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M-MATRICES SATISFY NEWTON'S INEQUALITIES

OLGA HOLTZ

ABSTRACT. Newton's inequalities $c_n^2 \geq c_{n-1}c_{n+1}$ are shown to hold for the normalized coefficients c_n of the characteristic polynomial of any M- or inverse M-matrix. They are derived by establishing first an auxiliary set of inequalities also valid for both of these classes.

1. INTRODUCTION

The goal of the paper is to prove a conjecture made in [4] on a set of inequalities satisfied by (the elementary symmetric functions of) the eigenvalues of any M- or inverse M-matrix.

Let $\langle n \rangle$ denote the collection of all increasing sequences with elements from the set $\{1, 2, \ldots, n\}$, let $\#\alpha$ denote the size of the sequence α, and let α' denote the complementary or 'dual' sequence whose elements are all the integers from $\{1, 2, \ldots, n\}$ not in α. Given a matrix $A \in \mathbb{C}^{n\times n}$, the notation $A(\alpha)$ ($A[\alpha]$) will be used for the principal submatrix (minor) of A whose rows and columns are indexed by α. A matrix A is called a P-matrix if $A[\alpha] > 0$ for all $\alpha \in \langle n \rangle$. A is called a (nonsingular) M-matrix if it is a P-matrix and its off-diagonal entries are nonpositive. If in this definition the positivity of all principal minors is relaxed to nonnegativity, one obtains the class of all M-matrices, including the singular ones. The class of inverse M-matrices consists of matrices whose inverses are M-matrices. The M-matrices are an important class arising in many contexts (see, for example, [2, Chapter 6]).

Given a matrix A, let $c_j(A)$ denote the normalized coefficients of its characteristic polynomial:

$$c_j(A) := \sum_{\#\alpha=j} A[\alpha] \Big/ \binom{n}{j}, \qquad j = 0, \ldots, n.$$

The inequalities

$$c_j^2(A) \geq c_{j-1}(A)c_{j+1}(A), \quad j = 1, \ldots, n-1 \tag{1}$$

are known for real diagonal matrices, i.e., simply for sequences of real numbers (see [7] and references therein), as was first proved by Newton. Since the numbers

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On leave from CS Department, Univ. of Wisconsin, Madison, WI 53706, USA. Supported by Alexander von Humboldt Foundation.

c_j are invariant under similarity, Newton's inequalities (1) also hold for all diagonalizable matrices with real spectrum, and therefore also for the closure of this set, viz. for *all* matrices with real spectrum.

It was conjectured in [4] that Newton's inequalities are also satisfied by M- and inverse M-matrices (and by matrices similar to those). The next section contains proofs of several results on M-matrices and symmetric functions culminating in the proof of this fact.

2. Results

Let us begin by establishing a set of auxiliary inequalities first. Given an $n \times n$-matrix A and nonnegative integers m_1, m_2, k, define functions $S_{m_1,m_2,k}$ as follows

$$S_{m_1,m_2,k}(A) := \sum_{\substack{\alpha \in \langle n \rangle, \#\alpha = m_1, \\ \beta \in \langle n \rangle, \#\beta = m_2, \#\alpha \cap \beta = k}} A[\alpha] A[\beta]. \tag{2}$$

Theorem 1. *For any M- or inverse M-matrix A of order n and nonnegative integers $m < n$, $k < m$,*

$$S_{m,m,k}(A)/S_{m,m,k}(I_n) \geq S_{m+1,m-1,k}(A)/S_{m+1,m-1,k}(I_n), \tag{3}$$

where I_n denote the identity matrix of order n.

Proof. by induction.

Case 1 (induction base). If $k = 0$, $n = 2m$, then (3) is a special case of Theorem 1.3 from [6]. Indeed, since $n = 2m$, the functions $S_{m.m,0}$ and $S_{m+1,m-1,0}$ are immanants, $\lambda := (m, m)$ and $\mu := (m+1, m-1)$ are partitions of n, and μ majorizes λ. Then the normalized immanant corresponding to μ does not exceed the one corresponding to λ (beware a typo in [6], where the sign is reversed). If an M-matrix A is nonsingular, then $A^{-1}[\alpha] = A[\alpha']/\det A$ (see, e.g., [3, Section 1.4]), hence $S_{m,m,0}(A^{-1}) = S_{m,m,0}(A)/(\det A)^2$, $S_{m+1,m-1,0}(A^{-1}) = S_{m+1,m-1,0}(A)/(\det A)^2$, so the inequality (3) holds for the matrix A^{-1} as well.

Now assume (3) holds for all M- and inverse M-matrices of order smaller than n.

Case 2. Suppose $2m - k < n$ and A is an M- or inverse M-matrix. Then both normalized functions $S_{m,m,k}(A)/S_{m,m,k}(I_n)$ and $S_{m+1,m-1,k}(A)/S_{m+1,m-1,k}(I_n)$ can be obtained by averaging the terms $A[\alpha]A[\beta]$ first over submatrices of order $n - 1$:

$$\begin{aligned} \frac{S_{m,m,k}(A)}{S_{m,m,k}(I_n)} &= \frac{1}{n} \sum_{\alpha \in \langle n \rangle, \#\alpha = n-1} \frac{S_{m,m,k}(A(\alpha))}{S_{m,m,k}(I_{n-1})} \\ \frac{S_{m+1,m-1,k}(A)}{S_{m+1,m-1,k}(I_n)} &= \frac{1}{n} \sum_{\alpha \in \langle n \rangle, \#\alpha = n-1} \frac{S_{m+1,m-1,k}(A(\alpha))}{S_{m+1,m-1,k}(I_{n-1})}. \end{aligned}$$

But principal submatrices of M- (inverse M-) matrices are again M- (inverse M-) matrices ([5, p.113, p.119]), therefore the inductive assumption holds for all submatrices $A(\alpha)$, $\#\alpha = n - 1$. This implies (3) for the matrix A itself.

Case 3. Let $2m - k = n$ and $k > 0$, and let A be a nonsingular M- or inverse M-matrix. Switch to the dual case: Each $A[\alpha]A[\beta]$ in the right-hand side of (2) equals $A^{-1}[\alpha']A^{-1}[\beta']/(\det A)^2$, the index sets α' and β' do not intersect, and

$\#\alpha' + \#\beta' = 2(n-m) < n$. Hence

$$S_{m,m,k}(A) = \frac{S_{n-m,n-m,0}(A^{-1})}{(\det A)^2}, \qquad S_{m+1,m-1,k}(A) = \frac{S_{n-m+1,n-m-1,0}(A^{-1})}{(\det A)^2}$$

and the functions $S_{n-m,n-m,0}(A^{-1})$, $S_{n-m+1,n-m-1,0}(A^{-1})$ are as in Case 2 above. Thus (3) holds for the matrix A^{-1} and hence for the matrix A. Finally, the set of all M- matrices is the closure of the set of nonsingular M-matrices (see, e.g., [5, p.119]), so the inequality (3) holds for singular M-matrices too.

With all possible cases considered, the theorem is proved. □

Now let us see what it implies.

Lemma 2. *Let Ψ denote the quadratic form*

(4)

$$t := (t_\alpha)_{\alpha\in\langle n\rangle} \mapsto t^*\Psi t := \sum_{j=0}^{m}\left(m(n-m)-(m+1)(n-m+1)\frac{m-j}{m-j+1}\right) \sum_{\substack{\#\alpha=\#\beta=m\\ \#\alpha\cap\beta=j}} \overline{t_\alpha} t_\beta.$$

If Ψ is nonnegative definite, then the inequalities (3) imply Newton's inequalities (1).

Proof. Expanding both sides of Newton's inequality yields

$$\begin{aligned} c_m^2(A) &= \sum_{j=0}^{m} S_{m,m,j}(A)/\binom{n}{m}^2, \qquad m = 1,\dots,n-1. \\ c_{m-1}(A)c_{m+1}(A) &= \sum_{j=0}^{m-1} S_{m+1,m-1,j}(A)/\binom{n}{m+1}\binom{n}{m-1}, \end{aligned}$$

So, Newton's inequalities are equivalent to

(5)

$$m(n-m)\sum_{j=0}^{m} S_{m,m,j}(A) \geq (m+1)(n-m+1)\sum_{j=0}^{m-1} S_{m+1,m-1,j}(A), \quad m = 1,\dots,n-1.$$

On the other hand, straightforward counting gives

$$\begin{aligned} S_{m,m,j}(I_n) &= \binom{n}{j}\binom{n-j}{m-j}\binom{n-m}{m-j}, \\ S_{m+1,m-1,j}(I_n) &= \binom{n}{j}\binom{n-j}{m-j-1}\binom{n-m+1}{m-j+1}, \end{aligned}$$

hence the inequalities (3) are equivalent to

$$(m-j)S_{m,m,j}(A) \geq (m-j+1)S_{m+1,m-1,j}(A).$$

Thus, upon replacing each $S_{m+1,m-1,j}$ in the right-hand side of (5) by $(m-j)/(m-j+1)S_{m,m,j}$, one obtains a set of inequalities stronger than Newton's. Precisely, these stronger inequalities assert that

$$a^*\Psi a \geq 0 \qquad \text{where} \quad a := (A[\alpha])_{\#\alpha=m}.$$

So, if Ψ is nonnegative definite, it follows that Newton's inequalities are satisfied. □

Thus, it remains to prove the following.

Lemma 3. *$t^*\Psi t \geq 0$ for all t.*

Proof. Consider first the quadratic form

$$\Phi : (t_\alpha)_{\#\alpha=m} \mapsto t^*\Phi t := \sum_{j=0}^{m} j \sum_{\substack{\#\alpha=\#\beta=m \\ \#\alpha\cap\beta=j}} \overline{t_\alpha} t_\beta.$$

The matrix of this quadratic form is the Gramian for the system of vectors $(v_\alpha)_\alpha$ where

$$v_\alpha(i) := \begin{cases} 1 & \text{if} \quad i \in \alpha \\ 0 & \text{otherwise,} \end{cases}$$

hence is nonnegative definite. Moreover, the vector e of all ones is an eigenvector of Φ. The form

$$\widetilde{\Phi} : (t_\alpha)_{\#\alpha=m} \mapsto t^*\widetilde{\Phi} t := \sum_{j=0}^{m} (m-j+1) \sum_{\substack{\#\alpha=\#\beta=m \\ \#\alpha\cap\beta=j}} \overline{t_\alpha} t_\beta$$

is obtained by subtracting Φ from a positive multiple of the Hermitian rank-one matrix ee^* (precisely $(m+1)ee^*$), therefore all of its eigenvalues are nonpositive except for the one corresponding to the eigenvector e, which is strictly positive. Therefore, by [1], the Hadamard inverse $\widetilde{\Psi}$ of the matrix $\widetilde{\Phi}$, i.e., the matrix

$$\left(\frac{1}{m - \#\alpha\cap\beta + 1} \right)_{\alpha,\beta}$$

is nonnegative definite. Finally, Ψ is obtained from $(m+1)(n-m+1)\widetilde{\Psi}$ by subtracting the rank-one matrix ee^* this time multiplied by $(n+1)$. The eigenvalue of Ψ corresponding to e is equal to zero, since

$$\begin{aligned} e^*\Psi e &= m(n-m)\sum_{j=0}^{m} S_{m,m,j}(I_n) - (m+1)(n-m+1)\sum_{j=0}^{m} \frac{m-j}{m-j+1} S_{m,m,j}(I_n) \\ &= m(n-m)\sum_{j=0}^{m} S_{m,m,j}(I_n) - (m+1)(n-m+1)\sum_{j=0}^{m-1} S_{m+1,m-1,j}(I_n) = 0. \end{aligned}$$

All the other eigenvalues of Ψ are nonnegative, so Ψ is nonnegative definite. $\square$

Note that a by-product of this Lemma is a binomial identity:

Corollary 4. $\sum_{j=0}^{m}(m(n-m) - (m+1)(n-m+1)\frac{m-j}{m-j+1})\binom{m}{j}\binom{n-m}{m-j} = 0.$

More importantly, Lemma 3 finishes the proof of Newton's inequalities.

Theorem 5. *Let A be similar to an M- or inverse M-matrix. Then the normalized coefficients of its characteristic polynomial satisfy Newton's inequalities (1).*

As possible applications of Theorem 5 one can envision eigenvalue localization for M- and inverse M-matrices as well as inverse eigenvalue problems.

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Institut für Mathematik, MA 4-5, Technische Universität Berlin, D-10623 Berlin, Germany

E-mail address: holtz@math.TU-Berlin.DE