not normal crossing.

4.2 Orientability

Let P be a cubical d-polytope ($d \geq 3$). The immersed dual manifolds in its boundary are crossed transversally by the edges of the polytope.
Thus we find that orientability of the dual manifolds is equivalent to the possibility to give *consistent* edge orientations to the edges of the P, that is, an edge-orientation such that in each 2-face of P opposite edges should get parallel (rather than antiparallel) orientations; compare Hetyei [30]. Figure 4.5 shows such an edge orientation for a cubical 3-polytope (whose derivative complex consists of three circles, so it has 8 consistent edge orientations in total):

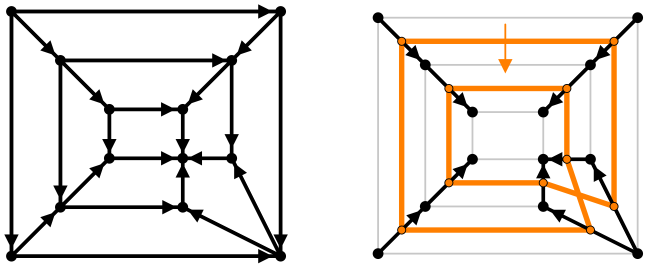


FIGURE 4.5: An edge-orientation of (the Schlegel diagram) of a cubical 3-polytope with 13 vertices and 3 dual circles.

One can attempt to assign such edge orientations by moving to from edge to edge across 2-faces. The obstruction to this arises if on a path moving from edge to edge across quadrilateral 2-faces we return to an already visited edge, with reversed orientation, that is, if we close a *cubical Möbius strip with parallel inner edges*, as displayed in the figure. (Such an immersion need not necessarily be embedded, that is, some 2-face may be used twice for the Möbius strip.)

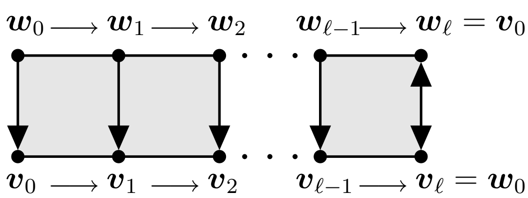


Figure 4.6: A cubical Möbius strip with parallel inner edges.

Proposition 4.2 *For every cubical d-polytope ($d \geq 3$), the following are equivalent:*

- *All dual manifolds of P are orientable.*
- *The 2-skeleton of P has a consistent edge orientation.*
- *The 2-skeleton of P does not admit an immersion of a cubical Möbius strip with parallel inner edges.*

Observation 4.3. All instances of the following classes of cubical polytopes are edge-orientable:

- cuboids,
- capped cubical polytopes,
- cubical zonotopes,
- neighborly cubical polytopes.

Conjecture 4.4 (Hetyei [30])
There are cubical 4-polytopes that are not edge-orientable.

4.3 From PL immersions to cubical PL spheres

The emphasis in this thesis is on cubical convex d-polytopes. In the more general setting of cubical PL $(d-1)$-spheres, one has more flexible tools available. In this setting, Babson & Chan [5] proved that “all PL codimension 1 normal crossing immersions appear.” The following sketch of proof is meant to explain the Babson-Chan theorem geometrically (it is pretended in a combinatorial framework and terminology in [5]), and to briefly indicate which parts of their construction are available in the setting of convex polytopes.

Construction 1: BABSON-CHAN [5]

Input: A normal crossing immersion $j : \mathcal{M}^{d-2} \looparrowright \mathcal{S}^{d-1}$ of a triangulated PL manifold $\mathcal{M}^{d-2}$ of dimension $d-2$ into a PL simplicial $(d-1)$-sphere.

Output: A cubical PL $(d-1)$-sphere with a dual manifold immersion PL-equivalent to j.

(1) Perform a barycentric subdivision on $\mathcal{M}^{d-2}$ and on $\mathcal{S}^{d-1}$.

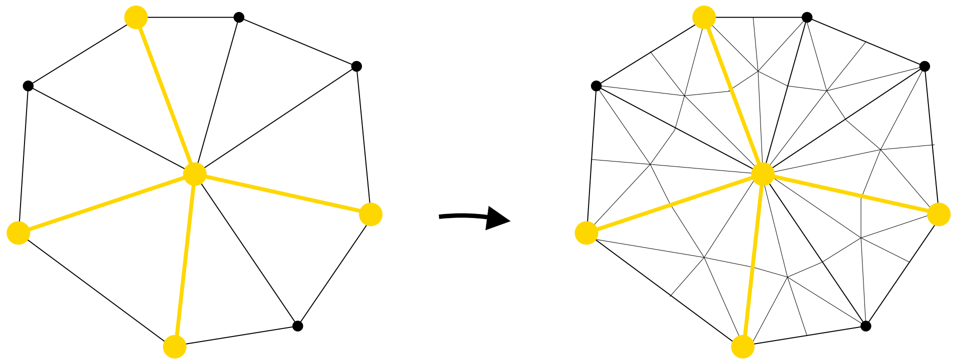


FIGURE 4.7: Step 1 of the Babson-Chan construction: Performing a barycentric subdivision. (We illustrate the impact of the construction on 2-ball, which might be part of the boundary of a 2-sphere. The immersion which is shown in bold has a single double-intersection point.)

(Here each i-simplex is replaced by $(i+1)!$ new i-simplices, which is an even number for $i > 0$. This step is done only to ensure parity conditions on the f-vector, especially that the number of facets of the final cubical sphere is congruent to the Euler characteristic of $\mathcal{M}^{d-2}$.
Barycentric subdivisions are easily performed in the polytopal category as well, see Ewald & Shephard [22].)

(2) Perform a cubical barycentric subdivision on $\mathcal{M}^{d-2}$ and $\mathcal{S}^{d-1}$. (This standard tool for passage from a simplicial complex to a PL-homeomorphic cubical complex; compare the description in Section 3.3. Recall that such cubations can be performed in the polytopal category according to Shephard [54]: If the starting triangulation of $\mathcal{S}^{d-1}$ was polytopal, the resulting cubation will be polytopal as well.)

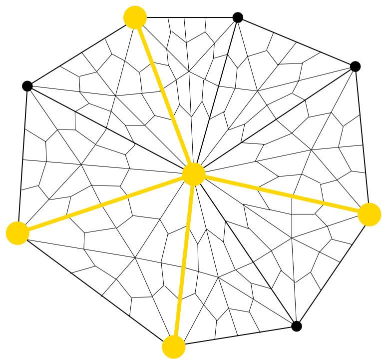


Figure 4.8: Step 2 of the Babson-Chan construction: Performing a cubical barycentric subdivision.

(3) "Thicken" the cubical $(d-1)$-sphere along the immersed $(d-2)$-manifold, to obtain the cubical $(d-1)$-sphere $BC(\mathcal{S}^{d-1}, j(\mathcal{M}^{d-2}))$.

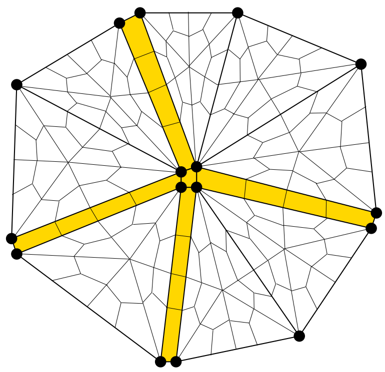


Figure 4.9: The outcome of the Babson-Chan construction: A cubical sphere with a dual manifold immersion that is PL-equivalent to the input immersion j.

(In this step, every $(d-1-i)$-cube in the i-fold multiple point locus results in a new $(d-1)$-cube. The original immersed manifold, in its cubified subdivided version, now appears as a dual manifold in the newly resulting $(d-1)$-cubes. This last step is the one that seems hard to perform for polytopes in any non-trivial instance.)

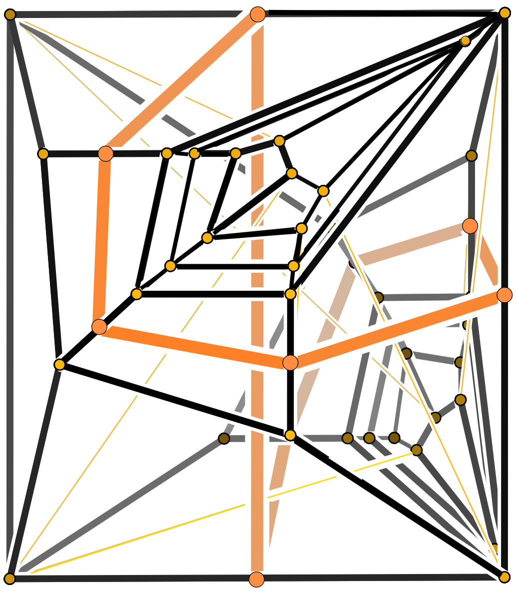
Theorem 4.5 (Babson & Chan [5])
Assume we are given a normal crossing immersion $j : \mathcal{M}^{d-2} \looparrowright \mathcal{S}^{d-1}$ of a triangulated PL manifold $\mathcal{M}^{d-2}$ of dimension $d-2$ into a PL simplicial $(d-1)$-sphere. Let $\mathcal{K} := BC(\mathcal{S}^{d-1}, j(\mathcal{M}^{d-2}))$ be the cubical PL $(d-1)$-sphere obtained by Construction 4.3, and $y : \mathcal{D}(\mathcal{K}) \looparrowright |K|$ the immersion of the derivative complex of $\mathcal{K}$.
Then, modulo two, the Euler characteristics for the multiple point loci of j are the same as for the immersion of the derivative complex y of $\mathcal{K}$. In particular, $\chi(\{s \in \mathbb{R}^d : |j^{-1}(s)| = k\} \equiv_2 f_k(\mathcal{K})$.

Using a Boy's surface, that is, an immersion of the real projective plane with one triple point and three double-intersection loops, they derive the result.

Corollary 4.6 *There is a cubical PL 3-sphere with an odd number of facets and with a non-orientable dual manifold.*

Part II

Elementary constructions



Chapter 5

Lifting polytopal subdivisions

5.1 Regular balls

In the following, the primary object we deal with is a *regular ball*: a regular polytopal subdivision $\mathcal{B}$ of a convex polytope $P = |\mathcal{B}|$. A polytopal subdivision $\mathcal{B}$ is *regular* (also known as *coherent* or *projective*) if it admits a concave lifting function $f : P \to \mathbb{R}$ whose domains of linearity are given by the facets of the subdivision. (A function $f : D \to \mathbb{R}$ is *concave* if for all $\boldsymbol{x}, \boldsymbol{y} \in D$ and $0 < \lambda < 1$ we have $f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y})$.)
In this definition, subdivisions of the boundary are allowed, that is, we do not necessarily require that the faces of $P = |\mathcal{B}|$ are themselves faces in $\mathcal{B}$.
In the sequel we focus on regular cubical balls. Only in some cases we consider regular non-cubical balls.

Example 5.1. If (P, F) is an almost cubical polytope, then the Schlegel diagram based on F, which we denote by $\textsc{Schlegel}(P, F)$, is a regular cubical ball (without subdivision of the boundary).

Lemma 5.2 *If $\mathcal{B}$ is a regular cubical d-ball, then there is a regular cubical ball $\mathcal{B}'$ without subdivision of the boundary, combinatorially isomorphic to $\mathcal{B}$.*

Proof. Using a positive lifting function $f : |\mathcal{B}| \to \mathbb{R}$, the d-ball $\mathcal{B}$ may be lifted to $\widetilde{\mathcal{B}}$ in $\mathbb{R}^{d+1}$, by mapping each $\boldsymbol{x} \in |\mathcal{B}|$ to $(\boldsymbol{x}, f(\boldsymbol{x})) \in \mathbb{R}^{d+1}$.
Viewed from $\boldsymbol{p} := \lambda \boldsymbol{e}_{d+1}$ for sufficiently large λ, this lifted ball will appear to be strictly convex. Thus one may look at the polytopal complex that consists of the cones spanned by faces of $\widetilde{\mathcal{B}}$ with apex $\boldsymbol{p}$. This polytopal complex is regular, since it appears convex when viewed from $\boldsymbol{p}$, which yields a lifting function for the restriction of $\widetilde{\mathcal{B}}$ to the hyperplane given by $x_{d+1} = 0$, which may be taken to be $\mathcal{B}'$. $\square$

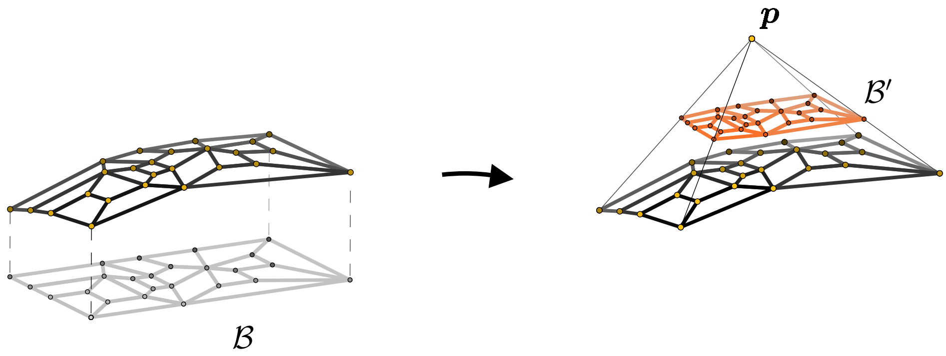


FIGURE 5.1: Illustration of the 'convexification' of a regular ball (Lemma 5.2).

5.2 The patching lemma

Often regular cubical balls are constructed from other regular balls. The following "patching lemma", which appears frequently in the construction of regular subdivisions (see [40, Cor. 1.12] or [17, Lemma 3.2.2]), helps to produce regular balls.

Notation. For a d-polytope $P \subset \mathbb{R}^{d'}$, a polytopal subdivision $\mathcal{T}$ of P and a hyperplane H in $\mathbb{R}^{d'}$, we denote by $\mathcal{T} \cap H$ the *restriction* of $\mathcal{T}$ to H, which is given by

$$\mathcal{T} \cap H := \{F \cap H \,:\, F \in \mathcal{T}\}\,.$$

For two d-polytopes P, Q with $Q \subset P$ and a polytopal subdivision $\mathcal{T}$ of P we denote by $\mathcal{T} \cap Q$ the *restriction* of $\mathcal{T}$ to Q, which is given by

$$\mathcal{T} \cap Q := \{F \cap Q \,:\, F \in \mathcal{T}\}\,.$$

Lemma 5.3 (Patching lemma)
Let Q be a d-polytope. Assume we are given the following data:

- *A regular polytopal subdivision $\mathcal{S}$ of Q (the "raw subdivision").*
- *For each facet F of $\mathcal{S}$, a regular polytopal subdivision $\mathcal{T}_F$ of F, such that $\mathcal{T}_F \cap F' = \mathcal{T}_{F'} \cap F$ for all facets F, F' of $\mathcal{S}$.*
- *For each facet F of $\mathcal{S}$, a concave lifting function h_F of $\mathcal{T}_F$, such that $h_F(\boldsymbol{x}) = h_{F'}(\boldsymbol{x})$ for all $\boldsymbol{x} \in F \cap F'$, where F, F' are facets of $\mathcal{S}$.*

Then this uniquely determines a regular polytopal subdivision $\mathcal{U} = \bigcup_F \mathcal{T}_F$ of Q (the "fine subdivision"). Furthermore, for every lifting function g of $\mathcal{S}$ there exists a small $\varepsilon_0 > 0$ such that for all $\varepsilon_0 > \varepsilon > 0$ the function $g + \varepsilon h$ is a lifting function of $\mathcal{U}$, where h is the piece-wise linear function $h : Q \to \mathbb{R}$ which on each $F \in \mathcal{S}$ is given by h_F.

Proof. Let g be a lifting function of $\mathcal{S}$. For a parameter $\varepsilon > 0$ we define a piece-wise linear function $\phi_\varepsilon : P \to \mathbb{R}$ that on $\boldsymbol{x} \in F \in \mathrm{fac}(\mathcal{S})$ takes the value $\phi_\varepsilon(\boldsymbol{x}) = g(\boldsymbol{x}) + \varepsilon h_F(\boldsymbol{x})$. (It is well-defined since the h_F coincide on the ridges of $\mathcal{S}$.) The domains of linearity of ϕ_ε are given by the facets of the "fine" subdivision $\mathcal{U}$. If ε tends to zero then ϕ_ε tends to the concave function g. This implies that there exists a small $\varepsilon_0 > 0$ such that ϕ_ε is concave and thus a lifting function of $\mathcal{U}$, for $\varepsilon_0 > \varepsilon > 0$. □

5.3 Products and prisms

Lemma 5.4 (Product lemma)
Let $(\mathcal{B}_1, h_1)$ be a lifted cubical d_1-ball in $\mathbb{R}^{d'_1}$ and $(\mathcal{B}_2, h_2)$ be a lifted cubical d_2-ball in $\mathbb{R}^{d'_2}$.
Then the product $\mathcal{B}_1 \times \mathcal{B}_2$ of $\mathcal{B}_1$ and $\mathcal{B}_2$ is a regular cubical $(d_1 + d_2)$-ball in $\mathbb{R}^{d'_1+d'_2}$.

Proof. Each cell of $\mathcal{B}_1 \times \mathcal{B}_2$ is a product of two cubes. Hence $\mathcal{B}_1 \times \mathcal{B}_2$ is a cubical complex. A lifting function h of $\mathcal{B}_1 \times \mathcal{B}_2$ is given by the sum of h_1 and h_2, that is, by $h((\boldsymbol{x}, \boldsymbol{y})) := h_1(\boldsymbol{x}) + h_2(\boldsymbol{y})$, for $\boldsymbol{x} \in |\mathcal{B}_1|, \boldsymbol{y} \in |\mathcal{B}_2|$. □

As a consequence, the *prism* $\mathrm{prism}(\mathcal{C})$ over a cubical d-complex $\mathcal{C}$ yields a cubical $(d+1)$-dimensional complex. Furthermore, the prism over a regular cubical ball $\mathcal{B}$ yields a regular cubical $(d+1)$-ball.

5.4 Piles of cubes

For integers $\ell_1, \ldots, \ell_d \geq 1$, the *pile of cubes* $P_d(\ell_1, \ldots, \ell_d)$ is the cubical d-ball formed by all unit cubes with integer vertices in the d-polytope

$$P := [0, \ell_1] \times \ldots \times [0, \ell_d],$$

that is, the cubical d-ball formed by the set of all d-cubes

$$C(k_1, \ldots, k_d) := [k_1, k_1 + 1] \times \ldots \times [k_d, k_d + 1]$$

for integers $0 \leq k_i \leq \ell_i - 1$ together with their faces [59, Sect. 5.1].
The pile of cubes $P_d(\ell_1, \ldots, \ell_d)$ is a product of 1-dimensional subdivisions, which are regular. Hence the product lemma implies that $P_d(\ell_1, \ldots, \ell_d)$ is a regular cubical subdivision of the d-polytope P.

5.5 Lifted balls

When constructing cubical complexes we often deal with regular cubical balls which are equipped with a lifting function. A *lifted d-ball* is a pair $(\mathcal{B}, h)$ consisting of a regular d-ball $\mathcal{B}$ and a lifting function h of $\mathcal{B}$. The *lifted boundary* of a lifted ball $(\mathcal{B}, h)$ is the pair $(\partial\mathcal{B}, h|_{\partial\mathcal{B}})$.
If $(\mathcal{B}, h)$ is a lifted d-ball in $\mathbb{R}^{d'}$ then $\mathrm{lift}(\mathcal{B}, h)$ denotes the copy of $\mathcal{B}$ in $\mathbb{R}^{d'+1}$ with vertices $(\boldsymbol{v}, h(\boldsymbol{v})) \in \mathbb{R}^{d'+1}$, $\boldsymbol{v} \in \mathrm{vert}(\mathcal{B})$. (In the sequel we sometimes do not distinguish between these two interpretations of a lifted ball.) We rely on Figure 5.2 for the illustration of this correspondence.

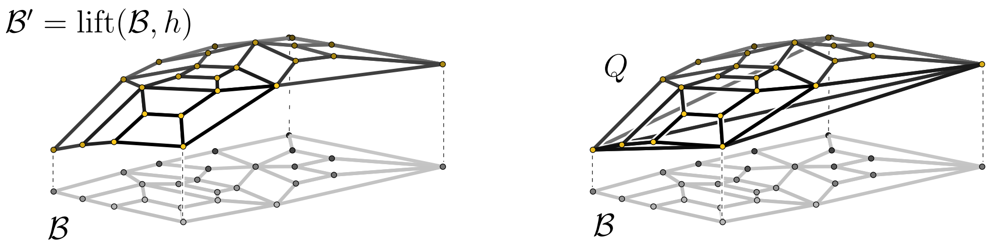


FIGURE 5.2: A lifted cubical ball $(\mathcal{B}, h)$ and its lifted copy $\mathrm{lift}(\mathcal{B}, h)$. The figure on the right shows the convex hull $Q = \mathrm{conv}(\mathrm{lift}(\mathcal{B}, h))$.

Notation. We identify $\mathbb{R}^d$ with $\mathbb{R}^d \times \{0\} \subset \mathbb{R}^{d+1}$, and decompose a point $\boldsymbol{x} \in \mathbb{R}^{d+1}$ as $\boldsymbol{x} = (\pi(\boldsymbol{x}), \gamma(\boldsymbol{x}))$ where $\gamma(\boldsymbol{x})$ is the last coordinate and $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is the projection that eliminates the last coordinate.

In the remaining of this thesis, often a lifted ball $(\mathcal{B}, \psi)$ is constructed as follows. Let P be a d-polytope (in $\mathbb{R}^d$) and $\mathcal{B}'$ a d-dimensional complex in $\mathbb{R}^{d+1}$ such that $|\pi(\mathcal{B}')| = |P|$. If $\mathcal{B}'$ is the set of upper faces of a polytope Q then $\mathcal{B}'$ determines a lifted polytopal subdivision $(\mathcal{B}, \psi)$ of P (where $\mathcal{B} := \pi(\mathcal{B}')$ and ψ is determined the vertex heights $\gamma(\boldsymbol{v})$, $\boldsymbol{v} \in \mathrm{vert}(\mathcal{B}')$). Hence $\mathrm{lift}(\mathcal{B}, \psi)$ equals $\mathcal{B}'$. Compare again Figure 5.2.
A *lifted boundary subdivision* of a d-polytope P is a pair $(\mathcal{S}^{d-1}, \psi)$ consisting of a polytopal subdivision $\mathcal{S}^{d-1}$ of the boundary of P and a piece-wise linear function $\psi : |\partial P| \to \mathbb{R}$ such that for each facet F of P the restriction of ψ to F is a lifting function of the induced subdivision $\mathcal{S}^{d-1} \cap F$.

5.6 Connector polytope

The following construction yields a "connector" polytope that may be used to attach cubical 4-polytopes resp. regular cubical 4-balls without the requirement that the attaching facets are projectively equivalent.

Lemma 5.5 *For any combinatorial* 3*-cube* F *there is a combinatorial* 4*-cube* C *which has both (a projective copy of)* F *and a regular* 3*-cube* F' *as (adjacent) facets.*

Proof. After a suitable projective transformation, we may assume that $F \subset \mathbb{R}^3$ has a unit square Q as a face. Now apply the prism $F \times I$ over F has a both F and $Q \times I$ as adjacent facets, where the latter is a unit cube. $\square$

5.7 Joining polytopal balls

As a final ingredient, we will need a quite simple construction of *joining* two regular polytopal d-balls. This is analogous to the operation of forming connected sums of polytopes as described in Section 2.2.

Construction 2: Join

Input: Two regular polytopal d-balls $\mathcal{B}$ and $\mathcal{B}'$ subdividing d-polytopes $P = |\mathcal{B}|$ and $P' = |\mathcal{B}'|$, where the facets F of P and F' of P' must be projectively equivalent and may not be subdivided in $\mathcal{B}$ resp. in $\mathcal{B}'$.

Output: A regular polytopal complex $\text{Join}(P, P')$ whose boundary is formed by projective copies of the Schlegel complexes $\partial\mathcal{B}\backslash\{F\}$ and $\partial\mathcal{B}'\backslash\{F'\}$.

To construct this, we proceed as follows. After a projective transformation (applied to P or to P') we may assume that $P \cup P'$ is a convex polytope that has all faces of P and of P' except for $F = F' = P \cap P'$. If (P, F) and (P', F') are almost cubical, then $\text{Join}(P, P')$ is a cubical polytope.

The proper faces of $F = F'$ are then also faces of $P \cup P'$. Now we take $\text{Join}(P, P') := \mathcal{B} \cup \mathcal{B}'$, the complex consisting of all faces of $\mathcal{B}$ and of $\mathcal{B}'$. In particular, this includes $F = F'$, which remains to be a face that separates the two "halves" of $\text{Join}(P, P')$.

5.8 From lifted balls to polytopes

5.8.1 Lifted prisms

While there appears to be no simple construction that would produce a cubical $(d + 1)$-polytope from a given cubical d-polytope, we do have a simple prism construction (suggested to us by Arnold Waßmer) that produces regular cubical $(d+1)$-balls from regular cubical d-balls. (A related approach was used by Jockusch in [33].)

Construction 3: LIFTED PRISM

Input: A lifted cubical d-ball $(\mathcal{B}, h)$.
Output: A lifted cubical $(d+1)$-ball $\textsc{LiftedPrism}(\mathcal{B}, h)$ which is combinatorially isomorphic to the prism over $\mathcal{B}$.

We may assume that the convex lifting function h defined on $P := |\mathcal{B}|$ is strictly positive. (Otherwise substitute h by $h - C + M$, where $M > 0$ is constant and $C := \max\{0, \min\{h(\boldsymbol{x}) : \boldsymbol{x} \in |\mathcal{B}|\}\}$.)

Then the lifted facets of $\textsc{LiftedPrism}(\mathcal{B}, h)$ may be taken to be the sets

$$\widetilde{F} \quad := \quad \{(\boldsymbol{x}, t, h(\boldsymbol{x})) : \boldsymbol{x} \in F,\ -h(\boldsymbol{x}) \le t \le +h(\boldsymbol{x})\}, \qquad F \in \operatorname{fac}(\mathcal{B}).$$

If $\mathcal{B}$ does not subdivide the boundary of P, then $\textsc{LiftedPrism}(\mathcal{B}, h)$ does not subdivide the boundary of $|\textsc{LiftedPrism}(\mathcal{B}, h)|$. In this case $\widehat{P} := |\textsc{LiftedPrism}(\mathcal{B}, h)|$ is a cubical $(d+1)$-polytope whose boundary complex is combinatorially isomorphic to the boundary of the prism over $\mathcal{B}$. The f-vector of $\widehat{P}$ is then given by

$$f_k(\widehat{P}) \quad = \quad \begin{cases} 2f_0(\mathcal{B}) & \text{for } k = 0, \\ 2f_k(\mathcal{B}) + f_{k-1}(\partial\mathcal{B}) & \text{for } 0 < k \le d. \end{cases}$$

Figure 5.3 shows the lifted prism over a lifted cubical 2-ball.

Since the outcome of a lifted prism construction is a lifted cubical ball, the lifted prism construction can be iterated:

Corollary 5.6 *For every lifted cubical d-ball $(\mathcal{B}, h)$ and for every $d' > d$, there is a cubical d'-polytope which is combinatorially isomorphic to the boundary of a $(d' - d + 1)$-fold prism over $\mathcal{B}$.*

However, it is an open problem whether a cubical d-polytope with an even number of facets has a (regular) cubification, even for dimension $d = 3$. Nevertheless, Schlegel diagrams help: Consider an almost cubical d-polytope (P, F). Then the Schlegel diagram of P based on the facet F gives a regular cubical subdivision of the $(d-1)$-polytope F. By iterated lifted prisms we obtain a cubical d'-polytope, for every $d' \ge d$.

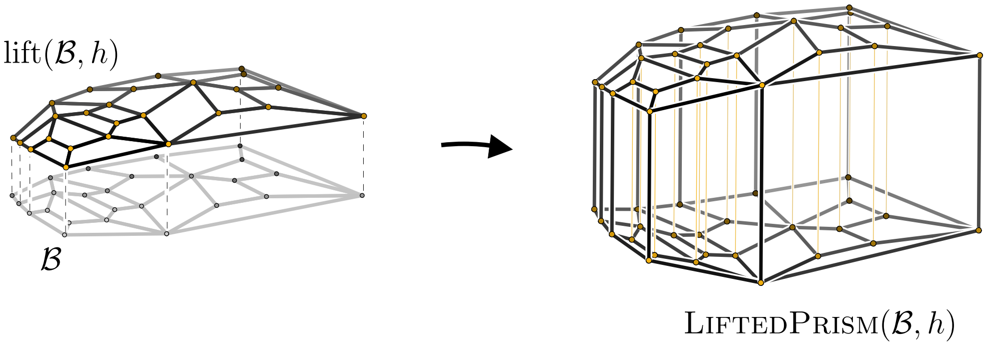


FIGURE 5.3: The lifted prism of a lifted cubical d-ball $(\mathcal{B}, h)$, displayed for $d = 2$. The outcome is a (regular) cubical $(d+1)$-ball which is isomorphic to the prism over $\mathcal{B}$. If $\mathcal{B}$ does not subdivide the boundary of $|\mathcal{B}|$ then the convex hull of the lifted prism yields a cubical $(d+1)$-polytope.

Corollary 5.7 *Let (P, F) be an almost cubical d-polytope.*
Then for every $d' > d$, there is a cubical d'-polytope that is combinatorially isomorphic to the boundary of a $(d' - d + 1)$-fold prism over the Schlegel diagram of P based on the facet F.

Since the class of almost cubical polytopes includes the class of cubical polytopes, this corollary gives an answer to the question due to Joswig and Ziegler whether there is a natural constuction of a a cubical $(d+1)$-polytope from any given cubical d-polytope (Question 2.4).
In the context of the Corollary above the following question arise:

Open Question 5.8 *Is every cubical d-polytope with even number of facets a facet of an almost cubical $(d+1)$-polytope?*

If a cubical d-polytope P is a facet of an almost cubical $(d+1)$-polytope Q, then the Schlegel diagram of Q based on the facet P yields a cubification $\mathcal{C}$ of P that is regular. Even more, this cubification is *Schlegel*, that is, it has a convex lifting function with height 0 on the boundary of $\mathcal{C}$. Hence this question is stronger than the "cubification question."

Open Question 5.9 (Schlegel cubification problem)
Has every cubical d-polytope with an even number of facets a cubification that is Schlegel?

Remark 5.10. (Dual manifolds)
Up to PL-homeomorphism, the cubical ball $\text{LIFTEDPRISM}(\mathcal{B}, h)$ has the following dual manifolds:

- $\mathcal{N} \times I$ for each dual manifold $\mathcal{N}$ of $\mathcal{B}$,
- one $(d-1)$-sphere PL isomorphic to $\partial\mathcal{B}$
  (each vertex of the dual sphere is contained in $\mathbb{R}^d \times \{0\}$).

Remark 5.11. (Modifications)
Assume we are given a lifted cubical d-ball $(\mathcal{B}, \psi)$ height 0 on the boundary such that $\psi(\boldsymbol{x}) = 0$ for all points on the boundary of $\mathcal{B}$. In this case $\mathcal{B}$ does not subdivide the boundary of $P = |\mathcal{B}|$, and the lifted polytope $\text{lift}(\mathcal{B}, \psi)$ has only one facet that is not a lifted facet of $\mathcal{B}$, namely the boundary of $\mathcal{B}$. Hence, $\mathcal{B}$ is projectively isomorphic to a Schlegel-diagram of the polytope $\text{lift}(\mathcal{B}, \psi)$ based on the facet $\partial\mathcal{B}$.
Then the convex hull

$$Q \;:=\; \text{conv}\left(\text{lift}(\mathcal{B}, \psi) \cup \text{lift}(\mathcal{B}, -\psi)\right)$$

gives a cubical $(d+1)$-polytope. Compare Figure 5.4.

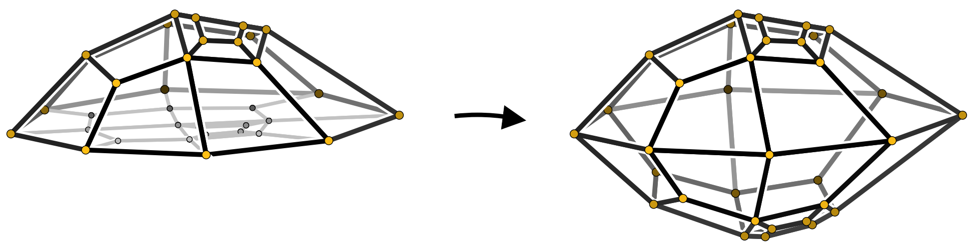


FIGURE 5.4: A variant of the lifted prism construction: If the input is a lifted cubical d-ball $(\mathcal{B}, \psi)$ with height 0 on the boundary, then $\text{conv}\left(\text{lift}(\mathcal{B}, \psi) \cup \text{lift}(\mathcal{B}, -\psi)\right)$ gives a cubical $(d+1)$-polytope – which looks like a lifted prism “without belt.”

5.8.2 Lifted prisms over two balls

Another modification of the lifted prism construction is to take two different lifted cubical balls $(\mathcal{B}_1, h_1), (\mathcal{B}_2, h_2)$ with the same lifted boundary complex (that is, $\partial\mathcal{B}_1 = \partial\mathcal{B}_2$ and $h_1(\boldsymbol{x}) = h_2(\boldsymbol{x})$ for all $\boldsymbol{x} \in \partial\mathcal{B}_1 = \partial\mathcal{B}_2$) as input. In this case the outcome is a cubical $(d+1)$-polytope which may not even have a cubical subdivision that wouldn't subdivide the boundary.

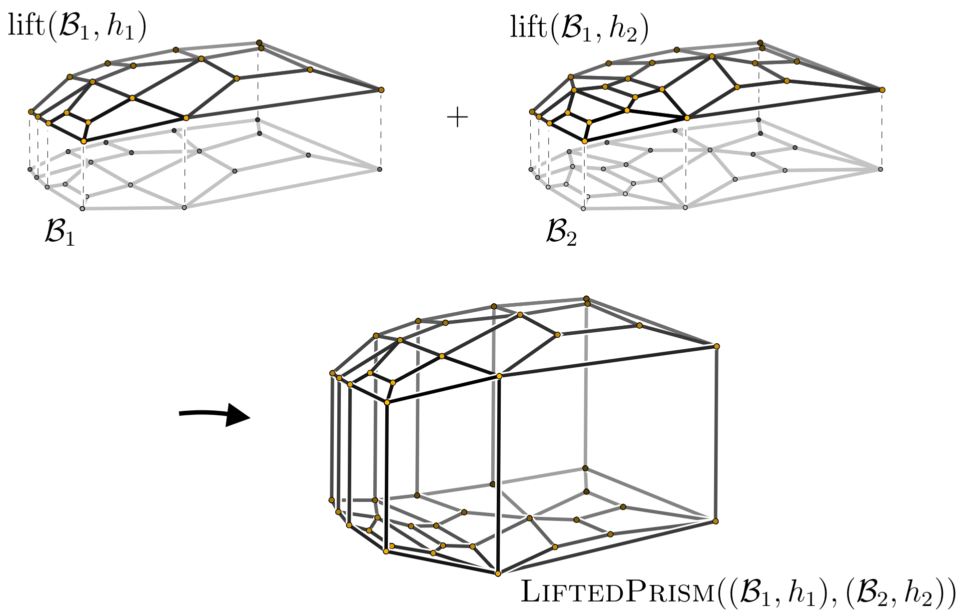


Figure 5.5: The lifted prism over two lifted cubical d-balls $(\mathcal{B}_1, h_1)$ and $(\mathcal{B}_2, h_2)$, displayed for $d = 2$. The outcome is a cubical $(d+1)$-polytope.

Construction 4: Lifted prism over two balls

Input: Two lifted cubical d-balls $(\mathcal{B}_1, h_1)$ and $(\mathcal{B}_2, h_2)$ with the same lifted boundary.

Output: A cubical $(d+1)$-polytope LiftedPrism$((\mathcal{B}_1, h_1), (\mathcal{B}_2, h_2))$ with lifted copies of $\mathcal{B}_1$ and $\mathcal{B}_1$ in its boundary.

If both balls do not subdivide their boundaries, we set $\mathcal{B}'_k := \mathcal{B}_k$ and $h'_k := h_k$ for $k \in \{1, 2\}$. Otherwise we apply the construction of the proof of Lemma 5.2 simultaneously to both lifted cubical balls $(\mathcal{B}_1, h_1)$ and $(\mathcal{B}_2, h_2)$ to obtain two lifted cubical d-balls $(\mathcal{B}'_1, h'_1)$ and $(\mathcal{B}'_2, h'_2)$ with the same support $Q = |\mathcal{B}_1| = |\mathcal{B}_2|$ which do not subdivide the boundary of Q.

Again we can assume that h'_1, h'_2 are strictly positive.

Then $\widehat{Q} :=$ LiftedPrism$((\mathcal{B}_1, h_1), (\mathcal{B}_2, h_2))$ is defined as the convex hull of the points

$$\{(\boldsymbol{x}, +h'_1(\boldsymbol{x})) : \boldsymbol{x} \in |\mathcal{B}'_1|\} \ \cup\ \{(\boldsymbol{x}, -h'_2(\boldsymbol{x})) : \boldsymbol{x} \in |\mathcal{B}'_2|\},$$

Since $\mathcal{B}_1'$ and $\mathcal{B}_2'$ both do not subdivide their boundaries, each of their proper faces yields a face of $\widehat{Q}$. Furthermore, $\widehat{Q}$ is a cubical $(d+1)$-polytope whose f-vector is given by

$$f_k(\widehat{Q}) \quad = \quad \begin{cases} f_0(\mathcal{B}_1) + f_0(\mathcal{B}_2) & \text{for } k = 0, \\ f_k(\mathcal{B}_1) + f_k(\mathcal{B}_2) + f_{k-1}(\partial\mathcal{B}_1) & \text{for } 0 < k \leq d. \end{cases}$$

5.8.3 Shephard patching

We remark that a modification of the Shepard construction (compare Section 3.4) yields the following result.

Proposition 5.12 *Assume we are given a simplicial d-polytope $P \subsetneq \mathbb{R}^d$ and a lifted cubical boundary subdivision $(\mathcal{S}, \psi)$ of P.*
Then there is a cubical d-polytope combinatorially isomorphic to $\mathcal{S}$.

Since this construction is not used in this thesis, the proof is left to the reader.

Chapter 6

Construction techniques for lifted cubical balls

6.1 Schlegel caps

The following construction is a projective variant of the prism construction, applied to a d-polytope P.

Construction 5: SCHLEGEL CAP

Input: An almost cubical d-polytope (P, F_0).

Output: A regular cubical d-ball $\text{SCHLEGELCAP}(P, F_0)$, with

$$P \quad \subset \quad |\,\text{SCHLEGELCAP}(P)| \quad \subset \quad \text{conv}(P \cup x_0),$$

which is combinatorially isomorphic to the prism over the Schlegel complex $\text{SCHLEGEL}(P, F_0)$.

The construction of the Schlegel cap depends on two further pieces of input data, namely a point $\boldsymbol{x}_0 \in \mathbb{R}^d$ beyond F_0 (and beneath all other facets of P), and a hyperplane H that separates $\boldsymbol{x}_0$ from P. It is obtained as follows:

(1) Apply a projective transformation that moves $\boldsymbol{x}_0$ to infinity while fixing H pointwise. This transformation moves the Schlegel complex $\mathcal{C}(\partial P)\backslash\{F_0\}$ to a new cubical complex $\mathcal{E}$.
(2) Reflect the image $\mathcal{E}$ of the Schlegel complex in H, and call its reflected copy $\mathcal{E}'$.
(3) Build the polytope bounded by $\mathcal{E}$ and $\mathcal{E}'$.
(4) Reverse the projective transformation of (1).

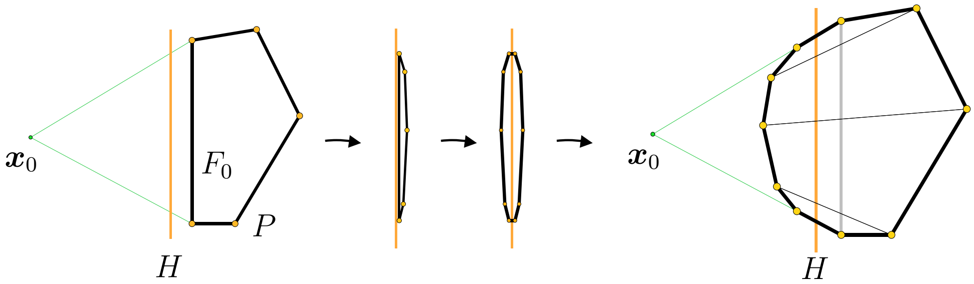


FIGURE 6.1: Construction steps of the Schlegel cap over an almost cubical polytope.

An alternative description, avoiding projective transformations, is as follows:

(1) For each point $\boldsymbol{x}$ in the Schlegel complex $\mathcal{C}(\partial P)\backslash\{F_0\}$ let $\bar{\boldsymbol{x}}$ be the intersection point of H and the segment $[\boldsymbol{x}_0, \boldsymbol{x}]$, and let $\boldsymbol{x}'$ be the point on the segment $[\boldsymbol{x}_0, \boldsymbol{x}]$ such that $[\boldsymbol{x}_0, \bar{\boldsymbol{x}}; \boldsymbol{x}', \boldsymbol{x}]$ form a harmonic quadruple (cross ratio -1).
That is, if $\vec{\boldsymbol{v}}$ is a direction vector such that $\boldsymbol{x} = \boldsymbol{x}_0 + t\vec{\boldsymbol{v}}$ for some $t > 1$ denotes the difference $\boldsymbol{x} - \boldsymbol{x}_0$, while $\bar{\boldsymbol{x}} = \boldsymbol{x}_0 + \vec{\boldsymbol{v}}$ lies on H, then $\boldsymbol{x}' = \boldsymbol{x}_0 + \frac{t}{2t-1}\vec{\boldsymbol{v}}$.

(2) For each face G of the Schlegel complex, $G' := \{\boldsymbol{x}' : \boldsymbol{x} \in G\}$ is the "projectively reflected" copy of G on the other side of H.

(3) The Schlegel cap $\textsc{SchlegelCap}(P, F_0)$ is the regular polytopal ball with faces G, G' and $\mathrm{conv}(G \cup G')$ for faces G in the Schlegel complex.

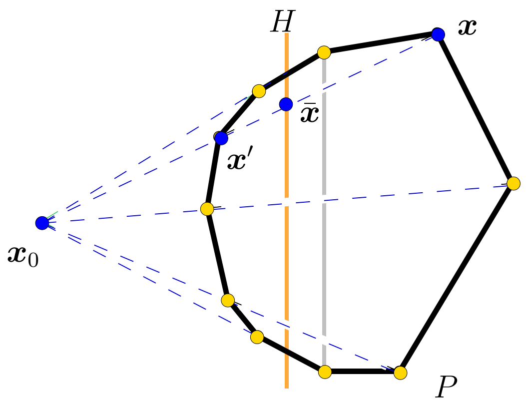


FIGURE 6.2: Constructing the Schlegel cap via cross ratios.

6.2 Truncated pyramids

The essential ingredient of most constructions presented in the following are *truncated pyramids*.

Construction 6: TRUNCATED PYRAMID OVER A CUBICAL SUBDIVISION

Input:

P a d'-polytope in $\mathbb{R}^d$ with $d' < d$,
$\mathcal{B}$ a cubical subdivision of P,
$\boldsymbol{p}$ a point in $\mathbb{R}^d$ with $\boldsymbol{p} \notin \operatorname{aff}(P)$, and
H a hyperplane in $\mathbb{R}^d$ that separates P and $\boldsymbol{p}$.

Output:

$\textsc{TruncPyr}(\mathcal{B}, \boldsymbol{p}, H)$
a cubical $(d'+1)$-ball which is projectively isomorphic to the prism over $\mathcal{B}$.

For each facet F of $\mathcal{B}$ construct the pyramid with apex $\boldsymbol{p}$ over F, and truncate it by the hyperplane H. This yields a $(d'+1)$-polytope C_F combinatorially isomorphic to the prism over F. The cubical $(d'+1)$-ball formed by all the resulting combinatorial d-cubes C_F is $\textsc{TruncPyr}(\mathcal{B}, \boldsymbol{p}, H)$.

Compare Figure 6.3 with illustrates the case $d' = 2$, $d = 3$.

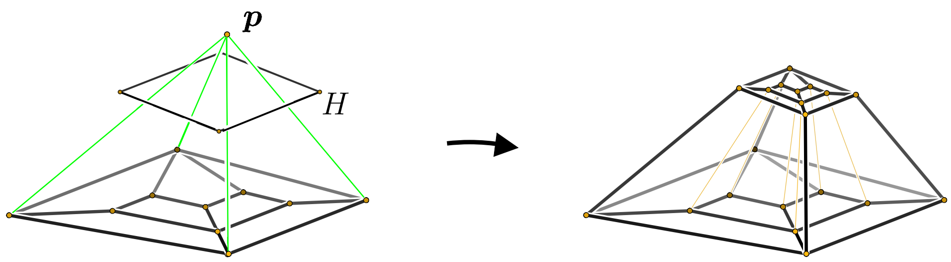


FIGURE 6.3: The truncated pyramid over a cubical subdivision.

We call $\boldsymbol{p}$ the *apex point* and H the *truncating hyperplane*.

6.3 Truncated pyramid over a lifted cubical subdivision

Construction 7: TRUNCATED PYRAMID OVER A LIFTED CUBICAL SUBDIVISION OF A POLYTOPE

Input:

P	a $(d-1)$-polytope in $\mathbb{R}^d$,
$(\mathcal{B}, \psi)$	a lifted cubical subdivision of P,
$\boldsymbol{p}$	a point in $\mathbb{R}^{d+1}$, and
H	a hyperplane in $\mathbb{R}^{d+1}$ that separates $\mathcal{B} \times \{0\}$ and $\boldsymbol{p}$.

Output:

$\textsc{TruncPyr}((\mathcal{B}, \psi), \boldsymbol{p}, H)$
a lifted cubical d-ball that is projectively isomorphic to the prism over $\mathcal{B}$.

Define $\textsc{TruncPyr}((\mathcal{B}, \psi), \boldsymbol{p}, H)$ as the lifted cubical subdivision of P determined by the cubical d-ball

$$\mathcal{T}' := \textsc{TruncPyr}(\operatorname{lift}(\mathcal{B}, \psi), \boldsymbol{p}, H).$$

Compare Figure 6.4 for $d = 2$.

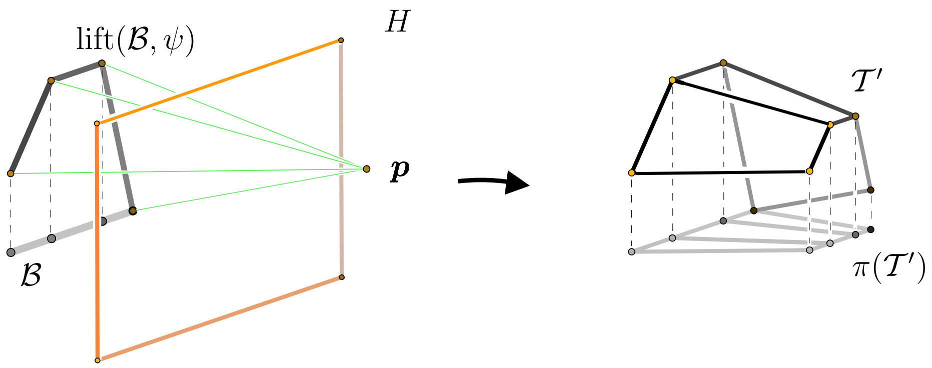


FIGURE 6.4: The truncated pyramid over a lifted cubical subdivision.

6.4 Iterated truncated pyramids

The *generalized regular Hexhoop* of Section 9.5 is based on the *iterated truncated pyramid* construction. This construction takes a lifted cubical subdivision of $(d-1)$-polytope in $\mathbb{R}^d$ and a sequence of apex points $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_\ell$ as well as a sequence of truncating hyperplanes $H_1, \ldots, H_\ell$. The outcome consists of

- a sequence $\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_\ell$ of lifted cubical $(d-1)$-balls combinatorially isomorphic to $\mathcal{B}$, and
- a sequence $\mathcal{T}_1, \ldots, \mathcal{T}_\ell$ of lifted cubical d-balls combinatorially isomorphic to a prism over $\mathcal{B}$,

as shown in the following figure.

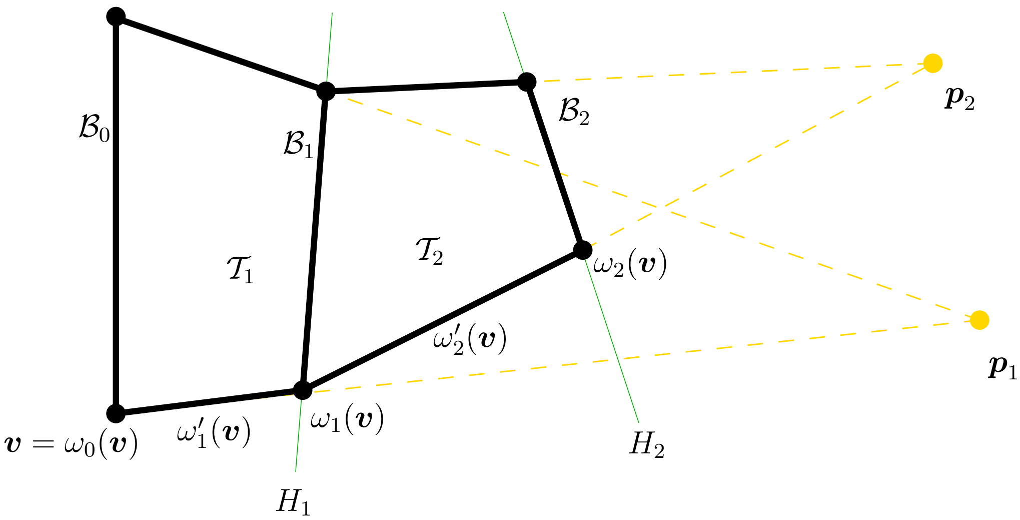


Figure 6.5: The iterated truncated pyramid over a lifted cubical subdivision. (Here the "top view", the projection into $\mathbb{R}^d$, is displayed.)

If the cubical balls $\mathcal{T}_1, \ldots, \mathcal{T}_\ell$ form a polytopal complex, that is, the intersection of two cells yields a face of both cells, we define the *iterated lifted truncated pyramid* over $(\mathcal{B}, \psi)$ with respect to $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_\ell$ and $H_1, \ldots, H_\ell$ as

$$\textsc{IteratedTruncPyr}((\mathcal{B}, \psi), (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_\ell; H_1, \ldots, H_\ell)) := \bigcup_{k=1}^{\ell} \mathcal{T}_k.$$

Construction 8: ITERATED TRUNCATED PYRAMID OVER A LIFTED CUBICAL SUBDIVISION OF A POLYLTOPE

Input:

P	a $(d-1)$-polytope in $\mathbb{R}^d$.
$(\mathcal{B}, \psi)$	a lifted cubical subdivision of P.
$\boldsymbol{p}_1, \ldots, \boldsymbol{p}_\ell$	a sequence of *apex points* in $\mathbb{R}^{d+1}$ with $\boldsymbol{p}_1 \notin \operatorname{aff}(P)$.
$H_1, \ldots, H_\ell$	a sequence of *truncating hyperplanes* in $\mathbb{R}^{d+1}$ with $\boldsymbol{p}_k \notin H_{k-1}$, for $k = 1, \ldots, \ell$.

Output:

$\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_\ell$	disjoint lifted cubical $(d-1)$-balls combinatorially isomorphic to $\mathcal{B}$.
$\mathcal{T}_1, \ldots, \mathcal{T}_\ell$	lifted cubical d-balls combinatorially isomorphic to a prism over $\mathcal{B}$ such that

- $\mathcal{B}_{k-1}, \mathcal{B}_k$ are (disjoint) subcomplexes of $\mathcal{T}_k$, for $k \in \{1, \ldots, \ell\}$,
- $\mathcal{T}_k \cap H_k = \mathcal{B}_k$, for $k \in \{1, \ldots, \ell\}$, and
- $\mathcal{T}_k \cap H_{k-1} = \mathcal{B}_{k-1}$, for $k \in \{1, \ldots, \ell\}$.

(1) Define $\mathcal{B}_0 := \operatorname{lift}(\mathcal{B}, \psi)$.

(2) For $k = 1$ to ℓ construct a lifted truncated prism over $\mathcal{B}_{k-1}$ with apex point $\boldsymbol{p}_k$ and truncating hyperplane H_k which yields the complex $\mathcal{T}_k$. Hence

$$\begin{aligned} \mathcal{T}_k &:= \textsc{TruncPyr}(\mathcal{B}_{k-1}, \boldsymbol{p}_k, H_k) \\ \mathcal{B}_k &:= \mathcal{T}_k \cap H_k. \end{aligned}$$

Assume that the cubical balls $\mathcal{T}_1, \ldots, \mathcal{T}_\ell$ form a polytopal complex. For the lifted cubical d-complex

$$\mathcal{T} := \textsc{IteratedTruncPyr}((\mathcal{B}, \psi), (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_\ell; H_1, \ldots, H_\ell)) = \bigcup_{k=1}^{\ell} \mathcal{T}_k$$

we define some functions

$$\begin{aligned} \omega'_1, \ldots \omega'_\ell : &\ \mathcal{F}(\operatorname{lift}(\mathcal{B}, \psi)) \to \mathcal{F}(\mathcal{T}), \\ \omega_0, \omega_1, \ldots \omega_\ell : &\ \mathcal{F}(\operatorname{lift}(\mathcal{B}, \psi)) \to \mathcal{F}_{d-1}(\mathcal{T}), \end{aligned}$$

where for each face $F \in \operatorname{lift}(\mathcal{B}, \psi)$ and $k \in \{1, \ldots, \ell\}$

$\omega_k(F)$ denotes the k-th copy of F in $\mathcal{T}$,

$\omega_0(F)$ denotes F (which is included in $\mathcal{T}$), and
$\omega'_k(F)$ denotes the k-th lifted truncated pyramid over F in $\mathcal{T}$.

Thus we have $\omega_0(F) := F$ for $F \in \mathcal{F}(\text{lift}(\mathcal{B}, \psi))$, and for $k = 1$ to ℓ set

$$\begin{array}{lcll} \omega'_k(F) & := & \text{TRUNCPYR}(\omega_{k-1}(F), \boldsymbol{p}_k, H_k) & \text{for } F \in \mathcal{F}(\text{lift}(\mathcal{B}, \psi)), \\ \omega_k(F) & := & H_k \cap \omega'_k(F) & \text{for } F \in \mathcal{F}(\text{lift}(\mathcal{B}, \psi)). \end{array}$$

6.5 Truncated Schlegel caps

Using truncated pyramids we can describe a variation of the Schlegel cap of Section 6.1, the *truncated Schlegel cap*.

Construction 9: TRUNCATED SCHLEGEL CAP

Input:

(P, F_0) an almost cubical d-polytope,
$\boldsymbol{p}$ a point in $\mathbb{R}^d$ such $\boldsymbol{p}$ is beyond F_0 and beneath all other facets of P, and
H a hyperplane that separates P and $\boldsymbol{p}$.

Output:

$\mathcal{B}$ a cubical d-ball combinatorially isomorphic to $\text{SCHLEGELCAP}(P, F_0)$.

The facets of cubical d-ball $\mathcal{B}$ may be taken to be the combinatorial d-cubes

$$\text{TRUNCPYR}(F, \boldsymbol{p}, H), \qquad F \in \text{fac}(P) \setminus \{F_0\}.$$

The "lifted version" of this construction can be described as follows.

Notation. Recall that we identify $\mathbb{R}^d$ with $\mathbb{R}^d \times \{0\} \subset \mathbb{R}^{d+1}$, and decompose a point $\boldsymbol{x} \in \mathbb{R}^{d+1}$ as $\boldsymbol{x} = (\pi(\boldsymbol{x}), \gamma(\boldsymbol{x}))$ where $\gamma(\boldsymbol{x})$ is the last coordinate and $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is the projection that eliminates the last coordinate.

Construction 10: LIFTED TRUNCATED SCHLEGEL CAP

Input:

(P, F_0) an almost cubical d-polytope,
$\boldsymbol{p}$ a point in $\mathbb{R}^{d+1}$ such that $\boldsymbol{p} \notin \text{aff}(P) \subset \mathbb{R}^d \subset \mathbb{R}^{d+1}$ and $\pi(\boldsymbol{p})$ is beyond F_0 and beneath all other facets of P, and

H a hyperplane parallel to $\mathbb{R}\boldsymbol{e}_{d+1}$ such that H separates P and $\boldsymbol{p}$.

Output:

$(\mathcal{B}, \psi)$ a lifted cubical d-ball combinatorially isomorphic to $\textsc{SchlegelCap}(P, F_0)$, given as a cubical d-ball $\mathcal{C}$.

The facets of cubical d-ball $\mathcal{C}$ may be taken to be the combinatorial d-cubes

$$\textsc{TruncPyr}(F, \boldsymbol{p}, H), \qquad F \in \mathrm{fac}(P) \setminus \{F_0\}.$$

The construction is illustrated in Firgure 6.6.

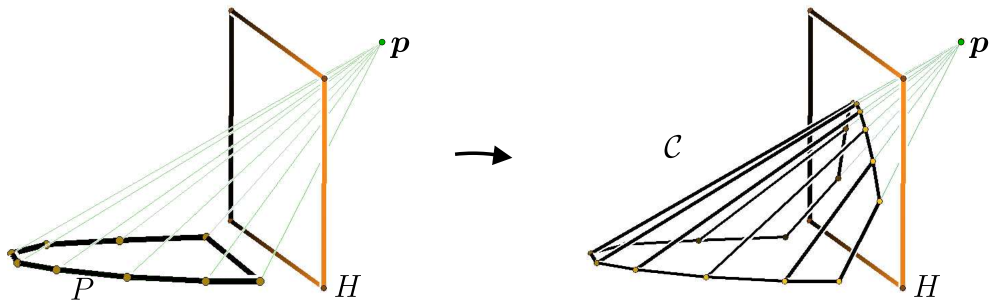


FIGURE 6.6: The lifted truncated Schlegel cap construction.

Remark. One can describe a similar construction which takes a lifted cubical boundary subdivision of a Schlegel complex of a polytope as input.

Chapter 7

A small cubical 4-polytope with a dual Klein bottle

In this chapter we present the first instance of a cubical 4-polytope with a non-orientable dual manifold. By Proposition 4.2 this instance is not edge-orientable. Hence, its existence also confirms the conjecture of Hetyei (Conjecture 4.4).

Theorem 7.1 *There is a cubical* 4*-polytope* P_{72} *with* f*-vector*

$$f(P_{72}) \quad = \quad (72,\ 196,\ 186,\ 62),$$

such that one of its dual manifolds is an immersed Klein bottle of f*-vector* $(80,\ 160,\ 80)$.

Step 1. We start with a cubical octahedron O_8, the smallest cubical 3-polytope that is not a cube, with f-vector

$$f(O_8) \quad = \quad (10,\ 16,\ 8).$$

We may assume that O_8 is already positioned in $\mathbb{R}^3$ with a regular square base facet Q and acute dihedral angles at this square base; compare Figure 7.1. The f-vector of any Schlegel diagram of this octahedron is

$$f(\textsc{Schlegel}(O_8, Q)) \quad = \quad (10,\ 16,\ 7).$$

Let O_8' be a congruent copy of O_8, obtained by reflection of O_8 in its square base followed by a 90° rotation around the axis orthogonal to the base; compare Figure 7.2. This results in a regular 3-ball with cubical 2-skeleton with f-vector

$$f(B_2) \quad = \quad (16,\ 28,\ 15,\ 2).$$

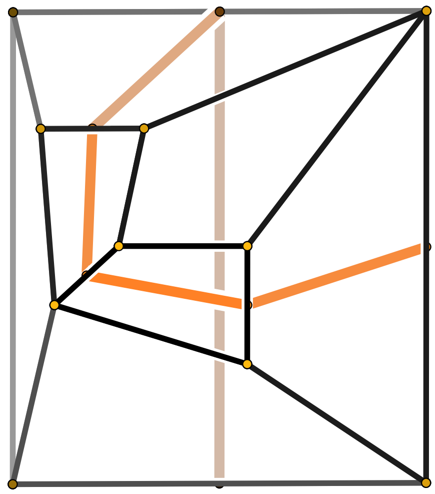


FIGURE 7.1: The cubical octahedron O_8 positioned in $\mathbb{R}^3$ with a regular square base facet Q and acute dihedral angles at this square base. A part of the dual manifold is highlighted – this dual curve 'travels' through one half of the Möbius strip with parallel inner edges that we construct in the second step.

The special feature of this complex is that it contains a cubical Möbius strip with parallel inner edges of length 9 in its 2-skeleton, as is illustrated in Figure 7.2.

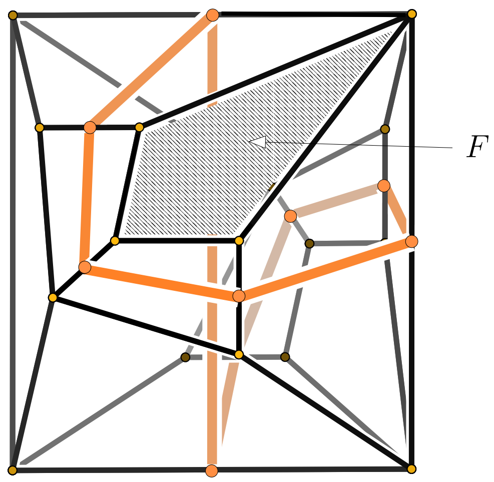


FIGURE 7.2: The outcome of step 1 of the construction: The 2-cubical convex 3-ball $\mathcal{B}_2$ which contains a Möbius strip with parallel inner edges in the 2-skeleton. (The dual manifold of the Möbius strip (an embedded S^1) is given by the bold orange edges.)

Step 2. Now we perform a Schlegel cap construction on O_8, based on the (unique) facet F of O_8 that is not contained in the Möbius strip mentioned above, and that is not adjacent to the square glueing facet Q. This Schlegel cap has the f-vector

$$f(S_7) = (20, 42, 30, 7),$$

while its boundary has the f-vector

$$f(\partial S_7) = (20, 36, 18).$$

Step 3. The same Schlegel cap operation may be performed on the second copy O'_8. Joining the two copies of the Schlegel cap results in a regular cubical 3-ball B_{14} with f-vector

$$f(B_{14}) = (36, 80, 59, 14)$$

whose boundary has the f-vector

$$f(\partial B_{14}) = (36, 68, 34).$$

The ball B_{14} again contains the cubical Möbius strip with parallel inner edges of length 9 as an embedded subcomplex in its 2-skeleton.

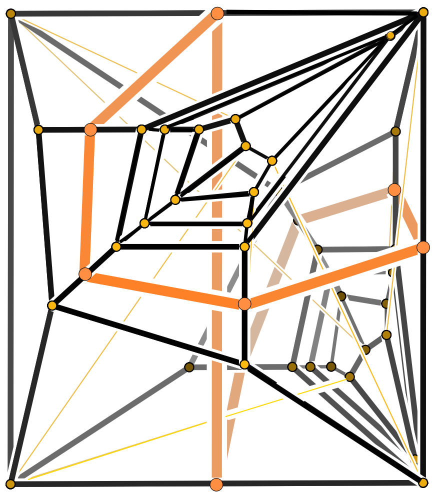


FIGURE 7.3: The outcome of step 2 of the construction: The cubical convex 3-ball $\mathcal{B}_{14}$ which contains a Möbius strip with parallel inner edges in the 2-skeleton. (Again, the dual manifold of the Möbius strip is given by the bold edges.)

Step 4. Now we build the prism over this regular cubical ball, resulting in a regular cubical 4-ball $\mathcal{B}$ whose f-vector is

$$f(\mathcal{B}) \quad = \quad (72,\ 196,\ 198,\ 87,\ 14)$$

and whose support is a cubical 4-polytope $P_{72} := |\mathcal{B}|$ with two copies of the cubical Möbius strip in its 2-skeleton. Its f-vector is

$$f(P_{72}) \quad = \quad (72,\ 196,\ 186,\ 62).$$

A further (computer-supported) analysis of the dual manifolds shows that there are six dual manifolds in total: one Klein bottle of f-vector (80, 160, 80), and five 2-spheres (four with f-vector (20, 36, 18), one with f-vector (36, 68, 34)). All the spheres are embedded, while the Klein bottle is immersed with five double-intersection curves (embedded 1-spheres), but with no triple points. $\square$

It is possible to reduce the number of double-intersection curves of the dual Klein bottle A cubical 4-polytope with an non-orientable dual manifold of genus 2 that is immersed with a *single* double-intersection curve is given in Section 12.3. (The drawback is that the number of faces increases significantly.)

Remark 7.2 (A small not edge-orientable 2-cubical 4-polytope).
The “by-product” $\mathcal{B}_2$ is a 2-cubical 3-ball with two facets. Thus its regular and the lifted prism yields a 2-cubical 4-polytope which does not admit an edge orientation. The f-vector of this polytope is (32, 32, 58, 18).

Remark 7.3 (A small not edge-orientable cubical 3-sphere).
A star-shaped not edge-orientable cubical 3-sphere, which is smaller than the cubical 4-polytope discussed above, can be obtained by a modification of our construction:
Instead of the octahedron, we take as ‘start object’ a cubical two-dimensional complex $\mathcal{C}$ with an embedded Möbius strip with seven quadrangles. This complex of f-vector $f(\mathcal{C}) = (14, 23, 10)$ is depicted in Figure 7.4.
Applying the truncated Schlegel cap construction to the “left” and the “right half” of $\mathcal{C}$ yields a not edge-orientable cubical 3-ball $\mathcal{B}'$ with $f(\mathcal{B}') = (34, 64, 32)$. It is not convex, but *star-shaped*, that is there is a point (in the “middle”) that is a view point of the boundary of $\mathcal{B}'$. This star-shaped cubical 3-ball is depicted in Figure 7.5.
The boundary of the prism over $\mathcal{B}'$ yields a not edge-orientable star-shaped cubical 3-sphere with f-vector (68, 180, 168, 56).

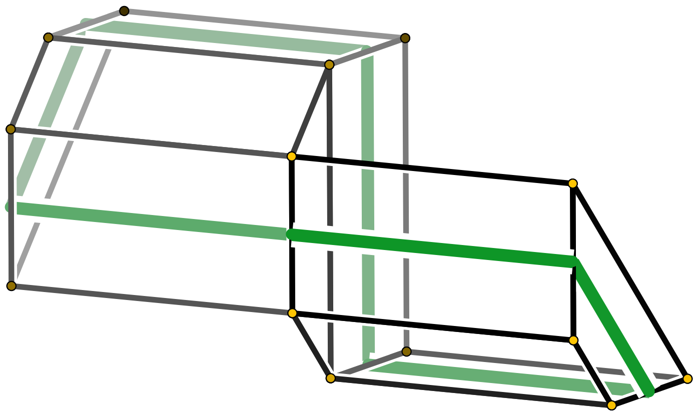


FIGURE 7.4: A two-dimensional cubical complex $\mathcal{C}$ of f-vector (14, 23, 10). This complex has an embedded Möbius strip with seven quadrangles.

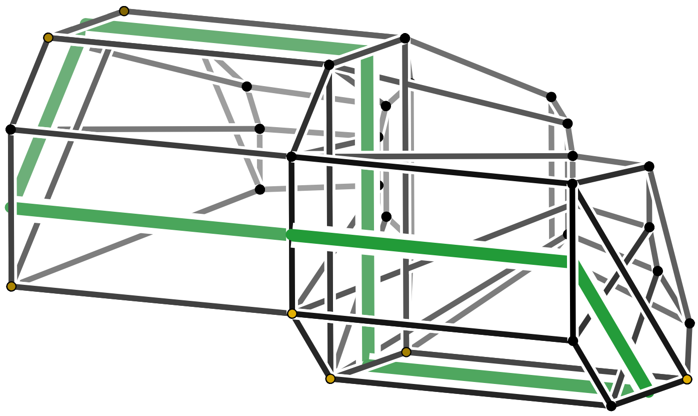


FIGURE 7.5: The star-shaped cubical 3-ball $\mathcal{B}'$ with an embedded Möbius strip with parallel interior edges. The f-vector is (34, 64, 32).

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Chapter 8

First consequences

8.1 Cubical 4-Polytopes with Orientable Dual Manifolds of Prescribed Genus

Cubical 4-polytopes with an orientable dual manifold of prescribed genus can be produced by means of connected sums of copies of the "neighborly cubical" 4-polytope C_4^5 with the graph of a 5-cube (compare Section 2.7.3).

Lemma 8.1 *For each $g > 0$, there is a cubical 4-polytope P_g with the following properties.*

(i) The polytope P_g has exactly one embedded orientable dual 2-manifold $\mathcal{M}$ of genus g with f-vector $f(\mathcal{M}) = (12g + 4,\ 28g + 4,\ 14g + 2)$.

(ii) There is a facet F of P that is not intersected by the image of the dual manifold $\mathcal{M}$, and that is affinely regular, that is there is an affine transformation between F and the standard cube $[-1, +1]^3$.

(iii) All other dual manifolds of P_g are embedded 2-spheres.

(iv) $f(P_g) = (24g + 8,\ 116g + 12,\ 138g + 6,\ 46g + 2)$.

Proof. The neighborly cubical 4-polytope $C_4^5 := \mathrm{conv}((Q \times 2Q) \cup (2Q \times Q))$, where $Q = [-1, +1] \times [-1, +1]$, has an embedded dual torus $\mathcal{M}$ (compare Sect. 2.7.3). Furthermore, C_4^5 has a facet $F := Q \times 2Q$ that is affinely regular, and that it is not intersected by the embedded dual manifold $\mathcal{M}$. Select an arbitrary facet G of C_4^5 which is intersected by $\mathcal{M}$. Due to the symmetry of C_4^5 there is an "opposite" facet G' of C_4^5 such that G' is intersected by $\mathcal{M}$, G and G' are disjoint, and G and G' are congruent. Let ϕ be the congruence which maps $(G, j(\mathcal{M}) \cap G)$ onto $(G', j(\mathcal{M}) \cap G')$.

Then take g copies of C_4^5 and glue them together such that each glueing operation merges a copy of G and a copy of G' (using ϕ). The resulting polytope P_g has an embedded dual manifold $\mathcal{M}_g$ which is obtained by glueing together g copies of $\mathcal{M}$. Furthermore, P_g has a facet F' that is projectively isomorphic to the facet F of C_4^5. ☐

Furthermore it is possible to prescribe the numbers of orientable dual manifolds for different genera. Let P be a cubical 4-polytope without non-orientable dual manifolds, and g the maximal genus of a dual manifold of P. Then the vector $(\ell_0, \ell_1, \ldots, \ell_g)$, where ℓ_i denotes the number of dual manifolds of P of genus i, is called the *ℓ-vector of P*.

Lemma 8.2 *Assume we are given a vector $(\ell_0, \ell_1, \ldots, \ell_g)$ of non-negative integers. Then there is a cubical 4-polytope P and an integer $\ell'_0 \geq 0$ such that $(\ell_0 + \ell'_0, \ell_1, \ell_2, \ldots, \ell_g)$ is the ℓ-vector of P.*

Proof. Take a cubical 4-polytope P with least $L := \ell_0 + \ell_1 + \ldots + \ell_g$ projectively regular facets, that is, for each facet F of P there is a projective transformation between F and the standard cube $[-1, +1]^3$. For instance a capped cubical 4-polytope which has at least L facets produced by the capping operation is appropriate.
Then glue Q together with ℓ_i copies of the polytope P_i (given by Lemma 8.1) for each $i \in \{1, \ldots, g\}$. Each connected sum operation involves one of the projectively regular facets of P and the projectively regular facet F of the copy P' of P_i which is not intersected by the dual manifold of P' of genus i. All dual manifolds of the resulting cubical 4-polytope are orientable, embedded, and the genus is at most g. Furthermore for every $i \in \{1, \ldots, \ell_g\}$ there are exactly ℓ_i dual manifolds of genus i, and all other dual manifolds are embedded 2-spheres. ☐

8.2 Cubical polytopes with few dual manifolds

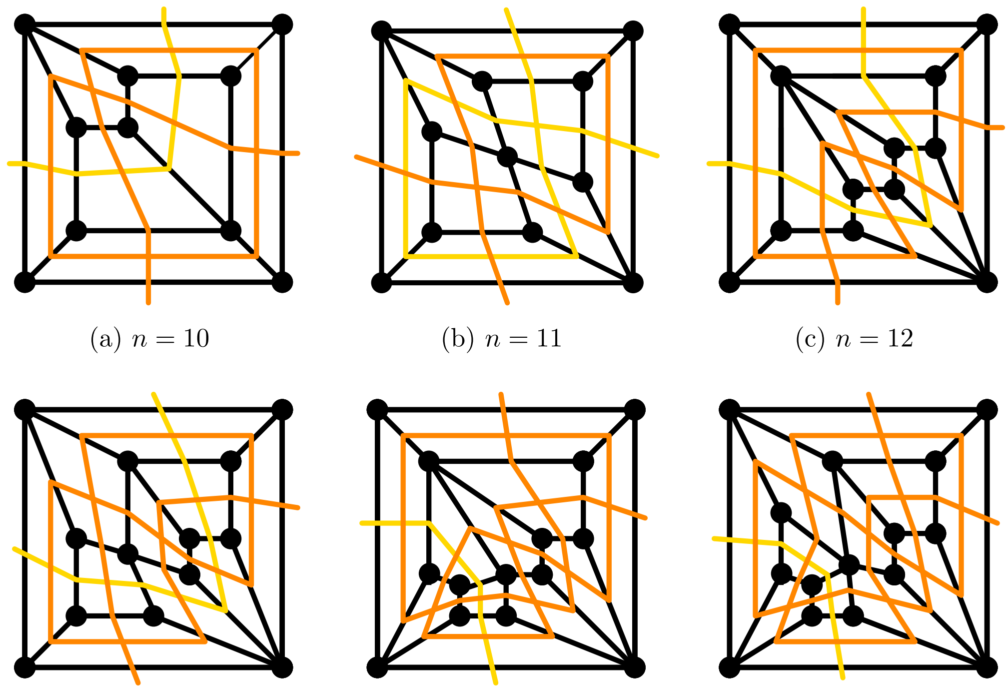
As mentionened in Section 4.1 there is a cubical 3-polytope with connected derivative complex, namely the *cubical octahedron* O_8, that is the only combinatiorial type of a cubical 3-polytope with 8 facets. Compare Figure 4.3. Let $\mu(P)$ denote the number of dual manifolds of a cubical d-polyope P. Clearly, every combinatorial d-cube Q satisfies $\mu(Q) = d$.

The following questions arise naturally:

Open Question 8.3 *Is there a cubical d-polytope P with $\mu(P) = 1$ for every dimension d? Furthermore, is there a cubical d-polytope P with $\mu(P) = 1$ and $f_0 > N$ vertices for every dimension d and every constant $N \in \mathbb{N}$?*

We give a positive answer for $d = 3$.

Lemma 8.4 *For every* $n \geq 10$*, there is a cubical* 3*-polytope* R^3_n *with* n *vertices and* $\mu(R_n) = 1$*.*



(a) $n = 10$ (b) $n = 11$ (c) $n = 12$

(d) $n = 13$ (e) $n = 14$ (f) $n = 15$

Figure 8.1: Cubical 3-polytopes with a single dual manifold.

Proof. The graphs of cubical 3-polytops with $f_0 \in \{10, \ldots, 15\}$ vertices with a single dual manifolds are depicted in Figure 8.1.

A cubical 3-polytope R_n with $n \geq 16$ vertices with connected derivative complex is constructed as follows: Take a cubical 3-polytope R^3_{n-6} with $n-6$ vertices and $\mu(R^3_{n-6}) = 1$. Perform a connected sum with the cubical octahedron O_8 (which has 8 vertices and $\mu(O_8) = 1$) such that the resulting cubical 3-polytope R^3_n satisfies $\mu(R^3_n) = 1$. (The connected sum of two d-polytopes with $\mu = 1$ has either one or two dual manifolds, depending on the identification of the attached facets.) ☐

Since we cannot answer the questions in higher dimensions, we consider the following (weaker) questions:

- Is there a cubical d-polytope P with $\mu(P) < d$, for any $d \geq 2$?
- Or, is there a cubical d-polytope P with $\mu(P) = L$ and more that n vertices, for a given $L \leq d$ and any $N > 2^d$?

Lemma 8.5 *For every $d \geq 4$ and every $n \geq 10$, there is a cubical d-polytope R^d_n with $\mu(R^d_n) = d$ and $f_0 = n2^{d-2}$ vertices.*

Proof. Let P be a cubical 3-polytope with n vertices that has extactly one dual manifold. Choose a facet F of P. Then the Schlegel complex $\mathcal{A} := \mathcal{C}(\partial P) \setminus \{F\}$ is a regular cubical 2-ball with two dual manifolds.
Let $\mathcal{B}^d$ be the $(d-2)$-fold lifted prisms over the Schlegel complex $\mathcal{C}(\partial P) \setminus \{F\}$. Then $\mathcal{B}^d$ has $n2^{d-2}$ vertices since each lifted prism operation doubles the number of vertices. By Remark 5.10 the d-ball $\mathcal{B}^d$ has $2 + (d-2) = d$ dual manifolds.
Hence the convex hull of $\mathcal{B}^d$ yields a cubical d-polytope R^d_n with $\mu(R^d_n) = d$ dual and $f_0(R^d_n) = n2^{d-2}$. □

8.3 Surfaces in the 2-skeleton of cubical polytopes

Now consider embeddings of surfaces in the 2-skeleton of cubical polytopes. Obviously, the boundary of a 3-cube is a 2-sphere, and there is a subset of the 2-faces of the 4-cube which forms a torus with 16 vertices, 32 edges and 16 quadrangles. Indeed, for every genus there is a high-dimensional cube such that a subset of the 2-faces of the cube is a surface of the given genus. This holds for both orientable and non-orientable surfaces, as the following result shows:

Theorem 8.6 (Köhler [41])

(i) *For every integer $g \geq 0$, there is an embedding of an orientable surface of genus g in the 2-skeleton of the cube of dimension $d(g) = g + 3$.*

(ii) *For every integer $g > 0$, there is an embedding of a non-orientable surface of genus g in the 2-skeleton of the cube of dimension $d'(g) = g + 5$.*

Joswig [35] observed that the dimension of the used polytopes can be reduced to dimension 6, if neighborly cubical polytopes are used instead of cubes:

Corollary 8.7

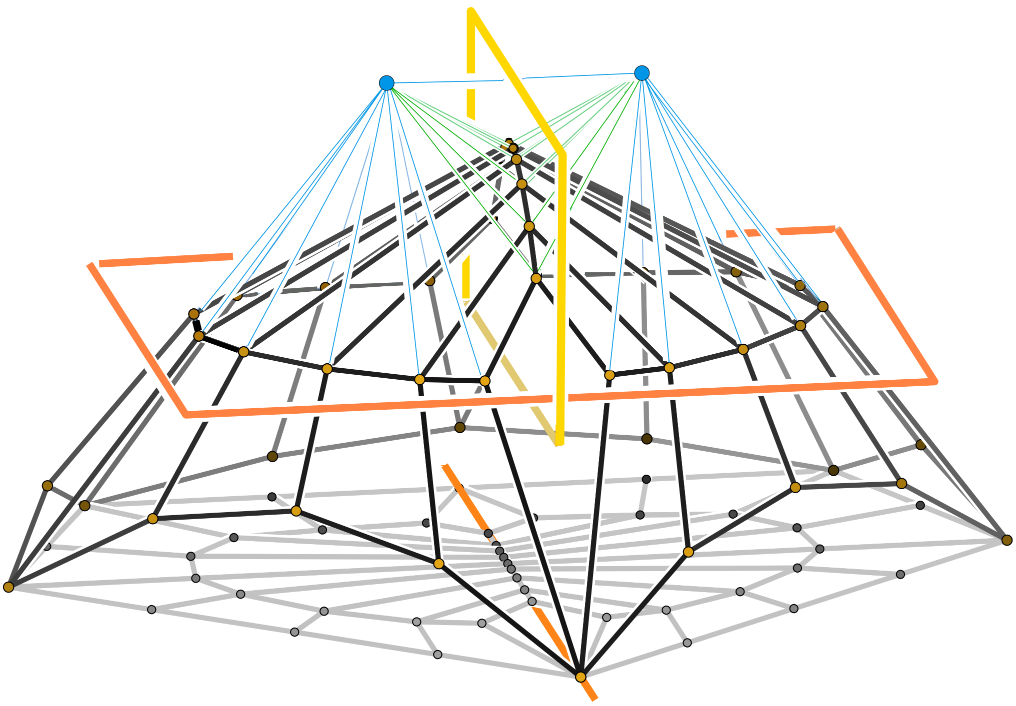
(i) *For every genus $g \geq 0$, there is a cubical 6-polytope polytope P such that there is an embedding of an orientable surface of genus g in the 2-skeleton of P.*

(ii) *For every genus $g > 0$, there is a cubical 6-polytope polytope P such that there is an embedding of a non-orientable surface of genus g in the 2-skeleton of P.*

This follows from Theorem 8.6 and the existence of neighborly cubical polytopes (Theorem 2.22).

Part III

Main results



Chapter 9

Constructing symmetric cubifications

9.1 Introduction

Let us recall that a *cubification* of a cubical PL $(d-1)$-sphere $\mathcal{S}^{d-1}$ is a cubical d-ball $\mathcal{B}^d$ with boundary $\mathcal{S}^{d-1}$. A double counting argument shows that every cubical $(d-1)$-sphere that admits a cubification has an even number of facets. Whether this condition is sufficient is a challenging open problem, even for $d = 3$ (compare e.g. [10], [21]). Compare the discussion in Section 3.2.

There is a construction of cubifications in dimension 3 called the *Hexhoop template* due to Yamakawa & Shimada [58], which is depicted in following figure.

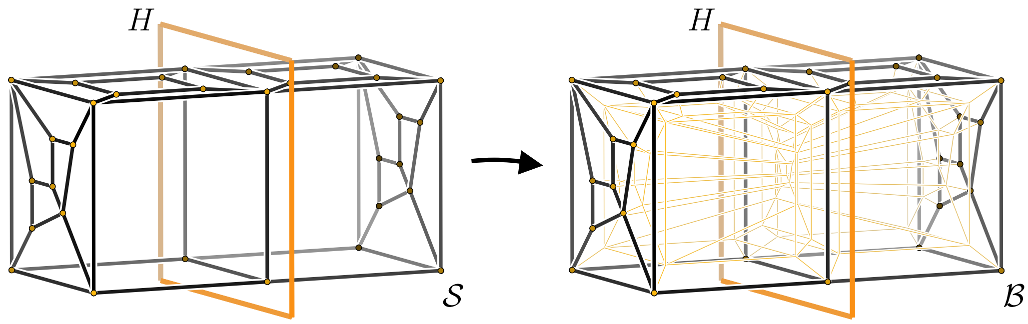


FIGURE 9.1: The Hexhoop template of Yamakawa & Shimada [58]. The input is a cubical boundary subdivision $\mathcal{S}$ of a 3-cube P such that both P and $\mathcal{S}$ are symmetric with respect to a hyperplane H and H intersects no facet of $\mathcal{S}$ in its relative interior. The outcome is a cubification of $\mathcal{S}$, that is a cubical 3-ball with boundary $\mathcal{S}$.

Their construction takes as input a 3-polytope P that is projectively isomorphic to a regular 3-cube, a hyperplane H and a cubical subdivision $\mathcal{S}$ of the boundary complex of P such that $\mathcal{S}$ is symmetric with respect to H and H intersects no facet of $\mathcal{S}$ in its relative interior. For such a cubical 2-complex $\mathcal{S}$ the Hexhoop template produces a cubification.
A 2-dimensional version of their construction looks as follows.

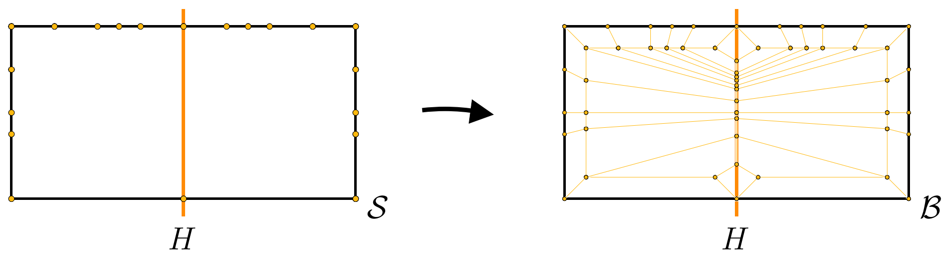


FIGURE 9.2: A two-dimensional version of the Hexhoop template.

The 'key insight' is that such a cubification can be constructed (in arbitrary dimensions $d \geq 2$) by means of iterated truncated pyramids. Alternatively, one can describe the construction using two truncated Schlegel cap constructions of Section 6.5 (where the input is a polytope with a boundary subdivision of a polytope, instead of a polytope).

The following questions about the Hexhoop template arise.

▷ Is the resulting cubification regular?
▷ More generally, is it possible to produce a lifted cubification with "prescribed heights on the boundary" (with a symmetry requirement)?
▷ Can we drop the requirement that the input polytope is a cube, and allow arbitrary input geometries (symmetric with respect to a hyperplane)?
▷ Is there a generalization of this technique that works in any dimension?

We give positive answers to these questions by introducing and analyzing the so-called *generalized regular Hexhoop-template*. This construction can be interpreted as a generalization of the Hexhoop template in several directions: Our approach admits arbitrary geometries, works in any dimension, and yields regular cubifications with "prescribed heights on the boundary" (with a symmetry requirement).

Figure 9.3 displays is a 2-dimensional cubification (of a boundary subdivision $\mathcal{S}$ of a 2-polytope such that $\mathcal{S}$ is symmetric with respect to a hyperplane H) obtained by our construction.

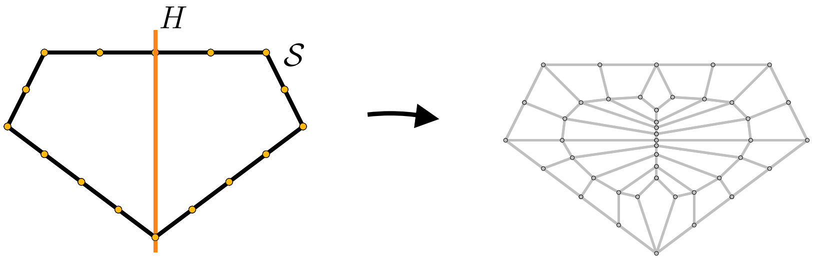


Figure 9.3: A cubification of a boundary subdivision of a pentagon produced by our *generalized regular Hexhoop* construction.

Not only we get a cubification, but we may also derive a symmetric lifting function for the cubification that may be quite arbitrarily prescribed on the boundary.

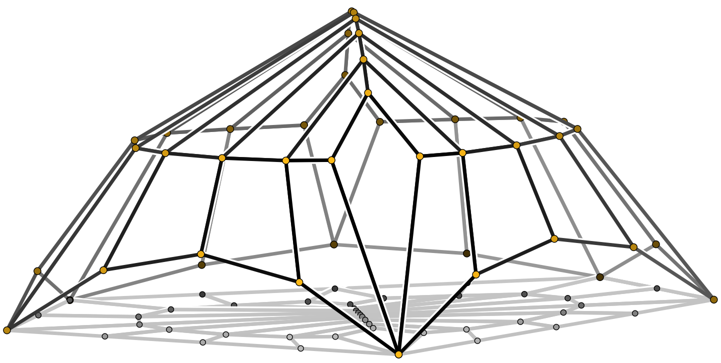


Figure 9.4: The lifted outcome of the *generalized regular Hexhoop* construction.

The input of our construction is a lifted cubical boundary subdivision $(\mathcal{S}^{d-1}, \psi)$ of a d-polytope P, such that both P and $(\mathcal{S}^{d-1}, \psi)$ are symmetric with respect to a hyperplane H.

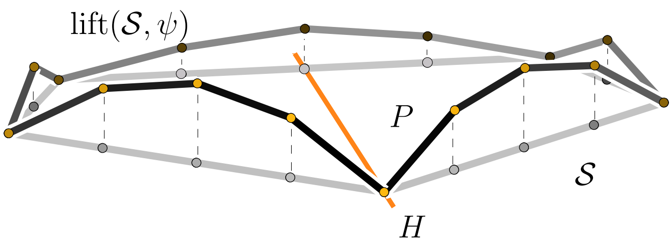


Figure 9.5: The input of the generalized regular Hexhoop construction is a lifted cubical boundary subdivision $(\mathcal{S}^{d-1}, \psi)$ of a d-polytope P, such that both P and $(\mathcal{S}^{d-1}, \psi)$ are symmetric with respect to a hyperplane H. (Here $d = 2$.)

9.2 Sketch of the generalized regular Hexhoop construction

Our approach goes roughly as follows.

(1) We first produce a $(d+1)$-polytope T that is a *symmetric tent* (defined in Section 9.4) over the given lifted boundary subdivision $(\mathcal{S}, \psi)$ of the input d-polytope P. Such a tent is the convex hull of all 'lifted vertices' $(\boldsymbol{v}, \psi(\boldsymbol{v})) \in \mathbb{R}^{d+1}$, $\boldsymbol{v} \in \mathrm{vert}(\mathcal{S})$, and of two *apex points* $\boldsymbol{p}_L, \boldsymbol{p}_R$ where the combinatorial structure of T is as depicted in Figure 9.6.

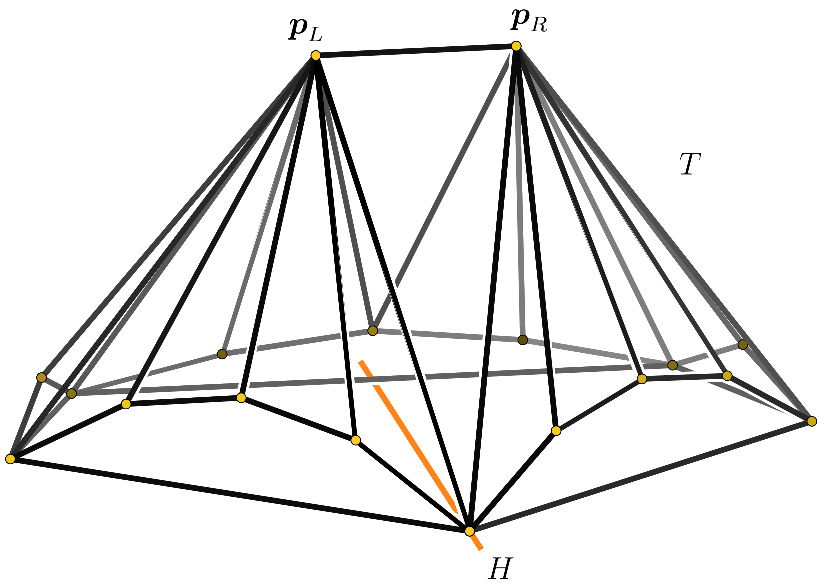


FIGURE 9.6: A *symmetric tent* over the lifted boundary subdivision $(\mathcal{S}, \psi)$ of the input d-polytope P.

(2) Truncate T by a hyperplane H' that separates the lifted points from the apex points, and remove the upper part.
(3) Add two truncated Schlegel caps as depicted in the following figure.
(4) Project the upper boundary of the resulting polytope to $\mathbb{R}^d$.

Throughout this section our figures illustrate the construction for a case of a 2-dimensional input polytope which yields 2-dimensional complexes in $\mathbb{R}^3$. Our running example will be the one of Figure 9.5. Nevertheless the construction works in higher dimensions. In particular, in Chapter 10 the generalized regular Hexhoop is used for 3-dimensional inputs.
A 3-dimensional cubification is shown at the end of this section, on page 104.

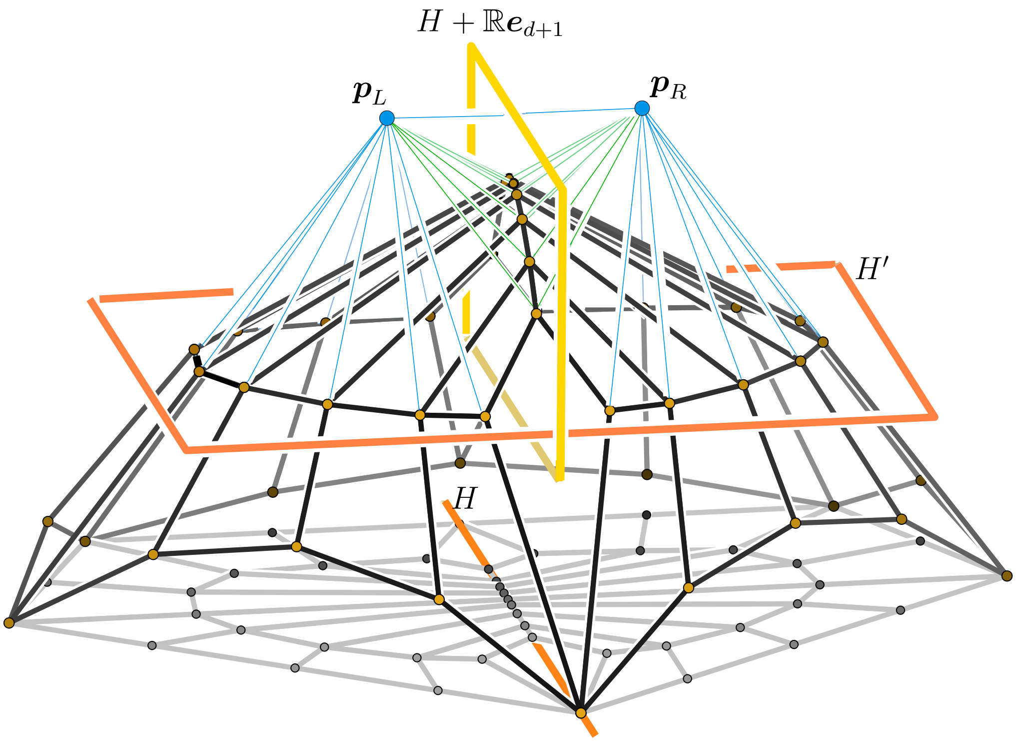


FIGURE 9.7: Sketch of the generalized regular Hexhoop construction.

9.3 Pyramids

Pyramids will be used for the discussion of *tents* in Section 9.4.

Pyramid over a polytope

If P is d-polytope $P \subset \mathbb{R}^{d'}$, $d' > d$, and $\boldsymbol{p} \in \mathbb{R}^{d'}$ is a point outside the affine hull of P, then the convex hull

$$\mathrm{pyr}(P) := \mathrm{conv}(P \cup \{\boldsymbol{p}\})$$

is a $(d+1)$-dimensional polytope called the *pyramid* over a P. Clearly the combinatorics of $\mathrm{pyr}(P)$ does not depend on the particular choice of $\boldsymbol{p}$. The facets of $\mathrm{pyr}(P)$ are P itself, and all the pyramids over facets of P.

Pyramid over a lifted polytopal subdivision

Assume $\mathcal{C}$ is a complex in $\mathbb{R}^{d+1}$, and $\boldsymbol{p} \in \mathbb{R}^{d+1}$ is a *view point* of $\mathcal{C}$, that is, a point which sees every point of $\mathcal{C}$ (i.e. $\boldsymbol{p} \notin \mathcal{C}$ and every line segment from $\boldsymbol{p}$ to a point $\boldsymbol{v} \in \mathcal{C}$ intersects $\mathcal{C}^d$ only in $\boldsymbol{v}$). Then the *pyramid* $\mathrm{pyr}(\mathcal{C}, \boldsymbol{p})$ with apex point $\boldsymbol{p}$ over $\mathcal{C}$ is the $(d+1)$-dimensional complex whose facets are all the pyramids (with apex point $\boldsymbol{p}$) over facets F of $\mathcal{C}$.

Let P be a d-polytope (in $\mathbb{R}^{d+1}$) and $(\mathcal{B}, \psi)$ a lifted polytopal subdivision of P. Furthermore let $\boldsymbol{p} \in \mathbb{R}^{d'}$ be a point outside the affine hull of P. Then the *pyramid* $\mathrm{pyr}((\mathcal{B}, \psi), \boldsymbol{p})$ over $(\mathcal{B}, \psi)$ is the lifted polytopal $(d+1)$-ball $(\mathcal{P}, \phi)$ given by $\mathcal{P}' := \mathrm{pyr}(\mathrm{lift}(\mathcal{B}, \psi), \boldsymbol{p})$.

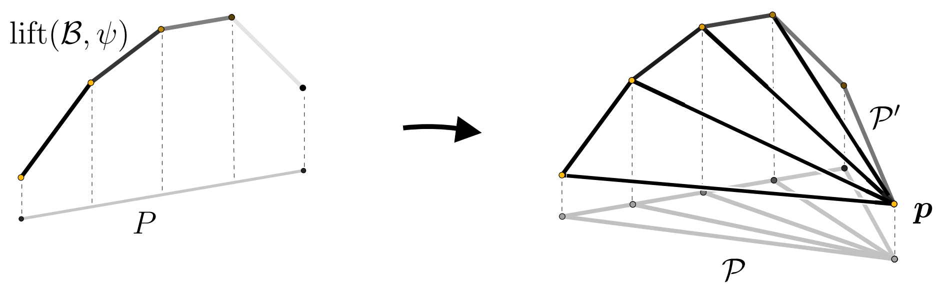


FIGURE 9.8: A pyramid over a lifted polytopal subdivision of a d-polytope P. The outcome is a lifted non-cubical $(d+1)$-ball which is a lifted polytopal subdivision of the pyramid over P.

9.4 Symmetric tents

A *tent* over a d-polytope $P \subset \mathbb{R}^{d+1}$ is the convex hull of P and a line segment $\mathrm{conv}\{\boldsymbol{p}_L, \boldsymbol{p}_R\}$ such that the affine hull of $\boldsymbol{p}_L, \boldsymbol{p}_R$ is parallel to the affine hull of P. In this section we consider tents over polytopes that are symmetric with respect to a given hyperplane. Furthermore, we discuss symmetric tents over lifted polytopal subdivisions or lifted polytopal boundary subdivisions of symmetric polytopes.

The goal is to prove that for a lifted polytopal subdivision that is symmetric with respect to a hyperplane H there is a tent (Proposition 9.3). The proof involves tents over polytopes and tents over lifted polytopal subdivisions of polytopes, as well as pyramids over polytopes and prisms over lifted polytopal subdivisions of polytopes.

Notation. We identify $\mathbb{R}^d$ with $\mathbb{R}^d \times \{0\} \subset \mathbb{R}^{d+1}$, and decompose a point $\boldsymbol{x} \in \mathbb{R}^{d+1}$ as $\boldsymbol{x} = (\pi(\boldsymbol{x}), \gamma(\boldsymbol{x}))$, where $\gamma(\boldsymbol{x})$ yields the last coordinate and $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is the projection which eliminates the last coordinate.

Symmetric tent over a polytope

Let $P \subset \mathbb{R}^d$ be a d-polytope that is symmetric with respect to a hyperplane H. Fix a positive halfspace H_+ of H. Furthermore, let $\boldsymbol{p}_L, \boldsymbol{p}_R \in \mathbb{R}^{d+1}$ two different points $\boldsymbol{p}_L, \boldsymbol{p}_R \in \mathbb{R}^{d+1} \setminus \operatorname{aff}(P)$ such that

(i) the affine hull of $\boldsymbol{p}_L, \boldsymbol{p}_R$ is parallel to the affine hull of P (which amounts to $\operatorname{aff}(\boldsymbol{p}_L, \boldsymbol{p}_R) \cap \operatorname{aff}(P) = \emptyset$), and

(ii) $\boldsymbol{p}_L, \boldsymbol{p}_R$ are symmetric to with respect to the symmetry hyperplane hyperplane $\widetilde{H}$ that contains H and is perpendicular to $\operatorname{aff}(P)$, and $\boldsymbol{p}_L \in \widetilde{H}_+, \boldsymbol{p}_R \in \widetilde{H}_-$.

Then the $(d+1)$-polytope $T := \operatorname{conv}\{P, \boldsymbol{p}_L, \boldsymbol{p}_R\}$ is called a *tent over* P with *apex points* $\boldsymbol{p}_L, \boldsymbol{p}_R$.

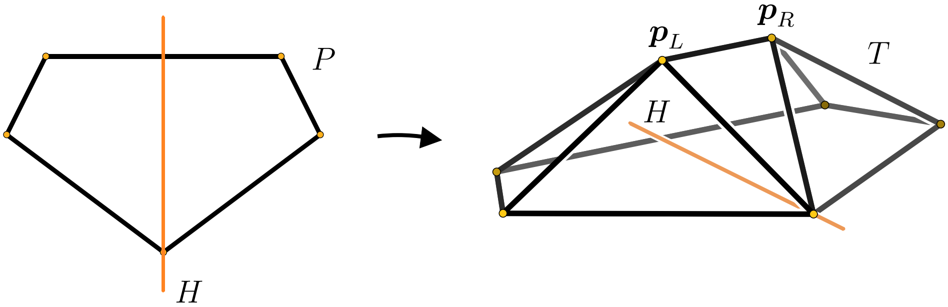


Figure 9.9: A symmetric tent over a polytope which is symmetric with respect to a given hyperplane H.

Observation 9.1. The set of facets of T consists of P and of

(a) a "left" pyramid $\operatorname{conv}\{F, \boldsymbol{p}_L\}$ for each facet F of P such that $F \subset H_+$,

(b) a "right" pyramid $\operatorname{conv}\{F, \boldsymbol{p}_R\}$ for each facet F of P such that $F \subset H_-$,

(c) a 2-fold pyramid $\operatorname{conv}\{R, \boldsymbol{p}_L, \boldsymbol{p}_R\}$ for each ridge R of P such that $R \subset H$, and

(d) a symmetric tent $\operatorname{conv}\{F, \boldsymbol{p}_L, \boldsymbol{p}_R\}$ for each facet $F \in \operatorname{fac}(P)$ with $\operatorname{relint}(F) \cap H \neq \emptyset$.

Observation 9.2. Assume $P \subset \mathbb{R}^d \times \{0\}$ and $\pi(\boldsymbol{p}_L), \pi(\boldsymbol{p}_R) \in \operatorname{relint}(P \cap H_+)$. Then the image of the projection π of the upper faces of T yields a lifted polytopal subdivision $(\mathcal{R}, \psi)$ of P.

Symmetric tent over a lifted polytopal subdivision

Let P be a d-polytope that is symmetric with respect to a hyperplane H. Fix a positive halfspace H_+ of H. Furthermore let $(\mathcal{B}, \psi)$ be a lifted polytopal subdivision of P that is symmetric with respect to H, and H intersects no

facet of $\mathcal{B}$ in its relative interior. Let T be a tent over P with apex points $\boldsymbol{p}_L$, $\boldsymbol{p}_R$ such that T is symmetric with respect to H.
Then the *tent* $(\mathcal{T}, \phi)$ over $(\mathcal{B}, \psi)$ is the lifted polytopal subdivision of T given as the $(d+1)$-ball $\mathcal{T}'$ whose facets are

- pyramids with apex point $\boldsymbol{p}_L$ over facets F of $\mathrm{lift}(\mathcal{B}, \psi)$ such that $\pi(F) \subset H_+$,
- pyramids with apex point $\boldsymbol{p}_R$ over facets F of $\mathrm{lift}(\mathcal{B}, \psi)$ such that $\pi(F) \subset H_-$, and
- 2-fold pyramids with apex points $\boldsymbol{p}_L, \boldsymbol{p}_R$ over ridges R of $\mathrm{lift}(\mathcal{B}, \psi)$ with $\pi(R) \subset H$.

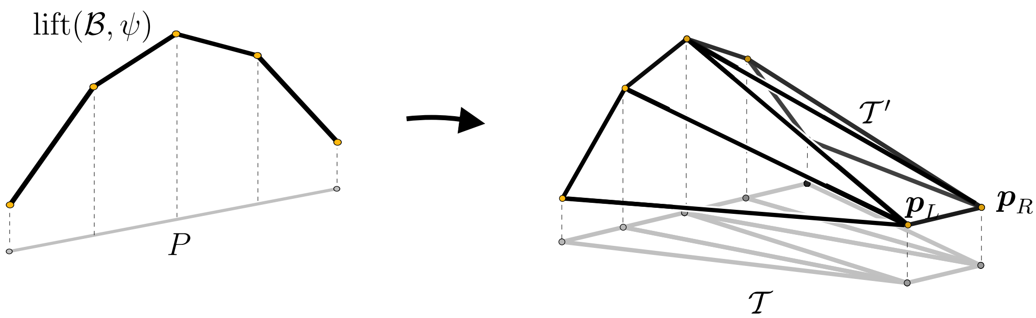


FIGURE 9.10: A symmetric tent over a lifted polytopal subdivision of a polytope. Both the input polytope and the lifted polytopal subdivision are required to be symmetric with respect to a given hyperplane H.

Symmetric tent over a lifted boundary subdivision

Let P be a d-polytope that is symmetric with respect to a hyperplane H in $\mathbb{R}^d$ (with a fixed normal vector), and $(\mathcal{S}, \psi)$ a lifted boundary subdivision of P such that $\mathcal{S} \cap H$ is a subcomplex of $\mathcal{S}$. Fix a positive halfspace H_+ of H. Furthermore, let $\boldsymbol{p}_L, \boldsymbol{p}_R \in \mathbb{R}^{d+1}$ be two points such that

- $\gamma(\boldsymbol{p}_L) = \gamma(\boldsymbol{p}_R)$,
- $\pi(\boldsymbol{p}_L) \in \mathrm{relint}(P \cap H_+)$, $\pi(\boldsymbol{p}_R) \in \mathrm{relint}(P \cap H_-)$, and
- $\boldsymbol{p}_L, \boldsymbol{p}_R$ are symmetric with respect to the symmetry hyperplane $\widetilde{H}$ that contains H and that is perpendicular to $\mathrm{aff}(P)$.

The *tent* over $(\mathcal{S}, \psi)$ is the lifted polytopal subdivision $(\mathcal{T}, \phi)$ of P that is given by the upper faces of a polytope

$$T := \mathrm{conv}(P \cup \{\boldsymbol{p}_L, \boldsymbol{p}_R \in \mathbb{R}^{d+1}\})$$

if the facets of T are

- pyramids with apex point $\boldsymbol{p}_L$ over facets F of $\operatorname{lift}(\mathcal{S}, \psi)$ such that $\pi(F) \subset H_+$,
- pyramids with apex point $\boldsymbol{p}_R$ over facets F of $\operatorname{lift}(\mathcal{S}, \psi)$ such that $\pi(F) \subset H_-$, and
- 2-fold pyramids with apex points $\boldsymbol{p}_L, \boldsymbol{p}_R$ over ridges R of $\operatorname{lift}(\mathcal{S}, \psi)$ with $\pi(R) \subset H$.

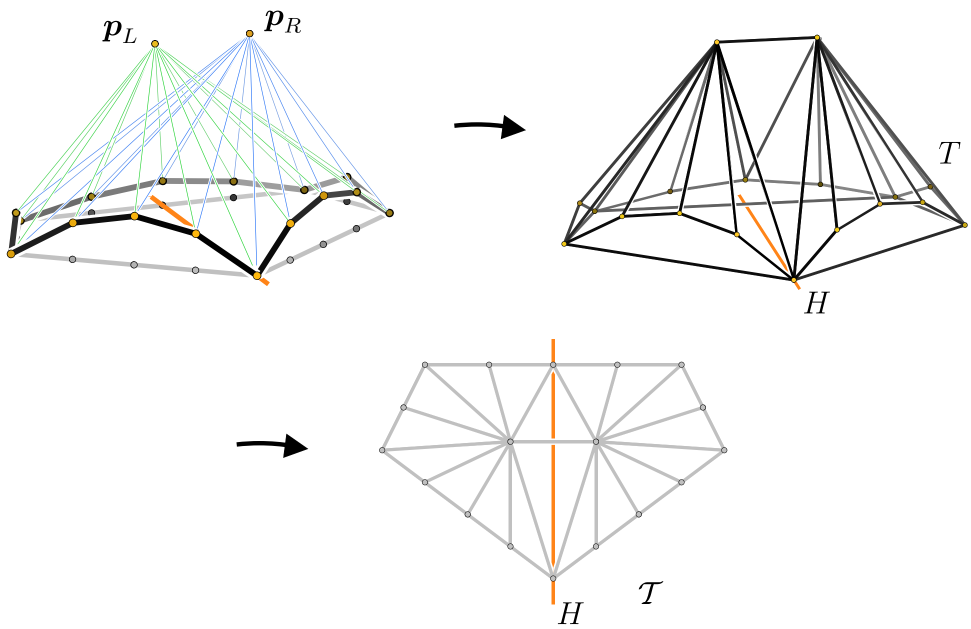


Figure 9.11: A symmetric tent over a lifted boundary subdivision.

In general $T := \operatorname{conv}(P \cup \{\boldsymbol{p}_L, \boldsymbol{p}_R \in \mathbb{R}^{d+1}\})$ can have a different combinatorial structure, as shown in the following figure.

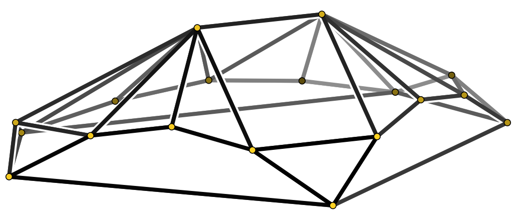


Figure 9.12: A convex hull $T := \operatorname{conv}(P \cup \{\boldsymbol{p}_L, \boldsymbol{p}_R \in \mathbb{R}^{d+1}\})$ since it has not all required facets.

By the patching lemma we obtain an existence statement for tents over lifted cubical subdivisions.

Proposition 9.3 *Assume we are given the following input.*

P	*a convex d-polytope in* $\mathbb{R}^d$,
$(\mathcal{S}, \psi)$	*a lifted boundary subdivision of* P, *and*
H	*a hyperplane in* $\mathbb{R}^d$ *(with a fixed normal vector) such that*

- P *and* $(\mathcal{S}, \psi)$ *are both symmetric with respect to* H, *and*
- $\mathcal{S} \cap H$ *is a subcomplex of* $\mathcal{S}$.

Then there are two points $\boldsymbol{p}_L, \boldsymbol{p}_R \in \mathbb{R}^{d+1}$ *such that* $T := \operatorname{conv}\{\operatorname{lift}(\mathcal{S}, \psi), \boldsymbol{p}_L, \boldsymbol{p}_R\}$ *is a tent over* $(\mathcal{S}, \psi)$ *which is symmetric with respect to* H.

Proof. First we construct a lifted "raw" polytopal subdivision $(\mathcal{R}, g)$ of P.

(1) Choose a point $\boldsymbol{q}_L \in \mathbb{R}^d$ such that $\boldsymbol{q}_L \in \operatorname{relint}(P \cap H_+)$. Let $\boldsymbol{q}_R$ be its mirrored copy with respect to H.

(2) Define $\boldsymbol{p}'_L := (\boldsymbol{q}_L, 1) \in \mathbb{R}^{d+1}$ and $\boldsymbol{p}'_R := (\boldsymbol{q}_R, 1) \in \mathbb{R}^{d+1}$.

(3) Let $Q := \operatorname{conv}\{P \times \{0\}, \boldsymbol{p}'_L, \boldsymbol{p}'_R\}$ be the symmetric tent over P. Hence the set of upper faces of Q yields a lifted polytopal subdivision $\mathcal{R}$ of P (Observation 9.2).

By Observation 9.1 there are four types of upper facets of Q, namely (a) left and (b) right pyramids, (c) tents and (d) 2-fold pyramids. Compare Figure 9.13.

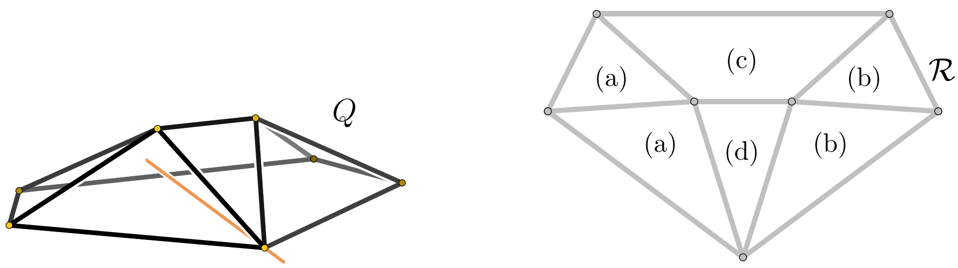


FIGURE 9.13: Illustration of the proof of Proposition 9.3: The "raw" polytopal subdivision $(\mathcal{R}, g)$ of P. Each facet is marked with its type.

For each of the facets G of $\mathcal{R}$ we construct a lifted polytopal subdivision $\mathcal{B}'_G$ as follows.

(a) *Left pyramid.* Assume G is a left pyramid, that is $G = \pi(\operatorname{conv}\{F, \boldsymbol{p}_L\})$, where $F \in \operatorname{fac}(P)$ with $F \subset H_+$. Then define $\mathcal{B}'_G$ as the pyramid over $(\mathcal{S} \cap F, \psi)$ with apex point $\boldsymbol{q}_L \in \mathbb{R}^d \times \{0\} \subset \mathbb{R}^{d+1}$.

(b) *Right pyramid.* Analogously, if G is a right pyramid, that is $G = \pi(\operatorname{conv}\{F, \boldsymbol{p}_R\})$, where $F \in \operatorname{fac}(P)$ with $F \subset H_-$, then $\mathcal{B}'_G$ is the pyramid over $(\mathcal{S} \cap F, \psi)$ with apex point $\boldsymbol{q}_R$.

(c) *Tent.* Assume $G := \pi(\operatorname{conv}\{F, \boldsymbol{p}_L, \boldsymbol{p}_R\})$, where $F \in \operatorname{fac}(P)$ with $\operatorname{relint}(F) \cap H \neq \emptyset$. Then $\mathcal{B}'_G$ is taken to be the tent over $(\mathcal{S} \cap F, \psi)$

(d) 2-*fold pyramid*. If $G = \pi(\mathrm{conv}\{R, \boldsymbol{p}_L, \boldsymbol{p}_R\})$, where $R \in \mathrm{ridges}(P)$ with $R \subset H$ then $\mathcal{B}'_G$ is the 2-fold pyramid over $(\mathcal{S} \cap R, \psi)$ with apex points $\boldsymbol{q}_L$ and $\boldsymbol{q}_R$.

In all these cases $\mathcal{B}'_G$ determines a lifted polytopal subdivision $(\mathcal{B}_G, h_G)$ of G, as illustrated in Figure 9.14. Hence all the $\mathcal{B}_G$ uniquely determine a “fine” polytopal subdivision $\mathcal{B}$ of $\mathcal{S}$ such that $\mathcal{B}$ is a refinement of $\pi(\mathcal{R})$; compare Figure 9.15.

Furthermore we have $h_G(\boldsymbol{x}) = h_{G'}(\boldsymbol{x})$ for all $\boldsymbol{x} \in G \cap G'$, where G, G' are facets of $\mathcal{R}$. Let h be the piece-wise linear function $h : Q \to \mathbb{R}$ that is uniquely determined by the h_F.

Then the patching lemma (Lemma 5.3) implies that there is an $\varepsilon > 0$ such that $\frac{1}{\varepsilon}g + h$ is a lifting function of $\mathcal{B}$. Therefore $\boldsymbol{p}_L := (\boldsymbol{q}_L, \frac{1}{\varepsilon})$, $\boldsymbol{p}_R := (\boldsymbol{q}_R, \frac{1}{\varepsilon})$ are two points such that $T := \mathrm{conv}\{\mathrm{lift}(\mathcal{S}, \psi), \boldsymbol{p}_L, \boldsymbol{p}_R\}$ is a tent over $(\mathcal{S}, \psi)$ which is symmetric with respect to H. □

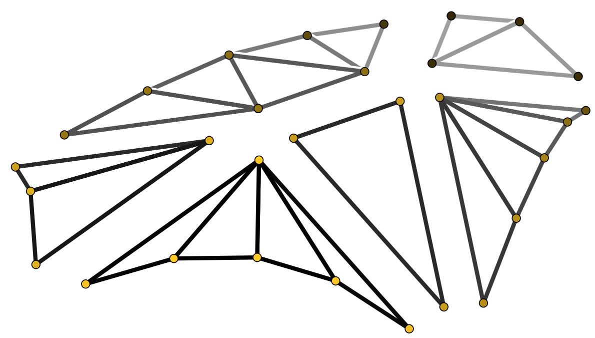


Figure 9.14: Illustration of the proof of Proposition 9.3: All lifted cubical subdivisions $(\mathcal{B}_G, h_G)$ of facets G of the “raw” subdivision $\mathcal{R}$.

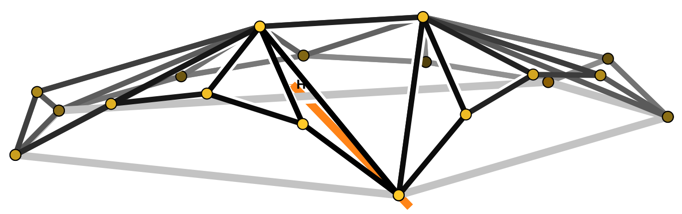


Figure 9.15: Illustration of the proof of Proposition 9.3: The “fine” polytopal subdivision $(\mathcal{B}, h)$ of P given by all $(\mathcal{B}_G, h_G)$.

9.5 Generalized regular Hexhoops

In this section we specify our generalization of the Hexhoop approach and prove the following existence statement for cubifications.

Theorem 9.4 *Assume we are given the following input.*

P	*a convex d-polytope in* $\mathbb{R}^d$,
$(\mathcal{S}^{d-1}, \psi)$	*a lifted cubical boundary subdivision of* P*, and*
H	*a hyperplane in* $\mathbb{R}^d$ *such that* • P *and* $(\mathcal{S}^{d-1}, \psi)$ *are symmetric with respect to* H*, and* • $\mathcal{S}^{d-1} \cap H$ *is a subcomplex of* $\mathcal{S}^{d-1}$.

Then there is a lifted cubification $(\mathcal{B}^d, \phi)$ *of* $(\mathcal{S}^{d-1}, \psi)$.

The proof relies on the following construction.

First description

There are several equivalent descriptions of the *Generalized regular Hexhoop construction.*

Construction 11: GENERALIZED REGULAR HEXHOOP

Input:

P	a convex d-polytope P in $\mathbb{R}^d$.
$(\mathcal{S}^{d-1}, \psi)$	a lifted cubical boundary subdivision of P. (In our figures the height $\psi(\boldsymbol{v})$ of a vertex $\boldsymbol{v}$ of P is zero, but this not required.)
H	a hyperplane in $\mathbb{R}^d$ such that • P and $(\mathcal{S}^{d-1}, \psi)$ are symmetric with respect to H, and • $\mathcal{S}^{d-1} \cap H$ is a subcomplex of $\mathcal{S}^{d-1}$.

Output:

$(\mathcal{B}^d, \phi)$	a symmetric lifted cubification of $(\mathcal{S}^{d-1}, \psi)$ given by a cubical d-ball $\mathcal{C}'$ in $\mathbb{R}^{d+1}$.

(1) Choose a positive halfspace H_+ with respect to H, and a point $\boldsymbol{q}_L \in \operatorname{relint}(P \cap H_+)$. Define $\boldsymbol{q}_R := \boldsymbol{p}_L^M$, where the upper index ^{M} denotes the mirrored copy with respect to $\widetilde{H} = H + \mathbb{R}\boldsymbol{e}_{d+1}$.
By Proposition 9.3 there is a height $h > 0$ such that

$$T := \operatorname{conv}\{\operatorname{lift}(\mathcal{S}^{d-1}, \psi), \boldsymbol{p}_L, \boldsymbol{p}_R\}$$

with $\boldsymbol{p}_L := (\boldsymbol{q}_L, h)$ and $\boldsymbol{p}_R := (\boldsymbol{q}_R, h)$ forms a symmetric tent over $(\mathcal{S}^{d-1}, \psi)$.

(2) Choose a hyperplane H' parallel to $\text{aff}(P) \subset \mathbb{R}^d$ that separates $\{\boldsymbol{p}_L, \boldsymbol{p}_R\}$ and $\text{lift}(\mathcal{S}^{d-1}, \psi)$. Let H'_+ be the halfspace with respect to H' that contains $\boldsymbol{p}_L$ and $\boldsymbol{p}_R$.

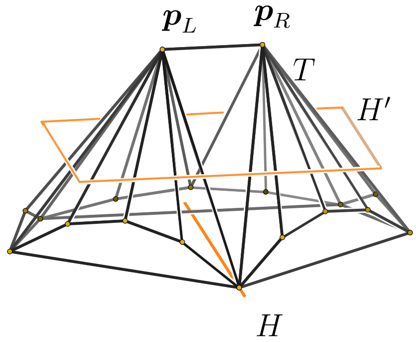


FIGURE 9.16: Step 2. The hyperplane H' separates $\{\boldsymbol{p}_L, \boldsymbol{p}_R\}$ from $\text{lift}(\mathcal{S}^{d-1}, \psi)$.

(3) Define the “lower half” of the tent T as

$$T_- \quad := \quad T \cap H'_-.$$

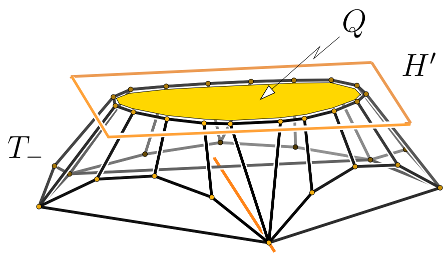


FIGURE 9.17: Step 3. The “lower half” T_- of T.

(4) Define the two d-polytopes

$$\begin{aligned} Q_L &:= \text{conv}(\{\boldsymbol{v} \in \text{vert}(Q) : \boldsymbol{v} \in H_+\}), \\ Q_R &:= \text{conv}(\{\boldsymbol{v} \in \text{vert}(Q) : \boldsymbol{v} \in H_-\}), \end{aligned}$$

where Q denotes the convex d-polytope $Q := T \cap H'$. Furthermore, let $F_L := H' \cap \text{conv}(\boldsymbol{p}_L, P \cap H)$ denote the unique facet of Q_L that is not a facet of Q.

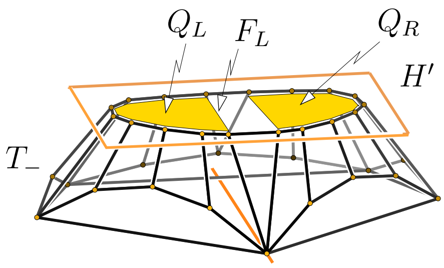


FIGURE 9.18: Step 4. Define Q_L and Q_R.

(5) Construct the polytope

$$R := \mathrm{cone}(\boldsymbol{p}_L, Q) \cap \mathrm{cone}(\boldsymbol{p}_R, Q) \cap H'_+.$$

(Recall that $\mathrm{cone}(\boldsymbol{p}_L, Q)$ is the cone through Q with apex point $\boldsymbol{p}_L$.)

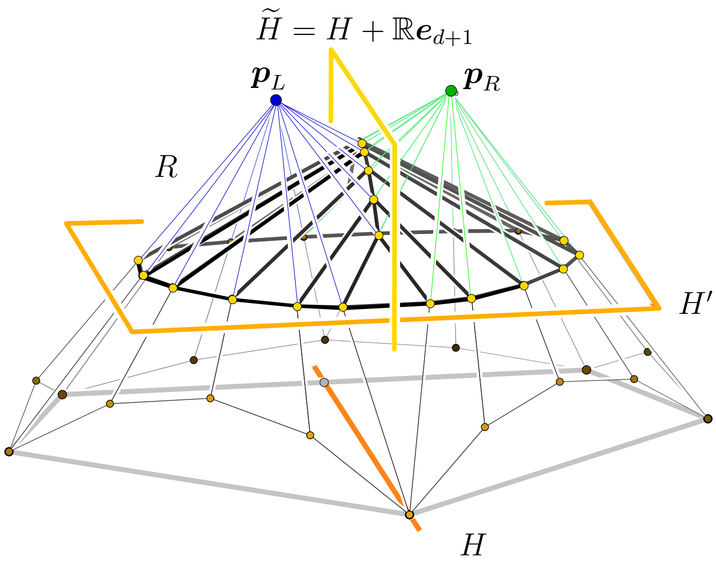


FIGURE 9.19: Step 5. The polytope $R := \mathrm{cone}(\boldsymbol{p}_L, Q) \cap \mathrm{cone}(\boldsymbol{p}_R, Q) \cap H'_+$.

The complex $\mathcal{C}'$ in question is given by the upper facets of the $(d+1)$-polytope

$$U := \mathrm{conv}(T_- \cup R).$$

Alternative description

There is an alternative description of the generalized regular Hexhoop based on iterated truncated pyramids. Later, in Section 14.2 we present a variation of the generalized regular Hexhoop construction using iterated truncated pyramids.

Construction 12: Generalized regular Hexhoop (alternative description)

Input:

P	a convex d-polytope P in $\mathbb{R}^d$.
$(\mathcal{S}^{d-1}, \psi)$	a lifted cubical boundary subdivision of P.
H	a hyperplane in $\mathbb{R}^d$ such that • P and $(\mathcal{S}^{d-1}, \psi)$ are symmetric with respect to H, and • $\mathcal{S}^{d-1} \cap H$ is a subcomplex of $\mathcal{S}^{d-1}$.

Output:

$(\mathcal{B}^d, \phi)$	a symmetric lifted cubification of $(\mathcal{S}^{d-1}, \psi)$ given as a cubical d-ball $\mathcal{C}'$ in $\mathbb{R}^{d+1}$.

The first two steps are identical to the first two steps of the first description of the generalized regular Hexhoop construction. The remaining steps are a follows:

(3) Partition the boundary subdivision $\mathcal{S}^{d-1}$ into two cubical $(d-1)$-balls, we fix a normal vector of the hyperplane H and define

$$\begin{array}{rcl} \mathcal{S}_L & := & \mathcal{S}^{d-1} \cap H_- \\ \mathcal{S}_R & := & \mathcal{S}^{d-1} \cap H_+. \end{array}$$

Furthermore, we define *waist* of $\mathcal{S}^{d-1}$ as $\mathcal{W}^{d-2} := \mathcal{S}^{d-1} \cap H$.

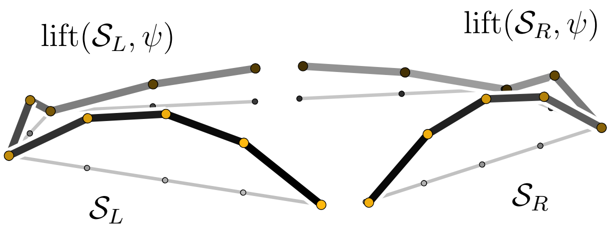


Figure 9.20: Step 3 of the generalized regular Hexhoop construction: Splitting the input in the “left” and “right” half.

(4) Construct iterated truncated pyramids over $\mathcal{S}_L$ using the input sequence $\mathcal{I} := (\boldsymbol{p}_L, \boldsymbol{p}_R; H', H+\mathbb{R}\boldsymbol{e}_{d+1})$. This means for every facet F of $P \cap H_+$ we construct the cubical d-ball

$\mathcal{C}_F := \textsc{IteratedTruncPyr}((F \cap \mathcal{S}_L, \psi), (\boldsymbol{p}_L, \boldsymbol{p}_R; H', H+\mathbb{R}\boldsymbol{e}_{d+1}))$.

The union of all these balls gives a cubical d-ball

$$\mathcal{C}_L \quad := \quad \bigcup \{\mathcal{C}_F \ : \ F \in \mathrm{fac}(P \cap H_+)\} \, .$$

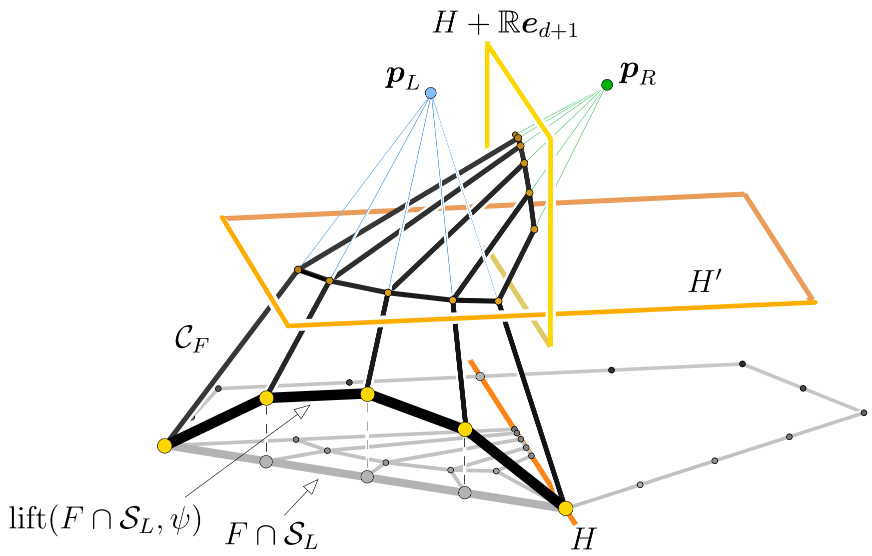


FIGURE 9.21: Step 4. Perform an iterated lifted truncated pyramid over the induced lifted subdivision of a facet F of $P \cap H_+$.

(5) Define $\mathcal{C}_R := \mathcal{C}_L^M$, where the upper index ^{M} denotes the mirrored copy with respect to $H + \mathbb{R}\boldsymbol{e}_{d+1}$.

(6) For every facet F of $P \cap H$ construct a truncated pyramid over $(F \cap \mathcal{W}^{d-2}, \psi)$ using the apex point $\boldsymbol{p}_L$ and the truncating hyperplane H'. The resulting cubical $(d-1)$-ball is

$$\mathcal{L}_F \quad := \quad \textsc{TruncPyr}((F \cap \mathcal{W}^{d-2}, \psi), \boldsymbol{p}_L, H').$$

Let H_F be a hyperplane through $\boldsymbol{p}_F$ and $\omega_1(\mathrm{lift}(F \cap \mathcal{W}^{d-2}, \psi))$. Observe that H_F^M separates $\mathcal{L}_F$ and $\boldsymbol{p}_R$. Compare Firgure 9.22.

(7) For every facet F of $P \cap H$ construct a truncated pyramid over $\mathcal{L}_F$ using the apex point $\boldsymbol{p}_R$ and the truncating hyperplane H_F^M. The resulting cubical d-ball is

$$\mathcal{A}_F \quad := \quad \textsc{TruncPyr}(\mathcal{L}_F, \boldsymbol{p}_R, H_F^M),$$

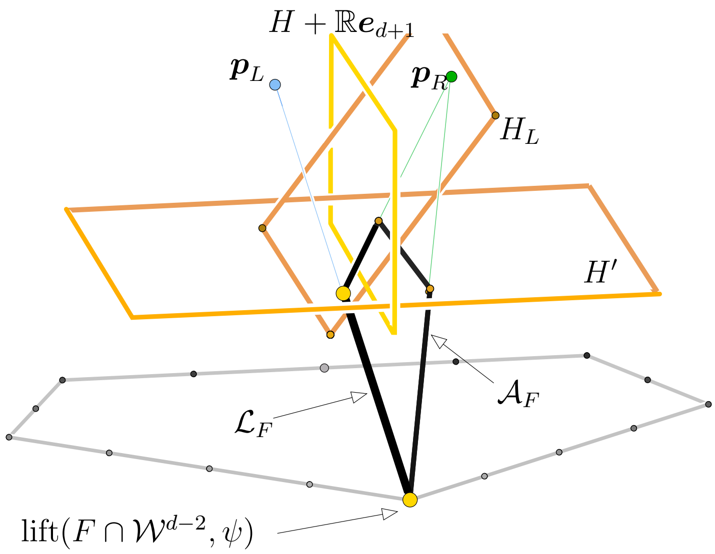


Figure 9.22: Step 6 and 7. 2-fold truncated pyramids over subdivisions of waist ridges.

and the union of all these cubical d-balls gives a cubical d-ball

$$\mathcal{A} \quad := \quad \bigcup \{\mathcal{A}_F \; : \; F \in \mathrm{fac}(P \cap H)\}\,.$$

The cubical ball $\mathcal{C}'$ in question will be given by the union of three cubical d-balls $\mathcal{C}_L$, $\mathcal{C}_R$ and $\mathcal{A}$ (all in $\mathbb{R}^{d+1}$).

The outcome is depicted in Figure 9.3.

Correctness proof

We now analyze the structure of the three polytopes Q, T_- and R. Afterwards we prove Theorem 9.4.

Lemma 9.5 (Combinatorial structure of Q)
The vertex set of Q consists of

- *the points* $\mathrm{conv}(\boldsymbol{p}_L, \boldsymbol{v}) \cap H'$ *for vertices* $\boldsymbol{v} \in \mathrm{vert}(\mathrm{lift}(\mathcal{S}, \psi))$ *such that* $\pi(\boldsymbol{v}) \subset H_+$*, and*
- *the points* $\mathrm{conv}(\boldsymbol{p}_R, \boldsymbol{v}) \cap H'$ *for vertices* $\boldsymbol{v} \in \mathrm{vert}(\mathrm{lift}(\mathcal{S}, \psi))$ *such that* $\pi(\boldsymbol{v}) \subset H_-$.

The set of facets of Q consists of

(a) *the combinatorial cubes* $\mathrm{conv}(\boldsymbol{p}_L, F) \cap H'$ *for facets F of* $\mathrm{lift}(\mathcal{S}, \psi)$ *such that* $F \subset \widetilde{H}_+$,

(b) *the combinatorial cubes* $\mathrm{conv}(\boldsymbol{p}_R, F) \cap H'$ *for facets* F *of* $\mathrm{lift}(\mathcal{S}, \psi)$ *such that* $F \subset \widetilde{H}_-$*,*
(c) *the combinatorial cubes* $\mathrm{conv}(\boldsymbol{p}_L, \boldsymbol{p}_R, F) \cap H'$ *for* $(d-2)$*-faces* F *of* $\mathrm{lift}(\mathcal{S}, \psi)$ *such that* $F \subset \widetilde{H}$*.*

Proof. By the definition of a symmetric tent, the set of upper facets of the symmetric tent T consists of

- pyramids with apex point $\boldsymbol{p}_L$ over facets F of $\mathrm{lift}(\mathcal{S}, \psi)$ such that $F \subset \widetilde{H}_+$,
- pyramids with apex point $\boldsymbol{p}_R$ over facets F of $\mathrm{lift}(\mathcal{S}, \psi)$ such that $F \subset \widetilde{H}_-$, and
- 2-fold pyramids with apex points $\boldsymbol{p}_L, \boldsymbol{p}_R$ over ridges R of $\mathrm{lift}(\mathcal{S}, \psi)$ with $R \subset \widetilde{H}$.

Since Q is the intersection of T with H, the polytope Q the vertices and facets listed above. It remains to show that the facets of type (c) are combinatorial cubes. Let F be a $(d-2)$-face of $\mathrm{lift}(\mathcal{S}, \psi)$ such that $F \subset \widetilde{H}$. Every point on the facet lies in the convex hull of F with a unique point on the segment $[\boldsymbol{p}_L, \boldsymbol{p}_R]$. Thus the facet is combinatorially isomorphic to a prism over F. $\square$

Let a *d-dimensional half-cube* be the product of a combinatorial $(d-2)$-cube and a triangle. A *combinatorial half-cube* is a polytope combinatorially isomorphic to a half-cube.

Lemma 9.6 (Combinatorial structure of T_-)
The vertices of T_- *are the vertices of* $\mathrm{lift}(\mathcal{S}, \psi)$ *and the vertices of* Q*. Furthermore, the set of upper facets of* T_- *consists of*
(a) *the combinatorial cubes* $\mathrm{cone}(\boldsymbol{p}_L, F) \cap H'_- \cap (\mathbb{R}^d \times \mathbb{R}_+)$ *for facets* F *of* Q *such that* $F \subset \widetilde{H}_+$*,*
(b) *the combinatorial cubes* $\mathrm{cone}(\boldsymbol{p}_R, F) \cap H'_- \cap (\mathbb{R}^d \times \mathbb{R}_+)$ *for facets* F *of* Q *such that* $F \subset \widetilde{H}_-$*,*
(c) *the combinatorial half-cubes* $\mathrm{cone}(\boldsymbol{p}_L, F) \cap \mathrm{cone}(\boldsymbol{p}_R, F) \cap H'_-$ *for facets* R *of* Q *such that* R *intersects* $\widetilde{H}$*, and*
(d) Q*.*

The set of facet defining hyperplanes of the upper facets of T_- *consists of*
(a) $\mathrm{aff}(\boldsymbol{p}_L, F)$ *for facets* F *of* Q *such that* $F \subset \widetilde{H}_+$*,*
(b) $\mathrm{aff}(\boldsymbol{p}_R, F)$ *for facets* F *of* Q *such that* $F \subset \widetilde{H}_-$*,*
(c) $\mathrm{aff}(\boldsymbol{p}_L, \boldsymbol{p}_R, F)$ *for facets* F *of* Q *such that* F *intersects* $\widetilde{H}$*, and*
(d) $\mathrm{aff}(Q)$*.*

Proof. Since T_- is the intersection of T with H'_-, the upper facets of T_- are given by Q plus the intersections of the upper facets of T with H'_-, and the vertices of T_- are the vertices of T and the vertices of Q. $\square$

Lemma 9.7 (Combinatorial structure of R)
The set of vertices of R consists of the vertices of Q and all points in $V'' := \mathrm{vert}(R) \setminus \mathrm{vert}(Q)$. Furthermore, the set of (all) facets of R consists of

(a) *the combinatorial cubes $\mathrm{conv}(\boldsymbol{p}_R, F) \cap \widetilde{H}_+$ for facets F of Q such that $F \subset \widetilde{H}_+$,*
(b) *the combinatorial cubes $\mathrm{conv}(\boldsymbol{p}_L, F) \cap \widetilde{H}_+$ for facets F of Q such that $F \subset \widetilde{H}_-$,*
(c) *the combinatorial half-cubes $\mathrm{conv}(\boldsymbol{p}_R, F) \cap \mathrm{conv}(\boldsymbol{p}_L, F)$ for facets F of Q such that F intersects $\widetilde{H}$, and*
(d) *Q.*

The set of facet defining hyperplanes of the facets of R consists of

(a) *$\mathrm{aff}(\boldsymbol{p}_R, F)$ for facets F of Q such that $F \subset \widetilde{H}_+$,*
(b) *$\mathrm{aff}(\boldsymbol{p}_L, F)$ for facets F of Q such that $F \subset \widetilde{H}_-$,*
(c) *$\mathrm{aff}(\boldsymbol{p}_L, \boldsymbol{p}_R, F)$ for facets F of Q such that F intersects $\widetilde{H}$, and*
(d) *$\mathrm{aff}(Q)$.*

Proof of Theorem 9.4. We show that the complex $\mathcal{C}'$ given by the upper facets of the polytope U given by Construction 9.5 determines a lifted cubification $(\mathcal{B}^d, \phi)$ of $(\mathcal{S}^{d-1}, \psi)$.
First observe that no vertex of T_- is beyond a facet of R, and no vertex of R is beyond a facet of T_-. Hence the boundary of $U = \mathrm{conv}(T_- \cup R)$ is the union of the two boundaries of the two polytopes, excluding the relative interior of Q.
Define the vertex sets $V := \mathrm{vert}(\mathrm{lift}(\mathcal{S}, \psi))$, $V' := \mathrm{vert}(Q)$ and $V'' := \mathrm{vert}(R) \setminus V'$. Then we have:

- Each vertex of V is beneath each facet of R that is of type (a) or (b).
- Each vertex of V'' is beneath each facet of T_- that is of type (a) or (b).

Hence these four types of facets are facets of U that are combinatorial cubes, and the set of vertices of U is given by the union of V, V' and V''. It remains to show that each hyperplane $\mathrm{aff}(\boldsymbol{p}_L, \boldsymbol{p}_R, F)$, where F is a facet of Q that intersects $\widetilde{H}$, is the affine hull of a cubical facet of U. To see this, observe that there are two facets F_+, F_- of R, T_- respectively, that are both contained in the affine hull of F. These two facets F_+, F_- are both half-cubes that intersect in a common $(d-1)$-cube, namely F. Furthermore, all vertices of F_+ and of F_- that are not contained in $\mathrm{aff}(F)$ are contained in $\widetilde{H}$. Hence the union of F_+ and F_- is a combinatorial cube.
Therefore, every upper facet of U is a combinatorial cube. Furthermore, $\pi(R) = \pi(Q)$ and $\pi(T_-) = |P|$. This implies that the upper facets of U determine a lifted cubical subdivision of $(\mathcal{S}^{d-1}, \psi)$. $\square$

Properties

Remark 9.8. (Face numbers)
The f-vector of $\mathcal{B}^d$ is given by

$$\begin{aligned}
f_0(\mathcal{B}^d) &= \tfrac{5}{2} f_0(\mathcal{S}^{d-1}) + \tfrac{3}{2} f_0(\mathcal{W}^{d-2}) \\
f_1(\mathcal{B}^d) &= \tfrac{5}{2} f_1(\mathcal{S}^{d-1}) + 2 f_0(\mathcal{S}^{d-1}) + \tfrac{3}{2} f_1(\mathcal{W}^{d-2}) + 2 f_0(\mathcal{W}^{d-2}) \\
f_k(\mathcal{B}^d) &= \tfrac{5}{2} f_k(\mathcal{S}^{d-1}) + 2 f_{k-1}(\mathcal{S}^{d-1}) \\
&\quad + \tfrac{3}{2} f_k(\mathcal{W}^{d-2}) + 2 f_{k-1}(\mathcal{W}^{d-2}) + f_{k-2}(\mathcal{W}^{d-2}) \qquad \text{for } 1 < k < d \\
f_d(\mathcal{B}^d) &= 2 f_{d-1}(\mathcal{S}^{d-1}) + f_{d-2}(\mathcal{W}^{d-2}).
\end{aligned}$$

Since the boundary subdivision $\mathcal{S}^{d-1}$ is symmetric with respect to H, the f-vector of $\mathcal{S}^{d-1}$ satisfies

$$f(\mathcal{S}^{d-1}) \;=\; 2f(\mathcal{S}_L) - f(\mathcal{W}^{d-2}).$$

Hence the f-vector of $\mathcal{B}^d$ can be written in terms of $f(\mathcal{S}_L)$ and $f(\mathcal{W}^{d-2})$ as

$$\begin{aligned}
f_0(\mathcal{B}^d) &= 5 f_0(\mathcal{S}_L) - f_0(\mathcal{W}^{d-2}) \\
f_1(\mathcal{B}^d) &= 5 f_1(\mathcal{S}_L) - f_1(\mathcal{W}^{d-2}) + 4 f_0(\mathcal{S}_L) \\
f_k(\mathcal{B}^d) &= 5 f_k(\mathcal{S}_L) - f_k(\mathcal{W}^{d-2}) + 4 f_{k-1}(\mathcal{S}_L) - f_{k-2}(\mathcal{W}^{d-2}) \qquad \text{for } 1 < k < d \\
f_d(\mathcal{B}^d) &= 4 f_{d-1}(\mathcal{S}_L) + f_{d-2}(\mathcal{W}^{d-2}).
\end{aligned}$$

Thus the Euler characteristic of $\mathcal{B}^d$ is

$$\begin{aligned}
\chi(\mathcal{B}^d) &= \sum_{k=0}^{d} (-1)^k f_k(\mathcal{B}^d) \\
&= \sum_{k=0}^{d-1} (-1)^k (\tfrac{5}{2} - 2) f_k(\mathcal{S}^{d-1}) \;+\; \sum_{k=0}^{d-2} (-1)^k (\tfrac{3}{2} - 2 + 1) f_k(\mathcal{W}^{d-2}) \\
&= \tfrac{1}{2} \sum_{k=0}^{d-1} (-1)^k f_k(\mathcal{S}^{d-1}) \;+\; \tfrac{1}{2} \sum_{k=0}^{d-2} (-1)^k f_k(\mathcal{W}^{d-2}) \\
&= \tfrac{1}{2} (\chi(\mathcal{S}^{d-1}) \;+ \chi(\mathcal{W}^{d-2})) \;=\; 1,
\end{aligned}$$

as expected.

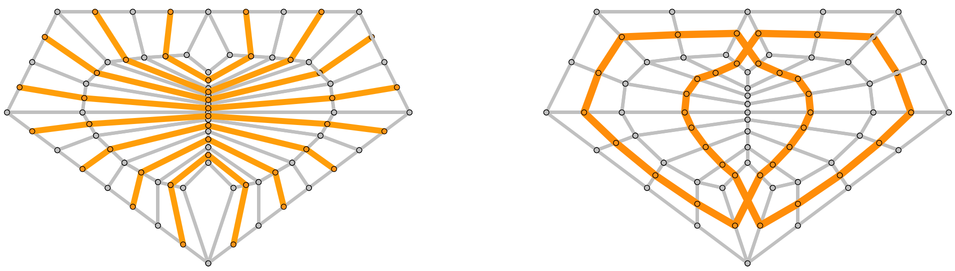
Remark 9.9. (Parity of face numbers)
Furthermore, $\mathcal{W}^{d-2}$ is the boundary of the cubical $(d-1)$-ball $\mathcal{S}_L$, which implies that $f_{d-2}(\mathcal{W}^{d-2})$ is even. Hence the f-vector of $\mathcal{B}^d$ satisfies the following parity constraints:

$$\begin{array}{lll} f_0(\mathcal{B}^d) & \equiv_2 & f_0(\mathcal{S}_L) + f_0(\mathcal{W}^{d-2}) \\ f_1(\mathcal{B}^d) & \equiv_2 & f_1(\mathcal{S}_L) + f_1(\mathcal{W}^{d-2}) \\ f_k(\mathcal{B}^d) & \equiv_2 & f_k(\mathcal{S}^{d-1}) + f_k(\mathcal{W}^{d-2}) + f_{k-2}(\mathcal{W}^{d-2}) \qquad \text{for } 1 < k < d \\ f_d(\mathcal{B}^d) & & \text{is even.} \end{array}$$

Proposition 9.10 (Dual manifolds) *Up to PL-homeomorphism, the generalized regular Hexhoop cubification $\mathcal{B}^d$ of $\mathcal{S}^{d-1}$ has the following dual manifolds:*

- *$\mathcal{N} \times I$ for each dual manifold $\mathcal{N}$ (with or without boundary) of $\mathcal{S}_L = \mathcal{S}^{d-1} \cap \widetilde{H}_+$,*
- *two $(d-1)$-spheres "around" Q, Q^M, respectively, where the upper index ^{M} denotes the mirrored copy.*

Proof. The "main part" of the complex $\mathcal{B}^d$ may be viewed as a prism of height 4, whose dual manifolds are of the form $\mathcal{N} \times I$, as well as four $(d-1)$-balls. This prism is then modified by glueing a full torus (product of the $(d-2)$-sphere $\mathcal{S}^{d-1} \cap \mathcal{H}$ with a square I^2) into its "waist." This extends the dual manifolds $\mathcal{N} \times I$ without changing the PL-homeomorphism type, while closing the four $(d-1)$-balls into two intersecting, embedded spheres.
We refer to Figure 9.23 (case $d = 2$) and Figure 9.24 ($d = 3$) for geometric intuition. $\square$



(a) dual product manifolds (b) dual spheres

Figure 9.23: The set of dual manifolds of a two-dimensional cubification produced by the generalized regular Hexhoop construction.

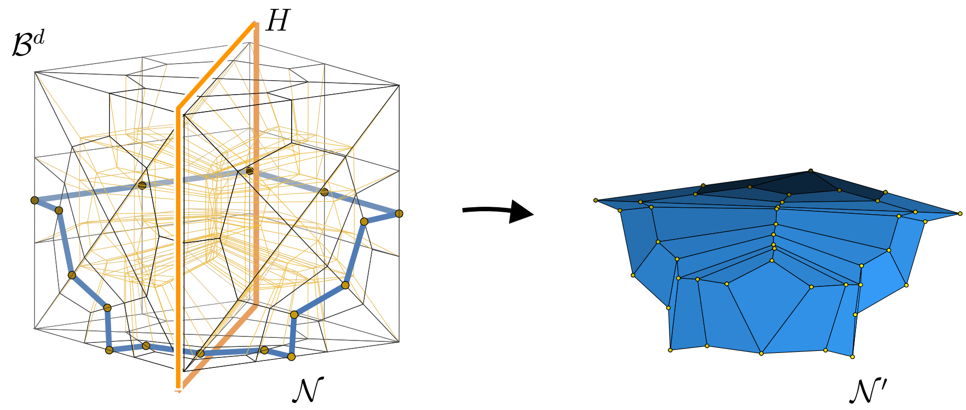


FIGURE 9.24: A three-dimensional cubification produced by the generalized regular Hexhoop construction. For every embedded dual circle $\mathcal{N}$ which intersects $H_+ \setminus H$ and $H_- \setminus H$, there is an embedded dual 2-ball $\mathcal{N}'$ with boundary $\mathcal{N}$ in the the cubification. (This is a cubification for the case "single 5" introduced in Chapter 10.)

Remark 9.11. (Symmetry)
Assume G is a hyperplane such that G is perpendicular to H, and P and $(\mathcal{S}^{d-1}, \psi)$ are symmetric with respect to G. Then the lifted cubification $(\mathcal{B}^d, \phi)$ of $(\mathcal{S}^{d-1}, \psi)$ is symmetric with respect G, too.
Furthermore, if $\mathcal{S}^{d-1} \cap G$ is a subcomplex of $\mathcal{S}^{d-1}$, then $\mathcal{B}^d \cap G$ is a subcomplex of $\mathcal{B}^d$.

Chapter 10

Cubical 4-polytopes with prescribed dual manifold immersions

In this section we combine the construction techniques developed in the previous sections and thus derive our main theorem.
As mentioned in Section 4.3, there seems to be no analog for the last step of the Babson-Chan construction in the polytopal category. For our alternative polytopal constructions we cannot start with arbitrary normal crossing PL-immersions, but we require immersions whose local geometric structure is rather simple. For the version of our constructions presented below, the following assumptions are needed and used (for $d = 3$):

(1) $\mathcal{M}^{d-1}$ is a $(d-1)$-dimensional cubical PL-manifold, and $j : \mathcal{M}^{d-1} \looparrowright \mathbb{R}^d$ is a *grid immersion*, that is, a cubical normal crossing codimension one immersion into $\mathbb{R}^d$ equipped with the standard unit cube structure (cf. [19]). In particular, for each k-face F of $\mathcal{M}^{d-1}$ the image $j(F)$ is a k-face of the skeleton of the standard unit cube structure on $\mathbb{R}^d$.

(2) Moreover, we require that the immersion is *locally symmetric*, that is, that at every vertex $\boldsymbol{w}$ of $j(\mathcal{M}^{d-1})$ there is hyperplane H through $\boldsymbol{w}$ such that for each vertex $\boldsymbol{v}$ with $j(\boldsymbol{v}) = \boldsymbol{w}$ the image of the vertex star of $\boldsymbol{v}$ is symmetric with respect to H. Thus, we require that H is a symmetry hyperplane separately for each of the (up to d) local sheets that intersect at $\boldsymbol{w}$.
Such a hyperplane H is necessarily of the form $x_i = k$, $x_i + x_j = k$ or $x_i - x_j = k$. In the first case we say H is a *coordinate hyperplane*, and in other cases it is *diagonal*.

10.1 From PL immersions to grid immersions

In view of triangulation and approximation methods available in PL and differential topology theory, the above assumptions are not so restrictive.

Proposition 10.1 *Every locally flat normal crossing immersion of a compact $(d-1)$-manifold into $\mathbb{R}^d$ is PL-equivalent to a codimension one grid immersion of a cubification of the manifold into the standard cube subdivision of $\mathbb{R}^d$.*

Proof. We may replace any PL-immersion of $\mathcal{M}^{d-1}$ by a simplicial immersion into a suitably fine triangulation of $\mathbb{R}^d$. Furthermore, the vertices of $j(\mathcal{M}^{d-1})$ may be perturbed into general position.
Now we overlay the polyhedron $j(\mathcal{M}^{d-1})$ with a cube structure of $\mathbb{R}^d$ of edge length ε for suitably small $\varepsilon > 0$, such that the vertices of $j(\mathcal{M}^{d-1})$ are contained in the interiors of distinct d-cubes.
Then working by induction on the skeleton, within each face of the cube structure, the restriction of $j(\mathcal{M}^{d-1})$ to a k-face — which by local flatness consists of one or several $(k-1)$-cells — is replaced by a standard cubical lattice version that is supposed to run through the interior of the respective cell, staying away distance ε' from the boundary of the cell; here we take different values for ε' in the situation where the immersion is not embedded at the vertex in question, that is, comes from several disjoint neighborhoods in $\mathcal{M}^{d-1}$. The resulting modified immersion into $\mathbb{R}^d$ will be cellular with re-

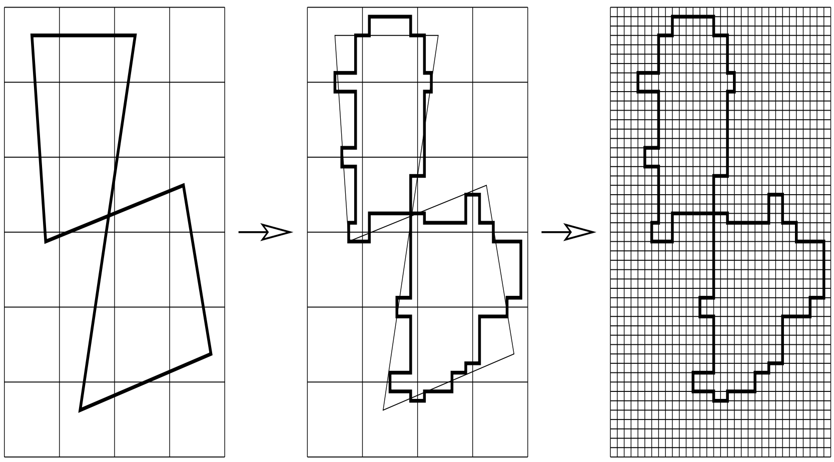
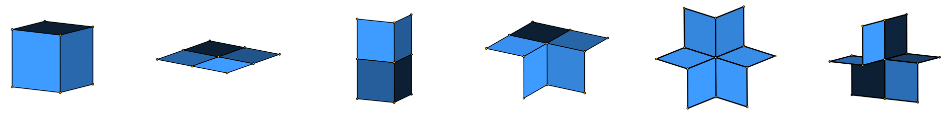


FIGURE 10.1: Illustration of the proof of Proposition 10.1.

spect to a standard cube subdivision of edge length $\frac{1}{N}\varepsilon$ for a suitable large N. Figure 10.1 illustrates the procedure for $d = 2$. □

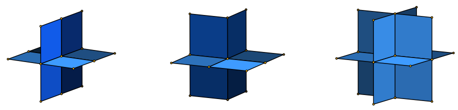
10.2 Vertex stars of grid immersions of surfaces

There are nine types of vertex stars of grid immersions of surfaces, namely the following five vertex stars of a regular vertex, plus two vertex stars



single 3 single 4a single 4b single 5 single 6a single 6b

with double intersection and the vertex star of a triple intersection point: All these vertex stars satisfy the local symmetry condition, with a single



double 8a double 8b triple 12

exception, namely the last depicted vertex star (single 6b) of a regular vertex with 6 adjacent quadrangles.

Proposition 10.2 *Any grid immersion of a compact cubical* 2*-manifold into* $\mathbb{R}^3$ *is equivalent to a locally symmetric immersion of the same type.*

Proof. There is only one type of vertex star that does not have the required symmetry, namely "case 6b". As indicated in Figure 10.2, a local modification of the surface solves the problem (with a suitable refinement of the standard cube subdivision). □

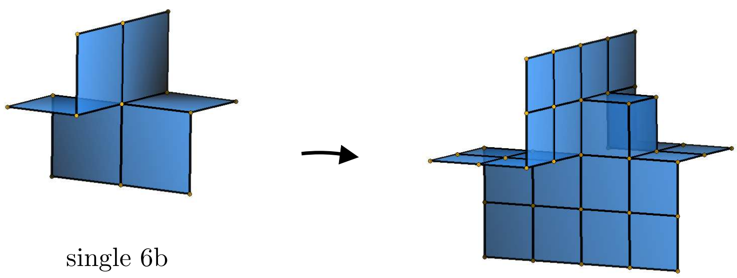


FIGURE 10.2: Local modification to "repair" the case "single 6a."

10.3 Main theorem (2-manifolds into cubical 4-polytopes)

Theorem 10.3 *Let $j\colon \mathcal{M} \looparrowright \mathbb{R}^3$ be a locally flat normal crossing immersion of a compact 2-manifold (without boundary) $\mathcal{M}$ into $\mathbb{R}^3$.*
Then there is a cubical 4-polytope P with a dual manifold $\mathcal{M}'$ and associated immersion $y : \mathcal{M}' \looparrowright |\partial P|$ such that the following conditions are satisfied:

(i) *$\mathcal{M}'$ is a cubical subdivision of $\mathcal{M}$, and the immersions j (considered as a map to $\mathbb{R}^3 \cup \{\infty\} \cong S^3$) and y are PL-equivalent.*

(ii) *The number of facets of P is congruent modulo two to the number $t(j)$ of triple points of the immersion j.*

(iii) *If the given surface $\mathcal{M}$ is non-orientable and of odd genus, then the cubical 4-polytope P has an odd number of facets.*

The main ingredient of our proof for this theorem is the following construction of a cubical 3-ball with a prescribed dual manifold immersion.

10.4 Cubical 3-balls with a prescribed dual manifold immersions

Construction 13: REGULAR CUBICAL 3-BALL WITH A PRESCRIBED DUAL MANIFOLD

Input: A 2-dimensional closed (that is, compact and without boundary) cubical PL-surface $\mathcal{M}$, and a locally symmetric codimension one grid immersion $j\colon \mathcal{M} \looparrowright |\mathcal{P}_3(\ell_1, \ell_2, \ell_3)| \subset \mathbb{R}^3$.
(Without loss of generality one can assume $j(\mathcal{M}) \subset |\mathcal{P}_3(\ell_1, \ell_2, \ell_3)|$.)

Output: A regular convex 3-ball $\mathcal{B}$ with a dual manifold $\mathcal{M}'$ and associated immersion $y : \mathcal{M}' \looparrowright |\mathcal{B}|$ such that the following conditions are satisfied:

(i) $\mathcal{M}'$ is a cubical subdivision of $\mathcal{M}$, and the immersions j and y are PL-equivalent.

(ii) The number of facets of $\mathcal{B}$ is congruent modulo two to the number $t(j)$ of triple points of the immersion j.

The Patching Lemma (Lemma 5.3) will be used to prove that the ball $\mathcal{B}$ is regular. Therefore, we first construct a raw complex $\mathcal{A}$.

(0) Raw complex.

As the raw complex $\mathcal{A}$ we take a copy of the pile of cubes $\mathcal{P}_3(\ell_1 + 1, \ell_2 + 1, \ell_3 + 1)$ with all vertex coordinates shifted by $-\frac{1}{2}\mathbb{1}$. (Hence $x_i \in \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots, \ell_i + \frac{1}{2}\}$ for each vertex $\boldsymbol{x} \in \mathrm{vert}(A)$.)

Due to the local symmetry of the immersion, and the choice of the vertex coordinates of $\mathcal{A}$, the following holds:

- Each vertex of $j(\mathcal{M})$ is the barycenter of a 3-cube C of $\mathcal{A}$.
- For each 3-cube C of $\mathcal{A}$ the restriction $(C, j(\mathcal{M}) \cap C)$ is locally symmetric.

The lifted cubical subdivision $\mathcal{B}$ of $\mathcal{A}$ is constructed by induction over the skeleton of $\mathcal{A}$:

For $k = 1$ to 3 we produce a lifted cubical subdivison $\mathcal{C}^k$ of the k-skeleton $\mathcal{F}_{\leq k}(\mathcal{A})$, that is, for any k-face F the restriction $\mathcal{C}^k \cap F$ is a lifted cubical subdivison of F. Each of these $\mathcal{C}^k \cap F$ is a called a *template*. The lifted cubical subdivision $\mathcal{B}$ of $\mathcal{A}$ arises as $\mathcal{B} := \mathcal{C}^3$. Let us define the following invariants (for $k \in \{1, 2, 3\}$).

($\mathrm{I}_k 1$) *Consistency requirement.*
For every k-face $Q \in \mathcal{F}_k(\mathcal{A})$ and every facet F of Q, the induced subdivision $\mathcal{C}^k \cap F$ equals $\mathcal{C}^{k-1} \cap F$.

($\mathrm{I}_k 2$) *PL equivalence requirement.*
For every k-face $Q \in \mathcal{F}_k(\mathcal{A})$ and every dual manifold $\mathcal{N}$ of Q (with boundary) the cubical subdivision $\mathcal{C}^k \cap Q$ has a dual manifold that is PL-equivalent to $j(\mathcal{N}) \cap Q$.

($\mathrm{I}_k 3$) *Symmetry requirement.*
Every symmetry of $(Q, j(\mathcal{M}) \cap Q)$ for a k-face $Q \in \mathcal{F}_k(\mathcal{A})$ that is a symmetry of each sheet of $j(\mathcal{M}) \cap Q$ separately is a symmetry of $(Q, \mathcal{C}^k \cap Q)$.

($\mathrm{I}_k 4$) *Subcomplex requirement.*
For every diagonal symmetry hyperplane H_Q of a facet Q of $\mathcal{A}$ and every facet F of Q the (lifted) induced subdivision $\mathcal{C}^k \cap (F \cap H)$ is a (lifted) subcomplex of $\mathcal{C}^k$.

All of these invariants are maintained while iteratively constructing $\mathcal{C}^1$ and $\mathcal{C}^2$. The resulting lifted cubical subdivision $\mathcal{C}^3$ of $\mathcal{A}$ satisfies the conditions ($\mathrm{I}_3 1$) and ($\mathrm{I}_3 2$) but not the other ones.

(1) Subdivision of edges.

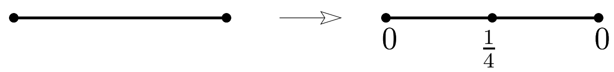
We construct a lifted cubical subdivision $\mathcal{C}^1$ of the 1-skeleton of $\mathcal{A}$ by providing for every edge e of $\mathcal{A}$ a lifted subdivision $\mathcal{C}^1_e$.

Let e be an edge of $\mathcal{A}$.

- If e is not intersected by the immersed manifold, then we subdivide the edge by an affine copy $\mathcal{C}_e^1$ of the lifted subdivision $\mathcal{U}_2 := (\mathcal{U}_2', h)$, where $\mathcal{U}_2'$ is a 1-dimensional complex affinely isomorphic to $\mathcal{P}_1(2)$. The vertices of $\operatorname{lift}(\mathcal{U}_2', h)$ are

$$\boldsymbol{v}_1 = (-\tfrac{1}{2}, 0),\ \ \boldsymbol{v}_2 = (0, \tfrac{1}{4}),\ \ \boldsymbol{v}_3 = (\tfrac{1}{2}, 0),$$

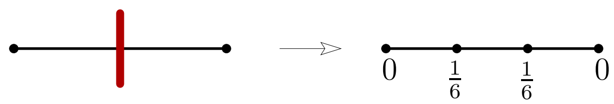
and the two edges are $(\boldsymbol{v}_1, \boldsymbol{v}_2)$ and $(\boldsymbol{v}_2, \boldsymbol{v}_3)$. The following figure depicts $(e, j(\mathcal{M}) \cap e)$ and $\mathcal{U}_2'$ where each vertex of the subdivision is labeled by its height.



- If e is intersected by the immersed manifold, then we subdivide the edge by an affine copy $\mathcal{C}_e^1$ of the lifted subdivision $\mathcal{U}_3 := (\mathcal{U}_3', h)$, where $\mathcal{U}_3'$ is a 1-dimensional complex affinely isomorphic to $\mathcal{P}_1(3)$. The vertices of $\operatorname{lift}(\mathcal{U}_3', h)$ are

$$\boldsymbol{v}_1 = (-\tfrac{1}{2}, 0),\ \ \boldsymbol{v}_2 = (-\tfrac{1}{6}, \tfrac{1}{6}),\ \ \boldsymbol{v}_3 = (\tfrac{1}{6}, \tfrac{1}{6}),\ \ \boldsymbol{v}_3 = (\tfrac{1}{2}.0),$$

and the edges are $(\boldsymbol{v}_i, \boldsymbol{v}_{i+1})$, for $i = 1, 2, 3$.

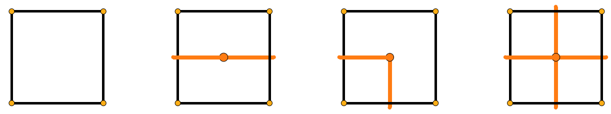


Observe that $(\mathrm{I}_1 1)$–$(\mathrm{I}_1 4)$ are satisfied.

(2) Subdivision of 2-faces.

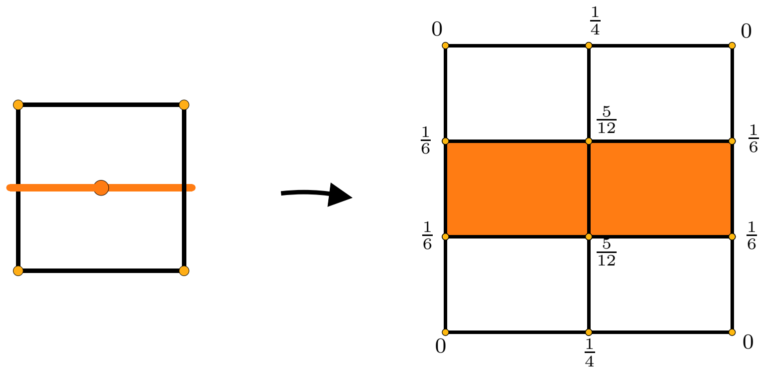
We construct the lifted cubical subdivision $\mathcal{C}^2$ of the 2-skeleton of $\mathcal{A}$ by providing for every 2-face (quadrangle) Q of $\mathcal{A}$ a lifted cubification $\mathcal{C}_Q^2$. Let Q be a quadrangle of $\mathcal{A}$, and $\boldsymbol{w}$ the unique vertex of $j(\mathcal{M})$ that is contained in Q. There are four possible types of vertex stars of grid immersions of curves:

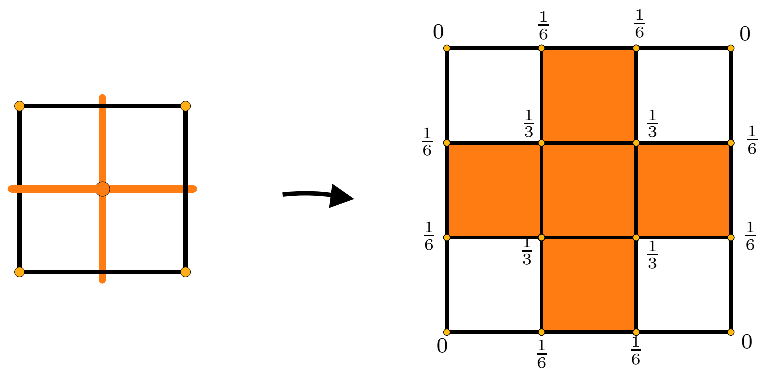
(a) In the cases "single 2a" and "double" there is a coordinate hyperplane H such that $(Q, j(\mathcal{M}) \cap Q)$ is symmetric with respect to H, and a vertex $\boldsymbol{v}$ of $\mathcal{M}$ such that $j(\boldsymbol{v}) = \boldsymbol{w}$ and the image of the vertex star is contained in H. Let F be a facet of Q that does not intersect H.



empty single 2a single 2b double

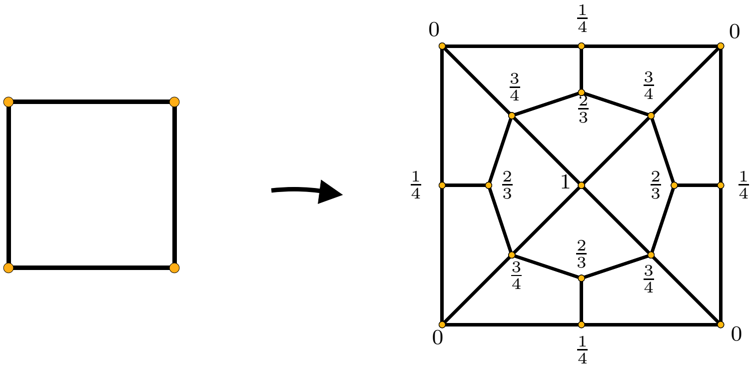
Then $\mathcal{C}_Q^2$ is taken to be an affine copy of the product $(\mathcal{C}^1 \cap F) \times \mathcal{U}_3$. The following figure depicts $(Q, j(\mathcal{M}) \cap Q)$ and the resulting cubification $\mathcal{C}_Q^2$ where each vertex of the subdivision is labeled by its height.





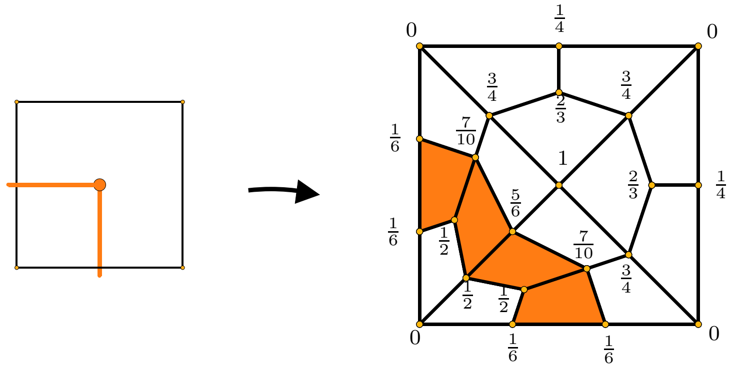
(b) Assume the immersion does not intersect Q. In this case $\mathcal{C}_Q^2$ is taken to be an affine copy of the lifted cubical 2-complex $\mathcal{V}$: This complex is given by the upper faces of the 3-polytope whose vertices are $(\pm\frac{1}{2}, \pm\frac{1}{2}, 0)$, $(\pm\frac{1}{4}, \pm\frac{1}{4}, \frac{3}{4})$, $(0, 0, 1)$, $(0, \pm\frac{1}{2}, \frac{1}{4})$, $(\pm\frac{1}{2}, 0, \frac{1}{4})$, $(0, \pm\frac{1}{3}, \frac{2}{3})$, and $(\pm\frac{1}{3}, 0, \frac{2}{3})$. ($\mathcal{V}$ arises as the cubical barycentric subdivision of the stellar subdivision of $[-\frac{1}{2}, \frac{1}{2}]^2$.)

(c) In the case "single 2b", we define $\mathcal{C}_Q^2$ as an affine copy of the lifted cubical 2-complex $\mathcal{V}'$: This complex is given by the upper faces of the 3-polytope whose vertices are $(\pm\frac{1}{2}, -\frac{1}{2}, 0)$, $(\frac{1}{2}, \frac{1}{2}, 0)$, $(0, 0, 1)$, $(\frac{1}{2}, 0, \frac{1}{4})$, $(\frac{1}{3}, 0, \frac{2}{3})$, $(\frac{1}{4}, \pm\frac{1}{4}, \frac{3}{4})$ $(\pm\frac{1}{6}, -\frac{1}{2}, \frac{1}{6})$, $(-\frac{1}{3}, -\frac{1}{3}, \frac{1}{2})$,



$(-\frac{3}{8}, -\frac{1}{8}, \frac{1}{2})$, $(-\frac{1}{6}, -\frac{1}{6}, \frac{5}{6})$, $(\frac{1}{10}, -\frac{3}{10}, \frac{7}{10})$ and their reflections on the hyperplane $x_1 - x_2 = 0$.

The complex $\mathcal{V}'$ is given by $\mathcal{V}$ truncated by the two hyperplanes $(1, 5, -3)^T \boldsymbol{x} = \frac{1}{2}$ and $(0, \frac{8}{3}, -1)\boldsymbol{x} = \frac{3}{2}$, and their reflections on the hyperplane $x_1 - x_2 = 0$.



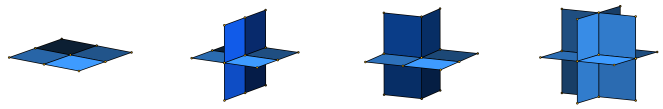
Observe that the conditions $(\mathrm{I}_2 1)$–$(\mathrm{I}_2 4)$ are satisfied.

(3) Subdivision of 3-cubes. We construct a lifted cubical subdivision $\mathcal{C}^3$ of the 3-skeleton of $\mathcal{A}$ by providing for every facet Q of $\mathcal{A}$ a lifted subdivision $\mathcal{C}^3_Q$.

Let Q be a facet of $\mathcal{A}$ and $\boldsymbol{w}$ the unique vertex of $j(\mathcal{M})$ that is mapped to the barycenter of Q. Let $\mathcal{S} := \mathcal{C}^2 \cap Q$ be the induced lifted cubical boundary subdivision of Q.

We construct a lifted cubification $\mathcal{C}^3_Q$ of $\mathcal{S}$ either as a generalized regular Hexhoop or as a product of $\mathcal{U}_3$ with a lifted cubical subdivision of a facet of Q.

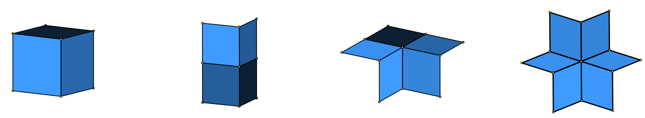
(a) For the following four types of vertex stars it is possible to take a product with $\mathcal{U}_3$: In all these cases there is a symmetry



single 4a double 8a double 8b triple 12

hyperplane H of Q of the form $x_i = k$ such that $H \cap Q$ is a sheet of $j(\mathcal{M}) \cap Q$.
Hence all facets of Q that intersect H are subdivided by $\mathcal{U}_3 \times \mathcal{U}_3$ or $\mathcal{U}_3 \times \mathcal{U}_2$. Let F be one of the two facets of Q that do not intersect H. Then the product $(\mathcal{C}^2 \cap F) \times \mathcal{U}_3$ yields the lifted subdivision $\mathcal{C}^3_Q$ of Q. Clearly $\mathcal{C}^3_Q$ satisfies ($\mathrm{I}_3 2$) restricted to Q.

(b) In the remaining five cases we take a generalized regular Hexhoop with a diagonal hyperplane of symmetry of Q to produce $\mathcal{C}^3_Q$. These cases include the following five cases:



empty single 3 single 4b single 5 single 6a

In each of these cases, $(Q, Q \cap j(\mathcal{M}))$ has a diagonal hyperplane H of symmetry. This hyperplane H intersects the relative interior of two facets of Q. Since ($\mathrm{I}_2 4$) holds, no facet of $\mathcal{S} = \mathcal{C}^2 \cap Q$ intersects H in its relative interior. By ($\mathrm{I}_2 3$) the lifted boundary subdivision $\mathcal{S}$ is symmetric with respect to H. Hence all preconditions of the generalized regular Hexhoop are satisfied. The resulting cubification $\mathcal{C}^3_Q$ satisfies ($\mathrm{I}_3 2$) restricted to Q (see Remark **??**).

10.4.1 Correctness

Proposition 10.4 *Let $\mathcal{M}$ be a 2-dimensional closed (that is, compact and without boundary) cubical PL-surface , and $j \colon \mathcal{M} \looparrowright \mathbb{R}^3$ a locally symmetric codimension one grid immersion.*

Then the cubical 3*-ball* $\mathcal{B}$ *given by Construction 10.4 has the following properties:*

(i) $\mathcal{B}$ *is regular, with a lifting function* ψ*.*

(ii) *There is a dual manifold* $\mathcal{M}'$ *of* $\mathcal{B}$ *and associated immersion* $y : \mathcal{M}' \looparrowright |\mathcal{B}|$ *such that* $\mathcal{M}'$ *is a cubical subdivision of* $\mathcal{M}$*, and the immersions* j *and* y *are PL-equivalent.*

(iii) *The number of facets of* $\mathcal{B}$ *is congruent modulo two to the number* $t(j)$ *of triple points of the immersion* j*.*

(iv) *There is a lifted cubification* $\mathcal{C}$ *of* $(\partial\mathcal{B}, \psi|_{\partial\mathcal{B}})$ *with an even number of facets.*

Proof. (i) *Regularity.* By construction the lifting functions ψ_F, $F \in \mathrm{fac}(\mathcal{A})$, satisfy the consistency precondition of the patching lemma. Since every pile of cube is regular (compare Section 5.4), the raw complex $\mathcal{A}$ is regular and the Patching Lemma implies that $\mathcal{B}$ is regular, too.

(ii) *PL-equivalence of manifolds.* Property ($\mathrm{I}_3 2$) implies that for every facet $Q \in \mathrm{fac}(\mathcal{A})$ the lifted cubical subdivision $\mathcal{C}^3 \cap Q$ has a dual manifold $\mathcal{M}_Q$ that is PL-equivalent to $j(\mathcal{M}) \cap Q$. The consistency requirements ($\mathrm{I}_1 1$) – ($\mathrm{I}_3 1$) imply that for each pair Q, Q' of facets of $\mathcal{A}$ the dual manifolds $\mathcal{M}_Q$, $\mathcal{M}_{Q'}$ coincide, that is, $j(\mathcal{M}_Q) \cap (Q \cap Q') = j(\mathcal{M}_{Q'}) \cap (Q \cap Q')$.

Hence the union of all $\mathcal{M}_Q$, $Q \in \mathrm{fac}(\mathcal{A})$ gives a dual manifold $\mathcal{M}'$ with associated immersion $y : \mathcal{M}' \looparrowright |\mathcal{B}|$ such that immersions j and y are PL-equivalent.

(iii) *Parity of the number of facets.* For each 3-cube Q of $\mathcal{A}$, its cubification $\mathcal{C}^3_Q$ is either a product $\mathcal{C}^2_F \times \mathcal{U}_3$ (where $\mathcal{C}^2_F$ is a cubification of a facet F of Q), or the outcome of a generalized regular Hexhoop construction. In the latter case the the number of facets of $\mathcal{C}^3_Q$ is even. In the first case the number of facets depends on the number of 2-faces of $\mathcal{C}^2_F$. The number of quadrangles of $\mathcal{C}^2_F$ is odd if and only if $j(\mathcal{M}) \cap F$ has a double intersection point. Hence, $f_3(\mathcal{C}^3_Q)$ is odd if and only if the immersion j has a triple point in Q.

(iv) *Alternative cubification.* Applying Construction 10.4 to $\mathcal{P}_3(\ell_1, \ell_2, \ell_3)$ without an immersed manifold yields a regular cubification $\mathcal{C}$ of $\partial\mathcal{B}$ with the same lifting function as $\mathcal{B}$ on the boundary. (It is possible to produce a smaller alternative cubification by the cubical barycentric subdivision or by a generalized Hexhoop. See the discussion in Section 11.2.) Since the immersion $\emptyset \looparrowright \mathbb{R}^3$ has no triple points the number of facets of $\mathcal{C}$ is even. $\square$

10.4.2 Proof of the main theorem

Let $j\colon \mathcal{M} \looparrowright \mathbb{R}^3$ be a locally flat normal crossing immersion of a compact $(d-1)$-manifold $\mathcal{M}$ into $\mathbb{R}^d$.
By Proposition 10.1 and Proposition 10.2 there is a cubical subdivision $\mathcal{M}'$ of $\mathcal{M}$ with a locally symmetric, codimension one grid immersion $j'\colon \mathcal{M}' \looparrowright \mathbb{R}^3$ such that j and j' are PL-equivalent.
We construct the convex cubical 3-ball $\mathcal{B}$ with prescribed dual manifold immersion j' as described above. By Proposition 10.4(i) the ball $\mathcal{B}$ is regular, and by Proposition 10.4(iv) there is a cubification $\mathcal{C}$ of $\partial\mathcal{B}$ with an even number of facets and the same lifting function on the boundary.
Perform a lifted prism over $\mathcal{B}$ and $\mathcal{C}$ (Construction 5.8.2 of Section 5.8.2). This yields a cubical 4-polytope P with

$$f_3(P) = f_3(\mathcal{B}) + f_3(\mathcal{C}) + f_2(\partial\mathcal{B}).$$

For every dual manifold of $\mathcal{B}$ there is a PL-equivalent dual manifold of P. Hence by Proposition 10.4 there is a dual manifold $\mathcal{M}''$ of P and associated immersion $y : \mathcal{M}'' \looparrowright |\partial P|$ such that y and j' are PL-equivalent. Condition (i) is satisfied since j' and j are PL-equivalent.
To see (ii), observe that for every cubical 3-ball the number of facets of the boundary is even. Hence $f_2(\partial\mathcal{B})$ is even. Since the number of facets of $\mathcal{C}$ is even, we obtain

$$f_3(P) \;\equiv_2\; f_3(\mathcal{B}) \;\equiv_2\; t(j).$$

Now consider (iii). By a famous theorem of Banchoff [6] the number of triple points of a normal crossing codimension one immersion of a surface has the same parity as the Euler characteristic. Hence, if $\mathcal{M}$ is a non-orientable surface of odd genus the number of triple points of j is odd, which implies that the cubical 4-polytope P has an odd number of facets. $\square$

10.5 Symmetric templates

The three-dimension templates constructed above, which we call the *standard templates*, do not satisfy the conditions ($I_3$3) and ($I_3$4). In particular, the condition ($I_3$3) is violated by the templates corresponding to the cases “empty”, “single 3”, and “single 6a” (and the condition is satisfied by all others). The cubification for the case “single 5” is illustrated in Figure 9.24. This cubifications satisfies ($I_3$3) since there is only one diagonal symmetry hyperplane.

It is possible to produce an alternative template for the "empty" case by means of the cubical barycentric subdivision. The resulting cubification satisfies both conditions ($I_3$3) and ($I_3$4), and furthermore, it has less faces — 96 facets, 149 vertices — than the original template.
An alternative cubification for the case "single 3" of full symmetry can be constructed from $\mathcal{C}''$ by truncating the lifted polytope corresponding to the lifted cubical ball $\mathcal{C}''$ by some additional hyperplanes.
It is open whether there is a cubification of full symmetry for the case "single 6a."

Chapter 11

Cubical 4-polytopes with odd numbers of facets

11.1 Introduction

Motivated by the success of the characterization of the f-vectors of simplicial 4-polytopes, one might tackle the following question.

Open question. What are the f-vectors of cubical 4-polytopes?

As mentioned in Section 2.5, this is a challenging open problem – and might be very difficult [60]. As a first approach one can consider the following question.

Open question. What is the $\mathbb{Z}$-affine span of f-vectors of cubical 4-polytopes?

The lattice of f-vectors of cubical 3-spheres is known: Babson & Chan [5] gave an existences proof for cubical PL 3-spheres with odd numbers of facets. (The proof involves Construction 4.3 which is described in Section 4.3). Hence the $\mathbb{Z}$-affine span of f-vectors of cubical 3-spheres is

$$f_0 \equiv_2 f_1 \equiv_2 f_2 + f_3 \equiv_2 0.$$

To prove a similar result for cubical 4-polytopes we have to answer the following question.

Open question. *(Eppstein 2002)*
Is there a cubical 4-polytope with an odd number of facets?

Applying the Babson & Chan construction (Construction 4.3) to an immersion of *Boy's surface* yields the existence of a cubical PL 3-sphere with an odd number of facets.

Corollary. (Babson & Chan [5])
There are cubical PL 3-spheres with odd numbers of facets.

11.2 An odd cubical 4-polytope with a dual Boy's surface

Cubical 4-polytopes with odd numbers of facets exist by our Main Theorem 10.3. In this section we describe the construction of a cubical 4-polytope with an odd number of facets in more detail. The data for the corresponding model are available from

http://www.math.tu-berlin.de/~schwartz/c4p

and will be submitted to the eg-models archive.

Theorem 11.1 *Cubical 4-polytopes with odd numbers of facets exist. In particular, there is a cubical 4-polytope of f-vector*

$$f = (19\,520,\ 56\,186,\ 54\,999,\ 18\,333)$$

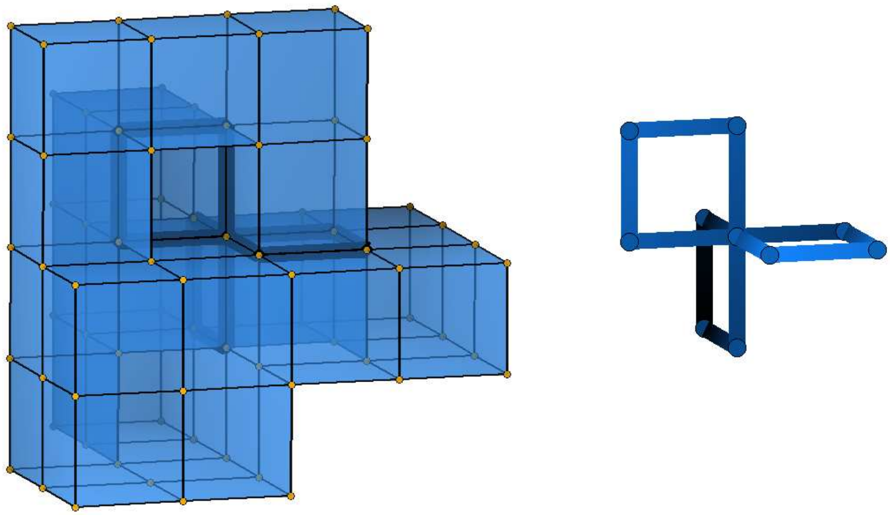
that has one dual manifold immersion that is a Boy's surface.

We prove this result by describingc in detail the construction of two instances of cubical 4-polytopes with odd numbers of facets. The first instance has 30 096 vertices and 27 933 facets, whereas the second one has the f-vector $f = (19\,520,\ 56\,186,\ 54\,999,\ 18\,333)$.

Note added in proof. It turns out that a small modification of the techniques presented in the here reduced the number of faces slightly. In particular, it is possible to produce an instance of cubical 4-polytope with odd numbers of facets with 17 718 vertices and 16 533 facets [53].

A grid immersion of Boy's surface.

The construction starts with a grid immersion (cf. [47]) of Boy's surface, that is, an immersion of the real projective plane with exactly one triple point and three double-intersection curves in a pattern of three self-loops [2][15]. This immersion $j : \mathcal{M} \looparrowright \mathbb{R}^3$ looks as follows.
The 2-manifold $\mathcal{M}$ has the f-vector $f(\mathcal{M}) = (85,\ 168,\ 84)$, whereas the image of the grid immersion has the f-vector $f(j(\mathcal{M})) = (74,\ 156,\ 84)$. The vertex coordinates can be chosen such that the image $j(\mathcal{M})$ is contained in a pile of cubes $P_3(4,4,4)$. The used vertex types of this grid immersion are listed in Table 11.1.



(a) grid immersion (b)

Figure 11.1: A grid immersion the Boy's surface, an immersion of the real projective plane with one triple point and three double-intersection curves in a pattern of three self-loops ("three-bladed propellor"). In this case, each double-intersection loop is of length four.

case		number of vertices
single	3	16
	4a	3
	4b	33
	5	12
	6	0
double	8a	0
	8b	9
triple	12	1
total		74

Table 11.1: The vertex types of the grid immersion of Boy's surface.

A cubical 3-ball with a dual Boy's surface

We apply Construction 10.4 to the grid immersion $j : \mathcal{M} \looparrowright \mathbb{R}^3$ to obtain a cubical 3-ball with a dual Boy's surface, and with an odd number of facets. Since the image $j(\mathcal{M})$ is contained in a pile of cubes $P_3(4,4,4)$, the raw complex $\mathcal{A}$ given by Construction 10.4 is combinatorially isomorphic to $P_3(5,5,5)$.

Hence we have $5^3 - 74 = 51$ vertices of $\mathcal{A}$ that are not vertices of $j(\mathcal{M})$.
We try to give an impression of the subdivision $\mathcal{C}^2$ of the 2-skeleton of $\mathcal{A}$ in Figure 11.2. The f-vector of $\mathcal{C}^2$ is $f = (4\,662,\, 9\,876,\, 5\,340)$.
The subdivision of the boundary of $\mathcal{A}$ consists of $150 = 6 \cdot 5 \cdot 5$ copies of the two-dimensional "empty pattern" template. Hence the subdivision of the boundary of $\mathcal{A}$ (given by $\mathcal{C}^2 \cap |\partial\mathcal{A}|$) has the f-vector $f = (1\,802,\, 3\,600,\, 1\,800)$.
The construction of the refinement $\mathcal{B}$ of $\mathcal{A}$ depends on the used set of templates for dimension 3. First consider the "symmetric" set of templates described in Chapter 10. In this case the f-vector of $\mathcal{B}$ is

$$f = (15\,915,\, 45\,080,\, 43\,299,\, 14\,133).$$

The calculation of the number of facets is shown in Table 11.2. The number of vertices of $\mathcal{B}$ can be calculated as

$$f_0(\mathcal{B}) = f_0(\mathcal{C}^2) + \sum_{Q \in \mathrm{fac}(\mathcal{A})} f_0(\mathcal{C}^3_Q) - \sum_{Q \in \mathrm{fac}(\mathcal{A})} f_0(\partial\mathcal{C}^3_Q).$$

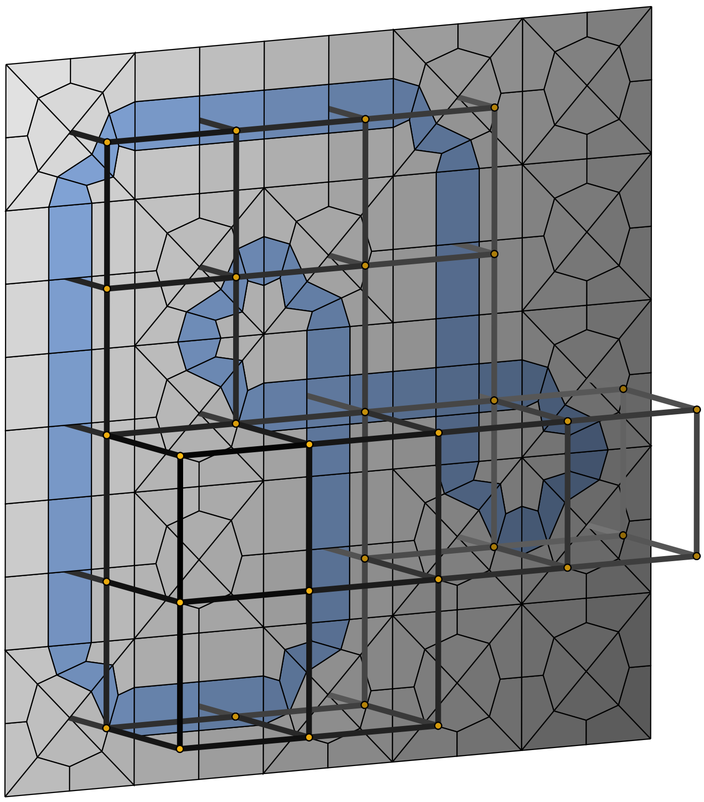


FIGURE 11.2: A sketch of the cubification of the 2-skeleton of $\mathcal{A}$.

case		#vert	type	bd. cub. f_0	f_1	f_2	cubification f_0	f_1	f_2	f_3	total f_3
empty	0	51	C	74	144	72	149	376	324	96	4 896
single	3	16	C	86	168	84	174	443	384	114	1 824
	4a	3	P	50	96	48	68	163	132	36	108
	4b	33	H	70	136	68	196	529	484	150	4 950
	5	12	H	74	144	72	206	557	510	158	1 896
	6a	0	H	98	192	96	266	725	666	282	0
double	8a	0	P	44	84	42	48	104	75	18	0
	8b	9	P	64	124	62	88	214	175	48	432
triple	12	1	P	56	108	54	64	144	108	27	27
total		74									14 133

Table 11.2: Calculation of the number of facets of the cubical 3-ball $\mathcal{B}$ that is given by Construction 10.4 using the symmetric set of templates.

In Figure 11.3 we illustrate the dual Boy's surface of the cubical 3-ball $\mathcal{B}$. It has the f-vector $f = (1\,998,\ 3\,994,\ 1\,997)$, and its set of multiple-intersection points has one triple point and three intersection loops of length 16. The ball $\mathcal{B}$ has 612 dual manifolds is total (339 of them without boundary).
Using the "standard set" of templates yields a cubical ball with 18 281 facets.

A cubical 4-polytope with a dual Boy's surface

An cubification $\mathcal{B}'$ of ∂B with an even number of facets is given by subdividing each facet of the raw ball $\mathcal{A}$ with a cubification for the empty pattern. Using the symmetric cubification for the empty pattern yields a regular cubical 3-ball $\mathcal{B}_2$ with

$$f(\mathcal{B}_2) = (14\,181,\ 39\,080,\ 36\,900,\ 12\,000).$$

In particular the number of facets is even. By Proposition 10.4 both cubical 3-balls $\mathcal{B}$ and $\mathcal{B}'$ have height functions that coincide on the common boundary $\partial\mathcal{B} = \partial\mathcal{B}'$.

Hence we can construct the lifted prism construction over $\mathcal{B}$ and $\mathcal{B}'$. The resulting cubical 4-polytope P has

$$f_0(P) = f_0(\mathcal{B}) + f_0(\mathcal{B}') = 15\,915 + 14\,181 = 30\,096 \quad \text{vertices,}$$

and

$$\begin{aligned} f_3(P) &= f_3(\mathcal{B}) + f_3(\mathcal{B}') + f_2(\partial\mathcal{B}) \\ &= 14\,133 + 12\,000 + 1\,800 = 27\,933 \quad \text{facets.} \end{aligned}$$

There are 1 102 dual manifolds in total: One dual Boy's surface of f-vector

$$f = (1\,998,\ 3\,994,\ 1\,997),$$

one immersed surface of genus 20 (immersed with 104 triple points) with f-vector

$$f = (13\,198,\ 26\,472,\ 13\,236),$$

and 1 100 embedded 2-spheres of different f-vectors.

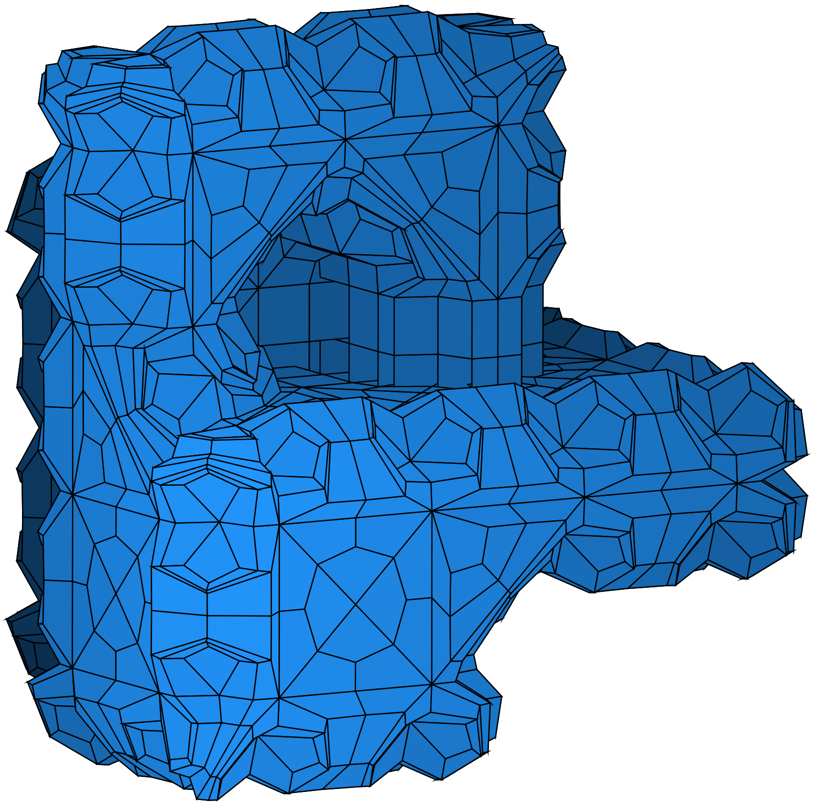


FIGURE 11.3: The dual Boy's surface of f-vector $f = (1\,998, 3\,994, 1\,997)$ of the cubical 3-ball $\mathcal{B}$.

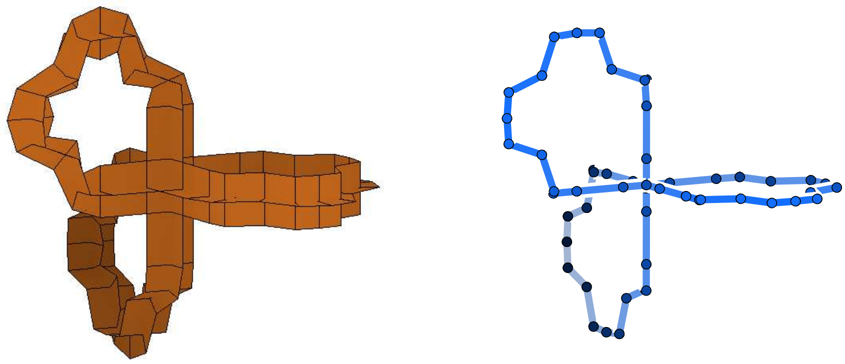


Figure 11.4: The intersecting quadrangles and the multiple-intersection curve of the dual Boy's surface of the cubical 3-ball $\mathcal{B}$.

A smaller cubical 4-polytope with a dual Boy's surface

To prove Theorem 11.1 we provide a significantly smaller alternative cubification $\mathcal{B}''$ of $\partial\mathcal{B}$ with an even number of facets. In the following we sketch the construction of a regular cubification $\mathcal{B}''$ of $\partial\mathcal{B}$ with

$$f(\mathcal{B}'') = (3\,605,\ 9\,304, 8\,100,\ 2\,400).$$

Again, the patching lemma is used.
To obtain a raw polytopal subdivision of $|\mathcal{B}|$ choose a point $\boldsymbol{p}$ in the relative interior of $|\mathcal{B}|$, and let $\mathcal{R}$ denote the 3-ball whose facets are the pyramids with apex point $\boldsymbol{p}$ over boundary facets F of $\mathcal{A}$. Clearly the piece-wise linear function h on $\mathcal{A}$ given by $h(\boldsymbol{p}) = 1$ and $h(\boldsymbol{v}) = 0$, for $\boldsymbol{v} \in \mathrm{vert}(\partial\mathcal{A})$, is a lifting function of $\mathcal{R}$.
For each facet G of $\mathcal{R}$ a lifted cubical subdivision $\mathcal{T}_G$ is given by the prism with apex point $(\boldsymbol{p}, 0) \in \mathbb{R}^{d+1}$ over the induced lifted cubical subdivision $\mathcal{B} \cap (G \cap \partial\mathcal{A})$.
Then the patching lemma implies that cubical 3-ball $\mathcal{B}''$ formed by all $\mathcal{T}_G$ is regular and the height on the boundary. Furthermore, there is a height function such that the boundary heights of $\mathcal{B}''$ and $\mathcal{B}$ coincide.
Hence we can construct the lifted prism construction over $\mathcal{B}$ and $\mathcal{B}''$. The resulting cubical 4-polytope P' has $f_0 = 19\,520$ vertices and $f_3 = 18\,333$ facets. A further analysis of the dual manifolds of P' shows that there are 614 dual manifolds in total: One dual Boy's surface of f-vector $f = (1\,998,\ 3\,994,\ 1\,997)$, one immersed surface of genus 20 (immersed with 104 triple points) with f-vector $(11\,902,\ 23\,880, 11\,940)$, and 612 embedded 2-spheres of different f-vectors.

The generalized regular Hexhoop construction can be used to produce an alternative cubification of $\partial\mathcal{B}$. However this requires more than $f_3(\mathcal{B}_2'')$ facets as the following argument shows. Let $\mathcal{H}$ be a three-dimensional cubification given by a generalized regular Hexhoop construction. Then the quotient $f_3(\mathcal{H})/f_2(\partial\mathcal{H})$ is roughly $2+\varepsilon$, for an $\varepsilon > 0$. In contrast, the quotient "facets/boundary quadrangles" equals $4/3$ for a cubification produced as the cubical barycentric subdivision of a stellar subdivision.

Verification of the instances

All the instances of the cubical 4-polytopes described above were produced electronically. This was done using the polymake system by Gawrilow & Joswig [25, 27], a system for the construction and analysis of convex polytopes. Furthermore a number of our own tools for handling cubical complexes are involved. They cover creation, verification, and visualization of cubical complexes (for $d \in \{2,3\}$).
The instances are available from

http://www.math.tu-berlin.de/~schwartz/c4p

(The smallest instance is submitted to *EG-Models*, an web archive of electronic geometry models [36].)

For each instance we provide

- a polymake file which contains the whole information about the polytope (including vertex coordinates, facet description, as well as the vertex-facet incidences),
- images in `gif` and encapsulated postscript format of the corresponding cubical 3-ball with a Boy's surface, and
- a javaview [50] `jvx` file of the same cubical 3-ball.

The correctness of an instance P of a cubical 4-polytope with an odd number of facets can be verified using the polymake system as follows.

(1) Take the set of points of P and calculate its convex hull using the `beneath_beyond` client [35] of the polymake system. The outcome of this convex hull code consists of the set of vertices, the list of fact defining hyperplanes, and the vertex-facet incidences.

(2) Check that the number of vertices f_0 and the number of facets f_3 coincide with the expected values. In particular, f_3 should be odd.
(We avoid to calculate the whole face lattice and the f-vector.)

(3) From the vertex-facet incidences the graph of P is calculated (using the polymake client `graph_from_incidence`).

(4) Check that the graph of P is triangle-free (using the polymake client `triangle_free`).

(5) Check that the number of vertex-facet incidences f_{03} equals $8f_3$. Furthermore, check that every facet has exactly 8 vertices.

Then the correctness of the instance follows by the following observation.

Observation 11.2. Let P be a 4-polytope such that the graph of P is triangle-free, and every facet of P has exactly 8 vertices.
Then P is cubical.

Whereas the construction of the instances involves new tools that were writted specifically for this purpose, the verification procedure uses only standard polymake tools. All tools used in the verification procedure are parts of polymake system which have been used (and thereby verified) by various users over the past years (using a rich variety of classes of polytopes).

The topology of the dual manifolds of our instances was examined using all the following tools:

- A homology calculation code based written by Frank Heckenbach [29].
- The homology calculation code of topaz, the *topological application zoo*, which is part of the polymake project mentioned above; it covers the construction and analysis of simplicial complexes.
- Our own tool for the calculation of the Euler characteristics.

Chapter 12

Consequences

12.1 Lattice of f-vectors of cubical 4-polytopes

Babson & Chan [5] have obtained a characterizazion of the $\mathbb{Z}$-affine span of the f-vectors of cubical 3-spheres: With the existence of cubical 4-polytopes with an odd number of facets this extends to cubical 4-polytopes.

Corollary 12.1 *The $\mathbb{Z}$-affine span of the f-vectors (f_0, f_1, f_2, f_3) of the cubical 4-polytopes is characterized by*
(i) *integrality ($f_i \in \mathbb{Z}$ for all i),*
(ii) *the cubical Dehn-Sommerville equations $f_0 - f_1 + f_2 - f_3 = 0$ and $f_2 = 3f_3$, and*
(iii) *the extra condition $f_0 \equiv 0 \bmod 2$.*

Note that this includes modular conditions such as $f_2 \equiv 0 \bmod 3$, which are not "modulo 2." The main result of Babson & Chan [5] says that for cubical d-spheres and $(d+1)$-polytopes, $d \geq 2$, "all congruence conditions are modulo 2." However, this refers only to the modular conditions *which are not implied by integrality and the cubical Dehn-Sommerville equations*. The first example of such a condition is, for $d = 4$, the congruence (iii) due to Blind & Blind [12].

12.2 Cubical 4-polytopes with orientable dual manifolds of prescribed genus

In Section 8.1 proved that a cubical 4-polytope with an orientable dual manifold of prescribed genus can be produced by means of connected sums of copies of the neighborly cubical 4-polytope C_4^5 with the graph of a 5-cube; compare Lemma 8.1.

Alternatively, a cubical 4-polytope with an orientable dual manifold of prescribed genus can be obtained using a grid immersion.

Corollary 12.2 *For each $g \geq 0$, there is a cubical 4-polytope that has a cubation of the orientable connected 2-manifold M_g of genus g as an embedded dual manifold.*

Proof. Let $g \geq 0$ be an integer. There is *grid torus* of f-vector $(32, 64, 32)$ and a *grid handle* of f-vector $(24, 44, 20)$, both depicted in Figure 12.2.
Take the grid torus and $g-1$ copies of the grid handle and glue them together. This yields a grid embedding of the oriented surface of genus g of f-vector $(16, 28, 14) + g(16, 36, 18)$.
Applying the construction in the proof of Theorem 10.3 to this grid embedding yields a cubical 4-polytope with an embedded dual manifold of genus g. □

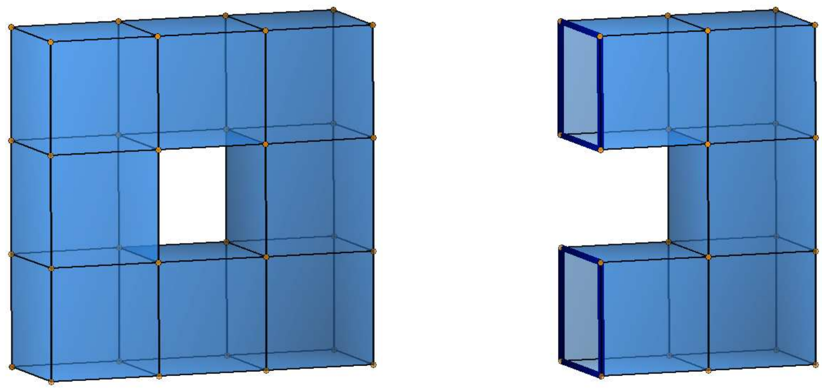


FIGURE 12.1: An grid embedding of a 2-torus of f-vector $f = (32, 64, 32)$ and a grid handle of f-vector $f = (24, 44, 20)$.

12.3 Cubical 4-polytopes with non-orientable dual manifolds of prescribed genus

Corollary 12.3 *For each even $g > 0$, there is a cubical 4-polytope that has a cubation of the non-orientable connected 2-manifold M'_g of genus g as a dual manifold (immersed without triple points and with one double-intersection curve of length 16).*

Proof. Let $g > 0$ be an even integer. Take the grid immersion of the Klein bottle of f-vector $f = (52, 108, 56)$ which is depicted in Figure 12.3. Glue the

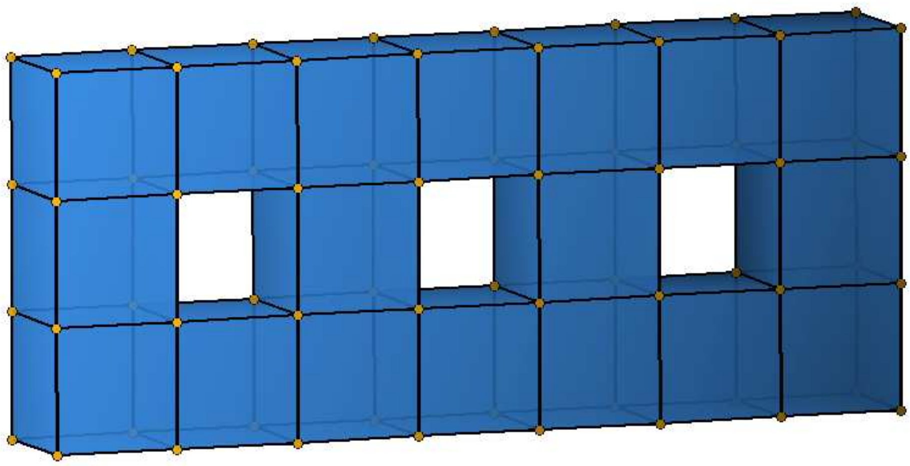


Figure 12.2: An grid embedding of an orientable surface of genus three, obtained as the connected sum of the grid handle and two grid tori glued.

grid immersion with $g-1$ copies of the grid handle. The resulting grid immersion is non-orientable and of genus g. Furthermore the immersion has no triple points and one double-intersection curve of length four. All four double-intersection points are of the case "double 8a." The double-intersection curve of our cubification made for this case consists of four edges. Hence the cubical 4-polytope given by the construction in the proof of Theorem 10.3 has an immersed non-orientable dual manifold of genus g, which is immersed without triple points and with one double-intersection curve of length 16. ☐

Smaller cubical 4-polytopes with non-orientable cubical dual manifolds can be produced by means of connected sums of the cubical 4-polytope P_{62} of Chapter 7 with a dual Klein bottle, and several copies of the neighborly cubical 4-polytope C_4^5. (Some "connector cubes" of Lemma 5.5 have to be used.) The resulting cubical 4-polytope has rather small f-vector entries, but the set of multiple-intersection points consists of five double-intersection curves.

Applying the same proof as above to the grid immersion of Boy's surface of the previous section yields the following result.

Corollary 12.4 *For each odd $g > 0$, there is a cubical 4-polytope that has a cubation of the non-orientable connected 2-manifold M'_g of genus g as a dual manifold (immersed with one triple point and three double-intersection curves of length 14).*

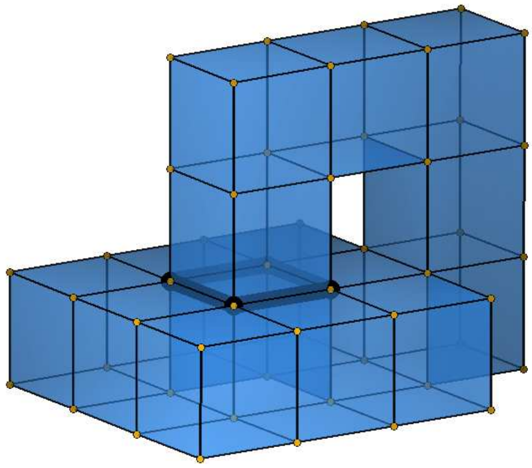


FIGURE 12.3: A grid immersion with f-vector $f = (52, 108, 56)$ of the Klein bottle, an immersion of the non-orientable 2-surface of genus 2 with one double-intersection curve (shown in bold) and without triple points.

12.4 Higher-dimensional cubical polytopes with non-orientable dual manifolds

Corollary 12.5 *For each $d \geq 4$ there are cubical d-polytopes with non-orientable dual manifolds.*

Proof. By construction, the 4-dimensional instance P_{62} of Chapter 7 comes with a subdivision into a regular cubical 4-ball. Since one of its dual manifolds is not orientable, its 2-skeleton is not edge orientable, i. e. it contains a cubical Möbius strip with parallel inner edges. So if we now iterate the lifted prism construction of Section 5.8.1, then the resulting cubical d-polytopes ($d \geq 4$) will contain the 2-skeleton of P_{62}. By Proposition 4.2 they must also have non-orientable dual manifolds. ☐

Chapter 13

Applications to hexa meshing

Hexa meshing is a nice application for cubical balls, and furthermore, we learned from the Hexhoop template, a well-known hexa meshing technique, how to produce symmetric cubifications.
On the other hand, our existence result about cubical 4-polytopes with odd numbers of facets has interesting consequences for some questions on hexa meshes.

13.1 Parity change

An interesting question deals with the parity of the number of facets of a mesh. For quad (2d) meshes there are several known parity changing operations, that is, operations that change the numbers of facets without changing the boundary mesh. In [10] Bern, Eppstein & Erickson raised the following questions:

Question 13.1 (Bern & Eppstein 2001)

(i) Are there geometric quad meshes with geometric hexa meshes of both parities?

(ii) Is there a parity changing operation *for geometric hexa meshes, that is, an operation that changes the parity of of the number of cubes of a cubical* 3*-ball without changing the boundary?*

From the existence of a cubical 4-polytope with odd number of facets we obtain positive answers to these questions.

Corollary 13.2

(i) *Every combinatorial* 3*-cube has a cubification with an even number of facets. Furthermore, this cubification is regular and even Schlegel.*

(ii) *Every combinatorial* 3*-cube is a facet of a cubical* 4*-polytope with an odd number of facets.*
(iii) *There is a parity changing operation for geometric hexa meshes.*

Proof. To see (ii), let F be a combinatorial 3-cube and P a cubical 4-polytope with an odd number of facets. By Lemma 5.5 there is a combinatorial 4-cube C that has both F and a projectively regular 3-cube G as facets. Let F' be an arbitrary facet of P. Then there is a combinatorial 4-cube C' that has both F' and a projectively regular 3-cube G' as facets. The connected sum of P and C based on the facet F' yields a cubical 4-polytope P' with an odd number of facets, and with a projectively regular 3-cube G'' as a facet. The connected sum of P' and C gluing the facets G and G'' yields a cubical 4-polytope with an odd number of facets, and with a projective copy of F as a facet.
The statements (i) and (iii) follow from (ii) via Schlegel diagrams. ☐

13.2 Flip graph connectivity

In analogy to the concept of flips for simplicial (pseudo-)manifolds one can define *cubical flips* for quad or hexa meshes; compare [10]. The list of quad and hexa mesh flips are illustrated in Figures 13.1 and 13.2.

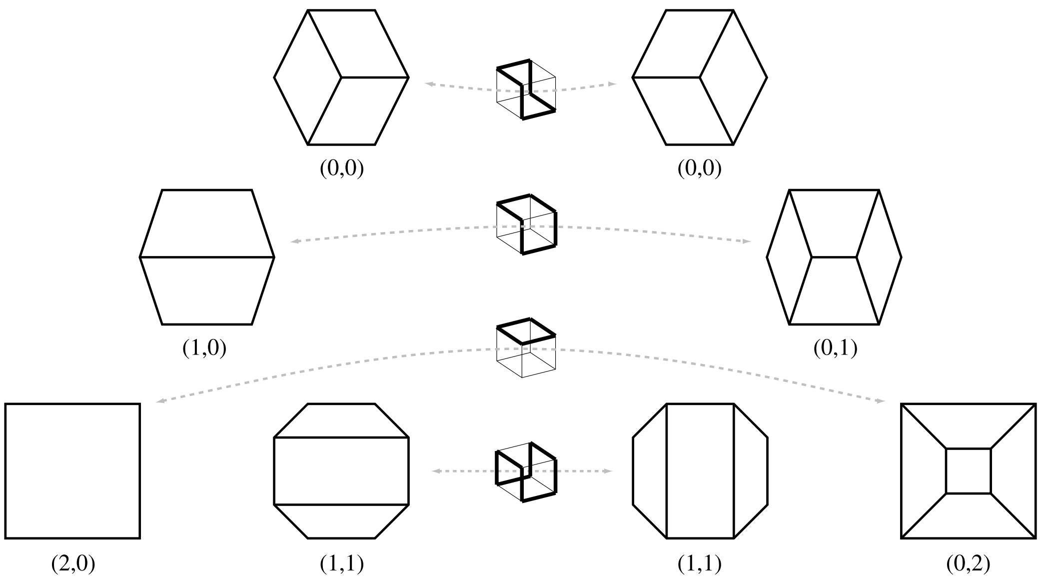


FIGURE 13.1: The list of quad flips. (Taken from [10].)

In the meshing terminology the flip graph is defined as follows. For any domain with boundary mesh, and a type of mesh to use for that domain, define

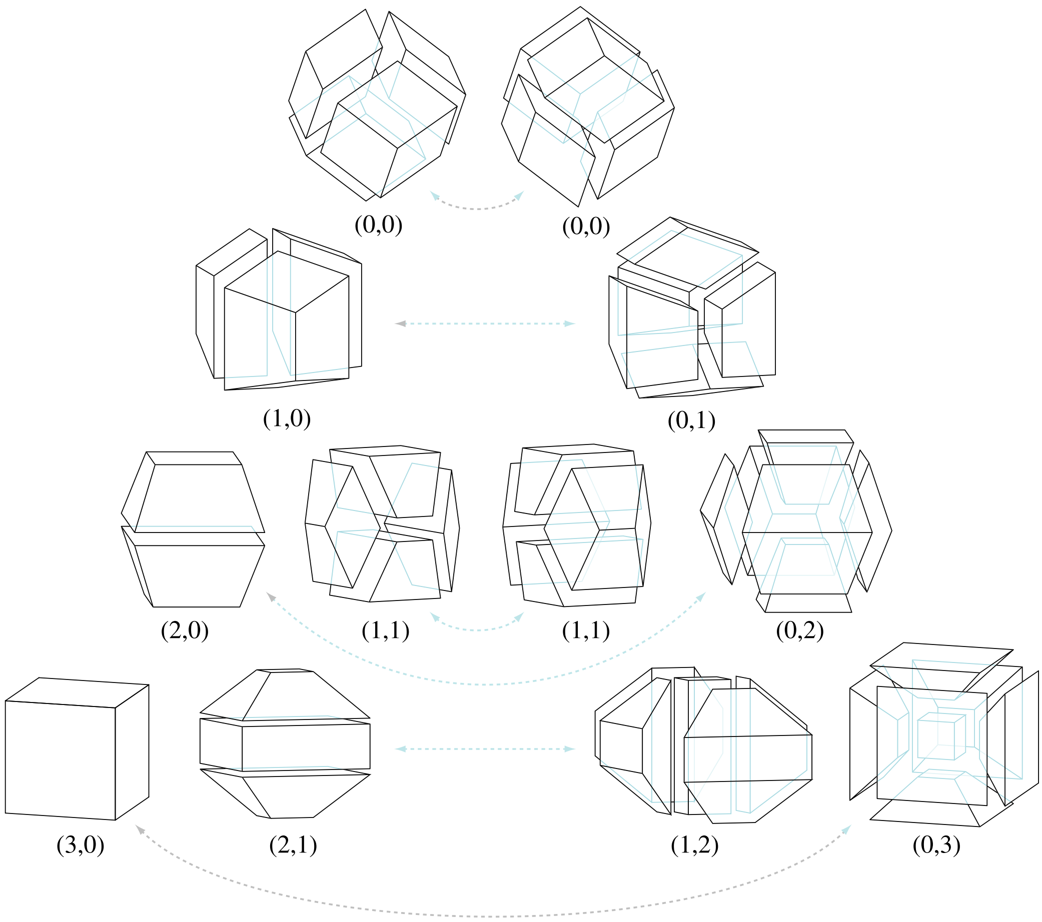


Figure 13.2: The list of hexa flips. (Taken from [10].)

the *flip graph* to be a graph with (infinitely many) vertices corresponding to possible meshes of the domain, and an edge connecting two vertices whenever the corresponding two meshes can be transformed into each other by a single flip.

In this framework, the question concerning a parity changing operation can be phrased as asking for a description of the connected components of the flip graph. Known results about flip graphs for *topological* cubical flips include the following ones:

Theorem 13.3 (Bern, Eppstein & Erickson [10])

(i) The flip graph for topological quad meshes of any simply-connected domain has exactly two connected components.

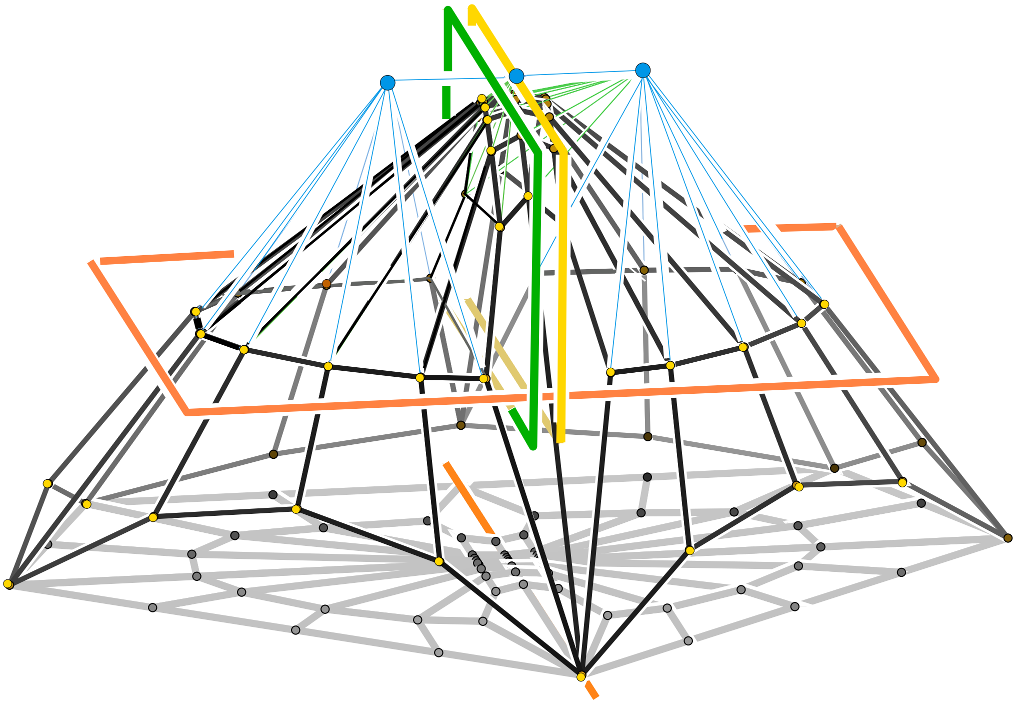
(ii) The flip graph for topological hexa meshes of any simply-connected domain has at least two connected components.

As an immediate consequence of Corollary 13.2 we obtain the following result.

Corollary 13.4 *The cubical flip graph for geometric hexa meshes of a has at least two connected components.*

Part IV

Towards higher dimensions



Chapter 14

Outlook: Higher-dimensional cubical polytopes with prescribed dual manifold immersions

In this thesis we are primarily concerned with the realization of 2-manifold immersions in terms of cubical 4-polytopes, but the higher-dimensional cases are interesting as well. In this chapter we indicate some possible further generalizations of our results to higher dimensions. The goal is to give a polytopal analog to the Babson-Chan construction for every dimension.

In the following we present two further constructions, namely the *encapsulated Schlegel cap* and the *mirrored encapsulated Schlegel cap*, and give some remarks on the construction of cubical 5-polytope with prescribed dual 3-manifolds. First we consider the existence of cubical d-polytopes or $(d-1)$-spheres with odd numbers of facets, for $d \geq 5$.

14.1 Odd number of facets

One would like to know whether there are cubical 5-polytopes with an odd number of facets. For this we have to realize a normal crossing immersion of 3-manifold into S^4 by a cubical 5-polytope with an odd number of quadruple points. Such immersions exist by an abstract result of Freedman [24] (see also Akhmetev [1]), but more concretely by John Sullivan's observation (personal communication) that there are regular sphere eversions of the 2-sphere with exactly one quadruple point [55, 48, 23, 42, 56], and from any such one obtains a normal-crossing immersion $S^3 \looparrowright S^4$ with a single quadruple point.

It is known that cubical polytopes or spheres with odd numbers of facets do not exist in all dimension. Babson & Chan [5] compiled the following list of restrictions (using Eccles [20]):

- If d is a multiple of 4 then a PL cubical d-sphere can have an odd number of facets if and only if $d = 4$.
- If d is odd, then a PL cubical d-sphere can have an odd number of facets if and only if $d = 1, 3$ or 7.
- If $d \equiv_4 2$, then a PL cubical d-sphere can have an odd number of facets only if $d = 2^n - 2$ for some n. Furthermore, examples of cubical d-spheres with an odd number of facets are known for $d = 2, 6, 14, 30$, and 62.
- Edge-orientable (in the sense of Hetyei [30]) cubical PL d-spheres can have an odd number of facets if and only if $d \in \{1, 2, 4\}$.

14.2 The mirrored encapsulated Schlegel cap

The *mirrored encapsulated Schlegel cap* is a modification of the *generalized regular Hexhoop* construction from Section 9.5. This construction can be interpreted as a mirrored *encapsulated Schlegel cap*, which is described in the next section. Both names are motivated by the fact that the resulting cubifications contains a Schlegel cap that is "encapsulated."

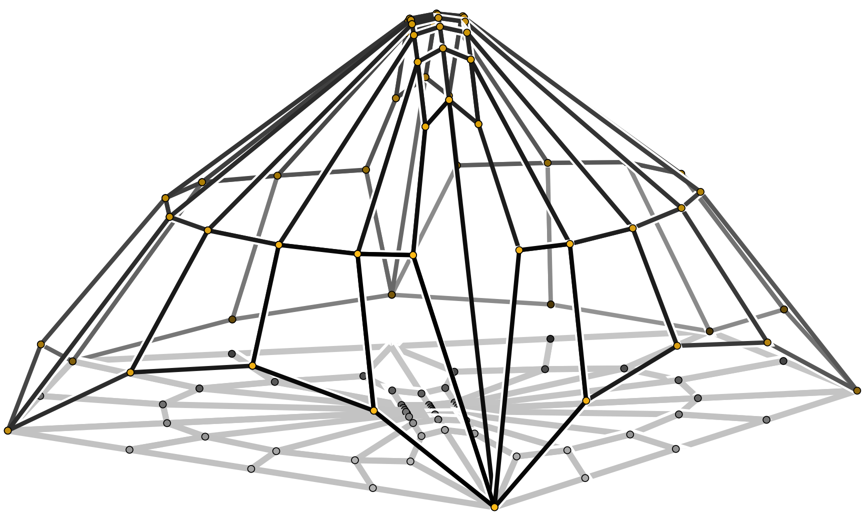


FIGURE 14.1: Outcome of the Mirrored encapsulated Schlegel cap.

The *mirrored encapsulated Schlegel cap* produces a lifted cubification $\mathcal{C}'$ for the same input as the generalized regular Hexhoop construction: A lifted boundary subdivision $(\mathcal{S}^{d-1}, \psi)$ of a polytope P that are both symmetric with respect to a hyperplane H. The outcome $\mathcal{C}'$ has a property that is not satisfied by the outcome of the generalized regular Hexhoop construction: H intersects no face of $\mathcal{C}'$ in its relative interior.

In Section 9.5 we gave a description of the generalized regular Hexhoop construction by iterated truncated pyramids (of length 2). The following description *mirrored encapsulated Schlegel cap* involves iterated truncated pyramids of length 3.

Construction 14: Mirrored encapsulated Schlegel cap

Input:

P	a convex d-polytope P in $\mathbb{R}^d$.
$(\mathcal{S}^{d-1}, \psi)$	a lifted cubical boundary subdivision of P.
H	a hyperplane in $\mathbb{R}^d$ such that • P and $(\mathcal{S}^{d-1}, \psi)$ are symmetric with respect to H, and • $\mathcal{S}^{d-1} \cap H$ is a subcomplex of $\mathcal{S}^{d-1}$.

Output:

$(\mathcal{B}^d, \phi)$	a symmetric lifted cubification of $(\mathcal{S}^{d-1}, \psi)$ given as the upper faces of a $(d+1)$-polytope P'.

The first three steps are identical to the first steps of the first description of the generalized regular Hexhoop construction. The construction is illustrated in Figure 14.2.

(1) Choose a positive halfspace H_+ with respect to H, and a point $\boldsymbol{q}_L \in \operatorname{relint}(P \cap H_+)$. Define $\boldsymbol{q}_R := \boldsymbol{p}_L^M$, where the upper index ^{M} denotes the mirrored copy with respect to $\widetilde{H} = H + \mathbb{R}\boldsymbol{e}_{d+1}$.
By Proposition 9.3 there is a height $h > 0$ such that

$$T := \operatorname{conv}\{\operatorname{lift}(\mathcal{S}^{d-1}, \psi), \boldsymbol{p}_L, \boldsymbol{p}_R\}$$

with $\boldsymbol{p}_L := (\boldsymbol{q}_L, h)$ and $\boldsymbol{p}_R := (\boldsymbol{q}_R, h)$ forms a symmetric tent over $(\mathcal{S}^{d-1}, \psi)$.

(2) Choose a hyperplane H' parallel to $\operatorname{aff}(P) \subset \mathbb{R}^d$ that separates $\{\boldsymbol{p}_L, \boldsymbol{p}_R\}$ and $\operatorname{lift}(\mathcal{S}^{d-1}, \psi)$. Let H'_+ be the closed halfspace with respect to H' that contains $\boldsymbol{p}_L$ and $\boldsymbol{p}_R$.

(3) Partition the boundary subdivision $\mathcal{S}^{d-1}$ into two cubical $(d-1)$-balls, we fix a normal vector of the hyperplane H and define

$$\begin{aligned}\mathcal{S}_L &:= \mathcal{S}^{d-1} \cap H_- \\ \mathcal{S}_R &:= \mathcal{S}^{d-1} \cap H_+.\end{aligned}$$

Furthermore, we define *waist* of $\mathcal{S}^{d-1}$ as $\mathcal{W}^{d-2} := \mathcal{S}^{d-1} \cap H$.

(4) Choose a point $\boldsymbol{p}'_R := \boldsymbol{p}_R + \varepsilon \boldsymbol{e}_{d+1}$ with $\varepsilon > 0$ such that $\boldsymbol{p}'_R$ is beneath all "left" facets of the tent, that are the facet contained in $\widetilde{H}_+$.

(5) Let $\boldsymbol{q}$ be the intersection of the segment $[\boldsymbol{p}_L, \boldsymbol{p}'_R]$ and the hyperplane $\widetilde{H}_+$.

(6) Choose a hyperplane H'' parallel to $\widetilde{H}$ that separates $\widetilde{H}$ and $H \cap \operatorname{conv}(\boldsymbol{p}_L, \operatorname{lift}(\mathcal{S}^{d-1}, \psi))$.

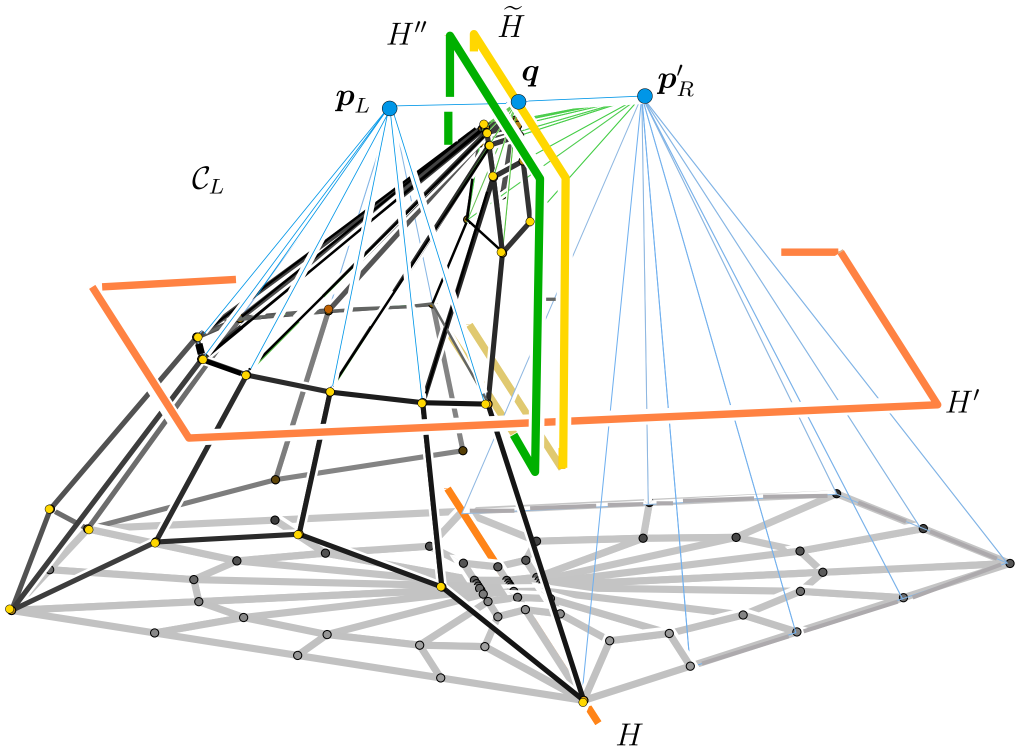


FIGURE 14.2: Mirrored encapsulated Schlegel cap

(7) Construct iterated truncated pyramids over $\mathcal{S}_L$ using the input sequence $\mathcal{I} := (\boldsymbol{p}_L, \boldsymbol{q}, \boldsymbol{p}'_R; H', H'', \widetilde{H})$. This means for every facet F of $P \cap H_+$ we construct the cubical d-ball

$$\mathcal{C}_F := \textsc{IteratedTruncPyr}((F \cap \mathcal{S}_L, \psi), (\boldsymbol{p}_L, \boldsymbol{q}, \boldsymbol{p}'_R; H', H'', \widetilde{H})).$$

The union of all these balls gives a cubical d-ball

$$\mathcal{C}_L \ := \ \bigcup \{\mathcal{C}_F \ : \ F \in \operatorname{fac}(P \cap H_+)\}\,.$$

Then the $(d+1)$-polytope P' is the convex hull of $\mathcal{C}_L$ and its mirrored copy $\mathcal{C}_L^M$ with respect to $\widetilde{H}$.

14.3 The encapsulated Schlegel cap

Roughly speaking, the *encapsulated Schlegel cap* is a half mirrored encapsulated Schlegel cap. If $\mathcal{B}'$ is a lifted cubical ball obtained by the mirrored encapsulated Schlegel cap of the lifted boundary subdivision $(\mathcal{S}^{d-1}, \psi)$, then the *encapsulated Schlegel cap* of $(\mathcal{S}^{d-1} \cap \widetilde{H}_+, \psi|_{\widetilde{H}_+})$ is given as

$$\textsc{EncapsulatedSchlegelCap}(\mathcal{S}^{d-1} \cap \widetilde{H}_+, \psi|_{\widetilde{H}_+}) := \mathcal{B}' \cap \widetilde{H}_+.$$

Hence the encapsulated Schlegel cap is a lifted cubification of a lifted subdivision of the "boundary of a polytope minus a single facet." For description above we obtain a precondition: Let P be a polytope, F a facet of P, and $(\mathcal{K}, \mu)$ a lifted cubical subdivision of the Schlegel complex $\mathcal{C}(\partial P)\backslash\{F\}$. Construct the complex $(\mathcal{S}^{d-1}, \psi)$ as the union of $(\mathcal{K}, \mu)$ and its mirrored copy with respect to the hyperplane $\widetilde{H} = H + \mathbb{R}\boldsymbol{e}_{d+1}$. Then $(\mathcal{S}^{d-1}, \psi)$ is required to be a lifted cubical subdivision of the convex hull of the union of P' and its mirrored copy with respect to $\widetilde{H}$.
By a slightly modification of the mirrored encapsulated Schlegel cap construction One can show, that this precondition can be dropped. (Substitute the used tent by a polytope with the same structure but different heights for $\boldsymbol{p}_L$ and $\boldsymbol{p}_R$.)

14.4 Cubical 5-polytopes

Theorem 14.1 *Assume that there is a cubical compact 3-manifold $\mathcal{M}$ and a codimension one immersion $j\colon \mathcal{M} \looparrowright \mathbb{R}^4$ such that all the following conditions are satisfied:*

- *The immersion j is a grid immersion.*
- *The immersion j is normal crossing.*
- *The immersion j is locally symmetric.*
- *The immersion j is locally symmetric in each edge, that is, for each edge e of $j(\mathcal{M})$ the grid immersion given as the restriction of the immersion j to the hyperplane H_e is locally symmetric, where H_e denotes the unique hyperplane through the barycenter of e and perpendicular to e.*
- *For each edge e of $j(\mathcal{M})$ the grid immersion given as the restriction of j to H_e is not PL-equivalent to the vertex star "single 6a."*

Then there is a cubical 5-polytope P with a dual manifold $\mathcal{M}'$ and associated immersion $y : \mathcal{M}' \looparrowright |\partial P|$ such that $\mathcal{M}'$ is a cubical subdivision of $\mathcal{M}$, and the immersions j (considered as a map to $\mathbb{R}^4 \cup \{\infty\} \cong S^4$) and y are PL-equivalent.

The proof is analog to our proof of our main theorem (Theorem 10.3).

Sketch of the proof. We show that there is a lifted cubical 4-ball $\mathcal{B}$ with a dual manifold immersion $y : \mathcal{M}' \looparrowright \mathbb{R}^4$ PL-equivalent to j. The cubical 5-polytope P is obtained as a lifted prism over $\mathcal{B}$.
Without loss of generality one can assume $j(\mathcal{M}) \subset |\mathcal{P}_4(\ell_1, \ell_2, \ell_3, \ell_4)|$. Then the ball $\mathcal{B}$ is constructed as a cubical subdivision of the raw complex $\mathcal{A}$ that is copy of the pile of cubes $\mathcal{P}_4(\ell_1 + 1, \ell_2 + 1, \ell_3 + 1, \ell_4 + 1)$ with all vertex coordinates shifted by $-\frac{1}{2}\mathbb{1}$.
For $k = 1$ to 4, we produce a lifted cubical subdivision $\mathcal{C}^k$ of the k-skeleton of $\mathcal{A}$, and the lifted cubical subdivision $\mathcal{B}$ of $\mathcal{A}$ arises as $\mathcal{B} := \mathcal{C}^4$. Recall the definition of our invariants (here for $k \in \{1, 2, 3, 4\}$).

(I_k1) *Consistency requirement.*
For every k-face $Q \in \mathcal{F}_k(\mathcal{A})$ and every facet F of Q, the induced subdivision $\mathcal{C}^k \cap F$ equals $\mathcal{C}^{k-1} \cap F$.

(I_k2) *PL equivalence requirement.*
For every k-face $Q \in \mathcal{F}_k(\mathcal{A})$ and every dual manifold $\mathcal{N}$ of Q (with boundary) the cubical subdivision $\mathcal{C}^k \cap Q$ has a dual manifold that is PL-equivalent to $j(\mathcal{N}) \cap Q$.

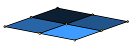
(I_k3) *Symmetry requirement.*
Every symmetry of $(Q, j(\mathcal{M}) \cap Q)$ for a k-face $Q \in \mathcal{F}_k(\mathcal{A})$ that is

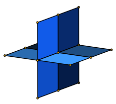
a symmetry of each sheet of $j(\mathcal{M}) \cap Q$ separately is a symmetry of $(Q, \mathcal{C}^k \cap Q)$.

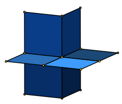
(I$_k$4) *Subcomplex requirement.*
For every diagonal symmetry hyperplane H_Q of a facet Q of $\mathcal{A}$ and every facet F of Q the (lifted) induced subdivision $\mathcal{C}^k \cap (F \cap H)$ is a (lifted) subcomplex of $\mathcal{C}^k$.

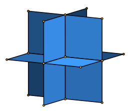
The lifted cubical subdivisions $\mathcal{C}^1$ and $\mathcal{C}^2$ are produced using our standard templates for these dimensions. Hence we can satisfy the all invariants (I$_k$1)–(I$_k$4) for $k \in \{1, 2\}$.
For dimension 3 we use the following set of templates:

- For following four types of vertex stars we take the cubification from our standard set of templates, namely a product with $\mathcal{U}_3$.


single 4a

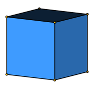

double 8a


double 8b

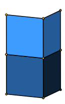

triple 12

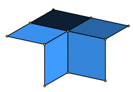
- For the cases "empty" and "single 3" we take the cubification from symmetric set of templates described in Section 10.5. (These two cubifications are constructed using cubical barycentric subdivisions.)

empty


single 3

- For the cases "single 4b" and "single 5" we construct a cubification using the mirrored encapsulated Schlegel cap.


single 4b


single 5

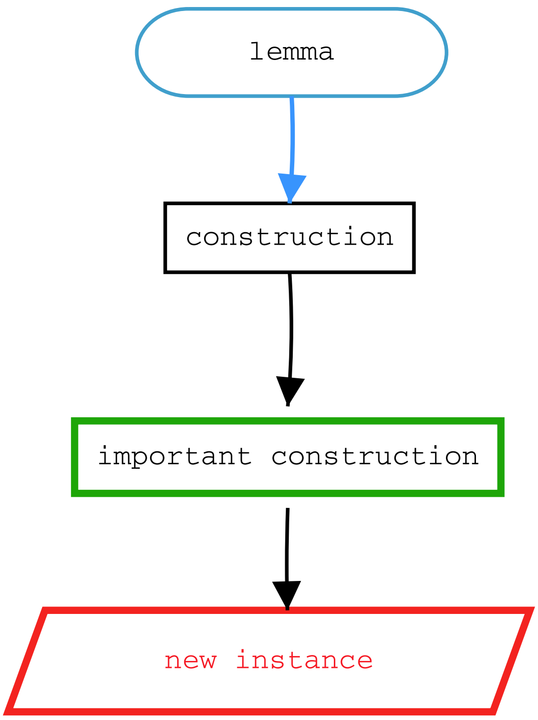
One can show that the invariants (I$_3$1)–(I$_3$4) are satisfied.
Hence the 4-dimensional cubifications can be constructed either as a product with $\mathcal{U}_3$, or using a mirrored encapsulated Schlegel cap. □

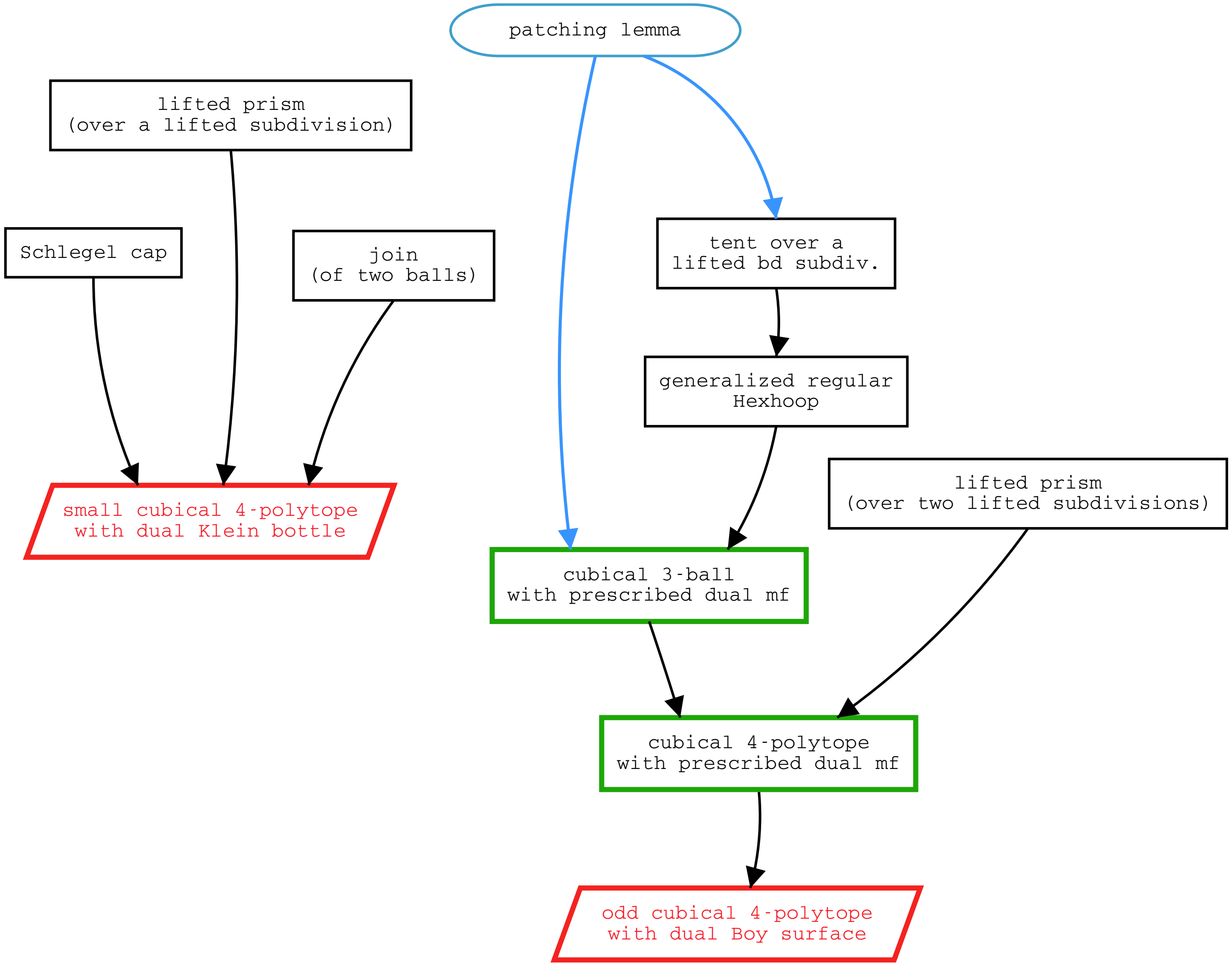
Part V

Appendix

“Big Picture” of constructions and results

Legend




patching lemma
lifted prism
(over a lifted subdivision)
Schlegel cap
join
(of two balls)
tent over a
lifted bd subdiv.
generalized regular
Hexhoop
lifted prism
(over two lifted subdivisions)
small cubical 4-polytope
with dual Klein bottle
cubical 3-ball
with prescribed dual mf
cubical 4-polytope
with prescribed dual mf
odd cubical 4-polytope
with dual Boy surface

Zusammenfassung

Polytope (konvexe, beschränkte Polyeder beliebiger Dimension) sind klassische Objekte der *Kombinatorischen Geometrie*. Eine der bekanntesten — und auch gut verstandenen — Klassen von Polytopen bilden die *simplizialen Polytope*, dies sind Polytope deren Seiten alle Simplices der entsprechenden Dimension sind. Anlog werden *kubische Polytope* definiert als Polytope deren Seiten alle die kombinatorische Struktur von Würfeln der entsprechenden Dimension besitzen.
Es ist bekannt, dass jedes kubische d-dimensionale Polytop eine Immersion einer abstrakten geschlossenen $(d-1)$-dimensionalen Mannigfaltigkeit in den Rand des Polytopes bestimmt. In dem Falle von kubischen 4-Polytopen sind die Zusammenhangskomponenten der dualen Mannigfaltigkeiten *Flächen*, geschlossene 2-dimensionale Mannigfaltigkeiten (d.h. kompakt und ohne Rand).
Für den allgemeineren Fall von kubischen PL $(d-1)$-Spähren haben Babson & Chan gezeigt, dass jeder PL-Typ einer normal-schneidenden Immersion einer geschlossenen $(d-2)$-Mannigfaltigkeit in eine $(d-1)$-Spähre als eine Komponente der dualen Mannigfaltigkeit einer kubischen PL $(d-1)$-Spähre auftaucht.
Für kubische Polytope war bisher kein analoges Resultat bekannt. Dies geht zurück auf einen Mangel an flexiblen Konstruktionstechniken für kubische Polytope und allgemeiner, für kubische Bälle, wie sie z.B. in der Hexaedernetzgenerierung (CAD) verwendet werden.
In dieser Arbeit entwickeln wir neue Konstruktionstechniken für kubische Polytope und kubische Bälle. Eine unserer kompliziertesten Konstruktionen verallgemeinert das "Hexhoop template" welches aus der Hexaedernetzgenerierung bekannt ist.

Mit den beschriebenen Konstruktionen erzielen wir folgende neue Resultate:

- Ein relativ elementare Konstruktion liefert ein kleines kubisches 4-Polytop (mit 72 Ecken und 62 Facetten) dessen duale Mannigfaltigkeit eine *Kleinsche Flasche* als eine Komponente enthält. Dies ist die erste bekannte Instanz eines kubischen 4-Polytops mit einer nicht orientierbaren dualen

Mannigfaltigkeit, was die Frage von Hetyei nach der Existenz von nicht "Kanten-orientierbaren" kubischen 4-Polytopen beantwortet.

- Allgemeiner können wir zeigen, dass jeder PL-Typ einer normal-schneidenden Immersion einer Fläche in eine 3-Spähre als eine Komponente der dualen Mannigfaltigkeit eines kubischen 4-Polytops auftaucht. Im Falle einer nicht orientierbaren Fläche ungeraden Geschlechts liefert dies ein kubisches 4-Polytop mit einer ungeraden Anzahl von Facetten.
- Wir konstruieren explizit eine Instanz eines kubischen 4-Polytops mit 19 520 Ecken und 18 333 Facetten mit einer dualen Boyschen Fläche, einer Immersion der projektiven Ebene mit einem Schnittpunkt vom Grad 3.
- Mit Hilfe von Schlegeldiagrammen folgt daraus, dass jeder geometrische Typ eines 3-dimensionalen Würfels eine Kubifizierung (d.h. eine Zerlegung in Würfel, ohne Unterteilung des Randes) mit einer geraden Anzahl von Würfeln besitzt. Dies impliziert, dass der kubische Flip-Graph für geometrische Hexaedernetze mindestens 2 Komponenten besitzt, was Fragen von Eppstein und Thurston beantwortet.

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