Discrete Versions of Gronwall's Lemma and Their Application to the Numerical Analysis of Parabolic Problems

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Abstract

The well-known Gronwall lemma often serves as a major tool for the analysis of time-dependent problems via energy methods. However, there is the need for a similar tool when considering temporal discretizations of evolutionary problems.

With the paper in hand, the author wishes to give an overview of some discrete versions of the Gronwall lemma and presents a unified approach. In particular, new discrete versions of the lemma in its differential form and their application, showing decay behaviour for discretized parabolic problems, are studied.

These versions give effective tools for the stability and error analysis of the temporal semi-discretization of parabolic problems covering non-homogeneous problems as well as approximations with variable step sizes.

Keywords Gronwall-type inequality, differential inequality, difference inequality, parabolic differential equation, temporal discretization, Euler method

Mathematics subject classification 26D10, 26D15, 39A12, 65M12

1 Introduction

Stability analysis and a priori estimates for the exact solution to parabolic problems often lead to an initial value problem for an ordinary differential inequality that reads as

$$\frac{d}{dt}|u(t)|^2 + \mu\,\|u(t)\|^2 \;\le\; \|g(t)\|_*^2, \quad t > 0\,, \tag{1.1a}$$

$$|u(0)| \;=\; |u_0|\,, \tag{1.1b}$$

where $u = u(t) : \mathbb{R}_0^+ \to V \subseteq H$ is the function analysed and $g = g(t) : \mathbb{R}_0^+ \to V^*$, $u_0 \in H$ are given. Here, the set of real numbers is denoted by $\mathbb{R}$ whereas its subset of nonnegative numbers is denoted by $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$. The set of positive integers will be denoted by $\mathbb{N}$. The spaces $(V, \|\cdot\|)$, $(H, |\cdot|)$ are suitable BANACH spaces with a continuous embedding (POINCARÉ-FRIEDRICHS inequality):

$$\exists \alpha > 0 \; \forall v \in V : \quad |v| \leq \alpha \, \|v\| \,. \tag{1.2}$$

The function g represents a right hand side of the underlying parabolic problem, measured in the dual norm $\|\cdot\|_*$ (V^* denotes the dual space to V). However, in some applications, $|g(t)|$ appears rather than $\|g(t)\|_*$ if g is somewhat "better", say $g \in H$.

Moreover, the parameter $\mu > 0$ comes from the ellipticity (strong positiveness) of the underlying differential operator and reflects in some sense the problem's complication: for a singular perturbed problem, $\mu \ll 1$ is characteristic, whereas diffusion dominates if $\mu \gg 1$.

For what we have in mind, the heat equation, describing the non-stationary process of heat conduction in a domain Ω, may serve as a simple example:

$$u_t - \nabla \cdot \mu \nabla u = f \,,$$

where $u = u(x,t)$ is the (scaled) temperature with a given initial temperature field $u(\cdot, 0) = u_0$ and vanishing temperature at the boundary, and $f = f(x,t)$ is the quotient of heat production per time and mass unit and the specific heat. The coefficient $\mu = \mu(x,t) > 0$ is the heat conductivity, a material constant that is positive due to the second law of thermodynamics, divided by specific heat and mass density.

Multiplying the equation by u, integrating over Ω with integration by parts, and applying CAUCHY-SCHWARZ', POINCARÉ-FRIEDRICHS', and YOUNG'S inequality leads, under suitable assumptions, to the differential inequality

$$\frac{d}{dt} \int_\Omega u^2 dx + \bar{\mu} \int_\Omega (\nabla u)^2 dx \leq \frac{\alpha^2}{\bar{\mu}} \int_\Omega f^2 dx \,,$$

where $\bar{\mu} := \inf_{(x,t)} \mu(x,t) > 0$.

This inequality is of structure (1.1) with $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, and norms chosen appropriately. Here, $L^2(\Omega)$ denotes the usual LEBESGUE space and $H_0^1(\Omega)$ the SOBOLEV space of functions in $L^2(\Omega)$ whose generalized first derivatives also lie in $L^2(\Omega)$.

By integration, we easily derive from (1.1) an a priori estimate at (almost) each time level $t \in \mathbb{R}_0^+$,

$$|u(t)|^2 + \mu \int_0^t \|u(s)\|^2 \, ds \leq |u_0|^2 + \int_0^t \|g(s)\|_*^2 \, ds \,. \tag{1.3}$$

This gives stability w. r. t. the maximum in time norm of $u = u(t)$, considered as an abstract function mapping into H, as well as the L^2-norm of $u = u(t)$, considered as a function mapping into V.

However, we may use the continuous embedding $V \hookrightarrow H$ in the course of the analysis: With (1.1) and (1.2), we have

$$\frac{d}{dt}\left(\mathrm{e}^{\mu t/\alpha^2} |u(t)|^2 \right) \;=\; \mathrm{e}^{\mu t/\alpha^2} \left(\frac{d}{dt} |u(t)|^2 + \frac{\mu}{\alpha^2} \, |u(t)|^2 \right)$$

$$\leq \quad \mathrm{e}^{\mu t/\alpha^2} \left(\frac{d}{dt} |u(t)|^2 + \mu \, \|u(t)\|^2 \right) \leq \mathrm{e}^{\mu t/\alpha^2} \, \|g(t)\|_*^2 \, .$$

By integration, we then arrive at the estimate desired

$$|u(t)|^2 \leq \mathrm{e}^{-\mu t/\alpha^2} \, |u_0|^2 + \int_0^t \mathrm{e}^{\mu(s-t)/\alpha^2} \, \|g(s)\|_*^2 \, ds \tag{1.4}$$

that holds for (almost) all $t \in \mathbb{R}_0^+$ and gives a somewhat improved estimate w. r. t. the norm $|\cdot|$. Simultaneously, the estimate shows exponential decay whenever the right-hand side g is suitable. To be more precise, the upper bound of $|u(t)|$ is decreasing with time t for suitable g, say $\|g(t)\|_*$ is bounded for (almost) all t in the time interval under consideration.

The estimate above can be only achieved by using the differential form (1.1) of the problem that provides more information than the integral inequality (1.3). We shall refer to (1.4) as an application of the GRONWALL lemma in differential form as it will be formulated in Proposition 2.2.

Let us now consider a discretization in time of the underlying parabolic problem or even of inequality (1.1) itself by means of the implicit EULER scheme. We then come up with an estimate of the form

$$\frac{|u^{n+1}|^2 - |u^n|^2}{\Delta t} + \mu \, \|u^{n+1}\|^2 \leq \|g^{n+1}\|_*^2 , \quad n = 0,\, 1,\, \ldots \, , \tag{1.5}$$

where $u^n \sim u(t_n)$, $n = 0$, 1, ..., $t_n = n\Delta t$, $\Delta t > 0$, and u^0, $g^n \sim g(t_n)$ are given. The question arising immediately is: Are there any estimates similar to those of the continuous problem?

Summation over n leads from (1.5) to

$$|u^n|^2 + \mu\Delta t \sum_{j=1}^{n} \|u^j\|^2 \leq |u^0|^2 + \Delta t \sum_{j=1}^{n} \|g^j\|_*^2 , \quad n = 1,\, 2,\, \ldots \tag{1.6}$$

that is similar to estimate (1.3) of the continuous case, and again gives stability, now for the discrete solution.

However, using the embedding (1.2) as well as some discrete version of GRONWALL's lemma, we are able to prove the estimate

$$|u^n|^2 \leq (1 + \frac{\mu}{\alpha^2}\Delta t)^{-n} \, |u^0|^2 + \Delta t \sum_{j=1}^{n} (1 + \frac{\mu}{\alpha^2}\Delta t)^{j-1-n} \, \|g^j\|_*^2 , \quad n = 1,\, 2,\, \ldots \tag{1.7}$$

that is analogous to estimate (1.4) of the continuous case, and indeed shows decay of the stability constants with increasing number of time steps.

Again, for this estimate, we have to use the original difference form (1.5) and, therefore, we will refer to it as an application of the GRONWALL lemma in difference form, that is a discrete version of the lemma in differential form. It will be stated in Proposition 3.1 and in the more general Proposition 3.3.

We are faced with the same demand for discrete analogues when considering not the discrete solution but the error $e^n = u(t_n) - u^n$. For the discretization error e^n, an inequality of essentially the same structure as (1.5) can be derived from estimating e^n in terms of the consistency error.

Besides the applications above, we often need to apply the GRONWALL lemma when studying nonlinear problems or parabolic problems for which the underlying differential operator is not strongly positive but satisfies a GÅRDING inequality. Instead of the estimates (1.1) or (1.3), we then have modified estimates with an additional term $\kappa\,|u(t)|^2$ or $\kappa\int_0^t |u(s)|^2 ds$, resp., on the right hand side, where $\kappa > 0$ is a given constant. Again, there is the need for counterparts of such estimates in the time discrete case. The lemmata, which have to be used here, are those in integral or sum form and will be stated in Proposition 2.1 and 4.1.

In the next section, we recall the GRONWALL lemma in its original and nowadays well-known *integral* as well as in its *differential* form and discuss the difference between them. In particular, we emphasize on a sign assumption that is necessary for the integral but *not* for the differential version. Although the latter one might be not found running explicitly under the name GRONWALL lemma, some authors use the idea behind, cf. e. g. TEMAM [10], THOMÉE [12]. The integral version and extensions of it have been studied in detail and in different context by many authors. We refer e. g. to BRUNNER/VAN DER HOUWEN [1] and the references cited there.

In Section 3, we turn to discrete versions of the lemma in its differential form. As to the knowledge of the author, those versions are not known from the literature. We consider *difference* inequalities that can be derived from an original differential inequality by means of the implicit or explicit EULER scheme or the more general θ-method. By this, the second order CRANK-NICOLSON scheme is covered, too.

In a further section, we deal with a discrete version of the lemma in its integral form. Indeed, the GRONWALL lemma in *sum* form is rather known from the literature and many authors make use of it, cf. e. g. GIRAULT/RAVIART [5], BRUNNER/VAN DER HOUWEN [1].

The results of both, Section 3 and Section 4, apply to the case of variable step sizes. However, as with the integral version, the sum form has a sign restriction which does not appear with the difference form. Moreover, there may appear restrictions on the step sizes we carefully take into account.

Finally, results will be applied to a linear parabolic problem in an abstract setting. This covers linear partial differential operators with constant in time coefficients satisfying a GÅRDING inequality and problems with non-homogeneous right-hand side. The numerical methods analysed are the backward EULER scheme with variable time steps as well as the general θ-method. Again, we will also focus on step size restrictions appearing. The application to problems with time-dependent coefficients is left out here and will be considered in forthcoming research.

Although the lemmata presented and their proofs might be simple, they give effective tools in the analysis of numerical approximations to evolutionary problems. It has been tried throughout the proofs of the propositions to employ a unified approach that shows the similarities of the continuous and the discrete case.

2 GRONWALL's lemma in integral and differential form

Let us first recall the GRONWALL lemma both in its integral, cf. e. g. WLOKA [13], DAUTRAY/LIONS [2], as well as differential form.[1]

In the following for an interval $S \subseteq \mathbb{R}$, let $L^1(S)$ and $L^\infty(S)$ denote the usual LEBESGUE spaces, and $W^{1,1}(S)$ the usual SOBOLEV space of all $L^1(S)$-functions whose generalized derivative lies in $L^1(S)$. Furthermore, $\mathcal{C}(S)$ denotes the space of continuous functions. Finally, if $T = \infty$, the interval is $[0, T)$ instead of $[0, T]$.

Proposition 2.1 (GRONWALL **lemma: integral form)**
Let $T \in \mathbb{R}^+ \cup \{\infty\}$, a, $b \in L^\infty(0,T)$ and $\lambda \in L^1(0,T)$, $\lambda(t) \geq 0$ for almost all $t \in [0,T]$. Then,

$$a(t) \leq b(t) + \int_0^t \lambda(s)a(s)ds \qquad \text{a. e. in } [0,T] \tag{2.1}$$

implies for almost all $t \in [0,T]$

$$a(t) \leq b(t) + \int_0^t \mathrm{e}^{\Lambda(t)-\Lambda(s)}\lambda(s)b(s)ds\,, \tag{2.2}$$

where $\Lambda(t) := \int_0^t \lambda(\tau)d\tau$. If $b \in W^{1,1}(0,T)$, it follows

$$a(t) \leq \mathrm{e}^{\Lambda(t)} \left(b(0) + \int_0^t \mathrm{e}^{-\Lambda(s)} b'(s)ds \right)\,. \tag{2.3}$$

Moreover, if b is a monotonically increasing, continuous function, it holds

$$a(t) \leq \mathrm{e}^{\Lambda(t)} b(t)\,. \tag{2.4}$$

Proof Since a, $b \in L^\infty(0,T)$ and $\lambda \in L^1(0,T)$, the integrals appearing are well-defined.

Let

$$\tilde{a}(t) := \mathrm{e}^{-\Lambda(t)} \int_0^t \lambda(s)a(s)ds\,.$$

Then, for almost all $t \in [0,T]$, the estimate

$$\tilde{a}'(t) = \mathrm{e}^{-\Lambda(t)}\lambda(t) \left(a(t) - \int_0^t \lambda(s)a(s)ds \right) \leq \mathrm{e}^{-\Lambda(t)}\lambda(t)b(t)$$

holds due to (2.1) and $\lambda(t) \geq 0$. With $\tilde{a}(0) = 0$ by definition, integration over t leads to

$$\tilde{a}(t) \leq \int_0^t \mathrm{e}^{-\Lambda(s)}\lambda(s)b(s)ds\,.$$

Recalling assumption (2.1) as well as the definition of $\tilde{a}$, we then arrive at

$$\mathrm{e}^{-\Lambda(t)} \left(a(t) - b(t) \right) \leq \mathrm{e}^{-\Lambda(t)} \int_0^t \lambda(s)a(s)ds = \tilde{a}(t) \leq \int_0^t \mathrm{e}^{-\Lambda(s)}\lambda(s)b(s)ds$$

[1]The lemma was first mentioned in GRONWALL [6] by T. Hakon Grönvall (1877 –1932) himself, originally in integral form.

that gives us the wanted estimate (2.2).

As b is differentiable, inequality (2.3) follows immediately via integration by parts:

$$\begin{aligned}\int_0^t \mathrm{e}^{-\Lambda(s)}\lambda(s)b(s)ds &= \left[-\mathrm{e}^{-\Lambda(s)}b(s)\right]_0^t + \int_0^t \mathrm{e}^{-\Lambda(s)}b'(s)ds \\ &= -\mathrm{e}^{-\Lambda(t)}b(t) + b(0) + \int_0^t \mathrm{e}^{-\Lambda(s)}b'(s)ds\,.\end{aligned}$$

Note here, that $W^{1,1}(0,t) \hookrightarrow \mathcal{C}([0,t])$ for all t, $0 < t < T$, cf. ZEIDLER [14].

Finally, let b be monotonically increasing and continuous. We then derive from (2.2), in virtue of $\lambda(t) \geq 0$, the estimate

$$\begin{aligned}a(t) &\leq b(t)\left(1 + \int_0^t \mathrm{e}^{\Lambda(t)-\Lambda(s)}\lambda(s)ds\right) = b(t)\left(1 + \mathrm{e}^{\Lambda(t)}\int_0^t \frac{d}{ds}\left(-\mathrm{e}^{-\Lambda(s)}\right)ds\right) \\ &= b(t)\left(1 + \mathrm{e}^{\Lambda(t)}\left(-\mathrm{e}^{-\Lambda(t)} + 1\right)\right) = \mathrm{e}^{\Lambda(t)}b(t)\end{aligned}$$

that proves (2.4). #

Obviously, the assumption $\lambda(t) \geq 0$ a. e. in $[0,T]$ is of essential importance and cannot be omitted as the following counterexample shows.

Example 2.1 (Negative λ) Let $\lambda(t) \equiv \lambda < 0$, $b(t) = b + \omega(t)$, where $b \in \mathbb{R}$ and $\omega = \omega(t) : [0,T] \to \mathbb{R}_0^+$, supp $\omega \subset (0,T)$, and $a(t) = b\,\mathrm{e}^{\lambda t}$. We then have

$$b(t) + \int_0^t \lambda(s)a(s)ds = b + \omega(t) + \lambda b\int_0^t \mathrm{e}^{\lambda s}ds = b\mathrm{e}^{\lambda t} + \omega(t) = a(t) + \omega(t) \geq a(t)$$

for all $t \in [0,T]$, and the assumption of Proposition 2.1 holds.

Furthermore, we have

$$\begin{aligned}& b(t) + \int_0^t \mathrm{e}^{\Lambda(t)-\Lambda(s)}\lambda(s)b(s)ds \\ &= b + \omega(t) + \lambda b\int_0^t \mathrm{e}^{\lambda(t-s)}ds + \lambda\int_{(0,t)\cap\mathrm{supp}\ \omega} \mathrm{e}^{\lambda(t-s)}\omega(s)ds \\ &= a(t) + \omega(t) + \lambda\int_{(0,t)\cap\mathrm{supp}\ \omega} \mathrm{e}^{\lambda(t-s)}\omega(s)ds\,.\end{aligned}$$

Let $t \in [0,T]$ be sufficiently large, such that supp $\omega \subset (0,t)$. Then, by construction, $\omega(t) = 0$, and, in virtue of $\lambda < 0$, it follows immediately

$$a(t) > a(t) + \lambda\int_{(0,t)\cap\mathrm{supp}\ \omega} \mathrm{e}^{\lambda(t-s)}\omega(s)ds = b(t) + \int_0^t \mathrm{e}^{\Lambda(t)-\Lambda(s)}\lambda(s)b(s)ds\,.$$

Hence, the proposition is violated. #

We now turn to a GRONWALL-type lemma in which the inequality assumed is not given in integral but in differential form. Hence, we have got more, not only global but local information.

Proposition 2.2 (GRONWALL **lemma: differential form)**
Let $T \in \mathbb{R}^+ \cup \{\infty\}$, $a \in W^{1,1}(0,T)$ *and* g, $\lambda \in L^1(0,T)$. *Then,*

$$a'(t) \le g(t) + \lambda(t)a(t) \qquad a.\ e.\ in\ [0,T] \tag{2.5}$$

implies for almost all $t \in [0,T]$

$$a(t) \le \mathrm{e}^{\Lambda(t)}a(0) + \int_0^t \mathrm{e}^{\Lambda(t)-\Lambda(s)}g(s)ds\,, \tag{2.6}$$

where $\Lambda(t) := \int_0^t \lambda(s)ds$.

Note, that integration of (2.5) gives, with $b(t) := a(0) + \int_0^t g(s)ds$, inequality (2.1). If Proposition 2.1 is applicable, i. e. $\lambda(t) \ge 0$, the resulting estimates (2.2) and (2.3), resp., coincide with inequality (2.6). However, we are about to prove Proposition 2.2 *without* any assumption on the sign of λ.

Proof Since $a \in W^{1,1}(0,t) \hookrightarrow \mathcal{C}([0,t])$ for all t, $0 < t < T$, cf. ZEIDLER [14], and g, $\lambda \in L^1(0,T)$, the terms appearing are well-defined.

Let

$$\tilde{a}(t) = \mathrm{e}^{-\Lambda(t)}a(t)\,. \tag{2.7}$$

Then it follows from (2.5)

$$\tilde{a}'(t) = \mathrm{e}^{-\Lambda(t)}\left(a'(t) - \lambda(t)a(t)\right) \le \mathrm{e}^{-\Lambda(t)}g(t)$$

and integration gives

$$\tilde{a}(t) - \tilde{a}(0) = \mathrm{e}^{-\Lambda(t)}a(t) - a(0) \le \int_0^t \mathrm{e}^{-\Lambda(s)}g(s)ds\,.$$

Thus, inequality (2.6) holds a. e. in $[0,T]$ #

The essential difference between the integral and differential version is the requirement for the function λ which has to be nonnegative in the integral but not necessarily in the differential version.

With Proposition 2.2, we may show some decay behaviour for negative λ (depending on how g behaves). This is not possible in the context of Proposition 2.1.

3 GRONWALL's lemma in difference form

The same difference as between Propositions 2.1 and 2.2 might be expected in the discrete case. In this section, we will consider discrete versions of the GRONWALL lemma in differential form, Proposition 2.2.

3.1 EULER method with constant time steps

Let us consider a sequence of inequalities

$$\frac{a_{n+1} - a_n}{\Delta t} \le g_{n+1} + \lambda\, a_{n+1}\,, \quad n = 0,\, 1,\, \ldots, \tag{3.1}$$

where $\{a_n\}$, $\{g_n\} \subseteq \mathbb{R}$ and a_0, $\{g_n\}$ are given. We then ask for an explicit inequality for a_n.

Inequality (3.1) might be interpreted as the *backward* EULER *discretization* of (2.5) with $\lambda(t) \equiv \lambda$ and is similar to (1.5) with $\lambda = -\mu/\alpha^2$.

Proposition 3.1 (Discrete GRONWALL lemma: backward difference form) *Let $\{a_n\}$, $\{g_n\} \subseteq \mathbb{R}$ and $1-\lambda\Delta t > 0$. Then, inequality (3.1) implies for $n = 1, 2, \ldots$*

$$a_n \le (1 - \lambda\Delta t)^{-n} \left(a_0 + \Delta t \sum_{j=0}^{n-1} (1 - \lambda\Delta t)^j\, g_{j+1} \right) . \tag{3.2}$$

Moreover, if $\{g_n\}$ is monotonically increasing, it holds

$$a_n \le (1 - \lambda\Delta t)^{-n}\, a_0 + \frac{1}{\lambda} \left((1 - \lambda\Delta t)^{-n} - 1 \right) g_n\,. \tag{3.3}$$

Proof Let $\tilde{a}_n := (1 - \lambda\Delta t)^n a_n$. We then achieve from (3.1) for $1 - \lambda\Delta t > 0$

$$\begin{aligned} \frac{\tilde{a}_{n+1} - \tilde{a}_n}{\Delta t} &= \frac{1}{\Delta t}(1 - \lambda\Delta t)^n \left((1 - \lambda\Delta t) a_{n+1} - a_n\right) \\ &= (1 - \lambda\Delta t)^n \left(\frac{a_{n+1} - a_n}{\Delta t} - \lambda\, a_{n+1} \right) \\ &\le (1 - \lambda\Delta t)^n\, g_{n+1}\,. \end{aligned}$$

Summation over n leads to

$$\frac{\tilde{a}_n - \tilde{a}_0}{\Delta t} \le \sum_{j=0}^{n-1} (1 - \lambda\Delta t)^j\, g_{j+1}$$

that gives us immediately estimate (3.2).

If $\{g_n\}$ is monotonically increasing, we may estimate $g_{j+1} \le g_n$ in the sum of the r. h. s. of (3.2). Evaluation of the remaining partial sum gives

$$\sum_{j=0}^{n-1} (1 - \lambda\Delta t)^j = \frac{1}{\lambda\Delta t} \left(1 - (1 - \lambda\Delta t)^n\right) ,$$

and the assertion follows. #

Let $\lambda \le 0$. Then, the assumption $1 - \lambda\Delta t > 0$ of Proposition 3.1 is trivially fulfilled and, for suitable $\{g_n\}$, decay of the upper bound for $\{a_n\}$ with increasing n can be observed analogously to the continuous case. For positive λ, we have to assume Δt to be sufficiently small in order to fulfil $1-\lambda\Delta t>0$ and growth can be observed.

Note, that for $\{g_n\}\subseteq\mathbb{R}_0^+$, $a_0\in\mathbb{R}_0^+$, with arbitrary $\gamma\geq-\frac{1}{\Delta t}\ln(1-\lambda\Delta t)$, we have from (3.2)

$$a_n\leq \mathrm{e}^{\gamma t_n}\left(a_0+\Delta t\sum_{j=0}^{n-1}\mathrm{e}^{-\gamma t_j}g_{j+1}\right) \tag{3.4}$$

that shows exponential growth or decay as in the continuous case but with $\gamma>\lambda$ since $\mathrm{e}^{-\lambda\Delta t}\geq 1-\lambda\Delta t\geq \mathrm{e}^{-\gamma\Delta t}$. It cannot be expected to improve this result to $\gamma=\lambda$ (except for $\lambda=0$). However, since

$$\mathrm{e}^{\lambda\Delta t/(1-\lambda\Delta t)}\geq 1+\frac{\lambda\Delta t}{1-\lambda\Delta t}=\frac{1}{1-\lambda\Delta t}\,,$$

we may choose

$$\gamma=\frac{\lambda}{1-\lambda\Delta t}=\lambda+\frac{\lambda^2\Delta t}{1-\lambda\Delta t}\geq\lambda\,.$$

We now proceed with the inequality

$$\frac{a_{n+1}-a_n}{\Delta t}\leq g_n+\lambda\,a_n\,,\quad n=0,\,1,\,\dots \tag{3.5}$$

with given a_0, $\{g_n\}$, that is similar to (3.1) and might be interpreted as the *forward* EULER *discretization* of (2.5).

Proposition 3.2 (Discrete GRONWALL lemma: forward difference form) *Let $\{a_n\}$, $\{g_n\}\subseteq\mathbb{R}$ and $1+\lambda\Delta t>0$. Then, inequality (3.5) implies for $n=1,\,2,\,\dots$*

$$a_n\leq(1+\lambda\Delta t)^n\left(a_0+\Delta t\sum_{j=0}^{n-1}(1+\lambda\Delta t)^{-(j+1)}\,g_j\right) \tag{3.6}$$

Moreover, if $\{g_n\}$ is monotonically increasing, it holds

$$a_n\leq(1+\lambda\Delta t)^n\,a_0+\frac{1}{\lambda}\,\left((1+\lambda\Delta t)^n-1\right)\,g_{n-1}\,. \tag{3.7}$$

Proof Let $\tilde a_n:=(1+\lambda\Delta t)^{-n}a_n$. Then it holds

$$\begin{aligned}\frac{\tilde a_{n+1}-\tilde a_n}{\Delta t}&=\frac{1}{\Delta t}(1+\lambda\Delta t)^{-(n+1)}\left(a_{n+1}-(1+\lambda\Delta t)a_n\right)\\&=(1+\lambda\Delta t)^{-(n+1)}\left(\frac{a_{n+1}-a_n}{\Delta t}-\lambda\,a_n\right)\\&\leq(1+\lambda\Delta t)^{-(n+1)}g_n\,.\end{aligned}$$

Summation over n leads to

$$\frac{\tilde a_n-\tilde a_0}{\Delta t}\leq\sum_{j=0}^{n-1}(1+\lambda\Delta t)^{-(j+1)}g_j$$

that gives estimate (3.6).

If $\{g_n\}$ is monotonically increasing, we may estimate $g_j \leq g_{n-1}$ in the sum of (3.6), and with

$$\sum_{j=0}^{n-1}(1+\lambda\Delta t)^{-(j+1)} = \frac{1}{\lambda\Delta t}\left(1-(1+\lambda\Delta t)^{-n}\right),$$

the assertion follows. #

Let $\lambda < 0$. Then, the assumption $1+\lambda\Delta t > 0$ is satisfied whenever Δt is sufficiently small. For nonnegative λ, the assumption is trivially fulfilled. The decay (or growth) behaviour is essentially the same as in the foregoing proposition.

Note further, that, under the assumption $1-\lambda\Delta t > 0$, the backward discretization (3.1) might be rewritten as

$$a_{n+1} \leq \omega(\lambda\Delta t)\,(a_n + \Delta t\, g_{n+1}), \tag{3.8}$$

where $\omega(z) = 1/(1-z)$. Analogously, we may rewrite the forward discretization (3.5) in the form

$$a_{n+1} \leq \omega(\lambda\Delta t)\, a_n + \Delta t\, g_n, \tag{3.9}$$

where $\omega(z) = 1+z$. Therefore, using the ansatz $\tilde{a}_n := \omega(\lambda\Delta t)^{-n} a_n$ in both the proof of the backward as well as forward difference version of the Gronwall lemma is not surprising.

These amplification factors ω are well-known in the context of one-step discretization methods. In comparison with the ansatz made for proving Proposition 2.2, we shall remark that the factors $\omega(\lambda\Delta t)$ are indeed approximations of $\mathrm{e}^{\lambda\Delta t}$ and in this sense the approaches for proving Proposition 2.2 as well as its discrete versions coincide.

3.2 Linear one-step methods with variable step sizes

In the following, we extend the foregoing results to situations in which λ as well as Δt may vary with each (time) level n. This is e. g. of essential importance with respect to discretizations using non-equidistant partitions of the time interval as adaptive methods do.

In addition, we generalize the approximation of $\mathrm{e}^{\lambda\Delta t}$ by $\omega(\lambda\Delta t)$ and turn to the well-known θ-scheme, described by

$$\omega_\theta(z) = \frac{1+(1-\theta)z}{1-\theta z}, \quad \theta \in [0,1].$$

By this, every linear one-step method is included, esp. the Euler forward ($\theta = 0$) and backward ($\theta = 1$) and the Crank-Nicolson (or trapezoidal) scheme ($\theta = 1/2$).

For convenience, we set in the following $\prod_{l=1}^{0} x_l := 1$ for any x_l whatsoever. Furthermore, let $t_{n+1} = t_n + \tau_{n+1} = \sum_{i=1}^{n+1}\tau_i$, $n = 0, 1, \ldots$, $t_0 = 0$, $\tau_i > 0$. We consider the sequence of inequalities

$$\frac{a_{n+1}-a_n}{\tau_{n+1}} \leq g_{n+1} + (1-\theta)\lambda_n a_n + \theta\lambda_{n+1}a_{n+1}, \quad n = 0, 1, \ldots, \tag{3.10}$$

which might be rewritten for $1-\theta\lambda_{n+1}\tau_{n+1}>0$ as

$$a_{n+1} \leq \frac{\tau_{n+1} g_{n+1}}{1-\theta\lambda_{n+1}\tau_{n+1}} + \omega_{n+1} a_n\,, \quad n=0,\,1,\,\ldots\,,$$

where

$$\omega_{n+1} := \frac{1+(1-\theta)\lambda_n\tau_{n+1}}{1-\theta\lambda_{n+1}\tau_{n+1}}\,. \tag{3.11}$$

Proposition 3.3 (Discrete Gronwall lemma: general difference form) *Let* $\{a_n\}$, $\{g_n\}$, $\{\lambda_n\} \subseteq \mathbb{R}$, $\{\tau_n\} \subseteq \mathbb{R}^+$ *and*

$$1-\theta\lambda_{n+1}\tau_{n+1}>0\,, \quad 1+(1-\theta)\lambda_n\tau_{n+1}>0\,, \quad n=0,\,1,\,\ldots\,. \tag{3.12}$$

Then, inequality (3.10) implies for $n=1,\,2,\,\ldots$

$$a_n \leq a_0 \prod_{l=1}^{n} \omega_l + \sum_{j=0}^{n-1} \frac{\tau_{j+1} g_{j+1}}{1+(1-\theta)\lambda_j\tau_{j+1}} \prod_{l=j+1}^{n} \omega_l\,. \tag{3.13}$$

Proof Let

$$\tilde{a}_n := a_n \prod_{l=1}^{n} \omega_l^{-1}\,, \quad n=0,\,1,\,\ldots\,.$$

Because of (3.12), we derive from (3.10)

$$\begin{aligned}
\frac{\tilde{a}_{n+1}-\tilde{a}_n}{\tau_{n+1}} &= \frac{1}{\tau_{n+1}} \left(\prod_{l=1}^{n} \omega_l^{-1}\right) \left(a_{n+1}\omega_{n+1}^{-1} - a_n\right) \\
&= (1+(1-\theta)\lambda_n\tau_{n+1})^{-1} \left(\prod_{l=1}^{n} \omega_l^{-1}\right) \left(\frac{a_{n+1}-a_n}{\tau_{n+1}} - \theta\lambda_{n+1}a_{n+1} - (1-\theta)\lambda_n a_n\right) \\
&\leq (1+(1-\theta)\lambda_n\tau_{n+1})^{-1} \left(\prod_{l=1}^{n} \omega_l^{-1}\right) g_{n+1}\,.
\end{aligned}$$

Summation over n leads to ($\tilde{a}_0 = a_0$)

$$\tilde{a}_n \leq a_0 + \sum_{j=0}^{n-1} \frac{\tau_{j+1} g_{j+1}}{1+(1-\theta)\lambda_j\tau_{j+1}} \prod_{l=1}^{j} \omega_l^{-1}\,,$$

and the assertion follows. #

Let $\lambda_n < 0$ for all $n=0,\,1,\,\ldots$. Then, the assumption (3.12) holds if τ_{n+1} is sufficiently small,

$$\tau_{n+1} < \frac{1}{(1-\theta)|\lambda_n|}\,,$$

or $\theta = 1$ (backward Euler). For $\lambda_n > 0$, $n=0,\,1,\,\ldots$, again sufficiently small step sizes have to be required,

$$\tau_{n+1} < \frac{1}{\theta\lambda_{n+1}}\,,$$

or $\theta = 0$ (forward EULER).

Regarding the requirement of sufficiently small time steps, we shall give some remarks related to the concept of A-stability, cf. STREHMEL/WEINER [9]. The θ-method is said to be A-stable if $|\omega_\theta(z)| \leq 1$ for all $z \in \mathbb{C}$ with $\Re(z) < 0$. This is the case for $\theta \geq 1/2$. Considering the simple test problem

$$\begin{aligned} u'(t) &= \lambda u(t)\,, \quad t > 0\,, \\ u(0) &= u_0\,, \end{aligned}$$

where $u : \mathbb{R}_0^+ \to \mathbb{C}$, $\lambda \in \mathbb{C}$, $\Re(\lambda) < 0$, the approximate solution $\{u^n\}$, computed by an A-stable method, fulfils $|u^{n+1}| \leq |u^n|$ for all $n = 0, 1, \ldots$ and behaves asymptotically as the exact solution $u(t) = u_0 \exp(\lambda t)$. If $\theta < 1/2$, this can be only achieved for small step sizes: $\Delta t < 2|\Re(\lambda)|/((1-2\theta)|\lambda|^2)$ (with equidistant time steps).

However, in order to adapt also the monotonicity behaviour in the real case ($u : \mathbb{R}_0^+ \to \mathbb{R}$, $\lambda \in \mathbb{R}$, $\lambda < 0$), the stricter inequality $0 < \omega_\theta(\lambda\Delta t) < 1$ has to be assumed. This leads to restrictions even for A-stable methods: $\Delta t < 1/((1-\theta)|\lambda|)$. A similar consideration for $\omega_\theta(\lambda\Delta t) > 0$ if $\lambda > 0$ leads to $\Delta t < 1/(\theta\lambda)$.

4 GRONWALL's lemma in sum form

In this section, we prove a discrete version of Proposition 2.1, the GRONWALL lemma in integral form. For this, we consider the inequalities

$$a_{n+1} \leq b_{n+1} + \sum_{j=0}^{n} \left((1-\theta)\lambda_j a_j + \theta\lambda_{j+1}a_{j+1}\right)\tau_{j+1}\,, \quad n = 0, 1, \ldots\,, \tag{4.1}$$

which can be interpreted as an approximation of (2.1) with

$$\int_0^{t_{n+1}} \lambda(s)a(s)ds \approx \sum_{j=0}^{n} \left((1-\theta)\lambda_j a_j + \theta\lambda_{j+1}a_{j+1}\right)\tau_{j+1} =: s_{n+1}$$

and variable time steps $\tau_{j+1} > 0$, $t_{j+1} = t_j + \tau_{j+1}$. Again, we assume $\theta \in [0,1]$.

Proposition 4.1 (Discrete GRONWALL lemma: general sum form)
Let $\{a_n\}$, $\{b_n\} \subseteq \mathbb{R}$, $\{\lambda_n\} \subseteq \mathbb{R}_0^+$, $\{\tau_n\} \subseteq \mathbb{R}^+$ and $1 - \theta\lambda_{n+1}\tau_{n+1} > 0$ for all $n = 0, 1, \ldots$. Then, inequality (4.1) with the initial inequality $a_0 \leq b_0$ implies for $n = 0, 1, \ldots$

$$a_{n+1} \leq b_{n+1} + \sum_{j=0}^{n} \frac{\tau_{j+1}}{1-\theta\lambda_{j+1}\tau_{j+1}} \left((1-\theta)\lambda_j b_j + \theta\lambda_{j+1}b_{j+1}\right) \prod_{l=j+1}^{n} \omega_{l+1}\,, \tag{4.2}$$

where ω_{l+1} is defined by (3.11). If $\lambda_n = \lambda = const$ and $\tau_n = \Delta t = const$, it holds

$$a_{n+1} \leq b_{n+1} + \frac{\lambda\Delta t}{1-\theta\lambda\Delta t} \sum_{j=0}^{n} \left(\frac{1+(1-\theta)\lambda\Delta t}{1-\theta\lambda\Delta t}\right)^{n-j} \left((1-\theta)b_j + \theta b_{j+1}\right). \tag{4.3}$$

Moreover, if $\{b_n\}$ is monotonically increasing, it follows

$$a_n \leq b_n \left(\frac{1+(1-\theta)\lambda\Delta t}{1-\theta\lambda\Delta t}\right)^n . \tag{4.4}$$

Proof Let

$$\tilde{a}_{n+1} = s_{n+1} \prod_{l=1}^{n+1} \omega_l^{-1} .$$

Since $1 - \theta\lambda_{n+1}\tau_{n+1} > 0$ and $\lambda_n \geq 0$, we have from (4.1) for $n = 1, 2, \ldots$

$$\begin{aligned}
\tilde{a}_{n+1} - \tilde{a}_n &= \left(\prod_{l=1}^{n} \omega_l^{-1}\right) \left(\omega_{n+1}^{-1} s_{n+1} - s_n\right) \\
&= \tau_{n+1}\left(1 + (1-\theta)\lambda_n\tau_{n+1}\right)^{-1} \left(\prod_{l=1}^{n} \omega_l^{-1}\right) \left(\frac{s_{n+1} - s_n}{\tau_{n+1}} - \theta\lambda_{n+1}s_{n+1} - (1-\theta)\lambda_n s_n\right) \\
&= \tau_{n+1}\left(1 + (1-\theta)\lambda_n\tau_{n+1}\right)^{-1} \left(\prod_{l=1}^{n} \omega_l^{-1}\right) \left((1-\theta)\lambda_n(a_n - s_n) + \theta\lambda_{n+1}(a_{n+1} - s_{n+1})\right) \\
&\leq \tau_{n+1}\left(1 + (1-\theta)\lambda_n\tau_{n+1}\right)^{-1} \left(\prod_{l=1}^{n} \omega_l^{-1}\right) \left((1-\theta)\lambda_n b_n + \theta\lambda_{n+1}b_{n+1}\right) .
\end{aligned}$$

Summation leads to

$$\tilde{a}_{n+1} \leq \tilde{a}_1 + \sum_{j=1}^{n} \tau_{j+1}\left(1 + (1-\theta)\lambda_j\tau_{j+1}\right)^{-1} \left(\prod_{l=1}^{j} \omega_l^{-1}\right) \left((1-\theta)\lambda_j b_j + \theta\lambda_{j+1}b_{j+1}\right) .$$

Since $a_0 \leq b_0$ and $a_1 \leq b_1 + s_1$, it holds

$$a_1 \leq \frac{b_1 + (1-\theta)\lambda_0\tau_1 a_0}{1 - \theta\lambda_1\tau_1} \leq \frac{b_1 + (1-\theta)\lambda_0\tau_1 b_0}{1 - \theta\lambda_1\tau_1}$$

and thus

$$s_1 \leq \frac{(1-\theta)\lambda_0\tau_1}{1 - \theta\lambda_1\tau_1}\, b_0 + \frac{\theta\lambda_1\tau_1}{1 - \theta\lambda_1\tau_1}\, b_1 .$$

With $\tilde{a}_1 = s_1/\omega_1$, we then arrive at

$$\tilde{a}_1 \leq \frac{\tau_1}{1 + (1-\theta)\lambda_0\tau_1} \left((1-\theta)\lambda_0 b_0 + \theta\lambda_1 b_1\right) .$$

The assertion follows with

$$a_{n+1} \leq b_{n+1} + s_{n+1} = b_{n+1} + \tilde{a}_{n+1} \prod_{l=1}^{n+1} \omega_l$$

and some changes in the indices.

For constant λ_n and τ_n, the resulting estimate (4.3) follows obviously from (4.2). If, moreover, $\{b_n\}$ is monotonically increasing, we use $(1-\theta)b_j + \theta b_{j+1} \leq b_{n+1}$ in the sum of (4.3) and estimate the remaining sum that is a partial sum of a geometric sequence. #

Again, in order to fulfil $1 - \theta\lambda_n\tau_{n+1} > 0$ for $\lambda_n \geq 0$, we have to choose the step sizes τ_n sufficiently small.

Comparing the foregoing proposition with its continuous counterpart, Proposition 2.1, as well as comparing their proofs, we find again direct accordance when recalling that $\omega_\theta(z)$ is an approximation of e^z.

Finally, the necessity of $\lambda_n \geq 0$ is obvious in the course of the proof and restricts the applicability of discrete versions of the GRONWALL lemma in integral form in opposite to those of the lemma in differential form.

For later applications, we will need slightly changed versions with $b_{n+1} = d_{n+1} - c_{n+1}$, where $\{c_n\}, \{d_n\} \subseteq \mathbb{R}_0^+$ and $\{d_n\}$ is monotonically increasing. The assumptions are as in the proposition above except that the initial inequality $a_0 \leq b_0$ is not given.

Without proof, we provide here the resulting estimates needed later. The proofs follow exactly the same steps as above, taking into account that c_n, d_n are nonnegative and $\{d_n\}$ is monotonically increasing.

Let us first consider $\lambda_n = \lambda = const$ and $\tau_n = \Delta t = const$ as it is the case for the θ-scheme with an equidistant partition of the time interval. We then have

$$a_n + c_n \leq \frac{\omega^n}{1 + (1-\theta)\lambda\Delta t} \left(d_n + (1-\theta)\lambda\Delta t a_0\right) \tag{4.5}$$

for $n = 1, 2, , \ldots$ with $\omega = (1 + (1-\theta)\lambda\Delta t)/(1 - \theta\lambda\Delta t)$.

Now, we consider $\lambda_n = \lambda = const$, variable τ_n, and $\theta = 1$ as it is the case for the backward EULER scheme with variable step sizes. The estimate sought for then reads as

$$a_n + c_n \leq d_n + \lambda \sum_{j=0}^{n-1} \tau_{j+1} d_{j+1} \prod_{l=j+1}^{n} (1 - \lambda\tau_l)^{-1} \leq (1 - \lambda\tau_{max})^n d_n \tag{4.6}$$

for $n = 1, 2, , \ldots$ with $\tau_{max} := \max\{\tau_n\}$.

5 Application to linear parabolic problems

Let V be a separable, reflexive BANACH space with norm $\|\cdot\|$ and H be a separable HILBERT space with inner product $(\cdot,\cdot)$ and induced norm $|\cdot|$. The dual space of V is denoted by V^* and equipped with the usual dual norm $\|f\|_* := \sup_{v\in V\setminus\{0\}}\langle f, v\rangle/\|v\|$, where $\langle\cdot,\cdot\rangle$ denotes the dual product between V^* and V (that gives the value $\langle f, v\rangle$ of a functional $f \in V^*$ at the element $v \in V$).[2]

Furthermore, V is assumed to be dense and continuously embedded in H. Identifying H with its dual, we have, due to the reflexivity of V, that H is dense and continuously embedded in V^*. Thus V, H and V^* form an evolutional or GELFAND triple. The dual pairing $\langle\cdot,\cdot\rangle$ then is an extension of the inner product in H. Owing to the continuous embeddings, there is a constant $\alpha > 0$ such that the POINCARÉ-FRIEDRICHS inequalities

$$|v| \leq \alpha\|v\| \quad \forall v \in V\,, \tag{5.1a}$$
$$\|v\|_* \leq \alpha\,|v| \quad \forall v \in H \tag{5.1b}$$

[2]Due to the reflexivity of V, $\langle\cdot,\cdot\rangle$ is also the dual pairing between V^{**} and V^* and in this sense symmetric.

hold.

For a BANACH space X with its dual X^* and a time interval $S \subseteq \mathbb{R}$, let $L^2(S;X)$ be the set of BOCHNER measurable functions $u : S \to X$ equipped with the norm $\|u\|^2_{L^2(S;X)} := \int_S \|u(s)\|^2_X ds < \infty$, let further $\mathcal{W}(S;X) := \{u \in L^2(S;X) : \ u' \in L^2(S;X^*) = (L^2(S;X))^*\}$ equipped with the norm $\|u\|^2_{\mathcal{W}(S;X)} = \|u\|^2_{L^2(S;X)} + \|u'\|^2_{L^2(S;X^*)}$, and $\mathcal{C}(S;X)$ be the set of continuous functions $u : S \to X$ equipped with the norm $\|u\|_{\mathcal{C}(S;X)} = \sup_{s\in S} \|u(s)\|_X$.[3]

For compact S, $\mathcal{C}(S;X)$ is a BANACH space. By interpolation, the continuous embedding $\mathcal{W}(S;X) \hookrightarrow \mathcal{C}(S;Y)$ holds true for compact S whenever $X \hookrightarrow Y \equiv Y^* \hookrightarrow X^*$ is an evolutional triple.

Finally, if $(X, \|\cdot\|_X)$ is a BANACH space, then in X^d, $d \in \mathbb{N}$, we use the EUCLIDEAN norm $\|u\|_{X^d} = \sqrt{\sum_{i=1}^d \|u_i\|^2_X}$, where $u = (u_1,\, u_2,\, \ldots,\, u_d) \in X^d$.

For more details on the foregoing abstract setting, in particular on the concept of BOCHNER integral and the related function spaces, we refer to GAJEWSKI/GRÖGER/ZACHARIAS [4], and WLOKA [13].

Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a continuous bilinear form satisfying a GÅRDING inequality. Thus we have constants β, $\mu > 0$ and $\kappa \geq 0$ such that for all u, $v \in V$

$$|a(u,v)| \leq \beta \|u\| \, \|v\| \,, \tag{5.2a}$$

$$a(v,v) \geq \mu \|v\|^2 - \kappa |v|^2 \,. \tag{5.2b}$$

The form $a(\cdot,\cdot)$ is said to be strongly positive iff $\kappa = 0$ can be chosen. Without loss of generality, we may assume $\mu \leq \alpha^2\kappa$ if $\kappa > 0$. Otherwise, $a(\cdot,\cdot)$ would be strongly positive with a constant $\overline{\mu} = \mu - \alpha^2\kappa > 0$ due to (5.1).

With the bilinear form $a(\cdot,\cdot)$, we associate the linear operator $A : V \to V^*$ via $\langle Au, v\rangle = a(u,v)$ for all u, $v \in V$. We should mention that A is the energetic extension of the underlying differential operator.

Finally, we mention that V is indeed a HILBERT space with the inner product $[u,v] := (a(u,v) + a(v,u))/2 + \kappa\,(u,v)$ and induced norm $|\!|\!|u|\!|\!| = \sqrt{a(u,u) + \kappa\,|u|^2}$. However, we will not make use of this structure in V.

We consider the weak formulation of the initial-boundary value problem for a linear parabolic equation in the time interval $[0,T]$ which can be written as

Problem (P) *For given $u_0 \in H$ and $f \in L^2(0,T;V^*)$, find $u \in L^2(0,T;V)$ s. t.*

$$\langle u'(t), v\rangle + a(u(t), v) = \langle f(t), v\rangle \qquad \forall v \in V\,, \ \textit{a. e. in } (0,T], \tag{5.3a}$$

$$u(0) = u_0\,. \tag{5.3b}$$

Problem (P) has a unique solution $u \in \mathcal{W}(0,T;V)$, cf. GAJEWSKI/GRÖGER/ZACHARIAS [4], WLOKA [13]. Since $u \in \mathcal{W}(0,T;V) \hookrightarrow \mathcal{C}([0,T];H)$, the initial condition makes sense.

We now give two concrete examples for Problem (P).

[3]With u', the derivative in the distributional sense is meant. We shall also use the notation $u \in \mathcal{C}(S;X)$ to say that a function u is a. e. in S equal to a continuous function. We then deal with the continuous representative, only.

Example 5.1 (Singularly perturbed problem) Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with locally LIPSCHITZ continuous boundary $\partial\Omega \in \mathcal{C}^{0,1}$. The space H is chosen to be the $L^2(\Omega)$ with the usual inner product $(\cdot,\cdot)$ and norm $|\cdot|$, the space V to be the $H_0^1(\Omega)$ with the inner product $((\cdot,\cdot)) \equiv (\nabla\cdot,\nabla\cdot)$ and norm $\|\cdot\| \equiv |\nabla\cdot|$.

For given $0 < \varepsilon \ll 1$, $b : \overline{\Omega} \to \mathbb{R}^d$, $c : \overline{\Omega} \to \mathbb{R}$, and $f : \overline{\Omega} \to \mathbb{R}$ (all sufficiently smooth), we consider the initial-boundary value problem

$$\begin{aligned} u_t(x,t) - \varepsilon\Delta u(x,t) + (b(x)\cdot\nabla)u(x,t) + c(x)u(x,t) &= f(x,t) && \text{in } \Omega\times(0,T]\,, \\ u(x,t) &= 0 && \text{on } \partial\Omega\times(0,T]\,, \\ u(x,0) &= u_0 && \text{in } \Omega\,, \end{aligned}$$

where $u : \overline{\Omega}\times[0,T] \to \mathbb{R}$ is wanted. Singularly perturbed problems of that kind arise in many applications, covering convection- or reaction-diffusion problems in fluid dynamics, heat conduction, or semiconductor device simulation. For more details, we refer to ROOS/STYNES/TOBISKA [8].

With

$$a(u,v) = \varepsilon((u,v)) + ((b\cdot\nabla)u, v) + (cu, v)\,,$$

Problem (P) then is a weak formulation for the foregoing initial-boundary value problem. The constants can be chosen as $\mu = \varepsilon$, $\kappa = \|\frac{1}{2}\nabla\cdot b - c\|_{L^\infty(\Omega)}$ or, if $\gamma := \operatorname{ess\,sup}_{x\in\Omega}(\frac{1}{2}\nabla\cdot b - c) < \varepsilon/\alpha^2$, $\mu = \varepsilon - \gamma\alpha^2$ and $\kappa = 0$, $\beta = \varepsilon + \alpha\,\|b\|_{L^\infty(\Omega)^d} + \alpha^2\|c\|_{L^\infty(\Omega)}$. The constant $\alpha = \operatorname{diam}\Omega$ comes from the POINCARÉ-FRIEDRICHS inequality representing the continuous embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$. #

Example 5.2 (STOKES problem) Let $\Omega \subseteq \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain with locally LIPSCHITZ continuous boundary $\partial\Omega \in \mathcal{C}^{0,1}$. We consider the initial-boundary value problem describing the non-stationary, isothermal motion of a viscous, incompressible, homogeneous NEWTONIAN fluid neglecting nonlinear phenomena,

$$\begin{aligned} u_t(x,t) - \frac{1}{\mathrm{Re}}\Delta u(x,t) + \nabla p(x,t) &= f(x,t) && \text{in } \Omega\times(0,T]\,, \\ \nabla\cdot u(x,t) &= 0 && \text{in } \Omega\times(0,T]\,, \\ u(x,t) &= 0 && \text{on } \partial\Omega\times(0,T]\,, \\ u(x,0) &= u_0 && \text{in } \Omega\,, \end{aligned}$$

where Re is the REYNOLDS number, u denotes the velocity and p the quotient of pressure and constant mass density. The function f describes a specific force.

Let H and V be defined as the following solenoidal spaces

$$\begin{aligned} H &= \left\{ v \in L^2(\Omega)^d :\ \nabla\cdot v = 0 \ \text{ in } H^{-1}(\Omega)\,,\ \gamma_n u = 0 \ \text{ in } H^{-1/2}(\partial\Omega) \right\}\,, \\ V &= \left\{ v \in H_0^1(\Omega)^d :\ \nabla\cdot v = 0 \ \text{ in } L^2(\Omega) \right\}\,, \end{aligned}$$

where γ_n is the trace operator mapping from a subset of $L^2(\Omega)^d$ on $H^{-1/2}(\partial\Omega)$ with $\gamma_n v = (v\cdot n)_{|\partial\Omega}$ for all smooth v. Here, n denotes the outer normal on $\partial\Omega$. The inner products and norms are as in Example 5.1, except that we deal here with vector-valued functions.

Let a be defined by $a(u,v) = \mathrm{Re}^{-1}\,(\!(u,v)\!)$ for all u, $v \in V$. Then with Problem (P), a weak formulation for the foregoing initial-boundary value problem is given, and the constants are $\mu = \mathrm{Re}^{-1}$, $\kappa = 0$, $\beta = \mu$. As in Example 5.1, α is the constant from the POINCARÉ-FRIEDRICHS inequality.

For more details see e. g. TEMAM [10]. #

At this point, it is worth to mention, that no assumptions on the dimension of V or H had to be made. Both spaces could be finite dimensional, too. This is particularly of interest when first discretizing in space and then discretizing in time. The spatial semi-discretization could be e. g. a conforming finite element method; for more details, we refer to THOMÉE [12], FUJITA/SUZUKI [3], and the references cited there. Hence, with the analysis in hand, we cover both the line method as well as the ROTHE method. However, in the finite dimensional case, the spaces of the GELFAND triple $V \hookrightarrow H \hookrightarrow V^*$ differ in particular in the use of different spatially discrete norms.

In the following, we consider discretizations in time of Problem (P) by means of *linear one-step methods (θ-method)*, covering the *backward* EULER *method* as well as the CRANK-NICOLSON *method*. Based upon energy methods, we provide a priori stability as well as error estimates assuming sufficiently smooth and compatible data.

To exemplify the approach, we firstly study the backward EULER method for the linear parabolic problem with a strongly positive bilinear form based upon an equidistant partition of the time interval $[0,T]$.

We then discuss the more general θ-method and take into consideration bilinear forms satisfying a GÅRDING inequality. It turns out, that parts of the analysis can be only carried out for so-called A-stable methods, i. e. $\theta \in [1/2, 1]$.

Finally, we consider the backward EULER method with variable step sizes.

The more general case of a linear parabolic problem with time-dependent form $a(t;\cdot,\cdot)$, arising from time-dependent coefficients in the underlying differential operator, as well as an error analysis for non-smooth or incompatible data that takes advantage of parabolic smoothing properties are considered partly in HUANG/THOMÉE [7], THOMÉE [12], and will be discussed in detail in a forthcoming paper.

5.1 Backward EULER method with constant step size

Let $N \in \mathbb{N}$ be given and $\Delta t = T/N$, $t_n = n\Delta t$, $n = 0, 1, 2, \ldots, N$. Since $f \in L^2(0,T;V^*)$, we can use the natural restriction

$$f^{n+1} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(t)\,dt\,. \tag{5.4}$$

The method under consideration is then defined as

Problem $(P_{\Delta t})$ *For given $u^0 \in H$ and $\{f^n\}_{n=1}^N \subseteq V^*$, find $\{u^n\}_{n=1}^N \subseteq V$ s. t. for $n = 0, 1, \ldots, N-1$ and all $v \in V$*

$$\frac{1}{\Delta t}(u^{n+1} - u^n, v) + a(u^{n+1}, v) \;=\; \langle f^{n+1}, v\rangle\,. \tag{5.5}$$

Here, the initial value u^0 might be taken to be the exact value u_0 or an approximation of it.

We assume $a(\cdot,\cdot)$ to be strongly positive. Due to the main theorem for monotone operators by Browder and Minty, cf. Gajewski/Gröger/Zacharias [4], Problem $(P_{\Delta t})$ has a unique solution if the bilinear form $(\cdot,\cdot)+\Delta t a(\cdot,\cdot)$ is strongly positive on V. Because of the strong positiveness of $a(\cdot,\cdot)$, this is obviously fulfilled without any restriction on the time step size Δt.[4]

Proposition 5.1 (Stability estimates for Problem $(P_{\Delta t})$)
The discrete solution $\{u^n\}$ to Problem $(P_{\Delta t})$ is stable in the following sense:

$$\max_{j=0,\dots,N}|u^j|^2+\sum_{j=0}^{N-1}|u^{j+1}-u^j|^2+\mu\Delta t\sum_{j=0}^{N-1}\|u^{j+1}\|^2\le 2\left(|u^0|^2+\frac{1}{\mu}\|f\|^2_{L^2(0,T;V^*)}\right) \tag{5.6}$$

$$|u^n|^2\le\left(1+\frac{\mu\Delta t}{\alpha^2}\right)^{-n}|u^0|^2+\frac{1}{\mu}\sum_{j=0}^{n-1}\left(1+\frac{\mu\Delta t}{\alpha^2}\right)^{j-n}\int_{t_j}^{t_{j+1}}\|f(t)\|_*^2\,dt \tag{5.7}$$

for $n=0,\,1,\,\dots,\,N$.

Here, $\max_{j=0,\dots,N}|u^j|^2$ can be interpreted as the square of the (time) discrete $l^\infty(0,T;H)$-norm, and $\Delta t\sum_{j=0}^{N-1}\|u^{j+1}\|^2$ as the square of the discrete $l^2(0,T;V)$-norm.

Indeed, (5.6) coincides with the usual stability estimate for the exact solution: With $v=u(t)$, we easily derive from Problem (P),

$$\operatorname*{ess\,sup}_{t\in[0,T]}|u(t)|^2+\mu\int_0^T\|u(t)\|^2\,dt\le 2\left(|u_0|^2+\frac{1}{\mu}\|f\|^2_{L^2(0,T;V^*)}\right). \tag{5.8}$$

The analysis presented for proving (5.6) will be more or less known from the literature, cf. e. g. Girault/Raviart [5]. However, there is the more refined estimate regarding the $l^\infty(0,T;H)$-norm of the discrete solution, estimate (5.7), that uses the continuous embedding and a discrete version of the Gronwall lemma in differential form.

This estimate again has a continuous counterpart (cf. also Proposition 2.2):

$$|u(t)|^2\le \mathrm{e}^{-\mu t/\alpha^2}|u_0|^2+\frac{1}{\mu}\int_0^t\mathrm{e}^{-\mu(t-s)/\alpha^2}\|f(s)\|_*^2ds\,,\quad t\in[0,T]\text{ a. e.} \tag{5.9}$$

However, for singularly perturbed problems (i. e. for small μ), a virtual improvement only turns up in the long-term behaviour for large t.

Proof Let us choose $v=u^{n+1}$ in (5.5) in order to profit by the strong positiveness of $a(\cdot,\cdot)$. However, we then have to deal with the term $(u^{n+1}-u^n,u^{n+1})$ that is of the structure $(a-b)a$. We have (using implicitly the Hilbert space structure)

$$(a-b)a=a^2-ab=\frac{a^2}{2}-\frac{b^2}{2}+\frac{1}{2}\left(a^2-2ab+b^2\right)=\frac{a^2}{2}-\frac{b^2}{2}+\frac{1}{2}(a-b)^2$$

[4] Since V is at first a Banach space, the Lax-Milgram lemma does not apply directly.

and analogously

$$\frac{1}{2}|u^{n+1}|^2 - \frac{1}{2}|u^n|^2 \le \frac{1}{2}|u^{n+1}|^2 - \frac{1}{2}|u^n|^2 + \frac{1}{2}|u^{n+1}-u^n|^2 = (u^{n+1}-u^n, u^{n+1})\,.$$

From this, we conclude (using CAUCHY-SCHWARZ' and YOUNG's inequality)

$$\frac{1}{2\Delta t}\left(|u^{n+1}|^2 - |u^n|^2 + |u^{n+1}-u^n|^2\right) + \mu\,\|u^{n+1}\|^2$$
$$= \langle f^{n+1}, u^{n+1}\rangle \le \|f^{n+1}\|_*\,\|u^{n+1}\| \le \frac{1}{2\mu}\|f^{n+1}\|_*^2 + \frac{\mu}{2}\|u^{n+1}\|^2\,.$$

After summing from 0 up to $n-1$, we end up with the a priori estimate

$$|u^n|^2 + \sum_{j=0}^{n-1}|u^{j+1}-u^j|^2 + \mu\Delta t\sum_{j=0}^{n-1}\|u^{j+1}\|^2 \le |u^0|^2 + \frac{\Delta t}{\mu}\sum_{j=0}^{n-1}\|f^{j+1}\|_*^2.$$

Furthermore, we have with (5.4)

$$\begin{aligned}\Delta t\sum_{j=0}^{n-1}\|f^{j+1}\|_*^2 &\le \frac{1}{\Delta t}\sum_{j=0}^{n-1}\left(\int_{t_j}^{t_{j+1}}\|f(t)\|_*dt\right)^2 \le \frac{1}{\Delta t}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}dt\int_{t_j}^{t_{j+1}}\|f(t)\|_*^2dt\\ &= \int_0^{t_n}\|f(t)\|_*^2dt \le \|f\|_{L^2(0,T;V^*)}^2\,,\end{aligned} \tag{5.10}$$

and the numerical solution is stable in the sense of (5.6). The factor 2 in the r. h. s. of (5.6) comes from splitting the inequality and taking the maximum on the l. h. s.

With the POINCARÉ-FRIEDRICHS inequality, we derive analogously

$$\frac{1}{\Delta t}\left(|u^{n+1}|^2 - |u^n|^2\right) + \frac{\mu}{\alpha^2}|u^{n+1}|^2 \le \frac{1}{\mu}\|f^{n+1}\|_*^2\,.$$

Applying the discrete GRONWALL lemma in backward difference form, Proposition 3.1, with $\lambda = -\mu/\alpha^2 < 0$, we come to (5.7). #

For the discretization error $e^n := u(t_n) - u^n$, the following result can be proved.

Proposition 5.2 (Error estimates for Problem $(P_{\Delta t})$)
For the discretization error e^n, the following estimates hold for $n = 1, 2, \ldots, N$ whenever the exact solution and the problem's data are smooth:

$$|e^n|^2 + \sum_{j=0}^{n-1}|e^{j+1}-e^j|^2 + \mu\Delta t\sum_{j=0}^{n-1}\|e^{j+1}\|^2 \le |e^0|^2 + \frac{(\Delta t)^2}{3\mu}\|f'-u''\|_{L^2(0,T;V^*)}^2\,, \tag{5.11}$$

$$|e^n|^2 \le \left(1+\frac{\mu\Delta t}{\alpha^2}\right)^{-n}|e^0|^2 + \frac{(\Delta t)^2}{3\mu}\sum_{j=0}^{n-1}\left(1+\frac{\mu\Delta t}{\alpha^2}\right)^{j-n}\int_{t_j}^{t_{j+1}}\|f'(t)-u''(t)\|_*^2\,dt\,. \tag{5.12}$$

As we will prove in the appendix, it holds for sufficiently smooth solutions and compatible data the a priori estimate

$$\|f' - u''\|^2_{L^2(0,T;V^*)} \leq \frac{\beta^2}{\mu}\left(|Au_0|^2 + \frac{\beta^2}{\mu}\|f\|^2_{L^2(0,T;V)}\right). \tag{5.13}$$

Proof Firstly, we derive an error equation that gives a relation between the discretization error e^n and the consistency error ρ^n, i. e. the error appearing when putting the exact solution into the scheme.

For all $v \in V$ and $n = 0, 1, \ldots, N-1$, we have

$$\frac{1}{\Delta t}(e^{n+1} - e^n, v) + a(e^{n+1}, v) = \langle \rho^{n+1}, v\rangle . \tag{5.14}$$

The consistency error ρ^{n+1} is given by

$$\langle \rho^{n+1}, v\rangle = -\left\langle u'(t_{n+1}) - \frac{u(t_{n+1}) - u(t_n)}{\Delta t}, v\right\rangle + \langle f(t_{n+1}) - f^{n+1}, v\rangle$$

for all $v \in V$. With integration by parts, we find with (5.4)

$$\rho^{n+1} = -\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t - t_n)u''(t)dt + \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t - t_n)f'(t)dt\,,$$

where the integrals shall be understood as BOCHNER integrals.

The error analysis now follows exactly the same steps as the stability analysis, so that we end up with the a priori error estimate for $n = 0, 1, \ldots, N$

$$|e^n|^2 + \sum_{j=0}^{n-1} |e^{j+1} - e^j|^2 + \mu\Delta t \sum_{j=0}^{n-1} \|e^{j+1}\|^2 \leq |e^0|^2 + \frac{\Delta t}{\mu}\sum_{j=0}^{n-1} \|\rho^{j+1}\|^2_* .$$

Furthermore, we have

$$\begin{aligned}
\|\rho^{j+1}\|^2_* &\leq \left\| \frac{1}{\Delta t}\int_{t_j}^{t_{j+1}} (t - t_j)(f'(t) - u''(t))dt \right\|^2_* \\
&\leq \frac{1}{(\Delta t)^2}\left(\int_{t_j}^{t_{j+1}} (t - t_j)\|f'(t) - u''(t)\|_* dt\right)^2 \\
&\leq \frac{1}{(\Delta t)^2}\int_{t_j}^{t_{j+1}} (t - t_j)^2 dt \int_{t_j}^{t_{j+1}} \|f'(t) - u''(t)\|^2_* dt \\
&= \frac{\Delta t}{3}\int_{t_j}^{t_{j+1}} \|f'(t) - u''(t)\|^2_* dt\,,
\end{aligned}$$

and hence

$$\begin{aligned}
\Delta t \sum_{j=0}^{n-1} \|\rho^{j+1}\|^2_* &\leq \frac{(\Delta t)^2}{3}\int_0^{t_n} \|f'(t) - u''(t)\|^2_* dt \\
&\leq \frac{(\Delta t)^2}{3}\|f' - u''\|^2_{L^2(0,T;V^*)} .
\end{aligned}$$

We end up with the a priori error estimate (5.11).

Again, we can apply Proposition 3.1 after estimating $\|e^{j+1}\| \geq |e^{j+1}|/\alpha$. This yields

$$\frac{1}{\Delta t}\left(|e^{n+1}|^2 - |e^n|^2\right) + \frac{\mu}{\alpha^2}\,|e^{n+1}|^2 \leq \frac{1}{\mu}\,\|\rho^{n+1}\|_*^2\,,$$

and hence (5.12). #

5.2 A-stable θ-method

As in the previous section, we proceed with an equidistant partition of the time interval. However, the use of variable time steps is (almost) straightforward. The method under consideration is now defined as

Problem $(P_{\Delta t}^{\theta})$ *For given $u^0 \in V$ and $\{f^{n+\theta}\}_{n=0}^{N-1} \subseteq V^*$, find $\{u^n\}_{n=1}^{N} \subseteq V$ s. t. for $n = 0, 1, \ldots, N-1$ and all $v \in V$*

$$\frac{1}{\Delta t}(u^{n+1} - u^n, v) + a(u^{n+\theta}, v) = \langle f^{n+\theta}, v\rangle\,. \tag{5.15}$$

Here, $\theta \in [0,1]$ is a parameter, $u^{n+\theta} := \theta u^{n+1} + (1-\theta)u^n$, and $f^{n+\theta}$ is an approximation of f at $t_{n+\theta} := \theta\, t_{n+1} + (1-\theta)\, t_n$, e. g. $f^{n+\theta} = \theta f(t_{n+1}) + (1-\theta) f(t_n)$ if f is sufficiently smooth or, which we will use in the following, $f^{n+\theta} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(t)dt$.

Note, that u^0 – the approximation of u_0 – has to be an element of V except for $\theta = 1$ where $u^0 \in H$ is sufficient.

It is known that the method applied to an ordinary differential equation with a smooth solution is convergent of first order, whereas second order is achieved for $\theta = 1/2$, only. Moreover, the method is A-stable for $\theta \in [1/2, 1]$ and strongly A-stable if, in addition, $\theta \neq 1/2$. For more details, we refer e. g. to Strehmel/Weiner [9].

The bilinear form $a(\cdot,\cdot)$ fulfils a Gårding inequality. Again, due to the main theorem for monotone operators by Browder and Minty, cf. Gajewski/Gröger/Zacharias [4], Problem $(P_{\Delta t}^{\theta})$ possesses a unique solution if $c(\cdot,\cdot) := (\cdot,\cdot) + \theta\Delta t a(\cdot,\cdot)$ is strongly positive. For all $v \in V$, we have

$$\begin{aligned} c(v,v) &= |v|^2 + \theta\Delta t a(v,v) \geq |v|^2 + \theta\Delta t(\mu\|v\|^2 - \kappa|v|^2) \\ &= (1-\kappa\theta\Delta t)|v|^2 + \mu\theta\Delta t\|v\|^2\,. \end{aligned}$$

If now $\kappa\theta\Delta t \leq 1$, i. e. $N \geq \kappa\theta T$, the form $c(\cdot,\cdot)$ is strongly positive. In the opposite case, if $\kappa\theta\Delta t > 1$, we have with the Poincaré-Friedrichs inequality

$$c(v,v) \geq \left(\alpha^2(1-\kappa\theta\Delta t) + \mu\theta\Delta t\right)\|v\|^2\,,$$

and $c(\cdot,\cdot)$ is strongly positive if $\alpha^2 + (\mu - \alpha^2\kappa)\theta\Delta t > 0$. Since $\mu \leq \alpha^2\kappa$ if $\kappa > 0$ by assumption, we arrive at

$$\frac{\alpha^2\kappa - \mu}{\alpha^2}\,\theta\Delta t < 1\,, \qquad \text{i. e. } N > \frac{\alpha^2\kappa - \mu}{\alpha^2}\,\theta T\,. \tag{5.16}$$

The latter case is less restrictive than $\kappa\theta\Delta t \leq 1$. Hence, Problem $(P_{\Delta t}^{\theta})$ has a unique solution if (5.16) is fulfilled, i. e. if the time steps are sufficiently small.

However, for the forward EULER method with $\theta = 0$ as well as for a positive $a(\cdot,\cdot)$ ($\kappa = 0$ or $\mu = \alpha^2\kappa$), there will be no restriction on the time step size.

We firstly come to stability estimates. There are many different ways to estimate the discrete solution in some norm. Hence, the following analysis can only present one possibility.

To derive the estimates desired, we choose $v = u^{n+\theta}$ in (5.15). We then have to estimate the term $(u^{n+1} - u^n, \theta u^{n+1} + (1-\theta)u^n)$ which is of the structure

$$
\begin{aligned}
(a-b)(\theta a + (1-\theta)b) &= \theta a^2 + (1-2\theta)ab - (1-\theta)b^2 \\
&= \frac{a^2}{2} - \frac{b^2}{2} + (\theta - \frac{1}{2})(a-b)^2 \\
&\geq \frac{a^2}{2} - \frac{b^2}{2} \qquad \text{if } \theta \geq 1/2.
\end{aligned}
$$

Obviously, there is no such estimate for $\theta \in [0, 1/2)$: Assume, there were constants δ, $\varepsilon > 0$ such that

$$\theta a^2 + (1-2\theta)ab - (1-\theta)b^2 \geq \delta a^2 - \varepsilon b^2 \,.$$

In order to be able to proceed with the analysis, we have to ensure $\delta > \varepsilon$. On the other hand, $b = 0$ and $a = 0$, resp., imply $\theta \geq \delta$ and $-(1-\theta) \geq \varepsilon$. Thus, we have $\theta \geq \delta \geq \varepsilon \geq 1 - \theta$ and hence $\theta \geq 1/2$ which is in contradiction to our assumption.

If, however, the form $a(\cdot,\cdot)$ is not strongly positive, then the GÅRDING inequality with $\kappa > 0$ effects in some sense a stabilization: The numerical solution need not to cover all the qualitative behaviour of the exact solution to a strongly positive problem. By this, the scheme can be stable even for $\theta < 1/2$ as we will show next.

Proposition 5.3 (Stability estimates for Problem $(P_{\Delta t}^\theta)$)
The discrete solution $\{u^n\}$ to Problem $(P_{\Delta t}^\theta)$ is stable in the following sense:

If $\theta \in [1/2, 1]$ and

$$2\theta\kappa\Delta t < 1 \tag{5.17}$$

then

$$
\begin{aligned}
&|u^n|^2 + \Big((2\theta - 1) + 2\kappa\theta(1-\theta)\Delta t\Big) \sum_{j=0}^{n-1} |u^{j+1} - u^j|^2 \\
&\quad + \mu(2\theta-1)^2\Delta t \sum_{j=0}^{n-1} \|u^{j+1}\|^2 + \mu(2\theta-1)(1-\theta)\Delta t\,\|u^n\|^2 \\
&\leq \omega^n \Big(|u^0|^2 + \mu(2\theta-1)(1-\theta)\Delta t\,\|u^0\|^2\Big) + \frac{\omega^n}{1 + 2(1-\theta)\kappa\Delta t}\,\frac{1}{\mu}\,\|f\|^2_{L^2(0,T;V^*)}
\end{aligned} \tag{5.18}
$$

for $n = 1, 2, \ldots, N$, where $\omega = (1 + 2(1-\theta)\kappa\Delta t)/(1 - 2\theta\kappa\Delta t)$.

If $\kappa \neq 0$ and

$$\frac{1-2\theta}{2\kappa\theta(1-\theta)} \leq \Delta t < \frac{1}{2\theta\kappa}\,, \tag{5.19}$$

where θ can be less than $1/2$, then the weaker stability estimate

$$|u^n|^2 + \Big((2\theta-1) + 2\kappa\theta(1-\theta)\Delta t\Big) \sum_{j=0}^{n-1} |u^{j+1} - u^j|^2 + \mu\Delta t \sum_{j=0}^{n-1} \|u^{j+\theta}\|^2$$
$$\le \omega^n |u^0|^2 + \frac{\omega^n}{1 + 2(1-\theta)\kappa\Delta t} \frac{1}{\mu} \|f\|^2_{L^2(0,T;V^*)}\,, \tag{5.20}$$

holds for $n = 1, 2, \ldots, N$ with ω as above.

If $\kappa = 0$ and

$$\mu\theta(1-\theta)\Delta t \le \alpha^2(2\theta - 1) \tag{5.21}$$

with $\theta \in (1/2, 1]$ or $\kappa \neq 0$ and

$$\frac{\alpha^2(1-2\theta)}{(2\alpha^2\kappa - \mu)\theta(1-\theta)} \le \Delta t < \frac{\alpha^2}{(2\alpha^2\kappa - \mu)\theta} \tag{5.22}$$

then

$$|u^n|^2 \le \omega^n\, |u^0|^2 + \frac{\Delta t}{\mu} \sum_{j=0}^{n-1} \frac{\omega^{n-j}}{1 + (1-\theta)\Delta t(2\kappa - \mu/\alpha^2)} \|f^{j+\theta}\|_*^2 \tag{5.23}$$

for $n = 1, 2, \ldots, N$, where ω is given by

$$\omega = \frac{1 + (1-\theta)\Delta t(2\kappa - \mu/\alpha^2)}{1 - \theta\Delta t(2\kappa - \mu/\alpha^2)}\,.$$

We firstly remark that step size restriction appearing are stronger than (5.16) which gave solvability of the discrete problem.

Furthermore, for $\kappa \neq 0$, it is not necessary to have $\theta \ge 1/2$, but Δt has to be appropriate: not to large and – if $\theta < 1/2$ – not to small.

The estimate (5.23) again is an improvement w. r. t. the $l^\infty(0,T;H)$-norm and will be proved by using the discrete Gronwall lemma in difference form. However, in the strongly positive case, this improvement can be only obtained for *strongly* A-stable schemes ($\theta \in (1/2, 1]$). Then, it holds $0 \le \omega < 1$, and some decaying of the stability constants can be observed. For $\kappa \neq 0$, we have $\omega > 1$ since $\mu \le \alpha^2\kappa$ by assumption.

Proof From (5.15), we have with the Gårding and Young inequality

$$|u^{n+1}|^2 - |u^n|^2 + (2\theta-1)|u^{n+1} - u^n|^2 + \mu\Delta t\,\|u^{n+\theta}\|^2 - 2\kappa\Delta t\,|u^{n+\theta}|^2 \le \frac{\Delta t}{\mu}\,\|f^{n+\theta}\|_*^2\,.$$

With the identity

$$|\theta u^{n+1} + (1-\theta)u^n|^2 = \theta|u^{n+1}|^2 + (1-\theta)|u^n|^2 - \theta(1-\theta)|u^{n+1} - u^n|^2$$

and the inequality

$$\begin{aligned}
\|u^{n+\theta}\|^2 &\ge \Big(\theta\,\|u^{n+1}\| - (1-\theta)\,\|u^n\|\Big)^2 \\
&= \theta^2\,\|u^{n+1}\|^2 - 2\theta(1-\theta)\,\|u^{n+1}\|\,\|u^n\| + (1-\theta)^2\,\|u^n\| \\
&\ge \theta^2\,\|u^{n+1}\|^2 - \theta(1-\theta)\,\Big(\|u^{n+1}\|^2 + \|u^n\|^2\Big) + (1-\theta)^2\,\|u^n\| \\
&= (2\theta-1)\,\Big(\theta\,\|u^{n+1}\|^2 - (1-\theta)\,\|u^n\|^2\Big)\,,
\end{aligned} \tag{5.24}$$

we obtain

$$
\begin{aligned}
|u^{n+1}|^2 - |u^n|^2 + \Big((2\theta-1) + 2\kappa\theta(1-\theta)\Delta t\Big)|u^{n+1}-u^n|^2 \\
+\mu(2\theta-1)\Delta t \left(\theta\,\|u^{n+1}\|^2 - (1-\theta)\,\|u^n\|^2\right) \\
\le \frac{\Delta t}{\mu}\,\|f^{n+\theta}\|_*^2 + 2\kappa\Delta t \left(\theta\,|u^{n+1}|^2 + (1-\theta)\,|u^n|^2\right) .
\end{aligned}
$$

Summation yields for $n = 1, 2, \ldots$

$$
\begin{aligned}
|u^n|^2 + \Big((2\theta-1) + 2\kappa\theta(1-\theta)\Delta t\Big) \sum_{j=0}^{n-1} |u^{j+1}-u^j|^2 \\
+\,\mu(2\theta-1)^2\Delta t \sum_{j=0}^{n-1} \|u^{j+1}\|^2 + \mu(2\theta-1)(1-\theta)\Delta t \left(\|u^n\|^2 - \|u^0\|^2\right) \\
\le\; |u^0|^2 + \frac{\Delta t}{\mu} \sum_{j=0}^{n-1} \|f^{j+\theta}\|_*^2 + 2\kappa\Delta t \sum_{j=0}^{n-1} \left(\theta\,|u^{j+1}|^2 + (1-\theta)\,|u^j|^2\right) .
\end{aligned}
$$

Applying the (slightly changed) discrete GRONWALL lemma in sum form, Proposition 4.1, with the resulting inequality (4.5), we end up with

$$
\begin{aligned}
|u^n|^2 + \Big((2\theta-1) + 2\kappa\theta(1-\theta)\Delta t\Big) \sum_{j=0}^{n-1} |u^{j+1}-u^j|^2 \\
+\,\mu(2\theta-1)^2\Delta t \sum_{j=0}^{n-1} \|u^{j+1}\|^2 + \mu(2\theta-1)(1-\theta)\Delta t\,\|u^n\|^2 \\
\le\; \omega^n \left(|u^0|^2 + \mu(2\theta-1)(1-\theta)\Delta t\,\|u^0\|^2\right) + \frac{\omega^n}{1+2(1-\theta)\kappa\Delta t}\,\frac{\Delta t}{\mu} \sum_{j=0}^{n-1} \|f^{j+\theta}\|_*^2
\end{aligned}
$$

under the assumption (5.17).

Analogously to (5.10), we have

$$
\Delta t \sum_{j=0}^{n-1} \|f^{j+\theta}\|_*^2 \le \|f\|^2_{L^2(0,T;V^*)} ,
$$

and estimate (5.18) follows.

If $\kappa \neq 0$, the estimate (5.24) makes no sense, and we have instead of (5.18) the (weaker) stability estimate (5.20) assuming (5.19).

We now come to the more refined estimate (5.23): With

$$
\alpha^2\,\|u^{n+\theta}\|^2 \ge |u^{n+\theta}|^2 = \theta|u^{n+1}|^2 + (1-\theta)|u^n|^2 - \theta(1-\theta)|u^{n+1}-u^n|^2 ,
$$

we find

$$
\begin{aligned}
\frac{1}{\Delta t}\left(|u^{n+1}|^2 - |u^n|^2\right) + \left(\frac{2\theta-1}{\Delta t} + \frac{2\alpha^2\kappa-\mu}{\alpha^2}\,\theta(1-\theta)\right)|u^{n+1}-u^n|^2 \\
\le \frac{1}{\mu}\,\|f^{n+\theta}\|_*^2 + \frac{2\alpha^2\kappa-\mu}{\alpha^2}\left(\theta\,|u^{n+1}|^2 + (1-\theta)\,|u^n|^2\right) .
\end{aligned}
$$

The assumptions of Proposition 3.3 are obviously fulfilled in the strongly positive case ($\kappa = 0$) for $\theta = 1$. Otherwise, the time step size Δt has to be small enough:

$$\Delta t < \begin{cases} \dfrac{\alpha^2}{\mu(1-\theta)} & \text{if } \kappa = 0 \\ \dfrac{\alpha^2}{(2\alpha^2\kappa - \mu)\theta} & \text{if } \kappa \neq 0 \end{cases} .$$

We repeat that by assumption $\mu \le \alpha^2\kappa$ if $\kappa \neq 0$.

In addition, the coefficient of $|u^{n+1} - u^n|^2$ has to be nonnegative in order to obtain stability. This is the case for

$$\Delta t \le \frac{\alpha^2(2\theta - 1)}{\mu\theta(1-\theta)} \quad \text{if } \kappa = 0$$

and

$$\Delta t \ge \frac{\alpha^2(1 - 2\theta)}{(2\alpha^2\kappa - \mu)\theta(1-\theta)} \quad \text{if } \kappa \neq 0 \,.$$

This shows that for strongly positive $a(\cdot,\cdot)$, it is necessary to have $\theta > 1/2$ and a sufficiently small step size. The case $\theta = 1/2$ is covered by (5.18).

Applying now the discrete Gronwall lemma, we obtain (after some simple inspections and omitting the term with $|u^{n+1} - u^n|$) estimate (5.23). #

Finally, we should remark that it is possible to avoid restrictions on the time step size in the strongly positive case. For this, we have to estimate

$$\langle f^{n+1}, u^{n+1} \rangle \le \frac{\gamma}{2}\, \|u^{n+1}\|^2 + \frac{1}{2\gamma}\, \|f^{n+1}\|_*^2 \,,$$

where $\gamma \in (0, 2\mu)$ has to be chosen appropriately. We then have to replace μ by $2\mu - \gamma$ in the above analysis (where we used $\gamma = \mu$). We would end up with a restriction on γ

$$\gamma > 2\mu - \frac{\alpha^2(2\theta - 1)}{\Delta t \theta(1-\theta)} \,. \tag{5.25}$$

However, this leads to worse stability constants.

Proposition 5.4 (Error estimates for Problem $(P^\theta_{\Delta t})$)
For the discretization error e^n, the following estimates hold whenever the exact solution and the problem's data are smooth: If $\theta \in [1/2, 1]$ and (5.17), then

$$\begin{aligned} &|e^n|^2 + \Big((2\theta - 1) + 2\kappa\theta(1-\theta)\Delta t\Big) \sum_{j=0}^{n-1} |e^{j+1} - e^j|^2 \\ &+ \mu(2\theta-1)^2 \Delta t \sum_{j=0}^{n-1} \|e^{j+1}\|^2 + \mu(2\theta - 1)(1-\theta)\Delta t\, \|e^n\|^2 \\ &\le \omega^n \left(|e^0|^2 + \mu(2\theta-1)(1-\theta)\Delta t\, \|e^0\|^2\right) \\ &+ \frac{\omega^n}{1 + 2(1-\theta)\kappa\Delta t}\, \frac{(\Delta t)^2}{3\mu}\, \|f' - u''\|^2_{L^2(0,T;V^*)} \end{aligned} \tag{5.26}$$

for $n = 1, 2, \ldots, N$, *where* $\omega = (1 + 2(1-\theta)\kappa\Delta t)/(1 - 2\theta\kappa\Delta t)$.

If $\kappa = 0$ *and (5.21) with* $\theta \in (1/2, 1]$ *or* $\kappa \neq 0$ *and (5.22) then*

$$|e^n|^2 \leq \omega^n\,|e^0|^2 + \frac{(\Delta t)^2}{3\mu}\sum_{j=0}^{n-1}\frac{\omega^{n-j}}{1 + (1-\theta)\Delta t(2\kappa - \mu/\alpha^2)}\int_{t_j}^{t_{j+1}}\|f'(t) - u''(t)\|_*^2\,dt \tag{5.27}$$

for $n = 1, 2, \ldots, N$, *where*

$$\omega = \frac{1 + (1-\theta)\Delta t(2\kappa - \mu/\alpha^2)}{1 - \theta\Delta t(2\kappa - \mu/\alpha^2)}\,.$$

Moreover, if $\theta = 1/2$ *second order accuracy is obtained: If* $\kappa\Delta t < 1$, *then*

$$|e^n|^2 + \frac{\kappa\Delta t}{2}\sum_{j=0}^{n-1}|e^{j+1} - e^j|^2 \leq \omega^n\,|e^0|^2 + \frac{\omega^n}{1+\kappa\Delta t}\,\frac{(\Delta t)^4}{120\mu}\,\|f'' - u'''\|_{L^2(0,T;V^*)}^2 \tag{5.28}$$

for $n = 1, 2, \ldots, N$, *where* $\omega = (1 + \kappa\Delta t)/(1 - \kappa\Delta t)$. *If in addition* $\kappa \neq 0$, *then*

$$|e^n|^2 \leq \omega^n\,|e^0|^2 + \frac{(\Delta t)^4}{120\mu}\sum_{j=0}^{n-1}\frac{2\omega^{n-j}}{2 + \Delta t(2\kappa - \mu/\alpha^2)}\int_{t_j}^{t_{j+1}}\|f''(t) - u'''(t)\|_*^2\,dt \tag{5.29}$$

for $n = 1, 2, \ldots, N$, *where*

$$\omega = \frac{2 + \Delta t(2\kappa - \mu/\alpha^2)}{2 - \Delta t(2\kappa - \mu/\alpha^2)}\,.$$

Again, we could also consider $\theta < 1/2$ if $\kappa \neq 0$ and prove a result similarly to (5.20) under the restriction (5.19). We omit this here.

Proof We start with an error equation giving the relation between the discretization error e^n and the consistency error $\rho^{n+\theta}$, $n = 0, 1, \ldots$,

$$\frac{1}{\Delta t}\,(e^{n+1} - e^n, v) + a(e^{n+\theta}, v) = \langle\rho^{n+\theta}, v\rangle \quad \forall v \in V\,, \tag{5.30}$$

with $e^{n+\theta} := \theta e^{n+1} + (1-\theta)e^n$ and

$$\begin{aligned}\langle\rho^{n+\theta}, v\rangle = \; & - \;\left\langle \theta u'(t_{n+1}) + (1-\theta)u'(t_n) - \frac{u(t_{n+1}) - u(t_n)}{\Delta t}, v\right\rangle \\ & + \;\left\langle \theta f(t_{n+1}) + (1-\theta)f(t_n) - f^{n+\theta}, v\right\rangle\end{aligned} \tag{5.31}$$

for all $v \in V$. With integration by parts and using the restriction

$$f^{n+\theta} := \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(t)dt\,,$$

we find

$$\rho^{n+\theta} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}(t - (1-\theta)t_{n+1} - \theta t_n)(f'(t) - u''(t))dt\,. \tag{5.32}$$

in the sense of a BOCHNER integral.

The error analysis follows the same steps as the stability analysis. Similarly to (5.18), we have under the restriction (5.17)

$$
\begin{aligned}
&|e^n|^2 + \Big((2\theta-1) + 2\kappa\theta(1-\theta)\Delta t\Big) \sum_{j=0}^{n-1} |e^{j+1} - e^j|^2 \\
&+\mu(2\theta-1)^2\Delta t \sum_{j=0}^{n-1} \|e^{j+1}\|^2 + \mu(2\theta-1)(1-\theta)\Delta t \, \|e^n\|^2 \\
&\leq \omega^n \Big(|e^0|^2 + \mu(2\theta-1)(1-\theta)\Delta t \, \|e^0\|^2\Big) + \frac{\omega^n}{1+2(1-\theta)\kappa\Delta t} \frac{\Delta t}{\mu} \sum_{j=0}^{n-1} \|\rho^{j+\theta}\|_*^2
\end{aligned}
$$

With (5.21) for $\kappa = 0$ and (5.22) for $\kappa \neq 0$, resp., we have similarly to estimate (5.23) the error estimate (5.27).

For the consistency error, we have

$$
\begin{aligned}
\|\rho^{j+\theta}\|_*^2 &\leq \frac{1}{(\Delta t)^2} \left(\int_{t_j}^{t_{j+1}} |t - (1-\theta)t_{j+1} - \theta t_j| \, \|f'(t) - u''(t)\|_* dt \right)^2 \\
&\leq \frac{1}{(\Delta t)^2} \int_{t_j}^{t_{j+1}} (t - (1-\theta)t_{j+1} - \theta t_j)^2 \, dt \int_{t_j}^{t_{j+1}} \|f'(t) - u''(t)\|_*^2 \, dt \\
&= \frac{\Delta t}{3} \left(\theta^3 + (1-\theta)^3\right) \int_{t_j}^{t_{j+1}} \|f'(t) - u''(t)\|_*^2 \, dt \, .
\end{aligned}
$$

Since $\phi(\theta) := \theta^3 + (1-\theta)^3$, $\theta \in [0,1]$, has maximum value $\phi(0) = \phi(1) = 1$, we come to

$$
\Delta t \sum_{j=0}^{n-1} \|\rho^{j+\theta}\|_*^2 \leq \frac{(\Delta t)^2}{3} \int_0^{t_n} \|f'(t) - u''(t)\|_*^2 dt \leq \frac{(\Delta t)^2}{3} \|f' - u''\|_{L^2(0,T;V^*)}^2 \tag{5.33}
$$

if the exact solution u and the right-hand side f are smooth enough.

However, if $\theta = 1/2$, we are able to improve the estimate in order to achieve second order accuracy. Again, integration by parts gives

$$
\rho^{n+1/2} = \frac{1}{2\Delta t} \int_{t_n}^{t_{n+1}} (t_{n+1} - t)(t - t_n)(f''(t) - u'''(t)) dt \tag{5.34}
$$

and thus

$$
\begin{aligned}
\|\rho^{j+1/2}\|_*^2 &\leq \frac{1}{4(\Delta t)^2} \int_{t_j}^{t_{j+1}} (t_{j+1} - t)^2 (t - t_j)^2 dt \int_{t_j}^{t_{j+1}} \|f''(t) - u'''(t)\|_*^2 dt \\
&= \frac{(\Delta t)^3}{120} \int_{t_j}^{t_{j+1}} \|f''(t) - u'''(t)\|_*^2 dt \, .
\end{aligned}
$$

Hence, it holds

$$
\Delta t \sum_{j=0}^{n-1} \|\rho^{j+1/2}\|_*^2 \leq \frac{(\Delta t)^4}{120} \int_0^{t_n} \|f''(t) - u'''(t)\|_*^2 dt \leq \frac{(\Delta t)^4}{120} \|f'' - u'''\|_{L^2(0,T;V^*)}^2 \tag{5.35}
$$

if the exact solution u and the right-hand side f are smooth enough. #

For the right-hand sides of the foregoing estimates (5.33) and (5.35), we have the following a priori estimates if the exact solution and the data u_0 and f are sufficiently smooth, i. e. if the equivalent compatibility conditions are fulfilled, cf. also Temam [11] or Wloka [13]:

$$\int_0^t \|f'(s) - u''(s)\|_*^2 ds$$

$$\leq \frac{\beta^2}{\mu} \mathrm{e}^{2\kappa t} \left(|Au_0|^2 + \frac{2}{\mu} \left(\beta^2 \|f\|_{L^2(0,T;V)}^2 + \frac{4\alpha^2\kappa - \mu}{\alpha^2} \kappa \|f\|_{L^2(0,T;V^*)}^2 \right) \right) \quad (5.36)$$

$$\int_0^t \|f''(s) - u'''(s)\|^2 ds$$

$$\leq \frac{\beta^2}{\mu} \mathrm{e}^{2\kappa t} \left(|A^2 u_0|^2 + \beta \left(1 + \frac{3\beta}{\mu} \right) \left(\|f'\|_{L^2(0,T;V)}^2 + \|f' - Af\|_{L^2(0,T;V)}^2 \right) \right.$$

$$\left. + \frac{2\kappa}{\alpha^2 \mu} (6\alpha^2\kappa - \mu) \|f' - Af\|_{L^2(0,T;V^*)}^2 \right) . \quad (5.37)$$

We shall give a proof in the appendix.

As we have introduced the incompressible Stokes problem as an example for Problem (P), we shall remark here that higher compatibility conditions on the data and thus higher regularity of the exact solution to the Stokes (and Navier-Stokes) problem are unrealistic. Therefore, the analysis presented for the second order Crank-Nicolson scheme is not applicable. Due to the divergence free constraint, higher regularity, as it is required for second order accuracy, leads to an overdetermined Neumann problem that is hardly ever fulfilled, and in most cases the conditions are uncheckable for given data, cf. Temam [11].

5.3 Backward Euler method with variable step sizes

We now consider the backward Euler scheme with a non-equidistant partition of the time interval. Let $0 = t_0 < t_1 < \ldots < t_N = T$ for $N \in \mathbb{N} \setminus \{0\}$ be given, and $\tau_{n+1} := t_{n+1} - t_n$, $n = 0, 1, \ldots, N-1$.

The method under consideration reads as

Problem (P_τ) *For given $u^0 \in H$ and $\{f^n\}_{n=1}^N \subseteq V^*$, find $\{u^n\}_{n=1}^N \subseteq V$ s. t. for $n = 0, 1, \ldots, N-1$ and all $v \in V$*

$$\frac{1}{\tau_{n+1}} (u^{n+1} - u^n, v) + a(u^{n+1}, v) = \langle f^{n+1}, v \rangle . \quad (5.38)$$

For $f \in L^2(0,T;V^*)$, we use again restriction (5.4),

$$f^{n+1} = \frac{1}{\tau_{n+1}} \int_{t_n}^{t_{n+1}} f(t) dt .$$

We assume that $a(\cdot,\cdot)$ fulfils a GÅRDING inequality.

Analogously to the previous case with $\theta = 1$, there is a unique solution to Problem (P_τ) if

$$\frac{\alpha^2\kappa - \mu}{\alpha^2}\,\tau_{max} < 1\,, \tag{5.39}$$

where $\tau_{max} = \max_{n=1,\dots,N} \tau_n$. However, there will be again no restriction on the time step size in the positive case ($\kappa = 0$ or $\kappa = \mu/\alpha^2$).

As we will see in the following, the estimates coincide with those of the previous section setting $\Delta t = \tau_{max}$ and $\theta = 1$. However, due to the variable step sizes, an improvement of the stability and error constants can be achieved.

Proposition 5.5 (Stability estimates for Problem (P_τ))
The discrete solution $\{u^n\}$ to Problem (P_τ) is stable in the following sense:

If $\kappa = 0$ or

$$\tau_{max} < \frac{1}{2\kappa}\,, \tag{5.40}$$

then

$$\begin{aligned} &|u^n|^2 + \sum_{j=0}^{n-1} |u^{j+1} - u^j|^2 + \mu \sum_{j=0}^{n-1} \tau_{j+1}\,\|u^{j+1}\|^2 \\ &\le (1 - 2\kappa\tau_{max})^{-n}\left(|u^0|^2 + \frac{1}{\mu}\|f\|^2_{L^2(0,T;V^*)}\right) \end{aligned} \tag{5.41}$$

for $n = 1, 2, \dots, N$.

If $\kappa = 0$ or

$$\tau_{max} < \frac{\alpha^2}{2\alpha^2\kappa - \mu}\,, \tag{5.42}$$

then

$$|u^n|^2 \le |u^0|^2 \prod_{l=1}^{n} \omega_l + \frac{1}{\mu}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \|f(t)\|_*^2\,dt \prod_{l=j+1}^{n} \omega_l\,, \tag{5.43}$$

for $n = 1, 2, \dots, N$, where

$$\omega_l = \left(1 - \frac{(2\alpha^2\kappa - \mu)\tau_l}{\alpha^2}\right)^{-1}. \tag{5.44}$$

Note, that (5.40) and (5.42) are more restrictive than (5.39), which gave solvability. Furthermore, we have $0 < \omega_l < 1$ in the strongly positive case whereas $\omega_l > 1$ for $\kappa \ne 0$. In addition, we have $\omega_l \le \omega_{max}$ if $\kappa \ne 0$, where

$$\omega_{max} = \left(1 - \frac{(2\alpha^2\kappa - \mu)\tau_{max}}{\alpha^2}\right)^{-1}.$$

Proof Setting $v = u^{n+1}$ in (5.38), we obtain with GÅRDING's inequality

$$\begin{aligned} &\frac{1}{2\tau_{n+1}}\left(|u^{n+1}|^2 - |u^n|^2 + |u^{n+1} - u^n|^2\right) + \mu\,\|u^{n+1}\|^2 - \kappa\,|u^{n+1}|^2 \\ &\le \|f^{n+1}\|_*\|u^{n+1}\| \le \frac{1}{2\mu}\,\|f^{n+1}\|_*^2 + \frac{\mu}{2}\|u^{n+1}\|^2 \end{aligned}$$

and after summation, it follows for $n = 1, 2, \ldots$

$$|u^n|^2 + \sum_{j=0}^{n-1} |u^{j+1} - u^j|^2 + \mu \sum_{j=0}^{n-1} \tau_{j+1} \|u^{j+1}\|^2$$
$$\leq |u^0|^2 + \frac{1}{\mu} \sum_{j=0}^{n-1} \tau_{j+1} \|f^{j+1}\|_*^2 + 2\kappa \sum_{j=0}^{n-1} \tau_{j+1} |u^{j+1}|^2 .$$

If (5.40), we can apply the discrete GRONWALL lemma Proposition 4.1 (slightly changed with resulting inequality (4.6)) and end up with the a priori stability estimate for $n = 1, 2, \ldots$

$$|u^n|^2 + \sum_{j=0}^{n-1} |u^{j+1} - u^j|^2 + \mu \sum_{j=0}^{n-1} \tau_{j+1} \|u^{j+1}\|^2 \leq |u^0|^2 + \frac{1}{\mu} \sum_{j=0}^{n-1} \tau_{j+1} \|f^{j+1}\|_*^2$$
$$+ 2\kappa \sum_{j=0}^{n-1} \tau_{j+1} \left(\prod_{l=j+1}^{n} (1 - 2\kappa\tau_l)^{-1} \right) \left(|u^0|^2 + \frac{1}{\mu} \sum_{l=0}^{j} \tau_{l+1} \|f^{l+1}\|_*^2 \right)$$
$$\leq (1 - 2\kappa\tau_{max})^{-n} \left(|u_0|^2 + \frac{1}{\mu} \sum_{j=0}^{n-1} \tau_{j+1} \|f^{j+1}\|_*^2 \right) .$$

Similarly to (5.10), it holds

$$\sum_{j=0}^{n-1} \tau_{j+1} \|f^{j+1}\|_*^2 \leq \sum_{j=0}^{n-1} \frac{1}{\tau_{j+1}} \left(\int_{t_j}^{t_{j+1}} \|f(t)\|_* dt \right)^2$$
$$\leq \sum_{j=0}^{n-1} \frac{1}{\tau_{j+1}} \int_{t_j}^{t_{j+1}} dt \int_{t_j}^{t_{j+1}} \|f(t)\|_*^2 dt = \int_0^{t_n} \|f(t)\|_*^2 dt \leq \|f\|_{L^2(0,T;V^*)}^2 ,$$

and the assertion (5.41) follows.

Using the continuous embedding $V \hookrightarrow H$ leads to

$$\frac{1}{\tau_{n+1}} \left(|u^{n+1}|^2 - |u^n|^2 \right) + \frac{\mu - 2\alpha^2\kappa}{\alpha^2} |u^{n+1}|^2 \leq \frac{1}{\mu} \|f^{n+1}\|_*^2 .$$

In order to apply the discrete GRONWALL lemma Proposition 3.3 with $\theta = 1$, we have to assume sufficiently small step sizes if $\kappa \neq 0$, i. e. (5.42). We then arrive at (5.43). #

Proposition 5.6 (Error estimates for Problem (P_τ))
For the discretization error e^n, the following estimates hold whenever the exact solution and the problem's data are smooth:

If $\kappa = 0$ or (5.40), then

$$|e^n|^2 + \sum_{j=0}^{n-1} |e^{j+1} - e^j|^2 + \mu \sum_{j=0}^{n-1} \tau_{j+1} \|e^{j+1}\|^2$$
$$\leq (1 - 2\kappa\tau_{max})^{-n} \left(|e^0|^2 + \frac{\tau_{max}^2}{3\mu} \|f' - u''\|_{L^2(0,T;V^*)}^2 \right) \tag{5.45}$$

for $n = 1, 2, \ldots, N$.

If $\kappa = 0$ *or (5.42), then*

$$|e^n|^2 \le |e^0|^2 \prod_{l=1}^{n} \omega_l + \frac{\tau_{max}^2}{3\mu} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \|f'(t) - u''(t)\|_*^2 \, dt \prod_{l=j+1}^{n} \omega_l \tag{5.46}$$

for $n = 1, 2, \ldots, N$, *where* ω_l *is given by (5.44).*

Again, we have $0 < \omega_l < 1$ in the strongly positive case and $\omega_l > 1$ otherwise. Also, we have $\omega_l \le \omega_{max}$ if $\kappa \ne 0$. Finally, we refer to the a priori estimate (5.36) for the exact solution that will be proved in the appendix.

Proof Starting with the error equation for $n = 0, 1, \ldots, N-1$

$$\frac{1}{\tau_{n+1}} (e^{n+1} - e^n, v) + a(e^{n+1}, v) = \langle \rho^{n+1}, v \rangle \qquad \forall v \in V\,, \tag{5.47}$$

where

$$\rho^{n+1} = \frac{1}{\tau_{n+1}} \int_{t_n}^{t_{n+1}} (t - t_n)(f'(t) - u''(t))dt$$

is the consistency error, we follow the steps of the stability analysis.

Then, we come with $\kappa = 0$ and (5.40) otherwise to the estimate

$$\begin{aligned} &|e^n|^2 + \sum_{j=0}^{n-1} |e^{j+1} - e^j|^2 + \mu \sum_{j=0}^{n-1} \tau_{j+1} \|e^{j+1}\|^2 \\ &\le (1 - 2\kappa\tau_{max})^{-n} \left(|e^0|^2 + \frac{1}{\mu} \sum_{j=0}^{n-1} \tau_{j+1} \|\rho^{j+1}\|_*^2 \right) . \end{aligned}$$

Using the continuous embedding $V \hookrightarrow H$ leads with $\kappa = 0$ and (5.42) otherwise to

$$|e^n|^2 \le |e^0|^2 \prod_{l=1}^{n} \omega_l + \frac{1}{\mu} \sum_{j=0}^{n-1} \tau_{j+1} \|\rho^{j+1}\|_*^2 \prod_{l=j+1}^{n} \omega_l \,.$$

For the consistency error, we have

$$\begin{aligned} \|\rho^{j+1}\|_*^2 &\le \left\| \frac{1}{\tau_{j+1}} \int_{t_j}^{t_{j+1}} (t - t_j)(f'(t) - u''(t))dt \right\|_*^2 \\ &\le \frac{1}{\tau_{j+1}^2} \int_{t_j}^{t_{j+1}} (t - t_j)^2 \, dt \int_{t_j}^{t_{j+1}} \|f'(t) - u''(t)\|_*^2 \, dt \\ &= \frac{\tau_{j+1}}{3} \int_{t_j}^{t_{j+1}} \|f'(t) - u''(t)\|_*^2 \, dt \,, \end{aligned}$$

and thus

$$\sum_{j=0}^{n-1} \tau_{j+1} \|\rho^{j+1}\|_*^2 \le \frac{\tau_{max}^2}{3} \|f' - u''\|_{L^2(0,T;V^*)}^2 \,.$$

The assertions follow immediately. #

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Appendix A priori estimates for Problem (P)

In the following, we prove the estimates (5.13), (5.36), and (5.37). For this, we shall assume higher regularity and hence (higher) compatibility of the data, cf. Temam [11] and Wloka [13] for more details.

However, there are smoothing a priori estimates and correlated smoothing error estimates even for rough data not satisfying compatibility and without assuming higher regularity of the exact solution. Such estimates rely on the parabolic smoothing property and use time weights. They will be discussed in detail in a forthcoming paper, cf. also Huang/Thomée [7] and Fujita/Suzuki [3].

Note furthermore, that the following analysis holds even for non-symmetric $a(\cdot,\cdot)$ and A, resp.

Let $u_0 \in V$, $Au_0 \in H$, $f \in L^2(0,T;V)$, and $f' \in L^2(0,T;V^*)$. Then, it holds u, u', $Au \in L^2(0,T;V) \cap \mathcal{C}([0,T];H)$, and u'', $Au' \in L^2(0,T;V^*)$.

Since $V \hookrightarrow_{\text{dense}} V^*$ and V reflexive, it follows from (5.3a)

$$\langle u''(t), v\rangle + a(u'(t), v) = \langle f'(t), v\rangle \tag{A.1}$$

for all $v \in V$ and almost all $t \in [0,T]$. Therefore, we have with the continuity of $a(\cdot,\cdot)$

$$\|f'(t) - u''(t)\|_* = \sup_{v\in V\setminus\{0\}} \frac{|a(u'(t),v)|}{\|v\|} \le \beta\,\|u'(t)\|\,.$$

Furthermore, Gårding's inequality leads to

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}|Au(t)|^2 - \kappa\,|Au(t)|^2 + \mu\,\|u'(t)\|^2 \\
\le\ & \frac{1}{2}\frac{d}{dt}|Au(t)|^2 - \kappa\,|Au(t)|^2 + a(u'(t),u'(t)) + \kappa\,|u'(t)|^2 \\
=\ & \langle Au'(t), Au(t)\rangle - \kappa\,\langle Au(t), Au(t)\rangle + \langle Au'(t), u'(t)\rangle + \kappa\,\langle u'(t), u'(t)\rangle \\
=\ & \langle Au'(t), u'(t) + Au(t)\rangle + \kappa\,\langle u'(t), u'(t)\rangle - \kappa\,\langle Au(t), Au(t)\rangle \\
=\ & \langle Au'(t), f(t)\rangle + \kappa\,\langle u'(t), u'(t)\rangle - \kappa\,\langle f(t) - u'(t), f(t) - u'(t)\rangle \\
=\ & a(u'(t), f(t)) + 2\kappa\,\langle u'(t), f(t)\rangle - \kappa\,|f(t)|^2 \\
\le\ & \beta\,\|u'(t)\|\,\|f(t)\| + 2\kappa\|u'(t)\|\,\|f(t)\|_* - \kappa\,|f(t)|^2 \\
\le\ & \frac{\mu}{4}\,\|u'(t)\|^2 + \frac{\beta^2}{\mu}\,\|f(t)\|^2 + \frac{\mu}{4}\,\|u'(t)\|^2 + \frac{4\kappa^2}{\mu}\,\|f(t)\|_*^2 - \kappa\,|f(t)|^2 \\
\le\ & \frac{\mu}{2}\,\|u'(t)\|^2 + \frac{\beta^2}{\mu}\,\|f(t)\|^2 + \left(\frac{4\kappa^2}{\mu} - \frac{\kappa}{\alpha^2}\right)\|f(t)\|_*^2\,.
\end{aligned}$$

We should mention that either $\kappa = 0$ (strongly positive A) and then the last term of the right-hand side vanishes or $\kappa \neq 0$ with $\mu \leq \alpha^2\kappa$ and it holds

$$\frac{3\kappa^2}{\mu} \leq \frac{4\kappa^2}{\mu} - \frac{\kappa}{\alpha^2} < \frac{4\kappa^2}{\mu}\,.$$

With

$$\frac{d}{dt}\left(\mathrm{e}^{-2\kappa t}|Au(t)|^2\right) = \mathrm{e}^{-2\kappa t}\left(\frac{d}{dt}|Au(t)|^2 - 2\kappa\,|Au(t)|^2\right)$$

we finally obtain

$$\frac{d}{dt}\left(\mathrm{e}^{-2\kappa t}|Au(t)|^2\right) + \mu\mathrm{e}^{-2\kappa t}\|u'(t)\|^2 \leq \frac{2}{\mu}\,\mathrm{e}^{-2\kappa t}\left(\beta^2\,\|f(t)\|^2 + \frac{4\alpha^2\kappa - \mu}{\alpha^2}\,\kappa\,\|f(t)\|_*^2\right)\,.$$

Integration over t gives

$$\begin{aligned}
&|Au(t)|^2 + \mu\int_0^t \mathrm{e}^{2\kappa(t-s)}\|u'(s)\|^2 ds \\
&\leq \mathrm{e}^{2\kappa t}|Au_0|^2 + \frac{2}{\mu}\int_0^t \mathrm{e}^{2\kappa(t-s)}\left(\beta^2\,\|f(s)\|^2 + \frac{4\alpha^2\kappa - \mu}{\alpha^2}\,\kappa\,\|f(s)\|_*^2\right) ds\,,
\end{aligned}$$

and with $1 \leq \mathrm{e}^{2\kappa(t-s)} \leq \mathrm{e}^{2\kappa t}$ for $t \geq s \geq 0$, it follows

$$\begin{aligned}
&|Au(t)|^2 + \mu\int_0^t \|u'(s)\|^2 ds \\
&\leq \mathrm{e}^{2\kappa t}\left(|Au_0|^2 + \frac{2}{\mu}\left(\beta^2\,\|f\|^2_{L^2(0,T;V)} + \frac{4\alpha^2\kappa - \mu}{\alpha^2}\,\kappa\,\|f\|^2_{L^2(0,T;V^*)}\right)\right)\,. \qquad \text{(A.2)}
\end{aligned}$$

The a priori estimate desired then reads as

$$\begin{aligned}
&\int_0^t \|f'(s) - u''(s)\|_*^2 ds \leq \beta^2\int_0^t \|u'(s)\|^2 ds \\
&\leq \frac{\beta^2}{\mu}\,\mathrm{e}^{2\kappa t}\left(|Au_0|^2 + \frac{2}{\mu}\left(\beta^2\,\|f\|^2_{L^2(0,T;V)} + \frac{4\alpha^2\kappa - \mu}{\alpha^2}\,\kappa\,\|f\|^2_{L^2(0,T;V^*)}\right)\right)\,, \qquad \text{(A.3)}
\end{aligned}$$

which is (5.36).

If $\kappa = 0$, we may prove in the same way estimate (5.13),

$$\int_0^t \|f'(s) - u''(s)\|_*^2 ds \leq \frac{\beta^2}{\mu}\left(|Au_0|^2 + \frac{\beta^2}{\mu}\,\|f\|^2_{L^2(0,T;V)}\right)\,. \qquad \text{(A.4)}$$

For the a priori estimate of $\|f'' - u'''\|_{L^2(0,T;V^*)}$, we assume in addition[5] $Au_0 \in V$, $A^2u_0 \in H$, f', $Af \in L^2(0,T;V)$, and f'', $Af' \in L^2(0,T;V^*)$. By this, we have in addition u'', $Au' \in L^2(0,T;V) \cap \mathcal{C}([0,T];H)$ and u''', $Au'' \in L^2(0,T;V^*)$. Furthermore, it holds (5.3a) and

$$\begin{aligned}
\langle Au'(t), v\rangle + \langle A^2u(t), v\rangle &= \langle Af(t), v\rangle && \text{(A.5a)}\\
\langle u''(t), v\rangle + \langle Au'(t), v\rangle &= \langle f'(t), v\rangle && \text{(A.5b)}
\end{aligned}$$

[5] We use the notation $A^2v := A(Av) \in V^*$ for all $v \in V$ with $Av \in V$.

for all $v \in V^*$ and almost all $t \in [0,T]$ as well as

$$\begin{aligned} \langle Au''(t), v\rangle + \langle A^2u'(t), v\rangle &= \langle Af'(t), v\rangle && \text{(A.5c)} \\ \langle u'''(t), v\rangle + \langle Au''(t), v\rangle &= \langle f''(t), v\rangle && \text{(A.5d)} \end{aligned}$$

for all $v \in V$ and almost all $t \in [0,T]$.

From (A.5d), we immediately obtain

$$\|f''(t) - u'''(t)\|_* \leq \beta \, \|u''(t)\| \, .$$

With (A.5a) and (A.5b), we have

$$\langle u''(t), v\rangle = \langle f'(t), v\rangle - \langle Af(t), v\rangle + \langle A^2u(t), v\rangle \qquad \text{(A.5e)}$$

for all $v \in V^*$. Moreover, Gårding's inequality gives

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|A^2u(t)|^2 - \kappa\,|A^2u(t)|^2 + \mu\,\|u''(t)\|^2 \\ \leq\; & \frac{1}{2}\frac{d}{dt}|A^2u(t)|^2 - \kappa\,|A^2u(t)|^2 + a(u''(t), u''(t)) + \kappa\,|u''(t)|^2 \\ =\; & \langle A^2u'(t), A^2u(t)\rangle - \kappa\,\langle A^2u(t), A^2u(t)\rangle + \langle Au''(t), u''(t)\rangle + \kappa\langle u''(t), u''(t)\rangle \, . \end{aligned}$$

Setting $v = A^2u(t)$ in (A.5c), it follows with (A.5e)

$$\begin{aligned} \langle A^2u'(t), A^2u(t)\rangle &= \langle Af'(t), A^2u(t)\rangle - \langle Au''(t), A^2u(t)\rangle = \langle A^2u(t), Af'(t) - Au''(t)\rangle \\ &= \langle u''(t), Af'(t) - Au''(t)\rangle - \langle f'(t) - Af(t), Af'(t) - Au''(t)\rangle \\ &= -\langle Au''(t), u''(t)\rangle + \langle Af'(t), u''(t)\rangle + \langle Au''(t), f'(t) - Af(t)\rangle \\ &\quad -\langle Af'(t), f'(t) - Af(t)\rangle \, . \end{aligned}$$

With (A.5e), we have furthermore

$$\begin{aligned} \langle A^2u(t), A^2u(t)\rangle &= \langle u''(t) - f'(t) + Af(t), A^2u(t)\rangle \\ &= \langle u''(t) - f'(t) + Af(t), u''(t) - f'(t) + Af(t)\rangle \\ &= \langle u''(t), u''(t)\rangle - 2\,\langle u''(t), f'(t) - Af(t)\rangle \\ &\quad +\langle f'(t) - Af(t), f'(t) - Af(t)\rangle \, . \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|A^2u(t)|^2 - \kappa\,|A^2u(t)|^2 + \mu\,\|u''(t)\|^2 \\ \leq\; & \langle Af'(t), u''(t)\rangle + \langle Au''(t), f'(t) - Af(t)\rangle - \langle Af'(t), f'(t) - Af(t)\rangle \\ & +2\kappa\,\langle u''(t), f'(t) - Af(t)\rangle - \kappa\,\langle f'(t) - Af(t), f'(t) - Af(t)\rangle \\ \leq\; & \beta\,\|u''(t)\|\,\|f'(t)\| + \beta\,\|u''(t)\|\,\|f'(t) - Af(t)\| + \beta\,\|f'(t)\|\,\|f'(t) - Af(t)\| \\ & +2\kappa\,\|u''(t)\|\,\|f'(t) - Af(t)\|_* - \kappa\,|f'(t) - Af(t)|^2 \end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{\mu}{6}\|u''(t)\|^2 + \frac{3\beta^2}{2\mu}\|f'(t)\|^2 + \frac{\mu}{6}\|u''(t)\|^2 + \frac{3\beta^2}{2\mu}\|f'(t)-Af(t)\|^2 + \frac{\beta}{2}\|f'(t)\|^2 \\
& + \frac{\beta}{2}\|f'(t)-Af(t)\|^2 + \frac{\mu}{6}\|u''(t)\|^2 + \frac{6\kappa^2}{\mu}\|f'(t)-Af(t)\|_*^2 - \frac{\kappa}{\alpha^2}\|f'(t)-Af(t)\|_*^2 \\
=\ & \frac{\mu}{2}\|u''(t)\|^2 + \frac{\beta}{2}\left(1+\frac{3\beta}{\mu}\right)\|f'(t)\|^2 + \frac{\beta}{2}\left(1+\frac{3\beta}{\mu}\right)\|f'(t)-Af(t)\|^2 \\
& + \frac{\kappa}{\alpha^2\mu}\left(6\alpha^2\kappa-\mu\right)\|f'(t)-Af(t)\|_*^2 .
\end{aligned}
$$

Integration over t leads to

$$
\begin{aligned}
& |A^2u(t)|^2 + \mu\int_0^t e^{2\kappa(t-s)}\|u''(s)\|^2 ds \\
\leq\ & e^{2\kappa t}|A^2u_0|^2 + \int_0^t e^{2\kappa(t-s)}\left(\beta\left(1+\frac{3\beta}{\mu}\right)\left(\|f'(s)\|^2 + \|f'(s)-Af(s)\|^2\right)\right. \\
& \left. + \frac{2\kappa}{\alpha^2\mu}\left(6\alpha^2\kappa-\mu\right)\|f'(s)-Af(s)\|_*^2\right) ds
\end{aligned}
$$

and we end up with the a priori estimate

$$
\int_0^t \|f''(s)-u'''(s)\|^2 ds \leq \beta^2\int_0^t \|u''(s)\|^2 ds
$$

$$
\begin{aligned}
\leq \frac{\beta^2}{\mu}e^{2\kappa t}\Bigg(&|A^2u_0|^2 + \beta\left(1+\frac{3\beta}{\mu}\right)\left(\|f'\|^2_{L^2(0,T;V)} + \|f'-Af\|^2_{L^2(0,T;V)}\right) \\
& + \frac{2\kappa}{\alpha^2\mu}\left(6\alpha^2\kappa-\mu\right)\|f'-Af\|^2_{L^2(0,T;V^*)}\Bigg), \qquad \text{(A.6)}
\end{aligned}
$$

which is (5.37).

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