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Cubic form geometry for surfaces in $\mathbb{S}^3(1)$

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Abstract. We consider the traceless part $\widetilde{C}$ of the difference tensor field C between the Levi-Civita connections of the first and the third fundamental forms for non-degenerate surface immersions in $\mathbb{S}^3(1)$. In analogy to affine differential geometry of $\mathbb{R}^{n+1}$ where quadrics are characterized by the vanishing of a traceless cubic form, we study the condition $\widetilde{C} \equiv 0$, give examples and classify non-degenerate surfaces in $\mathbb{S}^3(1)$ which satisfy this condition.

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1. Introduction

If $x\colon M^n \longrightarrow N^{n+1}$ is an n-dimensional hypersurface immersion into a space N^{n+1} of constant curvature which is non-degenerate, i.e. its shape operator S has maximal rank, then we can apply methods of affine differential geometry in the following way: Besides the first fundamental form I also the third fundamental form III is a Riemannian metric; they induce Levi-Civita connections ∇^{I} and ∇^{III}, respectively. Denoting the second fundamental form of x by II we obtain a conjugate triple $(\nabla^{\mathrm{I}}, \mathrm{II}, \nabla^{\mathrm{III}})$ (see below). The difference tensor field $C := \frac{1}{2}(\nabla^{\mathrm{I}} - \nabla^{\mathrm{III}})$ satisfies

$$2SC(u,v) = -(\nabla^{\mathrm{I}}_u S)v. \tag{1}$$

The associated tensor field

$$\widehat{C}\colon (u,v,w) \longmapsto \mathrm{II}(C(u,v),w) \tag{2}$$

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is the well known cubic form of the conjugate triple $(\nabla^{\mathrm{I}}, \mathrm{I\!I}, \nabla^{\mathrm{I\!I\!I}})$ which is totally symmetric. Because of (1) and (2) the following sequence of equivalences holds:

$$\widehat{C} \equiv 0 \Longleftrightarrow C \equiv 0 \Longleftrightarrow \nabla^{\mathrm{I}} S \equiv 0. \tag{3}$$

Furthermore, for $n = 2$, the parallelity of S is equivalent to the fact that x is an isoparametric surface.

In this paper we will investigate more general immersions by weakening condition (3). Namely, we study the case where only the traceless part $\widetilde{C}$ of C vanishes (for $\widetilde{C}$ see below). In the case that N^{n+1} is the Euclidean space $\mathbb{R}^{n+1}$ it is known that the condition $\widetilde{C} \equiv 0$ characterizes those non-degenerate immersions whose images lie on a quadric; see [6], p. 117-119, and the additional remark to the proof in [3], p. 208, (2.2.b). This result belonging to affine differential geometry is a generalization of the classical theorem of Maschke. Motivated by this result we ask for the geometric meaning of the condition $\widetilde{C} \equiv 0$ for other spaces N^{n+1} of constant curvature. It turns out that the situation is rather complicated. Thus we restrict our considerations to non-degenerate surfaces of the Euclidean sphere $\mathbb{S}^3(1)$ of radius 1. As mentioned above the isoparametric surface immersions, which are not totally geodesic are obviously in this class.

The purpose of this paper is to find further examples (see Example 3.7 and 3.6) and prove the following local classification:

Theorem 1.1. *Let $x\colon M^2 \longrightarrow \mathbb{S}^3(1) \subset \mathbb{R}^4$ be a non-degenerate surface immersion of a connected and orientable C^∞-manifold M^2 into $\mathbb{S}^3(1)$ satisfying the condition $\widetilde{C} \equiv 0$. Then there exist an open and dense subset U of M^2 and for every point $p \in U$ a neighbourhood $U(p) \subset U$, such that $x_{|U(p)}$ is one of the following types:*

(1) it is totally umbilical, i.e. it describes an open part of a small sphere in $\mathbb{S}^3(1)$,

(2) it is a part of a rotational surface with an arc of an ellipse or an arc of a hyperbola as profile curve (see Example 3.6 below),

(3) it is a part of a quadratic surface of the type described in Example 3.7 below.

It should be mentioned that (2) includes the isoparametric surfaces with two distinct principal curvatures; i.e. these are isometric to Clifford tori $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$ with $0 < r_1, r_2$ and $r_1^2 + r_2^2 = 1$.

From this theorem one can easily derive that the non-umbilical ($\widetilde{C} \equiv 0$)-surfaces lie on some quadrics of $\mathbb{R}^4$ centered at 0. In fact, in this way one can get another characterization of ($\widetilde{C} \equiv 0$)-surfaces, which will be presented in a forthcoming paper.

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2. Preliminaries

Let $x\colon M^2 \longrightarrow \mathbb{S}^3(1) \subset \mathbb{R}^4$ be an immersion of a connected, orientable 2-dimensional C^∞-manifold M^2 into $\mathbb{S}^3(1)$. Denote by y a unit normal vector field on $\mathbb{S}^3(1)$ along the immersion x, by $<,>$ the canonical inner product of the Euclidean structure, and by $\bar{\nabla}$ the flat connection of $\mathbb{R}^4$. To the immersion x are associated three fundamental forms, namely: I the first fundamental form (induced metric), II the second and III the third fundamental form related by

$$\mathrm{II}(u,v) = \mathrm{I}(Su,v), \qquad \mathrm{III}(u,v) = \mathrm{I}(Su,Sv), \tag{4}$$

where S is the Weingarten (shape) operator and $u, v \in \mathfrak{X}(M^2)$ (= the C^∞−module of vector fields on M^2). The immersion x is said to be non-degenerate or regular if the shape operator S has maximal rank. In this case, the second fundamental form II is a semi-Riemannian metric while the third fundamental form III is a Riemannian metric on M^2.
For all $u, v \in \mathfrak{X}(M^2)$, the structure (fundamental) equations of x as immersion into $\mathbb{R}^4$, namely the Gauß equation and the Weingarten equation, are given by:

$$\begin{aligned} \bar{\nabla}_u dx(v) &= dx(\nabla^{\mathrm{I}}_u v) + \mathrm{II}(u,v)y - \mathrm{I}(u,v)x; \\ dy(v) &= -dx(Sv). \end{aligned}$$

The structure equations above imply the following integrability conditions:

$$\begin{aligned} \left(\nabla^{\mathrm{I}}_u S\right) v &= \left(\nabla^{\mathrm{I}}_v S\right) u; \qquad (5)\\ R^{\mathrm{I}}(u,v)w &= \mathrm{I}(w,v)u - \mathrm{I}(u,w)v + \mathrm{II}(w,v)Su - \mathrm{II}(u,w)Sv, \qquad (6) \end{aligned}$$

where ∇^{I} and R^{I} denote the Levi-Civita connection and the Riemannian curvature tensor on M^2 of the first fundamental form, respectively.
Consequently, from the equation (4) and the Codazzi equation (5), one can check that the Levi-Civita connection ∇^{III} of the third fundamental form is given by:

$$\nabla^{\mathrm{III}}_u v = S^{-1}\nabla^{\mathrm{I}}_u(Sv). \tag{7}$$

Using (7), one can verify, for any $u, v, w \in \mathfrak{X}(M^2)$, that

$$w\mathrm{II}(u,v) = \mathrm{II}(\nabla^{\mathrm{I}}_w u, v) + \mathrm{II}(u, \nabla^{\mathrm{III}}_w v);$$

in other words: the triple $(\nabla^{\mathrm{I}}, \mathrm{II}, \nabla^{\mathrm{III}})$ is conjugate. Therefore the Levi-Civita connection ∇^{II} of the second fundamental form satisfies

$$\nabla^{\mathrm{II}} = \frac{1}{2}(\nabla^{\mathrm{I}} + \nabla^{\mathrm{III}}).$$

The difference tensor C defined by

$$C(u,v) := \frac{1}{2}(\nabla^{\mathrm{I}}_u v - \nabla^{\mathrm{III}}_u v) = -\frac{1}{2}S^{-1}(\nabla^{\mathrm{I}}_u S)v \tag{8}$$

is a symmetric $(1,2)$-tensor field and satisfies

$$\nabla^{\mathrm{I}} = \nabla^{\mathrm{II}} + C, \qquad \nabla^{\mathrm{III}} = \nabla^{\mathrm{II}} - C.$$

The Tchebychev vector field T is defined by: $\mathrm{II}(u,T) := \frac{1}{2}\mathrm{tr}[v \longmapsto C(u,v)]$. And finally, the $(1,2)$-tensor field $\widetilde{C}$, defined by

$$\widetilde{C}(u,v) := C(u,v) - \frac{1}{2}[\mathrm{II}(T,v)u + \mathrm{II}(T,u)v + \mathrm{II}(u,v)T]$$

is symmetric and traceless with respect to both variables. It is called the traceless part of C.

Proposition 2.1. *Denote by $K := \det S$ the Gauß-Kronecker curvature function of the non-degenerate immersion $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$. Then the Tchebychev vector field T satisfies*

$$T = -\frac{1}{4}S^{-1}\left(\mathrm{grad}^{\mathrm{I}} \ln |K|\right),$$

where $\mathrm{grad}^{\mathrm{I}}$ *is the gradient with respect to the first fundamental form.*

Proof. The Tchebychev vector field T satisfies [1]

$$T = -\frac{1}{4}\mathrm{grad}^{\mathrm{II}} \ln |K|,$$

where $\mathrm{grad}^{\mathrm{II}}$ is the gradient with respect to the second fundamental form. One has:

$$S(T) = -\frac{1}{4}S\left(\mathrm{grad}^{\mathrm{II}} \ln |K|\right) = -\frac{1}{4}\mathrm{grad}^{\mathrm{I}} \ln |K|.$$

□

3. The Equation $\widetilde{C} = 0$

The equation $C = 0$ implies the equation $\widetilde{C} = 0$, but both equations obviously are not equivalent. As mentioned in the introduction, an immersion $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$ satisfying the condition $C = 0$ is isoparametric, that means that $x(M^2)$ is an open part of a 2-dimensional subsphere or of a torus of $\mathbb{S}^3(1)$. In contrast, for $\widetilde{C} = 0$ the situation is much more difficult.
Assume from now on that the regular immersion $x\colon M^2 \longrightarrow \mathbb{S}^3(1) \subset \mathbb{R}^4$ has no umbilical points. Denote by λ_1 and λ_2 the two distinct principal curvature functions of the immersion x. Let (e_1, e_2) be a local I-orthonormal frame of principal vector fields on an open subset U of M^2: $Se_1 = \lambda_1 e_1$ and $Se_2 = \lambda_2 e_2$. There exist two differentiable functions $\alpha, \beta \in C^\infty(U)$ such that:

$$\nabla^{\mathrm{I}}_{e_1} e_1 = \alpha e_2, \quad \nabla^{\mathrm{I}}_{e_2} e_2 = \beta e_1.$$

Because of $\nabla^{\mathrm{I}}\mathrm{I} = 0, \mathrm{I}(e_1, e_1) = 1 = \mathrm{I}(e_2, e_2)$ and $\mathrm{I}(e_1, e_2) = 0$, one also has:

$$\nabla^{\mathrm{I}}_{e_1} e_2 = -\alpha e_1, \quad \nabla^{\mathrm{I}}_{e_2} e_1 = -\beta e_2.$$

So the Lie bracket of e_1 and e_2 is:

$$[e_1, e_2] = -\alpha e_1 + \beta e_2.$$

With respect to the frame (e_1, e_2), the structure equations of x as immersion into $\mathbb{R}^4$ are:

$$\begin{cases} \bar{\nabla}_{e_1} dx(e_1) = \alpha dx(e_2) + \lambda_1 y - x, \\ \bar{\nabla}_{e_1} dx(e_2) = -\alpha dx(e_1), \\ \bar{\nabla}_{e_2} dx(e_1) = -\beta dx(e_2), \\ \bar{\nabla}_{e_2} dx(e_2) = \beta dx(e_1) + \lambda_2 y - x, \\ dy(e_1) = -\lambda_1 dx(e_1); \qquad dy(e_2) = -\lambda_2 dx(e_2). \end{cases} \tag{9}$$

Proposition 3.1. *The functions α, β, λ_1 and λ_2 satisfy the following first order partial differential equations:*

$$e_1(\beta) + e_2(\alpha) = \alpha^2 + \beta^2 + 1 + \lambda_1\lambda_2, \tag{10}$$

$$e_1(\lambda_2) = \beta(\lambda_2 - \lambda_1), \tag{11}$$

$$e_2(\lambda_1) = \alpha(\lambda_1 - \lambda_2). \tag{12}$$

Proof. From the integrability condition (6), one has:

$$R^{\mathrm{I}}(e_1, e_2)e_2 = e_1 + \lambda_1\lambda_2 e_1 = (1 + \lambda_1\lambda_2)e_1.$$

Moreover,

$$\begin{aligned} R^{\mathrm{I}}(e_1, e_2)e_2 &= \nabla^{\mathrm{I}}_{e_1}\nabla^{\mathrm{I}}_{e_2}e_2 - \nabla^{\mathrm{I}}_{e_2}\nabla^{\mathrm{I}}_{e_1}e_2 - \nabla^{\mathrm{I}}_{[e_1,e_2]}e_2 \\ &= \nabla^{\mathrm{I}}_{e_1}(\beta e_1) - \nabla^{\mathrm{I}}_{e_2}(-\alpha e_1) - \nabla^{\mathrm{I}}_{-\alpha e_1 + \beta e_2}e_2 \\ &= \left(e_1(\beta) + e_2(\alpha) - \alpha^2 - \beta^2\right) e_1. \end{aligned}$$

So $1 + \lambda_1\lambda_2 = e_1(\beta) + e_2(\alpha) - \alpha^2 - \beta^2$.

From the Codazzi equation (5), one has:

$$\begin{aligned} 0 &= (\nabla^{\mathrm{I}}_{e_1}S)e_2 - (\nabla^{\mathrm{I}}_{e_2}S)e_1 \\ &= \nabla^{\mathrm{I}}_{e_1}Se_2 - S\nabla^{\mathrm{I}}_{e_1}e_2 - \nabla^{\mathrm{I}}_{e_2}Se_1 + S\nabla^{\mathrm{I}}_{e_2}e_1 \\ &= \nabla^{\mathrm{I}}_{e_1}(\lambda_2 e_2) - S(-\alpha e_1) - \nabla^{\mathrm{I}}_{e_2}(\lambda_1 e_1) + S(-\beta e_2) \\ &= e_1(\lambda_2)e_2 - \alpha\lambda_2 e_1 + \alpha\lambda_1 e_1 = -e_2(\lambda_1)e_1 + \beta\lambda_1 e_2 - \beta\lambda_2 e_2 \\ &= \left(\alpha(\lambda_1 - \lambda_2) - e_2(\lambda_1)\right)e_1 + \left(e_1(\lambda_2) - \beta(\lambda_2 - \lambda_1)\right)e_2. \end{aligned}$$

So $e_1(\lambda_2) = \beta(\lambda_2 - \lambda_1)$ and $e_2(\lambda_1) = \alpha(\lambda_1 - \lambda_2)$. □

Proposition 3.2. *With respect to the frame (e_1, e_2), the components of the tensor field C and the Tchebychev vector field T are given by:*

$$\begin{aligned} C^1_{11} &= -\frac{1}{2}\lambda_1^{-1}e_1(\lambda_1), & C^2_{11} &= -\frac{1}{2}\lambda_2^{-1}\alpha(\lambda_1 - \lambda_2), \\ C^1_{22} &= -\frac{1}{2}\lambda_1^{-1}\beta(\lambda_2 - \lambda_1), & C^2_{22} &= -\frac{1}{2}\lambda_2^{-1}e_2(\lambda_2), \\ C^1_{12} &= -\frac{1}{2}\lambda_1^{-1}\alpha(\lambda_1 - \lambda_2), & C^2_{12} &= -\frac{1}{2}\lambda_2^{-1}\beta(\lambda_2 - \lambda_1); \end{aligned} \tag{13}$$

$$T = -\frac{1}{4}[(\lambda_1^{-2}e_1(\lambda_1) + \lambda_1^{-1}\lambda_2^{-1}\beta(\lambda_2 - \lambda_1))e_1 + (\lambda_1^{-1}\lambda_2^{-1}\alpha(\lambda_1 - \lambda_2) + \lambda_2^{-2}e_2(\lambda_2))e_2]. \tag{14}$$

Proof. Using the relation (8) between C, S and ∇^{I}, one has:

$$C(e_1, e_1) = -\frac{1}{2}S^{-1}(\nabla^{\mathrm{I}}_{e_1} S)e_1 = -\frac{1}{2}(\lambda_1^{-1} e_1(\lambda_1)e_1 + \alpha\lambda_2^{-1}(\lambda_1 - \lambda_2)e_2).$$

Similarly,

$$\begin{aligned} C(e_2, e_2) = -\frac{1}{2}S^{-1}(\nabla^{\mathrm{I}}_{e_2} S)e_2 &= -\frac{1}{2}(\lambda_2^{-1} e_2(\lambda_2)e_2 + \beta\lambda_1^{-1}(\lambda_2 - \lambda_1)e_1), \\ C(e_1, e_2) = -\frac{1}{2}S^{-1}(\nabla^{\mathrm{I}}_{e_1} S)e_2 &= -\frac{1}{2}(\beta\lambda_2^{-1}(\lambda_2 - \lambda_1)e_2 + \alpha\lambda_1^{-1}(\lambda_1 - \lambda_2)e_1). \end{aligned}$$

For the Tchebychev vector field we have: $-\mathrm{grad}^{\mathrm{I}} \ln|\lambda_1\lambda_2| = 4T_1e_1 + 4T_2e_2$, where

$$\begin{aligned} 2T_1 = C^1_{11} + C^2_{12} &= -\frac{1}{2}\lambda_1^{-1} e_1(\lambda_1) - \frac{1}{2}\lambda_2^{-1}\beta(\lambda_2 - \lambda_1), \\ 2T_2 = C^1_{12} + C^2_{22} &= -\frac{1}{2}\lambda_2^{-1} e_2(\lambda_2) - \frac{1}{2}\lambda_1^{-1}\alpha(\lambda_1 - \lambda_2). \end{aligned}$$

So from Proposition 2.1,

$$\begin{aligned} T &= -\frac{1}{4}S^{-1}\left(\mathrm{grad}^{\mathrm{I}} \ln|\lambda_1\lambda_2|\right) \\ &= -\frac{1}{4}[(\lambda_1^{-2} e_1(\lambda_1) + \lambda_1^{-1}\lambda_2^{-1}\beta(\lambda_2 - \lambda_1))e_1 + (\lambda_1^{-1}\lambda_2^{-1}\alpha(\lambda_1 - \lambda_2) + \lambda_2^{-2} e_2(\lambda_2))e_2]. \end{aligned}$$

□

Proposition 3.3. *The condition $\widetilde{C} \equiv 0$ is equivalent to the following equations:*

$$e_1(\lambda_1) = 3\beta\lambda_1\lambda_2^{-1}(\lambda_2 - \lambda_1), \tag{15}$$

$$e_2(\lambda_2) = 3\alpha\lambda_2\lambda_1^{-1}(\lambda_1 - \lambda_2). \tag{16}$$

Proof. The immersion x satisfies the condition $\widetilde{C} \equiv 0$ if and only if

$$\begin{aligned} C^1_{11} &= \frac{1}{2}[T_1 + T_1 + T^1\mathrm{II}_{11}] = \frac{3}{2}T_1, \quad C^1_{12} = \frac{1}{2}T_2, \\ C^2_{11} &= \frac{1}{2}[T^2\mathrm{II}_{11}] = \frac{1}{2}\lambda_1\lambda_2^{-1}T_2, \\ C^1_{22} &= \frac{1}{2}[T^1\mathrm{II}_{22}] = \frac{1}{2}\lambda_2\lambda_1^{-1}T_1, \\ C^2_{22} &= \frac{1}{2}[T_2 + T_2 + T^2\mathrm{II}_{22}] = \frac{3}{2}T_2, \quad C^2_{12} = \frac{1}{2}T_1. \end{aligned}$$

So, $T_1 = 2C^2_{12} = -\beta\lambda_2^{-1}(\lambda_2 - \lambda_1)$, and then $C^1_{11} = \frac{3}{2}T_1$ and $C^1_{11} = -\frac{1}{2}\lambda_1^{-1}e_1(\lambda_1)$ implies the equation (15). For the equation (16), use $T_2 = 2C^1_{12} = -\alpha\lambda_1^{-1}(\lambda_1 - \lambda_2)$, $C^2_{22} = \frac{3}{2}T_2$, and $C^2_{22} = -\frac{1}{2}\lambda_2^{-1}e_2(\lambda_2)$. The converse is obvious. □

Corollary 3.4. *The condition $\widetilde{C} \equiv 0$ implies the following equations:*

$$\begin{aligned} e_1(\alpha) &= 3\alpha\beta, & (17)\\ e_2(\beta) &= 3\alpha\beta. & (18) \end{aligned}$$

Proof. From (11), (12), (15) and (16), one has:

$$\begin{aligned} 0 &= e_1e_2(\lambda_1) - e_2e_1(\lambda_1) - [e_1, e_2](\lambda_1) \\ &= (\lambda_2 - \lambda_1)\Big(-e_1(\alpha) - 3\lambda_1\lambda_2^{-1}e_2(\beta) + 3\alpha\beta(3\lambda_1\lambda_2^{-1} + 1)\Big) \end{aligned}$$

$$\begin{aligned} 0 &= e_1e_2(\lambda_2) - e_2e_1(\lambda_2) - [e_1, e_2](\lambda_2) \\ &= (\lambda_1 - \lambda_2)\Big(-e_2(\beta) - 3\lambda_2\lambda_1^{-1}e_1(\alpha) + 3\alpha\beta(3\lambda_2\lambda_1^{-1} + 1)\Big). \end{aligned}$$

So

$$\begin{aligned} 0 &= -e_1(\alpha) - 3\lambda_1\lambda_2^{-1}e_2(\beta) + 3\alpha\beta(3\lambda_1\lambda_2^{-1} + 1) & (19)\\ 0 &= -e_1(\beta) - 3\lambda_2\lambda_1^{-1}e_1(\alpha) + 3\alpha\beta(3\lambda_2\lambda_1^{-1} + 1). & (20) \end{aligned}$$

From the equations (19) and (20), one gets $e_1(\alpha) = 3\alpha\beta = e_2(\beta)$. □

Example 3.5. Open parts of small spheres in $\mathbb{S}^3(1)$ are trivial examples of immersions which satisfy the condition $\widetilde{C} \equiv 0$.

Example 3.6. (Surfaces of revolution with arcs of ellipses or arcs of hyperbolas as profile curves)
Let $\varepsilon \in \{-1, 1\}$. Define the functions $\cos_\varepsilon$ and $\sin_\varepsilon$ by:

$$\cos_\varepsilon := \begin{cases} \cos & \text{if } \ \varepsilon = 1 \\ \cosh & \text{if } \ \varepsilon = -1 \end{cases} \quad \text{and} \quad \sin_\varepsilon := \begin{cases} \sin & \text{if } \ \varepsilon = 1 \\ \sinh & \text{if } \ \varepsilon = -1. \end{cases} \tag{21}$$

Let (C_1, C_2, C_3, C_4) be an orthonormal basis of $\mathbb{R}^4$, $0 < a < 1, 0 < b$ two constant reals such that $b \neq 1$ if $\varepsilon = 1$, and $I \subset \mathbb{R}$ be a non-empty open interval such that the function r defined by

$$r(u) = \sqrt{1 - a^2\cos_\varepsilon^2(u) - b^2\sin_\varepsilon^2(u)}$$

is real and positive on I. Then the mapping $x\colon I \times \mathbb{R} \longrightarrow \mathbb{S}^3(1) \subset \mathbb{R}^4$:

$$x(u, v) = r(u)\cdot\big(\cos(v)C_1 + \sin(v)C_2\big) + A(u), \tag{22}$$

with

$$A(u) = a\cos_\varepsilon(u)C_3 + b\sin_\varepsilon(u)C_4,$$

is a $(\widetilde{C} \equiv 0)$-surface in $\mathbb{S}^3(1)$. We call x a surface of revolution with the profile curve A, which is in our cases an arc of an ellipse ($\varepsilon = 1$) or of a hyperbola ($\varepsilon = -1$). In the case $\varepsilon = 1$ and $a = b$, x describes an isoparametric surface with two distinct principal curvatures.

In order to show that the condition $\widetilde{C} \equiv 0$ is satisfied we compute

$$\begin{aligned} x_u &= r'(u) \cdot \big(\cos(v)C_1 + \sin(v)C_2\big) - a\sin_\varepsilon(u)C_3 + b\cos_\varepsilon(u)C_4, \\ x_v &= r(u) \cdot \big(-\sin(v)C_1 + \cos(v)C_2\big). \end{aligned}$$

Defining the function σ on I by $\sigma(u) := \sqrt{a^2 \sin_\varepsilon^2(u) + b^2 \cos_\varepsilon^2(u) - a^2 b^2}$, one has:

$$\begin{aligned} \mathrm{I}_{11} =< x_u, x_u > &= \frac{\sigma^2(u)}{r^2(u)}, \\ \mathrm{I}_{22} =< x_v, x_v > &= r^2(u), \qquad \mathrm{I}_{12} =< x_u, x_v >= 0. \end{aligned}$$

Since $\det(\mathrm{I}) = \mathrm{I}_{11}\mathrm{I}_{22} \neq 0$ for every $u \in I$, the mapping x defines an immersion. The vector field

$$y(u,v) = \frac{abr(u)}{\sigma(u)} \cdot \big(\cos(v)C_1 + \sin(v)C_2\big) + b(a^2-1)\frac{\cos_\varepsilon(u)}{\sigma(u)}C_3 + a(b^2-\varepsilon)\frac{\sin_\varepsilon(u)}{\sigma(u)}C_4, \quad (23)$$

is a unit normal vector field. As $< y_u, x_v >= 0 =< y_v, x_u >$, the coordinate lines are lines of curvatures; therefore the principal curvature functions are

$$\lambda_1 = -\frac{< y_u, x_u >}{< x_u, x_u >} = \frac{ab(1-a^2)(\varepsilon - b^2)}{\sigma^3}, \qquad \lambda_2 = -\frac{< y_v, x_v >}{< x_v, x_v >} = -\frac{ab}{\sigma}.$$

The principal curvature functions λ_1 and λ_2 above do not have zeros on I, i.e. the immersion x is non-degenerate. And the vector fields $e_1 = \frac{r}{\sigma}\frac{\partial}{\partial u}$ and $e_2 = \frac{1}{r}\frac{\partial}{\partial v}$ are orthonormal principal vector fields. They satisfy

$$\nabla^{\mathrm{I}}_{e_1} e_1 = 0 \quad \text{and} \quad \nabla^{\mathrm{I}}_{e_2} e_2 = \beta e_1,$$

where $\beta(u) = -\frac{r'(u)}{\sigma(u)}$. Moreover, one has:

$$e_2(\lambda_2) = 0 \quad \text{and} \quad e_1(\lambda_1) = \frac{3\beta(u)\lambda_1(\lambda_2 - \lambda_1)}{\lambda_2},$$

i.e. the equations (15) and (16) hold. Thus the immersion x satisfies $\widetilde{C} \equiv 0$.

Note finally that the surface (22) has no umbilical points since everywhere

$$\lambda_1 - \lambda_2 = \frac{\varepsilon abr^2}{\sigma^3} \neq 0.$$

Example 3.7. Let $\eta \in \{-1, 1\}$, $a_1, a_2, a_3, a_4 \in \mathbb{R}$ be distinct real numbers such that

$$a_1 < -\eta a_2 < 0 < a_3 < a_4, \quad a_2 < a_3 \quad \text{and} \quad a_1a_2a_3a_4 = -\eta;$$

and $C_1, C_2, C_3, C_4 \in \mathbb{R}^4$ be constant orthogonal vectors in $\mathbb{R}^4$ such that

$$\begin{aligned} < C_1, C_1 > &= \frac{a_1}{(a_1-a_2)(a_1-a_3)(a_1-a_4)}, \\ < C_2, C_2 > &= \frac{\eta a_2}{(a_2-a_3)(a_2-a_4)(a_2-a_1)}, \\ < C_3, C_3 > &= \frac{a_3}{(a_3-a_2)(a_4-a_3)(a_3-a_1)}, \\ < C_4, C_4 > &= \frac{a_4}{(a_4-a_2)(a_4-a_3)(a_4-a_1)}. \end{aligned}$$

Define the rectangle $R \subset \mathbb{R}^2$ by

$$R := \begin{cases}]a_3, a_4[\times]a_2, a_3[& \text{if} \quad \eta = 1 \\ \\ ]a_3, a_4[\times] - a_2, -a_1[& \text{if} \quad \eta = -1. \end{cases}$$

Then the mapping $x\colon R \longrightarrow \mathbb{S}^3(1) \subset \mathbb{R}^4$ defined by

$$\begin{aligned} x(u,v) \quad :=& \quad \sqrt{(u-a_1)(\eta v - a_1)}C_1 + \sqrt{(u-a_2)(v-\eta a_2)}C_2 + \sqrt{(u-a_3)(a_3-\eta v)}C_3 \\ & + \sqrt{(a_4-u)(a_4-\eta v)}C_4, \end{aligned} \tag{24}$$

is a $(\widetilde{C} \equiv 0)$-surface in $\mathbb{S}^3(1)$.
In fact,

$$\begin{aligned} x_u \quad &= \quad \frac{1}{2}\sqrt{\frac{\eta v - a_1}{u-a_1}}C_1 + \frac{1}{2}\sqrt{\frac{v-\eta a_2}{u-a_2}}C_2 + \frac{1}{2}\sqrt{\frac{a_3-\eta v}{u-a_3}}C_3 - \frac{1}{2}\sqrt{\frac{a_4-\eta v}{a_4-u}}C_4, \\ x_v \quad &= \quad \frac{\eta}{2}\sqrt{\frac{u-a_1}{\eta v - a_1}}C_1 + \frac{1}{2}\sqrt{\frac{u-a_2}{v-\eta a_2}}C_2 - \frac{\eta}{2}\sqrt{\frac{u-a_3}{a_3-\eta v}}C_3 - \frac{\eta}{2}\sqrt{\frac{a_4-u}{a_4-\eta v}}C_4. \end{aligned}$$

Since

$$\begin{aligned} \mathrm{I}_{12} = < x_u, x_v > \quad &= \quad 0, \\ \mathrm{I}_{11} = < x_u, x_u > \quad &= \quad -\frac{u(u-\eta v)}{4(u-a_1)(u-a_2)(u-a_3)(u-a_4)}, \\ \mathrm{I}_{22} = < x_v, x_v > \quad &= \quad \frac{\eta v(u-\eta v)}{4(v-\eta a_1)(v-\eta a_2)(v-\eta a_3)(v-\eta a_4)}, \end{aligned}$$

the mapping x defines an immersion. The vector field

$$\begin{aligned} y(u,v) \quad =& \quad -\eta\sqrt{\frac{(u-a_1)(\eta v - a_1)}{uv}}\frac{C_1}{a_1} - \eta\sqrt{\frac{(u-a_2)(v-\eta a_2)}{uv}}\frac{C_2}{a_2} - \eta\sqrt{\frac{(u-a_3)(a_3-\eta v)}{uv}}\frac{C_3}{a_3} \\ & -\eta\sqrt{\frac{(a_4-u)(a_4-\eta v)}{uv}}\frac{C_4}{a_4} \end{aligned} \tag{25}$$

is a unit normal vector field; it satisfies

$$y_u = -\frac{\eta}{\sqrt{u^3 v}}x_u, \qquad y_v = -\frac{1}{\sqrt{v^3 v}}x_v;$$

therefore the principal curvature functions of the immersion x are $\lambda_1(u,v) = \frac{\eta}{\sqrt{u^3 v}} \neq 0$ and $\lambda_2(u,v) = \frac{1}{\sqrt{v^3 u}} \neq 0$, thus x is non-degenerate. The vector fields $e_1 = \frac{1}{\sqrt{\mathrm{I}_{11}}}\frac{\partial}{\partial u}$ and $e_2 = \frac{1}{\sqrt{\mathrm{I}_{22}}}\frac{\partial}{\partial v}$ are orthonormal principal vector fields such that

$$\nabla^{\mathrm{I}}_{e_1} e_1 = \alpha e_2 \qquad \text{and} \qquad \nabla^{\mathrm{I}}_{e_2} e_2 = \beta e_1,$$

where α and β are functions on R given by

$$\begin{aligned} \alpha(u,v) &= \eta\sqrt{\frac{(v-\eta a_1)(v-\eta a_2)(v-\eta a_3)(v-\eta a_4)}{v(\eta u - v)^3}}, \\ \beta(u,v) &= -\sqrt{\frac{-(u-a_1)(u-a_2)(u-a_3)(u-a_4)}{u(u-\eta v)^3}}. \end{aligned}$$

Moreover the following equalities hold:

$$\frac{1}{\sqrt{\mathrm{I}_{11}}}\partial_u(\lambda_1) = \frac{3\beta\lambda_1(\lambda_2-\lambda_1)}{\lambda_2} \qquad \text{and} \qquad \frac{1}{\sqrt{\mathrm{I}_{22}}}\partial_v(\lambda_2) = \frac{3\alpha\lambda_2(\lambda_1-\lambda_2)}{\lambda_1},$$

i.e. the equations (15) and (16) are satisfied, which means that the regular immersion x satisfies the condition $\widetilde{C} \equiv 0$.

4. Classification of $(\widetilde{C} \equiv 0)$-surfaces in $\mathbb{S}^3(1)$

In this section we want to obtain a local description of the surface immersions in $\mathbb{S}^3(1)$ satisfying the condition $\widetilde{C} \equiv 0$. Let W be the open set of non-umbilical points on M^2. Assuming the situation described at the beginning of Section 3 we construct open subsets U_1, U_2 and U_3 of M^2 by

$$\begin{cases} U_1 := (M^2 \setminus W)^o, \\ U_3 := \{p \in W : \alpha(p) \neq 0 \neq \beta(p)\}, \\ U_2 := (W \setminus U_3)^o, \end{cases}$$

where P^o is the topological interior of P in M^2. Consider the disjoint union

$$U := U_1 \cup U_2 \cup U_3.$$

Obviously, U is an open dense subset of M^2. We will show in the following that on each of the subsets U_i the immersion x is locally of the type (i) described in Theorem 1.1. For $i = 1$ this is obvious.

The I-orthonormal frame (e_1, e_2) of principal vector fields is unique up to signs and to changes of the order of the vector fields e_1 and e_2. After the choice of (e_1, e_2) the functions α and β are uniquely determined. Consequently, the subsets U_1, U_2 and U_3 above are well defined.

4.1. Rotational $(\widetilde{C} \equiv 0)$-surface immersions in $\mathbb{S}^3(1)$

Assume that U_2 is non-empty. On each connected component of U_2, at least one of the functions α and β vanishes everywhere. Fix then $\alpha = 0$ (the case $\beta = 0$ is similar because of symmetries) everywhere on a connected component.

Lemma 4.1. *Let $x\colon (M^2, \mathrm{I}) \longrightarrow \mathbb{S}^3(1) \subset \mathbb{R}^4$ be a non-degenerate isometric immersion without umbilical points. If there exists an* I*-orthonormal frame (e_1, e_2) of principal vector fields satisfying*

$$\nabla^{\mathrm{I}}_{e_1} e_1 = 0 \quad \textit{and} \quad e_2(\lambda_2) = 0,$$

where λ_2 denotes the principal curvature function corresponding to the direction e_2, then for every point $p \in M^2$ there exists local coordinates (u, v) and an orthonormal basis C_1, C_2, C_3, C_4 of $\mathbb{R}^4$, such that x is described by the following parametrization of a surface of revolution

$$x(u, v) = r(u) \cdot \big(\cos(v) C_1 + \sin(v) C_2\big) + A(u),$$

where $A(u)$ is a curve in Span$\{C_3, C_4\}$ and r a differentiable function satisfying

$$r^2 + < A, A > \equiv 1. \tag{26}$$

Proof. From (10), (11) and (12) with $\alpha = 0$ on M^2, one has:

$$\begin{aligned} e_1(\beta) &= \beta^2 + 1 + \lambda_1\lambda_2, \\ e_1(\lambda_2) &= \beta(\lambda_2 - \lambda_1), \\ e_2(\lambda_1) &= 0 = e_2(\lambda_2). \end{aligned} \tag{27}$$

Furthermore,

$$\begin{aligned} 0 &= -\alpha e_1(\lambda_2) + \beta e_2(\lambda_2) \\ &= [e_1, e_2](\lambda_2) \\ &= e_1 e_2(\lambda_2) - e_2 e_1(\lambda_2) \\ &= -(\lambda_2 - \lambda_1) e_2(\beta); \end{aligned}$$

therefore the assumption of non-umbilicity on M^2 implies

$$e_2(\beta) = 0.$$

Define the function r on M^2 by:

$$r := \frac{1}{\sqrt{\beta^2 + 1 + \lambda_2^2}} \tag{28}$$

Using the equations (27), one has that the function r satisfies

$$e_2(r) = 0 \quad \text{and} \quad e_1(r) = -r\beta.$$

It follows that

$$[e_1, r e_2] = 0.$$

Consequently, there is a system of local coordinates (u, v) on U such that

$$e_1 = \frac{\partial}{\partial u} \quad \text{and} \quad e_2 = r^{-1} \frac{\partial}{\partial v}.$$

This means that the manifold M^2 locally is a warped product, with r as the warping function. The functions λ_1, λ_2, β and r depend only on u and satisfy the following system of first order ordinary differential equations, where we use the prime-notation $'$ to indicate the derivatives with respect to u:

$$\begin{cases} \beta' = \beta^2 + 1 + \lambda_1\lambda_2; \\ r' = -r\beta; \\ \lambda_2' = \beta(\lambda_2 - \lambda_1). \end{cases} \tag{29}$$

With respect to the frame $(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$, the structure equations are written as follows:

$$\begin{aligned} x_{uu} &= \lambda_1 y - x, && (30)\\ x_{uv} &= -\beta x_v = r' r^{-1} x_v = x_{vu}, && (31)\\ x_{vv} &= r^2\left(\beta x_u + \lambda_2 y - x\right), && (32)\\ y_v &= -\lambda_2 x_v; \qquad y_u = -\lambda_1 x_u. && (33) \end{aligned}$$

Differentiating (32) with respect to v, the immersion $x(u,v)$ then satisfies the following equation:

$$x_{vvv} = -r^2(\beta^2 + 1 + \lambda_2^2)x_v = -x_v;$$

for the last equality we use (28). Therefore there are vector valued functions A_1 and A_2 on U depending only on u such that

$$x_v = -A_1 \sin(v) + A_2 \cos(v). \tag{34}$$

Because of the linear independence of the functions 1, $\sin(2v)$ and $\cos(2v)$, and the fact that $< x_v, x_v >= r^2$, the vector valued functions A_1 and A_2 satisfy the following conditions:

$$< A_1(u), A_1(u) >=< A_2(u), A_2(u) >= r^2, \qquad < A_1(u), A_2(u) >= 0.$$

Differentiate the equation (34) with respect to u and use (31) to get

$$-A_1'(u)\sin(v) + A_2'(u)\cos(v) = \frac{r'(u)}{r(u)}[-A_1(u)\sin(v) + A_2(u)\cos(v)].$$

Using the fact that the functions $\cos(v)$ and $\sin(v)$ are linearly independent, we see that the vector valued functions A_1, A_2 which depend only on u satisfy the following equations:

$$A_1'(u) = \frac{r'(u)}{r(u)} A_1(u); \qquad A_2'(u) = \frac{r'(u)}{r(u)} A_2(u).$$

Hence there are constant vectors $C_1, C_2 \in \mathbb{R}^4$ such that $A_1(u) = r(u)C_1$, $A_2(u) = r(u)C_2$. Because of $< A_1(u), A_1(u) >= r^2(u) =< A_2(u), A_2(u) >$ and $< A_1(u), A_2(u) >= 0$, C_1 and C_2 are constant orthonormal vectors in $\mathbb{R}^4$.
For some vector valued function A depending only on u, the immersion x takes the form:

$$x(u,v) \equiv r(u) \cdot [C_1 \cos(v) + C_2 \sin(v)] + A(u). \tag{35}$$

Since $< x(u,v), x(u,v) >= 1$, one has:

$$\begin{aligned} 1 &= < x(u,v), x(u,v) > \\ &= < A(u), A(u) > + r^2(u) + 2r(u)\left[\cos(v) < C_1, A(u) > + \sin(v) < C_2, A(u) >\right]. \end{aligned}$$

By the linear independence of the functions 1, $\cos(v)$ and $\sin(v)$, the vector valued function A which depends only on u, satisfies $< C_1, A(u) >= 0 =< C_2, A(u) >$, and the condition (26) holds.

□

Remark 4.2. Another way to prove Lemma 4.1 is the following: by considering $\alpha = 0$, the λ_1-curvature lines are geodesics of M^2, and the λ_2-curvature lines are spherical bent in M^2 (see [5]). According to a result of Hiepko (see [2]), the manifold M^2 is then locally a warped product. The orthogonality of the vectors e_1 and e_2 with respect to the second fundamental form is equivalent to Nölker's Condition (D) in [4], p. 21 for an immersion of a warped product in a space of constant curvature to be a warped product of isometric immersions. So, in our situation, the case $\alpha = 0$ leads to rotational surfaces.

Lemma 4.3. *Let $(V, <,>)$ be a 2-dimensional Euclidean vector space and (w_1, w_2) a basis of V. For $\varepsilon \in \{-1, 1\}$, let $\cos_\varepsilon$ and $\sin_\varepsilon$ be the functions as defined by (21). Then there exist an orthonormal basis (v_1, v_2) of V, $s \in \mathbb{R}$ and $a, b \in \mathbb{R}_+$ such that*

$$\forall t \in \mathbb{R}\colon \ \cos_\varepsilon(t) w_1 + \sin_\varepsilon(t) w_2 = a\cos_\varepsilon(t-s) v_1 + b\sin_\varepsilon(t-s) v_2. \tag{36}$$

Proof. Let $A_\varepsilon\colon \mathbb{R} \longrightarrow V$ be the curve defined by:

$$A_\varepsilon(t) := \cos_\varepsilon(t) w_1 + \sin_\varepsilon(t) w_2,$$

and s a point where the real function $f_\varepsilon\colon t \longmapsto < A_\varepsilon(t), A_\varepsilon(t) >$ attains a minimum. One has:

$$0 = \frac{1}{2} f'_\varepsilon(s) =< A_\varepsilon(s), A'_\varepsilon(s) > .$$

So the vectors $v_1 := \frac{1}{\|A_\varepsilon(s)\|} A_\varepsilon(s)$ and $v_2 := \frac{1}{\|A'_\varepsilon(s)\|} A'_\varepsilon(s)$ constitute an orthonormal basis of V. By setting $a := \|A_\varepsilon(s)\|$ and $b := \|A'_\varepsilon(s)\|$, one verifies that the equality (36) holds. □

Theorem 4.4. *Let $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$ be a non-degenerate immersion without umbilical points of a connected and orientable C^∞-manifold M^2 of dimension 2 into $\mathbb{S}^3(1)$. Assume that $\alpha(p) = 0$, for all $p \in M^2$. Then the immersion x satisfies the condition $\widetilde{C} \equiv 0$ if and only if locally $x(M^2)$ can be represented by a surface of revolution (22).*

Proof. In Example (3.6) we proved that the surface (22) satisfies the condition $\widetilde{C} \equiv 0$. Conversely, assume that the immersion x satisfies the condition $\widetilde{C} \equiv 0$ with $\alpha = 0$. The equation (16) implies $e_2(\lambda_2) = 0$ while $\nabla^{\mathrm{I}}_{e_1} e_1 = 0$ from the definition of the function α in the beginning of Section 3.1. Thus, from Lemma 4.1 there are local coordinates (u, v) such that the immersion x is described by the following parametrization:

$$x(u,v) = r(u) \cdot Z(v) + A(u),$$

where $Z(v) = \cos(v)C_1 + \sin(v)C_2$ with C_1 and C_2 constant orthonormal vectors in $\mathbb{R}^4$, and $A(u)$ is a curve in $(\mathrm{Span}\{C_1, C_2\})^{\perp}$. One has:

$$
\begin{aligned}
x_u &= r'(u) \cdot Z(v) + A'(u), && (37)\\
x_v &= r(u) \cdot Z'(v), && (38)\\
x_{vv} &= -r(u) \cdot Z(v), && (39)\\
x_{uu} &= r''(u) \cdot Z(v) + A''(u). && (40)
\end{aligned}
$$

Eliminating y from the equations (30) and (32), we obtain:

$$\lambda_2 x_{uu} - \lambda_1 r^{-2} x_{vv} = -\lambda_1 \beta x_u + (\lambda_1 - \lambda_2)x. \quad (41)$$

The substitution of (37), (39) and (40) in (41) gives:

$$\left(\lambda_2 r'' + \lambda_1 \beta r' + \frac{\lambda_1}{r} + (\lambda_2 - \lambda_1)r\right) \cdot Z(v) = -\left(\lambda_2 A''(u) + \lambda_1 \beta A'(u) + (\lambda_2 - \lambda_1)A(u)\right).$$

Using respective derivatives of the functions β and r (see formulas (29)), one can check that

$$\lambda_2 r'' + \lambda_1 \beta r' + \frac{\lambda_1}{r} + (\lambda_2 - \lambda_1)r = 0.$$

Therefore $A(u)$ is a solution of the following second order ordinary differential equation

$$A''(u) - \left(-\frac{\lambda_1 \beta}{\lambda_2}\right) A'(u) + \frac{\lambda_2 - \lambda_1}{\lambda_2} A(u) = 0. \quad (42)$$

Note that for $\alpha = 0$ on M^2, by Proposition 3.3, the condition $\widetilde{C} \equiv 0$ is equivalent to the equation

$$\lambda_1' = 3\lambda_1 \lambda_2^{-1} \beta(\lambda_2 - \lambda_1).$$

Let ϕ be a differentiable function on M^2 depending only on u and satisfying

$$\phi' = \sqrt{\varepsilon \frac{\lambda_2 - \lambda_1}{\lambda_2}}, \quad \text{with} \quad \varepsilon = \mathrm{sign}\left(\frac{\lambda_2 - \lambda_1}{\lambda_2}\right).$$

One has:

$$
\begin{aligned}
\frac{\phi''}{\phi'} &= \frac{\varepsilon}{2\phi'^2}\left(\frac{\lambda_2(\lambda_2' - \lambda_1') - \lambda_2'(\lambda_2 - \lambda_1)}{\lambda_2^2}\right)\\
&= -\frac{\beta \lambda_1}{\lambda_2},
\end{aligned}
$$

and then the equation (42) takes the form

$$A''(u) - \frac{\phi''}{\phi'} A'(u) + \varepsilon \phi'^2 A(u) = 0.$$

Therefore there are constant vectors $\bar{C}_3, \bar{C}_4 \in \mathbb{R}^4$ such that the general solution $A(u)$ of the above equation is

$$A(u) = \cos_\varepsilon(\phi(u))\bar{C}_3 + \sin_\varepsilon(\phi(u))\bar{C}_4,$$

If the vectors $\bar{C}_3$ and $\bar{C}_4$ were not linearly independent, then there would exist a function ψ such that, up to an isometry, the immersion x would be represented by the form

$$x(u,v) = (r(u)\cos(v), r(u)\sin(v), \psi(u), 0)$$

which lies in a 2-dimensional great sphere, therefore it would be totally geodesic, so it would be degenerate. So the vectors $\bar{C}_3$ and $\bar{C}_4$ are linearly independent. Hence $A(u)$ describes an arc of an ellipse or an arc of a hyperbola in the plane orthogonal to C_1 and C_2 if $\varepsilon = 1$ or $\varepsilon = -1$, respectively.

According to Lemma 4.3 there exist constant orthonormal vectors $C_3, C_4 \in \mathrm{Span}\{\bar{C}_3, \bar{C}_4\}$, $s \in \mathbb{R}$ and $a, b \in \mathbb{R}_+$ such that

$$\cos_\varepsilon(\phi(u))\bar{C}_3 + \sin_\varepsilon(\phi(u))\bar{C}_4 = a\cos_\varepsilon(\phi(u) - s)C_3 + b\sin_\varepsilon(\phi(u) - s)C_4.$$

With the reparametrization $u^* = \phi(u) - s$, the immersion x takes the form of Example 3.6. $\square$

Remark 4.5. If $\varepsilon = 1$ and $a = b$ in (22), then the profile curve of the surface of revolution is a circle. This case, corresponding to the situation where both functions α and β vanish everywhere, describes isoparametric surface immersions in $\mathbb{S}^3(1)$ with two distinct principal curvatures. They are non-degenerate because of $1 + \lambda_1\lambda_2 = 0$.

4.2. Non-rotational ($\widetilde{C} \equiv 0$)-surface immersions in $\mathbb{S}^3(1)$

Assume now that the subset U_3 is not empty ($\alpha(p) \neq 0 \neq \beta(p)$, for all $p \in U_3 \neq \emptyset$). By assuming that the immersion has no umbilical points, the function $\lambda_2 - \lambda_1$ has constant sign on U_3. Denote by $\eta := \pm 1$ the sign of the Gauß-Kronecker curvature function. By a suitable choice of the unit normal vector field y and of the numbering of the principal curvatures, we fix

$$0 < \lambda_1 < \lambda_2 \quad \text{if} \quad \eta = 1, \quad \text{or} \quad \lambda_1 < 0 < \lambda_2 \quad \text{if} \quad \eta = -1.$$

Lemma 4.6. *Let $x\colon (M^2, \mathrm{I}) \longrightarrow \mathbb{S}^3(1) \subset \mathbb{R}^4$ be a non-degenerate isometric immersion without umbilical points and satifying the condition $\widetilde{C} \equiv 0$. If there is a* I*-orthonormal frame (e_1, e_2) of principal vector fields such that*

$$\nabla^{\mathrm{I}}_{e_1} e_1 = \alpha e_2 \quad \text{and} \quad \nabla^{\mathrm{I}}_{e_2} e_2 = \beta e_1,$$

where α and β are differentiable functions on M^2 such that $\alpha \neq 0 \neq \beta$ everywhere, then for every point $p \in M^2$ the positive functions

$$u := \sqrt[4]{\eta \cdot \frac{\lambda_2}{\lambda_1^3}} \quad \text{and} \quad v := \sqrt[4]{\eta \cdot \frac{\lambda_1}{\lambda_2^3}}. \tag{43}$$

define local parameters with $u - \eta v > 0$, and there exists a polynomial

$$P(t) = \prod_{k=1}^{4}(t - a_k)$$

with four distinct real roots $a_1 < a_2 < a_3 < a_4$ satisfying $P(0) = -\eta$ such that the functions α and β can be expressed by

$$\alpha^2(u,v) = \frac{\eta \cdot P(\eta v)}{v(u-\eta v)^3}, \tag{44}$$

and

$$\beta^2(u,v) = \frac{-P(u)}{u(u-\eta v)^3}. \tag{45}$$

Moreover, in all cases we have $0 < a_3 < u < a_4$ *and* $\begin{cases} 0 < a_2 < v < a_3 & \textit{for } \eta = 1 \\ 0 < -a_2 < v < -a_1 & \textit{for } \eta = -1 \end{cases}.$

Proof. The parameters u and v as defined by (43), together with λ_1 and λ_2, satisfy

$$\frac{\lambda_1}{\lambda_2} = \eta\frac{v}{u}, \quad \lambda_1 = \frac{\eta}{\sqrt{vu^3}}, \quad \lambda_2 = \frac{1}{\sqrt{uv^3}} \quad \text{and}$$

$$\lambda_1\lambda_2 = \frac{\eta}{u^2v^2}; \tag{46}$$

and because the functions λ_2 and $\lambda_2 - \lambda_1$ are strictly positiv, one also has $u - \eta v > 0$.

Furthermore, using (11), (12), (15), (16) to differentiate (43) we have

$$e_1(u) = -2(u-\eta v)\beta, \quad e_2(v) = 2\eta(u-\eta v)\alpha \quad \text{and} \quad e_1(v) = 0 = e_2(u); \tag{47}$$

therefore (u, v) defines a system of local curvature coordinates such that

$$\frac{\partial}{\partial u} = -\frac{1}{2(u-\eta v)\beta}e_1 \quad \text{and} \quad \frac{\partial}{\partial v} = \frac{\eta}{2(u-\eta v)\alpha}e_2. \tag{48}$$

Using (17) and (18), one gets the following first order partial differential equations for the function α and β:

$$\frac{\partial \alpha}{\partial u} = \frac{-3}{2(u-\eta v)} \cdot \alpha \quad \text{and} \quad \frac{\partial \beta}{\partial v} = \frac{3\eta}{2(u-\eta v)} \cdot \beta,$$

which actually are linear differential equations. The general solutions $\alpha \equiv \alpha(u,v)$ and $\beta \equiv \beta(u,v)$ for these equations, respectively, are

$$\alpha = \left(\frac{1}{u-\eta v}\right)^{\frac{3}{2}} g(v) \quad \text{and} \quad \beta = \left(\frac{1}{u-\eta v}\right)^{\frac{3}{2}} f(u), \tag{49}$$

where f and g are differentiable functions, each of one variable, taking values in $\mathbb{R} \setminus \{0\}$.

Using (47) to compute the derivatives $e_1(\beta)$ and $e_2(\alpha)$ in terms of u and v, one gets

$$e_1(\beta) = 3(u-\eta v)^{-3} f^2(u) - (u-\eta v)^{-2}(f^2)'(u), \tag{50}$$
$$e_2(\alpha) = 3(u-\eta v)^{-3} g^2(v) + \eta(u-\eta v)^{-2}(g^2)'(v). \tag{51}$$

Inserting the equations (46), (49), (50) and (51) into the Gauß-equation (10), we get the following equation:

$$\begin{aligned} 0 &= 1+\frac{\eta}{u^2v^2}-2(u-\eta v)^{-3}\tilde{g}(v)-2(u-\eta v)^{-3}\tilde{f}(u) \\ &\quad -\eta(u-\eta v)^{-2}\tilde{g}'(v)+(u-\eta v)^{-2}\tilde{f}'(u), \end{aligned}$$

where $\tilde{f}(u) := f^2(u)$ and $\tilde{g}(v) := g^2(v)$. From the equation above, we have

$$\eta\tilde{g}'(v) = \frac{(\eta+u^2v^2)(u-\eta v)^2}{u^2v^2}-\frac{2\tilde{g}(v)}{u-\eta v}-\frac{2\tilde{f}(u)}{u-\eta v}+\tilde{f}'(u). \tag{52}$$

Since $\tilde{g}(v)$ does not depend on u, differentiating both sides of the equation (52) with respect to u, one has

$$0=\frac{2(1+u^3v)(u-\eta v)}{vu^3}+\frac{2\tilde{g}(v)}{(u-\eta v)^2}-\frac{2\tilde{f}'(u)}{u-\eta v}+\frac{2\tilde{f}(u)}{(u-\eta v)^2}+\tilde{f}''(u).$$

From the equation above, one has

$$-2\tilde{g}(v)=\frac{2(1+u^3v)(u-\eta v)^3}{vu^3}+2\tilde{f}(u)-2(u-\eta v)\tilde{f}'(u)+(u-\eta v)^2\tilde{f}''(u). \tag{53}$$

Once again, differentiating both sides of this last equation with respect to u, we get that the function $\tilde{f}(u)$ satisfies the following third order ordinary differential equation:

$$6(\eta+u^4)+u^4\tilde{f}'''(u)=0.$$

Solving this differential equation, we get

$$f^2(u)=\tilde{f}(u)=\frac{-P(u)}{u} \tag{54}$$

where P is a fourth order polynomial with constant coefficients:

$$P(t)=\sum_{k=0}^{4}c_kt^k$$

such that $P(0)=c_0=-\eta$ and $c_4=1$. Substituting the expression (54) of $P(u)$ into (53), we get

$$g^2(v)=\tilde{g}(v)=\frac{\eta P(\eta v)}{v}.$$

Finally we have get the wanted expressions (44) and (45) for α^2 and β^2, respectively. We also have

$$0<-P(u) \quad \text{and} \quad 0<\eta P(\eta v)$$

on the range of u and v, respectively.

Consider now the four roots $a_1, a_2, a_3, a_4 \in \mathbb{C}$ of the polynomial equation $P(t)=0$. Following situations are possible (up to a numbering of the roots):

(i)
- all the four roots are complex (not real),
- a_1 and a_2 are complex (not real) and $a_3 = a_4$,
- the four roots are real and equal ($a_1 = a_2 = a_3 = a_4$),
- $a_1 = a_2 \neq a_3 = a_4$ real or not,

(ii)
- a_1 and a_2 are not real, and a_3 and a_4 are distinct real numbers,
- $a_1 = a_2 = a_3 \neq a_4$ are real,
- $a_1 = a_2 \neq a_3 \neq a_4 \neq a_1$;

(iii) the four roots are real and distinct, let say $a_1 < a_2 < a_3 < a_4$.

If the case (i) happens, the polynomial function $P(t)$ is nowhere negative; this gives a contradiction to $0 < -P(u)$. And for the case (ii), the real numbers $-P(t)$ and $\eta P(\eta t)$ have distinct signs, a contradiction to the fact that $f^2(u) > 0$ and $g^2(v) > 0$. This proves that only the case (iii) can occur, i.e. the four roots are real and distinct.

Since $-P(u) > 0$ and $\eta P(\eta v) > 0$ everywhere, we have that

$$0 < a_3 < u < a_4 \quad \text{and} \quad \begin{cases} 0 < a_2 < v < a_3 & \text{for } \eta = 1 \\ 0 < -a_2 < v < -a_1 & \text{for } \eta = -1 \end{cases}.$$

□

Theorem 4.7. *Let $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$ be a non-degenerate immersion of a connected and orientable C^∞-manifold M^2 of dimension 2 into $\mathbb{S}^3(1)$ without umbilical points. If x satisfies the condition $\widetilde{C} \equiv 0$ with $\alpha(p) \neq 0 \neq \beta(p)$ for all $p \in M^2$, then locally there are coordinates (u, v) such that x can be represented by a surface of the form (24).*

Proof. With respect to the coordinates introduced in Lemma 4.6, because of (48), the first fundamental form of x is given by

$$ds^2 = \frac{1}{4}(u - \eta v)\left(\frac{-u}{P(u)} \cdot du^2 + \frac{\eta v}{P(\eta v)} \cdot dv^2\right), \tag{55}$$

and the structure equations of x by

$$\begin{aligned}
x_{uu} &= \Gamma^1_{11} x_u + \Gamma^2_{11} x_v + \mathrm{II}_{11} y - \mathrm{I}_{11} x && (56)\\
x_{uv} &= \Gamma^1_{12} x_u + \Gamma^2_{12} x_v && (57)\\
x_{vv} &= \Gamma^1_{22} x_u + \Gamma^2_{22} x_v + \mathrm{II}_{22} y - \mathrm{I}_{22} x && (58)\\
y_u &= -\frac{\eta}{\sqrt{vu^3}} x_u && (59)\\
y_v &= -\frac{1}{\sqrt{uv^3}} x_v, && (60)
\end{aligned}$$

where Γ^k_{ij} are Christoffel symbols associated to the first fundamental form I given by (55).

We have:

$$
\begin{aligned}
\mathrm{I}_{11} &= -\frac{u(u-\eta v)}{4P(u)}, & \mathrm{I}_{22} &= \frac{\eta v(u-\eta v)}{4P(\eta v)},\\
\mathrm{II}_{11} &= \frac{\eta}{\sqrt{vu^3}}\mathrm{I}_{11}, & \mathrm{II}_{22} &= \frac{1}{\sqrt{uv^3}}\mathrm{I}_{22},\\
\Gamma^1_{11} &= \frac{(2u-\eta v)P(u)-u(u-\eta v)P'(u)}{2u(u-\eta v)P(u)}, &&\\
\Gamma^2_{22} &= \frac{(u-2\eta v)P(\eta v)-\eta v(u-\eta v)P'(\eta v)}{2v(u-\eta v)P(\eta v)}, &&\\
\Gamma^1_{12} &= \frac{-\eta}{2(u-\eta v)}, & \Gamma^2_{12} &= \frac{1}{2(u-\eta v)},\\
\Gamma^2_{11} &= \frac{-uP(\eta v)}{2v(u-\eta v)P(u)}, & \Gamma^1_{22} &= -\frac{\eta v P(u)}{2u(u-\eta v)P(\eta v)}.
\end{aligned}
$$

Using (57), (59) and (60) to differentiate (56) (respectively (58)) with respect to u (respectively to v), one can see that x_{uuu} and x_{uuuu} (respectively x_{uuu} and x_{uuuu}) are combinations of x_u, x_v, x and y with coefficients depending on both variables u and v. From the expressions of x_{uu} and x_{uuu} (respectively of x_{vv} and x_{vvv}), one can express x_v (respectively x_u) and y in terms of x_{uuu}, x_{uu}, x_u (respectively x_{vvv}, x_{vv}, x_v) and x. Inserting these expressions into the expression of x_{uuuu} (respectively into x_{vvvv}), one gets the following fourth order partial differential equations:

$$
\begin{aligned}
0 &= 128P(u)\cdot x_{uuuu}+320P'(u)\cdot x_{uuu}+240P''(u)\cdot x_{uu}\\
&\quad +40P'''(u)\cdot x_u-5P^{(4)}(u)\cdot x,
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
0 &= 128P(\eta v)\cdot x_{vvvv}+320P'(\eta v)\cdot x_{vvv}+240P''(\eta v)\cdot x_{vv}\\
&\quad +40P'''(\eta v)\cdot x_v-5P^{(4)}(\eta v)\cdot x.
\end{aligned}
\tag{62}
$$

For any fixed v, the equation (61) is a four order ordinary differential equation on the intervall $I_u=]a_3,a_4[$. One can check that

$$F_k(u)=\sqrt{|u-a_k|}\quad (k=1,...,4)$$

are particular solutions which are linearly independent. Therefore the general solution $x(u,v)$ is of the form

$$x(u,v)=\sum_{k=1}^{4}F_k(u)\phi_k(v), \tag{63}$$

where $\phi_k(v)\in\mathbb{R}^4$, $k=1,...,4$ are vector valued functions.

Similarly, for any fixed u, the equation (62) is a fourth order ordinary differential equation on the intervall $I_v=]a_2,a_3[$ if $\eta=1$ or $I_v=]-a_2,-a_1[$ if $\eta=-1$, with linearly independent solutions

$$G_k(v)=\sqrt{|v-\eta a_k|}\quad (k=1,...,4).$$

Hence there are vector valued functions $\psi_k(v) \in \mathbb{R}^4$, $k = 1, ..., 4$ such that

$$x(u,v) = \sum_{k=1}^{4} F_k(u)\phi_k(v) = \sum_{k=1}^{4} G_k(v)\psi_k(u).$$

From that we derive the following linear system of 4 equations with unkowns $\phi_k(v)$, $k = 1, ..., 4$:

$$\begin{cases} \sum_{k=1}^4 F_k(u)\phi_k(v) = \sum_{i=1}^4 G_i(v)\psi_i(u), \\ \sum_{k=1}^4 F_k'(u)\phi_k(v) = \sum_{i=1}^4 G_i(v)\psi_i'(u), \\ \sum_{k=1}^4 F_k''(u)\phi_k(v) = \sum_{i=1}^4 G_i(v)\psi_i''(u), \\ \sum_{k=1}^4 F_k'''(u)\phi_k(v) = \sum_{i=1}^4 G_i(v)\psi_i'''(u). \end{cases}$$

Because everywhere the Wronski-determinant for the functions F_k $(k = 1, ..., 4)$ does not vanish, there are unique functions $C_{ik}(u)$ such that

$$\phi_k(v) = \sum_{i=1}^{4} G_i(v)C_{ik}(u), \quad k = 1, ..., 4. \tag{64}$$

Differentiating the equation above with respect to u, one gets

$$0 = \sum_{k=1}^{4} G_k(v)C_{ik}'(u).$$

Using the linear independence of the functions $G_1(v), G_2(v), G_3(v), G_4(v)$, we have that the functions $C_{ik}(u) \equiv C_{ik}$, $k = 1, ..., 4$ are constant. Substituting (64) in (63) finally one gets

$$x(u,v) = \sum_{i,k} \sqrt{|u - a_i|}\sqrt{|v - \eta a_k|}C_{ik}. \tag{65}$$

Inserting this expression into the equation (57), one gets

$$0 = \sum_{i,k} \frac{1}{\sqrt{|u - a_i|}\sqrt{|v - \eta a_k|}} \cdot (a_k - a_i)C_{ik}.$$

By the linear independence of the functions $(u,v) \longmapsto \frac{1}{\sqrt{|u-a_i|}\sqrt{|v-\eta a_k|}}$, one deduces

$$(a_k - a_i)C_{ik} = 0, \quad i, k = 1..., 4.$$

Therefore,

$$C_{ik} = 0, \quad \text{for} \quad i \neq k.$$

Hence, (65) simplifies to

$$x(u,v) = \sum_{i=1}^{4} \sqrt{|u - a_i|}\sqrt{|v - \eta a_i|}C_i,$$

where $C_i \in \mathbb{R}^4$ $(i = 1, ..., 4)$ are constant vectors in $\mathbb{R}^5$.

Since $u \in I_u =]a_3, a_4[$ and $a_1 < a_2 < a_3$, we have

$$F_k(u) = \sqrt{u - a_k}, \quad k = 1, 2, 3 \quad \text{and} \quad F_4(u) = \sqrt{a_4 - u};$$

and by the fact that $v \in I_v$, we have

$$G_1(v) = \sqrt{\eta v - a_1}, \quad G_2(v) = \sqrt{v - \eta a_2}, \quad G_3(v) = \sqrt{a_3 - \eta v}, \qquad G_4(v) = \sqrt{a_4 - \eta v}.$$

One has

$$1 = < x, x > = \sum_{i=1}^{4} < C_i, C_i > F_i^2(u) G_i^2(v) + 2 \sum_{i<j} < C_i, C_j > F_i(u) G_i(v) F_j(u) G_j(v).$$

By the linear independence of the functions $1, u, v, uv$ and $F_i(u)G_i(v)F_j(u)G_j(v)$, $1 \le i < j \le 4$, we get following equations:

$$\begin{aligned}
0 &= < C_i, C_j > \quad (1 \le i < j \le 4), && (66)\\
0 &= -a_1 < C_1, C_1 > -\eta a_2 < C_2, C_2 > +a_3 < C_3, C_3 > -a_4 < C_4, C_4 >, && (67)\\
0 &= \eta < C_1, C_1 > + < C_2, C_2 > -\eta < C_3, C_3 > +\eta < C_4, C_4 >, && (68)\\
1 &= a_1^2 < C_1, C_1 > +\eta a_2^2 < C_2, C_2 > -a_3^2 < C_3, C_3 > +a_4^2 < C_4, C_4 > . && (69)
\end{aligned}$$

From the equations (66), we have $< C_i, C_j > = 0$ for $1 \le i \ne j \le 4$, i.e. the vectors C_i, $i = 1, ..., 4$ are orthogonal. Computing the normal vector field from the equation (56) we get

$$\begin{aligned}
y(u, v) &= \frac{1}{\mathrm{II}_{11}} \left(x_{uu} - \Gamma_{11}^1 x_u - \Gamma_{11}^2 x_v + \mathrm{I}_{11} x \right)\\
&= -\frac{\eta}{\sqrt{uv}} \Big(\sum_{i=1}^{4} \frac{1}{a_i} F_i(u) G_i(v) C_i \Big).
\end{aligned}$$

Since y is unit, we also get the following equation obtained by the same procedure as we did for x:

$$1 = \frac{\eta}{a_1^2} < C_1, C_1 > + \frac{1}{a_2^2} < C_2, C_2 > - \frac{\eta}{a_3^2} < C_3, C_3 > + \frac{\eta}{a_4^2} < C_4, C_4 > . \qquad (70)$$

Solving $< C_i, C_i >$ $(i = 1, .., 4)$ from the equations (67), (68), (69) and (70) we get the expressions for these entities described in Example 3.7. Thus, the proof of Theorem 4.7 is complete. □

Having in mind the beginning of this section we see now that only by combining Theorems 4.4 and 4.7 we get Theorem 1.1.

Remark 4.8. From the invariance of the condition $\widetilde{C} \equiv 0$ under polarization (see [1]), if a non-degenerate surface immersion $x \colon M^2 \longrightarrow \mathbb{S}^3(1) \subset \mathbb{R}^4$ satisfies the condition $\widetilde{C} \equiv 0$, then its Gauß map y is also a $(\widetilde{C} \equiv 0)$-surface in $\mathbb{S}^3(1)$; for the explicit expressions of y see Example 3.6 and 3.7.

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