A CLASS OF ABSTRACT BOUNDARY VALUE PROBLEMS WITH LOCALLY DEFINITIZABLE FUNCTIONS IN THE BOUNDARY CONDITION

Jussi Behrndt

Preprint No. 2004/28

A CLASS OF ABSTRACT BOUNDARY VALUE PROBLEMS WITH LOCALLY DEFINITIZABLE FUNCTIONS IN THE BOUNDARY CONDITION

Jussi Behrndt

Abstract. For a class of boundary value problems where the spectral parameter appears in the boundary condition in the form of a locally definitizable function linearizations are constructed and their local spectral properties are investigated.

1. Introduction

In this paper we study a class of boundary value problems with eigenvalue dependent boundary conditions. Let A be a closed symmetric operator or relation with defect one in a separable Krein space $\mathcal{K}$, let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary value space for the adjoint A^+ and let τ be a function locally holomorphic in some open subset of the extended complex plane which is symmetric with respect to the real line such that $\tau(\overline{\lambda}) = \overline{\tau(\lambda)}$ holds. We investigate boundary value problems of the following form: For a given $k \in \mathcal{K}$ find a vector $\hat{f} = \binom{f}{f'} \in A^+$ such that

$$f' - \lambda f = k \quad \text{and} \quad \tau(\lambda)\Gamma_0 \hat{f} + \Gamma_1 \hat{f} = 0 \tag{1.1}$$

holds. Under additional assumptions on τ and A a solution of this problem can be obtained with the help of the compressed resolvent of a selfadjoint extension $\widetilde{A}$ of A which acts in a larger Krein space. Making use of the coupling method from [6] we construct this so-called linearization and we study the spectral properties of $\widetilde{A}$ which are closely connected with the solvability of (1.1).

More precisely, let Ω be some domain in $\overline{\mathbb{C}}$ symmetric with respect to the real line such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersections of Ω with the upper and lower open half-planes are simply connected. We will assume that the selfadjoint extension $A_0 := \ker \Gamma_0$ of A is definitizable over Ω, i.e. for every subdomain Ω' of Ω with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, there exists a selfadjoint projection E which reduces A_0 such that $A_0 \cap (E\mathcal{K})^2$ is definitizable in the

Krein space $E\mathcal{K}$ and Ω' belongs to the resolvent set of $A_0 \cap ((1-E)\mathcal{K})^2$. With the help of approximative eigensequences or the local spectral function of A_0 the spectral points of A_0 in $\Omega \cap \overline{\mathbb{R}}$ can be classified in points of positive and negative type and critical points (cf. [13], [16]).

Further we assume that τ is a function which is definitizable in Ω, that is for every domain Ω' as Ω, $\overline{\Omega'} \subset \Omega$, the function τ can be written as the sum of a definitizable function (cf. [14], [15]) and a function holomorphic on Ω'. Similarly to selfadjoint operators and relations definitizable over Ω the points in $\Omega \cap \overline{\mathbb{R}}$ can be classified in points of positive and negative type and critical points. It was shown in [17] that τ can be represented with a selfadjoint relation T_0 definitizable over Ω' in some Krein space $\mathcal{H}$ such that the sign types of τ and T_0 coincide in $\Omega' \cap \overline{\mathbb{R}}$.

If, in addition, the sign types of A_0 and τ are "compatible" in $\Omega \cap \overline{\mathbb{R}}$ (see Definition 2.8) the selfadjoint relation $A_0 \times T_0$ in the Krein space $\mathcal{K} \times \mathcal{H}$ is definitizable over Ω'. The linearization $\widetilde{A}$ of the boundary value problem (1.1) turns out to be a two dimensional perturbation in resolvent sense of $A_0 \times T_0$. Under some additional minimality assumptions on the selfadjoint relations A_0 and T_0 and with the help of a recent result of T.Ya. Azizov and P. Jonas which states that the inverse of a matrix-valued locally definitizable function is again locally definitizable we prove in Theorem 3.6 that $\widetilde{A}$ is definitizable over Ω'.

The paper is organized as follows. In Section 2 we introduce the necessary notations and we recall the definitions of locally definitizable operators and relations and locally definitizable functions which can be found in e.g. [13], [16] and [17]. Section 3 deals with boundary value problems of the form (1.1). After some preparatory work in Sections 3.1 and 3.2 we formulate and prove the main result in Section 3.3: The linearization $\widetilde{A}$ of the eigenvalue dependent boundary value problem (1.1) is locally definitizable.

I thank P. Jonas for encouragement and critical help in the preparation of the manuscript.

2. Locally Definitizable Selfadjoint Relations and Locally Definitizable Functions

2.1. Notations and Definitions

Let $(\mathcal{K}, [\cdot,\cdot])$ be a separable Krein space. We study linear relations in $\mathcal{K}$, that is, linear subspaces of $\mathcal{K}^2$. The set of all closed linear relations in $\mathcal{K}$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{K})$. Linear operators in $\mathcal{K}$ are viewed as linear relations via their graphs. For the usual definitions of the linear operations with relations, the inverse etc., we refer to [9]. We denote the sum (direct sum) of subspaces in $\mathcal{K}^2$ by $\boldsymbol{+}$ (resp. $\dot{\boldsymbol{+}}$). The linear space of bounded linear operators defined on a Krein space $\mathcal{K}_1$ with values in a Krein space $\mathcal{K}_2$ is denoted by $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. In the case $\mathcal{K} := \mathcal{K}_1 = \mathcal{K}_2$ we simply write $\mathcal{L}(\mathcal{K})$.

If $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{H}})$ is another separable Krein space the elements of $\mathcal{K} \times \mathcal{H}$ will be written in the form $\{k, h\}$, $k \in \mathcal{K}$, $h \in \mathcal{H}$. $\mathcal{K} \times \mathcal{H}$ equipped with the inner product $[\cdot,\cdot]_{\mathcal{K}\times\mathcal{H}}$ defined by

$$[\{k,h\},\{k',h'\}]_{\mathcal{K}\times\mathcal{H}} := [k,k'] + [h,h']_{\mathcal{H}}, \quad k,k' \in \mathcal{K}, \quad h,h' \in \mathcal{H},$$

is also a Krein space. If S is a relation in $\mathcal{K}$ and T is a relation in $\mathcal{H}$ we shall write $S \times T$ for the direct product of S and T which is a relation in $\mathcal{K} \times \mathcal{H}$,

$$S \times T = \left\{ \begin{pmatrix} \{s,t\} \\ \{s',t'\} \end{pmatrix} \;\middle|\; \begin{pmatrix} s \\ s' \end{pmatrix} \in S,\ \begin{pmatrix} t \\ t' \end{pmatrix} \in T \right\}. \tag{2.1}$$

For the pair $\binom{\{s,t\}}{\{s',t'\}}$ on the right hand side of (2.1) we shall also write $\{\hat{s}, \hat{t}\}$, where $\hat{s} = \binom{s}{s'}$, $\hat{t} = \binom{t}{t'}$.

Let S be a closed linear relation in $\mathcal{K}$. The resolvent set $\rho(S)$ of S is the set of all $\lambda \in \mathbb{C}$ such that $(S-\lambda)^{-1} \in \mathcal{L}(\mathcal{K})$, the spectrum $\sigma(S)$ of S is the complement of $\rho(S)$ in $\mathbb{C}$. The extended spectrum $\widetilde{\sigma}(S)$ of S is defined by $\widetilde{\sigma}(S) = \sigma(S)$ if $S \in \mathcal{L}(\mathcal{K})$ and $\widetilde{\sigma}(S) = \sigma(S) \cup \{\infty\}$ otherwise. We say that $\lambda \in \mathbb{C}$ belongs to the *approximate point spectrum of* S, denoted by $\sigma_{ap}(S)$, if there exists a sequence $\binom{x_n}{y_n} \in S$, $n = 1, 2, \ldots$, such that $\|x_n\| = 1$ and $\lim_{n\to\infty} \|y_n - \lambda x_n\| = 0$. The *extended approximate point spectrum* $\widetilde{\sigma}_{ap}(S)$ *of* S is defined by

$$\widetilde{\sigma}_{ap}(S) := \begin{cases} \sigma_{ap}(S) \cup \{\infty\} & \text{if} \quad 0 \in \sigma_{ap}(S^{-1}) \\ \sigma_{ap}(S) & \text{if} \quad 0 \notin \sigma_{ap}(S^{-1}) \end{cases}.$$

We remark, that the boundary points of $\widetilde{\sigma}(S)$ in $\overline{\mathbb{C}}$ belong to $\widetilde{\sigma}_{ap}(S)$.

Next we recall the definitions of the spectra of positive and negative type of a closed linear relation (see [16]). For equivalent descriptions of the spectra of positive and negative type we refer to [16, Theorem 3.18].

Definition 2.1. Let S be a closed linear relation in $\mathcal{K}$. A point $\lambda \in \sigma_{ap}(S)$ is said to be of *positive type* (*negative type*) with respect to S, if for every sequence $\binom{x_n}{y_n} \in S$, $n = 1, 2 \ldots$, with $\|x_n\| = 1$, $\lim_{n\to\infty} \|y_n - \lambda x_n\| = 0$ we have

$$\liminf_{n\to\infty}\, [x_n, x_n] > 0 \quad (\text{resp. } \limsup_{n\to\infty}\, [x_n, x_n] < 0).$$

If $\infty \in \widetilde{\sigma}_{ap}(S)$, ∞ is said to be of *positive type* (*negative type*) with respect to S if 0 is of positive (resp. negative) type with respect to S^{-1}.

An open subset Δ of $\overline{\mathbb{R}}$ is said to be of *positive type* (*negative type*) with respect to S if each point $\lambda \in \Delta \cap \widetilde{\sigma}(S)$ is of positive (resp. negative) type with respect to S. Δ is called of *definite type* with respect to S if Δ is of positive or negative type with respect to S.

Let A be a linear relation in $\mathcal{K}$. The *adjoint relation* $A^+ \in \widetilde{\mathcal{C}}(\mathcal{K})$ is defined as

$$A^+ := \left\{ \begin{pmatrix} h \\ h' \end{pmatrix} \,\middle|\, [g', h] = [g, h'] \quad \text{for all} \quad \begin{pmatrix} g \\ g' \end{pmatrix} \in A \right\}.$$

A is said to be *symmetric* (*selfadjoint*) if $A \subset A^+$ (resp. $A = A^+$).

For a selfadjoint relation A in $\mathcal{K}$ the points of definite type introduced in Definition 2.1 are real. In fact, if e.g. $\lambda \neq \infty$ is of positive type with respect to A, and $\binom{x_n}{y_n} \in A$ is a sequence with $\|x_n\| = 1$ and $\lim_{n\to\infty} \|y_n - \lambda x_n\| = 0$, then

$$|\mathrm{Im}\,\lambda| \liminf_{n\to\infty} [x_n, x_n] = \liminf_{n\to\infty} |\mathrm{Im}\,[y_n - \lambda x_n, x_n]| \leq \lim_{n\to\infty} \|y_n - \lambda x_n\| = 0$$

implies $\lambda \in \mathbb{R}$.

2.2. Locally Definitizable Selfadjoint Relations in Krein Spaces

Let Ω be a domain in $\overline{\mathbb{C}}$ symmetric with respect to the real axis such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersections of Ω with the upper and lower open half-planes are simply connected.

Let A_0 be a selfadjoint relation in the Krein space $\mathcal{K}$ such that $\sigma(A_0) \cap (\Omega \backslash \overline{\mathbb{R}})$ consists of isolated points which are poles of the resolvent of A_0, and no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of the non-real spectrum of A_0. We say that A_0 belongs to the class $S^\infty(\Omega \cap \overline{\mathbb{R}})$, if for every open subset Δ in $\overline{\mathbb{R}}$, $\overline{\Delta} \subset \Omega \cap \overline{\mathbb{R}}$, there exists $m \geq 1$, $M > 0$ and an open neighbourhood $\mathcal{U}$ of $\overline{\Delta}$ in $\overline{\mathbb{C}}$ such that

$$\|(A_0 - \lambda)^{-1}\| \leq M(1 + |\lambda|)^{2m-2}\, |\mathrm{Im}\,\lambda|^{-m}$$

holds for all $\lambda \in \mathcal{U} \backslash \overline{\mathbb{R}}$.

Definition 2.2. Let A_0 be a selfadjoint relation in $\mathcal{K}$ and let Ω be a domain as above. A_0 is said to be definitizable over Ω, if $A_0 \in S^\infty(\Omega \cap \overline{\mathbb{R}})$ and every point $\mu \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighbourhood I_μ in $\overline{\mathbb{R}}$ such that both components of $I_\mu \backslash \{\mu\}$ are of definite type with respect to A_0.

The next theorem is a variant of [16, Theorem 4.8]. The simple modification of the proof is left to reader.

Theorem 2.3. *Let A_0 be a selfadjoint relation in $\mathcal{K}$ and let Ω be a domain as above. A_0 is definitizable over Ω if and only if for every domain Ω' with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, there exists a selfadjoint projection E in $\mathcal{K}$ such that A_0 can be decomposed in*

$$A_0 = \big(A_0 \cap (E\mathcal{K})^2\big) \dotplus \big(A_0 \cap ((1 - E)\mathcal{K})^2\big)$$

and the following holds.

(i) $A_0 \cap (E\mathcal{K})^2$ *is a definitizable relation in the Krein space* $E\mathcal{K}$.

(ii) $\widetilde{\sigma}\big(A_0 \cap ((1-E)\mathcal{K})^2\big) \cap \Omega' = \emptyset$.

Let A_0 be a selfadjoint relation in $\mathcal{K}$ definitizable over Ω, let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, and let E be a selfadjoint projection with the properties as in Theorem 2.3. If E' is the spectral function of the definitizable selfadjoint relation $A_0 \cap (E\mathcal{K})^2$ in the Krein space $E\mathcal{K}$ (cf. [15, page 71], [10] and [18]), then the mapping

$$\delta \mapsto E'(\delta)E =: E_{A_0}(\delta) \tag{2.2}$$

defined for all finite unions δ of connected subsets of $\Omega' \cap \overline{\mathbb{R}}$ the endpoints of which belong to $\Omega' \cap \overline{\mathbb{R}}$ and are of definite type with respect to $A_0 \cap (E\mathcal{K})^2$, is the spectral function of A_0 on $\Omega' \cap \overline{\mathbb{R}}$ (see [16, Section 3.4 and Remark 4.9]).

2.3. Locally Definitizable Functions

Let Ω be a domain as in the beginning of Section 2.2 and let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued piecewise meromorphic function in $\Omega\backslash\overline{\mathbb{R}}$ which is symmetric with respect to the real line, that is $\tau(\overline{\lambda}) = \tau(\lambda)^*$ for all points λ of holomorphy of τ. If, in addition, no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of nonreal poles of τ we write $\tau \in M^{n\times n}(\Omega)$. The set of the points of holomorphy of τ in $\Omega\backslash\overline{\mathbb{R}}$ and all points $\mu \in \Omega \cap \mathbb{R}$ such that τ can be analytically continued to μ and the continuations from $\Omega \cap \mathbb{C}^+$ and $\Omega \cap \mathbb{C}^-$ coincide, is denoted by $\mathfrak{h}(\tau)$.

In the next definition we introduce the sign type of open subsets in $\Omega \cap \overline{\mathbb{R}}$ with respect to functions from the class $M^{n\times n}(\Omega)$ (see [17]).

Definition 2.4. Let $\tau \in M^{n\times n}(\Omega)$. An open subset $\Delta \subset \Omega \cap \overline{\mathbb{R}}$ is said to be of *positive type* with respect to τ if for every $x \in \mathbb{C}^n$ and every sequence (μ_k) of points in $\Omega \cap \mathbb{C}^+ \cap \mathfrak{h}(\tau)$ which converges in $\overline{\mathbb{C}}$ to a point of Δ we have

$$\liminf_{k\to\infty} \operatorname{Im}\,(\tau(\mu_k)x, x) \geq 0.$$

An open subset $\Delta \subset \Omega \cap \overline{\mathbb{R}}$ is said to be of *negative type* with respect to τ if Δ is of positive type with respect to $-\tau$. Δ is said to be of *definite type* with respect to τ if Δ is of positive or of negative type with respect to τ.

Definition 2.5. A function $\tau \in M^{n\times n}(\Omega)$ is called *definitizable in* Ω if the following holds.

(i) Every point $\mu \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighbourhood I_μ in $\overline{\mathbb{R}}$ such that both components of $I_\mu\backslash\{\mu\}$ are of definite type with respect to τ.

(ii) For every open subset Δ in $\overline{\mathbb{R}}$, $\overline{\Delta} \subset \Omega \cap \overline{\mathbb{R}}$, there exists $m \geq 1$, $M > 0$ and an open neighbourhood $\mathcal{U}$ of $\overline{\Delta}$ in $\overline{\mathbb{C}}$ such that

$$\|\tau(\lambda)\| \leq M(1 + |\lambda|)^{2m} |\mathrm{Im}\,\lambda|^{-m}$$

holds for all $\lambda \in \mathcal{U} \backslash \overline{\mathbb{R}}$.

In [17] it is shown that a function $\tau \in M^{n\times n}(\Omega)$ is definitizable in Ω if and only if for every finite union Δ of open connected subsets of $\overline{\mathbb{R}}$ such that $\overline{\Delta} \subset \Omega \cap \overline{\mathbb{R}}$, τ can be written as the sum of an $\mathcal{L}(\mathbb{C}^n)$-valued definitizable function and an $\mathcal{L}(\mathbb{C}^n)$-valued function which is locally holomorphic on $\overline{\Delta}$.

The following theorem will be used in Sections 3.2 and 3.3. It states that a locally definitizable function can be represented with a locally definitizable selfadjoint relation. A proof can be found in [17].

Theorem 2.6. *Let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued function definitizable in Ω and let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$. Then there exists a Krein space $\mathcal{H}$, a selfadjoint relation T_0 in $\mathcal{H}$ definitizable over Ω' and a mapping $\gamma' \in \mathcal{L}(\mathbb{C}^n, \mathcal{H})$ with the following properties.*

(a) $\rho(T_0) \cap \Omega' = \mathfrak{h}(\tau) \cap \Omega'$.

(b) *For a fixed $\lambda_0 \in \rho(T_0) \cap \Omega'$ and all $\lambda \in \rho(T_0) \cap \Omega'$*

$$\tau(\lambda) = \mathrm{Re}\,\tau(\lambda_0) + \gamma'^{+}\big((\lambda - \mathrm{Re}\,\lambda_0) + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(T_0 - \lambda)^{-1}\big)\gamma'$$

holds.

(c) *For any finite union Δ of open connected subsets of $\overline{\mathbb{R}}$, $\overline{\Delta} \subset \Omega' \cap \overline{\mathbb{R}}$, such that the boundary points of Δ are of definite type with respect to τ the spectral projection $E_{T_0}(\Delta)$ is defined. If Ω'' is a domain with the same properties as Ω and Ω', $\overline{\Omega''} \subset \Omega'$, and if we set $E := E_{T_0}(\Delta) + E_{T_0}(\overline{\Omega''} \backslash \overline{\mathbb{R}})$, then the minimality condition*

$$E\mathcal{H} = \mathrm{clsp}\,\big\{\big(1 + (\lambda - \lambda_0)(T_0 - \lambda)^{-1}\big)E\gamma' x \,|\, \lambda \in \rho(T_0) \cap \Omega',\, x \in \mathbb{C}^n\big\}$$

is fulfilled. Δ is of positive (negative) type with respect to τ if and only if Δ is of positive (resp. negative) type with respect to T_0.

If τ and T_0 are as in Theorem 2.6 we shall say that T_0 is an Ω'-*minimal representing relation* for τ.

Remark 2.7. Let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued function definitizable in Ω and let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$. If, in addition, τ is the restriction of a definitizable function (see [14], [15]) or if, in addition, the boundary of Ω' is contained in $\mathfrak{h}(\tau)$, then the selfadjoint relation T_0 in Theorem 2.6 can be chosen such that the minimality condition

$$\mathcal{H} = \mathrm{clsp}\,\big\{\big(1 + (\lambda - \lambda_0)(T_0 - \lambda)^{-1}\big)\gamma' x \,|\, \lambda \in \rho(T_0) \cap \Omega',\, x \in \mathbb{C}^n\big\}$$

holds.

The next definition connects sign types of locally definitizable functions and sign types of spectral points of locally definitizable relations.

Definition 2.8. Let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued function definitizable in Ω and let A_0 be a selfadjoint relation in the Krein space $\mathcal{K}$ which is definitizable over Ω. We say that *the sign types of* τ *and* A_0 *are d-compatible in* Ω if for every point $\mu \in \Omega \cap \overline{\mathbb{R}}$ there exists an open connected neighbourhood $I_\mu \subset \Omega \cap \overline{\mathbb{R}}$ of μ such that each component of $I_\mu \backslash \{\mu\}$ is of the same sign type with respect to τ and A_0.

If τ is a function which is definitizable in Ω, Ω' is a domain as Ω, $\overline{\Omega'} \subset \Omega$, and T_0 is an Ω'-minimal representing relation for τ (see Theorem 2.6), then the sign types of τ and T_0 are d-compatible in Ω'.

3. Boundary Value Problems with Locally Definitizable Functions in the Boundary Condition

3.1. Boundary Value Spaces and Weyl Functions Associated with Symmetric Relations in Krein Spaces

Let $(\mathcal{K}, [\cdot,\cdot])$ be a separable Krein space, let J be a corresponding fundamental symmetry and let $A \in \widetilde{\mathcal{C}}(\mathcal{K})$ be a closed symmetric relation in $\mathcal{K}$. We say that A has *defect* $m \in \mathbb{N} \cup \{\infty\}$, if both deficiency indices

$$n_\pm(JA) = \dim \ker((JA)^* - \overline{\lambda}), \qquad \lambda \in \mathbb{C}^\pm,$$

of the symmetric relation JA in the Hilbert space $(\mathcal{K}, [J\cdot,\cdot])$ are equal to m. We remark, that this is equivalent to the fact that there exists a selfadjoint extension of A in $\mathcal{K}$ and that each selfadjoint extension $\hat{A}$ of A in $\mathcal{K}$ satisfies $\dim\big(\hat{A}/A\big) = m$.

We shall use the so-called boundary value spaces for the description of the selfadjoint extensions of closed symmetric relations in Krein spaces. The following definition is taken from [5].

Definition 3.1. Let A be a closed symmetric relation in the Krein space $(\mathcal{K}, [\cdot,\cdot])$. We say that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a *boundary value space for* A^+ if $(\mathcal{G}, (\cdot,\cdot))$ is a Hilbert space and there exist mappings $\Gamma_0, \Gamma_1 : A^+ \to \mathcal{G}$ such that $\Gamma = \binom{\Gamma_0}{\Gamma_1} : A^+ \to \mathcal{G} \times \mathcal{G}$ is surjective, and the relation

$$[f', g] - [f, g'] = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g}) - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g})$$

holds for all $\hat{f} = \binom{f}{f'}, \hat{g} = \binom{g}{g'} \in A^+$.

In the following we recall some basic facts on boundary value spaces which can be found in e.g. [4] and [5]. For the Hilbert space case we refer to [11], [7] and [8]. Let A be a closed symmetric relation in $\mathcal{K}$, let

$$\mathcal{N}_{\lambda,A^+} := \ker(A^+ - \lambda) = \operatorname{ran}\,(A - \overline{\lambda})^{[\perp]}$$

be the defect subspace of A and let

$$\hat{\mathcal{N}}_{\lambda,A^+} = \left\{ \left(\begin{smallmatrix} f_\lambda \\ \lambda f_\lambda \end{smallmatrix} \right) \middle| f_\lambda \in \mathcal{N}_{\lambda,A^+} \right\}.$$

When no confusion can arise we write $\mathcal{N}_\lambda$ and $\hat{\mathcal{N}}_\lambda$ instead of $\mathcal{N}_{\lambda,A^+}$ and $\hat{\mathcal{N}}_{\lambda,A^+}$. If there exists a selfadjoint extension A' of A such that $\rho(A') \neq \emptyset$ then we have

$$A^+ = A' \dotplus \hat{\mathcal{N}}_\lambda \qquad \text{for all} \quad \lambda \in \rho(A').$$

In this case there exists a boundary value space $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^+ such that $\ker \Gamma_0 = A'$ (cf. [5]).

Let in the following A, $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and Γ be as in Definition 3.1. It follows that the mappings Γ_0 and Γ_1 are continuous. The selfadjoint extensions

$$A_0 := \ker \Gamma_0 \quad \text{and} \quad A_1 := \ker \Gamma_1$$

of A are transversal, i.e. $A_0 \cap A_1 = A$ and $A_0 \,\hat{+}\, A_1 = A^+$. The mapping Γ induces, via

$$A_\Theta := \Gamma^{-1}\Theta = \left\{ \hat{f} \in A^+ \,|\, \Gamma \hat{f} \in \Theta \right\}, \quad \Theta \in \widetilde{\mathcal{C}}(\mathcal{G}), \tag{3.1}$$

a bijective correspondence $\Theta \mapsto A_\Theta$ between the set of all closed linear relations $\widetilde{\mathcal{C}}(\mathcal{G})$ in $\mathcal{G}$ and the set of closed extensions $A_\Theta \subset A^+$ of A. In particular (3.1) gives a one-to-one correspondence between the symmetric (selfadjoint) extensions of A and the symmetric (resp. selfadjoint) relations in $\mathcal{G}$. If Θ is a closed operator in $\mathcal{G}$, then the corresponding extension A_Θ of A is determined by

$$A_\Theta = \ker\big(\Gamma_1 - \Theta\Gamma_0\big). \tag{3.2}$$

Assume that $\rho(A_0) \neq \emptyset$ and denote by π_1 the orthogonal projection onto the first component of $\mathcal{K}^2$. For every $\lambda \in \rho(A_0)$ we define the operators

$$\gamma(\lambda) = \pi_1(\Gamma_0|\hat{\mathcal{N}}_\lambda)^{-1} \in \mathcal{L}(\mathcal{G}, \mathcal{K}) \quad \text{and} \quad M(\lambda) = \Gamma_1(\Gamma_0|\hat{\mathcal{N}}_\lambda)^{-1} \in \mathcal{L}(\mathcal{G}).$$

The functions $\lambda \mapsto \gamma(\lambda)$ and $\lambda \mapsto M(\lambda)$ are called the *γ-field* and *Weyl function* corresponding to A and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$. γ and M are holomorphic on $\rho(A_0)$ and the relations

$$\gamma(\zeta) = (1 + (\zeta - \lambda)(A_0 - \zeta)^{-1})\gamma(\lambda) \tag{3.3}$$

and

$$M(\lambda) - M(\zeta)^* = (\lambda - \overline{\zeta})\gamma(\zeta)^+\gamma(\lambda) \tag{3.4}$$

hold for all $\lambda, \zeta \in \rho(A_0)$ (cf. [5]).

If $\Theta \in \widetilde{\mathcal{C}}(\mathcal{G})$ and A_Θ is the corresponding extension of A (see (3.1)), then a point $\lambda \in \rho(A_0)$ belongs to $\rho(A_\Theta)$ if and only if 0 belongs to $\rho(\Theta - M(\lambda))$. For $\lambda \in \rho(A_\Theta) \cap \rho(A_0)$ the well-known resolvent formula

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+ \tag{3.5}$$

holds (for a proof see e.g. [5]).

3.2. Locally Definitizable Functions as Weyl Functions of Symmetric Relations

Let, as in Section 2.2, Ω be a domain in $\overline{\mathbb{C}}$ symmetric with respect to the real axis such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersections of Ω with the upper and lower open half-planes are simply connected.

In the next proposition we consider boundary value spaces and Weyl functions associated with symmetric relations in Krein spaces which have the additional property that there exists a locally definitizable selfadjoint extension. We restrict ourselves to the case of defect one.

Proposition 3.2. *Let A be a closed symmetric relation with defect one in the Krein space $\mathcal{K}$ and assume that there exists a selfadjoint extension A_0 of A which is definitizable over Ω. Let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary value space for A^+ such that $A_0 = \ker \Gamma_0$. Then the corresponding Weyl function M is definitizable in Ω. If Δ is an open subset of $\Omega \cap \overline{\mathbb{R}}$ which is of positive (negative) type with respect to A_0, then Δ is of positive (resp. negative) type with respect to M.*

Proof. As A_0 is a selfadjoint relation definitizable over Ω it follows that the Weyl function M corresponding to A and $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is piecewise meromorphic in $\Omega \backslash \overline{\mathbb{R}}$ and no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of nonreal poles of M. Let $\lambda_0 \in \rho(A_0)$. Making use of (3.3) and (3.4) we obtain that M is symmetric with respect to the real axis, $\operatorname{Im} M(\lambda_0) = \operatorname{Im} \lambda_0 \gamma(\lambda_0)^+\gamma(\lambda_0)$ and

(3.6)

$$\begin{aligned}
M(\lambda) &= \overline{M(\lambda_0)} + (\lambda - \overline{\lambda}_0)\gamma(\lambda_0)^+\gamma(\lambda) \\
&= \operatorname{Re} M(\lambda_0) - i \operatorname{Im} \lambda_0 \gamma(\lambda_0)^+\gamma(\lambda_0) \\
&\qquad + \gamma(\lambda_0)^+\big(\lambda - \overline{\lambda}_0 + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A_0 - \lambda)^{-1}\big)\gamma(\lambda_0) \\
&= \operatorname{Re} M(\lambda_0) + \gamma(\lambda_0)^+\big((\lambda - \operatorname{Re} \lambda_0) + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A_0 - \lambda)^{-1}\big)\gamma(\lambda_0)
\end{aligned}$$

holds for all $\lambda \in \rho(A_0)$. Since A_0 belongs to the class $S^\infty(\Omega \cap \overline{\mathbb{R}})$ it follows that M fulfils the second condition in Definition 2.5. Let $\mu \in \Omega \cap \overline{\mathbb{R}}$ and let $I_\mu \subset \Omega \cap \overline{\mathbb{R}}$ be an open connected neighbourhood of μ in $\overline{\mathbb{R}}$ such that both components of $I_\mu \backslash \{\mu\}$ are of definite type with respect to A_0. By [16, Theorem 3.18] a component of $I_\mu \backslash \{\mu\}$ is of positive (negative) type with respect to A_0

if and only if it is of positive (resp. negative) type with respect to the function

$$\lambda \mapsto (\lambda - \operatorname{Re} \lambda_0) + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A_0 - \lambda)^{-1}.$$

Therefore both components of $I_\mu \backslash \{\mu\}$ are of the same sign type with respect to A_0 and M and it follows that M is definitizable in Ω.

□

The next theorem is a variant of [3, Theorem 3.3]. For the convenience of the reader we sketch the proof.

Theorem 3.3. *Let τ be a complex valued function definitizable in Ω and assume that τ is not identically equal to a constant. Let Ω' be a domain with the same properties as Ω and $\overline{\Omega'} \subset \Omega$. Then there exists a Krein space $\mathcal{H}$, a closed symmetric relation T with defect one in $\mathcal{H}$ and a boundary value space $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ for T^+ such that τ coincides with the corresponding Weyl function on Ω' and $T_0 := \ker \Gamma_0'$ is an Ω'-minimal representing relation for τ.*

Sketch of the proof of Theorem 3.3. Let τ be represented with an Ω'-minimal selfadjoint relation T_0 in a Krein space $\mathcal{H}$ as in Theorem 2.6. Let $\gamma' \in \mathcal{L}(\mathbb{C}, \mathcal{H})$ be as in Theorem 2.6 and fix some $\lambda_0 \in \Omega' \cap \mathfrak{h}(\tau)$. For all $\lambda \in \Omega' \cap \mathfrak{h}(\tau)$ we define

$$\gamma'(\lambda) := \big(1 + (\lambda - \lambda_0)(T_0 - \lambda)^{-1}\big)\gamma' \in \mathcal{L}(\mathbb{C}, \mathcal{H}).$$

The linear functional $\gamma'(\lambda)c \mapsto c$ defined on ran $\gamma'(\lambda)$ is denoted by $\gamma'(\lambda)^{(-1)}$. The closed symmetric relation

$$T := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in T_0 \,\middle|\, [g - \overline{\mu} f, \gamma'(\mu)\, 1] = 0 \right\}$$

has defect one and does not depend on the choice of $\mu \in \Omega' \cap \mathfrak{h}(\tau)$. For every $\lambda \in \Omega' \cap \mathfrak{h}(\tau)$ we write the elements $\hat{f} \in T^+$ in the form $\hat{f} = \left(\begin{smallmatrix} f_0 \\ f_0' \end{smallmatrix}\right) + \left(\begin{smallmatrix} f_\lambda \\ \lambda f_\lambda \end{smallmatrix}\right)$, where $\left(\begin{smallmatrix} f_0 \\ f_0' \end{smallmatrix}\right) \in T_0$ and $f_\lambda \in \operatorname{ran} \gamma'(\lambda) = \mathcal{N}_{\lambda, T^+}$. As in the proof of [3, Theorem 3.3] (see also [7, Theorem 1]) one verifies that $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$, where

$$\begin{aligned} \Gamma_0' \hat{f} &:= \gamma'(\lambda)^{(-1)} f_\lambda, \\ \Gamma_1' \hat{f} &:= \gamma'(\lambda)^+ (f_0' - \overline{\lambda} f_0) + \tau(\lambda)\gamma'(\lambda)^{(-1)} f_\lambda, \end{aligned}$$

is a boundary value space for T^+ and the corresponding Weyl function coincides with τ on Ω'.

□

Remark 3.4. Let the function τ be definitizable in Ω and assume that τ is not identically equal to a constant. Let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, and assume that τ is the restriction of a definitizable function or that the boundary of Ω' is contained in $\mathfrak{h}(\tau)$. If we choose T_0 as in Remark 2.7 and $T \subset T_0 \subset T^+$ as in Theorem 3.3, then the condition

$$\mathcal{H} = \text{clsp}\,\{\text{ran}\,\gamma'(\lambda)\,|\,\lambda \in \rho(T_0) \cap \Omega'\} = \text{clsp}\,\{\mathcal{N}_{\lambda,T^+}\,|\,\lambda \in \rho(T_0) \cap \Omega'\}$$

is fulfilled.

In the following proposition we use Definition 2.8. The statements will be useful in the proof of our main result in Section 3.3.

Proposition 3.5. *Let A be a closed symmetric relation with defect one in the Krein space $\mathcal{K}$, let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary value space for A^+ and denote by M the corresponding Weyl function. Assume that the selfadjoint relation $A_0 = \ker \Gamma_0$ is definitizable over Ω and let τ be a function which is definitizable in Ω such that the sign types of τ and A_0 are d-compatible in Ω.*
Let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, and let T_0 be an Ω'-minimal representing relation for τ in some Krein space $\mathcal{H}$ (see Theorem 2.6). Then the following holds.

(i) *The selfadjoint relation $A_0 \times T_0 \in \widetilde{\mathcal{C}}(\mathcal{K} \times \mathcal{H})$ is definitizable over Ω' and the sign types of $A_0 \times T_0$ and the functions τ and M are d-compatible in Ω'.*

(ii) *$M + \tau$ is definitizable in Ω.*

(iii) *If $\tau(\eta) \neq 0$ and $(M+\tau)(\eta') \neq 0$ for some $\eta, \eta' \in \Omega$, then the functions*

$$\lambda \mapsto \begin{pmatrix} M(\lambda) & 0 \\ 0 & -\tau(\lambda)^{-1} \end{pmatrix} \quad \text{and} \quad \lambda \mapsto -\begin{pmatrix} M(\lambda) & -1 \\ -1 & -\tau(\lambda)^{-1} \end{pmatrix}^{-1} \tag{3.7}$$

are definitizable in Ω and their sign types are d-compatible with the sign types of the selfadjoint relation $A_0 \times T_0$ in Ω'.

Proof. (i) Since A_0 and T_0 belong to $S^\infty(\Omega \cap \overline{\mathbb{R}})$ and $S^\infty(\Omega' \cap \overline{\mathbb{R}})$, respectively, we conclude that $A_0 \times T_0$ belongs to $S^\infty(\Omega' \cap \overline{\mathbb{R}})$. Let $\mu \in \Omega' \cap \overline{\mathbb{R}}$ and let $I_\mu \subset \Omega' \cap \overline{\mathbb{R}}$ be an open connected neighbourhood of μ in $\overline{\mathbb{R}}$ such that each component of $I_\mu \backslash \{\mu\}$ is of the same sign type with respect to A_0 and τ. As T_0 is an Ω'-minimal representing relation for τ both components of $I_\mu \backslash \{\mu\}$ are of definite type with respect to $A_0 \times T_0$ and it follows that $A_0 \times T_0$ is definitizable over Ω'.

The assumption that the sign types of τ and A_0 are d-compatibel in Ω implies that the sign types of τ and $A_0 \times T_0$ as well as the sign types of M and $A_0 \times T_0$ are d-compatible in Ω.

(ii) For $\mu \in \Omega \cap \overline{\mathbb{R}}$ we choose an open connected neighbourhood $I_\mu \subset \Omega \cap \overline{\mathbb{R}}$ of μ such that both components of $I_\mu \backslash \{\mu\}$ are of the same sign type with respect to A_0 and τ. By Proposition 3.2 the sign types of M are the same as of A_0 and therefore both components of $I_\mu \backslash \{\mu\}$ are of the same sign type with respect to $M+\tau$. The growth properties of M and τ imply that $M+\tau$ fulfils the second condition in Definition 2.5 and therefore $M+\tau$ is definitizable in Ω.

(iii) By [1, Theorem 2.3] the function $-\tau^{-1}$ is definitizable in Ω and it follows from the proof of [1, Theorem 2.3] that each point $\mu \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighbourhood $I_\mu \subset \Omega \cap \overline{\mathbb{R}}$ such that both components of $I_\mu \backslash \{\mu\}$ are of the same sign type with respect to $-\tau^{-1}$ and τ. Therefore the sign types of $-\tau^{-1}$ and A_0 are d-compatible in Ω. Now it is easy to see that the first function in (3.7) is definitizable in Ω and its sign types are d-compatible with the sign types of $A_0 \times T_0$ in Ω'.

As the function

$$\lambda \mapsto \begin{pmatrix} M(\lambda) & -1 \\ -1 & -\tau(\lambda)^{-1} \end{pmatrix}$$

is also definitizable in Ω another application of [1, Theorem 2.3] shows that the second function in (3.7) is definitizable in Ω and its sign types are d-compatible with the sign types of $A_0 \times T_0$ in Ω'.

□

3.3. The Main Result

In this section we investigate the spectral properties of so-called linearizations of a class of abstract eigenvalue dependent boundary value problems with locally definitizable functions in the boundary condition. Similar problems with a local variant of generalized Nevanlinna functions in the boundary condition have been considered in [3]. The main feature in Theorem 3.6 below is that the linearization turns out to be locally definitizable.

Theorem 3.6. *Let Ω be a domain as in the beginning of Section* 3.2 *and let A be a closed symmetric relation with defect one in the Krein space $\mathcal{K}$ such that $\mathcal{K} = \operatorname{clsp}\{\mathcal{N}_{\lambda,A^+} \mid \lambda \in \Omega\}$ holds. Assume that there exists a selfadjoint extension A_0 of A which is definitizable over Ω. Let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary value space for A^+, $A_0 = \ker \Gamma_0$, and denote by γ and M the corresponding γ-field and Weyl function, respectively.*
Let τ be a nonconstant function which is definitizable in Ω, let Ω' be a domain as Ω, $\overline{\Omega'} \subset \Omega$, choose $\mathcal{H}$, $T \subset T_0 \subset T^+$ and $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ as in Theorem 3.3 and assume that the condition $\mathcal{H} = \operatorname{clsp}\{\mathcal{N}_{\lambda,T^+} \mid \lambda \in \Omega'\}$ is fulfilled.
Let the sign types of τ and A_0 be d-compatible in Ω, assume that the function $M+\tau$ is not identically equal to zero and define

$$\mathfrak{h}_0 := \mathfrak{h}(M) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}\big(\tau^{-1}\big) \cap \mathfrak{h}\big((M+\tau)^{-1}\big).$$

Then the relation

$$\widetilde{A} = \big\{\{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+ \,|\, \Gamma_1 \hat{f}_1 - \Gamma_1' \hat{f}_2 = \Gamma_0 \hat{f}_1 + \Gamma_0' \hat{f}_2 = 0\big\} \tag{3.8}$$

is a selfadjoint extension of A in $\mathcal{K}\times\mathcal{H}$ which is definitizable over Ω' and the sign types of $\widetilde{A}$ are d-compatible with the sign types of τ and M in Ω'. The set $\Omega'\backslash(\overline{\mathbb{R}}\cup\mathfrak{h}_0)$ is finite. For every $k\in\mathcal{K}$ and every $\lambda\in\mathfrak{h}_0$ a unique solution of the eigenvalue dependent boundary value problem

$$f_1'-\lambda f_1=k,\quad \tau(\lambda)\Gamma_0\hat{f}_1+\Gamma_1\hat{f}_1=0,\quad \hat{f}_1=\begin{pmatrix}f_1\\ f_1'\end{pmatrix}\in A^+, \tag{3.9}$$

is given by

$$\begin{aligned} f_1&=P_{\mathcal{K}}(\widetilde{A}-\lambda)^{-1}\{k,0\}=(A_0-\lambda)^{-1}k-\gamma(\lambda)\big(M(\lambda)+\tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+k,\\ f_1'&=\lambda f_1+k. \end{aligned}\tag{3.10}$$

Proof. **1.** In this step we show that (3.10) is a solution of the boundary value problem (3.9). We follow the lines of [3, Proof of Theorem 4.1] (see also [6, §5.2]).

As the functions M, τ and $M+\tau$ are definitizable in Ω (see Proposition 3.5) [1, Theorem 2.3] implies that $-\tau^{-1}$ and $-(M+\tau)^{-1}$ are also definitizable in Ω. Let Ω' and $\mathfrak{h}_0$ be as in the assumptions of the theorem. As the nonreal poles of the functions M, τ, τ^{-1} and $(M+\tau)^{-1}$ in Ω do not accumulate to $\Omega\cap\overline{\mathbb{R}}$ we conclude that the set $\Omega'\backslash(\overline{\mathbb{R}}\cup\mathfrak{h}_0)$ is finite.

Let $\mathcal{H}$, $T\subset T^+$ and $\{\mathbb{C},\Gamma_0',\Gamma_1'\}$ be as in Theorem 3.3. Then τ is the corresponding Weyl function and the selfadjoint relation $T_0=\ker\Gamma_0'$ is definitizable over Ω'. By γ' we denote the γ-field corresponding to $\{\mathbb{C},\Gamma_0',\Gamma_1'\}$ and we set $T_1:=\ker\Gamma_1'$. As $\{\mathbb{C},\Gamma_1',-\Gamma_0'\}$ is a boundary value space for T^+ with corresponding γ-field and Weyl function

$$\lambda\mapsto\gamma'(\lambda)\tau(\lambda)^{-1}\qquad\text{and}\qquad\lambda\mapsto-\tau(\lambda)^{-1}, \tag{3.11}$$

respectively, it follows without difficulty that $\{\mathbb{C}^2,\widetilde{\Gamma}_0,\widetilde{\Gamma}_1\}$, where $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_1$ are mappings from $A^+\times T^+$ into $\mathbb{C}^2$ defined by

$$\widetilde{\Gamma}_0\{\hat{f}_1,\hat{f}_2\}:=\begin{pmatrix}\Gamma_0\hat{f}_1\\ \Gamma_1'\hat{f}_2\end{pmatrix}\quad\text{and}\quad\widetilde{\Gamma}_1\{\hat{f}_1,\hat{f}_2\}:=\begin{pmatrix}\Gamma_1\hat{f}_1\\ -\Gamma_0'\hat{f}_2\end{pmatrix},$$

$\{\hat{f}_1,\hat{f}_2\}\in A^+\times T^+$, is a boundary value space for $A^+\times T^+$ with corresponding γ-field

$$\lambda\mapsto\widetilde{\gamma}(\lambda)=\begin{pmatrix}\gamma(\lambda)&0\\0&\gamma'(\lambda)\tau(\lambda)^{-1}\end{pmatrix},\quad\lambda\in\mathfrak{h}(M)\cap\mathfrak{h}(\tau)\cap\mathfrak{h}(\tau^{-1}), \tag{3.12}$$

and corresponding Weyl function

$$\lambda\mapsto\widetilde{M}(\lambda)=\begin{pmatrix}M(\lambda)&0\\0&-\tau(\lambda)^{-1}\end{pmatrix},\quad\lambda\in\mathfrak{h}(M)\cap\mathfrak{h}(\tau)\cap\mathfrak{h}(\tau^{-1}).$$

The selfadjoint relation $\widetilde{A}$ in $\mathcal{K}\times\mathcal{H}$ corresponding to $\Theta=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\in\mathcal{L}(\mathbb{C}^2)$ via (3.1) is given by

$$\begin{aligned}\widetilde{A}&=\ker(\widetilde{\Gamma}_1-\Theta\widetilde{\Gamma}_0)\\&=\Bigl\{\{\hat{f}_1,\hat{f}_2\}\in A^+\times T^+\,|\,\Gamma_1\hat{f}_1-\Gamma_1'\hat{f}_2=\Gamma_0\hat{f}_1+\Gamma_0'\hat{f}_2=0\Bigr\}.\end{aligned}\tag{3.13}$$

For $\lambda\in\mathfrak{h}_0$ the resolvent of $\widetilde{A}$ can be written as

$$(\widetilde{A}-\lambda)^{-1}=\begin{pmatrix}(A_0-\lambda)^{-1}&0\\0&(T_1-\lambda)^{-1}\end{pmatrix}+\widetilde{\gamma}(\lambda)\bigl(\Theta-\widetilde{M}(\lambda)\bigr)^{-1}\widetilde{\gamma}(\overline{\lambda})^+,\tag{3.14}$$

(see (3.5)). Calculating $(\Theta-\widetilde{M}(\lambda))^{-1}$ one verifies that the compressed resolvent of $\widetilde{A}$ onto $\mathcal{K}$ is given by

$$P_{\mathcal{K}}(\widetilde{A}-\lambda)^{-1}|\mathcal{K}=(A_0-\lambda)^{-1}-\gamma(\lambda)\bigl(M(\lambda)+\tau(\lambda)\bigr)^{-1}\gamma(\overline{\lambda})^+,\quad\lambda\in\mathfrak{h}_0.$$

For $k\in\mathcal{K}$ we set $f_1:=P_{\mathcal{K}}(\widetilde{A}-\lambda)^{-1}\{k,0\}$ and $f_2:=P_{\mathcal{H}}(\widetilde{A}-\lambda)^{-1}\{k,0\}$. Then

$$\begin{pmatrix}\{f_1,f_2\}\\\{\lambda f_1+k,\lambda f_2\}\end{pmatrix}\in\widetilde{A}\subset A^+\times T^+$$

and $\hat{f}_1:=\left(\begin{smallmatrix}f_1\\\lambda f_1+k\end{smallmatrix}\right)\in A^+$, $\hat{f}_2:=\left(\begin{smallmatrix}f_2\\\lambda f_2\end{smallmatrix}\right)\in\hat{\mathcal{N}}_{\lambda,T^+}$. From (3.13) and since τ is the Weyl function corresponding to $\{\mathbb{C},\Gamma_0',\Gamma_1'\}$ we get

$$\Gamma_1\hat{f}_1=\Gamma_1'\hat{f}_2=\tau(\lambda)\Gamma_0'\hat{f}_2=-\tau(\lambda)\Gamma_0\hat{f}_1,\quad\lambda\in\mathfrak{h}_0,$$

and it follows that $\hat{f}_1\in A^+$ is a solution of (3.9).

Let us verify that this solution $\hat{f}_1\in A^+$ is unique. Assume that the vector $\hat{g}_1=\left(\begin{smallmatrix}g_1\\\lambda g_1+k\end{smallmatrix}\right)\in A^+$ is also a solution of (3.9), $\lambda\in\mathfrak{h}_0$. Then $\hat{f}_1-\hat{g}_1$ belongs to $\hat{\mathcal{N}}_{\lambda,A^+}$ and

$$0=\tau(\lambda)\Gamma_0(\hat{f}_1-\hat{g}_1)+\Gamma_1(\hat{f}_1-\hat{g}_1)=\bigl(\tau(\lambda)+M(\lambda)\bigr)\Gamma_0(\hat{f}_1-\hat{g}_1)$$

implies $\hat{f}_1-\hat{g}_1\in A_0\cap\hat{\mathcal{N}}_{\lambda,A^+}$ as $\tau(\lambda)+M(\lambda)\neq 0$. Therefore $\hat{f}_1=\hat{g}_1$.

2. It remains to prove that $\widetilde{A}$ is definitizable over Ω' and that the sign types of $\widetilde{A}$ are d-compatible with the sign types of the functions τ and M in Ω'. In this step we show that for every point $\mu\in\Omega'\cap\overline{\mathbb{R}}$ there exists an open connected neighbourhood I_μ of μ in $\Omega'\cap\overline{\mathbb{R}}$ such that both components of $I_\mu\backslash\{\mu\}$ are of definite type with respect to $\widetilde{A}$.

As the sign types of τ and A_0 are d-compatible in Ω Proposition 3.5 implies that the selfadjoint relation $A_0\times T_0$ is definitizable over Ω'. It is straightforward to check that $\{\mathbb{C}^2,\widehat{\Gamma}_0,\widehat{\Gamma}_1\}$, where

$$\widehat{\Gamma}_0:=\widetilde{\Gamma}_1-\Theta\widetilde{\Gamma}_0,\qquad\widehat{\Gamma}_1:=-\widetilde{\Gamma}_0,$$

is a boundary value space for $A^+ \times T^+$ with $\ker \widehat{\Gamma}_0 = \widetilde{A}$, corresponding γ-field

$$\lambda \mapsto \widehat{\gamma}(\lambda) = \widetilde{\gamma}(\lambda)\big(\widetilde{M}(\lambda) - \Theta\big)^{-1}, \quad \lambda \in \mathfrak{h}_0, \tag{3.15}$$

and corresponding Weyl function

$$\lambda \mapsto \widehat{M}(\lambda) = -(\widetilde{M}(\lambda) - \Theta)^{-1} = -\begin{pmatrix} M(\lambda) & -1 \\ -1 & -\tau(\lambda)^{-1} \end{pmatrix}^{-1}, \quad \lambda \in \mathfrak{h}_0.$$

In particular

$$\begin{aligned} \widehat{M}(\lambda) = \operatorname{Re} \widehat{M}(\lambda_0) + \widehat{\gamma}(\lambda_0)^+ \big((\lambda - \operatorname{Re} \lambda_0) \\ + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(\widetilde{A} - \lambda)^{-1} \big) \widehat{\gamma}(\lambda_0) \end{aligned} \tag{3.16}$$

holds for a fixed $\lambda_0 \in \mathfrak{h}_0 \cap \Omega'$ and all $\lambda \in \mathfrak{h}_0 \cap \Omega'$ (see the proof of Proposition 3.2). By Proposition 3.5 the function $\widehat{M}$ is definitizable in Ω and the sign types of $\widehat{M}$ and $A_0 \times T_0$ are d-compatible in Ω'.

Let $\mu \in \Omega' \cap \overline{\mathbb{R}}$ and assume e.g. that a onesided open connected neighbourhood Δ_+ of μ in $\overline{\mathbb{R}}$, $\overline{\Delta_+} \subset \Omega' \cap \overline{\mathbb{R}}$, is of positive type with respect to $A_0 \times T_0$. As the sign types of $\widehat{M}$ and $A_0 \times T_0$ are d-compatible in Ω' it is no restriction to assume that Δ_+ is also of positive type with respect to $\widehat{M}$. Since $A_0 \times T_0$ and $\widetilde{A}$ are both selfadjoint extensions of the symmetric relation $A \times T$ in $\mathcal{K} \times \mathcal{H}$ we have

$$\dim\big(\operatorname{ran}\, \big((\widetilde{A} - \lambda)^{-1} - (A_0 \times T_0 - \lambda)^{-1}\big)\big) \leq 2$$

for all $\lambda \in \mathfrak{h}_0$. Let Ω_{Δ_+} be a domain with the same properties as Ω, $\overline{\Omega}_{\Delta_+} \subset \Omega'$, such that $\Omega_{\Delta_+} \cap \overline{\mathbb{R}} = \Delta_+$. It follows from [2, Corollary 2.5] that $\widetilde{A}$ is definitizable over Ω_{Δ_+} and that for every finite union δ of open connected subsets in Δ_+, $\overline{\delta} \subset \Delta_+$, such that the spectral projection $E_{\widetilde{A}}(\delta)$ is defined the space $E_{\widetilde{A}}(\delta)(\mathcal{K} \times \mathcal{H})$ equipped with the inner product $[\cdot,\cdot]_{\mathcal{K}\times\mathcal{H}}$ is a Pontryagin space with finite rank of negativity.

Let Ω'' be a domain with the same properties as Ω such that $\overline{\Omega''} \subset \Omega'$ and $\overline{\Delta_+} \subset \Omega'' \cap \overline{\mathbb{R}}$. By Theorem 2.6 there exists an Ω''-minimal representing relation S for $\widehat{M}$, that is S is a selfadjoint relation in some Krein space $\mathcal{G}$ which is definitizable over Ω'', $\rho(S) \cap \Omega'' = \mathfrak{h}(\widehat{M}) \cap \Omega''$, and with a suitable $\Lambda \in \mathcal{L}(\mathbb{C}^2, \mathcal{G})$ we have

$$\widehat{M}(\lambda) = \operatorname{Re} \widehat{M}(\lambda_0) + \Lambda^+ \big((\lambda - \operatorname{Re} \lambda_0) + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(S - \lambda)^{-1} \big) \Lambda \tag{3.17}$$

for a fixed $\lambda_0 \in \rho(S) \cap \Omega''$ and all $\lambda \in \rho(S) \cap \Omega''$. The spectral function of S on $\Omega'' \cap \overline{\mathbb{R}}$ will be denoted by E_S (comp. (2.2)).

In the following we will assume that the point λ_0 in (3.16) and (3.17) belongs to $\rho(\widetilde{A}) \cap \rho(S) \cap \Omega''$. This is no restriction. From (3.16) and (3.17) we obtain

$$\widehat{\gamma}(\lambda_0)^+ \widehat{\gamma}(\lambda_0) = \Lambda^+ \Lambda \quad \text{and} \quad \widehat{\gamma}(\lambda_0)^+ (\widetilde{A} - \lambda)^{-1} \widehat{\gamma}(\lambda_0) = \Lambda^+ (S - \lambda)^{-1} \Lambda$$

for all $\lambda \in \rho(\widetilde{A}) \cap \rho(S) \cap \Omega''$. Therefore the relation

$$V := \left\{ \begin{pmatrix} \sum_{k=1}^{n}(1+(\lambda_k-\lambda_0)(S-\lambda_k)^{-1})\Lambda x_k \\ \sum_{k=1}^{n}(1+(\lambda_k-\lambda_0)(\widetilde{A}-\lambda_k)^{-1})\widehat{\gamma}(\lambda_0)x_k \end{pmatrix} \,\middle|\, \begin{matrix} \lambda_k \in \rho(S)\cap\rho(\widetilde{A})\cap\Omega'' \\ x_k \in \mathbb{C}^2,\ k=1,\dots,n \end{matrix} \right\}$$

is isometric. The assumptions

$$\mathcal{K} = \operatorname{clsp}\{\mathcal{N}_{\lambda,A^+} \mid \lambda \in \Omega\} \quad \text{and} \quad \mathcal{H} = \operatorname{clsp}\{\mathcal{N}_{\lambda,T^+} \mid \lambda \in \Omega'\}$$

imply

$$\mathcal{K} = \operatorname{clsp}\left\{\operatorname{ran}\gamma(\lambda) \mid \lambda \in \rho(S)\cap\rho(\widetilde{A})\cap\Omega''\right\}$$

and

$$\mathcal{H} = \operatorname{clsp}\left\{\operatorname{ran}\gamma'(\lambda) \mid \lambda \in \rho(S)\cap\rho(\widetilde{A})\cap\Omega''\right\},$$

respectively. From (3.12) and (3.15) we obtain

$$\mathcal{K}\times\mathcal{H} = \operatorname{clsp}\left\{\widehat{\gamma}(\lambda)x \mid \lambda \in \rho(S)\cap\rho(\widetilde{A})\cap\Omega'',\, x \in \mathbb{C}^2\right\}$$

and therefore $\operatorname{ran} V$ is dense in $\mathcal{K}\times\mathcal{H}$. This implies that V is an isometric operator and the same holds for its closure $\overline{V}$.

Let δ be a finite union of open connected subsets in Δ_+, $\overline{\delta} \subset \Delta_+$, such that the boundary points of δ in $\overline{\mathbb{R}}$ are of definite type with respect to $\widetilde{A}$. Then $(E_{\widetilde{A}}(\delta)(\mathcal{K}\times\mathcal{H}), [\cdot,\cdot]_{\mathcal{K}\times\mathcal{H}})$ is a Pontryagin space with finite rank of negativity. As Δ_+ is of positive type with respect to S the spectral projection $E_S(\delta)$ is defined and $E_S(\delta)\mathcal{G}$ equipped with the inner product from $\mathcal{G}$ is a Hilbert space. Writing $E_S(\delta)$ and $E_{\widetilde{A}}(\delta)$ as strong limits of the resolvents of S and $\widetilde{A}$, respectively, one verifies that $\overline{V}$ is reduced by $E_S(\delta)\mathcal{G}\times E_{\widetilde{A}}(\delta)(\mathcal{K}\times\mathcal{H})$. Then

$$\overline{V}_\delta := \overline{V} \cap \big(E_S(\delta)\mathcal{G}\times E_{\widetilde{A}}(\delta)(\mathcal{K}\times\mathcal{H})\big)$$

is a closed isometric operator from the Hilbert space $E_S(\delta)\mathcal{G}$ with dense range in the Pontryagin space $E_{\widetilde{A}}(\delta)(\mathcal{K}\times\mathcal{H})$. By [12, Theorem 6.2] $\overline{V}_\delta$ is bounded and it follows that $E_{\widetilde{A}}(\delta)(\mathcal{K}\times\mathcal{H})$ is a Hilbert space.

Let $\xi \in \delta\cap\sigma(\widetilde{A})$ and choose a sequence $\binom{u_n}{v_n} \in \widetilde{A}$ with $\|u_n\| = 1$ and $\|v_n - \xi u_n\| \to 0$ for $n \to \infty$. From

$$\Big(\widetilde{A} \cap \big((I - E_{\widetilde{A}}(\delta))(\mathcal{K}\times\mathcal{H})\big)^2 - \xi\Big)^{-1} \in \mathcal{L}\big((I - E_{\widetilde{A}}(\delta))(\mathcal{K}\times\mathcal{H})\big)$$

and

$$\lim_{n\to\infty} \|(I - E_{\widetilde{A}}(\delta))(v_n - \xi u_n)\| = 0$$

we obtain $\|(I-E_{\widetilde{A}}(\delta))u_n\| \to 0$ and $\|E_{\widetilde{A}}(\delta)u_n\| \to 1$ for $n\to\infty$. As $E_{\widetilde{A}}(\delta)(\mathcal{K}\times\mathcal{H})$ is a Hilbert space we have

$$\liminf_{n\to\infty}\, [u_n,u_n]_{\mathcal{K}\times\mathcal{H}} = \liminf_{n\to\infty}\, [E_{\widetilde{A}}(\delta)u_n, E_{\widetilde{A}}(\delta)u_n]_{\mathcal{K}\times\mathcal{H}} > 0.$$

If ∞ belongs to $\delta \cap \widetilde{\sigma}(\widetilde{A})$ a similar reasoning shows that ∞ is of positive type with respect to $\widetilde{A}$. Therefore δ is of positive type with respect to $\widetilde{A}$. As this is true for every finite union δ of open connected subsets in Δ_+, $\overline{\delta} \subset \Delta_+$, such that the boundary points of δ in $\overline{\mathbb{R}}$ are of definite type with respect to $\widetilde{A}$ we conclude that Δ_+ is also of positive type with respect to $\widetilde{A}$.

Analogously one verifies that a onesided open connected neighbourhood Δ_- of μ in $\overline{\mathbb{R}}$, $\overline{\Delta_-} \subset \Omega' \cap \overline{\mathbb{R}}$ which is of negative type with respect to $A_0 \times T_0$ and $\widehat{M}$ is of negative type with respect to $\widetilde{A}$. We have shown that for every point $\mu \in \Omega \cap \overline{\mathbb{R}}$ there is an open connected neighbourhood I_μ in $\overline{\mathbb{R}}$ such that both components of $I_\mu \backslash \{\mu\}$ are of the same sign type with respect to $A_0 \times T_0$ and $\widetilde{A}$.

3. It remains to verify that $\widetilde{A}$ belongs to $S^\infty(\Omega' \cap \overline{\mathbb{R}})$. For this we use the relation (3.14). We show first that the selfadjoint relation $T_1 = \ker \Gamma_1'$ in $\mathcal{H}$ is definitizable over Ω'. As the function

$$\begin{aligned}-\tau(\lambda)^{-1} = \mathrm{Re}\,\big(-\tau(\lambda_0)^{-1}\big) + \tau(\overline{\lambda}_0)^{-1}\gamma'(\lambda_0)^+\big((\lambda - \mathrm{Re}\,\lambda_0)\\ + (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(T_1-\lambda)^{-1}\big)\gamma'(\lambda_0)\tau(\lambda_0)^{-1}\end{aligned}$$

(see (3.11) and the proof of Proposition 3.2) is definitizable in Ω and the selfadjoint relation T_0 is definitizable over Ω' the same considerations as in step 2 of the proof applied to $-\tau^{-1}$, T_0 and T_1 instead of $\widehat{M}$, $A_0 \times T_0$ and $\widetilde{A}$ show that every point $\mu \in \Omega' \cap \overline{\mathbb{R}}$ has an open connected neighbourhood I_μ in $\overline{\mathbb{R}}$ such that both components of $I_\mu \backslash \{\mu\}$ are of definite type with respect to T_1. By (3.5) we have

$$(T_1 - \lambda)^{-1} = (T_0 - \lambda)^{-1} + \gamma'(\lambda)\big(-\tau(\lambda)^{-1}\big)\gamma'(\overline{\lambda})^+$$

for all $\lambda \in \mathfrak{h}(\tau) \cap \mathfrak{h}(\tau^{-1})$. Since $-\tau^{-1}$ is definitizable in Ω the nonreal spectrum of T_1 in Ω' does not accumulate to points in $\Omega' \cap \overline{\mathbb{R}}$. The growth properties of $-\tau^{-1}$ (see Definition 2.5) and the resolvent of T_0 imply $T_1 \in S^\infty(\Omega' \cap \overline{\mathbb{R}})$ and therefore T_1 is definitizable over Ω'.

As the sign types of $-\tau^{-1}$ and A_0 are d-compatible in Ω (see the proof of Proposition 3.5 (iii)) it follows that $A_0 \times T_1$ is definitizable over Ω'. The relation

$$(\widetilde{A} - \lambda)^{-1} = (A_0 \times T_1 - \lambda)^{-1} + \widetilde{\gamma}(\lambda)\widehat{M}(\lambda)\widetilde{\gamma}(\overline{\lambda})^+, \quad \lambda \in \mathfrak{h}_0,$$

(cf. (3.14)) and the growth properties of $\widehat{M}$ and the resolvent of $A_0 \times T_0$ show $\widetilde{A} \in S^\infty(\Omega' \cap \overline{\mathbb{R}})$. This completes the proof of Theorem 3.6.

□

Remark 3.7. Let Ω, Ω', $A \subset A_0$, τ and T_0 be as in Theorem 3.6 and let Δ be an open connected subset in $\overline{\mathbb{R}}$, $\overline{\Delta} \subset \Omega' \cap \overline{\mathbb{R}}$, which is of positive (negative) type with respect to A_0 and τ. As T_0 is an Ω'-minimal representing relation for τ it follows that Δ is of positive (negative) type with respect to the selfadjoint relation $A_0 \times T_0$ in $\mathcal{K} \times \mathcal{H}$. From [2, Corollary 2.5] we obtain that for every finite union δ of open connected subsets in $\overline{\mathbb{R}}$, $\overline{\delta} \subset \Delta$, such that the spectral projection $E_{\widetilde{A}}(\delta)$ corresponding to $\widetilde{A}$ in (3.8) and the set δ is defined, $(E_{\widetilde{A}}(\delta)(\mathcal{K} \times \mathcal{H}), [\cdot,\cdot]_{\mathcal{K}\times\mathcal{H}})$ is a Pontryagin space with finite rank of negativity (positivity). This can also be deduced from [3, Theorem 4.1].

Remark 3.8. The assumption $\mathcal{H} = \operatorname{clsp}\{\mathcal{N}_{\lambda,T^+} \mid \lambda \in \Omega'\}$ in Theorem 3.6 implies that the selfadjoint extension $\widetilde{A}$ in (3.8) satisfies the minimality condition

$$\mathcal{K} \times \mathcal{H} = \operatorname{clsp}\left\{(1 + (\lambda - \lambda_0)(\widetilde{A} - \lambda)^{-1})\{k, 0\} \mid k \in \mathcal{K},\, \lambda \in \rho(\widetilde{A}) \cap \Omega'\right\} \tag{3.18}$$

for some fixed $\lambda_0 \in \rho(\widetilde{A}) \cap \Omega'$. This can be verified as in [3, Proof of Theorem 4.1]. Let $A \in \widetilde{\mathcal{C}}(\mathcal{K})$ be as in Theorem 3.6 and assume that $\widetilde{B}$ is a selfadjoint extension of A in some Krein space $\mathcal{K} \times \mathcal{H}'$ which is definitizable over Ω' such that the compressed resolvent of $\widetilde{B}$ onto $\mathcal{K}$ yields a solution of (3.9). Then $P_{\mathcal{K}}(\widetilde{B} - \lambda)^{-1}|_{\mathcal{K}}$ and $P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}}$ coincide. If $\widetilde{B}$ fulfils the minimality condition (3.18) with $\mathcal{K} \times \mathcal{H}$ and $\rho(\widetilde{A}) \cap \Omega'$ replaced by $\mathcal{K} \times \mathcal{H}'$ and $\rho(\widetilde{B}) \cap \Omega'$, respectively, and we choose $\lambda_0 \in \rho(\widetilde{A}) \cap \rho(\widetilde{B}) \cap \Omega'$, then

$$W := \left\{ \begin{pmatrix} \sum_{i=1}^n (1 + (\lambda_i - \lambda_0)(\widetilde{A} - \lambda_i)^{-1})\{k_i, 0\} \\ \sum_{i=1}^n (1 + (\lambda_i - \lambda_0)(\widetilde{B} - \lambda_i)^{-1})\{k_i, 0\} \end{pmatrix} \,\middle|\, \begin{matrix} \lambda_i \in \rho(\widetilde{A}) \cap \rho(\widetilde{B}) \cap \Omega', \\ k_i \in \mathcal{K},\, i = 1, 2, \dots, n \end{matrix} \right\}$$

is a densely defined isometric operator in $\mathcal{K} \times \mathcal{H}$ with dense range in $\mathcal{K} \times \mathcal{H}'$ and the same holds for its closure $\overline{W}$. We denote the local spectral functions of $\widetilde{A}$ and $\widetilde{B}$ by $E_{\widetilde{A}}$ and $E_{\widetilde{B}}$, respectively. Let $\Delta \subset \Omega' \cap \overline{\mathbb{R}}$ be a closed connected set such that $E_{\widetilde{A}}(\Delta)$ is defined. Then also $E_{\widetilde{B}}(\Delta)$ is defined and $\overline{W}$ is reduced by

$$E_{\widetilde{A}}(\Delta)(\mathcal{K} \times \mathcal{H}) \times E_{\widetilde{B}}(\Delta)(\mathcal{K} \times \mathcal{H}').$$

The closed isometric operator

$$\overline{W}_\Delta := \overline{W} \cap \left(E_{\widetilde{A}}(\Delta)(\mathcal{K} \times \mathcal{H}) \times E_{\widetilde{B}}(\Delta)(\mathcal{K} \times \mathcal{H}')\right)$$

intertwines the resolvents of

$$\widetilde{A}_1 := \widetilde{A} \cap \left(E_{\widetilde{A}}(\Delta)(\mathcal{K} \times \mathcal{H})\right)^2 \quad \text{and} \quad \widetilde{B}_1 := \widetilde{B} \cap \left(E_{\widetilde{B}}(\Delta)(\mathcal{K} \times \mathcal{H}')\right)^2,$$

i.e. for $\lambda \in \rho(\widetilde{A}_1) \cap \rho(\widetilde{B}_1) \cap \Omega'$ and $x \in \operatorname{dom} \overline{W}_\Delta$ we have

$$\overline{W}_\Delta(\widetilde{A}_1 - \lambda)^{-1}x = (\widetilde{B}_1 - \lambda)^{-1}\overline{W}_\Delta x.$$

In particular, the ranks of positivity and negativity of the inner products on the subspaces $E_{\widetilde{A}}(\Delta)(\mathcal{K} \times \mathcal{H})$ and $E_{\widetilde{B}}(\Delta)(\mathcal{K} \times \mathcal{H}')$ coincide.

If, in addition to the assumptions above, $(E_{\widetilde{A}}(\Delta)(\mathcal{K}\times\mathcal{H}),[\cdot,\cdot]_{\mathcal{K}\times\mathcal{H}})$ is a Pontryagin space, then $E_{\widetilde{B}}(\Delta)(\mathcal{K}\times\mathcal{H}')$ equipped with the inner product from $\mathcal{K}\times\mathcal{H}'$ is also a Pontryagin space and by [12, Theorem 6.2] the operator $\overline{W_\Delta}$ is an isometric isomorphism of $E_{\widetilde{A}}(\Delta)(\mathcal{K}\times\mathcal{H})$ onto $E_{\widetilde{B}}(\Delta)(\mathcal{K}\times\mathcal{H}')$, i.e. $\widetilde{A}_1$ and $\widetilde{B}_1$ are isometrically equivalent.

The case that the function τ is a real constant is excluded in Theorem 3.6. In this case we have the following theorem.

Theorem 3.9. *Let Ω be a domain as in the beginning of Section 3.2 and let A be a closed symmetric relation with defect one in the Krein space $\mathcal{K}$ such that $\mathcal{K}=\operatorname{clsp}\{\mathcal{N}_{\lambda,A^+}\,|\,\lambda\in\Omega\}$ holds. Assume that there exists a selfadjoint extension A_0 of A which is definitizable over Ω. Let $\{\mathbb{C},\Gamma_0,\Gamma_1\}$ be a boundary value space for A^+, $A_0=\ker\Gamma_0$, denote by γ and M the corresponding γ-field and Weyl function, respectively, and let τ be a real constant.*

Then the relation $A_{-\tau}=\ker\big(\Gamma_1+\tau\Gamma_0\big)$ is a selfadjoint extension of A in $\mathcal{K}$ which is definitizable over Ω. The sign types of M and $A_{-\tau}$ are d-compatible in Ω. For every $k\in\mathcal{K}$ and every $\lambda\in\Omega\cap\mathfrak{h}(M)\cap\mathfrak{h}((M+\tau)^{-1})$ a solution of the boundary value problem

$$f_1'-\lambda f_1=k,\quad \tau\Gamma_0\hat{f}_1+\Gamma_1\hat{f}_1=0,\quad \hat{f}_1=\begin{pmatrix}f_1\\ f_1'\end{pmatrix}\in A^+, \tag{3.19}$$

is given by

$$\begin{aligned}f_1&=(A_{-\tau}-\lambda)^{-1}k=(A_0-\lambda)^{-1}k-\gamma(\lambda)\big(M(\lambda)+\tau\big)^{-1}\gamma(\overline{\lambda})^+k,\\ f_1'&=\lambda f_1+k.\end{aligned}$$

Proof. The proof of Theorem 3.9 is a modification of the proof of Theorem 3.6. Note first that the relation (3.4) and $\mathcal{K}=\operatorname{clsp}\{\operatorname{ran}\gamma(\lambda)\,|\,\lambda\in\Omega\cap\rho(A_0)\}$ imply that the Weyl function M is not identically equal to a constant. Here it is obvious that the resolvent of $A_{-\tau}$ yields a solution of the boundary value problem (3.19) (compare (3.1), (3.2) and (3.5)). As in step 2 and step 3 of the proof of Theorem 3.6 the function $\lambda\mapsto-(M(\lambda)+\tau)^{-1}$, which by [1, Theorem 2.3] is definitizable in Ω, can be regarded as the Weyl function corresponding to a boundary value space $\{\mathbb{C},\hat{\Gamma}_0,\hat{\Gamma}_1\}$ for A^+ with $A_{-\tau}=\ker\hat{\Gamma}_0$. Now the same arguments as in the proof of Theorem 3.6 show that $A_{-\tau}$ is definitizable over Ω. The details are left to the reader.

□

References

[1] T.Ya. Azizov, P. Jonas: On Locally Definitizable Matrix Functions, *submitted*.

[2] J. Behrndt, P. Jonas: On Compact Perturbations of Locally Definitizable Selfadjoint Relations in Krein Spaces, *to appear in* Integral Equations Operator Theory.

[3] J. Behrndt, P. Jonas: Boundary Value Problems with Local Generalized Nevanlinna Functions in the Boundary Condition, *submitted*.

[4] V.A. Derkach: On Weyl Function and Generalized Resolvents of a Hermitian Operator in a Krein Space, Integral Equations Operator Theory **23** (1995), 387-415.

[5] V.A. Derkach: On Generalized Resolvents of Hermitian Relations in Krein Spaces, J. Math. Sci. (New York) **97** (1999), 4420-4460.

[6] V.A. Derkach, S. Hassi, M.M. Malamud, H.S.V. de Snoo: Generalized Resolvents of Symmetric Operators and Admissibility, Methods Funct. Anal. Topology **6** (2000), 24-53.

[7] V.A. Derkach, M.M. Malamud: Generalized Resolvents and the Boundary Value Problems for Hermitian Operators with Gaps, J. Funct. Anal. **95** (1991), 1-95.

[8] V.A. Derkach, M.M. Malamud: The Extension Theory of Hermitian Operators and the Moment Problem, J. Math. Sci. (New York) **73** (1995), 141-242.

[9] A. Dijksma, H.S.V. de Snoo: Symmetric and Selfadjoint Relations in Krein Spaces I, Operator Theory: Advances and Applications **24** (1987), Birkhäuser Verlag Basel, 145-166.

[10] A. Dijksma, H.S.V. de Snoo: Symmetric and Selfadjoint Relations in Krein Spaces II, Ann. Acad. Sci. Fenn. Math. **12**, 1987, 199-216.

[11] V.I. Gorbachuk, M.L. Gorbachuk: Boundary Value Problems for Operator Differential Equations, Kluwer Academic Publishers, Dordrecht (1991).

[12] I.S. Iohvidov, M.G. Krein, H. Langer: Introduction to the Spectral Theory of Operators in Spaces with an Indefinite Metric, Mathematical Research **9** (1982), Akademie-Verlag Berlin.

[13] P. Jonas: On a Class of Unitary Operators in Krein Space, Operator Theory: Advances and Applications **17** (1986), Birkhäuser Verlag Basel, 151-172.

[14] P. Jonas: A Class of Operator-Valued Meromorphic Functions on the Unit Disc, Ann. Acad. Sci. Fenn. Math. **17** (1992), 257-284.

[15] P. Jonas: Operator Representations of Definitizable Functions, Ann. Acad. Sci. Fenn. Math. **25** (2000), 41-72.

[16] P. Jonas: On Locally Definite Operators in Krein Spaces, in: Spectral Theory and Applications, Theta Foundation, (2003).

[17] P. Jonas: On Operator Representations of Locally Definitizable Functions, *submitted*.

[18] H. Langer: Spectral Functions of Definitizable Operators in Krein Spaces, Functional Analysis Proceedings of a Conference held at Dubrovnik, Yugoslavia, November 2-14, 1981, Lecture Notes in Mathematics, **948**, Springer Verlag Berlin-Heidelberg-New York, 1982, 1-46.