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MÖBIUS ISOPARAMETRIC HYPERSURFACES IN $\mathbf{S}^{n+1}$ WITH TWO DISTINCT PRINCIPAL CURVATURES

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ABSTRACT. A hypersurface $x : \mathbf{M} \to \mathbf{S}^{n+1}$ without umbilic point is called a Möbius isoparametric hypersurface if its Möbius form $\Phi = -\rho^{-2}\sum_i(e_i(H) + \sum_j(h_{ij} - H\delta_{ij})e_j(\log\rho))\omega_i$ vanishes and its Möbius shape operator $\mathbb{S} = \rho^{-1}(S - H\mathrm{id})$ has constant eigenvalues. Here $\{e_i\}$ is a local orthonormal basis for $I = dx \cdot dx$ with dual basis $\{\omega_i\}$, $II = \sum_{ij} h_{ij}\omega_i \otimes \omega_j$ is the second fundamental form, $H = \frac{1}{n}\sum_i h_{ii}$, $\rho^2 = \frac{n}{n-1}(||II||^2 - nH^2)$ and S is the shape operator of x. It is clear that any (Euclidean) isoparametric hypersurface is also a Möbius isoparametric hypersurface, but the converse is not true. In this paper we classify all Möbius isoparametric hypersurfaces in $\mathbf{S}^{n+1}$ with two distinct principal curvatures up to Möbius transformations. By using a theorem of Thorbergsson ([T]) we show also that the number of distinct principal curvatures of a compact Möbius isoparametric hypersurface embedded in $\mathbf{S}^{n+1}$ can only take the values 2, 3, 4, 6.

§1. Introduction.

An important class of hypersurfaces for Möbius differential geometry is the so-called Möbius isoparametric hypersurfaces in $\mathbf{S}^{n+1}$. It is a hypersurface $x : \mathbf{M} \to \mathbf{S}^{n+1}$ such that the Möbius invariant 1-form

$$\Phi = -\rho^{-2}\sum_i(e_i(H) + \sum_j(h_{ij} - H\delta_{ij})e_j(\log\rho))\omega_i \tag{1.1}$$

vanishes and all eigenvalues of the Möbius shape operator

$$\mathbb{S} := \rho^{-1}(S - H\mathrm{id}) \tag{1.2}$$

are constant. Standard examples of Möbius isoparametric hypersurfaces are the images of (Euclidean) isoparametric hypersurfaces in $\mathbf{S}^{n+1}$ under Möbius transformations. But there are some examples of (noncompact) Möbius isoparametric hypersurfaces which cann't obtained by this way. It is easy to show that (see Proposition 3.2) any Möbius isoparametric hypersurface

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variation formula for minimal surfaces, which is a special class of Willmore surfaces (cf. [9]). But it is too complicated to give the second variation formula for general Willmore surfaces by using euclidean invariants.

Since the Willmore functional defined by (0.1) is invariant under the Moebius group (cf. [3], [8]), one can use the framework of Moebius geometry and Moebius invariants to calculate the second variation formula. It is the key point of this paper. For any submanifold M in S^n we can introduce a Moebius invariant metric g on M. Then the Willmore functional is exactly the volume functional of g. The third author computed the first variation and got the Euler - Lagrange equations in [8]. Submanifolds in S^n satisfying these equations are called Willmore submanifolds or Moebius minimal submanifolds. In the paper we give the second variation formula of Willmore functional for submanifolds in S^n by using Moebius invariants. Although this formula looks very complicated, in case of surfaces in S^3 (which is the most important case) the formula is rather simple (cf. §2, (2.44)). Using the Euler-Lagrange equations we find the standard examples of Willmore hypersurfaces $\{W_k^n := S^k(\sqrt{(n-k)/n}) \times S^{n-k}(\sqrt{k/n}),\ 1 \le k \le n-1\}$ in S^{n+1}, which is (euclidean) minimal if and only if $2k = n-1$. It is somehow the dual hypersurface to the standard minimal hypersurface $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$ in S^{n+1}. We show that W_k^n are stable Willmore hypersurfaces.

We organize this paper as follows. In §1 we give Moebius invariants and local formulas in Moebius geometry for submanifolds in S^n. In §2 we calculate the second variation formula for Willmore submanifolds in S^n. As an application we prove in §3 that $\{W_k^n\}$ are stable Willmore hypersurfaces.

§1. Moebius invariants and local formulas for submanifolds in S^n

Let $x_0 : M \to S^n$ be an m-dimensional compact submanifold with boundary ∂M, $\{e_1, ..., e_m\}$ be a local orthonormal basis of TM with respective to the induced metric $dx_0 \cdot dx_0$ and $\{\theta_1, ..., \theta_m\}$ be its dual basis. Let $\{e_{m+1}, ..., e_n\}$ be the local normal orthonormal vector field. We make using of the following convention on the ranges of indices:

$$1 \le i, j, k, \cdots \le m; \quad m+1 \le \alpha, \beta, \gamma, \ldots \le n$$

and we shall agree that repeated indices are summed over the respective ranges. Then the structure equation of x_0 can be written as

$$\begin{aligned} dx_0 &= \theta_i e_i \\ de_i &= \theta_{ij} e_j + h_{ij}^\alpha \theta_j e_\alpha - \theta_i x_0 \\ de_\alpha &= -h_{ij}^\alpha \theta_j e_i + \theta_{\alpha\beta} e_\beta \end{aligned}$$

The quantities $I = dx_0 \cdot dx_0, II = h_{ij}^\alpha \theta_i \otimes \theta_j e_\alpha$ and $H = \frac{1}{m} h_{ii}^\alpha e_\alpha$ are the first, the second fundamental form and the mean curvature vector of x_0 in S^n, respectively. We define function $\rho : M \to R$ by

$$\rho = \sqrt{\frac{m}{m-1}} \|II - HI\| \tag{1.1}$$

The metric $g = \rho^2 dx_0 \cdot dx_0$ is called a Moebius metric which is invariant under Moebius transformations in S^n (cf.[8]) and is positive definite at any non-umbilical point. Then the Willmore functional in (0.1) is exactly the Moebius volume functional for g:

$$W(M) := \int_M \rho^m dM = Vol_g(M), \tag{1.2}$$

where dM is the volume element for the metric $dx_0 \cdot dx_0$. Our purpose is to calculate the second variation in the framework of Moebius geometry. We need the following notation and local formulas. For more detail we refer to [8].

Let R_1^{n+2} be the Lorentz space with the inner product $<,>$ given by

$$< X, Y >= -x^0y^0 + x^1y^1 + \cdots + x^{n+1}y^{n+1},$$

where $X = (x^0, x^1, \cdots, x^{n+1}), Y = (y^0, y^1, \cdots, y^{n+1}) \in R^{n+2}$. The half cone in R_1^{n+2} is defined as

$$C_+^{n+1} = \{X \in R_1^{n+2} | < X, X >= 0, x^0 > 0\}.$$

For the immersion $x_0 : M \to S^n$ we define

$$Y = \rho^2(1, x_0) : M \to C_+^{n+1}. \tag{1.3}$$

If $\tilde{x}_0 : M \to S^n$ is Moebius equivalent to x_0, then we have $\tilde{Y} = YT$ for some Lorentz matrix T. Thus

$$g =< dY, dY >= \rho^2 dx_0 \cdot dx_0 \tag{1.4}$$

is a Moebius invariant. In the following we assume that x_0 is an immersion without umbilical point, which implies that g is positive definite on M. Let $E_i = \rho^{-1}e_i$, then $\{E_i\}$ is an orthonormal basis with respect to metric g, with dual basis $\{\omega_i = \rho\theta_i\}$. Set

$$Y_i := E_i(Y),\ N := -\frac{1}{m}\Delta Y - \frac{1}{2m^2} < \Delta Y, \Delta Y > Y,$$

$$E_\alpha := (H^\alpha, e_\alpha + H^\alpha x_0), \tag{1.5}$$

where Δ is the Laplacian operator for metric g and $H = H^\alpha e_\alpha$ is the mean curvature vector.

Lemma 1.1([8]) *$\{Y, N, Y_i, E_\alpha\}$ satisfy conditions*

$$< Y, Y >=< N, N >= 0, < Y, N >= 1,\ < Y_i, Y_j >= \delta_{ij};$$

$$< Y, Y_i >=< N, Y_i >=< Y, E_\alpha >=< N, E_\alpha >= 0;$$

$$< E_\alpha, Y_i >= 0,\ < E_\alpha, E_\beta >= \delta_{\alpha\beta}.$$

Lemma 1.1 shows that $\{Y, N, Y_i, E_\alpha\}$ forms a Moebius moving frame in R_1^{n+2} along M. The structure equations can be written as

$$\begin{aligned} dY &= \omega_i Y_i \\ dN &= \psi_i Y_i + \phi_\alpha E_\alpha \\ dY_i &= -\psi_i Y - \omega_i N + \omega_{ij} Y_j + \omega_{i\alpha} E_\alpha \\ dE_\alpha &= -\phi_\alpha Y - \omega_{i\alpha} Y_i + \omega_{\alpha\beta} E_\beta \end{aligned} \tag{1.6}$$

By differenting these equations and using Cartan lemma, we obtain

$$\psi_i = A_{ij}\omega_j,\ A_{ij} = A_{ji};\ \omega_{i\alpha} = B_{ij}^\alpha \omega_j,\ B_{ij}^\alpha = B_{ji}^\alpha;\ \phi_\alpha = C_i^\alpha \omega_i.$$

We have the following equations:

$$\sum_i B_{ii}^\alpha = 0,\ \sum_{\alpha ij} \left(B_{ij}^\alpha\right)^2 = \frac{m-1}{m},\ \sum_j B_{ij,j}^\alpha = -(m-1)C_i^\alpha. \tag{1.7}$$

The relations between these Moebius invariants and euclidean invariants are given by

$$A_{ij} = -\rho^{-2}(Hess_{ij}(log\rho) - e_i(log\rho)e_j(log\rho) - H^\alpha h^\alpha_{ij})$$
$$-\frac{1}{2}\rho^{-2}(|\nabla log\rho|^2 - 1 + \sum_\alpha (H^\alpha)^2)\delta_{ij}, \tag{1.8}$$
$$B^\alpha_{ij} = \rho^{-1}(h^\alpha_{ij} - H^\alpha \delta_{ij}) \tag{1.9}$$
$$C^\alpha_i = -\frac{1}{m-1}B^\alpha_{ij,j} = -\rho^{-2}(H^\alpha_{,i} + (h^\alpha_{ij} - H^\alpha\delta_{ij})e_j(log\rho)). \tag{1.10}$$

§2. The second variation formula of the Moebius volume functional

In this section we caculate the second variation of the Willmore functional defined by (1.2) or (0.1). Since the volume variation depends only on the normal component of the variation vector field (cf. [8]), we will consider the normal variation.

Let $x : M \times R \to S^n$ be a smooth variation of x_0 such that $x(\cdot, t) = x_0$ and $dx_t(TM) = dx_0(TM)$ on ∂M for each (small) t. These two boundary conditions disppear if $\partial M = \emptyset$. For each t we denote by$\{e_i\}$ a local orthonormal basis for TM with respect to $dx_t \cdot dx_t$ with dual basis $\{\theta_i\}$ and by $\{e_\alpha\}$ a local orthonormal basis for the normal bundle of x_t. Let $Y = \rho(1, x) : M \times R \to C^{n+1}_+$ be the canonical lift of x_t and $g_t =< dY, dY >$ be the Moebius metric of x_t. Let $\{E_i := \rho^{-1}e_i\}$ be a local orthonormal basis for g_t with dual basis $\{\omega_i = \rho\theta_i\}$. Then the volume for g_t can be write as

$$W(t) := Vol_{g_t}(M) = \int_M \omega_1 \wedge \cdots \wedge \omega_m. \tag{2.1}$$

From Lemma 1.1 in section 1, we can choose a moving frame

$$\{Y, N, Y_1, \cdots, Y_m, E_{m+1}, \cdots, E_n\}$$

in R^{n+2}_1 along $M \times R$, which satisfy the conditions in Lamma 1.1 for each t. Let d denote the differential operator on $M \times R$, then we can find 1-forms

$$\{V, V_\alpha, \Psi_i, \Phi_\alpha, \Omega_i, \Omega_{ij}, \Omega_{i\alpha}, \Omega_{\alpha\beta}\}$$

on $M \times R$ with $\Omega_{ij} = -\Omega_{ji}$ and $\Omega_{\alpha\beta} = -\Omega_{\beta\alpha}$ such that

$$dY = VY + \Omega_i Y_i + V_\alpha E_\alpha, \tag{2.2}$$
$$dN = -VN + \Psi_i Y_i + \Phi_\alpha E_\alpha, \tag{2.3}$$
$$dY_i = -\Psi_i Y - \Omega_i N + \Omega_{ij} Y_j + \Omega_{i\alpha} E_\alpha, \tag{2.4}$$
$$dE_\alpha = -\Phi_\alpha - V_\alpha N - \Omega_{i\alpha} Y_i + \Omega_{\alpha\beta} E_\beta. \tag{2.5}$$

Taking differential of these equations we get

$$dV = \Psi_i \wedge \Omega_i + \Phi_\alpha \wedge V_\alpha, \tag{2.6}$$
$$d\Omega_i = \Omega_{ij} \wedge \Omega_j + V \wedge \Omega_i - V_\alpha \wedge \Omega_{i\alpha}, \tag{2.7}$$
$$dV_\alpha = \Omega_{\alpha\beta} \wedge V_\beta + \Omega_i \wedge \Omega_{i\alpha} + V \wedge V_\alpha, \tag{2.8}$$

$$d\Psi_i = \Omega_{ij} \wedge \Psi_j - \Phi_\alpha \wedge \Omega_{i\alpha} + \Psi_i \wedge V, \tag{2.9}$$

$$d\Phi_\alpha = \Omega_{\alpha\beta} \wedge \Phi_\beta + \Psi_i \wedge \Omega_{i\alpha} + \Phi_\alpha \wedge V, \tag{2.10}$$

$$d\Omega_{ij} = \Omega_{ik} \wedge \Omega_{kj} + \Omega_{i\alpha} \wedge \Omega_{\alpha j} - \Psi_i \wedge \Omega_j - \Omega_i \wedge \Psi_j, \tag{2.11}$$

$$d\Omega_{i\alpha} = \Omega_{ij} \wedge \Omega_{j\alpha} + \Omega_{i\beta} \wedge \Omega_{\beta\alpha} - \Psi_i \wedge V_\alpha - \Omega_i \wedge \Phi_\alpha, \tag{2.12}$$

$$d\Omega_{\alpha\beta} = \Omega_{\alpha\gamma} \wedge \Omega_{\gamma\beta} + \Omega_{\alpha i} \wedge_{i\beta} - \Phi_\alpha \wedge V_\beta - V_\alpha \wedge \Phi_\beta. \tag{2.13}$$

Since $Y = \rho(1, x)$, if we write the normal variation vector field of x in TS^n by

$$\frac{\partial x}{\partial t} = \rho^{-1} v_\alpha e_\alpha, \tag{2.14}$$

then by (1.5) we can find a function $v : M \times R \to R$ such that

$$\frac{\partial Y}{\partial t} = vY + v_\alpha E_\alpha. \tag{2.15}$$

From (2.2), (2.15) and the fact that $d = \omega_i E_i + dt\frac{\partial}{\partial t}$ on $C^\infty(M \times R)$ we get

$$V = vdt, \ V_\alpha = v_\alpha dt, \ \Omega_i = \omega_i. \tag{2.16}$$

Since $T^*(M \times R) = T^*M \oplus T^*R$ we can write

$$\Psi_i = \psi_i + a_i dt, \Phi_\alpha = \phi_\alpha + b_\alpha dt, \tag{2.17}$$

$$\Omega_{ij} = \omega_{ij} + P_{ij}dt, \ \Omega_{i\alpha} = \omega_{i\alpha} + L_{i\alpha}dt, \ \Omega_{\alpha\beta} = \omega_{\alpha\beta} + Q_{\alpha\beta}dt, \tag{2.18}$$

where $\{a_i, b_\alpha, P_{ij}, Q_{\alpha\beta}\}$ are local functions with $P_{ij} = -P_{ji}$ and $Q_{\alpha\beta} = -Q_{\beta\alpha}$. Let $\{B^\alpha_{ij}, A_{ij}, C^\alpha_i\}$ be Moebius invariants for x_t defined in §1. If we denote by d_M the exterior differential operator on T^*M, then we have $d = d_M + dt \wedge \frac{\partial}{\partial t}$ on $T^*(M \times R)$. It follows from (2.6), (2.16) and (2.17) (comparing the terms in $T^*(M) \wedge dt$) that

$$a_i = -v_{,i} + v_\alpha C^\alpha_i, \tag{2.19}$$

where $v_{,i} := E_i(v)$. Similarly we get from (2.8), (2.16), (2.17), (2.18) that

$$L_{i\alpha} = v_{\alpha,i} \tag{2.20}$$

and get from (2.7), (2.16), (2.18) that

$$\frac{\partial \omega_i}{\partial t} = (P_{ij} + v\delta_{ij} - v_\alpha B^\alpha_{ij})\omega_j. \tag{2.21}$$

By directly calculation in a similar way, we obtain from (2.9)$\sim$(2.13) that

$$\frac{\partial \omega_{ij}}{\partial t} = (P_{ij,k} + B^\alpha_{ik} v_{\alpha,j} - B^\alpha_{jk} v_{\alpha,i} - a_i \delta_{kj} + a_j \delta_{ik})\omega_k, \tag{2.22}$$

$$\frac{\partial \psi_i}{\partial t} = (a_{i,j} + P_{ik}A_{kj} + v_{\alpha,i}C^\alpha_j - b_\alpha B^\alpha_{ij} - vA_{ij})\omega_j \tag{2.23}$$

$$\frac{\partial \phi_\alpha}{\partial t} = (b_{\alpha,i} + Q_{\alpha\beta}C^\beta_i - A_{ij}v_{\alpha,j} + a_j B^\alpha_{ji} - vC^\alpha_i)\omega_i, \tag{2.24}$$

$$\frac{\partial \omega_{i\alpha}}{\partial t} = (v_{\alpha,ij} + P_{ik}B_{kj}^{\alpha} - B_{ij}^{\beta}Q_{\beta\alpha} + A_{ij}v_{\alpha} + b_{\alpha}\delta_{ij})\omega_j \tag{2.25}$$

$$\frac{\partial \omega_{\alpha\beta}}{\partial t} = (Q_{\alpha\beta,i} + v_{\beta,j}B_{ji}^{\alpha} - v_{\alpha,j}B_{ji}^{\beta} + v_{\beta}C_i^{\alpha} - v_{\alpha}C_i^{\beta})\omega_i, \tag{2.26}$$

where $\{v_{\alpha,i}\}$ are covariant derivatives of $\{v_{\alpha}\}$. Since $\phi_{\alpha} = C_i^{\alpha}\omega_i$ and $\psi_i = A_{ij}\omega_j$, from (2.21), (2.23) and (2.24) we have

$$\frac{\partial C_i^{\alpha}}{\partial t} = b_{\alpha,i} + Q_{\alpha\beta}C_i^{\beta} + a_jB_{ij}^{\alpha} - A_{ij}v_{\alpha,j} + P_{ij}C_j^{\alpha} + B_{ij}^{\beta}C_j^{\alpha}v_{\beta} - 2vC_i^{\alpha}, \tag{2.27}$$

$$\frac{\partial A_{ij}}{\partial t} = a_{i,j} + P_{ik}A_{kj} - P_{kj}A_{ki} + v_{\alpha,i}C_j^{\alpha} - B_{ij}^{\alpha}b_{\alpha} + A_{ik}B_{kj}^{\alpha}v_{\alpha} - 2vA_{ij}. \tag{2.28}$$

Since $\omega_{i\alpha} = B_{ij}^{\alpha}\omega_j$, from (2.21) and (2.25) we have

$$\frac{\partial B_{ij}^{\alpha}}{\partial t} = v_{\alpha,ij} - vB_{ij}^{\alpha} + P_{ik}B_{kj}^{\alpha} - P_{kj}B_{ki}^{\alpha} - B_{ij}^{\beta}Q_{\beta\alpha} + v_{\beta}B_{ik}^{\alpha}B_{kj}^{\beta} + A_{ij}v_{\alpha} + b_{\alpha}\delta_{ij}. \tag{2.29}$$

Multiplying B_{ij}^{α} to (2.29) and using (1.7) we get

$$\frac{m-1}{m}v = B_{ij}^{\alpha}v_{\alpha,ij} + B_{ik}^{\alpha}B_{kj}^{\beta}B_{ji}^{\alpha}v_{\beta} + A_{ij}B_{ij}^{\alpha}v_{\alpha}. \tag{2.30}$$

Making use of (2.21), (2.30), (1.7), Green's formula and the conditions on ∂M, we get (cf.[8])

$$W'(t) = \frac{m^2}{m-1}\int_M \left\{B_{ij,ij}^{\alpha} + B_{ik}^{\beta}B_{kj}^{\alpha}B_{ji}^{\beta} + A_{ij}B_{ij}^{\alpha}\right\} v_{\alpha}dM_t, \tag{2.31}$$

where dM_t is volume element of g_t. Formula (2.31) shows that $x_0 : M \to S^n$ is Moebius minimal (its Moebius volume is stationary) if and only if

$$B_{ij,ij}^{\alpha} + B_{ik}^{\beta}B_{kj}^{\alpha}B_{ji}^{\beta} + A_{ij}B_{ij}^{\alpha} = 0. \tag{2.32}$$

Since $\frac{\partial}{\partial t} \circ d_M - d_M \circ \frac{\partial}{\partial t} = 0$ (or equivalently $d^2 = (d_M + dt \wedge \frac{\partial}{\partial t})^2 = 0$) and

$$C_{i,j}^{\alpha}\omega_j = d_MC_i^{\alpha} + C_j^{\alpha}\omega_{ji} + C_i^{\beta}\omega_{\beta\alpha},$$

we get

$$\frac{\partial C_{i,j}^{\alpha}}{\partial t} = -C_{i,k}^{\alpha}\frac{\partial \omega_k}{\partial t}(E_j) + (\frac{\partial C_i^{\alpha}}{\partial t})_{,j} + C_k^{\alpha}\frac{\partial \omega_{ki}}{\partial t}(E_j) + C_i^{\beta}\frac{\partial \omega_{\beta\alpha}}{\partial t}(E_j). \tag{2.33}$$

By a direct calculation and using (2.21), (2.22), (2.26) and (2.33) we get

$$\begin{aligned}\frac{\partial C_{i,i}^{\alpha}}{\partial t}v_{\alpha} &= -(C_i^{\alpha}P_{ki})_{,k}v_{\alpha} - vC_{i,i}^{\alpha}v_{\alpha} + v_{\beta}B_{ki}^{\beta}C_{i,k}^{\alpha}v_{\alpha} + \left(\frac{\partial C_i^{\alpha}}{\partial t}\right)_{,i}v_{\alpha}\\ &+C_k^{\alpha}B_{ki}^{\beta}v_{\beta,i}v_{\alpha} + a_kC_k^{\alpha}v_{\alpha} - ma_kC_k^{\alpha}v_{\alpha} - C_i^{\beta}Q_{\alpha\beta,i}v_{\alpha} - B_{ki}^{\alpha}C_i^{\beta}v_{\beta,k}v_{\alpha}\\ &+B_{ki}^{\beta}C_i^{\beta}v_{\alpha,k}v_{\alpha} - C_i^{\beta}C_i^{\alpha}v_{\beta}v_{\alpha} + \sum_{i\beta}\left(C_i^{\beta}\right)^2\sum_{\alpha}v_{\alpha}^2\end{aligned} \tag{2.34}$$

By a straightforward calculation and using (2.27), (2.19) we get from (2.34) that

$$\begin{aligned}
&\frac{\partial C^\alpha_{i,i}}{\partial t}v_\alpha = -(C^\alpha_i P_{ki}v_\alpha)_{,k} + \left(\frac{\partial C^\alpha_i}{\partial t}v_\alpha\right)_{,i} + (vB^\alpha_{ki}v_{\alpha,i})_{,k} \\
&+(m-1)(vC^\alpha_k v_\alpha)_{,k} - (C^\beta_i Q_{\alpha\beta}v_\alpha)_{,i} - (b_\alpha v_{\alpha,i})_{,i} - mvC^\alpha_{i,i}v_\alpha \\
&+C^\beta_{i,i}Q_{\alpha\beta}v_\alpha + C^\alpha_{i,k}B^\beta_{ki}v_\alpha v_\beta + b_\alpha v_{\alpha,ii} - vB^\alpha_{ki}v_{\alpha,ik} + A_{ik}v_{\alpha,k}v_{\alpha,i} \\
&-2C^\alpha_k B^\beta_{ki}v_\beta v_{\alpha,i} + 2vC^\alpha_i v_{\alpha,i} - m\sum_k\left(\sum_\alpha C^\alpha_k v_\alpha\right)^2 \\
&+C^\beta_i B^\beta_{ik}v_{\alpha,k}v_\alpha + \sum_{i,\beta}(C^\beta_i)^2\sum_\alpha v^2_\alpha.
\end{aligned} \tag{2.35}$$

From the conditions that $x(.,t) = x_0$ and $dx_t(TM) = dx_0(TM)$ on ∂M for each t, we see that $v_\alpha|_{\partial M} = 0$ and

$$0 = \frac{\partial}{\partial t}(dx_t) = d(\frac{\partial x}{\partial t}) = \rho^{-1}dv_\alpha e_\alpha \tag{2.36}$$

on boundary ∂M. It follows from (2.35) and Green's fomula that

$$\begin{aligned}
&\Gamma_1 := \int_M \{\frac{\partial}{\partial t}|_{t=0}(C^\alpha_{i,i})\}v_\alpha dM \\
&= \int_M \{-mvC^\alpha_{i,i}v_\alpha + C^\beta_{i,i}Q_{\alpha\beta}v_\alpha + C^\alpha_{i,k}B^\beta_{ki}v_\alpha v_\beta + b_\alpha v_{\alpha,ii} \\
&-vB^\alpha_{ki}v_{\alpha,ik} + A_{ik}v_{\alpha,k}v_{\alpha,i} - 2C^\alpha_k B^\beta_{ki}v_\beta v_{\alpha,i} + 2vC^\alpha_i v_{\alpha,i} \\
&-m\sum_k\left(\sum_\alpha C^\alpha_k v_\alpha\right)^2 + C^\beta_i B^\beta_{ik}v_{\alpha,k}v_\alpha + \sum_{i,\beta}(C^\beta_i)^2\sum_\alpha v^2_\alpha\}dM.
\end{aligned} \tag{2.37}$$

Using (2.28), (2.29), (2.19) and the facts that $P_{ij} = -P_{ji}$ and $\sum_j B^\alpha_{ij,j} = -(m-1)C^\alpha_i$, we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(B^\beta_{ik}B^\alpha_{kj}B^\beta_{ji} + A_{ij}B^\alpha_{ij}\right)v_\alpha \\
&= (vB^\alpha_{ij}v_{\alpha,j})_{,i} - (m-1)(vC^\alpha_i v_\alpha)_{,i} + (a_i B^\alpha_{ij}v_\alpha)_{,j} \\
&-2(m-1)vC^\alpha_{i,i}v_\alpha + 2B^\beta_{ij}B^\alpha_{jk}v_{\beta,ik}v_\alpha + (B^\beta_{ik}B^\beta_{kj} + A_{ij})v_{\alpha,ij}v_\alpha - vB^\alpha_{ij}v_{\alpha,ij} \\
&+(m-1)C^\beta_{i,i}Q_{\alpha\beta}v_\alpha + 2(m-1)vC^\alpha_j v_{\alpha,j} + C^\gamma_i B^\alpha_{ij}v_{\gamma,j}v_\alpha - C^\gamma_i B^\alpha_{ij}v_{\alpha,j}v_\gamma \\
&+\left(3B^\beta_{ij}B^\alpha_{jk}B^\gamma_{kl}B^\beta_{li} + 4B^\alpha_{ik}B^\gamma_{kj}A_{ji} + (m-1)C^\alpha_i C^\gamma_i\right)v_\alpha v_\gamma \\
&+\sum_{ij}\sum_\beta(B^\beta_{ij}b_\beta)(\sum_\alpha B^\alpha_{ij}v_\alpha) + (\sum_\alpha b_\alpha v_\alpha)\left(\sum_{\beta ij}(B^\beta_{ij})^2 + tr(A)\right) \\
&+\sum_{ij}A^2_{ij}\sum_\alpha v^2_\alpha + B^\beta_{ik}A_{kj}B^\beta_{ji}(\sum_\alpha v^2_\alpha).
\end{aligned} \tag{2.38}$$

Using the boundary conditions and Green's theorem we have

$$\Gamma_2 := \int_M \{\frac{\partial}{\partial t}|_{t=0}(B^\beta_{ik}B^\alpha_{kl}B^\beta_{lj} + A_{ij}B^\alpha_{ij})\}v_\alpha dM$$

$$\begin{aligned}
&= \int_M \{-2(m-1)vC^\alpha_{i,i}v_\alpha + 2B^\beta_{ij}B^\alpha_{jk}v_{\beta,ik}v_\alpha + (B^\beta_{ik}B^\beta_{kj} + A_{ij})v_{\alpha,ij}v_\alpha \\
&-vB^\alpha_{ij}v_{\alpha,ij} + (m-1)C^\beta_{i,i}Q_{\alpha\beta}v_\alpha + 2(m-1)vC^\alpha_j v_{\alpha,j} + C^\gamma_i B^\alpha_{ij}v_{\gamma,j}v_\alpha \\
&-C^\gamma_i B^\alpha_{ij}v_{\alpha,j}v_\gamma + \left(3B^\beta_{ij}B^\alpha_{jk}B^\gamma_{kl}B^\beta_{li} + 4B^\alpha_{ik}B^\gamma_{kj}A_{ji} + (m-1)C^\alpha_i C^\gamma_i\right)v_\alpha v_\gamma \\
&+\sum_{ij}\sum_\beta(B^\beta_{ij}b_\beta)(\sum_\alpha B^\alpha_{ij}v_\alpha) + (\sum_\alpha b_\alpha v_\alpha)(\frac{m-1}{m} + tr(A)) \\
&+\sum_{ij}A^2_{ij}\sum_\alpha v^2_\alpha + B^\beta_{ik}A_{kj}B^\beta_{ji}(\sum_\alpha v^2_\alpha)\}dM,
\end{aligned} \tag{2.39}$$

where

$$b_\alpha = -\frac{1}{m}\left(\Delta v_\alpha + B^\alpha_{ik}B^\beta_{ki}v_\beta + \frac{1}{2m}(1+m^2\kappa)v_\alpha\right), \tag{2.40}$$

$$tr(A) = \frac{1}{2m} + \frac{m}{2}\kappa, \tag{2.41}$$

where κ is the normalized scalar curvature of the metric $g = \rho^2 dx_0 \cdot dx_0$. The function v is given by (2.15) and (2.30). Thus for a Moebius minimal submanifold x_0 we get from (2.31) and (1.7) that

$$\begin{aligned}
W''(0) &= -m^2\int_M \frac{\partial C^\alpha_{i,i}}{\partial t}|_{t=0}v_\alpha dM \\
&+\frac{m^2}{m-1}\int_M\left(\frac{\partial}{\partial t}|_{t=0}(B^\beta_{ik}B^\alpha_{kj}B^\beta_{ji} + A_{ij}B^\alpha_{ij})\right)v_\alpha dM.
\end{aligned}$$

Thus the formula of the second variation is given by (2.37) and (2.39). We conclude that

Theorem 2.1 *Let $x: M \to S^n$ be a compact Moebius minimal submanifold without boundary. Then the second variation formula of the Moebius volume functional is given* *by*

$$W''(0) = \int_M \left\{ m^2(m-2)vC^\alpha_{i,i}v_\alpha - m^2C^\alpha_{i,j}B^\beta_{ji}v_\alpha v_\beta + \frac{m^2(m-2)}{m-1}vB^\alpha_{ij}v_{\alpha,ij} \right.$$

$$-m^2A_{ij}v_{\alpha,i}v_{\alpha,j} + \frac{m^2(2m-1)}{m-1}C^\beta_i B^\alpha_{ij}v_{\beta,j}v_\alpha + m^2(m+1)\sum_i\left(\sum_\alpha C^\alpha_i v_\alpha\right)^2$$

$$-m^2C^\beta_i B^\beta_{ij}v_{\alpha,j}v_\alpha - m^2\sum_{i,\beta}\left(C^\beta_i\right)^2\sum_\alpha v_\alpha^2 + \frac{2m^2}{m-1}B^\beta_{ij}B^\alpha_{jk}v_{\beta,ik}v_\alpha$$

$$+\frac{m^2}{m-1}(B^\beta_{ik}B^\beta_{kj} + A_{ij})v_{\alpha,ij}v_\alpha - \frac{m^2}{m-1}C^\beta_i B^\alpha_{ij}v_{\alpha,j}v_\beta$$

$$+\frac{m^2}{m-1}\left(3B^\gamma_{ij}B^\alpha_{jk}B^\beta_{kl}B^\gamma_{li} + 4B^\alpha_{ik}B^\beta_{kj}A_{ji}\right)v_\alpha v_\beta + \frac{m^2}{m-1}\sum_{ij}A^2_{ij}\sum_\alpha v^2_\alpha$$

$$+\frac{m^2}{m-1}B^\beta_{ik}A_{kj}B^\beta_{ji}\left(\sum_\alpha v^2_\alpha\right) + m\sum_\alpha(\Delta v_\alpha)^2 + \frac{m(m-2)}{m-1}B^\alpha_{ij}B^\beta_{ij}\Delta v_\alpha v_\beta$$

$$+\left(\frac{m(m-2)}{m-1}tr(A) - 1\right)v_\alpha\Delta v_\alpha - \frac{m}{m-1}\sum_\alpha\left(\sum_{ij\beta}B^\alpha_{ij}B^\beta_{ij}v_\beta\right)^2$$

$$\left. -\left(1+\frac{2m}{m-1}tr(A)\right)\sum_{ij}\left(\sum_\alpha B^\alpha_{ij}v_\alpha\right)^2 - tr(A)\left(1+\frac{m}{m-1}tr(A)\right)\sum_\alpha v^2_\alpha \right\} dM. \tag{2.42}$$

In case of the surface in S^3, the formula is reduced to a simple form. We omit all α and β because the codimension now is one.

Corollary 2.2 *For a surface in S^3 the second variation formula is given by*

$$\begin{aligned} W''(0) &= \int_M \{2(\Delta f)^2 + 2f\Delta f + 12C_iB_{ij}f_jf + 4A_{ij}f_{ij}f \\ &\quad -4A_{ij}f_if_j + (4\sum_{ij}A^2_{ij} + 4\sum_i C^2_i + \frac{7}{8} + K - 2K^2)\}dM, \end{aligned} \tag{2.43}$$

where K is Gaussian curvature of the Moebius metric $g = \rho^2 dx_0 \cdot dx_0$.

Remark 2.3: *The second variation formula for Willmore surfaces in S^3 meight be important towards the Willmore conjecture. As we know sofar the only stable example of Willmore torus is the Clifford torus. Combining the existence result of L. Simon in [7] we know that the Willmore conjecture is true if one can show that the only stable Willmore torus embedded in S^3 is the Clifford torus.*

§3. Moebius tori in S^{m+1} and their stability

In this section we present a class of important examples of Moebius minimal hypersurfaces called Moebius tori. As an application of Theorem 2.1 we show that they are stable Willmore hypersurfaces.

Let R^{m+2} be (m+2)-dimensional Euclidean space with inner product $<,>$. We write $R^{m+2} = R^{k+1} \times R^{m-k+1}, 1 \le k \le m-1$. For any vector $\xi \in R^{m+2}$ there is unique decomposition $\xi = \xi_1 + \xi_2$ with $\xi_1 \in R^{k+1}$ and $\xi_2 \in R^{m-k+1}$. For another vector $\eta = \eta_1 + \eta_2$ the inner product of them can be written as $<\xi, \eta> = <\xi_1, \eta_1> + <\xi_2, \eta_2>$. Let $\xi_1 : S^k \to R^{k+1}$ and $\xi_2 : S^{m-k} \to R^{m-k+1}$ be standard embedding of unit spheres. Let $x : S^k(a_1) \times S^{m-k}(a_2) \to S^{m+1} \subset R^{m+2}$ be the embedded hypersurface $x = a_1\xi_1 + a_2\xi_2$ with $a_1^2 + a_2^2 = 1$. It is easy to check that

(i) the unit normal vector of $M := S^k(a_1) \times S^{m-k}(a_2)$ in S^{m+1} is given by

$$e_{m+1} = -a_2\xi_1 + a_1\xi_2;$$

(ii) the second fundamental form of M is given by

$$II = -<dx, de_{m+1}> = a_1a_2(<d\xi_1, d\xi_1> - <d\xi_2, d\xi_2>);$$

(iii) the induced metric of M is given by

$$I = a_1^2|d\xi_1|^2 + a_2^2|d\xi_2|^2.$$

If we take $\{e_i\}$ and $\{\omega_i\}$such that

$$d(a_1\xi_1) = \sum_{i=1}^{k} \omega_i e_i, \quad d(a_2\xi_2) = \sum_{j=k+1}^{n} \omega_j e_j,$$

then we have

$$I = \sum_{i=1}^{m} \omega_i^2, \quad II = \sum_{i=1}^{k} \frac{a_2}{a_1}\omega_i^2 - \sum_{j=k+1}^{m} \frac{a_1}{a_2}\omega_j^2 := h_{ij}\omega_i\omega_j, \tag{3.1}$$

where

$$h_{ij} = \begin{cases} \frac{a_2}{a_1}\delta_{ij}, & if\ 1 \le i,j \le k, \\ -\frac{a_1}{a_2}\delta_{ij}, & if\ k+1 \le i,j \le n \end{cases} \tag{3.2}$$

Theorem 3.1. *Let $W = S^k(a_1) \times S^{m-k}(a_2)$ be the hypersurface imbedded into S^{m+1}, where $a_1^2 + a_2^2 = 1$. Then W is Moebius minimal if and if*

$$a_1 = \sqrt{\frac{m-k}{m}}, \quad a_2 = \sqrt{\frac{k}{m}}.$$

Proof. From (3.1) we see that

$$S := \sum_{i,j}^{m} h_{ij}^2 = k\left(\frac{a_2}{a_1}\right)^2 + (m-k)\left(\frac{a_1}{a_2}\right)^2, \tag{3.3}$$

$$H := \frac{1}{m}\sum_{i=1}^{m} h_{ii} = \frac{1}{m}\left(k\frac{a_2}{a_1} - (m-k)\frac{a_1}{a_2}\right), \tag{3.4}$$

$$\rho^2 = \frac{m}{m-1}(S - mH^2). \tag{3.5}$$

Substituting (3.2) and (3.5) into (1.8), (1.9) we know that Moebius minimal condition

$$B_{ij,ij} + B_{ik}B_{kj}B_{ji} + A_{ij}B_{ji} = 0$$

is equivalent to that

$$(m-k)\left(\frac{a_2}{a_1}\right)^6 + (2m-3k)\left(\frac{a_2}{a_1}\right)^4 + (m-3k)\left(\frac{a_2}{a_1}\right)^2 - k = 0. \tag{3.6}$$

From the equations (3.6) and $a_1^2 + a_2^2 = 1$ we get $a_1 = \sqrt{\frac{m-k}{m}}$ and $a_2 = \sqrt{\frac{k}{m}}$. Q.E.D.

We call

$$W_k^m = S^k\left(\sqrt{\frac{m-k}{m}}\right) \times S^{m-k}\left(\sqrt{\frac{k}{m}}\right), \ 1 \le k \le m-1$$

Moebius tori.

Remark 3.1 *It is remarkable that Moebius torus W_k^m is (euclidean) minimal if and only if $2k = m$. We note that W_k^m can be obtained by exchanging radii a_1 and a_2 in the Clifford torus $S^k\left(\sqrt{\frac{k}{m}}\right) \times S^{m-k}\left(\sqrt{\frac{m-k}{m}}\right)$.*

Remark 3.2 *It is known that any minimal surface in S^{m+1} is also Moebius minimal. Theorem 3.1 show that for submanifolds of dimension great than 2 a minimal submanifold may not be Moebius minimal.*

From now on we study the stability of Moebius minimal torus W_k^m defined in Theorem 3.1. For W_k^m we get from (3.2), (3.3), (3.4) and (3.5) that

$$h_{ij} = \begin{cases} \sqrt{\frac{k}{m-k}}\delta_{ij}, & if\ 1 \le i,j \le k, \\ -\sqrt{\frac{m-k}{k}}\delta_{ij}, & if\ k+1 \le i,j \le m, \end{cases} \tag{3.7}$$

$$S = \frac{k^3 + (m-k)^3}{k(m-k)}, \quad H = -\frac{m-2k}{\sqrt{k(m-k)}}, \quad \rho^2 = \frac{m^2}{m-1}. \tag{3.8}$$

From (1.8) and (1.9) we have

$$A_{ij} = \rho^{-2}Hh_{ij} + \frac{1}{2}\rho^{-2}(1-H^2)\delta_{ij}$$

$$= \begin{cases} \rho^{-2}\frac{3km-k^2-m^2}{2k(m-k)}\delta_{ij}, & if\ 1 \le i,j \le k, \\ \rho^{-2}\frac{m^2-k^2-km}{2k(m-k)}\delta_{ij}, & if\ k+1 \le i,j \le m, \end{cases} \tag{3.9}$$

$$tr(A) = \frac{(m-1)(m^2-3km+3k^2)}{2km(m-k)}, \tag{3.10}$$

$$B_{ij} = \rho^{-1}(h_{ij} - H\delta_{ij}) = \begin{cases} \rho^{-1}\sqrt{\frac{m-k}{k}}\delta_{ij}, & if\ 1 \le i,j \le k \\ -\rho^{-1}\sqrt{\frac{k}{m-k}}\delta_{ij}, & if\ k+1 \le i,j \le m, \end{cases} \tag{3.11}$$

and $\sum B_{ij}^2 = \frac{m-1}{m}$. From the last equation in (1.7) we obtain $C_i = 0$. We are going to calculate

$$W''(0) = -m^2\Gamma_1 + \frac{m^2}{m-1}\Gamma_2,$$

where $\Gamma_1 = \int_M \frac{\partial C_{i,i}}{\partial t}|_{t=0} f dM, \Gamma_2 = \int_M \{\frac{\partial}{\partial t}|_{t=0}(B_{ik}B_{kl}B_{li} + A_{ij}B_{ji})\} f dM$ and $f \in C^\infty(M)$ is the normal component of the variation vector field. From (2.40) we have

$$b = -\frac{1}{m}\left(\Delta f + \frac{(m-1)(m^2+k^2-km)}{2km(m-k)} f\right). \tag{3.12}$$

Substituting (3.9), (3.11) and (3.12) into (2.37), we get

$$\begin{aligned}
-m^2\Gamma_1 = & \int_M \left(m(\Delta f)^2 + \frac{(m-1)(m^2+k^2-km)}{2k(m-k)} f\Delta f\right) dM \\
- & \int_M \left(\frac{(m-1)(3km-k^2-m^2)}{2k(m-k)}|\nabla_1 f|^2 + \frac{(m-1)(m^2-km-k^2)}{2k(m-k)}|\nabla_2 f|^2\right) dM \\
+ & m^2 \int_M vB_{ij}f_{ij} dM.
\end{aligned} \tag{3.13}$$

Substituting (3.9), (3.11) and (3.12) into (2.39), by a straightforward calculation we get

$$\begin{aligned}
& \tfrac{m^2}{m-1}\Gamma_2 \\
& = -\frac{m^2-k^2+km}{2k(m-k)}\int_M f\Delta f dM + \frac{5m^2+5k^2-9km}{2k(m-k)}\int_M f\Delta_1 f dM \\
& + \frac{5k^2+m^2-km}{2k(m-k)}\int_M f\Delta_2 f dM - \frac{m^2}{m-1}\int_M vB_{ij}f_{ij} dM \\
& + \Big(2km(m-k)(m^2+7km-7k^2) - m(m^2-k^2+km)(k^2+m^2-km) \\
& + k(3km-m^2-k^2)^2 + (m-k)(m^2-km-k^2)^2\Big)\frac{m-1}{4k^2m^2(m-k)^2}\int_M f^2 dM.
\end{aligned} \tag{3.14}$$

In (3.13) and (3.14) we denote by Δ the Laplacian operator on W_k^m with respect to the Moebius metric $g = \rho^2 dx \cdot dx$. We write $g = g_1 \oplus g_2$ according to the decomposition $M_k^m := M_1 \times M_2 := S^k\left(\sqrt{\frac{m-k}{m}}\right) \times S^{m-k}\left(\sqrt{\frac{k}{m}}\right)$. We denote by $\{\Delta_1, \Delta_2\}$ the Laplacian operators of $\{g_1, g_2\}$ and by $\{\nabla_1, \nabla_2\}$ the gradient operators on M_1 and M_2 respectively. From (2.30), (3.9) and (3.11), we have

$$vB_{ij}f_{ij} = \frac{1}{m}\left(\sqrt{\frac{m-k}{k}}\Delta_1 f - \sqrt{\frac{k}{m-k}}\Delta_2 f\right)^2 \tag{3.15}$$

From(3.13), (3.14) and (3.15) we get

$$\begin{aligned}W''(0) &= \int_M \left(m(\Delta f)^2 + \frac{m(k^2+m^2-km-2m)}{2k(m-k)} f\Delta f \right.\\ &+ \frac{m(m-2)}{m-1}\left(\sqrt{\frac{m-k}{k}}\Delta_1 f - \sqrt{\frac{k}{m-k}}\Delta_2 f\right)^2 \\ &+ \frac{m(3km-k^2-m^2)+6(m-k)^2}{2k(m-k)} f\Delta_1 f \\ &+ \left. \frac{m(m^2-km-k^2)+6k^2}{2k(m-k)} f\Delta_2 f + \frac{2(m-1)}{m} f^2 \right) dM.\end{aligned} \tag{3.16}$$

We denote the Laplacian operators with respect to Euclidean metric $dx \cdot dx$ by Δ_M, Δ_{M_1} and Δ_{M_2} on W_k^m, M_1 and M_2 respectively. Since $\rho = constant$ and Moebius metric $g = \rho^2 dx \cdot dx$, we have $\Delta_M = \rho^2\Delta, \Delta_{M_1} = \rho^2\Delta_1, \Delta_{M_2} = \rho^2\Delta_2$ and $\Delta_M = \Delta_{M_1} + \Delta_{M_2}$. From (3.8) and (3.16), we have

$$\begin{aligned}W''(0) &= \frac{m-1}{2k(m-k)m^3}\int_M \Big(2k(m-k)(m-1)(\Delta_M f)^2 \\ &+ m^2(k^2+m^2-km-2m) f\Delta_M f \\ &+ 2(m-2)\left((m-k)\Delta_{M_1} f - k\Delta_{M_2} f\right)^2 \\ &+ m\left(m(3km-k^2-m^2)+6(m-k)^2\right) f\Delta_{M_1} f \\ &+ m\left(m(m^2-km-k^2)+6k^2\right) f\Delta_{M_2} f \\ &+ 4k(m-k)m^2 f^2\Big)\, dM.\end{aligned} \tag{3.17}$$

Let λ_i, λ_i' and μ_{ij} be the eigenvalue of Laplacian operators $\Delta_{M_1}, \Delta_{M_2}$ and Δ_M respectively, then we have

$$\lambda_i = i(k+i-1)\frac{n}{n-k},\ \lambda_j' = j(n-k+j-1)\frac{n}{k},\ \mu_{ij} = \lambda_i + \lambda_j',$$

where i and j are nonnegative integers. Let f_i, f_i' be eigenfunctions corresponding to λ_i and λ_i' respectively, then $g_{ij}(p,q) = f_i(p)f_j'(q)((p,q) \in M_1 \times M_2)$ is an eigenfunction corresponding to μ_{ij}. For any $f \in C^\infty(M_k^m)$ we have the decomposition of f in eigenfuctions

$$f = \sum_{i+j\neq 0} c_{ij}g_{ij} + c_0, \tag{3.18}$$

where c_{ij} and c_0 are constants. Thus we have

$$\Delta_{M_1} f = -\sum_{i+j\neq 0}\lambda_i c_{ij}g_{ij},\ \Delta_{M_2} f = -\sum_{i+j\neq 0}\lambda_j' c_{ij}g_{ij},\ \Delta_M f = -\sum_{i+j\neq 0}\mu_{ij}c_{ij}g_{ij}. \tag{3.19}$$

Substituting (3.18) and (3.19) into formula (3.17) we get

$$
\begin{aligned}
W''(0) &\geq \tfrac{m-1}{2k(m-k)m^3} \int_M \sum_{i+j\neq 0} \Big(2k(m-k)(m-1)\mu_{ij}^2 \\
&-m^2(k^2+m^2-km-2m)\mu_{ij} + 2(m-2)\left((m-k)\lambda_i - k\lambda_j'\right)^2 \\
&-m\left(m(3km-k^2-m^2)+6(m-k)^2\right)\lambda_i \\
&-m\left(m(m^2-km-k^2)+6k^2\right)\lambda_j' + 4k(m-k)k^2\Big) c_{ij}^2 g_{ij}^2 dM
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
&= \tfrac{m-1}{2k(m-k)m^3} \sum_{i+j\neq 0} \int_M \{2(m-k)(m^2-2m+k)\lambda_i^2 \\
&-2(2m^3+km^3-6km^2+3k^2m)\lambda_i \\
&+2k(m^2-m-k)(\lambda_j')^2 - 2m(m^3-m^2-km^2+3k^2)\lambda_j' \\
&+4k(m-k)\lambda_i\lambda_j' + 4k(m-k)m^2\} c_{ij}^2 g_{ij}^2 dM.
\end{aligned}
\tag{3.21}
$$

If we set

$$
\begin{aligned}
A(i,j) &= 2(m-k)(m^2-2m+k)\lambda_i^2 - 2(2m^3+km^3-6km^2+3k^2m)\lambda_i \\
&+2k(m^2-m-k)(\lambda_j')^2 - 2m(m^3-m^2-km^2+3k^2)\lambda_j' \\
&+4k(m-k)\lambda_i\lambda_j' + 4k(m-k)m^2,
\end{aligned}
$$

then one can easily verify that

$$
\begin{aligned}
A(i,j) &= \left(\tfrac{km}{(m-k)^2}(m^2-2m+k)i(k+i-1) + (\tfrac{2km}{m-k}j(m-k+j-1) - 2km)\right) \\
&\cdot \left(\tfrac{m(m-k)}{k^2}(m^2-m-k)j(m-k+j-1) + (\tfrac{2(m-k)m}{k}i(k+i-1) - 2(m-k)m)\right) \\
&- \tfrac{m^2(m^2-2m+k)(m^2-m-k)}{k(m-k)} ij(k+i-1)(m-k+j-1).
\end{aligned}
\tag{3.22}
$$

From (3.22) it is not diffculty to see that $A(i,j) \geq 0$ and $A(i,j) = 0$ if and only if $(i,j) = (1,0), (0,1)$ or $(1,1)$. Thus we have proved the main result in this section:

Theorem 3.2 *All Moebius tori* $S^k\left(\sqrt{\frac{m-k}{m}}\right) \times S^{m-k}\left(\sqrt{\frac{k}{m}}\right) \to S^{m+1}, 1 \leq k \leq m-1$, *are stable.*

In the end of this paper we would like to pose the following generalized Willmore conjecture in S^{m+1}:

Generalized Willmore Conjecture: Let M be a m-dimensional manifold which is diffeomorphic to $S^k \times S^{m-k}$ and $x : M \to S^{m+1}$ be an imbedding, where $1 \leq k \leq m-1$. Set

$$
\tau_k(x) = \left(\frac{m}{m-1}\right)^{\frac{m}{2}} \int_{G_k} \left(S - mH^2\right)^{\frac{m}{2}} dM, \tag{3.23}
$$

where dM is the volum element for induced metric $dx \cdot dx$, S the square of the length of the second fundamental form and H the mean curvature of x. Then

$$\tau_k(x) \geq \frac{4\pi^{\frac{m+2}{2}}(m-k)^{\frac{k}{2}}k^{\frac{m-k}{2}}}{m^{\frac{m}{2}-2}(m-1)\Gamma(\frac{k+1}{2})\Gamma(\frac{m-k+1}{2})} \tag{3.24}$$

and equality holds if and only if $x(M)$ is Moebius equivalent to W_k^m. Here Γ is gamma function and the term on the right of (3.24) is the Moebius volume of W_k^m.

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