# Base Polytopes of Series-Parallel Posets: Linear Description and Optimization

by

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**Abstract**

We define the base polytope $B(P, g)$ of a partially ordered set $P$ and a supermodular function $g$ on the ideals of $P$ as the convex hull of the incidence vectors of all linear extensions of $P$. This new class of polytopes contains, among others, the base polytopes of supermodular systems and permutahedra as special cases. After introducing the notion of compatibility for $g$, we give a complete linear description of $B(P, g)$ for series-parallel posets and compatible functions $g$. In addition, we describe a greedy-type procedure which exhibits Sidney's job sequencing algorithm to minimize the total weighted completion time as a natural extension of the matroidal greedy algorithm from sets to posets.

## 1. Introduction

More than twenty years ago, Sidney [1975] published his article on the minimization of the weighted sum of completion times for one-machine scheduling problems with precedence constraints. The algorithm he proposed generalizes Smith's greedy-type rule for independent jobs (Smith [1956]). Sidney proved that there always is an optimal schedule starting with an ideal of maximum weight density. Hence, the problem can be solved recursively for this ideal and its complement.

While in general the only ideal with maximum weight density may be the whole set, and so nothing is gained, Sidney showed that in the case of series-parallel precedence constraints, we can always find a proper subideal to start with. Sidney's algorithm is a greedy algorithm extended from sets to special partially ordered sets. However, the precise relation of this algorithm to structures induced by sub- or supermodular functions, which many people would consider the natural setting for greedy algorithms, remained unclear. It is the purpose of this paper to shed some light on this relation.

We follow a common approach and study polyhedra induced by supermodular functions (the interested reader is referred to Fujishige's book [1991] and the references cited therein). It is well known that the linear minimization problem over the base polytope $B(g) = \{x \in \mathbb{R}^E : x(S) \geq g(S) \text{ for all } S \subset E, x(E) = g(E)\}$ of a supermodular function $g$ can be solved by the greedy algorithm. Every permutation $L = e_1 \dots e_n$ of $E$ induces

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by $x_{e_i} = g(\{e_1, \ldots, e_i\}) - g(\{e_1, \ldots, e_{i-1}\})$ a vertex of $B(g)$, and every vertex of the base polytope can be obtained in this way. While supermodular functions $g$ acting on families of subsets of a finite set seem to be well understood, at least as far as minimization of linear functions on $B(g)$ is concerned, much less is known if, in addition, the solutions have to respect a partial order relation. Let $P$ be a partially ordered set and $g$ a supermodular function on the ideals of $P$. To every linear extension of $P$, i.e. every permutation of $P$ which respects the partial ordering, we associate an incidence vector as before. The base polytope $B(P, g)$ of $P$ is the convex hull of the incidence vectors of all linear extensions of $P$. In Section 2 of this paper we introduce the notion of compatibility for a supermodular function $g$ on the ideals of $P$. In Section 3, we discuss some geometric properties of base polytopes of arbitrary posets, whereas in Section 4 we derive a complete linear description of $B(P, g)$ for series-parallel posets and supermodular and compatible functions $g$. Section 5 describes a greedy algorithm to minimize linear functions over $B(P, g)$. Instead of repeatedly picking elements with large weights, as is done in the matroid case, it follows Sidney's [1975] recipe and chooses ideals with large average weight. Section 6 gives some examples and extensions.

## 2. Notations and Definitions

Let $P = (E, <)$ be a partially ordered set (poset) on a finite set $E$ with $n$ elements. A *linear extension* $L = e_1 e_2 \ldots e_n$ of $P$ is a total ordering of the elements of $E$ which respects the partial ordering of $P$, i.e. $e_i < e_j$ in $P$ implies $i < j$. By $\mathcal{L}(P)$ we denote the set of all linear extensions of $P$.

Given two disjoint posets $(P_1, <_1)$ and $(P_2, <_2)$ we define two compositions on $P = P_1 \cup P_2$:

*Parallel composition*: $P = P_1 \parallel P_2$,

$$x < y \quad \text{if} \quad \begin{cases} x <_1 y & \text{and } x, y \in P_1, \\ x <_2 y & \text{and } x, y \in P_2. \end{cases}$$

*Series composition*: $P = P_1 \oplus P_2$,

$$x < y \quad \text{if} \quad \begin{cases} x <_1 y & \text{and } x, y \in P_1, \\ x <_2 y & \text{and } x, y \in P_2, \\ x \in P_1, & y \in P_2. \end{cases}$$

A poset is *series-parallel* if it can be constructed inductively from single elements by applying parallel and series compositions. Equivalently, a poset is series-parallel if no four elements induce an $N$ (Seinsche [1974]). Four elements $a, b, c, d$ induce an $N$, if $a < b, b > c$ and $c < d$ are the only comparabilities among them.

A subset $I \subseteq P$ is an *ideal* if $x \in I$ and $y < x$ implies $y \in I$. In particular $I(x) = \{y : y \leq x\}$ is the *principal ideal* induced by $x$. For $X \subseteq E$, let $I(X) = \bigcup(I(x) : x \in X)$. By $\mathcal{I}(P)$ we denote the set of all ideals of $P$. $\mathcal{I}(P)$ is a lattice, i.e. for ideals $I$ and $J$ in $\mathcal{I}(P)$, also $I \cap J$ and $I \cup J$ are in $\mathcal{I}(P)$.

A *convex* set $C \subseteq P$ is a subset which contains with $x, y \in C$ all elements $z \in C$ satisfying $x < z < y$. A convex set $C$ is called *series-reducible*, if it consists of two series components, i.e. $C = A \oplus B$.
Given a lattice $\mathcal{L}$, a function $g : \mathcal{L} \to \mathbb{R} \cup \{-\infty\}$ is *supermodular* if

$$g(X \cup Y) + g(X \cap Y) \geq g(X) + g(Y) \tag{1}$$

holds for all elements $X, Y \in \mathcal{L}$. The function $g$ is called *strictly* supermodular if, in addition, inequality (1) holds strictly for all $X \not\subseteq Y, Y \not\subseteq X$. We will assume throughout that supermodular functions are normalized, i.e. $g(\emptyset) = 0$.
Let $A, B \subseteq E$ be two disjoint subsets of $E$ (where $E$ is the ground-set of $P$). We call the tuple $(A, B)$ a *series-reducible convex set* if $A \cup B$ is convex and if $a < b$ holds for all $a \in A$ and $b \in B$. Note that if we fix one set, say $A$, then the collection of all sets $B$ which together with $A$ form a series-reducible convex set is a lattice.
Series-reducible convexity can be viewed as a covering relation on certain subsets of $P$. Two subsets $A$ and $B$ of a series-reducible convex set $(A, B)$ are covering in a linear extension $L$ of $P$, if $A \cup B$ is a chain in $L$, i.e. $L = L_I L_A L_B L_J$. We want linear inequalities tight at $(A, B)$ to be independent of the sets $I$ and $J$. For this purpose, we call a function $g : \mathcal{I}(P) \to \mathbb{R}$ *compatible (with $P$)* if for all series-reducible convex sets $(A, B)$ and any $I \in \mathcal{I}(P), I \subseteq E \setminus (A \cup B)$ such that $I \cup A \in \mathcal{I}(P)$ and $I \cup A \cup B \in \mathcal{I}(P)$ the term

$$|A|[g(I \cup A \cup B) - g(I \cup A)] - |B|[g(I \cup A) - g(I)] \tag{2}$$

is a constant $g_{A,B}$ independent of $I$. Any compatible function $g$ then induces a function $f$ on the series-reducible convex sets via $f(A, B) = g_{A,B}$. Modular functions $g$ are always compatible.

**Lemma 1** *Let $P$ be a poset and $g : \mathcal{I}(P) \to \mathbb{R}$ a function on the ideals of $P$. If $g$ is compatible and supermodular, then the function $f$ induced by $g$ on the series-reducible convex sets of $P$ is supermodular in both components.*

*Proof:* (i) Let $(A, B_1)$ and $(A, B_2)$ be two series-reducible convex sets. Choose $I = I(A \cup B_1 \cup B_2) \setminus (A \cup B_1 \cup B_2)$. Then

$$\begin{aligned}
&f(A, B_1) + f(A, B_2) \\
&\quad = \quad |A|[g(I \cup A \cup B_1) + g(I \cup A \cup B_2) - 2g(I \cup A)] - (|B_1| + |B_2|)[g(I \cup A) - g(I)] \\
&\quad \leq \quad |A|[g(I \cup A \cup (B_1 \cup B_2)) + g(I \cup A \cup (B_1 \cap B_2)) - 2g(I \cup A)] \\
&\qquad\qquad -(|B_1| + |B_2|)[g(I \cup A) - g(I)] \\
&\quad = \quad f(A, B_1 \cup B_2) + f(A, B_1 \cap B_2) \ .
\end{aligned}$$

(ii) Let $(A_1, B)$ and $(A_2, B)$ be two series-reducible convex sets. Choose $I = I(A_1 \cup A_2 \cup B) \setminus (A_1 \cup A_2 \cup B)$ and let $I_1 = I \cup (A_2 \setminus A_1), I_2 = I \cup (A_1 \setminus A_2)$ and $J = I_1 \cup I_2$. Observe that $I_1 \cup A_1 = I \cup (A_1 \cup A_2) = J \cup (A_1 \cap A_2) = I_2 \cup A_2$. Hence

$$f(A_1, B) + f(A_2, B)$$

$$
\begin{aligned}
&= |A_1|[g(I_1 \cup A_1 \cup B) - g(I_1 \cup A_1)] + |A_2|[g(I_2 \cup A_2 \cup B) - g(I_2 \cup A_2)] \\
&\quad -|B|[g(I_1 \cup A_1) + g(I_2 \cup A_2) - g(I_1) - g(I_2)] \\
&= |A_1 \cup A_2|[g(I \cup A_1 \cup A_2 \cup B) - g(I \cup A_1 \cup A_2)] - |B|[g(I \cup A_1 \cup A_2) - g(I_1)] \\
&\quad +|A_1 \cap A_2|[g(J \cup (A_1 \cap A_2) \cup B) - g(J \cup (A_1 \cap A_2))] \\
&\quad -|B|[g(J \cup (A_1 \cap A_2)) - g(I_2)] \\
&\leq f(A_1 \cup A_2, B) + f(A_1 \cap A_2, B) \text{ by supermodularity.}
\end{aligned}
$$

□

For a subset $S$ of $P$ and $x \in \mathbb{R}^P$, let $x(S) := \sum_{e \in S} x_e$. If $S = \{e\}$ is a singleton, we omit the brackets, i.e. $x(e) = x(\{e\}) = x_e$.

## 3. The Base Polytope of a Poset

Recall that the base polytope of a supermodular function $g$ is defined as $B(g) = \{x \in \mathbb{R}^E : x(S) \geq g(S) \text{ for all } S \subset E, x(E) = g(E)\}$. Equivalently, it is the convex hull of the incidence vectors $x(L)$ of all permutations $L = e_1 \ldots e_n$ of $E$. Here, the incidence vector $x(L)$ is defined as $x_{e_i} = g(\{e_1, \ldots, e_i\}) - g(\{e_1, \ldots, e_{i-1}\})$ for $1 \leq i \leq n$. A prominent member of the class of base polytopes is the *permutahedron*. It is defined as the convex hull of all permutations, perm $= \text{conv}\{(\pi(1), \ldots, \pi(n)) : \pi \text{ permutation of } E\}$. More general, the convex hull of those permutations that are extensions of a given poset $P$ on the ground set $E$ has achieved considerable interest in the last years, in particular for its application in scheduling, see v. Arnim, Faigle, Schrader [1990], Schulz [1995], Queyranne and Wang [1991], v. Arnim, Schrader, and Wang [1996], v. Arnim and Schulz [1994], and Queyranne and Schulz [1994]. In this section, we investigate which properties of these special polytopes carry over to the more general framework. To be more precise, let $P$ be a poset and $g : \mathcal{I}(P) \to \mathbb{R}$ a supermodular function on the ideals of $P$. Then, our object of study is the base polytope of the poset $P$ with respect to $g$ which is defined as

$$B(P, g) := \text{conv}\{x(L) : L \in \mathcal{L}(P)\} \ .$$

The following results which generalize the respective ones for the permutahedron of a poset (cf., e.g., Schulz [1996]) can be proved simply by using the (strict) supermodularity of $g$.

**Proposition 2** *Let $P$ be a poset with $P_1 \oplus P_2$, and let $g : \mathcal{I}(P) \to \mathbb{R}$ be a supermodular function. Then $B(P, g)$ is the Cartesian product of the polytopes $B(P_1, g)$ and $B(P_2, g')$ where $g'(S) = g(S \cup P_1) - g(P_1)$.*

Since a minimal description in terms of linear equations and inequalities for the Cartesian product of given polyhedra can be obtained by the juxtaposition of minimal linear systems of the given polyhedra, we may concentrate on posets $P$ that are not series decomposable, when studying $B(P, g)$. We will make use of this property in Section 4. With the help of Proposition 2 it is easy to determine the dimension of $B(P, g)$.

**Proposition 3** *Let $P$ be a poset with series decomposition $P_1 \oplus \cdots \oplus P_q$ (i.e., $P_i$ is not further series decomposable), and let $g : \mathcal{I}(P) \to \mathbb{R}$ be a strictly supermodular function. Then*

$$x(P_1 \cup \cdots \cup P_i) = g(P_1 \cup \cdots \cup P_i) \qquad \textit{for } i = 1, 2, \ldots, q$$

*is a minimal linear equation system defining the affine hull of $B(P, g)$. In particular,*

$$\dim(B(P, g)) = n - q \ .$$

Finally, for strictly supermodular functions $g$, we characterize the facet defining inequalities among those which naturally emerge from the base polytope $B(g)$ of $g$.

**Proposition 4** *Let $P$ be a poset with series decomposition $P_1 \oplus \cdots \oplus P_q$, and let $I = P_1 \oplus \cdots \oplus P_i \oplus \hat{I}$, $i \in \{0, \ldots, q-1\}$, be an ideal of $P$. If $\hat{I}_1 \oplus \cdots \oplus \hat{I}_r$ and $\hat{F}_1 \oplus \cdots \oplus \hat{F}_s$ are the series decompositions of $\hat{I}$ and $P_{i+1} \setminus \hat{I}$, respectively, then the face of $B(P, g)$ induced by $x(I) \geq g(I)$ is of dimension $n - (q + r + s) + 1$.*

## 4. Base Polytopes of Series-Parallel Posets

For a poset $P$ and a supermodular function $g : \mathcal{I}(P) \to \mathbb{R}$ compatible with $P$ let $\mathbb{P}(P, g)$ be the polytope defined by the inequalities

$$\begin{aligned} |A|x(B) - |B|x(A) &\geq f(A, B) \text{ for all series-reducible convex sets } (A, B), \\ & \phantom{\geq} A, B \text{ series-prime}, \\ x(I) &\geq g(I) \text{ for all ideals } I \in \mathcal{I}(P), \\ x(P) &= g(P). \end{aligned}$$

(Note that $\mathbb{P}(P, g)$ is not well-defined if $g$ is not compatible with $P$.) A subset $A$ of $P$ is called series-prime if $A$ does not allow a series-decomposition.
For series-parallel posets $P$ and compatible supermodular functions $g$ we will show that $\mathbb{P}(P, g)$ equals $B(P, g)$. In particular, $\mathbb{P}(P, g)$ is integral if $g$ is integral. Justified by Proposition 2, we assume throughout this chapter, when considering a series-parallel poset $P$, that $P$ is series-prime.
The following observation is immediate.

**Lemma 5** *Let $P$ be a poset and $g : \mathcal{I}(P) \to \mathbb{R}$ supermodular and compatible. Let $P' = P \setminus I$ for some ideal $I$ of $P$, $g' : \mathcal{I}(P') \to \mathbb{R}$ with $g'(J) := g(I \cup J) - g(I)$ and $f'$ the function induced by $g'$. Then $g'$ is supermodular and compatible with $P'$ and $f' = f$ on $P \setminus I$.* □

Given a vector $x \in \mathbb{P}(P, g)$, we call an ideal $I$ *tight* at $x$ if $x(I) = g(I)$ holds.

**Lemma 6** *Let $P$ be a poset and $g : \mathcal{I}(P) \to \mathbb{R}$ supermodular and compatible. Let $x \in \mathbb{P}(P, g)$ and let $I$ be a tight ideal. Then $y = (x_i : i \in I) \in \mathbb{P}(I, g)$ and $z = (x_i : i \in P \setminus I) \in \mathbb{P}(P \setminus I, g')$ where $g'(J) = g(I \cup J) - g(I)$, as above.*

*Proof:* The previous lemma implies that $z$ satisfies the convex set constraints and $y$ satisfies both ideal and convex set constraints. Since $I$ is tight, we also have $y(I) = g(I)$, i.e. $y \in \mathbb{P}(I, g)$. For an ideal $J \subseteq P \setminus I$, we have $z(J) = x(J) = x(I \cup J) - x(I) \geq g(I \cup J) - g(I) = g'(J)$, i.e. $z$ satisfies the ideal constraints induced by $g'$. Finally, $z(P \setminus I) = x(P \setminus I) = x(P) - x(I) = g(P) - g(I) = g'(P \setminus I)$, i.e. $z \in \mathbb{P}(P \setminus I, g')$. □

**Proposition 7** *Let $P$ be a series-parallel poset and $g : \mathcal{I}(P) \to \mathbb{R}$ supermodular and compatible. Then $\mathbb{P}(P, g) \subseteq B(P, g)$ holds.*

*Proof:* (i) We first show by induction that $B(P, g)$ contains $\mathbb{P}(P, g)$ if for any vertex of $\mathbb{P}(P, g)$ there exists a tight proper ideal. For $|P| = 1$ the claim obviously holds. Now let $x$ be a vertex of $\mathbb{P}(P, g)$ and $I$ be a tight proper ideal. By Lemma 6 and induction, $y = (x_i : i \in I) \in \mathbb{P}(I, g) \subseteq B(I, g)$ and $z = (x_i : i \in P \setminus I) \in \mathbb{P}(P \setminus I, g') \subseteq B(P \setminus I, g')$. Hence, $y$ and $z$ are convex combinations of incidence vectors of linear extensions of $I$ and $P \setminus I$, respectively. It follows that $x$ itself is a convex combination of incidence vectors of linear extensions of $P$ and thus contained in $B(P, g)$.
(ii) It remains to show that we can find tight proper ideals. Remember that we assumed that $P = P_1 \| P_2$. Suppose that $x$ is a vertex of $\mathbb{P}(P, g)$ with no tight proper ideal. Then $\varepsilon := \min\{x(I) - g(I) : I \text{ is a proper ideal of } P\}$ is positive. Choose a vector $c \in \mathbb{R}^E$, such that $x$ is the unique minimum for $\min\{cz : z \in \mathbb{P}(P, g)\}$. We may assume that $c(P_2)|P_1| - c(P_1)|P_2| \leq 0$, otherwise we can renumber $P_1$ and $P_2$. Now let $y$ be given by

$$y_i = \begin{cases} x_i + \varepsilon/|P_1|, & \text{for } i \in P_1, \\ x_i - \varepsilon/|P_2|, & \text{for } i \in P_2. \end{cases}$$

We claim that $y \in \mathbb{P}(P, g)$. Obviously, $y$ lies on the hyperplane $y(P) = g(P)$. Since $P = P_1 \| P_2$, any series-reducible convex set is contained in either $P_1$ or $P_2$. By using that the convex set constraints are invariant under adding the same constant to every component, it follows for any series-reducible convex set $(A, B)$ that $|A|y(B) - |B|y(A) = |A|x(B) - |B|x(A) \geq f(A, B)$. Finally, the ideal inequalities hold for any proper ideal $I \subsetneq P$ since

$$y(I) = \sum_{i \in I \cap P_1} \left( x_i + \frac{\varepsilon}{|P_1|} \right) + \sum_{i \in I \cap P_2} \left( x_i - \frac{\varepsilon}{|P_2|} \right) \geq x(I) - \varepsilon \geq g(I) \ .$$

But $cy = cx + \varepsilon\big(c(P_1)/|P_1| - c(P_2)/|P_2|\big) \leq cx$, contradicting the uniqueness of $x$. Hence there must exist a tight proper ideal. □

**Theorem 8** *Let $P$ be a series-parallel poset and $g : \mathcal{I}(P) \to \mathbb{R}$ supermodular and compatible. Then $\mathbb{P}(P, g)$ coincides with $B(P, g)$.*

*Proof:* Because of Proposition 7, it is sufficient to show $B(P, g) \subseteq \mathbb{P}(P, g)$. Let $x = x(L)$ be the incidence vector of a linear extension of $P$. Obviously, $x$ satisfies $x(P) = g(P)$ and the ideal constraints.
Now, let $(A, B)$ be a series-reducible convex set with $A$ and $B$ series-prime. Let $L = e_1 \dots e_n, j = \max\{i : e_i \in A\}$ and $J = \{e_1, \dots, e_j\}$. Then $x_J \in B(J, g)$ and $x_{P \setminus J} \in$

$B(P \setminus J, g')$. In particular, $x(B) \geq g'(B) = g(J \cup B) - g(J)$ holds if $B$ is an ideal in $P \setminus J$. If $J \setminus A$ is an ideal of $J$, too, then $x(A) = x(J) - x(J \setminus A) \leq g(J) - g(J \setminus A)$. Using $I = J \setminus A$, the wanted convex set inequality $|A|x(B) - |B|x(A) \geq |A|[g(I \cup A \cup B) - g(I \cup A)] - |B|[g(I \cup A) - g(I)]$ would follow. It remains to show that we can force $J \setminus A$ and $B$ to be ideals in $J$ and $P \setminus J$, respectively.
Suppose $J \cup B$ is no ideal of $P$. Then there exists an earliest element $e_i = c \in P \setminus (J \cup B)$ in $L$ with $c < b$ for some $b \in B$. From $i > j$ and the convexity of $A \cup B$ we conclude $c \parallel A$. Since $P$ is series-parallel and $B$ is series-prime, $c < B$ follows. Let $L'$ be the linear extension arising from $L$ by exchanging $e_i$ and $e_j$ with incidence vector $x'$. The supermodularity of $g$ implies $x'(A) \geq x(A)$, whereas $x'(B) = x(B)$ remains unchanged. We continue the exchange operations until $J \cup B$ is an ideal of $P$.
Suppose $I = J \setminus A$ is no ideal of $P$. Then there exists elements $a \in A$ and $y \in I$ with $a < y$. Since $e_j =: a'$ belongs to $A$, we know that $y < a'$ or $y \parallel a'$. Because $A$ is convex, $y$ must be parallel to $a'$. We again distinguish two cases.
(a) $a \parallel a'$. Let $b \in B$ be arbitrary. Then $a, b, a', y$ induce an $N$ in contradiction to the fact that $P$ is series-parallel.
(b) $a < a'$. Because $A$ is series-prime and $P$ series-parallel, there must be an element $a'' \parallel \{a, a'\}$. If $a'' \parallel y$, then $a, b, a'', y$ induce an $N$. If $a'' < y$, then $a'', b, a', y$ induce an $N$. Both possibilities contradict that $P$ is series-parallel. Since $A$ is convex, $y$ cannot be a predecessor of $a''$. □

## 5. Optimization

Let $P$ be a poset, $g : \mathcal{I}(P) \to \mathbb{R}$ a supermodular function and $c \in \mathbb{R}^P$. Consider the linear programming problem

$$\operatorname{opt}(P, g, c) := \min\{cx : x \in B(P, g)\}.$$

Since $x(P) = g(P)$, we may assume that the $c_i's$ are positive.
Call an ideal $I$ *$\rho$-maximal* if $\rho(I) = c(I)/|I| \geq c(J)/|J| = \rho(J)$ for all ideals $J \subseteq P$. The algorithm we propose to solve $\operatorname{opt}(P, g, c)$ for series-parallel posets and supermodular and compatible functions $g$ is a generalization of Sidney's [1975] algorithm for minimizing the weighted sum of completion times in a one-machine scheduling environment. It starts with some $\rho$-maximal ideal $I$ and solves the problem recursively on $I$ and $P \setminus I$. In general, $P$ itself may be the only $\rho$-maximal ideal and the approach does not work. However, if $P$ decomposes into two parallel components, there is always a $\rho$-maximal ideal which is contained in one of the components. This is implied by the following observation which holds for all $x, y \geq 0$ :

$$(X + Y)/(x + y) \leq X/x \iff Y/y \leq X/x \tag{3}$$

**Lemma 9** *Let $P$ be a poset, $g : \mathcal{I}(P) \to \mathbb{R}$ supermodular and $c \in \mathbb{R}^P$. Let $L = L_I L_{P \setminus I}$ be an optimal linear extension for* $\operatorname{opt}(P, g, c)$. *Then any linear extension $L' = L'_I L'_{P \setminus I}$ starting with $I$ is optimal for* $\operatorname{opt}(P, g, c)$ *if and only if $L'_I$ is optimal for* $\operatorname{opt}(I, g, c_I)$ *and $L'_{P \setminus I}$ is optimal for* $\operatorname{opt}(P \setminus I, g', c_{P \setminus I})$, *where $g'(J) = g(I \cup J) - g(I)$, as above.*

*Proof:* By definition of $B(P, g)$ and $g'$ we have $\mathrm{opt}(P, g, c) \leq \mathrm{opt}(I, g, c_I) + \mathrm{opt}(P \setminus I, g', c_{P \setminus I})$. Since $L$ is optimal for $\mathrm{opt}(P, g, c)$, equality follows. □

**Lemma 10** *Let $P$ be a poset and $g : \mathcal{I}(P) \to \mathbb{R}$ be supermodular and compatible. Let $L = L_I a_1 \dots a_r b_1 \dots b_s L_J$ be a linear extension of $P$ with incidence vector $x$ containing a series-reducible convex set $(A, B) = (\{a_1, \dots, a_r\}, \{b_1, \dots, b_s\})$. Then, using $A_j := \{a_j, \dots, a_r\}$ and $B_j := \{b_1, \dots, b_j\}$, we obtain*

$$\begin{aligned}
(4) \qquad x(b_j) &= x(a_1) + f(A, b_1) - f(A_2, b_1) \\
&\quad - f(A_r, b_1) + f(A_r, B_j) + f(A_r, B_{j-1}) \quad \textit{for } j = 1, \dots, s, \\
(5) \qquad x(a_j) &= x(a_1) + f(A, b_1) - f(A_2, b_1) \\
&\quad - f(A_j, b_1) + f(A_{j+1}, b_1) \quad \textit{for } j = 1, \dots, r.
\end{aligned}$$

*Proof:* The convex set constraints for $A_j \oplus \{b_1\}, 1 \leq j \leq r$, and $\{a_r\} \oplus B_j, 1 \leq j \leq s$, are tight at $x(L)$.
By induction on $j$ we first show $x(b_j) = x(a_r) + f(a_r, B_j) - f(a_r, B_{j-1})$. For $j = 1$, the tight convex set constraint for $\{a_r\} \oplus \{b_1\}$ gives $x(b_1) - x(a_r) = f(a_r, b_1) = f(a_r, B_1) - f(a_r, \emptyset)$. In the induction step we use the tight convex set constraint for $\{a_r\} \oplus B_j$. With $\sum_{i=1}^{j} x(b_i) - j \cdot x(a_r) = f(a_r, B_j)$ we get

$$\begin{aligned}
x(b_j) &= f(a_r, B_j) + j \cdot x(a_r) - \textstyle\sum_{i=1}^{j-1} x(b_i) \\
&= f(a_r, B_j) + j \cdot x(a_r) - (j-1)x(a_r) \\
&\quad + \textstyle\sum_{i=1}^{j-1}(-f(a_r, B_i) + f(a_r, B_{i-1})) \\
&= f(a_r, B_j) + x(a_r) - f(a_r, B_{j-1}) \; .
\end{aligned}$$

By symmetry, $x(a_j) = x(b_1) - f(A_j, b_1) + f(A_{j+1}, b_1)$ holds. Substitution of $x(a_1) = x(b_1) - f(A, b_1) + f(A_2, b_1)$ for $x(b_1)$ into the equation of $x(a_j)$ yields $x(a_j) = x(a_1) + f(A, b_1) - f(A_2, b_1) - f(A_j, b_1) + f(A_{j+1}, b_1)$. Substitution of this equation for $x(a_r)$ into the equation of $x(b_j)$ finally yields $x(b_j) = x(a_1) + f(A, b_1) - f(A_2, b_1) - f(A_r, b_1) + f(A_r, B_j) + f(A_r, B_{j-1})$. □

**Corollary 11** *Let $P$ and $g$ be as in Lemma 10, let $L = L_I e_1 \dots e_k L_J$ be a linear extension of $P$ with incidence vector $x$ and let $\{e_1, \dots, e_k\}$ be a series-reducible convex set of $P$. Then*

$$x(e_i) = x(e_1) + s_i \; ,$$

*and the values of $s_i$ are independent of $L$ on $P \setminus \{e_1, \dots, e_k\}$.* □

**Proposition 12** *Let $P$ be a poset, $g : \mathcal{I}(P) \to \mathbb{R}$ strictly supermodular and compatible and $c \in \mathbb{R}^P$. Let $L = L_I e_1 \dots e_k f_1 \dots f_l L_J$ and $L' = L_I f_1 \dots f_l e_1 \dots e_k L_J$ be two linear extensions of $P$ with incidence vectors $x$ and $y$ and series-reducible convex sets $E = \{e_1, \dots, e_k\}$ and $F = \{f_1, \dots, f_l\}$. Then $cx \leq cy$ holds if and only if $\rho(E) \geq \rho(F)$ holds.*

*Proof:* Using Corollary 11, we can write

$$\begin{aligned}
x(e_i) &= x(e_1) + t_i, \\
x(f_j) &= x(f_1) + s_j, \\
y(e_i) &= y(e_1) + t_i, \\
y(f_j) &= y(f_1) + s_j
\end{aligned}$$

for all $1 \leq i \leq k, 1 \leq j \leq l$. Then $cx \leq cy$ is equivalent to

$$\begin{array}{rrcl} & \sum_{i=1}^{k} c_{e_i} x(e_i) + \sum_{j=1}^{l} c_{f_j} x(f_j) & \leq & \sum_{i=1}^{k} c_{e_i} y(e_i) + \sum_{j=1}^{l} c_{f_j} y(f_j) \\ \Longleftrightarrow & c(E)x(e_1) + c(F)x(f_1) & \leq & c(E)y(e_1) + c(F)y(f_1) \\ \Longleftrightarrow & \multicolumn{3}{l}{c(E)[g(I \cup e_1) - g(I) - g(I \cup F \cup e_1) + g(I \cup F)]} \\ & \multicolumn{3}{l}{\leq \ c(F)[g(I \cup f_1) - g(I) - g(I \cup E \cup f_1) + g(I \cup E)] \ .} \end{array}$$

We have $g(I \cup E) - g(I) = \sum_{i=1}^{k} x(e_i) = k \cdot x(e_1) + \sum_{i=1}^{k} t_i = k \cdot g'(e_1) + \sum_{i=1}^{k} t_i$, and $g(I \cup F) - g(I) = l \cdot g'(f_1) + \sum_{j=1}^{l} s_j$. Using the linear extension $L_I e_1 f_1 \dots f_l e_2 \dots e_k L_J$ it follows that $g(I \cup F \cup e_1) - g(I \cup e_1) = l \cdot (g'(\{e_1, f_1\}) - g'(e_1)) + \sum_{j=1}^{l} s_j$, and similarly $g(I \cup E \cup f_1) - g(I \cup f_1) = k \cdot (g'(\{e_1, f_1\}) - g'(f_1)) + \sum_{i=1}^{l} t_i$. Substituted into the last inequality above we get

$$c(E)[l \cdot (g'(f_1) + g'(e_1) - g'(\{f_1, e_1\}))] \leq c(F)[k \cdot (g'(f_1) + g'(e_1) - g'(\{f_1, e_1\}))] \ .$$

Because of $g'(\{e_1, f_1\}) - g'(e_1) - g'(f_1) > 0$ ($g$ is strictly supermodular) the last inequality holds if and only if $c(E)/|E| \geq c(F)/|F|$ holds. □

Again, let $L = e_1 \dots e_n$ be a linear extension of $P$. We call an interval $I = e_i e_{i+1} \dots e_k$ of $L$ *series-reducible*, if $\{e_i, e_{i+1}, \dots, e_k\}$ is a series-reducible convex set of $P$. $I$ is called *maximal series-reducible* (in $L$), if no larger series-reducible interval $J \supsetneq I$ in $L$ exists.

**Proposition 13** *Let $P$ be a series-parallel poset. Let $L$ be a linear extension with (in this order, from the beginning of $L$ to its end) maximal series-reducible intervals $E_i, 1 \leq i \leq s$, i.e. $E_i$ is a singleton or $E_i = A_i \oplus B_i$ with nonempty subsets $A_i$ and $B_i$. Then*

$$E_i \parallel E_{i+1} \quad \text{for all} \quad 1 \leq i < s \ . \tag{6}$$

*Proof:* We proceed by induction on $n = |P|$. If $n = 1$ or $P = P_1 \oplus P_2$, we must have $s = 1$ and we are done. Now let $P = P_1 \parallel P_2$. Suppose the first element $e_1$ of $L$ belongs to $P_1$. Let $e_i$ be the first element in $L$ out of $P_2$. By induction hypothesis, the claim is valid for the linear extensions $e_1 \dots e_{i-1}$ and $e_i \dots e_n$. The last maximal series-reducible interval in $e_1 \dots e_{i-1}$ is a subset of $P_1$, and the first maximal series-reducible interval in $e_i \dots e_n$ is a subset of $P_2$, so they are parallel to each other, too. □

**Corollary 14** *Let $P$ be a series-parallel poset, $g : \mathcal{I}(P) \to \mathbb{R}$ strictly supermodular and compatible, and $c \in \mathbb{R}^P$. Let $L = L_I e_1 \dots e_k f_1 \dots f_l L_J$ be an optimal linear extension for* $\mathrm{opt}(P, g, c)$ *and $L' = L_I f_1 \dots f_l e_1 \dots e_k L_J$ be a linear extension of $P$, too. Let $E = \{e_1, \dots, e_k\}$ and $F = \{f_1, \dots, f_k\}$. Then, $L'$ is optimal for* $\mathrm{opt}(P, g, c)$ *if and only if $\rho(F) \geq \rho(E)$.*

*Proof:* Let $E = E_1 \parallel \dots \parallel E_s$ and $F = F_1 \parallel \dots \parallel F_t$ the decomposition of $E$ and $F$ in maximal series-reducible intervals in $L$. Since $L$ is optimal and $L'$ is valid, Propositions 13 and 12 imply

$$\rho(E_1) \geq \rho(E_2) \geq \dots \geq \rho(E_s) \geq \rho(F_1) \geq \rho(F_2) \geq \dots \geq \rho(F_t) \ . \tag{7}$$

Observe that $L$ can be transformed into $L'$ by swapping $E_i$ with $F_j$, $j = 1, \ldots, t, i = s, \ldots, 1$. By Proposition 12, every swap does not decrease the objective function value. Let $L'$ be optimal, too. Suppose $\rho(F) < \rho(E)$. Then $\rho(E_{i_0}) > \rho(F_{j_0})$ for at least one pair $(i_0, j_0)$. Swapping $E_{i_0}$ with $F_{j_0}$ increases the objective value, any other swap operation does not decrease the objective value in contradiction to the optimality of $L'$.
In the other direction, $\rho(F) \geq \rho(E)$ and the inequalities (3) and (7) imply $\rho(F) = \rho(E)$. Consequently, equality in (7) follows. Now, by Proposition 12 every swap operation keeps the objective function value constant. Hence $L'$ is optimal, too. □
This proves the "adjacent string interchange property" for optimal linear extensions.
Like Sidney we call an ideal $I$ *$\rho^\star$-maximal*, if it is $\rho$-maximal and does not contain a smaller ideal which is $\rho$-maximal, too.

**Theorem 15** *Let $P$ be a series-parallel poset, $g : \mathcal{I}(P) \to \mathbb{R}$ strictly supermodular and compatible and $c \in \mathbb{R}^P$. Let $L$ be an optimal linear extension for* $\mathrm{opt}(P, g, c)$. *Then $L$ starts with an $\rho^\star$-maximal ideal $I$.*

*Proof:* We proceed by induction on $n = |P|$. The case $n = 1$ is trivial. By Lemma 9, it is sufficient to consider the case $P = P_1 \parallel P_2$ in the induction step. Let $L_1$ be the $\rho^\star$-maximal initial ideal of $L = e_1 \ldots e_n$, i.e. among all initial ideals of $L$ the smallest one with maximal $\rho$-value. To be more precise, let $j_0 := \min\{\ j : \forall\ 1 \leq k \leq n : \rho(\{e_1, \ldots, e_j\}) \geq \rho(\{e_1, \ldots, e_k\})\ \}$. Then $L_1 = e_1, \ldots, e_{j_0}$ and we denote by $I_1$ the underlying set, i.e. $I_1 = \{e : e \in L_1\}$. By Propositions 13 and 12, $L_1$ is a singleton or series-reducible. Hence $I_1$ is entirely included in $P_1$ or in $P_2$. W.l.o.g., we assume that $I_1 \subseteq P_1$. By Lemma 9, $L_1$ is optimal for $\mathrm{opt}(I_1, g, c_{I_1})$ and $L \setminus L_1$ is optimal for $\mathrm{opt}(P \setminus I_1, g', c_{P \setminus I_1})$. We claim that $I_1$ is a $\rho^\star$-maximal in $P$.
By induction hypothesis, $L_1$ starts with a $\rho^\star$-maximal ideal of $I_1$. The choice of $I_1$ assures this is $I_1$. Hence, $I_1$ does not include a smaller ideal with the same or a bigger $\rho$-value. Again by induction hypothesis, $L \setminus L_1$ starts with a $\rho^\star$-maximal ideal $J \subseteq P \setminus I_1$. Now, if $\rho(I_1) < \rho(J)$ then equivalence (3) implies that $\rho(I_1) < \rho(I_1 \cup J)$. This contradicts the choice of $L_1$. Consequently, the $\rho$-value of any ideal in $P \setminus I_1$ is at most $\rho(I_1)$.
It remains to consider any ideal $K$ of $P$ such that $K \cap I_1 \neq \emptyset$ and $K \cap (P \setminus I_1) \neq \emptyset$. However, again by use of (3), $\rho(K \cap I_1) \leq \rho(I_1)$ and $\rho(K \cap (P \setminus I_1)) \leq \rho(J)$ imply $\rho(K) \leq \rho(I_1)$.
□

The next two lemmas are direct extensions of their counterparts in Sidney's theory. Lemma 17 corresponds to Sidney's "Main Decomposition Theorem".

**Lemma 16** *Let $P$ be a poset and $c \in \mathbb{R}^P$. Then, the $\rho^\star$-maximal ideals of $P$ are pairwise disjoint.*

*Proof:* Let $I$ and $J$ be two distinct $\rho^\star$-maximal ideals. Hence, $I \not\subseteq J$ and vice versa. Suppose that $K = I \cap J \neq \emptyset$. Then, $K$ is an ideal with $\rho(K) < \rho(I) = \rho(J)$. In this case, using equivalence (3), $\rho(I \setminus K) > \rho(K)$ and $\rho(I \setminus K) > \rho(I)$ follow. Again using equivalence (3), $\rho(J \cup (I \setminus K)) > \rho(I)$ would follow which contradicts the $\rho$-maximality of $I$. □

**Lemma 17** *Let $P$ be a poset and $c \in \mathbb{R}^P$. Let $S_1 S_2 \dots S_a$ and $T_1 T_2 \dots T_b$ be two sequences of $\rho^\star$-maximal ideals (i.e. $S_j$ is $\rho^\star$-maximal in $P \setminus \bigcup_{i=1}^{j-1} S_i$ for $1 \le j \le a$, and $T_j$ is $\rho^\star$-maximal in $P \setminus \bigcup_{i=1}^{j-1} T_i$ for $1 \le j \le b$). Then $a = b$, and there exists a permutation $\pi$ such that $S_i = T_{\pi(i)}$ for all $i$.*

*Proof:* We proceed by induction on $n = |P|$. The induction start is trivial. If $S_1 = T_1$ holds in the induction step, we are finished by induction hypothesis. Hence, assume $S_1 \neq T_1$. From Lemma 16 we know that $S_1 \cap T_1 = \emptyset$. Notice that $T_1$ is $\rho$-maximal in $P \setminus S_1$, too. By induction hypothesis for $P \setminus S_1$ we can conclude $T_1 \in \{S_2, \dots, S_a\}$. Now the exchange of $S_i := T_1$ and $S_1$ in the sequence $S_1 S_2 \dots S_a$ leads us back to the previous case. □

**Corollary 18** *Let $P$ be a poset and $c \in \mathbb{R}^P$. Every $\rho$-maximal ideal is the disjoint union of $\rho^\star$-maximal ideals.* □

**Theorem 19** *Let $P$ be a series-parallel poset, $g : \mathcal{I}(P) \to \mathbb{R}$ strictly supermodular and compatible and $c \in \mathbb{R}^P$. Let $S$ be a $\rho^\star$-maximal ideal. Then there exists an optimal linear extension of $P$ for* $\mathrm{opt}(P, g, c)$ *starting with $S$.*

*Proof:* Let $L = S_1 S_2 \dots S_a$ with $S = S_1$ be a linear extension of $P$ such that, for each $j = 1, \dots, a$, the set $S_j$ is a $\rho^\star$-maximal in $P \setminus \bigcup_{i=1}^{j-1} S_i$. By Theorem 15 there exists an linear extension $L'$ of $P$ optimal for $\mathrm{opt}(P, g, c)$ which starts with a $\rho^\star$-maximal ideal $T_1$ of $P$. By Lemma 9 $L' \setminus T_1$ is optimal for $P \setminus T_1$. By induction, $L' = T_1 T_2 \dots T_b$ with $T_j$ $\rho^\star$-maximal in $P \setminus \bigcup_{i=1}^{j-1} T_i$. Because of Lemma 17 there must be a $k$ with $T_k = S_1 = S$. In the case $k = 1$ we are done. Otherwise, let $L' = T_1 \dots T_{k-1} S_1 T_{k+1} \dots T_b = E_1 E_2 \dots E_l S_1 T_{k+1} \dots T_b$, where the $E_i$ are maximal series-reducible intervals in $L'$. From Proposition 13 and 12, $\rho(E_i) \ge \rho(E_{i+1})$ for all $i$. Since $T_1$ is $\rho^\star$-maximal, $\rho(S_1) = \rho(T_1) \ge \rho(E_1)$ holds. Successive swaps of $S_1$ with $E_i$ ($i = l, l-1, \dots, 1$) do not alter the objective value by Proposition 12. Consequently, $L'' = S_1 T_1 \dots T_{k-1} T_{k+1} \dots T_b$ is an optimal linear extension starting with $S$. □

Although Theorem 19 is apparently restricted to strictly supermodular functions, it already provides most ingredients for the general case.

**Corollary 20** *Let $P$ be a series-parallel poset, $g : \mathcal{I}(P) \to \mathbb{R}$ supermodular and compatible and $c \in \mathbb{R}^P$. Let $S$ be a $\rho^\star$-maximal ideal. Then there exists an optimal linear extension of $P$ for* $\mathrm{opt}(P, g, c)$ *starting with $S$.*

*Proof:* We define $h_\varepsilon : \mathcal{I}(P) \to \mathbb{R}$ by $h_\varepsilon(I) := \varepsilon \cdot \sum_{i=1}^{|I|} i$ for all $\varepsilon > 0$. The function $h_\varepsilon$ is strictly supermodular and compatible. Hence, $g + h_\varepsilon$ is strictly supermodular and compatible, too. Let $K = \max\{cx : x \in B(P, h_1)\} > 0$ (recall that we may assume that $c > 0$) and let $d \ge 0$ be a lower bound for the difference of the second best objective function value of a linear extension with $\mathrm{opt}(P, g, c)$. If all linear extension are optimal, there is nothing to show. Now we choose a positive $\varepsilon < d/K$. By Theorem 19 there exists an optimal linear extension $L$ for $\mathrm{opt}(P, g + h_\varepsilon, c)$ starting with $S$. Since

$$\begin{aligned} \mathrm{opt}(P, g, c) + \mathrm{opt}(P, h_\varepsilon, c) &\le \mathrm{opt}(P, g + h_\varepsilon, c) < \mathrm{opt}(P, g, c) + d/K \cdot K \\ &\le \min\{cx : x \in \mathrm{vert}(B(P, g)), cx > \mathrm{opt}(P, g, c)\}, \end{aligned}$$

$L$ must be optimal for $\text{opt}(P, g, c)$, too. □

Like in Sidney's paper the proof above shows that in general we can start with an arbitrary $\rho^*$-maximal ideal to construct an optimal linear extension. Hence, the algorithm of Lawler [1978] for optimizing over series-parallel ordered sets can be used here, too.

**Theorem 21** *Let $P$ be a series-parallel poset and $g : \mathcal{I}(P) \to \mathbb{R}$ be supermodular and compatible, let $c \in \mathbb{R}^P$. The optimization problem $\text{opt}(P, g, c)$ can be solved in $\mathcal{O}(n \log n)$ time.* □

In the case of an antichain $P$, the algorithm reduces to the greedy algorithm with the initial sorting phase done by mergesort.

## 6. Remarks and Open Questions

Base polytopes of series-parallel posets are a common generalization of base polytopes over sets (cf. Fujishige [1991]) and permutahedra of series-parallel posets which are defined as follows. With any permutation $\pi$ of an $n$-element set $E = \{1, \ldots, n\}$ we associate a *permutation vector* via $x(\pi) := (\pi(1), \ldots, \pi(n)) \in \mathbb{R}^E$. For a partially ordered set $P = (E, <_P)$, we consider only those permutations which are linear extensions of the poset and define the *permutahedron*

$$\text{perm}(P) = \text{conv}\{x(\pi) : \pi \text{ is a linear extension of } P\}.$$

In v. Arnim, Faigle and Schrader [1990] and Queyranne and Wang [1991] it is shown that the permutahedron of a series-parallel poset is given by the linear inequalities

$$\begin{aligned} |A|x(B) - |B|x(A) &\geq \tfrac{1}{2}|A||B|\,(|A| + |B|) \text{ for all series-reducible convex sets } (A, B), \\ x(I) &\geq \tfrac{1}{2}|I|(|I| + 1) \text{ for all ideals } I \in \mathcal{I}(P) \\ x(P) &= \tfrac{1}{2}|P|(|P| + 1). \end{aligned}$$

It is easily seen that $g(I) = \frac{1}{2}|I|(|I| + 1)$ is strictly supermodular and compatible and $f(A, B) = \frac{1}{2}|A||B|\,(|A| + |B|)$ is the function induced by $g$. Queyranne and Wang [1991] (see also v. Arnim and Schulz [1994]) extend this characterization to the *generalized permutahedron* which corresponds to the weighted case discussed below.

Queyranne and Schulz [1995] show that the problem of scheduling jobs with unit execution times and compatible release dates on $m$ machines with nonstationary speeds may be formulated in terms of optimizing linear functions over contra-polymatroids. E.g. for the case of zero release dates, let $P$ be an antichain of $n$ jobs, and suppose there are $m$ machines with processing rates $\sigma_i(\tau) \geq 0$. Define $t(i, 0) := 0$ and $t(i, k) := \min\{t : \int_{t(i,k-1)}^{t} \sigma_i(\tau) d\tau = 1\}$ as the earliest completion time of the $k$-th job on machine $i$. For $A \subseteq P$, let $\phi(A)$ be the sum of the $|A|$ smallest elements in the multiset $\{t(i, k) : 1 \leq i \leq m, 1 \leq k \leq n\}$. Then $\phi$ is supermodular, and the completion time vectors of all schedules (with minimum makespan) is $\{x \in \mathbb{R}^P : x(A) \geq \phi(A) \text{ for all } A \subseteq P, x(P) = \phi(P)\}$. By using the results of Section 3, we can immediately extend this description to *weak* orders. (A poset $P$ is a weak order, if it is the series-composition $A_1 \oplus \ldots \oplus A_s$ of a family of antichains $A_i$.) In this case, the compatibility requirement (2) is trivial because $I = I(B) \setminus (A \cup B)$ is unique, and the supermodularity of $\phi(A) := \min\{x(A) : x \text{ is a completion time vector of a minimum makespan schedule}\}$ follows from the supermodularity of $\phi$ on an antichain.

We have used the cardinality function in the convex set constraints to simplify the presentation. The arguments carry over to any positive weight function $w : E \to \mathbb{R}_+$. For a linear extension $L = e_1 \ldots e_n$ of $P$, let the *weighted incidence vector* $x$ of $L$ be the vector with components $x_{e_i} = 1/w_{e_i}\,(g(\{e_1, \ldots, e_i\}) - g(\{e_1, \ldots, e_{i-1}\}))$ for $1 \leq i \leq n$. Let $x_w(I) = \sum_{i \in I} w_i x_i$ and call a supermodular function $g : \mathcal{I}(P) \to \mathbb{R}$ *w-compatible* (on $P$) if for all series-reducible convex sets $(A, B)$ and any $I \in \mathcal{I}(P), I \subseteq E \setminus (A \cup B)$ such that $I \cup A \in \mathcal{I}(P)$ and $I \cup A \cup B \in \mathcal{I}(P)$ the term

$$w(A)[g(I \cup A \cup B) - g(I \cup A)] - w(B)[g(I \cup A) - g(I)] \tag{8}$$

is a constant independent of $I$. Consider the polytope $\mathbb{P}_w(P, g)$ defined by the inequalities

$$\begin{aligned}
w(A)x_w(B) - w(B)x_w(A) &\geq f(A, B) \text{ for all series-reducible convex sets } (A, B), \\
&\phantom{\geq} A, B \text{ series-prime}, \\
x_w(I) &\geq g(I) \text{ for all ideals } I \in \mathcal{I}(P), \\
x_w(P) &= g(P).
\end{aligned}$$

It is a technical exercise to derive the following corollary.

**Corollary 22** *Let $P$ be a series-parallel poset $P$ and $g : \mathcal{I}(P) \to \mathbb{R}$ be a supermodular and $w$-compatible function. Then a vector $x$ is a vertex of $\mathbb{P}_w(P, g)$ if and only if it is the weighted incidence vector of a linear extension.* □

We close with two open questions. First, the inequalities $|A|x(B) - |B|x(A) \geq f(A, B)$ closely resemble the defining system of pseudomatroids (cf. Chandrasekaran and Kabadi [1988]). Secondly, Faigle and Kern [1996] have introduced another type of greedy algorithm on posets, also generalizing the polymatroidal procedure. In both cases, it is not clear how these approaches relate to the base polytope of a poset.

## References


A. v. Arnim, U. Faigle, and R. Schrader: *The permutahedron of series-parallel posets*, Discrete Applied Mathematics 28 (1990), 3–9.

A. v. Arnim, R. Schrader, and Y. Wang: *The permutahedron of N-sparse posets*, Mathematical Programming 75 (1996), 1–18.

A. v. Arnim and A. S. Schulz: *Facets of the generalized permutahedron of a poset*, Preprint 386/1994, Department of Mathematics, University of Technology Berlin, 1994, to appear in Discrete Applied Mathematics.

R. Chandrasekaran and S. N. Kabadi, *Pseudomatroids*, Discrete Mathematics 71 (1988), 205–217.

U. Faigle and W. Kern, *Submodular linear programs on forests*, Mathematical Programming 72 (1996), 195–206.

A. Frank and É. Tardos: *Generalized polymatroids*, Mathematical Programming 42 (1988), 489–563.

S. Fujishige: *Submodular Functions and Optimization*, Annals of Discrete Mathematics 47, North-Holland, 1991.

E. L. Lawler: *Sequencing jobs to minimize total weighted completion time subject to precedence constraints*, Annals of Discrete Mathematics 2 (1978), 75–90.

M. Queyranne and A. S. Schulz: *Scheduling jobs with compatible release dates on parallel machines with nonstationary speeds*, in E. Balas and J. Clausen (eds.): Integer Programming and Combinatorial Optimization, Lecture Notes in Computer Science 920, Springer: Berlin, 1995, 307–320.

M. Queyranne and Y. Wang: *Single-machine scheduling polyhedra with precedence constraints*, Mathematics of Operations Research 16 (1991), 1–29.

A. S. Schulz: *The permutahedron of series-parallel posets*, Discrete Applied Mathematics 57 (1995), 85–90.

A. S. Schulz: *Polytopes and Scheduling*, PhD thesis, Department of Mathematics, University of Technology Berlin, 1996.

D. Seinsche: *On a property of the class of $n$-colourable graphs*, Journal of Combinatorial Theory Series B 16 (1974), 191–193.

J. B. Sidney: *Decomposition algorithms for single-machine sequencing with precedence relations and deferral costs*, Operations Research 23 (1975), 283–298.

W. E. Smith: *Various optimizers for single-stage production*, Naval Research and Logistics Quarterly 3 (1956), 59–66.