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Preprint No. 703 November 2000

Preprint Reihe Mathematik
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Fachbereich Mathematik
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Abstract

The temporal discretisation of a moderate semilinear parabolic problem in an abstract setting by means of the two-step backward differentiation formula with variable step sizes is analysed. Stability as well as optimal error estimates are derived for step size ratios bounded by 1.91.

Key words: Semilinear parabolic problem, time discretisation, backward differentiation formula, non-uniform grid, stability, error estimate

MSC (2000): 65M12, 65J15, 35K90, 47J35

1 Introduction

Whereas multistep methods with variable step sizes are widely used in numerical computations, their analysis is still not complete. Because of the non-uniform grid, non-constant coefficients appear in the resulting scheme. Theoretical tools developed for difference equations with constant coefficients are therefore not applicable.

Among the abundance of methods, the backward differentiation formulae (BDF) seem to be of particular interest. Especially the two-step BDF, which is strongly A-stable for constant time steps and of second order, plays an important rôle in the integration of non-stationary problems.

The stability of the variable two-step BDF has been studied by Grigorieff in a series of papers [2, 3, 4]. In particular, zero-stability has been shown for step size ratios less than $1 + \sqrt{2} \approx 2.414$. For ratios bounded by $(1 + \sqrt{3})/2 \approx 1.366$, A_0-stability as well as error estimates in the case of linear parabolic problems have been proven. Recently, Becker [1] could improve the bound up to $(2 + \sqrt{13})/3 \approx 1.868$. However, the stability and error constant may then depend on the sequence of step size ratios. A(θ)-stability type results with $\theta \leq \pi/3$ have been provided in [3]. So far, nonlinear problems have not been considered.

In this paper, we are concerned with the time discretisation of abstract semilinear parabolic problems with a moderate nonlinearity. We shall derive stability and optimal smooth data error estimates. These estimates can be obtained for step size ratios less than 1.910, which also slightly improves Becker's bound.

In the following, we set up the problem. We commence with the Gelfand triple $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$: Let V be a separable, reflexive Banach space with norm $\|\cdot\|$ and H be a separable Hilbert space with inner product $(\cdot,\cdot)$ and induced norm $|\cdot|$. The dual space of V is denoted by V^* and equipped with the usual norm $\|f\|_* := \sup_{v\in V\setminus\{0\}}\langle f,v\rangle/\|v\|$, where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between V^* and V. Furthermore, V is assumed to be dense and continuously embedded in H.

For a Banach space X and a time interval $S \subseteq \mathbb{R}$, let $L^p(S;X)$ be the usual spaces of Bochner integrable functions $u : S \to X$. The discrete counterparts for functions defined on a time grid are denoted by $l^p(0,T;X)$. We shall use the notation $u \in C(S;X)$ to say that a function u is almost everywhere in S equal to a continuous function. We then deal with the continuous representative, only. With u', the derivative in the distributional sense is meant. Note that $u \in L^2(0,T;V)$, $u' \in L^2(0,T;V^*)$ implies $u \in C([0,T];H)$.

Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a continuous, strongly positive bilinear form such that there are constants $\beta \geq \mu > 0$ with

$$|a(u,v)| \leq \beta\,\|u\|\,\|v\|\,, \quad a(v,v) \geq \mu\,\|v\|^2 \tag{1.1}$$

for all u, $v \in V$. For the skew-symmetric part of $a(\cdot,\cdot)$, we assume

$$|a(u,v) - a(v,u)| \leq \gamma\,\|u\|\,|v| \tag{1.2}$$

for all u, $v \in V$ with some $\gamma \geq 0$. If $a(\cdot,\cdot)$ is symmetric then $\gamma = 0$.

Furthermore, let $\mathcal{B}_M := \{v \in V : |v| \leq M\}$ and let $g : V \to V^*$ be a (possibly nonlinear) function that satisfies the following structural assumptions:

(H1) *There exist some $s_1 \in (0,1]$ and a constant $L_1 \geq 0$ such that for all $u \in V$*

$$\|g(u)\|_* \leq L_1\,(1 + |u|)^{s_1}\,\|u\|^{1-s_1}\,.$$

(H2) *There exists some $s_2 \in (0,1]$ such that for every $M > 0$ there is a constant $L_2 = L_2(M) \geq 0$ and, for all $u, v \in \mathcal{B}_M$, it holds*

$$\|g(u) - g(v)\|_* \leq L_2\,|u-v|^{s_2}\,\|u-v\|^{1-s_2}\,.$$

We remark that (H1) includes (with $s_1 = 1$) the case $\|g(u)\|_* \leq \mathit{const}$. Alternatively, we may assume $g : V \to H$ and the following:

$\widetilde{\text{(H1)}}$ *There exists a constant $\widetilde{L}_1 \geq 0$ such that for all $u \in V$*

$$|g(u)| \leq \widetilde{L}_1\,(1 + \|u\|)\,.$$

$\widetilde{\text{(H2)}}$ *For every $M > 0$ there is a constant $\widetilde{L}_2 = \widetilde{L}_2(M) \geq 0$ such that for all $u, v \in \mathcal{B}_M$*

$$|g(u) - g(v)| \leq \widetilde{L}_2\,\|u-v\|\,.$$

The problem we are concerned with then reads as

Problem (P) *For given $u_0 \in H$ and $f \in L^2(0,T;V^*)$, find $u \in L^2(0,T;V)$ such that for all $v \in V$ and almost everywhere in $(0,T)$*

$$\langle u'(t),v\rangle + a(u(t),v) + \langle g(u(t)),v\rangle = \langle f(t),v\rangle \tag{1.3}$$

holds with $u(0) = u_0$.

In the linear case $g(u) \equiv 0$, Problem (P) possesses a unique solution u with $u' \in L^2(0,T;V^*)$. In the nonlinear case, solvability has been studied by many authors in different situations. We refer to [6] and the references cited there.

We may remark that V and H can also be finite dimensional, which is the case when first discretising in space and afterwards in time. For the spatial semi-discretisation, a conforming finite element method might be used. For more details, we refer to [5] and the references cited there.

Let the time interval $[0,T]$ for given $N \in \mathbb{N}$ be partitioned via

$$0 = t_0 < t_1 < \cdots < t_N = T\,, \; \tau_n := t_n - t_{n-1}, \; r_n := \tau_n/\tau_{n-1}\,, \; \tau_{\max} := \max_{n=1,\ldots,N} \tau_n\,.$$

Throughout this paper, we assume that $r_n < R$ with $R > 1$. We denote by D_1 and D_2 the backward divided differences:

$$\mathrm{D}_1 u^n := \frac{u^n - u^{n-1}}{\tau_n}\,, \qquad \mathrm{D}_2 u^n := \frac{1}{\tau_n}\left(\frac{1+2r_n}{1+r_n}\,u^n - (1+r_n)\,u^{n-1} + \frac{r_n^2}{1+r_n}\,u^{n-2}\right).$$

We immediately find

$$\mathrm{D}_2 u^n = \frac{1+2r_n}{1+r_n}\mathrm{D}_1 u^n - \frac{r_n}{1+r_n}\mathrm{D}_1 u^{n-1}\,.$$

We consider the time discretisation of Problem (P) by means of the formally second order two-step backward differentiation formula with variable time steps:

Problem (P_τ) *For given $u^0,\ u^1 \in H$ and $\{f^n\} \in l^2(0,T;V^*)$, find $u^n \in V$ ($n = 2, 3, \ldots, N$) such that for all $v \in V$*

$$(\mathrm{D}_2 u^n, v) + a(u^n, v) + \langle g(u^n), v\rangle = \langle f^n, v\rangle\,. \tag{1.4}$$

We may suppose $u^0 = u_0$ and compute u^1 by means of the implicit Euler method. In the linear case $g(u) \equiv 0$, Problem (P_τ) possesses a unique solution as it can be shown via the Lax-Milgram lemma. For the solvability in the nonlinear case, we refer again to [6].

2 The linear case

In the following, let c be a generic constant that does not depend on problem parameters whereas C may depend on T, R, the sequence of step size ratios as well as on constants that appear in the assumptions on $a(\cdot,\cdot)$ and $g(\cdot)$. Note further the conventions $\sum_{j=m}^{n} x_j := 0$ and $\prod_{j=m}^{n} x_j := 1$ if $m > n$.

Theorem 2.1 *The solution to Problem (P_τ) with $g(u) \equiv 0$ is stable in $l^\infty(0,T;H)$ and $l^2(0,T;V)$ if $R < \bar{R} \approx 1.91$, where $\bar{R}$ is a root of*

$$\psi(R) := (R+1)^4 - 9R(R-1)^2\left(R+\tfrac{1}{3}\right)^2 = -9R^5 + 13R^4 + 6R^3 + 2R^2 + 3R + 1\,.$$

It holds the estimate ($n = 2, 3, \ldots, N$)

$$|u^n|^2 + \sum_{j=2}^{n} \tau_j\,\|u^j\|^2 \le C\left(|u^0|^2 + |u^1|^2 + \tau_2\,\|u^1\|^2 + \sum_{j=2}^{n}\frac{\tau_j}{1+r_j}\|f^j\|_*^2\right). \tag{2.1}$$

the term $a(u^n_{\hat\delta}, u^n_{\hat\delta})/(2(1+\hat\delta))$ will be absorbed within (2.7).

With (2.5) and (2.7), we now obtain

$$\begin{aligned}
&\frac{1+2R}{(1+R)^2}\,|u^n|^2 - \frac{R^2}{(1+R)^2}\,|u^{n-1}|^2 + \sigma_1(R)\,|u^n - u^{n-1}|^2 + \frac{\tau_n(1+\hat\delta)}{1+r_n}\,a(u^n,u^n) \\
&\qquad + \sum_{j=2}^{n-1} \frac{\tau_j}{1+r_j}\left(1+\hat\delta - \frac{\hat\delta^2}{1+\hat\delta}\,\frac{r_{j+1}(1+r_j)}{1+r_{j+1}}\right) a(u^j,u^j) \\
&\le |u^1|^2 + \sigma_1(R)\,|u^1-u^0|^2 + \frac{\hat\delta^2}{1+\hat\delta}\,\frac{\tau_2}{1+r_2}\,a(u^1,u^1) + \frac{8}{27}\sum_{j=2}^{n-2}[r_{j+2}-r_j]_-|u^j|^2 \\
&\qquad + \frac{\hat\delta^2\gamma^2}{2\mu(1+\hat\delta)}\sum_{j=2}^{n}\frac{\tau_j}{1+r_j}|u^{j-1}|^2 + \frac{2(1+\hat\delta)}{\mu}\sum_{j=2}^{n}\frac{\tau_j}{1+r_j}\,\|f^j\|_*^2 . \qquad (2.9)
\end{aligned}$$

Since $r_{j+1}(1+r_j)/(1+r_{j+1}) \le R$, we need that $1+\hat\delta > R\hat\delta^2/(1+\hat\delta)$, i. e.

$$R < \left(\frac{1+\hat\delta}{\hat\delta}\right)^2 = \frac{(R+1)^4}{9(R-1)^2\left(R+\frac{1}{3}\right)^2}, \qquad (2.10)$$

in order to prove the stability result. Unfortunately, relation (2.10), which is equivalent to $\psi(R) > 0$, is only satisfied for $R < \bar R \approx 1.91$, where $\psi(\bar R) = 0$. With

$$\sigma_2(R) := \frac{1}{1+R}\left(1+\hat\delta - R\frac{\hat\delta^2}{1+\hat\delta}\right) = \frac{\psi(R)}{2(1+R)^3(1+2R-R^2)},$$

and by virtue of (2.9), we come up with

$$\begin{aligned}
&\frac{1+2R}{(1+R)^2}\,|u^n|^2 + \sigma_1(R)\,|u^n-u^{n-1}|^2 + \frac{1+\hat\delta}{1+R}\,\mu\,\tau_n\,\|u^n\|^2 \\
&\qquad + \mu\,\sigma_2(R)\sum_{j=2}^{n-1}\tau_j\,\|u^j\|^2 \le \frac{R^2}{(1+R)^2}\,|u^{n-1}|^2 + \mathcal{K}_n , \qquad (2.11)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{K}_n \le C\Bigg(&\sum_{j=2}^{n-2}[r_{j+2}-r_j]_-|u^j|^2 + \sum_{j=2}^{n-1}\tau_{j+1}\,|u^j|^2 \\
&+ |u^0|^2 + |u^1|^2 + \tau_2\,\|u^1\|^2 + \sum_{j=2}^{n}\frac{\tau_j}{1+r_j}\,\|f^j\|_*^2\Bigg) .
\end{aligned}$$

Let $m^* = m^*(m)$ be such that $|u^{m^*}| = \max_{l=1,\dots,m}|u^l|$ for $m = 2,3,\dots,N$. If $m^* \ge 2$, it follows from (2.11) with $n = m^*$ and because of $\mathcal{K}_{m^*} \le \mathcal{K}_m$ the estimate

$$\frac{1+2R}{(1+R)^2}\,|u^{m^*}|^2 \le \frac{R^2}{(1+R)^2}\,|u^{m^*-1}|^2 + \mathcal{K}_{m^*} \le \frac{R^2}{(1+R)^2}\,|u^{m^*}|^2 + \mathcal{K}_m$$

that leads for $R < 1 + \sqrt{2}$ to

$$|u^{m^*}|^2 \le \frac{(1+R)^2}{1+2R-R^2}\,\mathcal{K}_m\,.$$

This last estimate holds also true if $m^* = 1$. With $n = m$ in (2.11) and $|u^{m-1}| \le |u^{m^*}|$, we thus have (changing m, n again) for $n = 2, 3, \ldots, N$

$$\frac{1+2R}{(1+R)^2}\,|u^n|^2 + \sigma_1(R)\,|u^n - u^{n-1}|^2 + \frac{1+\hat{\delta}}{1+R}\,\mu\,\tau_n\,\|u^n\|^2$$
$$+\,\mu\,\sigma_2(R)\sum_{j=2}^{n-1}\tau_j\,\|u^j\|^2 \le \left(\frac{R^2}{1+2R-R^2}+1\right)\mathcal{K}_n = \frac{1+2R}{1+2R-R^2}\,\mathcal{K}_n\,.$$

Taking into account that $1 < R < \bar{R} \approx 1.91$, estimates for the appearing coefficients yield

$$|u^n|^2 + |u^n - u^{n-1}|^2 + \mu\,\tau_n\,\|u^n\|^2 + \mu\,\psi(R)\sum_{j=2}^{n-1}\tau_j\,\|u^j\|^2 \le c\,\mathcal{K}_n\,. \tag{2.12}$$

The assertion now follows from Lemma A.1. #

Remark 2.1 With the natural restrictions

$$\mathrm{R}_1^n f := \frac{1}{\tau_n}\int_{t_{n-1}}^{t_n} f(t)dt\,, \quad \mathrm{R}_2^n f := \frac{1+2r_n}{1+r_n}\,\mathrm{R}_1^n f - \frac{r_n}{1+r_n}\,\mathrm{R}_1^{n-1} f\,,$$

we obtain for $f^j = \mathrm{R}_2^j f$ by standard arguments that

$$\sum_{j=2}^{n}\frac{\tau_j}{1+r_j}\,\|f^j\|_*^2 \le \frac{5}{4}\int_0^{t_n}\|f(t)\|_*^2\,dt\,.$$

Note that $\mathrm{R}_1^n u' = \mathrm{D}_1 u(t_n)$ and $\mathrm{R}_2^n u' = \mathrm{D}_2 u(t_n)$.

Remark 2.2 The constant in (2.1) is $C = C'\Lambda_n$ with $C' = C'(\beta, \gamma, \mu, R)$ and

$$\Lambda_n \le \left(1 + \frac{c\gamma^2\tau_n}{\mu}\right)\prod_{j=2}^{n-2}\left(1 + c\,[r_{j+2} - r_j]_- + \frac{c\gamma^2\tau_{j+1}}{\mu}\right) \le c\,\exp\left(\Gamma_n + \frac{\gamma^2 t_n}{\mu}\right),$$

where

$$\Gamma_n := \sum_{j=2}^{n-2}[r_{j+2} - r_j]_-\,, \quad n = 2,\,3,\,\ldots,\,N\,.$$

Note that $\Gamma_N = 0$ if $\{r_n\}$ is monotonically increasing. It can be proved that

$$\Lambda_n \le \prod_{j=2}^{n-2}\left(1 + c\,[r_{j+2} - r_j]_-\right) \le c\,\exp\Gamma_n \text{ if } \frac{\mu}{\beta}\,(R+1)^4 > 9R(R-1)^2\left(R + \tfrac{1}{3}\right)^2.$$

Theorem 2.2 *Let $g(u) \equiv 0$, $f^n = \mathrm{R}_2^n f$, and let $f'' - u''' \in L^2(0,T;V^*)$. If $R < \bar{R} \approx 1.91$ then the error $e^n = u(t_n) - u^n$ $(n = 2,3,\ldots,N)$ to Problem (P_τ) satisfies*

$$|e^n|^2 + \sum_{j=2}^{n} \tau_j \|e^j\|^2 \le C \left(|e^0|^2 + |e^1|^2 + \tau_2 \|e^1\|^2 + \sum_{j=1}^{n} \tau_j^4 \int_{t_{j-1}}^{t_j} \|f''(t) - u'''(t)\|_*^2 dt \right).$$

Proof We commence with the error equation

$$(\mathrm{D}_2 e^n, v) + a(e^n, v) = \langle \rho^n, v \rangle$$

that follows from (1.3) and (1.4) with the consistency error

$$\rho^n = \mathrm{D}_2 u(t_n) - u'(t_n) + f(t_n) - \mathrm{R}_2^n f = \mathrm{I}_2^n (f'' - u'''), \tag{2.13}$$

where

$$\mathrm{I}_2^n w := \frac{1}{2(1+r_n)} \left(\frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} (t_n - t)\left((1+2r_n)(t - t_{n-1}) + \tau_n\right) w(t)\, dt \right.$$
$$\left. + \frac{r_n}{\tau_{n-1}} \int_{t_{n-2}}^{t_{n-1}} (t - t_{n-2})^2 w(t)\, dt \right).$$

Note that $\mathrm{I}_2^n f'' = f(t_n) - \mathrm{R}_2^n f$. The assertion follows from Theorem 2.1 because of

$$\sum_{j=2}^{n} \frac{\tau_j}{1+r_j} \|\mathrm{I}_2^j w\|_*^2 \le c \sum_{j=1}^{n} \tau_j^4 \int_{t_{j-1}}^{t_j} \|w(t)\|_*^2\, dt \le c\, \tau_{\max}^4 \|w\|_{L^2(0,t_n;V^*)}^2 .$$

#

Remark 2.2 also applies here.

3 The nonlinear case

Theorem 3.1 *Under the assumption* (H1) *or* $(\widetilde{\mathrm{H1}})$, *the solution to Problem (P_τ) is stable in $l^\infty(0,T;H) \cap l^2(0,T;V)$ if $R < \bar{R} \approx 1.91$ and if $\tau_{\max}$ is sufficiently small. It holds for $n = 2,3,\ldots,N$*

$$|u^n|^2 + \sum_{j=2}^{n} \tau_j \|u^j\|^2 \le C \left(|u^0|^2 + |u^1|^2 + \tau_2 \|u^1\|^2 + \sum_{j=2}^{n} \frac{\tau_j}{1+r_j} \|f^j\|_*^2 + t_n \right). \tag{3.1}$$

Proof We reconsider the proof of Theorem 2.1. For the term $\langle g(u^n), u^n_{\hat\delta} \rangle$, we obtain from (H1) for arbitrary $\alpha > 0$ with Cauchy-Schwarz' and Young's inequality

$$|\langle g(u^n), u^n_{\hat\delta} \rangle| \le \|g(u^n)\|_* \, \|u^n_{\hat\delta}\| \le L_1 (1 + |u^n|)^{s_1} \|u^n\|^{1-s_1} \|u^n_{\hat\delta}\|$$
$$\le \frac{C_1}{2} (1 + |u^n|)^2 + \alpha\mu \left(\|u^n\|^2 + \|u^n_{\hat\delta}\|^2 \right) \le C_1 \left(1 + |u^n|^2\right) + \alpha \left(a(u^n, u^n) + a(u^n_{\hat\delta}, u^n_{\hat\delta}) \right),$$

where

$$C_1 = 2s_1 \left(\frac{L_1^2}{4\alpha\mu} \left(\frac{1-s_1}{\alpha\mu} \right)^{1-s_1} \right)^{1/s_1} . \tag{3.2}$$

Changing the coefficients in the right-hand side of (2.8) appropriately, the term $\alpha\, a(u_{\hat\delta}^n, u_{\hat\delta}^n)$ can be absorbed in view of (2.7). The additional term $\alpha\, a(u^n, u^n)$ requires, however, to modify the crucial condition (2.10) that came from (2.9). Here, we need that $1 + \hat\delta - \alpha > R\hat\delta^2/(1+\hat\delta)$. For any $R < \bar R \approx 1.91$, this can be fulfilled by taking α sufficiently small.

The remaining term $C_1\,(1 + |u^n|^2)$, which makes the application of Corollary A.1 and thus $C_1\tau_{\max} < 1$ necessary, leads to a change of the right-hand side in (2.12); here we have

$$\begin{aligned} \mathcal{K}_n \le C \Bigg(& \sum_{j=2}^{n-2} [r_{j+2} - r_j]_- |u^j|^2 + \sum_{j=2}^{n-1} (\tau_j + \tau_{j+1})\, |u^j|^2 + \tau_{\max}\, |u^n|^2 \\ & + t_n + |u^0|^2 + |u^1|^2 + \tau_2\, \|u^1\|^2 + \sum_{j=2}^{n} \frac{\tau_j}{1+r_j}\, \|f^j\|_*^2 \Bigg) . \end{aligned}$$

Suppose now that g satisfies $\widetilde{\text{(H1)}}$. We then have for arbitrary $\alpha > 0$

$$|\langle g(u^n), u_{\hat\delta}^n \rangle| \le |g(u^n)|\, |u_{\hat\delta}^n| \le \widetilde L_1 (1 + \|u^n\|)\, |u_{\hat\delta}^n| \le \alpha\, a(u^n, u^n) + 1 + \widetilde C_1\, |u_{\hat\delta}^n|^2 ,$$

where

$$\widetilde C_1 = \frac{\widetilde L_1^2}{4} \left(1 + \frac{1}{\alpha\mu} \right) . \tag{3.3}$$

Since

$$|u_{\hat\delta}^n|^2 \le 2 \max(1, \hat\delta^2)\, \left(|u^n|^2 + |u^n - u^{n-1}|^2 \right) ,$$

we can again apply Corollary A.1 if $\widetilde C_1 \tau_{\max} < 1$. #

Remark 3.1 The constant in (3.1) is $C = C'\Lambda_n$ with $C' = C'(\beta, \gamma, \mu, R)$ and

$$\begin{aligned} \Lambda_n & \le \frac{1}{1 - C_1\tau_{\max}} \left(1 + \frac{C_1\tau_{n-1} + c\gamma^2\tau_n/\mu}{1 - C_1\tau_{\max}} \right) \prod_{j=2}^{n-2} \left(1 + \frac{c[r_{j+2} - r_j]_- + C_1\tau_j + c\gamma^2\tau_{j+1}/\mu}{1 - C_1\tau_{\max}} \right) \\ & \le \frac{c}{1 - C_1\tau_{\max}} \exp\left(\frac{\Gamma_n + (C_1 + \gamma^2/\mu)t_n}{1 - C_1\tau_{\max}} \right) , \end{aligned}$$

where C_1 is given by (3.2) or, alternatively, by (3.3).

Theorem 3.2 *Let $f^n = \mathrm{R}_2^n f$, $u \in C([0,T]; H)$, $f'' - u''' \in L^2(0,T;V^*)$, and assume* (H1) *or* $\widetilde{\text{(H1)}}$ *and* (H2) *or* $\widetilde{\text{(H2)}}$. *Furthermore, let $R < \bar R \approx 1.91$ and $\tau_{\max}$ be* *sufficiently small. The error $e^n = u(t_n) - u^n$ $(n = 2, 3, \dots, N)$ to Problem (P_τ) then satisfies*

$$|e^n|^2 + \sum_{j=2}^{n} \tau_j \, \|e^j\|^2 \le C \left(|e^0|^2 + |e^1|^2 + \tau_2 \, \|e^1\|^2 + \sum_{j=1}^{n} \tau_j^4 \int_{t_{j-1}}^{t_j} \|f''(t) - u'''(t)\|_*^2 dt \right) .$$

Proof We again commence with the error equation that reads now as

$$(\mathrm{D}_2 e^n, v) + a(e^n, v) + \langle g(u(t_n)) - g(u^n), v\rangle = \langle \rho^n, v\rangle \,. \tag{3.4}$$

The consistency error ρ^n to the associated linear problem is given by (2.13). Due to Theorem 3.1 and since $u \in \mathcal{C}([0,T]; H)$, there is some $M > 0$, depending on problem data, such that $u(t_n)\,, u^n \in \mathcal{B}_M$ $(n = 2, 3, \dots, N)$. Because of (H2), we have for arbitrary $\alpha > 0$

$$\begin{aligned} &|\langle g(u(t_n)) - g(u^n), e^n_{\hat\delta}\rangle| \le L_2(M)\, |e^n|^{s_2} \, \|e^n\|^{1-s_2} \, \|e^n_{\hat\delta}\| \\ &\le C_2 \, |e^n|^2 + \alpha\mu \left(\|e^n\|^2 + \|e^n_{\hat\delta}\|^2 \right) \le C_2 \, |e^n|^2 + \alpha \left(a(e^n, e^n) + a(e^n_{\hat\delta}, e^n_{\hat\delta}) \right) , \end{aligned}$$

with some $C_2 > 0$ depending on s_2, $L_2(M)$, and $1/(\alpha\mu)$. Alternatively, we have with $\widetilde{\text{(H2)}}$ and some $\widetilde{C}_2 > 0$

$$|\langle g(u(t_n)) - g(u^n), e^n_{\hat\delta}\rangle| \le \widetilde{L}_2(M) \, \|e^n\| \, |e^n_{\hat\delta}| \le \alpha \, a(e^n, e^n) + \widetilde{C}_2 \, |e^n_{\hat\delta}|^2 \,.$$

We now follow the arguments in the proof of Theorem 3.1 and Theorem 2.1. #

We may finally remark that the error constant is essentially of the same structure as the stability constant (changing the subscript 1 to 2).

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Appendix: A discrete Gronwall lemma

Lemma A.1 *Let $a_n, b_n, c_n, \lambda_n \geq 0$ and $\{c_n\}$ be monotonically increasing. Then*

$$a_n + b_n \leq \sum_{j=2}^{n-1} \lambda_j a_j + c_n\,, \quad n = 2, 3, \ldots \tag{A.1}$$

implies for $n = 2, 3, \ldots$

$$a_n + b_n \leq c_n \prod_{j=2}^{n-1} (1 + \lambda_j) \leq c_n \exp\left(\sum_{j=2}^{n-1} \lambda_j\right).$$

Proof With

$$\tilde{a}_m := \sum_{j=2}^{m-1} \lambda_j a_j \prod_{j=2}^{m-1} (1 + \lambda_j)^{-1}$$

for $m = 2, 3, \ldots$, we have

$$\tilde{a}_{m+1} - \tilde{a}_m = \lambda_m \left(a_m - \sum_{j=2}^{m-1} \lambda_j a_j\right) \prod_{j=2}^{m} (1 + \lambda_j)^{-1} \leq c_m \lambda_m \prod_{j=2}^{m} (1 + \lambda_j)^{-1}.$$

Summation gives (because of $\tilde{a}_2 = 0$)

$$\tilde{a}_n \leq \sum_{m=2}^{n-1} c_m \lambda_m \prod_{j=2}^{m} (1 + \lambda_j)^{-1} \leq c_n \sum_{m=2}^{n-1} \lambda_m \prod_{j=2}^{m} (1 + \lambda_j)^{-1}.$$

We thus have from (A.1)

$$a_n + b_n \leq \tilde{a}_n \prod_{j=2}^{n-1} (1 + \lambda_j) + c_n \leq c_n \left(\sum_{m=2}^{n-1} \lambda_m \prod_{j=2}^{m} (1 + \lambda_j)^{-1} + \prod_{j=2}^{n-1} (1 + \lambda_j)^{-1}\right) \prod_{j=2}^{n-1} (1 + \lambda_j),$$

and the assertion follows with the identity

$$\sum_{m=2}^{n-1} \lambda_m \prod_{j=2}^{m} (1 + \lambda_j)^{-1} + \prod_{j=2}^{n-1} (1 + \lambda_j)^{-1} = 1.$$

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Corollary A.1 *Let, in addition to the assumptions of Lemma A.1, $0 \leq \mu < 1$. Then*

$$a_n + b_n \leq \sum_{j=2}^{n-1} \lambda_j a_j + \mu\, a_n + c_n\,, \quad n = 2, 3, \ldots \tag{A.2}$$

implies for $n = 2, 3, \ldots$

$$a_n + b_n \leq \frac{c_n}{1 - \mu} \prod_{j=2}^{n-1} \left(1 + \frac{\lambda_j}{1 - \mu}\right) \leq \frac{c_n}{1 - \mu} \exp\left(\frac{1}{1 - \mu} \sum_{j=2}^{n-1} \lambda_j\right).$$

Proof It immediately follows from (A.2) that

$$a_n + b_n \leq a_n + \frac{b_n}{1 - \mu} \leq \sum_{j=2}^{n-1} \frac{\lambda_j}{1 - \mu} a_j + \frac{c_n}{1 - \mu},$$

and we may apply Lemma A.1 with $\lambda_n := \lambda_n/(1 - \mu)$ and $c_n := c_n/(1 - \mu)$. #