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Intrinsic and Extrinsic Geometry of Ovaloids and Rigidity

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Dedicated to S.S. Chern

Abstract

Our main result is that an ovaloid with nowhere dense umbilics and prescribed Weingarten operator and spherical volume form is rigid in Euclidean 3-space. In case of an ovaloid of revolution we can drop the assumption on the volume form.

Keywords: ovaloids, global rigidity, Weingarten operator, surfaces of revolution.
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Introduction

Bonnet's uniqueness theorem for Euclidean hypersurfaces states that the first fundamental form I and the second fundamental form II together completely determine the hypersurface, that means that all geometric invariants can be derived from the two forms; thus I and II together completely describe the geometry of the hypersurface. Both forms are related by the Weingarten operator S:

$$\mathrm{II}(v, w) = \mathrm{I}(Su, v).$$

It is a trival consequence of this relation that the pair $\{\mathrm{I}, S\}$ forms another fundamental system of geometric invariants. In a standard terminology one calls all invariants *intrinsic* which belong to the Riemannian geometry of the first fundamental form metric; all invariants which depend on the immersion of the hypersurface into the ambient space belong to the *extrinsic* geometry. In particular, the Weingarten operator S and its invariants describe the extrinsic curvature properties of the hypersurface. The integrability conditions of the structure equations give relations between the two fundamental forms and thus they admit the study of relations between the intrinsic and the extrinsic geometry.
Review of some intrinsic results. One of the most famous results in this direction is the theorema egregium of Gauß. If we denote by $k_1, \dots, k_n$ the principal curvatures and by $H_1, \dots, H_n$ their (normed) elementary symmetric functions (in particular: $H := H_1$

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denotes the mean curvature, $K := H_n$ the Gauß-Kronecker curvature) of an n-dimensional hypersurface in Euclidean $(n+1)$-space, $n \geq 2$, then this result states (see e.g. [K-N-II], p.43 and [HEIL]):

Theorema egregium of GAUSS – extended version.

(i) *The curvature functions* H_{2r}, *for* $2 \leq 2r \leq n$, *are intrinsic invariants;*

(ii) *the curvature functions* $(H_{2r+1})^2$, *for* $3 \leq 2r+1 \leq n$, *are intrinsic invariants.*

In a rough terminology, the mean curvature is the only genuine extrinsic curvature invariant within the set of curvature functions $H_1, \ldots, H_n$; this explains the particular interest in the study of the mean curvature.

While Gauß' result is true for any hypersurface, there are other results which need additional assumptions; recall the following two well known results, one a local theorem, the other one global; they describe conditions under which the first fundamental form determines the second fundamental form.

We introduce the following notation: M denotes a connected, oriented C^∞-manifold of dimension $M = n \geq 2$ and $x, x^\#: M \to E^{n+1}$ hypersurface embeddings into Euclidean space such that $\varphi := x^\# \circ x^{-1}$ is a diffeomorphism between $x(M)$ and $x^\#(M)$. In an obvious notation $\mathrm{I}, \mathrm{II}, S$ and $\mathrm{I}^\#, \mathrm{II}^\#, S^\#$ denote the fundamental invariants of x and $x^\#$, resp., mentioned in the beginning.

Theorem of Beez (1876) – Killing (1885). *Let* $\dim M = n \geq 3$ *and* $x, x^\#$ *be isometric hypersurfaces, i.e.* $\mathrm{I} = \mathrm{I}^\#$ *on M. If* $\operatorname{rank}(S) \geq 3$, *then* $\mathrm{II}^\# = \pm\mathrm{II}$; *thus* $x, x^\#$ *are congruent in* E^{n+1}.

Theorem of Cohn-Vossen (1927). *Let* $x, x^\#: M \to E^3$ *be ovaloids (i.e. compact without boundary and with positive Gauß curvature). If* $x, x^\#$ *are isometric then they are congruent.*

Both results are a consequence of the integrability conditions; one proves that the metric form I uniquely determines the second fundamental form (modulo sign in the Beez-Killing theorem); then one applies Bonnet's rigidity result (see e.g. [K-N-II], p. 43, and [COHN-V]).

Remarks. We would like to recall some modest extensions of the foregoing uniqueness results.

(i) We weaken the assumption of the isometry and assume instead that only the Levi-Civita connections coincide: $\nabla = \nabla^\#$. According to the Ricci-Lemma both metrics are parallel

$$\nabla \mathrm{I} = 0 = \nabla \mathrm{I}^\#.$$

If we additionally assume that $x, x^\#$ are locally strongly convex, i.e. the Gauß curvatures are positive, then the Riemannian spaces (M, I) and $(M, \mathrm{I}^\#)$ are irreducible; thus parallelity implies $\mathrm{I}^\# = c\mathrm{I}$ for some positive constant $c \in \mathbb{R}$. This gives:
Extensions. *Let* $x, x^\#$ *be locally strongly convex and assume* $\nabla = \nabla^\#$:

(a) *If $n \geq 3$ and rank$(S) \geq 3$, then $x, x^\#$ are homothetic;*

(b) *if $n = 2$ and $x, x^\#$ are ovaloids, then $x, x^\#$ are homothetic.*

(ii) Another extension of Cohn-Vossen's result was proved by Hsü [HSÜ]: *If two ovaloids $x, x^\#\colon M \to E^3$ satisfy $K \cdot \mathrm{I} = K^\# \cdot \mathrm{I}^\#$ ($K, K^\#$ Gauß curvatures) at any $p \in M$, then $x, x^\#$ are homothetic* (and $K^\# = c \cdot K$ with $0 < c \in \mathbb{R}$). - Hsü's assumption is a particular conformal relation $\mathrm{I}^\# = q \cdot \mathrm{I}$ with $q = K \cdot K^{\#-1}$, where the proof finally gives $q = c^{-1}$. It is well known that the assertion is not any more true for a general conformal relation $\mathrm{I}^\# = q \cdot \mathrm{I}$ with $0 < q \in C^\infty(M)$, but without further restrictions on q.
Hsü's result generalizes the foregoing extension (i): $\nabla = \nabla^\#$ implies equality of the Ricci tensors; in dimension $n = 2$ that gives $K \cdot \mathrm{I} = \mathrm{Ric} = \mathrm{Ric}^\# = K^\# \cdot \mathrm{I}^\#$.

(iii) One might try other extensions of the foregoing results in (i). E.g., recall that a connection is determined by a natural parametrization of its autoparallel curves within the projective class. This raises, in particular, the question whether one can weaken the above assumption $\nabla = \nabla^\#$ and assume instead that there exists a diffeomorphism of the two ovaloids preserving the autoparallel curves. Are both still homothetic? The answer is in the negative; this follows from the examples recently given independently in [MATV], [TABA, Theorem 6], and [VOSS].

Review of some extrinsic results. As far as we know it was E. Cartan [CARTAN] in 1943 who startet a systematic investigation of the role of the second fundamental form and proved some local existence and uniqueness theorems in terms of the extrinsic geometry. So far any attempt failed of proving a global "extrinsic" analogue to Cohn-Vossen's result assuming that the second fundamental forms of two ovaloids $x, x^\#$ coincide: $\mathrm{II} = \mathrm{II}^\#$. But there exist several rigidity results under additional assumptions:

1.1. Theorem. *Let $x, x^\#\colon M \to E^3$ be ovaloids.*

(i) ([GROVE],1957). *If $\mathrm{II} = \mathrm{II}^\#$ and the Gauß curvatures coincide, $K = K^\#$, then $x, x^\#$ are congruent.*

(ii) *(1970-73; see* [HUCK et al] *pp.59-61). If $\mathrm{II} = \mathrm{II}^\#$ and $F(H, K) = F(H^\#, K^\#)$ for some C^1-function $F\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\partial_1 F \cdot \partial_2 F := \frac{\partial F}{\partial H} \cdot \frac{\partial F}{\partial K} > 0$ then $x, x^\#$ are congruent.*

Theorem 1.1(i) was generalized to dimension $n \geq 2$ by Gardner[1] [GARDNER-I] in 1969, using a new integral formula. Erard (1968) extended Cartan's investigations and additionally studied infinitesimal deformations preserving the second fundamental form. See [ERARD] and sections 2.3.a.B; 2.3.b.B; 3.4.2; 3.8.2 in [HUCK et al] for related results. So far, all similar global results for surfaces in Euclidean space E^3 have two extrinsic assumptions for proving an extrinsic uniqueness theorem.
The situation is different for closed surfaces in a non-flat space form $\bar{M}$. In [LIU-S-W] we recently proved a uniqueness theorem with only one extrinsic assumption.

[1] R. Gardner († 1998) was a Ph.D. student of S.S. Chern

1.2. Theorem. *Let $x, x^{\#}: M^2 \to \bar{M}^3$ be closed surfaces in a non-flat space form with positive definite Weingarten operators $S, S^{\#}$. If the third fundamental forms coincide,*

$$\mathrm{I}(Su, Sv) =: \mathrm{III}(u, v) = \mathrm{III}^{\#}(u, v) = \mathrm{I}^{\#}(S^{\#}u, S^{\#}v)$$

then $x, x^{\#}$ are congruent.

The proofs of the foregoing intrinsic and the extrinsic global results use the Codazzi equations as essential tool. All different proofs of the *intrinsic* Cohn-Vossen theorem consider the Codazzi equations for the difference tensor between the second fundamental forms $D := \mathrm{II} - \mathrm{II}^{\#}$ of two ovaloids $x, x^{\#}$ in Euclidean 3-space; one derives a *linear* equation for D. In case of the *extrinsic* rigidity results in Theorem 1.1 the Codazzi equations lead to a *nonlinear* elliptic equation for the difference tensor $E := \mathrm{I} - \mathrm{I}^{\#}$ (see [HUCK et al], l.c.).

It is the aim of this paper to present a new method of proof and the following new extrinsic result.

Theorem A. *Let $x, x^{\#}: M \to E^3$ be ovaloids in Euclidean 3-space with nowhere dense umbilics and with the property that, at any $p \in M$, the Weingarten operators $S, S^{\#}$ and the spherical volume forms $\omega(\mathrm{III})$, $\omega(\mathrm{III}^{\#})$ coincide:*

$$S = S^{\#}, \qquad \omega(\mathrm{III}) = \omega(\mathrm{III}^{\#}).$$

Then $x, x^{\#}$ are congruent.

1.3. Corollary. *Let $x, x^{\#}: M \to E^3$ be ovaloids such that $S = S^{\#}$ and $\omega(\mathrm{III}) = \omega(\mathrm{III}^{\#})$. If x is analytic then $x, x^{\#}$ are congruent up to a reparametrization.*

The basic idea for the proof of Theorem A (sections 2 and 3) is to consider the unique, I-selfadjoint, positive definite operator L defined by

$$\mathrm{I}^{\#}(v, w) =: \mathrm{I}(Lv, Lw)$$

and to study its algebraic and analytic properties. A second tool is to use the Codazzi equations for $S = S^{\#}$ in terms of the two Levi-Civita connections $\nabla = \nabla(\mathrm{I})$ and $\nabla^{\#} = \nabla(\mathrm{I}^{\#})$ to get relations for the symmetric (1.2) difference tensor $(\nabla - \nabla^{\#})$ between the connections which finally lead to *PDEs* for the operator L.
If one follows the proof it seems that one might drop the assumption on the volume forms. We would like to state the following

Conjecture. *Let $x, x^{\#}: M \to E^3$ be ovaloids with nowhere dense umbilics and with $S = S^{\#}$ at corresponding points. Then $x, x^{\#}$ are congruent.*

In section 4, we give another partial answer to this conjecture.

2 Codazzi operators in terms of different metrics

As before, let M be a connected, oriented C^∞-manifold of dimension $n \geq 2$. We recall the well known definition of a Codazzi operator ψ with respect to an affine connection ∇ on M

which we assume to be torsion free; if the pair $\{\nabla, \psi\}$ satisfies *Codazzi equations*

$$(\nabla_v \psi)w = (\nabla_w \psi)v;$$

we call $\{\nabla, \psi\}$ a *Codazzi pair.*

2.1. Lemma. *Let* $\dim M = n = 2$ *and consider two torsion free connections* $\nabla, \nabla^\#$ *and an operator* ψ *on* M*; let* $\{\nabla, \psi\}$, $\{\nabla^\#, \psi\}$ *be Codazzi pairs. Assume that* ψ *has two real eigenvalue functions* ν_1, ν_2 *on* M. *If, at* $p \in M$, $\nu_1(p) \neq \nu_2(p)$ *then there exists a local parametrization* (u^1, u^2) *of a chart* U *around* p *s.t. for the associated Gauß basis* $\{\partial_1, \partial_2\}$*:*

$$\psi(\partial_i) = \nu_i \partial_i. \tag{2.1.1}$$

In such coordinates we have $\Gamma^{\#r}_{12} = \Gamma^r_{12}$ *on* U *for* $r = 1, 2$.

Proof. In local terminology, the Codazzi equations for $\{\nabla, \psi\}$ read:

$$\partial_i \psi^r_j + \Gamma^r_{is} \psi^s_j = \partial_j \psi^r_i + \Gamma^r_{js} \psi^s_i.$$

Substract the analogous equation for $\{\nabla^\#, \psi\}$:

$$\psi^s_j (\Gamma^r_{is} - \Gamma^{\#r}_{is}) = \psi^s_i (\Gamma^r_{js} - \Gamma^{\#r}_{js}).$$

Equation (2.1.1) implies $(1 = j \neq i = 2)$:

$$\nu_1 (\Gamma^r_{21} - \Gamma^{\#r}_{21}) = \nu_2 (\Gamma^r_{12} - \Gamma^{\#r}_{12}).$$

This gives the assertion. $\square$

2.2. Calculation. *Let* $\nabla = \nabla(g)$ *be the Levi-Civita connection of a semi-Riemannian metric* g *on* M, $\dim M = n = 2$. *Assume that* g *has the local representation*

$$\begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$$

on a chart U *on* M. *Then the Christoffel symbols* Γ^k_{ij} *satisfy the relations*

$$2\Gamma^1_{11} = \partial_1 (\ln g_{11}); \qquad 2\Gamma^1_{12} = \partial_2 \ln g_{11}; \quad 2\Gamma^1_{22} = -(g_{11})^{-1} \partial_1 g_{22};$$
$$2\Gamma^2_{11} = -(g_{22})^{-1} \partial_2 g_{11}; \quad 2\Gamma^2_{12} = \partial_1 \ln g_{22}; \quad 2\Gamma^2_{22} = \partial_2 \ln(g_{22}).$$

2.3. Calculation. *Let* $\dim M = 2$ *and consider two Riemannian metrics* $g, g^\#$ *on* M. *Then there exists a unique* g*-self-adjoint, positive definite operator* L *such that*

$$g^\#(u, v) = g(Lu, Lv) \tag{2.3.1}$$

for tangent vectors u, v. *Denote the eigenvalue functions of* L *by* $\lambda_1, \lambda_2 > 0$ *and consider a local chart* U *with* $\lambda_1 \neq \lambda_2$ *such that the Gauß basis* $\{\partial_1, \partial_2\}$ *consists of eigendirections:* $L(\partial_i) = \lambda_i \partial_i$.

(i) *Then locally the metrics $g, g^{\#}$ are represented by matrices*

$$g : \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix} \quad \text{and} \quad g^{\#} : \begin{pmatrix} \lambda_1^2 g_{11} & 0 \\ 0 & \lambda_2^2 g_{22} \end{pmatrix}. \tag{2.3.2}$$

(ii) *Assume $\lambda_1 \neq \lambda_2$ on U. Then λ_1, λ_2 are differentiable and the Christoffel symbols of the metrics satisfy*

$$\begin{aligned}
&\Gamma^{\#1}_{11} - \Gamma^1_{11} = \partial_1 \ln \lambda_1; && \Gamma^{\#2}_{22} - \Gamma^2_{22} = \partial_2 \ln \lambda_2;\\
&\Gamma^{\#2}_{11} - \Gamma^2_{11} = -\frac{1}{2}\left\{\frac{\lambda_1^2 - \lambda_2^2}{\lambda_2^2}\right\}(g_{22})^{-1}\partial_2 g_{11} - \frac{\lambda_1}{\lambda_2^2}\partial_2\lambda_1 \cdot (g_{22})^{-1} g_{11}; &&\\
&\Gamma^{\#1}_{22} - \Gamma^1_{22} = -\frac{1}{2}\left\{\frac{\lambda_2^2 - \lambda_1^2}{\lambda_1^2}\right\}(g_{11})^{-1}\partial_1 g_{22} - \frac{\lambda_2}{\lambda_1^2}\cdot\partial_1\lambda_2 \cdot (g_{11})^{-1} g_{22}; &&\\
&\Gamma^{\#1}_{12} - \Gamma^1_{12} = \partial_2 \ln \lambda_1; && \Gamma^{\#2}_{12} - \Gamma^2_{12} = \partial_1 \ln \lambda_2.
\end{aligned}$$

2.4. Lemma. *Let $g, g^{\#}$ be Riemannian metrics on M satisfying (2.3.1); assume that there is an operator ψ on M which is selfadjoint with respect to g and $g^{\#}$ at the same time, denote the eigenvalues of ψ by ν_1, ν_2. Then, with the notation from 2.3:*

(i) $$L^2\psi = \psi L^2;$$

(ii) *If $\lambda_1 \neq \lambda_2$ and $\nu_1 \neq \nu_2$ then L and ψ have the same eigenspaces at any point of M.*

Proof. (i) $g(L^2\psi u, v) = g^{\#}(\psi u, v) = g^{\#}(u, \psi v) = g(L^2 u, \psi v) = g(\psi L^2 u, v)$ for all u, v; this implies (i); (ii) is an immediate consequence. □

2.5. Corollary. *Consider $g, g^{\#}$ and ψ as in 2.4 and assume that the two eigenvalues λ_1, λ_2 of L differ at a point $p \in M$. Then:*

(i) *there exists a chart U around p with $\lambda_1 \neq \lambda_2$ on U and with local coordinates (u^1, u^2) s.t. the operators and the metrics have the following local matrix representations:*

$$g : \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}, \qquad g^{\#} : \begin{pmatrix} \lambda_1^2 g_{11} & 0 \\ 0 & \lambda_2^2 g_{22} \end{pmatrix},$$

$$L : \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \psi : \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}.$$

(ii) *λ_1, λ_2 are differentiable on U;*

(iii) *the Christoffel symbols of g and $g^{\#}$ satisfy the relations in (2.2) and (2.3).*

2.6. Proposition. *Consider $g, g^\#$ and ψ as in (2.4) and assume additionally that ψ and the Levi-Civita connections $\nabla := \nabla(g)$ and $\nabla^\# := \nabla(g^\#)$ form two Codazzi pairs $\{\nabla, \psi\}$ and $\{\nabla^\#, \psi\}$. If $\lambda_1 \neq \lambda_2$ and $\nu_1 \neq \nu_2$ at p, consider the local parametrization (u^1, u^2) of a chart U as in (2.5) with $\lambda_1 \neq \lambda_2$ and $\nu_1 \neq \nu_2$ on U. Then:*

(i) *As before, λ_1, λ_2 are differentiable on U and*

$$\partial_2 \lambda_1 = 0 = \partial_1 \lambda_2.$$

(ii) *The relations for the Christoffel symbols in (2.3) simplify:*

$$\begin{aligned}
\Gamma^{\#1}_{11} - \Gamma^1_{11} &= \partial_1 \ln \lambda_1; \\
\Gamma^{\#2}_{22} - \Gamma^2_{22} &= \partial_2 \ln \lambda_2; \\
\Gamma^{\#2}_{11} - \Gamma^2_{11} &= -\frac{1}{2}(\lambda_2)^{-2}(\lambda_1^2 - \lambda_2^2) \cdot (g_{22})^{-1} \partial_2 g_{11}; \\
\Gamma^{\#1}_{22} - \Gamma^1_{22} &= -\frac{1}{2}(\lambda_1)^{-2}(\lambda_2^2 - \lambda_1^2) \cdot (g_{11})^{-1} \partial_1 g_{22}; \\
\Gamma^{\#r}_{12} - \Gamma^r_{12} &= 0.
\end{aligned}$$

Proof. (i) follows from Lemma 2.1 and 2.3.(ii). Insert now (i) into the other relation in 2.3.(ii); this gives (ii). □

3 Proof of Theorem A

We consider the two ovaloids $x, x^\# : M \to E^3$ with their metrics $g = \mathrm{I}, g^\# := \mathrm{I}^\#$, the associated Riemannian volume forms $\omega(g) = \omega(g^\#)$ and the Weingarten operators $S = S^\#$. We proceed with the following steps of the proof.

Step 1. Denote by N the (closed) set of umbilics of x on M. N is nonempty for an ovaloid. $S = S^\#$ implies $N = N^\#$. $M \setminus N$ is dense and open in M. We set $S = S^\# = \psi$ and adopt the notation from 2.1 - 2.6. Around any $p \in M \setminus N$ there exists a local chart U s.t. we have the matrix representations from 2.5; moreover, on U:

$$\partial_2 \lambda_1 = 0 = \partial_1 \lambda_2.$$

From $\omega(\mathrm{III}) = \omega(\mathrm{III}^\#)$ and $K = K^\#$ we get $\omega(g) = \omega(g^\#)$; together with (2.3.1) this gives $\det L = 1$. Thus $\lambda_1 \cdot \lambda_2 = 1$. Differentiation gives $\partial_1 \lambda_1 = 0 = \partial_2 \lambda_2$ on U, thus $\lambda_1 = const, \lambda_2 = const.$ on U and on any connected component of $M \setminus N$. As the eigenvalue functions λ_1, λ_2 are continuous on M and as N is nowhere dense λ_1, λ_2 must be positive constants on M.

Step 2. Assume that $\lambda_1 \neq \lambda_2$ on M. Then there exist two orthogonal eigenvectors e_1, e_2 at any point $p \in M$, and we get a pair of differentiable, nowhere vanishing tangent vector fields on M. But this contradicts the fact that the genus of M is zero.

Proof of Corollary 1.3. If x is a sphere then $x^\#$ must be a sphere of the same curvature and are therefore congruent up to a reparametrization. Otherwise the umbilics are isolated and the corollary follows from Theorem A. □

4 Ovaloids of revolution

In this section we want to prove the previously mentioned conjecture in the special case that x is an ovaloid of revolution. In particular we want to prove the following:

4.1. Theorem. *Let $x : M \to E^3$ be an ovaloid of revolution with nowhere dense umbilics and let $x^\# : M \to E^3$ be another ovaloid with $S = S^\#$ at corresponding points. Then $x, x^\#$ are congruent.*

Proof. Since $x : M \to E^3$ is an ovaloid of revolution we can write

$$x(u^1, u^2) = (r(u^1)\cos u^2, r(u^1)\sin u^2, s(u^1)), \tag{4.1.1}$$

where $0 \le u^1 \le \Lambda$ parametrizes the meridians as arc length parameter and $0 \le u^2 < 2\pi$ parametrizes the parallels of latitude with radius $r(u^1)$; thus we have

$$r(u^1) > 0 \text{ and } r'(u^1)^2 + s'(u^1)^2 = 1 \text{ for } 0 < u^1 < \Lambda\text{ , and } r(0) = r(\Lambda) = 0. \tag{4.1.2}$$

The two "poles" p_N and p_S ($u^1 = 0$ and $u^1 = \Lambda$) are umbilics.

It follows by a straightforward computation that $g = \mathrm{I}$ has the representation on $M\backslash\{p_N, p_S\}$:

$$g_{11} = 1,\ g_{12} = 0,\ g_{22} = r^2,$$

and r and s satisfy

$$S\partial_1 = (r's'' - r''s')\partial_1 \quad \text{and} \quad S\partial_2 = \tfrac{s'}{r}\partial_2.$$

Consider the two metrics g and $g^\#$ to be related as in (2.3.1). The curvature lines for x and $x^\#$ coincide on $M\backslash\{p_N, p_S\}$, on this set we get the following representation for $g^\# = \mathrm{I}^\#$:

$$g^\#_{11} = \lambda_1^2,\ g^\#_{12} = 0,\ g^\#_{22} = \lambda_2^2 r^2;$$

besides at the northpole p_N ($u^1 = 0$) and the southpole p_S ($u^1 = \Lambda$), we can write

$$L\partial_1 = \lambda_1\partial_1,\ L\partial_2 = \lambda_2\partial_2,$$

where the eigenvalue functions λ_1 and λ_2 are differentiable functions on $M\backslash\{p_N, p_S\}$ and continuous on M. The differentiability of λ_1, λ_2 implies that we can prove the PDE

$$\partial_2\lambda_1 = \partial_1\lambda_2 = 0 \tag{4.1.3}$$

on $M\backslash\{p_N, p_S\}$, similar to Proposition 2.6.

Now we look at what happens at p_N (and p_S). Suppose that L at p_N is not a multiple of the identity. Then there exists a unique differentiable orthonormal frame in a neighborhood of p_N such that

$$LX_1 = \lambda_1 X_1,\ LX_2 = \lambda_2 X_2.$$

Therefore, since L and S commute and the interior of the set of umbilic points is empty, it follows – if necessary after exchanging X_1 and X_2 – that in a neighborhood of p_N, except at p_N:

$$X_2 = (-\sin u^2, \cos u^2, 0).$$

Since the right hand side can not be differentiably extended to p_N, a contradiction follows. Thus $\lambda_1 = \lambda_2$ at p_N.

Again we use the continuity of λ_1 and λ_2 on M, thus $\lim_{u^1 \to 0} \lambda_1$ and $\lim_{u^1 \to \Lambda} \lambda_2$ exist. Recall the equations (4.1.3) on $M \backslash \{p_N, p_S\}$. Considering $\lim_{u^1 \to \Lambda} \lambda_2$, which exists and obviously must be independent of u^2, it follows that λ_2 is constant on the whole of M.

Now we introduce a function τ such that $\tau = \frac{1}{\lambda_1^2} - 1$. The condition $K = K^\#$ then implies that τ satisfies the following differential equation almost everywhere:

$$0 = r''\tau + \tfrac{1}{2} r' \tfrac{\partial \tau}{\partial u},$$

implying that there exist a constant c such that

$$(r')^2 \tau = c.$$

Since $r(0) = r(\Lambda) = 0$, it follows that c has to vanish. The fact that x is an ovaloid then implies that r cannot be constant on an open set and therefore that $\tau = 0$. Consequently $\lambda_1 = 1$. Since at p_N, $\lambda_1 = \lambda_2$, it follows that $L = id$ and therefore x and $x^\#$ are congruent. □

Remark. The condition that r' is somewhere zero is crucial in the proof of the previous theorem. Indeed, if we consider for example the paraboloid, parametrized by

$$x(u, v) = (u \cos v, u \sin v, \tfrac{1}{2} u^2),$$

we find, using the same technique as in the previous theorem, that all hyperbolioids of the one parameter family parametrized by

$$x_c(u, v) = \Big(\frac{u}{\sqrt{1+c}} \cos v, \frac{u}{\sqrt{1+c}} \sin v, \frac{\sqrt{(1+c)+cu^2}}{c\sqrt{1+c}} - \frac{1}{c}\Big),$$

where c is a nonzero real number satisfying $c > -1$, have the same shape operator given by

$$S\tfrac{\partial}{\partial v} = \frac{1}{\sqrt{1+u^2}} \tfrac{\partial}{\partial v}$$
$$S\tfrac{\partial}{\partial u} = \frac{1}{(1+u^2)^{\frac{3}{2}}} \tfrac{\partial}{\partial u}.$$

The above example also shows that the conjecture can not remain true for complete surfaces with positive Gaussian curvature.

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