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ABSTRACT. In [8] the first author and J.Szenthe proved, for a general homogeneous Riemannian manifold, some existence theorems on geodesics which are orbits of one-parameter groups of isometries. The aim of the present paper is to provide examples showing that the results from [8] are optimal in some sense.

KEYWORDS: Riemannian manifold, Homogeneous space, Geodesics as orbits, k-symmetric space.
1991 MATHEMATICS SUBJECT CLASSIFICATION: 53C20, 53C22, 53C30

1. Introduction.

A connected Riemannian manifold $(M, \langle , \rangle)$ is said to be *homogeneous* if its full isometry group $I(M)$ acts transitively on M. Then M can be always written in the form $M = G/H$, where $G \subset I(M)$ is a connected Lie group acting transitively and effectively on M and H is the isotropy subgroup at some point $o \in M$. In general, we have more than one choice for the group G.

A geodesic $\gamma(\tau)$ through the origin o is said to be *homogeneous* (w.r. to G) if it is an orbit of a one-parameter subgroup of G. This means that

$$\gamma(\tau) = \exp(\tau X)(o), \ \tau \in \mathbb{R} \tag{1}$$

where X is a nonzero vector from the corresponding Lie algebra $\mathfrak{g}$. Obviously, if $\gamma(\tau)$ is homogeneous with respect to some isometry group G then it is homogeneous also with respect to any enlarged group G' of isometries (but the converse is not true, in general).

There exist special homogeneous Riemannian manifolds, so-called g.o. spaces, on which every geodesic is homogeneous with respect to the largest connected group of isometries. For example, all symmetric spaces are g.o. spaces and, more generally, all naturally reductive spaces (see [4]) are g.o. spaces. Yet, the class of all g.o. spaces is broader than that of naturally reductive ones. The first example was given by A.Kaplan in 1983, [3]. Since that time an extensive research has been done in this direction. We mention especially [10] where all those g.o. spaces are classified up to dimension 6 which are in no way naturally

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reductive. Also in [10], many references are given to both naturally reductive spaces and g.o. spaces. From the recent time we mention the joint paper [7] of the first two authors and that of G.Gordon [1].

A natural related problem is the following one: consider a general homogeneous Riemannian manifold $(M, \langle , \rangle) = G/H$. How many homogeneous geodesics (if any) can be found in such a space? V.V.Kajzer [2] proved that there is at least one homogeneous geodesic on a Lie group with a left-invariant metric. Then the first author and J.Szenthe [8] generalized this result (using some ideas from [2]) to the general homogeneous Riemannian manifolds. The main result from [8] can be summarized in two theorems:

Theorem A. *Let $(M, \langle , \rangle) = G/H$ be a homogeneous Riemannian manifold (i.e., G acts on M transitively and effectively as a group of isometries). Then G/H admits at least one homogeneous geodesic through the origin $o \in M$.*

Theorem B. *If, in addition, the group G is semi-simple, then $M = G/H$ admits m mutually orthogonal homogeneous geodesics through the origin o, where $m = \dim M$.*

Now, some natural question arise. For the simplicity, a finite family of geodesics through o is said to be *linearly independent* if the corresponding initial tangent vectors are linearly independent. We put the following problems:

Problem 1. Let $(M, \langle , \rangle) = G/H$ be a homogeneous Riemannian manifold where G denotes the largest connected group of isometries and $\dim M \geq 3$. Does M always admit more than one homogeneous geodesic?

Problem 2. Suppose that $(M, \langle , \rangle) = G/H$ admits $m = \dim M$ linearly independent homogeneous geodesics through o. Does it admit m mutually orthogonal homogeneous geodesics?

In this paper we show that the answers to both problems are negative. From our examples we shall also see that a solvable Lie group with a left-invariant metric can admit m mutally orthogonal homogeneous geodesics, whereas there is a homogeneous space $(G/H, \langle , \rangle)$ satisfying $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ (thus G is "opposite to solvable" but not necessarily semi-simple) which admits m linearly independent homogeneous geodesics but not m orthogonal ones.

2. Technicalities.

Let, as before, $(M, \langle , \rangle) = G/H$ be a homogeneous Riemannian manifold with a given origin o and let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. Then G/H is a *reductive space* (cf.[4]) in the sense that there exists an $\mathrm{Ad}(H)$-invariant direct sum decomposition

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{h} \tag{2}$$

where $\mathfrak{m} \subset \mathfrak{g}$ is a linear subspace. (See [8], Proposition 1, for a rigorous proof.) There is a natural identification (canonical isomorphism) of $\mathfrak{m} \subset \mathfrak{g} \equiv T_eG$ with the tangent space T_oM via the projection $\pi : G \to G/H = M$. Using this natural identification and the scalar product $\langle , \rangle_o$ on T_oM we obtain a scalar product B on $\mathfrak{m}$, which is obviously $\mathrm{Ad}(H)$-invariant.

A nonzero vector $X \in \mathfrak{g}$ for which the trajectory (1) is a geodesic curve is called a *geodesic vector.* We shall use the following result (cf.[10], p.194, and also [1] and [7].)

Lemma 1.1. *Under the previous assumptions and notations, a nonzero vector $X \in \mathfrak{g}$ is geodesic if and only if*

$$B([X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}}) = 0 \tag{3}$$

holds for every $Z \in \mathfrak{m}$.

(Here $\mathfrak{m}$ written as subscript indicates the $\mathfrak{m}$-component of a vector from $\mathfrak{g}$ with respect to the decomposition (2).)

Now, the problem of finding all homogeneous geodesics for a given space $(M, \langle , \rangle)$ is reduced to the following steps:

(a) We calculate the connected component G of the full isometry group $I(M)$, or at least the corresponding Lie algebra $\mathfrak{g}$.

(b) We find a decomposition of the form (2).

(c) We choose a convenient basis $\{E_1, \ldots, E_m\}$ of $\mathfrak{m}$ and a basis $\{F_1, \ldots, F_r\}$ of $\mathfrak{h}$. Then we look for the geodesic vectors in the form

$$X = \sum_{i=1}^{m} x^i E_i + \sum_{j=1}^{r} a^j F_j \ . \tag{4}$$

The equation (3) gives a system of m quadratic equations for the variables x^i, a^j if we substitute $Z = E_i$ for $i = 1, \ldots, m$. (These equations can be linearly dependent in general, see [7] for the explicit formula.)

(d) We determine for which values of $x^1, \ldots, x^m$ and $a^1, \ldots a^r$ the algebraic equations obtained in (c) are satisfied. Such sets of values for which $x^1, \ldots, x^m$ are not all equal to zero define geodesic vectors.

Let us remark that $(M, \langle , \rangle)$ is a g.o. space if, and only if, for every nonzero m-tuplet $(x^1, \ldots, x^m)$ there is at least one r-tuplet $(a^1, \ldots, a^r)$ satisfying all basic quadratic equations. This is not the case in general.

The following proposition is obvious:

Proposition 2.1. *A finite family $(\gamma_1, \ldots, \gamma_k)$ of homogeneous geodesics through $o \in M$ is orthogonal, or linearly independent, respectively, if the $\mathfrak{m}$-components of the corresponding geodesic vectors are orthogonal, or linearly independent, respectively.*

3. Three-dimensional examples.

We start with the following result (see [11], p.321).

Proposition 3.1. *Let $(G, \langle , \rangle)$ be a 3-dimensional non-unimodular Lie group equipped with a left-invariant Riemannian metric and let $\mathfrak{g}$ be the corresponding Lie algebra.*

Then there is an orthonormal basis (X, Y, Z) of $\mathfrak{g}$ such that the multiplication table of $\mathfrak{g}$ has the form

$$[X,Y] = \alpha Y + \beta Z,\ [X,Z] = \gamma Y + \delta Z,\ [Y,Z] = 0, \tag{5}$$

where $\alpha, \beta, \gamma, \delta$ are real numbers such that $\alpha + \delta \neq 0$ and $\alpha\gamma + \beta\delta = 0$. The basis (X, Y, Z) also diagonalizes the Ricci form and the principal Ricci curvatures are given by the expressions

$$\begin{aligned} r(X) &= -\alpha^2 - \delta^2 - \frac{1}{2}(\beta+\gamma)^2 , \\ r(Y) &= -\alpha(\alpha+\delta) + \frac{1}{2}(\gamma^2 - \beta^2) , \\ r(Z) &= -\delta(\alpha+\delta) + \frac{1}{3}(\beta^2 - \gamma^2) . \end{aligned} \tag{6}$$

It is obvious that the principal Ricci curvatures (6) are distinct in general. Now we are going to prove

Proposition 3.2. *Let $\alpha, \beta, \gamma, \delta$ be such that all Ricci eigenvalues are distinct. Denote $D = (\beta+\gamma)^2 - 4\alpha\delta$. Then, up to a parametrization, the space $(G, \langle , \rangle)$ admits*

a) just one homogeneous geodesic through a point if $D < 0$,

b) just two homogeneous geodesics through a point if $D = 0$; they are mutually orthogonal,

c) just three homogeneous geodesics through a point if $D > 0$; they are linearly independent but never mutually orthogonal.

Proof. First we see that G itself acts on $(G, \langle , \rangle)$ from the left as the maximal connected group of isometries (because each isotropy group of $I(G)$ is finite). Hence each homogeneous geodesic in $(G, \langle , \rangle)$ is generated by a vector $U \in \mathfrak{g}$ of the form $U = aX + bY + cZ$. The condition (3) then leads to the system of equations

$$\alpha\beta^2 + \delta c^2 + (\beta+\gamma)bc = 0,\ a(\alpha b + \beta c) = 0,\ a(\gamma b + \delta c) = 0. \tag{7}$$

Obviously, X is a geodesic vector. Further, D is the discriminant of the first equation (7) and hence we have either zero, or one, or two additional geodesic vectors according to the sign of D. If $D = 0$, the additional geodesic vector is of the form $U = bY + cZ$, and hence orthogonal to X. If $D > 0$ then we get two additional geodesic vectors of the form $b_1 Y + c_1 Z$ and $b_2 Y + c_2 Z$. Then the orthogonality condition would imply $\alpha + \delta = 0$, which is forbidden by Proposition 3.1. □

Let us notice that a non-unimodular Lie group from Proposition 3.1 is always solvable.

Now, consider a solvable Lie algebra with the orthonormal basis (X, Y, Z) satisfying the multiplication table

$$[X,Y] = 0,\ [X,Z] = \alpha X,\ [Y,Z] = -\alpha Y,\ \alpha \neq 0. \tag{8}$$

The corresponding simply connected group G is unimodular and it is usually denoted as $E(1,1)$ (cf.[11]). We have a one-parameter system of invariant metrics on G. Two of the

Ricci eigenvalues are zero. Nevertheless, each isotropy subgroup is finite (see [6], p.19, for more details). An easy calculation shows that the only geodesic vectors are multiples of $X+Y$, $X-Y$ and Z, which form an orthogonal triplet. Hence there are just three homogeneous geodesics through each point and they are mutually orthogonal. This family of examples is interesting for two reasons:

1) These are the only generalized symmetric spaces in dimension 3 which are not locally symmetric (see [6]).

2) These are the only nonsymmetric homogeneous Riemannian manifolds which can be isometrically immersed (not embedded) as hypersurfaces in a space of constant negative curvature (see [12], [13]). It means, a.o., that there are not such examples in higher dimensions. T.Takahashi calls these spaces "B-spaces".

Remark. V.V.Kajzer in [2] studies in detail 3-dimensional Lie groups with left-invariant metrics using also the Milnor's paper [11]. But he is occupied with the problem of existence or non-existence of conjugate points on homogeneous geodesics.

4. Four-dimensional example.

Next we shall study a 4-dimensional example which has some more interesting properties. For example, it admits infinitely many quadruplets of linearly independent homogeneous geodesics but no orthogonal quadruplet. The underlying manifold is $\mathbb{R}^4[x,y,u,v]$ with the Riemannian metric

$$\begin{aligned} g =& (-x+\sqrt{x^2+y^2+1})\,du^2 + (x+\sqrt{x^2+y^2+1})\,dv^2 - 2y\,du\,dv \\ &+ \lambda^2\left[\frac{(1+y^2)\,dx^2 + (1+x^2)\,dy^2 - 2xy\,dx\,dy}{1+x^2+y^2}\right], \quad \text{where } \lambda > 0. \end{aligned} \tag{9}$$

The space $(\mathbb{R}^4, g)$ can be written as a homogeneous space G/H where G is the 5-dimensional group of equiaffine transformations of a Euclidean space and H is the subgroup of all rotations of the plane around the origin. (See [6], pp.136 and 139-140 for more details.) For the simplicity we choose $\lambda = 1$. Then there exists a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, an orthonormal basis (X_1, Y_1, X_2, Y_2) of $\mathfrak{m}$ and a generator B of $\mathfrak{h}$ such that the following multiplication table holds (cf.[5], p.19):

$$\begin{aligned} &[X_1, Y_1] = 0, \quad [X_1, X_2] = -X_1, \; [X_1, Y_2] = X_1, \\ &[Y_1, X_2] = Y_1, \quad [Y_1, Y_2] = X_1, \quad\; [X_2, Y_2] = -2B, \\ &[B, X_1] = -Y_1, \; [B, Y_1] = X_1, \quad\; [B, X_2] = 2Y_2, \quad [B, Y_2] = -2X_2. \end{aligned} \tag{10}$$

Obviously we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

By a routine but lengthy calculation one can prove that G is the maximal connected group of isometries of $(\mathbb{R}^4, g)$ (see [5], Theorem 13.5). Hence each geodesic vector must be an element of $\mathfrak{g}$, let us say $U = aX_1 + bY_1 + cX_2 + dY_2 + \alpha B$. From the condition (3) we obtain the following system of quadratic equations:

$$a(c-d) = \alpha b, \tag{11}$$

$$c(a+b) = \alpha(a+b), \tag{12}$$

$$a(a+b) = 2\alpha c, \tag{13}$$

$$(a-b)(a+b) = 2\alpha d\,. \tag{14}$$

We shall analyze the possible solutions for $(a, b, c, d) \neq (0,0,0,0)$.

Proposition 4.1. *Every nontrivial solution of (11)-(14) is either of the form*
(A) $b = (2c^2 - a^2)/a,\ d = 2c(a^2 - c^2)/a^2,\ \alpha = c,$ *where* $a \neq 0$ *and* $c \neq 0$ *are arbitrary, or of the form*
(B) $a = b = \alpha = 0$ *and* c, d *are arbitrary,* $(c, d) \neq (0,0)$, *or of the form*
(C) $b = -a, d = c,\ \alpha = 0,$ *where* $a \neq 0$ *and* c *are arbitrary.*

Proof. Suppose first that $a + b \neq 0$ and $\alpha \neq 0$. From (12) we obtain $c = \alpha \neq 0$ and (13) implies $a \neq 0$, $b = (2c^2 - a^2)/a$. From (14) we get $d = (a^2 - b^2)/2c$. Now, replacing b by its previous expression we get $d = 2c(a^2 - c^2)/a^2$. We check easily that this solves (11)–(14).

Suppose now $a + b \neq 0$ and $\alpha = 0$. From (13) and (14) we get $a = 0,\ b = 0$, which is a contradiction.

Next, suppose $a + b = 0$ and $\alpha \neq 0$. From (13) and (14) we get $c = d = 0$. Due to (11) we obtain $b = 0$ and consequently $a = 0$. We get only a trivial solution.

Finally, suppose $a + b = 0$ and $\alpha = 0$. From (11) we obtain $a(c - d) = 0$. We have two possibilities and hence the cases (B) and (C) arise. □

We shall need more preliminary results. For the sake of brevity, vectors from $\mathfrak{g}$ will be said to be *linearly independent* or *orthogonal* if their $\mathfrak{m}$-components are linearly independent or orthogonal, respectively.

Proposition 4.2. *At most two geodesic vectors of type (A) are mutually orthogonal.*

Proof. Let us introduce a new parameter $t = c/a$. Then each quadruplet (a, b, c, d) of type (A) can be written in the form $\big(a,\ a(2t^2 - 1),\ at,\ 2at(1 - t^2)\big),\ t \neq 0$. In particular, it is a scalar multiple of a vector

$$\mathbf{e}_t \ = \ \big(1,\ 2t^2 - 1,\ t,\ 2t(1 - t^2)\big),\ t \in \mathbb{R} \setminus \{0\}\ . \tag{15}$$

Assume that $t \neq 0$ is fixed. Let $\mathbf{e}_x = \big(1, 2x^2 - 1, x, 2x(1 - x^2)\big)$ be another vector of this type, $x \neq t$, and check the orthogonality condition $\langle \mathbf{e}_t, \mathbf{e}_x \rangle = 0$. We get for x the cubic equation

$$4x(1 - x^2)t(1 - t^2) + (2x^2 - 1)(2t^2 - 1) + xt + 1 \ = \ 0\ . \tag{16}$$

First we eliminate the special case $t = \pm 1$. Then (16) has only one nonzero solution, namely $x = -\frac{1}{2}$ or $x = \frac{1}{2}$, respectively. So an orthogonal triplet of geodesic vectors of type (A) cannot arise in this way.

Let now $t \neq \pm 1$. Then the equation (16) is cubic and it decomposes in the form

$$(2xt + 1)(2x^2t^2 - 2x^2 + xt - 2t^2 + 2) \ = \ 0\ . \tag{17}$$

The corresponding roots are the following:

$$x_1 = -\frac{1}{2t},\ x_{2,3} = \frac{-t \pm \sqrt{16(t^2 - 1)^2 + t^2}}{4(t^2 - 1)}\ .$$

Hence all three roots are real ones. Now let us denote by x_{ij} $(1 \le i < j \le 3)$ the scalar product $\langle \mathbf{e}_{x_i}, \mathbf{e}_{x_j} \rangle$. First we calculate

$$x_{12} = \frac{1+2t^2}{32t^2(t-1)^3(t+1)^3} \cdot \tilde{x}_{12}, \text{ where}$$
$$\tilde{x}_{12} = 10t^4 - 17t^2 + 8 + (t - 2t^3)\sqrt{16(t^2-1)^2 + t^2}\,.$$

Put $L = 10t^4 - 17t^2 + 8$, $P = (2t^3 - t)\sqrt{16(t^2-1)^2+t^2}$. Then $\tilde{x}_{12} = L - P$. On the other hand we get $L^2 - P^2 = -32\big(2(t^2-1)^2 + t^2\big)(t-1)^3(t+1)^3 \neq 0$. Hence $\tilde{x}_{12} \neq 0$ and $x_{12} \neq 0$. In the same way we show that $x_{13} \neq 0$. Finally we get $x_{23} = \frac{2(t^2-1)^2+t^2}{2(t+1)^2(t-1)^2} > 0$. We conclude that an orthonormal triplet of type (A) cannot exist. □

Proposition 4.3. *Three geodesic vectors of type (B), or those of type (C), respectively, are always linearly dependent.*

Proof. Obvious.

We now formulate the basic result about our 4-dimensional example $(\mathbb{R}^4, g)$ where g is given by (9) with $\lambda = 1$.

Theorem 4.4. $(\mathbb{R}^4, g)$ *admits a continuum of quadruplets of linearly independent homogeneous geodesics through the origin o but never an orthogonal quadruplet.*

Proof. The first part is the immediate consequence of the following

Proposition 4.5. *There is a one-parameter family $\mathcal{Z}$ of homogeneous geodesics through the origin $o \in \mathbb{R}^4$ such that every four elements $\gamma_1, \dots, \gamma_4 \in \mathcal{Z}$ are linearly independent.*

Proof. Define $\mathcal{Z}$ by a parametric expression for the corresponding geodesic vectors, namely put

$$a = 1,\ b = 2t^2 - 1,\ c = \alpha = t,\ d = 2t(1-t^2),\ t \in \mathbb{R}\,.$$

(For $t \neq 0$ we get geodesic vectors of type (A) and for $t = 0$ that of type (C) - see formula (15).) Then for every quadruplet (t_1, t_2, t_3, t_4) of distinct values of t the corresponding determinant $D[a,b,c,d](t_1,t_2,t_3,t_4)$ is equal to $4\prod_{1\le i<j\le 4}(t_j - t_i) \neq 0$. This means the linear independence of the corresponding geodesics. □

Now, we are going to prove the second part of Theorem 4.4. Suppose that there exists an orthogonal quadruplet $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ of geodesic vectors. According to Propositions 4.2 and 4.3 only the following cases are possible (up to a numeration):

(i) $\mathbf{e}_1, \mathbf{e}_2$ are of type (C) and $\mathbf{e}_3, \mathbf{e}_4$ are of type (B). We can write

$$\mathbf{e}_1 = (a_1,\ -a_1,\ c_1,\ c_1)\,,$$
$$\mathbf{e}_2 = (a_2,\ -a_2,\ c_2,\ c_2)\,,$$
$$\mathbf{e}_3 = (0,\ 0,\ c_3,\ d_3)\,,$$
$$\mathbf{e}_4 = (0,\ 0,\ c_4,\ d_4)$$

and these vectors are linearly dependent, which is a contradiction.

(ii) $\mathbf{e}_1$ is of type (A) and $\mathbf{e}_2, \mathbf{e}_3$ of type (B), e.g.,

$$\begin{aligned} \mathbf{e}_1 &= (1,\, 2t^2-1,\, t,\, 2t(1-t^2)),\ t \neq 0\ , \\ \mathbf{e}_2 &= (0,\, 0,\, c_2,\, d_2)\ , \\ \mathbf{e}_3 &= (0,\, 0,\, c_3,\, d_3)\ . \end{aligned}$$

Here $c_2d_3 - c_3d_2 \neq 0$ and the orthogonality conditions $\langle \mathbf{e}_1, \mathbf{e}_2\rangle = \langle \mathbf{e}_1, \mathbf{e}_3\rangle = 0$ imply $t = 0$, a contradiction.

(iii) $\mathbf{e}_1$ is of type (A) and $\mathbf{e}_2, \mathbf{e}_3$ are of type (C), e.g.,

$$\begin{aligned} \mathbf{e}_1 &= (1,\, 2t^2-1,\, t,\, 2t(1-t^2))\ , \\ \mathbf{e}_2 &= (a_2,\, -a_2,\, c_2,\, c_2)\ , \\ \mathbf{e}_3 &= (a_3,\, -a_3,\, c_3,\, c_3) \end{aligned}$$

where $a_2c_3 - a_3c_2 \neq 0,\ t \neq 0$.
The orthogonality conditions $\langle \mathbf{e}_1, \mathbf{e}_2\rangle = \langle \mathbf{e}_1, \mathbf{e}_3\rangle = 0$ mean

$$\begin{aligned} a_2(-2t^2+2) + c_2(3t-2t^3) &= 0\ , \\ a_3(-2t^2+2) + c_3(3t-2t^3) &= 0\ , \end{aligned}$$

i.e., $t^2 = 1$ and $2t^2 = 3$, which is a contradiction.

(iv) $\mathbf{e}_1, \mathbf{e}_2$ are of type (A), $\mathbf{e}_3$ is of type (B) and $\mathbf{e}_4$ is of type (C), i.e.,

$$\begin{aligned} \mathbf{e}_1 &= (1,\, 2x^2-1,\, x,\, 2x(1-x^2))\ , \\ \mathbf{e}_2 &= (1,\, 2y^2-1,\, y,\, 2y(1-y^2))\ , \\ \mathbf{e}_3 &= (0,\, 0,\, c_3,\, d_3)\ , \\ \mathbf{e}_4 &= (1,\, -1,\, c_4,\, c_4) \end{aligned}$$

where $x \neq y,\ x \neq 0,\ y \neq 0$ and $(c_3, d_3) \neq (0,0)$. The condition $\langle \mathbf{e}_3, \mathbf{e}_4\rangle = 0$ implies either $c_3 = -d_3 \neq 0$ or $c_4 = 0$. In the first case $\langle \mathbf{e}_1, \mathbf{e}_3\rangle = 0$ and $\langle \mathbf{e}_2, \mathbf{e}_3\rangle = 0$ imply $\{x, y\} = \{1/\sqrt{2}, -1/\sqrt{2}\}$ and $\langle \mathbf{e}_1, \mathbf{e}_4\rangle = \langle \mathbf{e}_2, \mathbf{e}_4\rangle = 0$ imply $1 + \sqrt{2}\, c_4 = 0,\ 1 - \sqrt{2}\, c_4 = 0$, which is a contradiction. In the second case $\langle \mathbf{e}_1, \mathbf{e}_4\rangle = \langle \mathbf{e}_2, \mathbf{e}_4\rangle = 0$ imply $\{x, y\} = \{1, -1\}$. Then $\langle \mathbf{e}_1, \mathbf{e}_2\rangle = 1 \neq 0$ which is a contradiction. □

Let us remark that orthogonal *triplets* of homogeneous geodesics always exist on $(\mathbb{R}^4, g)$. This is left to the reader as an easy exercise.

Remark. The family of spaces defined by (9) has some remarkable properties:

a) These are the only generalized symmetric spaces of dimension 4 which are not locally symmetric. More specifically, they are all 3-symmetric (see [5], [6]).

b) These are the only 4-dimensional homogeneous Riemannian manifolds which admit a self-dual or anti-self-dual homogeneous structure of class $\mathcal{T}_2$. (See [9] for the result and [14] for the general theory of homogeneous structures.)

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