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Preprint 2005/13

Preprint-Reihe des Instituts für Mathematik
Technische Universität Berlin

Report 2005/13 **April 2005**

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AMS subject classifications. 49K20, 35J65, 80M50

SECOND-ORDER SUFFICIENT OPTIMALITY CONDITIONS FOR A SEMILINEAR OPTIMAL CONTROL PROBLEM WITH NONLOCAL RADIATION INTERFACE CONDITIONS*

C. MEYER†

Abstract. We consider a control constrained optimal control problem governed by a semilinear elliptic equation with nonlocal interface conditions. These conditions occur during the modeling of diffuse-gray conductive-radiative heat transfer. After stating first-order necessary conditions, second-order sufficient conditions are derived that account for strongly active sets. These conditions ensure local optimality in a L^s-neighborhood whereby the underlying analysis allows to use weaker norms than L^∞.

Key words. Optimal control, semilinear elliptic equations, nonlocal interface conditions, second-order sufficient optimality conditions

AMS subject classifications. 49K20, 35J65, 80M50

1. Introduction. In this paper, we investigate an optimal control problem that arises from the sublimation growth of semiconductor single crystals by the physical vapor transport (PVT) method. Possible semiconductor materials, produced with this method, are silicon carbide (SiC) or aluminum nitrite (AlN). They are used in numerous industrial applications, e.g. the production of optoelectronic devices such as blue and green LEDs and lasers. For the PVT method, polycrystalline powder is placed under a low-pressure inert gas atmosphere at the bottom of a cavity inside a crucible. The crucible is heated up to 2000 till 3000 K by induction. Due to the high temperatures and the low pressure, the powder sublimates and crystallizes at a single-crystalline seed located at the cooled top of the cavity, such that the desired single crystal grows into the reaction chamber. See [6] for more details.

Here, we focus on the conductive-radiative heat transfer in the growth apparatus. Therefore, we consider a simplified setup of the growth apparatus, shown in Fig. 1.1, where Ω_s denotes the domain of the solid graphite crucible, whereas Ω_g is the domain of gas phase inside.

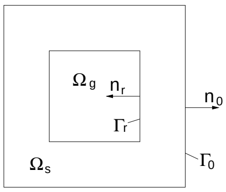

Fig. 1.1. *Exemplary domain for nonlocal radiative heat transfer.*

A very important determining factor for the crystal's quality and growth rate is the temperature gradient inside the gas phase [9]. Since we do not consider the electro-

*Supported by the DFG Research Center Matheon "Mathematics for key technologies" in Berlin.

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magnetic induction, we will optimize the temperature gradient in the gas phase Ω_g by directly controlling the heat source u in Ω_s.

The temperature y inside the growth apparatus arises as the solution of the conductive-radiative heat transfer problem in the growth apparatus. Accounting for radiative contributions is essential owing to the high temperatures. Thus, the problem is described by the stationary heat equation with radiation interface and boundary conditions on Γ_r and Γ_0, respectively. We take Ω_s to be entirely opaque, whereas Ω_g represents a transparent medium which does not interact with radiation. Furthermore, the radiative surfaces $\Gamma_0 := \partial\Omega$ and $\Gamma_\mathrm{r} := \overline{\Omega}_\mathrm{s} \cap \overline{\Omega}_\mathrm{g}$ are presumed to be diffuse-gray, i.e. the emissivity ε is independent of both the direction and the wavelength of the radiation. In particular, the local radiative heat exchange on Γ_0 can be modeled by the Boltzmann radiation condition with an external temperature y_0. Due to the heat exchange between points on Γ_r, we obtain an additional radiative heat flux on Γ_r, denoted by q_r.
In addition to the stationary semilinear heat equation with radiation interface and boundary conditions, we consider box constraints for the control function u. Thus, the optimal control problem, considered here, reads as follows:

$$
(\mathrm{P}) \quad \begin{cases}
\text{minimize} \quad J(y,u) := \dfrac{1}{2}\displaystyle\int_{\Omega_\mathrm{g}} |\nabla y - z|^2\,dx + \dfrac{\nu}{2}\displaystyle\int_{\Omega_\mathrm{s}} u^2\,dx \\
\text{subject to} \qquad -\mathrm{div}(\kappa_\mathrm{s}\,\nabla y) = u \quad \text{in } \Omega_\mathrm{s} \\
\qquad\qquad\qquad -\mathrm{div}(\kappa_\mathrm{g}\,\nabla y) = 0 \quad \text{in } \Omega_\mathrm{g} \\
\qquad \kappa_\mathrm{g}\left(\dfrac{\partial y}{\partial n_r}\right)_\mathrm{g} - \kappa_\mathrm{s}\left(\dfrac{\partial y}{\partial n_r}\right)_\mathrm{s} = q_\mathrm{r} \quad \text{on } \Gamma_\mathrm{r} \\
\qquad \kappa_\mathrm{s}\dfrac{\partial y}{\partial n_0} + \varepsilon\sigma\,|y|^3 y = \varepsilon\sigma\,y_0^4 \quad \text{on } \Gamma_0 \\
\text{and} \quad u_a \le u(x) \le u_b \quad \text{a.e. in } \Omega_\mathrm{s},
\end{cases}
$$

where n_0 is the outward unit normal on Γ_0, and n_r is the unit normal on Γ_r facing outward with respect to Ω_s (cf. Fig. 1.1). Furthermore, z denotes the desired temperature gradient and $\nu > 0$ is a Tikhonov regularization parameter. In the state equation, σ represents the Boltzmann radiation constant, and κ_s, κ_g denote the thermal conductivities in Ω_s, Ω_g, respectively.

In contrast to the boundary condition on Γ_0, the radiative heat transfer on Γ_r is nonlocal. The corresponding mathematical model used here is described in detail in [10]. It provides the additional radiative heat flux q_r on Γ_r given by

$$
q_\mathrm{r} = (I - K)(I - (1-\varepsilon)K)^{-1}\varepsilon\,\sigma|y|^3 y := G\,\sigma|y|^3 y, \tag{1.1}
$$

where K is an integral operator representing the irradiation on Γ_r. The nonlocal operators K and G will be specified in Section 3. The nonlocal radiation on Γ_r represents the main characteristic of the problem, since the nonlinearity in the state equation in (P) is in general not monotone due to nonpositivity of G (see [10]).

Problem (P) has already been investigated by Meyer, Philip, and Tröltzsch in [8], where first-order necessary conditions are proved. Based on these results, we establish second-order sufficient optimality conditions for (P). Due to the nonlinear interface and boundary conditions on Γ_r and Γ_0, (P) belongs to the class of semilinear elliptic optimal control problems. There are numerous publications which address second-order conditions for problems of such type. We only mention Casas, Tröltzsch, and Unger [4], Bonnans [1], and Casas and Mateos [3]. Here, we consider conditions that are sufficient for local optimality of a reference function in a L^s-neighborhood, where s is not necessarily equal to ∞. To that end, we use a technique, introduced for the Navier-Stokes equations by Tröltzsch and Wachsmuth [12]. In case of the Navier-Stokes equations, the situation is, in some sense, easier, since the nonlinearity in the state equation is only of quadratic type. Hence, under certain assumptions on the objective functional, it is possible to avoid the well-known two-norm discrepancy (see [12] for details). This is even valid, if one allows for strongly active sets as introduced by Dontchev, Hager, Poore, and Yang [5]. However, in our case, one has to deal with a two-norm discrepancy when using strongly active control constraints. Therefore, we modify the proof of Tröltzsch and Wachsmuth and follow an approach by Casas, Tröltzsch, and Unger [4], who consider a more general setting. This covers a class of optimal control problems with a semilinear elliptic state equation whose nonlinearity is monotone. However, although this is not the case here, main parts of the corresponding theory for second-order conditions can also be applied to (P).

The paper is organized as follows: After stating the mathematical setting in Section 2, we recall some results of [10], [7], and [8], concerning the semilinear state equation and first-order conditions for (P), see Sections 3 and 4. Then, in Section 5, our main result, i.e. the second-order sufficient conditions, are stated. Section 6 is devoted to some auxiliary results that are needed for the proof of the second order-conditions, that is presented in Section 7.

2. The mathematical setting. Throughout this paper, we assume the following conditions on the domain Ω and on the quantities and functions occurring in (P):

ASSUMPTION 1. *We assume that $\Omega \subset \mathbb{R}^3$ is a bounded simply connected domain with Lipschitz boundary Γ_0. The boundary of the simply connected subdomain $\overline{\Omega}_{\mathrm{g}} \subset \Omega$, denoted by Γ_{r}, is assumed to be a closed Lipschitz surface that is piecewise $C^{1,\delta}$. Notice that the distance of Γ_{r} to Γ_0 is positive. Then, Ω_{s} is defined by $\Omega_{\mathrm{s}} = \Omega \backslash \overline{\Omega}_{\mathrm{g}}$. The Boltzmann radiation constant is assumed to be positive, i.e. $\sigma \in \mathbb{R}^+$. For the thermal conductivity, we assume $\kappa \in L^\infty(\Omega)$ with*

$$\kappa(x) = \begin{cases} \kappa_{\mathrm{s}}(x) \ \text{in } \Omega_{\mathrm{s}} \\ \kappa_{\mathrm{g}}(x) \ \text{in } \Omega_{\mathrm{g}} \end{cases}$$

and $\kappa(x) \geq \kappa_{\min} > 0$ a.e. on Ω. Furthermore, the emissivity $\varepsilon \in L^\infty(\Gamma_0 \cup \Gamma_{\mathrm{r}})$ is bounded by $1 \geq \varepsilon \geq \varepsilon_{\min} > 0$ a.e. on $\Gamma_0 \cup \Gamma_{\mathrm{r}}$.

ASSUMPTION 2. *The desired temperature gradient z is given in $L^2(\Omega_{\mathrm{g}})$ and ν is a positive constant. For the box constraints, we assume $u_a, u_b \in L^\infty(\Omega_{\mathrm{s}})$ and $0 \leq u_a(x) < u_b(x)$ a.e. in Ω_{s}. The external temperature y_0 is a function in $L^{16}(\Gamma_0)$ and fulfills $y_0 \geq \vartheta$ a.e. on Γ_0 with a positive constant ϑ.*

Moreover, we use the following notations:

NOTATION. We introduce the set of admissible controls by

$$U_{ad} := \{u \in L^\infty(\Omega_{\mathrm{s}}) \,|\, u_a(x) \leq u(x) \leq u_b(x) \text{ a.e. in } \Omega_{\mathrm{s}}\}.$$

The identity operator in the respective function spaces is denoted by I. Moreover, τ_{r} is the trace operator on Γ_{r}, whereas τ_0 denotes the trace on Γ_0. Throughout this paper, c is a generic constant and φ denotes a generic function. Let W be a Banach space with its dual space W^*. Then, for $f \in W$ and $g \in W^*$, $\langle f\,,\,g\rangle$ denotes the associated pairing.

3. The semilinear state equations. In this section, we recall some results of Laitinen and Tiihonen [7], Tiihonen [10], [11], and Meyer, Philip, and Tröltzsch [8]. First, we present some properties of the nonlocal radiation operator G and the integral operator K.

DEFINITION 3.1. *The integral operator K, representing the irradiation on Γ_{r}, is given by*

$$(K\,y)(x) = \int_{\Gamma_{\mathrm{r}}} \omega(x,z)\,y(z)\,ds_z, \tag{3.1}$$

where the kernel ω is defined by

$$\omega(x,z) = \begin{cases} \Xi(x,z)\,\dfrac{[n_{\mathrm{r}}(z)\cdot(x-z)][n_{\mathrm{r}}(x)\cdot(z-x)]}{2|z-x|^3}\,, & \text{for } n=2 \\ \Xi(x,z)\,\dfrac{[n_{\mathrm{r}}(z)\cdot(x-z)][n_{\mathrm{r}}(x)\cdot(z-x)]}{\pi|z-x|^4}\,, & \text{for } n=3. \end{cases}$$

In this definition, x, z denote two points on Γ_{r}, and $n_{\mathrm{r}}(x)$ is the unit normal at x facing outward with respect to Ω_{s}. Here, Ξ represents the visibility factor which is given by

$$\Xi(x,z) = \begin{cases} 0 & \text{if } \overline{xz}\cap\Omega_{\mathrm{g}} \neq \emptyset, \\ 1 & \text{if } \overline{xz}\cap\Omega_{\mathrm{g}} = \emptyset, \end{cases}$$

with $\overline{xz}$ denotes the line between x and z.

In [11], it is proven that $\omega(x,z)$ has a singularity at x of type $|x-z|^{-(1-\delta)}$ in the two-dimensional and $|x-z|^{-2(1-\delta)}$ in the three-dimensional case, which is, in both cases, integrable. This is the key point to the following lemma derived in [11].

LEMMA 3.2.
(i) K maps $L^p(\Gamma_{\mathrm{r}})$ to $L^p(\Gamma_{\mathrm{r}})$ for all $1 \le p \le \infty$.
(ii) The operator $I-(1-\varepsilon)K : L^p(\Gamma_{\mathrm{r}}) \to L^p(\Gamma_{\mathrm{r}})$ is continuously invertible.

With the help of Lemma 3.2, Tiihonen and Laitinen proved the following property of $G = (I-K)(I-(1-\varepsilon)K)^{-1}\varepsilon$ (cf. [10, Lemma 6] and [7, Lemma 8]).

LEMMA 3.3. *G is a bounded linear operator from $L^p(\Gamma_{\mathrm{r}})$ to itself for all $1 \le p \le \infty$.*

Notice that the kernel ω is symmetric and hence, K is formally self-adjoint. Therefore, we obtain that $G^* = \varepsilon(I-(1-\varepsilon)K)^{-1}(I-K)$ is also linear and bounded from $L^p(\Gamma_{\mathrm{r}})$ to $L^p(\Gamma_{\mathrm{r}})$ for all $1 \le p \le \infty$.

With these results at hand, Laitinen and Tiihonen derived the existence of solutions to the state equation in (P) that is given by

$$\begin{aligned} -\mathrm{div}(\kappa_{\mathrm{s}}\,\nabla y) &= u && \text{in } \Omega_{\mathrm{s}} \\ -\mathrm{div}(\kappa_{\mathrm{g}}\,\nabla y) &= 0 && \text{in } \Omega_{\mathrm{g}} \\ \kappa_{\mathrm{g}}\left(\frac{\partial y}{\partial n_r}\right)_{\mathrm{g}} - \kappa_{\mathrm{s}}\left(\frac{\partial y}{\partial n_r}\right)_{\mathrm{s}} &= G\,\sigma|y|^3 y && \text{on } \Gamma_{\mathrm{r}} \\ \kappa_{\mathrm{s}}\frac{\partial y}{\partial n_0} + \varepsilon\sigma\,|y|^3 y &= \varepsilon\sigma\, y_0^4 && \text{on } \Gamma_0. \end{aligned} \tag{3.2}$$

Notice that, due to the non-positivity of G, the nonlinearity in (3.2) is not monotone. Therefore, Laitinen and Tiihonen used Brezis' existence theorem on the solution of equations with pseudomonotone operators to show the existence of solutions (see [7] for details). In the following, we consider y in the state space V that is defined by

$$V := \{v \in H^1(\Omega) \mid \tau_{\mathrm{r}}\, v \in L^5(\Gamma_{\mathrm{r}})\,,\, \tau_0\, v \in L^5(\Gamma_0)\}$$

where τ_{r} denotes the trace operator on Γ_{r}, whereas τ_0 is the trace on Γ_0. The space V is equipped with the norm

$$\|v\|_V = \|v\|_{H^1(\Omega)} + \|v\|_{L^5(\Gamma_{\mathrm{r}})} + \|v\|_{L^5(\Gamma_0)}.$$

Theorem 3.4. [7, Theorem 2] *Under Assumption 1, the semilinear equation* (3.2) *admits a unique solution in V for every $u \in H^1(\Omega_{\mathrm{s}})^*$ and $y_0 \in L^5(\Gamma_0)$.*

In [8], it is shown that, if the right-hand side is sufficiently regular, solutions to (3.2) belong to the following function space

$$V^\infty := H^1(\Omega) \cap L^\infty(\Omega), \tag{3.3}$$

equipped with the norm

$$\|v\|_{V^\infty} = \|v\|_{H^1(\Omega)} + \|v\|_{L^\infty(\Omega)}.$$

Notice that $y \in V^\infty$ implies $\tau_{\mathrm{r}} y \in L^\infty(\Gamma_{\mathrm{r}})$ and $\tau_0 y \in L^\infty(\Gamma_0)$ (see [8, Remark 3.5]).

Theorem 3.5. [8, Theorem 4.2] *Suppose that Assumption 1 is fulfilled and $u \in L^2(\Omega_{\mathrm{s}})$ and $y_0 \in L^{16}(\Gamma_0)$. Then, there exists a constant c only depending on Ω such that the solution of* (3.2) *fulfills*

$$\|y\|_{L^\infty(\Omega)} + \|y\|_{L^\infty(\Gamma_{\mathrm{r}} \cup \Gamma_0)} \le c(1 + \|u\|_{L^2(\Omega_{\mathrm{s}})} + \|y_0\|^4_{L^{16}(\Gamma_0)}\,). \tag{3.4}$$

For a fixed $y_0 \in L^{16}(\Gamma_0)$, we introduce the control-to-state operator $S : L^2(\Omega_{\mathrm{s}}) \to V^\infty$ that assigns y to u. The positivity of S is covered by the following maximum principle.

Theorem 3.6. [8, Theorem 4.3] *Suppose that Assumption 1 is fulfilled and $u(x) \ge 0$ a.e. in Ω_{s} and $y_0(x) \ge \vartheta > 0$ a.e. on Γ_0. If y is the solution of* (3.2)*, then $y(x) \ge \vartheta$ holds a.e. on Ω and a.e. on $\Gamma_{\mathrm{r}} \cup \Gamma_0$.*

The next theorem states the existence of an optimal solution for (P). It is also proven in [8] by rather standard arguments.

Theorem 3.7. [8, Theorem 5.2] *Under the Assumptions 1 and 2, there exists an optimal control $\bar{u} \in L^\infty(\Omega_{\mathrm{s}})$ with associated state $\bar{y} \in V^\infty$.*

4. First-order necessary optimality conditions. The key point in the proof of first-order necessary optimality conditions is to show the differentiability of the control-to-state operator $S : u \mapsto y$. In preparation of a corresponding theorem, we consider the following linear equation

$$\begin{aligned} -\operatorname{div}(\kappa\,\nabla y) &= f_\Omega && \text{in } \Omega\\ \kappa_{\mathrm{s}}\left(\frac{\partial y}{\partial n_{\mathrm{r}}}\right)_{\mathrm{s}} - \kappa_{\mathrm{g}}\left(\frac{\partial y}{\partial n_{\mathrm{r}}}\right)_{\mathrm{g}} &= f_{\mathrm{r}} && \text{on } \Gamma_{\mathrm{r}}\\ \kappa_{\mathrm{s}}\,\frac{\partial y}{\partial n_0} + 4\,\varepsilon\sigma|\bar y|^3 y &= f_0 && \text{on } \Gamma_0, \end{aligned} \tag{4.1}$$

with arbitrary functions $(f_\Omega, f_{\mathrm{r}}, f_0)$ in $L^2(\Omega)\times L^2(\Gamma_{\mathrm{r}})\times L^2(\Gamma_0)$ and $\bar y \in V^\infty$ with $\bar y > 0$ a.e. in Ω. It is easy to verify that the bilinear form associated to the left-hand side in (4.1) is bounded and coercive in $H^1(\Omega)$. Therefore, the Lax-Milgram lemma implies that (4.1) admits solutions in $H^1(\Omega)$ for every right-hand side in $H^1(\Omega)^*$. Embedding theorems give that, for $n \le 3$, f_Ω can be identified with an element of $H^1(\Omega)^*$, if $f_\Omega \in L^q(\Omega)$ with $q \ge 6/5$. Thus there exists a linear continuous operator $B_\Omega : L^q(\Omega) \to H^1(\Omega)$, $q \ge 6/5$, mapping f_Ω to y, if $f_{\mathrm{r}} = f_0 = 0$. Analogously, we have the existence of linear continuous operators $B_{\mathrm{r}} : L^2(\Gamma_{\mathrm{r}}) \to H^1(\Omega)$ and $B_0 : L^2(\Gamma_0) \to H^1(\Omega)$ such that the solution of (4.1) can be expressed as

$$y = B_\Omega\, f_\Omega + B_{\mathrm{r}}\, f_{\mathrm{r}} + B_0\, f_0. \tag{4.2}$$

Notice that the operators B_Ω, B_{r}, and B_0 depend on $\bar y$. However, to improve the readability, we simply write B_Ω instead of $B_\Omega(\bar y)$ (B_{r} and B_0 analogously). Next, we consider a slightly different PDE:

$$\begin{aligned} -\operatorname{div}(\kappa\,\nabla y) &= f_\Omega && \text{in } \Omega\\ \kappa_{\mathrm{s}}\left(\frac{\partial y}{\partial n_{\mathrm{r}}}\right)_{\mathrm{s}} - \kappa_{\mathrm{g}}\left(\frac{\partial y}{\partial n_{\mathrm{r}}}\right)_{\mathrm{g}} + 4\,G(\sigma|\bar y|^3 y) &= f_{\mathrm{r}} && \text{on } \Gamma_{\mathrm{r}}\\ \kappa_{\mathrm{s}}\,\frac{\partial y}{\partial n_0} + 4\,\varepsilon\sigma|\bar y|^3 y &= f_0 && \text{on } \Gamma_0. \end{aligned} \tag{4.3}$$

Since G is not positive, the bilinear form associated to this equation is in general not coercive. Thus, the Lax-Milgram lemma cannot be applied. However, (4.3) is equivalent to

$$y = B_\Omega\, f_\Omega + B_{\mathrm{r}}\,(f_{\mathrm{r}} - 4\,G(\sigma|\bar y|^3 y)) + B_0\, f_0. \tag{4.4}$$

Notice that it would be more appropriate to write $G(\sigma|\tau_{\mathrm{r}}\bar y|^3\,\tau_{\mathrm{r}} y)$ instead of $G(\sigma|\bar y|^3 y)$ in this context. However, for the purpose of readability, in all what follows, we supress the trace in arguments of operators with domain in $L^2(\Gamma_{\mathrm{r}})$ and $L^2(\Gamma_0)$, respectively. Applying the trace operator, (4.4) is transformed into

$$\tau_{\mathrm{r}} y + 4\,\tau_{\mathrm{r}}\, B_{\mathrm{r}}(G\sigma|\bar y|^3\, y) = \tau_{\mathrm{r}}(B_\Omega\, f_\Omega + B_{\mathrm{r}}\, f_{\mathrm{r}} + B_0\, f_0). \tag{4.5}$$

To show the existence of solutions of this equation and hence (4.3), we rely on the following assumption.

Assumption 3. *$\lambda = 1$ is neither an eigenvalue of*

$$B(\bar y)(\,\cdot\,) := 4\,\tau_{\mathrm{r}}\, B_{\mathrm{r}}(G(\sigma|\bar y|^3\ \cdot\,), \tag{4.6}$$

nor an eigenvalue of

$$\tilde{B}(\bar{y})(\,\cdot\,) := 4\,\tau_{\mathrm{r}}\,B_{\mathrm{r}}(\sigma|\bar{y}|^3\,G^*\,\cdot\,), \tag{4.7}$$

with $B(\bar{y}) : L^2(\Gamma_{\mathrm{r}}) \to L^2(\Gamma_{\mathrm{r}})$ *and* $\tilde{B}(\bar{y}) : L^2(\Gamma_{\mathrm{r}}) \to L^2(\Gamma_{\mathrm{r}})$*, respectively.*

Since $B_{\mathrm{r}} : L^2(\Gamma_{\mathrm{r}}) \to H^1(\Omega)$, we have that $\tau_{\mathrm{r}}\,B_{\mathrm{r}} : L^2(\Gamma_{\mathrm{r}}) \to H^{1/2}(\Gamma_{\mathrm{r}})$. Therefore, due to the compact embedding of $L^2(\Gamma_{\mathrm{r}})$ in $H^{1/2}(\Gamma_{\mathrm{r}})$, $B(\bar{y}) : L^2(\Gamma_{\mathrm{r}}) \to L^2(\Gamma_{\mathrm{r}})$ is a compact operator. Thus, thanks to Assumption 3, the theory of Fredholm operators ensures that $(I + B(\bar{y}))$ has a continuous inverse operator. Therefore, (4.5) admits a solution in $L^2(\Gamma_{\mathrm{r}})$, giving the existence of solutions to (4.3). An immediate consequence of this result is the following theorem (cf. [8]).

Theorem 4.1. *Under Assumptions 1–3,* $S : L^2(\Omega_{\mathrm{s}}) \to V^\infty$ *is twice continuously Fréchet-differenntiable at* $(\bar{y}, \bar{u})$*. Its first derivative, denoted by* $y = S'(\bar{u})h$, $h \in L^2(\Omega_{\mathrm{s}})$*, is given by*

$$\begin{aligned}
-\mathrm{div}(\kappa_{\mathrm{s}}\,\nabla y) &= h && \text{in } \Omega_{\mathrm{s}}\\
-\mathrm{div}(\kappa_{\mathrm{g}}\,\nabla y) &= 0 && \text{in } \Omega_{\mathrm{g}}\\
\kappa_{\mathrm{s}}\left(\frac{\partial y}{\partial n_r}\right)_{\mathrm{s}} - \kappa_{\mathrm{g}}\left(\frac{\partial y}{\partial n_r}\right)_{\mathrm{g}} + 4\,G(\sigma|\bar{y}|^3 y) &= 0 && \text{on } \Gamma_{\mathrm{r}}\\
\kappa_{\mathrm{s}}\frac{\partial y}{\partial n_0} + 4\,\varepsilon\sigma|\bar{y}|^3 y &= 0 && \text{on } \Gamma_0.
\end{aligned} \tag{4.8}$$

Moreover, the second derivative $w = S''(\bar{u})[h_1, h_2]$ *solves the equation*

$$\begin{aligned}
-\mathrm{div}(\kappa_{\mathrm{s}}\,\nabla w) &= 0 && \text{in } \Omega_{\mathrm{s}}\\
-\mathrm{div}(\kappa_{\mathrm{g}}\,\nabla w) &= 0 && \text{in } \Omega_{\mathrm{g}}\\
\kappa_{\mathrm{s}}\left(\frac{\partial w}{\partial n_r}\right)_{\mathrm{s}} - \kappa_{\mathrm{g}}\left(\frac{\partial w}{\partial n_r}\right)_{\mathrm{g}} + 4\,G(\sigma|\bar{y}|^3 w) &= -12\,G(\sigma|\bar{y}|\bar{y}\,y_1 y_2) && \text{on } \Gamma_{\mathrm{r}}\\
\kappa_{\mathrm{s}}\frac{\partial w}{\partial n_0} + 4\,\varepsilon\sigma|\bar{y}|^3 w &= -12\,\varepsilon\sigma|\bar{y}|\bar{y}\,y_1 y_2 && \text{on } \Gamma_0
\end{aligned} \tag{4.9}$$

with $y_i = S'(\bar{u})h_i$, $i = 1, 2$.

Proof: We follow the lines of [8, Theorem 7.1], where the Fréchet-differentiability of S is shown in detail. However, here we also need the second derivative of S, hence we shortly sketch the proof for convenience of the reader.

We reformulate (3.2) as

$$\begin{aligned}
-\mathrm{div}(\kappa_{\mathrm{s}}\,\nabla \bar{y}) &= \bar{u} && \text{in } \Omega_{\mathrm{s}}\\
-\mathrm{div}(\kappa_{\mathrm{g}}\,\nabla \bar{y}) &= 0 && \text{in } \Omega_{\mathrm{g}}\\
\kappa_{\mathrm{g}}\left(\frac{\partial \bar{y}}{\partial n_r}\right)_{\mathrm{g}} - \kappa_{\mathrm{s}}\left(\frac{\partial \bar{y}}{\partial n_r}\right)_{\mathrm{s}} &= G\,\sigma|\bar{y}|^3\bar{y} && \text{on } \Gamma_{\mathrm{r}}\\
\kappa_{\mathrm{s}}\frac{\partial \bar{y}}{\partial n_0} + \lambda\,\bar{y} &= \varepsilon\sigma(y_0^4 - |\bar{y}|^3\bar{y}) + \lambda\,\bar{y} && \text{on } \Gamma_0,
\end{aligned} \tag{4.10}$$

with some $\lambda > 0$ such that the bilinear form associated to the left-hand side in (4.10) is bounded an coercive in $H^1(\Omega)$. Thus, the Lax-Milgram lemma yields that (4.10) admits a solution in $H^1(\Omega)$ for every right-hand side in $H^1(\Omega)^*$. Moreover, in [8] it is shown that, if the right-hand side is sufficiently regular, i.e. in $L^2(\Omega_s) \times L^4(\Gamma_r) \times L^4(\Gamma_0)$, the solution is bounded in Ω and on $\Gamma_r \cup \Gamma_0$. Thus, linear continuous operators $\tilde{B}_\Omega : L^2(\Omega) \to V^\infty$, $\tilde{B}_r : L^4(\Gamma_r) \to V^\infty$, and $\tilde{B}_0 : L^4(\Gamma_0) \to V^\infty$ exist such that (4.10) is equivalent to

$$0 = \bar{y} - \tilde{B}_{\Omega_s}\, \bar{u} + \tilde{B}_r\,(G(\sigma|\bar{y}|^3\bar{y})) - \tilde{B}_0\,(\lambda\,\bar{y} + \varepsilon\sigma\, y_0^4 - \varepsilon\sigma\,|\bar{y}|^3\bar{y}) =: T(\bar{y}, \bar{u}), \tag{4.11}$$

with $T : V^\infty \times L^2(\Omega_s) \to V^\infty$. Since $\Phi(y) = |y|^3 y$ is twice Fréchet-differentiable in $L^\infty(\Gamma_r \cup \Gamma_0)$ and $\tilde{B}_\Omega$, $\tilde{B}_r$, and $\tilde{B}_0$ are linear continuous operators, the chain rule gives that T is twice continously differentiable from $V^\infty \times L^2(\Omega_s)$ to V^∞. Moreover, in [8] it is shown that, the equation $\frac{\partial T}{\partial y}(\bar{y}, \bar{u})y = f$ with some $f \in V^\infty$ corresponds to a linear PDE with the same differential operator as in (4.3). Hence, under Assumption 3, $\frac{\partial T}{\partial y}(\bar{y}, \bar{u})$ is continuously invertible in V^∞. Therefore, the implicit function theorem gives that S is as smooth as T and hence, $y = S(u)$ is twice continuously differentiable at $\bar{u}$.

It remains to derive the particular form of $S'(\bar{u})$ and $S''(\bar{u})$. Substituting $\bar{y} = S(\bar{u})$ in (4.11) and differentiating in direction h yield

$$\begin{aligned} S'(\bar{u})h = \tilde{B}_{\Omega_s}\, h &- \tilde{B}_r\,(G(4\sigma|S(\bar{u})|^3 S'(\bar{u})h)) \\ &+ \tilde{B}_0\,(\lambda\, S'(\bar{u})h - 4\varepsilon\sigma\,|S(\bar{u})|^3 S'(\bar{u})h). \end{aligned} \tag{4.12}$$

Now we replace $y = S'(\bar{u})h$ and $\bar{y} = S(\bar{u})$. Then, with the definitions of $\tilde{B}_\Omega$, $\tilde{B}_r$, and $\tilde{B}_0$, (4.12) is equivalent to the linearized equation (4.8). For the second derivative, we rename $h_1 = h$ in (4.12) and differentiate both sides in direction h_2

$$\begin{aligned} S''(\bar{u})[h_1, h_2] = &- \tilde{B}_r\,(G(12\sigma|S(\bar{u})|S(\bar{u})\,[S'(\bar{u})h_1\,,\, S'(\bar{u})h_2])) \\ &- \tilde{B}_r\,(G(4\sigma|S(\bar{u})|^3\, S''(\bar{u})[h_1\,,\, h_2])) \\ &+ \tilde{B}_0\,(\lambda\, S''(\bar{u})[h_1\,,\, h_2] - 12\varepsilon\sigma\,|S(\bar{u})|S(\bar{u})\,[S'(\bar{u})h_1\,,\, S'(\bar{u})h_2]) \\ &- \tilde{B}_0\,(4\varepsilon\sigma\,|S(\bar{u})|^3\, S''(\bar{u})[h_1\,,\, h_2]). \end{aligned}$$

By setting $\bar{y} = S(\bar{u})$, $y_i = S'(\bar{u})h_i$, $i = 1, 2$, and $w = S''(\bar{u})[h_1, h_2]$, the definitions of $\tilde{B}_\Omega$, $\tilde{B}_r$, and $\tilde{B}_0$ imply (4.9). ∎

Next we derive first-order necessary optimaliy consditions to (P). To that end, we introduce the reduced objective functional by

$$j(u) := J(S(u), u) = \frac{1}{2}\,\|\nabla\, S(u) - z\|^2_{L^2(\Omega_g)} + \frac{\nu}{2}\,\|u\|^2_{L^2(\Omega_s)}. \tag{4.13}$$

Furthermore, we define the set of admissible controls by

$$U_{ad} := \{u \in L^2(\Omega) \,|\, u_a(x) \le u(x) \le u_b(x) \text{ a.e. in } \Omega_s\}.$$

Due to Theorem 4.1 and the chain rule, we know that j is twice continuously Fréchet-differentiable from $L^2(\Omega_s)$ to $\mathbb{R}$. Thus, by standard arguments, an optimal solution $\bar{u}$ of (P) must satisfy the following variational inequality

$$j'(\bar{u})(u - \bar{u}) \ge 0 \quad \forall\, u \in U_{ad}. \tag{4.14}$$

For the derivative of j, one obtains

$$j'(\bar{u})h = (\nabla\bar{y} - z\,,\, \nabla y)_{L^2(\Omega_g)} + \nu(\bar{u}\,,\, h)_{L^2(\Omega_g)}, \tag{4.15}$$

with $\bar{y} = S(\bar{u})$ and $y = S'(\bar{u})h$. In [8], it is shown that

$$(\nabla \bar{y} - z \,,\, \nabla y)_{L^2(\Omega_{\rm g})} = (p \,,\, h)_{L^2(\Omega_{\rm s})} \tag{4.16}$$

holds true, where p solves the *adjoint equation* associated to (P) that is given by

$$\begin{aligned} \operatorname{div}(\kappa_{\rm g}\, \nabla p) &= \Delta \bar{y} - \operatorname{div} z && \text{in } \Omega_{\rm g} \\ \operatorname{div}(\kappa_{\rm s}\, \nabla p) &= 0 && \text{in } \Omega_{\rm s} \\ \kappa_{\rm s} \left(\frac{\partial p}{\partial n_{\rm r}} \right)_{\rm s} - \kappa_{\rm g} \left(\frac{\partial p}{\partial n_{\rm r}} \right)_{\rm g} + 4\sigma\, |\bar{y}|^3 G^* p &= \frac{\partial \bar{y}}{\partial n_{\rm r}} - z \cdot n_{\rm r} && \text{on } \Gamma_{\rm r} \\ \kappa_s \, \frac{\partial p}{\partial n_0} + 4\varepsilon\sigma\, |\bar{y}|^3 p &= 0 && \text{on } \Gamma_0. \end{aligned} \tag{4.17}$$

Notice that the right-hand site in (4.17) is well defined, since $\Delta \bar{y} \in H^1(\Omega)^*$. Similar to the discussion of (4.3), the Fredholm alternative implies the existence of solutions to (4.17), provided that Assumption 3 holds true.

THEOREM 4.2. [8, Theorem 7.2] *Under Assumptions 1–3, there exists a unique solution of* (4.17) *in* $H^1(\Omega)$.

With (4.16) at hand, (4.14) is equivalent to

$$(p + \nu\, \bar{u} \,,\, u - \bar{u})_{L^2(\Omega_{\rm s})} \geq 0 \quad \forall\, u \in U_{ad}. \tag{4.18}$$

A pointwise discussion of this inequality yields

$$\bar{u}(x) = \mathcal{P}_{ad} \left\{ -\frac{1}{\nu}\, p(x) \right\}, \tag{4.19}$$

where $\mathcal{P}_{ad}(x)$ denotes the pointwise projection operator on $[u_a(x), u_b(x)]$.

In this way, we have derived the following theorem:

THEOREM 4.3. *Suppose that Assumptions 1–3 are fulfilled and $\bar{u}$ is a locally optimal solution of* (P) *with associated state $\bar{y}$. Then there exists an adjoint state $p \in H^1(\Omega)$ such that the adjoint equation* (4.17) *and the condition* (4.19) *are satisfied.*

5. Second-order sufficient conditions. This section is devoted to our main result, second-order sufficient optimality conditions for (P). First, we establish second-order conditions that require a rather large subspace where the second derivative of j must be positive definit. These conditions are very easy to prove. Then, we shrink this subspace and formulate another sufficient condition that is less restrictive than the first one. The associated proof is performed in Section 7.

In the following, the subspace, where $j''(\bar{u})$ is assumed to be positive definit, is called *critical cone.* The "large" critical cone is defined by

$$\tilde{C}(\bar{u}) := \left\{ u \in L^2(\Omega_{\rm s}) \;\middle|\; \begin{array}{l} u(x) \geq 0 \text{ , where } \bar{u}(x) = u_a(x) \\ u(x) \leq 0 \text{ , where } \bar{u}(x) = u_b(x) \end{array} \right\},$$

and hence does not account for strongly active sets.

THEOREM 5.1. *Suppose that Assumptions 1–3 are fulfilled and that $(\bar{y}, \bar{u})$ satisfy the first-order necessary optimality conditions. Assume further that a constant $\tilde{\delta} > 0$ exists such that*

$$j''(\bar{u})u^2 \geq \tilde{\delta}\, \|u\|^2_{L^2(\Omega_{\rm s})} \tag{5.1}$$

is satisfied for all $u \in \tilde{C}(\bar{u})$*. Then positive constants* $\tilde{\varepsilon} > 0$ *and* $\tilde{\sigma} > 0$ *exist, such that the quadratic growth condition*

$$j(u) \geq j(\bar{u}) + \tilde{\sigma}\, \|u - \bar{u}\|^2_{L^2(\Omega_s)} \tag{5.2}$$

holds true for all $u \in U_{ad}$ *with* $\|u - \bar{u}\|_{L^2(\Omega_s)} \leq \tilde{\varepsilon}$*.*

Proof: The proof follows standard arguments. A Taylor expansion of j at $\bar{u}$ yields for an arbitrary $u \in U_{ad}$

$$j(u) = j(\bar{u}) + j'(\bar{u})(u - \bar{u}) + \frac{1}{2}\, j''(\bar{u})(u - \bar{u})^2 + r_j^{(2)} \tag{5.3}$$

$$\geq j(\bar{u}) + \frac{\tilde{\delta}}{2}\, \|u - \bar{u}\|^2_{L^2(\Omega_s)} - |r_j^{(2)}| \tag{5.4}$$

where we used the variational inequality (4.14). Moreover, $u \in U_{ad}$ implies $(u - \bar{u}) \in \tilde{C}(\bar{u})$, hence (5.1) applies to $j''(\bar{u})(u - \bar{u})^2$. Since j is twice continuously Fréchet-differentiable from $L^2(\Omega_s)$ to $\mathbb{R}$, we have that

$$\frac{|r_j^{(2)}|}{\|u - \bar{u}\|^2_{L^2(\Omega_s)}} \to 0 \text{ , if } \|u - \bar{u}\|_{L^2(\Omega_s)} \to 0. \tag{5.5}$$

Thus a constant $\tilde{\varepsilon}$ exists with $|r_j^{(2)}| \leq \tilde{\delta}/4\, \|u - \bar{u}\|^2_{L^2(\Omega_s)}$ for all $\|u - \bar{u}\|_{L^2(\Omega_s)} \leq \tilde{\varepsilon}$. Therefore, with $\tilde{\sigma} = \tilde{\delta}/4$, (5.4) implies (5.2). ∎

Next, we formulate less restrictive second-order sufficient conditions that consider strongly active sets. As mentioned in Section 1, in this case, we have to deal with a two-norm discrepancy. We establish a condition that gives local optimality in a L^s-neighborhood, where s is not necessarily equal to ∞, but can be chosen smaller. This gives some flexibility in the choice of the neighborhood where local optimality of a reference function is obtained. However, a "larger" neighborhood corresponds to a "weaker" growth condition (see Theorem 5.4).

We introduce the *strongly active set* as follows:

Definition 5.2. *Let* $\tau > 0$ *be given. Then the strongly active set* A_τ *is defined by*

$$A_\tau := \{x \in \Omega \,|\; |p(x) + \nu\, \bar{u}(x)| \geq \tau\}.$$

Moreover, the corresponding "small" τ*-critical cone* is defined in a standard way (cf. Dontchev et al. [5]).

Definition 5.3. *The critical cone belonging to* (P) *is given by*

$$C_\tau(\bar{u}) := \left\{ u \in L^2(\Omega) \;\middle|\; \begin{array}{l} u(x) = 0 \text{ , a.e. in } A_\tau \\ u(x) \geq 0 \text{ , where } \bar{u}(x) = u_a(x) \text{ and } x \notin A_\tau \\ u(x) \leq 0 \text{ , where } \bar{u}(x) = u_b(x) \text{ and } x \notin A_\tau \end{array} \right\}. \tag{5.6}$$

Now, we are in the position to state second order sufficient conditions for (P) with respect to the reduced critical cone $C_\tau(\bar{u})$.

$$\text{(SSC)} \quad \left\{ \begin{array}{l} \text{Let } \delta > 0 \text{ exist such that} \\ \qquad j''(\bar{u})\, u^2 \geq \delta\, \|u\|^2_{L^q(\Omega_s)} \quad \text{for all } u \in C_\tau(\bar{u}). \end{array} \right.$$

In Section 7, we show that (SSC) is indeed sufficient for local optimality of $\bar{u}$.

THEOREM 5.4. *Suppose that Assumptions 1–3 are fulfilled. Let $4/3 \le q \le 2$ be given. Define s by*

$$s := \begin{cases} q/(2-q) & , \text{for } q < 2 \\ \infty & , \text{for } q = 2. \end{cases} \tag{5.7}$$

Moreover, let $(\bar{y}, \bar{u})$ satisfy the first-order necessary optimality conditions for problem (P) *and assume that condition* (SSC) *is fulfilled with some $\delta > 0$, $\tau > 0$. Then there exist $\bar{\varepsilon} > 0$ and $\bar{\sigma} > 0$ such that*

$$j(u) \ge j(\bar{u}) + \bar{\sigma}\, \|u - \bar{u}\|^2_{L^q(\Omega_\mathrm{s})} \tag{5.8}$$

for all $u \in U_{ad}$ with $\|u - \bar{u}\|_{L^s(\Omega_\mathrm{s})} \le \bar{\varepsilon}$.

REMARK 5.5. *Setting $q = 4/3$, we obtain $s = 2$, and hence Theorem 5.4 gives a $L^{4/3}$-quadratic growth condition in a L^2-neighborhood of $\bar{u}$. Choosing $q = 2$ and thus $s = \infty$, we obtain L^2-quadratic growth of j in a L^∞-neighborhood of $\bar{u}$.*

6. Auxiliary results. Before we are in the position to prove Theorem 5.4, we show some properties of the linear PDEs (4.8), (4.9), and (4.17). In Section 6.2, we derive some results concerning the second derivative of j, that are also needed for the proof of Theorem 5.4. Throughout this section, we assume that $(\bar{y}, \bar{u})$ is a fixed stationary point of problem (P). Therefore, we have that $\bar{u} \in U_{ad}$ and $(\bar{y}, \bar{u})$ satisfy the state equation (3.2). This implies that $\|\bar{y}\|_{V^\infty}$ is bounded by a constant because of Theorem 3.5 and [8, Lemma 5.1]. This property is used several times in the proofs presented above. Notice that Lemma 3.3 implies the boundedness of G and G^* from $L^p(\Gamma_\mathrm{r})$ to $L^p(\Gamma_\mathrm{r})$ for all $1 \le p \le \infty$, what is also used in the subsequent proofs.

6.1. The linearized equations. We already know that (4.8), (4.9), and (4.17) admit solutions in $H^1(\Omega)$. Here, some estimates are derived that concern the $H^1(\Omega)$-norms of the solutions are derived.

LEMMA 6.1. *Let Assumptions 1–3 be fulfilled and $q \ge 6/5$ be given. Then the solution of* (4.8) *satisfies*

$$\|y\|_{H^1(\Omega)} \le c\, \|h\|_{L^q(\Omega_\mathrm{s})}, \tag{6.1}$$

with a positive constant c.

Proof: The differential operator in (4.8) has the same structure as the one in (4.3). Hence, analogously to (4.4), y can be expressed by

$$y = B_\Omega\, \tilde{h} - 4\, B_\mathrm{r}\, (G(\sigma|\bar{y}|^3 y)) \tag{6.2}$$

with $\tilde{h}|_{\Omega_\mathrm{s}} = h$ and $\tilde{h}|_{\Omega_\mathrm{g}} = 0$. The operator B_Ω is continuous from $L^q(\Omega)$ to $H^1(\Omega)$ as described in Section 4. Together with the continuity of $B_\Omega : L^q(\Omega) \to H^1(\Omega)$ and $B_\mathrm{r} : L^2(\Gamma_\mathrm{r}) \to H^1(\Omega)$, this implies

$$\begin{aligned} \|y\|_{H^1(\Omega)} &\le \|B_\Omega\|_{\mathcal{L}(L^q(\Omega), H^1(\Omega))}\, \|\tilde{h}\|_{L^q(\Omega)} \\ &\quad + 4\sigma\, \|B_\mathrm{r}\|_{\mathcal{L}(L^2(\Gamma_\mathrm{r}), H^1(\Omega))}\, \|G\|_{\mathcal{L}(L^2(\Gamma_\mathrm{r}))}\, \|\bar{y}\|^3_{L^\infty(\Gamma_\mathrm{r})} \|y\|_{L^2(\Gamma_\mathrm{r})} \\ &\le c\, (\|h\|_{L^q(\Omega_\mathrm{s})} + \|y\|_{L^2(\Gamma_\mathrm{r})}) \end{aligned} \tag{6.3}$$

because of Lemma 3.3. Similar to (4.5), we apply the trace operator τ_{r} to (6.2) and obtain

$$\tau_{\mathrm{r}}\, y = (I + B(\bar y))^{-1} \tau_{\mathrm{r}} B_\Omega\, \tilde h, \tag{6.4}$$

where $B(\bar y)$ is as defined in (4.6). This yields

$$\begin{aligned} \|y\|_{L^2(\Gamma_{\mathrm{r}})} &\le \|(I + B(\bar y))^{-1}\|_{\mathcal{L}(L^2(\Gamma_{\mathrm{r}}))} \|\tau_{\mathrm{r}}\, B_\Omega\|_{\mathcal{L}(L^q(\Omega), L^2(\Gamma_{\mathrm{r}}))} \|\tilde h\|_{L^q(\Omega)} \\ &\le c\, \|(I + B(\bar y))^{-1}\|_{\mathcal{L}(L^2(\Gamma_{\mathrm{r}}))}\, \|h\|_{L^q(\Omega_{\mathrm{s}})}, \end{aligned}$$

where Assumption 3 ensures that $\|(I + B(\bar y))^{-1}\|_{\mathcal{L}(L^2(\Gamma_{\mathrm{r}}))} < \infty$. Together with (6.3), this implies (6.1). ∎

LEMMA 6.2. *Suppose that Assumptions 1–3 are fulfilled and $q \ge 6/5$ is given. Then the solution of* (4.9) *satisfies*

$$\|w\|_{H^1(\Omega)} \le c\, \|h_1\|_{L^q(\Omega_{\mathrm{s}})}\, \|h_2\|_{L^q(\Omega_{\mathrm{s}})}, \tag{6.5}$$

with a positive constant c.

Proof: Since the differential operator in (4.9) is the same as in (4.3) and (4.8), we have analogously to (6.2)

$$w = -4\, B_{\mathrm{r}}\,(G(\sigma|\bar y|^3 w)) - 12\, B_{\mathrm{r}}\,(G(\sigma|\bar y|\bar y\, y_1\, y_2)) - 12\, B_0(\varepsilon\sigma|\bar y|\bar y\, y_1\, y_2)$$

and similarly to (6.4)

$$\tau_{\mathrm{r}}\, w = -(I + B(\bar y))^{-1} \tau_{\mathrm{r}}\big(12\, B_{\mathrm{r}}\,(G(\sigma|\bar y|\bar y\, y_1\, y_2)) + 12\, B_0(\varepsilon\sigma|\bar y|\bar y\, y_1\, y_2)\big),$$

where Assumption 3 again ensures the continuity of $(I + B(\bar y))^{-1}$. Therefore, we can argue as in the proof before and obtain

$$\|w\|_{H^1(\Omega)} \le\; c\,(\|G(\sigma\,|\bar y|\bar y\, y_1\, y_2)\|_{L^2(\Gamma_{\mathrm{r}})} + \|\varepsilon\sigma\,|\bar y|\bar y\, y_1\, y_2\|_{L^2(\Gamma_0)}). \tag{6.6}$$

The first addend on the right-hand side is estimated by

$$\|G(\sigma\,|\bar y|\bar y\, y_1\, y_2)\|_{L^2(\Gamma_{\mathrm{r}})} \le c\, \|G\|_{\mathcal{L}(L^2(\Gamma_{\mathrm{r}}))} \|\bar y\|^2_{L^\infty(\Gamma_{\mathrm{r}})} \|y_1\, y_2\|_{L^2(\Gamma_{\mathrm{r}})}.$$

Due to $\dim\,\Omega \le 3$, the embedding theorems imply for two arbitrary functions $v_1, v_2 \in H^1(\Omega)$:

$$\begin{aligned} \|v_1\, v_2\|_{L^2(\Gamma_{\mathrm{r}})} &\le (\, \|v_1^2\|_{L^2(\Gamma_{\mathrm{r}})}\, \|v_2^2\|_{L^2(\Gamma_{\mathrm{r}})}\,)^{1/2} = \|v_1\|_{L^4(\Gamma_{\mathrm{r}})}\, \|v_2\|_{L^4(\Gamma_{\mathrm{r}})} \\ &\le c\, \|v_1\|_{H^1(\Omega)}\, \|v_2\|_{H^1(\Omega)}. \end{aligned} \tag{6.7}$$

Thus, with $v_1 = y_1$ and $v_2 = y_2$, Lemma 6.1 yields

$$\|G(\sigma\,|\bar y|\bar y\, y_1\, y_2)\|_{L^2(\Gamma_{\mathrm{r}})} \le c\, \|h_1\|_{L^q(\Omega_{\mathrm{s}})}\, \|h_2\|_{L^q(\Omega_{\mathrm{s}})}. \tag{6.8}$$

Analogously, we obtain for the second addend in (6.6)

$$\|\varepsilon\sigma\,|\bar y|\bar y\, y_1\, y_2\|_{L^2(\Gamma_0)} \le c\, \|h_1\|_{L^q(\Omega_{\mathrm{s}})}\, \|h_2\|_{L^q(\Omega_{\mathrm{s}})}. \tag{6.9}$$

Inserting (6.8) and (6.9) in (6.6) finally gives the assertion. ∎

LEMMA 6.3. *Suppose that Assumptions 1–3 are fulfilled. Let p denote the solution of the adjoint equation* (4.17). *Then, there exists a positive constant c such that*

$$\|p\|_{L^q(\Gamma_r)} \le c$$

holds true for all $1 \le q \le 4$.

Proof: Formal integration by parts, also on the right-hand side, yields the weak formulation associated to (4.17):

$$\begin{aligned} \int_\Omega \kappa \nabla p \cdot \nabla v \, dx + 4 \int_{\Gamma_r} \sigma \, |\bar y|^3 \, G^*(p) \, v \, ds + 4 \int_{\Gamma_0} \varepsilon\sigma \, |\bar y|^3 \, p \, v \, ds \\ = \int_{\Omega_g} (\nabla \bar y - z) \cdot \nabla v \, dx =: \langle g, v \rangle \quad \forall \, v \in H^1(\Omega) \end{aligned} \tag{6.10}$$

with $g \in H^1(\Omega)^*$, since $\bar y \in V^\infty$ and $z \in L^2(\Omega_g)$ by Assumption 2. If we consider the operator B_Ω, introduced in Section 4, with domain in $H^1(\Omega)^*$, it is easy to see that (4.17) is equivalent to

$$p = B_\Omega \, g - 4 \, B_r \, (\sigma |\bar y|^3 \, G^* \, p). \tag{6.11}$$

Applying the trace operator to this equation gives

$$\tau_r \, p = (I - \tilde B(\bar y))^{-1} \tau_r B_\Omega \, g,$$

with $\tilde B(\bar y)$ as defined in (4.7). Therefore, we obtain

$$\|p\|_{L^2(\Gamma_r)} \le \|(I - \tilde B(\bar u))^{-1}\|_{\mathcal{L}(L^2(\Gamma_r))} \|\tau_r \, B_\Omega\|_{\mathcal{L}(H^1(\Omega)^*, L^2(\Gamma_r))} \|g\|_{H^1(\Omega)^*}, \tag{6.12}$$

and Assumption 3 again ensures that $\|(I - \tilde B(\bar u))^{-1}\|_{\mathcal{L}(L^2(\Gamma_r))} < \infty$. The definition of g in (6.10) implies

$$\|g\|_{H^1(\Omega)^*} \le \|\bar y\|_{H^1(\Omega)} + \|z\|_{L^2(\Omega_g)} \le c,$$

where the boundedness of $\|\bar y\|_{H^1(\Omega)}$ follows from [8, Lemma 5.1]. Hence, (6.12) yields $\|p\|_{L^2(\Gamma_r)} \le c$, and with the embedding theorems for $\dim \Omega \le 3$, (6.11) finally implies

$$\|p\|_{L^q(\Gamma_r)} \le c \, \|p\|_{H^1(\Omega)} \le c\big(\|g\|_{H^1(\Omega)^*} + \|p\|_{L^2(\Gamma_r)}\big) \le c. \qquad \blacksquare$$

6.2. The second derivative of j. As mentioned above, the reduced objective functional j is twice continuously Fréchet-differentiable from $L^2(\Omega_s)$ to $\mathbb{R}$. Due to the chain rule, its second derivative is given by

$$j''(\bar u)[h_1 \, , \, h_2] = (\nabla y_1 \, , \, \nabla y_2)_{L^2(\Omega_g)} + (\nabla \bar y - z \, , \, \nabla w)_{L^2(\Omega_g)} + \nu (h_1 \, , \, h_2)_{L^2(\Omega_s)},$$

with $y_i = S'(\bar u) h_i$, $i = 1, 2$, and $w = S''(\bar u)[h_1, h_2]$ defined by (4.9). The weak formulation of (4.9) is given by

$$\begin{aligned} \int_\Omega \kappa \nabla w \cdot \nabla v \, dx \, + \, 4 \int_{\Gamma_r} G(\sigma \, |\bar y|^3 w) v \, ds + 4 \int_{\Gamma_0} \varepsilon\sigma \, |\bar y|^3 \, w \, v \, ds = \\ - \, 12 \int_{\Gamma_r} G(\sigma \, |\bar y| \bar y \, y_1 \, y_2) v \, ds - 12 \int_{\Gamma_0} \varepsilon\sigma \, |\bar y| \bar y \, y_1 \, y_2 \, v \, ds \quad \forall \, v \in H^1(\Omega). \end{aligned}$$

Now, we insert $p \in H^1(\Omega)$ as test function in this equation and choose w as test function in the weak formulation of the adjoint equation (6.10). Subtracting both equations yields

$$(\nabla\bar{y} - z\,,\,\nabla w)_{L^2(\Omega_g)} = -\,12\int\limits_{\Gamma_r} G(\sigma\,|\bar{y}|\bar{y}\,y_1\,y_2)p\,ds - 12\int\limits_{\Gamma_0} \varepsilon\sigma\,|\bar{y}|\bar{y}\,y_1\,y_2\,p\,ds,$$

and hence

$$\begin{aligned} j''(\bar{u})[h_1\,,\,h_2] &= (\nabla y_1\,,\,\nabla y_2)_{L^2(\Omega_g)} + \nu(h_1\,,\,h_2)_{L^2(\Omega_s)} \\ &\quad - 12\Big(\int\limits_{\Gamma_r} G(\sigma\,|\bar{y}|\bar{y}\,y_1 y_2)p\,ds + \int\limits_{\Gamma_0} \varepsilon\sigma\,|\bar{y}|\bar{y}\,y_1\,y_2\,p\,ds\Big). \end{aligned} \tag{6.13}$$

Lemma 6.4. *Under Assumptions 1–3,*

$$\Big|\int\limits_{\Gamma_r} G(\sigma\,|\bar{y}|\bar{y}\,y_1\,y_2)p\,ds\Big| + \Big|\int\limits_{\Gamma_0} \varepsilon\sigma\,|\bar{y}|\bar{y}\,y_1\,y_2\,p\,ds\Big| \le c\,\|h_1\|_{L^q(\Omega_s)}\,\|h_2\|_{L^q(\Omega_s)}$$

holds true with a positive constant c and $y_i = S'(\bar{u})h_i$, $i = 1, 2$.

Proof: The Γ_r-integral is estimated as follows

$$\begin{aligned} \Big|\int\limits_{\Gamma_r} G(\sigma\,|\bar{y}|\bar{y}\,y_1\,y_2)p\,ds\Big| &\le \|p\|_{L^2(\Gamma_r)}\,\|G\|_{\mathcal{L}(L^2(\Gamma_r))}\,\sigma\,\|y\|^2_{L^\infty(\Gamma_r)}\,\|y_1\,y_2\|_{L^2(\Gamma_r)} \\ &\le c\,\|h_1\|_{L^q(\Omega_s)}\,\|h_2\|_{L^q(\Omega_s)}, \end{aligned} \tag{6.14}$$

where we used (6.8), Lemma 6.1, and Lemma 6.3 for the last inequality. Analogously, we obtain for the integral over Γ_0:

$$\Big|\int\limits_{\Gamma_0} \varepsilon\sigma\,|\bar{y}|\bar{y}\,y_1\,y_2\,p\,ds\Big| \le c\,\|h_1\|_{L^q(\Omega_s)}\,\|h_2\|_{L^q(\Omega_s)}.$$

Together with (6.14), this yields the assertion. ∎

The Taylor expansion of j is given by

$$j(u) = j(\bar{u}) + j'(\bar{u})(u-\bar{u}) + \frac{1}{2}\,j''(\bar{u})(u-\bar{u})^2 + r_j^{(2)}, \tag{6.15}$$

where the remainder term fulfills (5.5), since j is twice Fréchet-differentiable from $L^2(\Omega_s)$ to $\mathbb{R}$. Using the previous results, we show the following lemma that includes (5.5) as a special case.

Lemma 6.5. *Let Assumptions 1–3 be fulfilled and $q \ge 6/5$ be given. Then, the remainder term $r_j^{(2)}$ satisfies*

$$\frac{|r_j^{(2)}|}{\|h\|^2_{L^q(\Omega_s)}} \to 0 \tag{6.16}$$

for all h with $\bar{u} + h \in U_{ad}$ and $\|h\|_{L^2(\Omega_s)} \to 0$.

Proof: First we prove the assertion for $6/5 \le q \le 2$. Later we show, that (6.16) holds for every $q \ge 6/5$.

(i) Taylor expansions

With (6.15) at hand, one obtains for $r_j^{(2)}$

$$\begin{aligned} r_j^{(2)} &= j(\bar u + h) - j(\bar u) - j'(\bar u)h - \frac{1}{2} j''(\bar u)h^2 \\ &= \int_0^1 j'(\bar u + \beta\, h)h \, d\beta - j'(\bar u)h - \frac{1}{2} j''(\bar u)h^2 \\ &= \int_0^1 \int_0^\beta j''(\bar u + \theta\, h)h^2 - j''(\bar u)h^2 \, d\theta \, d\beta = \int_0^1 \int_0^\beta \rho_j \, d\theta \, d\beta. \end{aligned} \tag{6.17}$$

with $\rho_j := j''(\bar u + \theta\, h)h^2 - j''(\bar u)h^2$. Inserting (6.13) in the definition of ρ_j yields

$$\begin{aligned} \rho_j &= \|\nabla \eta_h\|^2_{L^2(\Omega_\mathrm{g})} - \|\nabla \eta\|^2_{L^2(\Omega_\mathrm{g})} \\ &\quad - 12 \int_{\Gamma_\mathrm{r}} G\big(s\,(|y_h| y_h\, \eta_h^2 - |\bar y| \bar y\, \eta^2)\big) p \, ds - 12 \int_{\Gamma_0} \varepsilon\sigma\, (|y_h| y_h\, \eta_h^2 - |\bar y| \bar y\, \eta^2) p \, ds \end{aligned} \tag{6.18}$$

with $\bar y = S(\bar u)$, $y_h = S(\bar u + \theta h)$, $\eta = S'(\bar u)h$, and $\eta_h = S'(\bar u + \theta h)h$. Straightforward computation shows that the first addend in (6.18) can be expressed as

$$\|\nabla \eta_h\|^2_{L^2(\Omega_\mathrm{g})} - \|\nabla \eta\|^2_{L^2(\Omega_\mathrm{g})} = J''(y_h, \bar u + \theta\, h)(\eta_h, h)^2 - J''(\bar y, \bar u)(\eta, h)^2 =: \rho_J.$$

For $y_h = S(\bar{+}\theta\, h)$, one obtains

$$y_h = S(\bar u) + \theta\, S'(\bar u)h + r_S^{(1)} = \bar y + \theta \eta + r_S^{(1)}. \tag{6.19}$$

This first-order remainder term satisfies

$$\|r_S^{(1)}\|_{H^1(\Omega)} \le \|r_S^{(1)}\|_{V^\infty} \le \varphi(\|h\|_{L^2(\Omega_\mathrm{s})})\, \|h\|_{L^2(\Omega_\mathrm{s})}, \tag{6.20}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ denotes a generic function with $\varphi(x) \to 0$ for every $x \downarrow 0$. Furthermore, since S is twice Fréchet-differentiable from $L^2(\Omega_\mathrm{s})$ to V^∞, we have

$$S'(\bar u + \theta h) = S'(\bar u) + \theta\, S''(\bar u)h + r_{S'}^{(1)} \tag{6.21}$$

with

$$\|r_{S'}^{(1)}\|_{\mathcal{L}(L^2(\Omega_\mathrm{s}), V^\infty)} \le \varphi(\|h\|_{L^2(\Omega_\mathrm{s})})\, \|h\|_{L^2(\Omega_\mathrm{s})}.$$

We apply both sides of (6.21) to h and obtain

$$\eta_h = S'(\bar u + \theta h)h = S'(\bar u)h + \theta\, S''(\bar u)h^2 + r_{S'}^{(1)} h = \eta + \theta\, w + \tilde r_S^{(2)}, \tag{6.22}$$

where $w = S''(\bar u)h^2$ solves equation (4.9) with the inhomogenities $-12\, G(\sigma\, |\bar y| \bar y\, \eta^2)$ and $-12\, \varepsilon\sigma\, |\bar y| \bar y\, \eta^2$, respectively. Moreover, $\tilde r_S^{(2)}$ is given by $\tilde r_S^{(2)} = r_{S'}^{(1)} h$ and thus fulfills

$$\begin{aligned}\|\tilde r_S^{(2)}\|_{H^1(\Omega)} &\le \|r_{S'}^{(1)}h\|_{V^\infty} \le \|r_{S'}^{(1)}\|_{\mathcal{L}(L^2(\Omega_\mathrm{s}),V^\infty)}\,\|h\|_{L^2(\Omega_\mathrm{s})}\\ &\le \varphi(\|h\|_{L^2(\Omega_\mathrm{s})})\,\|h\|^2_{L^2(\Omega_\mathrm{s})}\\ &\le \varphi(\|h\|_{L^2(\Omega_\mathrm{s})})\,\|h\|_{L^{q/(q-1)}(\Omega_\mathrm{s})}\|h\|_{L^q(\Omega_\mathrm{s})}\\ &\le c\,\varphi(\|h\|_{L^2(\Omega_\mathrm{s})})\,\|h\|_{L^q(\Omega_\mathrm{s})},\end{aligned}\tag{6.23}$$

since $|h(x)| \le u_b(x) - u_a(x)$ a.e. in Ω_s because of $(\bar u + h) \in U_{ad}$.

(ii) Estimation of ρ_J

With (6.22) and $\theta \le 1$, we find for ρ_J

$$\begin{aligned}|\rho_J| &= \big|\,\|\nabla\eta_h\|^2_{L^2(\Omega_\mathrm{g})} - \|\nabla\eta\|^2_{L^2(\Omega_\mathrm{g})}\big|\\ &\le \big|\,\|\eta_h\|^2_{H^1(\Omega)} - \|\eta\|^2_{H^1(\Omega)}\big|\\ &= \big|\,\|\eta + \theta\, w + \tilde r_S^{(2)}\|^2_{H^1(\Omega)} - \|\eta\|^2_{H^1(\Omega)}\big|\\ &\le 2\,\|w\|_{H^1(\Omega)}\|\eta\|_{H^1(\Omega)} + 2\,\|\tilde r_S^{(2)}\|_{H^1(\Omega)}\|\eta\|_{H^1(\Omega)} + 2\,\|w\|_{H^1(\Omega)}\|\tilde r_S^{(2)}\|_{H^1(\Omega)}\\ &\quad + \|\tilde r_S^{(2)}\|^2_{H^1(\Omega)} + \|w\|^2_{H^1(\Omega)},\end{aligned}\tag{6.24}$$

where (6.23) holds for $\|\tilde r_S^{(2)}\|_{H^1(\Omega)}$. Moreover, Lemma 6.1 and 6.2 give

$$\|\eta\|_{H^1(\Omega)} \le c\,\|h\|_{L^q(\Omega_\mathrm{s})} \quad\text{and}\quad \|w\|_{H^1(\Omega)} \le c\,\|h\|^2_{L^q(\Omega_\mathrm{s})}.$$

Therefore, by inserting these estimates together with (6.23), (6.24) results in

$$|\rho_J| \le \varphi(\|h\|_{L^2(\Omega_\mathrm{s})})\,\|h\|^2_{L^q(\Omega_\mathrm{s})}.\tag{6.25}$$

Notice that the assumption $q \le 2$ implies $\|h\|_{L^q(\Omega_\mathrm{s})} \le c\,\|h\|_{L^2(\Omega_\mathrm{s})}$. This is used for instance in the estimate $\|w\|_{H^1(\Omega)} \le \varphi(\|h\|_{L^2(\Omega_\mathrm{s})})\,\|h\|_{L^q(\Omega_\mathrm{s})}$.

(iii) Estimation of the boundary integrals

Next, we estimate the integral over Γ_r in (6.18). Using (6.22), one obtains

$$\begin{aligned}&\Big|\int_{\Gamma_\mathrm{r}} G\big(\sigma\,(|y_h|y_h\,\eta_h^2 - |\bar y|\bar y\,\eta^2)\big)p\,ds\Big|\\ &\qquad = \Big|\int_{\Gamma_\mathrm{r}} G\big[\sigma\,(|y_h|y_h\,\big(\eta + \theta\,w + \tilde r_S^{(2)}\big)^2 - |\bar y|\bar y\,\eta^2)\big]p\,ds\Big| \le I_1 + I_2\end{aligned}$$

with

$$I_1 := \Big|\int_{\Gamma_\mathrm{r}} \sigma\,(|y_h|y_h - |\bar y|\bar y)\eta^2\,G^*p\,ds\Big|$$

and

$$I_2 := \Big|\int_{\Gamma_\mathrm{r}} \sigma\,|y_h|y_h\,G^*p\,(2\theta\eta w + 2\tilde r_S^{(2)}\eta + 2\theta w\tilde r_S^{(2)} + (\tilde r_S^{(2)})^2 + \theta^2 w^2)\,ds\Big|.$$

We continue with

$$\begin{aligned} I_1 &\le \sigma\, \|\eta^2\|_{L^2(\Gamma_r)}\, \|(|y_h|y_h - |\bar y|\bar y)\, G^* p\|_{L^2(\Gamma_r)} \\ &\le \sigma\, \|\eta^2\|_{L^2(\Gamma_r)}\, \|G^*\|_{\mathcal{L}(L^4(\Gamma_r))}\, \|p\|_{L^4(\Gamma_r)} \|\, |y_h|y_h - |\bar y|\bar y\, \|_{L^4(\Gamma_r)}, \end{aligned} \tag{6.26}$$

where we used (6.7) for the last inequality. Thanks to $\bar u, \bar u + h \in U_{ad}$, the maximum principle in Theorem 3.6 implies $\bar y, y_h \ge \vartheta > 0$. Thus, together with (6.19), we have

$$|y_h|y_h - |\bar y|\bar y = y_h^2 - \bar y^2 = (y_h + \bar y)(y_h - \bar y) = (y_h + \bar y)(\eta + r_S^{(1)}).$$

Hence, (6.20) and Lemma 6.1 yield

$$\begin{aligned} \|\, |y_h|y_h - |\bar y|\bar y\, \|_{L^4(\Gamma_r)} &\le \|(y_h + \bar y)\|_{L^\infty(\Gamma_r)}\, (\|\eta\|_{L^4(\Gamma_r)} + \|r_S^{(1)}\|_{L^4(\Gamma_r)}) \\ &\le c\,(1 + \varphi(\|h\|_{L^2(\Omega_s)}))\|h\|_{L^2(\Omega_s)} = \varphi(\|h\|_{L^2(\Omega_s)}). \end{aligned}$$

Therefore, by applying Lemma 6.1 to $\|\eta\|^2_{L^2(\Gamma_r)}$ and Lemma 6.3 to $\|p\|_{L^4(\Gamma_r)}$, (6.26) results in

$$I_1 \le \varphi(\|h\|_{L^2(\Omega_s)})\, \|h\|^2_{L^q(\Omega_s)}. \tag{6.27}$$

Using again (6.7) and Lemma 6.3, the integral I_2 is estimated as follows:

$$\begin{aligned} I_2 \;&\le \|\sigma\, |y_h|y_h\, G^*\, p\|_{L^2(\Gamma_r)} \\ &\quad \big(2\|\eta w\|_{L^2(\Gamma_r)} + 2\|\tilde r_S^{(2)}\eta\|_{L^2(\Gamma_r)} + 2\|w\tilde r_S^{(2)}\|_{L^2(\Gamma_r)} + \|(\tilde r_S^{(2)})^2\|_{L^2(\Gamma_r)} + \|w^2\|_{L^2(\Gamma_r)}\big) \\ &\le \sigma\|y_h\|^2_{L^\infty(\Omega_s)}\|G^*\|_{L^2(\Gamma_r)}\|p\|_{L^2(\Gamma_r)} \\ &\quad \big(2\,\|w\|_{L^4(\Gamma_r)}\|\eta\|_{L^4(\Gamma_r)} + 2\,\|\tilde r_S^{(2)}\|_{L^4(\Gamma_r)}\|\eta\|_{L^4(\Gamma_r)} + 2\,\|w\|_{L^4(\Gamma_r)}\|\tilde r_S^{(2)}\|_{L^4(\Gamma_r)} \\ &\quad + \|\tilde r_S^{(2)}\|^2_{L^4(\Gamma_r)} + \|w\|^2_{L^4(\Gamma_r)}\big) \\ &\le c\,\big(2\,\|w\|_{H^1(\Omega)}\|\eta\|_{H^1(\Omega)} + 2\,\|\tilde r_S^{(2)}\|_{H^1(\Omega)}\|\eta\|_{H^1(\Omega)} + 2\,\|w\|_{H^1(\Omega)}\|\tilde r_S^{(2)}\|_{H^1(\Omega)} \\ &\quad + \|\tilde r_S^{(2)}\|^2_{H^1(\Omega)} + \|w\|^2_{H^1(\Omega)}\big). \end{aligned}$$

The expression on the right-hand side in the last inequality is the same as in (6.24). Hence, we argue as before and obtain

$$I_2 \le \varphi(\|h\|_{L^2(\Omega_s)})\, \|h\|^2_{L^q(\Omega_s)}.$$

Together with (6.27), this implies

$$\Big|\int_{\Gamma_r} G\big(\sigma\,(|y_h|y_h\,\eta_h^2 - |\bar y|\bar y\,\eta^2)\big)p\,ds\Big| \le \varphi(\|h\|_{L^2(\Omega_s)})\, \|h\|^2_{L^q(\Omega_s)}. \tag{6.28}$$

An analogous discussion for the integral over Γ_0 in (6.18) gives

$$\Big|\int_{\Gamma_0} \varepsilon\sigma\,(|y_h|y_h\,\eta_h^2 - |\bar y|\bar y\,\eta^2)p\,ds\Big| \le \varphi(\|h\|_{L^2(\Omega_s)})\, \|h\|^2_{L^q(\Omega_s)}.$$

Hence by inserting this estimate together with (6.28) and (6.25) in (6.18), we end up with

$$|\rho_j| \le \varphi(\|h\|_{L^2(\Omega_s)})\, \|h\|^2_{L^q(\Omega_s)}.$$

For the remainder term $r_j^{(2)}$, we finally obtain

$$\begin{aligned}\left|r_j^{(2)}\right| &\leq \int\limits_0^1 \int\limits_0^\beta |\rho_j|\, d\theta\, d\beta \leq \varphi(\|h\|_{L^2(\Omega_s)})\, \|h\|^2_{L^q(\Omega_s)} \int\limits_0^1 \int\limits_0^\beta d\theta\, d\beta \\ &\leq \varphi(\|h\|_{L^2(\Omega_s)})\, \|h\|^2_{L^q(\Omega_s)},\end{aligned}$$

with some $6/5 \leq q \leq 2$. Due to $\|h\|^2_{L^q(\Omega_s)} \leq c\, \|h\|^2_{L^{q'}(\Omega_s)}$ for every $q' \geq q$, (6.16) clearly holds for every $q \geq 6/5$. ∎

7. Proof of Theorem 5.4. As in the proof of Theorem 5.1, we start with the Taylor expansion of the reduced objective functional

$$j(u) = j(\bar u) + j'(\bar u)(u - \bar u) + \frac{1}{2}\, j''(\bar u)(u - \bar u)^2 + r_j^{(2)} \tag{7.1}$$

with $u \in U_{ad}$.

(i) Estimation of the first derivative $j'(\bar u)(u - \bar u)$

A pointwise evaluation of the necessary conditions in (4.18) yields

$$j'(\bar u)(x)\big(u(x) - \bar u(x)\big) = \big(p(x) + \nu \bar u(x)\big)\big(u(x) - \bar u(x)\big) \geq 0 \quad \text{a.e. in } \Omega_s, \quad \forall\, u \in U_{ad}.$$

This implies $(p(x) + \nu\bar u(x))(u(x) - \bar u(x)) = |p(x) + \nu\bar u(x)|\, |u(x) - \bar u(x)|$. Hence, with Definition 5.2, we obtain for the first derivative of j

$$\begin{aligned}j'(\bar u)(u - \bar u) &= \int\limits_{A_\tau} \big|p(x) + \nu\bar u(x)\big|\big|u(x) - \bar u(x)\big|\, dx + \int\limits_{\Omega_s \setminus A_\tau} \big|p(x) + \nu\bar u(x)\big|\big|u(x) - \bar u(x)\big|\, dx \\ &\geq \int\limits_{A_\tau} \tau |u(x) - \bar u(x)|\, dx = \tau\, \|u - \bar u\|_{L^1(A_\tau)}.\end{aligned} \tag{7.2}$$

(ii) Estimation of the second derivative $j''(\bar u)(u - \bar u)^2$

Let $\tilde u$ be defined by

$$\tilde u(x) = \begin{cases} \bar u(x)\ , \text{ for } x \in A_\tau \\ u(x)\ , \text{ for } x \notin A_\tau, \end{cases}$$

and thus $(\tilde u - \bar u) \in C_\tau(\bar u)$, thanks to Definition 5.3. We continue with

$$\begin{aligned}j''(\bar u)(u - \bar u)^2 &= j''(u - \tilde u + \tilde u - \bar u) \\ &= j''(\bar u)(u - \tilde u)^2 + 2\, j''(\bar u)[u - \tilde u, \tilde u - \bar u] + j''(\bar u)(\tilde u - \bar u)^2.\end{aligned} \tag{7.3}$$

In the following, we estimate the three addends on the right-hand side of (7.3) separately. To that end, define $y = S'(\bar u)u$ and $\tilde y = S'(\bar u)\tilde u$. Then, with (6.13), one obtains

$$\begin{aligned}j''(\bar u)(u - \tilde u)^2 &= \|\nabla(y - \tilde y)\|_{L^2(\Omega_g)} + \nu \|u - \tilde u\|_{L^2(\Omega_s)} \\ &\qquad - 12 \int\limits_{\Gamma_r} G\big(\sigma\, |\bar y| \bar y\, (y - \tilde y)^2\big) p\, ds - 12 \int\limits_{\Gamma_0} \varepsilon\sigma\, |\bar y| \bar y\, (y - \tilde y)^2\, p\, ds \\ &\geq -12 \Big| \int\limits_{\Gamma_r} G\big(\sigma\, |\bar y|\bar y\, (y - \tilde y)^2\big) p\, ds + \int\limits_{\Gamma_0} \varepsilon\sigma\, |\bar y|\bar y\, (y - \tilde y)^2\, p\, ds) \Big| \\ &\geq -c\, \|u - \tilde u\|^2_{L^q(\Omega_s)},\end{aligned} \tag{7.4}$$

where Lemma 6.4 is used for the last inequality. The second addend is transformed into

$$\begin{aligned} j''(\bar u)[u-\tilde u,\tilde u-\bar u] &= \big(\nabla(y-\tilde y)\,,\,\nabla(\tilde y-\eta)\big)_{L^2(\Omega_{\mathrm g})} + \nu\big(u-\tilde u\,,\,\tilde u-\bar u\big)_{L^2(\Omega_{\mathrm s})} \\ &\quad - 12\int_{\Gamma_{\mathrm r}} G\big(\sigma\,|\bar y|\bar y\,(y-\tilde y)\,(\tilde y-\eta)\big)p\,ds \\ &\quad - 12\int_{\Gamma_0} \varepsilon\sigma\,|\bar y|\bar y\,(y-\tilde y)\,(\tilde y-\eta)\,p\,ds, \end{aligned}$$

where y and $\tilde y$ are defined as above and η is given by $\eta = S'(\bar u)\bar u$. By the definition of $\tilde u$, we have $(\tilde u-\bar u)(x)=0$, if $x\in A_\tau$, and $(u-\tilde u)(x)=0$, if $x\in\Omega_{\mathrm s}\backslash A_\tau$, and hence $(u-\tilde u\,,\,\tilde u-\bar u)_{L^2(\Omega_{\mathrm s})}=0$. Moreover, Lemma 6.1 implies

$$\begin{aligned} -\big|\big(\nabla(y-\tilde y)\,,\,\nabla(\tilde y-\eta)\big)_{L^2(\Omega_{\mathrm g})}\big| &\ge -\|y-\tilde y\|_{H^1(\Omega)}\,\|\tilde y-\eta\|_{H^1(\Omega)} \\ &\ge -c\,\|u-\tilde u\|_{L^q(\Omega_{\mathrm s})}\,\|\tilde u-\bar u\|_{L^q(\Omega_{\mathrm s})}. \end{aligned}$$

The boundary integrals are again estimated with Lemma 6.4, and hence it follows that

$$j''(\bar u)[u-\tilde u,\tilde u-\bar u] \ge -c\,\|u-\tilde u\|_{L^q(\Omega_{\mathrm s})}\,\|\tilde u-\bar u\|_{L^q(\Omega_{\mathrm s})}.$$

With

$$\|\tilde u-\bar u\|_{L^q(\Omega_{\mathrm s})} \le \|u-\tilde u\|_{L^q(\Omega_{\mathrm s})} + \|u-\bar u\|_{L^q(\Omega_{\mathrm s})} \tag{7.5}$$

this results in

$$j''(\bar u)[u-\tilde u,\tilde u-\bar u] \ge -c\,(\|u-\tilde u\|^2_{L^q(\Omega_{\mathrm s})} + \|u-\tilde u\|_{L^q(\Omega_{\mathrm s})}\,\|u-\bar u\|_{L^q(\Omega_{\mathrm s})}). \tag{7.6}$$

Due to $(\tilde u-\bar u)\in C_\tau(\bar u)$, condition (SSC) yields for the last addend in (7.3)

$$j''(\bar u)(\tilde u-\bar u)^2 \ge \delta\,\|\tilde u-\bar u\|^2_{L^q(\Omega_{\mathrm s})}.$$

Using

$$\begin{aligned} \|u-\bar u\|^2_{L^q(\Omega_{\mathrm s})} &= \|u-\tilde u+\tilde u-\bar u\|^2_{L^q(\Omega_{\mathrm s})} \\ &\le \|u-\tilde u\|^2_{L^q(\Omega_{\mathrm s})} + 2\,\|u-\tilde u\|_{L^q(\Omega_{\mathrm s})}\,\|\tilde u-\bar u\|_{L^q(\Omega_{\mathrm s})} + \|\tilde u-\bar u\|^2_{L^q(\Omega_{\mathrm s})} \end{aligned}$$

and again (7.5), this is estimated by

$$j''(\bar u)(\tilde u-\bar u)^2 \ge \delta\,\|u-\bar u\|^2_{L^q(\Omega_{\mathrm s})} - 3\delta\,\|u-\tilde u\|^2_{L^q(\Omega_{\mathrm s})} - 2\delta\,\|u-\tilde u\|_{L^q(\Omega_{\mathrm s})}\,\|u-\bar u\|_{L^q(\Omega_{\mathrm s})}. \tag{7.7}$$

Now we insert (7.4), (7.6), and (7.7) in (7.3) and obtain

$$\begin{aligned} j''(\bar u)(u-\bar u)^2 &\ge \delta\,\|u-\bar u\|^2_{L^q(\Omega_{\mathrm s})} \\ &\quad - (3\delta+c)\,\|u-\tilde u\|^2_{L^q(\Omega_{\mathrm s})} - (2\delta+c)\,\|u-\tilde u\|_{L^q(\Omega_{\mathrm s})}\,\|u-\bar u\|_{L^q(\Omega_{\mathrm s})}. \end{aligned}$$

Then Young's inequality implies

$$\begin{aligned} &j''(\bar u)(u-\bar u)^2 \ge \\ &\delta\,\|u-\bar u\|^2_{L^q(\Omega_{\mathrm s})} - \left(3\delta+c+\frac{2\delta+c}{\kappa}\right)\|u-\bar u\|^2_{L^q(A_\tau)} - (2\delta+c)\kappa\,\|u-\bar u\|^2_{L^q(\Omega_{\mathrm s})}, \end{aligned} \tag{7.8}$$

with an arbitrary $\kappa > 0$. Notice that the definition of $\tilde{u}$ yields $\|u - \tilde{u}\|_{L^q(\Omega_s)} = \|u - \bar{u}\|_{L^q(A_\tau)}$.

(iii) The quadratic growth condition

Next, we insert (7.2) and (7.8) in the Taylor expansion (7.1) and obtain

$$\begin{aligned} j(u) \geq j(\bar{u}) + \tau \|u - \bar{u}\|_{L^1(A_\tau)} - \left(3\delta + c + \frac{2\delta + c}{\kappa}\right) \|u - \bar{u}\|^2_{L^q(A_\tau)} \\ + \frac{1}{2}\left(\delta - (2\delta + c)\kappa - 2\,\frac{|r_j^{(2)}|}{\|u - \bar{u}\|^2_{L^q(\Omega_s)}}\right) \|u - \bar{u}\|^2_{L^q(\Omega_s)}. \end{aligned} \tag{7.9}$$

The well-known interpolation inequality (cf. Brezis [2]) implies

$$\begin{aligned} \|u - \bar{u}\|^2_{L^q(A_\tau)} &\leq \|u - \bar{u}\|_{L^1(A_\tau)} \|u - \bar{u}\|_{L^s(A_\tau)} \\ &\leq \|u - \bar{u}\|_{L^1(A_\tau)} \|u - \bar{u}\|_{L^s(\Omega_s)}, \end{aligned}$$

with s as defined in (5.7). Then (7.9) results in

$$j(u) \geq j(\bar{u}) + a_1 \|u - \bar{u}\|_{L^1(A_\tau)} + \frac{1}{2}\, a_2 \|u - \bar{u}\|^2_{L^q(\Omega_s)}, \tag{7.10}$$

with

$$a_1 = \tau - \left(3\delta + c + \frac{2\delta + c}{\kappa}\right) \|u - \bar{u}\|_{L^s(\Omega_s)}$$

and

$$a_2 = \delta - (2\delta + c)\kappa - 2\,\frac{|r_j^{(2)}|}{\|u - \bar{u}\|^2_{L^q(\Omega_s)}}.$$

To derive the quadratic growth condition (5.8), we show that a_1 and a_2 are non negative, if $\|u - \bar{u}\|_{L^s(\Omega_s)}$ is sufficiently small. We start with a_2: Due to Lemma 6.5, for every $\varepsilon_r > 0$, there exists a constant $\varepsilon_1 > 0$ with $\|u - \bar{u}\|_{L^2(\Omega_s)} \leq \varepsilon_1$, such that

$$\frac{|r_j^{(2)}|}{\|u - \bar{u}\|^2_{L^q(\Omega_s)}} \leq \varepsilon_r.$$

Moreover, Lemma 6.5 implies that ε_r tends to zero if ε_1 is chosen sufficiently small. Therefore, if we also set κ sufficiently small, there exists a constant $\bar{\sigma}$ such that

$$a_2 \geq \delta - (2\delta + c)\kappa - 2\,\varepsilon_r \geq 2\,\bar{\sigma} > 0. \tag{7.11}$$

Furthermore, a_1 is non negative, if $\varepsilon_2 := \|u - \bar{u}\|_{L^s(\Omega_s)}$ is sufficiently small, i.e.

$$\varepsilon_2 \leq \frac{\tau}{3\delta + c + (2\delta + c)/\kappa}.$$

By assumption, we have $q \geq 4/3$ and hence $s \geq 2$. Therefore,

$$\|u - \bar{u}\|_{L^2(\Omega_s)} \leq c_s \|u - \bar{u}\|_{L^s(\Omega_s)} \leq c_s\, \varepsilon_2$$

holds true for every $q \geq 4/3$. Thus, if we set $\bar{\varepsilon} = \min\{\varepsilon_2\,;\,\varepsilon_1/c_s\}$, then (7.11) is satisfied and a_1 is positive. Therefore, for every $u \in U_{ad}$ with $\|u - \bar{u}\|_{L^s(\Omega_s)} \leq \bar{\varepsilon}$,

$$j(u) \geq j(\bar{u}) + \frac{1}{2}\,(\delta - (2\delta + c)\kappa - 2\,\varepsilon_r)\,\|u - \bar{u}\|^2_{L^q(\Omega_s)} \geq j(\bar{u}) + \bar{\sigma}\,\|u - \bar{u}\|^2_{L^q(\Omega_s)}$$

holds true. ∎

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