

Strong Nash Equilibria in Games with the Lexicographical Improvement Property

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Abstract. We introduce a class of finite strategic games with the property that every deviation of a coalition of players that is profitable to each of its members strictly decreases the lexicographical order of a certain function defined on the set of strategy profiles. We call this property the *Lexicographical Improvement Property (LIP)* and show that it implies the existence of a generalized strong ordinal potential function. We use this characterization to derive existence, efficiency and fairness properties of strong Nash equilibria. We then study a class of games that generalizes congestion games with bottleneck objectives that we call *bottleneck congestion games*. We show that these games possess the LIP and thus the above mentioned properties. For bottleneck congestion games in networks, we identify cases in which the potential function associated with the LIP leads to polynomial time algorithms computing a strong Nash equilibrium. Finally, we investigate the LIP for infinite games. We show that the LIP does not imply the existence of a generalized strong ordinal potential, thus, the existence of SNE does not follow. Assuming that the function associated with the LIP is continuous, however, we prove existence of SNE. As a consequence, we prove that bottleneck congestion games with infinite strategy spaces and continuous cost functions possess a strong Nash equilibrium.

1 Introduction

The theory of non-cooperative games is used to study situations that involve rational and selfish agents who are motivated by optimizing their own utilities rather than reaching some social optimum. In such a situation, a state is called a pure Nash equilibrium (PNE) if it is stable in the sense that no player has an incentive to unilaterally deviate from its strategy. While the PNE concept excludes the possibility that a single player can unilaterally improve her utility, it does not necessarily imply that a PNE is stable against coordinated deviations of a group of players if their joint deviation is profitable for each of its members. So when coordinated actions are possible, the Nash equilibrium concept is not sufficient to analyze stable states of a game.

To cope with this issue of coordination, we adopt the solution concept of a *strong equilibrium* (SNE for short) proposed by Aumann [4]. In a strong equilibrium, no coalition (of any size) can deviate and strictly improve the utility of each of its members (while possibly lowering the utility of players outside the coalition). Clearly, every SNE is a PNE, but the converse does not always hold. Thus, even though SNE may rarely exist, they form a very robust and appealing stability concept.

One of the most successful approaches in establishing existence of PNE (as well as SNE) is the potential function approach initiated by Monderer and Shapley [24] and later generalized by Holzman and Law-Yone [15] to strong equilibria: one defines a real-valued function P on the set of strategy profiles of the game and shows that every improving move of a coalition (which is profitable to each of its members) strictly reduces (increases) the value of P . Since the set of strategy profiles of such a (finite) game is finite, every sequence of improving moves reaches a SNE. In particular, the global minimum (maximum) of P is a SNE. The main difficulty is, however, that for most games it is hard to prove or disprove the existence of such a potential function.

Given that the existence of a real-valued potential is hard to detect, we derive in this paper an equivalent property (Theorem 1) that we call the *lexicographical improvement property (LIP)*. We consider strategic games $G = (N, X, \pi)$, where N is the set of players, X the strategy space, and players

experience private non-negative costs $\pi_i(x), i \in N$, for a strategy profile x . We say that G has the LIP if there exists a function $\phi : X \rightarrow \mathbb{R}_+^q, q \in \mathbb{N}$, such that every improving move (profitable deviation of an arbitrary coalition) from $x \in X$ strictly reduces the sorted lexicographical order of $\phi(x)$ (see Definition 4). We say that G has the π -LIP if G satisfies the LIP with $\phi_i(x) = \pi_i(x), i \in N$, that is, every improving move strictly reduces the lexicographical order of the players' private costs. Clearly, requiring $q = 1$ in the definition of the LIP reduces to the case of a generalized strong ordinal potential.

The main contribution of this paper is twofold. We first study desirable properties of arbitrary finite and infinite games having the LIP and π -LIP, respectively. These properties concern the existence of SNE, efficiency and fairness of SNE, and computability of SNE. Secondly, we identify an important class of games that we term bottleneck congestion games for which we can actually prove the π -LIP and, hence, prove that these games possess SNE with the above desirable properties. In the following, we will give an informal definition of bottleneck congestion games.

Let us first recall the definition of a standard congestion game. In a congestion game, there is a set of elements (facilities), and the pure strategies of players are subsets of this set. Each facility has a cost that is a function of its *load*, which is usually a function of the number (or total weight) of players that select strategies containing the respective facility. The private cost of a player's strategy in a standard congestion game is given by the *sum* over the costs of the facilities in the strategy. In a bottleneck congestion game, the private cost function of a player is equal to the cost of the *most expensive* facility that she uses (L_∞ -norm of the vector of players' costs of the strategy).

Before we outline our results, we briefly explain the importance of bottleneck objectives in congestion games with respect to real-world applications. Referring to previous work by Keshav [16], Cole et al. [9] pointed out that the performance of a communication network is closely related to the performance of its bottleneck (most congested) link. This behavior is also stressed by Banner and Orda [5], who studied Nash equilibria in routing games with bottleneck objectives. Similar observations are reported by Qiu et al. [26], who investigated the applicability of theoretical results of selfish routing models to realistic models of the Internet.

1.1 Our Results

We characterize games having the LIP by means of the existence of a generalized strong ordinal potential function. The proof of this characterization is constructive, that is, given a game G having the LIP for a function ϕ , we explicitly construct a generalized strong ordinal potential P . We further investigate games having the π -LIP with respect to efficiency and fairness of SNE. Our characterization implies that there are SNE satisfying various efficiency and fairness properties, e.g., bounds on the prices of stability and anarchy, Pareto optimality, and min-max fairness.

We establish that bottleneck congestion games possess the π -LIP and, thus, possess SNE with the above mentioned properties. Moreover, our characterization of games having the LIP implies that bottleneck congestion games possess the strong finite improvement property. For bottleneck congestion games in networks, we identify cases in which SNE can be computed in polynomial time.

It is worth noting that for singleton congestion games, Even-Dar et al. [12] and Fabrikant et al. [13], have already proved existence of PNE by arguing that the vector of facility costs decreases lexicographically for every improving move. Andelman et al. [3] used the same argument to establish even existence of SNE in this case. Our work generalizes these results to arbitrary strategy spaces and more general cost functions. In contrast to most congestion games considered so far, we require only that the cost functions on the facilities satisfy three properties: "non-negativity", "independence of irrelevant choices", and "monotonicity". Roughly speaking, the second and third condition assume that the cost of a facility solely depends on the set of players using the respective facility and that this cost decreases if some players leave this facility. Thus, this framework extends classical *load-based* models in which the cost of a facility depends on the number or total weight of players using the

respective facility. Our assumptions are weaker than in the load-based models and even allow that the cost of a facility may depend on the *set* of players using this facility.

We then study the LIP in *infinite* games, that is, games with infinite strategy spaces that can be described by compact subsets of \mathbb{R}^p , $p \in \mathbb{N}$. We first show that our characterization of finite games with the LIP does not hold anymore (essentially resembling Debreu's impossibility result [10]). We prove, however, that continuity of ϕ in the definition of LIP is sufficient for the existence of SNE. Our existence proof is constructive, that is, we outline an algorithm whose output is a SNE.

We consequently introduce *infinite bottleneck congestion games*. An infinite bottleneck congestion game arises from a bottleneck congestion game G by allowing players to fractionally distribute a certain *demand* over the pure strategies of G . We prove that these games have the π -LIP provided that the cost functions on the facilities are non-negative and non-decreasing. It turns out, however, that the function π may be discontinuous on the strategy space (even if the cost functions on the facilities are continuous). Thus, the existence of SNE does not immediately follow. We solve this difficulty by generalizing the LIP. As a consequence, we obtain for the first time the existence of SNE for infinite bottleneck congestion games with non-decreasing and continuous cost functions. For *bounded* cost functions on the facilities (that may be discontinuous), we show that α -approximate SNE exist for every $\alpha > 0$. Finally, we show that α -approximate SNE can be computed in polynomial time for bottleneck congestion games in networks.

In the final section, we show that our methods presented in this paper also apply to a more general framework.

1.2 Related Work

The SNE concept was introduced by Aumann [4] and refined by Bernheim et al. [6] to Coalition-Proof Nash Equilibrium (CPNE), which is a state that is stable against those deviations, which are themselves resilient to further deviations by subsets of the original coalition. This implies that every SNE is also a CPNE, but the converse need not hold.

Congestion games were introduced by Rosenthal [28] and further studied by Monderer and Shapley [24]. Holzman and Law-Yone [15] studied the existence of SNE in congestion games with monotone increasing cost functions. They showed that SNE need not exist in such games and gave a structural characterization of the strategy space for symmetric (and quasi-symmetric) congestion games that admit SNE. Based on the previous work of Monderer and Shapley [24], they also introduced the concept of a *strong potential function*: a function on the set of strategy profiles that decreases for every profitable deviation of a coalition. Rozenfeld and Tennenholtz [29] further explored the existence of (correlated) SNE in congestion games with non-increasing cost functions.

A further generalization of congestion games has been proposed by Milchtaich [23], where he allows for player-specific cost functions on the facilities (see also Mavronicolas et al. [21], Gairing et al. [14] and Ackermann et al. [1] for subsequent work on weighted congestion games with player-specific cost functions). Under restrictions on the strategy space (singleton strategies), Milchtaich proves existence of pure Nash equilibria. As shown by Voorneveld et al. [31], the model of Konishi et al. [17] is equivalent to that of Milchtaich, which is worth noting as Konishi et al. established the existence of SNE in such games.

Several authors studied the existence and efficiency (price of anarchy and stability) of PNE and SNE in various specific classes of congestion games. For example, Even-Dar et al. [12] showed that job scheduling games (on unrelated machines) admit a PNE by arguing that the load-lexicographically minimal schedule is a PNE. Fabrikant et al. [13] considered a scheduling model in which the processing time of a machine may depend on the set of jobs scheduled on the respective machine. For this model, they proved existence of PNE analogous to the proof of Even-Dar et al. Andelman et al. [3] considered scheduling games on unrelated machines and proved that the load-lexicographically minimal schedule is even a SNE. They further studied differences between PNE and SNE and derived

bounds on the (strong) price of anarchy and stability, respectively. Chien and Sinclair [8] recently studied the strong price of anarchy of SNE in general congestion games.

Bottleneck congestion games with network structure have been considered by Banner and Orda [5]. They studied existence of PNE in the unsplitable flow and in the splittable flow setting, respectively. They observed that standard techniques (such as Kakutani's fixed-point theorem) for proving existence of PNE do not apply to bottleneck routing games as the private cost functions may be discontinuous. They proved existence of PNE by showing that bottleneck games are better reply secure, quasi-convex, and compact. Under these conditions, they recall Reny's existence theorem [27] for better reply secure games with possibly discontinuous private cost functions. Banner and Orda, however, do not study SNE. We remark that our proof of the existence of SNE is direct and constructive.

Bottleneck routing with *non-atomic* players and elastic demands has been studied by Cole et al. [9]. Among other results they derived bounds on the price of anarchy in this setting. For subsequent work on the price of anarchy in bottleneck routing games with atomic and non-atomic players, we refer to the paper by Mazalo et al. [22].

2 Preliminaries

We consider strategic games $G = (N, X, \pi)$, where $N = \{1, \dots, n\}$ is the non-empty and finite set of players, $X = \times_{i \in N} X_i$ is the non-empty strategy space, and $\pi : X \rightarrow \mathbb{R}_+^n$ is the combined *private cost* function that assigns a private cost vector $\pi(x)$ to each strategy profile $x \in X$. These games are cost minimization games and we assume additionally that the private cost functions are non-negative. A strategic game is called *finite* if X is finite.

We use standard game theory notation; for a coalition $S \subseteq N$ we denote by $-S$ its complement and by $X_S = \times_{i \in S} X_i$ we denote the set of strategy profiles of players in S .

Definition 1 (Strong Nash equilibrium (SNE)). *A strategy profile x is a strong Nash equilibrium if there is no coalition $\emptyset \neq S \subseteq N$ that has an alternative strategy profile $y_S \in X_S$ such that $\pi_i(y_S, x_{-S}) - \pi_i(x) < 0$ for all $i \in S$.*

A pair $(x, (y_S, x_{-S})) \in X \times X$ is called an *improving move* (or *profitable deviation*) of coalition S if $\pi_i(x_S, x_{-S}) - \pi_i(y_S, x_{-S}) > 0$ for all $i \in S$. We denote by $I(S)$ the set of improving moves of coalition $S \subseteq N$ in a strategic game $G = (N, X, \pi)$ and we set $I := \bigcup_{S \subseteq N} I(S)$. We call a sequence of strategy profiles $\gamma = (x^0, x^1, \dots)$ an *improvement path* if every tuple $(x^k, x^{k+1}) \in I$ for all $k = 0, 1, 2, \dots$. One can interpret an improvement path as a path in the so called *improvement graph* $\mathcal{G}(G)$ derived from G , where every strategy profile $x \in X$ corresponds to a node in $\mathcal{G}(G)$ and two nodes x, x' are connected by a directed edge (x, x') if and only if $(x, x') \in I$. We are interested in conditions that assure that every improvement path is finite. A necessary and sufficient condition is the existence of a generalized *strong* ordinal potential function, which we define below (see also the potential function approach initiated by Monderer and Shapley [24], which has been generalized to strong potentials by Holzman and Law-Yone [15]).

Definition 2 (Generalized strong ordinal potential game). *A strategic game $G = (N, X, \pi)$ is called a generalized strong ordinal potential game if there is a function $P : X \rightarrow \mathbb{R}$ such that $P(x) - P(y) > 0$ for all $(x, y) \in I$. P is called a generalized strong ordinal potential of the game G .*

In recent years, much attention has been devoted to games admitting the finite improvement property (FIP), that is, each path of single-handed (one player) deviations is finite. Equivalently, we say that G has the strong finite improvement property (SFIP) if every improvement path is finite. Clearly, the SFIP implies the FIP, but the converse need not be true.

It is known that both the SFIP and the existence of a generalized strong ordinal potential are hard to prove or disprove for a particular game. We define a class of games that we call *games with the*

Lexicographical Improvement Property (LIP) and show that such games possess a generalized strong ordinal potential. For this purpose, we will first define the sorted lexicographical order.

Definition 3 (Sorted lexicographical order). Let $a, b \in \mathbb{R}_+^q$ and denote by $\tilde{a}, \tilde{b} \in \mathbb{R}_+^q$ be the sorted vectors derived from a, b by permuting the entries in non-increasing order, that is, $\tilde{a}_1 \geq \dots \geq \tilde{a}_q$ and $\tilde{b}_1 \geq \dots \geq \tilde{b}_q$. Then, a is strictly sorted lexicographically smaller than b (written $a < b$) if there exists an index m such that $\tilde{a}_i = \tilde{b}_i$ for all $i < m$, and $\tilde{a}_m < \tilde{b}_m$. The vector a is sorted lexicographically smaller than b (written $a \leq b$) if either $a < b$ or $\tilde{a} = \tilde{b}$.

Roughly speaking, the lexicographical improvement property of a strategic game requires that a vector-valued function $\phi : X \rightarrow \mathbb{R}_+^q$ is strictly decreasing with respect to the sorted lexicographical order on \mathbb{R}_+^q for every improvement step.

Definition 4 (Lexicographical improvement property, π -LIP). A finite strategic game $G = (N, X, \pi)$ possesses the lexicographical improvement property (LIP) if there exist $q \in \mathbb{N}$ and a function $\phi : X \rightarrow \mathbb{R}_+^q$ such that $\phi(x) > \phi(y)$ for all $(x, y) \in I$. G has the π -LIP if G has the LIP for $\phi = \pi$.

Clearly, the function ϕ is a generalized strong ordinal potential if $q = 1$. The next theorem (proof is given in the appendix) states that requiring the LIP is equivalent to requiring the existence of a generalized strong ordinal potential, regardless of q .

Theorem 1. Let $G = (N, X, \pi)$ be a finite strategic game. Then, the following statements are equivalent.

1. G has the SFIP
2. $\mathcal{G}(G)$ is acyclic (contains no directed cycles)
3. G has a generalized strong ordinal potential
4. G has the LIP
5. There exists $\phi : X \rightarrow \mathbb{R}_+^q$ and $M \in \mathbb{N}$ such that $P(x) = \sum_{i=1}^q \phi_i(x)^M$ is a generalized strong ordinal potential function for G .

Corollary 1. Every finite strategic game $G = (N, X, \pi)$ with the LIP possesses a strong Nash equilibrium.

Next, we provide an explicit formula to obtain a generalized strong ordinal potential function for a strategic game satisfying the LIP. The proof is given in the appendix.

Corollary 2. Let $G = (N, X, \pi)$ be a strategic game that satisfies the LIP for a function $\phi : X \rightarrow \mathbb{R}_+^q$. We set

$$\phi_{\max} := \max_{x \in X, 1 \leq i \leq q} \phi_i(x), \quad \epsilon_{\min} := \min_{\substack{(x,y) \in I \\ 1 \leq i \leq q: \phi_i(x) \neq \phi_i(y)}} \phi_i(x) - \phi_i(y),$$

and choose $M > \log(q) \phi_{\max} / \epsilon_{\min}$. Then, $P_M(x) = \sum_{i=1}^q \phi_i(x)^M$ is a generalized strong ordinal potential for G .

Corollary 3. Let $G = (N, X, \pi)$ be a finite game satisfying the LIP for a function $\phi : X \rightarrow \mathbb{R}_+^q$. Then, the number of arcs along any improvement path in $\mathcal{G}(G)$ is bounded from above by $\lceil q \phi_{\max}^M / \epsilon_{\min} \rceil$, where $M > \log(q) \frac{\phi_{\max}}{\epsilon_{\min}}$.

3 Properties of SNE in Games with the π -LIP

As the existence of SNE in games with the LIP is guaranteed, it is natural to ask which properties these SNE may satisfy. In recent years, several notions of efficiency have been discussed in the literature, see Koutsoupias and Papadimitriou [19]. We here cover the price of stability, the price of anarchy, Pareto optimality and min-max-fairness.

3.1 Price of Stability and Price of Anarchy

We study the efficiency of SNE with respect to the optimum of a predefined social cost function. In this context, two notions have evolved, the *strong price of anarchy* measures the ratio of the cost of the worst SNE and that of the social optimum. The *strong price of stability* measures the ratio of the cost of the best SNE and that of the social optimum. Given a game $G = (N, X, \pi)$ and a social cost function $C : X \rightarrow \mathbb{R}_+$, whose minimum is attained in a strategy profile $y \in X$, let $X^{\text{SNE}} \subseteq X$ denote the set of strong Nash equilibria. Then, the strong price of anarchy for G with respect to C is defined as $\sup_{x \in X^{\text{SNE}}} C(x)/C(y)$ and the strong price of stability for G with respect to C is defined as $\inf_{x \in X^{\text{SNE}}} C(x)/C(y)$.

We will consider the following natural social cost functions: the sum-objective or L_1 -norm defined as $L_1(x) = \sum_{i \in N} \pi_i(x)$, the L_p -objective or L_p -norm, $p \in \mathbb{N}$, defined as $L_p(x) = (\sum_{i \in N} \pi_i(x)^p)^{1/p}$, and the min-max objective or L_∞ -norm defined as $L_\infty(x) = \max_{i \in N} \{\pi_i(x)\}$.

Theorem 2. *Let G be a strategic game with the π -LIP. Then, the strong price of stability w.r.t. L_∞ is 1, and for any $p \in \mathbb{R}$, the strong price of stability w.r.t. L_p is strictly smaller than n .*

The proof of this theorem as well as a matching lower bound for $p = 1$ and a lower bound on the price of anarchy are given in the appendix.

3.2 Pareto Optimality

Pareto optimality, which we define below, is one of the fundamental concepts studied in economics, see Osborne and Rubinstein [25]. For a strategic game $G = (N, X, \pi)$, a strategy profile x is called *weakly Pareto efficient* if there is no $y \in X$ such that $\pi_i(y) < \pi_i(x)$ for all $i \in N$. A strategy profile x is *strictly Pareto efficient* if there is no $y \in X$ such that $\pi_i(y) \leq \pi_i(x)$ for all $i \in N$, where at least one inequality is strict.

So strictly Pareto efficient strategy profiles are those strategy profiles for which every improvement of a coalition of players is to the expense of at least one player outside the coalition. Pareto optimality has also been studied in the context of congestion games, see Chien and Sinclair [8] and Holzman and Law-Yone [15]. Clearly, every SNE is weakly Pareto optimal. We will show strict Pareto optimality of SNE in games having the π -LIP. For the proof of the next result, we refer to Section 3.3 in which an even stronger result is proved.

Corollary 4. *Let G be a finite strategic game having the π -LIP. Then, there exists a SNE that is strictly Pareto optimal.*

3.3 Min-Max-Fairness

We next define the notion of min-max fairness, which is a central topic in resource allocation in communication networks, see Srikant [30] for an overview and pointers to the large body of research in this area. While strict Pareto efficiency requires that there is no alternative profile which improves the cost for a single player without strictly deteriorating the costs of the other players, the notion of min-max-fairness is stricter. Formally, a profile x is called *min-max fair* if for any other strategy profile y with $\pi_i(y) < \pi_i(x)$ for some $i \in N$, there exists either $j \in N/\{i\}$ such that $\pi_j(x) \geq \pi_j(y)$ and $\pi_j(y) > \pi_j(x)$, or there exists $j \in N/\{i\}$ such that $\pi_j(x) < \pi_j(y)$ and $\pi_j(y) \geq \pi_j(x)$. Note that in contrast to Pareto efficiency, an improvement that increases the cost of a player with smaller original cost is allowed (up to the threshold $\pi_i(x)$). It is easy to see that every min-max-fair strategy profile constitutes a strict Pareto optimum, but the converse need not hold.

Corollary 5. *Let G be a finite strategic game having the π -LIP. Then, there exists a SNE that is min-max fair.*

The corollary is proved in the appendix.

4 Bottleneck Congestion Games

We now present a rich class of games satisfying the π -LIP. We call these games *bottleneck congestion games*. They are natural generalizations of variants of congestion games. In contrast to standard congestion games, we focus on *makespan-objectives*, that is, the cost of a player when using a set of facilities only depends on the highest cost of these facilities. For the sake of a clean mathematical definition, we introduce the general notion of a congestion model.

Definition 5 (Congestion model). A tuple $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$ is called a congestion model if $N = \{1, \dots, n\}$ is a non-empty, finite set of players, $F = \{1, \dots, m\}$ is a non-empty, finite set of facilities, and $X = \times_{i \in N} X_i$ is the set of strategies. For each player $i \in N$, her collection of pure strategies X_i is a non-empty, finite set of subsets of F . Given a strategy profile x , we define $\mathcal{N}_f(x) = \{i \in N : f \in x_i\}$ for all $f \in F$. Every facility $f \in F$ has a cost function $c_f : \times_{i \in N} X_i \rightarrow \mathbb{R}_+$ satisfying

Non-negativity: $c_f(x) \geq 0$ for all $x \in X$

Independence of Irrelevant Choices: $c_f(x) = c_f(y)$ for all $x, y \in X$ with $\mathcal{N}_f(x) = \mathcal{N}_f(y)$

Monotonicity: $c_f(x) \leq c_f(y)$ for all $x, y \in X$ with $\mathcal{N}_f(x) \subseteq \mathcal{N}_f(y)$.

Bottleneck congestion games generalize congestion games in the definition of the cost functions on the facilities. For bottleneck congestion games, the only requirements are that the cost $c_f(x)$ on facility f for strategy profile x only depends on the set of players using f in their strategy profile and that costs are increasing with larger sets.

Definition 6 (Bottleneck congestion game). Let $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$ be a congestion model. The corresponding bottleneck congestion game is the strategic game $G(\mathcal{M}) = (N, X, \pi)$ in which π is defined as $\pi = \times_{i \in N} \pi_i$ and $\pi_i(x) = \max_{f \in x_i} c_f(x)$.

A bottleneck congestion game with $|x_i| = 1$ for all $x_i \in X_i$ and $i \in N$ will be called *singleton bottleneck congestion game*. We remark that our condition "Independence of Irrelevant Choices" is weaker than the one frequently used in the literature. In Konishi et al. [7,17,18], the definition of "Independence of Irrelevant Choices" requires that the strategy spaces are symmetric and, given a strategy profile $x = (x_1, \dots, x_n)$, the utility of a player i depends only on her own choice x_i and the cardinality of the set of other players who also choose x_i . Our definition does neither require symmetry of strategies, nor that the utility of player i only depends on the set-cardinality of other players who also choose x_i . For the relationship between games considered by Konishi et al. and congestion games, see the discussion in Voorneveld et al. [31].

We are now ready to state our main result concerning bottleneck congestion games, providing a large class of games that satisfies the π -LIP.

Theorem 3. Let $G(\mathcal{M})$ be a bottleneck congestion game with allocation model \mathcal{M} . Then, G fulfills the LIP for the functions $\phi : X \rightarrow \mathbb{R}_+^n$ and $\psi : X \rightarrow \mathbb{R}_+^{mn}$ defined as

$$\phi_i(x) = \pi_i(x) \quad \text{for all } i \in N, \quad \psi_{i,f}(x) = \begin{cases} c_f(x) & \text{if } f \in x_i \\ 0 & \text{else} \end{cases} \quad \text{for all } i \in N, f \in F.$$

The proof is given in the appendix.

Note that the function $v : X \rightarrow \mathbb{R}^m$ with $x \mapsto (c_f(x))_{f \in F}$ does not fulfill $v(x) > v(y)$ for all $(x, y) \in I$. However, this property is satisfied if facility cost functions are strictly monotonic.

As a corollary of Theorem 3 we obtain that each bottleneck congestion game has the π -LIP and hence possesses the SFIP. In addition, the results on price of stability, Pareto optimality and min-max-fairness apply. We now proceed by giving some examples of bottleneck congestion games.

4.1 Scheduling Games

A special case of bottleneck congestion games are scheduling games on identical, restricted, related and unrelated machines. These games are allocation games by restricting the strategy space for every player to singletons and defining the appropriate cost functions on the facilities.

Corollary 6. *Scheduling games on identical, restricted, related and unrelated machines are bottleneck congestion games.*

The existence of SNE (as well as efficiency properties of SNE) has been established before by Andelman et al. [3] by arguing that the lexicographically minimal schedule is a strong Nash equilibrium.

4.2 Resource Allocation in Wireless Networks

Interference games are motivated by resource allocation problems in wireless networks. Consider a set of n terminals that want to connect to one out of m available base stations. Terminals assigned to the same base station impose interferences among each other as they use the same frequency band. We model the interference relations using an undirected interference graph $\mathcal{D} = (V, E)$, where $V = \{1, \dots, n\}$ is the set of vertices/terminals and an edge $e = (v, w)$ between terminals v, w has a non-negative weight $w_e \geq 0$ representing the level of pair-wise interference. We assume that the service quality of a base station j is proportional to the total interference $w(j)$, which is defined as $w(j) = \sum_{(v,w) \in E: x_v = x_w = j} W_{(v,w)}$.

We now obtain an interference game as follows. The nodes of the graph are the players, the set of strategies is given by $X_i = \{1, \dots, m\}, i = 1, \dots, n$, that is, the set of base stations, and the private cost function for every player is defined as $\pi_i(x) = w(x_i), i = 1, \dots, n$. Thus, interference games fit into the framework of singleton bottleneck congestion games with m facilities.

Corollary 7. *Interference games are bottleneck congestion games.*

Note that in interference games, we crucially exploit the property that cost functions on facilities depend on the set of players using the facility, that is, their *identity* determines the resulting cost. Most previous game-theoretic works addressing wireless networks only considered Nash equilibria, see for instance Liu et al. [20] and Etkin et al. [11].

4.3 Bottleneck Routing in Networks

A special case of bottleneck congestion games are bottleneck routing games. Here, the set of facilities are the edges of a directed graph $\mathcal{D} = (V, A)$. Every edge $a \in A$ has a load dependent cost function c_a . Every player is associated with a pair of vertices (s_i, t_i) (commodity) and a fixed demand $d_i > 0$ that she wishes to send along the chosen path in \mathcal{D} connecting s_i to t_i . The private cost for every player is the maximum arc cost along the path. Bottleneck routing games have been studied by Banner and Orda [5]. They, however, did not study existence of strong Nash equilibria. We state the following result.

Corollary 8. *Bottleneck routing games are bottleneck congestion games.*

This result establishes that these games possess the SFIP. To the best of our knowledge, our result establishes for the first time that bottleneck routing games possess the FIP. Banner and Orda [5] only proved that every improvement path of best-response dynamics is finite.

By using a reduction from *k-Directed Disjoint Paths*, Banner and Orda [5] proved that, given a value B , it is NP-hard to decide if a bottleneck routing game with k commodities and identical cost functions possesses a PNE x with $L_\infty(x) \leq B$. We can slightly strengthen this result by reducing from

2-Directed Disjoint Paths showing hardness already for 2 commodities. While every SNE is also a PNE, the above hardness result also carries over to SNE.

We will show that for single-commodity instances with unit demands and non-decreasing identical cost functions, there is a polynomial time algorithm for computing a SNE, see the appendix for the proof.

Theorem 4. *Consider a bottleneck routing game on a directed graph $\mathcal{D} = (V, A)$ with identical non-decreasing arc-cost functions and n players having unit demand each that have to be routed from a common source to a common sink. Then, there is a polynomial time algorithm computing a SNE.*

To the best of our knowledge, our result establishes for the first time an efficient algorithm computing a SNE in this setting. For the next result, we assume the unit cost model of arithmetic operations, that is, we assume that every arithmetic operation can be done in constant time, regardless of the required precision. Furthermore, we assume that cost functions on the facilities are bounded by a constant $C \in \mathbb{N}$ that is polynomially bounded with respect to the input size. Now, consider a bottleneck routing game G on a directed graph $\mathcal{D} = (V, A)$ with arc-costs c and constant C . By scaling every cost function of G with the factor $1/C$, we obtain the bottleneck routing game \tilde{G} . It is easy to see that SNE coincide in both games. Moreover, $c_a(x)/C \leq 1$ for all $a \in A$ and $x \in X$, thus, $(c_a(x)/C)^M \leq 1$ for every $M > 0$. This construction enables us to establish an efficient algorithm computing SNE even for non-identical arc-costs (proof can be found in the appendix).

Proposition 1. *Consider a class of bottleneck routing games on directed graphs $\mathcal{D} = (V, A)$ with convex arc-cost functions $c_a, a \in A$, that are bounded by a constant C , and players having a unit demand each that must be routed from a common source to a common sink. Then, there is a polynomial algorithm computing a SNE for every game in this class.*

Note that the SNE computed with the algorithm proposed in Proposition 1 need not to coincide with the strict Pareto optimal strategy as shown in Example 4 given in the appendix. We also illustrate in Example 5 structural differences between singleton bottleneck congestion games and bottleneck routing games.

5 Infinite Strategic Games

We now consider *infinite* strategic games in which the players' strategy sets are topological spaces and the private cost functions are defined on the product topology.

Formally, an infinite game is a tuple $G = (N, X, \pi)$, where $N = \{1, \dots, n\}$ is a set of players, and $X = X_1 \times \dots \times X_n$ is the set of pure strategies, where we assume that $X_i \subseteq \mathbb{R}^{n_i}, i \in N$ are compact sets, and $p = \sum_{i \in N} n_i$. The cost function for player i is defined by a non-negative real-valued function $\pi_i : X \rightarrow \mathbb{R}_+, i \in N$.

As in the previous section, we are interested in conditions for establishing existence of a generalized strong ordinal potential. Unfortunately, even if an infinite game has the LIP, Theorem 1 (and in particular Corollaries 1, 2, and 3) need not hold as they rely on the existence of a strictly positive parameter ϵ that is a lower bound on the minimal performance gain of a member of a coalition performing an improving move. In infinite games, however, such a constant need not exist since strategy sets are topological spaces and the minimal performance gain may be unbounded from below.

We recall a famous result of Debreu [10], who showed that the lexicographical ordering on an uncountable subset of \mathbb{R}^2 cannot be represented by a real-valued function. It is easy to derive that this also holds for the sorted lexicographical ordering as defined in Definition 3 as well. To see this, suppose there is a real-valued function α representing the sorted lexicographical order on $[0, 1] \times [0, 1]$, in particular α represents the sorted lexicographical ordering on $[2/3, 1] \times [0, 1/3]$ where the sorted lexicographical order and the lexicographical order coincide. Thus, we derive a contradiction.

This implies that a generalized strong ordinal potential need not exist in general. Still, we are able to prove existence of SNE in infinite games having the LIP for a *continuous* function $\phi : X \rightarrow \mathbb{R}_+^q$.

Theorem 5. *Let G be an infinite game that satisfies the LIP for a continuous function ϕ . Then G possesses a SNE.*

The proof can be found in the appendix. This result establishes the existence of a SNE in all infinite games $G = (N, X, \pi)$ with compact strategy spaces X that have the LIP for a continuous function $\phi : X \rightarrow \mathbb{R}_+^q$. Although the proof of Theorem 5 is constructive, the effort needed to compute a SNE in such a game is very high as it involves the calculation of up to $q!$ strategy profiles. We thus proceed by investigating some special cases of infinite games possessing the LIP and identify cases in which SNE can be computed efficiently.

5.1 Infinite Bottleneck Congestion Games

In this section, we introduce the "continuous counterpart" of bottleneck congestion games. We are given a congestion model $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$ with $X_i = \{x_{i1}, \dots, x_{in_i}\}$, $n_i \in \mathbb{N}$, $i \in N$, where as usual every x_{ij} is a subset of facilities of F .

From \mathcal{M} we derive a corresponding *infinite congestion model* $\mathcal{IM} = (N, F, X, d, \Delta, (c_f)_{f \in F})$, where $d \in \mathbb{R}_+^n$, $\Delta = \Delta_1 \times \dots \times \Delta_n$, and $\Delta_i = \{\xi_i = (\xi_{i1}, \dots, \xi_{in_i}) : \sum_{j=1}^{n_i} \xi_{ij} = d_i, \xi_{ij} \geq 0, j = 1, \dots, n_i\}$. The strategy profile $\xi_i = (\xi_{i1}, \dots, \xi_{in_i})$ of player i can be interpreted as a distribution of non-negative *intensities* over the elements in X_i satisfying $\sum_{j=1}^{n_i} \xi_{ij} = d_i$ for $d_i \in \mathbb{R}_+, i \in N$. Clearly, Δ_i is a compact subset of $\mathbb{R}_+^{n_i}$ for all $i \in N$. For a profile $\xi = (\xi_1, \dots, \xi_n)$, we define the set of used facilities of player i as $F_i(\xi) = \{f \in F : \text{there exists } j \in \{1, \dots, n_i\} \text{ with } f \in x_{ij} \text{ and } \xi_{ij} > 0\}$. We define the *load* of player i on f under profile ξ by $\xi_i^f = \sum_{x_{ij} \in X_i : f \in x_{ij}} \xi_{ij}$, $i \in N, f \in F$. In contrast to finite bottleneck congestion games, we assume that cost functions $c_f : X \rightarrow \mathbb{R}_+$ only depend on the *total load* defined as $\ell_f(\xi) = \sum_{i \in N} \xi_i^f$ and are non-decreasing.

Definition 7 (Infinite bottleneck congestion game). *Let $\mathcal{IM} = (N, F, X, d, \Delta, (c_f)_{f \in F})$ be an infinite congestion model derived from \mathcal{M} . The corresponding infinite bottleneck congestion game is the strategic infinite game $G(\mathcal{IM}) = (N, \Delta, \pi)$, where π is defined as $\pi = \times_{i \in N} \pi_i$ and $\pi_i(\xi) = \max_{f \in F_i(\xi)} c_f(\ell_f(\xi))$.*

Examples of such games are bottleneck routing game with splittable demands.

We are now ready to prove that infinite bottleneck congestion games have the π -LIP.

Theorem 6. *Let $G(\mathcal{IM}) = (N, \Delta, \pi)$ be an infinite bottleneck congestion game. Then, $G(\mathcal{IM})$ has the LIP for the functions $\phi : \Delta \rightarrow \mathbb{R}_+^n$ and $\psi : X \rightarrow \mathbb{R}_+^{mn}$ defined as*

$$\phi_i(\xi) = \pi_i(\xi), \quad \text{for all } i \in N, \quad \psi_{i,f}(\xi) = \begin{cases} c_f(\ell_f(\xi)), & \text{if } f \in F_i(\xi) \\ 0, & \text{else} \end{cases} \quad \text{for all } i \in N, f \in F.$$

The proof uses similar arguments as the proof of Theorem 3 in the previous section except, that we now use the fact that cost functions are load-dependent and non-decreasing.

We will now show that the above two functions ϕ and ψ may be discontinuous even if facility cost functions are continuous.

Example 1. Let $G(\mathcal{IM}) = (N, \Delta, \pi)$ be an infinite bottleneck congestion game with continuous cost functions c . Then, the function ϕ and ψ as defined in Theorem 6 may be discontinuous on Δ . Consider a bottleneck congestion game with one player having two facilities $\Delta_1 = \{\{r_1\}, \{r_2\}\}$ over which she has to assign a demand of size 1. The facility r_1 has a cost function equal to the load, while facility r_2 has a constant cost function equal to 2. Let $\xi^2(\epsilon) = \epsilon > 0$ be assigned on facility r_2 and the remaining demand $\xi^1(\epsilon) = 1 - \epsilon$ be assigned to r_1 . Then, for any $\epsilon > 0$ we have $\phi(\xi(\epsilon)) = 2$, while $\phi(\xi(0)) = 1$.

By assuming that cost functions are continuous and *strictly* increasing, we obtain, however, the LIP for a *continuous* function v .

Theorem 7. *Let $G(\mathcal{I}\mathcal{M}) = (N, \Delta, \pi)$ be an infinite bottleneck congestion game with strictly increasing cost functions. Then, $G(\mathcal{I}\mathcal{M})$ has the LIP for the function $v : \Delta \rightarrow \mathbb{R}_+^m$ defined as $v_f(\xi) = c_f(\ell_f(\xi))$ for all $f \in F$.*

We prove the theorem in the appendix.

This result relies on the strict monotonicity of c_f and cannot be generalized to non-strict monotonic functions. We illustrate this issue with the following example. Consider a bottleneck congestion game with two players having equal demands and two facilities $F = \{f, g\}$, where $c_f(\ell) = 10$ and $c_g(\ell) = \ell$. Both players may either choose f or g . Then $((f, f), (g, f))$ is an improving move for player 1 as his private cost decreases from 10 to 1. However $v((f, f)) = (10, 0)$ and $v((g, f)) = (10, 1)$ and hence v is not lexicographically decreasing along this improving move. This example shows that strict monotonicity is essential in the proof of the above theorem.

We overcome this problem by slightly generalizing the notion of lexicographical ordering to ordered sets that are different from (\mathbb{R}, \leq) . To this end, consider a totally ordered set $(\mathcal{A}, \leq_{\mathcal{A}})$. Similar to Definition 3, we introduce a lexicographical order on \mathcal{A} -valued vectors. For two vectors $a, b \in \mathcal{A}^q$, let \tilde{a} and \tilde{b} be two vectors that arise from a and b by ordering them w.r.t $\leq_{\mathcal{A}}$ in non-increasing order. We say that a is \mathcal{A} -lexicographically smaller than b , written $a <_{\mathcal{A}} b$ if there is $m \in \{1, \dots, q\}$ such that $\tilde{a}_i =_{\mathcal{A}} \tilde{b}_i$ for all $i < m$ and $\tilde{a}_m <_{\mathcal{A}} \tilde{b}_m$.

Consequently a strategic game satisfies the \mathcal{A} -LIP if there are $q \in \mathbb{N}$ and a function $\phi : X \rightarrow \mathcal{A}^q$ such that $\phi(x) >_{\mathcal{A}} \phi(y)$ for all $(x, y) \in I$.

The following theorem establishes the \mathcal{A} -LIP for infinite bottleneck congestion games, where $(\mathcal{A}, \leq_{\mathcal{A}}) = (\mathbb{R}^2, \leq_{lex})$ and \leq_{lex} denotes the ordinary lexicographical order (that does not involve any sorting of the entries) on \mathbb{R}^2 , that is, $(a_1, a_2) <_{lex} (b_1, b_2)$ if either $a_1 < b_1$ or $(a_1 = b_1 \text{ and } b_2 < b_2)$.

Theorem 8. *Let $(\mathcal{A}, \leq_{\mathcal{A}}) = (\mathbb{R}^2, \leq_{lex})$ and let $G(\mathcal{I}\mathcal{M}) = (N, \Delta, \pi)$ be an infinite bottleneck congestion game. Then, $G(\mathcal{I}\mathcal{M})$ has the \mathcal{A} -LIP for $\phi : \Delta \rightarrow \mathcal{A}^m$ defined as $\phi_f(\xi) = (c_f(\ell_f(\xi)), \ell_f(\xi))$ for all $f \in F$.*

The proof is similar to that of Theorem 7. Let (ξ, ν) be an improving move. Again, we denote by g one of the bottleneck facilities of that player of the coalition with highest cost before the improving move. Instead of the strict monotonicity of c_g we use that in the case $c_g(\ell(\xi)) = c_g(\ell(\nu))$, we still get $\ell(\xi) > \ell(\nu)$. Thus, the definition of \leq_{lex} implies $(c_g(\ell(\xi)), \ell(\xi)) > (c_g(\ell(\nu)), \ell(\nu))$, proving the result.

Corollary 9. *Let $G(\mathcal{I}\mathcal{M}) = (N, \Delta, \pi)$ be an infinite bottleneck congestion game with continuous cost functions. Then, $G(\mathcal{I}\mathcal{M})$ possesses an SNE.*

Consider the function ϕ as in Theorem 8. We observe that ϕ is continuous as $(c_f)_{f \in F}$ is continuous. Thus, a similar (but slightly more involved) proof as that of Theorem 5 implies that there is strategy profile $\xi \in \Delta$ (where Δ is compact) that minimizes ϕ w.r.t. the order defined on \mathcal{A} .

The above result establishes for the first time the existence of SNE for a variety of games such as scheduling games with malleable jobs, bottleneck routing games with splittable demands, etc. Note that this result gives also an alternative and constructive proof for the existence of PNE in bottleneck routing games with splittable demands compared to the proof of Banner and Orda [5].

5.2 Approximate SNE

We introduce the notion of α -approximate strong Nash equilibria for infinite games. We denote by $I^\alpha(S) \subset X \times X$ the set of tuples $(x, (y_S, x_{-S}))$ of α -improving moves for $S \subseteq N$ and define by I^α their union. A strategy profile x is an α -approximate strong Nash equilibrium if no coalition $\emptyset \neq S \subseteq N$ has an alternative strategy profile y_S such that $\pi_i(x) - \pi_i(y_S, x_{-S}) > \alpha$, for all $i \in S$. We call a function

$P : X \rightarrow \mathbb{R}$ an α -generalized strong ordinal potential if $(x, y) \in I^\alpha$ implies $P(x) - P(y) < 0$. We also define the α -lexicographical improvement property (α -LIP) and α - π -LIP similar to Definition 4, except that we replace I with I^α , respectively.

We will prove in the appendix that bottleneck congestion games with bounded cost functions possess an α -approximate SNE for every $\alpha > 0$.

Theorem 9. *Let G be an infinite bottleneck congestion game with bounded cost functions. Then, G possesses an α -approximate SNE for every $\alpha > 0$.*

Note that there is a fundamental difference to the result of Monderer and Shapley [24], who showed that every infinite game having an *exact* bounded potential (on a compact strategy space) possesses an α -PNE for every $\alpha > 0$. Monderer and Shapley use in their proof that the payoff difference of a deviating player is equal to the potential difference. This, however, is not true for generalized ordinal potentials and, in fact, we crucially exploit the combinatorial structure of infinite bottleneck congestion games in the proof of Theorem 9.

We finally show that there is a polynomial algorithm computing an α -approximate SNE for every $\alpha > 0$ for the class of bottleneck routing games with splittable flow and bounded convex cost functions, where the upper bound is polynomial in the input size.

Proposition 2. *Consider a class of splittable bottleneck routing games on multi-commodity graphs $\mathcal{D} = (V, A)$ with bounded convex arc-cost functions, that are bounded by a constant C , and players having arbitrary positive demands. Then, there is a polynomial algorithm computing an α -SNE for arbitrary $\alpha > 0$.*

Proof. The α -generalized strong ordinal potential function $P_{M(\alpha)}$ as defined in Lemma 2 is convex (as c is convex) and the flow polytope is compact. Thus, one can apply the same scaling argument as in Proposition 1 to obtain a polynomial time algorithm (e.g., the ellipsoid method) computing a splittable flow z that satisfies $P_{M(\alpha)}(z) \leq \min_{\xi \in \mathcal{A}} P_{M(\alpha)}(\xi) - \epsilon$ for arbitrary $\epsilon > 0$. \square

6 Extensions

A natural generalization of bottleneck congestion games can be obtained by assuming that players are *heterogeneous* with respect to the cost of the most expensive facility, that is, they attach different *values* to the cost of the most expensive facility. We can model this heterogeneity by introducing a player-specific function $\varpi_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that maps the cost of a facility to the private cost experienced by the player. Assuming that higher costs on facilities are associated with higher private costs, that is, ϖ_i is strictly increasing for all $i \in N$, we can actually show that these games possess the LIP (though not the π -LIP).

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Appendix

Proof of Theorem 1

Proof. We prove $1. \Rightarrow 2. \Rightarrow 3. \Rightarrow 4. \Rightarrow 5. \Rightarrow 1.$

1. \Rightarrow 2. is trivial.

2. \Rightarrow 3.: We use that every directed acyclic graph possesses a topological order, which gives rise to a generalized ordinal potential by simply assigning real-valued labels to strategy profiles according to their topological order.

3. \Rightarrow 4.: Let Q be a generalized ordinal potential function and let $Q_{\min} = \min_{x \in X} Q(x)$. We define $\phi(x) = Q(x) + Q_{\min}$ and it follows that G together with ϕ has the LIP.

4. \Rightarrow 5.: Let $(x, (y_S, x_{-S})) \in I(S), S \subseteq N$ be arbitrary and let ϕ be as in the definition of LIP. We will show that there is a constant $M > 0$ such that $P(x) - P(y_S, x_{-S}) = \sum_{i \in N} \phi_i(x)^{M'} - \phi_i(y_S, x_{-S})^{M'} > 0$ for all $M < M'$. To this end, we denote by $\tilde{\phi}(x)$ and $\tilde{\phi}(y_S, x_{-S})$ the vectors that arise by sorting $\phi(x)$ and $\phi(y_S, x_{-S})$ in non-increasing order. As $\phi(y_S, x_{-S}) < \phi(x)$, there is an index $m \in \{1, \dots, q\}$ such that $\tilde{\phi}_i(x) = \tilde{\phi}_i(y_S, x_{-S})$ for all $i < m$ and $\tilde{\phi}_m(x) < \tilde{\phi}_m(y_S, x_{-S})$. We then obtain

$$\begin{aligned} P_{M'}(x) - P_{M'}(y_S, x_{-S}) &= \sum_{i=1}^q \phi_i(x)^{M'} - \sum_{i=1}^q \phi_i(y_S, x_{-S})^{M'} \\ &= \tilde{\phi}_m(x)^{M'} - \tilde{\phi}_m(y_S, x_{-S})^{M'} + \sum_{i=m+1}^q \tilde{\phi}_i(x)^{M'} - \sum_{i=m+1}^q \tilde{\phi}_i(y_S, x_{-S})^{M'} \\ &\geq \tilde{\phi}_m(x)^{M'} - \tilde{\phi}_m(y_S, x_{-S})^{M'} - (q-m)\tilde{\phi}_m(y_S, x_{-S})^{M'} \\ &\geq \tilde{\phi}_m(x)^{M'} - q\tilde{\phi}_m(y_S, x_{-S})^{M'}. \end{aligned} \tag{1}$$

Standard calculus shows that the expression on the right hand side of (1) is positive if

$$M' > \log(q) / (\log(\tilde{\phi}_m(x)) - \log(\tilde{\phi}_m(y_S, x_{-S}))) > 0.$$

Clearly, M' depends on $(x, y) \in I$, but as the number of improvement steps is finite, we may chose $M := \max_{(x, y) \in I} M'((x, y))$ and get the claimed result.

5. \Rightarrow 1. is trivial. □

Proof of Corollary 2

Proof. In the proof of Theorem 1 we need $M > \log(q) / (\log(\tilde{\phi}_m(x)) - \log(\tilde{\phi}_m(y)))$ for every improving move $(x, y) \in I$. Here, m is the first index such that $\tilde{\phi}_m(x) > \tilde{\phi}_m(y)$. The mean value theorem implies that $\log(\tilde{\phi}_m(x)) - \log(\tilde{\phi}_m(y)) = (\tilde{\phi}_m(x) - \tilde{\phi}_m(y)) / \xi$ for some $\xi \in (\tilde{\phi}_m(x), \tilde{\phi}_m(y))$ and hence

$$M > \log(q) \frac{\phi_{\max}}{\epsilon_{\min}} \geq \frac{\log(q) \tilde{\phi}_m(x)}{\tilde{\phi}_m(x) - \tilde{\phi}_m(y)} \geq \frac{\log(q)}{\log(\tilde{\phi}_m(x)) - \log(\tilde{\phi}_m(y))},$$

establishing the result. □

Proof of Theorem 2

Proof. Let $p, q \in \mathbb{N}$ and $p < q$. As the L_p -norm is decreasing in p and by Hölder's inequality, we get $L_q(x) \leq L_p(x) \leq \sqrt[q-p]{n} L_q(x)$, and for the special case $q = \infty$ we get $L_\infty(x) \leq L_p(x) \leq \sqrt[p]{n} L_\infty(x)$. The latter inequality implies that $\lim_{p \rightarrow \infty} L_p(x) = L_\infty(x)$.

As G satisfies the π -LIP, $P_M(x) = \sum_{i \in N} \pi(x)^M$ is a generalized strong ordinal potential of G for all M large enough and hence the minimum x^* of P_M over X is a SNE. As the M th root is a monotone

function, x^* minimizes $L_M(x) = \sqrt[M]{\sum_{i \in N} \pi(x)^M}$ as well. As X is finite, we can choose M large enough so that the strategy profile x^* minimizes $L_\infty(x)$.

For proving the second claim, let x^* be a strong Nash equilibrium minimizing the generalized strong ordinal potential P_M and $L_M(x) = \sqrt[M]{\sum_{i \in N} \pi(x)^M}$ and let y be a strategy profile minimizing L_p . Then, we derive the inequalities $L_p(x^*) \leq \sqrt[M]{n^{M-p}} L_M(x^*) \leq \sqrt[M]{n^{M-p}} L_M(y) \leq \sqrt[M]{n^{M-p}} L_p(y)$. The second inequality is valid as x^* is a potential minimizer. Thus, $L_p(x^*) \leq n^{1-p/M} L_p(y) < n L_p(y)$, which proves the claimed result. \square

In the following we provide an example of a class of games with the π -LIP whose parameters can be chosen in a way such that the price of stability w.r.t. L_p is arbitrarily close to $\sqrt[p]{n}$, implying that the result of Theorem 2 w.r.t L_1 is tight.

Example 2 (Price of stability). We consider the game $G = (N, X, \pi)$ with $N = \{1, \dots, n\}$ with $X_1 = X_2 = \{0, 1\}$ and $X_i = \{0\}$ for $3 \leq i \leq n$. Private costs are shown in Fig. 1a.

It is straightforward to check that this game has the π -LIP. The unique SNE is the strategy profile $(0, \dots, 0)$ realizing a private cost vector of $(k - \epsilon, \dots, k - \epsilon)$. For any $p \in \mathbb{N}$, there is $\epsilon > 0$ such that $L_p(\cdot)$ is maximized in strategy profiles $(1, 0, 0, \dots, 0)$ realizing a cost vector of $(k, 0, \dots, 0)$. Hence the price of stability approaches $\sqrt[p]{n}$ arbitrarily close.

So far, our results concern the price of stability only. The next example shows that games with the π -LIP may have a price of anarchy that is unbounded.

Example 3 (Unbounded price of anarchy). Consider the game $G = (N, X, \pi)$ with $N = \{1, 2\}$, $X_1 = X_2 = \{0, 1\}$ and private costs given in Fig. 1b for any $k > 0$. It is straightforward to check that this game has the π -LIP and that both $(0, 0)$ and $(1, 1)$ are SNE. Hence, the price of anarchy w.r.t. any L_p norm is unbounded from above.

Proof of Corollary 5

We establish the result by proving the following Lemma.

Lemma 1. *Let G be a finite strategic game having the π -LIP. If the strategy profile x minimizes P_M as defined in Theorem 1 with $\phi = \pi$, then x is min-max fair.*

Proof. Let x minimize P_M as defined in Theorem 1 with $\phi = \pi$. Assume by contradiction that there is another strategy profile y such that $\pi_i(y) < \pi_i(x)$ for some $i \in N$ and the following two statements hold:

1. $\pi_j(y) \leq \pi_j(x)$ for all $j \in N/\{i\}$ with $\pi_j(x) \geq \pi_i(x)$
2. $\pi_j(y) < \pi_j(x)$ for all $j \in N/\{i\}$ with $\pi_j(x) < \pi_i(x)$.

Then, we can observe that every entry of $\pi(x)$, which is above $\pi_i(x)$ only decreases under y , while every entry strictly below $\pi_i(x)$ may only increase to a value strictly below the threshold $\pi_i(x)$. Because the value $\pi_i(x)$ strictly decreases under y , we obtain $P_M(x) - P_M(y) > 0$ for M large enough, contradicting the minimality of x . \square

	0	1		0	1	
0	$(k - \epsilon, k - \epsilon, k - \epsilon, \dots, k - \epsilon)$	(k, k, k, \dots, k)		0	$(0, 0)$	$(0, k)$
1	$(k, 0, 0, \dots, 0)$	$(k, \epsilon, k, \dots, k)$		1	(k, k)	$(0, k)$
	a)			b)		

Fig. 1. a) Private costs received by the players for strategy profiles $X_1 \times X_2$ of the game considered in Example 2. **b)** A game with unbounded price of anarchy w.r.t. any L_p -norm.

Proof of Theorem 3

Proof. We first prove the claim for ψ . Consider an improving move $(x, (y_S, x_{-S})) \in I$. Let $j \in S$ be a member of the coalition with highest cost before the improvement step, i.e., $j \in \arg \max_{i \in S} \pi_i(x)$. We set $\Psi^+ := \{(i, f) \in N \times F : \psi_{i,f}(x) \geq \pi_j(x)\}$ and claim that $\psi_{i,f}(x) \geq \psi_{i,f}(y_S, x_{-S})$ for all $(i, f) \in \Psi^+$. To see this, suppose there is $(k, g) \in \Psi^+$ such that $\psi_{k,g}(x) < \psi_{k,g}(y_S, x_{-S})$. The independence of irrelevant choices and the monotonicity of cost functions imply that a member $i \in S$ of the coalition uses g . So

$$\pi_j(x) \geq \pi_i(x) > \pi_i(y_S, x_{-S}) \geq \psi_{k,g}(y_S, x_{-S}),$$

which contradicts $(k, g) \in \Psi^+$.

Now we define $\Psi^- := \{(i, f) \in N \times F : \psi_{i,f}(x) < \pi_j(x)\}$ and claim that $\psi_{i,f}(y_S, x_{-S}) < \pi_j(x)$ for all $(i, f) \in \Psi^-$. To see this, suppose there is $(k, g) \in \Psi^-$ such that $\psi_{k,g}(y_S, x_{-S}) \geq \pi_j(x)$. Because of the monotonicity of the cost functions and the independence of irrelevant choices, there is a member $i \in S$ of the coalition using g in (y_S, x_{-S}) giving rise to

$$\pi_j(x) \geq \pi_i(x) > \pi_i(y_S, x_{-S}) \geq \psi_{k,g}(y_S, x_{-S}) \geq \pi_j(x),$$

which is a contradiction.

We remark that $N \times F = \Psi^+ \cup \Psi^-$ and that we have shown that $\psi_{i,f}(x) - \psi_{i,f}(y_S, x_{-S}) \geq 0$ for all $(i, f) \in \Psi^+$ and $\psi_{i,f}(y_S, x_{-S}) \leq \psi_{k,g}(y_S, x_{-S})$ for all $(i, f) \in \Psi^-$ and $(k, g) \in \Psi^+$. Since $j \in S$ and $\pi_j(x) > \pi_j(y_S, x_{-S})$, there exists $(j, f) \in \Psi^+$ with $\psi_{j,f}(x) > \psi_{j,f}(y_S, x_{-S})$. Hence, $\psi(x) > \psi(y_S, x_{-S})$, finishing the first part of the proof.

To prove the LIP for ϕ , we must show that $(\pi_i)_{i \in N} > (\pi_i(y_S, x_{-S}))_{i \in N}$ for all $(x, (y_S, x_{-S})) \in I$. We again consider a player $j \in \arg \max_{i \in S} \pi_i(x)$ and decompose the set of players into the sets

$$N^+ := \{i \in -S : \pi_i(x) \geq \pi_j(x)\} \quad N^- := \{i \in N : \pi_i(x) < \pi_j(x)\}.$$

A similar argument as in the first part shows that $\pi_i(x) \geq \pi_i(y_S, x_{-S})$ for all $i \in N^+$ and that $\pi_i(y_S, x_{-S}) < \pi_j(x)$ for all $i \in N^-$, establishing the result. \square

Proof of Theorem 4

Proof. We assign a uniform capacity of 1 to each arc $a \in A$. With the polynomial algorithm of Edmonds and Karp we obtain both a minimum (s, t) -cut C , and a maximum flow x . Let us say that $C := \{a_1, \dots, a_m\}$ for some $m \in \mathbb{N}$. As we may assume without loss of generality that x is integer and all capacities are 1, we may decompose the flow x into m arc-disjoint paths \mathcal{P}_j .

We set $k = m \lfloor \frac{n}{m} \rfloor - n$ and consider the strategy profile x in which $\lfloor \frac{n}{m} \rfloor$ players are routed along each path $P_j, j = 1, \dots, k$, and $\lceil \frac{n}{m} \rceil$ players are routed along each of the other paths. As all arcs have the same arc-cost function, we derive that $\pi_i(x) = \max_{a \in x_i} c_a(x) = c_{a_j}(x)$ for some $1 \leq j \leq m$. Since every other flow traverses the cut C , there can be no deviation of a coalition S that is profitable to all of its members, that is, strictly reduces their private costs. \square

Proof of Proposition 1

Proof. Instead of computing a SNE in the original game, we consider the game \tilde{G} with arc-costs $\tilde{c}_a(x) = c_a(x)/C$. Obviously, each strategy profile $x \in X$ establishes an integral (s, t) -flow with value n . Conversely, each such flow can be decomposed into n paths starting in s and ending in t , see [2]. So, there is a one-to-one correspondence between strategy profiles and integral (s, t) -flows with value n .

Let $n_a(x)$ denote the number of players using facility a under strategy profile x . As the bottleneck routing game has the π -LIP for the function ψ defined in Theorem 3, the function $P_M(x) :=$

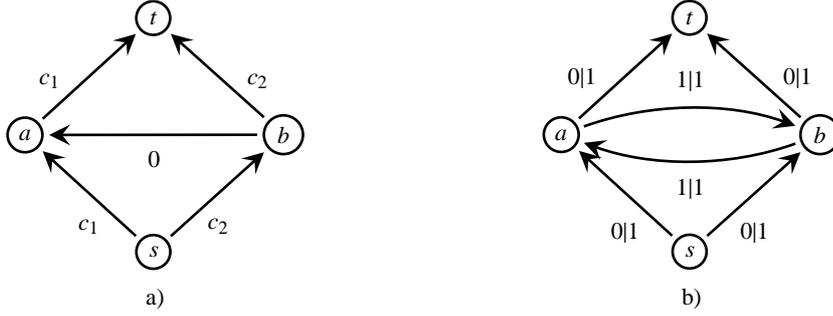


Fig. 2. a) Bottleneck routing game with multiple SNE b) Bottleneck routing game with non-strong PNE

$\sum_{a \in A} \tilde{c}_a(x)^M n_a(x)$ is a generalized strong potential function of the bottleneck routing game, see the construction of the generalized ordinal potential in Theorem 1. We can rewrite the potential using flow variables $x : \mathbb{R}^{|A|} \rightarrow \mathbb{R}_+$ and obtain $P_M(x) := \sum_{a \in A} \tilde{c}_a(x_a)^M x_a$, where x_a denotes the total flow on arc a . Moreover, note that the optimization problem $\min_{x \in X} P_M(x)$ of computing a minimal integral flow with convex arc-cost can be solved in polynomial time (given $P_M(x)$), see Ahuja et al. [2]. The optimal solution x^* of this problem minimizes the generalized strong potential function P_M and hence is a SNE. \square

Examples 4 and 5

Example 4. Consider the symmetric bottleneck routing game with players set $N = \{1, \dots, n\}$ depicted in Fig. 2. The strategy set X_i of each player $i \in N$ comprises all paths from s to t , that are $P_1 := \{(sa), (at)\}$, $P_2 := \{(sb), (bt)\}$ and $P_3 := \{(sb), (ba), (bt)\}$. In addition, we consider the cost functions

$$c_1(\ell) = \begin{cases} 0 & \text{if } \ell < n \\ 1 & \text{else,} \end{cases} \quad c_2(\ell) = \begin{cases} 1 & \text{if } \ell \leq 1 \\ 2 & \text{else.} \end{cases}$$

and the graph depicted in Fig. 2a.

There are two types of SNE. In the first type, $n-1$ players play P_1 and one player plays P_2 . The players on P_1 experience a cost of 0 while the single player on P_2 experiences a cost of 1. Thus, the sum of all costs equals 1. In the other type of SNE, again $n-1$ players play P_1 while one player plays P_3 . Hence all players experience a cost of 1 and the total cost of all players sums up to n . The two types of SNE actually correspond to the global minima of the generalized strong ordinal potential functions derived from the π -LIP w.r.t. ϕ and ψ as defined in Theorem 3.

Example 5. Consider the instance in Fig. 2b and assume there are two players with unit demand each. One can easily see that this instance admits a PNE (routing both demands along the two zig-zag paths) that is not a SNE. This example contrasts a result of Holzman and Law-Yone [15], who have shown that for singleton congestion games, every PNE is also a SNE. Thus, allowing more complex strategies (paths instead of single facilities) makes a structural difference.

Proof of Theorem 5

Proof. By assumption there exists $q \in \mathbb{N}$ and a function $\phi : X \rightarrow \mathbb{R}_+^q$ such that $\phi(y_S, x_{-S}) < \phi(x)$ for all $(x, (y_S, x_{-S})) \in I$. We will show that there exists $x_{\min} \in X$ with $\phi(x_{\min}) \leq \phi(y)$ for all $x, y \in X$. Our proof is constructive and proceeds in q phases. In the first phase, we solve the following program

$$\min_{x \in X} \alpha \text{ s.t. : } \phi_i(x) \leq \alpha, \text{ for all } i \in \{1, \dots, q\}. \quad (P_1)$$

Note that continuity of ϕ implies that the half-space $H_1 = \{x \in X : \phi_i(x) \leq \alpha\}$ is compact. To see this, observe that $H_1 = \phi_i^{-1}((-\infty, \alpha]) = (\phi_i^{-1}((\alpha, \infty)))^C$, where ϕ_i^{-1} denotes the pre-image of ϕ and $(A)^C$ denotes the complement of A with respect to X . As ϕ is continuous and (α, ∞) is open $\phi_i^{-1}((\alpha, \infty))$ is open and hence H_1 is closed. Hence, H_1 is a closed subset of the compact set X and thus compact. As the objective of (P_1) is continuous, the minimum is attained in H_1 with value $\alpha_1 \in \mathbb{R}_+$. Let $A_1 \subseteq X$ denote the set of optimal solutions and let $B_1 = \{j \in \{1, \dots, q\} : \text{there exists } x^* \in A_1 \text{ with } \phi_j(x^*) = \alpha_1\}$ denote the set of indices for which the optimal value is attained. Clearly, B_1 is non-empty. If B_1 contains a single element, say j , the lexicographical minimum x_{\min} fulfills $\phi_j(x_{\min}) = \alpha_1$ and we can proceed by solving

$$\min_{x \in X} \alpha \quad \text{s.t.: } \phi_j(x) = \alpha_1, \phi_i(x) \leq \alpha, \quad \text{for all } i \in \{1, \dots, q\} \setminus \{j\}. \quad (P_2^j)$$

If, in contrast, B_1 contains more than one element, we solve program P_2^j for every $j \in B_1$. Continuing this way we obtain at most $q!$ different solution vectors $\phi \in \mathbb{R}_+^q$. Taking the lexicographically smallest among them, we obtain x_{\min} , which is a SNE. \square

Proof of Theorem 7

Proof. We choose a deviating player $j \in \arg \max_{i \in S} \pi_i(x)$ with highest cost before the improving move and one of the facilities $g \in \arg \max_{f \in x_j} c_f(x)$ at which $\pi_j(x)$ is attained. Decompose F into F^+ and F^- defined as $F^+ := \{f \in F : c_f(x) \geq c_g\}$ and $F^- := \{f \in F : c_f(x) < c_g\}$. We claim that $c_f(y_S, x_{-S}) \leq c_f(x)$ for all $f \in F^+$ and that $c_f(y_S, x_{-S}) < c_g(x)$ for all $f \in F^-$, which establishes the result with similar arguments as in the proof of Theorem 3. Note that we use here that c_g is strictly decreasing as player j changes her strategy profile from x_j to y_j . \square

Proof of Theorem 9

We prove the theorem by stating a useful lemma.

Lemma 2. *Let the function $\psi : \Delta \rightarrow \mathbb{R}_+^{mn}$ be defined as*

$$\psi_{i,f}(\xi) = \begin{cases} c_f(\ell_f(\xi)), & \text{if } f \in F_i(\xi) \\ 0, & \text{else} \end{cases} \quad \text{for all } i \in N, f \in F.$$

Moreover, let $\alpha > 0$ and define $P_{M(\alpha)}(\xi) := \sum_{f \in F, i \in N} \psi_{i,f}(\xi)^{M(\alpha)}$, where $M(\alpha) \geq (2\psi_{\max}/\alpha + 1) \log(nm)$ and $\psi_{\max} := \sup_{\xi \in X, 1 \leq f \leq m} c_f(\ell_f(\xi))$. Then, $P_{M(\alpha)}$ is an α -generalized ordinal potential function satisfying

$$P_{M(\alpha)}(\xi) - P_{M(\alpha)}(\xi') \geq \left(\frac{\alpha}{2}\right)^{M(\alpha)} \text{ for all } (\xi, \xi') \in I^\alpha.$$

Proof. We must show that $P_{M(\alpha)}(\xi) - P_{M(\alpha)}(v_S, \xi_{-S}) \geq \left(\frac{\alpha}{2}\right)^{M(\alpha)}$ for an arbitrary α -improving move $(\xi, (v_S, \xi_{-S})) \in I^\alpha$. Let $j \in S$ with $j \in \arg \max_{i \in S} \pi_i(v_S, \xi_{-S})$.

We define

$$\Psi^+ := \{(i, f) \in -S \times F : \psi_{i,f}(\xi) \geq \pi_j(v_S, \xi_{-S})\} \text{ and } \Psi^- := \{(i, f) \in -S \times F : \psi_{i,f}(\xi) < \pi_j(v_S, \xi_{-S})\}.$$

Claim. 1. $\psi_{i,f}(\xi) \geq \psi_{i,f}(v_S, \xi_{-S})$ for all $(i, f) \in \Psi^+$

2. $\psi_{i,f}(v_S, \xi_{-S}) \leq \pi_j(v_S, \xi_{-S})$ for all $(i, f) \in \Psi^-$.

Proof. To prove the first claim, suppose there is $(k, g) \in \Psi^+$ such that $\psi_{k,g}(\xi) < \psi_{k,g}(v_S, \xi_{-S})$. Because of the monotonicity of cost functions there exists $i \in S$ with $g \in F_i(g)$ implying

$$\pi_j(v_S, \xi_{-S}) \leq \psi_{k,g}(\xi) < \psi_{k,g}(v_S, \xi_{-S}) \leq \pi_i(v_S, \xi_{-S}) \leq \pi_j(v_S, \xi_{-S}),$$

which is a contradiction.

For proving the second claim, suppose there is $(k, g) \in \Psi^-$ such that $\psi_{k,g}(v_S, \xi_{-S}) > \pi_j(v_S, \xi_{-S})$. Again, monotonicity of cost functions implies that there is $i \in S$ with $g \in F_i(g)$ giving rise to

$$\pi_i(v_S, \xi_{-S}) \geq \psi_{k,g}(v_S, \xi_{-S}) > \pi_j(v_S, \xi_{-S}) \geq \pi_i(v_S, \xi_{-S}),$$

which is a contradiction. This proves the claim. \square

We observe that $N \times F = \Psi^+ \cup \Psi^- \cup (F \times S)$.

Then,

$$\begin{aligned} P_{M(\alpha)}(\xi) - P_{M(\alpha)}(v_S, \xi_{-S}) &= \sum_{f \in \Psi^+ \cup \Psi^- \cup (F \times S)} \psi_{i,f}(\xi)^{M(\alpha)} - \psi_{i,f}(\xi, v_{-S})^{M(\alpha)} \\ &\geq \sum_{f \in \Psi^- \cup (F \times S)} \psi_{i,f}(\xi)^{M(\alpha)} - \psi_{i,f}(v_S, \xi_{-S})^{M(\alpha)}. \end{aligned}$$

The inequality follows from the first claim. We further derive

$$\begin{aligned} \sum_{f \in \Psi^- \cup (F \times S)} \psi_{i,f}(\xi)^{M(\alpha)} - \psi_{i,f}(v_S, \xi_{-S})^{M(\alpha)} &\geq \sum_{f \in (F \times S)} \psi_{i,f}(\xi)^{M(\alpha)} - \sum_{f \in \Psi^- \cup (F \times S)} \psi_{i,f}(v_S, \xi_{-S})^{M(\alpha)} \\ &\geq (\pi_j(v_S, \xi_{-S}) + \alpha)^{M(\alpha)} - (nm)\pi_j(v_S, \xi_{-S})^{M(\alpha)}, \end{aligned}$$

where the first inequality follows from the non-negativity of ψ . The second inequality follows from $\pi_j(\xi) \geq \pi_j(v_S, \xi_{-S}) + \alpha$ and the second claim. To this end, we obtain

$$\begin{aligned} P_{M(\alpha)}(\xi) - P_{M(\alpha)}(v_S, \xi_{-S}) &\geq (\alpha/2)^{M(\alpha)} + (\pi_j(v_S, \xi_{-S}) + \alpha/2)^{M(\alpha)} - (nm)\pi_j(v_S, \xi_{-S})^{M(\alpha)} \\ &\geq (\alpha/2)^{M(\alpha)}, \end{aligned}$$

where the last inequality follows from the choice of $M(\alpha)$. \square

Proof (Proof of Theorem 9). Fix $\alpha > 0$. Then, since Δ is compact and $P_{M(\alpha)}$ (as defined in Lemma 2) is bounded, there exists a strategy profile z satisfying $P_{M(\alpha)}(z) \leq \inf_{\xi \in \Delta} P_{M(\alpha)}(\xi) - \epsilon$ with $0 < \epsilon < (\frac{\alpha}{2})^{M(\alpha)}$. We claim that z is an α -approximate SNE. Suppose not. Then there exists a profitable deviation $v_S \in \Delta_S$ with $P_{M(\alpha)}(z) - P_{M(\alpha)}(v_S, z_{-S}) \geq (\alpha/2)^{M(\alpha)} > \epsilon$ (by Lemma 2), which contradicts the approximation guarantee of z . \square