Sturm-Liouville Operators with Indefinite Weight Functions and Eigenvalue Depending Boundary Conditions

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Abstract. We consider a class of boundary value problems for Sturm-Liouville operators with an indefinite weight function. The spectral parameter appears nonlinearly in the boundary condition in the form of a function τ which has the property that $\lambda \mapsto \lambda\tau(\lambda)$ is a generalized Nevanlinna function. We construct linearizations of these boundary value problems and study their spectral properties.

Keywords. boundary value problems, Sturm-Liouville operators, generalized Nevanlinna functions, operators with finitely many negative squares, definitizable operators, boundary value spaces, Weyl functions

1. Introduction

In this paper we consider a class of λ-dependent boundary value problems for differential operators with an indefinite weight function. In [5] B. Ćurgus and H. Langer showed that a differential expression of the form

$$\frac{1}{r}\big(-(pf')' + qf\big), \qquad p^{-1}, q, r \in L^1(0,1), \tag{1.1}$$

where r changes its sign, is connected with symmetric operators in the Krein space $(L^2_{|r|}(0,1), [\cdot,\cdot])$ (cf. Section 5), where the inner product $[\cdot,\cdot]$ is defined by

$$[f,g] := \int_0^1 f\,\overline{g}\,r\,dx, \qquad f,g \in L^2_{|r|}(0,1).$$

Under the assumption $p > 0$ the minimal operator $A_{\min}$ associated to (1.1) has a finite number κ of negative squares, that is, for some $\kappa \in \mathbb{N}_0$ there exists a κ-dimensional subspace in $\operatorname{dom}(A_{\min})$ such that the hermitian form $[A_{\min}\cdot,\cdot]$ is negative definite on this subspace, but there is no $\kappa+1$-dimensional subspace with this property. It turns out (cf. [5]) that all selfadjoint extensions of $A_{\min}$ in $(L^2_{|r|}(0,1), [\cdot,\cdot])$ are definitizable operators and therefore one can use the well developed spectral theory for these operator (see [28]) to investigate the spectral properties of the differential operators associated to (1.1).

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In this note we consider the equation

$$\frac{1}{r}\big(-(pf')' + qf\big) - \lambda f = k, \qquad f, k \in L^2_r(0,1), \tag{1.2}$$

subject to eigenparameter-dependent boundary conditions

$$\tau(\lambda) f(1) = (pf')(1) \quad \text{and} \quad f(0)\cos\alpha = (pf')(0)\sin\alpha, \quad \alpha \in [0,\pi). \tag{1.3}$$

For a positive function r and a generalized Nevanlinna function τ boundary value problems of the form (1.2)-(1.3) have been studied in a more or less abstract framework extensively in the last decades (see e.g. [3], [11], [17], [20], [30], [32] and the references quoted in [16]). Here we assume that τ belongs to some subclass $D_{\kappa'}$, $\kappa' \in \mathbb{N}_0$, of the so-called definitizable functions (see [22], [23]). The classes $D_{\kappa'}$ are introduced in Definition 3.2. It follows from results obtained by P. Jonas in [23] that each function $\tau \in D_{\kappa'}$ can be written with the help of a selfadjoint operator or relation T_0 with κ' negative squares in some Krein space $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{H}})$ in the form

$$\tau(\lambda) = \operatorname{Re}\tau(\lambda_0) + (\lambda - \operatorname{Re}\lambda_0)[e,e]_{\mathcal{H}} + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)\big[(T_0 - \lambda)^{-1} e, e\big]_{\mathcal{H}}, \tag{1.4}$$

where $\lambda_0 \in \rho(T_0)$ and $e \in \mathcal{H}$ are fixed. The formula (1.4) establishes a correspondence between the functions from $D_{\kappa'}$, $\kappa' = 0, 1, 2, \ldots$, and the selfadjoint operators and relations with finitely many negative squares. For generalized Nevanlinna functions and selfadjoint operators and relations in Pontryagin spaces such a correspondence is well known (see [18], [26]). As selfadjoint operators with finitely many negative squares appear in many applications (see e.g. [4], [5], [6], [7]) the classes $D_{\kappa'}$ are also of independent interest.

In order to solve (1.2)-(1.3) in Section 4 we investigate the abstract λ-dependent boundary value problem

$$f' - \lambda f = k, \quad \tau(\lambda)\Gamma_0 \hat{f} + \Gamma_1 \hat{f} = 0, \quad \hat{f} = \begin{pmatrix} f \\ f' \end{pmatrix} \in A^+. \tag{1.5}$$

Here A is a closed symmetric operator or relation with finitely many negative squares and defect one in a Krein space $\mathcal{K}$, $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is a boundary value space for A^+ (see Definition 2.1) and τ belongs to the class $D_{\kappa'}$, $\kappa' \in \mathbb{N}_0$. We assume that the selfadjoint extension $A_0 = \ker\Gamma_0$ of A has a nonempty resolvent set. Then the Weyl function corresponding to A and $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ (cf. (2.6)) belongs also to some class D_κ. Using the coupling method from [11] we construct a selfadjoint extension $\widetilde{A}$ of A which acts in the Krein space $\mathcal{K} \times \mathcal{H}$, where $\mathcal{H}$ is as in (1.4), such that its compressed resolvent $P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}}$ onto the basic space yields a solution of (1.5) (cf. [2]). Here we show that the linearization $\widetilde{A}$ also has a finite number $\tilde{\kappa}$ of negative squares and we obtain an estimate for this quantity.

In Section 5 we rewrite the λ-dependent boundary value problem (1.2)-(1.3) in the form (1.5) and we apply the general results from Section 4. Making use of results from the spectral and perturbation theory of definitizable operators, we describe the spectral properties of the linearization $\widetilde{A}$. It turns out that for all λ which are points of holomorphy of τ and do not belong to some discrete set and all $k \in L^2_r(0,1)$ the problem (1.2)-(1.3) has a unique solution. For a special function $\tau \in D_0$ we construct the linearization $\widetilde{A}$ in a more explicit form and give a criterion for $\widetilde{A}$ to be nonnegative.

The paper is organized as follows. In Section 2 we provide some basic facts on boundary value spaces and Weyl functions associated with symmetric relations in Krein spaces. The classes $D_{\kappa'}$ of complex valued functions defined and studied in Section 3 play an essential role in Section 4, where boundary value problems of the form (1.5) are solved with the help of compressed resolvents of selfadjoint operators and relations with finitely many negative squares. The problem (1.2)-(1.3) is studied in Section 5. Here we construct the linearization $\widetilde{A}$ and investigate its spectrum.

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2. Preliminaries

Let throughout this paper $(\mathcal{K}, [\cdot,\cdot])$ be a separable Krein space. The linear space of bounded linear operators defined on a Krein space $\mathcal{K}_1$ with values in a Krein space $\mathcal{K}_2$ is denoted by $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. If $\mathcal{K} := \mathcal{K}_1 = \mathcal{K}_2$ we write $\mathcal{L}(\mathcal{K})$. We study linear relations in $\mathcal{K}$, that is, linear subspaces of $\mathcal{K}^2$. The set of all closed linear relations in $\mathcal{K}$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{K})$. Linear operators are viewed as linear relations via their graphs. For the usual definitions of the linear operations with relations and the inverse we refer to [14].

If $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{H}})$ is another separable Krein space the elements of $\mathcal{K}\times\mathcal{H}$ will be written in the form $\{k,h\}$, $k\in\mathcal{K}$, $h\in\mathcal{H}$. $\mathcal{K}\times\mathcal{H}$ equipped with the inner product $[\cdot,\cdot]_{\mathcal{K}\times\mathcal{H}}$ defined by

$$[\{k,h\},\{k',h'\}]_{\mathcal{K}\times\mathcal{H}} := [k,k'] + [h,h']_{\mathcal{H}}, \quad k,k'\in\mathcal{K},\ \ h,h'\in\mathcal{H},$$

is also a Krein space. If S is a relation in $\mathcal{K}$ and T is a relation in $\mathcal{H}$ we shall write $S\times T$ for the direct product of S and T which is a relation in $\mathcal{K}\times\mathcal{H}$,

$$S\times T = \left\{ \begin{pmatrix} \{s,t\} \\ \{s',t'\} \end{pmatrix} \,\Big|\, \begin{pmatrix} s \\ s' \end{pmatrix} \in S,\ \begin{pmatrix} t \\ t' \end{pmatrix} \in T \right\}. \tag{2.1}$$

For the pair $\binom{\{s,t\}}{\{s',t'\}}$ on the right hand side of (2.1) we shall also write $\{\hat{s},\hat{t}\}$, where $\hat{s} = \binom{s}{s'}$, $\hat{t} = \binom{t}{t'}$.

Let S be a closed linear relation in $\mathcal{K}$. The resolvent set $\rho(S)$ of S is defined as the set of all $\lambda\in\mathbb{C}$ such that $(S-\lambda)^{-1}\in\mathcal{L}(\mathcal{K})$. If $\mu\in\rho(S)$ the relation S can be written as

$$S = \left\{ \begin{pmatrix} (S-\mu)^{-1}h \\ (I+\mu(S-\mu)^{-1})h \end{pmatrix} \,\Big|\, h\in\mathcal{K} \right\}.$$

The spectrum $\sigma(S)$ of S is the complement of $\rho(S)$ in $\mathbb{C}$. The extended spectrum $\widetilde{\sigma}(S)$ of S is defined by $\widetilde{\sigma}(S) = \sigma(S)$ if $S\in\mathcal{L}(\mathcal{K})$ and $\widetilde{\sigma}(S) = \sigma(S)\cup\{\infty\}$ otherwise. We set $\widetilde{\rho}(S) := \overline{\mathbb{C}}\backslash\widetilde{\sigma}(S)$. The *adjoint* S^+ of S is defined as

$$S^+ := \left\{ \begin{pmatrix} h \\ h' \end{pmatrix} \,\Big|\, [f',h] = [f,h'] \text{ for all } \begin{pmatrix} f \\ f' \end{pmatrix} \in S \right\}.$$

S is said to be *symmetric* (*selfadjoint*) if $S\subset S^+$ (resp. $S = S^+$).

A closed symmetric relation A in a Krein space $(\mathcal{K}, [\cdot,\cdot])$ is said to have κ *negative squares*, $\kappa \in \mathbb{N}_0$, if the hermitian form $\langle\cdot,\cdot\rangle$ on A, defined by

$$\left\langle \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} g \\ g' \end{pmatrix} \right\rangle := [f, g'], \qquad \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} g \\ g' \end{pmatrix} \in A,$$

has κ negative squares, that is, there exists a κ-dimensional subspace $\mathcal{M}$ in A such that $\langle \hat{v}, \hat{v}\rangle < 0$ if $\hat{v} \in \mathcal{M}$, $\hat{v} \neq 0$, but no $\kappa + 1$ dimensional subspace with this property. This holds if and only if for any positive integer n the symmetric matrix $(\langle \hat{v}_i, \hat{v}_j\rangle)_{i,j=1}^n$, $\hat{v}_1, \dots, \hat{v}_n \in A$, has at most κ negative eigenvalues and at least for one choice of n and $\hat{v}_1, \dots, \hat{v}_n \in A$ it has exactly κ negative eigenvalues. If A is selfadjoint and $\rho(A)$ is nonempty then A has κ negative squares if and only if the form

$$\left[\left(I + \lambda(A-\lambda)^{-1}\right)\cdot, (A-\lambda)^{-1}\cdot\right]$$

defined on $\mathcal{K}$ has κ negative squares.

We say that a closed symmetric relation A in $\mathcal{K}$ has *defect* $m \in \mathbb{N} \cup \{\infty\}$ if there exists a selfadjoint extension $\widehat{A}$ in $\mathcal{K}$ such that $\dim(\widehat{A}/A) = m$. It is not difficult to see that if A has κ negative squares and finite defect m each selfadjoint extension $\widehat{A}$ of A in $\mathcal{K}$ has κ', $\kappa \le \kappa' \le \kappa + m$, negative squares. If $\widehat{A}$ is a selfadjoint relation with κ negative squares and $A \subset \widehat{A}$, $A \in \widetilde{\mathcal{C}}(\mathcal{K})$, has finite defect m, then A is a symmetric relation with κ', $\kappa - m \le \kappa' \le \kappa$, negative squares.

For the description of the selfadjoint extensions of closed symmetric relations we use the so-called boundary value spaces.

Definition 2.1. Let A be a closed symmetric relation in the Krein space $(\mathcal{K}, [\cdot,\cdot])$. We say that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a *boundary value space for* A^+ if $(\mathcal{G}, (\cdot,\cdot))$ is a Hilbert space and there exist mappings $\Gamma_0, \Gamma_1 : A^+ \to \mathcal{G}$ such that $\Gamma := \binom{\Gamma_0}{\Gamma_1} : A^+ \to \mathcal{G} \times \mathcal{G}$ is surjective, and the relation

$$[f', g] - [f, g'] = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g}) - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g}) \tag{2.2}$$

holds for all $\hat{f} = \binom{f}{f'}, \hat{g} = \binom{g}{g'} \in A^+$.

If a closed symmetric relation A has a selfadjoint extension $\widehat{A}$ with $\rho(\widehat{A}) \neq \emptyset$, then there exists a boundary value space $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^+ such that $\widehat{A} = \ker \Gamma_0$ (see [9]).

For basic facts on boundary value spaces and further references see e.g. [8], [9], [12] and [13]. We recall only a few important consequences. For the rest of this section let A be a closed symmetric relation and assume that there exists a boundary value space $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^+. Then

$$A_0 := \ker \Gamma_0 \quad \text{and} \quad A_1 := \ker \Gamma_1 \tag{2.3}$$

are selfadjoint extensions of A. The mapping $\Gamma = \binom{\Gamma_0}{\Gamma_1}$ induces, via

$$A_\Theta := \Gamma^{-1}\Theta = \{\hat{f} \in A^+ | \Gamma\hat{f} \in \Theta\}, \quad \Theta \in \widetilde{\mathcal{C}}(\mathcal{G}), \tag{2.4}$$

a bijective correspondence $\Theta \mapsto A_\Theta$ between $\widetilde{\mathcal{C}}(\mathcal{G})$ and the set of closed extensions $A_\Theta \subset A^+$ of A. In particular (2.4) gives a one-to-one correspondence between the symmetric (selfadjoint) extensions of A and the symmetric (resp. selfadjoint) relations in $\mathcal{G}$. If Θ is a closed operator in $\mathcal{G}$, then the corresponding extension A_Θ of A is determined by

$$A_\Theta = \ker(\Gamma_1 - \Theta\Gamma_0). \tag{2.5}$$

Let $\mathcal{N}_\lambda := \ker(A^+ - \lambda) = \operatorname{ran}(A - \overline{\lambda})^{[\perp]}$ be the defect subspace of A and set

$$\hat{\mathcal{N}}_\lambda := \left\{ \left(\begin{smallmatrix} f \\ \lambda f \end{smallmatrix} \right) \middle| f \in \mathcal{N}_\lambda \right\}.$$

Now we assume that the selfadjoint relation A_0 in (2.3) has a nonempty resolvent set. For each $\lambda \in \rho(A_0)$ the relation A^+ can be written as a direct sum of (the subspaces) A_0 and $\hat{\mathcal{N}}_\lambda$ (see [9]). Denote by π_1 the orthogonal projection onto the first component of $\mathcal{K}^2$. The functions

$$\lambda \mapsto \gamma(\lambda) := \pi_1(\Gamma_0|\hat{\mathcal{N}}_\lambda)^{-1} \in \mathcal{L}(\mathcal{G},\mathcal{K}), \qquad \lambda \in \rho(A_0),$$

and

$$\lambda \mapsto M(\lambda) := \Gamma_1(\Gamma_0|\hat{\mathcal{N}}_\lambda)^{-1} \in \mathcal{L}(\mathcal{G}), \qquad \lambda \in \rho(A_0) \tag{2.6}$$

are defined and holomorphic on $\rho(A_0)$ and are called the *γ-field* and the *Weyl function* corresponding to A and $\{\mathcal{G},\Gamma_0,\Gamma_1\}$. For $\lambda,\zeta \in \rho(A_0)$ the relation (2.2) implies $M(\lambda)^* = M(\overline{\lambda})$ and

$$\gamma(\zeta) = (1 + (\zeta - \lambda)(A_0 - \zeta)^{-1})\gamma(\lambda) \tag{2.7}$$

and

$$M(\lambda) - M(\zeta)^* = (\lambda - \overline{\zeta})\gamma(\zeta)^+\gamma(\lambda) \tag{2.8}$$

hold (see [9]). If $\Theta \in \widetilde{\mathcal{C}}(\mathcal{G})$ and A_Θ is the corresponding extension of A then a point $\lambda \in \rho(A_0)$ belongs to $\rho(A_\Theta)$ if and only if 0 belongs to $\rho(\Theta - M(\lambda))$. For $\lambda \in \rho(A_\Theta) \cap \rho(A_0)$ the well-known resolvent formula

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+ \tag{2.9}$$

holds (for a proof see e.g. [9]).

3. Classes of functions connected with selfadjoint relations with finitely many negative squares

3.1. **The classes D_κ.** The class of all functions τ which are piecewise meromorphic in $\mathbb{C}\backslash\mathbb{R}$ and symmetric with respect to the real axis, that is $\tau(\overline{\lambda}) = \overline{\tau(\lambda)}$, is denoted by $M(\mathbb{C}\backslash\mathbb{R})$. By $\mathbb{C}^+$ ($\mathbb{C}^-$) we denote the open upper (resp. lower) half plane. For the extended real line and the extended complex plane we write $\overline{\mathbb{R}}$ and $\overline{\mathbb{C}}$, respectively. For a function $\tau \in M(\mathbb{C}\backslash\mathbb{R})$ the union of all points of holomorphy of τ in $\mathbb{C}\backslash\mathbb{R}$ and all points $\lambda \in \overline{\mathbb{R}}$ such that τ can be analytically continued to λ and the continuations from $\mathbb{C}^+$ and $\mathbb{C}^-$ coincide is denoted by $\mathfrak{h}(\tau)$.

Let $\tau \in M(\mathbb{C}\backslash\mathbb{R})$. We shall say that the *growth of τ near $\overline{\mathbb{R}}$ is of finite order* if there exist constants $M, m > 0$ and an open neighbourhood $\mathcal{U}$ of $\overline{\mathbb{R}}$ in $\overline{\mathbb{C}}$ such that

$\mathcal{U}\backslash\overline{\mathbb{R}} \subset \mathfrak{h}(\tau)$ and

$$|\tau(\lambda)| \leq \frac{M(1+|\lambda|)^{2m}}{|\mathrm{Im}\,\lambda|^m}$$

holds for all $\lambda \in \mathcal{U}\backslash\overline{\mathbb{R}}$. An open subset $\Delta \subset \overline{\mathbb{R}}$ is said to be of *positive type with respect to* τ if for every sequence $(\lambda_n) \subset \mathfrak{h}(\tau) \cap \mathbb{C}^+$ which converges in $\overline{\mathbb{C}}$ to a point of Δ we have

$$\liminf_{n\to\infty} \mathrm{Im}\,\tau(\lambda_n) \geq 0.$$

An open subset $\Delta \subset \overline{\mathbb{R}}$ is said to be of *negative type with respect to* τ if Δ is of positive type with respect to $-\tau$.

Let in the following the growth of $\tau \in M(\mathbb{C}\backslash\mathbb{R})$ near $\overline{\mathbb{R}}$ be of finite order. Let $\alpha \in \mathbb{R}$ and assume that there exists an open interval I_α, $\alpha \in I_\alpha$, such that $I_\alpha\backslash\{\alpha\}$ is of positive type with respect to τ. Let $\nu_\alpha \geq 0$ be the smallest integer such that

$$-\infty < \lim_{\lambda \widehat{\to} \alpha} (\lambda-\alpha)^{2\nu_\alpha+1}\tau(\lambda) \leq 0,$$

where $\lambda\widehat{\to}\alpha$ denotes the nontangential limit from $\mathbb{C}^+$. If $\nu_\alpha > 0$, then α is said to be a *generalized pole of nonpositive type of* τ *with multiplicity* ν_α. Assume that there exists a number $k_\infty > 0$ such that (k_∞, ∞) and $(-\infty, -k_\infty)$ are of positive type with respect to τ and let $\nu_\infty \geq 0$ be the smallest integer such that

$$0 \leq \lim_{\lambda \widehat{\to} \infty} \frac{\tau(\lambda)}{\lambda^{2\nu_\infty+1}} < \infty.$$

If $\nu_\infty > 0$, then ∞ is said to be a *generalized pole of nonpositive type of* τ *with multiplicity* ν_∞.

Let $\beta \in \mathbb{R}$ and assume that there exists an open interval I_β, $\beta \in I_\beta$, such that $I_\beta\backslash\{\beta\}$ is of positive type with respect to τ. Let $\eta_\beta \geq 0$ be the largest integer such that

$$-\infty < \lim_{\lambda \widehat{\to} \beta} \frac{\tau(\lambda)}{(\lambda-\beta)^{2\eta_\beta-1}} \leq 0.$$

If $\eta_\beta > 0$, then $\beta \in \mathbb{R}$ is said to be a *generalized zero of nonpositive type of* τ *with multiplicity* η_β. Assume that there exists a number $l_\infty > 0$ such that (l_∞, ∞) and $(-\infty, -l_\infty)$ are of positive type with respect to τ and let $\eta_\infty \geq 0$ be the largest integer such that

$$0 \leq \lim_{\lambda \widehat{\to} \infty} \lambda^{2\eta_\infty-1}\tau(\lambda) < \infty.$$

If $\eta_\infty > 0$, then ∞ is said to be a *generalized zero of nonpositive type of* τ *with multiplicity* η_∞.

The notions of generalized poles and generalized zeros of nonpositive type appear often in the investigation of the classes N_κ, $\kappa = 0, 1, 2, \ldots$, of generalized Nevanlinna functions. Recall that a function $G \in M(\mathbb{C}\backslash\mathbb{R})$ belongs to N_κ if the kernel N_G,

$$N_G(\lambda,\mu) := \frac{G(\lambda) - G(\overline{\mu})}{\lambda - \overline{\mu}},$$

has κ negative squares (see [26]). It follows from [22, Corollary 2.6] that a function $G \in M(\mathbb{C}\backslash\mathbb{R})$ is a generalized Nevanlinna function if and only if the growth of G near $\overline{\mathbb{R}}$ is of finite order and there exists a finite set $e \subset \overline{\mathbb{R}}$ such that $\overline{\mathbb{R}}\backslash e$ is of positive type with respect to G. The class N_0 coincides with the class of Nevanlinna functions. This class consists of functions which are holomorphic in $\mathbb{C}^+ \cup \mathbb{C}^-$ and have a nonnegative imaginary part.

Let $G \in N_\kappa$. Denote by α_j (β_i), $j = 1, \dots, r$ ($i = 1, \dots, s$) the poles (zeros) in $\mathbb{C}^+$ and the generalized poles (generalized zeros) of nonpositive type in $\mathbb{R}$ with multiplicities ν_j (η_i) of G (cf. [27], [29]). By [19] (see also [10]) there exists a Nevanlinna function G_0 such that

$$G(\lambda) = \frac{\prod_{i=1}^{s}(\lambda - \beta_i)^{\eta_i}(\lambda - \overline{\beta}_i)^{\eta_i}}{\prod_{j=1}^{r}(\lambda - \alpha_j)^{\nu_j}(\lambda - \overline{\alpha}_j)^{\nu_j}} G_0(\lambda). \tag{3.1}$$

Note that $G \in N_\kappa$ has poles in $\mathbb{C}^+$ and generalized poles of nonpositive type in $\mathbb{R} \cup \{\infty\}$ of total multiplicity κ (cf. [27]).

In Theorem 3.1 and Definition 3.2 below we define a special subclass of the so-called definitizable functions which were introduced and comprehensively studied by P. Jonas in [22] and [23]. Let again $\tau \in M(\mathbb{C}\backslash\mathbb{R})$, let the growth of τ near $\overline{\mathbb{R}}$ be of finite order and denote by $\kappa_{\mathbb{C}^+}$ the total multiplicity of the poles in $\mathbb{C}^+$ of τ and by $\kappa_{\mathbb{R}^+}$ ($\kappa_{\mathbb{R}^-}$) the total multiplicity of the generalized poles of nonpositive type in $(0,\infty)$ ($(-\infty,0)$) of the function τ ($-\tau$, respectively). If 0 (∞) is no generalized pole of nonpositive type of the function $\lambda \mapsto \lambda\tau(\lambda)$ ($\lambda \mapsto \frac{1}{\lambda}\tau(\lambda)$) we set $\kappa_0 = 0$ ($\kappa_\infty = 0$, respectively). Otherwise we denote by κ_0 and κ_∞ the multiplicity of the generalized pole 0 and ∞ of nonpositive type of the function $\lambda \mapsto \lambda\tau(\lambda)$ and $\lambda \mapsto \frac{1}{\lambda}\tau(\lambda)$, respectively.

Theorem 3.1. *For a function $\tau \in M(\mathbb{C}\backslash\mathbb{R})$ and $\kappa \in \mathbb{N}_0$ the following assertions are equivalent.*

(i) *There exists a point $\lambda_0 \in \mathfrak{h}(\tau)\backslash\{\infty\}$, a function $G \in N_\kappa$ holomorphic in λ_0 and a rational function g holomorphic in $\overline{\mathbb{C}}\backslash\{\lambda_0, \overline{\lambda}_0\}$ such that*

$$\frac{\lambda}{(\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)}\tau(\lambda) = G(\lambda) + g(\lambda)$$

holds for all points λ where τ, G and g are holomorphic.

(ii) *For every $z \in \mathfrak{h}(\tau)\backslash\{\infty\}$ there exists a function $G_z \in N_\kappa$ holomorphic in z and a rational function g_z holomorphic in $\overline{\mathbb{C}}\backslash\{z, \overline{z}\}$ such that*

$$\frac{\lambda}{(\lambda - z)(\lambda - \overline{z})}\tau(\lambda) = G_z(\lambda) + g_z(\lambda)$$

holds for all points λ where τ, G_z and g_z are holomorphic.

(iii) *The growth of τ near $\overline{\mathbb{R}}$ is of finite order, there exists a finite set $e \subset \mathbb{R}$ such that $(-\infty,0)\backslash e$ is of negative type and $(0,\infty)\backslash e$ is of positive type with respect to τ and*

$$\kappa = \kappa_{\mathbb{R}^+} + \kappa_{\mathbb{R}^-} + \kappa_{\mathbb{C}^+} + \kappa_0 + \kappa_\infty$$

holds.

Definition 3.2. For a function $\tau \in M(\mathbb{C}\backslash\mathbb{R})$ satisfying one of the equivalent assertions (i), (ii) and (iii) in Theorem 3.1 for some $\kappa \in \mathbb{N}_0$ we will write $\tau \in D_\kappa$.

The next lemma will be used in the proofs of Theorem 3.1 and Theorem 3.6.

Lemma 3.3. *Let G_+ and G_- be Nevanlinna functions holomorphic in $(-\infty, 0)$ and $(0, \infty)$, respectively. Then the functions $\lambda \mapsto \lambda G_+(\lambda)$ and $\lambda \mapsto -\lambda G_-(\lambda)$ belong to $N_0 \cup N_1$ and $\mathbb{R}$ is of positive type with respect to $\lambda \mapsto \lambda G_+(\lambda)$ and $\lambda \mapsto -\lambda G_-(\lambda)$.*

Proof. We prove the statement for the function G_+. A similar reasoning applies for G_-. By [1] there exists a positive measure σ, $\int_{0-}^{\infty} \frac{d\sigma(t)}{1+t^2} < \infty$, and constants $a \in \mathbb{R}$, $b \geq 0$, such that

$$G_+(\lambda) = a + b\lambda + \int_{0-}^{\infty} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d\sigma(t).$$

We claim that $\lambda \mapsto \lambda(1+\lambda^2)^{-1}G_+(\lambda)$ is a generalized Nevanlinna function. In fact, from

$$\frac{\lambda}{1+\lambda^2}\frac{1}{t-\lambda} = \frac{t}{(t-\lambda)(1+t^2)} + \frac{\lambda t - 1}{(1+t^2)(1+\lambda^2)}$$

we obtain

$$\frac{\lambda}{1+\lambda^2}\int_{0-}^{\infty} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d\sigma_+(t) = \int_{0-}^{\infty} \frac{d\widetilde{\sigma}(t)}{t-\lambda} - \frac{1}{1+\lambda^2}\int_{0-}^{\infty} \frac{d\sigma(t)}{1+t^2},$$

where $d\widetilde{\sigma}(t) = t(1+t^2)^{-1}d\sigma_+(t)$. This implies $\lambda \mapsto \lambda(1+\lambda^2)^{-1}G_{0+}(\lambda) \in N_0 \cup N_1$ and it follows that

$$\lambda G_+(\lambda) = a_+\lambda + b_+\lambda^2 - \int_{0-}^{\infty} \frac{d\sigma(t)}{1+t^2} + (1+\lambda^2)\int_{0-}^{\infty} \frac{d\widetilde{\sigma}(t)}{t-\lambda}$$

belongs to $N_0 \cup N_1$. The only possible generalized pole (of order one) of nonpositive type is ∞, i.e. $\mathbb{R}$ is of positive type with respect to $\lambda \mapsto \lambda G_+(\lambda)$.

$\square$

Proof of Theorem 3.1. Assume that (i) holds. Let $z \in \mathfrak{h}(\tau)\backslash\{\infty\}$ and assume that $z \neq \lambda_0, \overline{\lambda_0}$. By our assumption there exists a function $G \in N_\kappa$ holomorphic in λ_0 and z and a rational function g holomorphic in $\overline{\mathbb{C}}\backslash\{\lambda_0, \overline{\lambda_0}\}$ such that

$$\frac{\lambda}{(\lambda-z)(\lambda-\overline{z})}\tau(\lambda) = \frac{(\lambda-\lambda_0)(\lambda-\overline{\lambda_0})}{(\lambda-z)(\lambda-\overline{z})}G(\lambda) + \frac{(\lambda-\lambda_0)(\lambda-\overline{\lambda_0})}{(\lambda-z)(\lambda-\overline{z})}g(\lambda).$$

Define a function r_z by

$$r_z(\lambda) = \frac{(z-\lambda_0)(z-\overline{\lambda_0})}{(\lambda-z)(z-\overline{z})}G(z) + \frac{(\overline{z}-\lambda_0)(\overline{z}-\overline{\lambda_0})}{(\lambda-\overline{z})(\overline{z}-z)}G(\overline{z})$$

if $z \in \mathbb{C}\backslash\mathbb{R}$ and by

$$r_z(\lambda) = \frac{(z-\lambda_0)(z-\overline{\lambda_0})}{(\lambda-z)^2}G(z) + \frac{2(z-\operatorname{Re}\lambda_0)G(z) + (z-\lambda_0)(z-\overline{\lambda_0})G'(z)}{\lambda-z}$$

if z is real. Then

$$G_z(\lambda) = \frac{(\lambda-\lambda_0)(\lambda-\overline{\lambda_0})}{(\lambda-z)(\lambda-\overline{z})}G(\lambda) - r_z(\lambda)$$

belongs to the class N_κ and G_z is holomorphic in z. The function g_z,

$$g_z(\lambda) = r_z(\lambda) + \frac{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}{(\lambda-z)(\lambda-\overline{z})} g(\lambda),$$

is a rational function holomorphic in $\overline{\mathbb{C}}\backslash\{\lambda_0, \overline{\lambda}_0\}$ and (ii) holds.

Let us show that (ii) implies (iii). As the function $\lambda \mapsto \frac{1}{\lambda}\tau(\lambda)$ belongs to $N_{\kappa'}$ for some $\kappa' \in \mathbb{N}_0$ the growth of τ near $\overline{\mathbb{R}}$ is of finite order. Moreover, there exists a finite set $e \subset \overline{\mathbb{R}}$ such that $\overline{\mathbb{R}}\backslash e$ is of positive type with respect to the function $\lambda \mapsto \frac{1}{\lambda}\tau(\lambda)$. Let Δ be an open interval such that $\overline{\Delta} \subset (0,\infty)\backslash e$. Then [24, Proposition 2.6] implies the existence of a Nevanlinna function G_0 with $(-\infty, 0) \subset \mathfrak{h}(G_0)$ and a function G_1 locally holomorphic on $\overline{\Delta}$ such that

$$\frac{1}{\lambda}\tau(\lambda) = G_0(\lambda) + G_1(\lambda).$$

By Lemma 3.3 the interval Δ is of positive type with respect to τ. A similar reasoning shows that an open interval Δ, $\overline{\Delta} \subset (-\infty, 0)\backslash e$ is of negative type with respect to τ. We choose now $z \in \mathfrak{h}(\tau)\backslash\overline{\mathbb{R}}$ and G_z, g_z as in (ii). Then for some $a, b, c \in \mathbb{R}$ we have

$$\begin{aligned} \lambda\tau(\lambda) &= (\lambda-z)(\lambda-\overline{z})G_z(\lambda) + a\lambda^2 + b\lambda + c, \\ \frac{\tau(\lambda)}{\lambda} &= \frac{(\lambda-z)(\lambda-\overline{z})}{\lambda^2}G_z(\lambda) + a + \frac{b}{\lambda} + \frac{c}{\lambda^2}. \end{aligned} \tag{3.2}$$

By (3.2) the point 0 (∞) is a generalized pole of nonpositive type of multiplicity ν_0 (ν_∞) with respect to G if and only if 0 (∞) is a generalized pole of nonpositive type of multiplicity ν_0 (ν_∞) with respect to $\lambda \mapsto \lambda\tau(\lambda)$ ($\lambda \mapsto \frac{1}{\lambda}\tau(\lambda)$, respectively). An analogue relation holds between the poles in $\mathbb{C}^+$ (the generalized poles of nonpositive type in $(0,\infty)$ and $(-\infty,0)$) of G and the functions τ (τ and $-\tau$, respectively). Therefore we have

$$\kappa = \kappa_{\mathbb{R}^+} + \kappa_{\mathbb{R}^-} + \kappa_{\mathbb{C}^+} + \kappa_0 + \kappa_\infty.$$

Assume that (iii) holds. Then by similar arguments used before it follows that $(0,\infty)\backslash e$ and $(-\infty,0)\backslash e$ are of positive type with respect to the function $\lambda \mapsto \lambda\tau(\lambda)$. Hence this function belongs to some class $N_{\kappa'}$, $\kappa' \in \mathbb{N}_0$. Let $\lambda_0 \in \mathfrak{h}(\tau)\backslash\overline{\mathbb{R}}$ such that $\tau(\lambda_0) \neq 0$. Then the function

$$\widetilde{G}(\lambda) := \frac{\lambda}{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}\tau(\lambda)$$

belongs to the class $N_{\kappa'+1}$. As τ is holomorphic in λ_0 there exists a rational function g holomorphic in $\overline{\mathbb{C}}\backslash\{\lambda_0, \overline{\lambda}_0\}$ such that

$$\frac{\lambda}{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}\tau(\lambda) = (\widetilde{G}-g)(\lambda) + g(\lambda) \tag{3.3}$$

holds with $\widetilde{G} - g \in N_{\kappa'}$ and $\widetilde{G} - g$ is holomorphic in λ_0. It follows from (3.2), where z, G_z and g_z are replaced by λ_0, $\widetilde{G} - g$ and g, respectively, and relation (3.3) that $\kappa = \kappa'$. Hence (iii) implies (i).

□

Corollary 3.4. *Let κ and κ' be nonnegative integers, $\kappa \neq \kappa'$. Then $D_\kappa \cap D_{\kappa'} = \emptyset$.*

Note, that the sum of a function $\tau_1 \in D_{\kappa_1}$ and a function $\tau_2 \in D_{\kappa_2}$ belongs to some class $D_{\tilde{\kappa}}$ with $\tilde{\kappa} \leq \kappa_1 + \kappa_2$.

We emphasise that the classes D_κ, $\kappa \in \mathbb{N}_0$, introduced in Definition 3.2 are no subclasses of generalized Nevanlinna functions and vice versa a (generalized) Nevanlinna function in general does not belong to some class D_κ. Moreover, even if a function belongs to $D_\kappa \cap N_{\kappa'}$, $\kappa, \kappa' \in \mathbb{N}_0$, then there is in general no connection between κ and κ'.

Example 3.5. Consider two subsets $\mathcal{P}$ and $\mathcal{N}$ of $\mathbb{N}_0$ and let $(p_j)_{j\in\mathcal{P}} \subset \mathbb{R}$ and $(n_j)_{j\in\mathcal{N}} \subset \mathbb{R}$. Assume $\sum_{j\in\mathcal{P}} |p_j| < \infty$ and $\sum_{j\in\mathcal{N}} |n_j| < \infty$. Let $(\mu_j^+)_{j\in\mathcal{P}} \subset (0,\infty)$ and $(\mu_j^-)_{j\in\mathcal{N}} \subset (-\infty, 0)$ such that $\mu_j^+ \neq \mu_k^+$, $j \neq k$ and $\mu_j^- \neq \mu_k^-$, $j \neq k$. In the case that $\mathcal{P}$ ($\mathcal{N}$) consists of infinitely many elements we assume, in addition, that the sequence $(\mu_j^+)_{j\in\mathcal{P}}$ ($(\mu_j^-)_{j\in\mathcal{N}}$, respectively) converges to zero. Then the function τ,

$$\tau(\lambda) = \sum_{j\in\mathcal{P}} \frac{p_j}{\lambda - \mu_j^+} + \sum_{j\in\mathcal{N}} \frac{n_j}{\lambda - \mu_j^-},$$

belongs to D_κ if and only if the number of elements of $\{p_j \,|\, p_j > 0\} \cup \{n_j \,|\, n_j < 0\}$ equals to κ. This follows from Theorem 3.1 (iii) and the fact that 0 and ∞ are no generalized poles of nonpositive type of the functions $\lambda \mapsto \lambda\tau(\lambda)$ and $\lambda \mapsto \frac{1}{\lambda}\tau(\lambda)$, respectively.

If, in addition, zero is no generalized pole of nonpositive type of τ then τ belongs to $N_{\kappa'}$ if and only if the number of elements of $\{p_j \,|\, p_j > 0\} \cup \{n_j \,|\, n_j > 0\}$ is κ'.

Theorem 3.6. *A function τ belongs to some class D_κ if and only if τ can be written as the difference of two generalized Nevanlinna functions G_+ and G_-, $\tau = G_+ - G_-$, where G_+ is holomorphic on $(-\infty, 0)$ and G_- is holomorphic on $(0,\infty)$.*

Proof. Assume that τ belongs to the class D_κ and let $\lambda_0 \in \mathfrak{h}(\tau)\backslash\overline{\mathbb{R}}$. Then there exists a generalized Nevanlinna function $G \in N_\kappa$ holomorphic in λ_0 such that

$$\frac{\lambda}{(\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)}\tau(\lambda) = G(\lambda) + g(\lambda)$$

holds with some rational function g which is holomorphic in $\overline{\mathbb{C}}\backslash\{\lambda_0, \overline{\lambda}_0\}$. Let α_j (β_i), $j = 1, \ldots, r$ ($i = 1, \ldots, s$), be the poles (zeros) in $\mathbb{C}^+$ and the generalized poles (generalized zeros) of nonpositive type in $\mathbb{R}$ with multiplicities ν_j (η_i) of G and let $G_0 \in N_0$ such that

$$G(\lambda) = r(\lambda)G_0(\lambda), \qquad r(\lambda) = \frac{\prod_{i=1}^{s}(\lambda - \beta_i)^{\eta_i}(\lambda - \overline{\beta}_i)^{\eta_i}}{\prod_{j=1}^{r}(\lambda - \alpha_j)^{\nu_j}(\lambda - \overline{\alpha}_j)^{\nu_j}}, \tag{3.4}$$

holds (cf. (3.1)). We write G_0 as the sum of two Nevanlinna functions G_{0+} and G_{0-}, where G_{0+} is holomorphic on $(-\infty, 0)$ and G_{0-} is holomorphic on $(0,\infty)$. As $\lambda \mapsto \lambda G_{0+}(\lambda)$ and $\lambda \mapsto -\lambda G_{0-}(\lambda)$ are generalized Nevanlinna functions (cf. Lemma 3.3) the same holds true for

$$\lambda \mapsto \frac{(\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)}{\lambda}G_{0+}(\lambda) \quad \text{and} \quad \lambda \mapsto -\frac{(\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)}{\lambda}G_{0-}(\lambda).$$

We can assume that $\alpha_1,\dots,\alpha_l$ are negative and $\alpha_{l+1},\dots,\alpha_r \notin (-\infty,0)$. Let r_+ be a rational function with poles at α_j, $j=1,\dots,l$, such that the generalized Nevanlinna function

$$\widehat{G}_+(\lambda) := \frac{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}{\lambda} r(\lambda)G_{0+}(\lambda) - r_+(\lambda)$$

is holomorphic in $(-\infty,0)$ and let r_- be a rational function with poles in $(0,\infty)$ such that the generalized Nevanlinna function

$$\widehat{G}_-(\lambda) := -\frac{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}{\lambda} r(\lambda)G_{0-}(\lambda) - r_-(\lambda)$$

is holomorphic in $(0,\infty)$. Then $G_+(\lambda) := \widehat{G}_+(\lambda) - r_-(\lambda) + (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)\lambda^{-1}g(\lambda)$ and $G_-(\lambda) := \widehat{G}_-(\lambda) - r_+(\lambda)$ are generalized Nevanlinna functions holomorphic in $(-\infty,0)$ and $(0,\infty)$, respectively, and $\tau = G_+ - G_-$.

It remains to show that $G_+ - G_-$, where G_+ and G_- are generalized Nevanlinna functions holomorphic in $(-\infty,0)$ and $(0,\infty)$, respectively, belongs to some class D_κ. Clearly the growth of $G_+ - G_-$ near $\overline{\mathbb{R}}$ is of finite order. If e denotes the union of the generalized generalized poles of nonpositive type of G_+ and G_-, then e is finite and Theorem 3.1 (iii) implies $G_+ - G_- \in D_\kappa$ for some $\kappa \in \mathbb{N}_0$.

□

3.2. **Weyl functions of symmetric relations with finitely many negative squares.** In Lemma 3.7 and in Theorem 3.8 we establish a connection between the Weyl functions of symmetric relations with finitely many negative squares and the functions from the classes D_κ, $\kappa \in \mathbb{N}_0$. We remark, that Lemma 3.7 can also be deduced from [23, Theorem 1.12].

Lemma 3.7. *Let A be a closed symmetric relation with defect one in the Krein space $(\mathcal{K},[\cdot,\cdot])$ and assume that there exists a selfadjoint extension A_0 with κ negative squares and $\rho(A_0) \neq \emptyset$. Let $\{\mathbb{C},\Gamma_0,\Gamma_1\}$ be a boundary value space for A^+ such that $\ker\Gamma_0 = A_0$. Then the corresponding Weyl function M belongs to some class $D_{\kappa'}$, where $\kappa' \leq \kappa$. If, in addition, the condition*

$$\mathcal{K} = \operatorname{clsp}\left\{\mathcal{N}_\lambda \,|\, \lambda \in \rho(A_0)\right\}$$

is fulfilled, then $M \in D_\kappa$.

Proof. Let $\lambda,\lambda_0 \in \rho(A_0)$. Then, making use of (2.7) and (2.8), we find $\operatorname{Im} M(\lambda_0) = \operatorname{Im}\lambda_0\,\gamma(\lambda_0)^+\gamma(\lambda_0)$ and

$$\begin{aligned} M(\lambda) &= \overline{M(\lambda_0)} + (\lambda-\overline{\lambda}_0)\gamma(\lambda_0)^+\gamma(\lambda) \\ &= \operatorname{Re} M(\lambda_0) - i\operatorname{Im}\lambda_0\gamma(\lambda_0)^+\gamma(\lambda_0) \\ &\qquad + \gamma(\lambda_0)^+\big(\lambda-\overline{\lambda}_0 + (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(A_0-\lambda)^{-1}\big)\gamma(\lambda_0) \\ &= \operatorname{Re} M(\lambda_0) + \gamma(\lambda_0)^+\big(\lambda - \operatorname{Re}\lambda_0 + (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(A_0-\lambda)^{-1}\big)\gamma(\lambda_0). \end{aligned}$$

For $\lambda \in \rho(A_0)$ we set

$$G(\lambda) := \gamma(\lambda_0)^+(I + \lambda(A_0-\lambda)^{-1})\gamma(\lambda_0)$$

and

$$g(\lambda) := \frac{\lambda \operatorname{Re} M(\lambda_0) + (\lambda \operatorname{Re} \lambda_0 - |\lambda_0|^2)\gamma(\lambda_0)^+\gamma(\lambda_0)}{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}.$$

A simple calculation shows that

$$\frac{\lambda}{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)} M(\lambda) = G(\lambda) + g(\lambda)$$

holds. Now we regard $\gamma(\lambda_0)$ as an element in $\mathcal{K}$. Then

$$\frac{G(\lambda) - G(\overline{\mu})}{\lambda - \overline{\mu}} = \left[(I + \lambda(A_0-\lambda)^{-1})\gamma(\lambda_0), (A_0-\mu)^{-1}\gamma(\lambda_0)\right], \quad \lambda, \mu \in \rho(A_0), \tag{3.5}$$

and the assumption that A_0 has κ negative squares implies that G belongs to $N_{\kappa'}$ for some $\kappa' \leq \kappa$. Therefore $M \in D_{\kappa'}$.

Under the additional assumption $\mathcal{K} = \operatorname{clsp}\{\mathcal{N}_\lambda \,|\, \lambda \in \rho(A_0)\}$ we have

$$\mathcal{K} = \operatorname{clsp}\left\{(1 + (\lambda-\lambda_0)(A_0-\lambda)^{-1})\gamma(\lambda_0) \,|\, \lambda \in \rho(A_0)\right\}.$$

It is not difficult to see that the set

$$\operatorname{sp}\left\{\begin{pmatrix} (A_0-\lambda)^{-1}\gamma(\lambda_0) \\ (I+\lambda(A_0-\lambda)^{-1})\gamma(\lambda_0)\end{pmatrix} \,\Big|\, \lambda \in \rho(A_0)\right\}$$

is dense in A_0. Hence the kernel (3.5) has κ negative squares, i.e. $M \in D_\kappa$.

□

Let $\tau \in D_\kappa$ for some $\kappa \in \mathbb{N}_0$. By [23, Theorem 3.9] there exists a Krein space $(\mathcal{H}, [\cdot,\cdot])$, a definitizable selfadjoint relation T_0 in $\mathcal{H}$ and an element $e \in \mathcal{H}$ such that $\mathfrak{h}(\tau) = \widetilde{\rho}(T_0)$ and

$$\tau(\lambda) = \operatorname{Re}\tau(\lambda_0) + (\lambda - \operatorname{Re}\lambda_0)[e,e] + (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)\left[(T_0-\lambda)^{-1}e, e\right] \tag{3.6}$$

holds for a fixed $\lambda_0 \in \rho(T_0)$ and all $\lambda \in \rho(T_0)$. Recall that the definitizable selfadjoint relation T_0 possesses a spectral function with properties similar to the spectral function of a definitizable selfadjoint operator (cf. [23, page 71], [15] and [28]). It follows from [23, Theorem 3.9] that if the representation (3.6) is choosen minimal, i.e.

$$\mathcal{H} = \operatorname{clsp}\left\{(1 + (\lambda-\lambda_0)(T_0-\lambda)^{-1})e \,|\, \lambda \in \rho(T_0)\right\} \tag{3.7}$$

holds, then the relation T_0 has κ negative squares.

Making use of the representation (3.6) the next theorem is a variant of [2, Theorem 3.3]. For the convenience of the reader we sketch the proof.

Theorem 3.8. *Let $\tau \in D_\kappa$ be not equal to a constant. Choose $\mathcal{H}$, T_0 and $e \in \mathcal{H}$ as in (3.6) such that (3.7) holds. Then there exists a closed symmetric operator T with defect one and a boundary value space $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ for T^+ such that τ coincides with the corresponding Weyl function. Moreover $\mathcal{H} = \operatorname{clsp}\{\mathcal{N}_\lambda \,|\, \lambda \in \rho(T_0)\}$, $T_0 = \ker \Gamma_0'$ and T_0 has κ negative squares.*

Sketch of the proof of Theorem 3.8. Let $\mathcal{H}$, T_0, $e \in \mathcal{H}$ and λ_0 be as in (3.6). For all $\lambda \in \rho(T_0)$ we define $\gamma'(\lambda) := \big(1 + (\lambda - \lambda_0)(T_0 - \lambda)^{-1}\big)e$. The relation

$$T := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in T_0 \,\middle|\, [g - \overline{\lambda}_0 f, e] = 0 \right\}$$

is closed, symmetric and has defect one and $\ker(T^+ - \lambda) = \operatorname{sp}\gamma'(\lambda)$ holds for all $\lambda \in \rho(T_0)$. From $\mathcal{H} = \operatorname{clsp}\{\gamma'(\lambda) \,|\, \lambda \in \rho(T_0)\} = \operatorname{clsp}\{\mathcal{N}_\lambda \,|\, \lambda \in \rho(T_0)\}$ it follows that T is an operator without eigenvalues. We regard $\gamma'(\lambda)$ as the linear mapping $\mathbb{C} \ni c \mapsto c\,\gamma'(\lambda) \in \mathcal{H}$ and denote the linear functional $c\,\gamma'(\lambda) \mapsto c$ defined on $\operatorname{ran}\gamma'(\lambda)$ by $\gamma'(\lambda)^{(-1)}$. The elements $\hat{f} \in T^+$, for every $\lambda \in \rho(T_0)$, can be written in the form $\hat{f} = \binom{f_0}{f_0'} + \binom{f_\lambda}{\lambda f_\lambda}$, where $\binom{f_0}{f_0'} \in T_0$ and $f_\lambda \in \operatorname{ran}\gamma'(\lambda)$. One verifies as in the proof of [2, Theorem 3.3] (see also [12, Theorem 1]) that $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$, where

$$\begin{aligned} \Gamma_0'\hat{f} &:= \gamma'(\lambda)^{(-1)} f_\lambda, \\ \Gamma_1'\hat{f} &:= \gamma'(\lambda)^+ (f_0' - \overline{\lambda} f_0) + \tau(\lambda)\gamma'(\lambda)^{(-1)} f_\lambda, \end{aligned}$$

is a boundary value space for T^+ with corresponding Weyl function τ.

□

It is well known that for an invertible function $\tau \in N_\kappa$ the function $-\tau^{-1}$ belongs also to the class N_κ. For a function $\tau \in D_\kappa$ the index κ can change, e.g. if $\tau(\lambda) = \lambda^2 + 1$ then $\tau \in D_0$ but $-\tau^{-1}$ belongs to D_1.

Theorem 3.9. *Let $\tau \in D_\kappa$, $\kappa \in \mathbb{N}_0$, and assume that τ is not identically equal to zero. Then $-\tau^{-1}$ belongs to the class $D_{\kappa'}$, where $\kappa' \in \{\kappa - 1, \kappa, \kappa + 1\}$, $\kappa' \in \mathbb{N}_0$.*

Proof. We can assume that τ is not equal to a constant. Otherwise the statement is clear. By Theorem 3.8 there exists a closed symmetric operator T with defect one and a boundary value space $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ for T^+ such that τ is the Weyl function and $T_0 = \ker \Gamma_0'$ has κ negative squares. T has $\kappa - 1$ or κ negative squares and therefore the selfadjoint relation $T_1 := \ker \Gamma_1'$ has κ', $\kappa' \in \{\kappa - 1, \kappa, \kappa + 1\}$, negative squares.

It is easy to see that $\{\mathbb{C}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$, where $\widehat{\Gamma}_0 := -\Gamma_1'$, $\widehat{\Gamma}_1 := \Gamma_0'$, is also a boundary value space for T^+ and the corresponding Weyl function is given by

$$\widehat{\tau}(\lambda) = -\tau(\lambda)^{-1}, \quad \lambda \in \rho(T_0) \cap \mathfrak{h}(\tau^{-1}).$$

The statement of the theorem follows if we apply Lemma 3.7 to $\{\mathbb{C}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$.

□

4. A class of λ-dependent boundary value problems

In this section we study a class of boundary value problems where a function from some class $D_{\kappa'}$ appears in the boundary condition. Our investigation is based on the approach in [11] and [2].

Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space and let A be closed symmetric relation with κ negative squares and defect one. We assume that there exists a selfadjoint extension A_0 of A in $\mathcal{K}$ with $\rho(A_0) \neq \emptyset$. Let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary value space for

A^+ such that $A_0 = \ker\Gamma_0$, and let $\tau \in D_{\kappa'}$. The γ-field and the Weyl function corresponding to A and $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ will be denoted by γ and M. We consider the following λ-dependent boundary value problem: For a given element $k \in \mathcal{K}$ find a vector $\hat{f}_1 = \binom{f_1}{f_1'} \in A^+$ such that

$$f_1' - \lambda f_1 = k \quad \text{and} \quad \tau(\lambda)\Gamma_0\hat{f}_1 + \Gamma_1\hat{f}_1 = 0 \tag{4.1}$$

holds. We remark that in the case of a constant function τ in the boundary condition of (4.1) the boundary value problem can be solved with the help of the resolvent of the selfadjoint relation $\widetilde{A}_{-\tau} = \ker(\tau\Gamma_0 + \Gamma_1) \in \widetilde{\mathcal{C}}(\mathcal{K})$ (see (2.5)). As A has κ negative squares it follows that $\widetilde{A}_{-\tau}$ has κ or $\kappa+1$ negative squares.

Theorem 4.1. *Let $A \subset A^+$, $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ and M be as above. Let $\tau \in D_{\kappa'}$, $\kappa' \in \mathbb{N}_0$, be not equal to a constant, let $\mathcal{H}$, $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ and T^+ be as in Theorem* 3.8 *and assume that $M + \tau$ is not identically equal to zero. We define*

$$\mathfrak{h}_0 := \mathfrak{h}(M) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}\big(\tau^{-1}\big) \cap \mathfrak{h}\big((M+\tau)^{-1}\big).$$

Then the relation

$$\widetilde{A} = \Big\{\{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+ \,|\, \Gamma_1\hat{f}_1 - \Gamma_1'\hat{f}_2 = \Gamma_0\hat{f}_1 + \Gamma_0'\hat{f}_2 = 0\Big\}$$

is a selfadjoint extension of A in $\mathcal{K} \times \mathcal{H}$ with $\tilde{\kappa}$ negative squares, where

$$0 \le \tilde{\kappa} \in \{\kappa + \kappa' - 1, \dots, \kappa + \kappa' + 2\}.$$

The set $\mathbb{C}\backslash(\mathbb{R} \cup \mathfrak{h}_0)$ is finite and $\mathfrak{h}_0\backslash\{\infty\}$ is a subset of $\rho(\widetilde{A})$. For every $k \in \mathcal{K}$ and every $\lambda \in \mathfrak{h}_0\backslash\{\infty\}$ a unique solution of the λ-dependent boundary value problem (4.1) *is given by*

$$\begin{aligned} f_1 &= P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}\{k, 0\} = (A_0 - \lambda)^{-1}k - \gamma(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+k, \\ f_1' &= \lambda f_1 + k, \end{aligned}$$

where $P_{\mathcal{K}}$ is the projection onto the first component of $\mathcal{K} \times \mathcal{H}$. If, in addition, A is a densely defined operator, then $\widetilde{A}$ is a selfadjoint operator.

Proof. Here $A_0 = \ker\Gamma_0$ is a selfadjoint relation with κ or $\kappa + 1$ negative squares. Therefore, by Lemma 3.7 the Weyl function M corresponding to A and the boundary value space $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ belongs to some class $D_{\hat{\kappa}}$, $\hat{\kappa} \le \kappa + 1$. Since $\tau \in D_{\kappa'}$ the function $M + \tau$ belongs to some class D_η, $\eta \le \hat{\kappa} + \kappa'$, and Theorem 3.9 implies that $-\tau^{-1}$ and $-(M+\tau)^{-1}$ also belong to some classes D_μ, $\mu \in \mathbb{N}_0$. In particular these functions have only a finite number of nonreal poles and we conclude that the set $\mathbb{C}\backslash(\mathbb{R} \cup \mathfrak{h}_0)$ is finite.

Now we can proceed as in the proof of [2, Theorem 4.1]. Let $\mathcal{H}$, $T \subset T_0 \subset T^+$, $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ be as in Theorem 3.8 and set $T_1 := \ker\Gamma_1'$. We define mappings $\widetilde{\Gamma}_0$, $\widetilde{\Gamma}_1 : A^+ \times T^+ \to \mathbb{C}^2$ in the following way. For $\{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+$ we set

$$\widetilde{\Gamma}_0\{\hat{f}_1, \hat{f}_2\} := \begin{pmatrix} \Gamma_0\hat{f}_1 \\ \Gamma_1'\hat{f}_2 \end{pmatrix} \quad \text{and} \quad \widetilde{\Gamma}_1\{\hat{f}_1, \hat{f}_2\} := \begin{pmatrix} \Gamma_1\hat{f}_1 \\ -\Gamma_0'\hat{f}_2 \end{pmatrix}.$$

It is not difficult to verify that $\{\mathbb{C}^2, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is a boundary value space for $A^+ \times T^+$. The corresponding γ-field and Weyl function are denoted by $\widetilde{\gamma}$ and $\widetilde{M}$, respectively. Here we have

$$\widetilde{\gamma}(\lambda) = \begin{pmatrix} \gamma(\lambda) & 0 \\ 0 & \gamma'(\lambda)\tau(\lambda)^{-1} \end{pmatrix}, \quad \lambda \in \mathfrak{h}(M) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}(\tau^{-1}),$$

where γ and γ' are the γ-fields corresponding to A and $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ and to T and $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$, respectively, and

$$\widetilde{M}(\lambda) = \begin{pmatrix} M(\lambda) & 0 \\ 0 & -\tau(\lambda)^{-1} \end{pmatrix}, \quad \lambda \in \mathfrak{h}(M) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}(\tau^{-1}).$$

The selfadjoint relation $\widetilde{A}$ in $\mathcal{K} \times \mathcal{H}$ corresponding to $\Theta = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathcal{L}(\mathbb{C}^2)$ via (2.4) is given by

$$\begin{aligned} \widetilde{A} &= \ker(\widetilde{\Gamma}_1 - \Theta\widetilde{\Gamma}_0) \\ &= \Big\{ \{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+ \,|\, \Gamma_1 \hat{f}_1 - \Gamma_1' \hat{f}_2 = \Gamma_0 \hat{f}_1 + \Gamma_0' \hat{f}_2 = 0 \Big\}. \end{aligned} \tag{4.2}$$

For $\lambda \in \mathfrak{h}_0 \backslash \{\infty\}$ the resolvent of $\widetilde{A}$ can be written as

$$(\widetilde{A} - \lambda)^{-1} = (A_0 \times T_1 - \lambda)^{-1} + \widetilde{\gamma}(\lambda)\big(\Theta - \widetilde{M}(\lambda)\big)^{-1} \widetilde{\gamma}(\overline{\lambda})^+, \tag{4.3}$$

(see (2.9)). Calculating $(\Theta - \widetilde{M}(\lambda))^{-1}$ one verifies that the compressed resolvent of $\widetilde{A}$ onto $\mathcal{K}$ is given by

$$P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1} | \mathcal{K} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1} \gamma(\overline{\lambda})^+.$$

For $k \in \mathcal{K}$ we set

$$f_1 := P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}\{k, 0\} \quad \text{and} \quad f_2 := P_{\mathcal{H}}(\widetilde{A} - \lambda)^{-1}\{k, 0\}.$$

Then

$$\begin{pmatrix} \{f_1, f_2\} \\ \{\lambda f_1 + k, \lambda f_2\} \end{pmatrix} \in \widetilde{A} \subset A^+ \times T^+$$

and $\hat{f}_1 := \left(\begin{smallmatrix} f_1 \\ \lambda f_1 + k \end{smallmatrix}\right) \in A^+$, $\hat{f}_2 := \left(\begin{smallmatrix} f_2 \\ \lambda f_2 \end{smallmatrix}\right) \in T^+$, $f_2 \in \ker(T^+ - \lambda)$. From (4.2) and since τ is the Weyl function corresponding to $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ we obtain

$$\Gamma_1 \hat{f}_1 = \Gamma_1' \hat{f}_2 = \tau(\lambda)\Gamma_0' \hat{f}_2 = -\tau(\lambda)\Gamma_0 \hat{f}_1, \quad \lambda \in \mathfrak{h}_0 \backslash \{\infty\},$$

and it follows that $\hat{f}_1 \in A^+$ is a solution of (4.1).

Let us show that the solution $\hat{f}_1 \in A^+$ of (4.1) is unique. Assume that the vector $\hat{g}_1 = \left(\begin{smallmatrix} g_1 \\ \lambda g_1 + k \end{smallmatrix}\right) \in A^+$ is also a solution of (4.1), $\lambda \in \mathfrak{h}_0 \backslash \{\infty\}$. Then $\hat{f}_1 - \hat{g}_1$ belongs to $\hat{\mathcal{N}}_{\lambda, A^+} := \big\{ \left(\begin{smallmatrix} h \\ \lambda h \end{smallmatrix}\right) \big| \, h \in \ker(A^+ - \lambda) \big\}$ and

$$0 = \tau(\lambda)\Gamma_0(\hat{f}_1 - \hat{g}_1) + \Gamma_1(\hat{f}_1 - \hat{g}_1) = \big(\tau(\lambda) + M(\lambda)\big)\Gamma_0(\hat{f}_1 - \hat{g}_1)$$

implies $\hat{f}_1 - \hat{g}_1 \in A_0 \cap \hat{\mathcal{N}}_{\lambda, A^+}$ as $\tau(\lambda) + M(\lambda) \neq 0$. Therefore $\hat{f}_1 = \hat{g}_1$.

We claim that $\widetilde{A}$ has $\tilde{\kappa}$,

$$0 \leq \tilde{\kappa} \in \{\kappa + \kappa' - 1, \ldots, \kappa + \kappa' + 2\},$$

negative squares. In fact, since T has κ' or $\kappa'-1$ negative squares we obtain that $A\times T\in\widetilde{\mathcal{C}}(\mathcal{K}\times\mathcal{H})$ is a symmetric relation with $\kappa+\kappa'-1$ or $\kappa+\kappa'$ negative squares. Now the assertion follows from the fact that $A\times T$ has defect 2.

It remains to verify that $\widetilde{A}$ is an operator if A is a densely defined operator. As T is a symmetric operator (see Theorem 3.8) $A\times T$ is a (in general not densely defined) symmetric operator in $\mathcal{K}\times\mathcal{H}$. We denote the multivalued part of $A^+\times T^+$ by $\operatorname{mul}(A^+\times T^+)$. Let

$$\hat{\mathcal{N}_\infty} := \left\{ \begin{pmatrix} 0\\ h\end{pmatrix} \,\middle|\, h\in \operatorname{mul}(A^+\times T^+)\right\}.$$

By [9, Proposition 2.1] it is sufficient to show that $(\widetilde{\Gamma}_0\ \widetilde{\Gamma}_1)^\top\hat{\mathcal{N}_\infty}\cap\Theta=\{0\}$ holds. From $\operatorname{mul}(A^+\times T^+)=\{\{0,f\}\,|\,f\in\operatorname{mul}T^+\}$ we obtain

$$\begin{pmatrix}\widetilde{\Gamma}_0\\ \widetilde{\Gamma}_1\end{pmatrix}\hat{\mathcal{N}_\infty}\cap\Theta = \left\{ \begin{matrix}\{0,\Gamma_1'\binom{0}{f}\}\\ \{0,-\Gamma_0'\binom{0}{f}\}\end{matrix} \,\middle|\, f\in\operatorname{mul}T^+\right\}\cap\left\{\begin{matrix}\{x,y\}\\ \{y,x\}\end{matrix}\,\middle|\, x,y\in\mathbb{C}\right\}.$$

Hence $\Gamma_0'\binom{0}{f}=\Gamma_1'\binom{0}{f}=0$ and this implies $\binom{0}{f}\in T$. As T is an operator we have $f=0$ and therefore $(\widetilde{\Gamma}_0\ \widetilde{\Gamma}_1)^\top\hat{\mathcal{N}_\infty}\cap\Theta=\{0\}$, i.e. $\widetilde{A}$ is an operator. Theorem 4.1 is proved.

□

Since the representation of the function $\tau\in D_{\kappa'}$ in Theorem 4.1 is choosen minimal (see Section 3) one verifies as in the proof of [2, Theorem 4.1] that $\widetilde{A}$ fulfils the minimality condition

$$\operatorname{clsp}\left\{(1+(\lambda-\lambda_0)(\widetilde{A}-\lambda)^{-1})\{k,0\}\,|\,k\in\mathcal{K},\lambda\in\rho(\widetilde{A})\right\}=\mathcal{K}\times\mathcal{H}$$

for some $\lambda_0\in\rho(\widetilde{A})$. Let $\widetilde{B}$ be a selfadjoint extension of the symmetric relation $A\in\widetilde{\mathcal{C}}(\mathcal{K})$ which acts in some Krein space $\mathcal{K}\times\widetilde{\mathcal{H}}$ such that $\rho(\widetilde{B})\neq\emptyset$ and

$$P_{\mathcal{K}}(\widetilde{B}-\lambda)^{-1}\{k,0\}$$

yields a solution of (4.1). Then the compressed resolvents of $\widetilde{A}$ and $\widetilde{B}$ onto $\mathcal{K}$ coincide. Assume that $\widetilde{B}$ fulfils the minimality condition

$$\operatorname{clsp}\left\{(1+(\lambda-\lambda_0)(\widetilde{B}-\lambda)^{-1})\{k,0\}\,|\,k\in\mathcal{K},\lambda\in\rho(\widetilde{B})\right\}=\mathcal{K}\times\widetilde{\mathcal{H}}$$

for some $\lambda_0\in\rho(\widetilde{A})\cap\rho(\widetilde{B})$. Then

$$V:=\left\{\begin{pmatrix}\sum_{i=1}^n(1+(\lambda_i-\lambda_0)(\widetilde{A}-\lambda_i)^{-1})\{k_i,0\}\\ \sum_{i=1}^n(1+(\lambda_i-\lambda_0)(\widetilde{B}-\lambda_i)^{-1})\{k_i,0\}\end{pmatrix}\,\middle|\,\lambda_i\in\rho(\widetilde{A})\cap\rho(\widetilde{B}),k_i\in\mathcal{K}\right\}$$

is a densely defined isometric operator in $\mathcal{K}\times\mathcal{H}$ with dense range in $\mathcal{K}\times\widetilde{\mathcal{H}}$ such that

$$V(\widetilde{A}-\lambda)^{-1}x=(\widetilde{B}-\lambda)^{-1}Vx$$

is fulfilled for all $x \in \operatorname{dom} V$ and all $\lambda \in \rho(\widetilde{A}) \cap \rho(\widetilde{B})$. As

$$\begin{aligned}\big[(I + \lambda(\widetilde{B} - \lambda)^{-1})Vx, (\widetilde{B} - \lambda)^{-1}Vx\big]_{\mathcal{K}\times\widetilde{\mathcal{H}}}& \\ = \big[(I + \lambda(\widetilde{A} - \lambda)^{-1})x, (\widetilde{A} - \lambda)^{-1}x\big]_{\mathcal{K}\times\mathcal{H}}&\end{aligned}$$

holds for all $x \in \operatorname{dom} V$ we conclude that the number of negative squares of $\widetilde{A}$ and $\widetilde{B}$ coincide.

5. Sturm-Liouville operators with an indefinite weight function

In this section we show how Theorem 4.1 can be applied to Sturm-Liouville operators with an indefinite weight function and a λ-dependent boundary condition. Here we make use of results obtained by B. Ćurgus and H. Langer in [5]. For simplicity only regular second order differential expressions are considered. We remark that the following considerations can be generalized to higher order differential operators and to singular problems.

5.1. **The general case.** Let $p^{-1}, q, r \in L^1(a,b)$, $-\infty < a < b < \infty$, be real functions such that $p > 0$, $r \neq 0$ almost everywhere and assume that the sets

$$\{x \in (a,b) \,|\, r(x) > 0\} \qquad \text{and} \qquad \{x \in (a,b) \,|\, r(x) < 0\}$$

have positive Lebesgue measure. By $L^2_{|r|}(a,b)$ we denote the space of all equivalence classes of measurable functions f defined on (a,b) for which

$$\int_a^b |f(x)|^2 |r(x)| dx$$

is finite.

Let $\kappa' \in \mathbb{N}_0$ and let $\tau \in D_{\kappa'}$ be not equal to a constant. In this section we consider the following boundary value problem. For a given $k \in L^2_{|r|}(a,b)$ find $f \in L^2_{|r|}(a,b)$ such that f and pf' are absolutely continuous, the equation

$$-(pf')' + qf - \lambda r f = rk \tag{5.1}$$

and the boundary conditions

$$\tau(\lambda) f(b) = (pf')(b) \quad \text{and} \quad f(a)\cos\alpha = (pf')(a)\sin\alpha \tag{5.2}$$

are fulfilled for some $\alpha \in [0, \pi)$.

In order to apply Theorem 4.1 the boundary value problem (5.1)-(5.2) will be formulated in the form (4.1). For this we equip $L^2_{|r|}(a,b)$ with the inner products

$$[f,g] := \int_a^b f(x)\overline{g(x)} r(x) dx \quad \text{and} \quad (f,g) := \int_a^b f(x)\overline{g(x)}|r(x)| dx, \tag{5.3}$$

where $f, g, \in L^2_{|r|}(a,b)$. The corresponding Krein space $(L^2_{|r|}(a,b), [\cdot,\cdot])$ (Hilbert space $(L^2_{|r|}(a,b), (\cdot,\cdot))$) is denoted by $\mathcal{K}$ (resp. $\mathfrak{K}$). The fundamental symmetry connecting the inner products in (5.3) is given by

$$(Jf)(x) := (\operatorname{sgn} r(x)) f(x), \qquad f \in L^2_{|r|}(a,b).$$

Let $\widetilde{D}$ be the set of all $f \in L^2_{|r|}(a,b)$ such that f and pf' are absolutely continuous and the equation

$$-(pf')' + qf = |r|g$$

holds with a certain $g \in L^2_{|r|}(a,b)$. We define the operator $\widetilde{B}$ in $\mathfrak{K}$ by $\widetilde{B}f := g$. This definition makes sense since $f = 0$ (in $\mathfrak{K}$) implies $g = 0$ (in $\mathfrak{K}$). We set

$$\mathcal{D} := \{f \in \widetilde{\mathcal{D}} \,|\, f(a)\cos\alpha - (pf')(a)\sin\alpha = 0 = f(b) = (pf')(b)\}$$

and

$$\mathcal{D}' := \{f \in \widetilde{\mathcal{D}} \,|\, f(a)\cos\alpha - (pf')(a)\sin\alpha = 0\}.$$

The restrictions of the operator $\widetilde{B}$ to $\mathcal{D}$ and $\mathcal{D}'$ are denoted by B and B', respectively. As in [31] one verifys that B is a densely defined closed symmetric operator in $\mathfrak{K}$ which has defect $(1,1)$. The adjoint of B in $\mathfrak{K}$ is B'. It is obvious that $A := JB$ is a densely defined closed symmetric operator in $\mathcal{K}$ and its adjoint in $\mathcal{K}$ is given by $A^+ = JB^* = JB'$. We have $A^+f = g$, $f \in \operatorname{dom}(A^+) = \mathcal{D}'$ if and only if

$$-(pf')' + qf = rg.$$

It is well known that the assumption $p > 0$ implies that B is bounded from below and that the spectrum of an arbitrary selfadjoint extension of B in $\mathfrak{K}$ is discrete. By [5, Proposition 2.2] the form $[A\cdot,\cdot]$ has a finite number of negative squares. Let B_ϑ be a selfadjoint extension of B in $\mathfrak{K}$ such that $0 \in \rho(B_\vartheta)$. Then 0 belongs also to the resolvent set of the selfadjoint operator $A_\vartheta := JB_\vartheta$ in $\mathcal{K}$ which is an extension of A. From [5, Proposition 1.1] we obtain that each selfadjoint extension of A in $\mathcal{K}$ has a nonempty resolvent set. As B_ϑ^{-1} is a compact operator we conclude from $A_\vartheta^{-1} = B_\vartheta^{-1}J$ that the spectrum of A_ϑ is discrete. If A_θ is an arbitrary selfadjoint extension of A in $\mathcal{K}$ then the resolvents of A_θ and A_ϑ differ only by a rank one operator and therefore the spectrum of A_θ is also discrete.

A boundary value space $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ for A^+ is given by

$$\Gamma_0\hat{f} := -f(b) \quad \text{and} \quad \Gamma_1\hat{f} := (pf')(b), \quad \hat{f} = \begin{pmatrix} f \\ A^+f \end{pmatrix}. \tag{5.4}$$

The boundary value problem (5.1)-(5.2) takes the form

$$(A^+ - \lambda)f = k, \quad \tau(\lambda)\Gamma_0\hat{f} + \Gamma_1\hat{f} = 0, \qquad \hat{f} = \begin{pmatrix} f \\ A^+f \end{pmatrix}. \tag{5.5}$$

By Lemma 3.7 the Weyl function M corresponding to the operator A and the boundary value space $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ belongs to some class D_κ, $\kappa \in \mathbb{N}_0$. If M fulfils $M(\mu) \neq -\tau(\mu)$ for some $\mu \in \mathfrak{h}(M) \cap \mathfrak{h}(\tau)$ Theorem 4.1 can be applied to solve (5.5). Let T_0 be a minimal representing relation for τ (cf. (3.6) and (3.7)) in some Krein space $\mathcal{H}$ and choose $T \subset T_0 \subset T^+$ and $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ as in Theorem 3.8. By Theorem 4.1

$$\widetilde{A} = \Big\{ \{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+ \,|\, \Gamma_1\hat{f}_1 - \Gamma_1'\hat{f}_2 = \Gamma_0\hat{f}_1 + \Gamma_0'\hat{f}_2 = 0 \Big\} \tag{5.6}$$

is a selfadjoint operator in $\mathcal{K} \times \mathcal{H}$ with a finite number of negative squares and for all $\lambda \in \mathfrak{h}_0$, where

$$\mathfrak{h}_0 = \mathfrak{h}(M) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}\big(\tau^{-1}\big) \cap \mathfrak{h}\big((M+\tau)^{-1}\big), \tag{5.7}$$

the vector $f := P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}}\, k$ is the unique solution of the boundary value problem (5.1)-(5.2). By Theorem 4.1 there are at most finitely many points in $\mathbb{C}\backslash\mathbb{R}$ which do not belong to $\mathfrak{h}_0$. As the operator $A_0 = \ker \Gamma_0$ given by

$$\begin{aligned} A_0 f &= \frac{1}{r}\big((-pf')' + qf\big),\\ \operatorname{dom} A_0 &= \big\{f \in \widetilde{\mathcal{D}} \,|\, f(a)\cos\alpha - (pf')(a)\sin\alpha = f(b) = 0\big\}, \end{aligned}$$

has discrete spectrum more can be said about the solvability of the boundary value problem (5.1)-(5.2) for real λ.

Theorem 5.1. *Let $A \subset A^+$, M and τ be as above, assume that $M + \tau$ is not identically equal to zero and let $\widetilde{A}$ and $\mathfrak{h}_0$ be as in* (5.6) *and* (5.7)*. Assume that $\Delta \subset \mathbb{R}$ is a closed interval such that*

$$\Delta \subset \mathfrak{h}(\tau) \cup \{\textit{poles of } \tau\}$$

holds. Then there exists a finite set $e \subset \Delta$ such that $\Delta\backslash e \subset \mathfrak{h}_0$ and for every $\lambda \in \Delta\backslash e$ the vector $f = P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}}\, k$ is the unique solution of the boundary value problem (5.1)-(5.2)*.*

Proof. As τ is not equal to a constant and has only finitely many poles in Δ the function τ^{-1} also has only finitely many poles in Δ. Since the spectrum of A_0 is discrete the Weyl function M has at most finitely many poles in Δ and the assumption that $M + \tau$ is not identically equal to zero implies the same for the poles of $(M+\tau)^{-1}$. If e is the union of the poles of τ, τ^{-1}, M and $(M+\tau)^{-1}$ in Δ, then e is finite and therefore $\Delta\backslash e \subset \mathfrak{h}_0 \subset \rho(\widetilde{A})$.

□

In Theorem 5.2 below we describe the spectrum of the operator $\widetilde{A}$. For this recall that a point $\lambda \in \mathbb{C}$ belongs to the *essential spectrum* $\sigma_{ess}(C)$ of a densely defined closed operator C acting in a Banach space if $C - \lambda$ is not a Fredholm operator, i.e. the range of $C - \lambda$ has infinite codimension or $\dim(\ker(C - \lambda)) = \infty$.

Let D be a definitizable operator acting in a Krein space $(\mathcal{K}, [\cdot,\cdot])$ and let $E(\cdot)$ be the spectral function of D (cf. [28]). For the convenience of the reader we repeat some definitions. A point $\lambda \in \widetilde{\sigma}(D) \cap \overline{\mathbb{R}}$ belongs to $\sigma_{++}(D)$ ($\sigma_{--}(D)$) if there exists a connected open set $\delta \subset \overline{\mathbb{R}}$ with $\lambda \in \delta$ such that $E(\delta)$ is defined and $(E(\delta), [\cdot,\cdot])$ (resp. $(E(\delta), -[\cdot,\cdot])$) is a Hilbert space. If $E(\delta)\mathcal{H}$ (δ and λ as above) is a Pontryagin space with finite rank of negativity (positivity) then λ belongs to $\sigma_{\pi_+}(D)$ (resp. $\sigma_{\pi_-}(D)$).

A point $\lambda \in \overline{\mathbb{R}} \cap \widetilde{\sigma}(D)$ is called a *critical point* (*an essential critical point*) of D if $\lambda \notin \sigma_{++}(D) \cup \sigma_{--}(D)$ (resp. $\lambda \notin \sigma_{\pi_+}(D) \cup \sigma_{\pi_-}(D)$). A critical point λ is called regular if $\sup \|E(\Delta)\| < \infty$ where the supremum runs over all sufficiently small neighbourhoods Δ of λ. The set of critical points (essential critical points, regular critical points) of D is denoted by $c(D)$ (resp. $c_\infty(D)$, $c_r(D)$). The elements of $c_s(D) := c(D)\backslash c_r(D)$ are called *singular critical points.*

Theorem 5.2. *Let $A \subset A^+$, $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ and M be as above and denote by κ the number of negative squares of $[A\cdot,\cdot]$. Let $\tau \in D_{\kappa'}$, $\kappa' \in \mathbb{N}_0$, and let $\mathcal{H}$, $T \subset T_0 \subset T^+$ and $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ be as above. Assume that $M + \tau$ is not identically equal to zero. Then the selfadjoint operator $\widetilde{A}$ in (5.6) has*

$$0 \leq \widetilde{\kappa} \in \{\kappa + \kappa' - 1, \ldots, \kappa + \kappa' + 2\}$$

negative squares. The set $\mathbb{C}\backslash\mathbb{R}$ with the exception of at most $2\widetilde{\kappa}$ points belongs to the resolvent set of $\widetilde{A}$. Moreover, we have

$$(0,\infty) \subset \sigma_{\pi_+}(\widetilde{A}) \cup \rho(\widetilde{A}) \quad \text{and} \quad (-\infty, 0) \subset \sigma_{\pi_-}(\widetilde{A}) \cup \rho(\widetilde{A}), \tag{5.8}$$

and the set

$$\sigma(\widetilde{A}) \cap \big((\mathbb{C}\backslash\mathbb{R}) \cup \big((0,\infty)\backslash\sigma_{++}(\widetilde{A})\big) \cup \big((-\infty,0)\backslash\sigma_{--}(\widetilde{A})\big)\big) \tag{5.9}$$

is a subset of $\sigma_p(\widetilde{A})$ consisting of at most $2\widetilde{\kappa}$ points. The essential spectrum of $\widetilde{A}$ coincides with the essential spectrum of T_0,

$$\sigma_{ess}(\widetilde{A}) = \sigma_{ess}(T_0). \tag{5.10}$$

The set $c(\widetilde{A})$ is a subset of

$$\big((0,\infty)\backslash\sigma_{++}(\widetilde{A})\big) \cup \big((-\infty,0)\backslash\sigma_{--}(\widetilde{A})\big) \cup \{0,\infty\}. \tag{5.11}$$

The point 0 belongs to $c_\infty(\widetilde{A})$ if and only if $0 \in c_\infty(T_0)$. The point ∞ belongs to $c_\infty(\widetilde{A})$ and no other point belongs to $c_\infty(\widetilde{A})$, that is $c_\infty(\widetilde{A}) \subset \{0,\infty\}$. Further we have $c_s(\widetilde{A}) \cap \rho(T_0) = \emptyset$.

Proof. By Theorem 4.1 $\widetilde{A}$ is a selfadjoint operator with $\rho(\widetilde{A}) \neq \emptyset$ and $\widetilde{\kappa}$ negative squares. Therefore $\widetilde{A}$ is a definitizable operator with a definitizing polynomial p of the form

$$z \mapsto p(z) = zq(z)\overline{q(z)}, \tag{5.12}$$

where q is a monic polynomial (cf. [28]). This implies (5.8), (5.9) and (5.11).

For the operator $A_0 = \ker \Gamma_0$ we have $\sigma_{ess}(A_0) = \emptyset$, hence from (4.3) we obtain

$$\sigma_{ess}(\widetilde{A}) = \sigma_{ess}(A_0 \times T_0) = \sigma_{ess}(T_0)$$

and [25, Theorem 1] implies

$$c_\infty(\widetilde{A}) = c_\infty(A_0 \times T_0) = c_\infty(A_0) \cup c_\infty(T_0).$$

Observe that the assumptions on the function r and [5] imply $c_\infty(A_0) = \{\infty\}$ and, as T_0 has a κ' negative squares, $c_\infty(T_0) \subset \{0,\infty\}$ holds.

Moreover, for $\lambda \in \rho(T_0)$ it follows from above that λ is an isolated point of the spectrum of $\widetilde{A}$ or $\lambda \in \rho(\widetilde{A})$ and therefore $\lambda \notin c_s(\widetilde{A})$. The remaining assertions of the theorem follow from the fact that $\widetilde{A}$ has $\widetilde{\kappa}$ negative squares and that $\widetilde{A}$ has a definitizing polynomial p of the form (5.12).

□

5.2. **A special function $\tau \in D_0$ in the boundary condition.** For a special type of functions from the class D_0 we construct $\widetilde{A}$ in a more explicit form and investigate its spectral properties.

Assume that

$$\tau(\lambda) = m_1(\lambda) - m_2(\lambda), \qquad \lambda \in \mathfrak{h}(\tau), \tag{5.13}$$

holds with Nevanlinna functions m_1 and m_2 given by

$$m_1(\lambda) = a_1 + \int_0^\infty \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d\sigma_1(t),$$
$$m_2(\lambda) = a_2 + \int_{-\infty}^0 \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d\sigma_2(t),$$

where a_1, $a_2 \in \mathbb{R}$ and σ_1, σ_2 are positive measures such that

$$\int_0^\infty \frac{1}{1+t^2}\, d\sigma_1(t) < \infty, \qquad \int_{-\infty}^0 \frac{1}{1+t^2}\, d\sigma_2(t) < \infty.$$

In the next lemma we establish a simple operator model for τ. Equip the linear space $\mathcal{H} := L^2([0,\infty),\sigma_1) \times L^2((-\infty,0],\sigma_2)$ with the Krein space inner product $[\cdot,\cdot]$ defined by

$$\left[\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right] := \int_0^\infty g_1(t)\overline{h_1(t)}\, d\sigma_1(t) - \int_{-\infty}^0 g_2(t)\overline{h_2(t)}\, d\sigma_2(t),$$

$g_1, h_1 \in L^2([0,\infty),\sigma_1)$, $g_2, h_2 \in L^2((-\infty,0],\sigma_2)$, and define the operator T_0 in $\mathcal{H}$ on the dense linear subspace

$$\operatorname{dom} T_0 := \left\{ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{H} \,\middle|\, \int_0^\infty |t\, g_1(t)|^2 d\sigma_1(t) + \int_{-\infty}^0 |t\, g_2(t)|^2 d\sigma_2(t) < \infty \right\}$$

by

$$\left(T_0 \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right)(t) := t \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix},$$

set $\gamma : \mathbb{C} \to \mathcal{H}$, $c \mapsto c\left(\begin{smallmatrix} w_{i1} \\ w_{i2} \end{smallmatrix}\right)$, where $w_{i1}(t) := (t-i)^{-1}$, $t \in [0,\infty)$, $w_{i2}(t) := (t-i)^{-1}$, $t \in (-\infty, 0]$

Lemma 5.3. *T_0 is a selfadjoint nonnegative operator in the Krein space $(\mathcal{H},[\cdot,\cdot])$. For $\lambda \notin \operatorname{supp}\sigma_1 \cup \operatorname{supp}\sigma_2$ we have*

$$\tau(\lambda) = \operatorname{Re}\tau(i) + \gamma^+\big(\lambda + (\lambda^2+1)(T_0-\lambda)^{-1}\big)\gamma.$$

In particular the function τ belongs to the class D_0.

Proof. The adjoint of $\gamma \in \mathcal{L}(\mathbb{C},\mathcal{H})$ is given by

$$\gamma^+ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \int_0^\infty \frac{g_1(t)}{t+i}\, d\sigma_1(t) - \int_{-\infty}^0 \frac{g_2(t)}{t+i}\, d\sigma_2(t), \qquad \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{H}.$$

Now the statement of the lemma follows from an easy calculation. $\square$

Let T be the restriction of T_0 to the set

$$\operatorname{dom} T := \left\{ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \operatorname{dom} T_0 \,\Big|\, \int_0^\infty g_1(t)\, d\sigma_1(t) = \int_{-\infty}^0 g_2(t)\, d\sigma_2(t) \right\}.$$

Then T is a closed symmetric operator with defect one. The defect subspace $\mathcal{N}_\mu$, $\operatorname{Im}\mu \neq 0$, of T is spanned by $w_\mu = \left(\begin{smallmatrix} w_{\mu 1} \\ w_{\mu 2} \end{smallmatrix}\right) \in \mathcal{H}$, $w_{\mu 1}(t) := (t-\mu)^{-1}$, $t \in [0,\infty)$, $w_{\mu 2}(t) := (t-\mu)^{-1}$, $t \in (-\infty, 0]$. Since T^+ can be written as a direct sum of T_0 and $\hat{\mathcal{N}}_\mu = \operatorname{sp}\left\{ \left(\begin{smallmatrix} w_\mu \\ \mu w_\mu \end{smallmatrix}\right) \right\}$ a vector $\hat{g} \in T^+$ can be written in the form

$$\hat{g} = \begin{pmatrix} g_0 \\ T_0 g_0 \end{pmatrix} + c_{\hat{g}} \begin{pmatrix} w_\mu \\ \mu w_\mu \end{pmatrix}, \quad g_0 = \begin{pmatrix} g_{01} \\ g_{02} \end{pmatrix} \in \operatorname{dom} T_0, \tag{5.14}$$

with a suitable $c_{\hat{g}} \in \mathbb{C}$. We define the linear mappings $\Gamma_0', \Gamma_1' : T^+ \to \mathbb{C}$ by

$$\begin{aligned} \Gamma_0' \hat{g} &:= c_{\hat{g}}, \\ \Gamma_1' \hat{g} &:= \int_0^\infty g_{01}(t)\, d\sigma_1(t) - \int_{-\infty}^0 g_{02}(t)\, d\sigma_2(t) + \tau(\mu) c_{\hat{g}}. \end{aligned} \tag{5.15}$$

Let us show that $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ is a boundary value space for T^+ with corresponding Weyl function τ. The latter statement is clear. Let $\hat{g}, \hat{h} \in T^+$ be decomposed in the form (5.14). Then we have

$$\begin{aligned} &\big[T_0 g_0 + c_{\hat{g}} \mu w_\mu, h_0 + c_{\hat{h}} w_\mu\big] - \big[g_0 + c_{\hat{g}} w_\mu, T_0 h_0 + c_{\hat{h}} \mu w_\mu\big] = \\ &\quad = c_{\hat{g}} \overline{c_{\hat{h}}} (\mu - \overline{\mu}) [w_\mu, w_\mu] + \big[(T_0 - \overline{\mu}) g_0, c_{\hat{h}} w_\mu\big] - \big[c_{\hat{g}} w_\mu, (T_0 - \overline{\mu}) h_0\big] \\ &\quad = c_{\hat{g}} \overline{c_{\hat{h}}} \left(\tau(\mu) - \overline{\tau(\mu)}\right) + \overline{c_{\hat{h}}} \left(\int_0^\infty g_{01}(t)\, d\sigma_1(t) - \int_{-\infty}^0 g_{02}(t)\, d\sigma_2(t) \right) \\ &\qquad\qquad - c_{\hat{g}} \left(\int_0^\infty \overline{h_{01}(t)}\, d\sigma_1(t) - \int_{-\infty}^0 \overline{h_{02}(t)}\, d\sigma_2(t) \right) \\ &\quad = \Gamma_1' \hat{g} \overline{\Gamma_0' \hat{h}} - \Gamma_0' \hat{g} \overline{\Gamma_1' \hat{h}}. \end{aligned}$$

We check that $\left(\begin{smallmatrix} \Gamma_0' \\ \Gamma_1' \end{smallmatrix}\right) : T^+ \to \mathbb{C}^2$ is surjective. For a given vector $\left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right) \in \mathbb{C}^2$ we choose $k_0 \in \operatorname{dom} T_0$ such that $\gamma^+(T_0 + i) k_0 = c_2 - \tau(\mu) c_1$. Then we have

$$\begin{pmatrix} \Gamma_0' \\ \Gamma_1' \end{pmatrix} \left(\begin{pmatrix} k_0 \\ T_0 k_0 \end{pmatrix} + c_1 \begin{pmatrix} w_\mu \\ \mu w_\mu \end{pmatrix} \right) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

and it follows that $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ is a boundary value space for T^+.

The following theorem is a special case of Theorem 5.2 and Theorem 5.1.

Theorem 5.4. *Let τ be as in* (5.13)*, let $T \subset T_0 \subset T^+$ be as above and assume that the Weyl function M corresponding to A and $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ fulfils $M(\mu') \neq -\tau(\mu')$ for some $\mu' \in \mathfrak{h}(M) \cap \mathfrak{h}(\tau)$. Then the operator*

$$\begin{aligned} \widetilde{A} = \Bigg\{ \left\{ \begin{pmatrix} f \\ A^+ f \end{pmatrix}, \begin{pmatrix} g_0 \\ T_0 g_0 \end{pmatrix} + c_{\hat{g}} \begin{pmatrix} w_\mu \\ \mu w_\mu \end{pmatrix} \right\} \in A^+ \times T^+ \,\Big|\, & f(b) = c_{\hat{g}} \\ (pf')(b) = \int_0^\infty g_{01}(t)\, d\sigma_1(t) - \int_{-\infty}^0 g_{02}(t)\, d\sigma_2(t) + \tau(\mu) c_{\hat{g}} & \Bigg\} \end{aligned} \tag{5.16}$$

is a selfadjoint extension of A in $\mathcal{K} \times \mathcal{H}$ with $\widetilde{\kappa}$ negative squares, where

$$\widetilde{\kappa} \in \{\kappa, \kappa+1, \kappa+2\}.$$

For all $\lambda \in \mathfrak{h}_0$ the vector $f := P_{\mathcal{K}}(\widetilde{A}-\lambda)^{-1}\{k,0\}$ is the unique solution of the boundary value problem (5.1)-(5.2)*. The sets $\mathfrak{h}(\tau)$ and $\mathfrak{h}_0$ coincide with the exception of a discrete set which may accumulate only to nonisolated points of* $\operatorname{supp}\sigma_1 \cup \operatorname{supp}\sigma_2$.

The essential spectrum $\sigma_{ess}(\widetilde{A})$ of $\widetilde{A}$ coincides with the nonisolated points of $\operatorname{supp}\sigma_1 \cup \operatorname{supp}\sigma_2$. *Moreover* 0 *belongs to $c_\infty(\widetilde{A})$ if and only if* 0 *is an accumulation point of* $\operatorname{supp}\sigma_1$ *and* $\operatorname{supp}\sigma_2$.

Proof. It follows from the definition of $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ in (5.15) and Theorem 4.1 that $\widetilde{A}$ has the form (5.16). Since T_0 is a nonnegative operator we get $\widetilde{\kappa} \in \{\kappa, \kappa+1, \kappa+2\}$.

As the isolated points of $\operatorname{supp}\sigma_1 \cup \operatorname{supp}\sigma_2$ are poles of τ and the points not belonging to $\operatorname{supp}\sigma_1 \cup \operatorname{supp}\sigma_2$ are points of holomorphy of τ the same arguments as in the proof of Theorem 5.1 show that the sets $\mathfrak{h}(\tau)$ and $\mathfrak{h}_0$ coincide with the exception of a discrete set which has accumulation points only in the set of nonisolated points of $\operatorname{supp}\sigma_1 \cup \operatorname{supp}\sigma_2$.

Since $\sigma_{ess}(T_0)$ is the set of non-isolated points of $\operatorname{supp}\sigma_1$ and $\operatorname{supp}\sigma_2$ we obtain from (5.10) that $\sigma_{ess}(\widetilde{A})$ coincides with the non-isolated points of $\operatorname{supp}\sigma_1$ and $\operatorname{supp}\sigma_2$. If 0 is an eigenvalue of T_0 then the corresponding eigenspace has at most dimension 2. Hence $0 \in c_\infty(T_0)$ if and only if 0 is an accumulation point of $\operatorname{supp}\sigma_1$ and $\operatorname{supp}\sigma_2$. Now Theorem 5.2 implies the last statement of Theorem 5.4. □

In the next theorem we give a criterion for $\widetilde{A}$ to be nonnegative.

Theorem 5.5. *Let the assumptions be as in Theorem* 5.4 *and let $\widetilde{A}$ be as in* (5.16)*. For $\mathrm{h} := \{f, g_0 + c_{\hat{g}} w_\mu\} \in \operatorname{dom}\widetilde{A}$ we have*

$$\begin{aligned}[\widetilde{A}\mathrm{h}, \mathrm{h}] = (pf')(a)\overline{f(a)} + \int_a^b (p|f'|^2 + q|f|^2)dx + [T_0 g_0, g_0] \\ + 2\operatorname{Re}\bigl(c_{\hat{g}}\mu[w_\mu, g_0]\bigr) + |c_{\hat{g}}|^2\bigl(\operatorname{Re}\mu\,[w_\mu, w_\mu] - \operatorname{Re}\tau(\mu)\bigr).\end{aligned}$$

If $\alpha \in [0, \frac{\pi}{2}]$ and the function q is nonnegative, $0 \in \mathfrak{h}(\tau)$ and $\tau(0) \le 0$ then $\widetilde{A}$ is a nonnegative operator in the Krein space $\mathcal{K} \times \mathcal{H}$.

Proof. The first assertions follow from Theorem 4.1, (5.4) and the fact that T_0 and A are densely defined operators which implies (see the proof of Theorem 4.1) that $\widetilde{A}$ is a selfadjoint operator.

Let $\mathrm{h} := \{f, g_0 + c_{\hat{g}} w_\mu\} \in \operatorname{dom}\widetilde{A}$. Then

$$-(pf')(b)\overline{f(b)} = -\overline{c}_{\hat{g}}[T_0 g_0, w_\mu] + \overline{c}_{\hat{g}}[\overline{\mu} g_0, w_\mu] - |c_{\hat{g}}|^2\tau(\mu)$$

and $\operatorname{Im}\tau(\mu) = \operatorname{Im}\mu[w_\mu, w_\mu]$ implies

$$\begin{aligned}
[\widetilde{A}\mathrm{h}, \mathrm{h}] &= [A^+ f, f] + [T_0 g_0 + c_{\hat{g}}\mu w_\mu, g_0 + c_{\hat{g}} w_\mu] \\
&= -(pf')\overline{f}\big|_a^b + \int_a^b \big(p|f'|^2 + q|f|^2\big)dx + [T_0 g_0 + c_{\hat{g}}\mu w_\mu, g_0 + c_{\hat{g}} w_\mu] \\
&= (pf')(a)\overline{f(a)} + \int_a^b \big(p|f'|^2 + q|f|^2\big)dx + [T_0 g_0, g_0] \\
&\qquad + 2\operatorname{Re}\big(c_{\hat{g}}\mu[w_\mu, g_0]\big) + |c_{\hat{g}}|^2\big(\operatorname{Re}\mu[w_\mu, w_\mu] - \operatorname{Re}\tau(\mu)\big).
\end{aligned}$$

If $0 \in \mathfrak{h}(\tau)$ we have $0 \in \rho(T_0)$ and therefore we can choose $\mu = 0$ in (5.14) and (5.16). Then

$$[\widetilde{A}\mathrm{h}, \mathrm{h}] = (pf')(a)\overline{f(a)} + \int_a^b \big(p|f'|^2 + q|f|^2\big)dx + [T_0 g_0, g_0] - |c_{\hat{g}}|^2\tau(0),$$

the nonnegativity of T_0, the relation $f(a)\cos\alpha = (pf')(a)\sin\alpha$ and the assumptions $\alpha \in [0, \frac{\pi}{2}]$, $p > 0$, $q \geq 0$ and $\tau(0) \leq 0$ imply that the operator $\widetilde{A}$ is nonnegative.

$\square$

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