Computation of State Reachable Points of Linear Time Invariant Descriptor Systems

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Abstract

This paper considers the problem of computing the state reachable points, from the origin, of a linear constant coefficient first or higher order descriptor system. A method is proposed that allows to compute the reachable set in a numerically stable way. The original descriptor system is transformed into a strangeness-free system within the behavioral framework followed by a projection that separates the system into differential and algebraic equations while keeping the original state variables. For first order systems it is shown that the computation of the image space of two matrices, associated with the projected system, is enough to compute the reachable set (from the origin). Moreover, a characterization is presented of all the inputs by which one can reach an arbitrary point in the reachable set. The results are extended to second order systems and the effectiveness of the proposed approach is demonstrated through some elementary examples.

Keywords: Linear time invariant descriptor system, behavior formulation, strangeness-free formulation, reachability, derivative array, second order system

MSC Subject classification: 93C05, 93C15, 93B05

1 Introduction

Due to the wide availability of automated modeling tools such as Dymola [13], Matlab Simscape [20], or Spice [1], the modeling of dynamical systems via descriptor systems, where the system equation is a differential-algebraic equation (DAE) has become the industrial standard in many application domains.

In the linear time-invariant case descriptor systems have the form

$$E\dot{x} = Ax + Bu, \tag{1}$$

where, if algebraic constraints are present, $E \in \mathbb{R}^{\ell \times n}$ does not have full row rank, $A \in \mathbb{R}^{\ell \times n}$, and $B \in \mathbb{R}^{\ell \times m}$, $x : \mathbb{R} \to \mathbb{R}^{\ell}$ is the state vector and $u : \mathbb{R} \to \mathbb{R}^m$ is an input (control) to the system. Typically, also an initial condition $x(t_0) = x_0$ is considered. In this paper we only discuss the case of square coefficient matrices E, A, i.e., we assume $\ell = n$ and we also assume that the pencil $\lambda E - A$ is regular, i.e., there exists a complex number λ such $\det(\lambda E - A) \neq 0$. It should be noted, however, that the systems generated in automated modeling tools are often over- and under-determined. This case can, however, be reduced to the regular square case, see [9] for a detailed remodeling and regularization procedure.

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Practical examples of descriptor systems are, e.g., power system models, where the differential equations describe the dynamic behavior of electrical machines and other energy storage components while the algebraic constraints represent the power flow in the network [23]. Similarly, in spacecraft and robot trajectory planning, in addition to the differential equations, algebraic relations define specific trajectories [5]. When modeling constrained mechanical systems like cranes and earth moving vehicles, a servo-constraint is described via an algebraic relation [4]. For further examples, see [9, 11, 16].

Descriptor systems are different from ordinary state-space systems in the sense that the system integrates and differentiates. The number of differentiations that are needed is usually described by an *index*. There are many index concepts, see [21] for a comparison. In all cases (except for different counting) the index characterizes the smoothness requirements of the inhomogeneity, i.e., in descriptor systems that of the control function in the range space of B. If the inputs are piecewise continuous in this space, then due to the differentiation, impulsive responses may arise [10, 11, 16, 25, 29]. This should be avoided and from the control theoretic perspective, an important question is to determine for a given system a control input which transfers from one state (say the origin) to another state in finite time. A related question is to compute, for a given descriptor system, the set of all state reachable points via some chosen set of input functions.

One of the traditional ways to carry out the analysis of descriptor systems is to transform the system into an ordinary state space system governed by an ordinary differential equation (ODE). This is typically done by resolving all the algebraic equations. For instance, in linearized power systems, an ODE model is obtained from a given descriptor system by first expressing the algebraic variables in terms of the remaining variables and then eliminating them from the system, see e.g. the procedure in [23, 26]. However, in this way the algebraic constraints are not available any longer in the dynamical system and the system may violate the constraints when numerical methods are used for simulation and control, see a discussion of several drawbacks in [9, 16]. Solving the algebraic equations gives very bad results, in particular, when the solution of the algebraic equations is ill-conditioned, i.e., small perturbations lead to drastic changes.

Another classical approach for the analysis of descriptor systems, presented, e.g., in [11], is to use the Weierstraß canonical form to decouple the original system into the fast (algebraic) and slow (differential) subsystem. But the transformation to Weierstraß canonical form may be arbitrarily ill-conditioned [6], with the effect that small perturbations (such as measurement or round-off errors) in the data of the original descriptor system may lead to largely perturbed subsystems (slow and fast). An alternative to the computation of the Weierstraß canonical form is to use so-called Wong sequences, see e.g. the recent work [3], but also this approach is not suited for numerical implementation.

To avoid the discussed difficulties, in [6] a numerically stable approach is presented that derives a staircase form [12, 28] of a least generic system close to the original system under orthogonal transformations, via a sequence of rank decisions. However, this is very subtle in finite precision arithmetic, and a complete error analysis is not available.

In this paper we discuss a *derivative array approach* [8, 9, 16] which takes the point of view that, since derivatives of equations are needed, it is best to differentiate the original equations and not equations with numerically computed quantities. For this approach, one adds a sufficient number of derivatives of the original descriptor to the system, so that the system becomes overdetermined, but all necessary derivative information is available. From the derivative array then two orthogonal projections are computed that allow to identify the differential and the algebraic equations. In this way inconsistent initial conditions can be identified and possibly be made consistent. The derivative array approach is very robust to perturbations, and it has been implemented successfully in numerical simulation codes for linear and nonlinear differential algebraic systems [16]. Another surplus of this approach is that no changes of basis are carried out and thus the physical meaning of all the variables is preserved in the equivalent system. This is particularly important when the physical control has a direct influence on the original physical variables.

For the derivative array approach we first express the original system in a behavior framework, making no distinction between the variables as states and inputs [24]. Then, as in [17] we construct an equivalent system (with the same solution set), called *strangeness-free* behavior system, by

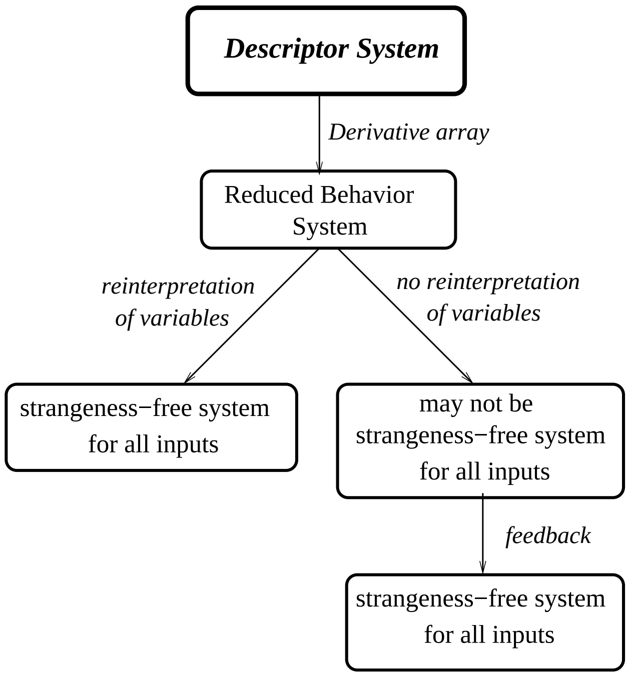

Figure 1: Pictorial diagram to obtain strangeness-free descriptor system.

using the derivative array consisting of the original descriptor system and its derivatives. The minimum number of derivatives required to obtain a strangeness-free behavior model is referred to as *strangeness-index* of the behavior system [16, 17]. From this derivative array then, via orthogonal transformations from the left, we filter out the separate sets of differential and algebraic equations, respectively. This new system is then strangeness-free in the behavior sense but in general not as a free system (with $u = 0$). However, since the behavior system is strangeness-free, it is *impulse controllable (or controllable at infinity)* [11], and so there exists a state feedback to make the system strangeness-free [6]. To identify the reachable set from this system, we use a recently developed projection representation [2] to obtain the solution as the sum of the solution of a purely differential and a purely algebraic system.

A pictorial diagram to obtain a strangeness-free system is shown in Figure 1. To illustrate the procedure we present the following example.

Example 1.1 Consider the descriptor system $E\dot{x} = Ax + Bu$, given by

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Solving the second equation yields $x_2 = -u$ and inserting it into the first equation yields $x_1 = x_2 + \dot{x}_2 = -u - \dot{u}$ and, hence, if the input is, e.g., a Heaviside function, then the solution contains an impulse. However, via an appropriate state feedback this behavior can be changed. Setting $u = u_f + \widetilde{u}$, with a feedback $u_f = Fx$, where $F = [1 \; -1]$, we obtain the closed loop system $E\dot{x} = (A + BF)x + B\widetilde{u}$ given by

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \widetilde{u}.$$

Here we obtain $x_1 = -\tilde{u}$ and inserting this, the dynamics of the system is characterized by the standard state space system $\dot{x}_2 = -x_2 - \tilde{u}$, which does not contain any impulsive behavior.

The paper is organized as follows. In Section 2 we briefly discuss the *derivative array* approach to obtain a strangeness-free behavior model associated with the original descriptor system. The coefficient matrices associated with the transformed system are used in Section 2.1 to define two projection matrices that are used to separate the descriptor system additively into its differential and algebraic parts. We discuss the computation of the state reachable points of the original descriptor system in Section 3. The results are then extended to second order systems in Section 4. Some elementary model problems, a circuit example and a mechanical multi-body system, are presented in Section 5 followed by concluding remarks in Section 6.

2 Strangeness-free Formulation of Descriptor Systems

In this section we briefly recall the construction of a *strangeness-free* behavior model associated with (1) via the *derivative array* approach. For this we formally rewrite (1) as

$$\mathbf{E}\dot{z}(t) = \mathbf{A}z(t), \tag{2}$$

where

$$\mathbf{E} := [E \;\; 0] \in \mathbb{R}^{n\times(n+m)}, \quad \mathbf{A} := [A \;\; B] \in \mathbb{R}^{n\times(n+m)}, \tag{3}$$

by introducing a new variable $z = \begin{bmatrix} x \\ u \end{bmatrix}$. For the time being let us formally assume that x and u are at least $\mu > 0$ times differentiable, we will see later that we can drop this assumption.

By performing a sequence of differentiations of (2) and stacking the original system and its derivatives up to order μ on top of each other, we get a *derivative array*

$$\mathcal{M}_\mu \dot{z}_\mu = \mathcal{N}_\mu z_\mu, \tag{4}$$

where

$$\mathcal{M}_\mu \;=\; \begin{bmatrix} \mathbf{E} & 0 & \cdots & 0 & 0 \\ -\mathbf{A} & \mathbf{E} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\mathbf{A} & \mathbf{E} \end{bmatrix}; \; \mathcal{N}_\mu = \begin{bmatrix} \mathbf{A} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}; \;\; z_\mu = \begin{bmatrix} z \\ \dot{z} \\ \vdots \\ z^{(\mu)} \end{bmatrix}.$$

The process of adding derivatives in (4) is continued until one is able to determine integers d and a with $d + a = n$ such that $(\mathcal{M}_\mu, \mathcal{N}_\mu)$ in (4) has the following properties, see [16].

1. We have $\operatorname{rank}(\mathcal{M}_\mu) = (\mu+1)n - a$, which implies the existence of a matrix $Z_2 \in \mathbb{R}^{(\mu+1)n\times a}$ with orthonormal columns and maximal rank a, satisfying $Z_2^T \mathcal{M}_\mu = 0$.

2. Setting
$$\widehat{A}_2 := Z_2^T \mathcal{N}_\mu \begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix} \in \mathbb{R}^{a\times(n+m)},$$
we have $\operatorname{rank}(\widehat{A}_2) = a$, and there exists $T_2 \in \mathbb{R}^{(n+m)\times d}$ with orthonormal columns and maximal rank satisfying $\widehat{A}_2 T_2 = 0$.

3. We have $\operatorname{rank}(\mathbf{E}T_2) = d$, which implies the existence of a matrix $Z_1 \in \mathbb{R}^{n\times d}$ with orthonormal columns and maximal $\operatorname{rank}(\widehat{E}_1) = d$, such that $\widehat{E}_1 = Z_1^T \mathbf{E} \in \mathbb{R}^{d\times(n+m)}$.

The smallest $\mu \geq 0$ for which these conditions hold is called the *strangeness-index* of (2). Once the orthonormal matrices Z_1, Z_2 and T_2 are constructed, we obtain a strangeness-free system in behavior form associated with (1) given by

$$\begin{bmatrix} \widehat{E}_1 \\ 0 \end{bmatrix} \dot{z} = \begin{bmatrix} \widehat{A}_1 \\ \widehat{A}_2 \end{bmatrix} z \tag{5}$$

where the coefficients $\widehat{E}_1 \in \mathbb{R}^{d\times(n+m)}$, $\widehat{A}_1 \in \mathbb{R}^{d\times(n+m)}$, $\widehat{A}_2 \in \mathbb{R}^{a\times(n+m)}$ are given by

$$\widehat{E}_1 = Z_1^T \mathbf{E},\ \widehat{A}_1 = Z_1^T \mathbf{A},\ \widehat{A}_2 = Z_2^T \mathcal{N}_\mu \begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix} \in \mathbb{R}^{a\times(n+m)}.$$

Moreover, the differential-algebraic system (2) and the strangeness-free system (5) have the same solution set.

Example 2.1 Consider the descriptor system $E\dot{x} = Ax + Bu$ from Example 1.1, which is already strangeness-free in the behavior sense and thus with $Z_2 = [0\ 1]^T$ and $Z_1 = [1\ 0]^T$ the strangeness-free behavior system is obtained.

Once the strangeness-free behavior system is available, it is possible to check the consistency of initial conditions, which is the case if $x(t_0) = x_0$ is consistent with the algebraic equation $\widehat{A}_2 z_0 = 0$ for all possible input functions u. If this is not the case, then the input function will be restricted in its initial value through the equation $\widehat{A}_2 z(t_0) = \widehat{A}_2 [x_0^T \ \ u(t_0)^T]^T = 0$.

Using (3) and the fact that no changes of variable have been performed, the strangeness-free system (5) can be represented as

$$\begin{aligned} E_1\dot{x} &= A_1 x + B_1 u, && \text{(6a)} \\ 0 &= A_2 x + B_2 u, && \text{(6b)} \end{aligned}$$

where $E_1 = Z_1^T E \in \mathbb{R}^{d\times n}$, and

$$\begin{aligned} A_1 &= \widehat{A}_1 \begin{bmatrix} I_n \\ 0 \end{bmatrix} \in \mathbb{R}^{d\times n},\ \ A_2 = \widehat{A}_2 \begin{bmatrix} I_n \\ 0 \end{bmatrix} \in \mathbb{R}^{a\times n}, \\ B_1 &= \widehat{A}_1 \begin{bmatrix} 0 \\ I_m \end{bmatrix} \in \mathbb{R}^{d\times m},\ \ B_2 = \widehat{A}_2 \begin{bmatrix} 0 \\ I_m \end{bmatrix} \in \mathbb{R}^{a\times m}. \end{aligned}$$

Note that the resulting transformed system (6) may not be strangeness-free as a free system with $u = 0$, but it is *impulse controllable* (or *controllable at infinity*), which means that $\operatorname{rank}[E, AS_\infty, B] = n$, where the columns of the matrix S_∞ span the kernel of E. Under this condition it is well-known [6] that there exist a state feedback $u = Fx$ such that the closed loop system $E\dot{x} = (A + BF)x$ is strangeness-free. This will be important in the context of computing the set of *reachable points* of system (1).

Definition 2.2 *A vector $x_1 \in \mathbb{R}^n$ is said to be* reachable *from the initial condition $x(t_0) = x_0$, if there exists a (sufficiently smooth) control input u and a finite time $t_1 > 0$ such that $x(t_1) = x_1$.*

Note that for reachability, the initial condition has to be consistent and the set of initial values for the control may be restricted as well, because state and input together have to satisfy the algebraic constraint in the strangeness-free behavior formulation (6).

In the following section we will use a projection method to determine the reachable set of system (1). More details on projector based analysis of differential algebraic systems can be found in [18].

2.1 An Equivalent Projected System

Let us rewrite (6) as

$$\mathcal{E}\dot{x} = \mathcal{A}x + \mathcal{B}u,$$

where $\mathcal{E} = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}$, $\mathcal{A} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, and define the two subspaces

$$\mathbb{E}_d := \operatorname{Im}(\mathcal{E}^T),\ \mathbb{E}_a := \ker(\mathcal{E}),$$

where Im and ker denote the *image* and *kernel*, respectively. Note that $\mathbb{E}_d$ and $\mathbb{E}_a$ are orthogonal complements of each other. Corresponding to these two subspaces, we can now partition the state

x additively into two parts x_d and x_a, which we will call *differential and algebraic parts* of x, respectively. To obtain x_d and x_a, let us define the two projectors

$$P_d = \mathcal{E}^+\mathcal{E}, \quad \text{and} \quad P_d' = I - \mathcal{E}^+\mathcal{E}, \tag{7}$$

where $\mathcal{E}^+$ is the Moore-Penrose inverse of $\mathcal{E}$, [14]. Note that P_d is an orthogonal projector onto the subspace $\mathbb{E}_d$, whereas P_a is an orthogonal projector onto the subspace $\mathbb{E}_a$. Setting

$$x_d := P_d x, \; x_a(t) = P_d' x, \tag{8}$$

then, according to definition (8), we have

$$x = x_d + x_a.$$

In addition, let us define another projector Q onto $\mathrm{Im}(\mathcal{E})$ via

$$Q := \mathcal{E}\mathcal{E}^+ = \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Q' = I - \mathcal{E}\mathcal{E}^+. \tag{9}$$

Then, we have the following result.

Theorem 2.3 *[2] Let the projectors P_d and P_d' be defined as in (7) and the variables x_d and x_a as in (8). Then, $x = x_d + x_a$ is a solution of (1) if and only if x_d and x_a are solutions of the system*

$$\dot{x}_d = G_d x_d + B_d u, \tag{10a}$$
$$x_a = G_a x_d + B_a u, \tag{10b}$$

where

$$G_a := -(Q'\mathcal{A}P_d')^+(Q'\mathcal{A}P_d), \; G_d := \mathcal{E}^+\mathcal{A}(P_d + G_a), \tag{11a}$$
$$B_a := -(Q'\mathcal{A}P_d')^+\mathcal{B}, \; B_d := \mathcal{E}^+\mathcal{B} + \mathcal{E}^+\mathcal{A}B_a. \tag{11b}$$

Moreover, an initial value x_0 is consistent if and only if it satisfies

$$(P_d' - G_a)x_0 = B_a u(t_0)$$

at the initial time t_0.

The proof of the Theorem 2.3 follows from the proof for general differential-algebraic systems, presented in [2]. Theorem 2.3 states that a solution x of (1) can be computed by solving system (10) and forming $x = x_d + x_a$. Hence, the set of reachable points of system (1) can be determined by computing the state reachable points of system (10), i.e., a vector $x_1 := x_{d_1} + x_{a_1}$ is reachable from a consistent initial condition $x(t_0) = x_0$, if there exists a (sufficiently smooth) control input u and a finite time $t_1 > 0$ such that $x_d(t_1) = x_{d_1}$ and $x_a(t_1) = x_{a_1}$.

3 Computing the Reachable Set

Following the discussion in Section 2, it is clear that a reachable point x_1 of (1) can be determined by computing its differential complement x_{d_1} and algebraic component x_{a_1} via the relations given in (10). In this section, we first compute the differential component x_{d_1} of a reachable point x_1 from the relation $\dot{x}_d = G_d x_d + B_d u$. Then we use $x_a = G_a x_d + B_a u$ to determine the algebraic component x_{a_1} of x_1.

Let us assume that the initial state x_0 of (1) is consistent. Then, the differential component x_{d_0} of x_0 would be $x_{d_0} = P_d x_0$, which is an initial condition for the standard state space system

$\dot{x}_d = G_d x_d + B_d u$. The differential component x_d of a state trajectory x, i.e., the solution of the differential equation in (10) then has the form

$$x_d = e^{G_d(t-t_0)} x_{d_0} + \int_{t_0}^{t} e^{G_d(t-\tau)} B_d u(\tau)\, d\tau \tag{12}$$

and the resulting algebraic component x_a of x is

$$x_a = G_a \left[e^{G_d(t-t_0)} x_{d_0} + \int_{t_0}^{t} e^{G_d(t-\tau)} B_d u(\tau)\, d\tau \right] + B_a u.$$

This shows that the state responses x_d and x_a are uniquely determined by the initial condition x_0, the control input $u(\tau)$ for $t_0 \le \tau \le t$ and the initial time t_0. If the initial time is $t_0 = 0$ and the initial condition is $x_0 = 0$, then we have $x_d(0) = P_d x_0 = 0$ and hence the solution (12) is given by

$$x_d = \int_{0}^{t} e^{G_d(t-\tau)} B_d u(\tau)\, d\tau. \tag{13}$$

Let us define an operator $\mathcal{L}_d(u,t)$ via

$$\mathcal{L}_d(u,t) := \int_{0}^{t} e^{G_d(t-\tau)} B_d u(\tau)\, d\tau,$$

then the set of reachable points x_{d_1} from the origin is essentially the set of points in the image space of $\mathcal{L}_d(u,t)$ (denoted as $\mathrm{Im}[\mathcal{L}_d(u,t)]$) which, in the following result, we show to be the image space of the *symmetric Gramian* matrix $W(p,t)$ given by

$$W(p,t) := \int_{0}^{t} p(\tau)^2 e^{G_d(t-\tau)} B_d B_d^T e^{G_d^T(t-\tau)}\, d\tau, \tag{14}$$

where $p(\tau)$ is a polynomial which is not identically zero.

Lemma 3.1 *Let $W(p,t)$ be as in* (14)*. Then, we have*

$$\mathrm{Im}[W(p,t)] = \mathrm{Im}[\mathcal{L}_d(u,t)].$$

Proof. We show first that $\mathrm{Im}[W(p,t)] \subseteq \mathrm{Im}[\mathcal{L}_d(u,t)]$. Consider a vector x_{d_1} such that $x_{d_1} \in \mathrm{Im}[W(p,t)]$, i.e., there exists a vector $w \in \mathbb{R}^n$ such that $W(p,t)w = x_{d_1}$. Choosing the input

$$u(\tau) = p^2(\tau) B_d^T e^{G_d^T(t-\tau)} w,\ \ 0 \le \tau \le t,$$

we have

$$\begin{aligned}\mathcal{L}_d(u,t) &= \left[\int_{0}^{t} p(\tau)^2 e^{G_d(t-\tau)} B_d B_d^T e^{G_d^T(t-\tau)}\, d\tau \right] w \\ &= W(p,t)w = x_{d_1},\end{aligned}$$

and hence, $x_{d_1} \in \mathrm{Im}[\mathcal{L}_d(u,t)]$. Since x_{d_1} is chosen arbitrarily, it follows that $\mathrm{Im}[W(p,t)] \subseteq \mathrm{Im}[\mathcal{L}_d(u,t)]$.

To show the converse implication, $\mathrm{Im}[\mathcal{L}_d(u,t)] \subseteq \mathrm{Im}[W(p,t)]$, let $x_{d_1} \in \mathrm{Im}[\mathcal{L}_d(u,t)]$. Then there exists an input u_1 such that $\mathcal{L}_d(u_1,t) = x_{d_1}$. Suppose that $x_{d_1} \notin \mathrm{Im}[W(p,t)]$, then $\ker[W(p,t)]$ is non-empty and we can find a vector $k \in \ker[W(p,t)]$ such that $k^T x_{d_1} \neq 0$, i.e. $k^T W(p,t) k = 0$. This implies that

$$\int_{0}^{t} \left[k^T p(\tau) e^{G_d(t-\tau)} B_d \right] \left[k^T p(\tau) e^{G_d(t-\tau)} B_d \right]^T d\tau = 0,$$

and therefore

$$\int_{0}^{t} \| k^T p(\tau) e^{G_d(t-\tau)} B_d \|_2^2\, d\tau = 0,$$

which means that $k^T p(\tau) e^{G_d(t-\tau)} B_d = 0$ for all $\tau \in [0,\ t]$. Since $p(\tau)$ is not identically zero, we obtain

$$k^T e^{G_d(t-\tau)} B_d = 0, \text{ for all } \tau \in [0,t],$$

which implies that

$$k^T x_{d_1} = k^T \mathcal{L}_d(u_1, t) = \int_0^t k^T e^{G_d(t-\tau)} B_d u_1 \, d\tau = 0,$$

which is a contradiction, since k was chosen so that $k^T x_{d_1} \neq 0$. Hence $\mathrm{Im}[\mathcal{L}_d(u,t)] \subseteq \mathrm{Im}[W(p,t)]$ and the proof is complete. According to Lemma 3.1 the set of all reachable points x_{d_1} (from the origin) due to the state response $x_d(t)$ is essentially the set of points which belong to $\mathrm{Im}[W(p,t)]$. Moreover, by defining the subspace

$$\mathcal{X}_d := \mathrm{Im}\,(\mathcal{C}_d)\,, \tag{15}$$

where $\mathcal{C}_d = \begin{bmatrix} B_d & G_d B_d & \cdots & G_d^{n-1} B_d \end{bmatrix}$ is the controllability matrix of the dynamical part, it follows from the proof of [11] that

$$\mathrm{Im}[W(p,t)] = \mathcal{X}_d. \tag{16}$$

So far we have characterized the reachability set for the dynamical part, but we also need to discuss the restrictions that are forced by the algebraic part. Recall that the algebraic component of a state x is determined by $x_a = G_a x_d + B_a u$. Hence, to compute the algebraic component x_{a_1} of a reachable point x_1, assume that x_{d_1} is a reachable point due to the state response x_d for some input u_1 at a finite time $t_1 > 0$. Then, at t_1, the algebraic component x_{a_1} is given by

$$x_{a_1} = G_a x_{d_1} + B_a u_1(t_1).$$

Since $x_{d_1} \in \mathcal{X}_d$, we can write $x_{d_1} = \mathcal{C}_d z$ for some $z \in \mathbb{R}^{nm}$. Hence, x_{a_1} takes the form

$$x_{a_1} = G_a \mathcal{C}_d z + B_a c,$$

where $c = u_1(t_1) \in \mathbb{R}^m$ and this determines the algebraic component of a reachable point x_1. Defining the set

$$\mathcal{X}_a := \{x_{a_1} \in \mathbb{R}^n \mid x_{a_1} = G_a \mathcal{C}_d z + B_a c,\ c = u(t_1)\}\,, \tag{17}$$

where $z \in \mathbb{R}^{mn}$ and u is an input to system (10), then, $\mathcal{X}_a$ consists of all the algebraic components of the reachable points x_1. Note that $\mathcal{X}_a = \mathrm{Im}[G_a \mathcal{C}_d \quad B_a]$, and thus we have the complete characterization of the set $\mathcal{R}_0$ of reachable points of system (1) from an initial condition $x_0 = 0$ at $t_0 = 0$.

Theorem 3.2 *Let $\mathcal{X}_d$ and $\mathcal{X}_a$ be as in* (15) *and* (17)*, respectively. Then,*

$$\mathcal{R}_0 = \mathcal{X}_d + \mathcal{X}_a = \{x_{d_1} + x_{a_1} \mid x_{d_1} \in \mathcal{X}_d,\ x_{a_1} \in \mathcal{X}_a\}\,.$$

Proof. We will show first, that $\mathcal{R}_0 \subseteq \mathcal{X}_d + \mathcal{X}_a$. If $x_1 \in \mathcal{R}_0$, then there exists an input u and a finite time t_1 such that $x_d(t_1) = x_{d_1}$ and $x_a(t_1) = x_{a_1}$. In addition to this, $x_1 = x_{d_1} + x_{a_1}$. Hence, we only have to show that $x_{d_1} \in \mathcal{X}_d$ and $x_{a_1} \in \mathcal{X}_a$.

Since $x_d = \int_0^t e^{G_d(t-\tau)} B_d u(\tau)\, d\tau$, and since it is well known, see e.g., [11], that corresponding to a matrix $G_d \in \mathbb{R}^{n\times n}$ there exists continuous functions $\alpha_i(t)$, $i = 1, 2, \cdots, n-1$, such that

$$e^{G_d t} = \alpha_0(t) I + \alpha_1(t) G_a + \cdots + \alpha_{n-1}(t) G_a^{n-1}, \tag{18}$$

we obtain

$$\begin{aligned} x_d(t) &= \int_0^t \alpha_0(t-\tau) B_d u(\tau)\, d\tau + \int_0^t \alpha_1(t-\tau) G_d B_d u(\tau)\, d\tau + \\ &\qquad \cdots + \int_0^t \alpha_{n-1}(t-\tau) G_d^{n-1} B_d u(\tau)\, d\tau \\ &= \begin{bmatrix} B_d & G_d B_d & \cdots & G_d^{n-1} B_d \end{bmatrix} \begin{bmatrix} \int_0^t \alpha_0(t-\tau) u(\tau)\, d\tau \\ \int_0^t \alpha_1(t-\tau) u(\tau)\, d\tau \\ \vdots \\ \int_0^t \alpha_{n-1}(t-\tau) u(\tau)\, d\tau \end{bmatrix}. \end{aligned}$$

This implies that $x_{d_1} \in \mathcal{X}_d$. Since $x_{d_0} = 0$, the algebraic component x_a in (13) has the form

$$\begin{aligned} x_a &= G_a \left[\int_0^t e^{G_d(t-\tau)} B_d u(\tau)\, d\tau \right] + B_a u(t) \\ &= [G_a \mathcal{C}_d \quad B_a] \begin{bmatrix} \int_0^t \alpha_0(t-\tau) u(\tau)\, d\tau \\ \int_0^t \alpha_1(t-\tau) u(\tau)\, d\tau \\ \vdots \\ \int_0^t \alpha_{n-1}(t-\tau) u(\tau)\, d\tau \\ u(t) \end{bmatrix}. \end{aligned}$$

Therefore, $x_{a_1} \in \mathcal{X}_a$, which proves that $\mathcal{R}_0 \subseteq \mathcal{X}_d + \mathcal{X}_a$.

To show that $\mathcal{X}_d + \mathcal{X}_a \subseteq \mathcal{R}_0$, let us consider a vector $x_1 \in \mathcal{X}_d + \mathcal{X}_a$. Then, according to the definition of $\mathcal{X}_d + \mathcal{X}_a$, we can find $x_{d_1} \in \mathcal{X}_d$ and $x_{a_1} \in \mathcal{X}_a$ such that $x_1 = x_{d_1} + x_{a_1}$. Hence, we only have to show that there exists an input u and a finite time $t_1 > 0$ such that $x_d(t_1) = x_{d_1}$ and $x_a(t_1) = x_{a_1}$.

Since $x_{d_1} \in \mathcal{X}_d$, it follows form (16) that $x_{d_1} \in \mathrm{Im}\,[W(p,t)]$. Hence, we can find a vector $w \in \mathbb{R}^n$ such that $x_{d_1} = W(p,t)w$. Let us choose $p(\tau) = \tau^\mu$ for some fixed $\tau > 0$. Then, corresponding to the chosen w, to obtain x_{d_1}, we construct an input via

$$u = \tau^{2\mu} B_d^T e^{G_d^T(t-\tau)} w, \ 0 \le \tau \le t. \tag{19}$$

The resulting state response x_d then is given by

$$\begin{aligned} x_d &= \int_0^t e^{G_d(t-\tau)} B_d u(\tau)\, d\tau \\ &= \left[\int_0^t \tau^\mu e^{G_d(t-\tau)} B_d B_d^T e^{G_d^T(t-\tau)} \tau^\mu \, d\tau \right] w. \end{aligned} \tag{20}$$

Since $p(\tau) = \tau^\mu$ is not identically zero, the matrix $W(p,t)$ is well defined, see also (14), and it follows from (20) that $x_d = W(p,t)w = x_{d_1}$.

Now, since $x_{a_1} \in \mathcal{X}_a$, there exist two vectors $z \in \mathbb{R}^{mn}$ and $c \in \mathbb{R}^m$ such that $x_{a_1} = G_a \mathcal{C}_d z + B_a c$. Corresponding to the input u defined in (19), the algebraic component of the state variable x then is

$$\begin{aligned} x_a(t) &= G_a \int_0^t e^{G_d(t-\tau)} B_d u(\tau)\, d\tau + B_a u(t) \\ &= G_a \left[\int_0^t \tau^\mu e^{G_d(t-\tau)} B_d B_d^T e^{G_d^T(t-\tau)} \tau^\mu \, d\tau \right] w + t^{2\mu} B_a B_d^T e^{G_d^T(t-t)} w \\ &= G_a W(p,t) w + t^{2\mu} B_a B_d^T w \\ &= G_a x_{d_1} + t^{2\mu} B_a B_d^T w. \end{aligned}$$

Setting $c = t^{2\mu} B_d^T w$, we have $x_a(t) = G_a x_{d_1} + B_a c$, hence we only have to show that there exists a vector z such that $\mathcal{C}_d z = x_{d_1}$. Since x_{d_1} is in the image space of $W(p,t)$, and since $\mathrm{Im}(\mathcal{C}_d) = \mathrm{Im}[W(p,t)]$, we can find z such that $\mathcal{C}_d z = x_{d_1}$, and the proof is complete. Theorem 3.2 characterizes the reachable set of the descriptor system (1) from the origin. In addition, the characterization of all the inputs u by which one can reach an arbitrary point in the reachable set is given by (19). Since, $\mathcal{X}_d = \mathrm{Im}(\mathcal{C}_d)$ and $\mathcal{X}_a = \mathrm{Im}[G_a \mathcal{C}_d \quad B_a]$, the set $\mathcal{R}_0$ can be directly computed using the matrices defined in (11). Moreover, the dimension of the subspace $\mathcal{R}_0$ is $\dim(\mathcal{R}_0) = \dim(\mathcal{X}_d) + \dim(\mathcal{X}_a)$, where dim refers to the dimension of a subspace.

When we start from an arbitrary consistent initial condition $x(t_0) = x_0$, then by performing a time shift and considering $x - x_0$ we can apply Theorem 3.2 and obtain as reachable set the affine space $x_0 + \mathcal{R}_0$.

In the following sections we extend these results to compute the set of reachable points for a second order descriptor system.

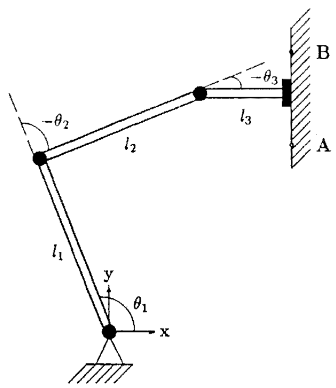

Figure 2: Three link mobile manipulator model [15].

4 Second Order Descriptor Systems

Consider a second order linear constant coefficient descriptor system of the form

$$M\ddot{x} + D\dot{x} + Kx = Bu, \tag{21}$$

where $M, D, K \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x : \mathbb{R} \to \mathbb{R}^n$ is the state and $u : \mathbb{R} \to \mathbb{R}^m$ is input (control) to the system. Furthermore, initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = \hat{x}_0$ are considered. Here, i general the matrix M is singular.

Example 4.1 A simple example of such a descriptor system is the model of a two-dimensional, three-link mobile manipulator, from [15]. The task of the manipulator is to clean a surface by moving the end-effector with a specified contact force. Assume that the flat cleaning surface is a rigid body and the end of the third arm is smooth and a rigid plate. Then the constraints on the manipulator are the restrictions of the motion in the x direction and the orthogonality of the third arm to the cleaning surface. A linearized model of the manipulator in the Cartesian coordinates ([15]) has the form:

$$\begin{aligned} M_0\ddot{\delta} + D_0\dot{\delta} + K_0\delta &= F_0^T\Lambda + S_0u, \\ F_0\delta &= 0, \end{aligned}$$

where M_0 is the mass matrix, D_0 describes the damping, K_0 is the stiffness matrix, Λ is a Lagrange multiplier, and $F_0^T\Lambda$ is the generalized constraint force. Setting $x = \begin{bmatrix}\delta^T & \Lambda^T\end{bmatrix}^T$, we can represent the system as the second order DAE

$$\begin{bmatrix} M_0 & 0 \\ 0 & 0 \end{bmatrix}\ddot{x} + \begin{bmatrix} D_0 & 0 \\ 0 & 0 \end{bmatrix}\dot{x} + \begin{bmatrix} K_0 & -F_0^T \\ F_0 & 0 \end{bmatrix} x = \begin{bmatrix} S_0 \\ 0 \end{bmatrix} u.$$

The traditional approach to deal with second order systems is to transform it to a first order system by introducing a new variable $v = \dot{x}$. However, this approach has several drawbacks, which we illustrate via the following example.

Example 4.2 Consider the second order descriptor system from [7],

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\ddot{x} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\dot{x} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u,$$

which has the solution

$$x_1 = b_2u, \quad x_2 = b_1u - b_2(u + \dot{u} + \ddot{u}).$$

If $b_2 \neq 0$ and if the input u is twice continuously differentiable, then we will get a continuous solution, but if we introduce a new variable $v = \dot{x}$ and transfer the second order system to first order, then the solution is

$$\begin{aligned} x_1 &= b_2 u, \\ x_2 &= b_1 u - b_2 (u + \dot{u} + \ddot{u}), \\ v_1 &= b_2 \dot{u}, \\ v_2 &= b_1 \dot{u} - b_2 \left(\dot{u} + \ddot{u} + u^{(3)} \right), \end{aligned}$$

and hence for the existence of a continuous solution, u has to be three times differentiable. One solution to avoid this dilemma is to only introduce $v_1 = b_2 \dot{u}$ as a new variable. This process is called *trimmed linearization*, see [7].

In Example 4.2 we have seen that the transformation to first order may require extra smoothness of the input function, and it may happen that there is no continuous solution of the first order formulation, even though there exist continuous solutions to the original second order system. In fact, if we consider the three-link manipulator, then the velocity component in the x direction is restricted, and hence it is not necessary to introduce the velocity component in that direction.

In the following, we distinguish between the set of state values that can be reached for the second order descriptor system (21) and the set of state values that can be reached for a first order formulation of the second order system, see also [19].

Definition 4.3 *Consider the second order descriptor system* (21). *A vector $x_1 \in \mathbb{R}^n$ is said to be* reachable from the initial condition $x(t_0) = x_0$, *if there exists a (sufficiently smooth) control input u and a finite time $t_1 > 0$ such that $x(t_1) = x_1$.*

A first order formulation of (21) *with an extended state vector $w_1 := [x_1^T \;\; v_{t_1}^T]^T \in \mathbb{R}^{n+\tilde{n}}$ is said to be* reachable from the initial condition $w(t_0) := [x(t_0)^T,\; v_{t_1}(t_0)^T]^T$, *if there exists a (sufficiently smooth) control input u and a finite time $t_1 > 0$ such that $w(t_1) = w_1$.*

To characterize the reachable set of system (21) and also to transform the second order system (21) to first order without introducing further smoothness requirements, we again first regularize the strangeness-free second order system by a derivative array approach. Then, we transform the resulting second order system to first order by following the *trimmed linearization* procedure introduced in [7].

To obtain a strangeness-free behavior model associated with (21) via the *derivative array* approach, see [22, 27], we rewrite (21) in behavior form as

$$\mathbf{M}\ddot{z}(t) + \mathbf{D}\dot{z}(t) = \mathbf{K}z(t), \tag{22}$$

where $\mathbf{M} := [M \;\; 0] \in \mathbb{R}^{n\times(n+m)}$, $\mathbf{D} := [D \;\; 0] \in \mathbb{R}^{n\times(n+m)}$, $\mathbf{K} := [-K \;\; B] \in \mathbb{R}^{n\times(n+m)}$, by introducing the new variable $z = \begin{bmatrix} x^T & u^T \end{bmatrix}^T$. Then, by performing a sequence of differentiations of (22) and stacking all equations on top of each other, we get

$$\mathcal{M}_\mu \ddot{z}_\mu + \mathcal{L}_\mu \dot{z}_\mu = \mathcal{N}_\mu z_\mu, \tag{23}$$

where

$$\mathcal{M}_\mu = \begin{bmatrix} \mathbf{M} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mathbf{D} & \mathbf{M} & 0 & \cdots & 0 & 0 & 0 \\ -\mathbf{K} & \mathbf{D} & \mathbf{M} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\mathbf{K} & \mathbf{D} & \mathbf{M} \end{bmatrix}, \quad \mathcal{L}_\mu = \begin{bmatrix} \mathbf{D} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\mathbf{K} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{N}_\mu = \begin{bmatrix} \mathbf{K} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \quad z_\mu = \begin{bmatrix} z \\ \dot{z} \\ \ddot{z} \\ \vdots \\ z^{(\mu)} \end{bmatrix}.$$

We then obtain a strangeness-free model (in behavior form) by using projection matrices Z_0, Z_1, Z_2 and Z_3 (see [27] and also Appendix 7) such that

$$\begin{bmatrix} \widehat{M}_1 \\ 0 \\ 0 \end{bmatrix} \ddot{z} + \begin{bmatrix} \widehat{D}_1 \\ \widehat{D}_2 \\ 0 \end{bmatrix} \dot{z} = \begin{bmatrix} \widehat{K}_1 \\ \widehat{K}_2 \\ \widehat{K}_3 \end{bmatrix} z, \tag{24}$$

where

$$\begin{aligned} \widehat{M}_1 &= Z_0^T \mathbf{M} \in \mathbb{R}^{d_1 \times (n+m)}, \quad \widehat{D}_1 = Z_0^T \mathbf{D} \in \mathbb{R}^{d_1 \times (n+m)}, \quad \widehat{K}_1 = Z_0^T \mathbf{K} \in \mathbb{R}^{d_1 \times (n+m)}, \\ \widehat{D}_2 &= Z_1^T Z_2^T \mathcal{L}_\mu \begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix} \in \mathbb{R}^{d_2 \times (n+m)}, \quad \widehat{K}_2 = Z_1^T Z_2^T \mathcal{N}_\mu \begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix} \in \mathbb{R}^{d_2 \times (n+m)}, \\ \widehat{K}_3 &= Z_3^T \mathcal{N}_\mu \begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix} \in \mathbb{R}^{a \times (n+m)}. \end{aligned}$$

In the strangeness-free system (24), the matrices $\widehat{M}_1$, $\widehat{D}_2$ and $\widehat{K}_3$ are of full row rank and, furthermore, also the matrix

$$\widehat{H} := \begin{bmatrix} \widehat{M}_1 \\ \widehat{D}_2 \\ \widehat{K}_3 \end{bmatrix} \in \mathbb{R}^{n,n+m} \tag{25}$$

is of full row rank n. Moreover, (24) can be represented (in the original variables) as

$$\begin{aligned} M_1 \ddot{x} + D_1 \dot{x} + K_1 x &= B_1 u, & \text{(26a)} \\ D_2 \dot{x} + K_2 x &= B_2 u, & \text{(26b)} \\ K_3 x &= B_3 u, & \text{(26c)} \end{aligned}$$

where M_i, D_i and K_i, B_i are obtained from $\widehat{M}_i$, $\widehat{D}_i$ and $\widehat{K}_i$, respectively, by inserting the partitioned vector z and taking all state terms to the left..

System (26) may, however, not be strangeness-free as a free system with $u = 0$, since the matrix

$$H := \begin{bmatrix} M_1 \\ D_2 \\ K_3 \end{bmatrix} \in \mathbb{R}^{n,n}$$

may not be of full row rank n. But since the behavior system is strangeness-free, there exists a feedback $u = -G\dot{x} - Fx + \tilde{u}$ such that in the closed loop system

$$\begin{aligned} M_1 \ddot{x} + (D_1 + B_1 G)\dot{x} + (K_1 + B_1 F)x &= B_1 \tilde{u}, & \text{(27a)} \\ (D_2 + B_2 G)\dot{x} + (K_2 + B_2 F)x &= B_2 \tilde{u}, & \text{(27b)} \\ (K_3 + B_3 F)x &= B_3 \tilde{u}, & \text{(27c)} \end{aligned}$$

$$H_F := \begin{bmatrix} M_1 \\ D_2 + B_2 G \\ K_3 + B_3 F \end{bmatrix} \in \mathbb{R}^{n,n}$$

is nonsingular.

System (27) allows to transform the second order system into a first-order equivalent system that does not need unnecessary smoothness requirements. For this, we determine an orthogonal matrix W (via a singular value or QR decomposition with pivoting [14]) such that in

$$H_F W = \begin{bmatrix} \tilde{M}_{11} & 0 & 0 \\ \tilde{D}_{21} & \tilde{D}_{22} & 0 \\ \tilde{K}_{31} & \tilde{K}_{32} & \tilde{K}_{33} \end{bmatrix}$$

the matrices $\tilde{M}_{11} \in \mathbb{R}^{d_1,d_1}$, $\tilde{D}_{22} \in \mathbb{R}^{d_2,d_2}$, $\tilde{K}_{33} \in \mathbb{R}^{a,a}$ are square and nonsingular. Then, by performing a change of basis, i.e., by setting $x = Ww$, we obtain $M_1W = [\tilde{M}_{11}\, 0\, 0]$, $(D_1 + B_1G)W = [\tilde{D}_{11}\, \tilde{D}_{12}\, \tilde{D}_{13}]$, $(D_2 + B_2G)W = [\tilde{D}_{21}\, \tilde{D}_{22}\, 0]$ and $(K_j + B_jF)W = [\tilde{K}_{j1}\, \tilde{K}_{j2}\, \tilde{K}_{j3}]$, $j = 1, 2, 3$. In this setting, system (27) can be represented as

$$\begin{bmatrix} \tilde{M}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \\ \ddot{w}_3 \end{bmatrix} + \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}_{13} \\ \tilde{D}_{21} & \tilde{D}_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} + \begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} & \tilde{K}_{13} \\ \tilde{K}_{21} & \tilde{K}_{22} & \tilde{K}_{23} \\ \tilde{K}_{31} & \tilde{K}_{32} & \tilde{K}_{33} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \tilde{u}.$$

By introducing a new variable $v = \dot{w}_1$ we obtain the strangeness-free first order system

$$\begin{bmatrix} \tilde{M}_{11} & \tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}_{13} \\ 0 & \tilde{D}_{21} & \tilde{D}_{22} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I_{d_1} & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} + \begin{bmatrix} 0 & \tilde{K}_{11} & \tilde{K}_{12} & \tilde{K}_{13} \\ 0 & \tilde{K}_{21} & \tilde{K}_{22} & \tilde{K}_{23} \\ 0 & \tilde{K}_{31} & \tilde{K}_{32} & \tilde{K}_{33} \\ -I_{d_1} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ 0 \end{bmatrix} u. \quad (28)$$

Note that v can be interpreted as the *velocity vector* of the dynamic part. Expressing this system by inserting $w = W^T x$ in terms of the original variables one obtains

$$\begin{bmatrix} \tilde{M}_{11} & (D_1 + B_1G) \\ 0 & (D_2 + B_2G) \\ 0 & 0 \\ 0 & W_1^T \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 & K_1 + B_1F \\ 0 & K_2 + B_2F \\ 0 & K_3 + B_3F \\ -I_{d_1} & 0 \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ 0 \end{bmatrix} u,$$

where $W_1^T = [I_{d_1}\, 0\, 0]W^T$. Reordering the block rows we obtain

$$\begin{bmatrix} \tilde{M}_{11} & (D_1 + B_1G) \\ 0 & (D_2 + B_2G) \\ 0 & W_1^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 & K_1 + B_1F \\ 0 & K_2 + B_2F \\ -I_{d_1} & 0 \\ 0 & K_3 + B_3F \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ B_3 \end{bmatrix} \tilde{u}.$$

This system is strangeness-free as a free system with $\tilde{u} = 0$ and we write this as

$$L\dot{y} = Sy + \tilde{B}\tilde{u}, \quad (29)$$

where

$$L = \begin{bmatrix} \tilde{M}_{11} & (D_1 + B_1G) \\ 0 & (D_2 + B_2G) \\ 0 & W_1 \\ 0 & 0 \end{bmatrix}, \; S = -\begin{bmatrix} 0 & K_1 + B_1F \\ 0 & K_2 + B_2F \\ -I_{d_1} & 0 \\ 0 & K_3 + B_3F \end{bmatrix}, \; \tilde{B} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ B_3 \end{bmatrix}, \; y = \begin{bmatrix} v \\ x \end{bmatrix}.$$

We then apply the procedure discussed in Section 2.1 to this system and obtain

$$\dot{w}_d(t) = G_d w_d(t) + B_d \tilde{u}(t), \quad (30a)$$
$$w_a(t) = G_a w_d(t) + B_a \tilde{u}(t), \quad (30b)$$

where

$$G_a := -(Q' S P_d')^+ Q' S P_d, \; G_d := L^+ S(P_d + G_a), \quad (31a)$$
$$B_a := -(Q' S P_d')^+ \tilde{B}, \; B_d := L^+ \tilde{B} + L^+ S B_a, \quad (31b)$$

with projectors

$$\begin{aligned}P_d &= L^+L \text{ and } P_d' = I - L^+L,\\ Q &:= LL^+ = \begin{bmatrix} I_d & 0\\ 0 & 0\end{bmatrix},\ Q' = I - LL^+.\end{aligned}$$

Moreover, an initial value $y_0 = \begin{bmatrix} v_0 \\ x_0 \end{bmatrix}$ is consistent if and only if it satisfies

$$(P_d' - G_a)y_0 = B_a\tilde{u}(t_0)$$

at initial time t_0.

Hence the reachable points due to a control input $\tilde{u}$ can be determined from the relations in (30). Note that the modified closed loop model due to the control input $u = -G\dot{x} - Fx + \tilde{u}$ is strangeness-free. Hence, we may drop the extra smoothness requirement for the input function $\tilde{u}(t)$, when forming the derivative array. Recall that we are interested in finding the reachable position and (for the dynamic part) velocity points from the zero initial conditions, i.e., $x(0) = x_0 = 0$ and $v(0) = v_0 = 0$. Corresponding to these initial conditions the solution of the differential equation (30a) is given by

$$w_d = \int_0^t e^{G_d(t-\tau)} B_d\tilde{u}(\tau)d\tau,$$

and hence,

$$v = \begin{bmatrix} I_{d_1} & 0 \end{bmatrix}\left(\int_0^t e^{G_d(t-\tau)} B_d\tilde{u}(\tau)d\tau\right), \tag{32a}$$

$$x_d = \begin{bmatrix} 0 & I_n \end{bmatrix}\left(\int_0^t e^{G_d(t-\tau)} B_d\tilde{u}(\tau)d\tau\right). \tag{32b}$$

If we partition the matrix S as

$$S = \left[\begin{array}{c|ccc} 0 & K_{11} & K_{12} & K_{13} \\ \hline 0 & K_{21} & K_{22} & K_{23} \\ I_{d_1} & 0 & 0 & 0 \\ 0 & K_{31} & K_{32} & K_{33} \end{array}\right],$$

and partition the projectors Q' and P_d' accordingly, then we obtain the following consistency conditions from the algebraic relation (30b):

$$v_a = 0, \tag{33a}$$

$$x_a = \begin{bmatrix} 0 & I_n \end{bmatrix}\left[G_a\left(\int_0^t e^{G_d(t-\tau)} B_d\tilde{u}(\tau)d\tau\right) + B_a\tilde{u}\right]. \tag{33b}$$

Hence, the reachable position and velocity points are determined from (32) and (33). Furthermore, since $v_a = 0$, the reachable vectors v are determined by (32a). With this, we obtain a reformulation of the reachability condition.

- A position vector $x_1 := x_{d_1} + x_{a_1}$ is *reachable from a consistent initial condition* $x(t_0) = x_0$, if there exists a control input $\tilde{u}(t)$ and a finite time $t_1 > 0$ such that $x_d(t_1) = x_{d_1}$ and $x_a(t_1) = x_{a_1}$.
- A velocity vector v_1 is reachable from a consistent initial velocity $v_1(t_0) = v(t_0)$, if there exists a control input $\tilde{u}(t)$ and a finite time $t_1 > 0$ such that $v(t_1) = v_1$.

Then, following Section 3, the reachable set $\mathcal{R}_0$ associated with the trimmed first order system (28) is given by

$$\mathcal{R}_{w_0} = \mathcal{W}_d + \mathcal{W}_a,$$

where

$$\begin{aligned}\mathcal{W}_d &:= \operatorname{Im}(\mathcal{C}_{d_m}), \text{ with } \mathcal{C}_{d_m} = \begin{bmatrix} B_{d_m} & G_{d_m}B_{d_m} & \cdots & G_{d_m}^{n-1}B_{d_m}\end{bmatrix}, \\ \mathcal{W}_a &:= \operatorname{Im}\left(\begin{bmatrix} G_{a_m}\mathcal{C}_{d_m} & B_{a_m}\end{bmatrix}\right).\end{aligned}$$

Since the system (29) is strangeness-free, we can drop the smoothness requirements for $\tilde{u}$ and hence the control input defined in (19) is given by

$$\tilde{u} = B_d^T e^{G_d^T(t-\tau)}w, \; 0 \le \tau \le t. \tag{34}$$

With the subspaces

$$\begin{aligned}\mathcal{X}_{d_m} &:= \operatorname{Im}([0 \;\; I_n]\,\mathcal{C}_{d_m}), \\ \mathcal{X}_{a_m} &:= \operatorname{Im}\left([0 \;\; I_n]\begin{bmatrix} G_{a_m}\mathcal{C}_{d_m} & B_{a_m}\end{bmatrix}\right),\end{aligned}$$

then the set of reachable position vectors is contained in the subspace

$$\mathcal{R}_{x_0} = \mathcal{X}_{d_m} + \mathcal{X}_{a_m}. \tag{35}$$

Furthermore, with the subspace

$$\mathcal{R}_{v_0} := \operatorname{Im}[I_{d_1} \;\; 0]\,\mathcal{C}_{d_m},$$

the set of reachable velocity vectors of system (21) is contained in the subspace $\mathcal{R}_{v_0}$. Hence, $\mathcal{R}_{x_0}$ and $\mathcal{R}_{v_0}$ describe the reachable spaces for position and velocity vectors, respectively.

In this section we have characterized the reachable sets for second order systems and it is clear that the procedure can be extended in a similar way for systems of order higher that two.

Remark 4.4 We have discussed at several places that the strangeness-free behavior model may not be strangeness-free for $u(t) = 0$. This situation can be circumvented by using feedback controls of the form $u(t) = Fx(t) + \tilde{u}(t)$ and $u(t) = -G\dot{x}(t) + Fx(t) + \tilde{u}(t)$ for first and second order descriptor systems, respectively. Recall that while computing the reachable space for the first order descriptor system we have used the strangeness-free behavior model and the control input $u(t)$ instead of $\tilde{u}(t)$. Hence, we have considered a smooth control input $u(t)$ (see (19)). On the other hand, by using a feedback control, we do not need smoothness conditions on the input function $\tilde{u}(t)$. This situation is demonstrated in the second order descriptor system, where the control input $\tilde{u}(t)$ takes the form $\tilde{u}(t) = B_d^T e^{G_d^T(t-\tau)}w, \; 0 \le \tau \le t$ for first as well as second order descriptor systems.

5 Examples

To illustrate the proposed procedure, in this section, we compute the reachable set of an electrical circuit and a simple multi-body system.

Example 5.1 Let us consider an electrical circuit as shown in Fig. 3, see [11]. Assuming that the voltages in capacitors v_{c_1}, v_{c_2} and currents i_1, i_2 as states, the descriptor system associated with the electrical circuit can be represented as

$$E\dot{x}(t) = Ax(t) + Bv_s(t),$$

where $x(t) = [v_{c_1}(t) \;\; v_{c_2}(t) \;\; i_2(t) \;\; i_1(t)]^T$, $v_s(t)$ is the input voltage to the circuit and

$$E = \begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & -L & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}, \; A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & R\end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1\end{bmatrix}$$

v_{c_2}

i_2

with the concrete data

$$M_0 = \begin{bmatrix} 18.7532 & -7.94493 & 7.94494 \\ -7.94493 & 31.8182 & -26.8182 \\ 7.94493 & -26.8182 & 26.8182 \end{bmatrix}, \quad D_0 = \begin{bmatrix} -1.52143 & -1.55168 & 1.55168 \\ 3.22064 & 3.28467 & -3.28467 \\ -3.22064 & -3.28467 & 3.28467 \end{bmatrix},$$

$$K_0 = \begin{bmatrix} 67.4894 & 69.2392 & -69.2392 \\ 69.8124 & 1.68624 & -1.68617 \\ -69.8124 & -1.68617 & -68.2707 \end{bmatrix}, \quad S_0 = \begin{bmatrix} -0.216598 & -0.338060 & 0.554659 \\ 0.458506 & -0.845154 & 0.386648 \\ -0.458506 & 0.845154 & 0.613353 \end{bmatrix},$$

$$F_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Carrying out the procedure discussed in Section 4, see Appendix 7, it follows that this system has strangeness-index $\mu = 2$ in the behavior setting, and we obtain a strangeness-free behavior model (24), where

$$\widehat{M}_1 = \begin{bmatrix} -14.5135 & 42.3643 & -38.6090 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\widehat{D}_1 = \begin{bmatrix} 4.7430 & 4.8373 & -4.8373 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\widehat{K}_1 = \begin{bmatrix} -83.9704 & 10.6511 & -54.9365 & -0.1875 & -0.6330 & 0.6752 & -1.1064 & -0.2019 \end{bmatrix},$$

$$\widehat{D}_2 = 10^{-13} \begin{bmatrix} -0.0008 & -0.0438 & 0.5359 & 0.0003 & 0.0070 & -0.0005 & 0.0007 & 0.0068 \end{bmatrix},$$

$$\widehat{K}_2 = \begin{bmatrix} 0.8620 & 0.4826 & -0.7789 & -0.0069 & 0.0098 & -0.0000 & 0.0051 & 0.0047 \end{bmatrix},$$

$$\widehat{K}_3 = \begin{bmatrix} 0.0276 & -0.1237 & 0.9269 & 0.0018 & -0.0025 & 0.0000 & -0.0013 & -0.0012 \\ -0.7527 & 0.1570 & 0.2713 & -0.0022 & 0.0032 & -0.0000 & 0.0017 & 0.0015 \\ -6.3728 & -4.4142 & -1.8547 & 0.0630 & -0.0893 & 0.0000 & -0.0465 & -0.0429 \end{bmatrix}.$$

The matrix $\widehat{H}$ in (25) is full row rank, i.e., 5, when setting the tolerance equal to the machine precision *eps* in computing the rank. The resulting system is also strangeness-free for $u = 0$, since the rank of H is 5. However, since the matrix H is nearly singular, we may improve the robustness of the system by using a preliminary feedback control $u = -G\dot{x} - Fx + \tilde{u}$, with

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \end{bmatrix}, \quad F = 0.$$

which leads to an H_F which is robustly of full row rank.

To obtain a first order formulation we choose the following orthogonal matrix

$$W = \begin{bmatrix} -0.2455 & 0.0804 & 0.2188 & 0.9409 & 0.0064 \\ 0.7165 & -0.2347 & -0.5633 & 0.3379 & 0.0051 \\ -0.6530 & -0.2877 & -0.7004 & 0.0171 & 0.0032 \\ 0 & -0.9250 & 0.3798 & -0.0093 & -0.0017 \\ 0 & 0 & 0.0044 & -0.0079 & 1.0000 \end{bmatrix}$$

which is obtained by a QR decomposition of H_F^T. Then following the discussion in Section 4 we obtain an associated first order model $L\dot{y} = Sy + \tilde{B}u$, where

$$L = \begin{bmatrix} 59.1273 & 4.7430 & 4.8373 & -10.3692 & -2.0190 & 0 \\ 0 & 0 & 0 & 0.0254 & 0.0469 & 0 \\ 0 & -0.2455 & 0.7165 & -0.6530 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} 0 & 83.9704 & -10.6511 & 54.9365 & 0.1875 & 0.6330 \\ 0 & -0.8620 & -0.4826 & 0.7789 & 0.0069 & -0.0098 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0276 & 0.1237 & -0.9269 & -0.0018 & 0.0025 \\ 0 & 0.7527 & -0.1570 & -0.2713 & 0.0022 & -0.0032 \\ 0 & 6.3728 & 4.4142 & 1.8547 & -0.0630 & 0.0893 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} 0.6752 & -1.1064 & -0.2019 \\ 0 & 0.0051 & 0.0047 \\ 0 & 0 & 0 \\ 0 & -0.0013 & -0.0012 \\ 0 & 0.0017 & 0.0015 \\ 0 & -0.0465 & -0.0429 \end{bmatrix}.$$

This gives

$$\begin{aligned} G_d &= \begin{bmatrix} -0.1286 & 0.2251 & -0.6976 & 0.6520 & 0.0106 & 0 \\ -0.2862 & -0.0041 & -0.0783 & 0.0956 & 0.0012 & 0 \\ 0.7949 & 0.0832 & 0.0083 & -0.0730 & -0.0001 & 0 \\ -0.5516 & -0.0729 & 0.0385 & 0.0100 & -0.0006 & 0 \\ 0.2991 & 0.0395 & -0.0209 & -0.0054 & 0.0003 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ G_a &= \begin{bmatrix} 0.0164 & -0.0022 & 0.0117 & -0.0280 & -0.0295 & 0 \\ -0.1105 & -0.0791 & 0.1960 & -0.1310 & 0.0711 & 0 \\ 0.0117 & -0.1465 & 0.4314 & -0.5117 & -0.2167 & 0 \\ 0.0543 & -0.1310 & 0.3997 & -0.5122 & -0.2645 & 0 \\ -0.0295 & 0.0711 & -0.2167 & 0.2777 & 0.1434 & 0 \\ -0.0000 & 16.9301 & -49.4183 & 45.4203 & 0.7056 & 0 \end{bmatrix}, \\ B_d &= \begin{bmatrix} 0.0112 & -0.0129 & 0.0017 \\ 0.0013 & -0.0015 & 0.0002 \\ -0.0001 & 0.0002 & 0 \\ -0.0006 & 0.0007 & -0.0001 \\ 0.0003 & -0.0004 & 0.0001 \\ 0 & 0 & 0 \end{bmatrix}, \ B_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.5203 & 0.4797 \end{bmatrix}, \end{aligned}$$

and so the subspaces $\mathcal{X}_{d_m}$ and $\mathcal{X}_{a_m}$ are spanned by the columns of

$$X_{d_m} = \begin{bmatrix} 0 & 0 \\ -0.2718 & 0.8649 \\ 0.7553 & -0.0801 \\ -0.5243 & -0.4356 \\ 0.2843 & 0.2362 \\ 0 & 0 \end{bmatrix}, \ X_{a_m} = \begin{bmatrix} 0 & 0 & 0 \\ -0.0037 & 0.2870 & -0.9224 \\ -0.0078 & 0.6569 & -0.0067 \\ -0.0072 & 0.6129 & 0.3394 \\ 0.0039 & -0.3323 & -0.1840 \\ 0.9999 & 0.0119 & -0.0003 \end{bmatrix},$$

respectively, while the subspace $\mathcal{R}_{v_0}$ is spanned by $[1 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0]^T$.

6 Conclusion

In this article we have presented a new algorithm to compute the reachable space for the first order and second order linear time invariant descriptor systems. We have obtained a strangeness-free behavior model corresponding to the original model via a derivative array approach. The first order descriptor system is then decoupled into a differential part and an algebraic part with the help of a projection method. The coefficient matrices of the projected system are used to define two subspaces and finally we have shown that the addition of these two subspaces is the reachable space, from the origin, of the original descriptor system.

For second order systems, the constructed strangeness-free behavior model is transformed into a first order system by a trimmed first order formulation, which is then used to compute the reachable position and velocity vectors following the procedure developed for first order descriptor systems. The new method is illustrated by an electrical circuit and a three-link manipulator.

References

[1] P. Antognetti and G. Massobrio. *Semiconductor device modeling with SPICE*. McGraw-Hill, New York, NY, 1997.

[2] A. K. Baum and V. Mehrmann. Positivity characterization of nonlinear DAEs, part II : decomposition of DAEs using projections. Preprint, Institut für Mathematik, TU Berlin, Berlin, Germany, 2013.

[3] T. Berger. *On differential-algebraic control systems*. PhD thesis, Technische Universität Ilmennau, 2014. Dissertation.

[4] Wojciech Blajer and Krzysztof Kołodziejczyk. Control of underactuated mechanical systems with servo-constraints. *Nonlinear Dynamics*, 50(4):781–791, 2007.

[5] K. E. Brenan, S. L. Campbell, and L. R. Petzold. *Numerical Solution of Initial-Value Problems in Differential Algebraic Equations*. SIAM Publications, Philadelphia, PA, 2nd edition, 1996.

[6] A. Bunse-Gerstner, R. Byers, V. Mehrmann, and N. K. Nichols. Feedback design for regularizing descriptor systems. *Lin. Alg. Appl.*, 299:119–151, 1999.

[7] R. Byers, V. Mehrmann, and H. Xu. Trimmed linearizations for structured matrix polynomials. *Lin. Alg. Appl.*, 429(10):2373–2400, 2008.

[8] S. L. Campbell. A general form for solvable linear time varying singular systems of differential equations. *SIAM J. Math. Anal.*, 18:1101–1115, 1987.

[9] S. L. Campbell, P. Kunkel, and V. Mehrmann. *Control and Optimization with Differential-Algebraic Constraints*, chapter Regularization of Linear and Nonlinear Descriptor Systems. SIAM, 2012.

[10] D. Cobb. Controllability, observability and duality in singular systems. *IEEE Trans. Automat. Control*, 29:1076–1082, 1984.

[11] L. Dai. *Singular Control Systems*. Springer-Verlag, Berlin, Germany, 1989.

[12] J. W. Demmel and B. Kågström. Computing stable eigendecompositions of matrix pencils. *Lin. Alg. Appl.*, 88:139–186, 1987.

[13] Dynasim AB, Ideon Research Park - SE-223 70 Lund - Sweden. *Dymola,Multi-Engineering Modelling and Simulation*, 2006.

[14] G. H. Golub and C. F. Van Loan. *Matrix Computations*. The Johns Hopkins University Press, Baltimore, MD, 3rd edition, 1996.

[15] M. Hou and P. C. Müller. *LQ* and tracking control of descriptor systems with application to constrained manipulator. Technical report, Sicherheitstechnische Regelungs- und Meßtechnik, Universität Wuppertal, Gauß-Straße 20, D-5600 Wuppertal 1, Germany, 1994.

[16] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations. Analysis and Numerical Solution*. EMS Publishing House, Zürich, Switzerland, 2006.

[17] P. Kunkel, V. Mehrmann, and W. Rath. Analysis and numerical solution of control problems in descriptor form. *Math. Control, Signals, Sys.*, 14:29–61, 2001.

[18] R. Lamour, R. Marz, and C. Tischendorf. *Differential-Algebraic Equations*: *A Projector Based Analysis*. Springer-Verlag, Berlin, 2013.

[19] P. Losse and V. Mehrmann. Controllability and observability of second order descriptor systems. *SIAM J. Cont.*, 47:1351–1379, 2008.

[20] The MathWorks, Inc., Natick, MA. *Simscape*, 2013. http://www.mathworks.de/products/simscape/.

[21] V. Mehrmann. Index concepts for differential-algebraic equations. Preprint 03–2012, Institut für Mathematik, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, FRG, 2012.

[22] V. Mehrmann and C. Shi. Analysis of higher order linear differential-algebraic systems. Preprint 17/2004, Institut für Mathematik, TU Berlin, Berlin, Germany, 2004.

[23] B. Pal and B. Chaudhuri. *Robust control in power systems.* Springer : New York, Power electronics and power systems, 2005.

[24] J. W. Polderman and J. C. Willems. *Introduction to Mathematical Systems Theory: A Behavioural Approach.* Springer-Verlag, New York, NY, 1998.

[25] P. J. Rabier and W. C. Rheinboldt. Classical and generalized solutions of time-dependent linear differential-algebraic equations. *Lin. Alg. Appl.*, 245:259–293, 1996.

[26] P.W. Sauer and M.A. Pai. *Power System Dynamics and Stability.* Prentice Hall, Upper Saddle River, NJ, 1998.

[27] L. Scholz. A derivative array approach for linear second order differential-algebraic systems. *Electr. J. Lin. Alg.*, 22:310–347, 2011.

[28] P. Van Dooren. The computation of Kronecker's canonical form of a singular pencil. *Lin. Alg. Appl.*, 27:103–141, 1979.

[29] G. C. Verghese, B. C. Lévy, and T. Kailath. A general state space for singular systems. *IEEE Trans. Automat. Control*, AC-26:811–831, 1981.

7 Appendix

Stepwise procedure to obtain strangeness free behavior model for second order descriptor systems [27]:

1. Start with $\mu = 0$.
2. Form the derivative array according to (23).
3. Compute a from the relation: $\operatorname{rank}([\mathcal{M}_\mu \;\; \mathcal{N}_\mu]) = (\mu+1)n - a$.
4. Compute Z such that $Z^T\mathcal{M}_\mu = 0$ and Z_3 such that $Z_3^T[\mathcal{M}_\mu \;\; \mathcal{N}_\mu] = 0$.
5. Let R_z be a basis matrix for the range space of $Z^T Z_3$. Then, compute $Z_2 = ZR_z$.
6. Compute T_3 such that $Z_3^T\mathcal{N}_\mu[I \;\; 0 \;\; \cdots \;\; 0]^T T_3 = 0$.
7. Compute the rank of $Z_2^T\mathcal{L}_\mu[I \;\; 0 \;\; \cdots \;\; 0]^T T_3$ and denote it as d_2.
8. Compute SVD of $Z_2^T\mathcal{L}_\mu[I \;\; 0 \;\; \cdots \;\; 0]^T$ and write it as $Z_2^T\mathcal{L}_\mu[I \;\; 0 \;\; \cdots \;\; 0]^T = U_{z_2}S_{z_2}V_{z_2}^T$. Then, form Z_1 by taking the first d_2 columns of U_{z_2}.
9. Compute T_2 such that $Z_2^T\mathcal{L}_\mu[I \;\; 0 \;\; \cdots \;\; 0]^T T_3 T_2 = 0$.
10. Compute SVD of $\mathbf{M}T_3T_2$ and write it as $\mathbf{M}T_3T_2 = U_m S_m V_m^T$. Denote d_1 as the rank of S_m. Then form Z_0 by taking the first d_1 columns of U_m.
11. Check if $a + d_2 + d_1 = n$ then stop, otherwise increase the value of μ by one and follow the procedure from Step 2.

Computed results for Example 5.2: Corresponding to $\mu = 2$, the matrices Z and Z_3 such that $Z^T \mathcal{M}_\mu = 0$ and $Z_3^T[\mathcal{M}_\mu \ \ \mathcal{N}_\mu] = 0$ are as follows:

$$Z = \begin{bmatrix} -0.0014 & 0.0626 & -0.0045 & 0.0067 & 0.0025 & 0 \\ -0.0105 & -0.0738 & -0.1359 & 0.1111 & 0.0033 & 0 \\ -0.0120 & -0.1061 & -0.1599 & 0.1299 & 0.0032 & 0 \\ 0.0347 & -0.0378 & 0.6417 & 0.7607 & 0.0833 & 0 \\ 0.9972 & 0.0154 & -0.0720 & 0.0160 & -0.0003 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0001 & 0.0323 & 0.0290 & 0.0862 & -0.9953 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0385 & 0.9169 & -0.2759 & 0.2751 & 0.0455 & 0 \\ -0.0523 & -0.3688 & -0.6797 & 0.5557 & 0.0164 & 0 \end{bmatrix}, \ Z_3 = \begin{bmatrix} 0.0018 & -0.0022 & 0.0630 \\ -0.0017 & 0.0021 & -0.0595 \\ -0.0025 & 0.0032 & -0.0893 \\ 0.2051 & -0.9779 & -0.0405 \\ 0.9783 & 0.2060 & -0.0201 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.0265 & -0.0336 & 0.9454 \\ -0.0083 & 0.0106 & -0.2977 \end{bmatrix}.$$

Furthermore, the matrices Z_2 and T_3 are

$$Z_2 = \begin{bmatrix} 0.0056 & -0.0628 & -0.0018 \\ -0.0053 & 0.0594 & 0.0017 \\ -0.0079 & 0.0890 & 0.0025 \\ -0.9754 & -0.0811 & -0.2051 \\ 0.2020 & 0.0455 & -0.9783 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.0839 & -0.9423 & -0.0265 \\ -0.0264 & 0.2968 & 0.0083 \end{bmatrix}, \ T_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.0143 & -0.0202 & 0 & -0.0105 & -0.0097 \\ 0 & 0 & 0 & 0 & 0 \\ 0.9999 & 0.0001 & 0 & 0.0001 & 0.0001 \\ 0.0001 & 0.9998 & 0 & -0.0001 & -0.0001 \\ 0 & 0 & 1.0000 & 0 & 0 \\ 0.0001 & -0.0001 & 0 & 0.9999 & -0.0001 \\ 0.0001 & -0.0001 & 0 & -0.0001 & 1.0000 \end{bmatrix}$$

and hence $d_2 = \text{rank}\left(Z_2^T \mathcal{L}_\mu [I \ \ 0 \ \ \cdots \ \ 0]^T T_3\right) = 1$. Then, $Z_1 = [-0.3730 \ \ 0.0504 \ \ 0.9265]^T$ is chosen such that $\text{rank}\left(Z_1^T Z_2^T \mathcal{L}_\mu [I \ \ 0 \ \ \cdots \ \ 0]^T\right) = 1$. The matrix T_2 is computed as follows:

$$T_2 = \begin{bmatrix} 0.7310 & -0.0474 & 0.1064 & 0.6719 \\ 0.4792 & 0.0337 & -0.0758 & -0.4787 \\ 0.0337 & 0.9978 & 0.0049 & 0.0310 \\ -0.0758 & 0.0049 & 0.9890 & -0.0697 \\ -0.4787 & 0.0310 & -0.0697 & 0.5600 \end{bmatrix}.$$

Then we have $d_1 = \text{rank}\left(\mathbf{M} T_3 T_2\right) = 1$. The matrix $Z_0 = [-0.1875 \ \ 0.7511 \ \ -0.6330 \ \ 0 \ \ 0]^T$ is computed from the SVD of $\mathbf{M} T_3 T_2$ such that $\text{rank}\left(Z_0^T \mathbf{M}\right) = 1$.