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Abstract

An embedding domain method for the stationary incompressible Navier-Stokes equations is presented. The method is useful for solving the equations or complicated-shaped or varying domains. The original domain is embedded in a so-called fictitious one. On the latter an equivalent formulation of the Navier-Stokes equation is derived. Existence and uniqueness results of its solution(s) are presented. The structure of and solution methods for the discrete systems and numerical issues are discussed. An algorithm for finding the trace of the original boundary in the fictitious domain are given. Numerical results for the fictitious domain method are compared with those from computations on the original domain. As test case the 2-D flow around a circular cylinder in a channel is used.

Keywords: Navier-Stokes equations, embedding domain technique

1 Introduction

Efficient numerical solution of fluid problems is still a challenging task in many applications and a topic of current mathematical research. One aspect is the generation of appropriate (or optimal) grids for complex and/or complicated-shaped domain. This becomes even more important if quantities shall be evaluated near or at the boundaries of the domain. This is the case for example if pressure differences or gradients near body walls in a flow shall be computed.

The effort of discretization and assembling of the grid-dependent system matrices and operators can be high, specifically in 3-D or if this procedure has to be repeated due to changes in the geometry, for example during a design or optimization process.

An alternative approach is based on the idea of embedding the current computational domain into a fixed one. This *fictitious domain* is discretized only once in the beginning. Local grid-refinement may be used to adjust the grid to the original, embedded domain. If this original domain is changed in a design process, only a part of the discrete system has to be reassembled. This technique we call *Fictitious Domain (FD) method.*

FD methods have been used widely for the abovementioned reasons, among others for example by Glowinski, Pan and Periaux [1] and Myslinki and Zochovski [2] for elliptic equations, Börgers [3] for the Stokes, and again Glowinski, Pan and Periaux [4] for the Navier-Stokes equations. FD methods have also been used as preconditioners, for example by Hakopian and Kuznetsov [5]. In shape optimization problems they have been used by Dankova and Haslinger, see e.g. [6], by Kunisch and Peichl [7] for the stationary heat equation, by Slawig for the stationary Stokes and Navier-Stokes system [8],[9].

In this paper we present a FD method for the incompressible stationary Navier-Stokes equations. We state the equations in weak form in the second section and recall some standard results of existence, uniqueness and regularity we will use later on. Then we present the motivation for the FD formulation of the equations, show its equivalence to the original problem and discuss existence and uniqueness. We continue by presenting the discretized system using Finite Elements and discuss possible solution methods and preconditioning techniques. We discuss the special structure of the discrete system of the FD formulation. In the next section we treat technical details of the method, namely the determination of the intersection between the former boundary of the original domain and the new fictitious domain. In the last section we show numerical results for a test case, namely the flow around a circular cylinder. We compared the solutions obtained by computations on the original domain and by the FD method with results given by a benchmark organized by the German research foundation DFG.

2 The stationary Navier-Stokes Equations

In this section we recall the weak formulation of the stationary Navier-Stokes equations and summarize well-known results on existence, uniqueness, and regularity of its solutions. The weak formulation is standard (compare e.g. [10, IV.(2.20)]). We concentrate on pure Dirichlet boundary conditions for the velocity. The case of free outflow (or "do nothing") boundary conditions (see [11]) is discussed briefly at the end of this section.

The stationary Navier-Stokes equations in a domain $\Omega \subset \mathbb{R}^d, d = 2, 3$, in weak form read: Find $(\mathbf{u}, p) \in H^1(\Omega)^d \times L_0^2(\Omega)$ such that

$$\begin{aligned} \nu(\nabla \mathbf{u}, \nabla \mathbf{v})_\Omega + (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v})_\Omega - (p, \operatorname{div} \mathbf{v})_\Omega &= (\mathbf{f}, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in H_0^1(\Omega)^d \\ (\operatorname{div} \mathbf{u}, q)_\Omega &= 0 \quad\quad\quad \forall q \in L_0^2(\Omega) \\ \mathbf{u} &= \Phi \quad\quad\quad \text{on } \Gamma := \partial\Omega \end{aligned} \tag{1}$$

where $(\cdot, \cdot)_\Omega$ here denotes the inner products on $L^2(\Omega), L^2(\Omega)^d$, and $L^2(\Omega)^{d \times d}$, respectively. The parameter $\nu > 0$ represents the inverse of the Reynolds number. The nonlinearity is defined by the operator $(\mathbf{u} \cdot \nabla) := \sum_{j=1}^2 u_j \frac{\partial}{\partial x_j}$. The inhomogeneity $\mathbf{f}$ shall be in $L^2(\Omega)^d$. For the boundary values on $\Gamma := \partial\Omega = \cup_i \bar{\Gamma}_i$ we assume

$$\Phi \in H^{1/2}(\partial\Omega)^d, \qquad \int_{\Gamma_i} \Phi \cdot \eta \, ds = 0. \tag{2}$$

The pressure is uniquely determined in $L^2(\Omega)$ only up to an additive constant, thus in $L^2(\Omega)/\mathbb{R}$. Endowed with the norm

$$\|q\|_{L^2(\Omega)/\mathbb{R}} \quad := \quad \min_{c \in \mathbb{R}} \|q + c\|_{L^2(\Omega)}$$

this space is isometrically isomorph to the space we use here, namely

$$L_0^2(\Omega) \quad := \quad \left\{ q \in L^2(\Omega) : \int_\Omega q \, dx = 0 \right\},$$

which (endowed with the $L^2(\Omega)$ inner product) is a Hilbert space as well.

We summarize some classical results on existence, uniqueness and regularity.

Theorem 1 *Let Ω be Lipschitz, $\mathbf{f} \in L^2(\Omega)^d$ and Φ satisfy (2). Then the following results hold.*

(a) Problem (1) has at least one solution $(\mathbf{u}, p) \in H^1(\Omega)^d \times L_0^2(\Omega)$.

(b) For every $\mathbf{u} \in H^1(\Omega)^d$ solving (1) the corresponding p is unique in $L_0^2(\Omega)$.

(c) For $\nu > \nu_0(\Omega, \mathbf{f}, \Phi)$ problem (1) has a unique solution $(\mathbf{u}, p) \in H^1(\Omega)^d \times L_0^2(\Omega)$. In the linear case of the Stokes equations or if $\mathbf{f} = \mathbf{0}$ and $\Phi = \mathbf{0}$ uniqueness is given without any restriction on ν.

(d) If Ω is of class C^2 and $\Phi \in H^{3/2}(\partial\Omega)^d$, then every solution $(\mathbf{u}, p)$ of (1) is in $H^2(\Omega)^d \times \left(H^1(\Omega) \cap L_0^2(\Omega)\right)$. There exists $C > 0$ independent of $\mathbf{f}$ and Φ such that

$$\|\mathbf{u}\|_{H^2(\Omega)^d} + \|p\|_{H^1(\Omega)} \quad \le \quad C \left(\|\mathbf{f}\|_{L^2(\Omega)^d} + \|\Phi\|_{H^{3/2}(\partial\Omega)^d} \right).$$

(e) If $d = 2$ and $\partial\Omega$ is a convex polygon, or a combination of a C^2 boundary with a convex polygon, then the results of (d) remain valid.

Proof: (a-c) can be found in [10]: (a) Theorem IV.2.3, (b) Theorem IV.1.4, Corollary I.2.4, (c) Theorem IV.2.4. The regularity results in (d) are based on those for the linear Stokes equations by treating the non-linearity as an additional inhomogeneity and using embedding and function space interpolation theorems, see for example [12, Prop. II.1.1, Remarks II.1.4,II.1.6]. For (e) see [13, Theorem pp.403-404]. □

2.1 Weak formulation with free outflow boundary conditions

From the physical point of view a given inflow on one boundary part (let us say Γ_{out}) into a bounded region induces a determined outflow on some other boundary part, due to the law of mass conservation. Thus it should not be necessary to prescribe the flow profile on Γ_{out} in the mathematical model. This is the motivation for the free outflow boundary conditions. The counterpart of (1) in this case reads: Find $(\mathbf{u}, p) \in H^1(\Omega)^d \times L^2(\Omega)$ such that

$$\begin{array}{rcll} \nu(\nabla \mathbf{u}, \nabla \mathbf{v})_\Omega + (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v})_\Omega - (p, \operatorname{div} \mathbf{v})_\Omega & = & (\mathbf{f}, \mathbf{v})_\Omega & \forall \mathbf{v} \in W \\ (\operatorname{div} \mathbf{u}, q)_\Omega & = & 0 & \forall q \in L^2(\Omega) \\ \mathbf{u} & = & \Phi & \text{on } \Gamma := \partial\Omega \setminus \bar{\Gamma}_{out} \end{array} \tag{3}$$

where we test with functions in

$$W := \{\mathbf{v} \in H^1(\Omega)^d, \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega \setminus \Gamma_{out}\}. \tag{4}$$

Thus we implicitly prescribe the condition

$$\nu \partial_\eta \mathbf{u} = p\eta \quad \text{ on } \Gamma_{out}, \tag{5}$$

where η denotes the outer unit normal vector with respect to $\partial\Omega$. The pressure space is now $L^2(\Omega)$ without normalization. For theoretical results for these boundary conditions we refer to the work of Rannacher, Heywood, Turek, see for example [11].

3 The Fictitious Domain Method

In this section we introduce an equivalent formulation of the Navier-Stokes equations (1) on a fixed domain. Furthermore we show existence and uniqueness of a solution. Again we treat Dirichlet boundary conditions and give only a brief remark on differences for the " do nothing" boundary conditions at the end of the section.

We choose the *fictitious domain* $\hat{\Omega}$ such that $\hat{\Omega} \supset \Omega$, and usually with a simple geometric shape, for example a rectangle in two space dimensions. Depending on their geometry Ω and $\hat{\Omega}$ may have some common boundary parts. We use the following notations:

$$\Gamma := \partial\Omega, \quad \Gamma_0 := \Gamma \cap \partial\hat{\Omega}, \quad \Gamma_1 := \Gamma \setminus \bar{\Gamma}_0, \quad \hat{\Gamma} := \partial\hat{\Omega} \setminus \partial\Omega, \quad \Omega^c := \hat{\Omega} \setminus \bar{\Omega}.$$

Here Γ_0 or $\hat{\Gamma}$ may be empty. We assume that both Ω and Ω^c are Lipschitz, which is not trivial in every case, compare the example in the left-hand picture in Fig. 1.

For simplicity we assume the following boundary conditions in (1):

$$\mathbf{u} \quad = \quad \Phi_0 \text{ on } \Gamma_0, \quad \mathbf{u} = \Phi_1 \text{ on } \Gamma_1$$

where both parts shall be connected. Condition (2) is slightly modified, i.e. we now assume

$$\Phi_i \in H_{00}^{1/2}(\Gamma_i)^d = \left\{\mathbf{h} \in H^{1/2}(\Gamma_i)^d : \text{there exists } \tilde{\mathbf{h}} \in H^{1/2}(\partial\Omega)^d : \tilde{\mathbf{h}}|_{\Gamma_i} = \mathbf{h}, \tilde{\mathbf{h}}|_{\partial\Omega\setminus\Gamma_i} = \mathbf{0}\right\},$$

compare [14, VII.§ 2 Section 2.1 Remark 1].

To motivate the FD formulation we consider at first the linear Stokes equations: Find $(\mathbf{u}, p) \in H^1(\Omega)^d \times L_0^2(\Omega)$ such that

$$\begin{array}{rcll} \nu(\nabla \mathbf{u}, \nabla \mathbf{v})_\Omega - (p, \operatorname{div} \mathbf{v})_\Omega & = & (\mathbf{f}, \mathbf{v})_\Omega & \forall \mathbf{v} \in H_0^1(\Omega)^d \\ (\operatorname{div} \mathbf{u}, q)_\Omega & = & 0 & \forall q \in L_0^2(\Omega) \\ \mathbf{u} & = & \Phi_0 & \text{on } \Gamma_0 \\ \mathbf{u} & = & \Phi_1 & \text{on } \Gamma_1 \end{array} \tag{6}$$

Their solution $\mathbf{u}$ also solves the constrained optimization problem

$$\min_{\substack{\mathbf{v} \in H^1(\Omega)^d, \mathbf{v}|_{\Gamma_0} = \Phi_0 \\ \mathbf{v}|_{\Gamma_1} = \Phi_1}} F(\mathbf{v}) := \frac{1}{2}(\nabla \mathbf{v}, \nabla \mathbf{v})_\Omega - (\mathbf{f}, \mathbf{v})_\Omega \qquad \text{s.t.} \qquad \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega. \tag{7}$$

Introducing the Lagrangian with the multiplier p corresponding to the zero divergence constraint and computing the necessary condition for a saddle point gives (6).

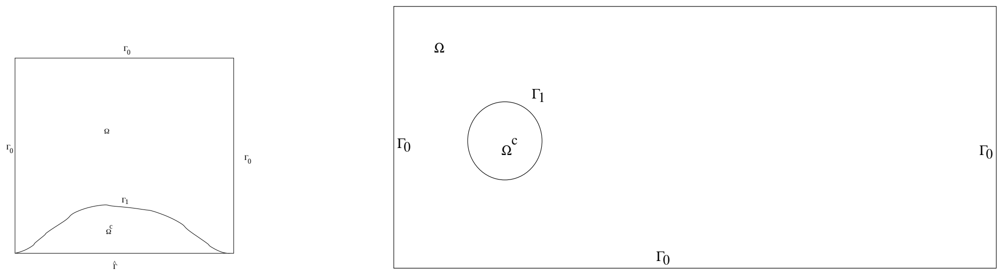

Figure 1: Two possible geometrical settings. The one on the right is suitable for the flow around a circular cylinder in a channel. On the right vertical boundary there may also free outflow conditions (5) be imposed, i.e. it may belong to Γ_{out} instead of Γ_0 (see Section 22.1).

We now formulate problem (7) on the fictitious domain $\hat{\Omega}$ and add the former boundary condition on Γ_1 as additional constraint, since this boundary part is an inner line or surface in $\hat{\Omega}$. On the boundary part $\hat{\Gamma}$ we impose homogeneous boundary Dirichlet conditions on the velocity. We thus introduce the space

$$\hat{V} := \{\hat{\mathbf{v}} \in H^1(\Omega)^d, \hat{\mathbf{v}} = \mathbf{0} \text{ on } \hat{\Gamma}\}, \tag{8}$$

and study the problem

$$\min_{\substack{\hat{\mathbf{v}} \in \hat{V} \\ \hat{\mathbf{v}}|_{\Gamma_0} = \Phi_0}} \hat{F}(\hat{\mathbf{v}}) := \frac{1}{2}(\nabla\hat{\mathbf{v}}, \nabla\hat{\mathbf{v}})_{\hat{\Omega}} - (\hat{\mathbf{f}}, \hat{\mathbf{v}})_{\hat{\Omega}} \qquad \text{s.t.} \quad \left\{ \begin{array}{rcll} \operatorname{div}\hat{\mathbf{v}} & = & 0 & \text{in } \hat{\Omega} \\ \hat{\mathbf{v}} & = & \Phi_1 & \text{on } \Gamma_1 \end{array} \right. \tag{9}$$

where $\hat{\mathbf{f}}$ is an appropriate extension of $\mathbf{f}$ onto $\hat{\Omega}$. The Lagrangian is now given as

$$\hat{L}(\hat{\mathbf{u}}, \hat{p}, g) \quad := \quad \hat{F}(\hat{\mathbf{u}}) - (\hat{p}, \operatorname{div}\hat{\mathbf{u}})_{\hat{\Omega}} - \langle g, \tau_1\hat{\mathbf{u}}\rangle_{\Gamma_1}.$$

Here g is the additional multiplier corresponding to the second constraint in (9) and $\tau_1\hat{\mathbf{v}} := \hat{\mathbf{v}}|_{\Gamma_1}$ denotes the inner trace operator on Γ_1. The exact choice of the spaces, e.g. in the dual pairing $\langle\cdot,\cdot\rangle_{\Gamma_1}$ will be discussed below. The necessary conditions for a saddle point $(\hat{\mathbf{u}}, \hat{p}, g)$ of $\hat{L}$ are given as

$$\begin{array}{rcll} \nu(\nabla\hat{\mathbf{u}}, \nabla\hat{\mathbf{v}})_{\hat{\Omega}} - (\hat{p}, \operatorname{div}\hat{\mathbf{v}})_{\hat{\Omega}} - \langle g, \tau_1\hat{\mathbf{v}}\rangle_{\Gamma_1} & = & (\hat{\mathbf{f}}, \hat{\mathbf{v}})_{\hat{\Omega}} & \forall\hat{\mathbf{v}} \in H_0^1(\hat{\Omega})^d \\ (\operatorname{div}\hat{\mathbf{u}}, \hat{q})_{\hat{\Omega}} & = & 0 & \forall\hat{q} \in L_0^2(\hat{\Omega}) \\ \hat{\mathbf{u}} & = & \Phi_0 & \text{on } \Gamma_0 \\ \hat{\mathbf{u}} & = & \Phi_1 & \text{on } \Gamma_1. \end{array} \tag{10}$$

Even though the representation of a solution to the weak Navier-Stokes equations as a minimizer as in (7) is not valid anymore, we nevertheless formulate an analogous system by just adding the nonlinear term. The resulting *FD formulation of the Navier-Stokes equations* reads: Find $(\hat{\mathbf{u}}, \hat{p}, g)$ such that

$$\begin{array}{rcll} \nu(\nabla\hat{\mathbf{u}}, \nabla\hat{\mathbf{v}})_{\hat{\Omega}} + (\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, \hat{\mathbf{v}})_{\hat{\Omega}} - (\hat{p}, \operatorname{div}\hat{\mathbf{v}})_{\hat{\Omega}} - \langle g, \tau_1\hat{\mathbf{v}}\rangle_{\Gamma_1} & = & (\hat{\mathbf{f}}, \hat{\mathbf{v}})_{\hat{\Omega}} & \forall\hat{\mathbf{v}} \in H_0^1(\hat{\Omega})^d \\ (\operatorname{div}\hat{\mathbf{u}}, \hat{q})_{\hat{\Omega}} & = & 0 & \forall\hat{q} \in L_0^2(\hat{\Omega}) \\ \hat{\mathbf{u}} & = & \Phi_0 & \text{on } \Gamma_0 \\ \hat{\mathbf{u}} & = & \Phi_1 & \text{on } \Gamma_1. \end{array} \tag{11}$$

We now show that the restriction of a solution to (11) onto the original domain Ω solves (1).

Theorem 2 *Let* $(\hat{\mathbf{u}}, \hat{p}, g) \in \hat{V} \times L^2(\hat{\Omega}) \times \left(H_{00}^{1/2}(\Gamma_1)^d\right)^*$ *be a solution to (11). Then* $(\mathbf{u}, p) := (\hat{\mathbf{u}}, \hat{p})|_\Omega \in H^1(\Omega)^d \times L^2(\Omega)$ *is a solution to (1).*

Proof: We take any $(\mathbf{v}, q) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ and denote by $(\tilde{\mathbf{v}}, \tilde{q})$ its extension by zero onto $\hat{\Omega}$ which clearly is in $H_0^1(\hat{\Omega})^d \times L_0^2(\hat{\Omega})$. Testing (11) with this pair we obtain

$$\begin{aligned} \nu(\nabla\hat{\mathbf{u}}, \nabla\mathbf{v})_\Omega + (\hat{\mathbf{u}} \cdot \nabla\hat{\mathbf{u}}, \nabla\mathbf{v})_\Omega - (\hat{p}, \operatorname{div}\mathbf{v})_\Omega &= (\mathbf{f}, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in H_0^1(\Omega)^d \\ (\operatorname{div}\hat{\mathbf{u}}, q)_\Omega &= 0 \qquad\quad\ \forall q \in L_0^2(\Omega). \end{aligned} \tag{12}$$

Since the boundary conditions are satisfied we have that $(\mathbf{u}, p) := (\hat{\mathbf{u}}, \hat{p})|_\Omega \in H^1(\Omega)^d \times L^2(\Omega)$ is a solution to (1). $\square$

Note that in this case $p \in L^2(\Omega)$ only, not in $L_0^2(\Omega)$. Clearly $(\mathbf{u}, \bar{p}) := (\hat{\mathbf{u}}, \hat{p} - c)|_\Omega$ is a solution to (1) in $H^1(\Omega)^d \times L_0^2(\Omega)$ if

$$c = \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega \hat{p}\, dx. \tag{13}$$

Using the same technique as in the proof of the above theorem on Ω^c we deduce the following result.

Remark 1 *If* Ω^c *is Lipschitz, then* $(\hat{\mathbf{u}}, \hat{p})|_{\Omega^c} \in H^1(\Omega^c)^d \times L^2(\Omega^c)$ *solves the weak Navier-Stokes equations (1) on* Ω^c *with inhomogeneity* $\hat{\mathbf{f}}|_{\Omega^c} \in L^2(\Omega^c)^d$ *and the boundary conditions* $\mathbf{u} = \Phi_0$ *on* $\Gamma_0, \mathbf{u} = \mathbf{0}$ *on* $\hat{\Gamma}$.

We obtain the following representation of the Lagrange multiplier g.

Theorem 3 *Let* $(\hat{\mathbf{u}}, \hat{p}, g) \in \hat{V} \times L^2(\hat{\Omega}) \times \left(H_{00}^{1/2}(\Gamma_1)^d\right)^*$ *be a solution to (11). Then the Lagrange multiplier* g *satisfies*

$$\langle g, \mathbf{h}\rangle_{\Gamma_1} \quad = \quad ([\nu\partial_\eta\mathbf{u} - p\eta]_{\Gamma_1}, \mathbf{h})_{\Gamma_1} \quad \forall \mathbf{h} \in H_{00}^{1/2}(\Gamma_1)^d \tag{14}$$

where $[\,\cdot\,]_{\Gamma_1}$ *denotes the jump along* Γ_1.

Proof: From (12) we deduce that

$$-\nu\triangle\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u} + \nabla p \quad = \quad \mathbf{f} \quad \text{in } L^2(\Omega)^d \tag{15}$$

and the boundary condition (5). With a similar argument we get

$$-\nu\triangle\mathbf{u}^c + \mathbf{u}^c \cdot \nabla\mathbf{u}^c + \nabla p^c \quad = \quad \mathbf{f}^c \quad \text{in } L^2(\Omega^c)^d \tag{16}$$

for $(\mathbf{u}^c, p^c) := (\hat{\mathbf{u}}, \hat{p})|_{\Omega^c}, \mathbf{f}^c := \hat{\mathbf{f}}|_{\Omega^c}$. We now test the first equation of (11) with arbitrary $\hat{\mathbf{v}} \in H_0^1(\hat{\Omega})^d$ and split up the integrals over $\hat{\Omega}$ in the L^2 inner products into ones over Ω and Ω^c, respectively. This gives

$$\begin{aligned} &\nu(\nabla\mathbf{u}, \nabla\hat{\mathbf{v}})_\Omega + (\mathbf{u} \cdot \nabla\mathbf{u}, \hat{\mathbf{v}})_\Omega - (p, \operatorname{div}\hat{\mathbf{v}})_\Omega \\ &+\nu(\nabla\mathbf{u}^c, \nabla\hat{\mathbf{v}})_{\Omega^c} + (\mathbf{u}^c \cdot \nabla\mathbf{u}^c, \hat{\mathbf{v}})_{\Omega^c} - (p^c, \operatorname{div}\hat{\mathbf{v}})_{\Omega^c} \\ &\qquad\qquad -\langle g, \tau_1\hat{\mathbf{v}}\rangle_{\Gamma_1} \quad = \quad (\hat{\mathbf{f}}, \hat{\mathbf{v}})_\Omega + (\mathbf{f}^c, \hat{\mathbf{v}})_{\Omega^c} \quad \forall \hat{\mathbf{v}} \in H_0^1(\hat{\Omega})^d. \end{aligned}$$

Applying Green's formula (see [10, Lemma I.1.4, I.(2.17)]) for both subdomains we obtain using (5), (15), (16), and $\hat{\mathbf{v}} = \mathbf{0}$ on $\Gamma_0 \cup \hat{\Gamma}$ that

$$(\nu\partial_\eta\mathbf{u} - p\eta, \hat{\mathbf{v}})_{\Gamma_1} + (\nu\partial_{\eta^c}\mathbf{u}^c - p^c\eta^c, \hat{\mathbf{v}})_{\Gamma_1} \quad = \quad \langle g, \tau_1\hat{\mathbf{v}}\rangle_{\Gamma_1} \quad \forall \hat{\mathbf{v}} \in H_0^1(\hat{\Omega})^d.$$

Here η, η^c are the unit outer normal vectors on Γ_1, where "outer" refers to Ω and Ω^c, respectively. Thus $\eta^c = -\eta$ and because of the definition of $H_{00}^{1/2}(\Gamma_1)^d$ we get (17). $\square$

In shape optimization problems the following consequence is quite useful, see for example [7],[8],[9].

Theorem 4 *If the inhomogeneity is extended by zero onto* $\hat{\Omega}$, *i.e.* $\mathbf{f}^c = \hat{\mathbf{f}}|_{\Omega^c} = \mathbf{0}$, *and* $\Phi_1 = \mathbf{0}$, *then* $(\hat{\mathbf{u}}, \hat{p})|_{\Omega^c} = (\mathbf{0}, c)$ *in* $H_0^1(\Omega^c)^d \times L^2(\Omega^c)$ *with some* $c \in \mathbb{R}$ *and* g *satisfies*

$$\langle g, \mathbf{h}\rangle_{\Gamma_1} \quad = \quad (\nu\partial_\eta\mathbf{u} - p\eta, \mathbf{h})_{\Gamma_1} \quad \forall \mathbf{h} \in H_{00}^{1/2}(\Gamma_1)^d. \tag{17}$$

Proof: From the uniqueness result of the Navier-Stokes equations on Ω^c (see Theorem 1(c)) it follows that $\hat{\mathbf{u}}|_{\Omega^c} = \mathbf{0}$ and $\hat{p}|_{\Omega^c} = 0$ in $L^2_0(\Omega^c)$ which implies that $\hat{p}$ is constant in $L^2(\Omega^c)$. $\square$

Representation (17) motivates to replace the dual pairing in the first equation of (10) by an $L^2(\Gamma_1)^d$ inner product. We then obtain the following FD formulation: Find $(\hat{\mathbf{u}}, \hat{p}, g) \in \hat{V} \times L^2(\hat{\Omega}) \times L^2(\Gamma_1)^d$ such that

$$\begin{array}{rcll} \nu(\nabla\hat{\mathbf{u}}, \nabla\hat{\mathbf{v}})_{\hat{\Omega}} + (\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, \hat{\mathbf{v}})_{\hat{\Omega}} - (\hat{p}, \operatorname{div}\hat{\mathbf{v}})_{\hat{\Omega}} - (g, \hat{\mathbf{v}})_{\Gamma_1} &=& (\hat{\mathbf{f}}, \hat{\mathbf{v}})_{\hat{\Omega}} & \forall \hat{\mathbf{v}} \in H^1_0(\hat{\Omega})^d \\ (\operatorname{div}\hat{\mathbf{u}}, \hat{q})_{\hat{\Omega}} &=& 0 & \forall \hat{q} \in L^2_0(\hat{\Omega}) \\ \hat{\mathbf{u}} &=& \mathbf{0} & \text{on } \Gamma_1 \end{array} \tag{18}$$

We now study existence and uniqueness of the Lagrange multiplier pair $(\hat{p}, g)$. Problem (11) fits in the following framework. Let X, M be Hilbert spaces and

$$b \quad : \quad X \times M \to \mathbb{R}, \quad d : X \times X \times X \to \mathbb{R}$$

two multi-linear and continuous mappings. For $l \in X^*$ we consider the following problem: Find $(x, \lambda) \in X \times M$ such that

$$\begin{array}{rcll} d(x; x, y) + b(y, \lambda) &=& \langle l, y\rangle_{X^*, X} & \text{for all } y \in X \\ b(x, g) &=& 0 & \text{for all } g \in M. \end{array} \tag{19}$$

Introducing the operator

$$B : X \to M^*, \qquad \langle By, \mu\rangle_{M^*, M} := b(y, \mu), \quad y \in X, \mu \in M. \tag{20}$$

and projecting on $\mathcal{V} = \ker B$ we obtain the problem: Find $x \in \mathcal{V}$ such that

$$d(x; x, y) \quad = \quad \langle l, y\rangle_{\mathcal{V}^*, \mathcal{V}} \qquad \text{for all } y \in \mathcal{V} \tag{21}$$

The crucial property for the existence of a unique Lagrange multiplier is the inf-sup or LBB condition stated below.

Theorem 5 *Let b satisfies the inf-sup condition, i.e. there exists β such that*

$$\inf_{g \in M} \sup_{y \in X} \frac{b(y, g)}{\|y\|_X \|g\|_M} \quad \geq \quad \beta > 0.$$

Then for every solution x of (21) there exists a unique $\lambda \in M$ such that (x, λ) is a solution to (19). The inf-sup condition is equivalent to the surjectivity of the operator B defined in (20).

Proof: See [10, Theorem IV.1.4.]. By [10, Lemma I.4.1] the inf-sup condition is equivalent to the fact that the adjoint operator B^* is an isomorphism from M onto the set $\mathcal{V}^0 := \{l \in X^* : \langle l, y\rangle_{X^*, X} = 0 \text{ for all } y \in \ker B\}$ and that for some $\beta > 0$

$$\|B^*\mu\|_{X^*} \quad \geq \quad \beta\|\mu\|_M \qquad \text{for all } \mu \in M \tag{22}$$

holds. Since B is continuous it is closed the Closed Range Theorem (see [15, Theorem II.19]) implies that B is surjective if and only if (22) holds. $\square$

In the FD formulation of the Navier-Stokes equations the trilinear form d has all necessary properties such that a solution

$$x = \hat{\mathbf{u}} \in \mathcal{V} \quad := \quad \{\hat{\mathbf{v}} \in H^1_0(\hat{\Omega})^d : \operatorname{div}\hat{\mathbf{v}} = 0 \text{ in } \hat{\Omega}, \hat{\mathbf{v}} = \mathbf{0} \text{ on } \Gamma_1\}$$

of (21) exists, i.e. that

$$d(\hat{\mathbf{u}}; \hat{\mathbf{u}}, \hat{\mathbf{v}}) := \nu(\nabla\hat{\mathbf{u}}, \nabla\hat{\mathbf{v}})_{\hat{\Omega}} + (\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, \hat{\mathbf{v}})_{\hat{\Omega}} \quad = \quad (\hat{\mathbf{f}}, \hat{\mathbf{v}})_{\hat{\Omega}} \quad \forall \hat{\mathbf{v}} \in \mathcal{V}. \tag{23}$$

The form b, given by

$$b(\hat{\mathbf{v}}; \hat{q}, \mathbf{h}) \quad := \quad -(\hat{q}, \operatorname{div}\hat{\mathbf{v}})_{\hat{\Omega}} - (\mathbf{h}, \hat{\mathbf{v}})_{\Gamma_1},$$

is continuous on $X \times L^2_0(\hat{\Omega}) \times L^2(\Gamma_1)^d$. The space M will be chosen as an appropriate subspace of $L^2_0(\hat{\Omega}) \times L^2(\Gamma_1)^d$ such that the operator B in (20), here given by

$$\langle B\hat{\mathbf{v}}, (\hat{q}, \mathbf{h})\rangle_{M^*, M} \quad := \quad -(\hat{q}, \operatorname{div}\hat{\mathbf{v}})_{\hat{\Omega}} - (\mathbf{h}, \tau_1\hat{\mathbf{v}})_{\Gamma_1}, \ \hat{\mathbf{v}} \in H^1_0(\hat{\Omega})^d, (\hat{q}, \mathbf{h}) \in M, \tag{24}$$

becomes surjective. The right choice is stated in the following theorem.

Theorem 6 *The operator B defined in (24) maps $H_0^1(\hat{\Omega})^d$ onto the space*

$$M \quad := \quad \left\{ (\hat{q}, \mathbf{h}) \in L_0^2(\hat{\Omega}) \times H_{00}^{1/2}(\Gamma_1)^d : \int_\Omega \hat{q}\, dx = \int_{\Gamma_1} \mathbf{h} \cdot \eta \, ds \right\}.$$

Proof: Let $\hat{q} \in L_0^2(\hat{\Omega})$. By [10, Corollary I.2.4] there exists $\hat{\mathbf{v}} \in H_0^1(\hat{\Omega})^d$ satisfying $\operatorname{div} \hat{\mathbf{v}} = \hat{q}$ in $\hat{\Omega}$. Thus $\hat{\mathbf{v}}|_\Omega \in H^1(\Omega)^d$ and the trace of $\hat{\mathbf{v}}$ on $\partial\Omega$ is in $H_{00}^{1/2}(\partial\Omega)^d$. This implies $\tau_1 \hat{\mathbf{v}} \in H_{00}^{1/2}(\Gamma_1)^d$ by definition of this space. For any $\mathbf{h} \in H_{00}^{1/2}(\Gamma_1)^d$ thus also $\mathbf{h} - \tau_1 \hat{\mathbf{v}} \in H_{00}^{1/2}(\Gamma_1)^d$. Again by definition there exists $\mathbf{w} \in H^1(\Omega)^d$ with $\tau_1 \mathbf{w} = \mathbf{h} - \tau_1 \hat{\mathbf{v}}$. Analogously the trace of $\hat{\mathbf{v}}$ on $\partial\Omega^c$ is in $H_{00}^{1/2}(\partial\Omega^c)^d$, and there exists $\mathbf{w}^c \in H^1(\Omega^c)^d$ with $\tau_1 \mathbf{w}^c = \mathbf{h} - \tau_1 \hat{\mathbf{v}}$. Thus the function

$$\hat{\mathbf{w}} \quad := \quad \left\{ \begin{array}{lll} \mathbf{w} & \text{on} & \Omega \\ \mathbf{w}^c & \text{on} & \Omega^c \end{array} \right\} \in H_0^1(\hat{\Omega})^d$$

satisfies $\tau_1 \hat{\mathbf{w}} \; = \; \mathbf{h} - \tau_1 \hat{\mathbf{v}}$. If additionally

$$\int_{\Gamma_1} (\mathbf{h} - \tau_1 \hat{\mathbf{v}}) \cdot \eta \, ds \quad = \quad 0, \tag{25}$$

then by [10, Lemma I.2.2] both $\mathbf{w}$ and $\mathbf{w}^c$ can be chosen divergence-free in Ω and Ω^c, respectively. Thus $\operatorname{div} \hat{\mathbf{w}} = 0$ in $\hat{\Omega}$. This implies that $\hat{\mathbf{u}} := \hat{\mathbf{v}} + \hat{\mathbf{w}} \in H_0^1(\hat{\Omega})^d$ satisfies

$$\operatorname{div} \hat{\mathbf{u}} = \hat{q} \text{ in } \hat{\Omega}, \qquad \tau_1 \hat{\mathbf{u}} = \mathbf{h}.$$

Condition (25) implies

$$\int_{\Gamma_1} \mathbf{h} \cdot \eta \, ds \quad = \quad \int_{\Gamma_1} \hat{\mathbf{v}} \cdot \eta \, ds = \int_{\partial\Omega} \hat{\mathbf{v}} \cdot \eta \, ds = \int_\Omega \operatorname{div} \hat{\mathbf{v}} \, dx = \int_\Omega \hat{q} \, dx.$$

□

3.1 FD formulation for free outflow boundary conditions

Here we may keep the basic notations of the section if we set $\Gamma := \partial\Omega \setminus \Gamma_{out}$. The FD formulation reads: Find $(\hat{\mathbf{u}}, \hat{p}, g) \in \hat{V} \times L^2(\hat{\Omega}) \times L^2(\Gamma_1)^d$ such that

$$\begin{array}{rcll} \nu(\nabla \hat{\mathbf{u}}, \nabla \hat{\mathbf{v}})_{\hat{\Omega}} + (\hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}}, \hat{\mathbf{v}})_{\hat{\Omega}} - (\hat{p}, \operatorname{div} \hat{\mathbf{v}})_{\hat{\Omega}} - (g, \hat{\mathbf{v}})_{\Gamma_1} & = & (\hat{\mathbf{f}}, \hat{\mathbf{v}})_{\hat{\Omega}} & \forall \hat{\mathbf{v}} \in \hat{W} \\ (\operatorname{div} \hat{\mathbf{u}}, \hat{q})_{\hat{\Omega}} & = & 0 & \forall \hat{q} \in L_0^2(\hat{\Omega}) \\ \hat{\mathbf{u}} & = & \mathbf{0} & \text{on } \Gamma_1 \end{array} \tag{26}$$

where

$$\hat{W} := \{ \hat{\mathbf{v}} \in H^1(\Omega)^d, \hat{\mathbf{v}} = \mathbf{0} \text{ on } \partial\hat{\Omega} \setminus \Gamma_{out} \}. \tag{27}$$

4 The discrete systems and their solution

In this section we discuss the forms of and solution and preconditioning techniques for the systems in the FD method obtained by a Finite Element discretization. We start with a standard formulation on the original domain Ω and discuss the changes that occur in the fictitious domain version (on $\hat{\Omega}$) afterwards.

We begin with the Stokes problem, since its solution can be used as starting point for an iterative scheme to solve the non-linear Navier-Stokes equations. Moreover the latter themselves can be solved in a gradient or conjugate gradient least squares algorithm which consists of a sequence of Stokes-like systems, see below. This is also the case when solving the unsteady Navier-Stokes problem via operator splitting techniques, see [4].

Discretization of the weak Stokes equations (6) results in a linear system of the form

$$\begin{pmatrix} \nu A & B^T \\ B & -\beta C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} \quad = \quad \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{28}$$

Here u is the discrete velocity vector, p the pressure, and f the vector-valued inhomogeneity in the Stokes equation, The matrices A, B are discretizations of the vector-valued negative weak Laplacian of the negative weak divergence operator, respectively. The entry $-\beta C$ is only present if a pressure stabilization is used, i.e. if the choice of velocity and pressure basis functions does not satisfy the discrete inf-sup condition. This is for example the case if elementwise linear basis functions are used for both u and p. Then C is chosen as a positive definite matrix and $\beta > 0$. A suitable choice for C in any case is the mass matrix M_p of the pressure basis functions. Some stabilization schemes also add an inhomogeneity in the second equation of (28) that we omit here for simplicity. For more details see [16],[17]. For stable ansatz spaces (Taylor-Hood elements etc.) the stabilization parameter β is set to zero.

The system can be solved working on the Schur complement

$$(B(\nu A)^{-1}B^T + \beta C)p = B(\nu A)^{-1}f \tag{29}$$

that is obtained by resolving the first equation in (28) for u and inserting it into the second. Using a stable element or an appropriate stabilization the system matrix is positive definite and (29) can be efficiently solved by a preconditioned conjugate gradient (*pcg*) method. In [18],[19] Wathen and Silvester compared different preconditioners, among them simple diagonal ones. Recalling that $\triangle = \text{div grad}$ and thus approximating

$$BA^{-1}B^T \approx M_p \tag{30}$$

a suitable choice for a preconditioner is $Q = \nu^{-1}M_p + \tau C$. If a diagonal preconditioner is desired one can use $diag(sum(Q))$ where the sum of Q is taken column- or row-wise.

For the fictitious domain formulation the system matrices A, B, C now are assembled for a grid on $\hat{\Omega}$, and additionally the negative weak trace operator τ_1 has to be discretized, resulting in a matrix D. The FD system retains the form of (28) with B, C replaced by

$$\tilde{B} := \begin{pmatrix} B \\ D \end{pmatrix}, \quad \tilde{C} := \begin{pmatrix} -\beta C & 0 \\ 0 & 0 \end{pmatrix}, \tag{31}$$

respectively. We present the structure of the matrix D for a general Ritz-Galerkin approximation, i.e. test and ansatz spaces are equal. Let $\{\varphi_i\}_{i=1,\ldots,N_v}$ denote the one-dimensional ansatz/test functions for every velocity component and $\{\psi_j\}_{j=1,\ldots,N_g}$ those for every component of the Lagrange multiplier g, then for

$$\begin{aligned} \hat{\mathbf{u}} &:= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, u_1 = \sum_{i=1}^{N_v} u_{1i}\varphi_i, u_2 = \sum_{i=1}^{N_v} u_{2i}\varphi_i, \\ \mathbf{g} &:= \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, g_1 = \sum_{i=1}^{N_v} g_{1i}\psi_i, g_2 = \sum_{i=1}^{N_v} g_{2i}\psi_i \end{aligned}$$

and the vector-valued test functions

$$\begin{aligned} &(\hat{\mathbf{v}}_i)_{i=1,\ldots,2N_v}, \quad \hat{\mathbf{v}}_i := \begin{pmatrix} \varphi_i \\ 0 \end{pmatrix}, \hat{\mathbf{v}}_{N_v+i} := \begin{pmatrix} 0 \\ \varphi_i \end{pmatrix}, i = 1, \ldots, N_v, \\ &(\mathbf{h}_i)_{i=1,\ldots,2N_g}, \quad \mathbf{h}_i := \begin{pmatrix} \psi_i \\ 0 \end{pmatrix}, \mathbf{h}_{N_g+i} := \begin{pmatrix} 0 \\ \psi_i \end{pmatrix}, i = 1, \ldots, N_g, \end{aligned}$$

we obtain the discrete equations

$$\begin{aligned} (\hat{\mathbf{u}}, \mathbf{h}_i)_{\Gamma_1} &= \left(\hat{\mathbf{u}}, \begin{pmatrix} \psi_i \\ 0 \end{pmatrix}\right)_{\Gamma_1} = (u_1, \psi_i)_{\Gamma_1} = \sum_{j=1}^{N_v} u_{1j}(\varphi_j, \psi_i)_{\Gamma_1} = \sum_{j=1}^{N_v} d_{ij}u_{1j}, \\ (\hat{\mathbf{u}}, \mathbf{h}_{N_g+i})_{\Gamma_1} &= \left(\hat{\mathbf{u}}, \begin{pmatrix} 0 \\ \psi_i \end{pmatrix}\right)_{\Gamma_1} = (u_2, \psi_i)_{\Gamma_1} = \sum_{j=1}^{N_v} u_{2j}(\varphi_j, \psi_i)_{\Gamma_1} = \sum_{j=1}^{N_v} d_{ij}u_{2j}, \end{aligned}$$

both for $i = 1, \ldots, N_g$. The block matrix D is thus given as

$$D = \begin{pmatrix} -(d_{ij})_{i=1,\ldots,N_g,j=1,\ldots,N_v} & 0 \\ 0 & -(d_{ij})_{i=1,\ldots,N_g,j=1,\ldots,N_v} \end{pmatrix} \in \mathbb{R}^{2N_g \times 2N_v} \tag{32}$$

with

$$d_{ij} = (\psi_i, \varphi_j)_{\Gamma_1}, \quad i = 1, \ldots, N_g, j = 1, \ldots, N_v.$$

The nonlinear Navier-Stokes system can be solved in different ways. One opportunity is to use a fix point iteration, see [10, Remarks IV.1.2,1.3,2.2]:
Algorithm 1:

(i) Choose $\hat{\mathbf{u}}$ with $\operatorname{div}\hat{\mathbf{u}} = 0, \tau_1\hat{\mathbf{u}} = \mathbf{0}$.

(ii) Set $\hat{\mathbf{w}} := \hat{\mathbf{u}}$ and compute $(\hat{\mathbf{u}}, \hat{p}, g)$ from (1) where the first equation is replaced by the linear equation

$$\nu(\nabla\hat{\mathbf{u}}, \nabla\hat{\mathbf{v}})_{\hat{\Omega}} + (\hat{\mathbf{w}}\cdot\nabla\hat{\mathbf{u}}, \hat{\mathbf{v}})_{\hat{\Omega}} - (\hat{p}, \operatorname{div}\hat{\mathbf{v}})_{\hat{\Omega}} - (g, \hat{\mathbf{v}})_{\Gamma_1} \quad = \quad (\hat{\mathbf{f}}, \hat{\mathbf{v}})_{\hat{\Omega}} \quad \forall\hat{\mathbf{v}} \in H_0^1(\hat{\Omega})^d$$

Repeat step (ii) until convergence is reached.

In the discrete version in each iteration step the following system has to be solved.

$$\begin{pmatrix} \nu A + N(w) & B^T & D^T \\ B - S_\beta(w) & -C_\beta & 0 \\ D & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ p \\ g \end{pmatrix} = \begin{pmatrix} f \\ k_\beta \\ 0 \end{pmatrix}$$

where $S_\beta, C_\beta, k_\beta$ are only present if pressure stabilization is used. In this case the system loses its symmetry. In any case the upper left block matrix $\nu A + N(w)$ is no longer constant (but depends on the current iterate w) nor positive definite (due to the linearized convective term). The linear system may be solved with a sparse direct or iterative solver (as *GMRES* for example).

A second method is an operator splitting technique where the divergence constraint is decoupled from the nonlinearity. Computing on the original domain Ω this scheme consists of the following two steps (compare [10, IV.§6.2]).
(i) Compute $(\mathbf{u}^{m+1/2}, p^{m+1/2})$ from

$$\left.\begin{array}{r} \nu(\nabla\mathbf{u}^{m+1/2}, \nabla\mathbf{v})_\Omega + r_m(\mathbf{u}^{m+1/2}, \mathbf{v})_\Omega \\ -(p^{m+1/2}, \operatorname{div}\mathbf{v})_\Omega \end{array}\right\} = \left\{\begin{array}{l} (\mathbf{f}, \mathbf{v})_\Omega + r_m(\mathbf{u}^m, \mathbf{v})_\Omega \\ -(\mathbf{u}^m\cdot\nabla\mathbf{u}^m, \mathbf{v})_\Omega \end{array}\right\} \quad \forall\mathbf{v} \in W$$
$$(\operatorname{div}\mathbf{u}^{m+1/2}, q)_\Omega = 0 \qquad \forall q \in L_0^2(\Omega)$$

(ii) Compute $\mathbf{u}^{m+1}$ from

$$\left.\begin{array}{r} \nu(\nabla\mathbf{u}^{m+1}, \nabla\mathbf{v})_\Omega + r_m(\mathbf{u}^{m+1}, \mathbf{v})_\Omega \\ +(\mathbf{u}^{m+1}\cdot\nabla\mathbf{u}^{m+1}, \mathbf{v})_\Omega \end{array}\right\} = \left\{\begin{array}{l} (\mathbf{f}, \mathbf{v})_\Omega + r_m(\mathbf{u}^{m+1/2}, \mathbf{v})_\Omega \\ +(p^{m+1/2}, \operatorname{div}\mathbf{v})_\Omega \end{array}\right\} \quad \forall\mathbf{v} \in W$$

The parameter r_m has to be chosen appropriately.

To use this scheme for the FD formulation one can either treat the additional constraint $\tau_1\hat{\mathbf{u}} = 0$ like the divergence constraint, or derive FD formulations of both steps in the algorithm above. The first variant leads to the following scheme.
Algorithm 2a:
(i) Compute $(\hat{\mathbf{u}}^{m+1/2}, \hat{p}^{m+1/2}, g^{m+1/2})$ from

$$\left.\begin{array}{r} \nu(\nabla\hat{\mathbf{u}}^{m+1/2}, \nabla\hat{\mathbf{v}})_{\hat{\Omega}} + r_m(\hat{\mathbf{u}}^{m+1/2}, \hat{\mathbf{v}})_{\hat{\Omega}} \\ -(\hat{p}^{m+1/2}, \operatorname{div}\hat{\mathbf{v}})_{\hat{\Omega}} - (g^{m+1/2}, \hat{\mathbf{v}})_{\Gamma_1} \end{array}\right\} = \left\{\begin{array}{l} (\hat{\mathbf{f}}, \hat{\mathbf{v}})_{\hat{\Omega}} + r_m(\hat{\mathbf{u}}^m, \hat{\mathbf{v}})_{\hat{\Omega}} \\ -(\hat{\mathbf{u}}^m\cdot\nabla\hat{\mathbf{u}}^m, \hat{\mathbf{v}})_{\hat{\Omega}} \end{array}\right\} \quad \forall\mathbf{v} \in H_0^1(\hat{\Omega})^d$$
$$(\operatorname{div}\hat{\mathbf{u}}^{m+1/2}, \hat{q})_\Omega = 0 \qquad \forall\hat{q} \in L_0^2(\hat{\Omega})$$
$$(\hat{\mathbf{u}}^{m+1/2}, \mathbf{h})_{\Gamma_1} = 0 \qquad \forall\mathbf{h} \in L^2(\Gamma_1)^d$$

(ii) Compute $\hat{\mathbf{u}}^{m+1}$ from

$$\left.\begin{array}{r} \nu(\nabla\hat{\mathbf{u}}^{m+1}, \nabla\hat{\mathbf{v}})_{\hat{\Omega}} + r_m(\hat{\mathbf{u}}^{m+1}, \hat{\mathbf{v}})_{\hat{\Omega}} \\ +(\hat{\mathbf{u}}^{m+1}\cdot\nabla\hat{\mathbf{u}}^{m+1}, \hat{\mathbf{v}})_{\hat{\Omega}} \end{array}\right\} = \left\{\begin{array}{l} (\hat{\mathbf{f}}, \hat{\mathbf{v}})_\Omega + r_m(\hat{\mathbf{u}}^{m+1/2}, \hat{\mathbf{v}})_{\hat{\Omega}} \\ +(\hat{p}^{m+1/2}, \operatorname{div}\hat{\mathbf{v}})_{\hat{\Omega}} \\ +(g^{m+1/2}, \hat{\mathbf{v}})_{\Gamma_1} \end{array}\right\} \quad \forall\hat{\mathbf{v}} \in H_0^1(\hat{\Omega})^d$$

The alternative one results in a different second step.

Algorithm 2b:
(i) as above.
(ii) Compute $(\hat{\mathbf{u}}^{m+1}, g^{m+1})$ from

$$\begin{array}{rcll} \left.\begin{array}{r} \nu(\nabla\hat{\mathbf{u}}^{m+1}, \nabla\hat{\mathbf{v}})_{\hat{\Omega}} + r_m(\hat{\mathbf{u}}^{m+1}, \hat{\mathbf{v}})_{\hat{\Omega}} \\ +(\hat{\mathbf{u}}^{m+1}\cdot\nabla\hat{\mathbf{u}}^{m+1}, \hat{\mathbf{v}})_{\hat{\Omega}} - (g^{m+1}, \hat{\mathbf{v}})_{\Gamma_1} \end{array}\right\} & = & \left\{\begin{array}{l} (\hat{\mathbf{f}}, \hat{\mathbf{v}})_{\Omega} + r_m(\hat{\mathbf{u}}^{m+1/2}, \hat{\mathbf{v}})_{\hat{\Omega}} \\ +(\hat{p}^{m+1/2}, \mathrm{div}\,\hat{\mathbf{v}})_{\hat{\Omega}} \end{array}\right\} & \forall\hat{\mathbf{v}} \in H_0^1(\hat{\Omega})^d \\ (\hat{\mathbf{u}}^{m+1}, \mathbf{h})_{\Gamma_1} & = & 0 & \forall\mathbf{h} \in L^2(\Gamma_1)^d. \end{array}$$

This second variant can also be viewed as just a different splitting of the Navier-Stokes operator on the fictitious domain with the additional constraint $\tau_1\hat{\mathbf{u}} = 0$.

The nonlinearity in the second step of each version can be treated for example by a fix point iteration as in Algorithm 1. An Alternative is to write (ii) as $F(\hat{\mathbf{u}}^{m+1}) = 0$ (in Algorithm 2a) or $F(\hat{\mathbf{u}}^{m+1}, g^{m+1}) = 0$ (in Algorithm 2b), use a least squares formulation and apply a gradient or conjugate gradient minimization algorithm to it, see [10, IV.§6.2] and [20], respectively. Then only linear quasi-Stokes problems (four with system matrix $\nu A + r_m M$ per minimization step) have to solved which can be treated in a similar way as the Stokes problems mentioned above. Here lies the difference between Algorithms 2a and 2b. In Algorithm 2a the system matrix does not depend on Ω since the Lagrange multiplier g is treated explicitly. In Algorithm 2b the matrix D (see (32)) representing the trace operator on Γ_1 has to be incorporated.

5 Determining the trace of $\partial\Omega$ in $\hat{\Omega}$

A main task in the FD method is to determine the intersection of the (former) boundary part Γ_1 with the mesh of the fictitious domain $\hat{\Omega}$. Furthermore it is necessary to integrate the product of the basis functions of the velocity vector and the Lagrange multiplier g along Γ_1.

We consider here the case $d = 2$ where Γ_1 is the graph of a real-valued function. For a rectangular, structured mesh on $\hat{\Omega}$ the determination of the intersection with Γ_1 is more ore less trivial. Below we present an algorithm to determine the trace if the mesh is triangular, but may be unstructured. For unstructured rectangular or combined rectangular and triangular meshes the algorithm has to be modified.

We assume that Γ_1 is approximated by a polygon $\tilde{\Gamma}_1$ with nodes $\{\gamma_n\}_{n=0,\ldots,N}, \gamma_n \in \mathbb{R}^2$. By $[\delta_0, \delta_1]$ we denote the line between $\delta_0, \delta_1 \in \mathbb{R}^d, d = 2$, by $\int_{\delta_0}^{\delta_1} dx$ the integral along this line, and by h_{min} the minimal meshsize in the vicinity of Γ_1. The algorithm below has the following properties.

- It computes a list of the nodes of the polygon $\tilde{\Gamma}_1$ and the intersection between the polygon itself and all triangle edges, i.e. the set $\{\gamma_n\}_{n=0,\ldots,N} \cup (\tilde{\Gamma}_1 \cap (\cup_k \partial T_k))$,
- It computes the contribution of all segments of the polygon to the system matrix D which is a discretization of the trace operator τ_1.
- It adjusts the supports of the Lagrange multiplier basis functions ψ_l such that a discrete inf-sup (or LBB) condition is satisfied.

Two list are used, the first one describes the relations between global and local node numbers for all triangles, i.e.

$GlobNode(k, j) = i \Longleftrightarrow$ local node j in triangle T_k has global number i.

The second one describes the neighborhood relations of the triangles, i.e.

$NeiTri(k, j) = i \Longleftrightarrow$ local node j in triangle T_k is opposite of triangle T_i.

The latter is often (not always) provided by the mesh generator or matrix assembling routines, respectively. The algorithm now is the following.
Algorithm 3:

(1) *Initialization:*

(i) $\delta_1 := \gamma_0, n := 1$.
(ii) $l := 1, L := 0$, choose $h_\Gamma \geq 2h_{min}$
(iii) Find first triangle $T_k \ni \delta_1$.

(2) *Main loop (over all points γ_n on Γ_1):*

While $n \leq N$:

(i) $\delta_0 := \delta_1$.

(ii) *Determine whether the next point γ_n lies inside the current triangle:*

If $\gamma_n \in T_k$:

$\delta_1 := \gamma_n$

else:

$\delta_1 := \partial T_k \cap [\delta_0, \gamma_n]$,

(iii) *If the support of the Lagrange multiplier basis function is big enough to satisfy the discrete inf-sup condition, go to the next basis function:*

$L := L + \|\delta_1 - \delta_0\|_2$.
If $L \geq h_\Gamma$:

$l := l + 1, L := 0$.

(iv) *Add the contribution of $[\delta_0, \delta_1]$ to the system matrix D:*

For $j = 1, 2, 3$:

$$d_{l,GlobNode(k,j)} := d_{l,GlobNode(k,j)} + \int_{\delta_0}^{\delta_1} \psi_l(x)\varphi_{GlobNode(k,j)}(x)\, dx$$

(v) *Continue the with next point on Γ_1 or the neighborhood triangle:*

If $\gamma_n \in T_k$:

$n := n + 1$

else:

(a) find edge $E_j \subset \partial T_k$ with $\delta_1 \in E_j$,

(b) find neighborhood triangle T_i with $E_j = \partial T_k \cap \partial T_i$ ($i = NeiTri(k, j)$),

(c) $k := i$.

We end this section with some remarks:

- Only in the initialization (1,iii) a loop over all triangles has to be performed to find the triangle containing the starting point of $\tilde{\Gamma}_1$. In step (v,a) the neighborhood relations are exploited to avoid this quadratic (with respect to the numbers of triangles) effort.
- The line integrals in (iv) are computed by appropriate Gaussian quadrature rules that are exact for the chosen degree of the product $\varphi\psi$ of ansatz functions. Choosing for example linear velocity basis functions (with pressure stabilization) and piecewise constant Lagrange multipliers the midpoint rule is exact.
- The support of the Lagrange multipliers generated by the algorithm is a polygon itself, not a straight line. The validity of the inf-sup condition is proved in [21] only for straight lines with $h_\Gamma \geq 2h_{min}$. Numerical experiments with our choice showed stable behavior (i.e. no oscillations) if this restriction is retained. This result however is not theoretical proved.
- The location of a point with respect to a given triangle (to decide whether the point lies inside or not, in (ii) and (iv)) can be easily computed by using barycentric coordinates. For points inside the triangle all of them are between 0 and 1. They can also be used to determine the location of a point outside a triangle more exactly.
- The algorithm in the form above fails if two triangles that are intersected by $\tilde{\Gamma}_1$ consecutively have only one common node (namely the intersection point with $\tilde{\Gamma}_1$), but no common edge, i.e. if $\gamma_{n-1} \in T_k, \gamma_{n+1} \in T_m$ and $T_k \cap T_m = \{\gamma_n\}$. The failure of the algorithm in this case is due to the fact that T_m is not a neighborhood triangle of T_k in the sense of the list $NeiTri$. The easiest remedy in such a case (where two barycentric coordinates are zero) is to move the point slightly along $\tilde{\Gamma}_1$, i.e. to set $\delta_1 := \gamma_n + \varepsilon(\gamma_n - \gamma_{n-1})$ with some small $\varepsilon > 0$ in step (ii).

6 Benchmark results

To investigate the effectivity of the FD method we computed the 2D-1 (steady) test case of the DFG benchmark [22] where the flow around a circular cylinder in a non-symmetric channel in two dimensions is studied. The computational domain $\Omega = R \setminus C$ is given as the rectangle $R := (0, 2.2)m \times (0, 0.41)m$ without the circle C with diameter $d = 0.1m$ and center at $(0.2, 0.2)m$. We computed the solution both on Ω and by the FD method on $\hat{\Omega} := R$, i.e. on the whole rectangle.

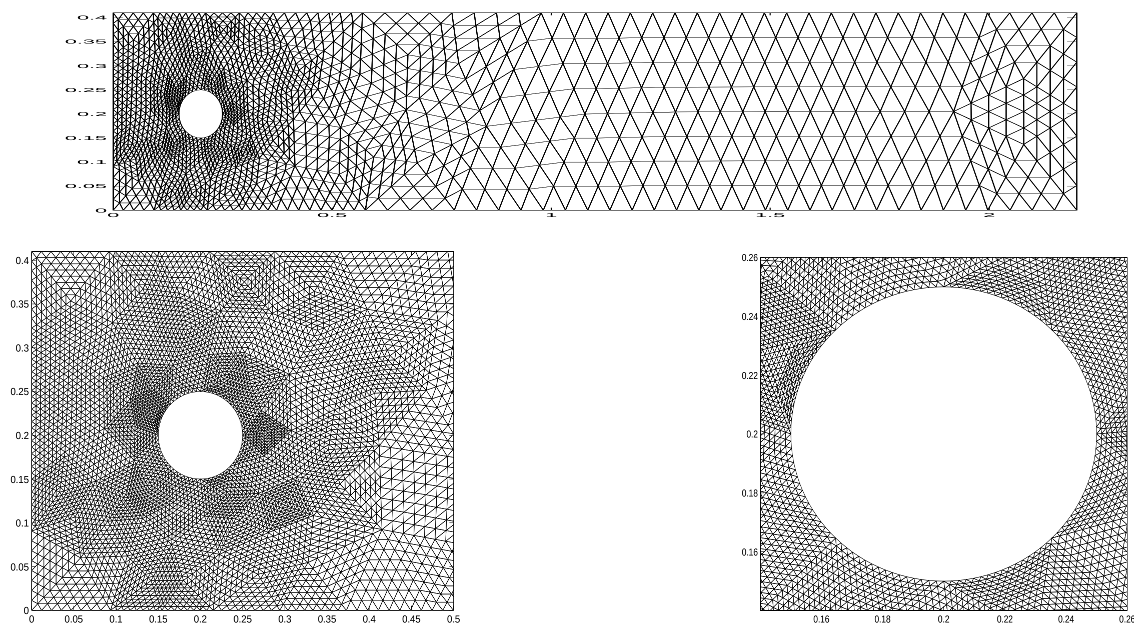

Figure 2: Meshes on Ω, top: mesh 1, bottom: part of mesh 2 (left) and mesh 3 (right) in the vicinity of the cylinder.

On the left boundary of the channel (i.e. the set $\Gamma_{in} = \{0\} \times (0, 0.41)m$, $\Gamma_{in} \subset \Gamma_0$ in our notation) a parabolic inflow

$$\mathbf{u} := (u, v), \quad u(0, y) = 4U_m(H - y)/H^2, \quad v = 0$$

with $H = 0.41m$ and $U_m = 0.3m/s$ was prescribed, leading to a Reynolds number $Re = \bar{U}d/\bar{\nu} = 20$, computed with $\bar{U} = 2u(0, H/2)/3 = 2U_m/3$ and viscosity $\bar{\nu} = 10^{-3}m^2/s$ and density $\rho = 1.0kg/m^3$. Note that in our dimensionless formulation of the equation we have used the notation $\nu = 1/Re$.

On the cylinder wall ∂C ($= \Gamma_1$ in our notation) and on the top and bottom wall of the cylinder homogeneous Dirichlet boundary conditions for $\mathbf{u}$ are given. On the channel outlet $\{2.2\} \times (0, 0.41)m(\subset \Gamma_0$ or Γ_{out} in our notation) the boundary conditions were not restricted, we tested both Dirichlet (prescribing a parabolic outflow) and free outflow conditions. The function $\mathbf{f}$ was set to zero.

6.1 Mesh generation

For the computations on the original domain Ω we used three unstructured meshes generated by Matlab's PDE Toolbox. On the fictitious domain $\hat{\Omega}$ we constructed three structured, locally adjusted meshes generated by refining a given basic triangulation of the rectangular channel by triangle bisection. The mesh data are given in Table 6.2, the meshes can be seen in Figures 2 and 3, respectively. The two groups of meshes (on Ω and $\hat{\Omega}$, respectively) were designed such that their minimal meshsizes h_{min} is somehow comparable. The mesh generator

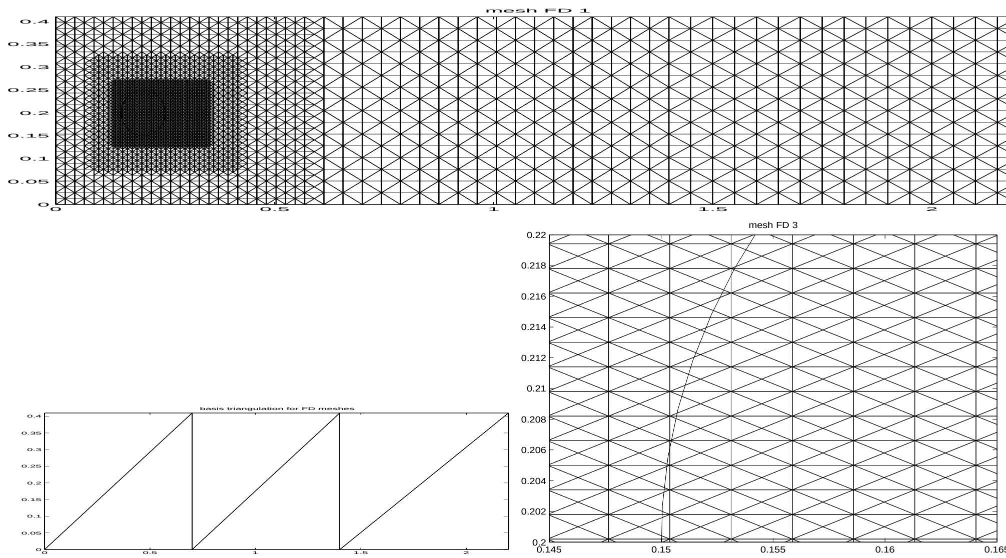

Figure 3: Meshes on fictitious domain $\hat{\Omega}$, top: mesh FD1, bottom left: basic triangulation, right: part of mesh FD3 in the vicinity of the cylinder.

used on Ω automatically refines locally in the vicinity of the boundary of the cylinder along with the better approximation of its circular geometry. Thus meshes 1-3 on Ω are somehow locally refined. We tried to achieve a similar local refinement for the FD meshes by defining some simple rectangular areas where additional triangle bisections are performed. To reduce the number of gridpoints further more more sophisticated refinement strategies for the FD meshes can be used. Moreover the inner part of the cylinder, i.e. the purely fictitious part Ω^c, can be discretized with a coarser mesh. On the other hand the idea of FD is not to adjust the mesh completely to the geometry, otherwise the method would lose some of its benefits. We thus only performed a small number of local refinements which are independent of moderate geometry changes, compare Figure 3.

6.2 Discussion

Our focus here is to compare the results between the standard method, i.e. computation on Ω, and the FD method, but not between the different numerical solvers and preconditioners. In both cases the equations were discretized by pressure-stabilized linear finite elements both for velocity and pressure. The stabilization parameter was chosen following Tezduyar, see [23]. The results on the original domain Ω and on the fictitious domain $\hat{\Omega}$ were both computed using a fixpoint iteration (Algorithm 1). The other algorithms were tested, produced similar results, but needed more computing time. The linear Stokes and quasi-Stokes systems were solved by a diagonally preconditioned conjugate gradient algorithm for the Schur complement. For the non-symmetric systems in Algorithm 1 we used a sparse direct solver.

We computed both pure Dirichlet and free outflow conditions. The overall flow profile shows no big differences, see Figure 4, neither between the two boundary conditions nor between computations on Ω and on $\hat{\Omega}$.

The length of the recirculation area behind the cylinder was computed by

$$L_a := \min\{x_i : u_i > 0\} - 0.2, \tag{33}$$

where x_i is the x-component of node i of the mesh and u_i the value of the x-component of the velocity at this point. Thus L_a was only determined with the accuracy of the meshsize. As can be seen in Figures 5, 6, and Table 6.2 its value is captured quite good by the FD method, and even better than by the computations on Ω. The values for the free outflow conditions

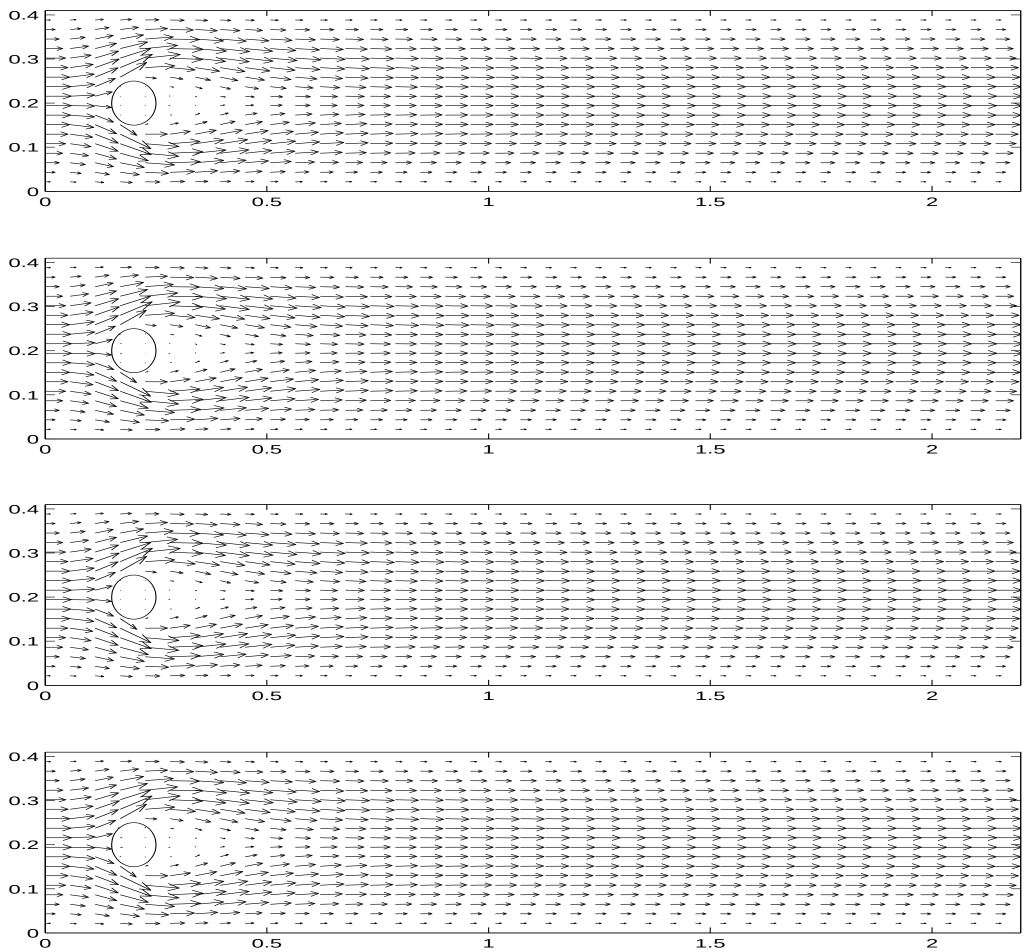

Figure 4: Upper two pictures: Velocity fields on Ω (mesh 3, top) and $\hat{\Omega}$ (mesh FD3, bottom), computed with Dirichlet boundary condition on the channel outlet. Lower two pictures: same for free outflow boundary conditions.

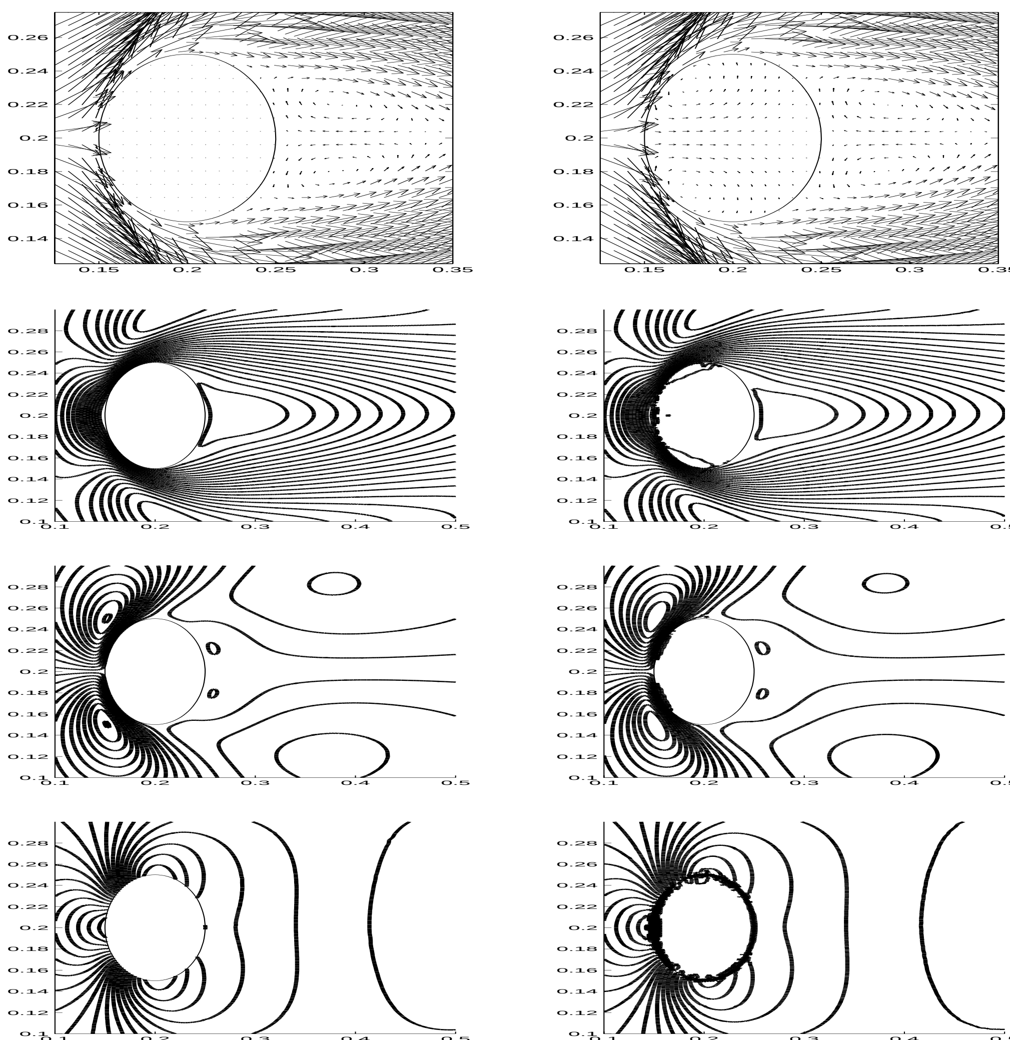

Figure 5: From top to bottom: Velocity field, isolines of its x and y components and of the pressure, computed on Ω (mesh 3, left) and $\hat{\Omega}$ (mesh FD3, right) with Dirichlet boundary condition on the channel outlet. Pressure is normalized to satisfy $p, \hat{p}|_{\Omega} \in L^2_0(\hat{\Omega})$.

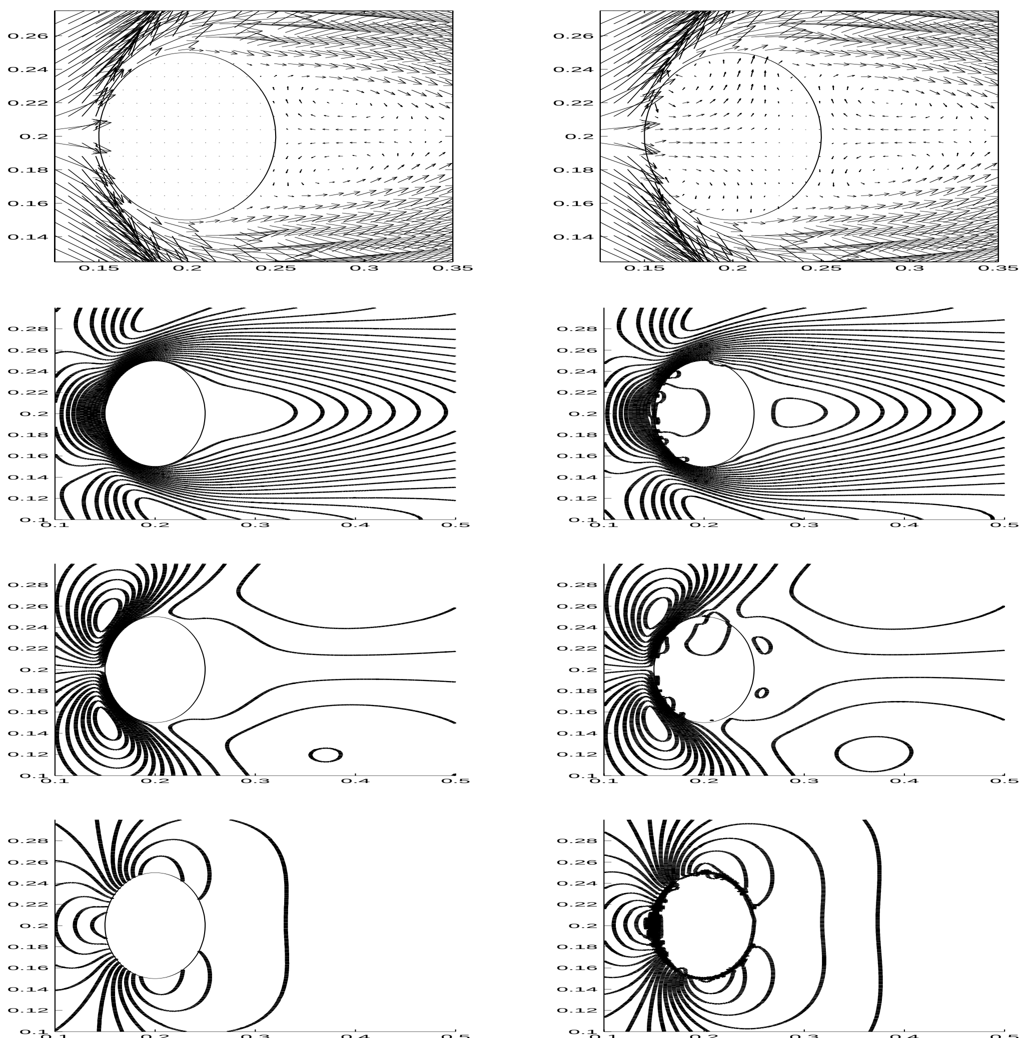

Figure 6: From top to bottom: Velocity field, isolines of its x and y components and of the pressure, computed on Ω (mesh 3, left) and $\hat{\Omega}$ (mesh FD3, right) with free outflow boundary condition on the channel outlet.

are slightly different. Moreover Figures 5, 6, and Table 6.2 numerically prove the result of Theorem 4, specifically that the velocity vanishes in Ω^c.

In flow applications often the behavior close to the domain boundary is crucial and interesting. In the DFG benchmark the pressure difference Δp between the leftmost and rightmost points P_a, P_b of the cylinder boundary should be computed. As can be seen from Table 6.2 the values computed on Ω again are quite accurate. Here P_a, P_b both are nodes of the mesh. This is not the case for the FD meshes where P_a and P_b lie in the interior of two triangles. For the linear finite elements we used this implies that the pressure value at P_a and P_b, respectively, has to be interpolated by the three values at the nodes of the corresponding triangle. By Theorem 4 we deduce that p usually has a jump at the cylinder boundary such that naturally the interpolated value is not very accurate. This is due to the fact that some nodes of the triangle lie in Ω, and the others in Ω^c. The same effect is to be expected for piecewise constant pressure ansatz functions, since then the constant pressure value on the triangle containing P_a and P_b, respectively, will be also some kind of mean value. Table 6.2 also shows values (denoted by $\Delta_* p$) of the pressure difference computed at a certain distance (depending on the meshsize) in x-direction away from the cylinder boundary, where the pressure values are quite accurate and thus the pressure difference is much better.

Concerning the convergence behavior of the linear and nonlinear solvers the results in Table 6.2 the incorporation of the additional matrix D in the FD method increases the number of required iteration steps until the methods converged, but the use of the rather simple diagonal preconditioning techniques proposed in Section 4 results in iteration numbers that are nearly independent of the meshsize.

	on Ω			on $\hat{\Omega}$ (FD)			bounds in [22]
mesh	1	2	3	FD1	FD2	FD3	lower...upper
triangles	3392	13568	54272	7638	30060	60188	
nodes	1788	6968	27504	3905	15202	30266	
minimal meshsize	0.0045	0.0021	0.0010	0.0032	0.0016	0.0016	
maximal meshsize	0.0667	0.0333	0.0167	0.0562	0.0281	0.0250	
parabolic outflow boundary condition							
unknowns	5363	20903	82511	11808	45605	90997	
Stokes it.	7	10	11	14	13	13	
nonlinear it.	4	5	6	7	6	6	
$\|\hat{\mathbf{u}}\|_{L^2(\Omega^c)^2}$	–	–	–	1.6e-3	9.6e-4	6.6e-4	
L_a	0.0814	0.0815	0.0837	0.0836	0.0836	0.0850	0.0842...0.0852
Δp (exact)	0.1178	0.1177	0.1176	0.0629	0.0610	0.0609	0.1172...0.1176
$\Delta_* p$ (best,				0.1172	0.1175	0.1174	
distance in x)				0.0067	0.0036	0.0037	
free outflow boundary condition on outflow							
unknowns	5364	20904	82512	11809	45606	90998	
Stokes it.	12	14	15	23	8	8	
nonlinear it.	4	5	5	7	4	6	
$\|\hat{\mathbf{u}}\|_{L^2(\Omega^c)^2}$	–	–	–	1.6e-3	1.2e-3	1.0e-3	
L_a	0.0815	0.0815	0.0842	0.0836	0.0890	0.0904	0.0842...0.0852
Δp (exact)	0.1192	0.1182	0.1177	0.0631	0.0581	0.0618	0.1172...0.1176
$\Delta_* p$ (best,				0.1176	0.1176	0.1174	
distance in x)				0.0067	0.0031	0.0018	

Table 1: Mesh properties, convergence behavior, and numerical results for the different meshes on Ω and $\hat{\Omega}$. The length of the recirculation area was computed by (33). The bounds in the last column refer to "optimal" values as proposed in [22].

mesh on Ω	1	2	2	3	3
mesh on $\hat{\Omega}$	FD1	FD1	FD2	FD2	FD3
$\|u - \hat{u}\|_\Omega\|_{L^2(\Omega)}$	0.003097	0.003346	0.002127	0.002136	0.002823
$\|v - \hat{v}\|_\Omega\|_{L^2(\Omega)}$	0.001442	0.001529	0.000680	0.000784	0.000462
$\|p - \hat{p}\|_\Omega\|_{L^2(\Omega)}$	0.001199	0.001769	0.001187	0.002263	0.001633

Table 2: Differences between velocity components u, v and pressure computed on Ω and FD solutions on $\hat{\Omega}$, both with Dirichlet boundary conditions, interpolated on the grid on Ω. Grid numbers refer to those in Table 6.2.

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