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ABSTRACT. Let $x : \mathbf{M}^m \to \mathbf{S}^n$ be a submanifold without umbilics. Two basic invariants of x under the Möbius transformation group in $\mathbf{S}^n$ are a 1-form Φ called Möbius form and a symmetric (0,2) tensor $\mathbb{A}$ called Blaschke tensor. x is said to be Möbius isotropic if $\Phi \equiv 0$ and $\mathbb{A} = \lambda dx \cdot dx$ for some smooth function $\lambda : \mathbf{M}^m \to \mathbf{R}$. An interesting property for a Möbius isotropic submanifold is that its conformal Gauss map $f : \mathbf{M}^m \to G^+_{n-m}(\mathbf{R}^{n+2}_1)$ is harmonic. The main result in this paper is the classification of Möbius isotropic submanifolds in $\mathbf{S}^n$. We show that (i) if $\lambda > 0$, then x is Möbius equivalent to a minimal submanifold with constant scalar curvature in $\mathbf{S}^n$; (ii) if $\lambda = 0$, then x is Möbius equivalent to the pre-image of a stereographic projection of a minimal submanifold with constant scalar curvature in $\mathbf{R}^n$; (iii) if $\lambda < 0$, then x is Möbius equivalent to the image of the standard conformal map $\sigma : \mathbf{H}^n \to \mathbf{S}^n_+$ of a minimal submanifold with constant scalar curvature in $\mathbf{H}^n$. This result shows that one can use Möbius differential geometry to unify the three different classes of minimal submanifolds with constant scalar curvature in $\mathbf{S}^n$, $\mathbf{R}^n$ and $\mathbf{H}^n$.

§1. Introduction.

Let $x : \mathbf{M} \to \mathbf{S}^n$ be a submanifold without umbilics. Let $\{e_i\}$ be a local orthonormal basis for the first fundamental form $I = dx \cdot dx$ with dual basis $\{\theta_i\}$. Let $II = \sum_{ij\alpha} h^\alpha_{ij}\theta_i\theta_j e_\alpha$ be the second fundamental form of x and let $H = \sum_\alpha H^\alpha e_\alpha$ be the mean curvature vector of x, where $\{e_\alpha\}$ is a local orthonormal basis for the normal bundle of x. We define $\rho^2 = \frac{m}{m-1}(||II||^2 - m||H||^2)$. Then two basic Möbius invariants of x, the Möbius form $\Phi = \sum_i C^\alpha_i \theta_i e_\alpha$ and the Blaschke tensor $\mathbb{A} = \rho^2 \sum_{ij} A_{ij}\theta_i\theta_j$, are defined by (cf. [W])

$$C^\alpha_i = -\rho^{-2}(H^\alpha_{,i} + \sum_j (h^\alpha_{ij} - H^\alpha \delta_{ij}) e_j(\log \rho)), \tag{1.1}$$

$$\begin{aligned} A_{ij} = &-\rho^{-2}(\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - \sum_\alpha H^\alpha h^\alpha_{ij}) \\ &-\frac{1}{2}\rho^{-2}(||\nabla \log\rho||^2 - 1 + ||H||^2)\delta_{ij}, \end{aligned} \tag{1.2}$$

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Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX where Hess_{ij} and ∇ are the Hessian-matrix and the gradient of $dx \cdot dx$. A submanifold $x : \mathbf{M} \to \mathbf{S}^n$ is called Möbius isotropic if $\Phi \equiv 0$ and $\mathbb{A} = \lambda dx \cdot dx$ for some function λ.

Let $\sigma : \mathbf{R}^n \to \mathbf{S}^n \backslash \{(-1, 0)\}$ and $\tau : \mathbf{H}^n \to \mathbf{S}^n_+$ be the following conformal diffeomorphisms:

$$\sigma(u) = \left(\frac{1 - ||u||^2}{1 + ||u||^2}, \frac{2u}{1 + ||u||^2}\right), \quad u \in \mathbf{R}^n \tag{1.3}$$

$$\tau(y) = \left(\frac{1}{y_0}, \frac{y_1}{y_0}\right), \quad y_0 > 0, \quad -y_0^2 + y_1 \cdot y_1 = -1, \quad y_1 \in \mathbf{R}^n. \tag{1.4}$$

Then we can state our main result as follows

Classification Theorem. *Any Möbius isotropic submanifold in $\mathbf{S}^n$ is Möbius equivalent to one of the following Möbius isotropic submanifolds:*
(i) *minimal submanifolds with constant scalar curvature in $\mathbf{S}^n$;*
(ii) *the images of σ of minimal submanifolds with constant scalar curvature in $\mathbf{R}^n$;*
(iii) *the images of τ of minimal submanifolds with constant scalar curvature in $\mathbf{H}^n$.*

This paper is organized as follows. In §2 we give Möbius invariants and structure equations for submanifolds in $\mathbf{S}^n$. In §3 we show that the conformal Gauss map of an isotropic submanifold in $\mathbf{S}^n$ is harmonic. In §4 we give conformal invariants for submanifolds in $\mathbf{R}^n$ and $\mathbf{H}^n$ and relate them to the Möbius invariants of submanifolds in $\mathbf{S}^n$. Using these relations we show that all submanifolds in (i), (ii) and (iii) of the classification theorem are Möbius isotropic submanifolds . Then in §5 we prove the classification theorem for Möbius isotropic submanifolds .

§2. Möbius invariants for submanifolds in $\mathbf{S}^n$.

In this section we give Möbius invariants and structure equations for submanifolds in $\mathbf{S}^n$. For more detail we refer to [W].

Let $\mathbf{R}^{n+2}_1$ be the Lorentzian space with the inner product

$$< x, w > = -x_0 w_0 + x_1 w_1 + \cdots + x_{n+1} w_{n+1}, \tag{2.1}$$

where $x = (x_0, x_1, \cdots, x_{n+1})$ and $w = (w_0, w_1, \cdots, w_{n+1})$. Let $x : \mathbf{M} \to \mathbf{S}^n$ be a m-dimensional submanifold without umbilics. We define the Möbius position vector $Y : \mathbf{M} \to \mathbf{R}^{n+2}_1$ of x by

$$Y = \rho(1, x), \quad \rho^2 = \frac{m}{m-1}(||II||^2 - m||H||^2) > 0. \tag{2.2}$$

Then we have the following

Theorem 2.1.([W]) *Two submanifolds $x, \tilde{x} : \mathbf{M} \to \mathbf{S}^n$ are Möbius equivalent if and only if there exists T in the Lorentz group $O(n+1, 1)$ in $\mathbf{R}^{n+2}_1$ such that $Y = \tilde{Y}T$.*

As a matter of fact the Möbius group in $\mathbf{S}^n$ is isomorphic to the subgroup $O^+(n+1, 1)$ which preserves the positive part of the light cone in $\mathbf{R}^{n+2}_1$. It follows immediately from Theorem 2.1 that

$$g = < dY, dY > = \rho^2 dx \cdot dx \tag{2.3}$$

is a Möbius invariant (cf. [CH]). We call it induced Möbius metric for x. Now let Δ be the Laplacian operator of g. Then there is an identity given by

$$< \Delta Y, \Delta Y >= 1 + m^2\kappa,$$

where κ is the normalized scalar curvature of g (cf. [W]). We define

$$N = -\frac{1}{m}\Delta Y - \frac{1}{2m^2}(1 + m^2\kappa)Y. \tag{2.4}$$

Then we have

$$< Y, Y >=< N, N >= 0, \quad < Y, N >= 1. \tag{2.5}$$

Moreover, if we take a local orthonormal basis $\{E_i\}$ for the Möbius metric g with dual basis $\{\omega_i\}$, then we have

$$< E_i(Y), E_j(Y) >= \delta_{ij}, \quad < E_i(Y), Y >=< E_i(Y), N >= 0, \quad 1 \le i, j \le m. \tag{2.6}$$

Let $\mathbb{V}$ be the orthogonal complement space to the subspace in $\mathbf{R}^{n+2}_1$ spanned by $\{Y, N, E_i(Y)\}$. Then we have the following orthogonal decomposition:

$$\mathbf{R}^{n+2}_1 = \text{span}\{Y, N\} \oplus \text{span}\{E_1(Y), \cdots, E_m(Y)\} \oplus \mathbb{V}. \tag{2.7}$$

$\mathbb{V}$ is called the Möbius normal bundle of x. A local orthonormal basis $\{E_\alpha\}$ for $\mathbb{V}$ can be written as

$$E_\alpha = (H^\alpha, H^\alpha x + e_\alpha), \quad m + 1 \le \alpha \le n. \tag{2.8}$$

The conformal Gauss map $f : \mathbf{M} \to \mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1) \subset \wedge^{n-m}(\mathbf{R}^{n+2}_1)$ is defined by

$$f = E_{m+1} \wedge E_{m+2} \wedge \cdots \wedge E_n. \tag{2.9}$$

Since $\{Y, N, E_1(Y), \cdots, E_m(Y), E_{m+1}, \cdots, E_n\}$ are Möbius invariant moving frame in $\mathbf{R}^{n+2}_1$ along $\mathbf{M}$, we can write the structure equations as

$$E_i(N) = \sum_j A_{ij}E_j(Y) + \sum_\alpha C^\alpha_i E_\alpha \tag{2.10}$$

$$E_j(E_i(Y)) = -A_{ij}Y - \delta_{ij}N + \sum_k \Gamma^k_{ij}E_k(Y) + \sum_\alpha B^\alpha_{ij}E_\alpha \tag{2.11}$$

$$E_i(E_\alpha) = -C^\alpha_i Y - \sum_j B^\alpha_{ij}E_j(Y) + \sum_\beta \Gamma^\beta_{\alpha i}E_\beta, \tag{2.12}$$

where $\{\Gamma^k_{ij}\}$ is the Levi-Civita connection of the Möbius metric g; $\{\Gamma^\beta_{\alpha i}\}$ is the normal connection for $x : \mathbf{M} \to \mathbf{S}^n$ which is a Möbius invariant; $\mathbb{A} = \sum_{ij} A_{ij}\omega_i \otimes \omega_j$ and $\Phi = \sum_{i\alpha} C^\alpha_i \omega_i(\rho^{-1}e_\alpha)$ are called the Blaschke tensor and the Möbius form respectively; and

$\mathbb{B} = \sum_{ij\alpha} B^\alpha_{ij}\omega_i\omega_j(\rho^{-1}e_\alpha)$ is called the Möbius second fundamental form of x. The relations between $\mathbb{A}$, Φ, $\mathbb{B}$ and the Euclidean invariants of x are given by (1.1), (1.2) and

$$B^\alpha_{ij} = \rho^{-1}(h^\alpha_{ij} - H^\alpha\delta_{ij}). \tag{2.13}$$

The integrability conditions for the structure equations (2.10)-(2.12) are gived by (cf. [W])

$$A_{ij,k} - A_{ik,j} = \sum_\alpha (B^\alpha_{ik}C^\alpha_j - B^\alpha_{ij}C^\alpha_k) \tag{2.14}$$

$$C^\alpha_{i,j} - C^\alpha_{j,i} = \sum_k (B^\alpha_{ik}A_{kj} - B^\alpha_{kj}A_{ki}) \tag{2.15}$$

$$B^\alpha_{ij,k} - B^\alpha_{ik,j} = \delta_{ij}C^\alpha_k - \delta_{ik}C^\alpha_j \tag{2.16}$$

$$R_{ijkl} = \sum_\alpha (B^\alpha_{ik}B^\alpha_{jl} - B^\alpha_{il}B^\alpha_{jk}) + (\delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}) \tag{2.17}$$

$$R_{\alpha\beta ij} = \sum_k (B^\alpha_{ik}B^\beta_{kj} - B^\beta_{ik}B^\alpha_{kj}) \tag{2.18}$$

$$\sum_i B^\alpha_{ii} = 0, \quad \sum_{ij\alpha}(B^\alpha_{ij})^2 = \frac{m-1}{m}, \quad \mathrm{tr}\mathbb{A} = \sum_i A_{ii} = \frac{1}{2m}(1 + m^2\kappa), \tag{2.19}$$

where κ is the normalized scalar curvature of g. From (2.16) and (2.19) we get

$$B^\alpha_{ij,i} = (1-m)C^\alpha_j. \tag{2.20}$$

Definition 2.2. *Let $x : \mathbf{M} \to \mathbf{S}^n$ be a submanifold without umbilics. We call x a Möbius isotropic submanifold in $\mathbf{S}^n$ if $\Phi \equiv 0$ and there exists a function $\lambda : \mathbf{M} \to \mathbf{R}$ such that $\mathbb{A} = \lambda g$.*

Proposition 2.3. *Let $x : \mathbf{M} \to \mathbf{S}^n$ be a Möbius isotropic submanifold. Then the function λ in Definition 2.2 has to be a constant.*

Proof. Since $\Phi \equiv 0$ and $\mathbb{A} = \lambda g$, we can write (2.10) as $\mathrm{d}N = \lambda \mathrm{d}Y$, which implies that $\mathrm{d}\lambda \wedge \mathrm{d}Y = 0$. Since $\{E_1(Y), \cdots, E_m(Y)\}$ are linearly independent, we get $\lambda =$ constant. □

§3. Conformal Gauss map of submanifolds in $\mathbf{S}^n$.

Let $x : \mathbf{M} \to \mathbf{S}^n$ be a submanifold. We assume that $\mathbf{M}$ is oriented. Then we can give the normal bundle $\mathbf{N}(\mathbf{M})$ of x an orientation. Let $\{e_\alpha\}$ be a local orthonormal basis for $\mathbf{N}(\mathbf{M})$ which gives the orientation. Using the bundle isometry $\tau : \mathbf{N}(\mathbf{M}) \to \mathbb{V}$ defined by $e_\alpha \to (H^\alpha, H^\alpha x + e_\alpha)$ we can give $\mathbb{V}$ an orientation. We define the conformal Gauss map $f : \mathbf{M} \to \mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1) \subset \wedge^{n-m}(\mathbf{R}^{n+2}_1)$ by

$$f = E_{m+1} \wedge E_{m+2} \wedge \cdots \wedge E_n, \tag{3.1}$$

where $\{E_\alpha\}$ is an oriented orthonormal basis for $\mathbb{V}$ and $\mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1)$ is the Grassmannian manifold of all positive definite oriented $(n-m)$-planes in $\mathbf{R}^{n+2}_1$. We denote by I_G the induced metric of the standard embedding $\mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1)$ in $\wedge^{n-m}(\mathbf{R}^{n+2}_1)$. Our goal in this section is to prove the following

Theorem 3.1. *Let $x : \mathbf{M} \to \mathbf{S}^n$ be a Möbius isotropic submanifold. Then its conformal Gauss map $f : (\mathbf{M}, g) \to (\mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1), I_G)$ is harmonic.*

Let $(\mathbf{M}, g)$ and $(\mathbf{N}, h)$ be two Semi-Riemannian manifolds. We assume that g is positive definite and h is a metric of type (r, s). Then locally we can write

$$g = \sum_{i=1}^{m} \theta_i^2, \quad h = -\sum_{\alpha=1}^{r} \theta_\alpha^2 + \sum_{\lambda=r+1}^{r+s} \theta_\lambda^2. \tag{3.2}$$

We denote by $\{\theta_{ij}\}$ the connection forms of g with respect to $\{\theta_i\}$ and denote by $\{\theta_{\alpha\beta}, \theta_{\alpha\lambda}, \theta_{\lambda\mu}\}$ the connection forms of h with respect to h, here we use the following ranges of the indices:

$$1 \le i, j \le m, \quad 1 \le \alpha, \beta \le r, \quad r+1 \le \lambda, \mu \le r+s. \tag{3.3}$$

Then we have

$$d\theta_i = \sum_j \theta_{ij} \wedge \theta_j \tag{3.4}$$

$$d\theta_\alpha = -\sum_\beta \theta_{\alpha\beta} \wedge \theta_\beta + \sum_\lambda \theta_{\alpha\lambda} \wedge \theta_\lambda, \quad d\theta_\lambda = -\sum_\beta \theta_{\lambda\beta} \wedge \theta_\beta + \sum_\mu \theta_{\lambda\mu} \wedge \theta_\mu. \tag{3.5}$$

Now let $f : \mathbf{M} \to \mathbf{N}$ be a smooth map. We define $\{f_{\alpha i}, f_{\lambda i}\}$ by

$$f^*\theta_\alpha = \sum_i f_{\alpha i}\theta_i, \quad f^*\theta_\lambda = \sum_i f_{\lambda i}\theta_i. \tag{3.6}$$

The so-called tension-field $\{f_{\alpha i,j}, f_{\lambda i,j}\}$ of $f : \mathbf{M} \to \mathbf{N}$ is defined by

$$df_{\alpha i} + \sum_j f_{\alpha j}\theta_{ji} - \sum_\beta f_{\beta i} f^*\theta_{\beta\alpha} + \sum_\lambda f_{\lambda i} f^*\theta_{\lambda\alpha} = \sum_j f_{\alpha i,j}\theta_j \tag{3.7}$$

$$df_{\lambda i} + \sum_j f_{\lambda j}\theta_{ji} - \sum_\alpha f_{\alpha i} f^*\theta_{\alpha\lambda} + \sum_\mu f_{\mu i} f^*\theta_{\mu\lambda} = \sum_j f_{\mu i,j}\theta_j. \tag{3.8}$$

Then $f : \mathbf{M} \to \mathbf{N}$ is harmonic if and only if

$$\sum_i f_{\alpha i,i} = 0, \quad \sum_i f_{\lambda i,i} = 0, \quad 1 \le \alpha \le r, \quad r+1 \le \lambda \le r+s. \tag{3.9}$$

To prove the Theorem 3.1 we study first the geometry of the Grassmannian manifold $\mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1)$ as a submanifold in the pseudo-Euclidean space $\wedge^{n-m}(\mathbf{R}^{n+2}_1)$ with the inner product induced by $(\mathbf{R}^{n+2}_1, <\ ,\ >)$. Let $\tilde{O}(n+1,1)$ be the manifold defined by

$$\tilde{O}(n+1,1) = \{T \in GL(n+2, \mathbf{R}) \mid {}^tT I_1 T = J\}, \tag{3.10}$$

where $I_1 = \operatorname{diag}\{-1, 1, \cdots, 1\}$ and $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \operatorname{diag}\{1, \cdots, 1\}$. Then

$$T = (\xi_{-1}, \xi_0, \xi_1, \cdots, \xi_n) \in \tilde{O}(n+1,1)$$

if and only if

$$< \xi_{-1}, \xi_{-1} > = < \xi_0, \xi_0 > = 0, \quad < \xi_{-1}, \xi_0 > = 1 \tag{3.11}$$

$$< \xi_a, \xi_{-1} > = < \xi_a, \xi_0 > = 0, \quad < \xi_a, \xi_b > = \delta_{ab}, \quad 1 \leq a, b \leq n. \tag{3.12}$$

Let $\pi : \tilde{O}(n+1,1) \to \mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1)$ be the fibre bundle defined by

$$\pi(T) = \xi_{m+1} \wedge \cdots \wedge \xi_n. \tag{3.13}$$

Then around each point in $\mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1)$ there exists an open set $U \subset \mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1)$ such that we have a local section

$$T = (\xi_{-1}, \xi_0, \xi_1, \cdots, \xi_n) : U \to \tilde{O}(n+1,1). \tag{3.14}$$

Thus the embedding of $\mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1)$ in $\wedge^{n-m}(\mathbf{R}^{n+2}_1)$ can be written locally by the position vector

$$\xi = \xi_{m+1} \wedge \cdots \wedge \xi_n : U \to \wedge^{n-m}(\mathbf{R}^{n+2}_1). \tag{3.15}$$

Since $\{\xi_{-1}, \xi_0, \xi_1, \cdots, \xi_n\}$ is a moving frame in $\mathbf{R}^{n+2}_1$ along $U \subset \mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1)$, we can write the structure equations as

$$\mathrm{d}\xi_A = \sum_B \theta_{AB} \xi_B, \quad -1 \leq A, B \leq n, \tag{3.16}$$

where d is the differential operator on $\mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1)$ and $\{\theta_{AB}\}$ are local 1-forms on $\mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1)$. The integrability conditions for (3.16) are given by

$$\mathrm{d}\theta_{AB} = \sum_C \theta_{AC} \wedge \theta_{CB}, \quad -1 \leq A, B, C \leq n. \tag{3.17}$$

Since (3.11) and (3.12) hold on U, we get from (3.16) that

$$\theta_{0(-1)} = \theta_{(-1)0} = 0, \quad \theta_{00} = -\theta_{(-1)(-1)} \tag{3.18}$$

$$\theta_{0a} = -\theta_{a(-1)}, \quad \theta_{(-1)a} = -\theta_{a0}, \quad \theta_{ab} = -\theta_{ba}, \quad 1 \leq a, b \leq n. \tag{3.19}$$

We make the following convention on the range of indices:

$$1 \leq i, j, k \leq m, \quad m+1 \leq \alpha, \beta, \gamma \leq n, \quad -1 \leq A, B, C \leq n.$$

Then from (3.15) we get

$$\begin{aligned} \mathrm{d}\xi &= \sum_\alpha \xi_{m+1} \wedge \cdots \wedge \mathrm{d}\xi_\alpha \wedge \cdots \wedge \xi_n \\ &= \sum_\alpha (-1)^{\alpha-m-1} \theta_{\alpha(-1)} \xi_{-1} \wedge \xi_{m+1} \wedge \cdots \wedge \hat{\xi_\alpha} \wedge \cdots \wedge \xi_n \\ &\quad + \sum_\alpha (-1)^{\alpha-m-1} \theta_{\alpha 0} \xi_0 \wedge \xi_{m+1} \wedge \cdots \wedge \hat{\xi_\alpha} \wedge \cdots \wedge \xi_n \\ &\quad + \sum_\alpha (-1)^{\alpha-m-1} \theta_{\alpha i} \xi_i \wedge \xi_{m+1} \wedge \cdots \wedge \hat{\xi_\alpha} \wedge \cdots \wedge \xi_n. \end{aligned} \tag{3.20}$$

Thus the induced metric I_G of $\mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1)$ in $\wedge^{n-m}(\mathbf{R}^{n+2}_1)$ is given by

$$I_G = < \mathrm{d}\xi, \mathrm{d}\xi >= \sum_\alpha (\theta_{\alpha(-1)} \otimes \theta_{\alpha 0} + \theta_{\alpha 0} \otimes \theta_{\alpha(-1)}) + \sum_{\alpha i} \theta^2_{\alpha i}. \tag{3.21}$$

If we define

$$\phi_{\alpha(-1)} = \frac{1}{\sqrt{2}}(\theta_{\alpha(-1)} - \theta_{\alpha 0}), \quad \phi_{\alpha 0} = \frac{1}{\sqrt{2}}(\theta_{\alpha(-1)} + \theta_{\alpha 0}), \tag{3.22}$$

then we can write

$$I_G = -\sum_\alpha \phi^2_{\alpha(-1)} + \sum_\alpha \phi^2_{\alpha 0} + \sum_{\alpha i} \theta^2_{\alpha i}. \tag{3.23}$$

Thus $\{\phi_{\alpha(-1)}, \phi_{\alpha 0}, \theta_{\alpha i}\}$ is a local orthonormal basis of $T^*\mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1)$, which implies that I_G is a semi-Riemannian metric on $\mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1)$ of type $((n-m),(n-m)(m+1))$. From (3.22), (3.17), (3.18) and (3.19) we get

$$\mathrm{d}\phi_{\alpha(-1)} = \sum_\beta \theta_{\alpha\beta} \wedge \phi_{\beta(-1)} + \theta_{00} \wedge \phi_{\alpha 0} + \sum_k \frac{1}{\sqrt{2}}(\theta_{k0} - \theta_{k(-1)}) \wedge \theta_{\alpha k} \tag{3.24}$$

$$\mathrm{d}\phi_{\alpha 0} = \theta_{00} \wedge \phi_{\alpha(-1)} + \sum_\beta \theta_{\alpha\beta} \wedge \theta_{\beta 0} - \sum_k \frac{1}{\sqrt{2}}(\theta_{k(-1)} + \theta_{k0}) \wedge \theta_{\alpha k} \tag{3.25}$$

$$\begin{aligned} \mathrm{d}\theta_{\alpha k} =& \frac{1}{\sqrt{2}}(\theta_{k0} - \theta_{k(-1)}) \wedge \phi_{\alpha(-1)} + \frac{1}{\sqrt{2}}(\theta_{k(-1)} + \theta_{k0}) \wedge \phi_{\alpha 0} \\ &+ \sum_{j\beta} (\theta_{jk}\delta_{\alpha\beta} + \theta_{\alpha\beta}\delta_{jk})\theta_{\beta j}. \end{aligned} \tag{3.26}$$

By (3.5) we obtain the following connection forms of I_G with respect to the orthonormal basis $\{\phi_{\alpha(-1)}, \phi_{\alpha 0}, \theta_{\alpha i}\}$:

$$\Omega_{\alpha(-1)\beta(-1)} = -\theta_{\alpha\beta}, \quad \Omega_{\alpha(-1)\beta 0} = \theta_{00}\delta_{\alpha\beta}, \quad \Omega_{\alpha(-1)\beta k} = \frac{1}{\sqrt{2}}(\theta_{k0} - \theta_{k(-1)})\delta_{\alpha\beta} \tag{3.27}$$

$$\Omega_{\alpha 0\beta(-1)} = -\theta_{00}\delta_{\alpha\beta}, \quad \Omega_{\alpha 0\beta 0} = \theta_{\alpha\beta}, \quad \Omega_{\alpha 0\beta k} = -\frac{1}{\sqrt{2}}(\theta_{k(-1)} + \theta_{k0})\delta_{\alpha\beta} \tag{3.28}$$

$$\begin{aligned} \Omega_{\alpha k\beta(-1)} =& \frac{1}{\sqrt{2}}(\theta_{k(-1)} - \theta_{k0})\delta_{\alpha\beta}, \quad \Omega_{\alpha k\beta 0} = \frac{1}{\sqrt{2}}(\theta_{k(-1)} + \theta_{k0})\delta_{\alpha\beta} \\ \Omega_{\alpha k\beta j} =& \theta_{jk}\delta_{\alpha\beta} + \theta_{\alpha\beta}\delta_{jk}. \end{aligned} \tag{3.29}$$

Now let $f : \mathbf{M} \to \mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1)$ be the conformal Gauss map of a submanifold $x : \mathbf{M} \to \mathbf{S}^n$. Let $\{Y, N, E_1(Y), \cdots, E_m(Y), E_{m+1}, \cdots, E_n\}$ be the Möbius moving frame in $\mathbf{R}^{n+2}_1$ along $\mathbf{M}$. Then we can find a local section T of $\pi : \tilde{O}(n+1,1) \to \mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1)$ given by (3.14) such that

$$(Y, N, E_1(Y), \cdots, E_m(Y), E_{m+1}, \cdots, E_n) = T \circ f = (f^*\xi_{-1}, \cdots, f^*\xi_n). \tag{3.30}$$

It follows from (2.10), (2.11), (2.12) and (3.16) that

$$f^*\theta_{00}=0,\quad f^*\theta_{k(-1)}=-\sum_j A_{kj}\omega_j,\quad f^*\theta_{k0}=-\omega_k \tag{3.31}$$

$$f^*\theta_{ij}=\omega_{ij}:=\sum_k \Gamma^j_{ik}\omega_k,\quad f^*\theta_{\alpha\beta}=\omega_{\alpha\beta}:=\sum_i \Gamma^\beta_{\alpha i}\omega_i \tag{3.32}$$

$$f^*\theta_{\alpha(-1)}=-\sum_i C^\alpha_i\omega_i,\quad f^*\theta_{\alpha 0}=0,\quad f^*\theta_{\alpha k}=-\sum_j B^\alpha_{kj}\omega_j. \tag{3.33}$$

If we define $\{f_{\alpha(-1)i}, f_{\alpha 0i}, f_{\alpha ki}\}$ by

$$f^*\phi_{\alpha(-1)}=\sum_i f_{\alpha(-1)i}\omega_i,\quad f^*\phi_{\alpha 0}=\sum_i f_{\alpha 0i}\omega_i,\quad f^*\theta_{\alpha k}=\sum_i f_{\alpha ki}\omega_i. \tag{3.34}$$

Then by (3.22) and (3.33) we have

$$f_{\alpha(-1)i}=-\frac{1}{\sqrt{2}}C^\alpha_i,\quad f_{\alpha 0i}=-\frac{1}{\sqrt{2}}C^\alpha_i,\quad f_{\alpha ki}=-B^\alpha_{ki}. \tag{3.35}$$

By definition (cf. (3.7) and (3.8)) the tension field $\{f_{\alpha(-1)i,j}, f_{\alpha 0i,j}, f_{\alpha ki,j}\}$ are defined by the following formulas

$$\begin{aligned} df_{\alpha(-1)i}&+\sum_j f_{\alpha(-1)j}\omega_{ji}-\sum_\beta f_{\beta(-1)i}f^*\Omega_{\beta(-1)\alpha(-1)}+\sum_\beta f_{\beta 0i}f^*\Omega_{\beta 0\alpha(-1)}\\ &+\sum_{\beta k} f_{\beta ki}f^*\Omega_{\beta k\alpha(-1)}=\sum_j f_{\alpha(-1)i,j}\omega_j \end{aligned} \tag{3.36}$$

$$\begin{aligned} df_{\alpha 0i}&+\sum_j f_{\alpha 0j}\omega_{ji}-\sum_\beta f_{\beta(-1)i}f^*\Omega_{\beta(-1)\alpha 0}+\sum_\beta f_{\beta 0i}f^*\Omega_{\beta 0\alpha 0}\\ &+\sum_{\beta k} f_{\beta ki}f^*\Omega_{\beta k\alpha 0}=\sum_j f_{\alpha 0i,j}\omega_j \end{aligned} \tag{3.37}$$

$$\begin{aligned} df_{\alpha ki}&+\sum_j f_{\alpha kj}\omega_{ji}-\sum_\beta f_{\beta(-1)i}f^*\Omega_{\beta(-1)\alpha k}+\sum_\beta f_{\beta 0i}f^*\Omega_{\beta 0\alpha k}\\ &+\sum_{\beta j} f_{\beta ji}f^*\Omega_{\beta j\alpha k}=\sum_j f_{\alpha ki,j}\omega_j. \end{aligned} \tag{3.38}$$

It follows from (3.27)-(3.29) and (3.31)-(3.35) that

$$f_{\alpha(-1)i,j}=-\frac{1}{\sqrt{2}}(C^\alpha_{i,j}-\sum_k B^\alpha_{ik}A_{kj}+B^\alpha_{ij}) \tag{3.39}$$

$$f_{\alpha 0i,j}=-\frac{1}{\sqrt{2}}(C^\alpha_{i,j}-\sum_k B^\alpha_{ik}A_{kj}-B^\alpha_{ij}) \tag{3.40}$$

$$f_{\alpha ki,j}=-(B^\alpha_{ki,j}+C^\alpha_i\delta_{kj}). \tag{3.41}$$

Thus we know from (2.19) and (2.20) that the conformal Gauss map $f : \mathbf{M} \to \mathbf{G}^+_{n-m}(\mathbf{R}^{n+2}_1)$ is harmonic if and only if

$$\sum_i C^\alpha_{i,i} - \sum_k B^\alpha_{ik} A_{ki} = 0, \quad (m-2)C^\alpha_k = 0, \quad 1 \le k \le m, 1 \le \alpha \le n. \tag{3.42}$$

In case $m = 2$ the first equation of (3.42) is exactly the Euler-Lagrange equation for the Willmore functional (it is the Möbius volume functional, cf. [W]). The surfaces in $\mathbf{S}^n$ satisfying this equation are known as Willmore surfaces in $\mathbf{S}^n$. The conformal Gauss map of a surface in $\mathbf{S}^n$ has been studied by Bryant ([BR]) for $n = 3$ and Rigoli ([R]) for $n > 3$ by using complex coordinate on the surface. It follows immediately from (3.42) that

Theorem 3.2.([BR], [R]) *A surface $x : \mathbf{M} \to \mathbf{S}^n$ is Willmore if and only if its conformal Gauss map is harmonic.*

In case $m > 2$ we know that the conformal Gauss map of $x : \mathbf{M} \to \mathbf{S}^n$ is harmonic if and only if x satisfies

$$C^\alpha_k \equiv 0, \quad \sum_k B^\alpha_{ik} A_{ki} \equiv 0, \quad 1 \le k \le m, m+1 \le \alpha \le n. \tag{3.43}$$

Since for any Möbius isotropic submanifold we have $C^\alpha_k \equiv 0$ and $A_{ki} \equiv \lambda \delta_{ki}$ for some λ, which implies (3.42). Thus we complete the proof of Theorem 3.1.

§4. Conformal invariants for submanifolds in $\mathbf{R}^n$ and $\mathbf{H}^n$.

Let $\sigma : \mathbf{R}^n \to \mathbf{S}^n$ and $\tau : \mathbf{H}^n \to \mathbf{S}^n_+$ be the conformal maps definded by (1.3) and (1.4). Using σ and τ we can regard submanifolds in $\mathbf{R}^n$ and $\mathbf{H}^n$ as submanifolds in $\mathbf{S}^n$. In this section we give the conformal invariants for submanifolds in $\mathbf{R}^n$ and $\mathbf{H}^n$ and relate them to the Möbius invariants for submanifolds in $\mathbf{S}^n$. By using these relations we show that any minimal submanifolds with constant scalar curvature in $\mathbf{R}^n$, $\mathbf{H}^n$ and $\mathbf{S}^n$ are Möbius isotropic.

Let $x : \mathbf{M} \to \mathbf{S}^n$ be a minimal submanifold with constant scalar curvature. Then by the Gauss equation we know that $\rho^2 = \frac{m}{m-1}(||II||^2 - m||H||^2)$ is a constant. Thus from (1.1) and (1.2) we get

$$C^\alpha_i = 0 \quad A_{ij} = \frac{1}{2}\rho^{-2}\delta_{ij}.$$

By definition x is a Möbius isotropic submanifold in $\mathbf{S}^n$.

Let $u : \mathbf{M} \to \mathbf{R}^n$ be a submanifold without umbilics. Let $\{\tilde{e}_i\}$ be a local orthonormal basis for the first fundamental form $\tilde{I} = du \cdot du$ with the dual basis $\{\tilde{\theta}_i\}$. Let $\tilde{II} = \sum_{ij\alpha} \tilde{h}^\alpha_{ij}\tilde{\theta}_i\tilde{\theta}_j\tilde{e}_\alpha$ be the second fundamental form of u and $\tilde{H} = \sum_\alpha \tilde{H}^\alpha \tilde{e}_\alpha$ be the mean curvature vector of u, where $\{\tilde{e}_\alpha\}$ is a local orthonormal basis for the normal bundle of u. We define

$$\tilde{g} = \tilde{\rho}^2 du \cdot du, \quad \tilde{\rho}^2 = \frac{m}{m-1}(||\tilde{II}||^2 - m||\tilde{H}||^2) \tag{4.1}$$

$$\tilde{B}^\alpha_{ij} = \tilde{\rho}^{-1}(\tilde{h}^\alpha_{ij} - \tilde{H}_\alpha \delta_{ij}) \tag{4.2}$$

$$\tilde{C}^\alpha_i = -\tilde{\rho}^{-2}(\tilde{H}^\alpha_{,i} + \sum_j (\tilde{h}^\alpha_{ij} - \tilde{H}^\alpha \delta_{ij})\tilde{e}_j(\log \tilde{\rho})) \tag{4.3}$$

$$\begin{aligned}\tilde{A}_{ij} =& -\tilde{\rho}^{-2}(\mathrm{Hess}_{ij}(\log\tilde{\rho}) - \tilde{e}_i(\log\tilde{\rho})\tilde{e}_j(\log\tilde{\rho}) - \sum_\alpha \tilde{H}^\alpha \tilde{h}^\alpha_{ij}) \\ & - \frac{1}{2}\tilde{\rho}^{-2}(||\nabla \log\tilde{\rho}||^2 + \sum_\alpha (\tilde{H}^\alpha)^2)\delta_{ij}.\end{aligned} \tag{4.4}$$

We call the globally defined tensors $\tilde{g}$, $\tilde{\Phi} = \sum_{i\alpha} \tilde{C}^\alpha_i \tilde{\theta}_i \tilde{e}_\alpha$, $\tilde{\mathbb{A}} := \tilde{\rho}^2 \sum_{ij} \tilde{A}_{ij}\tilde{\theta}_i\tilde{\theta}_j$ and $\tilde{\mathbb{B}} = \tilde{\rho}\sum_{ij\alpha} \tilde{B}^\alpha_{ij}\tilde{\theta}_i\tilde{\theta}_j\tilde{e}_\alpha$ the Möbius metric, the Möbius form, the Blaschke tensor and the Möbius second fundamental form of $u : \mathbf{M} \to \mathbf{R}^n$. Now let $\sigma : \mathbf{R}^n \to \mathbf{S}^n$ be the conformal map given by (1.3). We define $x := \sigma \circ u : \mathbf{M} \to \mathbf{S}^n$. Then x is a submanifold in $\mathbf{S}^n$ without umbilics. We denote by Φ and $\mathbb{A}$ the Möbius form and the Blaschke tensor of x defined by (1.1) and (1.2) and denote by g and $\mathbb{B}$ the Möbius metric and the Möbius second fundamental form defined by (2.3) and (2.13) for $x = \sigma \circ u$. Our goal in this section is to prove the following

Theorem 4.1. *$g = \tilde{g}$, $\mathbb{B} = d\sigma(\tilde{\mathbb{B}})$, $\Phi = d\sigma(\tilde{\Phi})$ and $\mathbb{A} = \tilde{\mathbb{A}}$. In particular, $\{\tilde{g}, \tilde{\mathbb{B}}, \tilde{\Phi}, \tilde{\mathbb{A}}\}$ are conformal invariants for submanifolds in $\mathbf{R}^n$.*

Let $\sigma : \mathbf{R}^n \to \mathbf{S}^n$ be the conformal map given by

$$x = \sigma(u) = \left(\frac{1 - ||u||^2}{1 + ||u||^2}, \frac{2u}{1 + ||u||^2}\right), \quad u \in \mathbf{R}^n. \tag{4.5}$$

Then for any vector $V \in T_u\mathbf{R}^n$ we have

$$d\sigma(V) = \frac{2}{1 + ||u||^2}\{-(u \cdot V)x + (-u \cdot V, V)\}. \tag{4.6}$$

Thus we get

$$dx \cdot dx = \frac{4}{(1 + ||u||^2)^2} du \cdot du. \tag{4.7}$$

Now let $u : \mathbf{M} \to \mathbf{R}^n$ be a submanifold and $x = \sigma \circ u : \mathbf{M} \to \mathbf{S}^n$. We denote by $\{\tilde{e}_i\}$ and $\{\tilde{e}_\alpha\}$ local orthonormal basis for $du \cdot du$ and the norml bundle of u respectively and define

$$e_i = \frac{1 + ||u||^2}{2}\tilde{e}_i, \quad e_\alpha = \frac{1 + ||u||^2}{2} d\sigma(\tilde{e}_\alpha). \tag{4.8}$$

Then $\{e_i\}$ is a local orthonormal basis for $dx \cdot dx$ with dual basis $\{\theta_i\}$ and $\{e_\alpha\}$ is a local orthonormal basis for the normal bundle of x in $\mathbf{S}^n$. It follows from (4.6) that

$$e_i(x) = \frac{1 + ||u||^2}{2} d\sigma(\tilde{e}_i(u)) = -(u \cdot \tilde{e}_i(u))x + (-u \cdot \tilde{e}_i(u), \tilde{e}_i(u)) \tag{4.9}$$

$$\begin{aligned} e_\alpha =& \frac{1 + ||u||^2}{2} d\sigma(\tilde{e}_\alpha) = -\frac{2u \cdot \tilde{e}_\alpha}{1 + ||u||^2}(1, u) + (0, \tilde{e}_\alpha) \\ =& -(u \cdot \tilde{e}_\alpha)x + (-u \cdot \tilde{e}_\alpha, \tilde{e}_\alpha).\end{aligned} \tag{4.10}$$

By (4.9) we get

$$e_ie_j(x) = \frac{1+||u||^2}{2}((-\delta_{ij}, \tilde{e}_i(u)) + (-u\cdot\tilde{e}_j\tilde{e}_i(u), \tilde{e}_j\tilde{e}_i(u))) \mod (x, e_i(x)). \tag{4.11}$$

Thus (4.10) and (4.11) yield

$$h^\alpha_{ij} = \frac{1+||u||^2}{2}\tilde{h}^\alpha_{ij} + \tilde{e}_\alpha\cdot u\delta_{ij}, \quad H^\alpha = \frac{1+||u||^2}{2}\tilde{H}^\alpha + \tilde{e}_\alpha\cdot u. \tag{4.12}$$

It follows from (4.12) and (4.7) that

$$\rho^2 = \frac{(1+||u||^2)^2}{4}\tilde{\rho}^2 \tag{4.13}$$

$$g = \rho^2\mathrm{d}x\cdot\mathrm{d}x = \tilde{\rho}^2\mathrm{d}u\cdot\mathrm{d}u := \tilde{g}. \tag{4.14}$$

We call $\tilde{g}$ the Möbius metric for submanifolds in $\mathbf{R}^n$. It is clear that $\tilde{g}$ is a conformal invariant. By (4.12) and (4.13) we get

$$B^\alpha_{ij} = \rho^{-1}(h^\alpha_{ij} - H^\alpha\delta_{ij}) = \tilde{\rho}^{-1}(\tilde{h}^\alpha_{ij} - \tilde{H}^\alpha\delta_{ij}) := \tilde{B}^\alpha_{ij}. \tag{4.15}$$

By (4.10) we get

$$\mathrm{d}e_\alpha = (-u\cdot\mathrm{d}\tilde{e}_\alpha, \mathrm{d}\tilde{e}_\alpha) \mod (x, \mathrm{d}x),$$

which implies that

$$\theta_{\alpha\beta} = \mathrm{d}e_\alpha\cdot e_\beta = \mathrm{d}\tilde{e}_\alpha\cdot\tilde{e}_\beta = \tilde{\theta}_{\alpha\beta}. \tag{4.16}$$

Let $\{H^\alpha_{,i}\}$ and $\{\tilde{H}^\alpha_{,i}\}$ be the covariant derivative of the mean curvature vector in the normal bundle of $x = \sigma\circ u : \mathbf{M}\to\mathbf{S}^n$ and $u : \mathbf{M}\to\mathbf{R}^n$ respectively. By definition we have

$$\mathrm{d}H^\alpha + \sum_\beta H^\beta\theta_{\beta\alpha} = \sum_i H^\alpha_{,i}\theta_i, \quad \mathrm{d}\tilde{H}^\alpha + \sum_\beta \tilde{H}^\beta\tilde{\theta}_{\beta\alpha} = \sum_i \tilde{H}^\alpha_{,i}\tilde{\theta}_i.$$

Since $\tilde{\theta}_i = \frac{1+||u||^2}{2}\theta_i$, from (4.12) and (4.16) we get

$$H^\alpha_{,i} = \left(\frac{1+||u||^2}{2}\right)^2\tilde{H}^\alpha_{,i} - \frac{1+||u||^2}{2}\sum_j(\tilde{h}^\alpha_{ij} - \tilde{H}^\alpha\delta_{ij})(\tilde{e}_j(u)\cdot u). \tag{4.17}$$

By (4.13) we get

$$e_j(\log\rho) = \frac{1+||u||^2}{2}\tilde{e}_j(\log\tilde{\rho}) + \tilde{e}_j(u)\cdot u. \tag{4.18}$$

We define $\{C^\alpha_i\}$ and $\{\tilde{C}^\alpha_i\}$ by (1.1) and (4.3) respectively. It follows from (4.17) and (4.18) that

$$C^\alpha_i = \tilde{C}^\alpha_i. \tag{4.19}$$

Let $\{\theta_{ij}\}$ and $\{\tilde{\theta}_{ij}\}$ be the Levi-Civita connections of $dx \cdot dx$ and $du \cdot du$ with respect to the basis $\{e_i\}$ and $\{\tilde{e}_i\}$. Then by (4.7) we have

$$\theta_{ij} = \tilde{\theta}_{ij} + \frac{2u \cdot \tilde{e}_j(u)}{1+||u||^2}\tilde{\theta}_i - \frac{2u \cdot \tilde{e}_i(u)}{1+||u||^2}\tilde{\theta}_j. \tag{4.20}$$

We define the $\mathrm{Hess}_{ij}(\log \rho)$ and $\mathrm{Hess}_{ij}(\log \tilde{\rho})$ by

$$de_i(\log \rho) + \sum_j e_j(\log \rho)\theta_{ji} = \sum_j \mathrm{Hess}_{ij}(\log \rho)\theta_j$$
$$d\tilde{e}_i(\log \tilde{\rho}) + \sum_j \tilde{e}_j(\log \tilde{\rho})\tilde{\theta}_{ji} = \sum_j \mathrm{Hess}_{ij}(\log \tilde{\rho})\tilde{\theta}_j.$$

Using (4.18) and (4.20) we get

$$\begin{aligned}\mathrm{Hess}_{ij}(\log \rho) &= \left(\frac{1+||u||^2}{2}\right)^2 \mathrm{Hess}_{ij}(\log \tilde{\rho}) + \frac{1+||u||^2}{2}\Big(\sum_\alpha \tilde{h}^\alpha_{ij}(\tilde{e}_\alpha \cdot u)\\ &+ (u \cdot \tilde{e}_j(u))\tilde{e}_i(\log \tilde{\rho}) + (u \cdot \tilde{e}_i(u))\tilde{e}_j(\log \tilde{\rho}) + (u \cdot \tilde{e}_i(u))(u \cdot \tilde{e}_j(u))\Big)\\ &+ \left(\frac{1+||u||^2}{2} - \frac{1+||u||^2}{2}\sum_k (u \cdot \tilde{e}_k(u))\tilde{e}_k(\log \tilde{\rho}) - \sum_k (u \cdot \tilde{e}_k(u))^2\right)\delta_{ij}.\end{aligned} \tag{4.21}$$

Using (4.12) and (4.18) we get

$$\begin{aligned}& e_i(\log \rho)e_j(\log \rho) + \sum_\alpha H^\alpha h^\alpha_{ij}\\ &= \left(\frac{1+||u||^2}{2}\right)^2 \Big(\tilde{e}_i(\log \tilde{\rho})\tilde{e}_j(\log \tilde{\rho})\\ &+ \sum_\alpha \tilde{H}^\alpha \tilde{h}^\alpha_{ij}\Big) + \frac{1+||u||^2}{2}\left(\tilde{e}_i(\log \tilde{\rho})(\tilde{e}_j(u) \cdot u) + \tilde{e}_j(\log \tilde{\rho})(\tilde{e}_i(u) \cdot u)\right)\\ &+ (\tilde{e}_i(u) \cdot u)(\tilde{e}_j(u) \cdot u) + \frac{1+||u||^2}{2}\tilde{h}^\alpha_{ij}(\tilde{e}_\alpha \cdot u)\\ &+ \left(\sum_\alpha (\tilde{e}_\alpha \cdot u)^2 + \frac{1+||u||^2}{2}\sum_\alpha (\tilde{e}_\alpha \cdot u)\tilde{H}^\alpha\right)\delta_{ij}\end{aligned} \tag{4.22}$$

$$\begin{aligned}\frac{1}{2}(||\nabla \log \rho||^2 - 1 + \sum_\alpha (H^\alpha)^2) &= \left(\frac{1+||u||^2}{2}\right)^2\left(\frac{1}{2}||\nabla \log \tilde{\rho}||^2 + \sum_\alpha (\tilde{H}^\alpha)^2\right)\\ &+ \frac{(1+||u||^2)}{2}\Big(\sum_k \tilde{e}_k(\log \tilde{\rho})(\tilde{e}_k(u) \cdot u) + \sum_\alpha \tilde{H}^\alpha(\tilde{e}_\alpha \cdot u)\Big)\\ &+ \frac{1}{2}\sum_k (u \cdot \tilde{e}_k(u))^2 + \frac{1}{2}\sum_\alpha (u \cdot \tilde{e}_\alpha(u))^2 - \frac{1}{2}.\end{aligned} \tag{4.23}$$

Let $\{A_{ij}\}$ and $\{\tilde{A}_{ij}\}$ be the tensor defined by (1.2) and (4.4), we get from (4.13), (4.21), (4.22) and (4.23) that

$$A_{ij} = \tilde{A}_{ij}. \tag{4.24}$$

Now we come to the proof of Theorem 4.1. It follows from (4.14) that $g = \tilde{g}$. We take $\omega_i = \rho\theta_i = \tilde{\rho}\tilde{\theta}_i$. Then by (4.24) we get $\mathbb{A} = \tilde{\mathbb{A}}$. From (4.8) and (4.13) we get $d\sigma(\tilde{\rho}^{-1}\tilde{e}_\alpha)) = \rho^{-1}e_\alpha$. Thus we get from (4.15) and (4.19) that $d\sigma(\tilde{\mathbb{B}}) = \mathbb{B}$ and $d\sigma(\tilde{\Phi}) = \Phi$. We complete the proof of Theorem 4.1.

It follows from (4.3) and (4.4) that

Theorem 4.2. *The images of σ of minimal submanifolds with constant scalar curvature in $\mathbf{R}^n$ are Möbius isotropic submanifolds in $\mathbf{S}^n$.*

Let $\mathbf{R}^{n+1}_1$ be the Lorentzian space with inner product

$$< y, w >= -y_0w_0 + y_1w_1 + \cdots + y_nw_n, \quad y = (y_0, \cdots, y_n), w = (w_0, \cdots, w_n).$$

Let $\mathbf{H}^n = \{y \in R^{n+1}_1 \mid < y, y >= -1, y_0 > 0\}$ be the hyperbolic space. We define now the conformal invariants for the submanifolds in $\mathbf{H}^n$. Let $y : \mathbf{M} \to \mathbf{H}^n$ be a submanifold without umbilics. Let $\{\hat{e}_i\}$ be a local orthonormal basis for $< dy, dy >$ with dual basis $\{\hat{\theta}_i\}$. Let $\hat{II} = \sum_{\alpha ij} \hat{h}^\alpha_{ij}\hat{\theta}_i\hat{\theta}_j\hat{e}_\alpha$ be the second fundamental form of y and $\hat{H} = \sum_\alpha \hat{H}^\alpha\hat{e}_\alpha$ the mean curvature vector of y, where $\{\hat{e}_\alpha\}$ is a local orthonormal basis for the normal bundle of y. Wedefine

$$\hat{g} =\hat{\rho}^2 < dy, dy >, \quad \hat{\rho}^2 = \frac{m}{m-1}(||\hat{II}||^2 - m||\hat{H}||^2) \tag{4.25}$$

$$\hat{B}^\alpha_{ij} =\hat{\rho}^{-1}(\hat{h}^\alpha_{ij} - \hat{H}_\alpha\delta_{ij}) \tag{4.26}$$

$$\hat{C}^\alpha_i =-\hat{\rho}^{-2}(\hat{H}^\alpha_{,i} + \sum_j(\hat{h}^\alpha_{ij} - \hat{H}^\alpha\delta_{ij})\hat{e}_j(\log\hat{\rho})) \tag{4.27}$$

$$\begin{aligned}\hat{A}_{ij} =& -\hat{\rho}^{-2}(\mathrm{Hess}_{ij}(\log\hat{\rho}) - \hat{e}_i(\log\hat{\rho})\hat{e}_j(\log\hat{\rho}) - \sum_\alpha \hat{H}^\alpha\hat{h}^\alpha_{ij}) \\ &-\frac{1}{2}\hat{\rho}^{-2}(||\nabla\log\hat{\rho}||^2 + 1 + \sum_\alpha(\hat{H}^\alpha)^2)\delta_{ij}.\end{aligned} \tag{4.28}$$

We call $\hat{g}$ the Möbius metric of y, $\hat{\mathbb{B}} = \hat{\rho}\sum_{ij\alpha}\hat{B}^\alpha_{ij}\hat{\theta}_i\hat{\theta}_j\hat{e}_\alpha$ the Möbius second fundamental form of y, $\hat{\Phi} = \sum_{i\alpha}\hat{C}^\alpha_i\hat{\theta}_i\hat{e}_\alpha$ the Möbius form of y and $\hat{\mathbb{A}} = \sum_{ij}\hat{\rho}^2\hat{A}_{ij}\hat{\theta}_i\hat{\theta}_j$ the Blaschke tensor of y.

Set $D^n = \{u \in \mathbf{R}^n \mid ||u||^2 < 1\}$. Let $\mu : D^n \to \mathbf{H}^n$ be the conformal diffeomorphism given by

$$\mu(u) = \left(\frac{1+||u||^2}{1-||u||^2}, \frac{2u}{1-||u||^2}\right). \tag{4.29}$$

Then $u = \mu^{-1}\circ y : \mathbf{M} \to D^n$ is a submanifold without umbilics. We denote by $\{\tilde{g}, \tilde{\mathbb{B}}, \tilde{\Phi}, \tilde{\mathbb{A}}\}$ the basic Möbius invariants for $u = \mu^{-1}\circ y : \mathbf{M} \to D^n \subset \mathbf{R}^n$. Using the same method as in the proof of Theorem 4.1 we can prove that

Theorem 4.3. *$\hat{g} = \tilde{g}$, $\hat{\mathbb{B}} = d\mu(\tilde{\mathbb{B}})$, $\hat{\Phi} = d\mu(\tilde{\Phi})$ and $\hat{\mathbb{A}} = \tilde{\mathbb{A}}$. In particular, $\{\hat{g}, \hat{\mathbb{B}}, \hat{\Phi}, \hat{\mathbb{A}}\}$ are conformal invariants for submanifolds in $\mathbf{H}^n$.*

Let $\tau : \mathbf{H}^n \to \mathbf{S}^n_+$ be the conformal diffeomorphism defined by (1.4). Then we have $\tau = \sigma \circ \mu^{-1}$. Thus from Theorem 4.1 and Theorem 4.3 we get

Theorem 4.4. *Let $y : \mathbf{M} \to \mathbf{H}^n$ be a submanifold without umbilics. Let $x = \tau \circ y : \mathbf{M} \to \mathbf{S}^n_+$. Then we have*

$$g = \hat{g}, \quad \mathbb{B} = d\tau(\hat{\mathbb{B}}), \quad \Phi = d\tau(\hat{\Phi}), \quad \mathbb{A} = \hat{\mathbb{A}}.$$

In particular, $\{\hat{g}, \hat{\mathbb{B}}, \hat{\Phi}, \hat{\mathbb{A}}\}$ are conformal invariants for submanifolds in $\mathbf{H}^n$.

It follows immediately from (4.27) and (4.28) that

Theorem 4.5. *The images of τ of minimal submanifolds with constant scalar curvature in $\mathbf{H}^n$ are Möbius isotropic submanifolds in $\mathbf{S}^n$.*

§5. The classification of Möbius isotropic submanifolds in $\mathbf{S}^n$.

In this section we prove the classification theorem mentioned in §1.

Let $x : \mathbf{M} \to \mathbf{S}^n$ be a Möbius isotropic submanifold. By definition we have

$$A_{ij} = \lambda\delta_{ij}, \quad C_i^\alpha \equiv 0. \tag{5.1}$$

It follows from (2.10) and Proposition 2.3 that

$$dN = \lambda dY, \tag{5.2}$$

for some constant λ. Using (5.1) and the last equation in (2.19) we get

$$A_{ij} = \frac{1}{2m^2}(1 + m^2\kappa)\delta_{ij}, \quad \kappa = \text{constant}, \tag{5.3}$$

where κ is the normalized scalar curvature of the Möbius metric. By (5.2) we can find a constant vector $\mathbf{c} \in \mathbf{R}^{n+2}_1$ such that

$$N = \frac{1}{2m^2}(1 + m^2\kappa)Y + \mathbf{c}. \tag{5.4}$$

It follows from (5.4) and (2.5) that

$$< \mathbf{c}, \mathbf{c} >= -\frac{1}{m^2}(1 + m^2\kappa), \quad < Y, \mathbf{c} >= 1. \tag{5.5}$$

Then we consider the following three cases: (i) $\mathbf{c}$ is timelike; (ii) $\mathbf{c}$ is lightlike; (iii) $\mathbf{c}$ is spacelike.

First we consider the case (i) that $< \mathbf{c}, \mathbf{c} >= -r^2$ with $r = \frac{1}{m}\sqrt{1 + m^2\kappa} > 0$. By (2.2) and $< Y, N >= 1$ we know that the first coordinate of Y is positive and of N is negative. Thus by (5.4) we know that the first coordinate of $\mathbf{c}$ is negative. So there exists a $T \in O^+(n+1,1)$ such that

$$(-r,0) = \mathbf{c}T = NT - \frac{r^2}{2}YT. \tag{5.6}$$

Let $\tilde{x} : \mathbf{M} \to \mathbf{S}^n$ be the submanifold which is Möbius equivalent to x such that $\tilde{Y} = YT$ (cf. Theorem 2.1). Then we have $\tilde{N} = NT$. Since

$$\mathbf{c}T = (-r,0), \quad <\tilde{Y}, \mathbf{c}T>= 1, \quad \tilde{Y} = \tilde{\rho}(1,\tilde{x}), \tag{5.7}$$

we get

$$\tilde{\rho} = r^{-1} = \text{constant}. \tag{5.8}$$

It follows from (5.6) and (2.4) that

$$(-r,0) = \tilde{N} - \frac{r^2}{2}\tilde{Y}, \quad \tilde{N} = -\frac{1}{m}\tilde{\Delta}\tilde{Y} - \frac{1}{2}r^2\tilde{Y}. \tag{5.9}$$

Since $\tilde{\rho} = r^{-1}$, we know from $\tilde{g} = \tilde{\rho}^2 d\tilde{x} \cdot d\tilde{x}$ that the Laplacian operator $\Delta_{\mathbf{M}}$ of $d\tilde{x} \cdot d\tilde{x}$ is given by $\Delta_{\mathbf{M}} = \tilde{\rho}^2\tilde{\Delta}$. Thus by (5.9) we get

$$\Delta_{\mathbf{M}}\tilde{x} + m\tilde{x} = 0. \tag{5.10}$$

By Takahashi theorem ([T]) we know that $\tilde{x} : \mathbf{M} \to \mathbf{S}^n$ is a minimal submanifold. The normalized scalar curvature $\tilde{\kappa}$ of $d\tilde{x} \cdot d\tilde{x}$ is a constant given by

$$\tilde{\kappa} = \tilde{\rho}^2\kappa = \frac{m^2\kappa}{1+m^2\kappa}. \tag{5.11}$$

Next we consider the case (ii) that $<\mathbf{c},\mathbf{c}>= 0$. By making a Möbius transformation if necessary we may assume that $\mathbf{c} = (-1,1,0)$. Thus by (5.4) and (2.4) we have

$$\mathbf{c} = (-1,1,0) = N = -\frac{1}{m}\Delta Y. \tag{5.12}$$

We write $x = (x_0, x_1)$. Then $Y = (\rho, \rho x_0, \rho x_1)$. By (5.5) and (5.12) we get $<Y,\mathbf{c}>= \rho(1+x_0) = 1$, which implies that $x_0 \neq -1$ and $x(\mathbf{M}) \subset \mathbf{S}^n \backslash \{(-1,0)\}$. Now let $\sigma^{-1} : \mathbf{S}^n \backslash \{(-1,0)\} \to \mathbf{R}^n$ be the stereographic projection from the point $(-1,0) \in \mathbf{S}^n$. We define $u = \sigma^{-1} \circ x : \mathbf{M} \to \mathbf{R}^n$. Then by (1.3) we have

$$Y = \rho(1,x) = \left(\rho, \frac{\rho(1-||u||^2)}{1+||u||^2}, \frac{2\rho u}{1+||u||^2}\right). \tag{5.13}$$

From $<Y,\mathbf{c}>= 1$ we get $\rho = \frac{1+||u||^2}{2}$. Thus we get from (5.13) that

$$Y = \left(\frac{1+||u||^2}{2}, \frac{1-||u||^2}{2}, u\right).$$

The Möbius metric of x is given by

$$g =< \mathrm{d}Y, \mathrm{d}Y >= \mathrm{d}u \cdot \mathrm{d}u, \tag{5.14}$$

which is exactly the first fundamental form of $u = \sigma^{-1} \circ x : \mathbf{M} \to \mathbf{R}^n$. In particular, the Laplacian operator Δ of g is equal to the Laplacian operator of $\mathrm{d}u \cdot \mathrm{d}u$. Comparing the last coordinate in (5.12) we get $\Delta u = 0$. Thus $u = \sigma^{-1} \circ x : \mathbf{M} \to \mathbf{R}^n$ is a minimal submanidold. By (5.14) and (5.4) we know that the normalized scalar curvature of u is exactly the scalar curvature κ of g. Since $< \mathbf{c}, \mathbf{c} >= -\frac{1}{m^2}(1 + m^2\kappa) = 0$, we get $\kappa = -\frac{1}{m^2}$.

Finally we consider the case that $< \mathbf{c}, \mathbf{c} >= r^2$ with $r = \frac{1}{m}\sqrt{-(1 + m^2\kappa)} > 0$. By making a Möbius transformation if necessary we may assume that $\mathbf{c} = (0, r, 0)$. We write $x = (x_0, x_1)$. Then $Y = (\rho, \rho x_0, \rho x_1)$. It follows from (5.5) that $< Y, \mathbf{c} >= \rho r x_0 = 1$, which implies that $x_0 > 0$ and $x(\mathbf{M}) \subset \mathbf{S}^n_+$. Now let $\tau : \mathbf{H}^n \to \mathbf{S}^n_+$ be the conformal diffeomorphsim defined by (1.4) and $y = \tau^{-1} \circ x : \mathbf{M} \to \mathbf{H}^n \subset \mathbf{R}^{n+1}_1$. Since $< Y, \mathbf{c} >= \rho r x_0 = 1$, we get $x_0 = \frac{1}{r\rho}$. By (1.4) we get $y_0 = \frac{1}{x_0} = r\rho$ and

$$Y = (\rho, \rho x_0, \rho x_1) = (\frac{y_0}{r}, \frac{1}{r}, \frac{y_1}{r}). \tag{5.15}$$

It follows that

$$g =< \mathrm{d}Y, \mathrm{d}Y >= r^{-2} < \mathrm{d}y, \mathrm{d}y > . \tag{5.16}$$

The Laplacian operator $\Delta_{\mathbf{M}}$ of $< \mathrm{d}y, \mathrm{d}y >$ is given by $\Delta_{\mathbf{M}} = r^{-2}\Delta$. By (5.4) and (2.4) we have

$$-\frac{1}{m}\Delta Y + \frac{r^2}{2}Y = -\frac{r^2}{2}Y + (0, r, 0), \tag{5.17}$$

which is equivalent to the equation

$$\Delta_{\mathbf{M}} y - my = 0. \tag{5.18}$$

Thus $y = \tau^{-1} \circ x : \mathbf{M} \to \mathbf{H}^n$ is a minimal submanifold. Since the Möbius metric g has constant scalar curvature, we know from (5.16) that $y : \mathbf{M} \to \mathbf{H}^n$ has constant scalar curvature.

Thus we complete the proof of the classification theorem.

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