Quantitative Error Estimates for the Implicit Euler Scheme for Linear Evolutionary Problems with Rough Data

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Abstract For the initial-boundary value problem for a non-homogeneous linear parabolic differential equation with time-dependent coefficients, the discretization in time by the backward Euler method is considered. The method is shown to be convergent of first order even for rough data. Attention is directed, in particular, to estimates of the appearing constants as well as to restrictions on the step size in dependence on the problem's parameters. In addition, the temporal discretization of the incompressible Stokes problem and the dependence of the error on the Reynolds number is analysed.

Keywords Linear parabolic PDE, parabolic smoothing, discretization in time, backward Euler, a priori error estimates, incompressible Stokes equation

Classification 65M15, 65J10, 76D07, 34G10

1 Introduction

Considering approximations of fluid flow problems, we are concerned with the search for optimal error estimates under suitable assumptions. As the fluid flow is mainly described by the Reynolds number Re, the dependence of the error on it is of essential importance.

Let $\Omega \subseteq \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain with locally Lipschitz continuous boundary $\partial\Omega$. The non-stationary, isothermal motion of a viscous, incompressible, homogeneous Newtonian fluid neglecting nonlinear phenomena can be modelled by the initial-boundary value problem

$$\begin{gathered} u_t(x,t) - \mathrm{Re}^{-1}\,\Delta u(x,t) + \nabla p(x,t) = f(x,t)\,, \quad \nabla\cdot u(x,t) = 0 \quad \text{in } \ \Omega\times(0,T]\,, \\ u(x,t) = 0 \quad \text{on } \ \partial\Omega\times(0,T]\,, \quad u(\cdot,0) = u_0 \quad \text{in } \ \Omega\,, \end{gathered}$$

where u denotes the velocity vector, p the quotient of pressure and constant density, and f a specific force.

We have studied the simplest temporal discretization by means of the backward Euler scheme and asked for quantitative error estimates for rough initial data and right hand side. However, this problem can be embedded in the more general context of quantitative smoothing error estimates for general linear parabolic problems.

From the general result we shall present in this paper, it immediately follows for the error e^n between the exact and the time-discrete velocity at time $t_n = n\Delta t$

$$\|t_n e^n\|_{L^2}^2 + \mathrm{Re}^{-1}\Delta t \sum_{j=1}^{n} \|\nabla t_j e^j\|_{L^2}^2 \le (4\Delta t)^2 \left(\|u_0\|_{L^2}^2 + \mathrm{Re}\,\|f\|_{L^2(0,T;H^{-1})}^2 + \mathrm{Re}\,\|tf_t\|_{L^2(0,T;H^{-1})}^2 \right) .$$

This shows that the dependence on Re is exactly the same as in usual stability estimates for the exact solution, the term with tf_t excepted.

We now turn to general parabolic problems. Due to a lack of the solution's regularity, unrealistic assumptions on the problem's data or having hardly realizable restrictions on the discretization parameter, standard error estimates may fail. For parabolic problems, higher regularity corresponds to compatibility conditions on the initial values and right hand side, cf. Wloka [9], Temam [7]. In the appearance of additional constraints, as the divergence free constraint in the incompressible Stokes or Navier-Stokes problem, higher compatibility is hard to verify and mostly violated, cf. Heywood [3] , Temam [7].

Therefore, we ask for so-called smoothing error estimates that take advantage of the parabolic smoothing property and hold even for rough data. However, only strongly A-stable (or G-stable) time discretizations seem to profit by this smoothing, cf. the summarizing work by FUJITA/SUZUKI [1] (esp. the résumé in Theorem 18.1). Besides the question of regularity, appearing constants as well as restrictions on the step size and their dependence on the problem's parameters have to be quantified for having relevant estimates.

We shall derive a quantified smoothing a priori error estimate for the implicit Euler scheme approximating an initial-boundary value problem for a linear parabolic equation with time-dependent coefficients in an abstract setting. The analysis restricts itself to the temporal discretization and is independent of a possible spatial approximation. The error estimated will be of order $\mathcal{O}(\Delta t/t)$ in norms natural for the problem.

The same order of convergence has been obtained by HUANG/THOMÉE [4] and LUSKIN/RANNACHER [5] in similar situations: The first authors consider abstract parabolic problems, though with homogeneous right hand side and without having a stronger look to the appearing constants and step size bounds. They employ an elliptic auxiliary problem for estimating the error in the dual norm. LUSKIN/RANNACHER [5] firstly consider a spatial finite element approximation of a scalar second-order parabolic partial differential equation and afterwards the temporal discretization. The underlying bilinear form is assumed to be strongly positive. For the estimate of the error in the dual norm, a "backward in time" parabolic duality argument is used. For homogeneous right hand side, their analysis results also in the order $\mathcal{O}(\Delta t/t)$. For vanishing initial values but non-homogeneous right hand side, the error is shown to be of order $\mathcal{O}(\Delta t \ln(1/\Delta t))$ whenever the right hand side is in $\mathcal{C}([0,T];L^2)$.

Our analysis relies on energy methods and duality arguments, too. Thus, there is no need to assume the self-adjointness of the underlying differential operator. Instead of being strongly positive, it suffices to assume that the bilinear form satisfies a Gårding inequality. Moreover, we cover non-homogeneous initial values and right hand side. By means of a priori estimates for the exact solution, we prove, under suitable assumptions, the regularity required by the error estimate.

Attention is directed to the appearing constants and restrictions on the time step size that arise essentially from the application of a discrete Gronwall lemma. We show explicitly the dependence of the constants on the problem's parameters.

The discrete Gronwall lemma we use will be in difference form, which gives a more general but also simpler statement than the sum versions known from the literature.

For estimating the error in the dual norm, we firstly use the elliptic auxiliary problem by HUANG/THOMÉE [4] and alternatively a parabolic duality argument similar to the one used by LUSKIN/RANNACHER [5].

2 Main result

By $\mathbb{R}$, we denote the real numbers whereas $\mathbb{R}_0^+$ denotes the nonnegative real numbers. Let V be a separable, reflexive, real Banach space with norm $\|\cdot\|$ and H be a separable, real Hilbert space with inner product $(\cdot,\cdot)$ and induced norm $|\cdot|$. The dual space of V is denoted by V^* and equipped with the usual dual norm $\|f\|_* := \sup_{v\in V\setminus\{0\}} \langle f,v\rangle/\|v\|$, where $\langle\cdot,\cdot\rangle$ denotes the dual product between V^* and V. Due to the reflexivity of V, $\langle\cdot,\cdot\rangle$ is also the dual pairing between $V = V^{**}$ and V^*, and in this sense symmetric.

Furthermore, V is assumed to be dense and continuously embedded in H. Identifying H with its dual, H will be dense and continuously embedded in V^*. Thus, V, H, and V^* form an evolutional (Gelfand) triple, and the dual pairing is the extension of the inner product in H. Owing to the continuous embeddings, there is a constant $\alpha > 0$ s. t. Poincaré-Friedrichs inequalities hold:

$$|v| \quad \leq \alpha\,\|v\| \quad \forall v \in V\,, \quad \|v\|_* \leq \alpha\,|v| \quad \forall v \in H\,. \tag{2.1}$$

For a time interval $[0,T] \subset \mathbb{R}_0^+$, let $L^2(0,T;V)$ be the set of Bochner measurable functions $u:[0,T] \to V$ with $\|u\|^2_{L^2(0,T;V)} := \int_0^T \|u(s)\|^2 ds < \infty$, and $\mathcal{W}(0,T;V) := \{u \in L^2(0,T;V) : u' \in L^2(0,T;V^*)\}$ equipped with the graph norm. With u', the derivative in the distributional sense is meant. By $\mathcal{C}([0,T];H)$ with $\|u\|_{\mathcal{C}([0,T];H)} := \sup_{s\in[0,T]} |u(s)|$, we denote the Banach space of continuous functions $u:[0,T] \to H$. By interpolation, the continuous embedding $\mathcal{W}(0,T;V) \hookrightarrow \mathcal{C}([0,T];H)$ holds true, cf. GAJEWSKI/GRÖGER/ZACHARIAS [2], WLOKA [9].

For any $t \in [0,T]$, let $a(t;\cdot,\cdot) : V \times V \to \mathbb{R}$ be a (uniform in time) continuous bilinear form satisfying (uniformly in time) a Gårding inequality and being continuously differentiable with respect to t. The derivative is denoted by a_t and is assumed to be a (uniform in time) continuous bilinear form, too. Thus, we have constants $\mu > 0$, $\beta > 0$, $\beta_t \geq 0$, $\kappa \geq 0$, independent on t, s. t. for all $t \in [0,T]$ and for all $u,\, v \in V$

$$a(t;v,v) \geq \mu \|v\|^2 - \kappa |v|^2 , \quad |a(t;u,v)| \leq \beta \|u\| \|v\| , \quad |a_t(t;u,v)| \leq \beta_t \|u\| \|v\| . \tag{2.2a}$$

The form $a(t;\cdot,\cdot)$ is said to be strongly positive iff $\kappa = 0$ can be chosen. Without loss of generality, we may assume $\mu \leq \alpha^2\kappa$ if $\kappa > 0$. Otherwise, $a(t;\cdot,\cdot)$ would be strongly positive with a constant $\overline{\mu} = \mu - \alpha^2\kappa > 0$ due to (2.1).

For the skew-symmetric part of $a(t;\cdot,\cdot)$, we shall assume

$$|a(t;u,v) - a(t;v,u)| \leq c \|u\| |v| \qquad \forall u, v \in V,\ t \in [0,T] \tag{2.2b}$$

with some constant $c > 0$. This allows us to prove the solution's regularity we need, see Proposition 7. For a usual second-order differential operator, (2.2b) is fulfilled.

With $a(t;\cdot,\cdot)$, we associate for each $t \in [0,T]$ a linear operator $A(t) : V \to V^*$ via $\langle A(t)u, v\rangle = a(t;u,v)$ for all $u,\, v \in V$. Actually, $A(t)$ is the energetic extension of the underlying differential operator. In addition, we have linear operators $A'(t)$ via $\langle A'(t)u, v\rangle = a_t(t;u,v)$.

We consider the weak formulation of the initial-boundary value problem for a linear parabolic equation in the time interval $[0,T]$ that can be written as

Problem (P) *For given $u_0 \in H$ and $f \in L^2(0,T;V^*)$, find $u \in \mathcal{W}(0,T;V)$ s. t.*

$$\begin{aligned} \langle u'(t), v\rangle + a(t;u(t),v) &= \langle f(t), v\rangle \qquad \forall v \in V\,,\ a.\ e.\ in\ (0,T]\,, \\ u(0) &= u_0\,. \end{aligned}$$

The Stokes problem fits into this context by the following observations: Let H and V be the solenoidal function spaces

$$\begin{aligned} H &= \left\{ v \in L^2(\Omega)^d : \ \nabla\cdot v = 0 \ \text{ in } H^{-1}(\Omega)\,,\ \gamma_n u = 0 \ \text{ in } H^{-1/2}(\partial\Omega) \right\}, \\ V &= \left\{ v \in H_0^1(\Omega)^d : \ \nabla\cdot v = 0 \ \text{ in } L^2(\Omega) \right\}, \end{aligned}$$

where γ_n is the trace operator mapping from $\{v \in L^2(\Omega)^d : \nabla\cdot v \in L^2(\Omega)\}$ onto $H^{-1/2}(\partial\Omega)$ with $\gamma_n v = (v\cdot n)_{|\partial\Omega}$ for all smooth v. By n, we denote the outer normal on $\partial\Omega$. Furthermore, $L^2(\Omega)$ denotes the usual Lebesgue space with its natural inner product and norm (denoted by $|\cdot|$), and $H_0^1(\Omega)$ is the usual Sobolev space normed by $\|\cdot\| = |\nabla\cdot|$. With $a(t;\cdot,\cdot)$ independent on t and defined by $a(u,v) = \mathrm{Re}^{-1}(\nabla u, \nabla v)$ for $u,\, v \in V$, Problem (P) then is the weak formulation for the Stokes problem with eliminated pressure. The constants are $\mu = \mathrm{Re}^{-1}$, $\beta = \mu$, and $\kappa = \beta_t = c = 0$. The constant $\alpha \sim \operatorname{diam}\Omega$ comes from the Poincaré-Friedrichs inequality for $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$. Moreover, A is the energetic extension of the classical Stokes operator, i. e. $A = -\mu P\Delta$ where $P : L^2(\Omega)^d \overset{\text{onto}}{\longrightarrow} H$ is the ortho-projector of the Weyl decomposition. For more details see e. g. TEMAM [6].

Problem (P) possesses a unique solution $u \in \mathcal{W}(0,T;V) \hookrightarrow \mathcal{C}([0,T];H)$. Therefore, the initial condition makes sense, cf. GAJEWSKI/GRÖGER/ZACHARIAS [2], WLOKA [9]. The at this used fact that $u' \in L^2(0,T;V^*)$ relies on the linearity of the problem and is not trivial: For the three-dimensional Navier-Stokes equation, only $u' \in L^{4/3}(0,T;V^*)$ holds true. In addition to $u_0 \in H$, $f \in L^2(0,T;V^*)$, we shall assume

$$tf \in L^2(0,T;V)\,, \quad tf' \in L^2(0,T;V^*)\,. \tag{2.3}$$

Thus, we have $tf \in \mathcal{C}([0,T];H)$. However, the assumption $tf \in L^2(0,T;V)$ can be replaced by $\sqrt{t}f \in L^2(0,T;H)$.

We remark that we need no assumptions on the dimension of V and H. Both could be finite dimensional. This is of particular interest if the differential equation is firstly discretized in space and afterwards in time. For the spatial semi-discretization, a conformal finite element method can be used. For more details, we refer to LUSKIN/RANNACHER [5], THOMÉE [8], FUJITA/SUZUKI [1], and the references cited there.

We now consider the discretization in time by means of the *backward Euler method* based upon an equidistant distribution of the time interval $[0,T]$. Let N be a given positive integer and $\Delta t = T/N$, $t_n = n\Delta t$, $u^n \sim u(t_n)$ for $n = 0,\,1,\,\ldots,\,N$. For any x^n, we will use the abbreviation $\tilde{x}^n := t_n x^n$. The method under consideration is then defined as

Problem $(P_{\Delta t})$ *For given $u_0 \in H$ and $\{f^n\}_{n=1}^N \subseteq V^*$, find $\{u^n\}_{n=1}^N \subseteq V$ s. t.*

$$\begin{aligned}\frac{1}{\Delta t}(u^{n+1} - u^n, v) + a(t_{n+1}; u^{n+1}, v) &= \langle f^{n+1}, v\rangle \quad \forall v \in V,\ n = 0,\,1,\,\ldots,\,N-1\,,\\ u^0 &= u_0\,.\end{aligned}$$

We shall use the natural restriction

$$f^n = \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} f(t)\,dt\,. \tag{2.4}$$

Due to the main theorem on monotone operators by Browder and Minty, Problem $(P_{\Delta t})$ has a unique solution if $(\cdot,\cdot) + \Delta t a(t;\cdot,\cdot)$ is strongly positive on V. For this, we may assume $\kappa\Delta t \le 1$, or $\mu\Delta t + \alpha^2(1-\kappa\Delta t) > 0$ if $\kappa\Delta t > 1$. Both relations represent bounds on the step size. Since $\mu \le \alpha^2\kappa$ if $\kappa \ne 0$ by assumption, $(P_{\Delta t})$ is uniquely solvable for

$$N > \left(\kappa - \frac{\mu}{\alpha^2}\right) T\,. \tag{2.5}$$

This bound here is somewhat weaker than the restriction $\kappa\Delta t < 1$.

For the implicit Euler method, which is supposed to be convergent of first order, standard error estimates require $u,\, u' \in L^2(0,T;V)$, $u'' \in L^2(0,T;V^*)$ (and hence $u \in \mathcal{C}([0,T];V)$, $u' \in \mathcal{C}([0,T];H)$) but this is mostly unrealistic: Higher regularity is equivalent to conditions on the compatibility of the data and relies on the search for the "best" space in which $u(t) \to u_0$ as $t \to 0$, cf. WLOKA [9], TEMAM [7]. Here, $f,\, f' \in L^2(0,T;V^*)$ (and thus $f \in \mathcal{C}([0,T];V^*)$), $u_0 \in V$, and $u'(0) := f(0) - A(0)u_0 \in H$ would be required.

However, in virtue of the so-called parabolic smoothing property, the solution to a parabolic problem is smooth whenever $t > 0$, even for rough data. So, as we will show in Proposition 7, $tu' \in L^2(0,T;V)$, $tu'' \in L^2(0,T;V^*)$ holds true rather than $u' \in L^2(0,T;V)$, $u'' \in L^2(0,T;V^*)$.

Our main result will be

Theorem 1 *Let u and $\{u^n\}$, resp., be the solution to (P) and $(P_{\Delta t})$, resp., Then for the error $e^n := u(t_n) - u^n$, $n = 1,\,2,\,\ldots,\,N$, it holds*

$$\begin{aligned}&|t_n e^n|^2 + \frac{\mu\Delta t}{1-2\kappa\Delta t}\sum_{j=0}^{n-1}\|t_{j+1}e^{j+1}\|^2 \le \frac{2(\Delta t)^2}{\mu^2}\,\mathrm{e}^{\Theta(t_n)}\times\\ &\times\left(\mathcal{A}\,|u_0|^2 + \mathcal{B}\int_0^{t_n}\mathrm{e}^{-2\kappa t}\|f(t)\|_*^2 dt + \mathcal{C}\int_0^{t_n}\mathrm{e}^{-2\kappa t}\|tf'(t)\|_*^2 dt\right)\end{aligned} \tag{2.6}$$

i) if $\kappa = 0$ (strongly positive case) or $\Delta t < 1/2\kappa$, and

$$\mu^4 > \alpha^2(\beta + \alpha^2\kappa)^2\beta_t\,; \tag{2.7}$$

ii) if

$$\Delta t < 1/2\lambda\,, \quad \lambda = \kappa + \alpha^2(\beta+\alpha^2\kappa)^2\beta_t^2/(4\eta\mu^5) \quad (0 \ll \eta < 1 \text{ arbitrary}); \tag{2.8}$$

iii) if $\kappa = 0$ (strongly positive case) or $\Delta t < 1/2\kappa$.

The exponents are $\Theta(t_n) = 2\kappa\Delta t + \omega t_n$ in i), $\Theta(t_n) = 2\kappa\Delta t + \omega_\lambda t_n$ in ii), and $\Theta = 2\omega t_n$ in iii), where $\omega \geq -\frac{1}{\Delta t}\ln(1-2\kappa\Delta t)$ and $\omega_\lambda \geq -\frac{1}{\Delta t}\ln(1-2\lambda\Delta t)$ are arbitrary. In i) and ii), the constants are given by

$$\mathcal{A} = cc_3 + 2(1+cc_5)c_6\,,\ \mathcal{B} = \frac{cc_3 + 2(1+cc_5)(4\beta^2+c_6)}{\mu} + \mu cc_4\,,\ \mathcal{C} = \frac{8\beta^2(1+cc_5)}{\mu}$$

with $c = c_1$ in i) and $c = c_2$ in ii). In iii), it is

$$\mathcal{A} = a\left((1+b)c_5 + \beta^2 c_7\right) + 2c_6 e\,,\ \mathcal{B} = \frac{a}{\mu}\left((5+b)c_5 + (\beta^2+\mu^2)c_7\right) + \frac{2}{\mu}(4\beta^2+c_6)e\,,$$
$$\mathcal{C} = \frac{4}{\mu}(ac_5 + 2\beta^2 e)\,,\ a = \frac{4(\beta+\alpha^2\kappa)^4}{3\mu^2}\,,\ b = \frac{4(\beta^2+\beta_t^2t^2)}{\mu^2}\,,\ e = \mathrm{e}^{2\kappa\Delta t - \omega t_n}\,.$$

Finally, it is

$$c_1 = \left(\frac{\mu^2}{(\beta+\alpha^2\kappa)^2} - \frac{\alpha^2\beta_t}{\mu^2}\right)^{-2},\ c_2 = \frac{(\beta+\alpha^2\kappa)^4}{(1-\eta)^2\mu^4}\,,\ c_3 = \left(\frac{\alpha^2\beta\beta_t}{\mu^2} + \frac{\alpha^4\kappa\beta_t}{\mu^2} + \frac{\alpha^2\kappa\beta}{\mu} + \beta\right)^2,$$
$$c_4 = \left(\frac{\alpha^2\kappa}{\mu}+1\right)^2,\ c_5 = \frac{\alpha^4\beta_t^2}{\mu^4}\,,\ c_6 = \frac{4\beta^2(\beta^2+\beta_t^2t^2)}{\mu^2} + 2\beta_t^2t^2,\ c_7 = \frac{2}{\mu^4}\left(\beta^2 + \frac{(\Delta t)^2\alpha^4\beta_t^2}{\mu^2}\right).$$

Note that (2.7) is fulfilled whenever $a(t;\cdot,\cdot)$ is independent on t. Otherwise, remember the anyhow holding relations $\mu \leq \beta$ if $\kappa = 0$ and $\mu \leq \alpha^2\kappa$ if $\kappa \neq 0$. Note further that $\omega_\lambda \geq \omega \geq 2\kappa$, and $\omega = 0$ if $\kappa = 0$. For ω, we can choose $\omega = 2\kappa/(1-2\kappa\Delta t)$.

Besides the time step restriction (2.5) for the solvability of $(P_{\Delta t})$, we have a second, more restrictive bound that comes from the application of the discrete Gronwall lemma.

Case iii) provides, without the restriction (2.7) or (2.8), the same result as i) and ii) but, in general, with greater error constants. The cases i) and ii) will be proved using an elliptic auxiliary problem whereas for iii) a discrete duality argument will be employed.

The result can be strengthened for the case $\beta = \mu$ with a time-independent, strongly positive bilinear form ($\beta_t = \kappa = 0$). To this end, we shall revisit the proof of Theorem 1 step by step in order to obtain optimal estimates. We emphasize that, nevertheless, the analysis is a worst case scenario.

Because of the strong positiveness, there is *no bound on the step size.* Since $\beta_t = 0$, assumption (2.7) is trivially fulfilled. We immediately come up with

Corollary 2 *Let u and $\{u^n\}$, resp., be the solution to (P) and $(P_{\Delta t})$, resp., with $\kappa = \beta_t = 0$ and $\mu = \beta$. Then for the error $e^n := u(t_n) - u^n$, $n = 1, 2, \ldots, N$, it holds*

$$|t_n e^n|^2 + \mu\Delta t\sum_{j=0}^{n-1}\|t_{j+1}e^{j+1}\|^2 \leq 2(\Delta t)^2\left(4\,|u_0|^2 + \frac{8}{\mu}\int_0^{t_n}\|f(t)\|_*^2dt + \frac{3}{\mu}\int_0^{t_n}\|tf'(t)\|_*^2dt\right).$$

The estimate given for the Stokes problem is a direct consequence.

3 Proof of the main result

In the course of the proof, we have to apply a discrete version of Gronwall's lemma. Although we often find sum versions in the literature, we present here a more general version (without sign condition for λ) based upon the difference structure of the inequality given.

Lemma 3 (Discrete Gronwall lemma) *Let $\{a_n\}$, $\{b_n\} \subseteq \mathbb{R}$ and $1 - \lambda\Delta t > 0$. Then,*

$$\frac{a_{n+1} - a_n}{\Delta t} \leq b_{n+1} + \lambda\, a_{n+1}\,, \quad n = 0,\, 1,\, \ldots, \tag{3.1}$$

implies for $n = 1,\, 2,\, \ldots$

$$a_n \leq (1 - \lambda\Delta t)^{-n} \left(a_0 + \Delta t \sum_{j=0}^{n-1} (1 - \lambda\Delta t)^j\, b_{j+1} \right). \tag{3.2}$$

Proof Let $\tilde{a}_n := (1 - \lambda\Delta t)^n a_n$. We then obtain from (3.1) for $1 - \lambda\Delta t > 0$

$$\frac{\tilde{a}_{n+1} - \tilde{a}_n}{\Delta t} = \frac{1}{\Delta t}(1 - \lambda\Delta t)^n \left((1 - \lambda\Delta t)a_{n+1} - a_n\right) \leq (1 - \lambda\Delta t)^n\, b_{n+1}\,.$$

Summation over n leads to

$$\frac{\tilde{a}_n - \tilde{a}_0}{\Delta t} \leq \sum_{j=0}^{n-1} (1 - \lambda\Delta t)^j\, b_{j+1}$$

that gives estimate (3.2). #

Proof of Theorem 1 Problems (P) and $(P_{\Delta t})$ lead straightforward to the *error equation*

$$\frac{1}{\Delta t}(e^{n+1} - e^n, v) + a(t_{n+1}; e^{n+1}, v) = \langle \rho^{n+1}, v \rangle\,, \tag{3.3}$$

where

$$\langle \rho^{n+1}, v \rangle = \frac{1}{\Delta t} \langle u(t_{n+1}) - u(t_n) - \Delta t\, u'(t_{n+1}), v \rangle$$

for all $v \in V$ and $n = 0,\, 1,\, \ldots,\, N-1$. With integration by parts, we find for the consistency error

$$\rho^{n+1} = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (t - t_n)(f'(t) - u''(t))\, dt \tag{3.4}$$

in the sense of Bochner integrals. The integral is well-defined since $t(f' - u'') \in L^2(0, T; V^*)$ as we will show later (see Proposition 7).

We now multiply (3.3) by t_{n+1} and arrive at (remembering $\tilde{x}^n := t_n x^n$)

$$\frac{1}{\Delta t}(\tilde{e}^{n+1} - \tilde{e}^n, v) + a(t_{n+1}; \tilde{e}^{n+1}, v) = \langle \tilde{\rho}^{n+1}, v \rangle + (e^n, v)\,. \tag{3.5}$$

Testing with $v = \tilde{e}^{n+1}$ and taking advantage of the frequently used inequality[1]

$$(a - b, a) = \frac{1}{2}|a|^2 - \frac{1}{2}|b|^2 + \frac{1}{2}|a - b|^2 \geq \frac{1}{2}|a|^2 - \frac{1}{2}|b|^2 \quad \forall a,\, b \in H \tag{3.6}$$

leads, with Gårding's inequality for $a(t; \cdot, \cdot)$, Cauchy-Schwarz' and Young's inequality, to

[1] This inequality, indeed, expresses the A and G-stability, resp., of the implicit Euler scheme.

$$\frac{1}{2\Delta t}\left(|\tilde{e}^{n+1}|^2-|\tilde{e}^n|^2\right)+\mu\|\tilde{e}^{n+1}\|^2\le\kappa|\tilde{e}^{n+1}|^2+\langle\tilde{\rho}^{n+1},\tilde{e}^{n+1}\rangle+(e^n,\tilde{e}^{n+1})$$
$$\le\kappa|\tilde{e}^{n+1}|^2+\frac{1}{\mu}\left(\|\tilde{\rho}^{n+1}\|_*^2+\|e^n\|_*^2\right)+\frac{\mu}{2}\|\tilde{e}^{n+1}\|^2\,.$$

If $\Delta t<1/2\kappa$, application of the discrete Gronwall lemma (Lemma 3) gives

$$|\tilde{e}^n|^2+\frac{\mu\Delta t}{1-2\kappa\Delta t}\sum_{j=0}^{n-1}\|\tilde{e}^{j+1}\|^2\le\frac{2\Delta t}{\mu}\sum_{j=0}^{n-1}(1-2\kappa\Delta t)^{j-n}\left(\|\tilde{\rho}^{j+1}\|_*^2+\|e^j\|_*^2\right)\,. \tag{3.7}$$

In the strongly positive case ($\kappa=0$), this estimate holds without an assumption on Δt.

Inequality (3.7), together with Propositions 4, 5, and 6, gives

$$|\tilde{e}^n|^2+\frac{\mu\Delta t}{1-2\kappa\Delta t}\sum_{j=0}^{n-1}\|\tilde{e}^{j+1}\|^2\le\frac{2(\Delta t)^2}{\mu}\mathcal{R}$$

i) under the assumptions $2\kappa\Delta t<1$ and (2.7) with

$$\mathcal{R}=\mathrm{e}^{2\kappa\Delta t+(\omega-2\kappa)t_n}\int_0^{t_n}\mathrm{e}^{2\kappa(t_n-t)}\left(c_1c_3\,\|u(t)\|^2+c_1c_4\,\|f(t)\|_*^2+(1+c_1c_5)\,\|t(f'(t)-u''(t))\|_*^2\right)\,dt\,,$$

ii) under the assumption (2.8) with

$$\mathcal{R}=\mathrm{e}^{2\kappa\Delta t+(\omega_\lambda-2\kappa)t_n}\int_0^{t_n}\mathrm{e}^{2\kappa(t_n-t)}\left(c_2c_3\,\|u(t)\|^2+c_2c_4\,\|f(t)\|_*^2+(1+c_2c_5)\,\|t(f'(t)-u''(t))\|_*^2\right)\,dt\,,$$

iii) under the assumption $2\kappa\Delta t<1$ with

$$\mathcal{R}=\mathrm{e}^{(\omega-2\kappa)t_n}\int_0^{t_n}\mathrm{e}^{2\kappa(t_n-t)}\Big(\frac{4(\beta+\alpha^2\kappa)^4}{3\mu^2}\,\mathrm{e}^{\omega t_n}\left((c_5+\beta^2c_7)\|u(t)\|^2+c_5\|tu'(t)\|^2+c_7\|f(t)\|_*^2\right)$$
$$+\mathrm{e}^{2\kappa\Delta t}\|t(f'(t)-u''(t))\|_*^2\Big)\,dt\,.$$

With the remaining a priori estimates for the exact solution, given by Proposition 7 and (3.24), the proof will be completed. #

The crucial term in (3.7) is $\sum_j\|e^j\|_*^2$ and comes from splitting $t_{n+1}e^n$ into t_ne^n and Δte^n in (3.5). For leading back this term to the consistency error, we firstly consider the following elliptic auxiliary problem:

Problem ($\tilde{P}$) *For given $t\in[0,T]$ and $\psi\in V^*$, find $\phi(t)=:\mathcal{L}(t)\psi\in V$ s. t.*

$$a(t;\phi(t),v)+\kappa(\phi(t),v)=\langle\psi,v\rangle\qquad\forall v\in V\,.$$

Due to the Browder-Minty theorem, ($\tilde{P}$) has a unique solution. However, $\mathcal{L}(t):V^*\to V$ is the inverse of $A(t)+\kappa I$, where $I:V\to V^*$ is the identity, and is bijective, linear and continuous for each $t\in[0,T]$.

Proposition 4 *For $n=1,\,2,\,\ldots,\,N$, it holds with $c_{1,2}$ given in Theorem 1*

$$\sum_{j=0}^{n-1}(1-2\kappa\Delta t)^{j-n}\,\|e^{j+1}\|_*^2\le\begin{cases}c_1\displaystyle\sum_{j=0}^{n-1}(1-2\kappa\Delta t)^{j-n}\,\|\mathcal{L}(t_{j+1})\rho^{j+1}\|_*^2 & \textit{if } 2\kappa\Delta t<1 \textit{ and (2.7)}\\ c_2\displaystyle\sum_{j=0}^{n-1}(1-2\lambda\Delta t)^{j-n}\,\|\mathcal{L}(t_{j+1})\rho^{j+1}\|_*^2 & \textit{if (2.8)}\end{cases} \tag{3.8}$$

Proof We have the equivalence of certain norms:

$$\|\cdot\|_* \sim \langle\,\cdot, \mathcal{L}(t)\cdot\,\rangle^{1/2} \sim \|\mathcal{L}(t)\cdot\|\,.$$

With the boundedness of $a(t;\cdot,\cdot)$, we have with $(\tilde{P})$ and for arbitrary $t \in [0,T]$, $\psi \in V^*$

$$\|\psi\|_* = \sup_{v\in V\setminus\{0\}} \frac{\langle\psi, v\rangle}{\|v\|} = \sup_{v\in V\setminus\{0\}} \frac{a(t;\mathcal{L}(t)\psi, v) + \kappa(\mathcal{L}(t)\psi, v)}{\|v\|} \leq (\beta + \alpha^2\kappa)\|\mathcal{L}(t)\psi\|\,. \tag{3.9a}$$

With Cauchy-Schwarz' inequality and the preceding result, we find

$$\langle\psi, \mathcal{L}(t)\psi\rangle \leq \|\psi\|_*\|\mathcal{L}(t)\psi\| \leq (\beta+\alpha^2\kappa)\|\mathcal{L}(t)\psi\|^2\,. \tag{3.9b}$$

Because of Gårding's inequality for $a(t;\cdot,\cdot)$, we have

$$0 \leq \mu\|\mathcal{L}(t)\psi\|^2 \leq a(t;\mathcal{L}(t)\psi, \mathcal{L}(t)\psi) + \kappa|\mathcal{L}(t)\psi|^2 = \langle\psi, \mathcal{L}(t)\psi\rangle \leq \|\psi\|_*\|\mathcal{L}(t)\psi\|\,,$$

and hence

$$\|\mathcal{L}(t)\psi\| \leq \frac{1}{\sqrt{\mu}}\,\langle\psi, \mathcal{L}(t)\psi\rangle^{1/2}\,, \qquad \|\mathcal{L}(t)\psi\| \leq \frac{1}{\mu}\|\psi\|_*\,. \tag{3.9c}$$

With the error equation (3.3), we conclude

$$e^{n+1} = \mathcal{L}(t_{n+1})\left(\rho^{n+1} - \frac{1}{\Delta t}(e^{n+1}-e^n) + \kappa e^{n+1}\right)\,,$$

and therefore

$$\begin{aligned}
&\frac{1}{\Delta t}\left(\mathcal{L}(t_{n+1})e^{n+1} - \mathcal{L}(t_n)e^n\right) + e^{n+1} \\
&= \mathcal{L}(t_{n+1})\rho^{n+1} + \frac{1}{\Delta t}\left(\mathcal{L}(t_{n+1}) - \mathcal{L}(t_n)\right)e^n + \kappa\mathcal{L}(t_{n+1})e^{n+1}\,,
\end{aligned} \tag{3.10}$$

both equations hold in V^*. Note that $\mathcal{L}(t)$ is differentiable with respect to t since $a(t;\cdot,\cdot)$ is differentiable. We have $\mathcal{L}(t_{n+1})-\mathcal{L}(t_n) = \Delta t\,\mathcal{L}'(t^*)$ for a $t^* \in (t_n, t_{n+1})$. Moreover, differentiation of $(\tilde{P})$ implies for all $\psi \in V^*$, $v \in V$ and $t \in [0,T]$

$$a_t(t;\mathcal{L}(t)\psi, v) + a(t;\mathcal{L}'(t)\psi, v) + \kappa(\mathcal{L}'(t)\psi, v) = 0\,,$$

and hence with $v = \mathcal{L}'(t)\psi$, we obtain

$$\|\mathcal{L}'(t)\psi\| \leq \frac{\beta_t}{\mu}\,\|\mathcal{L}(t)\psi\| \leq \frac{\beta_t}{\mu^2}\,\|\psi\|_*\,. \tag{3.11}$$

Testing (3.10) with $\mathcal{L}(t_{n+1})e^{n+1} \in V$, using (3.6) as well as the norm equivalence (3.9), and (3.11) yield

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(|\mathcal{L}(t_{n+1})e^{n+1}|^2 - |\mathcal{L}(t_n)e^n|^2\right) + \frac{\mu}{(\beta+\alpha^2\kappa)^2}\,\|e^{n+1}\|_*^2 \\
\leq\ &\left(\mathcal{L}(t_{n+1})\rho^{n+1}, \mathcal{L}(t_{n+1})e^{n+1}\right) + \left(\mathcal{L}'(t^*)e^n, \mathcal{L}(t_{n+1})e^{n+1}\right) + \kappa|\mathcal{L}(t_{n+1})e^{n+1}|^2 \\
\leq\ &\frac{1}{\mu}\,\|\mathcal{L}(t_{n+1})\rho^{n+1}\|_*\,\|e^{n+1}\|_* + \frac{\alpha\beta_t}{\mu^2}\,\|e^n\|_*\,|\mathcal{L}(t_{n+1})e^{n+1}| + \kappa|\mathcal{L}(t_{n+1})e^{n+1}|^2 \\
\leq\ &\frac{1}{4\mu\gamma}\,\|\mathcal{L}(t_{n+1})\rho^{n+1}\|_*^2 + \frac{\gamma}{\mu}\,\|e^{n+1}\|_*^2 + \frac{\alpha^2\beta_t\delta}{\mu^3}\,\|e^n\|_*^2 + \frac{\beta_t}{4\mu\delta}\,|\mathcal{L}(t_{n+1})e^{n+1}|^2 + \kappa|\mathcal{L}(t_{n+1})e^{n+1}|^2\,,
\end{aligned} \tag{3.12}$$

where γ, $\delta > 0$ are arbitrary. We now have a few possibilities to proceed with the analysis, depending on the estimate for $|\mathcal{L}(t_{n+1})e^{n+1}|$. However, estimating $|\mathcal{L}(t_{n+1})e^{n+1}| \leq (\alpha/\mu)\|e^{n+1}\|_*$ in the last term of the r. h. s. is not possible since $\mu^3 - \alpha^2\kappa(\beta + \alpha^2\kappa)^2 \leq 0$ if $\kappa \neq 0$. Estimating the last but one term in this way and applying the discrete Gronwall lemma (that again requires the restriction $1 - 2\kappa\Delta t > 0$ if $\kappa \neq 0$) leads to (remembering $e^0 = 0$)

$$|\mathcal{L}(t_n)e^n|^2 + 2\sigma\Delta t \sum_{j=0}^{n-1}(1 - 2\kappa\Delta t)^{j-n} \|e^{j+1}\|_*^2 \leq \frac{\Delta t}{2\mu\gamma} \sum_{j=0}^{n-1}(1 - 2\kappa\Delta t)^{j-n} \|\mathcal{L}(t_{j+1})\rho^{j+1}\|_*^2 ,$$

where

$$\sigma := \frac{1}{\mu}\left(A - \gamma - B(\delta + \frac{1}{4\delta})\right) , \quad A := \frac{\mu^2}{(\beta + \alpha^2\kappa)^2} , \quad B := \frac{\alpha^2\beta_t}{\mu^2} .$$

We then have to ensure $\sigma > 0$. For this, it is necessary to assume $A > B$. In addition, $1/(4\gamma\mu\sigma)$ should be small which leads us to choose $\gamma = (A - B)/2$, $\delta = 1/2$. By this optimal choice, it is $\sigma = \gamma/\mu = (A - B)/2\mu$. So $A > B$, i. e. (2.7), implies the positiveness of σ. The assertion immediately follows.

Leaving the last but one term of the r. h. s. of (3.12) unchanged and combining it with the last term to $\lambda|\mathcal{L}(t_{n+1})e^{n+1}|^2$ with $\lambda := \beta_t/(4\mu\bar{\delta}) + \kappa$ (the bar serves to distinguish from the constants γ, δ, σ used just before) is another way to analyse (3.12). In order to ensure $\bar{\sigma} := (A - \bar{\gamma} - B\bar{\delta})/\mu > 0$ (with the same abbreviations for A, B) and a small error constant, it is nearby to choose $\bar{\gamma} = (A - B\bar{\delta})/2$, $\bar{\delta} = \eta A/B$ with some $0 \ll \eta < 1$. The assertion follows after applying the discrete Gronwall lemma which requires $1 - 2\lambda\Delta t > 0$, i. e. (2.8). #

The advantage of the second strategy, which goes back to HUANG/THOMÉE [4], is that there is no need for additional assumptions on the problem's parameters as with (2.7). But we have the more intrusive bound (2.8), and the error constant will be greater (due to $\lambda \geq \kappa$). The choice of $\bar{\gamma}$, $\bar{\delta}$ might be non-optimal.

Instead of the elliptic auxiliary problem $(\tilde{P})$, we can employ a parabolic duality argument giving an alternative analysis for estimating $\sum_j \|e^j\|_*^2$. The duality trick might be fruitful in the context of nonlinear problems. Moreover, the additional assumptions (2.7), (2.8) can be avoided.

For fixed $n = 1, 2, ..., N$, let us consider the following auxiliary problem.

Problem $(\tilde{P}_{\Delta t})$ *For $\phi^n = 0$ and $\{g^j\}_{j=1}^{n-1} \subseteq V$, given by*

$$a(t_j; w, g^j) + \kappa(w, g^j) = \langle e^j, w\rangle \quad \forall w \in V ,$$

find $\{\phi^j\}_{j=1}^{n-1} \subseteq V$ s. t. for $j = n-1,\ n-2,\ \dots,\ 1$

$$-\frac{1}{\Delta t}(w, \phi^{j+1} - \phi^j) + a(t_j; w, \phi^j) = \langle w, g^j\rangle \quad \forall w \in V .$$

We remark that $(A(t_j)^* + \kappa I)g^j = e^j$ with $A(t_j)^*$ being the dual operator to $A(t_j)$. Thus g^j is well-defined. The problem again is uniquely solvable if (2.5) holds. Since $g^j \in V$, ϕ^j will be better than V.

Proposition 5 *If $\kappa = 0$ or $\Delta t < 1/2\kappa$ then, with constants given in Theorem 1,*

$$\sum_{j=0}^{n-1} \|e^j\|_*^2 \leq \frac{4\Delta t}{3\mu^2}(\beta + \alpha^2\kappa)^4 \mathrm{e}^{\omega t_n} \int_0^{t_n} \mathrm{e}^{-2\kappa t} \left((c_5 + \beta^2 c_7)\|u(t)\|^2 + c_5\|tu'(t)\|^2 + c_7\|f(t)\|_*^2\right) dt .$$

Proof From the definition of g^j, we have analogously to (3.9)

$$\|e^j\|_*^2 \le (\beta+\alpha^2\kappa)^2\|g^j\|^2 \le \frac{(\beta+\alpha^2\kappa)^2}{\mu}\langle e^j, g^j\rangle \le \frac{(\beta+\alpha^2\kappa)^2}{\mu^2}\|e^j\|_*^2 .$$

With $(\tilde{P}_{\Delta t})$ and the error equation (3.3) (remembering $e^0 = \phi^n = 0$), we then have

$$\sum_{j=0}^{n-1}\|e^j\|_*^2 \le \frac{(\beta+\alpha^2\kappa)^2}{\mu}\sum_{j=1}^{n-1}\langle e^j, g^j\rangle = \frac{(\beta+\alpha^2\kappa)^2}{\mu}\sum_{j=1}^{n-1}\langle \rho^j, \phi^j\rangle .$$

With (3.4) and differentiation of (P) with respect to t, it follows

$$\langle \rho^j, \phi^j\rangle = \frac{1}{\Delta t}\int_{t_{j-1}}^{t_j}(t-t_{j-1})\left(a_t(t;u(t),\phi^j) + a(t;u'(t),\phi^j)\right)\,dt .$$

Problem (P) immediately gives

$$\|f(t)-u'(t)\|_* = \sup_{v\in V\setminus\{0\}}\frac{a(t;u(t),v)}{\|v\|} \le \beta\,\|u(t)\|\,, \quad \|u'(t)\|_* \le \beta\,\|u(t)\| + \|f(t)\|_* . \tag{3.13}$$

Because of $a(t;u'(t),\phi^j) = a(t_j;u'(t),\phi^j) - (t_j - t)\,a_t(t^*;u'(t),\phi^j)$ for a $t^*\in(t,t_j)$ and

$$a(t_j;u'(t),\phi^j) = \langle u'(t), A(t_j)^*\phi^j\rangle \le \|u'(t)\|_*\|A(t_j)^*\phi^j\|\,,$$

we then find

$$a_t(t;u(t),\phi^j) + a(t;u'(t),\phi^j) \le$$
$$\beta_t\left(\|u(t)\| + (t_j-t)\|u'(t)\|\right)\|\phi^j\| + \left(\beta\|u(t)\| + \|f(t)\|_*\right)\|A(t_j)^*\phi^j\| .$$

By using Cauchy-Schwarz' and Hölder's inequality, we come up with

$$\begin{aligned}\sum_{j=1}^{n-1}\langle\rho^j,\phi^j\rangle \le\ & \frac{\beta_t}{\sqrt{3}}\left(\left(\int_0^{t_{n-1}} \mathrm{e}^{-2\kappa t}\|u(t)\|^2 dt\right)^{1/2} + \left(\int_0^{t_{n-1}} \mathrm{e}^{-2\kappa t}\|tu'(t)\|^2 dt\right)^{1/2}\right)\times \\ & \times\left(\Delta t\sum_{j=1}^{n-1}(1-2\kappa\Delta t)^{-j}\|\phi^j\|^2\right)^{1/2} + \frac{1}{\sqrt{3}}\left(\beta\left(\int_0^{t_{n-1}} \mathrm{e}^{-2\kappa t}\|u(t)\|^2 dt\right)^{1/2} + \right. \\ & \left. + \left(\int_0^{t_{n-1}} \mathrm{e}^{-2\kappa t}\|f(t)\|_*^2 dt\right)^{1/2}\right)\left(\Delta t\sum_{j=1}^{n-1}(1-2\kappa\Delta t)^{-j}\|A(t_j)^*\phi^j\|^2\right)^{1/2}\end{aligned} \tag{3.14}$$

With $w=\phi^j$ in $(\tilde{P}_{\Delta t})$, we obtain

$$\frac{1}{2\Delta t}\left(|\phi^j|^2 - |\phi^{j+1}|^2\right) + \mu\,\|\phi^j\|^2 \le \kappa\,|\phi^j|^2 + \frac{\mu}{2}\,\|\phi^j\|^2 + \frac{1}{2\mu}\,\|g^j\|_*^2 .$$

Since $\|g^j\|_* \le \alpha^2\|g^j\| \le (\alpha^2/\mu)\,\|e^j\|_*$, the discrete Gronwall lemma, applied in a backward manner, gives for $1-2\kappa\Delta t > 0$ with $\phi^n = 0$

$$\mu\sum_{j=1}^{n-1}(1-2\kappa\Delta t)^{-j}\|\phi^j\|^2 \le \frac{\alpha^4}{\mu^3}\sum_{j=1}^{n-1}(1-2\kappa\Delta t)^{-j}\|e^j\|_*^2 . \tag{3.15}$$

The remaining estimate for $\|A(t_j)^*\phi^j\|$ is somewhat more subtle: We shall set $w = A(t_j)A(t_j)^*\phi^j$ in $(\tilde{P}_{\Delta t})$. With $A(t_{j+1}) - A(t_j) = \Delta t A'(t^*)$ for a $t^*\in(t_j,t_{j+1})$ and (3.6), it follows

$$\begin{aligned}\langle A(t_j)^*\phi^j, A(t_j)^*(\phi^j - \phi^{j+1})\rangle &= \langle A(t_j)^*\phi^j, A(t_j)^*\phi^j - A(t_{j+1})^*\phi^{j+1}\rangle + \Delta t\langle A'(t^*)A(t_j)^*\phi^j, \phi^{j+1}\rangle \\ &\ge \frac{1}{2\Delta t}\left(|A(t_j)^*\phi^j|^2 - |A(t_{j+1})^*\phi^{j+1}|^2\right) - \Delta t\beta_t\|A(t_j)^*\phi^j\|\,\|\phi^{j+1}\|\end{aligned}$$

Hence, we have (with $\|g^j\| \le \|e^j\|_*/\mu$)

$$\frac{1}{2\Delta t}\left(|A(t_j)^*\phi^j|^2 - |A(t_{j+1})^*\phi^{j+1}|^2\right) + \mu\,\|A(t_j)^*\phi^j\|^2$$
$$\le \kappa\,|A(t_j)^*\phi^j|^2 + \Delta t\beta_t\,\|A(t_j)^*\phi^j\|\,\|\phi^{j+1}\| + \frac{\beta}{\mu}\,\|A(t_j)^*\phi^j\|\,\|e^j\|_*$$
$$\le \kappa\,|A(t_j)^*\phi^j|^2 + \frac{\mu}{2}\,\|A(t_j)^*\phi^j\|^2 + \frac{(\Delta t)^2\beta_t^2}{\mu}\,\|\phi^{j+1}\|^2 + \frac{\beta^2}{\mu^3}\,\|e^j\|_*^2 .$$

For $1 - 2\kappa\Delta t \ge 0$, application of the (backward) discrete Gronwall lemma gives ($\phi^n = 0$)

$$\mu\sum_{j=1}^{n-1}(1-2\kappa\Delta t)^{-j}\|A(t_j)^*\phi^j\|^2$$
$$\le \frac{2(\Delta t)^2\beta_t^2}{\mu}\sum_{j=1}^{n-1}(1-2\kappa\Delta t)^{-j}\|\phi^{j+1}\|^2 + \frac{2\beta^2}{\mu^3}\sum_{j=1}^{n-1}(1-2\kappa\Delta t)^{-j}\|e^j\|_*^2 . \tag{3.16}$$

The assertion follows from (3.14) together with (3.15) and (3.16) after some calculations. #

Proposition 6 *Let $t(f' - u'') \in L^2(0,T;V^*)$. If $2\kappa\Delta t < 1$ and $2\lambda\Delta t < 1$, resp., then for $n = 1, 2, \ldots, N$ with constants given in Theorem 1, it holds*

$$\sum_{j=0}^{n-1}(1-2\kappa\Delta t)^{j-n}\,\|\tilde\rho^{j+1}\|_*^2 \le \Delta t\,\mathrm{e}^{2\kappa\Delta t+(\omega-2\kappa)t_n}\int_0^{t_n}\mathrm{e}^{2\kappa(t_n-t)}\|t(f'(t)-u''(t))\|_*^2\,dt \tag{3.17}$$

$$\sum_{j=0}^{n-1}(1-2\kappa\Delta t)^{j-n}\,\|\mathcal{L}(t_{j+1})\rho^{j+1}\|_*^2 \le \Delta t\,\mathrm{e}^{2\kappa\Delta t+(\omega-2\kappa)t_n}\times$$
$$\times\int_0^{t_n}\mathrm{e}^{2\kappa(t_n-t)}\left(c_3\|u(t)\|^2 + c_4\|f(t)\|_*^2 + c_5\|t(f'(t)-u''(t))\|_*^2\right)dt \tag{3.18}$$

$$\sum_{j=0}^{n-1}(1-2\lambda\Delta t)^{j-n}\,\|\mathcal{L}(t_{j+1})\rho^{j+1}\|_*^2 \le \Delta t\,\mathrm{e}^{2\kappa\Delta t+(\omega_\lambda-2\kappa)t_n}\times$$
$$\times\int_0^{t_n}\mathrm{e}^{2\kappa(t_n-t)}\left(c_3\|u(t)\|^2 + c_4\|f(t)\|_*^2 + c_5\|t(f'(t)-u''(t))\|_*^2\right)dt\,. \tag{3.19}$$

Since these estimates are based upon (2.1) and (3.11), e. g. $\|\mathcal{L}'(t)u(t)\|_* \le \alpha^2\,\|\mathcal{L}'(t)u(t)\| \le (\alpha^2\beta_t/\mu^2)\,\|u(t)\|_* \le (\alpha^4\beta_t/\mu^2)\,\|u(t)\|$, they are not optimal, and $\|\mathcal{L}(t)\cdot\|_*$, $\|\mathcal{L}'(t)\cdot\|_*$ are, indeed, norms weaker than $\|\cdot\|_*$. If e. g. $A(t) \equiv -\Delta$ in a sufficiently smooth domain, we would have $\|\cdot\|_* \sim \|\cdot\|_{H^{-1}}$ but $\|\mathcal{L}(t)\cdot\|_* \sim \|\cdot\|_{H^{-3}}$ with the usual Sobolev spaces.

Proof With Hölder's inequality, we obtain from (3.4)

$$(1-2\kappa\Delta t)^{j-n}\|\tilde\rho^{j+1}\|_*^2 \le K_{j+1}^n\int_{t_j}^{t_{j+1}}\mathrm{e}^{2\kappa(t_n-t)}\|t(f'(t)-u''(t))\|_*^2\,dt\,,$$

where

$$K_{j+1}^n := \frac{(1-2\kappa\Delta t)^{j-n}}{(\Delta t)^2}\int_{t_j}^{t_{j+1}}\mathrm{e}^{-2\kappa(t_n-t)}\left(\frac{t_{j+1}(t-t_j)}{t}\right)^2 dt\,.$$

Some elementary calculations lead to $K_{j+1}^n \le \Delta t\,\exp((\omega-2\kappa)t_n + 2\kappa\Delta t)$ for $j = 0, \ldots, n-1$, and hence (3.17) follows.

Because of the commutativity of $\mathcal{L}(t_{j+1})$ for a fixed t_{j+1} with the integral, it holds

$$\|\mathcal{L}(t_{j+1})\rho^{j+1}\|_*^2 \le \left(\frac{1}{\Delta t}\int_{t_j}^{t_{j+1}} (t-t_j)\,\|\mathcal{L}(t_{j+1})(f'(t)-u''(t))\|_*\,dt\right)^2 .$$

Again, there is a $t^* \in (t, t_{j+1})$ s. t. $\mathcal{L}(t_{j+1}) = \mathcal{L}(t) + (t_{j+1}-t)\mathcal{L}'(t^*)$. Moreover, Problem (P) can be rewritten as

$$\mathcal{L}(t)u'(t) - \kappa\mathcal{L}(t)u(t) + u(t) = \mathcal{L}(t)f(t)\,,$$

and differentiation gives

$$\mathcal{L}(t)(f'(t)-u''(t)) = -\mathcal{L}'(t)(f(t)-u'(t)) - \kappa\mathcal{L}'(t)u(t) - \kappa\mathcal{L}(t)u'(t) + u'(t)\,.$$

With (2.1) and (3.11), we obtain for $t \in [t_j, t_{j+1}]$

$$\begin{aligned}\|\mathcal{L}(t_{j+1})(f'(t)-u''(t))\|_* \;&\le\; \|\mathcal{L}'(t)(f(t)-u'(t))\|_* + \kappa\,\|\mathcal{L}'(t)u(t)\|_* \\ &+ \kappa\,\|\mathcal{L}(t)u'(t)\|_* + \|u'(t)\|_* + (t_{j+1}-t)\,\|\mathcal{L}'(t^*)(f'(t)-u''(t))\|_* \\ \le \frac{\alpha^2\beta_t}{\mu^2}\,\|f(t)-u'(t)\|_* + \frac{\alpha^4\kappa\beta_t}{\mu^2}\,\|u(t)\| &+ \left(\frac{\alpha^2\kappa}{\mu}+1\right)\,\|u'(t)\|_* + \frac{\alpha^2\beta_t(t_{j+1}-t)}{\mu^2}\,\|f'(t)-u''(t)\|_*\,.\end{aligned}$$

Observing (3.13), we then come to

$$\|\mathcal{L}(t_{j+1})(f'(t)-u''(t))\|_* \le \sqrt{c_3}\,\|u(t)\| + \sqrt{c_4}\,\|f(t)\|_* + \sqrt{c_5}\,(t_{j+1}-t)\,\|f'(t)-u''(t)\|_*\,.$$

With Cauchy-Schwarz' and Hölder's inequality, and observing

$$\int_{t_j}^{t_{j+1}} \mathrm{e}^{-2\kappa(t_n-t)}\left(\frac{(t_{j+1}-t)(t-t_j)}{t^2}\right)^2 dt \le \frac{(\Delta t)^3}{3}\,\mathrm{e}^{-2\kappa(t_n-t_j)+2\kappa\Delta t}\,,$$

the assertion follows with some elementary calculations. #

Proposition 7 (Smoothing a priori estimates) *Let $u \in \mathcal{W}(0,T;V)$ be the solution to (P) with (2.3). The coming estimates then hold true for $t \in [0,T]$:*

$$|u(t)|^2 + \mu\int_0^t \mathrm{e}^{2\kappa(t-s)}\|u(s)\|^2 ds \le \mathrm{e}^{2\kappa t}|u_0|^2 + \frac{1}{\mu}\int_0^t \mathrm{e}^{2\kappa(t-s)}\|f(s)\|_*^2 ds \tag{3.20}$$

$$\begin{aligned}\frac{\mu}{2}\int_0^t \mathrm{e}^{2\kappa(t-s)}\|s(f'(s)-u''(s))\|_*^2\,ds &\le c_6\,\mathrm{e}^{2\kappa t}|u_0|^2 + \\ + \frac{1}{\mu}(4\beta^2+c_6)\int_0^t \mathrm{e}^{2\kappa(t-s)}\|f(s)\|_*^2 ds &+ \frac{4\beta^2}{\mu}\int_0^t \mathrm{e}^{2\kappa(t-s)}\|sf'(s)\|_*^2 ds\,.\end{aligned} \tag{3.21}$$

Such smoothing a priori estimates are known for the homogeneous case, cf. Huang/Thomée [4], Fujita/Suzuki [1], and Luskin/Rannacher [5].

Although the assumptions (2.2b) and $tf \in L^2(0,T;V)$ (or $\sqrt{t}f \in L^2(0,T;H)$) do not enter the estimate, they ensure that the appearing integrals are well-defined.

Proof Setting $v = u(t)$ in (P) and using Gårding's inequality yields

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \mu\,\|u(t)\|^2 - \kappa\,|u(t)|^2 \le \|f(t)\|_*\,\|u(t)\| \le \frac{1}{2\mu}\|f(t)\|_*^2 + \frac{\mu}{2}\|u(t)\|^2\,.$$

Multiplying by $e^{-2\kappa t}$ and observing $\frac{d}{dt}\left(\mathrm{e}^{-2\kappa t}|u(t)|^2\right) = \mathrm{e}^{-2\kappa t}\left(\frac{d}{dt}|u(t)|^2 - 2\kappa|u(t)|^2\right)$ gives, after integration, (3.20). Note that $u \in \mathcal{C}([0,T];H)$ and therefore $|u(t)| \to |u_0|$ as $t \to 0$.

From differentiating (P) with respect to t, we see that

$$\langle u''(t), v\rangle + a_t(t; u(t), v) + a(t; u'(t), v) = \langle f'(t), v\rangle \qquad \forall v \in V\,,\ \text{a. e. in } (0, T] \tag{3.22}$$

holds (in the distributional sense). It follows

$$\|f'(t) - u''(t)\|_* \leq \beta_t \, \|u(t)\| + \beta \, \|u'(t)\| \,. \tag{3.23}$$

Inserting $v = tu'(t)$ into (P) and $v = t^2u'(t)$ into (3.22), using Cauchy-Schwarz' and Young's inequality as well as the assumptions on the bilinear form lead to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\mathrm{e}^{-2\kappa t}|tu'(t)|^2\right) + \mu\, \mathrm{e}^{-2\kappa t}\|tu'(t)\|^2 \\ &\leq \mathrm{e}^{-2\kappa t}\left(\langle u'(t), tu'(t)\rangle + \langle u''(t), t^2u'(t)\rangle + a(t; u'(t), t^2u'(t))\right) \\ &\leq \mathrm{e}^{-2\kappa t}\left(\frac{2}{\mu}\left(\|f(t)\|_*^2 + \|tf'(t)\|_*^2 + (\beta^2 + \beta_t^2 t^2)\,\|u(t)\|^2\right) + \frac{\mu}{2}\,\|tu'(t)\|^2\right). \end{aligned} \tag{3.24}$$

Integration gives, together with (3.23), the estimate wanted.

For completing the proof, we need to verify that integration and limit process $t \to 0$ are allowed. For this, we have to show that $\sqrt{t}u' \in L^2(0, T; H)$. Then for every positive integer n, there is a $s_n \in (0, 1/n)$ s. t. $|s_n u'(s_n)|^2 < 1/n$. Hence, $|s_n u'(s_n)|^2 \to 0$ as $n \to \infty$, and we may integrate (3.24) over $[s_n, t]$.

With $v = tu'(t)$ in (P), we find

$$\begin{aligned} 2t|u'(t)|^2 + \frac{d}{dt}\left(t\, a(t; u(t), u(t))\right) \;&=\; 2t\langle f(t), u'(t)\rangle + a(t; u(t), u(t)) + t\, a_t(t; u(t), u(t)) \\ &\quad - \left(a(t; u(t), tu'(t)) - a(t; tu'(t), u(t))\right) \end{aligned}$$

In view of $tf \in L^2(0, T; V)$, we have $2t\langle f(t), u'(t)\rangle \leq \|tf(t)\|^2 + \|u'(t)\|_*^2$. With (2.2b), we can split the appearing skew-symmetric part of a and absorb the resulting term $t|u'(t)|^2$ in the left hand side.

Since $u \in L^2(0, T; V)$, there is again a null sequence $\{s_n\}$ s. t. $s_n\|u(s_n)\|^2 \to 0$ as $n \to \infty$. Together with Gårding's inequality and the boundedness of a, integration over $[s_n, t]$ gives

$$\int_{s_n}^t s|u'(s)|^2 ds \leq \kappa t|u(t)|^2 + \beta s_n\|u(s_n)\|^2 + const\,\left(\|tf\|^2_{L^2(0,T;V)} + \|u\|^2_{L^2(0,T;V)} + \|u'\|^2_{L^2(0,T;V^*)}\right).$$

Since $u \in \mathcal{C}([0, T]; H)$, $t|u(t)|^2$ remains bounded. It follows that $\sqrt{t}u' \in L^2(0, T; H)$. #

If $\sqrt{t}f \in L^2(0, T; H)$ instead of $tf \in L^2(0, T; V)$ is assumed, we can estimate $2t\langle f(t), u'(t)\rangle \leq 4t|f(t)|^2 + t|u'(t)|^2/2$, and the rest of the proof remains unchanged giving the same result.

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