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V. Mehrmann and C. Shi

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# Analysis of higher order linear differential-algebraic systems

Volker Mehrmann,[*] [†] Chunchao Shi [*] [‡]

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**Abstract**

We study linear over- or under-determined differential-algebraic systems of order larger than 1. We analyze the classical procedure of turning the system into a first order system. We show that this approach leads to solutions that may have different smoothness requirements. We derive canonical and condensed forms as well as general existence and uniqueness results for differential-algebraic systems of arbitrary order and index. We also show how to identify exactly those variables for which the order reduction to first order does not lead to extra smoothness requirements. Finally we discuss some consequences for the analysis of matrix polynomials.

**Keywords:** differential-algebraic equation, higher order system, order reduction, index reduction, strangeness-index, matrix polynomial.

**AMS(MOS) subject classification:** 65L80, 65L05, 34A30

## Notation

| | |
|---|---|
| $\mathbb{N}_0$ | the integers including 0 |
| $\mathbb{R}$ ($\mathbb{C}$) | the real (complex) numbers |
| $\mathbb{C}^{m,n}$, ( $\mathbb{C}^n = \mathbb{C}^{n,1}$) | space of complex matrices of size $m, n$ |
| $\mathcal{C}^q(\mathbb{I}, \mathbb{C}^{m,n})$ | set of all $q$-times continuously differentiable matrix-valued functions mapping from the real interval $\mathbb{I}$ to $\mathbb{C}^{m,n}$, where $q \in \mathbb{N}_0$ |
| $I$ or $I_n$ | identity matrix of size $n \times n$ |
| $A^T$ ($A^H$) | (conjugate) transpose of a matrix $A$ |
| $\mathcal{R}(\cdot)$ | column space of a matrix $A$ |
| $\mathcal{N}(\cdot)$ | null space of a matrix $A$ |
| rank$(\cdot)$ | rank of a matrix or a matrix-valued function, |
| dim$(\cdot)$ | dimension of a subspace |

[1]Institut für Mathematik, MA 4-5 TU Berlin, Straße des 17. Juni 136, D-10623 Berlin, FRG; {mehrmann,shi}@math.tu-berlin.de.

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## 1 Introduction

We study linear $l$-th order systems of Differential-Algebraic Equations (DAEs) with variable coefficients

$$A_l(t)x^{(l)}(t) + A_{l-1}(t)x^{(l-1)}(t) + \cdots + A_0(t)x(t) = f(t), \tag{1}$$

in a real interval $\mathbb{I} \subset \mathbb{R}$, together with an initial condition

$$x(t_0) = x_0^{[0]},\ \ldots,\ x^{(l-2)}(t_0) = x_0^{[l-2]},\ x^{(l-1)}(t_0) = x_0^{[l-1]},\ t_0 \in \mathbb{I}. \tag{2}$$

Here, $A_i(t) \in \mathcal{C}(\mathbb{I}, \mathbb{C}^{m,n})$, $i = 0, 1, \ldots, l$, $A_l(t) \not\equiv 0$, $x(t)$ is an unknown vector-valued function in $\mathcal{C}(\mathbb{I}, \mathbb{C}^n)$, and the right-hand side $f(t)$ is a given vector-valued function in $\mathcal{C}^\mu(\mathbb{I}, \mathbb{C}^m)$, where $\mathcal{C}^\mu(\mathbb{I}, \mathbb{C}^{m,n})$, $\mu \in \mathbb{N}_0$, denotes the set of all $\mu$-times continuously differentiable matrix-valued functions from the real interval $\mathbb{I}$ to the complex vector space $\mathbb{C}^{m,n}$. In the following we will refer to DAEs with order $l$ greater than 1 simply as *higher order* systems.

As the name "DAE" indicates, a system of DAEs consists of Ordinary Differential Equations (ODEs) coupled with purely algebraic equations; or put in others words, DAEs are everywhere singular implicit ODEs. Therefore, in the case $m = n$, if not otherwise specified, we assume that the leading coefficient function $A_l(t)$ in the system (1) satisfies $\text{rank}(A_l(t)) < n$ for all $t \in \mathbb{I}$; in the case $m > n$ (or $m < n$, respectively), the system (1) becomes over-determined (or under-determined, respectively).

Systems of DAEs play a key role in the modeling and simulation of constrained dynamical systems in numerous applications. Such systems have been intensively studied, theoretically as well as numerically, in the past three decades. For a systematic and comprehensive exposition of important aspects regarding the theory and the numerical treatment of first order DAEs, see e.g. [3, 4, 5, 8, 13, 16, 17, 21, 22, 27] and the references therein. Typical applications where higher order DAEs arise naturally are multi-body systems, see [8, 27] or models of electrical circuits [14, 15].

Usually, in the classical theory of ordinary differential equations, higher order systems are turned into first order systems by introducing new variables for the derivatives up to order $l - 1$. There is no unique way of performing this transformation and, in particular for DAEs, this classical textbook approach has been questioned frequently in the literature [1, 7, 28], mostly due to instabilities arising in numerical solution methods.

But, as we will demonstrate below, if the degree of differentiability of the right-hand side $f(t)$ in the higher order system is limited, then the transformation to first order may be mathematically incorrect, in the sense that there may not exist any continuous solution to the resulting first order system, whereas there exist continuous solutions to the original higher order system. The reason for this possible difference in solvability is the coupling between differential and algebraic equations in DAEs.

**Example 1** Consider the linear second order constant coefficient DAE

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \ddot{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) = f(t), \quad t \in \mathbb{I}, \tag{3}$$

where $x(t) = [x_1(t), x_2(t)]^T$, and $f(t) = [f_1(t), f_2(t)]^T$. System (3) has the unique solution

$$\begin{cases} x_1(t) = f_2(t), \\ x_2(t) = f_1(t) - \dot{f}_2(t) - \ddot{f}_2(t). \end{cases} \tag{4}$$

Using the classical transformation to first order

$$v(t) = [v_1(t), v_2(t)]^T = [\dot{x}_1(t), \dot{x}_2(t)]^T, \; y(t) = [v_1(t), v_2(t), x_1(t), x_2(t)]^T,$$

we obtain the first order system

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \dot{y}(t) + \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} y(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ 0 \\ 0 \end{bmatrix}, \tag{5}$$

which has the unique solution

$$\begin{cases} x_1(t) = f_2(t), \\ x_2(t) = f_1(t) - \dot{f}_2(t) - \ddot{f}_2(t), \\ v_1(t) = \dot{f}_2(t), \\ v_2(t) = \dot{f}_1(t) - \ddot{f}_2(t) - f_2^{(3)}(t). \end{cases} \tag{6}$$

□

We see from Example 1 that the classical approach of introducing the derivatives of the unknown vector-valued function $x(t)$ as new variables may lead to higher smoothness requirements for the inhomogeneity. On the other hand, introducing only some derivatives may avoid this difficulty. At first look, it is not clear which variables this may be in Example 1. It will be one of the main results of this paper to determine these variables for general linear DAEs of higher order. Let us illustrate this idea with a second example.

**Example 2** Consider the linear second order constant coefficient DAE

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \ddot{x}(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} x(t) = f(t), \quad t \in \mathbb{I}, \tag{7}$$

where $x(t) = [x_1(t), x_2(t)]^T$, and $f(t) = [f_1(t), f_2(t)]^T$.

System (7) has the general solution

$$\begin{bmatrix} x_1(t) \\ x_2 \end{bmatrix} = \begin{bmatrix} \int_{t_0}^{t} \int_{t_0}^{s} f_1(\tau) + f_2(\tau)\; d\tau \; ds + c_1 t + c_2, \\ x_2(t) = f_2(t) \end{bmatrix},$$

where the integration constants $c_1, c_2$ may be determined by assigning two initial or boundary conditions. It follows, that it is sufficient to have a continuous inhomogeneity to get a continuous solution (actually $f_1$ may be even less smooth).

Transforming to first order by introducing $v(t) = [v_1(t), v_2(t)]^T = [\dot{x}_1(t), \dot{x}_2(t)]^T$, we would need the derivative $\dot{x}_2 = \dot{f}_2$, while introducing only $v_1 = \dot{x}_1$ is sufficient to reduce the order and does not lead to extra differentiability restrictions. □

From a purely analytical point of view, the distinction between the different smoothness requirements for higher order systems and associated first order systems may seem artificial, since one could argue that the solution as a whole should be $l$ times differentiable if we want to write down $x^{(l)}$. But examples 1 and 2 show that to obtain continuous solutions of a system of DAEs, some parts of the coefficients and of the right-hand side $f(t)$ may be required to be more smooth than others. To analyze the exact degree of smoothness required for differential-algebraic systems is currently an important research topic see [2, 24, 25]. This means that, in general, we do not need a solution space $\mathcal{C}^l(\mathbb{I}, \mathbb{C}^n)$. For this reason, in the following, we consider the differential operator $d^l/dt^l, d^{l-1}/dt^{l-1}, \ldots, d/dt$ only formally, i.e., we do not necessarily mean that the complete vector-valued function $x(t)$ must be $l$-times continuously differentiable.

Another difficulty that arises in practice, is that the system may be badly scaled and one also has disturbances and perturbations in the data (see the concept of perturbation index in [17]), so that the transformation to first order may lead to very different solutions in the perturbed system.

**Example 3** Consider the second order system

$$\epsilon_1 \ddot{x}(t) + \epsilon_2 \dot{x}(t) + \epsilon_3 x(t) = \epsilon_4 f(t), \quad t \in \mathbb{I}, \tag{8}$$

with coefficients $\epsilon_i$, $i = 1, \ldots, 4$ of absolute value close or smaller than the machine precision. If we transform (8) to first order in the classical way by introducing

$$y(t) = [y_1(t), y_2(t)]^T := [\dot{x}(t), x(t)]^T,$$

then we obtain the system

$$\begin{bmatrix} \epsilon_1 & 0 \\ 0 & 1 \end{bmatrix} \dot{y}(t) + \begin{bmatrix} \epsilon_2 & \epsilon_3 \\ -1 & 0 \end{bmatrix} y(t) = \begin{bmatrix} \epsilon_4 f(t) \\ 0 \end{bmatrix}. \tag{9}$$

For different values of $\epsilon_i$, in finite precision arithmetic we may decide that the system (9) has a unique solution, no solutions at all or is actually a singular system. Compare also the different conditioning of different linearizations of matrix polynomials in [30]. □

After having seen that the classical transformation of a higher order system of DAEs to first order may lead to difficulties, we have to find another approach to investigate the analytic properties of higher order DAEs. We will carry out the analysis by generalizing the algebraic techniques derived for first order systems in [18, 19].

We discuss the following questions:

1. Does system (1)–(2) always have continuous solutions? Under which conditions does it have a unique solution?

2. If system (1) has a continuous solution, how smooth is the right-hand side $f(t)$ required to be?

3. How can we transform the system to a first order system without changing the smoothness requirements for the coefficients or the inhomogeneity.

In order to answer these questions, in Section 2 we discuss different types of equivalence transformations for matrix tuples and tuples of matrix functions and we recall some results on condensed forms for matrix pairs and pairs of matrix functions from [18, 19]. In Section 3 we then present condensed forms for matrix triples and triples of matrix functions that are associated with systems of second order DAEs. These forms allow to answer the above questions for systems of second order. In Section 4, the results for second order systems are extended to general higher order systems, and finally in Section 5 we discuss implications of these results for the linearization of matrix polynomials.

# 2 Preliminaries

In this section we discuss different types of equivalence relations and associated condensed forms and we recall some results for pairs of matrices and pairs of matrix functions.

It is well-known that the nature of the solutions of linear first order constant coefficient DAEs

$$A_1\dot{x}(t) + A_0x(t) = f(t), \quad A_0, A_1 \in \mathbb{C}^{m,n}$$

can be determined by the algebraic properties of the corresponding matrix pencil $\lambda A_1 - A_0$, which follow from the canonical forms for matrix pencils

$$\lambda(PA_1Q) - (PA_0Q), \tag{10}$$

where $P \in \mathbb{C}^{m,m}$ and $Q \in \mathbb{C}^{n,n}$ are any nonsingular matrices; see, for example, [3, 9]. In particular, these are the well-known *Weierstrass canonical form* for regular matrix pencils, and the *Kronecker canonical form* for general singular matrix pencils see e.g. [9], from which one can directly read off the properties of the corresponding DAEs.

Unfortunately, it is not so easy to extend these results to higher order systems, since it is a well-known open problem [12, 31] to find a canonical form for quadratic matrix polynomials, let alone for higher degree matrix polynomials.

But for the analysis and solution of linear first order DAEs the complete information from these canonical forms is not necessary. It has been shown in [18, 19] for matrix pairs that it suffices to study condensed forms rather than canonical forms. We will extend these condensed forms to matrix tuples $(A_l, \ldots, A_1, A_0)$ and tuples $(A_l(t), \ldots, A_1(t), A_0(t))$ of matrix functions.

To obtain condensed forms we need three types of equivalence relations. The first one generalizes the classical equivalence for matrix pairs to matrix tuples.

**Definition 4** *Two tuples of matrices* $(A_l, \ldots, A_1, A_0)$ *and* $(B_l, \ldots, B_1, B_0)$, $A_i, B_i \in \mathbb{C}^{m,n}$, $i = 0, 1, \ldots, l$, $l \in \mathbb{N}_0$, *are called* strongly equivalent *if there exist nonsingular matrices* $P \in \mathbb{C}^{m,m}$ *and* $Q \in \mathbb{C}^{n,n}$ *such that*

$$B_i = PA_iQ, \quad i = 0, 1, \ldots, l. \tag{11}$$

*If this is the case, we write*

$$(A_l, \dots, A_1, A_0) \sim (B_l, \dots, B_1, B_0).$$

In the case of $l$-th order systems, i.e. for tuples of matrix valued functions we have the following definition of *global equivalence*, which results from scaling the equation by a nonsingular matrix $P(t)$ and carrying out a change of basis with a nonsingular matrix $Q(t)$.

**Definition 5** *Two tuples of matrix-valued functions $(A_l(t), \dots, A_1(t), A_0(t))$ and $(B_l(t), \dots, B_1(t), B_0(t))$ with $A_i(t), B_i(t) \in \mathcal{C}(\mathbb{I}, \mathbb{C}^{m,n})$, $i = 0, 1, \dots, l$ are called* globally equivalent *if there exist pointwise nonsingular matrix-valued functions $P(t) \in \mathcal{C}(\mathbb{I}, \mathbb{C}^{m,m})$ and $Q(t) \in \mathcal{C}^l(\mathbb{I}, \mathbb{C}^{n,n})$ such that*

$$\begin{aligned}&[B_l(t), \dots, B_0(t)] \\ &= P(t)[A_l(t), \dots, A_0(t)] \begin{bmatrix} Q(t) & \binom{l}{1}\frac{d}{dt}Q(t) & \cdots & \cdots & \binom{l}{l}\frac{d^l}{dt^l}Q(t) \\ & Q(t) & \binom{l-1}{1}\frac{d}{dt}Q(t) & \cdots & \binom{l-1}{l-1}\frac{d^{l-1}}{dt^{l-1}}Q(t) \\ & & \ddots & \ddots & \vdots \\ & & & Q(t) & \binom{1}{1}\frac{d}{dt}Q(t) \\ & & & & Q(t) \end{bmatrix},\end{aligned} \tag{12}$$

*$i, j \in \mathbb{N}$, $i \leq j$. If this it is clear from the context, then we still write $(A_l(t), \dots, A_1(t), A_0(t)) \sim (B_l(t), \dots, B_1(t), B_0(t))$.*

It is well-known, see e.g. [11, 18], that strong equivalence is not the associated local (at a fixed point $\hat{t}$) version of global equivalence for variable coefficient systems. Since due to the Theorem of Hermite interpolation, see e.g. [29], for any nonsingular matrices $\hat{P} \in \mathbb{C}^{m,m}$, $\hat{Q} \in \mathbb{C}^{n,n}$ and any matrices $R_1, \dots, R_l \in \mathbb{C}^{n,n}$, we can always find pointwise nonsingular matrix functions $P(t)$ and $Q(t)$ such that $P(\hat{t}) = \hat{P}$, $Q(\hat{t}) = \hat{Q}$ and $\frac{d^i}{dt^i}Q(t) = R_i$, $i = 1, \dots, l$, we obtain the following definition.

**Definition 6** *Two tuples of matrices $(A_l, \dots, A_1, A_0)$ and $(B_l, \dots, B_1, B_0)$ with $A_i, B_i \in \mathbb{C}^{m,n}$, $i = 0, 1, \dots, l$ are called* locally equivalent *if there exist matrices $P \in \mathbb{C}^{m,m}$ nonsingular, $Q \in \mathbb{C}^{n,n}$ nonsingular and $R_1, \dots, R_l \in \mathbb{C}^{n,n}$ such that*

$$\begin{aligned}&[B_l, \dots, B_0] \\ &= P[A_l, \dots, A_0] \begin{bmatrix} Q & \binom{l}{1}R_1 & \cdots & \cdots & \binom{l}{l}R_l \\ & Q & \binom{l-1}{1}R_1 & \cdots & \binom{l-1}{l-1}R_{l-1} \\ & & \ddots & \ddots & \vdots \\ & & & Q & \binom{1}{1}R_1 \\ & & & & Q \end{bmatrix}.\end{aligned} \tag{13}$$

As already suggested by the definitions, the relations (11), (12) and (13) are equivalence relations, see Appendix A.

The canonical form for pairs of matrix functions under local and global equivalence has been derived in [18, 19].

**Theorem 7** *[18, 19] Let $A_0, A_1 \in \mathbb{C}^{m,n}$ and let the columns of*

$$\begin{array}{ll} (a) & T \text{ form a basis of } \operatorname{kernel} A_1, \\ (b) & Z \text{ form a basis of } \operatorname{corange} A_1 = \operatorname{kernel} A_1^H, \\ (c) & T' \text{ form a basis of } \operatorname{cokernel} A_1 = \operatorname{range} A_1^H, \\ (d) & V \text{ form a basis of } \operatorname{corange}(Z^H A_0 T). \end{array}$$

*Then the quantities (with the convention* $\operatorname{rank}\emptyset = 0$*)*

$$\begin{array}{lll} (a) & r = \operatorname{rank} A_1 & \text{(rank)} \\ (b) & a = \operatorname{rank}(Z^H A_0 T) & \text{(algebraic part)} \\ (c) & s = \operatorname{rank}(V^H Z^H A_0 T') & \text{(strangeness)} \\ (d) & d = r - s & \text{(differential part)} \\ (e) & u = n - r - a - s & \text{(undetermined part)} \\ (f) & v = m - d - a - s & \text{(vanishing part)} \end{array}$$

*are invariant under local equivalence (13) and $(A_1, A_0)$ is equivalent to the canonical form*

$$\left( \begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{array}{c} s \\ d \\ a \\ s \\ v \end{array}, \tag{14}$$

*where the last block column in both matrices has width $u$.*

A slight modification of the proof of Theorem 7 leads to the following condensed form for pairs of matrices under strong equivalence.

**Theorem 8** *Let $A_0, A_1 \in \mathbb{C}^{m,n}$, and let the columns of*

$$\begin{array}{lll} (a) & Z_1 \in \mathbb{C}^{m,m-r} & \text{form a basis for } \mathcal{N}(A_1^T), \\ (b) & Z_2 \in \mathbb{C}^{n,n-r} & \text{form a basis for } \mathcal{N}(A_1). \end{array} \tag{15}$$

*Then, the matrix pair $(A_1, A_0)$ is strongly equivalent to a matrix pair of the form*

$$\left( \begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & 0 & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{array}{c} s \\ d \\ a \\ s \\ v \end{array}, \tag{16}$$

*where $s, d, a, u, v \in \mathbb{N}_0$, the last block column has size $u$, and the quantities*

$$\begin{array}{ll} (a) & r = \operatorname{rank}(E) \\ (b) & a = \operatorname{rank}(Z_1^T A Z_2) \\ (c) & s = \operatorname{rank}(Z_1^T A) - a \\ (d) & d = r - s \\ (e) & v = m - r - a - s \\ (f) & u = n - r - a \end{array} \tag{17}$$

*are invariant under the strong equivalence relation* (11)*.*

*Proof.* The proof is similar to the proof of Theorem 7 in [19], just leaving out all the elimination steps that are not allowed in strong equivalence. For completeness we give the proof in Appendix B. □

Applying the local canonical form (14) globally in a neighborhood of a fixed point $\hat{t}$, one has the following result.

**Theorem 9** *[18, 19] Let $A_1, A_0 \in \mathcal{C}(\mathbb{I}, \mathbb{C}^{m,n})$ be sufficiently smooth and suppose that*

$$r(t) \equiv r, \quad a(t) \equiv a, \quad s(t) \equiv s \tag{18}$$

*for the local characteristic values of $(A_1(t), A_0(t))$ in the neighborhood of a fixed point $\hat{t} \in \mathbb{I}$. Then, $(A_1(t), A_0(t))$ is globally equivalent to the condensed form*

$$\left( \begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{1,2(t)} & 0 & A_{1,4}(t) \\ 0 & 0 & 0 & A_{2,4}(t) \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{matrix} s \\ d \\ a \\ s \\ v \end{matrix} . \tag{19}$$

*All entries $A_{i,j}(t)$ are matrix functions on $\mathbb{I}$ and the last block column in both matrices has size $u = n - s - d - a$.*

Unfortunately (see [18]) Theorem 9 is not yet sufficient to explain the solution behavior of linear differential algebraic systems of first order. Another type of equivalence transformations is needed. This can be seen by writing down the system of differential-algebraic equations that corresponds to (19) (in the transformed variables). We get

$$\begin{array}{ll} (a) & \dot{x}_1(t) = A_{12}(t)x_2(t) + A_{14}(t)x_4(t) + f_1(t) \\ (b) & \dot{x}_2(t) = A_{24}(t)x_4(t) + f_2(t) \\ (c) & 0 = x_3(t) + f_3(t) \\ (d) & 0 = x_1(t) + f_4(t) \\ (e) & 0 = f_5(t). \end{array} \tag{20}$$

Here, we can insert the derivative of equation (20d) in (20a), which then becomes an algebraic equation. This corresponds to passing from (19) to

$$\left( \begin{bmatrix} \mathbf{0} & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12}(t) & 0 & A_{14}(t) \\ 0 & 0 & 0 & A_{24}(t) \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right), \tag{21}$$

for which we again compute characteristic values $r, a, s, d, u, v$.

Applying this step inductively, we obtain an inductive definition of a sequence of pairs of matrix functions $(A_1^i(t), A_0^i(t))$, $i \in \mathbb{N}_0$, where $(A_0^0(t), A_1^0(t)) = (A_1(t), A_0(t))$ and $(A_1^{i+1}(t), A_0^{i+1}(t))$ is derived from $(A_1^i(t), A_0^i(t))$ by bringing it into the form (19) and passing then to (21). Here we must assume (18) for every occurring pair of matrices. Connected with this sequence, we then have sequences $r_i(t) \equiv r_i$, $a_i(t) \equiv a_i$, $s_i(t) \equiv s_i$, $d_i(t) \equiv d_i$, $u_i(t) \equiv u_i$, $v_i(t) \equiv v_i$, which are characteristic for the given pair $(A_1(t), A_0(t))$, that is, they do not depend on the specific way they are obtained. Note again that we have to assume that all these quantities are constant in the considered interval $\mathbb{I}$. Furthermore, the sequence stops after finitely many (say $\mu(t) \equiv \mu$) steps with $s_i = 0$. The quantity $\mu$ is called the *strangeness-index* or shorter *s-index* of the pencil $(A_1(t), A_0(t))$.

Note that for square systems, for which $n = m$ and therefore also $u_\mu = v_\mu$, and for which the same smoothness and rank assumptions hold, the s-index is a generalization of the differentiation index [3]. For a detailed analysis of the relationship between these two index concepts see [20].

After having recalled some results for pairs of matrices or matrix functions, in the next section we treat the case of matrix triples and triples of matrix valued functions arising from second order systems.

# 3 Condensed forms for triples of matrices and matrix functions

We begin the analysis with systems of linear second order DAEs with constant coefficients

$$A_2\ddot{x}(t) + A_1\dot{x}(t) + A_0x(t) = f(t), \; t \in \mathbb{I}, \tag{22}$$

with $A_2, A_1, A_0 \in \mathbb{C}^{m,n}$, $f(t) \in \mathcal{C}(\mathbb{I}, \mathbb{C}^m)$ sufficiently smooth, together with an initial condition

$$x(t_0) = x_0^{[0]}, \; \dot{x}(t_0) = x_0^{[1]}, \quad x_0^{[0]}, x_0^{[1]} \in \mathbb{C}^n. \tag{23}$$

Similarly as in the case of first order systems, the behavior of the system (22) (and the initial value problem (22)–(23)) depends on the properties of the quadratic matrix polynomial

$$A(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0.$$

If we apply a strong equivalence transformation with nonsingular matrices $P \in \mathbb{C}^{m,m}$ and $Q \in \mathbb{C}^{n,n}$ then we obtain a transformed quadratic matrix polynomial

$$\hat{A}(\lambda) = \lambda^2 \hat{A}_2 + \lambda \hat{A}_1 + \hat{A}_0 := \lambda^2(PA_2Q) + \lambda(PA_1Q) + (PA_0Q). \tag{24}$$

As we have already mentioned in the introduction, it is a well-known open problem [12, 31] to find a canonical form for quadratic matrix polynomials (24) under strong equivalence. However, as for first order systems, we do not need the complete canonical form to understand the solution behavior of the corresponding DAE. We will now use the ideas developed for first order systems to derive a condensed form for quadratic matrix polynomials.

**Theorem 10** *Let $A_2, A_1, A_0 \in \mathbb{C}^{m,n}$. Then, $(A_2, A_1, A_0)$ is strongly equivalent to a matrix triple $(\hat{A}_2, \hat{A}_1, \hat{A}_0)$ of the following form*

$$
\left(
\begin{bmatrix}
I_{s^{(0,1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{s^{(1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{s^{(0,2)}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{d^{(2)}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & * & * & 0 & 0 & * & * \\
0 & 0 & * & * & 0 & 0 & * & * \\
0 & 0 & * & * & 0 & 0 & * & * \\
0 & 0 & * & * & 0 & 0 & * & * \\
0 & 0 & 0 & * & I_{s^{(0,1)}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{d^{(1)}} & 0 & 0 \\
I_{s^{(0,1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{s^{(1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & I_a & 0 \\
0 & 0 & 0 & 0 & I_{s^{(0,1)}} & 0 & 0 & 0 \\
0 & 0 & I_{s^{(1,2)}} & 0 & 0 & 0 & 0 & 0 \\
I_{s^{(0,1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\right)
\begin{array}{l}
s^{(0,1,2)} \\
s^{(1,2)} \\
s^{(0,2)} \\
d^{(2)} \\
s^{(0,1)} \\
d^{(1)} \\
s^{(0,1,2)} \\
s^{(1,2)} \\
a \\
s^{(0,1)} \\
s^{(0,2)} \\
s^{(0,1,2)} \\
v
\end{array}, \tag{25}
$$

*where $s^{(0,1,2)}$, $s^{(1,2)}$, $s^{(0,2)}$, $s^{(0,1)}$, $d^{(2)}$, $d^{(1)}$, $a$ and $v$ are in $\mathbb{N}_0$, and the entries marked with $*$ are blocks of matrices which are not specified.*

*Proof.* The proof is given in Appendix C. □

Each of the integer quantities in Theorem 10 has an expression in terms of dimensions of column spaces or ranks of matrices, and is invariant under strong equivalence as the next lemma shows.

**Lemma 11** *Let $A_2, A_1, A_0 \in \mathbb{C}^{m,n}$ and let the columns of*

$$\begin{array}{lll} (a) & Z_1 & \textit{form a basis for } \mathcal{N}(A_2^T), \\ (b) & Z_2 & \textit{form a basis for } \mathcal{N}(A_2), \\ (c) & Z_3 & \textit{form a basis for } \mathcal{N}(A_2^T) \cap \mathcal{N}(A_1^T), \\ (d) & Z_4 & \textit{form a basis for } \mathcal{N}(A_2) \cap \mathcal{N}(Z_1^T A_1). \end{array} \tag{26}$$

*Then the quantities*

$$\begin{array}{lll} (a) & r = \operatorname{rank}(A_2) & \text{(rank of } A_2) \\ (b) & a = \operatorname{rank}(Z_3^T A_1 Z_4) & \text{(algebraic part)} \\ (c) & s^{(0,1,2)} = \dim\big(\mathcal{R}(A_2^T) \cap \mathcal{R}(A_1^T Z_1) & \\ & \qquad\qquad \cap \mathcal{R}(A_0^T Z_3)\big) & \text{(strangeness due to } A_2, A_1, A_0) \\ (d) & s^{(0,1)} = \operatorname{rank}(Z_3^T A_0 Z_2) - a & \text{(strangeness due to } A_1, A_0) \\ (e) & d^{(1)} = \operatorname{rank}(Z_1^T A_1 Z_2) - s^{(0,1)} & \text{(1st order differential part)} \\ (f) & s^{(1,2)} = \operatorname{rank}(Z_1^T A_1) - s^{(0,1,2)} & \\ & \qquad\qquad -s^{(0,1)} - d^{(1)} & \text{(strangeness due to } A_2, A_1) \\ (g) & s^{(0,2)} = \operatorname{rank}(Z_3^T A_0) - a & \\ & \qquad\qquad -s^{(0,1,2)} - s^{(0,1)} & \text{(strangeness due to } A_2, A_0) \\ (h) & d^{(2)} = r - s^{(0,1,2)} - s^{(1,2)} - s^{(0,2)} & \text{(2nd order differential part)} \\ (i) & v = m - r - 2s^{(0,1)} - d^{(1)} - 2s^{(0,1,2)} & \\ & \qquad\qquad -s^{(1,2)} - a - s^{(0,2)} & \text{(vanishing equations)} \\ (j) & u = n - r - s^{(0,1)} - d^{(1)} - a & \text{(undetermined part)} \end{array} \tag{27}$$

*are invariant under the strong equivalence relation* (11) *and* $(A_2, A_1, A_0)$ *is strongly equivalent to the form* (25).

*Proof.* The proof follows in a similar fashion as the proof of strong equivalence for matrix pairs. For this reason we only present a partial proof.

*Step 1.* First, we show that the quantities in (27) are well-defined with respect to the choices of the bases in (26). We take $a = \operatorname{rank}(Z_3^T A_0 Z_4)$ as an example. Every change of basis can be represented by

$$\tilde{Z}_3 = Z_3 Q_1, \quad \tilde{Z}_4 = Z_4 Q_2$$

with nonsingular matrices $Q_1$, $Q_2$. From

$$\operatorname{rank}(\tilde{Z}_3^T A_0 \tilde{Z}_4) = \operatorname{rank}(Q_1^T Z_3^T A_0 Z_4 Q_2) = \operatorname{rank}(Z_3^T K Z_4),$$

it then follows that $\operatorname{rank}(Z_1^T A_1 Z_2)$ is well-defined. Similarly, we can prove that the other quantities in (27) are also well–defined.

*Step 2.* Next, we show that the quantities in (27) are invariant under strong equivalence. Here, we just take $s^{(0,1,2)}$ as an example. Let $(A_2, A_1, A_0)$ and $(\tilde{A}_2, \tilde{A}_1, \tilde{A}_0)$ be strongly equivalent, i.e., there exist nonsingular matrices $P$ and $Q$, such that

$$\tilde{A}_2 = P A_2 Q, \quad \tilde{A}_1 = P A_1 Q, \quad \tilde{A}_0 = P A_0 Q. \tag{28}$$

Let the columns of $\tilde{Z}_1$ form a basis for $\mathcal{N}(\tilde{A}_2^T)$, and let the columns of $\tilde{Z}_3$ form a basis for $\mathcal{N}(\tilde{A}_2^T) \cap \mathcal{N}(\tilde{A}_1^T)$. Then, from (28) it follows that the columns of $Z_1 := P^T \tilde{Z}_1$ form a basis for $\mathcal{N}(A_2^T)$, and the columns of $Z_3 := P^T \tilde{Z}_3$ form a basis for $\mathcal{N}(A_2^T) \cap \mathcal{N}(A_1^T)$. Thus, the invariance of $s^{(0,1,2)}$ follows from

$$\begin{aligned}
\tilde{s}^{(0,1,2)} &= \dim\left(\mathcal{R}(\tilde{A}_2^T) \cap \mathcal{R}(\tilde{A}_1^T \tilde{Z}_1) \cap \mathcal{R}(\tilde{A}_0^T \tilde{Z}_3)\right) \\
&= \dim\left(\mathcal{R}(Q^T A_2^T P^T) \cap \mathcal{R}(Q^T A_1^T P^T \tilde{Z}_1) \cap \mathcal{R}(Q^T A_0^T P^T \tilde{Z}_3)\right) \\
&= \dim\left(\mathcal{R}(A_2^T P^T) \cap \mathcal{R}(A_1^T P^T \tilde{Z}_1) \cap \mathcal{R}(A_0^T P^T \tilde{Z}_3)\right) \\
&= \dim\left(\mathcal{R}(A_2^T) \cap \mathcal{R}(A_1^T Z_1) \cap \mathcal{R}(A_0^T Z_3)\right) \\
&= s^{(0,1,2)}.
\end{aligned}$$

Similarly, the invariance of the other quantities in (27) can be proved.

*Step 3.* Finally, we show that the quantities in the equivalent form (25) of $(A_2, A_1, A_0)$ are identical with those defined in (27). Let $P \in \mathbb{C}^{m,m}$, $Q \in \mathbb{C}^{n,n}$ be nonsingular matrices such that

$$(\hat{A}_2, \hat{A}_1, \hat{A}_0) = (PA_2Q, PA_1Q, PA_0Q),$$

where $(\hat{A}_2, \hat{A}_1, \hat{A}_0)$ is of the form (25). Furthermore, let $P$ and $Q$ be partitioned as $P := \left[P_1^T, P_2^T, \ldots, P_{13}^T\right]^T$ and $Q := [Q_1, Q_2, \ldots, Q_8]$ conformably with (25), respectively. Then, by (25), we have

$$\begin{aligned}
\left[P_5^T, \ldots, P_{13}^T\right]^T A_2 &= 0, \\
A_2\left[Q_5, \ldots, Q_8\right] &= 0, \\
\left[P_9^T, \ldots, P_{13}^T\right]^T A_1 &= 0, \\
\left[P_5^T, \ldots, P_{13}^T\right]^T A_1\left[Q_7, Q_8\right] &= 0,
\end{aligned}$$

namely (using MATLAB notation [26]), the columns of $P^T(:, 5:13) := \left[P_5^T, \ldots, P_{13}^T\right]$ form a basis of $\mathcal{N}(A_2^T)$, the columns of $Q(:, 5:8) := [Q_5, \ldots, Q_8]$ form a basis of $\mathcal{N}(A_2)$, the columns of $P^T(:, 9:13) := \left[P_9^T, \ldots, P_{13}^T\right]$ form a basis of $\mathcal{N}(A_2^T) \cap \mathcal{N}(A_1^T)$, and the columns of $Q(:, 7:8)$ form a basis of $\mathcal{N}(A_2) \cap \mathcal{N}((P^T(:, 5:13))^T A_1)$. Observing that, by (25),

$$\left(P^T(:, 9:13)\right)^T A_0 Q(:, 7:8) = \begin{bmatrix} P_9 \\ \vdots \\ P_{13} \end{bmatrix} K\left[Q_7, Q_8\right] = \begin{bmatrix} I_a & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

we have $a = \operatorname{rank}\left(\left(P^T(:, 9:13)\right)^T A_0 Q(:, 7:8)\right) = \operatorname{rank}(Z_3^T A_0 Z_4)$. Similarly, we can prove that the other quantities in the equivalent form (25) are equal to those defined in (27). $\square$

For linear second order DAEs with variable coefficients

$$A_2(t)\ddot{x}(t) + A_1(t)\dot{x}(t) + A_0(t)x(t) = f(t), \quad t \in \mathbb{I}, \tag{29}$$

where $A_2(t), A_1(t), A_0(t) \in \mathcal{C}(\mathbb{I}, \mathbb{C}^{m,n})$, $f(t) \in \mathcal{C}(\mathbb{I}, \mathbb{C}^m)$, we first derive the local condensed form. To study this case, we derive a set of invariants and a condensed form under (local and global) equivalence transformations for the triple $(A_2(t), A_1(t), A_0(t))$ of matrix-valued functions. We first discuss local equivalence for matrix triples.

**Lemma 12** *Under the assumptions of Theorem 10, the quantities in (27) are invariant under the local equivalence relation (13) (l=2) and the triple* $(A_2, A_1, A_0)$ *is locally equivalent to the form* (25).

*Proof.* Since the strong equivalence relation (11) is a special case of the local equivalence relation (13) by setting $R_i = 0$, $i = 1, \ldots, 2$, it follows form Theorem 10 that $(A_2, A_1, A_0)$ is locally equivalent to the form (25). It remains to show that the quantities in (27) are invariant under the local equivalence relation (13). Consider $s^{(0,1,2)}$ and let $(A_2, A_1, A_0)$ and $(\tilde{A}_2, \tilde{A}_1, \tilde{A}_0)$ be locally equivalent. Let the columns of $\tilde{Z}_1$ form a basis for $\mathcal{N}(\tilde{A}_2^T)$, and let the columns of $\tilde{Z}_3$ form a basis for $\mathcal{N}(\tilde{A}_2^T) \cap \mathcal{N}(\tilde{A}_1^T)$. Then, from (13) it follows that the columns of $Z_1 := P^T \tilde{Z}_1$ form a basis for $\mathcal{N}(A_2^T)$. Since for any $z \in \tilde{Z}_3$, and any matrix $R_1$ of appropriate size

$$Q^T A_2^T P^T z = 0, \quad Q^T A_1^T P^T z + 2R_1^T A_2^T P^T z = 0,$$

if and only if

$$A_2^T P^T z = 0, \quad A_1^T P^T z = 0,$$

it follows that the columns of $Z_3 := P^T \tilde{Z}_3$ form a basis for $\mathcal{N}(A_2^T) \cap \mathcal{N}(A_1^T)$. Thus, the invariance of $s^{(0,1,2)}$ follows from

$$\begin{aligned}
\tilde{s}^{(0,1,2)} &= \dim \Big( \mathcal{R}(\tilde{A}_2^T) \cap \mathcal{R}(\tilde{A}_1^T \tilde{Z}_1) \cap \mathcal{R}(\tilde{A}_0^T \tilde{Z}_3) \Big) \\
&= \dim \Big( \mathcal{R}(Q^T A_1^T P^T) \cap \mathcal{R}(Q^T A_0^T P^T \tilde{Z}_1 + 2R_1^T A_2^T P^T \tilde{Z}_1) \\
&\qquad \cap \mathcal{R}(Q^T A_0^T P^T \tilde{Z}_3 + R_1^T A_1^T P^T \tilde{Z}_3 + R_2^T A_2^T P^T \tilde{Z}_3) \Big) \\
&= \dim \Big( \mathcal{R}(Q^T A_2^T P^T) \cap \mathcal{R}(Q^T A_1^T P^T \tilde{Z}_1) \cap \mathcal{R}(Q^T A_0^T P^T \tilde{Z}_3) \Big) \\
&= \dim \Big( \mathcal{R}(A_2^T P^T) \cap \mathcal{R}(A_1^T P^T \tilde{Z}_1) \cap \mathcal{R}(A_0^T P^T \tilde{Z}_3) \Big) \\
&= \dim \big( \mathcal{R}(A_2^T) \cap \mathcal{R}(A_1^T Z_1) \cap \mathcal{R}(A_0^T Z_3) \big) \\
&= s^{(0,1,2)}.
\end{aligned}$$

Similarly, the invariance of the other quantities in (27) can be proved. □

For triples $(A_2(t), A_1(t), A_0(t))$ of matrix-valued functions we can then calculate, based on Lemma 12, the characteristic quantities in (27) for $(A_2(\hat{t}), A_1(\hat{t}), A_0(\hat{t}))$ at any fixed value $\hat{t} \in \mathbb{I}$. We obtain functions

$$r, a, s^{(0,1,2)}, s^{(0,1)}, d^{(1)}, s^{(1,2)}, s^{(0,2)}, d^{(2)}, u, v : \mathbb{I} \to \mathbb{N}_0.$$

For the triple $(A_2(t), A_1(t), A_0(t))$ of matrix-valued functions, to derive a condensed form which is similar in form to the condensed form (25) for the matrix triple $(A_2, A_1, A_0)$, we then need the following *regularity assumptions* for the triple $(A_2(t), A_1(t), A_0(t))$ in $\mathbb{I}$:

$$\begin{array}{c} r(t) \equiv r,\ a(t) \equiv a,\ s^{(0,1,2)}(t) \equiv s^{(0,1,2)},\ s^{(0,1)}(t) \equiv s^{(0,1)}, \\ d^{(1)}(t) \equiv d^{(1)},\ s^{(1,2)}(t) \equiv s^{(1,2)},\ s^{(0,2)}(t) \equiv s^{(0,2)}. \end{array} \tag{30}$$

By (27) and (30), it then follows that $d^{(2)}(t), u(t), v(t)$ are also constant on $\mathbb{I}$.

Conditions (30) imply that the sizes of the blocks in the condensed form (25) do not depend on $t \in \mathbb{I}$. We then obtain the following global condensed form for triples of matrix-valued functions via global equivalence transformations (12). For convenience of expression, in the condensed form and the following proof, we drop the subscripts of the blocks and omit the argument $t$ unless they are needed for clarification.

**Lemma 13** *Let $A_2(t), A_1(t), A_0(t) \in \mathcal{C}(\mathbb{I}, \mathbb{C}^{m,n})$ be sufficiently smooth, and suppose that the conditions (30) hold for the local characteristic values of $(A_2(t), A_1(t), A_0(t))$. Then, $(A_2(t), A_1(t), A_0(t))$ is globally equivalent to a matrix-valued triple $(\hat{A}_2(t), \hat{A}_1(t), \hat{A}_0(t))$ of the condensed form*

$$\left( \begin{bmatrix} I_{s^{(0,1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^{(1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{s^{(0,2)}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{d^{(2)}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & I_{s^{(0,1)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{d^{(1)}} & 0 & 0 \\ I_{s^{(0,1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^{(1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{31}$$

$$
\left.\left[\begin{array}{cccccccc}
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & I_a & 0 \\
0 & 0 & 0 & 0 & I_{s^{(0,1)}} & 0 & 0 & 0 \\
0 & 0 & I_{s^{(0,2)}} & 0 & 0 & 0 & 0 & 0 \\
I_{s^{(0,1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]\right)
\begin{array}{l}
s^{(0,1,2)} \\ s^{(1,2)} \\ s^{(0,2)} \\ d^{(2)} \\ s^{(0,1)} \\ d^{(1)} \\ s^{(0,1,2)} \\ s^{(1,2)} \\ a \\ s^{(0,1)} \\ s^{(0,2)} \\ s^{(0,1,2)} \\ v
\end{array} .
$$

*All blocks except the identity matrices in (31) are again matrix-valued functions on $\mathbb{I}$.*

*Proof.* The proof of Lemma 13 is given in Appendix D. □

Note that the block in position $(5,4)$ of $\hat{A}^1(t)$ in (31) satisfies $A^1_{5,4}(t) \equiv 0$, whereas the corresponding block $A^1_{5,4}$ in (25) may be a nonzero matrix, which is the major difference between condensed forms (31) and (25). This difference is due to the global equivalence which allows to eliminate $A^1_{5,4}(t)$ by solving an initial value problem for a linear ordinary differential equations. If we consider the associated system of DAEs

$$\hat{A}_2(t)\ddot{y}(t) + \hat{A}_1(t)\dot{y}(t) + \hat{A}_0(t)y(t) = \hat{f}(t), \tag{32}$$

then we obtain the equations

$$
\begin{array}{ll}
(a) & \ddot{y}_1(t) + \sum_{i=3,4,7,8} A^1_{1,i}(t)\dot{y}_i(t) + \sum_{i=2,4,6,8} A^0_{1,i}(t)y_i(t) = \hat{f}_1(t), \\
(b) & \ddot{y}_2(t) + \sum_{i=3,4,7,8} A^1_{2,i}(t)\dot{y}_i(t) + \sum_{i=2,4,6,8} A^0_{2,i}(t)y_i(t) = \hat{f}_2(t), \\
(c) & \ddot{y}_3(t) + \sum_{i=3,4,7,8} A^1_{3,i}(t)\dot{y}_i(t) + \sum_{i=2,4,6,8} A^0_{3,i}(t)y_i(t) = \hat{f}_3(t), \\
(d) & \ddot{y}_4(t) + \sum_{i=3,4,7,8} A^1_{4,i}(t)\dot{y}_i(t) + \sum_{i=2,4,6,8} A^0_{4,i}(t)y_i(t) = \hat{f}_4(t), \\
(e) & \dot{y}_5(t) + \sum_{i=2,4,6,8} A^0_{5,i}(t)y_i(t) = \hat{f}_5(t), \\
(f) & \dot{y}_6(t) + \sum_{i=2,4,6,8} A^0_{6,i}(t)y_i(t) = \hat{f}_6(t), \\
(g) & \dot{y}_1(t) + \sum_{i=2,4,6,8} A^0_{7,i}(t)y_i(t) = \hat{f}_7(t), \\
(h) & \dot{y}_2(t) + \sum_{i=2,4,6,8} A^0_{8,i}(t)y_i(t) = \hat{f}_8(t), \\
(i) & y_7(t) = \hat{f}_9(t), \\
(j) & y_5(t) = \hat{f}_{10}(t), \\
(k) & y_3(t) = \hat{f}_{11}(t), \\
(l) & y_1(t) = \hat{f}_{12}(t), \\
(m) & 0 = \hat{f}_{13}(t).
\end{array} \tag{33}
$$

Immediately we recognize the coupling (strangeness due to $A_2, A_1, A_0$) between the algebraic equations (33-l) and the differential equations (33-g) and (33-a), the coupling (strangeness due to $A_2, A_0$) between the algebraic equations (33-k) and the differential equations (33-c), the coupling (strangeness due to $A_1, A_0$) between the algebraic equations (33-j) and the differential equations (33-e), the coupling (strangeness due to $A_2, A_1$) between the differential equations (33-h) and the differential equations (33-b), and the possible coupling between the algebraic equations (33-i) and the differential equations (33-a)–(33-d).

Similar to the case of matrix pairs, we

1. insert the derivative of equation (33-l) in (33-g) to eliminate $\dot{y}_1(t)$;
2. insert the second derivative of equation (33-l) in (33-a) to eliminate $\ddot{y}_1(t)$;
3. insert the second derivative of equation (33-k) in (33-c) to eliminate $\ddot{y}_3(t)$;
4. insert the derivative of equation (33-j) in (33-e) to eliminate $\dot{y}_5(t)$;
5. insert the derivative of equation (33-h) in (33-b) to eliminate $\ddot{y}_2(t)$; and
6. insert the derivative or if possible only parts of equation (33-i) in (33-a)–(33-d) to eliminate possibly existent $\dot{y}_7(t)$.

These steps correspond to transforming the systems (32) into an equivalent second order system of DAEs

$$A_2^{\langle 1\rangle}(t)\ddot{y}(t) + A_1^{\langle 1\rangle}(t)\dot{y}(t) + A_0^{\langle 1\rangle}(t)y(t) = f^{\langle 1\rangle}(t), \tag{34}$$

with $(A_2^{\langle 1\rangle}(t), A_1^{\langle 1\rangle}(t), A_0^{\langle 1\rangle}(t); f^{\langle 1\rangle})$ being of the following form

$$\left(\begin{bmatrix} \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{d^{(2)}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$
$$\begin{bmatrix} 0 & 0 & * & * & 0 & 0 & \mathbf{0} & * \\ 0 & * & * & * & 0 & * & \mathbf{0} & * \\ 0 & 0 & * & * & 0 & 0 & \mathbf{0} & * \\ 0 & 0 & * & * & 0 & 0 & \mathbf{0} & * \\ 0 & 0 & 0 & * & \mathbf{0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{d^{(1)}} & 0 & 0 \\ \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^{(1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{35}$$

$$
\begin{bmatrix}
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & I_a & 0 \\
0 & 0 & 0 & 0 & I_{s^{(0,1)}} & 0 & 0 & 0 \\
0 & 0 & I_{s^{(0,2)}} & 0 & 0 & 0 & 0 & 0 \\
I_{s^{(0,1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
;
\begin{bmatrix}
\hat{f}_1(t) - \ddot{\hat{f}}_{12}(t) \\
\hat{f}_2(t) - \dot{\hat{f}}_8(t) \\
\hat{f}_3(t) - \ddot{\hat{f}}_{11}(t) \\
\hat{f}_4(t) \\
\hat{f}_5(t) - \dot{\hat{f}}_{10}(t) \\
\hat{f}_6(t) \\
\hat{f}_7(t) - \dot{\hat{f}}_{12}(t) \\
\hat{f}_8(t) \\
\hat{f}_9(t) \\
\hat{f}_{10}(t) \\
\hat{f}_{11}(t) \\
\hat{f}_{12}(t) \\
\hat{f}_{13}(t)
\end{bmatrix}
\Bigg)
$$

It should be noted that the procedure of using derivatives of some equations to eliminate in coefficients other equations only involves the absolutely necessary derivatives of the right-hand side $\hat{f}(t)$. Moreover, after the transformation from the system (32) to the system (34), the solution sets of the two systems are still the same.

We then continue the *index reduction procedure* as follows. For the triple $(A_2^{\langle 1\rangle}(t), A_1^{\langle 1\rangle}(t), A_0^{\langle 1\rangle}(t))$ in (35), we can again transform to the condensed form (31), and then apply Steps (1)–(6) to pass it to the form (35). Proceeding inductively, we obtain a sequence of triples of matrix functions (matrices) $(A_2^{\langle i\rangle}(t), A_1^{\langle i\rangle}(t), A_0^{\langle i\rangle}(t))$, $i \in \mathbb{N}_0$, where $(A_2^{\langle 0\rangle}(t), A_1^{\langle 0\rangle}(t), A_0^{\langle 0\rangle}(t)) = (A_2(t), A_1(t), A_0(t))$ and $(A_2^{\langle i+1\rangle}(t), A_1^{\langle i+1\rangle}(t), A_0^{\langle i+1\rangle}(t))$ is derived from $(A_2^{\langle i\rangle}(t), A_1^{\langle i\rangle}(t), A_0^{\langle i\rangle}(t))$ by bringing it into the form (31) and then applying Steps (1)–(6) again. In the $j-$th step we assume that

$$
\begin{aligned}
&s_j^{(0,1,2)}(t), s_j^{(1,2)}(t), s_j^{(0,2)}(t), d_j^{(2)}(t), s_j^{(0,1)}(t), d_j^{(1)}(t), \\
&a_j(t), v_j(t), u_j(t) \text{ are constant in } \mathbb{I}.
\end{aligned}
\tag{36}
$$

Comparing $\hat{A}_2(t)$ in (31) with $A_2^{\langle 1\rangle}(t)$ in (35), we have

$$
\begin{aligned}
\operatorname{rank}(A_2^{\langle 1\rangle}(t)) &= \operatorname{rank}(\hat{A}_2(t)) - s_{\langle 0\rangle}^{(0,1,2)} - s_{\langle 0\rangle}^{(0,2)} - s_{\langle 0\rangle}^{(1,2)} \\
&= \operatorname{rank}(A_2^{\langle 0\rangle}(t)) - s_{\langle 0\rangle}^{(0,1,2)} - s_{\langle 0\rangle}^{(0,2)} - s_{\langle 0\rangle}^{(1,2)},
\end{aligned}
\tag{37}
$$

where $s_{\langle 0\rangle}^{(0,1,2)}$, $s_{\langle 0\rangle}^{(0,2)}$, and $s_{\langle 0\rangle}^{(1,2)}$ denote the strangeness due to $A_2^{\langle 0\rangle}(t), A_1^{\langle 0\rangle}(t), A_0^{\langle 0\rangle}(t)$, the strangeness due to $A_2^{\langle 0\rangle}(t), A_0^{\langle 0\rangle}(t)$, and the strangeness due to $A_2^{\langle 0\rangle}(t), A_1^{\langle 0\rangle}(t)$, respectively. Since after the differentiation-and-elimination step 4, equation (33-j) becomes an un-coupled purely algebraic equation, it follows that

$$
\operatorname{rank}(A_0) \geq a_{\langle 1\rangle} \geq \left(a_{\langle 0\rangle} + s_{\langle 0\rangle}^{(0,1)}\right),
\tag{38}
$$

where $a_{\langle 1\rangle}$, $a_{\langle 0\rangle}$, and $s_{\langle 0\rangle}^{(0,1)}$ denotes the size of the algebraic part of $(A_2^{\langle 1\rangle}, A_1^{\langle 1\rangle}, A_0^{\langle 1\rangle})$, the size of the algebraic part of $(A_2^{\langle 0\rangle}, A_1^{\langle 0\rangle}, A_0^{\langle 0\rangle})$, and the strangeness due to $A_1^{\langle 0\rangle}, A_0^{\langle 0\rangle}$, respectively. Hence, the relations in (37) and (38) guarantee that after a finite number (say $\mu(t) \equiv \mu$) of steps, the strangeness $s_{\langle\mu\rangle}^{(0,1,2)}$ due to $A_2(t), A_1(t), A_0(t)$, the strangeness $s_{\langle\mu\rangle}^{(0,2)}$ due to $A_2(t), A_0(t)$, the strangeness $s_{\langle\mu\rangle}^{(0,1)}$ due to $A_1(t), A_0(t)$, and the strangeness $s_{\langle\mu\rangle}^{(1,2)}$ due to $A_2(t), A_1(t)$ must vanish. We call $\mu$ the *strangeness-index* or *s-index* of the second order system of DAEs and we call the final equivalent second order system of DAEs *strangeness-free.*

**Theorem 14** *Consider the system (29), suppose that (36) holds and let $\mu$ be the s-index of (29). If $f(t) \in \mathcal{C}^\mu(\mathbb{I}, \mathbb{C}^m)$, then system (29) is equivalent (in the sense that there is a one-to-one correspondence between the solution sets) to a system of second order differential-algebraic equations $\tilde{A}_2(t)\ddot{\tilde{x}}(t) + \tilde{A}_1(t)\dot{\tilde{x}}(t) + \tilde{A}_0(t)\tilde{x}(t) = \tilde{f}(t)$ of the form*

$$
\begin{aligned}
(a)\quad & \ddot{\tilde{x}}_1(t) + \tilde{A}^1_{1,1}(t)\dot{\tilde{x}}_1(t) + \tilde{A}^1_{1,4}(t)\dot{\tilde{x}}_4(t) + \\
& \qquad\qquad \tilde{A}^0_{1,1}(t)\tilde{x}_1(t) + \tilde{A}^0_{1,2}(t)\tilde{x}_2(t) + \tilde{A}^0_{1,4}(t)\tilde{x}_4(t) = \tilde{f}_1(t), \\
(b)\quad & \dot{\tilde{x}}_2(t) + \tilde{A}^0_{2,1}(t)\tilde{x}_1(t) + \tilde{A}^0_{2,2}(t)\tilde{x}_2(t) + \tilde{A}^0_{2,4}(t)\tilde{x}_4(t) = \tilde{f}_2(t), \\
(c)\quad & \tilde{x}_3(t) = \tilde{f}_3(t), \\
(d)\quad & 0 = \tilde{f}_4(t),
\end{aligned}
\tag{39}
$$

*where the inhomogeneity $\tilde{f}(t) := \left[\tilde{f}_1^T(t), \ldots, \tilde{f}_4^T(t)\right]^T$ is determined by $f^{(0)}(t)$, $\ldots$, $f^{(\mu)}(t)$. In particular, $d_\mu^{(2)}(t) \equiv d_\mu^{(2)}$, $d_\mu^{(1)}(t) \equiv d_\mu^{(1)}$, and $a_\mu(t) \equiv a_\mu$ are the number of second order differential, first order differential, and algebraic components of the unknown $\tilde{x}(t) := \left[\tilde{x}_1^T(t), \ldots, \tilde{x}_4^T(t)\right]^T$ in (39-a), (39-b), and (39-c), respectively, while $u_\mu(t) \equiv u_\mu$ is the dimension of the undetermined vector $\tilde{x}_4(t)$ in (39-a) and (39-b), and $v_\mu(t) \equiv v_\mu$ is the number of conditions in (39-d).*

*Proof.* Inductively transforming $(A_2(t), A_1(t), A_0(t))$ to the condensed form (31), and then applying steps (1)–(6) until $s_\mu^{(0,1,2)} = s_\mu^{(1,2)} = s_\mu^{(0,2)} = s_\mu^{(0,1)} = 0$ yields a triple of matrix functions $(\tilde{A}_2(t), \tilde{A}_1(t), \tilde{A}_0(t))$ of the form

$$
\left(\begin{bmatrix} I_{d_\mu^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}^1_{1,1}(t) & 0 & 0 & \tilde{A}^1_{1,4}(t) \\ 0 & I_{d_\mu^1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\right.
$$
$$
\left.\begin{bmatrix} \tilde{A}^0_{1,1}(t) & \tilde{A}^0_{1,2}(t) & 0 & \tilde{A}^0_{1,4}(t) \\ \tilde{A}^0_{2,1}(t) & \tilde{A}^0_{2,2}(t) & 0 & \tilde{A}^0_{2,4}(t) \\ 0 & 0 & I_{a_\mu} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right),
$$

with block size $v_\mu$ for the last block row and $u_\mu$ for the last block column. We know that the transformation from $(A_2(t), A_1(t), A_0(t))$ to $(\hat{A}_2(t), \hat{A}_1(t), \hat{A}_0(t))$ in the condensed form (31) establishes a one-to-one correspondence between the solution sets of the two corresponding systems of DAEs. Hence, for any solution

$x(t)$ of the system (29) (if existent), there exists a solution of the system (39) such that

$$x(t) = \tilde{Q}(t)\tilde{x}(t),$$

where $\tilde{Q}(t)$ is a nonsingular matrix function, and vice versa. □

**Remark 15** In order to derive the condensed form (31) we have to assume the regularity condition (36) in each step of the index reduction procedure. This seems a rather strong assumption. But it follows from a result for matrix pairs in [20] (see also [6]) that for a closed interval $\mathbb{I}$ and sufficiently smooth coefficients functions $A_i(t)$, there exist open intervals $\mathbb{I}_j$, $j \in \mathbb{N}$, such that

$$\overline{\bigcup_{j\in\mathbb{N}} \mathbb{I}_j} = \mathbb{I}, \quad \mathbb{I}_i \cap \mathbb{I}_j = \emptyset \text{ for } i \neq j, \tag{40}$$

where (36) holds, i.e., the s-index $\mu(t)$ is defined on a dense subset of $\mathbb{I}$. □

**Example 16** Consider again the system (3) of second order DAEs. By the described index reduction procedure, system (3) can be equivalently transformed to the following strangeness-free system

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \ddot{\tilde{x}}(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \dot{\tilde{x}}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tilde{x}(t) = \tilde{f}(t),$$

where $\tilde{x}(t) = [x_2(t), x_1(t)]^T$, and $\tilde{f}(t) = [f_1(t) - \dot{f}_2(t) - \ddot{f}_2(t), f_2(t)]^T$. Hence, by Theorem 14, the s-index of system (3) is $\mu = 2$.

In contrast to this, the first order system (5) can be equivalently transformed to the strangeness-free system

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{\tilde{y}}(t) + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tilde{y}(t) = \begin{bmatrix} f_2(t) \\ f_1(t) - \dot{f}_2(t) - \ddot{f}_2(t) \\ \dot{f}_2(t) \\ \dot{f}_1(t) - \ddot{f}_2(t) - f_2^{(3)}(t) \end{bmatrix},$$

where $\tilde{y}(t) = [x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t)]^T$. By Theorem 14, the s-index of the first order version (5) is $\mu = 3$, which is larger by 1 than the s-index of the original second order system (3).

We can now answer the questions concerning the existence and uniqueness of solutions and consistency of initial conditions.

**Corollary 17** *Under the assumptions of Theorem 14 the following statements hold.*

1. *The system (29) has a continuous solution if and only if one of the following two cases happens.*

   *(i)* $v_\mu = 0$.

*(ii) If $v_\mu > 0$, then the $v_\mu$ functional consistency conditions*

$$\tilde{f}_4(t) = 0$$

*are satisfied.*

2. *If the system (29) has a continuous solution, then it is uniquely solvable without providing any initial conditions if and only if the conditions*

$$d_\mu^{(2)} = d_\mu^{(1)} = u_\mu = 0$$

*hold.*

3. *If the system (29) is solvable, then initial conditions (23) are consistent if and only if one of the following two cases happens.*

*(i) $a_\mu = 0$.*

*(ii) If $a_\mu > 0$, then the $a_\mu$ conditions*

$$\tilde{x}_3(t_0) = \tilde{f}_3(t_0),\ \dot{\tilde{x}}_3(t_0) = \left.\frac{d\tilde{f}_3(t)}{dt}\right|_{t_0+}$$

*are implied by (23).*

4. *If the initial value problem (29)–( 23) is solvable, then it is uniquely solvable if and only if*

$$u_\mu = 0$$

*holds.*

An immediate Corollary of Theorem 14 is the identification of those second derivatives of variables that can be replaced to obtain a first order system without changing the smoothness requirements.

**Corollary 18** *Under the assumptions of Theorem 14, let $\mu$ be the s-index of the matrix triple associated with the system (29) and let $f(t) \in \mathcal{C}^\mu(\mathbb{I}, \mathbb{C}^m)$. Then, the solution set of system (29) is in one-to-one correspondence (without further smoothness requirements) to the partial solution set given by the components $\tilde{x}_1(t)$–$\tilde{x}_4(t)$ of the system of first order differential-algebraic equations*

$$\begin{array}{ll}(a) & \dot{\tilde{x}}_5(t) + \tilde{A}^1_{1,1}\dot{\tilde{x}}_1(t) + \tilde{A}^1_{1,4}\dot{\tilde{x}}_4(t) + \\ & \qquad \tilde{A}^0_{1,1}\tilde{x}_1(t) + \tilde{A}^0_{1,2}\tilde{x}_2(t) + \tilde{A}^0_{1,4}\tilde{x}_4(t) = \tilde{f}_1(t), \\ (b) & \dot{\tilde{x}}_2(t) + \tilde{A}^0_{2,1}\tilde{x}_1(t) + \tilde{A}^0_{2,2}\tilde{x}_2(t) + \tilde{A}^0_{2,4}\tilde{x}_4(t) = \tilde{f}_2(t), \\ (c) & \tilde{x}_3(t) = \tilde{f}_3(t), \\ (d) & 0 = \tilde{f}_4(t), \\ (e) & \dot{\tilde{x}}_1 = \tilde{x}_5. \end{array} \tag{41}$$

*Proof.* The proof follows immediately from (39) and it is clear that no extra smoothness requirements are needed. □

**Example 19** Consider again the system (7) in Example 2 which already is in the condensed form (39) and hence, by Corollary 18, we only introduce the new variable $x_3 = \dot{x}_1$. After some permutations we obtain the system

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \dot{x}_3 \\ \dot{x}_1 \\ \hline \dot{x}_2 \end{array}\right] + \left[\begin{array}{cc|c} 0 & 0 & -1 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} x_3 \\ x_1 \\ \hline x_2 \end{array}\right] = \left[\begin{array}{c} f_1(t) \\ 0 \\ \hline f_2(t) \end{array}\right], \tag{42}$$

which is strangeness-free and has the solution

$$\begin{array}{l} x_1(t) = \int_{t_0}^{t} \int_{t_0}^{s} f_1(\tau) + f_2(\tau)\ d\tau\ ds + c_1 t + c_2 \\ x_2(t) = f_2(t), \\ x_3(t) = \int_{t_0}^{s} f_1(\tau) + f_2(\tau)\ d\tau + c_1. \end{array} \tag{43}$$

□

After having derived the results for second order systems the extension to arbitrary systems of order $l$ is obvious. We present these results in the next section.

# 4 Linear $l$-th Order DAEs

As we have seen in Section 3, we can get a condensed form via strong equivalence transformations for matrix triples. Using induction, this form can be extended to $(l+1)$-tuples $(A_l(t), \ldots, A_1(t), A_0(t))$ of matrix functions and we obtain for (1) new sets of invariant quantities, i.e., those that characterize the algebraic part, 1st order, 2nd order, …, and $l$-th order differential parts, and strangeness parts due to each two, each three, …, each $l$, and $l+1$ matrices out of $A_l(t), \ldots, A_1(t)$, and $A_0(t)$, respectively.

Then, based on the condensed form for $(l+1)$-tuples of matrices functions, we can write down the system of differential-algebraic equations after strong equivalence transformations. Analogous to the treatment of systems of second order, we can design steps of inserting derivatives of some equations into others to decouple those equations that are coupled to each other to reduce it to a simpler but equivalent system, which we can again transform to the condensed form. Inductively, by this procedure we obtain a sequence of $(l+1)$-tuples of matrix functions, and after a finite number (called again $\mu$) of steps, we obtain that all strangeness parts of the corresponding system have vanished, in other words, in the final form the system becomes strangeness-free.

Here we only state the essential results and its main consequences.

**Theorem 20** *Let $f(t) \in \mathcal{C}^{\mu}(\mathbb{I}, \mathbb{C}^m)$. Then, under appropriate constant rank assumptions (analogous to those in (36) for second order systems), system (1) is equivalent (in the sense that there is a one-to-one correspondence between the solution sets) to a system of $l$-th order differential-algebraic equations*

$$A^{\langle l \rangle}(t)\tilde{x}^{(l)}(t) + A^{\langle l-1 \rangle}(t)\tilde{x}^{(l-1)}(t) + \cdots + A^{\langle 0 \rangle}(t)\tilde{x}(t) = \tilde{f}(t)$$

*of the form*

$$
\begin{aligned}
&(1) && \tilde{x}_1^{(l)}(t) + \sum_{i=0}^{l-1}\sum_{j=i}^{l-1} A_{1,l-j}^{\langle i\rangle}(t)\tilde{x}_{l-j}^{(i)}(t) \\
&&& + \sum_{i=0}^{l-1} A_{1,l+2}^{\langle i\rangle}(t)\tilde{x}_{l+2}^{(i)}(t) = \tilde{f}_1(t), \\
&(2) && \tilde{x}_2^{(l-2)}(t) + \sum_{i=0}^{l-2}\sum_{j=i}^{l-2} A_{2,l-1-j}^{\langle i\rangle}(t)\tilde{x}_{l-1-j}^{(i)}(t) \\
&&& + \sum_{i=0}^{l-2}\left(A_{2,1}^{\langle i\rangle}(t)\tilde{x}_1^{(i)}(t) + A_{2,l+2}^{\langle i\rangle}(t)\ \tilde{x}_{l+2}^{(i)}(t)\right) = \tilde{f}_2(t), \\
&\vdots && \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (44) \\
&(k) && \tilde{x}_{l-k+1}^{(l-k+1)}(t) + \sum_{i=0}^{l-k}\sum_{j=i}^{l-k} A_{k,l-k+1-j}^{\langle i\rangle}(t)\tilde{x}_{l-k+1-j}^{(i)}(t) \\
&&& + \sum_{i=0}^{l-k}\left(\sum_{j=1}^{k} A_{k,j}^{\langle i\rangle}(t)\tilde{x}_1^{(i)}(t) + A_{k,l+2}^{\langle i\rangle}(t)\tilde{x}_{l+2}^{(i)}(t)\right) = \tilde{f}_k(t), \\
&&& \qquad\qquad\qquad\qquad (1 \le k \le l) \\
&\vdots && \vdots \\
&(l+1) && \tilde{x}_{l+1}(t) = \tilde{f}_{l+1}(t), \\
&(l+2) && 0 = \tilde{f}_{l+2}(t),
\end{aligned}
$$

*where* $A_{p,q}^{\langle i\rangle}(t)$, $1 \le p \le (l+2)$, $1 \le q \le (l+2)$, *denotes a sub-matrix of* $A^{\langle i\rangle}(t)$, *and the inhomogeneity* $\tilde{f}(t) := \left[\tilde{f}_1^T(t), \dots, \tilde{f}_{l+2}^T(t)\right]^T$ *is determined by* $f^{(0)}(t)$, $\dots$, $f^{(\mu)}(t)$. *In particular,* $d_\mu^{(l)}$, $\dots$, $d_\mu^{(1)}$, *and* $a_\mu$ *are the number of l-th order differential, …, first order differential, algebraic components of the unknown* $\tilde{x}(t) := \left[\tilde{x}_1^T(t), \dots, \tilde{x}_{l+2}^T(t)\right]^T$ *in (44-1), …, (44-(l+1)), respectively, while* $u_\mu$ *is the dimension of the undetermined vector* $\tilde{x}_{l+2}(t)$ *in (44-1), …, (44-l), and* $v_\mu$ *is the number of conditions in (44-(l+2)).*

*Proof.* The proof is analogous to the proof of Theorem 14 and follows by induction. □

**Corollary 21** *Under the assumption of Theorem 20, the following statements hold.*

1. *The system (1) is solvable if and only if one of the following two cases happens.*

   *(i)* $v_\mu = 0$.

   *(ii) If* $v_\mu > 0$, *then the* $u_\mu$ *functional consistency conditions*

   $$\tilde{f}_{l+2}(t) = 0$$

   *are satisfied.*

2. *If the system (1) is solvable, then it is uniquely solvable without providing any initial condition if and only if the conditions*

$$d_\mu^{(l)} = \cdots = d_\mu^{(2)} = d_\mu^{(1)} = u_\mu = 0$$

*hold.*

3. *If the system (1) is solvable, then initial conditions (2) are consistent if and only if one of the following two cases happens.*

   *(i)* $a_\mu = 0$.

   *(ii) If* $a_\mu(t) > 0$*, then the* $a_\mu$ *conditions*

$$\begin{array}{l} \tilde{x}_{l+1}(t_0) = \tilde{f}_{l+1}(t_0), \\ \dot{\tilde{x}}_{l+1}(t_0) = \frac{d\tilde{f}_{l+1}(t)}{dt}\Big|_{t_0+}, \ldots, \tilde{x}_{l+1}^{(l-1)}(t_0) = \tilde{f}_{l+1}^{(l-1)}(t)\Big|_{t_0+} \end{array}$$

   *are implied by (1).*

4. *If the initial value problem (1)–(2) is solvable, then it is uniquely solvable if and only if*

$$u_\mu = 0$$

*holds.*

*Proof.* The proof is analogous to the proof of Corollary 17. □

**Corollary 22** *Under the assumptions of Theorem 20, let* $\mu$ *be the s-index of the matrix tuple associated with the system (1) and let* $f(t) \in \mathcal{C}^\mu(\mathbb{I}, \mathbb{C}^m)$*. Then, the solution set of system (1) is in one-to-one correspondence (without further smoothness requirements) to the partial solution set given by the components* $\tilde{x}_1(t)$*–*$\tilde{x}_{l+2}(t)$ *of the system of first order differential-algebraic equations that is obtained by replacing in (44) the derivatives* $\tilde{x}_{l-k+1}^{(l-k+1)}(t)$ *by new variables* $v_{l-k+1}$*,* $k = 1, \ldots, l$*.*

*Proof.* The proof follows as in the case of matrix triples. □

## 5 Constant coefficient systems and matrix polynomials

For constant coefficient systems we obtain some further consequences.

Recalling the close relationship of a constant coefficient system to the associated matrix polynomials, it is a natural question to ask what is the relation between the matrix polynomials associated with the system (32) and (34)? The following lemma answers this question for the case of quadratic matrix polynomials.

**Lemma 23** *Consider the quadratic matrix polynomials* $\hat{A}(\lambda) = \lambda^2 \hat{A}_l + \lambda \hat{A}_1 + \hat{A}_0$*, and* $A^{\langle 1 \rangle}(\lambda) = \lambda^2 A_2^{\langle 1 \rangle} + \lambda A_1^{\langle 1 \rangle} + A_0^{\langle 1 \rangle}$*, where* $\hat{A}_2$*,* $\hat{A}_1$ *and* $\hat{A}_0$ *are as in (32), and* $A_2^{\langle 1 \rangle}$*,* $A_1^{\langle 1 \rangle}$ *and* $A_0^{\langle 1 \rangle}$ *are as in (34). Then,*

$$A^{\langle 1 \rangle}(\lambda) = E(\lambda)\hat{A}(\lambda),$$

*where* $E(\lambda)$ *is a* unimodular matrix polynomial*, i.e., the determinant of* $E(\lambda)$ *is a nonzero constant.*

*Proof.* In terms of elementary row operations for matrix polynomials, the procedure of inserting the derivatives of one equation into another in Steps (1)–(6) corresponds to premultiplying $\hat{A}(\lambda)$ by an elementary matrix polynomial $E_1(\lambda)$, premultiplying $E_1(\lambda)\hat{A}(\lambda)$ by an elementary matrix polynomial $E_2(\lambda)$, premultiplying $E_2(\lambda)E_1(\lambda)\hat{A}(\lambda)$ by an elementary matrix polynomial $E_3(\lambda)$, premultiplying $E_3(\lambda)\cdots\ E_1(\lambda)\hat{A}(\lambda)$ by an elementary matrix polynomial $E_4(\lambda)$, premultiplying $E_4(\lambda)\cdots E_1(\lambda)\hat{A}(\lambda)$ by an elementary matrix polynomial $E_5(\lambda)$, and premultiplying $E_5(\lambda)\cdots E_1(\lambda)\hat{A}(\lambda)$ by an elementary matrix polynomial $E_6(\lambda)$, respectively, where

$$E_1(\lambda) = \begin{bmatrix} I & & & & & \\ & \ddots & & & & \\ & & I & \cdots & -\lambda I & \\ & & & \ddots & \vdots & \\ & & & & I & \\ & & & & & I \end{bmatrix} \begin{matrix} s^{(0,1,2)} \\ \vdots \\ s^{(0,1,2)} \\ \vdots \\ s^{(0,1,2)} \\ v \end{matrix},$$

$$E_2(\lambda) = \begin{bmatrix} I & \cdots & & \cdots & -\lambda^2 I & \\ & \ddots & & & \vdots & \\ & & I & & & \\ & & & \ddots & \vdots & \\ & & & & I & \\ & & & & & I \end{bmatrix} \begin{matrix} s^{(0,1,2)} \\ \vdots \\ s^{(0,1,2)} \\ \vdots \\ s^{(0,1,2)} \\ v \end{matrix},$$

$$E_3(\lambda) = \begin{bmatrix} I & & & & & & \\ & \ddots & & & & & \\ & & I & \cdots & -\lambda^2 I & & \\ & & & \ddots & \vdots & & \\ & & & & I & & \\ & & & & & \ddots & \\ & & & & & & I \end{bmatrix} \begin{matrix} s^{(0,1,2)} \\ \vdots \\ s^{(0,2)} \\ \vdots \\ s^{(0,2)} \\ \vdots \\ v \end{matrix},$$

$$E_4(\lambda) = \begin{bmatrix} I & & & & & & \\ & \ddots & & & & & \\ & & I & \cdots & -\lambda I & & \\ & & & \ddots & \vdots & & \\ & & & & I & & \\ & & & & & \ddots & \\ & & & & & & I \end{bmatrix} \begin{matrix} s^{(0,1,2)} \\ \vdots \\ s^{(0,1)} \\ \vdots \\ s^{(0,1)} \\ \vdots \\ v \end{matrix},$$

$$E_5(\lambda) = \begin{bmatrix} I & & & & & \\ & I & \cdots & -\lambda I & & \\ & & \ddots & \vdots & & \\ & & & I & & \\ & & & & \ddots & \\ & & & & & I \end{bmatrix} \begin{matrix} s^{(0,1,2)} \\ s^{(1,2)} \\ \vdots \\ s^{(1,2)} \\ \vdots \\ v \end{matrix},$$

$$E_6(\lambda) = \begin{bmatrix} I & \cdots & & \cdots & -\lambda A^1_{1,7} & & \\ & I & \cdots & \cdots & -\lambda A^1_{2,7} & & \\ & & I & \cdots \;\cdots & -\lambda A^1_{3,7} & & \\ & & & I \;\; \cdots & -\lambda A^1_{4,7} & & \\ & & & \ddots & \vdots & & \\ & & & & I & & \\ & & & & & \ddots & \\ & & & & & & I \end{bmatrix} \begin{matrix} s^{(0,1,2)} \\ s^{(1,2)} \\ s^{(0,2)} \\ d^{(2)} \\ \vdots \\ a \\ \vdots \\ v \end{matrix},$$

and $E_i(t)$, $i = 1, \ldots, 6$ are partitioned conformably with (25). Let $E(\lambda) = E_6(\lambda) \cdots E_1(\lambda)$. Then we have $A^{\langle 1 \rangle}(\lambda) = E(\lambda)\hat{A}(\lambda)$. Since the determinant of each $E_i(t)$, $i = 1, \ldots, 6$, is a nonzero constant, the determinant of $E(\lambda)$ is a nonzero constant. $\square$

This result and its obvious generalization to higher degree matrix polynomials has some immediate consequences for the *linearization of matrix polynomials*

$$R(\lambda) = \sum_{i=0}^{l} A_i \lambda^i, \tag{45}$$

with coefficient matrices $A_i \in \mathbb{C}^{n,n}$ $i = 1, \ldots, l$.

**Definition 24** *Consider a matrix polynomial $R(\lambda)$ as in (45). A pencil $L(\lambda) = \lambda X + Y$ with $X, Y \in \mathbb{C}^{ln,ln}$ is called a* linearization *of $R(\lambda)$ if there exist nonsingular, unimodular matrix polynomials $P(\lambda), Q(\lambda) \in \mathbb{C}^{ln,ln}$ such that*

$$P(\lambda)L(\lambda)Q(\lambda) = \operatorname{diag}(R(\lambda), I_n, \ldots, I_n).$$

We then have the following corollary of Theorem 20.

**Corollary 25** *Consider a quadratic matrix polynomial as in (45). Then there exist nonsingular unimodular matrix polynomials $P(\lambda) \in \mathbb{C}^{m,m}$, $Q(\lambda) \in \mathbb{C}^{n,n}$ such that*

$$\begin{aligned} P(\lambda)R(\lambda)Q(\lambda) &= \lambda^2 \begin{bmatrix} I_{d^2_\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} \tilde{A}^1_{1,1} & 0 & 0 & \tilde{A}^1_{1,4} \\ 0 & I_{d^1_\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} \tilde{A}^0_{1,1} & \tilde{A}^0_{1,2} & 0 & \tilde{A}^0_{1,4} \\ \tilde{A}^0_{2,1} & \tilde{A}^0_{2,2} & 0 & \tilde{A}^0_{2,4} \\ 0 & 0 & I_{a_\mu} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned} \tag{46}$$

*with block sizes $v_\mu$ for the last block row and $u_\mu$ for the last block column.*

Corollary 18 implies, in particular, that the structure of the Jordan chain associated with the eigenvalue $\lambda = \infty$ can be modified by nonsingular, unimodular transformations in such a way that no Jordan chains (higher index blocks) associated with the eigenvalue $\lambda = \infty$ occur, see also [10, 12].

A special case of interest is the case of the system (22) of DAEs with which a *regular* quadratic matrix polynomial is associated. Here, we call a matrix polynomial $A(\lambda)$ of size $m \times n$ *regular* if $m = n$ and the determinant of $A(\lambda)$ is not identically equal to zero.

**Theorem 26** *Consider a system (22) of s-index $\mu$, with $A_2, A_1, A_0 \in \mathbb{C}^{n,n}$, and suppose that the matrix polynomial $A(\lambda) := \lambda^2 A_2 + \lambda A_1 + A_0$ is regular. If $f(t) \in \mathcal{C}^{\mu}(\mathbb{I}, \mathbb{C}^n)$, then there exists a unique solution of the initial value problem (22)-(23), provided that the given initial conditions (23) are consistent.*

*Proof.* Let $\tilde{A}(\lambda) := \lambda^2 \tilde{A}_2 + \lambda \tilde{A}_1 + \tilde{A}_0$, where $\tilde{A}_2, \tilde{A}_1, \tilde{A}_0 \in \mathbb{C}^{n,n}$ are associated with the system (39). Then we have

$$\tilde{A}(\lambda) = E_r(\lambda) P_r E_{r-1}(\lambda) P_{r-1} \cdots E_1(\lambda) P_1 A(\lambda) Q_1 Q_2 \cdots Q_r, \tag{47}$$

where $P_i, Q_i$, $i = 1, \ldots, r$ are nonsingular matrices, and $E_i(\lambda)$, $i = 1, \ldots, r$ are unimodular matrix polynomials. From (47) it follows that $\det(\tilde{A}(\lambda)) = c \det(A(\lambda))$, where $c$ is a nonzero constant. Since $\det(A(\lambda)) \not\equiv 0$, we have $\det(\tilde{A}(\lambda)) \not\equiv 0$, in other words, $\tilde{A}(\lambda)$ is regular. This immediately implies

$$u_\mu = 0, \quad v_\mu = 0$$

in (39). Then, under the condition that the given initial conditions (23) are consistent, the existence and uniqueness of solutions of the initial value problem (22)–(23) directly follows from Corollary 17. □

If the matrix polynomial associated with the second order is singular, then we have the following result.

**Theorem 27** *Consider a system (22) of s-index $\mu$, with $A_2, A_1, A_0 \in \mathbb{C}^{m,n}$, and suppose that the matrix polynomial $A(\lambda) := \lambda^2 A_2 + \lambda A_1 + A_0$ is singular.*

1. *If $\operatorname{rank}(A(\lambda)) < n$ for all $\lambda \in \mathbb{C}$, then the homogeneous initial value problem*

$$A_2 \ddot{x}(t) + A_1 \dot{x}(t) + A_0 x(t) = 0, \; x(t_0) = \dot{x}(t_0) = 0 \tag{48}$$

   *has a nontrivial solution.*

2. *If $\operatorname{rank}(A(\lambda)) = n$ for some $\lambda \in \mathbb{C}$ and hence $m > n$, then there exist arbitrary smooth inhomogeneities $f(t)$ for which the corresponding system (22) of DAEs is not solvable.*

*Proof.*

1. Suppose that $\operatorname{rank}(A(\lambda)) < n$ for all $\lambda \in \mathbb{C}$. Let $\lambda_i$, $i = 1, \ldots, n+1$, be pairwise different complex numbers. Then, for each $\lambda_i$, there exists a nonzero vector $v_i \in \mathbb{C}^n$ satisfying $(\lambda_i^2 A_2 + \lambda_i A_1 + A_0)v_i = 0$, and clearly the vectors $v_i$, $i = 1, \ldots, n+1$, are linearly dependent. Hence, there exist $\alpha_i \in \mathbb{C}$, $i = 1, \ldots, n+1$, not all of them being zero, such that
$$\sum_{i=1}^{n+1} \alpha_i v_i = 0.$$
For the function $x(t)$ defined by
$$x(t) = \sum_{i=1}^{n+1} \alpha_i v_i e^{\lambda_i (t-t_0)},$$
we then have $x(t_0) = 0$ as well as
$$A_2 \ddot{x}(t) + A_1 \dot{x}(t) + A_0 x(t) = \sum_{i=1}^{n+1} \alpha_i (\lambda_i^2 A_2 + \lambda_i A_1 + A_0) v_i e^{\lambda_i (t-t_0)} = 0.$$
Since $x(t)$ is not the zero function, it is a nontrivial solution of the homogeneous initial value problem (48).

2. Suppose that there is a $\lambda \in \mathbb{C}$ such that $\operatorname{rank}(A(\lambda)) = n$. Because $A(\lambda)$ is assumed to be singular, we have $m > n$. We set
$$x(t) = e^{\lambda t} \tilde{x}(t),$$
and therefore
$$\dot{x}(t) = e^{\lambda t} \left( \dot{\tilde{x}}(t) + \lambda \tilde{x}(t) \right), \ \ddot{x}(t) = e^{\lambda t} \left( \ddot{\tilde{x}}(t) + 2\lambda \dot{\tilde{x}}(t) + \lambda^2 \tilde{x}(t) \right),$$
such that (22) is transformed to
$$A_2 (\ddot{\tilde{x}}(t) + 2\lambda \dot{\tilde{x}}(t)) + A_1 \dot{\tilde{x}}(t) + (\lambda^2 A_2 + \lambda A_1 + A_0) \tilde{x}(t) = e^{-\lambda t} f(t).$$
Since $\lambda^2 A_2 + \lambda A_1 + A_0$ has full column rank, there exists a nonsingular matrix $P \in \mathbb{C}^{m,m}$, such this equation premultiplied by $P$ gives
$$\begin{bmatrix} A_{21} \\ A_{22} \end{bmatrix} (\ddot{\tilde{x}}(t) + 2\lambda \dot{\tilde{x}}(t)) + \begin{bmatrix} A_{11} \\ A_{12} \end{bmatrix} \dot{\tilde{x}}(t) + \begin{bmatrix} I \\ 0 \end{bmatrix} \tilde{x}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}.$$
Obviously the matrix polynomial $\xi^2 A_{21} + \xi(2\lambda A_{21} + A_{11}) + I$ in $\xi$ is regular. By Theorem 26, the initial value problem
$$A_{21} \ddot{\tilde{x}}(t) + (2\lambda A_{21} + A_{11}) \dot{\tilde{x}}(t) + \tilde{x}(t) = f_1(t), \ \tilde{x}(t_0) = \tilde{x}_0^{[0]}, \dot{\tilde{x}}(t_0) = \tilde{x}_0^{[1]}$$
has a unique solution for every sufficiently smooth inhomogeneity $f_1(t)$ and for every consistent initial value. But then
$$f_2(t) = A_{22} (\ddot{\tilde{x}}(t) + 2\lambda \dot{\tilde{x}}(t)) + A_{12} \dot{\tilde{x}}(t)$$
is a consistency condition for the inhomogeneity $f_2(t)$ that must hold for a solution to exist. This immediately shows that there are arbitrary smooth functions $f(t)$ for which this consistency condition is not satisfied. □

The extension of these results to higher order systems is obvious.

**Corollary 28** *Consider a matrix polynomial as in (45). Then there exist nonsingular unimodular matrix polynomials $P(\lambda) \in \mathbb{C}^{m,m}$, $Q(\lambda) \in \mathbb{C}^{n,n}$ such that*

$$P(\lambda)R(\lambda)Q(\lambda) = \sum_{i=0}^{l} \tilde{A}_i \lambda^i$$

*with*

$$\tilde{A}_l = \begin{bmatrix} I_{d_\mu^l} & 0 \\ 0 & 0 \end{bmatrix}, \; \tilde{A}_{l-1} = \begin{bmatrix} \tilde{A}_{1,1}^{l-1} & 0 & 0 & \tilde{A}_{1,l+2}^{l-1} \\ 0 & I_{d_\mu^{l-1}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; \ldots,$$

$$\tilde{A}_1 = \begin{bmatrix} \tilde{A}_{1,1}^1 & \ldots & \tilde{A}_{1,l-1}^1 & 0 & 0 & \tilde{A}_{1,l+2}^0 \\ \tilde{A}_{2,1}^1 & \ldots & \tilde{A}_{1,l-1}^2 & 0 & 0 & \tilde{A}_{2,l+2}^1 \\ \vdots & & \vdots & \vdots & \vdots & \\ \tilde{A}_{l,1}^1 & \ldots & \tilde{A}_{l,l-1}^1 & 0 & 0 & \tilde{A}_{l,l+2}^1 \\ 0 & \ldots & 0 & I_{d_\mu^1} & 0 & 0 \\ 0 & \ldots & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{A}_0 = \begin{bmatrix} \tilde{A}_{1,1}^0 & \ldots & \tilde{A}_{1,l}^0 & 0 & \tilde{A}_{1,l+2}^0 \\ \tilde{A}_{2,1}^0 & \ldots & \tilde{A}_{2,l}^0 & 0 & \tilde{A}_{2,l+2}^0 \\ \vdots & & \vdots & \vdots & \vdots \\ \tilde{A}_{l,1}^0 & \ldots & \tilde{A}_{l,l}^0 & 0 & \tilde{A}_{l,l+2}^0 \\ 0 & \ldots & 0 & I_{a_\mu} & 0 \\ 0 & \ldots & 0 & 0 & 0 \end{bmatrix},$$

*with block sizes $v_\mu$ for the last block row and $u_\mu$ for the last block column.*

**Theorem 29** *Consider a system (1) of s-index $\mu$, with $f(t) \in \mathcal{C}^\mu(\mathbb{I}, \mathbb{C}^n)$, and constant coefficients $A_i \in \mathbb{C}^{n,n}$, $i = 0, 1, \ldots, l$. If the matrix polynomial $A(\lambda) := \sum_{i=0}^{l} \lambda_i A_i$ is regular, then there exists a unique solution of the initial value problem (1)–(2), provided that the given initial conditions are consistent.*

*Proof.* The proof is analogous to the proof of Theorem 26. □

**Theorem 30** *Consider a system (1) of s-index $\mu$, with $f(t) \in \mathcal{C}^\mu(\mathbb{I}, \mathbb{C}^n)$, and constant coefficients $A_i \in \mathbb{C}^{n,n}$, $i = 0, 1, \ldots, l$. Suppose that the matrix polynomial $A(\lambda) := \sum_{i=0}^{l} \lambda_i A_i$ is singular.*

1. *If* $\operatorname{rank}(A(\lambda)) < n$ *for all* $\lambda \in \mathbb{C}$*, then the homogeneous initial value problem*

$$A_l x^{(l)}(t) + A_{l-1} x^{(l-1)}(t) + \cdots + A_0 x(t) = 0,$$
$$x(t_0) = \dot{x}(t_0) = \cdots = x^{(l-1)}(t_0) = 0$$

   *has a nontrivial solution.*

2. *If* $\operatorname{rank}(A(\lambda)) = n$ *for some* $\lambda \in \mathbb{C}$ *and hence* $m > n$*, then there exist arbitrary smooth inhomogeneities* $f(t)$ *for which the the corresponding system (1) of DAEs is not solvable.*

*Proof.* The proof is analogous to the proof of Theorem 27. □

## 6 Conclusions

We have presented the theoretical analysis of linear systems of differential-algebraic equations of higher order. We have presented condensed forms for tuples of matrices and tuples of matrix-valued functions which are associated with the systems of constant and variable coefficients, respectively. We have derived a system of invariant quantities and a set of regularity conditions to ensure that the condensed form can be obtained. Based on the condensed forms, we have converted the systems into equivalent strangeness-free systems, from which the solution behavior with respect to solvability, uniqueness of solutions and consistency of initial conditions can be directly read off.

We have demonstrated that if one turns a higher order problem in the traditional way into a first order system of DAEs, then, to get the solvability and uniqueness of solutions, more smoothness of the right-hand side $f(t)$ may be required. The condensed forms, however, allow to do the transformation to first order without extra smoothness requirements.

Several issues remain to be analyzed. These include the perturbation theory for higher order systems of DAEs, in particular how the decision making in the condensed forms influences the transformation to first order as well as the construction of appropriate numerical methods for the treatment of high order, high index differential-algebraic systems.

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# Appendix A.

**Proposition 31** *Relations (12) and (13) are equivalence relations.*

*Proof.* We show that relation (12) has the three properties required of an equivalence relation.

1. Reflexivity: Let $P(t) = I_m$ and $Q(t) = I_n$. Then, we have $(A_l(t), \ldots, A_1(t), A_0(t)) \sim (A_l(t), \ldots, A_1(t), A_0(t))$.
2. Symmetry: Assume that $(A_l(t), \ldots, A_1(t), A_0(t)) \sim (B_l(t), \ldots, B_1(t), B_0(t))$ with pointwise nonsingular matrix-valued functions $P(t)$ and $Q(t)$ that satisfy (12). We prove that

$(B_l(t), \ldots, B_1(t), B_0(t)) \sim (A_l(t), \ldots, A_1(t), A_0(t))$. Since every derivative of $Q(t)Q^{-1}(t) = I$ with respect to $t$ is identically zero, it is immediate that

$$\begin{bmatrix} Q(t) & \binom{l}{1}\frac{d}{dt}Q(t) & \cdots & \cdots & \binom{l}{l}\frac{d^l}{dt^l}Q(t) \\ & Q(t) & \binom{l-1}{1}\frac{d}{dt}Q(t) & \cdots & \binom{l-1}{l-1}\frac{d^{l-1}}{dt^{l-1}}Q(t) \\ & & \ddots & \ddots & \vdots \\ & & & Q(t) & \binom{1}{1}\frac{d}{dt}Q(t) \\ & & & & Q(t) \end{bmatrix} \times$$
$$\begin{bmatrix} Q^{-1}(t) & \binom{l}{1}\frac{d}{dt}Q^{-1}(t) & \cdots & \cdots & \binom{l}{l}\frac{d^l}{dt^l}Q^{-1}(t) \\ & Q^{-1}(t) & \binom{l-1}{1}\frac{d}{dt}Q^{-1}(t) & \cdots & \binom{l-1}{l-1}\frac{d^{l-1}}{dt^{l-1}}Q^{-1}(t) \\ & & \ddots & \ddots & \vdots \\ & & & Q^{-1}(t) & \binom{1}{1}\frac{d}{dt}Q^{-1}(t) \\ & & & & Q^{-1}(t) \end{bmatrix} = I. \tag{49}$$

Hence, by (12) and (49), we have

$$[A_l(t), \ldots, A_0(t)] = P^{-1}(t)[B_l(t), \ldots, B_0(t)] \times$$
$$\begin{bmatrix} Q^{-1}(t) & \binom{l}{1}\frac{d}{dt}Q^{-1}(t) & \cdots & \cdots & \binom{l}{l}\frac{d^l}{dt^l}Q^{-1}(t) \\ & Q^{-1}(t) & \binom{l-1}{1}\frac{d}{dt}Q^{-1}(t) & \cdots & \binom{l-1}{l-1}\frac{d^{l-1}}{dt^{l-1}}Q^{-1}(t) \\ & & \ddots & \ddots & \vdots \\ & & & Q^{-1}(t) & \binom{1}{1}\frac{d}{dt}Q^{-1}(t) \\ & & & & Q^{-1}(t) \end{bmatrix},$$

namely, $(B_l(t), \ldots, B_1(t), B_0(t)) \sim (A_l(t), \ldots, A_1(t), A_0(t))$.

3. Transitivity: Assume that $(A_l(t), \ldots, A_0(t)) \sim (B_l(t), \ldots, B_0(t))$ with pointwise nonsingular matrix-valued functions $P_1(t)$ and $Q_1(t)$ and that $(B_l(t), \ldots, B_0(t)) \sim (C_l(t), \ldots, C_0(t))$ with pointwise nonsingular matrix-valued functions $P_2(t)$ and $Q_2(t)$, which satisfy (12), respectively. We prove that $(A_l(t), \ldots, A_0(t)) \sim (C_l(t), \ldots, C_0(t))$. By the product role and Leibniz identity for differentiation, we can immediately verify that

$$\begin{bmatrix} Q_1(t) & \binom{l}{1}\frac{d}{dt}Q_1(t) & \cdots & \cdots & \binom{l}{l}\frac{d^l}{dt^l}Q_1(t) \\ & Q_1(t) & \binom{l-1}{1}\frac{d}{dt}Q_1(t) & \cdots & \binom{l-1}{l-1}\frac{d^{l-1}}{dt^{l-1}}Q_1(t) \\ & & \ddots & \ddots & \vdots \\ & & & Q_1(t) & \binom{1}{1}\frac{d}{dt}Q_1(t) \\ & & & & Q_1(t) \end{bmatrix} \times$$
$$\begin{bmatrix} Q_2(t) & \binom{l}{1}\frac{d}{dt}Q_2(t) & \cdots & \cdots & \binom{l}{l}\frac{d^l}{dt^l}Q_2(t) \\ & Q_2(t) & \binom{l-1}{1}\frac{d}{dt}Q_2(t) & \cdots & \binom{l-1}{l-1}\frac{d^{l-1}}{dt^{l-1}}Q_2(t) \\ & & \ddots & \ddots & \vdots \\ & & & Q_2(t) & \binom{1}{1}\frac{d}{dt}Q_2(t) \\ & & & & Q_2(t) \end{bmatrix} \tag{50}$$
$$= \begin{bmatrix} Q_1(t)Q_2(t) & \binom{l}{1}\frac{d}{dt}\left(Q_1(t)Q_2(t)\right) & \cdots & \binom{l}{l}\frac{d^l}{dt^l}\left(Q_1(t)Q_2(t)\right) \\ & Q_1(t)Q_2(t) & \cdots & \binom{l-1}{l-1}\frac{d^{l-1}}{dt^{l-1}}\left(Q_1(t)Q_2(t)\right) \\ & & \ddots & \vdots \\ & & \ddots & \binom{1}{1}\frac{d}{dt}\left(Q_1(t)Q_2(t)\right) \\ & & & Q_1(t)Q_2(t) \end{bmatrix}.$$

Thus, by the assumptions and (50), we have

$$[C_l(t),\ldots,C_0(t)] = P_1(t)P_2(t)[A_l(t),\ldots,A_0(t)]\times$$
$$\begin{bmatrix} Q_1(t)Q_2(t) & \binom{l}{1}\frac{d}{dt}(Q_1(t)Q_2(t)) & \cdots & \binom{l}{l}\frac{d^l}{dt^l}(Q_1(t)Q_2(t)) \\ & Q_1(t)Q_2(t) & \cdots & \binom{l-1}{l-1}\frac{d^{l-1}}{dt^{l-1}}(Q_1(t)Q_2(t)) \\ & & \ddots & \vdots \\ & & \ddots & \binom{1}{1}\frac{d}{dt}(Q_1(t)Q_2(t)) \\ & & & Q_1(t)Q_2(t) \end{bmatrix},$$

namely, $(A_l(t),\ldots,A_1(t),A_0(t)) \sim (C_l(t),\ldots,C_1(t),C_0(t))$. The proof for (13) is analogous. □

# Appendix B

To prove Theorem 8 we need the following lemma on the well-known canonical form for a single matrix under equivalence relation (11).

**Lemma 32** *([23], p. 51) Let $A \in \mathbb{C}^{m,n}$. Then there exist nonsingular matrices $P := \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \in \mathbb{C}^{m,m}$ and $Q := [Q_1, Q_2] \in \mathbb{C}^{n,n}$ such that*

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \tag{51}$$

*where $P_1 \in \mathbb{C}^{r,m}$, $Q_1 \in \mathbb{C}^{n,r}$. Moreover, we have*

$$r = \operatorname{rank}(A),\ \mathcal{N}(A) = \mathcal{R}(Q_2),\ \mathcal{N}(A^T) = \mathcal{R}(P_2^T). \tag{52}$$

**Proof of Theorem 8**. In the following, the word "new" on top of the equivalence operator denotes that the subscripts of the entries are adapted to the new block structure of the matrices. Using Lemma 32, we obtain the following sequence of equivalent matrix pairs.

$$(E,A) \sim \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) \overset{\text{new}}{\sim} \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & I_a & 0 \\ A_{31} & 0 & 0 \end{bmatrix}\right)$$

$$\overset{\text{new}}{\sim} \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & I_a & 0 \\ A_{31} & 0 & 0 \end{bmatrix}\right)$$

$$\sim \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & I_a & 0 \\ A_{31} & 0 & 0 \end{bmatrix}\right)$$

$$\overset{\text{new}}{\sim} \left(\begin{bmatrix} P_{11} & P_{12} & 0 & 0 \\ P_{21} & P_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & 0 & A_{14} \\ A_{21} & A_{22} & 0 & A_{24} \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$\left(\text{where the matrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \text{is nonsingular}\right)$$

$$\overset{\text{new}}{\sim} \left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & 0 & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right).$$

It remains to show that the quantities $r, s, d, a, v, u$ are well-defined by (17) and invariant under the equivalence relation (11). In the case of $r = \text{rank}(E)$, this is clear. For the other quantities, indeed, we only need to show that two quantities $a$ and $s$ are well-defined and invariant under equivalence relation (11). Since we have proved (16), let $P := \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \in \mathbb{C}^{m,m}$ and $Q := [Q_1, Q_2] \in \mathbb{C}^{n,n}$ be nonsingular matrices, where $P_1 \in \mathbb{C}^{r,m}$, $Q_1 \in \mathbb{C}^{n,r}$, such that

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} E[Q_1, Q_2] = \begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} A[Q_1, Q_2] = \begin{bmatrix} 0 & A_{12} & 0 & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{53}$$

By Lemma 32, we have

$$\mathcal{N}(E^T) = \mathcal{R}(P_2^T), \quad \mathcal{N}(E) = \mathcal{R}(Q_2), \tag{54}$$

namely, the columns of $P_2^T$ span $\mathcal{N}(E^T)$, and the columns of $Q_2$ span $\mathcal{N}(E)$. From (53), it immediately follows that

$$P_2AQ = \begin{bmatrix} 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; P_2AQ_2 = \begin{bmatrix} I_a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{55}$$

Hence, by (55), we have

$$a = \operatorname{rank}(P_2 A Q_2), \quad s = \operatorname{rank}(P_2 A Q) - a = \operatorname{rank}(P_2 A) - a. \tag{56}$$

From (15) and (54) it follows that there exist nonsingular matrices $T_1 \in \mathbb{C}^{m-r,m-r}$ and $T_2 \in \mathbb{C}^{n-r,n-r}$ such that

$$P_2^T = Z_1 T_1, \quad Q_2 = Z_2 T_2. \tag{57}$$

Then, from (56) and (57) it follows that

$$a = \operatorname{rank}(P_2 A Q_2) = \operatorname{rank}\left(T_1^T Z_1^T A Z_2 T_2\right) = \operatorname{rank}(Z_1^T A Z_2),$$

and that

$$s = \operatorname{rank}(P_2 A) - a = \operatorname{rank}\left(T_1^T Z_1^T A\right) - a = \operatorname{rank}(Z_1^T A) - a.$$

Thus, $a$ and $s$ are indeed well-defined by (17) and therefore so are the quantities $d$, $v$ and $u$.

At last, we prove that $a$ and $s$ are invariant. Let $(E_i, A_i)$, $i = 1, 2$, be equivalent, and let $Z_1^{(i)}$ be a matrix whose columns form a basis for $\mathcal{N}(E_i^T)$ and let $Z_2^{(i)}$ be a matrix whose columns form a basis for $\mathcal{N}(E_i)$. Since there exist nonsingular matrices $P$ and $Q$ such that $E_1 = P E_2 Q$ and $A_1 = P A_2 Q$, from $E_1^T Z_1^{(1)} = 0$ and $E_1 Z_2^{(1)} = 0$ it follows that

$$Q^T E_2^T P^T Z_1^{(1)} = 0, \quad P E_2 Q Z_2^{(1)} = 0,$$

and therefore

$$E_2^T P^T Z_1^{(1)} = 0, \quad E_2 Q Z_2^{(1)} = 0.$$

Thus, the columns of $P^T Z_1^{(1)}$ form a basis for $\mathcal{N}(E_2^T)$ and the columns of $Q Z_2^{(1)}$ form a basis for $\mathcal{N}(E_2)$. Therefore, there exist nonsingular matrices $\hat{T}_1 \in \mathbb{C}^{m-r,m-r}$ and $\hat{T}_2 \in \mathbb{C}^{n-r,n-r}$ such that

$$P^T Z_1^{(1)} = Z_1^{(2)} \hat{T}_1, \quad Q Z_2^{(1)} = Z_2^{(2)} \hat{T}_2.$$

Then, the proof follows since

$$\operatorname{rank}({Z_1^{(1)}}^T A_1) = \operatorname{rank}({Z_1^{(2)}}^T P^{-1} P A_2 Q) = \operatorname{rank}({Z_1^{(2)}}^T A_2 Q) = \operatorname{rank}({Z_1^{(2)}}^T A_2),$$
$$\operatorname{rank}({Z_1^{(1)}}^T A_1 Z_2^{(1)}) = \operatorname{rank}({Z_1^{(2)}}^T P^{-1} P A_2 Q Q^{-1} Z_2^{(2)} \hat{T}_2) = \operatorname{rank}({Z_1^{(2)}}^T A_2 Z_2^{(2)}).$$

□

# Appendix C

**Proof of Theorem 10.** The proof proceeds by successively using the transformations to strong canonical form for pairs of matrices. We do not denote the blocks of the matrices by specific letters but use 0, $I$ and $*$ to denote 0 blocks, identity matrices and unspecified blocks. The sequence of transformations is as follows:

$$(A_2, A_1, A_0)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & * & * \\ 0 & * & 0 & * & * \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & I & 0 \\ * & * & * & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & * & * \\ 0 & * & 0 & * & * \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & * & I & 0 \\ * & * & * & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & * & 0 & * \\ 0 & 0 & 0 & I & 0 \\ * & * & * & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & * & * \\ 0 & * & 0 & * & * \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & * & 0 & * \\ 0 & 0 & 0 & I & 0 \\ * & * & * & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & 0 & * & * \\ 0 & * & 0 & 0 & * & * \\ 0 & 0 & P_{11} & P_{12} & 0 & 0 \\ 0 & 0 & P_{21} & P_{22} & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} * & * & * & * & 0 & * \\ * & * & * & * & 0 & * \\ * & * & * & * & 0 & * \\ * & * & * & * & 0 & * \\ * & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & I & 0 \\ * & * & I & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

(where the matrix $\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ is nonsingular)

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & 0 & * & * \\ 0 & * & 0 & 0 & * & * \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & I & 0 \\ * & * & I & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & 0 & * & * \\ 0 & * & 0 & 0 & * & * \\ * & * & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & 0 & * & * \\ 0 & * & 0 & 0 & * & * \\ 0 & * & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Q_{11} & Q_{12} & 0 & 0 & 0 & 0 \\ 0 & Q_{21} & Q_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & * & 0 & 0 & * & * \\ 0 & * & * & 0 & 0 & * & * \\ 0 & * & * & 0 & 0 & * & * \\ 0 & * & * & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} * & * & * & 0 & * & 0 & * \\ * & * & * & 0 & * & 0 & * \\ * & * & * & 0 & * & 0 & * \\ * & * & * & 0 & * & 0 & * \\ * & * & * & 0 & * & 0 & * \\ * & * & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ * & I & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

(where the matrix $\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$ is nonsingular)

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & * & 0 & 0 & * & * \\ 0 & * & * & 0 & 0 & * & * \\ 0 & * & * & 0 & 0 & * & * \\ 0 & * & * & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ * & I & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & 0 & 0 & * & * \\ * & * & * & 0 & 0 & * & * \\ * & * & * & 0 & 0 & * & * \\ * & * & * & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & * & 0 & 0 & * & * \\ 0 & * & * & 0 & 0 & * & * \\ 0 & * & * & 0 & 0 & * & * \\ 0 & * & * & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} R_{11} & R_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ R_{21} & R_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ R_{11} & R_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ R_{21} & R_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} * & * & 0 & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

(where the matrix $\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$ is nonsingular)

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$
\left.\begin{bmatrix}
* & * & 0 & * & 0 & * & 0 & * \\
* & * & 0 & * & 0 & * & 0 & * \\
* & * & 0 & * & 0 & * & 0 & * \\
* & * & 0 & * & 0 & * & 0 & * \\
* & * & 0 & * & 0 & * & 0 & * \\
* & * & 0 & * & 0 & * & 0 & * \\
* & * & 0 & * & 0 & * & 0 & * \\
* & * & 0 & * & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}\right)
$$

$$
\sim \left(\begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & * & * & 0 & 0 & * & * \\
0 & 0 & * & * & 0 & 0 & * & * \\
0 & 0 & * & * & 0 & 0 & * & * \\
0 & 0 & * & * & 0 & 0 & * & * \\
0 & 0 & * & * & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},\right.
$$

$$
\left.\begin{bmatrix}
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}\right)
$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & * & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & * & 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & * & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left.\begin{bmatrix} 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right) . \square$$

# Appendix D

**Proof of Lemma 13.** By the global equivalence relation (12) and using Lemma 8 of [18] we obtain the following sequence of globally equivalent triples of matrix-valued functions. (Here we denote unspecified blocks by $*$ and derivatives of unspecified blocks by $\dot{*}$ or $\ddot{*}$, respectively.)

$$
\begin{aligned}
&(A_2, A_1, A_0) \\
&\sim \left( \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * \\ * & A_{22}^1 \end{bmatrix}, \begin{bmatrix} * & * \\ * & A_{22}^0 \end{bmatrix} \right) \\
&\sim \left( \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * \\ * & U_1^H A_{22}^1 V_1 \end{bmatrix} \right. \\
&\qquad\qquad \left. +2 \begin{bmatrix} I & 0 \\ 0 & U_1^H \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \dot{V}_1 \end{bmatrix}, \begin{bmatrix} * & * \\ * & A_{22}^0 \end{bmatrix} \right) \\
&\sim \left( \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & I & 0 \\ * & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & A_{23}^0 \\ * & * & A_{33}^0 \end{bmatrix} \right) \\
&\sim \left( \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & 0 & * \\ * & I & 0 \\ * & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & A_{23}^0 \\ * & * & A_{33}^0 \end{bmatrix} \right) \\
&\sim \left( \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & 0 & * \\ 0 & I & 0 \\ * & 0 & 0 \end{bmatrix} \right. \\
&\qquad\qquad \left. -2 \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ \dot{*} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & A_{23}^0 \\ * & * & A_{33}^0 \end{bmatrix} \right)
\end{aligned}
$$

$$\sim \left( \begin{bmatrix} V_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & 0 & * \\ 0 & I & 0 \\ U_2^H A_{31}^0 V_2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & A_{23}^0 \\ * & * & A_{33}^0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} V_{11} & V_{12} & 0 & 0 \\ V_{21} & V_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & A_{54}^0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} V_{11} & V_{12} & 0 & 0 \\ V_{21} & V_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & A_{54}^0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & A_{54}^0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & Q_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & * \\ 0 & A_1 Q_1 + 2\dot{Q}_1 & 0 & * \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & A_{54}^0 \end{bmatrix} \right)$$

(where the pointwise nonsingular matrix-valued function $Q_1(t)$ is chosen as the solution of the initial value problem $\dot{Q}_1(t) = -\frac{1}{2} A_{2,2}^1(t) Q_1(t), t \in \mathbb{I}, Q_1(t_0) = I$)

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & A_{54}^0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dot{V}_3 \end{bmatrix}, \right.$$

$$
\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & U_3^H A_{54}^0 V_3 \end{bmatrix} + \begin{bmatrix} 0 & * & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dot{V}_3 \end{bmatrix}
$$

$$
+ \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddot{V}_3 \end{bmatrix} \Bigg)
$$

$$
\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & I & 0 \\ * & * & * & 0 & 0 \end{bmatrix} \right)
$$

$$
\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & * & I & 0 \\ * & * & * & 0 & 0 \end{bmatrix} \right)
$$

$$
\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} - 2 \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right.
$$

$$
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dot{*} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & 0 & I & 0 \\ * & * & * & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & * & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\left. \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dot{*} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddot{*} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)
$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & 0 & I & 0 \\ * & * & * & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & 0 & I & 0 \\ * & * & * & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & * & 0 & * \\ * & * & * & 0 & * \\ 0 & 0 & 0 & I & 0 \\ * & * & * & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & V_{11} & V_{12} & 0 & 0 \\ 0 & 0 & V_{21} & V_{22} & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} * & * & * & * & 0 & * \\ * & * & * & * & 0 & * \\ * & * & * & * & 0 & * \\ * & * & * & * & 0 & * \\ * & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & I & 0 \\ * & * & I & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & I & 0 \\ * & * & I & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$\sim\left(\begin{bmatrix} I&0&0&0&0&0\\ 0&I&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix},\begin{bmatrix} 0&*&0&0&*&*\\ 0&0&0&0&*&*\\ 0&0&I&0&0&0\\ 0&0&0&I&0&0\\ I&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix},\begin{bmatrix} *&*&0&*&0&*\\ *&*&0&*&0&*\\ *&*&0&*&0&*\\ *&*&0&*&0&*\\ *&*&0&*&0&*\\ 0&0&0&0&I&0\\ *&0&I&0&0&0\\ *&*&0&0&0&0 \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I&0&0&0&0&0\\ 0&I&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix},\begin{bmatrix} 0&*&0&0&*&*\\ 0&0&0&0&*&*\\ 0&0&I&0&0&0\\ 0&0&0&I&0&0\\ I&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix},\begin{bmatrix} *&*&0&*&0&*\\ *&*&0&*&0&*\\ *&*&0&*&0&*\\ *&*&0&*&0&*\\ *&*&0&*&0&*\\ 0&0&0&0&I&0\\ 0&0&I&0&0&0\\ *&*&0&0&0&0 \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I&0&0&0&0&0&0\\ 0&V_{11}&V_{12}&0&0&0&0\\ 0&V_{21}&V_{22}&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix},\begin{bmatrix} 0&*&*&0&0&*&*\\ 0&0&0&0&0&*&*\\ 0&0&0&0&0&*&*\\ 0&0&0&I&0&0&0\\ 0&0&0&0&I&0&0\\ I&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix}+\right.$$

$$2\begin{bmatrix} 0&0&0&0&0&0&0\\ 0&\dot V_{11}&\dot V_{12}&0&0&0&0\\ 0&\dot V_{21}&\dot V_{22}&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix},\begin{bmatrix} *&*&*&0&*&0&*\\ *&*&*&0&*&0&*\\ *&*&*&0&*&0&*\\ *&*&*&0&*&0&*\\ *&*&*&0&*&0&*\\ *&*&*&0&*&0&*\\ 0&0&0&0&0&I&0\\ 0&0&0&I&0&0&0\\ *&I&0&0&0&0&0\\ *&0&0&0&0&0&0 \end{bmatrix}+$$

$$\begin{bmatrix} 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dot{V}_{11} & \dot{V}_{12} & 0 & 0 & 0 & 0 \\ 0 & \dot{V}_{21} & \dot{V}_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddot{V}_{11} & \ddot{V}_{12} & 0 & 0 & 0 & 0 \\ 0 & \ddot{V}_{21} & \ddot{V}_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Bigg)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & * & 0 & 0 & * & * \\ 0 & * & * & 0 & 0 & * & * \\ 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} * & * & * & 0 & * & 0 & * \\ * & * & * & 0 & * & 0 & * \\ * & * & * & 0 & * & 0 & * \\ * & * & * & 0 & * & 0 & * \\ * & * & * & 0 & * & 0 & * \\ * & * & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ * & I & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$
\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & * & 0 & 0 & * & * \\ 0 & * & * & 0 & 0 & * & * \\ 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.
$$

$$
\left. \begin{bmatrix} * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ * & I & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)
$$

$$
\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & 0 & 0 & * & * \\ * & * & * & 0 & 0 & * & * \\ * & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right. -
$$

$$
2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \dot{*} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} +
$$

$$\begin{bmatrix} 0 & \ast & 0 & 0 & 0 & 0 & 0 \\ 0 & \ast & 0 & 0 & 0 & 0 & 0 \\ 0 & \ast & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \ddot{\ast} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\Bigg)\Bigg)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ \ast & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \ast & \ast & \ast & 0 & 0 & \ast & \ast \\ \ast & \ast & \ast & 0 & 0 & \ast & \ast \\ \ast & \ast & \ast & 0 & 0 & \ast & \ast \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} \ast & \ast & \ast & 0 & \ast & 0 & \ast \\ \ast & \ast & \ast & 0 & \ast & 0 & \ast \\ \ast & \ast & \ast & 0 & \ast & 0 & \ast \\ \ast & 0 & \ast & 0 & \ast & 0 & \ast \\ \ast & 0 & \ast & 0 & \ast & 0 & \ast \\ \ast & 0 & \ast & 0 & \ast & 0 & \ast \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ \ast & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \ast & \ast & 0 & 0 & \ast & \ast \\ 0 & \ast & \ast & 0 & 0 & \ast & \ast \\ 0 & \ast & \ast & 0 & 0 & \ast & \ast \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$
\left.\begin{bmatrix}
* & 0 & * & 0 & * & 0 & * \\
* & 0 & * & 0 & * & 0 & * \\
* & 0 & * & 0 & * & 0 & * \\
* & 0 & * & 0 & * & 0 & * \\
* & 0 & * & 0 & * & 0 & * \\
* & 0 & * & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}\right)
$$

$$
\sim \left( \begin{bmatrix}
V_{11} & V_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
V_{21} & V_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 & * & * & 0 & 0 & * & * \\
0 & 0 & * & * & 0 & 0 & * & * \\
0 & 0 & * & * & 0 & 0 & * & * \\
0 & 0 & * & * & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
V_{11} & V_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
V_{21} & V_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}\right.
$$

$$
+2 \begin{bmatrix}
\dot{V}_{11} & \dot{V}_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
\dot{V}_{21} & \dot{V}_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
* & * & 0 & * & 0 & * & 0 & * \\
* & * & 0 & * & 0 & * & 0 & * \\
* & * & 0 & * & 0 & * & 0 & * \\
* & * & 0 & * & 0 & * & 0 & * \\
* & * & 0 & * & 0 & * & 0 & * \\
* & * & 0 & * & 0 & * & 0 & * \\
* & * & 0 & * & 0 & * & 0 & * \\
* & * & 0 & * & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
+\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\dot{V}_{11} & \dot{V}_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
\dot{V}_{21} & \dot{V}_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
+\begin{bmatrix}
\ddot{V}_{11} & \ddot{V}_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
\ddot{V}_{21} & \ddot{V}_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}\Bigg)
$$

$$
\sim\left(\begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
\ast & \ast & \ast & \ast & 0 & 0 & \ast & \ast \\
\ast & \ast & \ast & \ast & 0 & 0 & \ast & \ast \\
0 & 0 & \ast & \ast & 0 & 0 & \ast & \ast \\
0 & 0 & \ast & \ast & 0 & 0 & \ast & \ast \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},\right.
$$

$$
\left.\begin{bmatrix}
\ast & \ast & 0 & \ast & 0 & \ast & 0 & \ast \\
\ast & \ast & 0 & \ast & 0 & \ast & 0 & \ast \\
\ast & \ast & 0 & \ast & 0 & \ast & 0 & \ast \\
\ast & \ast & 0 & \ast & 0 & \ast & 0 & \ast \\
\ast & \ast & 0 & \ast & 0 & \ast & 0 & \ast \\
\ast & \ast & 0 & \ast & 0 & \ast & 0 & \ast \\
A_0 & \ast & 0 & \ast & 0 & \ast & 0 & \ast \\
\ast & \ast & 0 & \ast & 0 & \ast & 0 & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}\right)
$$

$$\sim \left( \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right). \quad \square$$