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Randomized Construction of Complexes with Large Diameter

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Abstract
We consider the question of the largest possible combinatorial diameter among pure dimensional and strongly connected $(d-1)$-dimensional simplicial complexes on n vertices, denoted $H_s(n, d)$. Using a probabilistic construction we give a new lower bound on $H_s(n, d)$ that is within an $O(d^2)$ factor of the upper bound. This improves on the previously best known lower bound which was within a factor of $e^{\Theta(d)}$ of the upper bound. We also make a similar improvement in the case of pseudomanifolds.

Keywords Simplicial complex · Diameter · Hirsch conjecture · Probabilistic method

Mathematics Subject Classification 05C12 · 05C15 · 05D40 · 05E45

1 Introduction

Given a pure simplicial complex C, one may define the *dual graph* $\mathrm{G}(C)$ as the graph whose vertices are the top-dimensional faces of C and whose edges are pairs of top-dimensional faces that intersect at a face of codimension 1. Using standard terminology we refer to top-dimensional faces of C as facets and codimension 1 faces as ridges. From the definition of $\mathrm{G}(C)$, one defines the combinatorial diameter of C as the graph diameter of $\mathrm{G}(C)$. Of course, in general this may be infinite as $\mathrm{G}(C)$ need

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not be connected. Thus, here we consider the case where C is *strongly connected*, that is exactly the case that $\mathrm{G}(C)$ is connected.

The best known situation in which the combinatorial diameter is considered is in the case when C is a simplicial polytope. In this situation there is the now-disproved [9] Hirsch conjecture which stated that if C is a simplicial polytope of dimension d with n vertices, then the diameter of C is at most $n-d$. While this is now known to be false, the polynomial Hirsch conjecture remains open. The polynomial Hirsch conjecture asserts that the combintorial diameter of a simplicial d-polytope on n vertices is bounded by a polynomial in n and d.

On the other hand, the question can also be considered for other classes of simplicial complexes as a purely combinatorial question. Following the notation of [3], we define $H_s(n, d)$ to be the largest combinatorial diameter of any strongly connected, pure $(d-1)$-dimensional simplicial complex on n vertices. Observe that d is not the dimension of the complex, rather $d-1$. This is to respect the notation for polyhedra, in which d refers to the dimension of the ambient space. In general, one could think of the d as the number of vertices in a facet. In [3], Criado and Santos prove the following:

Theorem 1.1 [3, Thm. 1.2] *For every $d \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ such that*

$$H_s(n, d) \geq \frac{n^{d-1}}{(d+2)^{d-1}} - 3.$$

Combined with a trivial upper bound of $n^{d-1}/((d-1)(d-1)!)$, which appears as [10, Corr. 2.7], this shows that $H_s(n, d) = \Theta_d(n^{d-1})$. This result of [3] gives the previously best known upper and lower bounds on $H_s(n, d)$, but as d tends to infinity the ratio between the upper bound and the lower bound grows like $e^{\Theta(d)}$. Here we take a step further toward establishing the true value of $H_s(n, d)$ in decreasing this ratio to $\Theta(d^2)$. Specifically, we prove the lower bound of the following; the upper bound is still the trivial upper bound:

Theorem 1.2 *Fix $d \geq 3$, then $H_s(n, d)$ satisfies*

$$\frac{1-o_n(1)}{4ed^2} \leq \frac{H_s(n, d)d!}{n^{d-1}} \leq \frac{d}{d-1}.$$

Moreover, our proof takes a different approach than the deterministic construction in [3]. We instead use the probabilistic method in a way similar to the main result of [7].

We make the comparison between what we do here and what is done in [7] more precise when we outline the proof below. At a basic level, there is a deterministic step and then a probabilistic step. We start with the dimension $d-1 \geq 2$ and a positive integer L that we want to realize as the combinatorial diameter of our construction. In the deterministic step we build a complex on $\Theta(L)$ vertices that has diameter L. In the probabilistic step we take a quotient of the complex that preserves the diameter, drops the number of vertices to $\Theta(L^{1/(d-1)})$, and remains a simplicial complex.

This approach also works for the class of pseudomanifolds, which is also considered in [3]. A pseudomanifold without boundary is a pure dimensional and strongly connected simplicial complex so that every ridge is contained in exactly two facets.

We denote by $H_{\text{pm}}(n, d)$ the maximum diameter of all $(d-1)$-dimensional pseudo-manifolds on n vertices. A result of [3] shows that $H_{\text{pm}}(n, d) = \Theta_d(n^{d-1})$, but again the ratio between the upper bound and the lower bound is $e^{\Theta(d)}$. Here we improve this to $\Theta(d^3)$:

Theorem 1.3 *Fix $d \geq 3$, then $H_{\text{pm}}(n, d)$ satisfies*

$$\frac{1 - o_n(1)}{4ed^4} \leq \frac{H_{\text{pm}}(n, d)d!}{n^{d-1}} \leq \frac{6}{d+1}.$$

In this case we slightly improve on the upper bound too by using the fact that $\mathrm{G}(C)$ is d-regular when C is a $(d-1)$-pseudomanifold.

2 Proof of the Main Result

The approach will be as in [7], the goal of which is to construct simplicial complexes on few vertices with large torsion groups in homology. The main result of [7] shows that for any finite abelian group G and dimension d, there exists a simplicial complex on $O_d(\log^{1/d}|G|)$ vertices which realizes G as its top cohomology group. The construction in [7] is partially deterministic and partially probabilistic. The deterministic piece constructs a simplical complex C on $O_d(\log|G|)$ vertices that realizes G as the top cohomology group. The probabilistic piece is to use the Lovász local lemma to color the vertices of C in a way that allows us to take a quotient of the complex by the coloring to obtain a simplicial complex on the right number of vertices, but without changing the top cohomology group. This technique was further refined in [6]. Both in [6,7] and here, once we have found a good coloring and a good initial construction the quotient is taken according the following, combinatorial definition.

Definition 2.1 [7] If C is a simplicial complex with a coloring f of $V(C)$ we define the *pattern* of a face to be the multiset of colors on its vertices. If f is a proper coloring, in the sense that no two vertices connected by an edge receive the same color, we define the *pattern complex* C/f to be the simplicial complex on the set of colors of f so that a subset P of the colors of f is a face of C/f if and only if there is a face of C with P as its pattern.

We also introduce the term *pattern classes of f on C* to refer to equivalence classes under the equivalence relation given by faces of C with the same pattern under f.

Our initial construction, that is the deterministic step, for Theorem 1.2 is quite simple. For dimension $d-1$ fixed, we define the *straight* $(d-1)$*-corridor on N vertices* to be the pure complex $\mathrm{SC}(N, d)$ on $[N] := \{1, 2, \ldots, N\}$ where the facets are given by $\{1, \ldots, d\}, \{2, \ldots d+1\}, \{3, \ldots, d+2\}, \ldots, \{N-d+1, \ldots, N\}$. Clearly, $\mathrm{SC}(N, d)$ has N vertices and its dual graph is a path of length $N-d$, so the diameter of $\mathrm{SC}(N, d)$ is $N-d$. For the probabilistic step we want to color the vertices by a coloring f with $O_d(N^{1/(d-1)})$ colors so that $\mathrm{SC}(N, d)/f$ still has diameter $N-d$. In fact, we take a quotient so that the resulting complex will still have the same dual graph as $\mathrm{SC}(N, d)$. The number of vertices will decrease drastically after this identification.

The rule that the coloring f should satisfy is that it should be a proper coloring and it should assign no pair of ridges the same pattern. This rule for coloring vertices is exactly the same as the rule for the result in [7]. If C is a simplicial complex and f is a coloring that colors every ridge with a unique pattern, then C and C/f will have the same diameter; in fact, they will have the same dual graph. Also under this condition, if C is a pseudomanifold then so is C/f. These facts can be proven by comparing the $(d-1)$st boundary matrix of C and of C/f.

Recall that the i-dimensional boundary matrix ∂_i over $\mathbb{Z}/2\mathbb{Z}$ of a simplicial complex C is a matrix over $\mathbb{Z}/2\mathbb{Z}$ with columns indexed by the i-dimensional faces of C, rows indexed by $(i-1)$-dimensional faces of C, and the (σ, τ)-entry equal to 1 if and only if $\sigma \subseteq \tau$.

Lemma 2.2 *If C is a $(d-1)$-simplicial complex and f is a proper coloring of C such that every ridge of C has a unique pattern, then the top-dimensional boundary matrix of C over $\mathbb{Z}/2\mathbb{Z}$ is the same as the top-dimensional boundary matrix of C/f. In particular, if C is a pseudomanifold then so is C/f, and C and C/f have the same dual graph.*

Proof We have that $\phi\colon V(C) \to V(C/f)$, $v \mapsto f(v)$, induces a simplicial map and by the assumption on f it is injective on the set of ridges and on the set of facets. Moreover, it is clear that for any ridge σ and any facet τ one has $\sigma \subseteq \tau$ if and only if $\phi(\sigma) \subseteq \phi(\tau)$. From this it is immediate that over $\mathbb{Z}/2\mathbb{Z}$, both C and C/f have the same top-dimensional boundary matrix.

A $(d-1)$-pseudomanifold is a $(d-1)$-complex C such that then every row of ∂_{d-1} of C has exactly two non-zero entries, so if C is a pseudomanifold then so is C/f. The dual graph of C may be read off from ∂_{d-1} as the off-diagonal entries of $\partial_{d-1}^T \partial_{d-1}$ match those of the adjacency matrix of $\mathrm{G}(C)$, so C and C/f have the same dual graph. □

For the probabilistic step of our proof we want to properly color the vertices in $\mathrm{SC}(N,d)$ using $O_d(N^{1/(d-1)})$ colors so that no pair of ridges receives the same pattern. Then the pattern complex will still have diameter $N-d$. As in [6,7] the coloring is done in steps. In particular, there will be two steps in the coloring process. The main reason for this is that the first step allows us to handle pairs of ridges intersecting one another while the second step will handle ridges that do not intersect one another.

For fixed dimension $d-1$, the two step approach first colors $\mathrm{SC}(N,d)$ by a coloring f with $O_d(1)$ colors so that no vertices that are at distance at most two from one another in the 1-skeleton of $\mathrm{SC}(N,d)$ receive the same color. This condition implies that there are no pair of intersecting ridges receiving the same pattern. In the second step, one uses the Lovász local lemma to color the vertices of $\mathrm{SC}(N,d)$ by a coloring g having $O_d(N^{d-1})$ colors, so that no pair of disjoint ridges receive the same pattern. The product of the two colorings (f,g), assigning the pair $(f(v), g(v))$ to each vertex v, is then the final coloring that we use.

There is another advantage of this two step approach as well. Namely, we may regard the second coloring as a refinement of the first. Indeed, if two disjoint ridges receive different patterns by f, then it does not matter what happens on them under the coloring by g as they will still receive different patterns. What this allows for is that if we use *more* colors for f, though still only a number depending on d, we can save on the number of colors we need for g in a way that is a net reduction in the number of colors in (f, g).

It is not too hard to check that the 1-skeleton of $\mathrm{SC}(N, d)$ has maximum vertex degree $2(d-1)$, and so by a greedy coloring with at most $[2(d-1)]^2+1$ colors we may color the vertices so that no pair at distance two receives the same color. This can be refined using a coloring g obtained randomly with the Lovász local lemma, but this turns out to give a worse lower bound on $H_s(n, d)$ than the bound in [3], though it is still $\Theta_d(n^{d-1})$.

Here, instead, it is better to take a random coloring for f which we describe below. This will ultimately allow for fewer colors for g and overall.

2.1 The First Coloring

The purpose of this section is to prove the following:

Proposition 2.3 *Let $d \geq 3$, $c_1 > 6(d-1)$, and $\epsilon > 0$ be fixed. There is a coloring of $\mathrm{SC}(N, d)$ with c_1 colors such that all pattern classes of ridges have size at most $(1+\epsilon)N(d-1)\big/\binom{c_1}{d-1}$ and no pair of intersecting ridges have the same pattern for N large enough.*

Our proof makes use of the following result on Markov chains [8, Thm. 1.10.2]:

Theorem 2.4 *Let $(X_n)_{n\geq 0}$ be an irreducible Markov chain on a finite set of states S, that is (X_n) is a Markov chain on S in which every state can reach any other with nonzero probability in an arbitrary number of steps. Then*

$$\Pr\left(\frac{V_i(n)}{n} \to \frac{1}{m_i} \text{ as } n \to \infty\right) = 1$$

for every $i \in S$, where m_i is the expected number of steps to go from state i to itself, and $V_i(n)$ is the ***number of visits to*** i ***before*** n ***steps****.*

Proof of Proposition 2.3 We assign a sequence of colors to the vertices $v_1, \ldots, v_N$ of $\mathrm{SC}(N, d)$ by a greedy randomized approach. To each vertex we assign a uniformly random color of $[c_1]$ that was not used in the last $2(d-1)$ vertices. This is always possible, as $c_1 > 6(d-1) > 2(d-1)$.

If two intersecting ridges share a pattern, in particular it means that two vertices, one from each ridge but not from the intersection, have the same color. Since the facets of $\mathrm{SC}(N, d)$ are consecutive d-sequences of vertices, two vertices from two intersecting ridges have to be at most $2(d-1)$ units apart with respect to the ordering on the vertices. This proves that the random procedure will always produce a coloring such that no pair of intersecting ridges have the same pattern.

It remains to prove that the pattern classes have the claimed bounded size. Observe that the randomized coloring procedure defined above defines a Markov chain on the set

$$S = \left\{x \in [c_1]^{2(d-1)} : x_i \neq x_j \ \forall i, j \in [2(d-1)],\ i \neq j\right\},$$

coming from the tuple of colors on $2(d-1)$ consecutive vertices. In this Markov chain, we can get from any state $x \in S$ to any other $y \in S$ by first replacing the colors of x with colors not in x or y, which is possible because $c_1 > 6(d-1)$ and each state has $2(d-1)$ colors, and then replacing these new colors with the colors of y. This takes $4(d-1)$ steps independently of the initial and final states. Therefore, the Markov chain is irreducible.

By Theorem 2.4, as N grows large enough, the proportion of visits to each state approximates the expected value arbitrarily, with probability 1. By symmetries via permutations of colors, this proportion of visits has to be the same for each state, that is,

$$\frac{1}{\binom{c_1}{2(d-1)}(2(d-1))!} = \frac{(c_1-2(d-1))!}{c_1!}.$$

This means that for $\epsilon > 0$ and N large enough there is a coloring f of $\mathrm{SC}(N,d)$ by c_1 colors such that for every tuple of $2(d-1)$ colors of $[c_1]$, the proportion of tuples of $2(d-1)$ consecutive vertices colored by that tuple of colors is at most $(1+\epsilon)(c_1-2(d-1))!/c_1!$

Now each ridge σ of $\mathrm{SC}(N,d)$ is given by $\sigma_{i,j} := \{i+1,\dots,i+d\} \setminus \{i+j\}$ for some $i \in \{0,\dots,N-d\}$ and $j \in \{1,\dots,d-1\}$ except for the ridge $\{1,\dots,d-1\}$. For each ridge $\sigma_{i,j}$ in $\mathrm{SC}(N,d)$ with $i \geq d-2$ we may associate the set of $2(d-1)$ consecutive vertices ending at $i+d$. This canonically associates almost all ridges of $\mathrm{SC}(N,d)$ with a list of $2(d-1)$ consecutive vertices and every list of $2(d-1)$ consecutive vertices with a set of $d-1$ ridges.

It follows that if we pick a tuple of $2(d-1)$ vertices $[v_1,\dots,v_{2(d-1)}]$ uniformly at random and then pick one of the $d-1$ ridges obtained by deleting a vertex from $[v_{d-1},\dots,v_{2(d-1)}]$ different from $v_{2(d-1)}$, this procedure generates a ridge, uniformly at random, from all ridges $\sigma_{i,j}$ with $i \geq d-2$ of $\mathrm{SC}(N,d)$. For any fixed pattern π for a ridge we therefore have that under f the proportion of ridges $\sigma_{i,j}$ with $i \geq d-2$ colored by π is at most

$$(d-1)(d-1)! \cdot \frac{(c_1-(d-1))!}{(c_1-2(d-1))!} \cdot \frac{(1+\epsilon)(c_1-2(d-1))!}{c_1!} \cdot \frac{1}{d-1}.$$

This can be seen because we may specify first which ridge we will choose from a selected list of consecutive vertices, that is which vertex from among $v_{d-1},\dots,v_{2(d-1)-1}$ we will remove; there are $d-1$ choices. From here there will be $(d-1)!$ ways to map π to the selected positions and then $(c_1-(d-1))!/(c_1-2(d-1))!$ choices for the remaining positions to build a valid colored tuple of consecutive vertices. Each of these colored tuples will be selected with probability at most $(1+\epsilon)(c_1-2(d-1))!/c_1!$ by the assumptions on f. But every tuple is associated with $d-1$ ridges, so we divide by $d-1$ to correct for this overcount. Now we further simplify:

$$(d-1)(d-1)! \cdot \frac{(c_1-(d-1))!}{(c_1-2(d-1))!} \cdot \frac{(1+\epsilon)(c_1-2(d-1))!}{c_1!} \cdot \frac{1}{d-1} = \frac{1+\epsilon}{\binom{c_1}{d-1}}.$$

So no pattern on ridges will appear with proportion greater than $(1+\epsilon)/\binom{c_1}{d-1}$ among ridges $\sigma_{i,j}$ with $i \geq d-2$. Of course, this ignores patterns on ridges $\sigma_{i,j}$ with $i < d-2$ and $\{1, \ldots, d-1\}$, but this is a negligible proportion of ridges as N tends to infinity. The claim follows as we observe that $\mathrm{SC}(N, d)$ has $1 + (N-d+1)(d-1)$ ridges. □

We can generalize this result for faces of any other codimension. This will be relevant for the proof of Theorem 1.3.

Corollary 2.5 *Fix d, and set $k \in \{1, \ldots, d-1\}$, $c_1 > 6(d-1)$, and $\epsilon > 0$. There is a coloring of $\mathrm{SC}(N, d)$ with c_1 colors such that all pattern classes of codimension k faces have at most size $(1+\epsilon)N\binom{d-1}{k}/\binom{c_1}{d-k}$ and no pair of intersecting $(d-k-1)$-faces have the same pattern for N large enough.*

Proof We replicate the strategy of the proof of Proposition 2.3. As before, a randomized greedy algorithm picks a color that has not been used in the last $2(d-1)$ vertices as before, which is enough to guarantee that no pair of intersecting $(d-k-1)$-faces have the same pattern.

The Markov chain remains unchanged from the proof of Proposition 2.3; it is still defined on tuples of $2(d-1)$ colors. So each state corresponds to a coloring of $2(d-1)$ consecutive vertices, but now each state of S is associated to $\binom{d-1}{k}$ $(d-k-1)$-faces that end at the final vertex of the consecutive list of $2(d-1)$ vertices. This comes from the fact that a $(d-k-1)$-face is a sequence of d consecutive vertices with k of them removed, but we do not remove the last one in order to associate each $(d-k-1)$-face to a unique state.

By the same reasoning as the proof of Proposition 2.3, which is the $k = 0$ case, we have that for N large enough, every pattern class will have at most $(1+\epsilon)N\binom{d-1}{k}/\binom{c_1}{d-k}$ faces of codimension k. □

2.2 The Second Coloring

We are now ready to refine a coloring of $\mathrm{SC}(N, d)$ satisfying the conclusions of Proposition 2.3 so that no pair of ridges receive the same pattern. We do this via Proposition 2.6, which we state in a fairly general way in order to apply it to the pseudomanifold case later.

Proposition 2.6 *If C is a $(d-1)$-simplicial complex and there is a coloring of C with at most c_1 colors such that no pattern class of ridges has size more than S, no intersecting ridges receive the same pattern, and for any ridge σ, there are at most t other ridges that intersect σ, then there is a refinement of the coloring with at most $c_1\left\lceil \sqrt[d-1]{e(2tS+1)}\right\rceil$ colors and every ridge colored uniquely.*

As is the strategy in [6,7], we will prove this proposition from the Lovász local lemma, originally due to Erdős and Lovász [4], which we recall in the symmetric version from [1] below:

Theorem 2.7 (Lovász local lemma) *Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is independent of all other events A_j except for some subset of $\{A_1, \ldots, A_n\}$ with at most m elements. Suppose also that $\Pr(A_i) \le p$ for all $1 \le i \le n$. If $pe(m+1) \le 1$ then* $\Pr\left(\bigwedge_{i=1}^{n} \overline{A}_i\right) > 0$.

Proof of Proposition 2.6 Let C be the complex described and colored by $f\colon V(C) \to [c_1]$ so that every pattern class of ridges has size at most S and no intersecting ridges receive the same pattern. We will find a second coloring g so that the final coloring satisfying the conclusions of the statements will be (f, g). The coloring g will be constructed randomly by choosing for each vertex a random color uniformly from a set of c_2 colors to be determined later.

For σ, τ, two ridges of C, with $f(\sigma) = f(\tau)$, let $A_{\sigma,\tau}$ denote the event that σ and τ receive the same color by (f, g). Clearly, if

$$\Pr\left(\bigwedge_{\substack{(\sigma,\tau)\text{ ridges of C}\\ \text{s.t. } f(\sigma)=f(\tau)}} \overline{A}_{\sigma,\tau}\right) > 0,$$

then there exists a choice for g with c_2 colors such that (f, g) colors every ridge uniquely. Indeed, for σ, τ with $f(\sigma) \neq f(\tau)$, refining the coloring by g will not cause σ and τ to receive the same pattern under (f, g). We apply the local lemma to $A_{\sigma,\tau}$.

First, for disjoint σ, τ such that $f(\sigma) = f(\tau)$, there is a bijection $\phi\colon \sigma \to \tau$ sending each vertex in σ to the unique vertex in τ that receives the same coloring under f. Hence (f, g) gives σ and τ the same pattern if and only if $g(v) = g(\phi(v))$ for all $v \in \sigma$. Then we have the following bound on the probability of $A_{\sigma,\tau}$:

$$\Pr(A_{\sigma,\tau}) \le \left(\frac{1}{c_2}\right)^{d-1}.$$

We now bound for fixed (σ, τ), still with $f(\sigma) = f(\tau)$, the number of pairs $A_{\sigma',\tau'}$ such that $A_{\sigma',\tau'}$ is not independent of $A_{\sigma,\tau}$ *and such that* $f(\sigma') = f(\tau')$ also. Clearly if $(\sigma \cup \tau) \cap (\sigma' \cup \tau') = \emptyset$, then $A_{\sigma,\tau}$ is independent of $A_{\sigma',\tau'}$. For (σ, τ) fixed we bound from above the number of pairs (σ', τ') such that $(\sigma \cup \tau) \cap (\sigma' \cup \tau') \neq \emptyset$. If $\sigma' \cup \tau'$ is to intersect $\sigma \cup \tau$, then without loss of generality we may assume that σ' intersects either σ or τ. There are at most t choices for σ' such that σ' intersects σ, and from here τ' may be selected to be any ridge with the same pattern as σ', and there are S such ridges. So for (σ, τ) there are at most tS choices for (σ', τ') such that σ' intersects σ and another at most tS choices such that σ' intersects τ. Therefore, $A_{\sigma,\tau}$ is independent of all but at most $2tS$ other events $A_{\sigma',\tau'}$. It follows that choosing c_2 large enough so that

$$e\frac{2tS+1}{c_2^{d-1}} \le 1,$$

will imply by the Lovász local lemma that with positive probability $A_{\sigma,\tau}$ fails to hold simultaneously for all pairs of ridges σ, τ with $f(\sigma) = f(\tau)$. By setting $c_2 = \left\lceil \sqrt[d-1]{e(2tS+1)} \right\rceil$, we complete the proof. □

With Propositions 2.3 and 2.6 now proven we are ready to prove Theorem 1.2.

Proof of the lower bound from Theorem 1.2 Let $d \geq 3$ be fixed. The result is asymptotic in n, so here we fix $c_1 > 6(d-1)$ and $\epsilon > 0$, and we will show the constant factor in the lower bound emerges as $c_1 \to \infty$ and $\epsilon \to 0$.

Let N be large enough so that there exists a coloring of $\mathrm{SC}(N, d)$ with c_1 colors such that all pattern classes of ridges have size at most $S = (1+\epsilon)(d-1)N\big/\binom{c_1}{d-1}$, and let $f\colon V(\mathrm{SC}(N, d)) \to [c_1]$ be such a coloring. To apply Proposition 2.6 we need to find an upper bound for the maximum number of ridges that intersect any given ridge, that is, in the notation of the proposition, we should find a suitable value for t in the case of $\mathrm{SC}(N, d)$.

The facets of $\mathrm{SC}(N, d)$ are d consecutive vertices in $[N]$, so the ridges are obtained by taking any d consecutive elements of $[N]$ and removing exactly one element. For any fixed ridge σ, there is a facet $F := F(\sigma)$ such that σ is contained in F. From the structure of $\mathrm{SC}(N, d)$, we have that F intersects at most $2d$ other facets and each of these facets contains d ridges. Thus σ intersects at most $2d^2$ other ridges.

With c_1, S, and t determined we apply Proposition 2.6 to say that there is a coloring of $\mathrm{SC}(N, d)$ by at most c_1c_2 colors where $c_2 = \left\lceil \sqrt[d-1]{e(2tS+1)} \right\rceil$ such that every ridge has a unique pattern. Let $(f, g)\colon V(\mathrm{SC}(N, d)) \to [c_1c_2]$ be such a coloring. By Lemma 2.2 the complex $C := \mathrm{SC}(N, d)/(f, g)$ has diameter equal to the diameter of $\mathrm{SC}(N, d)$. Therefore, the diameter of C is $N - d$ and the number of vertices of C is at most

$$\begin{aligned} c_1c_2 &= c_1\left\lceil \sqrt[d-1]{e(2tS+1)} \right\rceil \\ &= c_1\left\lceil \sqrt[d-1]{e\left(2 \cdot 2d^2(1+\epsilon)(d-1)N\Big/\binom{c_1}{d-1} + 1\right)} \right\rceil. \end{aligned}$$

Now as c_1 tends to infinity, $c_1^{d-1}\big/\binom{c_1}{d-1}$ tends to $(d-1)!$ Thus, given $\delta > 0$ we may set ϵ small enough and c_1 large enough so that for all N large enough we have a complex on at most $(1+\delta)\sqrt[d-1]{(d-1)!4d^3(N-d)e}$ vertices with diameter $N - d$. Letting n be the number of vertices in this complex, we have a complex on n vertices with diameter at least

$$\left(\frac{1}{1+\delta}\right)^{d-1} \frac{n^{d-1}}{4ed^2d!}.$$

This proves the theorem as δ is arbitrary. □

3 Pseudomanifold Case

Here we prove Theorem 1.3. In this case we establish a tighter upper bound than in the general case based on the observation that the dual graph of a $(d-1)$-dimensional pseudomanifold is a d-regular graph. For arbitrary regular graphs there is the following bound on the diameter; this result is a special case of Theorem 5 of Caccetta and Smyth [2] and it also appears in a different form earlier in work of Moon [5].

Theorem 3.1 [2,5] *Let G be a connected d-regular graph on n vertices. Then*

$$\operatorname{diam}(G) \leq \frac{3n}{d+1}.$$

From this it follows that we can give an upper bound to the combinatorial diameter of a pseudomanifold as in Theorem 1.3.

Proof of the upper bound from Theorem 1.3 Here we use the notation f_i for a complex to denote the usual f-vector entry. Let C be a $(d-1)$-dimensional pseudomanifold on n vertices, then $\mathrm{G}(C)$ is a d-regular graph on $f_{d-1}(C)$ vertices. By Theorem 3.1 we have that

$$\operatorname{diam}(\mathrm{G}(C)) \leq \frac{3f_{d-1}(C)}{d+1}.$$

Now, since C is a $(d-1)$-dimensional pseudomanifold on n vertices,

$$df_{d-1}(C) = 2f_{d-2}(C) \leq 2\binom{n}{d-1}.$$

Therefore,

$$\operatorname{diam}(\mathrm{G}(C)) \leq \frac{6}{d(d+1)}\binom{n}{d-1} \leq \frac{6n^{d-1}}{(d+1)!}. \qquad \square$$

We prove the lower bound of Theorem 1.3 by the same method as Theorem 1.2, but with a different starting complex. For given d, we will start with the top-dimensional boundary of $\mathrm{SC}(N, d+1)$, denoted $\partial\mathrm{SC}(N, d+1)$ and color the vertices as in Lemma 2.2. It is clear that $\partial\mathrm{SC}(N, d+1)$ is a pseudomanifold. In fact it is a triangulated sphere as it has the combinatorial type of a stacked polytope.

We now proceed to describe the facets of $\partial\mathrm{SC}(N, d+1)$. These facets are the ridges of $\mathrm{SC}(N, d+1)$ that are contained in exactly one facet of $\mathrm{SC}(N, d+1)$. Recall that facets of $\mathrm{SC}(N, d+1)$ are sequences of $d+1$ consecutive vertices. Then, ridges of $\mathrm{SC}(N, d+1)$ are sequences of $d+1$ vertices with one of them missing. If we remove the first or the last vertex of a facet, then the corresponding ridge of $\mathrm{SC}(N, d+1)$ is contained in two facets, unless it is the first d vertices or the last d vertices. This gives us three types of facets of $\partial\mathrm{SC}(N, d+1)$. The first two types each correspond to a single facet and are

$$\alpha = \{1, \ldots, d\} \quad \text{and} \quad \omega = \{N - d + 1, \ldots, N\}.$$

We call these facets of $\partial \mathrm{SC}(N, d+1)$ respectively the *initial facet* and the *final facet*. We refer to the remaining facets of $\partial \mathrm{SC}(N, d+1)$ as *middle facets*. These are given by

$$\tau_{i,j} = \{i, i+1, i+2, \ldots, i+d\} \setminus \{i+j\},$$

for $i \in [N-d]$ and $j \in \{1, \ldots d-1\}$,

From now on we are working with $\partial \mathrm{SC}(N, d+1)$, so when we talk about *facets* or *ridges*, we are referring to that complex; facets are of dimension $d-1$ and ridges are of dimension $d-2$. The diameter of $\partial \mathrm{SC}(N, d+1)$ is less obvious than that of $\mathrm{SC}(N, d)$, so we first prove the following lemma to establish a lower bound on the diameter of $\partial \mathrm{SC}(N, d+1)$.

Lemma 3.2 *The diameter of* $\partial \mathrm{SC}(N, d+1)$ *is at least* $(d-1)N/d - d$.

Proof We will use a potential function to provide a lower bound on the length in the dual graph of any path from α to ω. Observe that any path from α to ω will only have middle facets between α and ω. We set a potential function p over the middle facets of $\partial \mathrm{SC}(N, d+1)$, defined by $p(\tau_{i,j}) = i - j/(d-1)$. We will show that every move from one middle facet to an adjacent middle facet increases this potential at most by $1 + 1/(d-1)$.

A move in the dual graph corresponds to removing one vertex from a facet, and adding a new vertex. The set of vertices shared by the initial and final facets then form the ridge in $\partial \mathrm{SC}(N, d+1)$ connecting the two.

Since $\partial \mathrm{SC}(N, d+1)$ is a pseudomanifold, every facet is adjacent to exactly d facets, corresponding to d choices of a vertex to remove. Moreover, the new vertex to add is uniquely determined because in a pseudomanifold every ridge is in exactly two facets. For a move between middle facets along a path from α to ω there are three cases to consider:

- We remove the last vertex of $\tau_{i,j}$. In this case, the adjacent facet is $\tau_{i-1,j+1}$ if $j \neq d-1$, or $\tau_{i-2,1}$ if $j = d-1$. In both cases, the potential decreases.
- We remove the first vertex. If $j = 1$, the adjacent facet is $\tau_{i+2,d-1}$. If $j \neq 1$, the next facet is $\tau_{i+1,j-1}$. In both cases, the increase in potential is $1 + 1/(d-1)$.
- We remove a vertex $i + j' \in \{i+1, \ldots, i+d-1\}$ different from the first or the last vertex. Then, the adjacent facet is $\tau_{i,j'}$, and the potential increases at most by $(d-1)/(d-1) = 1$.

From these three cases we see that any move between middle facets of $\partial \mathrm{SC}(N, d+1)$ increases the potential at most by $1 + 1/(d-1) = d/(d-1)$. Observe that α is adjacent only to facets with potential at most one, since its neighborhood is $N(\alpha) = \{\tau_{1,j} : j = 1, \ldots, d-1\} \cup \{\tau_{2,d-1}\}$. Similarly, the neighborhood of ω is $N(\omega) = \{\tau_{N-d,j} : j = 2, \ldots, d\} \cup \{\tau_{N-d-1,1}\}$, so ω is adjacent only to facets with potential at least $N - d - 2$.

Any path from α to ω passes only through middle facets, and increases the potential from a starting value on the first middle facet at most 1 to a final value on the final middle facet of at least $N - d - 2$ potential. And every move between these facets increases the potential at most by $d/(d-1)$. Hence, the middle part of the path takes at least $((N-d-2)-1)(d-1)/d$ moves. And the path from α to ω is at least two moves longer than that. Therefore, the shortest paths between α and ω have length at least

$$((N-d-2)-1)\frac{d-1}{d}+2 = N\frac{d-1}{d} - d + \frac{3}{d} > N\frac{d-1}{d} - d.$$

The diameter of $\partial \mathrm{SC}(N, d+1)$, as the largest distance between any pair of vertices, is at least as large. □

This lower bound on the diameter of $\partial \mathrm{SC}(N, d+1)$ is actually sharp for N large enough, but we do not need to prove that for the purposes of this article.

Now we want to color $\partial \mathrm{SC}(N, d+1)$ as in the proof of the general case to apply Lemma 2.2.

Proof of the lower bound from Theorem 1.3 As in the proof of Theorem 1.2, fix $d \geq 2$ as well as $c_1 > 6(d-1)$ and $\epsilon > 0$. We let N be large enough so that we may apply Corollary 2.5 to color $\mathrm{SC}(N, d+1)$ by a coloring f with c_1 colors so that all pattern classes of $(d-2)$-dimensional faces have size at most

$$S = (1+\epsilon)N\frac{\binom{(d+1)-1}{2}}{\binom{c_1}{(d+1)-2}}.$$

This coloring then induces a coloring of $\partial \mathrm{SC}(N, d+1)$ so that no pattern class of ridges of $\partial \mathrm{SC}(N, d+1)$ has size more than S and no intersecting ridges receive the same pattern.

In order to apply Proposition 2.6 to $\partial \mathrm{SC}(N, d+1)$, we need to determine an upper bound t for the maximum number of ridges that intersect any given ridge. We will show now that $t = (d+1)^3$ is such a bound.

Recall that facets of $\partial \mathrm{SC}(N, d+1)$ are sequences of $d+1$ consecutive vertices with one of them removed. Then ridges are sequences of $d+1$ consecutive vertices with two of them removed. For a fixed ridge, the number of choices of such potential intervals is at most $2d+1$. And for each choice of an interval of consecutive vertices, we have to remove two of them. This gives us a total of at most $(2d+1)\binom{d+1}{2} \leq (d+1)^3$ ridges intersecting the fixed one.

With c_1, S, and t determined we apply Proposition 2.6 to color $\partial \mathrm{SC}(N, d+1)$. The number of colors required is at most

$$c_1\left\lceil \sqrt[d-1]{e\left(2(d+1)^3(1+\epsilon)N\binom{d}{2}\Big/\binom{c_1}{d-1}+1\right)}\right\rceil.$$

The proposition guarantees that no pair of ridges receive the same pattern. Thus, exactly as in the proof of Theorem 1.2, for any $\delta > 0$ we set c_1 large enough and ϵ small enough so that for N large enough we have a complex on at most

$$(1+\delta)\sqrt[d-1]{e(d-1)!(d+1)^3 N d^2}$$

vertices, which has diameter at least

$$(1-\delta)\frac{d-1}{d}N.$$

Letting n denote the number of vertices in this complex, we have a complex on n vertices whose diameter is at least

$$(1-\delta)\left(\frac{1}{1+\delta}\right)^{d-1}\frac{(d-1)n^{d-1}}{e(d+1)^3 d^3 (d-1)!} \geq (1-\delta)\left(\frac{1}{1+\delta}\right)^{d-1}\frac{n^{d-1}}{4ed^4 d!}.$$

As δ is arbitrary, this proves the claim. □

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