On Hilbert's tenth problem: Is classical set theory inconsistent?

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Abstract

We consider cartesian categorical (free-variables) theory **PR** of primitive recursion and arithmetise (gödelise) it into the natural numbers set of a classical **set** theory **T**. We evaluate the map codes of the coded theory by a general recursive **T** map and construct a μ-recursive decision algorithm based on evaluation of primitive recursive map codes. Within theory **T** strengthend by p. r. internal inconsistency **axiom,** the predicate decision algorithm turns out to be total, terminating. It decides in a uniform way all diophantine equations and contradicts within the strengthend theory Matiyasevich's

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negative solution of Hilbert's 10th problem. But by Gödel's second incompleteness theorem the strengthend theory is relative consistent to **T**. This is to show inconsistency of classical **set** theorie(s).

Contents

1 Overview

1. Axioms of cartesian categorical free-variables theory **PR** of primitive recursion with equality definability theorem are recalled. Mentioned is embedding extension of **PR** into theory **PRa** with abstraction of primitive recursive ("p. r.") predicates into subsets of objects of **PR**.

2. Consider a classical, quantified arithmetical **set** theory **T** with quantifiers which has in particular terms for all primitive recursive maps; **T** is to be one of Principia Math-

ematica **PM** or Zermelo-Fraenkel **set** theory **ZF**, or v. Neumann Gödel Bernays **set** theory **NGB**.

3. Theory **PRa** is gödelised into internal theory PRa $\subset \mathbb{N}$ within classical **set** theorie(s) **T**.

4. This latter theory admits (general) recursive evaluation of internal, gödelised theory PRa into **T**.

5. By use of evaluation it is shown that theory **T** admits a μ-recursive decision map/algorithm for decision of p. r. predicates/subsets.

6. Theory **T** has a strengthening $\tilde{\mathbf{T}} = \mathbf{T} + \neg\mathrm{Con}_{\mathbf{T}}$ of **T** by **axiom** $\neg\mathrm{Con}_{\mathbf{T}}$ of internal, *arithmetised* inconsistency.

7. Within theory $\tilde{\mathbf{T}}$ the μ-recursive predicate decision algorithm *terminates,* is totally defined. It decides there all p. r. predicates into availability of counterexamples vs. overall truth. It decides there in particular – in a uniform way – diophantine equations in the sense of Hilbert's 10th problem as stated by Matiyasevich.

8. This author's negative solution of that problem within **T**, taken as theorem in stronger theory $\tilde{\mathbf{T}}$, contradicts there (uniform) decidability of diophantine equations.

9. So $\tilde{\mathbf{T}}$ must be inconsistent and so are classical **set** theories **T** as well, by Gödel's second incompleteness theorem stating relative consistency of $\neg\mathrm{Con}_{\mathbf{T}}$ over **T**.

2 Primitive Recursion

2.1 Cartesian language

Free-variables cartesian "but" categorical language starts here with cartesian basic one object $\mathbb{1}$ and *natural numbers object* "NNO" $\mathbb{N}$, and their (nested) formal cartesian products, coming with (formal) left and right projections

$$\ell = \ell_{A,B} : A \times B \to B \text{ and } \mathrm{r} = \mathrm{r}_{A,B} : A \times B \to B.$$

We ***define/interpret*** *free variables* as *identity maps* resp. left or right *projections* – possibly nested – out of cartesian products, onto their *factors*.

A special rôle is played by *terminal object* $\mathbb{1}$. It works as the *empty cartesian product* $\mathbb{N}^0$, comes with a (unique) "projection" map $\Pi : A \to \mathbb{1}$ for each object A, and is the *domain* object for *concrete "elements"* $\boldsymbol{a} : \mathbb{1} \to A$ of A, in particular for *(concrete) numbers* $\boldsymbol{n} : \mathbb{1} \to \mathbb{N}$.

We first state the **axioms** for *cartesian theory* **CA** :

$$\frac{f : A \to B}{\begin{gathered} f \circ \mathrm{id} = f \circ \mathrm{id}_A = f; \\ \mathrm{id} \circ f = \mathrm{id}_B \circ f = f \end{gathered}}$$

neutrality *of identities to composition.*

$$\frac{f : A \to B;\ g : B \to C;\ h : C \to D}{\begin{gathered} (h \circ g) \circ f = h \circ (g \circ f) : A \to D \\ = h \circ g \circ f = h\, g\, f = h(g(f(a)))) \end{gathered}}$$

associativity *of composition.*

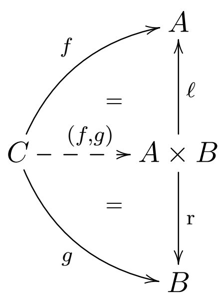
$$\frac{f : A \to \mathbb{1}}{f = \Pi_A}$$

uniqueness of terminal map.

$$\frac{f : C \to A,\ g : C \to B}{(f, g) : C \to A \times B}$$

(unique) ***induced*** *map into product:*

$\ell \circ (f, g) = f,\ \mathrm{r} \circ (f, g) = g$



Godement's diagram

2.2 Theory PR of primitive recursion

Add to cartesian theory **CA** *iteration* **axioms:**

$f : A \to A$ (endomap)

$f^{\S} = f^{\S}(a, n) : A \times \mathbb{N} \to A$ (*iterated*);

$f^{\S}(a, 0) := f^{\S}(\mathrm{id}_A, 0_A) = f^{\S}(\mathrm{id}_A, 0\,\Pi_A) = a = \mathrm{id}_A :$

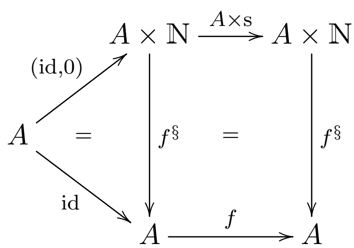
$A \to A \times \mathbb{N}$ (*anchoring*);

$f^{\S} \circ (A \times \mathrm{s}) = f^{\S}(a, \mathrm{s}\,n) = f \circ f^{\S} = f(f^{\S}(a, n)) :$

$A \times \mathbb{N} \to A \to A$ (*iteration step*);

$f^n(a) := f^{\S}(a, n)$

apply *iteratively endomap f to initial argument a, iterate n times.*



Iteration DIAGRAM

$f : A \to B;\ g : B \to B;\ h : A \times \mathbb{N} \to B;$

$h(a, 0) = f(a);$

$h(a, \mathrm{s}\, n) = g\, h(a, n)$

$h = g^{\S}\, (f \times \mathrm{id}_{\mathbb{N}})$ i. e.

$h(a, n) = g^n(f(a)) : A \times \mathbb{N} \to B :$

Freyd's ***uniqueness*** *of the iterated endomap g*

initialised *by map f*

3 Evaluation

Crucial for present approach to Hilbert's decidability problem is availability – within $\mathbf{T}$ as well as in $\tilde{\mathbf{T}} = \mathbf{T} + \neg\mathrm{Con}_{\mathbf{T}}$ – of a (general) recursive *evaluation* map

$$\boldsymbol{ev} = \boldsymbol{ev}(\chi, n) : [\mathbb{N}, 2]_{\mathbf{PRa}} \times \mathbb{N} \to 2 = \{0, 1\}$$

on the $\mathbf{T}$-internal (primitive recursively decidable) *code set* (gödel numbers set) $[\mathbb{N}, 2]_{\mathbf{PRa}}$,

$$\chi \in [\mathbb{N}, 2]_{\mathbf{PRa}} \subset \mathrm{PRa} = \bigcup_{A,B} [A, B]_{\mathbf{PRa}} \subset \mathbb{N}$$

Explication: Primitive recursive *predicates* are viewed[1] as p. r. *maps* with codomain $2 \subset \mathbb{N}$ within cartesian categorical (free-variables) theory

$$\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$$

[1] Reiter 1982

of primitive recursion with interpretation of **PR** predicates as additional objects, "subsets". Theory **PRa** is an embedding extension of **PR**.

Evaluation map ***ev*** is defined in **T** by *(nested) double recursion* à la Ackermann (see PÉTER 1967), and satisfies – for p. r. predicate $\varphi = \varphi(n) : \mathbb{N} \to \mathbb{2}$ – the characteristic equation

$$\boldsymbol{ev}(\ulcorner \varphi \urcorner, n) = \varphi(n)$$

Evaluation in **detail:**

Evaluation family $\boldsymbol{ev} = [\boldsymbol{ev}_{A,B} : [A, B]_{\mathbf{PRa}} \times A \to B]$ is recursively **defined** by

$$\boldsymbol{ev}(\ulcorner \mathrm{ba} \urcorner, x) = \mathrm{ba}(x)$$

for $\mathrm{ba} \in \mathrm{bas} = \{0, \mathrm{s}\} \cup \{\Pi_A, \ell_{A,B}, \mathrm{r}_{A,B} : A, B \text{ objects}\}$

in particular

$$\boldsymbol{ev}_{\mathbb{N},\mathbb{N}}(\mathrm{s}, n) = \mathrm{s}(n)$$

$$\boldsymbol{ev}_{A \times B, A}(\ell_{A,B}, (a, b)) = \ell_{A,B}(a, b) = a$$

as well as recursively

$$\boldsymbol{ev}_{A,C}(g \ulcorner \circ \urcorner f, a) = \boldsymbol{ev}_{B,C}(g, \boldsymbol{ev}_{A,B}(f, a))$$

$$\boldsymbol{ev}_{C,A \times B}(\langle f; g \rangle, c) = (\boldsymbol{ev}_{C,A}(f, c), \boldsymbol{ev}_{C,B}(g, c))$$

$$\boldsymbol{ev}_{A \times \mathbb{N},A}(f^{\ulcorner \S \urcorner}, (a, 0)) = \mathrm{id}_A(a) = a$$

$$\boldsymbol{ev}_{A \times \mathbb{N},A}(f^{\ulcorner \S \urcorner}, (a, \mathrm{s}\, n)) = \boldsymbol{ev}_{A,A}(f, \boldsymbol{ev}_{A \times \mathbb{N}}(f^{\ulcorner \S \urcorner}, (a, n))$$

Objectivity theorem

For $f : A \to B$ in **PRa**

$$\boldsymbol{ev}_{A,B}(\ulcorner f \urcorner, a) = f(a) : A \to B$$

Proof by nested recursion:

anchor: The theorem holds for $f \in \mathrm{bas}$ by definition of $\boldsymbol{ev}$.

steps:

$\boldsymbol{ev}_{A,C}(\ulcorner g \circ f \urcorner, a) \ =_{\text{by def}} \ \boldsymbol{ev}_{A,C}(\ulcorner g \urcorner \ulcorner \circ \urcorner \ulcorner f \urcorner, a)$

$= \boldsymbol{ev}_{B,C}(\ulcorner g \urcorner, \boldsymbol{ev}_{A,B}(\ulcorner f \urcorner, a))$

$= g(f(a)) = (g \circ f)(a)$

by recursion hypothesis

$\boldsymbol{ev}(\ulcorner (f, g) \urcorner, c) \ =_{\text{by def}} \ \boldsymbol{ev}(\langle \ulcorner f \urcorner; \ulcorner g \urcorner \rangle)$

$= (\boldsymbol{ev}(\ulcorner f \urcorner, c), \boldsymbol{ev}(\ulcorner g \urcorner, c)) = (f(c), g(c)) = (f, g)(c)$

by recursion hypothesis

as well as – inner induction on $n \in \mathbb{N}$:

anchor:

$\boldsymbol{ev}(\ulcorner f^{\S} \urcorner, (a, 0)) \ =_{\text{by def}} \ \boldsymbol{ev}(\ulcorner f \urcorner^{\ulcorner \S \urcorner}, (a, 0))$

$= a = f^{\S}(a, 0)$

step:

$\boldsymbol{ev}(\ulcorner f^{\S} \urcorner, (a, \mathrm{s}\, n)) = \boldsymbol{ev}(\ulcorner f \urcorner^{\ulcorner \S \urcorner}, (a, \mathrm{s}\, n))$

$= \boldsymbol{ev}(\ulcorner f \urcorner, \boldsymbol{ev}(\ulcorner f \urcorner^{\ulcorner \S \urcorner}, (a, n))$

$= \boldsymbol{ev}(\ulcorner f \urcorner, f^{\S}(a, n))$ by induction hypothesis on n

$= (f \circ f^{\S})(a, n)$ by recursion hypothesis on f

$= f^{\S}(a, \mathrm{s}\, n)$ **q. e. d.**

4 Decision

Define the a priori *partial* μ-recursive *decision* map

$$\mathrm{decis} = \mathrm{decis}(\varphi) : [\mathbb{N}, 2]_{\mathbf{PRa}} \rightharpoonup 2 = \{0, 1\} \text{ in } \mathbf{T},$$
$$\varphi \in [\mathbb{N}, 2]_{\mathbf{PR}} \subset \mathit{formulae}_{\mathbf{T}} \subset \mathbb{N}$$

via two antagonistic *termination indices*

$$\mu_{ex}(\varphi),\ \mu_{\mathrm{thm}_{\mathbf{T}}}(\varphi) : [\mathbb{N}, 2]_{\mathbf{PR}} \to \mathbb{N} \cup \{\infty\} \text{ within } \mathbf{T} \text{ as follows:}$$
$$\mu_{ex}(\varphi) := \mu\{n : \boldsymbol{ev}(\varphi, n) = 0\} \quad \text{“minimal counter}\mathit{ex}\text{ample”}$$
$$= \begin{cases} \min\{n : \boldsymbol{ev}(\varphi, n) = 0\} & \text{if} \quad \exists n[\boldsymbol{ev}(\varphi, n) = 0] \\ \infty \text{ (undefined)} & \text{if} \quad \forall n[\boldsymbol{ev}(\varphi, n) = 1] \end{cases}$$

Theorem index $\mu_{\mathrm{thm}_{\mathbf{T}}}(\varphi) \in \mathbb{N} \cup \{\infty\}$ of $\varphi \in [\mathbb{N}, 2]_{\mathbf{PRa}}$ is defined by

$$\mu_{\mathrm{thm}_{\mathbf{T}}}(\varphi) = \mu\{k : \mathrm{thm}_{\mathbf{T}}(k) = \varphi\}$$

Here the p. r. enumeration

$$\mathrm{thm}_{\mathbf{T}} = \mathrm{thm}_{\mathbf{T}}(k) : \mathbb{N} \to \mathit{formulae}_{\mathbf{T}} \subset \mathbb{N}$$

is the **T** *internal* version of the p. r. enumeration of all (Gödel numbers of) **T** theorems; enumeration is *lexicographic* by “length of shortest *proof*”.

Finally, we define the – a priori partial – *μ-recursive* **T** map

$$\mathrm{decis} = \mathrm{decis}(\varphi) : [\mathbb{N}, 2]_{\mathbf{PR}} \rightharpoonup 2 \text{ by}$$

$$\mathrm{decis}(\varphi) = \begin{cases} 0 \text{ if } \mu_{ex}(\varphi) < \infty \quad (\text{“counterexample”}) \\ 1 \text{ if } \mu_{ex}(\varphi) = \infty \text{ and } \mu_{\mathrm{thm}_{\mathbf{T}}}(\varphi) < \infty \quad (\mathbf{T}\ \textit{theorem}) \\ \infty \text{ otherwise, i. e. if } \mu_{\mathrm{thm}_{\mathbf{T}}}(\varphi) = \mu_{ex}(\varphi) = \infty. \end{cases}$$

For proof of decis to be *totally defined* within $\tilde{\mathbf{T}}$ we rely on the following (trivial)

Lemma (*Semantical completeness of* $\tilde{\mathbf{T}}$ *rel. p. r. predicates*):

$$\tilde{\mathbf{T}} \vdash \ \forall n[\boldsymbol{ev}(\varphi, n) = 1] \implies \exists k[\mathrm{thm}_{\mathbf{T}}(k) = \varphi],$$

φ free on $[\mathbb{N}, 2]_{\mathbf{PRa}}$

Proof: One of the equivalent $\mathbf{T}$ formulae expressing internal inconsistency of $\mathbf{T}$ is

$$\neg\mathrm{Con}_{\mathbf{T}} = (\forall f \in \mathit{formulae}_{\mathbf{T}})(\exists k \in \mathbb{N})[\mathrm{thm}_{\mathbf{T}}(k) = f]$$

"Every internal $\mathbf{T}$ formula – Gödel number – is *provable*."
This gives in particular

$$\tilde{\mathbf{T}} \vdash \ \exists k[\mathrm{thm}_{\mathbf{T}}(k) = \varphi],$$

φ free on $[\mathbb{N}, 2]_{\mathbf{PR}} \subset \mathit{formulae}_{\mathbf{T}}$ **q. e. d.**

Decision theorem

(i) within $\tilde{\mathbf{T}} = \mathbf{T} + \neg\mathrm{Con}_{\mathbf{T}}$, the (a priori partial) μ-recursive *decision algorithm*

$$\mathrm{decis}(\varphi) : [\mathbb{N}, 2]_{\mathbf{PR}} \rightharpoonup 2$$

is in fact *totally defined*, in other words it *terminates* on all internal Gödel numbers $\varphi \in [\mathbb{N}, 2]_{\mathbf{PR}}$.

(ii) For $\varphi = \varphi(n)$ a p. r. predicate, $\ulcorner\varphi\urcorner \in [\mathbb{N}, 2]_{\mathbf{PR}} \subset \mathbb{N}$ its gödel number, $\mathrm{decis}(\ulcorner\varphi\urcorner)$ gives – in $\tilde{\mathbf{T}}$ – the *correct* result:

- $\tilde{\mathbf{T}} \vdash \mathrm{decis}(\ulcorner\varphi\urcorner) = 0 \iff \exists n[\neg\varphi(n)],$
- $\tilde{\mathbf{T}} \vdash \ \mathrm{decis}(\ulcorner\varphi\urcorner) = 1 \implies \forall n\, \varphi(n).$

Proof of (i):

$$\tilde{\mathbf{T}} \vdash \mu_{ex}(\varphi) = \infty$$
$$\iff \forall n[\boldsymbol{ev}(\varphi, n) = 1]$$
$$\implies \exists k[\mathrm{thm}_{\mathbf{T}}(k) = \varphi]$$
by internal semantical completeness of $\tilde{\mathbf{T}}$ above
$$\iff \mu_{\mathrm{thm}_{\mathbf{T}}}(\varphi) < \infty]$$

Hence not both of $\mu_{ex}(\varphi), \mu_{\mathrm{thm}_{\mathbf{T}}}(\varphi)$ can be undefined within $\tilde{\mathbf{T}}$.

This shows *termination* $\mathrm{decis}(\varphi) \in \{0, 1\}$ of decis within $\tilde{\mathbf{T}}$ for all internal p. r. predicates φ.

Proof of (ii):

$$\tilde{\mathbf{T}} \vdash \mathrm{decis}(\ulcorner\varphi\urcorner) = 0$$
$$\iff \mu_{ex}[\ulcorner\varphi\urcorner < \infty]$$
$$\iff \exists n[\boldsymbol{ev}(\ulcorner\varphi\urcorner, n) = 0]$$
$$\iff \exists n[\varphi(n) = 0]$$ by $\boldsymbol{ev}$'s evaluation property
$$\iff \exists n[\neg\varphi(n)]$$

as well as

$$\tilde{\mathbf{T}} \vdash \mathrm{decis}(\ulcorner\varphi\urcorner) = 1$$
$$\implies \mu_{ex}\ulcorner\varphi\urcorner = \infty$$
$$\iff \forall n[\boldsymbol{ev}(\ulcorner\varphi\urcorner, n) = 1]$$
$$\iff \forall n\, \varphi(n)$$ **q. e. d.**

[if here $\mathrm{decis}(\ulcorner\varphi\urcorner) = 0 = 1$ then $\tilde{\mathbf{T}}$ is inconsistent and we are done.]

5 Hilbert's 10th Problem revisited

A *diophantine equation*

$$[D_L(x_1, \ldots, x_m) = D_R(x_1, \ldots, x_m)] :$$
$$\mathbb{N}^m \to \mathbb{N}^m \times \mathbb{N}^m \to \mathbb{N} \times \mathbb{N} \xrightarrow{=} 2 = \{0, 1\}$$

is equivalent to p. r. predicate

$$\varphi_D = \varphi_D(n)$$
$$= [D_L(x_1, \ldots, x_m) = D_R(x_1, \ldots, x_m)] \circ \mathrm{cantor}_{\mathbb{N}^m} :$$
$$\mathbb{N} \xrightarrow{\cong} \mathbb{N}^m \to 2$$

decided as $\mathrm{decis}(\ulcorner \varphi_D \urcorner)$ *defined* within theory $\tilde{\mathbf{T}}$:

$$\tilde{\mathbf{T}} \vdash \mathrm{decis}(\ulcorner \varphi_D \urcorner) < \infty \quad (\bullet)$$

Consider now countable family

$$[D^L_\alpha(x_1, \ldots, x_{m(\alpha)}) = D^R_\alpha(x_1, \ldots, x_{m(\alpha)})]_{\alpha \in \mathbb{N}}$$

of *all* diophantine equations: The equations are counted lexicographically by their (finite) polynome-coefficient lists.

Cf. Matiyasevich 1993, 1.1, 1.2, and 1.3. This family gives rise to *p. r. predicates*

$$\varphi_\alpha = [D^L_\alpha(x_1, \ldots, x_{m(\alpha)}) \neq D^R_\alpha(x_1, \ldots, x_{m(\alpha)})] : \mathbb{N}^{m(\alpha)} \to 2$$

which has property that

$(x_1, \ldots, x_{m(\alpha)}) \in \mathbb{N}^{m(\alpha)}$ is a solution to $\varphi(\alpha)$

iff it is a *counterexample to*

$$D_\alpha = [D^L_\alpha(x_1, \ldots, x_{m(\alpha)}) = D^R_\alpha(x_1, \ldots, x_{m(\alpha)})] : \mathbb{N}^{m(\alpha)} \to 2$$

and D_α has *no solution* (in natural numbers)

iff φ_α *holds* for $(x_1, \ldots, x_{m(\alpha)})$ free in $\mathbb{N}^{m(\alpha)}$

From Decision Lemma (for p. r. predicates) above we obtain

Decision Theorem

1. $\tilde{\mathbf{T}} \vdash \mathrm{decis}\ulcorner\varphi_\alpha\urcorner < \infty, \ \alpha \in \mathbb{N}$ free.

   Within the – somewhat strange – theory $\tilde{\mathbf{T}} = \mathbf{T} + \neg\mathrm{Con}_{\mathbf{T}}$ the (partial) μ-recursive map (the "*algorithm*")

   $\mathrm{decis} : [\mathbb{N}, 2]_{\mathbf{PRa}} \rightharpoonup 2$

   decides in fact all primitive recursive predicates, in particular all diophantine predicates as considered above, uniformally.

2. Since μ-recursion and Turing machines have equal *computation power* – by the verified part of Church's thesis – this means: Within $\tilde{\mathbf{T}}$, decis gives rise to a Turing machine TM deciding all diophantine equations, i. e. $\tilde{\mathbf{T}}$ admits a *positive* solution to Hilbert's 10th problem.

3. On the other hand, Matiyasevich's *negative* solution to this problem works in **set** theory $\mathbf{T}$,

   a fortiori in theory $\tilde{\mathbf{T}} = \mathbf{T} + \neg\mathrm{Con}_{\mathbf{T}}$.

4. The latter two results – Matiyasevich's *negative* $\mathbf{T}$ *theorem* and our *positive* $\tilde{\mathbf{T}}$ *theorem* contradict each other in stronger theory $\tilde{\mathbf{T}}$. This shows $\tilde{\mathbf{T}}$ to be *inconsistent.*

5. Gödel's consistency of $\neg\mathrm{Con}_{\mathbf{T}}$ relative to $\mathbf{T}$ – second incompleteness theorem – then entails inconsistency of classical **set** theorie(s) $\mathbf{T}$.

Outlook: Since Matiyasevich 1993 makes essential use of formal (existential) quantification for "unsolving" Hilbert's 10th problem, this only decidability problem on Hilbert's list is again open – for treatment within the framework of suitable *constructive* foundations for Arithmetic.

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