# Optimal investments for robust utility functionals in complete market models

Alexander Schied*
Institut für Mathematik, MA 7-4
TU Berlin, Strasse des 17. Juni 136
10623 Berlin, Germany
e-mail: schied@math.tu-berlin.de

April 4, 2003, this version: March 19, 2004

**Abstract:** We introduce a systematic approach to the problem of maximizing the robust utility of the terminal wealth of an admissible strategy in a general complete market model, where the robust utility functional is defined by a set $\mathcal{Q}$ of probability measures. Our main result shows that this problem can be reduced to determining a "least favorable" measure $Q_0 \in \mathcal{Q}$, which is universal in the sense that it does not depend on the particular utility function. The robust problem is thus equivalent to a standard utility maximization problem with respect to the "subjective" probability measure $Q_0$. By using the Huber-Strassen theorem from robust statistics, it is shown that $Q_0$ always exists if $\mathcal{Q}$ is the core of a 2-alternating upper probability. We also discuss the problem of robust utility maximization with uncertain drift in a Black-Scholes market and the case of "weak information" as studied by Baudoin (2002).

## 1 Introduction

The problem of constructing utility-maximizing investment strategies in complete and incomplete market models has been a major theme of mathematical finance throughout the past decade. Today, the problem is very well understood, in particular through the efforts of Kramkov and Schachermayer [18], [19]; see also [23] and Karatzas and Shreve [17] for the history of the problem and an overview of further developments.

Economists, however, have long been arguing that the paradigm of von Neumann-Morgenstern expected utility, in both its objective and subjective forms, has various deficiencies. In its objective form, it requires precise knowledge of the probability distribution governing the market evolution, but this distribution is typically subject to

*Supported by a Heisenberg fellowship and the DFG Research Center "Mathematics for key technologies" (FZT 86).

uncertainty. In its subjective form, uncertainty is taken into account by means of a "subjective probability measure", but this approach is open to criticism from the celebrated Ellberg paradox. In the late 1980's, Gilboa and Schmeidler [11], [25], [12] and Yaari [26] formulated natural axioms which should be satisfied by a preference order on payoff profiles in order to account for both risk and uncertainty aversion. They also showed that such a preference order can be numerically represented by a *robust utility functional* of the form

$$X \longmapsto \inf_{Q \in \mathcal{Q}} E_Q[\, U(X)\,]\,, \tag{1}$$

where $\mathcal{Q}$ is a set of probability measures and $U$ is a utility function.

For a special set $\mathcal{Q}$, Baudoin [4] solved the problem of maximizing the robust utility of the terminal wealth in a complete financial market model. In this note, we propose a systematic approach to the construction of optimal investment strategies for robust utility functionals. More precisely, we give a complete solution of the problem of maximizing the robust utility of the terminal wealth in a complete market model, under the condition that the set $\mathcal{Q}$ admits a "least favorable measure" $Q_0$, which is independent of the utility function. The result is that the robust problem is then equivalent to the standard utility maximization problem with respect to $Q_0$. Thus, although the preference order associated with (1) does not satisfy the axioms of (subjective) expected utility, optimal investment decisions are still made in accordance with the Savage/Anscombe-Aumann theory, provided that one takes $Q_0$ as "subjective" probability measure. By means of the measure $Q_0$, we will be able to translate the results by Kramkov and Schachermayer [18] and others to our robust setting.

We also discuss the existence and construction of the least favorable measure $Q_0$, which typically arises from $\mathcal{Q}$ in a non-linear way. For instance, if the set $\mathcal{Q}$ is the core of a 2-alternating Choquet capacity, then $Q_0$ is obtained by an application of the Neyman-Pearson lemma for capacities. This result was developed thirty years ago by Huber and Strassen [15] with the purpose of constructing optimal statistical tests for composite hypotheses and alternatives. The assumption that $\mathcal{Q}$ arises from a 2-alternating capacity is quite natural and includes examples such as convex distortions of probability measures or neighborhoods with respect to many standard probability metrics. We will also show that Baudoin's "weak information" [4] fits into this situation.

We also consider the problem of robust utility maximization in a standard Black-Scholes market with uncertain drift. Here, the set $\mathcal{Q}$ is not related to a 2-alternating capacity. Nevertheless, a least favorable measure $Q_0$ can be constructed by comparing option prices under uncertain *volatility* as in El Karoui et al. [9]. Huber [14] Augustin [1] give further examples of least favorable measures for sets that do not necessarily fall into the range of the Huber-Strassen theorem.

This note is organized as follows. In the next section, we describe our model and the main results. Explicit examples are provided in Section 3: First we discuss robust utility maximization in a Black-Scholes market with uncertain drift. Then we recall the notion of a Radon-Nikodym derivative for capacities, and discuss several examples within the framework of the Huber-Strassen theory. In particular, we prove that the case of "weak information" corresponds to a 2-alternating capacity. Then we briefly review further examples from robust statistics. The proofs of our main results are given in Section 4.

## 2 Main results

We make the standard assumptions on our market model. That is, we consider a complete market model consisting of one bond and $d$ risky assets, whose price processes are denoted $S = (S_t^i)_{0\le t\le T, i=1,\dots,d}$. We may assume without loss of generality that the price of the bond is constant. The process $S$ is assumed to be a semimartingale on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$, and we emphasize that this includes the case of a discrete-time market model, in which prices are adjusted only at times $t = 0, 1, \dots, T$: just set $S_t := S_{[t]}$ and $\mathcal{F}_t := \mathcal{F}_{[t]}$ for arbitrary $t \in [0, T]$. We assume that $\mathcal{F}_0$ is $P$-trivial and that the market is complete in the sense that there exists a unique probability measure $\widehat{P}$ that is equivalent to $P$ and under which $S$ is a $d$-dimensional local martingale. In a discrete-time setting, market completeness implies that $\Omega$ can be chosen as a finite set, and this will simplify certain assumptions on our set $\mathcal{Q}$.

A self-financing trading strategy can be regarded as a pair $(x, \xi)$, where $x \in \mathbb{R}$ is the initial investment and $\xi = (\xi_t^i)_{0\le t\le T, i=1,\dots,d}$ is a predictable and $S$-integrable process. The value process $X$ associated with $(x, \xi)$ is given by $X_0 = x$ and

$$X_t = X_0 + \int_0^t \xi_r \, dS_r \,, \qquad 0 \le t \le T \,.$$

For $x \in \mathbb{R}$ given, we denote by $\mathcal{X}(x)$ the set of all such processes $X$ with $X_0 \le x$ which are admissible in the sense that $X_t \ge 0$ for $0 \le t \le T$ and whose terminal wealth $X_T$ has a well-defined robust utility

$$\inf_{Q\in\mathcal{Q}} E_Q[\, U(X_T)\,] \tag{2}$$

in the sense that

$$\inf_{Q\in\mathcal{Q}} E_Q[\, U(X_T) \wedge 0\,] > -\infty \,. \tag{3}$$

Here, $U : (0, \infty) \to \mathbb{R}$ is an increasing and strictly concave utility function. Now we can state our main problem:

$$\text{maximize } \inf_{Q\in\mathcal{Q}} E_Q[\, U(X_T)\,] \text{ among all } X \in \mathcal{X}(x). \tag{4}$$

**Definition 2.1** *$Q_0 \in \mathcal{Q}$ is called a* least favorable measure *with respect to $\widehat{P}$ if the density $\pi = d\widehat{P}/dQ_0$ (taken in the sense of the Lebesgue decomposition) satisfies*

$$Q_0[\, \pi \le t\,] = \inf_{Q\in\mathcal{Q}} Q[\, \pi \le t\,] \qquad \textit{for all } t > 0. \tag{5}$$

The preceding definition makes sense without assuming any relations of absolute continuity between $\widehat{P}$ and the members of $\mathcal{Q}$. However, we will assume throughout this note that $\mathcal{Q}$ is equivalent to $P$ in the following sense:

$$P[\,A\,]=0 \qquad \Longleftrightarrow \qquad Q[\,A\,]=0 \text{ for all } Q\in\mathcal{Q}. \tag{6}$$

Note that our problem (4) would not be well-posed without the implication "$\Rightarrow$" in this assumption. The converse implication is economically natural, since a position with a positive price should lead to a non-vanishing utility.

Now we can state our first main result. It reduces the robust utility maximization problem to a standard utility maximization problem plus the computation of a least favorable measure, which is *independent* of the utility function.

**Theorem 2.2** *Suppose that $\mathcal{Q}$ admits a least favorable measure $Q_0\approx\widehat{P}$. Then the robust utility maximization problem* (4) *is equivalent to the standard utility maximization problem with respect to $Q_0$, i.e., to* (4) *with $\mathcal{Q}$ replaced by $\mathcal{Q}_0:=\{Q_0\}$.*

This result has the following striking economic consequence. Let $\succ$ denote the preference order induced by our robust utility functional, i.e.,

$$X\succ Y \qquad \Longleftrightarrow \qquad \inf_{Q\in\mathcal{Q}} E_Q[\,U(X)\,] > \inf_{Q\in\mathcal{Q}} E_Q[\,U(Y)\,]\,.$$

Then, although $\succ$ does not satisfy the axioms of (subjective) expected utility theory, optimal investment decisions with respect to $\succ$ are still made in accordance with the Savage/Anscombe-Aumann version of expected utility, provided that we take $Q_0$ as the subjective probability measure.

By combining Theorem 2.2 with Proposition 3.1 below, we are now able to translate Theorem 2.0 of [18] to our situation. To this end, we have to assume that $U$ is continuously differentiable and satisfies the Inada conditions

$$U'(0):=\lim_{x\downarrow 0}U'(x)=+\infty \qquad \text{and} \qquad U'(\infty):=\lim_{x\uparrow\infty}U'(x)=0\,.$$

We denote by

$$u(x)=\sup_{X\in\mathcal{X}(x)}\inf_{Q\in\mathcal{Q}} E_Q[\,U(X_T)\,]\,, \qquad x>0\,,$$

the value function of the problem (4). Since $u(x)\ge U(x)$ for all $x$, our condition (3) on $\mathcal{X}(x)$ poses no restriction. Let

$$V(y)=\sup_{x>0}\big[U(x)-xy\big]\,, \qquad y>0\,,$$

denote the convex conjugate of $U$ and define the function

$$I:=-V'=(U')^{-1}\,.$$

We also define the convex function

$$v(y)=\inf_{Q\in\mathcal{Q}} E_Q[\,V(y\cdot\pi)\,]\,, \qquad y>0\,.$$

**Corollary 2.3** *Suppose that $\mathcal{Q}$ admits a least favorable measure $Q_0$ which is equivalent to $P$ and that $u(x)$ is finite for some $x > 0$. Then:*

*(a) $u(x)$ is finite for all $x > 0$, and $v(y) < \infty$ for $y > 0$ sufficiently large. The function $v$ is continuously differentiable in the interior $(y_0, \infty)$ of its effective domain. The function $u$ is continuously differentiable on $(0, \infty)$ and strictly concave on $(0, x_0)$, where $x_0 := -\lim_{y\downarrow y-0} v'(y)$. For $x, y > 0$,*

$$v(y) = \sup_{x>0} \big[u(x) - xy\big] \qquad \textit{and} \qquad u(x) = \inf_{y>0} \big[v(y) + xy\big] .$$

*Moreover, $u'(0) := \lim_{x\downarrow 0} u'(x) = \infty$ and $v'(\infty) = \lim_{y\uparrow\infty} v'(y) = 0$.*

*(b) For $x < x_0$ there exists a unique solution $X^*(x) \in \mathcal{X}(x)$ of (4), and its terminal wealth is of the form*

$$X_T^*(x) = I(y \cdot \pi) , \qquad \textit{for } y = u'(x).$$

*(c) For $0 < x < x_0$ and $y < y_0$,*

$$u' = x^{-1} \sup_{Q\in\mathcal{Q}} E_Q\big[\, X_T^*(x) U'(X_T^*(x)) \,\big] \qquad \textit{and} \qquad v'(y) = \widehat{E}[\, V'(y \cdot \pi) \,] .$$

Kramkov and Schachermayer [18], [19] give further results on optimal investment strategies, in particular those involving the asymptotic elasticity of $U$ and necessary conditions for the validity of the duality theorem. We leave it to the reader to translate the complete-market versions of these theorems to our robust setting.

Let us conclude this section with a comment of the condition that the least favorable measure $Q_0$ is equivalent to $\widehat{P}$. Under condition (6), the set $\mathcal{Q}$ is closed in total variation if and only if the set $\{\, dQ/d\widehat{P} \,|\, Q \in \mathcal{Q} \,\}$ is closed in $L^1(\widehat{P})$.

**Lemma 2.4** *Suppose that $\mathcal{Q}$ is convex and and closed in total variation. Then every least favorable measure $Q_0$ is equivalent to $\widehat{P}$.*

**Proof:** Suppose that $\widehat{P} \not\ll Q_0$ so that $\widehat{P}[\, \pi < \infty \,] < 1$. Due to our assumptions and the Halmos-Savage theorem, $\mathcal{Q}$ contains a measure $Q_1 \approx \widehat{P}$. We get

$$1 = Q_0[\, \pi < \infty \,] = \lim_{t\uparrow\infty} Q_0[\, \pi \le t \,] = \lim_{t\uparrow\infty} \inf_{Q\in\mathcal{Q}} Q[\, \pi \le t \,] \le Q_1[\, \pi < \infty \,] < 1 ,$$

which clearly is a contradiction. □

# 3 Examples

In this section, we will discuss three classes of examples in which least favorable measures can be determined. The first is a Black-Scholes market with uncertain drift. The second is provided by the classical Huber-Strassen theory. Here, $\mathcal{Q}$ is is the core of a 2-alternating capacity. The third class is given by extensions of the Huber-Strassen theory due to Huber [14] and Augustin [1].

First, let us state the following elementary characterization of least favorable measures, which is a variant of Theorem 6.1 in [15].

**Proposition 3.1** *For $Q_0 \in \mathcal{Q}$ with $Q_0 \approx \widehat{P}$, the following conditions are equivalent.*

*(a) $Q_0$ is a least favorable measure for $\widehat{P}$.*

*(b) For $\pi = d\widehat{P}/dQ_0$ and for all decreasing functions $f : (0,\infty] \to \mathbb{R}$ such that $\inf_{Q\in\mathcal{Q}} E_Q[\, f(\pi) \wedge 0\,] > -\infty$,*

$$\inf_{Q\in\mathcal{Q}} E_Q[\, f(\pi)\,] = E_{Q_0}[\, f(\pi)\,]\,.$$

*(c) For $\pi = d\widehat{P}/dQ_0$ and for all increasing functions $g : (0,\infty] \to \mathbb{R}$ such that $\sup_{Q\in\mathcal{Q}} E_Q[\, g(\pi) \vee 0\,] < \infty$,*

$$\sup_{Q\in\mathcal{Q}} E_Q[\, g(\pi)\,] = E_{Q_0}[\, g(\pi)\,]\,.$$

*(d) $Q_0$ minimizes*

$$I_\Phi(\widehat{P}|Q) := \int \Phi\Big(\frac{dQ}{d\widehat{P}}\Big)\, d\widehat{P}$$

*among all $Q \in \mathcal{Q}$, for all convex functions $\Phi : [0,\infty) \to \mathbb{R}$ such that $I_\Phi(\widehat{P}|Q)$ is finite for some $Q \in \mathcal{Q}$.*

**Proof:** (a)$\Leftrightarrow$(b): According to the definition, $Q_0$ is a least favorable measure if and only if $Q_0\circ\pi^{-1}$ stochastically dominates $Q\circ\pi^{-1}$ for all $Q \in \mathcal{Q}$. Hence, if $f$ is bounded, then the equivalence of (a) and (b) is just the standard characterization of stochastic dominance (see, e.g., Theorem 2.71 in [10]). If $f$ is unbounded but satisfies $\inf_{Q\in\mathcal{Q}} E_Q[\, f(\pi)\wedge 0\,] > -\infty$, then assertion (b) holds for $f_N := (-N) \vee f \wedge 0$. Thus, for all $Q \in \mathcal{Q}$ and $N \in \mathbb{N}$,

$$E_Q[\, f_N(\pi)\,] \geq E_{Q_0}[\, f_N(\pi)\,] \geq E_{Q_0}[\, f(\pi)\wedge 0\,] > -\infty\,.$$

By sending $N$ to infinity, it follows that $E_Q[\, f(\pi)\wedge 0\,] \geq E_{Q_0}[\, f(\pi)\wedge 0\,]$ for every $Q \in \mathcal{Q}$. By using a similar argument on $0 \vee f(\pi)$, we get

$$E_Q[\, f(\pi)\,] = E_Q[\, f(\pi)\vee 0\,] + E_Q[\, f(\pi)\wedge 0\,] \geq E_{Q_0}[\, f(\pi)\,] \qquad \text{for all } Q \in \mathcal{Q}.$$

(b)$\Leftrightarrow$(c) follows by changing signs.

(b)$\Rightarrow$(d): Clearly, $I_\Phi(\widehat{P}|Q)$ is well-defined and larger than $\Phi(1)$ for each $Q \ll P$. Now take a $Q_1 \in \mathcal{Q}$ with $I_\Phi(\widehat{P}|Q_1) < \infty$, and denote by $\Phi'_+(x)$ the right-hand derivative of $\Phi$ at $x \geq 0$. Since $\Phi(y) - \Phi(x) \geq \Phi'_+(x)(y-x)$, we have

$$I_\Phi(\widehat{P}|Q_1) - I_\Phi(\widehat{P}|Q_0) \geq \int \Phi'_+\big(\pi^{-1}\big)\Big(\frac{dQ_1}{d\widehat{P}} - \frac{dQ_0}{d\widehat{P}}\Big)\, d\widehat{P} = \int f(\pi)\, dQ_1 - \int f(\pi)\, dQ_0\,,$$

where $f(x) := \Phi'_+(1/x)$ is a decreasing function, which is bounded from below. Therefore $\int f(\pi)\, dQ_1 \geq \int f(\pi)\, dQ_0$, and $Q_0$ minimizes $I_\Phi(\widehat{P}|\,\cdot\,)$ on $\mathcal{Q}$.

(d)$\Rightarrow$(b): It is enough to prove (b) for bounded decreasing functions $f$. For such a function $f$ let $\Phi(x) := \int_1^x f(1/t)\, dt$. Then $\Phi$ is convex. For $Q_1 \in \mathcal{Q}$ we let $Q_t := tQ_1 + (1-t)Q_0$ and $h(t) := I_\Phi(\widehat{P}|Q_t)$. The right-hand derivative of $h$ satisfies $0 \leq h'_+(0) = \int f(\pi)\, dQ_1 - \int f(\pi)\, dQ_0$, and the proof is complete. $\square$

**Remark 3.2** By taking a strictly convex function $\Phi$ in (d) it follows that there exists at most one equivalent least favorable measure $Q_0$. If the condition of equivalence is dropped, then there may be several least favorable measures; see the proof of Proposition 3.14 for examples.

## 3.1 Utility maximization with uncertain drift

Consider a Black-Scholes market model with a riskless bond, $B_t$, of which we assume $B_t \equiv 1$ and with $d$ risky assets $S_t = (S_t^1, \dots, S_t^d)$ that satisfy an SDE of the form

$$dS_t^i = S_t^i \sum_{j=1}^{d} \sigma_t^{ij}\, dW_t^j + \alpha_t^i S_t^i\, dt \tag{7}$$

with a $d$-dimensional Brownian motion $W$ and a volatility matrix $\sigma_t$ that has full rank. Now suppose the investor is uncertain about the "true" future drift $\alpha_t$ in the market: any drift $\widetilde{\alpha}$ is possible that is adapted and such that $\widetilde{\alpha}_t \in C_t$, where $C_t$ is a nonrandom bounded closed convex subset of $\mathbb{R}^d$ for $0 \le t \le T$. Let us denote by $\mathcal{A}$ the set of all such processes $\widetilde{\alpha}$. This uncertainty in the choice of the drift can be expressed by the set

$$\mathcal{Q} := \Big\{ Q \,\Big|\, S \text{ has drift } \alpha^Q \in \mathcal{A} \text{ under } Q \Big\},$$

while under $\widehat{P}$ the drift $\alpha$ in (7) vanishes. It turns out that the optimal investment problem with *uncertain drift* can be solved by transforming it into a problem for *uncertain volatility*. To this end, we denote by $\alpha_t^0$ the element in $C_t$ that minimizes the norm $|\sigma_t^{-1}x|$ among all $x \in C_t$

**Proposition 3.3** *Suppose that $\sigma_t$ is deterministic and that both $\alpha_t^0$ and $\sigma_t$ are continuous in $t$. Then $\mathcal{Q}$ admits a least favorable measure $Q_0$ with respect to $\widehat{P}$ which is characterized by having the drift $\alpha^0$.*

**Proof:** We will use arguments from [9] to check condition (d) of Proposition 3.1. The density process of $Q \in \mathcal{Q}$ with respect to $\widehat{P}$ has the form

$$Z_t^Q := \frac{dQ}{d\widehat{P}}\Big|_{\mathcal{F}_t} = \exp\Big( \int_0^t \lambda_s\, d\widehat{W}_s - \frac{1}{2}\int_0^t \lambda_s^2\, ds \Big),$$

where $\lambda_s = \sigma_s^{-1}\alpha_s^Q$ and $\widehat{W}$ is a $d$-dimensional $\widehat{P}$-Brownian motion. Similarly, the density process $Z := Z^{Q_0}$ will involve the deterministic integrand $\gamma_s := \sigma_s^{-1}\alpha_s^0$. Let $\Phi$ be a convex function. We may assume without loss of generality that $\Phi$ is bounded by a polynomial. Then $v(t,x) := E^*[\,\Phi(xZ_t)\,]$ is a solution to the Black-Scholes equation $v_t = \frac{1}{2}|\gamma_t|^2 x^2 v_{xx}$. This fact and Itô's formula show that

$$dv(T-t, Z_t^Q) = v_x(T-t, Z_t^Q)\, dZ_t^Q + \frac{1}{2}(Z_t^Q)^2 v_{xx}(T-t, Z_t^Q)(|\lambda_t|^2 - |\gamma_t|^2)\, dt\,.$$

One easily checks that the first term on the right is a martingale. Moreover, $v$ is convex and $|\lambda_t|^2 \ge |\gamma_t|^2$ by definition of $\alpha^0$. Hence, $v(T-t, Z_t^Q)$ is a submartingale and

$$E^*[\,\Phi(Z_T^Q)\,] = E^*[\,v(0, Z_T^Q)\,] \ge v(T, Z_0^Q) = E^*[\,\Phi(Z_T)\,]\,.$$

□

An obvious question is whether the strong condition that the volatility $\sigma_t$ is deterministic can be relaxed. The most interesting case would be a local volatility model in which the equation (7) for $d = 1$ is replaced by the SDE

$$dS_t = \sigma(t, S_t)S_t\,dW_t + \alpha_t S_t\,dt\,. \tag{8}$$

In this case, however, the density process $Z$ appearing in the preceding proof involves the integrand $\gamma_t = \sigma(t, S_t)^{-1}\alpha_t^0$, which depends in a nontrivial way on the whole path of $W$. The discussion in Section 4 of [9] shows that the convexity of the function $v$ may be lost for path-dependend integrands. Moreover, $\sigma(t, S_t)$ is not Hölder continuous of order 1/2, and so the method of Hobson [13] and Janson and Tysk [16] does not apply. Therefore, it is apparently not clear whether the preceding proposition remains true for the SDE (8).

**Remark 3.4** When $d = 1$ and $\mathcal{A}$ is of the form

$$\mathcal{A} = \left\{\, \widetilde{\alpha} \,\middle|\, |\lambda_t - \widetilde{\alpha}_t/\sigma_t| \le \beta_t \text{ a.e.} \,\right\},$$

the upper and lower expectations induced by the corresponding set $\mathcal{Q}$ can be interpreted as *g-expectations* in the sense of Peng [22]; see, e.g., Example 1 of Chen and Sulem [5].

## 3.2 Examples within the Huber-Strassen theory

In order to formulate our conditions on $\mathcal{Q}$, let us define a set function $\gamma$ by

$$\gamma(A) := \sup_{Q \in \mathcal{Q}} Q[\,A\,]\,, \qquad A \in \mathcal{F}_T.$$

**Assumption 3.5** *Consider the following set of conditions.*

(a) *$\mathcal{Q}$ is maximal in the sense that it contains every measure $Q$ with $Q[\,A\,] \le \gamma(A)$ for all $A \in \mathcal{F}_T$.*

(b) *$\gamma$ is 2-alternating:*

$$\gamma(A \cup B) + \gamma(A \cap B) \le \gamma(A) + \gamma(B) \qquad \textit{for } A, B \in \mathcal{F}_T. \tag{9}$$

(c) *There exists a Polish topology on $\Omega$ such that $\mathcal{F}_T$ is the corresponding Borel field and $\mathcal{Q}$ is compact.*

**Remark 3.6** Let us comment on the conditions in Assumption 3.5.

(a) This condition implies that $\mathcal{Q}$ is convex and closed in total variation. Hence, Lemma 2.4 yields that the least favorable measure must be equivalent to $\widehat{P}$. Moreover, under assumption (b), (a) is equivalent to the weaker condition:

$$\text{if } E_Q[\,X\,] \le \int_0^\infty \gamma(X > t)\,dt \text{ for all } X \in L_+^\infty \text{ then } Q \in \mathcal{Q};$$

see Section 5 of [8].

(b) This condition can be justified economically by assuming that the preference order that induces the robust utility functional (2) satisfies the axiom of "comonotonic independence"; see Schmeidler [25].

(c) This condition guarantees that $\gamma$ is a capacity in the sense of Choquet [6]. Thus, the results of Huber and Strassen [15] apply to our situation.

◇

Consider the 2-alternating set function

$$\nu_t(A) := t\gamma(A) - \widehat{P}[\,A\,]\,, \qquad A \in \mathcal{F}_T\,. \tag{10}$$

It is shown in Lemmas 3.1 and 3.2 of [15] that, under parts (a) and (b) of Assumption 3.5, there exists a decreasing family $(A_t)_{t>0} \subset \mathcal{F}_T$ such that $A_t$ minimizes $\nu_t$ and such that $A_t = \bigcup_{s>t} A_s$.

**Definition 3.7 (Huber and Strassen)** *The function*

$$\frac{d\widehat{P}}{d\gamma}(\omega) = \inf\{\, t \,|\, \omega \notin A_t \,\}\,, \qquad \omega \in \Omega\,,$$

*is called the* Radon-Nikodym derivative *of $\widehat{P}$ with respect to $\gamma$.*

The terminology "Radon-Nikodym derivative" comes from the fact that $d\widehat{P}/d\gamma$ coincides with the usual Radon-Nikodym derivative $d\widehat{P}/dQ$ in case where $\mathcal{Q} = \{Q\}$; see [15]. We will need the following simple lemma:

**Lemma 3.8** *Condition* (6) *implies that* $P\big[\,0 < \frac{d\widehat{P}}{d\gamma} < \infty\,\big] = 1$.

**Proof:** Let $\nu_t$ be as in (10). Clearly, $\frac{d\widehat{P}}{d\gamma}(\omega) = \infty$ if and only if $\omega \in A_\infty := \bigcap_{0<t<\infty} A_t$. Since $\nu_t(A_t) \leq \nu_t(\emptyset) = 0$, we have $\gamma(A_t) \leq 1/t$. It follows that $\gamma(A_\infty) = 0$, which by (6) implies that $P[\,A_\infty\,] = 0$.

Letting $A_0 := \bigcup_{0<t<\infty} A_t$, we see that $\frac{d\widehat{P}}{d\gamma}(\omega) = 0$ if and only if $\omega \in A_0^c$. From $\nu_t(A_t) \leq \nu_t(\Omega) = t - 1$, we find that $\widehat{P}[\,A_t^c\,] \leq t(1 - \gamma(A_t))$. As $t \downarrow 0$ we thus get $\widehat{P}[\,A_0^c\,] = 0$. □

Let us now state the Huber-Strassen theorem from [15] in a form in which it will be needed here.

**Theorem 3.9 (Huber-Strassen)** *Under Assumption 3.5, $\mathcal{Q}$ admits a least favorable measure $Q_0$ with respect to any probability measure $R$ on $(\Omega, \mathcal{F}_T)$. Moreover, if $R = \widehat{P}$ and $\mathcal{Q}$ satisfies* (6) *, then $Q_0$ is equivalent to $\widehat{P}$ and given by*

$$dQ_0 = \Big(\frac{d\widehat{P}}{d\gamma}\Big)^{-1} d\widehat{P}\,.$$

Together with Theorem 2.2, we get a complete solution of the robust utility maximization problem within the large class of utility functionals that arise from sets $\mathcal{Q}$ as in Assumption 3.5. Before discussing particular examples, let us state the following converse of the Huber-Strassen theorem in order to clarify the role of condition (b) in Assumption 3.5.

**Theorem 3.10** *Suppose $\Omega$ is a Polish space with Borel field $\mathcal{F}_T$ and $\mathcal{Q}$ is a compact set of probability measures. If every probability measure on $(\Omega, \mathcal{F}_T)$ admits a least favorable measure $Q_0 \in \mathcal{Q}$, then $\gamma(A) = \sup_{Q\in\mathcal{Q}} Q[\,A\,]$ is 2-alternating.*

For finite probability spaces, Theorem 3.10 is due to Huber and Strassen [15]. In the form stated above, it was proved by Lembcke [21]. An alternative formulation was given earlier by Bednarski [3].

Let us now turn to the discussion of examples. The following Examples 3.11 and 3.12 were first studied by Bednarski [2] under slightly different conditions than here. They are also closely connected to the concept of a comonotone law-invariant coherent measure of risk as introduced by Kusuoka [20].

**Example 3.11** For $\lambda \in (0, 1)$, consider the set

$$\mathcal{Q} := \left\{ Q \ll P \,\middle|\, \frac{dQ}{dP} \le \frac{1}{\lambda} \right\},$$

which satisfies Assumption 3.5. Let $\varphi := d\widehat{P}/dP$, suppose that $\varphi$ has a continuous and strictly increasing distribution function $F_\varphi$ under $P$, and denote by $q_\varphi$ the corresponding quantile function (i.e., the inverse of $F_\varphi$). Then the function

$$(0, 1] \ni y \longmapsto \frac{y + \lambda - 1}{\int_0^y q_\varphi(t)\, dt}$$

has a unique maximizer $y_\lambda \in (1 - \lambda, 1]$, and the Radon-Nikodym derivative of $\widehat{P}$ with respect to $\gamma$ is given by

$$\pi = \frac{d\widehat{P}}{d\gamma} = \lambda \cdot (\varphi \vee q_\varphi(y_\lambda)),$$

as proved in Remark 4.6 of [24]. If $\|\varphi\|_{L^\infty} > \lambda^{-1}$, then $y_\lambda$ is the unique solution of the equation

$$q_\varphi(y)(y + \lambda - 1) = \int_0^y q_\varphi(t)\, dt.$$

◇

**Example 3.12** The utility functionals constructed from the cores of the following class of 2-alternating set functions generalize Example 3.11 and arise as numerical representations in Yaari's [26] "dual theory of choice under risk". Let $\psi : [0, 1] \to [0, 1]$ be an increasing concave function with $\psi(0) = 0$ and $\psi(1) = 1$. In particular, $\psi$ is continuous on $(0, 1]$. We define $\gamma$ by

$$\gamma(A) := \psi\big(P[\,A\,]\big), \qquad A \in \mathcal{F}.$$

Then $\gamma$ is 2-alternating, and the set $\mathcal{Q}$ of all probability measures $Q$ on $(\Omega, \mathcal{F}_T)$ with $Q[\,A\,] \leq \gamma(A)$ for all $A \in \mathcal{F}_T$ satisfies part (b) of Assumption 3.5. If $\psi$ is continuous at 0, then $\mathcal{Q}$ can be described in terms of $\psi$; see Carlier and Dana [7].

Example 3.11 corresponds to the choice $\psi(t) = (t\lambda^{-1}) \wedge 1$. Apart from this special case, an explicit formula for $\pi = d\widehat{P}/d\gamma$ is not known to the writer, but $\pi$ can be computed (in principal and numerically) by solving a certain non-linear variational problem in two real parameters; see Section 4 of [24]. $\diamond$

**Example 3.13 (Weak information)** Let $Y$ be a measurable function on $(\Omega, \mathcal{F}_T)$, and denote by $\mu$ its law under $\widehat{P}$. For $\nu \approx \mu$ given, let

$$\mathcal{Q} := \left\{ Q \ll \widehat{P} \,\middle|\, Q \circ Y^{-1} = \nu \right\}.$$

The robust utility maximization problem for this set $\mathcal{Q}$ was studied by Baudoin [4], who coined the terminology “weak information”. The interpretation behind the set $\mathcal{Q}$ is that an investor has full knowledge about the pricing measure $\widehat{P}$ but is uncertain about the true distribution $P$ of market prices and only knows that a certain functional $Y$ of the stock price has distribution $\nu$. Write $\widehat{P} = \mu \otimes \widehat{K}$, where $\widehat{K}(y, \cdot) = \widehat{P}[\,\cdot\,|\,Y = y\,]$ is the regular conditional expectation given $Y$. Then

$$Q_0 := \nu \otimes \widehat{K}$$

is a least favorable measure. To see this, note first that $Q_0$ is equivalent to $\widehat{P}$ and belongs to $\mathcal{Q}$. Moreover, we have

$$\pi = \frac{d\widehat{P}}{dQ_0} = \frac{d\mu}{d\nu}(Y)\,.$$

Hence, $Q[\,\pi \leq t\,] = \nu(d\mu/d\nu \leq t)$ is independent of $Q \in \mathcal{Q}$, and it follows that $Q_0$ satisfies the definition of a least favorable measure. The same procedure can be applied to *any* measure $R \approx \widehat{P}$. This fact implies that $\mathcal{Q}$ fits into the framework of the Huber-Strassen theory, as is shown in the following proposition. $\diamond$

**Proposition 3.14** *Suppose $(\Omega, \mathcal{F}_T)$ is a standard Borel space. Then the set $\mathcal{Q}$ defined in Example 3.13 satisfies Assumption 3.5. In particular, $\gamma(A) := \sup_{Q \in \mathcal{Q}} Q[\,A\,]$ is 2-alternating.*

**Proof:** If $Q$ is a probability measure with $Q[\,\cdot\,] \leq \gamma(\,\cdot\,)$, then

$$Q[\,Y \leq t\,] \leq \gamma(Y \leq t) = \nu((-\infty, t])\,.$$

Using the same argument on $\{Y > t\}$ shows that $Y$ has law $\nu$ under $Q$. Hence, $\mathcal{Q}$ is maximal in the sense of part (a) of Assumption 3.5.

To prove that part (b) holds we will use Theorem 3.10. To this end, we may choose a compact metric topology on $\Omega$ such that $Y$ is continuous and $\mathcal{F}_T$ is the Borel $\sigma$-algebra. If $R \ll \widehat{P}$, then $\eta := R \circ Y^{-1} \ll \nu$ and $R$ can be written as $\eta \otimes K_R$, where $K_R$ is a stochastic kernel such that $K_R(y, \cdot) \ll \widehat{K}(y, \cdot)$ for $\eta$-a.e. $y$. Let $\nu = \nu_a + \nu_s$ be the

Lebesgue decomposition of $\nu$ with respect to $\eta$ into the absolutely continuous part $\nu_a \ll \eta$ and into the singular part $\nu_s$. If we let $Q_0 := \nu_a \otimes K_R + \nu_s \otimes \widehat{K}$, then $Q_0 \in \mathcal{Q}$ and

$$\pi = \frac{dR}{dQ_0} = \frac{d\eta}{d\nu}(Y)\,.$$

Again, the distribution of $\pi$ is the same for all $Q \in \mathcal{Q}$, and it follows that $Q_0$ is a least favorable measure. If $R \not\ll \widehat{P}$, then it is clear that any measure $Q_0$ will be least favorable for $R$ if it is least favorable for the absolutely continuous part of $R$. □

In the 1970's and 1980's, explicit formulas for Radon-Nikodym derivatives with respect to capacities were found in a number of examples such as sets $\mathcal{Q}$ defined in terms of $\varepsilon$-contamination or via probability metrics like total variation or Prohorov distance; we refer to Chapter 10 in the book [14] by Huber and the references therein. But, unless $\Omega$ is finite, these examples fail to satisfy either implication in (6) (see, however, Example 3.15 below). Nevertheless, they are still interesting for discrete-time market models.

## 3.3 Further examples from robust statistics

In this section, we briefly discuss further example classes that may or may not lead to 2-alternating capacities but for which least favorable measures are available.

**Example 3.15 (Huber [14])** Let $Y$ be a real-valued random with distributions $\mu$ and $\widehat{\mu}$ under $P$ and $\widehat{P}$, respectively. Suppose that $d\widehat{\mu}/d\mu$ is an increasing function on the real line. For $\varepsilon, \delta \in [0,1)$, we define

$$\mathcal{Q} := \big\{\, Q \ll P \,\big|\, Q[\,Y < t\,] \geq (1-\varepsilon)P[\,Y < t\,] - \delta \text{ for all } t \,\big\}\,.$$

This class of examples includes absolutely continuous restrictions of $\varepsilon$-contamination and of neighborhoods with respect to the following probability metrics: total variation, Prohorov metric, Kolmogorov distance, and Lévy metric; see [14], p. 271. A possible financial interpretation would be similar to the case of "weak information" in Example 3.13. Under the above conditions, one can show that $\mathcal{Q}$ admits a least favorable measure $Q_0$, and $\pi$ is proportional to $c' \vee d\widehat{\mu}/d\mu(X) \wedge c''$ for certain constants $c'$ and $c''$. We refer to Section 10.3 of [14] for details. ◇

**Example 3.16 (Augustin [1])** Here, one starts with any set $\mathcal{Q}$ that admits an equivalent least favorable measure $Q_0$ and applies a distortion function $\psi$ to the upper probability arising from $\mathcal{Q}$:

$$\overline{\mathcal{Q}} := \Big\{\, \overline{Q} \,\Big|\, \overline{Q}[\,A\,] \leq \psi\Big(\sup_{Q\in\mathcal{Q}} Q[\,A\,]\Big) \text{ for all } A \in \mathcal{F}_T \,\Big\}\,.$$

Here, $\psi : [0,1] \to [0,1]$ is increasing and concave with $\psi(0) = 0$ and $\psi(1) = 1$ as in Example 3.12. Augustin [1] gives various conditions under which a least favorable measure $\overline{Q}_0$ for the core of the 2-alternating set function $\psi(Q_0[\,\cdot\,])$ is also a least favorable measure for $\overline{\mathcal{Q}}$. ◇

# 4 Proof of Theorem 2.2

Let $X^*$ be a solution of the standard utility maximization problem for the least-favorable measure $Q_0$. Then it is well known that $X_T^* = I(y\pi)$ for some constant $y > 0$. Thus, one easily checks via Proposition 3.1 that $X^*$ is also a solution of the robust utility maximization problem. However, in order to show the full equivalence of the two problems, we must also take care of the situation in which the standard problem has no solution. Our key result is the following proposition.

**Proposition 4.1** *Let $Q_0 \approx \widehat{P}$ be a least favorable measure and $\pi = d\widehat{P}/dQ_0$.*

(a) *For any $X \in \mathcal{X}(x)$ there exists $\widetilde{X} \in \mathcal{X}(x)$ such that*

$$\inf_{Q\in\mathcal{Q}} E_Q[\,U(\widetilde{X}_T)\,] \geq \inf_{Q\in\mathcal{Q}} E_Q[\,U(X_T)\,]$$

*and such that $\widetilde{X}_T = f(\pi)$ for some deterministic decreasing function $f : (0,\infty) \to [0,\infty)$.*

(b) *The terminal wealth of any solution $X^*$ of (4) is of the form $X_T^* = f^*(\pi)$ for a deterministic decreasing function $f^*(0,\infty) \to [0,\infty)$.*

The proof of this proposition is based on ideas from [24] and on the following version of the classical Hardy-Littlewood inequalities, which we recall here for the convenience of the reader. See, e.g., Theorem 2.76 of [10] for a proof.

**Theorem 4.2 (Hardy-Littlewood)** *Let $X$ and $Y$ be two non-negative random variables on $(\Omega, \mathcal{F}_T, Q)$, and let $q_X$ and $q_Y$ denote quantile functions of $X$ and $Y$ with respect to $Q$. Then,*

$$\int_0^1 q_X(1-t)q_Y(t)\,dt \leq E_Q[\,XY\,] \leq \int_0^1 q_X(t)q_Y(t)\,dt\,.$$

*If $X = f(Y)$, then the lower (upper) bound is attained if and only if $f$ can be chosen as a decreasing (increasing) function.*

**Proof of Proposition 4.1:** (a) By market completeness, it suffices to construct a decreasing function $f \geq 0$ such that $\widehat{E}[\,f(\pi)\,] \leq x$ and

$$\inf_{Q\in\mathcal{Q}} E_Q[\,U(f(\pi))\,] \geq \inf_{Q\in\mathcal{Q}} E_Q[\,U(X_T)\,]\,. \tag{11}$$

To this end, we denote by $F_Y(x) := Q_0[\,Y \leq x\,]$ the distribution function and by $q_Y(t)$ a quantile function of a random variable $Y$ with respect to the probability measure $Q_0$. We will need the the following basic property of quantile functions: If $f$ is a decreasing or increasing function and $Y \geq 0$, then the quantile $q_{f(Y)}$ of $f(Y)$ satisfies for a.e. $t \in (0,1)$

$$q_{f(Y)}(t) = \begin{cases} f(q_Y(1-t)) & \text{if } f \text{ is decreasing,} \\ f(q_Y(t)) & \text{if } f \text{ is increasing;} \end{cases} \tag{12}$$

see, e.g., Lemma 2.77 in [10].

Let us define a function $f$ by

$$f(t) := \begin{cases} q_{X_T}(1 - F_\pi(t)) & \text{if } F_\pi \text{ is continuous at } t, \\ \\ \dfrac{1}{F_\pi(t) - F_\pi(t-)} \displaystyle\int_{F_\pi(t-)}^{F_\pi(t)} q_{X_T}(1-s)\,ds & \text{otherwise.} \end{cases} \tag{13}$$

Then $f$ is decreasing and satisfies $f(q_\pi) = E_\lambda[\,h\,|\,q_\pi\,]$, where $\lambda$ is the Lebesgue measure and $h(t) := q_{X_T}(1-t)$. Hence, Jensen's inequality for conditional expectations and (12) show that

$$\begin{aligned} \inf_{Q\in\mathcal{Q}} E_Q[\,U(X_T)\,] &\leq E_{Q_0}[\,U(X_T)\,] = \int_0^1 U(h(t))\,dt \\ &\leq \int_0^1 U\big(E_\lambda[\,h\,|\,q_\pi\,](t)\big)\,dt = \int_0^1 U(q_{f(\pi)}(1-t))\,dt \\ &= E_{Q_0}[\,U(f(\pi))\,] = \inf_{Q\in\mathcal{Q}} E_Q[\,U(f(\pi))\,], \end{aligned} \tag{14}$$

where we have used Proposition 3.1 in the last step. Thus, $f$ satisfies (11).

It remains to show that $f(\pi)$ satisfies the capital constraint. To this end, we first use the lower Hardy-Littlewood inequality:

$$x \geq \widehat{E}[\,X_T\,] = E_{Q_0}[\,\pi X_T\,] \geq \int_0^1 q_\pi(t)\,q_{X_T}(1-t)\,dt\,. \tag{15}$$

Here we may replace $q_{X_T}(1-t) = h(t)$ by $E_\lambda[\,h\,|\,q_\pi\,](t) = f(q_\pi(t))$. We then get

$$\int_0^1 q_\pi(t)\,q_{X_T}(1-t)\,dt = \int_0^1 q_\pi(t)\,f(q_\pi(t))\,dt = E_{Q_0}[\,\pi f(\pi)\,] = \widehat{E}[\,f(\pi)\,]\,. \tag{16}$$

Thus, $f$ is as desired.

(b) Now suppose $X^*$ solves (4). If $X^*_T$ is not $Q_0$-a.s. $\sigma(\pi)$-measurable, then $Y := E_{Q_0}[\,X^*_T\,|\,\pi\,]$ must satisfy

$$E_{Q_0}[\,U(Y)\,] > E_{Q_0}[\,U(X^*_T)\,]\,, \tag{17}$$

due to the strict concavity of $U$. If we define $\widetilde{f}$ as in (13) with $Y$ replacing $X_T$, then the proof of part (a) yields that

$$\widehat{E}[\,\widetilde{f}(\pi)\,] = E_{Q_0}[\,\pi\widetilde{f}(\pi)\,] \leq E_{Q_0}[\,\pi Y\,] = E_{Q_0}[\,\pi X^*_T\,] \leq x\,,$$

and by (14) and (17),

$$\inf_{Q\in\mathcal{Q}} E_Q[\,U(\widetilde{f}(\pi))\,] \geq E_{Q_0}[\,U(Y)\,] > E_{Q_0}[\,U(X^*_T)\,] \geq \inf_{Q\in\mathcal{Q}} E_Q[\,U(X^*_T)\,]\,,$$

in contradiction to the optimality of $X^*$. Thus, $X^*_T$ is necessarily $\sigma(\pi)$-measurable and can hence be written as a (not yet necessarily decreasing) function of $\pi$.

If we define $f^*$ as in (13) with $X^*_T$ replacing $X_T$, then $f^*(\pi)$ is the terminal wealth of yet another solution in $\mathcal{X}(x)$. Clearly, we must have $\widehat{E}[\,X^*_T\,] = x = \widehat{E}[\,f^*(\pi)\,]$. Thus, (15) and (16) yield that $E_{Q_0}[\,\pi X^*_T\,] = \int_0^1 q_\pi(t)\,q_{X^*_T}(1-t)\,dt$. But then the "only if" part of the lower Hardy-Littlewood inequality together with the $\sigma(\pi)$-measurability of $X^*_T$ imply that $X^*_T$ is a decreasing function of $\pi$. $\square$

Thus, in solving the robust utility maximization problem (4), we may restrict ourselves to strategies whose terminal wealth is a decreasing function of $\pi$. By Propositions 3.1, the robust utility of a such a terminal wealth is the same as the expected utility with respect to $Q_0$. On the other hand, taking $\mathcal{Q}_0 := \{Q_0\}$ in Proposition 4.1 implies that the standard utility maximization problem for $Q_0$ also requires only strategies whose terminal wealth is a decreasing function of $\pi$. Therefore, the two problems are equivalent, and Theorem 2.2 is proved. $\square$

# References

[1] Augustin, T. *Neyman-Pearson testing under interval probability by globally least favorable pairs reviewing Huber-Strassen theory and extending it to general interval probability.* Imprecise probability models and their applications (Ghent, 1999). J. Statist. Plann. Inference 105 , no. 1, 149–173 (2002).

[2] Bednarski, T. *On solutions of minimax test problems for special capacities.* Z. Wahrsch. Verw. Gebiete 58, no. 3, 397–405 (1981).

[3] Bednarski, T. *Binary experiments, minimax tests and 2-alternating capacities.* Ann. Statist. 10, no. 1, 226–232 (1982).

[4] Baudoin, F. *Conditioned stochastic differential equations: theory, examples and application to finance.* Stochastic Process. Appl. 100, 109–145 (2002).

[5] Chen, Z., Sulem, A. *An integral representation theorem of g-expectations.* Inria rapport de recherche no. 4284 (2001).

[6] Choquet, G. *Theory of capacities.* Ann. Inst. Fourier 5, 131-292 (1953/1954).

[7] Dana, R.-A., Carlier, G. *Core of convex distortions of a probability on an non atomic space.* Preprint, Ceremade, Université Paris-Dauphine (2002).

[8] Delbaen, F. *Coherent risk measures.* Lecture notes, Scuola Normale di Pisa (2000).

[9] El Karoui, N., Jeanblanc-Picqué, M., Shreve, S. *Robustness of the Black and Scholes formula.* Math. Finance 8, no. 2, 93–126 (1998).

[10] Föllmer, H., Schied, A. *Stochastic Finance: An Introduction in Discrete Time.* Berlin: de Gruyter Studies in Mathematics 27 (2002).

[11] Gilboa, I. *Expected utility with purely subjective non-additive probabilities.* J. Math. Econ. 16, 65-88 (1987).

[12] Gilboa, I., Schmeidler, D. *Maxmin expected utility with non-unique prior.* J. Math. Econ. 18, 141-153 (1989).

[13] Hobson, D. *Volatility mis-specification, option pricing and super-replication via coupling.* Ann. Appl. Probab. 8, 193-205 (1998).

[14] Huber, P. *Robust statistics.* Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York (1981).

[15] Huber, P., Strassen, V. *Minimax tests and the Neyman-Pearson lemma for capacities.* Ann. Statistics 1, no. 2, 251-263 (1973). *Correction:* Ann. Statistics 2, no. 1, 223-224 (1974).

[16] Janson, S., Tysk, J. *Volatility time and properties of option prices.* Ann. Appl. Probab. 13, 890-913 (2003).

[17] Karatzas, I., Shreve, S. *Methods of mathematical finance.* Applications of Mathematics. Berlin: Springer (1998).

[18] Kramkov, D., Schachermayer, W. *The asymptotic elasticity of utility functions and optimal investment in incomplete markets.* Ann. Appl. Probab. 9, no. 3, 904–950 (1999).

[19] Kramkov, D., Schachermayer, W. *Necessary and sufficient conditions in the problem of optimal investment in incomplete markets.* Ann. Appl. Probab., Vol.13, no. 4 (2003).

[20] Kusuoka, S. *On law invariant coherent risk measures.* Adv. Math. Econ. 3, 83-95 (2001).

[21] Lembcke, J. *The necessity of strongly subadditive capacities for Neyman-Pearson minimax tests.* Monatsh. Math. 105, no. 2, 113–126 (1988).

[22] Peng, S. *Backward SDE and related $g$-expectation.* In: Backward stochastic differential equations (Paris, 1995–1996), 141–159, Pitman Res. Notes Math. Ser., 364, Longman, Harlow, (1997).

[23] Schachermayer, W. *Optimal investment in incomplete financial markets.* Proceedings of the first World Congress of the Bachelier Society, Paris 2000, Springer-Verlag, Berlin (2001).

[24] Schied, A. *On the Neyman-Pearson problem for law-invariant risk measures and robust utility functionals.* To appear in the Annals of Applied Probability.

[25] Schmeidler, D. *Subjective probability and expected utility without additivity.* Econometrica 57, no. 3, 571-587 (1989).

[26] Yaari, M. *The dual theory of choice under risk.* Econometrica 55, no. 1, 95-115 (1987).