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Robust numerical methods for robust control

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Abstract

We present numerical methods for the solution of the optimal H_∞ control problem. In particular, we investigate the iterative part often called the γ-iteration. We derive a method with better robustness in the presence of rounding errors than other existing methods. It remains robust in the presence of rounding errors even as γ approaches its optimal value. For the computation of a suboptimal controller, we avoid solving algebraic Riccati equations with their problematic matrix inverses and matrix products by adapting recently suggested methods for the computation of deflating subspaces of skew-Hamiltonian/Hamiltonian pencils. These methods are applicable even if the pencil has eigenvalues on the imaginary axis. We compare the new method with older methods and present several examples.

Keywords. H_∞ control, algebraic Riccati equation, CS decomposition, Lagrangian subspaces, skew-Hamiltonian/Hamiltonian pencil,
AMS subject classification. 93B40, 93B36, 65F15, 93B52, 93C05.

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1 Introduction

The optimal infinite-horizon output (or measurement) feedback H_∞ control problem is one of the central tasks in robust control, see, e.g., [22, 34, 42, 45], but the development of robust numerical methods for the H_∞ control is unusually difficult [41] and remains a major open problem [12]. This paper derives a numerical method with better robustness in the presence of rounding errors than other methods.

For a matrix valued rational function $F(s)$ that is analytic and bounded in the open right-half plane, the H_∞ norm is given by

$$\|F\|_\infty = sup_{\omega\in\mathbb{R}}\sigma_{max}[F(i\omega)],$$

where $\sigma_{max}[F(i\omega)]$ denotes the maximal singular value of the matrix $F(i\omega)$. In robust control, $\|F\|_\infty$ is used as a measure of the worst case influence of the disturbances on the output where in this case, F is the transfer function mapping noise or disturbance inputs to error signals. The optimal H_∞ control problem is the task of designing a dynamic controller that minimizes (or at least approximately minimizes) this measure.

Put more rigorously, the optimal H_∞ control problem is the following. Consider the linear system

$$\begin{array}{rcl} \dot{x} &=& Ax + B_1w + B_2u, \qquad x(t_0) = x^0, \\ z &=& C_1x + D_{11}w + D_{12}u, \\ y &=& C_2x + D_{21}w + D_{22}u, \end{array} \tag{1}$$

where $A \in \mathbb{R}^{n,n}$, $B_i \in \mathbb{R}^{n,m_i}$, $C_i \in \mathbb{R}^{p_i,n}$, and $D_{ij} \in \mathbb{R}^{p_i,m_j}$ for $i,j = 1,2$. (By $\mathbb{R}^{n,k}$ we denote the set of real $n \times k$ matrices.) As usual, see [22, 45], we assume $p_1 \geq m_2$ and $m_1 \geq p_2$. In this system, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{m_2}$ is the control input vector, and $w(t) \in \mathbb{R}^{m_1}$ is an exogenous input that may include noise, linearization errors and unmodeled dynamics. The vector $y(t) \in \mathbb{R}^{p_2}$ contains measured outputs, while $z(t) \in \mathbb{R}^{p_1}$ is a regulated output or an estimation error.

Definition 1.1 The Optimal H_∞ control problem: Determine a controller (dynamic compensator)

$$\begin{array}{rcl} \dot{\hat{x}} &=& \hat{A}\hat{x} + \hat{B}y, \\ u &=& \hat{C}\hat{x} + \hat{D}y, \end{array} \tag{2}$$

with $\hat{A} \in \mathbb{R}^{N,N}$, $\hat{B} \in \mathbb{R}^{N,p_2}$, $\hat{C} \in \mathbb{R}^{m_2,N}$, $\hat{D} \in \mathbb{R}^{m_2,p_2}$ and transfer function $K(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}$ such that the closed-loop system resulting from

(1) and (2),

$$\begin{aligned}
\dot{x} &= (A + B_2\hat{D}Z_1C_2)x + (B_2Z_2\hat{C})\hat{x} + (B_1 + B_2\hat{D}Z_1D_{21})w, \\
\dot{\hat{x}} &= \hat{B}Z_1C_2x + (\hat{A} + \hat{B}Z_1D_{22}\hat{C})\hat{x} + \hat{B}Z_1D_{21}w, \\
z &= (C_1 + D_{12}Z_2\hat{D}C_2)x + D_{12}Z_2\hat{C}\hat{x} + (D_{11} + D_{12}\hat{D}Z_1D_{21})w,
\end{aligned} \tag{3}$$

with $Z_1 = (I - D_{22}\hat{D})^{-1}$ and $Z_2 = (I - \hat{D}D_{22})^{-1}$,

1. is *internally stable*, i.e., the solution of the system with $w \equiv 0$ is asymptotically stable, and

2. the closed-loop transfer function $T_{zw}(s)$ from w to z is minimized in the H_∞ norm.

In principle, there is no restriction on the dimension N of $\hat{x}$ in (2), although, smaller dimensions N are preferred for practical implementation and computation.

The configuration of the closed-loop system is illustrated by the block diagram in Figure 1.

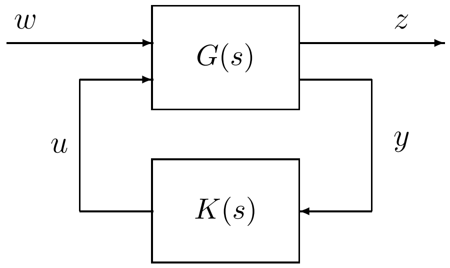

Figure 1: Closed-loop diagram for the linear system (1) with transfer function $G(s)$ [45] and dynamic compensator (2) with transfer function $F(s)$.

The solution of the problem is, in general, difficult. Solving the H_∞ control problem by directly minimizing $\|T_{zw}\|_\infty$ over the complicated set of internally stabilizing controllers (2) is intractable by conventional optimization methods. It is often unclear whether a minimizing controller exists [45, p.414]. When a minimizing controller or an approximately minimizing controller does exist, it is typically not unique.

The well-known state-space solution to the H_∞ control problem [15], relating H_∞ control to algebraic Riccati equations, provides a way to solve many H_∞ control problems despite the above difficulties.

We review the theorem (as presented in [45]) as Theorem 3.7 in Section 3. However, in summary, for each number $\gamma > 0$, the theorem gives an explicit computational test for whether there is an internally stabilizing dynamic controller (3) whose closed-loop transfer function $T_{zw}(s)$ satisfies $\gamma > \|T_{zw}\|_\infty$. Explicit but complicated formulae in terms of γ for a dynamic controller that achieves $\gamma > \|T_{zw}\|_\infty$ (when one exists) appear, e.g., in [23, 45] and are discussed in Section 8. Hence, at least in principle, the theorem can be used to solve the H_∞ control problem by bisection on γ.

Here, we view the theorem as a tool to divide the optimal H_∞ problem into two subproblems that we call the *modified optimal H_∞ control problem* and the *suboptimal H_∞ control problem.*

Definition 1.2 The Modified optimal H_∞ control problem: Let Γ be the set of numbers $\gamma > 0$ for which there exists an internally stabilizing dynamic controller with transfer function $T_{zw}(s)$ satisfying $\gamma > \|T_{zw}\|_\infty$. Determine $\gamma_{mo} = \inf \Gamma$. (If no internally stabilizing controller exists, then $\Gamma = \emptyset$ and $\gamma_{mo} = \infty$.)

Because there may be no dynamic controller whose transfer function actually achieves H_∞ norm equal to γ_{mo}, in general, one must necessarily use a controller whose transfer function has larger H_∞ norm, i.e., one must use a *suboptimal controller.*

Definition 1.3 The Suboptimal H_∞ control problem: For a given value $\gamma \in \Gamma$, find an internally stabilizing dynamic controller such that the closed loop transfer function satisfies $\|T_{zw}\|_\infty < \gamma$. We call such a controller a *γ-suboptimal controller* or simply a *suboptimal controller* when γ is implicitly understood.

The process of solving the modified optimal H_∞ control problem is sometimes called *γ-iteration.* Once a sufficiently accurate approximation to γ_{mo} has been determined, a suboptimal controller may be constructed using the formulae suggested in [23, 45] or by the more robust formulae in Section 8. In this paper we present rounding-error robust numerical methods for the γ-iteration and suboptimal H_∞ control problem.

Note that for each $\gamma \in \Gamma$, there are typically many internally stabilizing controllers that achieve $\|T_{zw}\|_\infty < \gamma$. In any application, some are likely to be more robust or less expensive or more elegant than others. See the discussion in Section 8.

The solution of the optimal H_∞ control problem, when it exists, is a dynamic controller. So is the solution of the suboptimal H_∞ problem. The solution to the modified optimal H_∞ control problem is the number γ_{mo}. The optimal H_∞ problem may or may not have a solution. If an optimal H_∞ controller exists and $T_{zw}(s)$ is the corresponding closed-loop transfer function, then $\|T_{zw}\|_\infty = \gamma_{mo}$. The modified optimal H_∞ control problem always has a solution if $\gamma_{mo} = \infty$ is allowed. The suboptimal H_∞ control problem has a solution for every $\gamma \in \Gamma$. The suboptimal H_∞ control problem may or may not have a solution corresponding to $\gamma = \gamma_{mo} = \inf \Gamma$, depending on whether the infimum is attained in Γ. It should be noted that the modified optimal H_∞ control problem is an optimization in the single independent variable γ, while the optimal H_∞ control problem requires optimization over the complicated set of stabilizing controllers.

The modified optimal H_∞ control problem is well analyzed, [22, 34, 42, 45] and numerical methods have been implemented in several software packages [6, 19, 13, 35]. The accuracy of conventional methods like these is limited by the empirical fact that as γ approaches γ_{mo}, the Riccati equations and other subproblems tend to become so ill-conditioned that rounding errors make it impossible to use the test in Theorem 3.7. The procedures often fail before an accurate approximation of γ_{mo} is obtained. This paper presents numerical methods that mostly overcome this problem.

We briefly recall some of the theory surrounding H_∞ control in Section 3. In Section 4 we discuss some of the existing numerical methods pointing out where numerical difficulties may arise. In Section 5 we present a formulation of the modified optimal H_∞ control problem chosen to avoid such numerical difficulties. The formulation incorporates ideas from singular H_∞ control [14, 18] in combination with numerical methods designed especially for skew-Hamiltonian/Hamiltonian eigenvalue problems [5]. Exploiting skew-Hamiltonian/Hamiltonian methods, we derive a numerically robust γ-iteration in Section 6. The procedure applies in situations where classical γ-iterations fail. Consequently, it allows the H_∞ approach to be used on a broader range of problems. Throughout the paper and in particular in Section 7, we present several examples which challenge existing numerical methods. In Section 8 we present formulae for the suboptimal H_∞ problem after the γ-iteration is complete. These formulae avoid some of the numerical difficulties of the original formulae [35, 41, 45]. Finally, in Section 8, we also discuss nonclassical optimal or near optimal controls.

2 Preliminaries

In this section we introduce some notation and definitions. By $\mathbb{R}^{n,k}$ we denote the set of real $n \times k$ matrices. For symmetric matrices A and B, $A \geq B$ and $A > B$ mean that $A - B$ is positive semidefinite and positive definite, respectively. By $A^\dagger \in \mathbb{R}^{k,n}$ we denote the Moore-Penrose inverse of the matrix $A \in \mathbb{R}^{n,k}$ [11].

Definition 2.1

a) Let $A \in \mathbb{R}^{n,n}$ and $B \in \mathbb{R}^{n,m}$.

   i) (A, B) is *controllable* if $\operatorname{rank}[A - \lambda I, B] = n$ for all complex λ;

   ii) (A, B) is *stabilizable* if $\operatorname{rank}[A - \lambda I, B] = n$ for all λ in the closed right half complex plane;

b) Let $A \in \mathbb{R}^{n,n}$ and $C \in \mathbb{R}^{m,n}$. (A, C) is *observable* (*detectable*) if (A^T, C^T) is controllable (stabilizable).

c) An eigenvalue λ of A is *stable (semi-stable)* if its real part is negative (zero).

d) A square matrix A is *stable* (*semi-stable*) if all the eigenvalues of A are in the open (closed) left half complex plane.

Definition 2.2 Let $\mathcal{J} := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, where I_n is the $n \times n$ identity matrix.

a) A matrix $\mathcal{H} \in \mathbb{R}^{2n,2n}$ is *Hamiltonian* if $(\mathcal{H}\mathcal{J})^T = \mathcal{H}\mathcal{J}$ and $\mathcal{H} \in \mathbb{R}^{2n,2n}$ is *skew-Hamiltonian* if $(\mathcal{H}\mathcal{J})^T = -\mathcal{H}\mathcal{J}$.

b) A matrix $\mathcal{Z} \in \mathbb{R}^{2n,2n}$ is *symplectic* if $\mathcal{Z}\mathcal{J}\mathcal{Z}^T = \mathcal{J}$ and $\mathcal{U} \in \mathbb{R}^{2n,2n}$ is *orthogonal symplectic* if $\mathcal{U}\mathcal{J}\mathcal{U}^T = \mathcal{J}$ and $\mathcal{U}^T\mathcal{U} = I_{2n}$.

c) An invariant subspace $\mathcal{L}$ of a Hamiltonian matrix $\mathcal{H} \in \mathbb{R}^{2n,2n}$ is *Lagrangian* if it is n-dimensional and $x^H J y = 0$, for all x, $y \in \mathcal{L}$.

d) An invariant subspace $\mathcal{L}$ of a Hamiltonian matrix $\mathcal{H} \in \mathbb{R}^{2n,2n}$ is *stable (semi-stable) Lagrangian* if it is Lagrangian and corresponds to the eigenvalues of $\mathcal{H}$ in the open (closed) left half complex plane.

Real Hamiltonian matrices take the form

$$\mathcal{H} = \begin{bmatrix} F & -G \\ -K & -F^T \end{bmatrix}, \tag{4}$$

where F, G, $K \in \mathbb{R}^{n,k}$, $G = G^T$ and $K = K^T$. To each Hamiltonian matrix there corresponds an *algebraic Riccati equation*

$$F^T X + XF + K - XGX = 0, \tag{5}$$

which is often encountered in control design.

Definition 2.3 A matrix X is a *stabilizing* (*semi-stabilizing*) solution of (5) if $X = X^T$ and $F - GX$ is stable (semi-stable).

It is well known [25, 30] and easy to verify that if X is a stabilizing (semi-stabilizing) solution of the algebraic Riccati equation (5), then the columns of $\begin{bmatrix} I_n \\ X \end{bmatrix}$ span a stable (semi-stable) Lagrangian invariant subspace of the Hamiltonian matrix (4). Conversely, if the columns of $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ span a stable (semi-stable) Lagrangian invariant subspace of the Hamiltonian matrix (4) and X_1 is nonsingular, then $X = X_2 X_1^{-1}$ is a stabilizing (semi-stabilizing) solution of the algebraic Riccati equation (5).

If G and K are positive semidefinite, (F, G) is stabilizable and (F, K) is detectable, then the Hamiltonian matrix (4) has a unique stable Lagrangian invariant subspace and a corresponding unique, stabilizing positive semidefinite solution of the algebraic Riccati equation (5). For more details and weaker assumptions see [25, 30]. However, if stabilizability or detectability or semidefiniteness fails, then the correspondence between the Lagrangian invariant subspaces and solutions of (5) is not necessarily one-to-one; the Lagrangian subspace $\mathcal{L} = \operatorname{span} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ corresponds to a solution of the algebraic Riccati equation if and only if X_1 is nonsingular. In particular, it is possible for the Riccati equation to have no semi-stabilizing solution despite the corresponding Hamiltonian matrix having a unique, stable, Lagrangian subspace [17].

Example 2.4 The one-dimensional algebraic Riccati equation $0x^2 + 2x + 1 = 0$ has no positive semidefinite solution, but the corresponding Hamiltonian matrix $\mathcal{H} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ has the unique Lagrangian invariant subspace $\mathcal{L} = \operatorname{span} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, corresponding to the eigenvalue -1. (In this example $K = [1]$, $F = [1]$ and $G = [0]$, so the pair (K, G) is not stabilizable.)

Example 2.5 The one dimensional algebraic Riccati equation $x^2 + 3x + 2 = 0$ has no positive semidefinite solution, but its corresponding Hamiltonian matrix $\mathcal{H} = \begin{bmatrix} 1.5 & -1 \\ 2 & -1.5 \end{bmatrix}$ has the unique Lagrangian invariant subspace $\mathcal{L} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ corresponding to the eigenvalue $-1/2$. In this example $G = [-1]$ which is not positive semidefinite.

Conventional numerical methods for the modified optimal H_∞ control problem require the computation of the stabilizing solution of Riccati equations of the form (5) in which F and/or G are not necessarily semidefinite or for which (K, F) is not stabilizable or (K, G) is not detectable [25]. Such algebraic Riccati equations may have no positive semidefinite solution. This is one source of trouble in numerical methods for the modified optimal H_∞ control problem. This paper presents a numerical method that circumvents this problem by working directly with the stabilizing Lagrangian subspace.

3 Theoretical Background

In this section we discuss the theoretical background for the modified optimal H_∞ problem. We start with a typical set of assumptions [23, 22, 34, 45].

Assumptions:

A1. The pair (A, B_2) is *stabilizable* and the pair (A, C_2) is *detectable.*

A2. $D_{22} = 0$ and both D_{12} and D_{21} have full rank.

A3. The matrix $\begin{bmatrix} A-i\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all real ω.

A4. The matrix $\begin{bmatrix} A-i\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all real ω.

Remark 3.1 The requirement that $D_{22} = 0$ (Assumption A2) is for convenience. It is not a fundamental restriction, since systems that have a direct link from input to output, i.e., for which $D_{22} \neq 0$, can be synthesized by first studying the problem without this term, see [45].

In the literature, it is often assumed that $D_{12} = \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}$ and $D_{21} = [0,\ I_{p_2}]$ and that $D_{11} = 0$. In principle, this particular form can be obtained from a more general system by transforming the system in advance, i.e., by choosing suitable bases for the state, inputs and outputs. This special form for D_{12}, D_{21} and D_{11} is not required for the numerical method that we discuss here, and reducing the system to this form may require ill-conditioned transformations that lead to unnecessary numerical errors. For this reason we allow general D_{12}, D_{21} and D_{11} subject to Assumption A2.

Note that this leads to slightly different solution formulae for the optimal feedbacks and the closed-loop system than those given in [23, 45], see Section 8.

To formulate the basic theorem of H_∞ control, we introduce the following two symmetric matrices formed from the matrices D_{ij} and a parameter $\gamma \in \mathbb{R}$,

$$\begin{aligned} R_H(\gamma) &:= \begin{bmatrix} D_{11}^T \\ D_{12}^T \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}, \\ R_J(\gamma) &:= \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} \begin{bmatrix} D_{11}^T & D_{21}^T \end{bmatrix} - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{6}$$

These matrices play an important role in the theory of optimal H_∞ control problems, and the classical numerical methods require both $R_H(\gamma)$ and $R_J(\gamma)$ to be nonsingular. To check when this is the case, we use the following lemma.

Proposition 3.2 *If Assumption A2 is not satisfied, then either D_{12} or D_{21} is rank deficient. In case D_{12} is rank deficient, then R_H is singular for all $\gamma \in \mathbb{R}$ and if D_{21} is rank deficient then R_J is singular for all $\gamma \in \mathbb{R}$.*

If Assumption A2 holds, then there exist only a finite number of nonnegative values γ for which one or both of the matrices $R_H(\gamma)$ or $R_J(\gamma)$ is singular.

Definition 3.3 Let

$$\begin{aligned} \hat{\gamma}_H &:= \max\{\gamma \in \mathbb{R} \mid R_H(\gamma) \text{ is singular }\}, \\ \hat{\gamma}_J &:= \max\{\gamma \in \mathbb{R} \mid R_J(\gamma) \text{ is singular }\}, \\ \hat{\gamma} &:= \max\{\hat{\gamma}_H, \hat{\gamma}_J\}. \end{aligned}$$

Denote the set of all values $\gamma \geq 0$, for which both $R_H(\gamma)$ and $R_J(\gamma)$ are nonsingular by Γ_{Rns}.

Remark 3.4 The set Γ_{Rns} may not be connected and may have elements $\gamma < \hat{\gamma}$, see Example 7.4.

Remark 3.5 If $D_{11} = 0$, then $\hat{\gamma} = 0$. If $D_{11} \neq 0$, $\hat{\gamma}$ is typically positive.

Let

$$\begin{aligned} D_{12} &= U_{12} \begin{bmatrix} 0 \\ \Sigma_{12} \end{bmatrix} V_{12}^T, \\ D_{21} &= V_{21} \begin{bmatrix} 0 & \Sigma_{21} \end{bmatrix} U_{21}^T, \end{aligned}$$

be (slightly permuted) singular value decompositions of D_{12} and D_{21} with real orthogonal matrices U_{12}, U_{21}, V_{12}, V_{21} and nonnegative diagonal matrices Σ_{12} and Σ_{21}. The diagonal entries of Σ_{12} and Σ_{21} are the singular values of D_{12} and D_{21}, respectively. We may use the orthogonal equivalence transformation

$$\left[\begin{array}{c|c} U_{12}^T & 0 \\ \hline 0 & V_{21}^T \end{array}\right] \left[\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & 0 \end{array}\right] \left[\begin{array}{c|c} U_{21} & 0 \\ \hline 0 & V_{12} \end{array}\right] = \left[\begin{array}{cc|c} D_1 & D_2 & 0 \\ D_3 & D_4 & \Sigma_{12} \\ \hline 0 & \Sigma_{21} & 0 \end{array}\right] \tag{7}$$

to define D_1, D_2, D_3 and D_4.

Proposition 3.6 *[45] If Assumption A2 holds, then*

$$\hat{\gamma}_H = \sigma_{max} \left[\begin{array}{cc} D_1 & D_2 \end{array} \right], \ \hat{\gamma}_J = \sigma_{max} \left[\begin{array}{c} D_1 \\ D_3 \end{array} \right],$$

and the following equivalences hold.

i) $R_H(\gamma)$ is invertible if and only if $D_1 D_1^T + D_2 D_2^T - \gamma^2 I$ is invertible.

ii) $R_J(\gamma)$ is invertible if and only if $D_1^T D_1 + D_3^T D_3 - \gamma^2 I$ is invertible.

The next theorem gives the theoretical basis for the γ-iteration.

Theorem 3.7 *[45]. Consider system (1), with R_H and R_J as in (6). Under assumptions A1–A4, there exists an internally stabilizing controller such that the transfer function from w to z satisfies $\|T_{zw}\|_\infty < \gamma$ if and only if the following four conditions hold.*

1. *$\gamma > \hat{\gamma}$ with $\hat{\gamma}$ as in Definition 3.3.*

2. *There exists a stabilizing positive semidefinite solution $X_H = X_H(\gamma)$ of the algebraic Riccati equation associated with the Hamiltonian matrix*

$$H(\gamma) = \left[\begin{array}{cc} A & 0 \\ -C_1^T C_1 & -A^T \end{array} \right] - B_H R_H^{-1}(\gamma) B_H^T \mathcal{J}, \tag{8}$$

where $B_H = \left[\begin{smallmatrix} B_1 & B_2 \\ -C_1^T D_{11} & -C_1^T D_{12} \end{smallmatrix}\right]$.

3. *There exists a stabilizing positive semidefinite solution $X_J = X_J(\gamma)$ of the algebraic Riccati equation associated with the Hamiltonian matrix*

$$J(\gamma) = \left[\begin{array}{cc} A^T & 0 \\ -B_1 B_1^T & -A \end{array} \right] - \mathcal{J} C_J^T R_J^{-1}(\gamma) C_J, \tag{9}$$

where $C_J := \left[\begin{smallmatrix} D_{11} B_1^T & C_1 \\ D_{21} B_1^T & C_2 \end{smallmatrix}\right]$.

4. $\gamma^2 > \rho(X_H X_J)$ *(where* $\rho(X_H X_J)$ *is the spectral radius of* $X_H X_J$*).*

The solution to the suboptimal control problem, γ_{mo}, is the infimum of the set of all $\gamma > 0$ such that the four conditions in Theorem 3.7 hold. Equivalently, γ_{mo} is the supremum of all $\gamma \geq 0$ for which at least one of the conditions in Theorem 3.7 fails.

4 Current Numerical Methods

This section discusses finite precision arithmetic hazards encountered by typical numerical methods for checking the four conditions in Theorem 3.7. Finite precision hazards are also discussed in [18, 23, 41].

Currently, numerical methods for the solution of the modified optimal H_∞ problem [35, 41] fall into two categories. The first embeds the problem into a linear matrix inequality [9] and then employs methods of semidefinite programming to find γ_{mo}. This is attractive, because easy-to-use methods for semidefinite programming are available, see, e.g., [19, 32]. However, the computational complexity of this approach is up to $\mathcal{O}(n^6)$ which is unfavorably high, so it is practical only for low-dimensional problems.

The second category is the category of *Riccati methods.* A typical Riccati method uses Theorem 3.7 to find upper and lower bounds on γ_{mo} which are then refined by bisection also using Theorem 3.7. A quadratically convergent method based on Newton's method can be found in [38]. Each iterative step includes checking whether $\gamma > \hat{\gamma}$, using a Riccati solver like those discussed in [2, 3, 30, 39] to compute stabilizing solutions X_H and X_J (if they exist) to the Riccati equations associated with the Hamiltonian matrices $H(\gamma)$ and $J(\gamma)$ in (8) and (9), and then checking whether $\gamma^2 > \rho(X_H X_J)$. This approach has complexity $\mathcal{O}(n^3)$ per step and is currently the only practical choice for higher dimensional problems.

Unfortunately there are several numerical difficulties associated with the Riccati method. Primary among these is the fact that often as γ approaches γ_{mo}, one of the Riccati solutions X_H or X_J either diverges to ∞ or becomes highly ill-conditioned, i.e., tiny errors in the Hamiltonian matrices $H(\gamma)$ or $J(\gamma)$ may lead to large errors in X_H or X_J. The following example, which we will use frequently, demonstrates this effect.

Example 4.1 Consider the system

$$\left[\begin{array}{c|c|c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{cc|cc|c} -1 & 0 & \epsilon_1 & 0 & 1 \\ 0 & -1 & 0 & \epsilon_2 & 1 \\ \hline \alpha & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \beta & 0 & \frac{1}{2} & 1 \\ \hline \delta & \eta & 0 & 1 & 0 \end{array}\right].$$

If $\epsilon_1 = \epsilon_2 = 0$, then (6) becomes

$$R_H(\gamma) = R_J(\gamma) = \begin{bmatrix} \frac{1}{4} - \gamma^2 & 0 & 0 \\ 0 & \frac{1}{4} - \gamma^2 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{bmatrix},$$

and $\hat{\gamma} = \frac{1}{2}$. With $\zeta(\gamma) := 1 - \frac{1}{4}\gamma^{-2}$, the Hamiltonian matrices (8) and (9) become

$$H(\gamma) \quad = \quad \left[\begin{array}{cc|cc} -1 & -\beta & -\zeta(\gamma) & -\zeta(\gamma) \\ 0 & -1-\beta & -\zeta(\gamma) & -\zeta(\gamma) \\ \hline -\frac{\alpha^2}{\zeta(\gamma)} & 0 & 1 & 0 \\ 0 & 0 & \beta & 1+\beta \end{array}\right]$$

and

$$J(\gamma) \quad = \quad \left[\begin{array}{cc|cc} -1 & 0 & \alpha^2\gamma^{-2}\zeta^{-1}(\gamma) - \delta^2\zeta(\gamma) & -\frac{\beta\delta}{2}\gamma^{-2} - \delta\eta\zeta(\gamma) \\ 0 & -1 & -\frac{\beta\delta}{2}\gamma^{-2} - \delta\eta\zeta(\gamma) & (\beta-\eta)\beta\gamma^{-2} - \eta^2\zeta(\gamma) \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

The matrix $J(\gamma)$ has the double stable eigenvalue -1 and the corresponding positive semidefinite Riccati solution is $X_J = 0$. The matrix $H(\gamma)$ has two eigenvalues -1, $-\sqrt{(1+\beta)^2+\alpha^2}$ in the open left half complex plane. When $\gamma > \frac{1}{2}$, setting $\nu := 1 + \sqrt{(1+\beta)^2+\alpha^2}$, the positive semidefinite Riccati solution corresponding to $H(\gamma)$ is

$$X_H = \frac{\alpha^2}{\zeta(\gamma)(2\beta+\beta^2+\alpha^2)} \cdot \begin{bmatrix} \frac{\beta(2+\beta)}{2} + \frac{\alpha^2}{\nu} & \beta(2+\beta)\left(\frac{1}{\nu} - \frac{1}{2}\right) \\ \beta(2+\beta)\left(\frac{1}{\nu} - \frac{1}{2}\right) & \beta^2\left(\frac{1}{2} - \frac{(2+\beta)}{\nu(\nu+\beta)}\right) \end{bmatrix}.$$

(If $\beta^2 + 2\beta + \alpha^2 = 0$, then

$$X_H = \frac{\alpha^2}{8\zeta(\gamma)} \begin{bmatrix} 4-\alpha^2 & \alpha^2 \\ \alpha^2 & \beta^2\left(1 + \frac{2}{2+\beta}\right) \end{bmatrix}.$$

Note in this case $|\alpha| \leq 1$ and $\beta = -1 \pm \sqrt{1-\alpha^2}$. Moreover, $H(\gamma)$ has the double eigenvalues 1 and -1.)

Since the semi-stabilizing Riccati solutions X_H and X_J exist and $\rho(X_J X_H) = 0$ for all $\gamma > \hat{\gamma}$, we have $\gamma_{mo} = \hat{\gamma} = \frac{1}{2}$. As γ approaches γ_{mo}, $\zeta(\gamma)$ approaches 0. R_H and R_J become singular, the Hamiltonian matrices $H(\gamma)$ and $J(\gamma)$ become ill-defined, and the Riccati solution X_H converges to infinity.

Typical numerical Riccati solvers are unable to succeed on problems as extreme as those in Example 4.1 with $\gamma \approx \hat{\gamma}$. Failing to solve a Riccati equation may cause a computation to abort before attaining a close approximation to γ_{mo} [23]. In the most extreme case, $H(\gamma)$, $J(\gamma)$ or a Riccati solution may have entries larger than the overflow threshold and not be representable in the working floating point number system, thus guaranteeing failure of any numerical method that explicitly constructs any of the matrices in Theorem 3.7!

A more subtle and more likely problem (also observed in [18]) is that explicitly forming the Hamiltonian matrices themselves may lead to large inaccuracies. If the matrices $R_H(\gamma)$ or $R_J(\gamma)$ are ill-conditioned or if cancellation errors occur in computing the blocks of $H(\gamma)$ and $J(\gamma)$, then the input data for the Riccati solvers may be corrupted. Example 4.1 demonstrates how the matrices $R_H(\gamma)$ and $R_J(\gamma)$ become nearly singular and highly ill-conditioned as γ approaches $\hat{\gamma} = \gamma_{mo}$.

As suggested in [14, 18], a suitable embedding of the Hamiltonian matrices into a matrix pencil may avoid problems caused by explicitly forming the Hamiltonian matrices. In Section 5, we will discuss an approach similar to the structured embedding technique introduced by [4]. The method discussed in [18] avoids much of the trouble with the Riccati method, but it has some drawbacks. First of all, it still computes explicit solutions to the Riccati equations. Also, since it uses the general QZ algorithm to compute deflating subspaces, it does not make use or preserve the special structure of the Hamiltonian problem. This becomes critical when there are eigenvalues close to or on the imaginary axis as may happen near γ_{mo}. Unstructured numerical methods are not reliable when there are eigenvalues on or near the imaginary axis, see [17].

One way to overcome the problem of exploding Riccati solutions is to replace the Riccati equation solutions by Lagrangian invariant subspaces, see [18, 31, 45]. We will discuss and modify this approach in Section 5.

To facilitate our discussion, we need notation for several critical points of γ that play roles determining γ_{mo}.

Definition 4.2 Define $\hat{\gamma}_H^R$, $\hat{\gamma}_J^R$ and $\hat{\gamma}^R$ as

$$\begin{aligned}
\hat{\gamma}_H^R &= \inf\left\{\gamma \geq \hat{\gamma} \;\middle|\; \begin{array}{l}\text{The Riccati equation corresponding to } H(\gamma) \\ \text{in (8) has a positive semi-definite, semi-} \\ \text{stabilizing solution.}\end{array}\right\} \\
\hat{\gamma}_J^R &= \inf\left\{\gamma \geq \hat{\gamma} \;\middle|\; \begin{array}{l}\text{The Riccati equation corresponding to } J(\gamma) \\ \text{in (9) has a positive semi-definite, semi-} \\ \text{stabilizing solution.}\end{array}\right\} \\
\hat{\gamma}^R &= \max(\hat{\gamma}_H^R, \hat{\gamma}_J^R)
\end{aligned}$$

Let Γ_{ns} be the set of all $\gamma > 0$ for which both the Riccati equations associated with $H(\gamma)$ and $J(\gamma)$ in (8) and (9) have stabilizing or semi-stabilizing positive semidefinite solutions.

Definition 4.3 Define $\hat{\gamma}_H^L$, $\hat{\gamma}_J^L$ and $\hat{\gamma}^L$ as

$$\begin{aligned}
\hat{\gamma}_H^L &= \inf\left\{\gamma \geq \hat{\gamma} \;\middle|\; \begin{array}{l}\text{The Hamiltonian matrix } H(\gamma) \text{ in (8) has a} \\ \text{semi-stable Lagrangian invariant subspace.}\end{array}\right\} \\
\hat{\gamma}_J^L &= \inf\left\{\gamma \geq \hat{\gamma} \;\middle|\; \begin{array}{l}\text{The Hamiltonian matrix } J(\gamma) \text{ in (9) has a} \\ \text{semi-stable Lagrangian invariant subspace.}\end{array}\right\} \\
\hat{\gamma}^L &= \max(\hat{\gamma}_H^L, \hat{\gamma}_J^L)
\end{aligned}$$

Let Γ_L be the set of all $\gamma > 0$ for which both $H(\gamma)$ and $J(\gamma)$ in (8) and (9) have semi-stable Lagrangian invariant subspaces.

Definition 4.4 Define $\hat{\gamma}_H^I$, $\hat{\gamma}_J^I$ and $\hat{\gamma}^I$ as

$$\begin{aligned}
\hat{\gamma}_H^I &= \sup\left\{\gamma > \hat{\gamma} \;\middle|\; \begin{array}{l}\text{The Hamiltonian matrix } H(\gamma) \text{ in (8) has an} \\ \text{eigenvalue on the imaginary axis.}\end{array}\right\} \\
\hat{\gamma}_J^I &= \sup\left\{\gamma > \hat{\gamma} \;\middle|\; \begin{array}{l}\text{The Hamiltonian matrix } J(\gamma) \text{ in (9) has an} \\ \text{eigenvalue on the imaginary axis.}\end{array}\right\} \\
\hat{\gamma}^I &= \max(\hat{\gamma}_H^I, \hat{\gamma}_J^I)
\end{aligned}$$

If both $H(\gamma), J(\gamma)$ have no eigenvalues on the imaginary axis for all $\gamma > \hat{\gamma}$, $\hat{\gamma}^I$ does not exist.

Let Γ_{ni} be the set of all $\gamma > 0$ for which neither $H(\gamma)$ in (8) nor $J(\gamma)$ in (9) have an eigenvalue on the imaginary axis.

The eigenvalues of real Hamiltonian matrices are symmetric about both the real axis and the imaginary axis, see [25, 27, 30]. Eigenvalues with nonzero real and imaginary parts occur in quadruples consisting of two $\pm$ pairs, λ, $-\lambda$, $\bar{\lambda}$, $-\bar{\lambda}$. Real eigenvalues and pure imaginary eigenvalues appear in $\pm$ pairs. If a Hamiltonian matrix has no eigenvalue on the imaginary axis, then the corresponding Riccati solution can be constructed from the invariant subspace corresponding the eigenvalues in the open left-half plane. (See the comments following Definition 2.3.) It is now common for numerical methods for solving Riccati equations to do exactly that [3, 26, 30, 39]. Unfortunately, if γ_{mo} is near to $\hat{\gamma}^I$ (if it exists) then, due to roundoff errors, it may be difficult to determine whether an eigenvalue has zero real part. Non-Hamiltonian structured perturbations introduced by rounding errors may move ill-conditioned or near-imaginary axis eigenvalues into the wrong half plane.

When $\gamma = \hat{\gamma}^I$, $H(\gamma)$ or $J(\gamma)$ have eigenvalues on the imaginary axis. Even with otherwise robust numerical methods like the QR algorithm, rounding errors made while calculating eigenvalues and invariant subspaces may introduce non-Hamiltonian perturbations of the Hamiltonian matrix. Unstructured, non-Hamiltonian rounding errors may destroy the uniqueness of the semi-stable Lagrangian invariant subspace [36, 37] causing a Riccati solver to fail. Even the number of eigenvalues in the closed left-half plane may drop below its theoretical minimum of n. The following example illustrates this effect. (See also Section 7.)

Example 4.5 Consider the Hamiltonian matrix with non-Hamiltonian perturbation

$$H(\varepsilon) := H + E := \left[\begin{array}{rr|rr} -2 & 1 & 1 & 0 \\ -4 & 2 & 0 & 1 \\ \hline 0 & 0 & 2 & 4 \\ 0 & 0 & -1 & -2 \end{array}\right] + \left[\begin{array}{cc|cc} \varepsilon & 0 & 0 & 0 \\ \varepsilon & \varepsilon & 0 & 0 \\ \hline 0 & 0 & \varepsilon & \varepsilon \\ 0 & 0 & 0 & \varepsilon \end{array}\right],$$

The unperturbed matrix ($\varepsilon = 0$) has eigenvalue $\lambda = 0$ with algebraic multiplicity 4 and geometric multiplicity 1. There is a unique, Lagrangian semi-stable, two-dimensional invariant subspace which is spanned by the first two columns of the identity matrix. The situation is visualized in Figure 2.

The perturbed matrix ($\varepsilon \neq 0$) has eigenvalues $\varepsilon \pm \sqrt{\varepsilon}$ and $\varepsilon \pm \sqrt{-\varepsilon}$. The symmetry in the spectrum is lost. If $-1 < \varepsilon < 0$, then three eigenvalues lie in the open left-half plane and there are three two-dimensional stable invariant subspaces. (Despite the non-Hamiltonian perturbation, one of the two-dimensional stabilizing invariant subspaces is Lagrangian.) If $1 > \varepsilon > 0$, then only one eigenvalue lies in the open left-half plane and there is no two-dimensional stable invariant subspace.

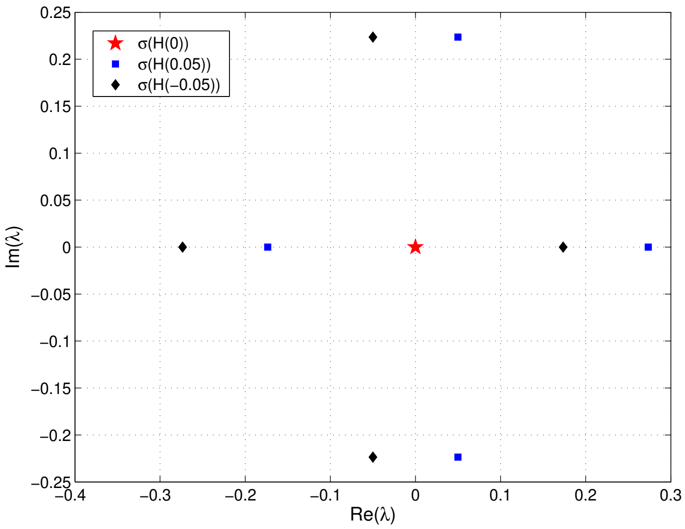

Figure 2: The spectra of the unperturbed matrix $H(0)$ and the perturbed matrices $H(\pm\varepsilon)$ for $\varepsilon = 0.05$.

Many Riccati equation solvers begin their work by extracting the stable invariant subspace of a Hamiltonian matrix [3, 26, 30, 39]. A naive algorithm applied to this Hamiltonian matrix may select an incorrect invariant subspace and either conclude that there is no solution to the Riccati equation or simply return a far-from-symmetric and/or non-stabilizing solution. Note that the signs of the real parts of eigenvalues are of little help in determining which of the two-dimensional invariant subspaces might be used—even when $\varepsilon = 0$! The unperturbed Hamiltonian matrix has a quadruple eigenvalue; the loss of the eigensymmetry in the perturbed matrix obscures the semi-stable Lagrangian invariant subspace.

Note that a Hamiltonian perturbation of H in Example 4.5 could not create confusion, because the $\pm$ pairing of eigenvalues would be preserved. A numerically stable algorithm that fully exploits Hamiltonian structure would introduce Hamiltonian-structured rounding errors and avoid some of the discussed numerical difficulties. The design of such a method is a difficult problem which is still partially unsolved. But recent progress has lead to new methods [1, 5, 7, 8] that are almost ideal in the sense that they are numerically stable and exploit the structure of the Hamiltonian matrices to a very large extent.

Another problem that arises from non-Hamiltonian rounding errors perturbing a Hamiltonian matrix is that the uniqueness of the semi-stable Lagrangian invariant subspace may be lost [17, 36, 37].

Example 4.6 Consider Example 4.1 with $\alpha = \beta = \delta = \eta = \epsilon_2 = 1$ and $\epsilon_1 = 0$. In this case, the Riccati solution associated with (9) has semi-stabilizing solution $X_J = 0$, independent of γ. We constructed $H(\gamma)$ in (8) for 91 values of γ equally spaced in the interval $[0.1, 1]$ and used the MATLAB builtin function `eig` (based on the QR algorithm, see, e.g., [21]) to calculate the eigenvalues of each $H(\gamma)$. In no case did any computed eigenvalue have zero real part. If a Hamiltonian matrix has no eigenvalue with zero real part, then there is a unique stabilizing solution of the corresponding algebraic Riccati equation. A naive program to calculate γ_{mo} might use this to conclude that the algebraic Riccati equation corresponding to each $H(\gamma)$ has a stabilizing solution for $\gamma \in [.5, 1]$. Such a program might even construct "solutions" X_H, calculate $\rho(X_H X_J) = 0$ and ultimately conclude that $\gamma_{mo} = \hat{\gamma} = 1/2$.

In fact, $\gamma_{mo} = \hat{\gamma}^I \approx 0.806$. In this example, the algebraic Riccati equation corresponding to (8) has a stabilizing positive semidefinite solution if and only if $\gamma > \hat{\gamma}^I$. As γ approaches $\hat{\gamma}^I$, a $\pm\lambda$ pair of real eigenvalues of the Hamiltonian matrix $H(\gamma)$ in (8) coalesces into a double eigenvalue at 0 corresponding to a 2-by-2 Jordan block. As γ decreases further, this double eigenvalue splits into two complex conjugate eigenvalues with zero real part.

Rounding errors constructing $H(\gamma)$ and computing its eigenvalues perturb eigenvalues off the imaginary axis. If these rounding errors are of magnitude ε then the eigenvalues of the 2-by-2 Jordan block are perturbed by $O(\sqrt{\varepsilon})$. Similar eigenvalue perturbations result from perturbations of γ near $\hat{\gamma}^I$. Thus, eigenvalues may be relatively distant from the imaginary axis even when $\hat{\gamma} \approx \hat{\gamma}^I$. Consequently, it is problematical to use the computed eigenvalues to determine whether $H(\gamma)$ has eigenvalues with zero real part and whether the corresponding algebraic Riccati equation has a stabilizing solution.

Definition 4.7 Let $X_H = X_H(\gamma)$, $X_J = X_J(\gamma)$ be the positive semidefinite stabilizing solutions of the Riccati equations associated with $H(\lambda)$ and $J(\lambda)$ in Theorem 3.7, respectively. Define $\hat{\gamma}^\rho$ to the largest number $\gamma \geq \hat{\gamma}$ satisfying $\gamma^2 = \rho(X_H X_J)$. If no such number γ exists, then $\hat{\gamma}^\rho$ does not exist.

Under Assumptions A1-A4 ,$\hat{\gamma}$'s defined in this section satisfy

$$0 \leq \hat{\gamma} \leq \hat{\gamma}^L \leq \hat{\gamma}^R.$$

Table 1: Summary definitions of the $\hat{\gamma}$'s. Here $X_H = X_H(\gamma)$ and $X_J = X_J(\gamma)$ are the positive semi-definite stabilizing solutions of the Riccati equations corresponding to the Hamiltonian matrices $H(\gamma)$ and $J(\gamma)$ in Theorem 3.7. The $\hat{\gamma}$'s satisfy $0 \leq \hat{\gamma} \leq \hat{\gamma}^L \leq \hat{\gamma}^R$. If $\hat{\gamma}^I$ exists, $\hat{\gamma}^I = \hat{\gamma}^L > \hat{\gamma}$. If $\hat{\gamma}^\rho$ exists, then $\hat{\gamma}^\rho \geq \hat{\gamma}^R$.

$\hat{\gamma}$	$\sup\{\gamma \geq 0 \,\vert\,$ either R_H or R_J is singular. $\}$
$\hat{\gamma}^L$	$\inf\left\{\gamma \geq \hat{\gamma} \,\middle\vert\, \text{Both } H(\gamma) \text{ and } J(\gamma) \text{ have a semi-stable Lagrangian invariant subspace.}\right\}$
$\hat{\gamma}^R$	$\inf\{\gamma \geq \hat{\gamma} \,\vert\,$ Both $X_H(\gamma)$ and $X_J(\gamma)$ exist. $\}$
$\hat{\gamma}^I$	$\sup\left\{\gamma > \hat{\gamma} \,\middle\vert\, \text{Either } H(\gamma) \text{ or } J(\gamma) \text{ has an eigenvalue with zero real part. Note that } \hat{\gamma}^I \text{ may or may not exist.}\right\}$
$\hat{\gamma}^\rho$	Largest number $\gamma \geq \hat{\gamma}$ satisfying $\gamma^2 = \rho(X_H X_J)$. Note that $\hat{\gamma}^\rho$ may or may not exist.

If $\hat{\gamma}^I$ exists, then $\hat{\gamma}^I = \hat{\gamma}^L > \hat{\gamma}$. If $\hat{\gamma}^\rho$ exists, then $\hat{\gamma}^\rho \geq \hat{\gamma}^R$. Note that $\gamma_{mo} = \max(\hat{\gamma}, \hat{\gamma}^L, \hat{\gamma}^R, \hat{\gamma}^\rho)$. Table 1 summarizes the definitions of the different $\hat{\gamma}$'s.

5 Reformulations

The following section reviews the properties of Lagrangian invariant subspaces and Riccati solutions associated with $H(\gamma)$ and $J(\gamma)$ along with the relationship between γ_{mo} and the various $\hat{\gamma}$'s. This section also reformulates Theorem 3.7 in order to overcome numerical difficulties.

5.1 Avoiding explicit solution of Riccati equations

The solution of the algebraic Riccati equations is only an intermediate step toward solving the H_∞ control problem. Avoiding explicit solution of algebraic Riccati equations is the only way to avoid numerical instabilities like those in Example 4.1. A similar situation occurs in the linear quadratic regulator problem and, more generally speaking, in H_2 control problems. There, the solution of algebraic Riccati equations is an intermediate step toward the closed-loop matrix and optimal feedbacks. Explicit Riccati solutions may be avoided by computing deflating subspaces of matrix pencils. This idea was first suggested in [43] and has been modified to a structure-preserving method in [4, 5].

A reformulation of Theorem 3.7 suggested in [45, Theorem 16.4, p. 419] employs this idea. Instead of the existence of positive semidefinite solutions of the two algebraic Riccati equations, it suffices to determine the existence of certain Lagrangian invariant subspaces for the Hamiltonian matrices $H(\gamma)$ and $J(\gamma)$. In particular, as stated in [45], Conditions 2 and 3 in Theorem 3.7 may be replaced by the following alternative conditions.

2'. There exist matrices $X_{H,1}$, $X_{H,2} \in \mathbb{R}^{n,n}$ with $X_{H,1}$ nonsingular such that the columns of $\left[\begin{smallmatrix} X_{H,1} \\ X_{H,2} \end{smallmatrix}\right]$, span a semi-stable Lagrangian invariant subspace of $H(\gamma)$, i.e., there exists a semi-stable matrix T_H for which

$$H(\gamma)\begin{bmatrix} X_{H,1} \\ X_{H,2} \end{bmatrix} = \begin{bmatrix} X_{H,1} \\ X_{H,2} \end{bmatrix} T_H. \tag{10}$$

3'. There exist matrices $X_{J,1}, X_{J,2} \in \mathbb{R}^{n,n}$ with $X_{J,1}$ nonsingular such that the columns of $\left[\begin{smallmatrix} X_{J,1} \\ X_{J,2} \end{smallmatrix}\right]$, span a semi-stable Lagrangian invariant subspace of $J(\gamma)$, i.e., there exists a semi-stable matrix T_J for which

$$J(\gamma)\begin{bmatrix} X_{J,1} \\ X_{J,2} \end{bmatrix} = \begin{bmatrix} X_{J,1} \\ X_{J,2} \end{bmatrix} T_J. \tag{11}$$

(Below, we further reformulate the invariant subspace approach and remove the non-singularity requirement for $X_{H,1}$ and $X_{J,1}$.)

It is well-known that these Lagrangian subspaces could be used to compute the Riccati solutions $X_H = X_{H,2}X_{H,1}^{-1}$ and $X_J = X_{J,2}X_{J,1}^{-1}$, see, e.g., [25, 26]. Hence, at first glance, there seems to be little difference between the subspace formulation and the Riccati formulation. However, the reformulation is helpful, because when $\gamma = \hat{\gamma}_H^I$ (or $\gamma = \hat{\gamma}_J^I$), then $H(\gamma)$ (or

$J(\gamma)$) may have a unique semi-stable Lagrangian subspace but no positive semi-stabilizing solution to the associated Riccati equation. Furthermore, as illustrated in Example 2.4, there exist many Hamiltonian matrices for which the computation of the unique semi-stable Lagrangian invariant subspace is well-conditioned, but the solution of the Riccati equation is ill-conditioned.

5.2 Avoiding the spectral radius condition

In order to avoid explicit Riccati solutions entirely, we must also reformulate Condition 4 of Theorem 3.7, $\rho(X_H X_J) < \gamma^2$, in terms of the invariant semi-stable, Lagrangian invariant subspaces (10) and (11). (See also [45, Section 16.11].) For this purpose, we introduce the following symmetric matrix. Let $\left[\begin{smallmatrix} X_{H,1}(\gamma) \\ X_{H,2}(\gamma) \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} X_{J,1}(\gamma) \\ X_{J,2}(\gamma) \end{smallmatrix}\right]$ be as in (10) and (11), respectively. Set

$$\mathcal{Y}(\gamma) := \left[\begin{array}{cc} \gamma X_{H,2}^T X_{H,1} & X_{H,2}^T X_{J,2} \\ X_{J,2}^T X_{H,2} & \gamma X_{J,2}^T X_{J,1} \end{array}\right]. \tag{12}$$

Note that all the blocks of $\mathcal{Y}$ are functions of γ, even if γ does not appear explicitly in the off-diagonanl blocks. If there is no semi-stable, Lagrangian invariant subspace, then $\mathcal{Y}(\gamma)$ is undefined. We will show that $\mathcal{Y}(\gamma)$ is positive semidefinite with a particular rank if and only if the Riccati solutions X_H and X_J in Theorem 3.7 exist and $\gamma^2 > \rho(X_H X_J)$. First recall that the ranks of $X_H = X_H(\gamma)$ and $X_J = X_J(\gamma)$ are constant for $\gamma > \gamma_{mo}$.

Theorem 5.1 *[15, 24, 44] Under assumptions A1-A4, γ_{mo} exists. The solutions $X_H = X_H(\gamma)$ and $X_J = X_J(\gamma)$ of the algebraic Riccati equations associated with $H(\gamma)$ and $J(\gamma)$ in (8) and (9) as well as the spectral radius $\rho(X_H X_J) = \rho(X_H(\gamma) X_J(\gamma))$ are monotone decreasing functions of γ on the infinite interval $\mathcal{I} = (\gamma_{mo}, \infty)$, i.e., if $\gamma_{mo} < \gamma_1 \le \gamma_2$, then $X_H(\gamma_2) \le X_H(\gamma_1)$, $X_J(\gamma_2) \le X_J(\gamma_1)$ and $\rho(X_H(\gamma_2)X_J(\gamma_2)) \le \rho(X_H(\gamma_1)X_J(\gamma_1))$. In addition, the ranks of $X_H = X_H(\gamma)$ and $X_J = X_J(\gamma)$ are constant on $\mathcal{I}$.*

Proof. See [15, 44]. For particularly complete proofs, see [24, Theorems 2.4, 4.1, 5.1]. □

The following well-known theorem on the *CS decomposition* of orthonormal bases of Lagrangian subspaces helps display the internal structure of $\mathcal{Y}(\gamma)$.

Lemma 5.2 [33] *If X_1, $X_2 \in \mathbb{R}^{n,n}$ and the columns of $\left[\begin{smallmatrix} X_1 \\ X_2 \end{smallmatrix}\right]$ form an orthonormal basis of a Lagrangian subspace, then there exist orthogonal matrices $U \in \mathbb{R}^{n,n}$ and $V \in \mathbb{R}^{n,n}$ such that $U^T X_1 V = C$ and $U^T X_2 V = S$ are both diagonal and $C^2 + S^2 = I$.*

Apply Lemma 5.2 to $\begin{bmatrix} X_{H,1}(\gamma) \\ X_{H,2}(\gamma) \end{bmatrix}$, $\begin{bmatrix} X_{J,1}(\gamma) \\ X_{J,2}(\gamma) \end{bmatrix}$ separating any zero and one diagonal elements of C and S to get

$$U_H^T X_{H,1} V_H = C_H \quad =: \quad \begin{array}{c} \\ r_H \\ k_H \\ n-t_H \end{array} \begin{array}{c} \begin{array}{ccc} r_H & k_H & n-t_H \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Sigma_H & 0 \\ 0 & 0 & I \end{bmatrix} \end{array}, \tag{13}$$

$$U_H^T X_{H,2} V_H = S_H \quad =: \quad \begin{array}{c} \\ r_H \\ k_H \\ n-t_H \end{array} \begin{array}{c} \begin{array}{ccc} r_H & k_H & n-t_H \end{array} \\ \begin{bmatrix} I & 0 & 0 \\ 0 & \Delta_H & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}, \tag{14}$$

$$U_J^T X_{J,1} V_J = C_J \quad =: \quad \begin{array}{c} \\ r_J \\ k_J \\ n-t_J \end{array} \begin{array}{c} \begin{array}{ccc} r_J & k_J & n-t_J \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Sigma_J & 0 \\ 0 & 0 & I \end{bmatrix} \end{array}, \tag{15}$$

$$U_J^T X_{J,2} V_J = S_J \quad =: \quad \begin{array}{c} \\ r_J \\ k_J \\ n-t_J \end{array} \begin{array}{c} \begin{array}{ccc} r_J & k_J & n-t_J \end{array} \\ \begin{bmatrix} I & 0 & 0 \\ 0 & \Delta_J & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}, \tag{16}$$

where $k_H + r_H = t_H$, $k_J + r_J = t_J$, Σ_H, Δ_H, Σ_J and Δ_J are diagonal, nonsingular and satisfy $\Sigma_H^2 + \Delta_H^2 = I$ and $\Sigma_J^2 + \Delta_J^2 = I$. In this notation, the semi-stabilizing Riccati solutions X_H in Theorem 3.7 exists and has rank $k_H = t_H$ if and only if $r_H = 0$. The semi-stabilizing Riccati solution X_J in Theorem 3.7 exists and has rank $k_J = t_J$ if and only if $r_J = 0$. Theorem 5.1 shows that $k_H = \mathrm{rank}(X_H)$ and $k_J = \mathrm{rank}(X_J)$ are constant for $\gamma > \gamma_{mo}$. If $r_H = r_J = 0$, then $k_H = t_H$, $k_J = t_J$ and

$$U_H^T X_H U_H \quad = \quad \begin{array}{c} \\ k_H \\ n-k_H \end{array} \begin{array}{c} \begin{array}{cc} k_H & n-k_H \end{array} \\ \begin{bmatrix} \Delta_H \Sigma_H^{-1} & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \tag{17}$$

$$U_J^T X_J U_J \quad = \quad \begin{array}{c} \\ k_J \\ n-k_J \end{array} \begin{array}{c} \begin{array}{cc} k_J & n-k_J \end{array} \\ \begin{bmatrix} \Delta_J \Sigma_J^{-1} & 0 \\ 0 & 0 \end{bmatrix} \end{array}. \tag{18}$$

Define $\tilde{\mathcal{Y}}(\gamma)$ by

$$\tilde{\mathcal{Y}}(\gamma) = \begin{bmatrix} V_H & 0 \\ 0 & V_J \end{bmatrix} \mathcal{Y}(\gamma) \begin{bmatrix} V_H^T & 0 \\ 0 & V_J^T \end{bmatrix} \tag{19}$$

$$= \begin{array}{c} \\ r_H \\ k_H \\ n-t_H \\ r_J \\ k_J \\ n-t_J \end{array} \begin{array}{c} \begin{array}{cccccc} r_H & k_H & n-t_H & r_J & k_J & n-t_J \end{array} \\ \left[\begin{array}{cccccc} 0 & 0 & 0 & Q_{11} & Q_{12}\Delta_J & 0 \\ 0 & \gamma\Delta_H\Sigma_H & 0 & \Delta_H Q_{21} & \Delta_H Q_{22}\Delta_J & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ Q_{11}^T & Q_{21}^T\Delta_H & 0 & 0 & 0 & 0 \\ \Delta_J Q_{12}^T & \Delta_J Q_{22}^T\Delta_H & 0 & 0 & \gamma\Delta_J\Sigma_J & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

where the blocks Q_{11}, Q_{12}, Q_{21}, and Q_{22} are sub-blocks of the orthogonal matrix

$$U_H^T U_J = \begin{array}{c} \\ r_H \\ k_H \\ n-t_H \end{array} \begin{array}{c} \begin{array}{ccc} r_J & k_J & n-t_J \end{array} \\ \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \end{array}. \tag{20}$$

The following lemma shows the relationship between $\mathcal{Y}(\gamma)$, X_H, X_J, and $\gamma^2 - \rho(X_H X_J)$.

Lemma 5.3 *Let $\hat{k}_H$ and $\hat{k}_J$ be the ranks of $X_{H,1}(\gamma)$ and $X_{J,1}(\gamma)$ when $\gamma = \infty$.*

i) $\mathcal{Y}(\gamma) \geq 0$ if and only if each of the blocks Q_{11}, Q_{12}, Q_{21} in (20) are either zero or void and

$$\begin{bmatrix} \gamma\Delta_H\Sigma_H & \Delta_H Q_{22}\Delta_J \\ \Delta_J Q_{22}^T \Delta_H & \gamma\Delta_J\Sigma_J \end{bmatrix} \geq 0.$$

ii) $\mathcal{Y}(\gamma) \geq 0$ and $\operatorname{rank}\mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$ if and only if the (semi-)stabilizing, positive semidefinite Riccati solutions X_H and X_J in Theorem 3.7 exist and $\gamma^2 > \rho(X_H X_J)$.

Proof. The matrix $\mathcal{Y}(\gamma)$ is a congruence transformation of $\tilde{\mathcal{Y}}(\gamma)$ in (19). Hence $\mathcal{Y}(\gamma)$ is positive (semi)definite if and only if $\tilde{\mathcal{Y}}$ is positive (semi)definite. Statement *i)* now follows immediately from (19).

ii) Note that $n-t_H$ and $n-t_j$ are independent of γ, [24]. By Theorem 5.1, when $\gamma = \infty$, $\hat{k}_H = n-t_H$, $\hat{k}_J = n-t_J$. If $\mathcal{Y}(\gamma) \geq 0$ and $\operatorname{rank}\mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$, then it follows from (19) that $r_H = r_J = 0$. If $X_H = X_H(\gamma)$ and $X_J = X_J(\gamma)$ exist, then it follows from (13)–(16) that $r_H = r_J = 0$. So, in either the forward hypothesis of Statement *ii)* or the converse hypothesis, it holds that $r_H = r_J = 0$ and that Q_{11}, Q_{12} and Q_{21} are void.

Using (17) and (18), the product $X_H X_J$ can be written as

$$\begin{aligned} U_J^T X_H X_J U_J &= U_J^T U_H \begin{bmatrix} \Sigma_H^{-1}\Delta_H & 0 \\ 0 & 0 \end{bmatrix} U_H^T U_J \begin{bmatrix} \Sigma_J^{-1}\Delta_J & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} Q_{22}^T \Sigma_H^{-1}\Delta_H Q_{22}\Sigma_J^{-1}\Delta_J & 0 \\ Q_{23}^T \Sigma_H^{-1}\Delta_H Q_{22}\Sigma_J^{-1}\Delta_J & 0 \end{bmatrix}. \end{aligned}$$

Hence, $\rho(X_H X_J) = \rho(Q_{22}^T \Sigma_H^{-1}\Delta_H Q_{22}\Sigma_J^{-1}\Delta_J)$, and

$$\begin{aligned} &\gamma^2 - \rho(X_H X_J) > 0 \\ &\iff \gamma^2 - \rho(Q_{22}^T \Sigma_H^{-1}\Delta_H Q_{22}\Sigma_J^{-1}\Delta_J) > 0 \\ &\iff \gamma^2 - \rho((\Sigma_J^{-1}\Delta_J)^{\frac{1}{2}} Q_{22}^T \Sigma_H^{-1}\Delta_H Q_{22}(\Sigma_J^{-1}\Delta_J)^{\frac{1}{2}}) > 0 \\ &\iff \gamma^2 I - (\Sigma_J^{-1}\Delta_J)^{\frac{1}{2}} Q_{22}^T \Sigma_H^{-1}\Delta_H Q_{22}(\Sigma_J^{-1}\Delta_J)^{\frac{1}{2}} > 0 \\ &\iff \gamma^2 \Sigma_J^{-1}\Delta_J - \Sigma_J^{-1}\Delta_J Q_{22}^T \Sigma_H^{-1}\Delta_H Q_{22}\Sigma_J^{-1}\Delta_J > 0. \end{aligned} \tag{21}$$

The matrix $\tilde{\mathcal{Y}}(\gamma)$ factors as

$$\tilde{\mathcal{Y}}(\gamma) = T \left[\begin{array}{cc|cc} \gamma\Sigma_H^{-1}\Delta_H & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Y_{33} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] T^T \tag{22}$$

where

$$Y_{33} = \gamma\Sigma_J^{-1}\Delta_J - \gamma^{-1}\Sigma_J^{-1}\Delta_J Q_{22}^T \Delta_H \Sigma_H^{-1} Q_{22}\Delta_J \Sigma_J^{-1},$$

and

$$T = \left[\begin{array}{cc|cc} \Sigma_H & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline \gamma^{-1}\Delta_J Q_{22}^T & 0 & \Sigma_J & 0 \\ 0 & 0 & 0 & I \end{array}\right]$$

Hence,

$$\begin{aligned} &\mathcal{Y}(\gamma) \geq 0, \quad \text{and} \quad \operatorname{rank}\mathcal{Y} = \hat{k}_H + \hat{k}_J \\ &\iff \tilde{\mathcal{Y}}(\gamma) \geq 0, \quad \text{and} \quad \operatorname{rank}\tilde{\mathcal{Y}} = \hat{k}_H + \hat{k}_J \\ &\iff \Sigma_H^{-1}\Delta_H > 0, \quad \text{and} \quad \Sigma_J^{-1}\Delta_J > 0, \quad \text{and} \quad Y_{33} > 0 \\ &\iff X_H \geq 0 \quad \text{and} \quad X_J \geq 0 \quad \text{and} \\ &\qquad\qquad \gamma^2 I - (\Sigma_J^{-1}\Delta_J)^{\frac{1}{2}} Q_{22}^T \Delta_H \Sigma_H^{-1} Q_{22}(\Delta_J \Sigma_J^{-1})^{\frac{1}{2}} > 0 \\ &\iff X_H \geq 0 \quad \text{and} \quad X_J \geq 0 \quad \text{and} \quad \gamma^2 - \rho(X_H X_J) > 0. \end{aligned}$$

$\square$

In summary, the problem of finding γ_{mo} reduces to the problem of finding the largest value of $\gamma \geq \hat{\gamma}$ at which $\mathcal{Y}(\gamma)$ changes rank or fails to exist. The following theorem summarizes these observations.

Theorem 5.4 *For all $\gamma > \gamma_{mo}$, $\mathcal{Y}(\gamma) \geq 0$ and* $\operatorname{rank}\mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$ *is constant. For all $\hat{\gamma} < \gamma < \gamma_{mo}$, either $\mathcal{Y}(\gamma)$ is not defined,* $\operatorname{rank}\mathcal{Y}(\gamma) < \hat{k}_H + \hat{k}_J$ *or $\mathcal{Y}(\gamma)$ is not positive semidefinite.*

Proof. Immediate from Theorem 3.7, Lemma 5.3 and Theorem 5.1. $\square$

In general, detecting rank changes using finite precision arithmetic is problematic, because, at least in principle, rounding errors can change rank. Mitigating this difficulty are the facts that $\mathcal{Y}$ is positive semidefinite and has constant rank for all $\gamma > \gamma_{mo}$ and the possibility of using the congruent matrix $\tilde{\mathcal{Y}}$ obtained via the CS decomposition to detect rank changes in $\mathcal{Y}$.

Example 5.5 Returning to Example 4.1, observe that checking the semidefiniteness of X_H and X_J and the spectral radius $\rho(X_J X_H)$ may not be a viable procedure as γ approaches γ_{mo}, because $X_H = X_H(\gamma)$ diverges to infinity. In contrast, $\mathcal{Y}(\gamma)$ and $\tilde{\mathcal{Y}}(\gamma)$ remain bounded as γ approaches γ_{mo}. Using the CS decomposition to check the rank of $\tilde{\mathcal{Y}}(\gamma)$ is reliable as long as orthogonal bases of the semistable Lagrangian invariant subspaces are accurately computed.

Remark 5.6 In Theorem 5.4 we have seen the relationship between the rank and semidefiniteness of $\mathcal{Y}(\gamma)$ and X_H, X_J and $\rho(X_J X_H)$. Theorem 5.1 states that $X_H = X_H(\gamma)$, $X_J = X_J(\gamma)$ and $\rho(X_H X_J)$ are monotone in γ. However, it should be noted that neither $\mathcal{Y}(\gamma)$ nor $\tilde{\mathcal{Y}}(\gamma)$ are monotone in γ. See the figures in Section 6.

Remark 5.7 Let $f(\gamma)$ be the $(\hat{k}_H + \hat{k}_J)$-th largest eigenvalue of $\mathcal{Y}(\gamma)$. Theorem 5.4 shows that γ_{mo} is often the largest root of $f(\gamma)$. In principle, rapidly convergent one dimensional root finding methods can be applied. However, it is our observation that the paths of the eigenvalues of $\mathcal{Y}(\gamma)$ often intersect near γ_{mo} creating a discontinuity in the first derivative of $f(\gamma)$. See the figures in Section 7. Consequently, rapidly converging methods like the secant method accelerate convergence only after a more slowly converging method like bisection has already attained a good approximation to γ_{mo}.

5.3 Avoiding R_H^{-1} and R_J^{-1}

The formulae (8) and (9) of the Hamiltonian matrices $H(\gamma)$ and $J(\gamma)$ involve inverses of matrices that may be ill-conditioned along with many matrix products and matrix sums that may involve subtractive cancellation of significant digits. The Hamiltonian matrices constructed in the presence of finite precision arithmetic may become so corrupted by rounding errors that accurate calculation of the semi-stable invariant subspaces is impossible, see Example 4.1.

In order to avoid these difficulties we employ a structured version of the embedding introduced in [14, 18]. Here, we embed the Hamiltonian eigenvalue problems into generalized eigenvalue problems with skew-Hamiltonian/Hamiltonian pencils, see [4]. Essentially, the embedding process reformulates the original control problem as a descriptor system control problem [30]. This embedding is only needed for the computation of the Lagrangian subspaces and γ_{mo}; it is not relevant for the computation of the controller.

If $m_1 + m_2 + p_1$ is odd then set $r = (m_1 + m_2 + p_1 + 1)/2$ and enlarge C_1, D_{11} and D_{12} by one row of zeros, i.e., we add a regulated output that is constant and zero. Note that Assumptions A1–A4 as well as the value $\hat{\gamma}$ from Definition 3.3 are not affected by this. Set

$$\begin{aligned}
\begin{bmatrix} B_3 & B_4 \end{bmatrix} &:= \begin{bmatrix} B_1 & B_2 & 0 \end{bmatrix} \in \mathbb{R}^{n,2r},\\
\begin{bmatrix} S_3 & S_4 \end{bmatrix} &:= \begin{bmatrix} 0 & 0 & C_1^T & 0 \end{bmatrix} \in \mathbb{R}^{n,2r},\\
\hat{R}_H(\gamma) &:= \begin{bmatrix} R_{11}(\gamma) & R_{12} \\ R_{12}^T & R_{22}(\gamma) \end{bmatrix} = \begin{bmatrix} \gamma^2 I_{m_1} & 0 & D_{11}^T & 0 \\ 0 & 0 & D_{12}^T & 0 \\ D_{11} & D_{12} & I & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{2r,2r}.
\end{aligned} \tag{23}$$

If $m_1 + m_2 + p_1$ is even then set

$$\begin{aligned}
\begin{bmatrix} B_3 & B_4 \end{bmatrix} &:= \begin{bmatrix} B_1 & B_2 & 0 \end{bmatrix} \in \mathbb{R}^{n,2r},\\
\begin{bmatrix} S_3 & S_4 \end{bmatrix} &:= \begin{bmatrix} 0 & 0 & C_1^T \end{bmatrix} \in \mathbb{R}^{n,2r},\\
\hat{R}_H(\gamma) &:= \begin{bmatrix} R_{11}(\gamma) & R_{12} \\ R_{12}^T & R_{22}(\gamma) \end{bmatrix} = \begin{bmatrix} \gamma^2 I_{m_1} & 0 & D_{11}^T \\ 0 & 0 & D_{12}^T \\ D_{11} & D_{12} & I \end{bmatrix} \in \mathbb{R}^{2r,2r}.
\end{aligned} \tag{24}$$

In both cases $R_{11}(\gamma) \in \mathbb{R}^{r,r}$, $B_3, S_3 \in \mathbb{R}^{n,r}$.

Similarly, if $m_1 + p_1 + p_2$ is odd then we set $\tilde{r} = (m_1 + p_1 + p_2 + 1)/2$, enlarge B_1, D_{11} and D_{21} by one column of zeros. This corresponds to adding a zero disturbance input which again has no effect on Assumptions A1.–A4. or the value $\hat{\gamma}$. We then set

$$\begin{aligned}
\left[\begin{array}{cc} C_3^T & C_4^T \end{array}\right] &:= \left[\begin{array}{ccc} C_1^T & C_2^T & 0 \end{array}\right] \in \mathbb{R}^{n,2\tilde{r}},\\
\left[\begin{array}{cc} T_3^T & T_4^T \end{array}\right] &:= \left[\begin{array}{cccc} 0 & 0 & B_1 & 0 \end{array}\right] \in \mathbb{R}^{n,2r},\\
\hat{R}_J(\gamma) &:= \left[\begin{array}{cc} W_{11}(\gamma) & W_{12} \\ W_{12}^T & W_{22}(\gamma) \end{array}\right] = \left[\begin{array}{cccc} \gamma^2 I_{p_1} & 0 & D_{11} & 0 \\ 0 & 0 & D_{21} & 0 \\ D_{11}^T & D_{21}^T & I & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \in \mathbb{R}^{2\tilde{r},2\tilde{r}}.
\end{aligned} \tag{25}$$

Otherwise we set $\tilde{r} = (m_1 + p_1 + p_2)/2$ and

$$\begin{aligned}
\left[\begin{array}{cc} C_3^T & C_4^T \end{array}\right] &:= \left[\begin{array}{ccc} C_1^T & C_2^T & 0 \end{array}\right] \in \mathbb{R}^{n,2\tilde{r}},\\
\left[\begin{array}{cc} T_3^T & T_4^T \end{array}\right] &:= \left[\begin{array}{ccc} 0 & 0 & B_1 \end{array}\right] \in \mathbb{R}^{n,2r},\\
\hat{R}_J(\gamma) &:= \left[\begin{array}{cc} W_{11}(\gamma) & W_{12} \\ W_{12}^T & W_{22}(\gamma) \end{array}\right] = \left[\begin{array}{ccc} \gamma^2 I_{p_1} & 0 & D_{11} \\ 0 & 0 & D_{21} \\ D_{11}^T & D_{21}^T & I \end{array}\right] \in \mathbb{R}^{2\tilde{r},2\tilde{r}},
\end{aligned} \tag{26}$$

where $W_{11}(\gamma) \in \mathbb{R}^{\tilde{r},\tilde{r}}$, $C_3^T, T_3^T \in \mathbb{R}^{n,\tilde{r}}$.

With this repartitioning of the data we introduce the skew Hamiltonian/Hamiltonian matrix pencils

$$\tilde{H}(\gamma) - \lambda N := \left[\begin{array}{cc|cc} A & B_4 & 0 & -B_3 \\ S_3^T & R_{12} & -B_3^T & -R_{11}(\gamma) \\ \hline 0 & S_4 & -A^T & -S_3 \\ S_4^T & R_{22}(\gamma) & -B_4^T & -R_{12}^T \end{array}\right] - \lambda \left[\begin{array}{cc|cc} I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{27}$$

and

$$\tilde{J}(\gamma) - \lambda N := \left[\begin{array}{cc|cc} A^T & C_4^T & 0 & -C_3^T \\ T_3 & W_{12} & -C_3 & -W_{11}(\gamma) \\ \hline 0 & C_4^T & -A & -T_3^T \\ C_4 & W_{22}(\gamma) & -C_4 & -W_{12}^T \end{array}\right] - \lambda \left[\begin{array}{cc|cc} I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{28}$$

Remark 5.8 A well-known property, of real skew-Hamiltonian/Hamiltonian pencils like (27) and (28) is that they have the Hamiltonian eigensymmetry, i.e., the eigenvalues occur in quadruples $(\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda})$, see [29].

Definition 5.9 For E, $A \in \mathbb{R}^{n,n}$, the pencil $\lambda E - A$ is *regular* if $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$. If $\lambda E - A$ is not regular, then it is said to be *singular.* The index of $\lambda E - A$ is the size of the largest block associated with the eigenvalue ∞ in the Kronecker canonical form of $\lambda E - A$, see [20]. If there are no infinite eigenvalues, then the pencil $\lambda E - A$ has index zero.

Let us consider the pencils $\hat{H} - \lambda N$ and $\hat{J} - \lambda N$ in more detail.

Proposition 5.10

i) The pencil $\tilde{H}(\gamma) - \lambda N$ is regular and of index at most one if and only if $R_H(\gamma)$ is invertible. In this case, $\tilde{H}(\gamma) - \lambda N$ has exactly $2n$ finite eigenvalues.

ii) The pencil $\tilde{J}(\gamma) - \lambda N$ is regular and of index at most one if and only if $R_J(\gamma)$ is invertible. In this case, $\tilde{J}(\gamma) - \lambda N$ has exactly $2n$ finite eigenvalues.

Proof. See, e.g. [30]. □

This leads us to a characterization for the existence and uniqueness of deflating subspaces.

Theorem 5.11 *Suppose that the assumptions A1–A4 are satisfied by the control system (1). Let $\hat{\gamma}_H^I$, $\hat{\gamma}_J^I$ be as in Definition 4.4, and let $\hat{\gamma}_H$, $\hat{\gamma}_J$ be as in Definition 3.3 and Proposition 3.6.*

i) If $\hat{\gamma}_H^I$ exists, then for all $\gamma > \hat{\gamma}_H^I$ the skew-Hamiltonian/Hamiltonian pencil $\tilde{H}(\gamma) - \lambda N$ has a unique stable deflating subspace. At $\gamma = \hat{\gamma}_H^I$, $\tilde{H}(\hat{\gamma}_H^I) - \lambda N$ has a unique semi-stable deflating subspace.

If $\hat{\gamma}_H^I$ does not exist, then for all $\gamma > \hat{\gamma}_H$, $\tilde{H}(\gamma) - \lambda N$ has a unique stable deflating subspace.

ii) If $\hat{\gamma}_J^I$ exists, then for all $\gamma > \hat{\gamma}_J^I$ the skew-Hamiltonian/Hamiltonian pencil $\tilde{J}(\gamma) - \lambda N$ has a unique stable deflating subspace. At $\gamma = \hat{\gamma}_J^I$, $\tilde{H}(\hat{\gamma}_J^I) - \lambda N$ has a unique semi-stable deflating subspace.

If $\hat{\gamma}_J^I$ does not exist, then for all $\gamma > \hat{\gamma}_J$, $\tilde{J}(\gamma) - \lambda N$ has a unique stable deflating subspace.

Furthermore, if

$$Q_H = \begin{bmatrix} Q_{H,1} \\ Q_{H,2} \\ Q_{H,3} \\ Q_{H,4} \end{bmatrix} \in \mathbb{R}^{2r,n}, \qquad Q_J = \begin{bmatrix} Q_{J,1} \\ Q_{J,2} \\ Q_{J,3} \\ Q_{J,4} \end{bmatrix} \in \mathbb{R}^{2\tilde{r},n} \tag{29}$$

are matrices partitioned conformably with (27) and (28) and whose columns span the unique semi-stable deflating subspaces of $\tilde{H} - \lambda N$ and $\tilde{J} - \lambda N$, then the columns of

$$\begin{bmatrix} X_{H,1} \\ X_{H,2} \end{bmatrix} = \begin{bmatrix} Q_{H,1} \\ Q_{H,3} \end{bmatrix}, \qquad \begin{bmatrix} X_{J,1} \\ X_{J,2} \end{bmatrix} = \begin{bmatrix} Q_{J,1} \\ Q_{J,3} \end{bmatrix} \tag{30}$$

span the semi-stable Lagrangian invariant subspaces of $H(\gamma)$ and $J(\gamma)$.

Proof. We only prove i), the proof of ii) is analogous.

i) The matrix R_H is invertible if and only if $\hat{R}_H$ in (23) or (24) is invertible and therefore, when $\hat{\gamma}_H^I$ exists, since $\hat{\gamma}_H^I \geq \hat{\gamma}^I \geq \hat{\gamma}$, the pencil $\tilde{H} - \lambda N$ is regular and has index at most one for all $\gamma > \hat{\gamma}_H^I$. By Proposition 5.10 both pencils have exactly $2n$ finite eigenvalues. If $\hat{\gamma}_H^I$ does not exist, the same is true for all $\gamma > \hat{\gamma}$. Because $\tilde{H} - \lambda N$ is a skew-Hamiltonian/Hamiltonian pencil, these finite eigenvalues have the Hamiltonian eigensymmetry. Hence there exists an n-dimensional deflating subspace associated with all eigenvalues in the open left half plane and half of the eigenvalues on the imaginary axis. If the columns of Q_H span such a subspace, i.e. $\tilde{H}Q_H = NQ_HT_H$ for some matrix T_H with eigenvalues in the closed left half plane, then after some permutation of block rows and columns we obtain

$$\left[\begin{array}{cc|cc} A & 0 & B_3 & B_4 \\ 0 & -A^T & S_3 & S_4 \\ \hline S_3^T & -B_3^T & R_{11} & R_{12} \\ S_4^T & -B_4^T & R_{12}^T & R_{22} \end{array}\right] \left[\begin{array}{c} Q_{H,1} \\ Q_{H,3} \\ \hline -Q_{H,4} \\ Q_{H,2} \end{array}\right] = \left[\begin{array}{c} Q_{H,1} \\ Q_{H,3} \\ \hline 0 \\ 0 \end{array}\right] T_H.$$

Since $\hat{R}_H$ is invertible, we can use it to eliminate the upper right 2×2 block and obtain (in the form of (23)) that

$$\left(\begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} - \begin{bmatrix} B_1 & B_2 & 0 & 0 \\ 0 & 0 & -C_1^T & 0 \end{bmatrix} \hat{R}_H^{-1} \begin{bmatrix} 0 & -B_1 \\ 0 & -B_2 \\ C_1 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} Q_{H,1} \\ Q_{H,3} \end{bmatrix} = \begin{bmatrix} Q_{H,1} \\ Q_{H,3} \end{bmatrix} T_H. \tag{31}$$

A simple calculation shows that this is exactly (10). The same is true using (24).

At $\hat{\gamma}_H^I$, as shown in [17], the pencil has a unique semi-stable deflating subspace. $\square$

It follows from this theorem that in the computation of γ_{mo} it suffices to compute deflating subspaces of the skew-Hamiltonian/Hamiltonian pencils in (27) and (28) associated with the closed left half plane eigenvalues. It is important that the deflating subspaces be computed with a skew-Hamiltonian/Hamiltonian structure preserving numerical method. It has been shown in [36, 37] that the uniqueness of a Lagrangian invariant subspace is not invariant under non-structured perturbations, see also [17]. Also, rounding errors in a non-structure preserving method may destroy the eigenvalue symmetry. In particular, if eigenvalues lie near or on the imaginary axis, rounding errors in a non-structure preserving method like the QZ algorithm may cause the numerical method to find fewer than n eigenvalues in the closed left half plane. This in turn makes it difficult or impossible to determine the desired Lagrangian invariant subspace, see Example 4.5. In contrast, structure-preserving methods typically compute a nearby Lagrangian subspace even when eigenvalues are near or on the imaginary axis.

Remark 5.12 The columns of $\left[\begin{smallmatrix} Q_{H,1} \\ Q_{H,3} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} Q_{J,1} \\ Q_{J,3} \end{smallmatrix}\right]$ in (30) may not be orthonormal even when the matrices Q_H and Q_J in (29) do have orthonormal columns. A numerically stable, structure preserving numerical method for extracting an orthonormal basis of a Lagrangian subspace is the symplectic QR decomposition, see [10]. The symplectic QR decomposition determines orthogonal symplectic matrices

$$S_H = \begin{bmatrix} S_{H,1} & -S_{H,2} \\ S_{H,2} & S_{H,1} \end{bmatrix}, \; S_J = \begin{bmatrix} S_{J,1} & -S_{J,2} \\ S_{J,2} & S_{J,1} \end{bmatrix},$$

such that

$$S_H \begin{bmatrix} Q_{H,1} \\ Q_{H,3} \end{bmatrix} = \begin{bmatrix} M_H \\ 0 \end{bmatrix}, \; S_J \begin{bmatrix} Q_{J,1} \\ Q_{J,3} \end{bmatrix} = \begin{bmatrix} M_J \\ 0 \end{bmatrix}.$$

The matrix $\tilde{\mathcal{Y}}(\gamma)$ may then be constructed from the CS decompositions of $\left[\begin{smallmatrix} S_{H,1} \\ S_{H,2} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} S_{J,1} \\ S_{J,2} \end{smallmatrix}\right]$.

A difficulty that could arise here, is that $\left[\begin{smallmatrix} Q_{H,1} \\ Q_{H,3} \end{smallmatrix}\right]$ and/or $\left[\begin{smallmatrix} Q_{J,1} \\ Q_{J,3} \end{smallmatrix}\right]$ may be ill-conditioned or may be small norm sections of the matrices with orthonormal columns Q_H and Q_J in (29). Such a problem may be either traced back to an ill-conditioning of the problem of computing the invariant subspace or to a near failure of one or some of assumptions A1-A4.

In both cases we cannot expect a solution to be accurate, but clearly then the same or worse problems arise in the reduced pencils such as (31).

If $\hat{R}_H(\gamma)$ or $\hat{R}_J(\gamma)$ are nearly singular, then the pencils (27) and (28) are close to pencils that are either not regular or have index greater than one. In this case we are close to a situation, where the dimension of the deflating subspace associated with the left half plane eigenvalues becomes less than n. If $\hat{\gamma}_H < \gamma_{mo}$ and $\hat{\gamma}_J < \gamma_{mo}$, then this does not happen for $\gamma \geq \gamma_{mo}$. Example 4.1 demonstrates that $\gamma_{mo} = \hat{\gamma}_H$ is possible and the pencil $\tilde{H} - \lambda N$ becomes singular near γ_{mo}.

In summary, numerical computations based on the skew-Hamiltonian/Hamiltonian pencils (27) and (28) avoid unnecessary rounding errors caused by explicitly forming $H(\gamma)$, $J(\gamma)$, and the corresponding algebraic Riccati solutions. Deflating subspaces of the skew-Hamiltonian/Hamiltonian pencils (27) and (28) provide the desired Lagrangian subspaces, and the factors of $\mathcal{Y}(\gamma)$ and $\tilde{\mathcal{Y}}(\gamma)$ without explicitly forming the inverses, sums and products that occur in (8) and (9).

5.4 A summary of critical cases

In the course of this section we have discussed several computational hazards that may be encountered when computing γ_{mo}. We demonstrate many of them in Section 7. Each hazard is associated with one of the $\hat{\gamma}$'s listed in Table 1.

$\gamma_{mo} = \hat{\gamma}^{\rho}$: Since $\hat{\gamma}^{\rho} > \hat{\gamma}^R \geq \hat{\gamma}^L (= \hat{\gamma}^I) \geq \hat{\gamma}$, most of the numerical hazards discussed here do not occur and typical numerical methods perform well.

$\gamma_{mo} = \hat{\gamma}^R > \hat{\gamma}^L$: One or both of the Riccati solutions do not exist and numerical methods that explicitly solve algebraic Riccati equations usually fail. Hence the Riccati approach will fail. An invariant/deflating subspace approach should be used.

$\gamma_{mo} = \hat{\gamma}^R = \hat{\gamma}^L = \hat{\gamma}^I > \hat{\gamma}$: At least one of the Hamiltonian matrices or the skew-Hamiltonian/Hamiltonian pencils has purely imaginary eigenvalues. A Hamiltonian structure preserving algorithm for the solution of the Hamiltonian or the skew-Hamiltonian/Hamiltonian eigenvalue problem should be used, see [5].

$\gamma_{mo} = \hat{\gamma}^R = \hat{\gamma}^L = \hat{\gamma}$: The pencil (27) or (28) (or both) has index greater than 1 or is not regular. At least one n-dimensional Lagrangian subspace fails to exist. This case can be detected by checking the condition number of R_H, R_J. If γ is close to γ_{mo}, then (27) or (28) may have eigenvalues close to the imaginary axis, and, possibly, one or both algebraic Riccati equation solutions may have large norm.

A Hamiltonian structure preserving algorithm should be used. If γ is close to γ_{mo}, then a subspace approach should be used. This is a computationally difficult case.

A numerical method designed for the general modified H_∞ control problem, might use a structured method for the skew-Hamiltonian/Hamiltonian embedding, without explicitly constructing the Riccati solutions and with a check on the condition number of the matrices R_H and R_J.

6 Computation of γ_{mo}

In this section we synthesize the above observations in a new numerical method for the modified optimal H_∞ control problem.

The simplest approach to finding γ_{mo} is to use a bisection method. Given a number $\gamma \geq 0$, the following procedure may be used to determine whether $\gamma \leq \gamma_{mo}$ or $\gamma \geq \gamma_{mo}$.

Algorithm 1 (Basic bisection procedure)

1. *Form the skew-Hamiltonian/Hamiltonian pencils $\tilde{H}(\gamma)-\lambda N$ and $\tilde{J}(\gamma)-\lambda N$ as in (27) and (28).*
2. *Use the algorithm from [5] to compute the deflating subspaces Q_H and Q_L associated with the eigenvalues in the closed left half plane.*
3. *If the dimension of one or both of these subspaces is less than n, then report $\gamma < \gamma_{mo}$ and STOP.*
4. *Compute the symplectic QR decomposition of the two matrices in (30) followed by the CS decompositions (13)–(16).*
5. *If any diagonal element of $\Delta_H\Sigma_H$ or $\Delta_J\Sigma_J$ is negative, then report $\gamma < \gamma_{mo}$ and STOP.*
6. *Form the matrix $\tilde{\mathcal{Y}}$.*
7. *If $\tilde{\mathcal{Y}}$ is not positive semidefinite, then report $\gamma < \gamma_{mo}$ and STOP.*
8. *If $\tilde{\mathcal{Y}}$ is positive semidefinite and* $\operatorname{rank}\tilde{\mathcal{Y}} < \hat{k}_H+\hat{k}_J$ *then report $\gamma = \gamma_{mo}$ and STOP*

   ($\hat{k}_H$ and $\hat{k}_J$ can be computed with a sufficiently large γ.)

9. Report $\gamma > \gamma_{mo}$.

Often, γ_{mo} is a root of the function $f(\gamma)$ described in Remark 5.7. Since the eigenvalues of a symmetric matrix are continuous functions of the entries of the matrix (hence also of γ) and continuously differentiable as long as the eigenvalue is simple [40], the secant method applies. We then have the following basic structure of the optimization procedure.

Algorithm 2 (Basic optimization procedure)

1. *Compute upper and lower bounds γ_{low} and γ_{up} for γ_{mo}.*
2. *Use the bisection method (Algorithm 1) to determine a sufficiently small interval $[\gamma_0, \gamma_1]$ in which γ_{mo} lies.*
3. *Use a quadratically convergent method such as the secant method to determine γ.*

This algorithm needs to fall back upon the bisection procedure in case the secant method produces an approximate root γ for which $\mathcal{Y}(\gamma)$ does not exist.

We implemented this procedure based on the root finding method in MATLAB [28]. In the following section we present several numerical examples.

7 Numerical Examples

In this section we solve several H_∞ control problems and compare our experimental implementation of Algorithm 2 with `Hinfopt` (version 1.8) from the MATLAB Robust Control Toolbox (version 2.0.7) [13]. We used the same highly demanding stopping criterion $\texttt{tolX} = 10^{-14}$ for stopping the γ iteration in both programs.

All the numerical examples were run on a Dell 530 workstation using MATLAB (version 6.0.0.88) with IEEE754 conforming floating point arithmetic. The unit round is approximately 2.22×10^{-16}.

Example 7.1 For

$$A = \begin{bmatrix} -a & 0 & 1 & -2 & 1 \\ 0 & -100 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2a & a \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ a \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -90 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$
\begin{aligned}
C_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \ D_{11} = 0, \ D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
C_2 &= \begin{bmatrix} 0 & 0 & 1 & -2 & 1 \end{bmatrix}, \ D_{21} = 1, \ D_{22} = 0
\end{aligned}
$$

γ_{mo} is independent of the choice of a. As is typical, $\hat{\gamma}^\rho$ is greater than $\hat{\gamma}$, $\hat{\gamma}^R$, $\hat{\gamma}^I$ and $\hat{\gamma}^L$, so $\gamma_{mo} = \hat{\gamma}^\rho$. Our experimental program determined $\gamma_{mo} = \hat{\gamma}^\rho = 7.853923684022$ which is correct to roughly thirteen significant digits. The experimental program computed the same optimal value of γ to at least thirteen significant digits for values of a between 1 and 10^{-7}. When $a = 10^{-8}$, then the pencil $\hat{H} - \lambda N$ has finite eigenvalues of magnitude comparable to (and possibly smaller than) the unit round of the floating point arithmetic. At that point, eigenvalue based numerical methods are no longer able to reliably extract the stable deflating subspace. The experimental program delivers an error message. `Hinfopt` gets the same accuracy for a as small as 10^{-10}. (It does not recognize that the computed eigenvalues are unreliable for $a \leq 10^{-8}$, but by luck uses the correct subspaces.)

Figure 3 shows the nonzero eigenvalues of $\tilde{\mathcal{Y}}(\gamma)$ as a function of γ for $a = 1$. In this example, $\mathcal{Y}(\gamma)$ and $\tilde{\mathcal{Y}}(\gamma)$ have an eigenvalue of magnitude roughly 10^{-6} in the neighborhood of γ, but it is one of the other, relatively larger eigenvalues that changes sign at $\hat{\gamma}^\rho$. This example demonstrates that, counter to intuition, a relatively small eigenvalue of $\mathcal{Y}(\gamma)$ or $\tilde{\mathcal{Y}}(\gamma)$ does not necessarily imply that $\gamma \approx \hat{\gamma}^\rho$.

Example 7.2 (Example 4.1 continued) In this example $\gamma_{mo} = \hat{\gamma}$. With $\alpha = \beta = \delta = \eta = 1$ and $\epsilon_1 = \epsilon_2 = 0$, the experimental program determined $\gamma_{mo} = \hat{\gamma} = .5000000000000$ which agrees with the theoretical value to thirteen significant digits.

Note that $R_H(\gamma)$ is singular at $\gamma = \gamma_{mo} = \hat{\gamma}$. `Hinfopt` fails on this example because, as it starts up, it explicitly inverts a singular matrix probably related to $R_H(\gamma)$.

Example 7.3 (Example 4.1 continued) Example 4.1 with $\alpha = \beta = \delta = \eta = \epsilon_2 = 1$ and $\epsilon_1 = 0$ demonstrates a case in which $\gamma_{mo} = \hat{\gamma}^L$. As shown in Figure 5, $\hat{\mathcal{Y}}(\gamma)$ does not change rank at $\gamma = \gamma_{mo}$. The Riccati solution to (9) is $X_J = 0$ independent of γ. The Riccati solution to (8) is not constant, but remains positive definite in a one sided neighborhood to the right of γ_{mo}. In a neighborhood to the left of γ_{mo}, the Hamiltonian matrix $H(\gamma)$ (8) and the pencil $\hat{H} - \lambda N$ have eigenvalues with zero real part and the required Lagrangian invariant subspaces fail to exist. Our experimental code reports $\gamma_{mo} = \hat{\gamma}^L = .8062257748299$. `Hinfopt` fails on this example because, as it starts up, it explicitly inverts $R_H(\gamma)$.

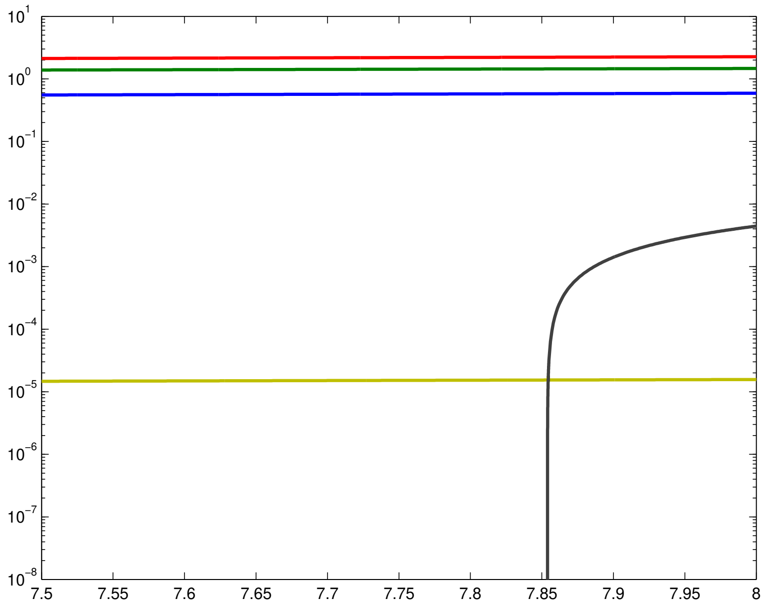

Figure 3: Nonzero eigenvalues of $\tilde{\mathcal{Y}}(\gamma)$ from Example 7.1 with $a = 1$ as a function of γ. Graphs of the eigenvalues of $\mathcal{Y}(\gamma)$ are similar. Here, $\gamma_{mo} = \hat{\gamma}^{\rho} \approx 7.853923684022$.

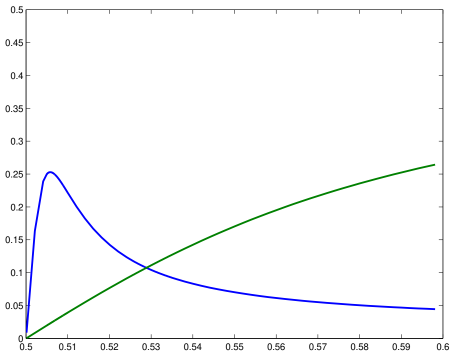

Figure 4: Nonzero eigenvalues of $\tilde{\mathcal{Y}}(\gamma)$ from Example 7.2 as a function of γ. Graphs of the eigenvalues of $\mathcal{Y}(\gamma)$ are similar. Here, $\gamma_{mo} = \hat{\gamma} = .5$.

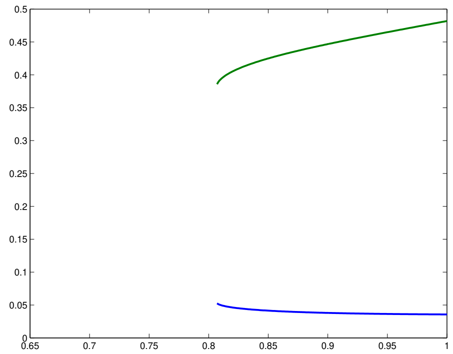

Figure 5: Nonzero eigenvalues of $\tilde{\mathcal{Y}}(\gamma)$ from Example 7.3 as a function of γ. Graphs of the eigenvalues of $\mathcal{Y}(\gamma)$ are similar. Here, $\gamma_{mo} = \hat{\gamma}^L = .806\ldots$.

Example 7.4 In this example the H_∞ norm of T_{zw} is nearly minimized by a large range of values γ using the γ-parametrization of Theorem 3.7, including a region below γ_{mo}. That is, using any of these γ's to construct a controller, nearly the same H_∞ norm of T_{zw} is attained. Let

added a bit of explanation. don't know if it helps

$$\left[\begin{array}{c|c|c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{cc|cc|c} 2 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 & -2 \\ \hline 1 & 0 & \alpha & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ \hline 4 & -2 & 0 & 1 & 0 \end{array}\right].$$

Then $\hat{\gamma} = \gamma_{mo} = \alpha$. Taking $\alpha = 3$ one can verify that, except for $\gamma \in [2.7, 3]$, the Lagrangian subspaces and Riccati solutions exist. But note that for $\gamma < 3$, Condition 1. of Theorem 3.7 is not satisfied, so $\|T_{zw}\|_\infty < 3$ cannot be achieved. Using the formulae in [45] we constructed a controller for each $\gamma \in [1.5, 4] \setminus [2.7, 3]$ and found that $\|T_{zw}\|_\infty = 3.00$ to three significant digits independent of γ.

Figure 6 shows the nonzero eigenvalues of $\tilde{\mathcal{Y}}(\gamma)$ for $\gamma \in [.5, 3.5]$. The Riccati solutions X_H of (8) and X_J of (9) have the peculiar property that $X_J(\gamma) \equiv 0$ and $\lim_{\gamma \to \gamma_{mo}+} X_H(\gamma) = 0$, so $\rho(X_H X_J) = 0$ independent of γ. When $\gamma \approx \gamma_{mo}$, a small error in X_J may lead to a relatively large error in the computed spectral radius $\rho(X_J X_H)$. An inaccurately computed spectral radius may limit the accuracy attainable by conventional algorithms that rely on Theorem 3.7 and explicit calculation of Riccati solutions. Nevertheless, `Hinfopt` correctly determined γ_{mo} to within an absolute error of 10^{-13} as did our experimental algorithm described in this paper.

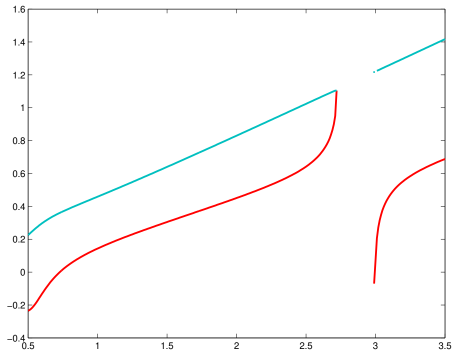

Figure 6: Eigenvalues of $\tilde{\mathcal{Y}}(\gamma)$ from Example 7.4. Graphs of the eigenvalues of $\mathcal{Y}(\gamma)$ are similar. Here, $\gamma_{mo} = \hat{\gamma} = 3$.

8 Computing the Optimal Controller

This section briefly discusses the construction of a (near) optimal H_∞ controller (2) from γ_{mo} or a close enough approximation to it. Note that for each $\gamma \in \Gamma$, there are typically many internally stabilizing controllers that achieve $\|T_{zw}\|_\infty < \gamma$. In any application, some are likely to be more robust or less expensive or more elegant than others. Explicit formulae for a particular choice (the 'so-called "central controller") of $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ in (2) appear in [23, 45]. Unfortunately, there formulae are not well suited to finite precision computations. We do not present them here but note that they use several inverses of matrices that are ill-conditioned or even singular. For example, using $\gamma = \gamma_{mo}$ in Example 4.1 the formulae call for the inverse of a singular matrix. The new formulae that we present here are quite technical, but they are formulated in terms of generalized inverses, hence avoid these difficulties and are therefore better suited to finite precision computations.

Remark 8.1 *Note that in case it can be guaranteed that failure of condition* 4. *of Theorem 3.7 will yield γ_{mo}, a method suggested in [16] based on balancing the Riccati solutions is also a viable alternative to the formulae in [23, 45]. But this approach is restricted to special situations and balancing the Riccati solutions itself may be an ill-conditioned operation.*

Let $\gamma^* \geq \hat{\gamma}$ be γ_{mo} or an approximation of it. The first step in the generation of a (sub)optimal controller corresponding to γ^* is the factorization

(7) given by

$$\left[\begin{array}{cc} U_{12}^T & 0 \\ 0 & V_{21}^T \end{array}\right]\left[\begin{array}{cc} D_{11} & D_{12} \\ D_{21} & 0 \end{array}\right]\left[\begin{array}{cc} U_{21} & 0 \\ 0 & V_{12} \end{array}\right] = \left[\begin{array}{cc|c} D_1 & D_2 & 0 \\ D_3 & D_4 & \Sigma_{12} \\ \hline 0 & \Sigma_{21} & 0 \end{array}\right].$$

Extract the singular value decomposition of D_1

$$D_1 = \tilde{U}_1 \left[\begin{array}{cc} \Sigma_1 & 0 \\ 0 & 0 \end{array}\right] \tilde{V}_1^T = \tilde{U}_1 \bar{D}_1 \tilde{V}_1^T,$$

then partition

$$\bar{D}_2 = \tilde{U}_1^T D_2 = \left[\begin{array}{c} \Delta_{13} \\ \Delta_{23} \end{array}\right], \bar{D}_3 = D_3 \tilde{V}_1 = \left[\begin{array}{cc} \Delta_{31} & \Delta_{32} \end{array}\right]$$

conformably.

As in Proposition 3.2, a small singular value of D_{12} or D_{21} demonstrates that a perturbation of the data would violate Assumption A2, i.e., we are near to a singular H_∞ problem. Recall that if Σ_{21} or Σ_{12} are singular, then R_H or R_J is singular regardless of the value of γ. According to our definition, in this case $\gamma_{mo} = \infty$.

Let

$$\begin{array}{rcl} \bar{D}_{11} & = & U_{12}^T D_{11} U_{21} = \left[\begin{array}{cc} D_1 & D_2 \\ D_3 & D_4 \end{array}\right], \\ \bar{D}_{12} & = & U_{12}^T D_{12} V_{12} \Sigma_{12}^\dagger = \left[\begin{array}{c} 0 \\ \Sigma_{12}\Sigma_{12}^\dagger \end{array}\right], \\ \bar{D}_{21} & = & \Sigma_{21}^\dagger V_{21}^T D_{21} U_{21}^T = \left[\begin{array}{cc} 0 & \Sigma_{21}^\dagger \Sigma_{21} \end{array}\right], \\ \bar{B}_1 & = & B_1 U_{21},\ \bar{B}_2 = B_2 V_{12} \Sigma_{12}^\dagger, \\ \bar{C}_1 & = & U_{12}^T C_1,\ \bar{C}_2 = \Sigma_{21}^\dagger V_{21} C_2, \end{array} \tag{32}$$

where the superscript † denotes the Moore-Penrose pseudo inverse. Form the spectral decompositions

$$\begin{array}{rcl} \bar{R}_H(\gamma^*) & = & \left[\begin{array}{c} \bar{D}_{11}^T \\ \bar{D}_{12}^T \end{array}\right]\left[\begin{array}{cc} \bar{D}_{11} & \bar{D}_{12} \end{array}\right] - \left[\begin{array}{cc} \gamma^{*2} I_{m_1} & 0 \\ 0 & 0 \end{array}\right] = U_{R,H} \Sigma_{R,H} U_{R,H}^T, \\ \bar{R}_J(\gamma^*) & = & \left[\begin{array}{c} \bar{D}_{11} \\ \bar{D}_{21}^T \end{array}\right]\left[\begin{array}{cc} \bar{D}_{11}^T & \bar{D}_{21}^T \end{array}\right] - \left[\begin{array}{cc} \gamma^{*2} I_{p_1} & 0 \\ 0 & 0 \end{array}\right] = U_{R,J} \Sigma_{R,J} U_{R,J}^T. \end{array} \tag{33}$$

If the diagonal matrices $\Sigma_{R,H}$ or $\Sigma_{R,J}$ are very close to being singular, then γ^* is near $\hat{\gamma}$.

Set

$$
\begin{aligned}
\tilde{D}_{11} &= \left[\begin{array}{cc} \Delta_{31} & \Delta_{32} \end{array}\right] \bar{D}_1(\gamma^{*2} I - \bar{D}_1 \bar{D}_1^T)^\dagger \left[\begin{array}{c} \Delta_{13} \\ \Delta_{23} \end{array}\right] - D_4 \\
&= \Delta_{31}\Sigma_1(\gamma^{*2} I - \Sigma_1^2)^\dagger \Delta_{13} - D_4 \\
\tilde{D}_{12}\tilde{D}_{12}^T &= I - \left[\begin{array}{cc} \Delta_{31} & \Delta_{32} \end{array}\right] (\gamma^{*2} I - \bar{D}_1 \bar{D}_1^T)^\dagger \left[\begin{array}{c} \Delta_{31}^T \\ \Delta_{32}^T \end{array}\right] \\
\tilde{D}_{21}\tilde{D}_{21}^T &= I - \left[\begin{array}{cc} \Delta_{13}^T & \Delta_{23}^T \end{array}\right] (\gamma^{*2} I - \bar{D}_1^T \bar{D}_1)^\dagger \left[\begin{array}{c} \Delta_{13} \\ \Delta_{23} \end{array}\right].
\end{aligned}
\tag{34}
$$

where $\tilde{D}_{21}$, $\tilde{D}_{12}$ are full-rank factors. Since $\gamma^* \geq \hat{\gamma}$, it follows that inverted terms are always semidefinite, but they may become singular if $\gamma^* = \hat{\gamma}$. Since $\hat{\gamma}$ is the maximal singular value of $\left[\begin{array}{c} D_1 \\ D_3 \end{array}\right]$ or $\left[\begin{array}{cc} D_1 & D_2 \end{array}\right]$, by looking at the matrices we see that the formulae are still viable if we replace the inverses by generalized inverses. Set

$$
\hat{M} = \left[\begin{array}{c} \bar{D}_{11}^T \\ \bar{D}_{12}^T \end{array}\right] \bar{C}_1 U_H^T C_H + \left[\begin{array}{c} \bar{B}_1^T \\ \bar{B}_2^T \end{array}\right] U_H^T S_H
$$

and compute

$$
\begin{aligned}
M_B &= U_{R,J}\Sigma^\dagger U_{R,J}^T \left[\begin{array}{cc} 0 & 0 \\ -\tilde{D}_{11} & -\tilde{D}_{12} \\ I_{p_2} & 0 \end{array}\right], \\
M_C &= \left[\begin{array}{ccc} 0 & -\tilde{D}_{11} & I_{m_2} \\ 0 & -\tilde{D}_{21} & 0 \end{array}\right] U_{R,H}\Sigma_{R,H}^\dagger U_{R,H}^T, \\
M_{CB} &= U_{R,H}\Sigma_{R,H}^\dagger U_{R,H}^T \hat{M}.
\end{aligned}
\tag{35}
$$

Set

$$
\begin{aligned}
W &= (C_J U_J U_H^T C_H - \gamma^{*-2} S_J U_J U_H^T S_H) C_H^\dagger \\
&= \begin{array}{c} \\ r_J \\ t_J \\ n - k_J \end{array}
\left[\begin{array}{ccc} r_H & t_H & n - k_H \\ 0 & 0 & 0 \\ 0 & (\Sigma_J Q_{22}^T \Sigma_H - \gamma^{*-2}\Delta_J Q_{22}^T \Delta_H)\Delta_H^{-1} & 0 \\ 0 & 0 & 0 \end{array}\right]
\end{aligned}
\tag{36}
$$

and define the abbreviations

$$
\begin{aligned}
\left[\begin{array}{cc} P_{11} & P_{12} \end{array}\right] &= U_J^T(\bar{B}_2 \left[\begin{array}{cc} \tilde{D}_{11} & \tilde{D}_{12} \end{array}\right] + \bar{B}_1 \left[\begin{array}{cc} \bar{D}_{11}^T & \bar{D}_{21}^T \end{array}\right] M_B), \\
\left[\begin{array}{cc} P_{21} & P_{22} \end{array}\right] &= U_J^T \left[\begin{array}{cc} \bar{C}_1^T & \bar{C}_2^T \end{array}\right] M_B.
\end{aligned}
$$

Using

$$\begin{aligned}
\hat{D}_{21} &= V_{21}\Sigma_{21}\tilde{D}_{21},\\
\hat{D}_{12} &= \tilde{D}_{12}\Sigma_{12}V_{12},\\
\hat{D}_{11} &= \tilde{D}_{11},
\end{aligned}$$

determine the coefficients matrices of the dynamic controller (2) as

$$\begin{aligned}
\hat{A} &= WU_H^T A U_H C_H - WU_H^T \begin{bmatrix} \bar{B}_1^T & \bar{B}_2^T \end{bmatrix} M_{CB}\\
&\quad -(C_J P_{11} + S_J P_{21})(\bar{C}_2 U_H C_H - \begin{bmatrix} 0 & I_{p_2} & 0 \end{bmatrix} M_{CB}),\\
\hat{B} &= \begin{bmatrix} \hat{B}_1 & \hat{B}_2 \end{bmatrix} = \begin{bmatrix} C_J P_{11} + S_J P_{21} & (C_J P_{12} + S_J P_{22})\Sigma_{12} V_{12}^T \end{bmatrix},\\
\hat{C} &= \begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \end{bmatrix} = -\begin{bmatrix} \hat{D}_{11} \\ \hat{D}_{21} \end{bmatrix} \bar{C}_2 U_H C_H - \begin{bmatrix} I_{p_1} & 0 \\ 0 & V_{21}\Sigma_{21} \end{bmatrix} M_C \hat{M},\\
\hat{D} &= \begin{bmatrix} \hat{D}_{11} & \hat{D}_{12} \\ \hat{D}_{21} & 0 \end{bmatrix}\\
\hat{E} &= WU_H^T.
\end{aligned}$$

The controller (2) is then given by the descriptor system

$$\begin{aligned}
\hat{E}\dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}\hat{u},\\
\hat{y} &= \hat{C}\hat{x} + \hat{D}\hat{u}.
\end{aligned} \tag{37}$$

One of the conditions in Theorem 3.7 requires $\gamma_{mo} \geq \hat{\gamma}$. Recall that $\hat{\gamma}$ is the supremum of all γ for which one of $R_H(\gamma)$ or $R_J(\gamma)$ are nonsingular. However both $R_H(\gamma)$ and $R_J(\gamma)$ may sometimes be nonsingular for some $\gamma < \hat{\gamma}$. Thus, one may sometimes construct a controller (2) using the formulae in [23, 45] or the formulae in this section with $\gamma < \hat{\gamma}$.

One may ask in that case what happens to the H_∞ norm of the closed-loop transfer function T_{zw}.

Consider the set Γ of all values of $\gamma \geq 0$ for which there exists an internally stabilizing controller so that the transfer function T_{zw} can be formed according to the formulae in Section 8. This requires the following conditions

i) $R_H(\gamma)$ and $R_J(\gamma)$ are invertible;

ii) the pencils $\tilde{H} - \lambda N$ and $\tilde{J} - \lambda N$ have exactly n eigenvalues in the left half plane and n eigenvalues in the right half plane;

iii) $\mathcal{Y}(\gamma) \geq 0$.

According to Definitions 3.3, 4.2, 4.4, and 4.3 we have that

$$\Gamma = \Gamma_{Rns} \cap \Gamma_{ns} \cap \Gamma_{ni} \cap \Gamma_L.$$

The set Γ may consist of the union of several intervals as we see by a closer look at Example 7.4.

Example 8.2 In Example 7.4 with $\alpha = 3$, we have

$$\Gamma = \Gamma_1 \cup \Gamma_2 := (.7279748414803329, \sqrt{\frac{37}{5}}) \cup (3, \infty).$$

In both subintervals the Riccati solutions exist as $X_H > 0$, $X_J = 0$. Here $\hat{\gamma} = 3$ and hence $\gamma_{mo} = 3$.

As Example 7.4 indicates there may exist values of $\gamma \leq \gamma_{mo}$ where we get the same minimal value of the transfer function if we construct the controller in the same way. It may still be better to use $\gamma < \gamma_{mo}$ if this leads to a smaller feedback gain or a better conditioned closed loop system. In Example 7.4, for example, one might maximize the distance to instability over all $\gamma \in \Gamma$ or try to achieve other design goals.

9 Conclusion

This paper discusses the design of a robust numerical method for the modified H_∞ control problem. The proposed method avoids matrix sums, products and inverses needed to construct Hamiltonian matrices and avoids potentially ill-conditioned algebraic Riccati equations by working with skew-Hamiltonian/Hamiltonian pencils and its deflating subspaces. The computation of the optimal γ reduces to a one-dimensional optimization problem for which, in principle, one can apply quadratically convergent methods. Several examples illustrate the numerical hazards and the properties of the proposed numerical method. The new approach effectively increases the set of problems to which H_∞ control may be applied.

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