# ROBUST CONTROL OF DESCRIPTOR SYSTEMS

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**Abstract.** The $\mathcal{H}_\infty$ control problem is studied for linear constant coefficient descriptor systems. Necessary and sufficient optimality conditions are derived in terms of deflating subspaces of even matrix pencils for index one systems as well as for higher index problems. It is shown that this approach leads to a more robust method in computing the optimal value $\gamma$ in contrast to other methods such as the widely used Riccati based approach. The results are illustrated by a numerical example.

**Key words.** descriptor system, $\mathcal{H}_\infty$-control, algebraic Riccati equation, even matrix pencil, $\gamma$-iteration, deflating subspace

**AMS subject classifications.** 34A09, 93B40, 93B36, 65F15, 65L80, 93B52, 93C05.

## 1. Introduction.

The optimal infinite-horizon output (or measurement) feedback $\mathcal{H}_\infty$ control problem is one of the central tasks in robust control, see, e.g., [12, 13, 24, 36, 37]. For standard state space systems, where the dynamics of the system are modeled by a linear constant coefficient ordinary differential equation, the analysis of this problem is well studied and numerical methods have been developed and integrated in control software packages such as [1, 4, 14, 25]. These methods work well for a wide range of problems in computing close to optimal (suboptimal) controllers but the exact computation of the optimal value $\gamma$ in $\mathcal{H}_\infty$ control is considered a challenge [7]. In order to avoid some of the numerical difficulties that arise when approaching the optimum, in [2, 3] several improvements of the previously known methods were presented. These are based on the solution of structured eigenvalue problems with structured methods.

In this paper we study the more general case that the dynamics is constrained, i.e. described by a *differential-algebraic equation (DAE) or descriptor system.* Descriptor systems arise in the control of constrained mechanical systems, see e.g. [10, 26, 31, 33, 34], in electrical circuit simulation, see e.g. [15, 16], and in particular in heterogeneous systems, where different models are coupled [23].

Robust control for descriptor systems has been studied in [27–29] using linear matrix inequalities (LMIs) and in [35] via generalized Riccati equations and J-spectral factorization. In contrast to these approaches, we extend the analysis and the robust numerical methods that were derived via deflating subspaces in [2, 3]. We discuss descriptor systems of the form

$$\mathbf{P}: \quad \begin{aligned} E\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \qquad x(t_0) = x^0, \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t), \\ y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t), \end{aligned} \tag{1.1}$$

where $E, A \in \mathbb{R}^{n,n}$, $B_i \in \mathbb{R}^{n,m_i}$, $C_i \in \mathbb{R}^{p_i,n}$, and $D_{ij} \in \mathbb{R}^{p_i,m_j}$ for $i, j = 1, 2$. (Here, by $\mathbb{R}^{k,l}$ we denote the set of real $k \times l$ matrices.)

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In this system, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{m_2}$ is the control input vector, and $w(t) \in \mathbb{R}^{m_1}$ is an exogenous input that may include noise, linearization errors and un-modelled dynamics. The vector $y(t) \in \mathbb{R}^{p_2}$ contains measured outputs, while $z(t) \in \mathbb{R}^{p_1}$ is a regulated output or an estimation error. Our approach can also be extended to rectangular systems and systems in behavior formulation, using a remodelling as it was suggested in [17, 19], see also [18], but here we only study the formulation in (1.1).

The optimal $\mathcal{H}_\infty$ control problem is typically formulated in frequency domain. For this we need the following notation. The space $\mathcal{H}_\infty^{p,m}$ consists of all $\mathbb{C}^{p,m}$-valued functions that are analytic and bounded in the complex half plane $\mathbb{C}^+ = \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$. For $F \in \mathcal{H}_\infty^{p,m}$ the $\mathcal{H}_\infty$-norm is given by

$$\|F\|_\infty = \sup_{s \in \mathbb{C}^+} \sigma_{\max}(F(s)),$$

where $\sigma_{\max}(F(s))$ denotes the maximal singular value of the matrix $F(s)$.

In robust control, $\|F\|_\infty$ is used as a measure of the worst case influence of the disturbances $w$ on the output $z$, where in this case $F$ is the transfer function mapping noise or disturbance inputs to error signals [37].

The optimal $\mathcal{H}_\infty$ control problem is the task of designing a dynamic controller as presented in Fig. 1.1 that minimizes (or at least approximately minimizes) this measure.

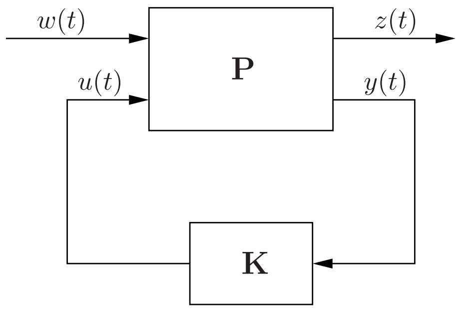


FIG. 1.1. *Interconnection with controller*

Put more rigorously, the optimal $\mathcal{H}_\infty$ control problem is the following.

DEFINITION 1.1 (The Optimal $\mathcal{H}_\infty$ control problem). *For the descriptor system (1.1), determine a controller (dynamic compensator)*

$$\mathbf{K}: \quad \begin{aligned} \hat{E}\dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{B}y(t), \\ u(t) &= \hat{C}\hat{x}(t) + \hat{D}y(t) \end{aligned} \tag{1.2}$$

*with* $\hat{E}, \hat{A} \in \mathbb{R}^{N,N}$, $\hat{B} \in \mathbb{R}^{N,p_2}$, $\hat{C} \in \mathbb{R}^{m_2,N}$, $\hat{D} \in \mathbb{R}^{m_2,p_2}$, *and transfer function* $K(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} + \hat{D}$ *such that the closed-loop system resulting from the combination of (1.1) and (1.2), that is given by*

$$\begin{aligned} E\dot{x}(t) &= (A + B_2\hat{D}Z_1C_2)x(t) + (B_2Z_2\hat{C})\hat{x}(t) + (B_1 + B_2\hat{D}Z_1D_{21})w(t), \\ \hat{E}\dot{\hat{x}}(t) &= \hat{B}Z_1C_2x(t) + (\hat{A} + \hat{B}Z_1D_{22}\hat{C})\hat{x}(t) + \hat{B}Z_1D_{21}w(t), \\ z(t) &= (C_1 + D_{12}Z_2\hat{D}C_2)x(t) + D_{12}Z_2\hat{C}\hat{x}(t) + (D_{11} + D_{12}\hat{D}Z_1D_{21})w(t) \end{aligned} \tag{1.3}$$

*with* $Z_1 = (I_{p_2} - D_{22}\hat{D})^{-1}$ *and* $Z_2 = (I_{m_2} - \hat{D}D_{22})^{-1}$*, has the following properties.*

**1.)** *System (1.3) is* internally stable*, that is, the solution* $\begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$ *of the system with* $w \equiv 0$ *is* asymptotically stable*, i.e.* $\lim_{t\to\infty} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} = 0$.

**2.)** *The closed-loop transfer function* $T_{zw}(s)$ *from* $w$ *to* $z$ *satisfies* $T_{zw} \in \mathcal{H}_\infty^{p_1,m_1}$ *and is minimized in the* $\mathcal{H}_\infty$*-norm.*

In principle, there is no restriction on the dimension $N$ of the auxiliary state $\hat{x}$ in (1.2), although, smaller dimensions $N$ are preferred for practical implementation and computation.

As in the case of the optimal $\mathcal{H}_\infty$ control problems for ordinary state space systems it is also necessary to study two closely related optimization problems, the *modified optimal* $\mathcal{H}_\infty$ *control problem* and the *suboptimal* $\mathcal{H}_\infty$ *control problem.*

Definition 1.2 (The modified optimal $\mathcal{H}_\infty$ control problem.). *For the descriptor system (1.1) let* $\Gamma$ *be the set of positive real numbers* $\gamma$ *for which there exists an internally stabilizing dynamic controller of the form (1.2) so that the transfer function* $T_{zw}(s)$ *of the closed loop system (1.3) satisfies* $T_{zw} \in \mathcal{H}_\infty^{p_1,m_1}$ *with* $\|T_{zw}\|_\infty < \gamma$*. Determine* $\gamma_{mo} = \inf \Gamma$ *If no internally stabilizing dynamic controller exits, we set* $\Gamma = \emptyset$ *and* $\gamma_{mo} = \infty$*.*

Note that it is possible that there is no internally stabilizing dynamic controller with the property $\|T_{zw}\|_\infty = \gamma_{mo}$. In this case one solves the suboptimal $\mathcal{H}_\infty$ control problem.

Definition 1.3 (The suboptimal $\mathcal{H}_\infty$ control problem.). *For the descriptor system (1.1) and* $\gamma \in \Gamma$ *with* $\gamma > \gamma_{mo}$*, determine an internally stabilizing dynamic controller of the form (1.2) such that the closed loop transfer function satisfies* $T_{zw} \in \mathcal{H}_\infty^{p_1,m_1}$ *with* $\|T_{zw}\|_\infty < \gamma$*. We call such a controller* $\gamma$-suboptimal controller *or simply* suboptimal controller*.*

The outline of the paper is as follows: In the forthcoming section we present the notation and some definitions that are used throughout the paper. Section 3 contains the main result of the paper and states conditions for the existence of an appropriate controller in terms of deflating subspaces of matrix pencils. The proof is given in three parts. First we briefly discuss the standard state space case. The results are then generalized to descriptor systems of index 1 and, thereafter, to systems with arbitrary index. In Section 4 we present the algorithmic framework for the $\gamma$-iteration to compute the optimal $\Gamma$ and we illustrate the presented theory by means of a numerical example.

**2. Preliminaries.** In this section we introduce some notation and definitions. For symmetric matrices $A$ and $B$, by $A \geq B$ and $A > B$ we denote that $A - B$ is positive semi-definite and positive definite, respectively. The spectral radius of a matrix $A \in \mathbb{R}^{n,n}$ is denoted by $\rho(A)$. The set of complex numbers with positive real part is denoted by $\mathbb{C}^+$ and the set of positive real numbers by $\mathbb{R}^+$.

Let $\lambda E - A$ be a matrix pencil with $E, A \in \mathbb{R}^{n,n}$. Then $\lambda E - A$ is called *regular* if $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$.

A pencil $P(\lambda) = \lambda E - A$ is called *even* if $P(-\lambda)^T = P(\lambda)$, i.e. if $E = -E^T$ and $A = A^T$.

For regular pencils, *generalized eigenvalues* are the pairs $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$ for which $\det(\alpha E - \beta A) = 0$. If $\beta \neq 0$, then the pair represents the finite eigenvalue $\lambda = \alpha/\beta$. If $\beta = 0$, then $(\alpha, \beta)$ represent the eigenvalue infinity. In the following we use the notation with $\lambda$.

The solution and many properties of the *free descriptor system (with $u, w = 0$)* can be characterized in terms of the *Weierstraß canonical form (WCF).*

THEOREM 2.1. *[11] For a regular matrix pencil $\lambda E - A$, there exist matrices $W_f, V_f \in \mathbb{R}^{n,n_f}$, $W_\infty, V_\infty \in \mathbb{R}^{n,n_\infty}$ with the property that $W = \begin{bmatrix} W_f & W_\infty \end{bmatrix}$, $V = \begin{bmatrix} V_f & V_\infty \end{bmatrix}$ are square and invertible, with*

$$W^T E V = \begin{bmatrix} W_f^T \\ W_\infty^T \end{bmatrix} E \begin{bmatrix} V_f & V_\infty \end{bmatrix} = \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix}, \tag{2.1a}$$

$$W^T A V = \begin{bmatrix} W_f^T \\ W_\infty^T \end{bmatrix} A \begin{bmatrix} V_f & V_\infty \end{bmatrix} = \begin{bmatrix} A_f & 0 \\ 0 & I_{n_\infty} \end{bmatrix}, \tag{2.1b}$$

*$A_f \in \mathbb{R}^{n_f,n_f}$ is in real Jordan canonical form and $N \in \mathbb{R}^{n_\infty,n_\infty}$ is a nilpotent matrix, also in Jordan canonical form. We call $n_f, n_\infty$ the number of finite or infinite eigenvalues, respectively.*

The index of nilpotency of the nilpotent matrix $N$ in (2.1a) is called the *index* of the system and if $E$ is nonsingular, then the pencil is said to have *index zero.*

DEFINITION 2.2. *A subspace $\mathcal{L} \subset \mathbb{R}^n$ is called* deflating subspace *for the pencil $\lambda E - A$ if for a matrix $X_{\mathcal{L}} \in \mathbb{R}^{n,k}$ with full column rank and $\operatorname{im} X_{\mathcal{L}} = \mathcal{L}$ there exists matrices $Y_{\mathcal{L}} \in \mathbb{R}^{n,k}$, $R_{\mathcal{L}} \in \mathbb{R}^{k,k}$, and $U_{\mathcal{L}} \in \mathbb{R}^{k,k}$ such that*

$$E X_{\mathcal{L}} = Y_{\mathcal{L}} R_{\mathcal{L}}, \quad A X_{\mathcal{L}} = Y_{\mathcal{L}} U_{\mathcal{L}}. \tag{2.2}$$

*A deflating subspace $\mathcal{L}$ of $\lambda E - A$ is called* stable (semi-stable) *if all finite eigenvalues of $\lambda R_{\mathcal{L}} - U_{\mathcal{L}}$ are in the open (closed) left half plane.*

*Let $\mathcal{J} = \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right]$, where $I_n$ is the $n \times n$ identity matrix. A subspace $\mathcal{L} \subset \mathbb{R}^{2n}$ is called* isotropic *if $x^T \mathcal{J} y = 0$ for all $x$, $y \in \mathcal{L}$. An isotropic subspace with $\dim \mathcal{L} = n$ is called* Lagrangian.

In the notation of (2.1a)–(2.1b) with

$$\begin{aligned} B_{i,f} &= W_f^T B_i, \quad & B_{i,\infty} &= W_\infty^T B_i, \\ C_{i,f} &= C_i V_f, \quad & C_{i,\infty} &= C_i V_\infty, \quad i = 1, 2 \end{aligned} \tag{2.3}$$

classical solutions of (1.2) take the form $x(t) = V_f x_f(t) + V_\infty x_\infty(t)$, where $x_f$ and $x_\infty$ satisfy

$$\dot{x}_f(t) = A_f x_f(t) + B_{1,f}\, w(t) + B_{2,f} u(t), \tag{2.4a}$$

$$N \dot{x}_\infty(t) = \quad x_\infty(t) + B_{1,\infty} w(t) + B_{2,\infty} u(t). \tag{2.4b}$$

If the pencil $\lambda E - A$ has index $\nu$, then this system has the explicit solution

$$x_f(t) = e^{A_f(t-t_0)} x_f(t_0) + \int_{t_0}^{t} e^{A_f(t-\tau)} \left( B_{1,f} w(\tau) + B_{2,f} u(\tau) \right) d\tau, \tag{2.5a}$$

$$x_\infty(t) = -\sum_{i=0}^{\nu-1} \tfrac{d^i}{dt^i} N^i \left( B_{1,\infty} w(t) + B_{2,\infty} u(t) \right). \tag{2.5b}$$

In contrast to standard state space systems, this shows that the initial condition $x_\infty(t_0)$ is restricted by (2.5b). Moreover, if $\nu > 1$, then the solution will depend on derivatives of the input $u$ and the disturbance $w$.

Note further that for the closed loop system (1.3) to be *internally stable*, the controller has to be designed so that both $x_f$ and $x_\infty$ are asymptotically stable.

While for the finite part this can be guaranteed if the spectrum of the matrix $A_f$ lies in the open left half plane, for the infinite part this has to be explicitly achieved by the construction of the controller.

As in the case of standard state space systems, certain conditions will be needed to guarantee the existence of optimal $\mathcal{H}_\infty$ controls. First of all these are stabilizability and detectability conditions, which for descriptor systems are the following, see [5,8].

Definition 2.3. *Let $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$ and $C \in \mathbb{R}^{p,n}$. Further, let $T_\infty, S_\infty$ be matrices with $\operatorname{im} T_\infty = \ker E^T$ and $\operatorname{im} S_\infty = \ker E$.*

*i)* *The triple $(E, A, B)$ is called* finite dynamics stabilizable *if* $\operatorname{rank}[\lambda E - A, B] = n$ *for all* $\lambda \in \mathbb{C}^+$;

*ii)* $(E, A, B)$ *is* impulse controllable *if* $\operatorname{rank}[E, AS_\infty, B] = n$;

*iii)* $(E, A, B)$ *is* strongly stabilizable *if it is both finite dynamics stabilizable and impulse controllable;*

*iv)* *The triple $(E, A, C)$ is* finite dynamics detectable *if* $\operatorname{rank}[\lambda E^T - A^T, C^T] = n$ *for all* $\lambda \in \mathbb{C}^+$;

*v)* $(E, A, C)$ *is* impulse observable *if* $\operatorname{rank}[E^T, A^T T_\infty, C^T] = n$;

*vi)* $(\lambda E - A, C)$ *is* strongly detectable *if it is both finite dynamics detectable and impulse observable.*

After introducing our notation and giving some preliminary results, we derive the theoretical basis for the optimal $\mathcal{H}_\infty$ control problem for descriptor systems in the next section.

## 3. The Modified optimal $\mathcal{H}_\infty$ control problem.

In this section we approach the problem of determining $\gamma_{mo}$ for a given system (1.1). As in the case of standard state space systems, see [12, 13, 24, 37], we need several assumptions on the system matrices. In the following we set $r = \operatorname{rank} E$.

**Assumptions:**

**A1)** The triple $(E, A, B_2)$ is strongly stabilizable and the triple $(E, A, C_2)$ is strongly detectable, see Definition 2.3.

**A2)** $\operatorname{rank}\begin{bmatrix} A - i\omega E & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2$ for all $\omega \in \mathbb{R}$.

**A3)** $\operatorname{rank}\begin{bmatrix} A - i\omega E & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2$ for all $\omega \in \mathbb{R}$.

**A4)** For matrices $T_\infty, S_\infty \in \mathbb{R}^{n,n-r}$ with $\operatorname{im} S_\infty = \ker E$ and $\operatorname{im} T_\infty = \ker E^T$ the rank conditions

$$\operatorname{rank}\begin{bmatrix} T_\infty^T A S_\infty & T_\infty^T B_2 \\ C_1 S_\infty & D_{12} \end{bmatrix} = n + m_2 - r,$$
$$\operatorname{rank}\begin{bmatrix} T_\infty^T A S_\infty & T_\infty^T B_1 \\ C_2 S_\infty & D_{21} \end{bmatrix} = n + p_1 - r$$

hold.

It is well known for standard state space systems that Assumption **A1)** is essential for the existence of a controller that internally stabilizes the system. We will see that a similar result holds for the descriptor case. Assumptions **A2)** and **A3)** correspond to the typical claim that the system does not have transmission zeros on the imaginary axis. This is assumed in many works about $\mathcal{H}_\infty$-control of standard state space systems, since eigenvalues on the imaginary axis of the Hamiltonian matrices that are used in the computation of an optimal controller usually lead to problems in the computation of a semi-stable subspace, see [21, 30].

Further typical assumptions in the $\mathcal{H}_\infty$-control of standard state space systems are that $D_{12}, D_{21}^T$ have full column rank, see [13,24,37]. The conditions in **A4)** reduce to these rank conditions if $E$ is invertible.

For the construction of optimal or suboptimal controllers we will make use of the following two even matrix pencils, which generalize the pencils constructed in [2, 3]. Let

$$\lambda N_H + M_H(\gamma) = \\ \lambda \left[\begin{array}{cc|ccc} 0 & -E^T & 0 & 0 & 0 \\ E & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{cc|ccc} 0 & -A^T & 0 & 0 & -C_1^T \\ -A & 0 & -B_1 & -B_2 & 0 \\ \hline 0 & -B_1^T & -\gamma^2 I_{m_1} & 0 & -D_{11}^T \\ 0 & -B_2^T & 0 & 0 & -D_{12}^T \\ -C_1 & 0 & -D_{11} & -D_{12} & -I_{p_1} \end{array}\right] \tag{3.1}$$

and

$$\lambda N_J + M_J(\gamma) = \\ \lambda \left[\begin{array}{cc|ccc} 0 & -E & 0 & 0 & 0 \\ E^T & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{cc|ccc} 0 & -A & 0 & 0 & -B_1 \\ -A^T & 0 & -C_1^T & -C_2^T & 0 \\ \hline 0 & -C_1 & -\gamma^2 I_{p_1} & 0 & -D_{11} \\ 0 & -C_2 & 0 & 0 & -D_{21} \\ -B_1^T & 0 & -D_{11}^T & -D_{21}^T & -I_{m_1} \end{array}\right]. \tag{3.2}$$

Our approach is based on considering deflating subspaces of the matrix pencils (3.1) and (3.2), where the subspaces are spanned by the columns of the matrices $X_H$ and $X_J$ that are partitioned conformably with the pencils, i.e.

$$X_H(\gamma) = \begin{bmatrix} X_{H,1}(\gamma) \\ X_{H,2}(\gamma) \\ X_{H,3}(\gamma) \\ X_{H,4}(\gamma) \\ X_{H,5}(\gamma) \end{bmatrix}, \quad X_J(\gamma) = \begin{bmatrix} X_{J,1}(\gamma) \\ X_{J,2}(\gamma) \\ X_{J,3}(\gamma) \\ X_{J,4}(\gamma) \\ X_{J,5}(\gamma) \end{bmatrix}, \tag{3.3}$$

with

$$\begin{aligned} &X_{H,1}(\gamma),\ X_{H,2}(\gamma), X_{J,1}(\gamma),\ X_{J,2}(\gamma) \in \mathbb{R}^{n,r}, X_{H,4}(\gamma) \in \mathbb{R}^{m_2,r}, \\ &X_{J,4}(\gamma) \in \mathbb{R}^{p_2,r}, \quad X_{H,3}(\gamma), X_{J,5}(\gamma) \in \mathbb{R}^{m_1,r}, X_{H,5}(\gamma), X_{J,3}(\gamma) \in \mathbb{R}^{p_1,r}. \end{aligned}$$

We extend the results in [2, 3] to general descriptor systems and use deflating subspaces of the even pencils (3.1) and (3.2) to characterize the elements of the set $\Gamma$ in Definition 1.1. For this we introduce the following conditions which will be shown to be necessary for the existence of a controller with the desired properties associated with a parameter $\gamma \in \Gamma$.

**C1)** The index of both pencils (3.1) and (3.2) is at most one.

**C2)** There exists a matrix $X_H(\gamma)$ as in (3.3) such that

- **C2.a)** the space $\operatorname{im} X_H(\gamma)$ is a semi-stable deflating subspace of $\lambda N_H + M_H(\gamma)$ and $\operatorname{im} \begin{bmatrix} EX_{H,1} \\ X_{H,2} \end{bmatrix}$ is an $r$-dimensional isotropic subspace of $\mathbb{R}^{2n}$;
- **C2.b)** $\operatorname{rank} EX_{H,1}(\gamma) = r$.

**C3)** There exists a matrix $X_J(\gamma)$ as in (3.3) such that

**C3.a)** the space $\operatorname{im} X_J(\gamma)$ is a semi-stable deflating subspace of $\lambda N_J + M_J(\gamma)$ and $\operatorname{im}\begin{bmatrix} E^T X_{J,1} \\ X_{J,2} \end{bmatrix}$ is an $r$-dimensional isotropic subspace of $\mathbb{R}^{2n}$;

**C3.b)** $\operatorname{rank} E^T X_{J,1}(\gamma) = r$.

Based on these conditions on the pencils, we introduce the following sets.

Definition 3.1. *Consider system (1.1) and the associated even pencils $\lambda N_H + M_H(\gamma)$ in (3.1) and $\lambda N_J + M_J(\gamma)$ in (3.2). Define the sets*

$$\Gamma_H = \{\gamma \in \mathbb{R}^+ \mid \text{the index of } \lambda N_H + M_H(\gamma) \ \text{ is greater than one}\},$$
$$\Gamma_J = \{\gamma \in \mathbb{R}^+ \mid \text{the index of } \lambda N_J + M_J(\gamma) \ \text{ is greater than one}\},$$

*and set $\hat{\gamma}_H = \sup \Gamma_H$, $\hat{\gamma}_J = \sup \Gamma_J$ and $\hat{\gamma} = \max\{\hat{\gamma}_H, \hat{\gamma}_J\}$.*

Note that in general the sets $\Gamma_H$ and $\Gamma_J$ may be all of $\mathbb{R}^+$, but as we will show later it follows from the assumptions **A1)** – **A4)** that $\hat{\gamma}_H$ and $\hat{\gamma}_J$ and therefore also $\hat{\gamma}$ are finite. If $\gamma > \hat{\gamma}$ then, since both $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$ have index at most one, it follows that these pencils have $2r$ finite eigenvalues, where $r = \operatorname{rank} E$. Due to the fact that the pencils are even, and thus the eigenvalues occur in pairs $\lambda, -\lambda$, see [20], it follows that there exist at least $r$ eigenvalues in the closed left half complex plane and at most $r$ eigenvalues in the open left half plane.

The next group of sets are related to the conditions **C2.a)** and **C2.b)**.

Definition 3.2. *Consider (1.1) and the associated even pencils $\lambda N_H + M_H(\gamma)$ in (3.1) and $\lambda N_J + M_J(\gamma)$ in (3.2). Define the sets*

$$\Gamma_H^L = \{\gamma \geq \hat{\gamma} \mid \text{the pencil } \lambda N_H + M_H(\gamma) \text{ satisfies } \mathbf{C2.a)}\},$$
$$\Gamma_J^L = \{\gamma \geq \hat{\gamma} \mid \text{the pencil } \lambda N_J + M_J(\gamma) \text{ satisfies } \mathbf{C3.a)}\},$$
$$\Gamma^L = \Gamma_J^L \cap \Gamma_H^L,$$
$$\Gamma_H^R = \{\gamma \geq \hat{\gamma} \mid \text{the pencil } \lambda N_H + M_H(\gamma) \text{ satisfies condition } \mathbf{C2)}\},$$
$$\Gamma_J^R = \{\gamma \geq \hat{\gamma} \mid \text{the pencil } \lambda N_J + M_J(\gamma) \text{ satisfies condition } \mathbf{C3)}\},$$
$$\Gamma^R = \Gamma_J^R \cap \Gamma_H^R$$

*and set*

$$\hat{\gamma}_H^L = \inf \Gamma_H^L, \quad \hat{\gamma}_J^L = \inf \Gamma_J^L, \quad \hat{\gamma}^L = \inf \Gamma^L,$$
$$\hat{\gamma}_H^R = \inf \Gamma_H^R, \quad \hat{\gamma}_J^R = \inf \Gamma_J^R, \quad \hat{\gamma}^R = \inf \Gamma^R.$$

For the numerical method to compute the optimal or suboptimal $\mathcal{H}_\infty$ it will turn out to be essential to figure out these extremal values. Finally we discuss the situation of finite purely imaginary eigenvalues.

Definition 3.3. *Consider (1.1) and the associated even pencils $\lambda N_H + M_H(\gamma)$ in (3.1) and $\lambda N_J + M_J(\gamma)$ in (3.2). Define the sets*

$$\Gamma_H^I = \left\{\gamma \geq \hat{\gamma} \;\middle|\; \begin{array}{l}\text{the pencil } \lambda N_H + M_H(\gamma) \text{ has at least one finite} \\ \text{eigenvalue on the imaginary axis}\end{array}\right\},$$
$$\Gamma_J^I = \left\{\gamma \geq \hat{\gamma} \;\middle|\; \begin{array}{l}\text{the pencil } \lambda N_J + M_J(\gamma) \text{ has at least one finite} \\ \text{eigenvalue on the imaginary axis}\end{array}\right\},$$
$$\Gamma^I = \Gamma_J^I \cap \Gamma_H^I.$$

*and set*

$$\hat{\gamma}_H^I = \inf \Gamma_H^I, \quad \hat{\gamma}_J^I = \inf \Gamma_J^I, \quad \hat{\gamma}^I = \inf \Gamma^I.$$

*In the case where $\Gamma^I_H$, $\Gamma^I_J$ or $\Gamma^I$ are empty, we set $\hat{\gamma}^I_H = \infty$, $\hat{\gamma}^I_J = \infty$ or $\hat{\gamma}^I = \infty$, respectively.*

As in classical $\mathcal{H}_\infty$ control problem for state space systems, see [3], we also need some further rank conditions which are characterized in the following theorem that is proven in full generality in Subsection 3.3.

THEOREM 3.4. *Consider a system of the form (1.1) satisfying assumptions* **A1)** – **A4)**. *Let $X_H(\gamma)$ and $X_J(\gamma)$ be deflating subspace matrices of the form (3.3) that satisfy conditions* **C2)** *and* **C3)***), respectively. Then there exist parameters $\hat{\gamma}^k_H \geq \hat{\gamma}^L_H$, $\hat{\gamma}^k_J \geq \hat{\gamma}^L_J$ and $\hat{k}_H, \hat{k}_J \in \mathbb{N}$ with the property that for all $\gamma_{H,1}, \gamma_{H,2} > \hat{\gamma}^k_H$, $\gamma_{J,1}, \gamma_{J,2} > \hat{\gamma}^k_J$ the rank conditions*

$$\begin{aligned} \operatorname{rank} E^T X_{H,2}(\gamma_{H,1}) &= \operatorname{rank} E^T X_{H,2}(\gamma_{H,2}) = \hat{k}_H, \\ \operatorname{rank} E X_{J,2}(\gamma_{J,1}) &= \operatorname{rank} E X_{J,2}(\gamma_{J,2}) = \hat{k}_J \end{aligned} \tag{3.4}$$

*hold.*

This rank property will be the fundament for the formulation of a further condition on the pencils in (3.1), (3.2) and on the blocks of the deflating subspace matrices $X_H(\gamma) \in \mathbb{R}^{2n+m_1+m_2+p_1,r}$, $X_J(\gamma) \in \mathbb{R}^{2n+p_1+p_2+m_1,r}$ satisfying **C2)** (resp. **C3)**).

**C4)** The matrix

$$\mathcal{Y}(\gamma) = \begin{bmatrix} -\gamma X^T_{H,2}(\gamma) E X_{H,1}(\gamma) & X^T_{H,2}(\gamma) E X_{J,2}(\gamma) \\ X^T_{J,2}(\gamma) E^T X_{H,2}(\gamma) & -\gamma X_{J,2}(\gamma)^T E^T X_{J,1}(\gamma) \end{bmatrix} \tag{3.5}$$

is symmetric, positive semi-definite and satisfies $\operatorname{rank} \mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$.

Since $X_H(\gamma)$ and $X_J(\gamma)$ are unique up to a multiplication from the right with invertible matrices, $\mathcal{Y}(\gamma)$ is unique up to a block-diagonal congruence transformation. Therefore, the value $\operatorname{rank} \mathcal{Y}(\gamma)$ is well-defined.

Note that if we consider $\mathcal{Y}(\gamma)$ in the standard case $E = I_n$, then it slightly differs from the matrix $\mathcal{Y}(\gamma)$ used in [3]. This is due to the fact that the pencils (3.1) and (3.2) are expressed in a slightly different form in the generalization to descriptor systems.

Condition **C4)** then leads to another set that has to be considered.

DEFINITION 3.5. *Consider a system of the form (1.1) that satisfies assumptions* **A1)** – **A4)**. *Then we define*

$$\Gamma^\rho = \left\{ \gamma \geq \hat{\gamma} \;\middle|\; \begin{array}{l} \text{the matrix } \mathcal{Y}(\gamma) \text{ is positive semi-definite} \\ \text{with } \operatorname{rank} \mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J \end{array} \right\}$$

*and we set $\hat{\gamma}^\rho := \inf \Gamma^\rho$.*

In this section we have introduced several assumptions and conditions as well as sets of $\gamma$-parameters that will be used in the next section to derive conditions for the optimal and suboptimal $\gamma$-parameters.

We proceed in three steps, first recalling the standard state space case in Subsection 3.1, then considering the index one case in Subsection 3.2 and finally the general case in Subsection 3.3.

**3.1. The standard state space case.** In the first step, we briefly review the results from [3, 9] for the standard state space, that is $E = I_n$. The relation between the values introduced in Definitions 3.1–3.3 is given by the following proposition.

PROPOSITION 3.6 ( [3]). *Consider a system of the form (1.1) with $E = I_n$. Then the following inequality holds:*

$$0 \leq \hat{\gamma} \leq \hat{\gamma}^L \leq \hat{\gamma}^R. \tag{3.6}$$

*If $\hat{\gamma}^I < \infty$, then $\hat{\gamma}^I = \hat{\gamma}^L > \hat{\gamma}$. If $\hat{\gamma}^\rho$ exists, then $\hat{\gamma}^\rho \geq \hat{\gamma}^R$.*
Furthermore it was shown in [3] that Theorem 3.4 holds if $E = I_n$. Therefore, **C4)** represents a well-defined condition and we can present the main result for the modified optimal $\mathcal{H}_\infty$ control problem of standard systems.

PROPOSITION 3.7. *Consider system (1.1) with $E = I_n$ and the even pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$ as in (3.1) and (3.2), respectively. Suppose that assumptions* **A1)** – **A4)** *hold.*

*Then there exists an internally stabilizing controller such that the transfer function from $w$ to $z$ satisfies $T_{zw} \in \mathcal{H}_\infty^{p_1,m_1}$ with $\|T_{zw}\|_\infty < \gamma$ if and only if $\gamma$ is such that the conditions* **C1)** – **C4)** *hold.*

*Furthermore, the set of $\gamma$ satisfying the conditions* **C1)** – **C4)** *is non-empty.*

*Proof.* In the standard state space case, see [3, 9], the assumptions **A1)** – **A4)** reduce to

- stabilizability of the pair $(A, B_2)$,
- detectability of the pair $(A, C_2)$,
- $\operatorname{rank}\begin{bmatrix} A - i\omega I_n & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2$ for all $\omega \in \mathbb{R}$,
- $\operatorname{rank}\begin{bmatrix} A - i\omega I_n & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2$ for all $\omega \in \mathbb{R}$,
- $\operatorname{rank} D_{12} = m_2$,
- $\operatorname{rank} D_{21} = p_1$.

Under these assumptions, it is shown in [9, 37] that the set of internally stabilizing controllers is non-empty and there exists an internally stabilizing controller such that the transfer function from $w$ to $z$ satisfies $T_{zw} \in \mathcal{H}_\infty^{p_1,m_1}$ with $\|T_{zw}\|_\infty < \gamma$ if and only if $\gamma$ is such that the following four conditions hold.

**$\mathbf{C_{st}1}$)** The matrices

$$R_H(\gamma) = \begin{bmatrix} D_{11}^T D_{11} - \gamma^2 I_{p_1} & D_{11}^T D_{12} \\ D_{12}^T D_{11} & D_{12}^T D_{12} \end{bmatrix},$$
$$R_J(\gamma) = \begin{bmatrix} D_{11} D_{11}^T - \gamma^2 I_{m_1} & D_{11} D_{21}^T \\ D_{21} D_{11}^T & D_{21} D_{21}^T \end{bmatrix}$$

are invertible.

**$\mathbf{C_{st}2}$)** There exists a matrix $X_H(\gamma) = \left[\begin{smallmatrix} X_{H,1} \\ X_{H,2} \end{smallmatrix}\right]$ with $X_{H,1}, X_{H,2} \in \mathbb{R}^{n,n}$ such that

**$\mathbf{C_{st}2.a}$)** for $B_H = \left[\begin{smallmatrix} B_1 & B_2 \\ -C_1^T D_{11} & -C_1^T D_{12} \end{smallmatrix}\right]$, the matrix

$$H(\gamma) = \begin{bmatrix} A & 0 \\ -C_1^T C_1 & -A^T \end{bmatrix} - B_H R_H^{-1}(\gamma) B_H^T \mathcal{J}$$

has a semi-stable Lagrangian invariant subspace $\operatorname{im} X_H(\gamma)$.

**$\mathbf{C_{st}2.b}$)** $\operatorname{rank} X_{H,1}(\gamma) = n$.

**$\mathbf{C_{st}3}$)** There exists a matrix $X_J(\gamma) = \left[\begin{smallmatrix} X_{J,1} \\ X_{J,2} \end{smallmatrix}\right]$ with $X_{J,1}, X_{J,2} \in \mathbb{R}^{n,n}$ such that

**$\mathbf{C_{st}3.a}$)** for $C_J = \left[\begin{smallmatrix} D_{11} B_1^T & C_1 \\ -D_{11} B_1^T & C_2 \end{smallmatrix}\right]$, the matrix

$$J(\gamma) = \begin{bmatrix} A^T & 0 \\ -B_1 B_1^T & -A \end{bmatrix} - C_H^T R_J^{-1}(\gamma) C_H \mathcal{J}$$

has a semi-stable Lagrangian invariant subspace $\operatorname{im} X_J$.

**$\mathbf{C_{st}3.b}$)** $\operatorname{rank} X_{J,1}(\gamma) = n$.

**$\mathbf{C_{st}4}$)** The inequality $\rho(X_{H,2}X_{H,1}^{-1}X_{J,2}X_{J,1}^{-1}) < \gamma^2$ holds.

For $E = I_n$, it was shown in [3] that for every $\mathbf{i} \in \{\mathbf{1}, \mathbf{2.a}, \mathbf{2.b}, \mathbf{3}, \mathbf{4}\}$, the property **$\mathbf{C_{st}i}$)** is equivalent to the corresponding **Ci)** for the pencils in (3.1) and (3.2). ∎

**3.2. The Index One Case.** To extend Proposition 3.7 to the case that the index of $\lambda E - A$ is $\nu = 1$, we will make use of the Weierstraß canonical form in Theorem 2.1. Transforming system (1.1) and using the notation introduced in (2.3), the explicit solution (2.5b) reduces to $x_\infty(t) = -B_{1,\infty}w(t) - B_{2,\infty}u(t)$. Inserting this into the transformed out equations, we obtain the standard state space system (often called the slow or finite dynamics subsystem)

$$\begin{aligned}
\dot{x}_f(t) &= A_f x_f(t) + B_{1,f}w(t) + B_{2,f}u(t),\\
z(t) &= C_{1,f}x_f(t) + (D_{11} - C_{1,\infty}B_{1,\infty})w(t) + (D_{12} - C_{1,\infty}B_{2,\infty})u(t),\\
y(t) &= C_{2,f}x_f(t) + (D_{21} - C_{2,\infty}B_{1,\infty})w(t) + (D_{22} - C_{2,\infty}B_{2,\infty})u(t).
\end{aligned} \tag{3.7}$$

LEMMA 3.8. *Consider system (1.1) and suppose that the index of $\lambda E - A$ is at most one. Then for* $\mathbf{i} \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$*, system (1.1) satisfies* **Ai)** *if and only if the slow subsystem (3.7) satisfies* **Ai)**.

*Proof.* Any system of index at most one is both impulse controllable and observable, see [8, 21] and, furthermore, finite dynamics stabilizability (detectability) is equivalent to stabilizability (detectability) of the slow subsystem obtained from the Weierstraß canonical form. Then from Theorem 2.1, see also [8, 21], the equivalence of the corresponding conditions **A1)** is immediate.

The equivalence for the corresponding conditions **A2)** is obtained by using the transformation matrices to Weierstraß canonical form, since

$$\begin{bmatrix} W_f^T & 0 \\ -C_{1,\infty}W_\infty^T & I_{p_1} \\ W_\infty^T & 0 \end{bmatrix} \begin{bmatrix} A - i\omega E & B_2 \\ C_1 & D_{12} \end{bmatrix} \begin{bmatrix} V_f & -V_\infty B_{2,\infty} & V_\infty \\ 0 & I_{m_2} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} A_f - i\omega I_{n_f} & B_{2,f} & 0 \\ C_{1,f} & D_{12} - C_{2,\infty}B_{2,\infty} & 0 \\ 0 & 0 & I_{n_\infty} \end{bmatrix}.$$

The proof for the equivalence of the corresponding conditions **A3)** is analogous.

We now consider condition **A4)**. By definition, the columns of the matrices $T_\infty, S_\infty$ span the left and right nullspace of $E$. Thus there exist invertible matrices $M_l, M_r \in \mathbb{R}^{n_\infty,n_\infty}$ such that $W_\infty = T_\infty M_l$, $V_\infty = S_\infty M_r$. The assertion then follows from

$$\begin{bmatrix} M_l^T & 0 \\ -C_{1,\infty}M_l^T & I_{p_1} \end{bmatrix} \begin{bmatrix} T_\infty^T A S_\infty & T_\infty^T B_2 \\ C_1 S_\infty & D_{12} \end{bmatrix} \begin{bmatrix} M_r & -M_r B_{2,\infty} \\ 0 & I_{m_2} \end{bmatrix}$$
$$= \begin{bmatrix} I_{n_\infty} & 0 \\ 0 & D_{12} - C_{1,\infty}B_{2,\infty} \end{bmatrix},$$
$$\begin{bmatrix} M_l^T & 0 \\ -C_{2,\infty}M_l^T & I_{p_2} \end{bmatrix} \begin{bmatrix} T_\infty^T A S_\infty & T_\infty^T B_1 \\ C_2 S_\infty & D_{21} \end{bmatrix} \begin{bmatrix} M_r & -M_r B_{1,\infty} \\ 0 & I_{m_1} \end{bmatrix}$$
$$= \begin{bmatrix} I_{n_\infty} & 0 \\ 0 & D_{21} - C_{2,\infty}B_{1,\infty} \end{bmatrix}.$$

∎

After proving the equivalence of the conditions **Ai)**, we now show that the $\Gamma$ sets and $\gamma$

parameters introduced in Definitions 3.1–3.3 and 3.5 are those of the slow subsystem. We denote by $\lambda N_{H,st} + M_{H,st}(\gamma)$ and $\lambda N_{J,st} + M_{J,st}(\gamma)$ the even pencils (3.1) and (3.2) constructed from the data of system (3.7).

LEMMA 3.9. *Consider the system (1.1) and assume that the index of $\lambda E - A$ is at most one. Let $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$ be the even pencils constructed from the data of (1.1) and let $\lambda N_{H,st} + M_{H,st}(\gamma)$, $\lambda N_{J,st} + M_{J,st}(\gamma)$ be the corresponding pencils constructed from the data of (3.7).*

*Let $\Gamma_H$, $\Gamma_J$, $\Gamma_H^L$, $\Gamma_J^L$, $\Gamma_H^R$, $\Gamma_J^R$, $\Gamma_H^I$ and $\Gamma_J^I$ be the sets introduced in Definitions 3.1–3.3 and 3.5 and let $\mathcal{Y}(\gamma)$ be the matrix introduced in (3.5).*

*Let analogously $\Gamma_{H,st}$, $\Gamma_{J,st}$, $\Gamma_{H,st}^L$, $\Gamma_{J,st}^L$, $\Gamma_{H,st}^R$, $\Gamma_{J,st}^R$, $\Gamma_{H,st}^I$, $\Gamma_{J,st}^I$ and $\mathcal{Y}_{st}(\gamma)$ be correspondingly defined for the slow subsystem (3.7). Then,*

$$\begin{array}{llll} \Gamma_{J,st} = \Gamma_H, & \Gamma_{H,st}^L = \Gamma_H^L, & \Gamma_{H,st}^R = \Gamma_H^R, & \Gamma_{H,st}^I = \Gamma_H^I, \\ \Gamma_{J,st} = \Gamma_J, & \Gamma_{J,st}^L = \Gamma_J^L, & \Gamma_{J,st}^L = \Gamma_J^L, & \Gamma_{J,st}^I = \Gamma_H^I, \end{array}$$

*and*

$$\operatorname{rank} \mathcal{Y}(\gamma) = \operatorname{rank} \mathcal{Y}_{st}(\gamma).$$

*Proof.* First we consider the pencil $\lambda N_H + M_H(\gamma)$ and introduce the transformation matrix

$$P_H = \begin{bmatrix} V_f^T & 0 & 0 & 0 & 0 \\ 0 & W_f^T & 0 & 0 & 0 \\ B_{1,\infty}^T V_\infty^T & 0 & I_{m_1} & 0 & 0 \\ B_{2,\infty}^T V_\infty^T & 0 & 0 & I_{m_2} & 0 \\ 0 & -C_{1,\infty} W_\infty^T & 0 & 0 & I_{p_2} \\ V_\infty^T & 0 & 0 & 0 & 0 \\ 0 & W_\infty^T & 0 & 0 & 0 \end{bmatrix}^T. \tag{3.8}$$

We obtain that

$$\lambda P_H^T N_H P_H - P_H^T M_H(\gamma) P_H = \begin{bmatrix} \lambda N_{H,st} + M_{H,st}(\gamma) & 0 & 0 \\ 0 & I_{n_\infty} & 0 \\ 0 & 0 & I_{n_\infty} \end{bmatrix}. \tag{3.9}$$

This directly implies $\Gamma_{H,st} = \Gamma_H$ and $\Gamma_{H,st}^I = \Gamma_H^I$. Analogously, we can show that $\Gamma_{J,st} = \Gamma_J$ and $\Gamma_{H,st}^I = \Gamma_H^I$. Furthermore, it can be concluded from (3.9) that the columns of a matrix

$$X_{H,st} = \begin{bmatrix} X_{H,st,1}^T & X_{H,st,2}^T & X_{H,st,3}^T & X_{H,st,4}^T & X_{H,st,5}^T \end{bmatrix}^T$$

partitioned conformably to the block structure of $\lambda N_{H,st} + M_{H,st}(\gamma)$ span a semi-stable deflating subspace if and only if the columns of

$$\begin{bmatrix} X_{H,1} \\ X_{H,2} \\ X_{H,3} \\ X_{H,4} \\ X_{H,5} \end{bmatrix} = \begin{bmatrix} V_f X_{H,st,1} + V_\infty B_{1,\infty} X_{H,st,3} + V_\infty B_{2,\infty} X_{H,st,4} + V_\infty X_{H,st,6} \\ W_f X_{H,st,2} - W_\infty C_{1,\infty}^T X_{H,st,5} + W_\infty X_{H,st,7} \\ X_{H,st,3} \\ X_{H,st,4} \\ X_{H,st,5} \end{bmatrix} \tag{3.10}$$

span the semi-stable deflating subspace of $\lambda N_H + M_H(\gamma)$.
Analogously, we can show that matrices spanning the semi-stable deflating subspace of $\lambda N_J + M_J(\gamma)$ and $\lambda N_{J,st} + M_{J,st}(\gamma)$ are related by

$$\begin{bmatrix} X_{J,1} \\ X_{J,2} \\ X_{J,3} \\ X_{J,4} \\ X_{J,5} \end{bmatrix} = \begin{bmatrix} W_f X_{J,st,1} + W_\infty C_{1,\infty}^T X_{J,st,3} + W_\infty C_{2,\infty}^T X_{J,st,4} + W_\infty X_{J,st,6} \\ V_f X_{J,st,2} - V_\infty B_{1,\infty} X_{J,st,5} + V_\infty X_{J,st,7} \\ X_{J,st,3} \\ X_{J,st,4} \\ X_{J,st,5} \end{bmatrix}. \tag{3.11}$$

Using the fact that $EV_\infty = 0$, $W_\infty^T E = 0$ and $W_f^T E T_f = I_{n_f}$, it follows that

$$\operatorname{rank} E X_{H,1}(\gamma) = \operatorname{rank} X_{H,st,1}(\gamma), \tag{3.12a}$$
$$\operatorname{rank} E^T X_{J,1}(\gamma) = \operatorname{rank} X_{J,st,1}(\gamma), \tag{3.12b}$$
$$\operatorname{rank} E^T X_{H,2}(\gamma) = \operatorname{rank} X_{H,st,2}(\gamma), \tag{3.12c}$$
$$\operatorname{rank} E X_{J,2}(\gamma) = \operatorname{rank} X_{J,st,2}(\gamma), \tag{3.12d}$$
$$X_{H,2}(\gamma)^T E X_{H,1}(\gamma) = X_{H,st,2}(\gamma)^T X_{H,st,1}(\gamma), \tag{3.12e}$$
$$X_{H,2}(\gamma)^T E X_{J,2}(\gamma) = X_{H,st,2}(\gamma)^T X_{J,st,2}(\gamma), \tag{3.12f}$$
$$X_{J,2}(\gamma)^T E^T X_{J,1}(\gamma) = X_{J,st,2}(\gamma)^T X_{J,st,1}(\gamma). \tag{3.12g}$$

The relations (3.11), (3.12e) and (3.12g) imply that $\Gamma_{H,st}^L = \Gamma_H^L$ and $\Gamma_{J,st}^L = \Gamma_J^L$. Additionally, from (3.12a), (3.12b), we obtain $\Gamma_{H,st}^R = \Gamma_H^R$ and $\Gamma_{J,st}^R = \Gamma_J^R$.
By using (3.12e)-(3.12g) we then obtain that the matrices $\mathcal{Y}(\gamma)$ and $\mathcal{Y}_{st}(\gamma)$ coincide, in particular, we have $\operatorname{rank} \mathcal{Y}(\gamma) = \operatorname{rank} \mathcal{Y}_{st}(\gamma)$. ∎
An immediate consequence is that Proposition 3.6 holds for systems of index at most one. Furthermore, from (3.12c) and (3.12b) and the corresponding fact for standard systems, we can conclude that Theorem 3.4 holds for systems of index at most one.

With these preparations we can formulate the following extension of Proposition 3.7 for systems of index at most one.

Proposition 3.10. *Consider system (1.1) such that the index of the pencil $\lambda E - A$ is at most one, and the even pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$ as in (3.1) and (3.2), respectively. Suppose that assumptions* **A1) – A4)** *hold.*

*Then there exists an internally stabilizing controller such that the transfer function from $w$ to $z$ satisfies $T_{zw} \in \mathcal{H}_\infty^{p_1,m_1}$ with $\|T_{zw}\|_\infty < \gamma$ if and only if $\gamma$ is such that the conditions* **C1) – C4)** *hold.*

*Furthermore, the set of $\gamma$ satisfying the conditions* **C1) – C4)** *is nonempty.*

*Proof.* The closed-loop transfer function $T_{zw}(s)$ of the system (3.7) with a controller of the form (1.2) is equal to the closed-loop transfer function of the system (1.1) with the same controller.

Since (1.1) is strongly stabilizable (strongly detectable), if and only if system (3.7) is stabilizable (detectable), a controller that internally stabilizes (3.7) also stabilizes the finite dynamics of (1.1).

Therefore, the existence of a controller with desired properties for (1.1) is equivalent to the existence of such a controller for (3.7). Since by Lemma 3.8 the validity of assumptions **A1) – A4)** for (3.7) is equivalent to those of (1.1) and, furthermore, also by Lemma 3.9 the corresponding conditions **C1) – C4)** of these two systems are equivalent, the assertion follows. ∎

We have seen so far that the standard state space case and the index one case follow after some simple transformation. In the next subsection we now study the general case.

**3.3. The General Case.** In this section we formulate the results for the modified optimal $\mathcal{H}_\infty$-control problem for descriptor systems of arbitrary index. A key tool in the proof will be an a priori static output feedback $u(t) = Ky(t) + \bar{u}(t)$ resulting in a system

$$\begin{aligned} E\dot{x}(t) &= (A + B_2KC_2)x(t) + (B_1 + B_2KD_{21})w(t) + B_2\bar{u}(t), \qquad x(t_0) = x^0, \\ z(t) &= (C_1 + D_{12}KC_2)x(t) + (D_{11} + D_{12}KD_{21})w(t) + D_{12}\bar{u}(t), \\ y(t) &= C_2x(t) + D_{21}w(t). \end{aligned} \tag{3.13}$$

The feedback matrix $K$ will be constructed in a way that system (3.13) has index one. Then we are able to apply the results of the previous section. If (1.2) is a controller for (3.13) then a controller for the system (1.1) is given by

$$\begin{aligned} \hat{E}\dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{B}y(t), \\ u(t) &= \hat{C}\hat{x}(t) + (\hat{D} - K)y(t). \end{aligned} \tag{3.14}$$

To proceed, we need the following results about the existence of a static output feedback $K$ that leads to a system of index at most one.

LEMMA 3.11. *[6, 8] Consider matrices $C \in \mathbb{R}^{p,n}$, $B \in \mathbb{R}^{n,m}$ and a regular matrix pencil $\lambda E - A$. Then there exists $K \in \mathbb{R}^{p,m}$ such that the pencil $\lambda E - (A + BKC)$ is regular and has index at most one if and only if the triple $(E, A, B)$ is impulse controllable and the triple $(E, A, C)$ is impulse observable, see Definition 2.3.*

To make use of this result, we show that a static output feedback does not change the assumptions **A1)** – **A4)**.

LEMMA 3.12. *Consider system (1.1) and let $K \in \mathbb{R}^{m_2,p_2}$ such that the pencil $\lambda E - (A + B_2KC_2)$ is regular. Then for every $\mathbf{i} \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$ the system (1.1) satisfies* **Ai)** *if and only if the system (3.13) satisfies* **Ai)**.

*Proof.* The invariance of strong stabilizability and strong detectability under output feedback is trivial. The proof for the equivalence of the corresponding assumptions **A2)** follows from the identity

$$\begin{bmatrix} A - i\omega E & B_2 \\ C_1 & D_{12} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ KC_2 & I_{m_2} \end{bmatrix} = \begin{bmatrix} A + B_2KC_2 - i\omega E & B_2 \\ C_1 + D_{12}KC_2 & D_{12} \end{bmatrix},$$

while the equivalence statement for **A3)** can be shown analogously. The fact that (3.13) satisfies **A4)** if and only if (1.1) satisfies **A4)** is a consequence of

$$\begin{bmatrix} T_\infty^T AS_\infty & T_\infty^T B_2 \\ C_1S_\infty & D_{12} \end{bmatrix} \begin{bmatrix} I_{n-r} & 0 \\ KC_2S_\infty & I_{m_2} \end{bmatrix} = \begin{bmatrix} T_\infty^T(A + B_2KC_2)S_\infty & T_\infty^T B_2 \\ (C_1 + D_{12}KC_2)S_\infty & D_{12} \end{bmatrix},$$

$$\begin{bmatrix} I_{n-r} & T_\infty^T B_2K \\ 0 & I_{p_2} \end{bmatrix} \begin{bmatrix} T_\infty^T AS_\infty & T_\infty^T B_1 \\ C_2S_\infty & D_{21} \end{bmatrix} = \begin{bmatrix} T_\infty^T(A + B_2KC_2)S_\infty & T_\infty^T(B_1 + B_2KD_{21}) \\ C_2S_\infty & D_{21} \end{bmatrix}.$$

□

In the following Lemma we show that the sets introduced in Definitions 3.1–3.3 and 3.5 are invariant under output feedback as well.

LEMMA 3.13. *Consider the system (1.1) and let* $K \in \mathbb{R}^{m_2,p_2}$ *be such that the pencil* $\lambda E - (A + BKC)$ *is regular. Let* $\Gamma_H$, $\Gamma_J$, $\Gamma_H^L$, $\Gamma_J^L$, $\Gamma_H^R$, $\Gamma_J^R$, $\Gamma_H^I$ *and* $\Gamma_J^I$ *be the sets introduced in Definitions 3.1–3.3 and 3.5 and let* $\mathcal{Y}(\gamma)$ *be the matrix introduced in (3.5). Furthermore, let* $\Gamma_{H,K}$, $\Gamma_{J,K}$, $\Gamma_{H,K}^L$, $\Gamma_{J,K}^L$, $\Gamma_{H,K}^R$, $\Gamma_{J,K}^R$, $\Gamma_{H,K}^I$, $\Gamma_{J,K}^I$ *and* $\mathcal{Y}_K(\gamma)$ *be the corresponding quantities for the system (3.13). Then,*

$$\begin{aligned} &\Gamma_{J,K} = \Gamma_H, \qquad &&\Gamma_{H,K}^L = \Gamma_H^L, \qquad &&\Gamma_{H,K}^R = \Gamma_H^R, \qquad &&\Gamma_{H,K}^I = \Gamma_H^I, \\ &\Gamma_{J,K} = \Gamma_J &&\Gamma_{J,K}^L = \Gamma_J^L, &&\Gamma_{J,K}^L = \Gamma_J^L, &&\Gamma_{J,K}^I = \Gamma_H^I, \end{aligned}$$

*and*

$$\operatorname{rank} \mathcal{Y}(\gamma) = \operatorname{rank} \mathcal{Y}_K(\gamma).$$

*Proof.* Let $\lambda N_{H,K} + M_{H,K}(\gamma)$ be the even pencil associated to the system (3.13). Then, with the transformation matrices

$$T_{H,K} = \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & I_{m_1} & 0 & 0 \\ KC_2 & 0 & KD_{21} & I_{m_2} & 0 \\ 0 & 0 & 0 & 0 & I_{p_1} \end{bmatrix}, \quad T_{J,K} = \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & I_{p_1} & 0 & 0 \\ K^T B_2^T & 0 & K^T D_{12}^T & I_{p_2} & 0 \\ 0 & 0 & 0 & 0 & I_{m_1} \end{bmatrix},$$

we have the identities

$$\begin{aligned} \lambda T_{H,K}^T N_H T_{H,K} + T_{H,K}^T M_H(\gamma) T_{H,K} &= \lambda N_{H,K} + M_{H,K}(\gamma), \\ \lambda T_{J,K}^T N_J T_{J,K} + T_{J,K}^T M_J(\gamma) T_{J,K} &= \lambda N_{J,K} + M_{J,K}(\gamma). \end{aligned}$$

Thus, we have that the pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_{H,K} + M_{H,K}(\gamma)$ have the same index and eigenvalues. Similarly, this holds for $\lambda N_J + M_J(\gamma)$ and $\lambda N_{J,K} + M_{J,K}(\gamma)$. Therefore we have $\Gamma_{H,K} = \Gamma_H$, $\Gamma_{J,K} = \Gamma_J$, $\Gamma_{H,K}^I = \Gamma_H^I$, $\Gamma_{J,K}^I = \Gamma_J^I$. The relations $\Gamma_{H,K}^R$, $\Gamma_{J,K}^R$, $\Gamma_{H,K}^L$, $\Gamma_{J,K}^L$ follow from the facts that

$$\begin{aligned} &\operatorname{im} \begin{bmatrix} X_{H,1}^T & X_{H,2}^T & X_{H,3}^T & X_{H,4}^T & X_{H,5}^T \end{bmatrix}^T, \\ &\operatorname{im} \begin{bmatrix} X_{J,1}^T & X_{J,2}^T & X_{J,3}^T & X_{J,4}^T & X_{J,5}^T \end{bmatrix}^T \end{aligned}$$

are semi-stable deflating subspaces of $\lambda N_{H,K} + M_{H,K}(\gamma)$ and $\lambda N_{J,K} + M_{J,K}(\gamma)$, respectively, if and only if

$$\begin{aligned} &\operatorname{im} \begin{bmatrix} X_{H,1}^T & X_{H,2}^T & X_{H,3}^T & (X_{H,4} - KC_2 X_{H,1} + KD_{21} X_{H,3})^T & X_{H,5}^T \end{bmatrix}^T, \\ &\operatorname{im} \begin{bmatrix} X_{J,1}^T & X_{J,2}^T & X_{J,3}^T & (X_{J,4} - K^T B_2^T X_{J,1} + K^T D_{12}^T X_{H,3})^T & X_{H,5}^T \end{bmatrix}^T \end{aligned} \tag{3.15}$$

are semi-stable deflating subspace of $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$. From (3.15), we further obtain that $\mathcal{Y}(\gamma) = \mathcal{Y}_K(\gamma)$ and thus, their ranks coincide. ∎

With these auxiliary results, we are now in a position to prove Theorem 3.4.

*Proof of Theorem 3.4.* First we apply an priori feedback $K \in \mathbb{R}^{m_2,p_2}$ to (1.1) such that the resulting system (3.13) has index at most one. Then we know from (3.15) that for the corresponding matrices $X_{H,1}$, $X_{H,2}$, $X_{J,1}$, $X_{J,2}$ of (1.1) and (3.13) are equal. Since Theorem 3.4 holds for systems of index one, the assertion follows. ∎

Lemma 3.13 also implies that the assertion of Proposition 3.6 still holds for the general case, i.e., for (1.1), the inequality

$$0 \leq \hat{\gamma} \leq \hat{\gamma}^L \leq \hat{\gamma}^R.$$

is valid. In the case where $\hat{\gamma}^I < \infty$, we have $\hat{\gamma}^I = \hat{\gamma}^L > \hat{\gamma}$ and if $\hat{\gamma}^\rho$ exists, then $\hat{\gamma}^\rho \geq \hat{\gamma}^R$.

With the described framework, we can now formulate the main result for the modified $\mathcal{H}_\infty$ control problem for descriptor systems.

THEOREM 3.14. *Consider system (1.1) and the even pencils* $\lambda N_H + M_H(\gamma)$ *and* $\lambda N_J + M_J(\gamma)$ *as in (3.1) and (3.2), respectively. Suppose that assumptions* **A1)** – **A4)** *hold.*

*Then there exists an internally stabilizing controller such that the transfer function from* $w$ *to* $z$ *satisfies* $T_{zw} \in \mathcal{H}_\infty^{p_1,m_1}$ *with* $\|T_{zw}\|_\infty < \gamma$ *if and only if* $\gamma$ *is such that the conditions* **C1)** – **C4)** *hold.*

*Furthermore, the set of* $\gamma$ *satisfying the conditions* **C1)** – **C4)** *is nonempty.*

*Proof.* Due to Lemma 3.11, there exists a matrix $K \in \mathbb{R}^{m_2,p_2}$ such that the system (3.13) has index at most one. Lemma 3.12 implies that (3.13) satisfies **A1)** – **A4)** as well. Furthermore, by Lemma 3.13, the validity of the conditions **C1)** – **C4)** for the system (1.1) are equivalent to the respective conditions for system (3.13).

Proposition 3.10 then implies that conditions **C1)** – **C4)** for (3.13) are fulfilled if and only if there exists a desired controller for (3.13).

Since an application of the controller (1.2) to (3.13) results in the same closed loop system as controlling (3.13) with (3.14), the desired result follows immediately. □

THEOREM 3.15. *Consider system (1.1) and suppose that assumptions* **A1)** – **A4)** *hold. Then the set* $\Gamma^\rho$ *is non-empty and optimal* $\gamma$ *for the modified optimal* $\mathcal{H}_\infty$ *control problem is given by*

$$\gamma_{mo} = \hat{\gamma}^\rho. \tag{3.16}$$

*Proof.* Let $\Gamma$ be the set of $\gamma > 0$ for which an internally stabilizing controller exists such that the transfer function from $w$ to $z$ satisfies $\|T_{zw}\|_\infty < \gamma$.

We know from Theorem 3.14 that $\Gamma$ is non-empty and for some $\gamma > 0$, we have $\gamma \in \Gamma$ if and only if the conditions **C1)** – **C4)** are fulfilled. By the definition of $\Gamma_H$, $\Gamma_J$, $\Gamma^R$ and $\Gamma^\rho$, the existence of a controller with desired properties is therefore equivalent to

$$\gamma \in \Gamma_H \cap \Gamma_J, \quad \gamma \in \Gamma^R, \quad \gamma \in \Gamma^\rho. \tag{3.17}$$

Especially, we have that $\Gamma^\rho$ is non-empty. By the definition of $\hat{\gamma}$, $\hat{\gamma}^R$ and $\hat{\gamma}^\rho$, condition (3.17) is the same as

$$\gamma > \hat{\gamma}, \quad \gamma > \hat{\gamma}^R, \quad \gamma \in \hat{\gamma}^\rho. \tag{3.18}$$

Hence, $\gamma \in \Gamma$ is equivalent to

$$\gamma > \max\{\hat{\gamma}, \hat{\gamma}^R, \hat{\gamma}^\rho\}. \tag{3.19}$$

However, since by Lemma 3.13 we have that Proposition 3.6 still holds for arbitrary descriptor systems, the equation $\hat{\gamma}^\rho = \max\{\hat{\gamma}, \hat{\gamma}^R, \hat{\gamma}^\rho\}$ holds. Thus we have that $\hat{\gamma}_{mo} = \inf \Gamma = \hat{\gamma}^\rho$. □

**4. Computation of $\gamma_{mo}$.** In this section we give a numerical method for the computation of $\gamma_{mo}$ that is similar to the procedure proposed in [3] and uses a bisection method.

**Procedure 1: (Classification of $\gamma$)**
**Input:** Data of system (1.1), value $\gamma \geq 0$.
**Output:** Decision whether $\gamma < \gamma_{mo}$ or $\gamma \geq \gamma_{mo}$.

1. Form the pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$.
2. Compute the deflating subspace matrices $X_H$ and $X_J$ associated with the eigenvalues in the closed left half plane.
3. IF the dimension of one/both of these subspaces is less than $r$, then $\gamma < \gamma_{mo}$,
   ELSE
   IF the rank of $EX_{H,1}$ and/or $E^T X_{J,1}$ is less than r, then $\gamma < \gamma_{mo}$,
   ELSE
   Form the matrix $\hat{\mathcal{Y}}$.
   IF $\hat{\mathcal{Y}}$ is not positive semi-definite and/or rank $\hat{\mathcal{Y}} < \hat{k}_H + \hat{k}_J$, then $\gamma < \gamma_{mo}$,
   ELSE $\gamma \geq \gamma_{mo}$.

To determine $\gamma_{mo}$ we use the following bisection method.

**Procedure 2: (Bisection method)**
**Input:** upper and lower bounds $\gamma_{up}$ and $\gamma_{lo}$ for $\gamma_{mo}$, tolerance Tol.

**Output:** Approximation to $\gamma_{mo}$.

1. IF $\gamma_{up} - \gamma_{lo} < Tol$, then set $\gamma_{mo} = \gamma_{up}$,
   ELSE
   IF $\gamma < \gamma_{mo}$, then set $\gamma_{lo} = \gamma$, $\gamma = (\gamma_{mo} + \gamma_{up})/2$ and test whether $\gamma < \gamma_{mo}$ or $\gamma \geq \gamma_{mo}$ with Procedure 1.
   IF $\gamma > \gamma_{mo}$, then set $\gamma_{up} = \gamma$, $\gamma = (\gamma_{mo} + \gamma_{up})/2$ and test whether $\gamma < \gamma_{mo}$ or $\gamma \geq \gamma_{mo}$ with Procedure 1.
2. GOTO Step 1.

To illustrate the functionality of our approach, consider the following example from [35] which is also discussed in [27]. The descriptor system is given by (1.1) with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix},$$

$$D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{21} = 1, \quad D_{11} = D_{22} = 0.$$

This system is of index 2 and the associated pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$ have index 1 for $\gamma \neq 0$. The goal is to find the minimum value $\gamma$ that satisfies the conditions **C1)** $-$ **C4)**. Using the $QZ$-Algorithm in MATLAB to calculate the eigenvalues of the pencils and the deflating subspaces associated with eigenvalues in the closed left half plane and, using the Procedure 2 to determine the optimal value for gamma, we computed $\gamma_{opt}$ given by $\gamma^{\rho} = 0.7397$, which is smaller than the suboptimal value obtained in [27, 35].

The $QZ$-Algorithm does not make use of the special even structure of the matrix pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$ and thus the decision whether eigenvalues are purely imaginary is sometimes difficult. As the perturbation theory for the standard state space case indicates, see [22], it can be expected that even better results can be obtained with a structure preserving method that takes the even structure of the pencil into account. A production code for this is currently under development, see [32].

In this paper we have developed conditions for optimal and suboptimal $\mathcal{H}_\infty$-control for descriptor systems of arbitrary index. We have expressed criteria for the existence of an internally stabilizing controller in terms of even pencils. Furthermore we have presented the framework for the $\gamma$-iteration applied to general descriptor systems. We have illustrated the theoretical results with a numerical example.

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