Tverberg-type theorems and the Fractional Helly property

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Introduction

In this thesis we study some classical problems from discrete geometry of convex sets via topological methods. More precisely, we start with a classical problem from convex geometry and ask the following question:

Does the problem admit a topological generalization?

Here *topological generalization* means for example: What happens if we replace *affine map* with *continuous map*? What happens if we replace *convex sets* with *contractible sets*? In doing so, we obtain on the one hand solutions of problems from convex geometry which haven't been solved using classical methods. On the other hand, we get more general statements often involving nice proofs.

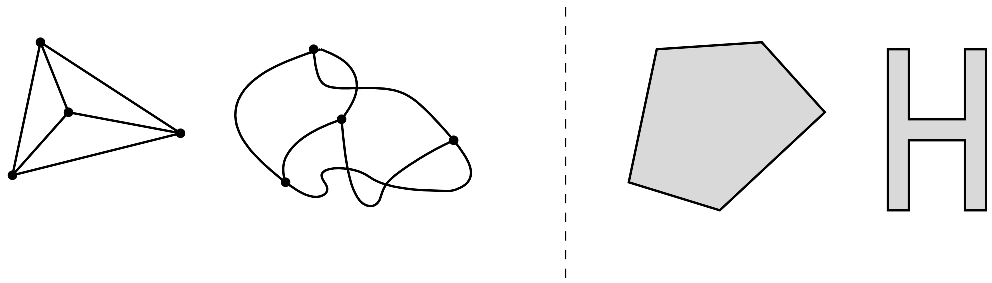

Figure 1: Topological generalizations.

The results of this thesis deal with new generalizations of Radon's and of Helly's theorem: Tverberg-type theorems and Fractional Helly theorems.

A rough outline of this thesis is as follows. In Chapter 1, we summarize the tools from discrete geometry and topology. In addition, we give a short introduction to the equivariant method in combinatorics and discrete geometry including a summary of Tverberg-type theorems. Chapters 2 and 3 contain the Tverberg-type theorems. In Chapter 4, we discuss the outcome of a computer project that led to many examples, and that stimulated the results of Chapters 2 and 3. Finally, Chapter 5 is independent of the Chapters 2, 3, and 4. It is entirely devoted to the Fractional Helly property.

For the remaining part of this introduction, we give an overview on the main results of this thesis hoping that this equips the reader with the curiosity to read on.

Tverberg-type results

In the 1960's, young Helge Tverberg generalized Radon's theorem, a basic result in convex geometry, during a cold winter night while staying in a small hotel in England:

> Any set of $(d+1)(q-1)+1$ points in $\mathbb{R}^d$ can be partitioned into q disjoint subsets such that their convex hulls have a non-empty intersection.

This result was published in [66], and it is nowadays known as Tverberg's theorem. The problem had been suggested to him by B. J. Birch, nowadays known for his conjecture together with Swinnerton-Dyer, who had solved it in [17] for dimension $d = 2$ in 1959.

Tverberg's theorem implies the existence of one Tverberg partition. It is very natural in mathematics to ask: If there is one, are there more than one? How many? In the case of Tverberg partitions, this question has first been asked by Sierksma in 1979. He conjectured that every set of $(d+1)(q-1)+1$ points in $\mathbb{R}^d$ has many Tverberg partitions, in fact at least $((q-1)!)^d$ many. This conjecture is still unresolved in nearly all cases, except for the two trivial cases $q = 2$, or $d = 1$.

In 1981, Bárány, Shlosman, and Szücs [11] generalized Tverberg's theorem for primes q to the topological Tverberg theorem for primes q. The place, the weather conditions, or stellar constellations of this event are not known to me.[1] This result has been extended to prime powers q by many people, the first by Özaydin in an unpublished preprint [52] in 1986. All known proofs of the topological Tverberg theorem use the equivariant method, a beautiful application of equivariant topology, in the form of the Borsuk-Ulam theorem, to discrete geometry. The origins of this method are attributed to Lovász who applied it in 1979 to settle an old problem of graph theory, Kneser's conjecture. Since then it has been applied to obtain a number of important results in combinatorics, discrete geometry, and theoretical computer science. See Matoušek's textbook [46] "Using the Borsuk-Ulam theorem" for a captivating introduction and survey.

Vućić and Živaljević [72] showed in 1993 the first non-trivial lower bound for the number of Tverberg partitions for primes q. Their proof is a nice application of the equivariant method.

According to Matoušek [46], "the validity of the topological Tverberg theorem for arbitrary q is one of the most challenging problems in topological combinatorics". Recently, Schöneborn and Ziegler [62] have shown that this

[1] Promise, this is the only joke of this introduction.

problem is equivalent to the winding number conjecture. The case $d = 2$ of the winding number conjecture is a problem on drawings of complete graphs on $3q - 2$ vertices in the plane.

In Chapter 2 we obtain Tverberg-type theorems for prime powers q using the equivariant method. First we show a new lower bound for the number of Tverberg partitions extending the result of Vućić and Živaljević [72]. The lower bound holds in the general setting of the topological Tverberg theorem.

Motivated by the recent of work Schöneborn and Ziegler [62], we generalize the topological Tverberg theorem to the topological Tverberg theorem with constraints. For this, we determine a list of constraint graphs C in the 1-skeleton of $\sigma^{(d+1)(q-1)}$ such that every continuous map $f : \sigma^{(d+1)(q-1)} \longrightarrow \mathbb{R}^d$ admits a Tverberg partition not using any edge of C. The list of constraint graphs reads as follows:

- Complete graphs K_l for $l \leq \frac{q+1}{2}$,
- complete bipartite graphs $K_{1,l}$ for $l < q - 1$,
- paths P_l on l vertices for $q > 3$,
- cycles C_l on l vertices for $q > 4$,
- disjoint unions of graphs from above.

In the case $C = K_l$, our result implies that all l vertices end up in l disjoint partition sets. This result is optimal for complete bipartite graphs $K_{1,l}$ for $l < q - 1$, as Schöneborn and Ziegler [62] observed that $K_{1,q-1}$ is not a constraint graph. In addition, the class of constraint graphs is closed under taking subgraphs. It is thus a monotone graph property. In the proof of the topological Tverberg theorem with constraints, we show new connectivity results for chessboard-type complexes using the nerve theorem. The topological Tverberg theorem with constraints is used in Chapter 3 to obtain a lower bound for the number of Tverberg points. All results carry over to winding partitions introduced by Schöneborn and Ziegler [62].

Moreover, we study the problem of fairly splitting a generic necklace for q thieves. We extend the result of Vućić and Živaljević [72], a lower bound on the number of fair splittings of a generic necklace for primes q.

Parts of the results of Chapter 2 have already been accepted by the *European Journal of Combinatorics* for publication [35] in 2005: The lower bound for the number of Tverberg partitions, and the lower bound on the number of fair splittings of a generic necklace.

In Chapter 3 we show the first non-trivial lower bound for the number of Tverberg partitions that holds for arbitrary q. In a first step, we study Birch partitions into k subsets to a fixed point in $\mathbb{R}^d$. We then obtain two properties for the number of Birch partitions: Evenness, and a lower bound. The proof of the lower bound uses induction on k: The induction base follows from the evenness, and for the induction step we do not make use of convexity. Birch partitions and Tverberg partitions are closely related: Tverberg partitions are Birch partitions to the Tverberg point. Hence the results for the number of Birch partitions imply evenness, and a lower bound for the number of Tverberg partitions. These result are partly motivated by a computer project which will be discussed in Chapter 4.

Using the topological Tverberg theorem with constraints from Chapter 2, we show a lower bound for the number of Tverberg points in the setting of the topological Tverberg theorem for prime powers q. The lower bounds for the number of Tverberg points differ for Tverberg points, and for topological Tverberg points.

Combining the new lower bounds for the number of Tverberg partitions and for the Tverberg points from this chapter, we improve this lower bound once more for the number of Tverberg partitions for prime powers q. This settles Sierksma's conjecture for a wide class of sets of points in $\mathbb{R}^2$ for $q = 3$.

In a next step, we extend the definition of Birch partitions to topological Birch partitions, and to winding Birch partitions. We show that the result for Birch partitions do not carry over to topological resp. winding partitions. In the case $d = 2$, the problem for winding partitions reduces to a problem on drawing complete graphs on $3k$ vertices in the plane. The result is true for rectilinear drawings, but it does in general not hold for arbitrary drawings. A counter-example, a piecewise linear drawing of K_6, was found by our computer project from Chapter 4. Starting with this example, we construct counter-examples for arbitrary dimension d. This negative result is interesting on its own: It shows for the first time that there might be a difference between the number of Tverberg partitions, and the number of topological Tverberg partitions.

Finally, we discuss two conjectures each of which imply the winding number conjecture in dimension $d = 2$. Based on results by Roudneff [56], we settle both conjectures for pseudoconfigurations of points in the plane. This discussion also leads us to a geodesical Tverberg theorem on the 2-dimensional sphere for arbitrary q.

In Chapter 4 we report on the setup, and on the outcome of our computer project in discrete geometry. We implemented several algorithms in Java for various settings:

- Determining the number of Birch resp. Tverberg partitions for a given set of points in $\mathbb{R}^d$, with $d = 2, 3$.

- Determining the number of Birch resp. Tverberg partitions for planar maps that are projections of geodesical maps on the two-dimensional sphere.
- Determining the number of winding resp. Tverberg partitions for a special class of piecewise linear planar maps.

In a first step, all programs generate point sets resp. maps uniformly at random, then they determine the corresponding number using a brute force approach.

The programs assisted us in looking at many examples. Part of the data has motivated the results of Chapter 3. It also led to a list of open problems. For example, the programs suggest that the results for the number of Birch resp. Tverberg partitions shown in Chapter 3 carry over to geodesical maps on the sphere.

FRACTIONAL HELLY PROPERTY

In 1979, Katchalski and Liu extended Helly's theorem for families of convex sets in $\mathbb{R}^d$, another basic theorem in convex geometry, as follows. For every $\alpha \in (0, 1]$ there is a $\beta = \beta(\alpha) > 0$ such that the following implication holds:

> If the intersection of a fraction α of all $(d+1)$-subsets of $\mathcal{F}$ is non-empty, then the intersection of a fraction β of $\mathcal{F}$ is non-empty.

Recently, several Fractional Helly theorems in different settings have been shown. Alon et al. [2] showed a topological Fractional Helly theorem for families of good covers based on a Fractional Helly theorem for d^*-Leray families. Matoušek [50] obtained one for families of semialgebraic sets of bounded description complexity. This result is a corollary of a Fractional Helly theorem for families of bounded VC-dimension, a well-studied concept from computational geometry. Bárány and Matoušek [8] showed a version for families of convex lattice sets in $\mathbb{Z}^d$ using a Ramsey-type argument.

Alon et al. [2] showed in a quite general and abstract setting that the Fractional Helly Theorem implies a (p, q)-theorem, another far-reaching generalization of Helly's theorem.

During my stay in Prague, Matoušek introduced me to these recent developments. At that time, he proposed to me the project of further extending the Fractional Helly theorem of Alon et al. [2]. I learned that this is also a long term project of Kalai and Meshulam, outlined in the technical report [41].

Chapter 5 starts with a short introduction on spectral sequences. Then we reprove the key result of the Fractional Helly theorem for d^*-Leray families, the upper bound theorem for d-Leray complexes of Kalai using algebraic shifting, and the classification of f-vectors of Cohen-Macaulay complexes due to Stanley. On the one hand, this is motivated by the fact that the original proof of Kalai is based on results published in a series of several papers [37], [38], [39], and [40]. On the other hand, we appreciated very much the interplay of results from various fields of mathematics.

Our main result of this chapter is a topological Fractional Helly theorem for $k_{\mathcal{G}}$-acyclic families extending the result of Alon et al. [2]. The proof is based on a spectral sequence argument, suggested to me by Carsten Schultz. On our way, we obtain a nice and short proof of a homological version of the nerve theorem due to Björner [19]. The Fractional Helly theorem for $k_{\mathcal{G}}$-acyclic families then implies a (p, q)-theorem for $k_{\mathcal{G}}$-acyclic families. These results are contained in my preprint "On a topological Fractional Helly theorem" [34].

Finally, we study the relation of the Fractional Helly property and homological VC-dimension, as outlined by Kalai [41]. We obtain partial answers towards an upper bound theorem for families of bounded homological VC-dimension.

Nearly all sections of Chapters 2, 3, and 5 end with open problems and conjectures pointing in further directions of research. In addition, a list of open problems that arose from our computer project can be found in the last section of Chapter 4.

THANKS

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Chapter 1

Foundations

In this chapter, we give some background both in convex geometry and in algebraic topology. This serves to fix our notation, and to prepare the tools for this thesis. Section 1.1 is on convex geometry, Section 1.2 summarizes definitions and facts from (algebraic) topology that we need in this thesis, and how it is applied to obtain combinatorial results. In Section 1.3, we give a survey on how Radon's theorem has been generalized in many ways with a focus on Tverberg-type theorems.

1.1 Discrete geometry of convex sets

We will go back to the origins of discrete geometry, and start with three classical theorems on convex sets which were discovered during the first quarter of the twentieth century. A more detailed treatment can be found in the textbook [49] by Jiří Matoušek. Jürgen Eckhoff's survey article [29] in the Handbook of Convex Geometry is a good starting point.

Theorem 1.1 (Radon's theorem)**.** *Let X be a set of $d+2$ points in $\mathbb{R}^d$. Then there are two disjoint subsets of X whose convex hulls have a common point.*

Theorem 1.2 (Carathéodory's theorem)**.** *Let X be a set of points in $\mathbb{R}^d$, and let p be a point in the convex hull of X. Then there is a subset Y of X consisting of at most $d+1$ points such that p is in the convex hull of Y.*

Theorem 1.3 (Helly's theorem)**.** *Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^d$, and suppose that $\mathcal{F}$ is finite or each member of $\mathcal{F}$ is compact. If every $d+1$ or fewer members of $\mathcal{F}$ have a common point, then there is a point common to all members of $\mathcal{F}$.*

All three of them have been generalized in many ways and will reappear again and again in this thesis in different guises. Radon's theorem is an easy consequence of the fact that any system of $d+2$ homogeneous linear equations in $d+1$ unknowns has a non-trivial solution; see e. g. [49]. Radon's theorem obviously also holds for $(d+2)$-element multi-sets. Carathéodory's and Helly's theorem can be proved with the help of Radon's theorem.

Our focus in this thesis is on Radon's and on Helly's theorem. As Chapters 2 and 3 are devoted to Tverberg-type theorems which are generalizations of Radon's theorem, so that we include a proper introductory Section 1.3.

Basic definitions. Before proceeding with Carathéodory's theorem, let's review some standard notations and definitions. If X is a set, then $|X|$ denotes the number of elements resp. the cardinality of X. For non-negative integers k, we write $\binom{X}{k}$ for the set of all k-element subsets of X. For $n \in \mathbb{N}$, $[n]$ denotes the n-element set $\{1, 2, \dots, n\}$. If $\mathcal{F}$ is a family of sets, then $\bigcup \mathcal{F}$ denotes the union and $\bigcap \mathcal{F}$ the intersection of all members of $\mathcal{F}$. For a set X, we denote its power set by 2^X. For $x \in \mathbb{R}^d$, $\|x\|$ denotes the *Euclidean norm* of x. The *unit d-sphere* S^d is the set of all points in $\mathbb{R}^{d+1}$ of norm 1. The *unit d-ball* B^d is the set of all points in $\mathbb{R}^d$ of norm ≤ 1.
Let X be a subset of the d-dimensional space $\mathbb{R}^d$. Then the *affine hull* $\mathrm{aff}(X)$ *of* X is the intersection of all affine subspaces of $\mathbb{R}^d$ containing X. For $x_1, x_2, \dots, x_n \in \mathbb{R}^d$, an expression of the form

$$t_1x_1 + t_2x_2 + \cdots + t_nx_n, \text{ where } t_1, t_2, \dots, t_n \in \mathbb{R} \text{ and } \sum_{i=1}^{n} t_i = 1,$$

is an *affine combination* of the points $x_1, x_2, \dots, x_n$. It is easily checked that the affine hull of a set X equals the set of all affine combinations of points in X. Points $x_1, x_2, \dots, x_n \in \mathbb{R}^d$ are *affinely dependent* if one of the points can be written as an affine combination of the others. This is the same as the existence of real numbers $t_1, t_2, \dots, t_n$, at least one of them non-zero, such that

$$t_1x_1 + t_2x_2 + \cdots + t_nx_n = 0, \text{ and } t_1 + t_2 + \cdots + t_n = 0.$$

If a set $X \subset \mathbb{R}^d$ contains for every two points $x, y \in X$ the segment xy, then X is *convex*. Any intersection of convex sets is again convex. So we can define the *convex hull* $\mathrm{conv}(X)$ *of* X as the intersection of all convex sets in $\mathbb{R}^d$ containing X. It is easily checked that

$$\mathrm{conv}(X) = \{t_1x_1 + t_2x_2 + \cdots + t_nx_k \,|\, \text{all } x_i \in X, \text{ all } t_i \geq 0, \text{ and } \sum_{i=1}^{n} t_i = 1\}$$

holds for sets X in $\mathbb{R}^d$. Every point in the convex hull of X can therefore be expressed as a *convex combination* of a finite number of points in X. Carathéodory's theorem now implies that it can be expressed as a convex combination of at most $d+1$ points. The convex hull of $d+1$ affinely independent points in some $\mathbb{R}^n$ is a *d-simplex*.
The expression *simplex* is also used in other contexts, e. g. in algebraic

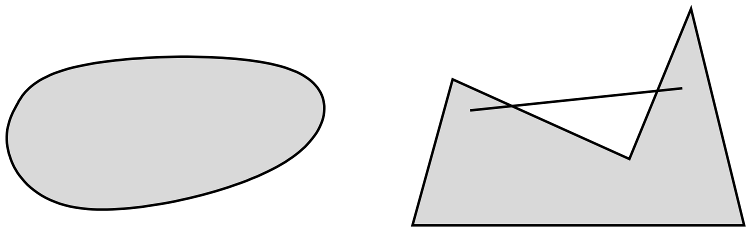

Figure 1.1: Example of a convex and a non-convex set.

topology. In this section, we use the expression *a point* p *is contained in a simplex* S. In this case S is a set of points, and we mean $p \in \mathrm{conv}(S)$ even if the points of S might not be affinely independent.
A non-empty set $X \subset \mathbb{R}^d$ is a *cone* if it contains with any finite set of vectors also all their linear combinations with non-negative coefficients. In particular, every cone contains the origin. For a non-empty set $X \subset \mathbb{R}^d$, we define its *conical hull* $\mathrm{cone}(X)$ as the intersection of all cones containing X. Clearly, $\mathrm{cone}(X)$ is a cone. Geometrically, $\mathrm{cone}(X)$ is the union of all rays starting at the origin and passing through a point of $\mathrm{conv}(X)$, see also Figure 1.2 for an example of a cone in $\mathbb{R}^2$.

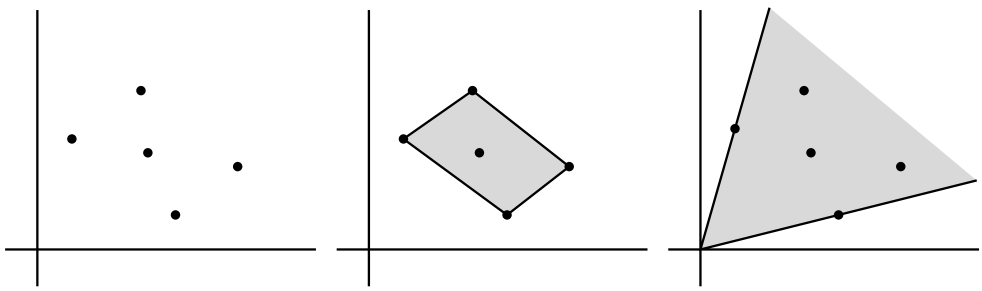

Figure 1.2: A set of points, its convex hull, and its conical hull.

Carathéodory's theorem is one of the fundamental results of discrete geometry. It is often convenient to assume that the point $p \in \mathbb{R}^d$ in the statement equals the origin. The following generalization is due to Bárány [4].

Theorem 1.4 (Colorful Carathéodory theorem)**.** *Consider* $d+1$ *finite point sets* $X_1, X_2, \ldots, X_{d+1}$ *in* $\mathbb{R}^d$ *such that the convex hull of each* X_i *contains the origin. Then there is a* $(d+1)$*-point set* $S \subset X_1 \cup \ldots \cup X_{d+1}$ *with* $|X_i \cap S| = 1$ *for each* i*, such that the origin is in its convex hull.*

If we imagine that the points in X_i are of color i, then the theorem guarantees the existence of a multi-colored d-simplex containing the origin. Here multi-colored means that every color is used. It is also common to call multi-colored simplices *rainbow simplices*.

In the plane it asserts the following: Given a red, a blue, and a green triangle as in Figure 1.3, each of them containing the (black) origin then there is a triangle using all three colors containing the origin. The multi-colored triangle is drawn with broken lines.

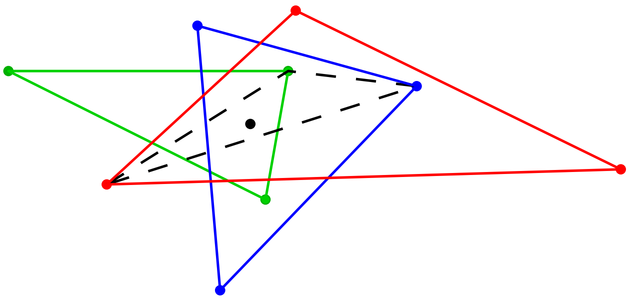

Figure 1.3: The colorful Carathéodory theorem in the plane.

Theorem 1.4 has found numerous applications, see e. g. [9]. We mention it here for two reasons. First, it is used in our favorite proof of the Tverberg theorem in Section 1.3. Secondly, recent progress on the number of rainbow simplices is due to Deza et al. [26], Bárány and Matoušek [7], and Stephan and Thomas [65].

Remark 1.5. Both Carathéodory's theorem and the colorful Carathéodory theorem can be seen as existence results: They guarantee the existence of at least one (rainbow) simplex containing the origin.

Another natural question is:

Problem 1.6. What is the number of simplices having this property? What about lower bounds for this number?

A first step to answer this question is to state precisely what one wants to count. For this, we define the following notion of general position.

Definition 1.7 (General position with respect to the origin)**.** The sets X_i as in the colorful Carathéodory theorem are *in general position with respect to the origin* 0 if $X_i \cap X_j = \emptyset$ for $i \neq j$, and if no $k+1$ points of $X_1 \cup X_2 \cup \ldots \cup X_{d+1}$ are on a common k-dimensional linear subspace of $\mathbb{R}^d$, for all $k = 0, 1, \ldots, d-1$.

In this situation $0 \in \text{conv}(X_i)$ implies that $|X_i| \geq d+1$. In the following, we often write *(rainbow) simplex* instead of *(rainbow) d-simplex containing the origin*, as we are not interested in any others. The number of rainbow simplices has been studied in [4], [26], [7], and [65]. We summarize these results in the following.

The first lower bound for the number of rainbow simplices goes back to the original article [4] of Bárány. Using the following stronger version of Theorem 1.4 one gets a lower bound of $d+1$ for the the number of rainbow simplices for points in general position with respect to the origin.

Theorem 1.8 (Stronger version of Theorem 1.4)**.** *Consider d finite point sets $X_1, X_2, \dots, X_d$ in $\mathbb{R}^d$ such that the convex hull of each X_i contains the origin 0, and let $x \in \mathbb{R}^d$ be arbitrary. Then there is a d-point set $S \subset X_1 \cup \ldots \cup X_d$ with $|X_i \cap S| = 1$ for each i, such that $0 \in \text{conv}(S \cup \{x\})$.*

Both [26] and [7] make use of the following basic lemma for proving their results on the number of rainbow simplices. We reuse it in Chapter 3 while looking at Tverberg partitions.

Lemma 1.9. *If $X = \{p_1, p_2, \dots, p_{d+1}\} \subset \mathbb{R}^d$ is a set of points in general position with respect to the origin 0, then $0 \in \text{conv}(X)$ if and only if $-p_{d+1} \in \text{cone}(X \setminus \{p_{d+1}\})$.*

Deza et al. [26] prove the following parity result for the number of rainbow simplices using what they call “a variational approach”.

Theorem 1.10 (Parity of the number of rainbow simplices)**.** *Consider $d+1$ finite point sets $X_1, X_2, \dots, X_{d+1}$ in $\mathbb{R}^d$ such that the convex hull of each X_i contains the origin 0. Assume that $|X_i|$ is even for every i, and that the set $\bigcup_{i=1}^{d+1} X_i$ is in general position with respect to the origin. Then the number of rainbow simplices containing the origin is even.*

Theorem 1.10 holds for example if the sets are of cardinality $d+1$ in odd dimension. The proof is based on the analysis of the following question: What happens if one moves one of the colored points at a time? There is also a monochromatic version Theorem 1.12 of this result, which we furnish with a proof.

In the same article, Deza et al. prove a lower bound of $2d$ for the number of rainbow simplices, and conjecture that the optimal lower bound equals d^2+1. Their conjecture is based on an explicit construction of sets X_i. The currently best lower bound is due to [7] and [65].

Theorem 1.11 (Lower bound for the number of rainbow simplices). *Consider $d+1$ finite point sets $X_1, X_2, \ldots, X_{d+1}$ in $\mathbb{R}^d$ in general position with respect to the origin such that the convex hull of each X_i contains the origin. Then there are at least $\max\{3d, \lfloor\frac{(d+2)^2}{4}\rfloor\}$ rainbow simplices containing the origin for $d \geq 3$.*

The above conjecture has been confirmed for $d = 2$ in [26]. The linear bound of $3d$ is due to [7]; and together with the above parity result it implies the conjecture for $d = 3$. Both [7] and [65] include a quadratic lower bound, the currently best quadratic lower bound, from [65], is exceeded by the linear bound for $3 \leq d \leq 7$.

Using the same technique as for Theorem 1.10, Deza et al. [26] also show a parity result in the monochromatic case.

Theorem 1.12 (Parity in the monochromatic case). *Let X be a set of n points in $\mathbb{R}^d$ such that $n - d$ is even, and that the set X is in general position with respect to the origin. Then the number of d-simplices containing the origin is even.*

Proof. The general position implies that no $k+1$ points in X lie on a k-dimensional linear subspace. The set of the normed points in X is again in general position in respect to the origin. Hence we can assume without loss of generality that all points are of norm 1, and therefore lie on the $(d-1)$-dimensional sphere S^{d-1}. Remember, we write *d-simplex* instead of *d-simplex containing the origin*, as we are not interested in any others. Let's start with the configuration that all points are clustered around the north pole of S^{d-1}. In that case $0 \notin \text{conv}(X)$, so that the number of d-simplices is clearly even. We now move one point after the other to its position while the others remain fixed. Hence only one point moves at a time. Let's call this moving point p, and let $z(p)$ be the number of d-simplices. Instead of following p, we look at its antipodal point $-p$, as we know from Lemma 1.9 that for a d-element subset S of $X \setminus \{p\}$ one has

$$0 \in \text{conv}(S \cup \{p\}) \quad \text{iff} \quad -p \in \text{cone}(S).$$

Moving $-p$ does not affect the number of d-simplices not involving p. Every d-element subset of $X \setminus \{p\}$ defines a cone, and they altogether define a decomposition of the sphere S^{d-1} into cells. One can attribute to every cell the number of cones containing it. See Figure 1.4, where an example of a set $X \setminus \{p\}$ of 5 points on S^1 is defined through 5 rays starting at the origin. Moreover, the attribution of numbers to every cell is shown in this example. Moving $-p$ inside a cell doesn't change $z(p)$. The boundary of a

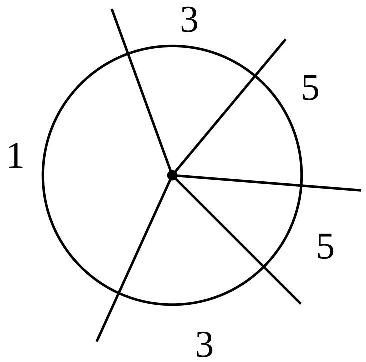

Figure 1.4: The cell decomposition defined by $X \setminus \{p\}$.

cell is defined through hyperplanes passing through $(d-1)$-element subsets of $X \setminus \{p\}$ and the origin. At some point we are forced to move $-p$ transversally from one side of a boundary hyperplane defined by such a $(d-1)$-element subset T to the other side. The number $z(p)$ changes in the following way: Our point $-p$ leaves a certain number m of cones involving T. On the other side, $-p$ enters some cones also involving T. In fact, there are $n-d-2-m$ cones left which involve T, and $-p$ enters all of them. The number $z(p)$ thus changes by $n-2-2m$, which is even. In Figure 1.4, the number of d-simplices involving p is everywhere odd, but its parity doesn't change, and the total number of d-simplices is even for every choice of p.

We start with $z(p)$ even, and every change is again even. In our proof we use that our decomposition is nice: We can move $-p$ to every position on the sphere while crossing hyperplanes in a transversal way. This is a well-known fact from topology, see also the original paper [26]. □

The colorful Carathéodory theorem also admits a cone version due to Bárány [4]. The cone version of Carathéodory's theorem was already mentioned by Danzer, Grünbaum, and Klee [24].

Let's end this section with some comments on Helly's theorem 1.3. Matoušek [49] notes that *Helly's theorem inspired a whole industry of Helly-type theorems*. For an extensive treatment see e. g. Eckhoff [29],

Chapter 5 is on topological generalizations of Helly-type theorems. To get an idea of the spirit of these generalizations, remember that the first topological version of Helly's theorem was established by Helly himself. A space X is *acyclic* if it is non-empty, and if its reduced singular homology vanishes in all dimensions.

Theorem 1.13 (Topological Helly theorem)**.** *Let $\mathcal{F}$ be a finite family of open sets in $\mathbb{R}^d$ such that the intersection of every k members of $\mathcal{F}$ is a acyclic for $k \leq d$, and is non-empty for $k = d+1$. Then $\bigcap \mathcal{F}$ is acyclic.*

1.2 On topological methods in combinatorics

This section comes with a crash course on topological methods in combinatorics. We also give some general topological background. In Chapters 2 and 5 we use topology as a tool to prove combinatorial results. Our first contact with topological methods will be in the introductory Section 1.3. In our crash course, we follow very much Matoušek's textbook *Using the Borsuk-Ulam Theorem* [46], where an extensive treatment on topological methods in combinatorics can be found. Anders Björner's survey article [18], and Rade Živaljević's survey articles [75], [73], and [74] also give a lot of background and motivation for topological methods. Spanier's book [63] is a good reference for (algebraic) topology. We start with some definitions, then we give a short introduction into topological methods à la Matoušek [46] mixed with some ideas from Živaljević [75].

Geometric simplicial complexes. In Section 1.1, we have introduced the notion of a *d-simplex* σ as the convex hull of $d+1$ affinely independent points in some $\mathbb{R}^n$. The number d is the *dimension of* σ, and the affinely independent points are the *vertices* of σ. The convex hull of an arbitrary subset of the set of vertices is a *face* of the simplex σ.
A *geometric simplicial complex* is a non-empty finite family Δ of simplices in some $\mathbb{R}^n$ such that:

1. Each face of any simplex of Δ is also a simplex in Δ.
2. The intersection of any two simplices σ_1, σ_2 of Δ is a face of both σ_1 and σ_2.

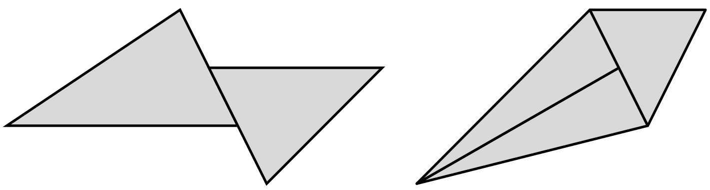

Figure 1.5: Examples of forbidden intersections.

The *dimension* $\dim(\Delta)$ of a simplicial complex Δ is the maximal dimension occurring in Δ. The *vertex set* $V(\Delta)$ is the union of all vertex sets of all

simplices in Δ. The union of all simplices of a simplicial complex Δ is the *polyhedron* of Δ, and it is denoted as $\|\Delta\|$. The polyhedron of Δ is a subset of $\mathbb{R}^n$ which inherits the subspace topology, so that all topological standard terms like *interior, closure, continuity,* ... are well-defined.
All simplicial complexes in this thesis will be finite. This is from a topologist's point of view quite restrictive, but it is sufficient for us. Moreover, we would like to keep our objects as simple as possible.

Examples 1.14 (Simplicial complexes)**.** 1. The family of all faces of a d-simplex is a d-dimensional simplicial complex.

2. Any finite set of p points in some $\mathbb{R}^n$ is a 0-dimensional simplicial complex which we will denote by $[p]$.

For a simplicial complex Δ, any subset that is again a simplicial complex (that is, closed under taking faces) is a *subcomplex of* Δ. An important example of a subcomplex is the *k-skeleton* $\Delta^{\leq k}$. It consists of all simplices of Δ of dimension at most k. A special case of a subcomplex is the *induced subcomplex* $\Delta[W]$ *on the vertex set* W, where W is a subset of the vertex set of Δ, and $\Delta[W]$ is obtained by taking all simplices whose vertices are in W. See Figure 1.6 for an example of a 2-dimensional simplicial complex on 5 vertices, its 1-skeleton (which is not an induced subcomplex), and an induced subcomplex on 4 vertices. For a face F of Δ, the *link* $\mathrm{lk}(F, \Delta)$ is the subcomplexes of all faces G in Δ such that $F \cup G \in \Delta$, but $F \cap G = \emptyset$. This definition implies $\mathrm{lk}(\emptyset, \Delta) = \Delta$. In our example of Figure 1.6, the link of the right-most vertex consists of the two neighboring vertices.

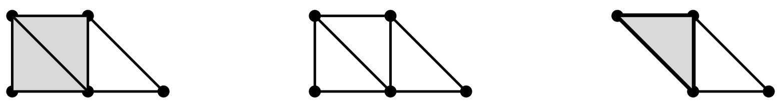

Figure 1.6: A simplicial complex, its 1-skeleton, and an induced subcomplex.

Abstract simplicial complexes. We will see that an abstract simplicial complex is a combinatorial object which describes the same mathematical object as a geometric simplicial complex. So we can speak of a simplicial complex, and we can focus both on its combinatorial and its geometric nature as convenient.
An *abstract simplicial complex* is a pair (V, K), where V is a finite set and $\mathsf{K} \subset 2^V$ is a family of sets satisfying: $F \in \mathsf{K}$ and $G \subset F$ implies $G \in \mathsf{K}$. The sets in K are *abstract simplices*, and its *dimension* $\dim(\mathsf{K})$ is defined as $\max\{|F|-1\,;\, F \in \mathsf{K}\}$. Usually, we may assume $V = \bigcup \mathsf{K}$. Therefore, we often

write K instead of (V, K). Let $f_i(\mathsf{K})$ denote the number of i-dimensional faces for an (abstract) simplicial complex. Let σ^p be the standard p-dimensional simplex with vertex set $[p+1]$.
A geometric simplicial complex Δ determines an abstract simplicial complex (V, K): The set V is the vertex set $V(\Delta)$, and the sets in K are just the vertex sets of the simplices in Δ. If (V, K) is an abstract simplicial complex, and if a geometric simplicial complex Δ determines (V, K) in the just described way, then we call Δ a *geometric realization* of (V, K).
Given an abstract simplicial complex (V, K), one can easily obtain a geometric realization as a subcomplex of the geometric n-simplex in $\mathbb{R}^n$ for $n = |V| - 1$. In the next paragraph, the polyhedron of the geometric realization will turn out to be unique up to homeomorphism.

Simplicial mappings. Let K and L be two abstract simplicial complexes. A *simplicial mapping* of K into L is a mapping $f : V(\mathsf{K}) \to V(\mathsf{L})$ that maps simplices to simplices, or precisely: for every $F \in \mathsf{K}$ one has $f(F) \in \mathsf{L}$. A bijective simplicial mapping whose inverse mapping is also simplicial is called an *isomorphism of abstract simplicial complexes.* The existence of such that isomorphism between abstract simplicial complexes K and L will be denoted by $\mathsf{K} \cong \mathsf{L}$.
For geometric simplicial complexes Δ_1 and Δ_2, let K_1 and K_2 be their associated abstract simplicial complex, and let $f : V(\mathsf{K}_1) \to V(\mathsf{K}_2)$ be a simplicial mapping. The *geometric realization of* f is the continuous map $\|f\| : \|\Delta_1\| \to \|\Delta_2\|$ which is obtained by extending f affinely from the vertex set to the simplices. It can be checked that this well-defined, and that $\|f\|$ is a homeomorphism if f is an isomorphism. Therefore, the *polyhedron of an abstract simplicial complex* is a well-defined topological space up to homeomorphism. Another way of viewing the geometric realization of f is:

Remark 1.15. Vertices are mapped to vertices. The image of a geometric simplex of Δ_1 equals the convex hull of the images of its vertices.

Connectivity. Let $k \geq -1$. A topological space X is *k-connected* if for every $l = -1, 0, 1, \ldots, k$, each continuous map $f : S^l \to X$ can be extended to a continuous map $\bar{f} : B^l \to X$. Here S^{-1} is interpreted as the empty set and B^0 as a single point, so (-1)-connected means non-empty. We write $\mathrm{conn}(X)$ for the maximal k such that X is k-connected.

Joins of topological spaces. The *join* $X * Y$ of spaces X and Y is a standard construction in topology. One way of looking at it is to identify it

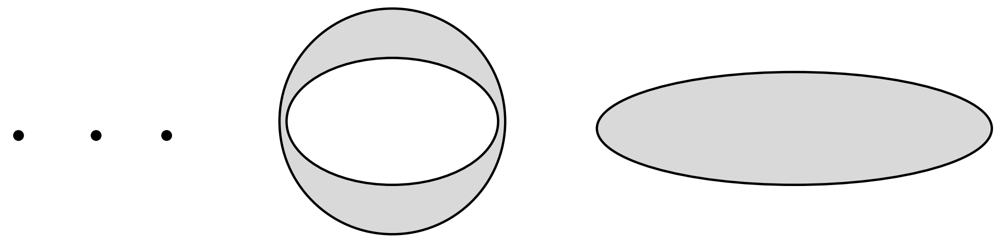

Figure 1.7: Examples of spaces X with $\mathrm{conn}(X) = -1, 0$, and $+\infty$.

with the set of formal convex combinations $tx \oplus (1-t)y$, where $t \in [0,1]$, $x \in X$, $y \in Y$. We use the symbol $\oplus$ to underline that the sum is formal and does not commute for $X = Y$. With this identification the n-fold join X^{*n} becomes the set of all formal convex combinations $t_1x_1 \oplus t_2x_2 \oplus \cdots \oplus t_nx_n$, where $t_1, t_2, \ldots, t_n$ are non-negative reals summing to 1 and $x_1, x_2, \ldots, x_n \in X$. For compact subsets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ of Euclidean spaces the join can be represented *geometrically* in the following way: Embed $A \subset \mathbb{R}^n \subset \mathbb{R}^{n+m+1}$ in the standard way, and embed $B \subset \mathbb{R}^m \subset \mathbb{R}^{n+m+1}$ such that the first n coordinates are equal to 0 and the last one is equal to 1. The subspace $C \subset \mathbb{R}^{n+m+1}$ defined as the union of all segments joining a point of A with a point of B is homeomorphic to $A * B$. This geometric representation of the join also works if A and B are skew affine subspaces of some $\mathbb{R}^n$. There is an inequality for the connectivity of the join $X * Y$ for topological spaces X and Y which we will use very often:

$$\mathrm{conn}(X * Y) \geq \mathrm{conn}(X) + \mathrm{conn}(Y) + 2; \tag{1.1}$$

see also [46, Section 4.4].

Let $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ be continuous maps, then the *join* $f * g$ *of maps* is again a continuous map from $X_1 * X_2$ into $Y_1 * Y_2$ given through: $tx_1 \oplus (1-t)x_2 \mapsto tf(x_1) \oplus (1-t)g(x_2)$.

The *join of simplicial complexes* is again a simplicial complex. We define it here for abstract complexes, but the above geometric interpretation can also be adapted to geometric simplicial complexes. For abstract simplicial complexes K and L the join is defined as the set of simplices $\{F \uplus G \,|\, F \in \mathsf{K}, G \in \mathsf{L}\}$, where $F \uplus G = (F \times \{1\}) \cup (G \times \{2\})$ is the disjoint union of F and G. The join is of dimension $\dim(\mathsf{K}) + \dim(\mathsf{L}) + 1$. Using the geometric interpretation, one can show for finite simplicial complexes that $\|\mathsf{K} * \mathsf{L}\| \cong \|\mathsf{K}\| * \|\mathsf{L}\|$.

Example 1.16 (Join of points)**.** The n-simplex – seen as a simplicial complex – can be obtained by starting with a point – in other words a 0-simplex – and taking joins iteratively: $\sigma^n \cong (\sigma^0)^{*(n+1)}$.

Deleted joins. Let $n \geq k \geq 2$. The *n-fold k-wise deleted join* of a topological space X is

$$X^{*n}_{\Delta(k)} := X^{*n} \setminus \{\tfrac{1}{n}x_1 \oplus \tfrac{1}{n}x_2 \oplus \cdots \oplus \tfrac{1}{n}x_n \mid \ k \text{ of the } x_i \in X \text{ are equal}\}.$$

In the case $k = n$, we remove the diagonal elements from n-fold cartesian product X^{*n}, and we write X^{*n}_{Δ} for $X^{*n}_{\Delta(n)}$. For $k_1 < k_2$ we have $X^{*n}_{\Delta(k_1)} \subset X^{*n}_{\Delta(k_2)}$. On the left the side of Figure 1.8, we have the twofold pairwise deleted join of a unit interval I: a simplex with a hole drilled at its center. For a simplicial complex K we define its *n-fold k-wise deleted join* as the following set of simplices:

$$\mathsf{K}^{*n}_{\Delta(k)} := \{F_1 \uplus F_2 \uplus \cdots \uplus F_n \in \mathsf{K}^{*n} \mid (F_1, F_2, \ldots, F_n) \ k\text{-wise disjoint}\},$$

where an n-tuple $(F_1, F_2, \ldots, F_n)$ is called k-wise disjoint if no k among them have a non-empty intersection. For simplicial complexes K, we have $\|\mathsf{K}^{*n}_{\Delta(k)}\| \subset \|\mathsf{K}\|^{*n}_{\Delta(k)}$. On the right side of Figure 1.8, we have the twofold pairwise deleted join of a 1-simplex σ^1. Most of the time we are interested in the special cases $k = 2$, or $k = n$.

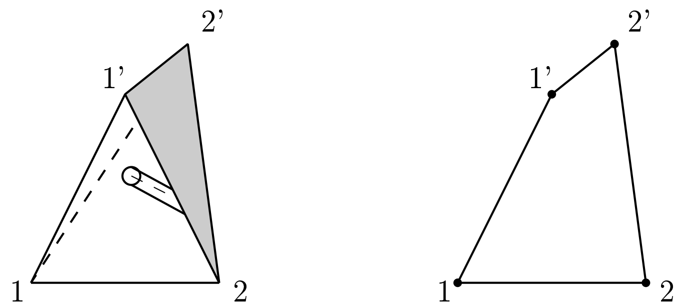

Figure 1.8: The twofold pairwise deleted joins $I^{*2}_{\Delta(2)}$ and $(\sigma^1)^{*2}_{\Delta(2)}$.

Join and deleted join are commutative in the following way.

Lemma 1.17. *For $n \geq 2$, let K and L be simplicial complexes. Then*

$$(\mathsf{K} * \mathsf{L})^{*n}_{\Delta(2)} \cong \mathsf{K}^{*n}_{\Delta(2)} * \mathsf{L}^{*n}_{\Delta(2)}.$$

The p-fold pairwise deleted join of the n-simplex σ^n shows up in many applications of topological methods.

Corollary 1.18. *For $n \geq 0$, and $p \geq 2$, one has*

$$(\sigma^n)^{*p}_{\Delta(2)} \cong ([p])^{*(n+1)}.$$

In particular, the simplicial complex $(\sigma^n)^{*p}_{\Delta(2)}$ is n-dimensional, and $(n-1)$-connected.

Proof.

$$(\sigma^n)^{*p}_{\Delta(2)} \overset{(1.16)}{\cong} ((\sigma^0)^{*(n+1)})^{*p}_{\Delta(2)} \overset{(1.17)}{\cong} ((\sigma^0)^{*p}_{\Delta(2)})^{*(n+1)} \cong ([p])^{*(n+1)}.$$

The first step is due to Example 1.16, and the last step follows from the definition of the deleted join: The p-fold pairwise deleted join of the 0-simplex σ^0 consists of p copies of this point, and no other faces. The complex $([p])^{*(n+1)}$ is clearly n-dimensional, and the $(n-1)$-connectivity follows from inequality (1.1). □

The chessboard complexes form another class of examples that arises in many interesting situations.

Example 1.19 (Chessboard complexes)**.** The *chessboard complex* $\Delta_{m,n}$ is defined as the simplicial complex $([n])^{*m}_{\Delta(2)}$. Its vertex set is the set $[n] \times [m]$, and its simplices can be interpreted as placements of rooks on an $n \times m$ chessboard such that no rook threatens any other; see also Figure 1.9. The roles of m and n are hence symmetric. $\Delta_{m,n}$ is an $(n-1)$-dimensional simplicial complex with $\binom{m}{n}n!$ maximal faces for $m \geq n$. See also Figure 1.9, every maximal face corresponds to a placement of 3 rooks on a 3×5 chessboard.

Using the nerve theorem 1.39 and induction, Björner, Lovász, Vrećica, and Živaljević proved in [22] the following connectivity result.

Theorem 1.20. *The chessboard complex $\Delta_{m,n}$ is $(\nu - 2)$-connected, for*

$$\nu := \min\{m, n, \lfloor \tfrac{1}{3}(m+n+1) \rfloor\}.$$

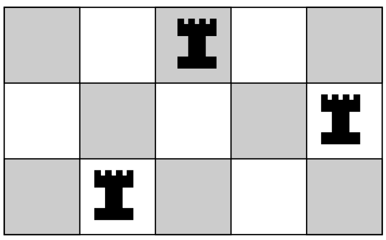

Figure 1.9: A maximal face of the chessboard complex $\Delta_{3,5}$.

G-**actions and** *G*-**spaces.** Let $(G, \cdot)$ be a finite group with $|G| > 1$. A topological space X equipped with a (left) G-action via a group homomorphism $\Phi : (G, \cdot) \to (\mathrm{Homeo}(X), \circ)$ is a *G-space* (X, Φ); we write $g\,x$

for $\Phi(g)(x)$. Here Homeo(X) is the group of homeomorphisms on X, the product of two homeomorphisms h_1 and h_2 is their composition $h_2 \circ h_1$. A continuous map f between G-spaces (X, Φ) and (Y, Ψ) that commutes with the G-actions of X and Y is called a *G-map*, or a *equivariant map*. If there is no G-map from a G-space X into a G-space Y, we write $(X, \Phi) \overset{G}{\nrightarrow} (Y, \Psi)$, and in the other case $(X, \Phi) \overset{G}{\rightarrow} (Y, \Psi)$. In the following, we will write *a G-space* X when it is clear which action Φ is fixed on X.
$\mathbb{Z}_n$ denotes the *cyclic group*, represented as $\{0, 1, \ldots, n-1\}$ with addition modulo n. The *symmetric group* of all permutations of $[n]$ is denoted by S_n. The subgroup of S_n that is generated by the permutation of $[n]$ that shifts cyclicly every element by one to the left is isomorphic to $\mathbb{Z}_n$.
A simplicial complex K is a *simplicial G-complex* if $(\|\mathsf{K}\|, \Phi)$ is a G-space, and if each homeomorphism $\Phi(g)$ is a simplicial mapping.

Examples 1.21 (G-spaces)**.**

1. The unit sphere S^d and $\mathbb{R}^d$ equipped with the "antipodal" map $x \mapsto -x$ are examples of $\mathbb{Z}_2$-spaces $(S^d, -)$ resp. $(\mathbb{R}^d, -)$.

2. Let (X, Φ) be a G-space, and H a non-trivial subgroup of G. Then X together with the induced H-action $\Phi_{|_H}$ is a H-space.

3. Let X be G-space, and $Y \subset X$ be invariant under the G-action, then Y with the induced G-action is a G-space.

4. For any space X, the symmetric group S_n (and all its subgroups) acts on the n-fold join X^{*n} by permuting the n coordinates. A permutation $\pi \in S_n$ acts on the element $t_1x_1 \oplus \cdots \oplus t_nx_n \in X^{*n}$ through:
$$\pi(t_1x_1 \oplus \cdots \oplus t_nx_n) = (t_{\pi^{-1}(1)}x_{\pi^{-1}(1)} \oplus \cdots \oplus t_{\pi^{-1}(n)}x_{\pi^{-1}(n)})$$
For the cyclic subgroup $\mathbb{Z}_n$, this leads to a cyclic shifting of the n coordinates. This also leads to an S_n-action on n-fold k-wise deleted joins, as the deleted subspaces are S_n-invariant.

5. Let (X, Φ) and (Y, Ψ) be G-spaces. Then $(X * Y, \Phi * \Psi)$ with the action $(\Phi * \Psi)(g) = \Phi(g) * \Psi(g)$ is again a G-space.

6. The symmetric group S_n (and all its subgroups) act as in Example 4 on the n-fold Cartesian product X^n by permuting the n coordinates.

7. Let G be a group, then G as a discrete space together with the left multiplication is a G-space. Using construction of Example 5, the join $G^{*(n+1)}$ becomes a n-dimensional, $(n-1)$-connected simplicial G-complex.

8. Suppose (X, Φ) is a $\mathbb{Z}_n$-space. Then the $\mathbb{Z}_n$-action is uniquely determined via the homeomorphism $\Phi(1)$, as $\Phi(i) = \Phi(1)^i$.

Remark 1.22. We will see that applying topology to combinatorics often reduces to the problem to prove $X \overset{G}{\nrightarrow} Y$ for some well-chosen G-spaces X, Y. The action of a subgroup H of G as in Example 1.21.2 can be very useful for our purposes:

$$X \overset{H}{\nrightarrow} Y \text{ implies } X \overset{G}{\nrightarrow} Y.$$

It is often sufficient to verify this for $H = \mathbb{Z}_p$ due to the following well-known fact from algebra : Every non-trivial finite group contains a subgroup isomorphic to $\mathbb{Z}_p$ for some prime p.

Remark 1.23. The space $G^{*(n+1)}$ of Example 1.21.7 shows up in many applications. It can be shown that it is homotopy equivalent to a wedge of $(|G|-1)^{n-1}$ many n-spheres. The space G^{*2} is the complete bipartite graph $K_{|G|,|G|}$; see Figure 1.10 for any group of order 6.

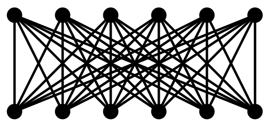

Figure 1.10: The space G^{*2} for any group of order 6.

Remark 1.24 (Composition of G-maps)**.** The composition $g \circ f : X \to Z$ of G-maps $f : X \to Y$ and $g : Y \to Z$ is again a G-map.

A number of important results in combinatorics, discrete geometry, and theoretical computer science have been proved by application of algebraic topology. A big portion of these results was obtained by using, implicitly or explicitly, the Borsuk-Ulam theorem. The Borsuk-Ulam theorem is part of the standard curriculum for every algebraic topology course. It has many proofs, equivalent versions, extensions, generalizations, and applications; see Matoušek's textbook *Using the Borsuk-Ulam theorem* [46] for many details. One of the many equivalent versions can now be written as:

Theorem 1.25 (Borsuk-Ulam theorem)**.** $(S^d, -) \overset{\mathbb{Z}_2}{\nrightarrow} (S^{d-1}, -)$.

Fixed point free and free G-spaces. For $x \in X$, the set $O_x = \{g\,x \,|\, g \in G\}$ is the *orbit* of x. A G-space (X, Φ) where every O_x has at least two elements is *fixed point free*, i. e. no point of X is fixed by all group elements. A G-space (X, Φ) is *free* if none of the homeomorphisms $\Phi(g)$, $g \neq e$, has a fixed-point.

Observation 1.26. 1. In a free G-space the orbit of any point x is a $|G|$-element set. Free G-spaces are thus fixed point free.

2. Let p be a prime number. A $\mathbb{Z}_p$-space (X, Φ) is free iff $\Phi(1)$ has no fixed point. In particular, the notion of free and of fixed point free G-spaces coincide for $G = \mathbb{Z}_2$.

Examples 1.27 (G-spaces revisited)**.** 1. The antipodal $\mathbb{Z}_2$-action on the d-sphere S^d is free. The action is not free on $\mathbb{R}^d$.

2. The restriction of a (fixed point) free G-action to a subgroup H leads to a (fixed point) free H-action.

3. The restriction of a (fixed point) free G-action to an invariant subspace Y is a (fixed point) free G-action.

4. The S_n-action on joins is not fixed point free. The S_n-actions on n-fold n-wise deleted joins are not free for $n \geq 3$. The cyclic $\mathbb{Z}_n$-action on n-fold n-wise deleted joins is free iff n is prime. We will see in Chapter 2 that the cyclic $\mathbb{Z}_n$-action on n-fold n-wise deleted joins is fixed point free for any n.

5. The S_n-actions on n-fold products are not free.

6. The join of (fixed point) free actions is (fixed point) free again.

7. G acts freely on itself, so that G^{*n+1} becomes a free G-space.

In the next paragraph, we define one of the most important tools in topological combinatorics: the G-index for G-spaces. The G-index is of no interest in the case of non-free actions. Nevertheless, similar results can be obtained for fixed point free actions.

The G-index. The G-index is a $(\mathbb{N} \cup \{\infty\})$-valued invariant which enables us sometimes to show that there is no G-map between G-spaces X and Y. Again [46] comes with a detailed introduction to this topological tool which is based on an extension of the Borsuk-Ulam theorem known as Dold's theorem. In this thesis, we introduce it both for the sake of completeness, and for its simplicity.

Definition 1.28 (G-index)**.** For a G-space (X, Φ), its *G-index* is:

$$\mathrm{ind}_G(X) = \min\{n \mid X \xrightarrow{G} G^{*(n+1)}\},$$

where $G^{*(n+1)}$ is the n-dimensional, $(n-1)$-connected, and free simplicial G-complex from Example 1.21.7.

In the case $G = \mathbb{Z}_2$, this is the n-dimensional sphere $(S^n, -)$. For us, it is sufficient to keep the following properties in mind.

Proposition 1.29 (Properties of the G-index)**.** *Let G be a non-trivial finite group.*

1) $\mathrm{ind}_G(X) > \mathrm{ind}_G(Y)$ *implies* $X \overset{G}{\not\to} Y$*.*

2) *If X is a non-free G-space, then* $\mathrm{ind}_G(X) = \infty$*.*

3) $\mathrm{ind}_G(G^{*(n+1)}) = n$*.*

4) $\mathrm{ind}_G(X * Y) \leq \mathrm{ind}_G(X) + \mathrm{ind}_G(Y) + 1$*.*

5) *If X is $(n-1)$-connected, then* $\mathrm{ind}_G(X) \geq n$*.*

6) *If K is a free simplicial G-complex of dimension n, then* $\mathrm{ind}_G(\mathsf{K}) \leq n$*.*

Proof. For detailed proofs see e. g. [46, Section 6.2]. To prove Property 2 remember that for a non-free G-space X some $g \neq e$ has a fixed point $x_0 = gx_0$. The orbit O_{x_0} is therefore not a copy of G, i. e. $|O_{x_0}| < |G|$. On the other hand, every orbit of the free G-space $G^{*(n+1)}$ contains $|G|$ many distinct elements. Moreover, any G-map maps every orbit onto an orbit surjectively, and this can not happen for O_{x_0}. □

Remark 1.30. Property 1 of the G-index enables us to show $X \overset{G}{\not\to} Y$ for free G-space Y.

Remark 1.31. Determining the G-index is in general not easy. For our applications, the lower bound implied by the connectivity due to Property 5 is often sufficient.

For this thesis, it is sufficient to know the index for two types of G-spaces given in the next proposition.

Proposition 1.32. *Let $p \geq 2$ be an integer.*

1) $\mathrm{ind}_{\mathbb{Z}_p}((\sigma^n)^{*p}_{\Delta(2)}) = n$*.*

2) *If p is prime, then* $\mathrm{ind}_{\mathbb{Z}_p}((\mathbb{R}^d)^{*p}_{\Delta}) = (d+1)(p-1) - 1$*.*

See [46, Section 6.3] for a proof.

The non-free, fixed point free case. Now our focus is on the fixed point free case where one still has similar tools to show the non-existence of

G-maps. The following result due to Volovikov [69] is sufficient for our purposes in Chapter 2. It has first appeared as Corollary 3.4 in an unpublished preprint of Özaydin [52]. The proof of Proposition 1.33 uses deeper results from bundle cohomology.
A cohomology n-sphere over $\mathbb{Z}_p$ is a CW-complex having the same cohomology groups with $\mathbb{Z}_p$-coefficients as the n-dimensional sphere S^n.

Proposition 1.33 (Volovikov's Lemma). *Set $G = (\mathbb{Z}_p)^r$, and let X and Y be fixed point free G-spaces such that Y is a finite-dimensional cohomology n-sphere over $\mathbb{Z}_p$ and $\tilde{H}^i(X, \mathbb{Z}_p) = 0$ for all $i \leq n$. Then there is no G-map from X to Y.*

An alternative approach in the non-free case is the ideal-valued cohomological index introduced by Fadell and Husseini in [31]; see also [74] for details.

Applying topology to discrete geometry. As noted by many authors, many proofs in topological combinatorics share a common scheme called *configuration space/test map paradigm* which we outline in the following inspired by Živaljević [75]. This abstract scheme consists of two major steps:

Step 1: The problem is rephrased in topological terms.

Step 2: Solving the rephrased problem using (standard) techniques from topology.

To illustrate this scheme, we prove on our way a generalization of Radon's theorem called the *Topological Radon theorem*.

Theorem 1.34 (Topological Radon theorem). *For every continuous map $f : \|\sigma^{d+1}\| \to \mathbb{R}^d$ there exist two disjoint faces F_1, F_2 of the $(d+1)$-simplex σ^{d+1} such that $f(\|F_1\|) \cap f(\|F_2\|) \neq \emptyset$.*

Proof. **Step 1 for the topological Radon theorem:** We go back to Radon's theorem, and we realize that it can be rephrased equivalently as in the topological Radon theorem restricted to affine instead of continuous maps.

Having this in mind, we wonder whether the statement is true using the more general setting of continuous maps. This involves looking at examples – as always in mathematics. See Figure for an example of dimension $d = 2$.

After this, we look for a natural *configuration space* X. This configuration space models all possible configurations of our problem. For Radon's

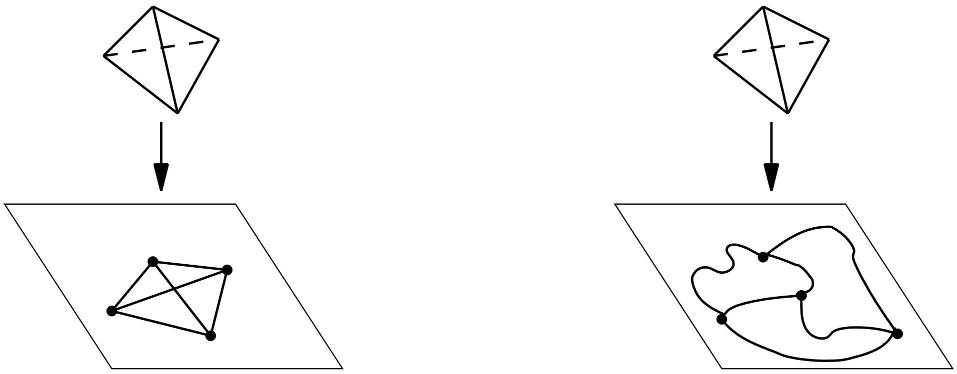

Figure 1.11: Affine versus continuous problem.

theorem, the twofold pairwise deleted join of the $(d+1)$-simplex comes up: It models all possibilities to partition $d+2$ vertices into two subsets. Faces of $(\sigma^{d+1})^{*2}_{\Delta(2)}$ are of the form $F_1 \cup F_2$ for some disjoint faces F_1, F_2 of σ^{d+1} – the $(d+1)$-simplex with vertex set $[d+2]$. A point in $(\sigma^{d+1})^{*2}_{\Delta(2)}$ can be written as $tx \oplus (1-t)y$ with $t \in [0,1]$, $x \in \|F_1\|$, $y \in \|F_2\|$, and $F_1 \cap F_2 = \emptyset$, if $F_1, F_2 \neq \emptyset$. The remaining points can be written as $1x \oplus 0$ resp. $0 \oplus 1y$.

Subsequently, we search for a *test map* $T : X \to V$ which tells us whether a point from our configuration space is a solution of our problem, or not:

$$T(x) \in Z \ \text{ iff } x \text{ is a solution,}$$

for a given *test subspace* Z of our *test space* V. Looking at $(\sigma^{d+1})^{*2}_{\Delta(2)}$ leads to the test map

$$f^{*2} : \underbrace{(\sigma^{d+1})^{*2}_{\Delta(2)}}_{=:X} \to \underbrace{(\mathbb{R}^d)^{*2}}_{=:V},\ tx \oplus (1-t)y \mapsto tf(x) \oplus (1-t)f(y).$$

Suppose there are two points x_1 resp. x_2 in disjoint faces F_1 resp. F_2 which are mapped under f to the same point z in $\mathbb{R}^d$. Then our test map f^{*2} maps the point $\frac{1}{2}x_1 \oplus \frac{1}{2}x_2$ to the diagonal $\Delta = \{\frac{1}{2}z \oplus \frac{1}{2}z \mid z \in \mathbb{R}^d\}$ of $(\mathbb{R}^d)^{*2}$. If f^{*2} meets this diagonal this then leads conversely to a solution of our problem. The diagonal Δ of $(\mathbb{R}^d)^{*2}$ is therefore our test subspace Z.

Additionally, our problem comes with an inherent symmetry showing up in an action of its symmetry group G on the configuration space, and on the test space, and turning our test map into a G-map. In Radon's theorem, we look for unordered partitions into two subsets. The symmetry group is thus $S_2 = \mathbb{Z}_2$, and it can easily be checked that f^{*2} is a $\mathbb{Z}_2$-map.

Finally, assume that there is no solution to our problem. Hence the test map does not meet the test subspace, and one obtains G-maps

$$T : X \xrightarrow{G} V \setminus Z, \qquad \text{and} \qquad f^{*2} : (\sigma^{d+1})^{*2}_{\Delta(2)} \xrightarrow{\mathbb{Z}_2} (\mathbb{R}^d)^{*2} \setminus \Delta = (\mathbb{R}^d)^{*2}_{\Delta}.$$

Step 2 for the topological Radon theorem: $X \overset{G}{\not\rightarrow} V \setminus Z$.
To prove the existence of a solution, it remains to show that such a G-map cannot exist. For this, we can use what we have summarized above: a G-index argument, Volovikov's Lemma, etc. Let's put the results from Proposition 1.32 together to obtain the topological Radon theorem:

$$\operatorname{ind}_{\mathbb{Z}_2}((\sigma^{d+1})^{*2}_{\Delta(2)}) = d + 1 > d = \operatorname{ind}_{\mathbb{Z}_2}((\mathbb{R}^d)^{*2}_{\Delta}).$$

□

Remark 1.35.
1. Step 2 looks simple, but if an G-index argument does not work – e. g. in case of a non-free action on $V \setminus Z$ – the results (and their proofs) needed from equivariant topology may be quite involved. We see in Section 1.3 that this part can also break down.
2. If a G-map as in Step 2 exists, this does not imply that our statement is false. This would mean for our example: If there was a G-map $h : (\sigma^{d+1})^{*2}_{\Delta(2)} \xrightarrow{\mathbb{Z}_2} (\mathbb{R}^d)^{*2}_{\Delta(2)}$, then it need not be of the form $g = f^{*2}$ for some map f as above. During the rephrasing process of Step 1, some information gets lost.

The nerve theorem. Another very useful tool in topological combinatorics is the nerve theorem, e. g. it can be used to determine the connectivity of a given topological space, or simplicial complex.

Definition 1.36 (Nerve complex)**.** The *nerve* $N(\mathcal{F})$ of a family of sets $\mathcal{F}$ is the abstract simplicial complex with vertex set $\mathcal{F}$ whose simplices are all $\sigma \subset \mathcal{F}$ such that $\bigcap_{F\in\sigma} F \neq \emptyset$.

Example 1.37 (Nerve complex)**.** Take a family of 6 convex black sets as in Figure 1.12. Its nerve complex (in blue) has 6 blue vertices, one for each set from our family. Two vertices form a blue edge if they intersect. There are no faces of dimension greater than one as no three or more sets intersect. Version I of the nerve theorem below states that the union of the convex black sets (seen as a subspace of $\mathbb{R}^2$), and the polyhedron of its nerve complex are homotopy equivalent: Both consist of two components, the right one is trivial, and the left one is homotopy equivalent to a one-dimensional sphere.

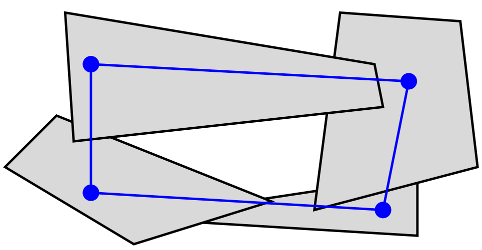

Figure 1.12: Example of a nerve complex.

The nerve complex often helps to simplify a given topological space. It was first obtained by Leray [45], and it has many versions; see Björner [18] for a survey on nerve theorems. In its most popular form, it says:

Theorem 1.38 (Nerve theorem, version I)**.** *Let $\mathcal{F}$ be a family of open sets in some topological space X such that $\bigcap \mathcal{G}$ is empty or contractible for all non-empty subfamilies $\mathcal{G} \subset \mathcal{F}$. Then the topological space $\bigcup \mathcal{F}$ and the nerve complex $\|N(\mathcal{F})\|$ are homotopy equivalent: $\bigcup \mathcal{F} \simeq \|N(\mathcal{F})\|$.*

A non-empty topological space X is *contractible* if the identity $i : X \to X$ is homotopic to a constant map $c : X \to X$. See Figure 1.13 for examples of (non)-contractible spaces.

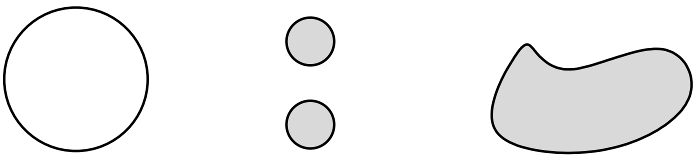

Figure 1.13: Two non-contractible spaces, and a contractible space.

In Chapter 5, we get a short proof of the homological version of the following nerve theorem of Björner [19].

Theorem 1.39 (Nerve theorem, version II)**.** *For $k \geq 0$ let $\mathcal{F}$ be a family of open sets in some topological space X such that $\bigcap \mathcal{G}$ is empty or $(k - |\mathcal{G}| + 1)$-connected for all non-empty subfamilies $\mathcal{G} \subset \mathcal{F}$. Then the topological space $\| \bigcup \mathcal{F} \|$ is k-connected iff the nerve complex $\|N(\mathcal{F})\|$ is k-connected.*

Both nerve theorems also hold if one replaces families of open sets with families of subcomplexes of a finite simplicial complexes, even CW-complexes.

1.3 From Radon's theorem to the colored Tverberg theorem

This section comes with a discussion on one of the major generalizations of Radon's theorem: Tverberg-type theorems. As before, a lot of background can be found in Matoušek [46], [49], and Živaljević [75].

Radon's theorem for dimension 2 claims that four points in general position in the plane either are in convex position, or one point is in the convex hull of the others. Any point in the intersection of the convex hulls of two

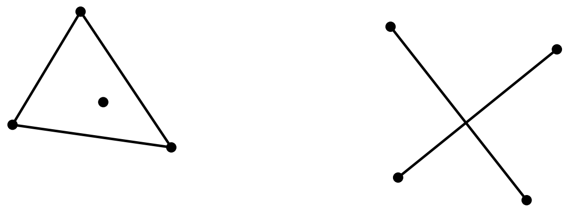

Figure 1.14: Four points in the plane.

disjoint subsets of the initial $d+2$ points is a *Radon point*. The Radon point is unique for sets in general position.

A natural question that arises while studying Radon's theorem is the following:

Problem 1.40. What is the minimal number $T(d,q)$ such that any set X of $T(d,q)$ points in $\mathbb{R}^d$ admits a partition into $q \geq 2$ disjoint subsets $F_1, F_2, \ldots, F_q$ whose convex hulls have a common point?

Radon's theorem states that $T(d,2) = d+2$. The existence of $T(d,q)$ is not hard to prove. One obtains a strict lower bound of $(d+1)(q-1)$ for $T(d,q)$ by analyzing the following configuration. For arbitrary q and d, $(d+1)(q-1)$ points are arranged in $\mathbb{R}^d$ in the following way: Cluster $q-1$ points around every vertex of a d-simplex in $\mathbb{R}^d$, see Figure 1.17 for $d=2$ and $q=4$. The points of each cluster can end up in most $q-1$ different sets. Hence there is no way to partition this point configuration into q sets as desired.

This lower bound for $T(d,q)$ was already known to Birch [17] who settled this problem for $d=2$. In 1966, Helge Tverberg [66] showed that $T(d,q)$ equals $(d+1)(q-1)+1$.

Theorem 1.41 (Tverberg theorem)**.** *Let X be a set of $(d+1)(q-1)+1$ points in $\mathbb{R}^d$. Then there are q pairwise disjoint subsets $X_1, X_2, \ldots, X_q$ of X whose convex hulls have a common point.*

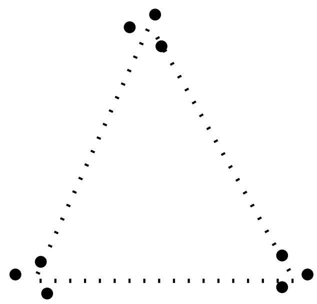

Figure 1.15: A point configurations of $3(q-1)$ points in the plane.

Any point in $\bigcap_{i=1}^{q} \operatorname{conv}(X_i)$ is a *Tverberg point* of X, and the partition $X_1, X_2, \ldots, X_q$ is a *Tverberg partition of X into q subsets.* We will write *Tverberg q-partition* for short. Most of the time, we will study Tverberg q-partitions of $(d+1)(q-1)+1$ points in $\mathbb{R}^d$ so that we even drop the letter q and speak of Tverberg partitions.

Remark 1.42. For X in general position, the Tverberg point is unique for any given Tverberg partition. The set of Tverberg points is then a finite set of points.

Tverberg's first proof has been regarded as difficult. Fifteen years later, Tverberg published a simpler proof [67]. Sarkaria [59] invented another, very nice and simple proof. This proof has been streamlined by Onn [10], and by Matoušek [49]. This proof is my current favorite. It is based on the colorful Carathéodory theorem, and the following equivalent conic version of Theorem 1.41 is shown. Other people claim that the currently most beautiful proof is due to Tverberg and Vrécica [68].

Theorem 1.43 (Conic Tverberg theorem)**.** *Let X be a set of $(d+1)(q-1)+1$ points in $\mathbb{R}^{d+1}$ such that $0 \notin \operatorname{conv}(X)$. Then there are q pairwise disjoint subsets $X_1, X_2, \ldots, X_q$ of X such that $\bigcap_{i=1}^{q} \operatorname{cone}(X_i) \neq \{0\}$.*

Both versions 1.41 and 1.43 also hold for multi-sets instead of sets. Theorem 1.43 is a special case of Roudneff's conic Tverberg theorem from [57].

Proof. Put $N = (d+1)(q-1)$. Before proving Theorem 1.43, let's see that the original Tverberg Theorem 1.41 and its conic version are equivalent. For this, assume first the conic version, and let X be a set of $N+1$ points in $\mathbb{R}^d$. Embed $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$ as the hyperplane $h = \{(x,1) \mid x \in \mathbb{R}^d\}$. The set X becomes a subset of $\mathbb{R}^{d+1}$, and its convex hull is in h, so that X satisfies the hypothesis of the conic version. By Theorem 1.43, there is a partition $X_1, X_2, \ldots, X_q$ such that $\bigcap_{i=1}^{q} \operatorname{cone}(X_i)$ has point $p \neq 0$ in common. The ray

through p starting at 0 is thus in every $\text{cone}(X_i)$. This ray intersects h in some point p'. The point p' and the partition $X_1, X_2, \ldots, X_q$ are what we were looking for.
Assume now that the Tverberg theorem is true, and let X be a set of $N+1$ points in $\mathbb{R}^{d+1}$ such that $0 \notin \text{conv}(X)$. Then there is a hyperplane h' separating $\text{conv}(X)$ from the origin. Let X' be the (multi)-set of points obtained by intersecting the rays trough every point in X with h'. Then X' is a $(N+1)$-element subset of some $\mathbb{R}^d$. It can easily be checked that every Tverberg partition of X' leads to a conic Tverberg partition of X. In fact, the set of Tverberg partitions of X and the set of conic Tverberg partitions of X' are thus in one-to-one correspondence.

To prove Theorem 1.43, define linear maps $\phi_i : \mathbb{R}^{d+1} \to \mathbb{R}^N$ for $i \in [q]$ via

$$\phi_i(x) = (0, \ldots, 0, x, 0, \ldots, 0) \in (\mathbb{R}^{d+1})^{q-1}, \text{ for } i < q,\ 0 \in \mathbb{R}^{d+1} \text{ and } x \in \mathbb{R}^{d+1},$$

where x is in ith position. Moreover, set $\phi_q(x) = (-x, -x, \ldots, -x)$ for $x \in \mathbb{R}^{d+1}$. These linear maps satisfy: For any q vectors $v_1, v_2, \ldots, v_q \in \mathbb{R}^{d+1}$

$$\sum_{i=1}^{q} \phi_i(v_i) = 0 \text{ iff } v_1 = v_2 = \cdots = v_q. \tag{1.2}$$

In the next step, we make use of these maps to get colored sets $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_{N+1}$ in $\mathbb{R}^N$ as in the colored Carathéodory theorem 1.4. Let X be the initial set of $N+1$ points in $\mathbb{R}^{d+1}$. We consider the set $\tilde{X} = \phi_1(X) \cup \phi_2(X) \cup \cdots \cup \phi_q(X)$ in $\mathbb{R}^N$, and we color the points in $\tilde{X}$ with colors $1, 2, \ldots, N+1$: For every $x_i \in X$ all q images $\phi_1(x_i), \phi_2(x_i), \ldots \phi_q(x_i)$ get color i, and call this colored set $\tilde{X}_i$. Using (1.2) we get $0 \in \text{conv}(\tilde{X}_i)$ for every $i = 1, 2, \ldots, q$ as wanted.

The colored Carathéodory theorem leads to a rainbow simplex containing the origin. In other words, there is an $(N+1)$-element set S of the form $\{\phi_{i_1}(x_1), \phi_{i_2}(x_2), \ldots, \phi_{i_{N+1}}(x_{N+1})\}$, and non-negative real numbers α_1, α_2, …, α_{N+1} summing up to 1 such that

$$\sum_{j=1}^{N+1} \alpha_j \phi_{i_j}(x_j) = 0.$$

See also Figure 1.16 for an example in $d = 1$ and $q = 2$. Setting $I_j = \{k \in [N+1] \,|\, i_k = j\}$, for $j \in [q]$, partitions the index set $[N+1]$ – and at the time the original point set X – into q subsets. It remains to prove that this partition $X_1, X_2, \ldots, X_q$ defines a conic Tverberg partition of X.
For $l \in [q]$, consider $u_l = \sum_{k \in I_l} \alpha_k x_k$. This defines a point in $\text{cone}(X_l)$, and

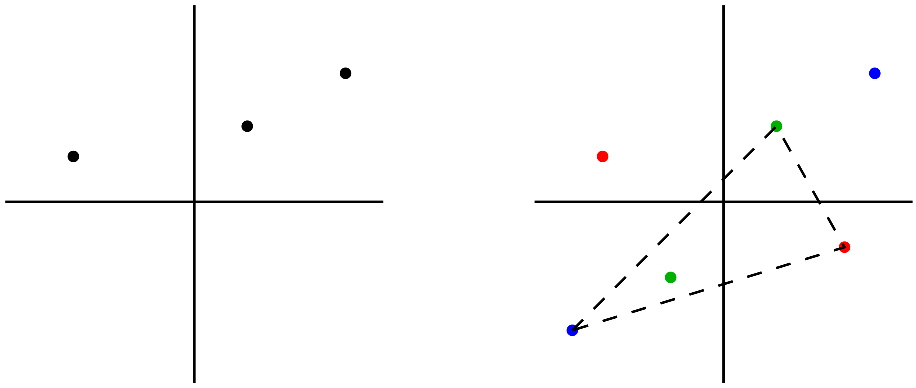

Figure 1.16: The initial set $X \subset \mathbb{R}^{d+1}$, and the colored sets in $\mathbb{R}^{(d+1)(q-1)}$.

we can rearrange our sum

$$0 = \sum_{j=1}^{N+1} \alpha_j \phi_{i_j}(x_j) = \sum_{l=1}^{q} \sum_{k \in I_l} \alpha_k \phi_l(x_k) \stackrel{(*)}{=} \sum_{l=1}^{q} \phi_l(\sum_{k \in I_l} \alpha_k x_k) = \sum_{l=1}^{q} \phi_l(u_l),$$

where we use the linearity of ϕ_l in step $(*)$. By (1.2), we obtain that

$$u_1 = u_2 = \cdots = u_q$$

which is then a point common to all $\mathrm{cone}(X_l)$. The assumption $u_l = 0$ implies $0 \in \mathrm{conv}(X)$. $\square$

Having established the existence of one Tverberg partition, it is natural to ask for the number of Tverberg partitions:

Problem 1.44. Given $(d+1)(q-1)+1$ points in $\mathbb{R}^d$, how many unordered ways are there to partition them into q subsets?
What is the minimal number of Tverberg partitions over all configurations of $(d+1)(q-1)+1$ points in $\mathbb{R}^d$?

In 1979, Gerard Sierksma came up with the following conjecture on the minimal number of Tverberg partitions.

Conjecture 1.45 (Sierksma's conjecture)**.** Let $q \geq 2$, $d \geq 1$, and put $N = (d+1)(q-1)$. For every $N+1$ points in $\mathbb{R}^d$ the number of unordered Tverberg partitions is at least $((q-1)!)^d$.

This conjecture obviously holds for $q = 2$, and the case $d = 1$ holds in a more general setting via the intermediate value theorem for continuous maps. For $d > 1$, and for $q > 2$ the conjecture is still open. In the literature, this conjecture is also known as the *Dutch cheese problem.* Sierksma has offered a Dutch cheese[1] for a solution of this problem. Sarkaria's preprint [58] comes with an attempt, but his argument has serious gaps.

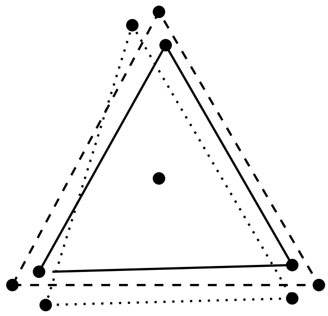

Figure 1.17: A planar configuration with 24 Tverberg 4-partitions.

The conjectured lower bound is attained by the following construction from Figure 1.15: Start with the configuration that led to $T(d,q) > (d+1)(q-1)$, and add a point in its center.

Remark 1.46. One check that the point in the center of the simplex from Figure 1.17 is the only Tverberg point of this point configuration. All Tverberg partitions consist of this point, and of $q-1$ many d-simplices which are obtained by choosing one point from each cluster. Now focus on one cluster, every point of this cluster will end up in one d-simplex. For the first point, one has $q-1$ possibilities to choose from every of the remaining d clusters, for the second, there are $q-2$ possibilities to do so. Finally, you end up with $((q-1)!)^d$ many ways to partition. In Chapter 4, we present many more examples attaining this conjectured lower bound.

Remark 1.47. It is one of the main objectives of this thesis to prove new lower bounds for the number of Tverberg partitions.

In Chapters 2 and 3, we discuss in detail old and new results for the number of Tverberg partitions. Up to now, all known proofs make use of topological methods. Before starting with the setup for a topological version of Tverberg's theorem, let's go back to the proof of 1.43 once more.

Observation 1.48.

1. The proof of Theorem 1.43 does not immediately imply a lower bound for the number of Tverberg partitions using the lower bounds on the number of rainbow simplices from Section 1.1.

[1] For my wife, it is obvious that this problem is still unsolved: "Who wants Dutch cheese? Why not Tomme de Savoie, or Camembert?" – She is French.

2. Starting with a point configuration of $N+1$ points in $\mathbb{R}^{d+1}$ with n conic Tverberg partitions leads to a colored point configuration of $q(N+1)$ points in $\mathbb{R}^N$ with $q!n$ rainbow simplices.

Proof. (of Observation 1.48) The correspondence in the proof of 1.43 between rainbow simplices containing the origin and conic Tverberg partitions is not one-to-one. Every conic Tverberg partition corresponds to $q!$ rainbow simplices. The colored sets $\tilde{X}_i$ contain $q < N+1$ points in $\mathbb{R}^N$, so that they are not in general position. Only the stronger version 1.8 of the colorful Carathéodory can be applied in this case, and gives the lower bound $|\tilde{X}_i| = q$ for the number of rainbow simplices. □

Let's apply the topological method from Section 1.2 which has led to the topological Radon theorem to Tverberg's theorem. As for Radon's theorem, there is an equivalent version of the Tverberg's Theorem 1.41 using affine maps instead of convexity.

Theorem 1.49 (Affine Tverberg theorem). *Let $q \geq 2$, $d \geq 1$, and put $N = (d+1)(q-1)$. For every affine map $f : \|\sigma^N\| \to \mathbb{R}^d$ there are q pairwise disjoint faces $F_1, F_2, \ldots, F_q$ of the standard N-simplex σ^N whose images under f intersect: $\bigcap_{i=1}^q f(\|F_i\|) \neq \emptyset$.*

Again, any point in $\bigcap_{i=1}^q f(\|F_i\|)$ is a *Tverberg point of f*, and the set of disjoint faces $F_1, F_2, \ldots, F_q$ form a *Tverberg partition of f into q pairwise disjoint faces*. We will write *Tverberg partition* for short. For readability reasons, we stop distinguishing between the polyhedron $\|\sigma^N\|$, and the underlying simplicial complex σ^N.

Theorem 1.50 (Topological Tverberg theorem). *Let $q \geq 2$ be a prime power, $d \geq 1$. For every continuous map*

$$f : \sigma^{(d+1)(q-1)} \longrightarrow \mathbb{R}^d$$

there is a Tverberg partition in the standard $(d+1)(q-1)$-simplex $\sigma^{(d+1)(q-1)}$.

For q prime the result is due to Bárány, Shlosman, and Szücs [11]. The prime power case was first proved by Özaydin in an unpublished preprint [52] in 1987. Different proofs can be found in Volovikov [69], and Sarkaria [60], see also de Longueville [25]. The case $d = 1$ and arbitrary q is implied by the intermediate value theorem for continuous maps.

Problem 1.51. According to Matoušek [46], "the validity of the topological Tverberg theorem for arbitrary q is one of the most challenging problems in topological combinatorics".

Proof. (of Theorem 1.50) Put $N = (d+1)(q-1)$, and choose $(\sigma^N)^{*q}_{\Delta(2)}$ as configuration space. This simplicial complex models all possibilities to partition the $N+1$ vertices into q faces. Define

$$f^{*q} : (\sigma^N)^{*q}_{\Delta(2)} \longrightarrow (\mathbb{R}^d)^{*q}$$

as test map. One can check that the diagonal $\Delta = \{\frac{1}{q}y \oplus \cdots \oplus \frac{1}{q}y \,|\, x \in \mathbb{R}^d\}$ of $(\mathbb{R}^d)^{*q}$ is our test subspace:

$$x \text{ is a solution iff } f^{*q}(x) \in \Delta.$$

We look for unordered partitions into q subsets so that the symmetric group S_q comes up as natural symmetry group of our problem. This symmetry group acts on the configuration space and the test space by permutation, as explained in Section 1.2. Our test map f^{*q} is thus turned into a S_q-map. The test subspace Δ is invariant under this S_q-action. It is therefore sufficient to prove in a second step that

$$(\sigma^N)^{*q}_{\Delta(2)} \stackrel{S_q}{\not\to} (\mathbb{R}^d)^{*q} \setminus \Delta = (\mathbb{R}^d)^{*q}_{\Delta}.$$

Recall from Remark 1.22 that it is sufficient to show

$$(\sigma^N)^{*q}_{\Delta(2)} \stackrel{H}{\not\to} (\mathbb{R}^d)^{*q}_{\Delta}$$

for some subgroup H of S_q. For q prime choose $H = \mathbb{Z}_q$ so that the H-action acts by shifting the q coordinates cyclically. Let's put the results from Proposition 1.32 on the $\mathbb{Z}_q$-indices of the above spaces together to prove the prime case:

$$\operatorname{ind}_{\mathbb{Z}_q}((\sigma^N)^{*q}_{\Delta(2)}) = N > N-1 = \operatorname{ind}_{\mathbb{Z}_q}((\mathbb{R}^d)^{*q}_{\Delta}).$$

The prime power case needs a little more effort. For $q = p^r$ choose $H = (\mathbb{Z}_p)^r$ as subgroup, then one has to check that the assumptions of Volovikov's lemma are satisfied. More details can be found in Section 2.1 where we prove a lower bound for the number of Tverberg partitions in the prime power case. □

Remark 1.52. Özaydin proved in [52] that this topological method breaks down for non-prime powers q:

$$(\sigma^N)^{*q}_{\Delta(2)} \stackrel{S_q}{\longrightarrow} (\mathbb{R}^d)^{*q}_{\Delta} \text{ iff } q \text{ is not a prime power.}$$

However, this does not imply that the topological Tverberg theorem does not hold for arbitrary q, because not every S_q-map $g : (\sigma^N)^{*q}_{\Delta(2)} \longrightarrow (\mathbb{R}^d)^{*q}_{\Delta}$ is of the form $g = f^{*q}$ for some continuous map $f : \sigma^N \to \mathbb{R}^d$; see also [25].

In many applications, e. g. the problem of splitting necklaces, the result for the prime (power) case is sufficient. The arbitrary case follows via an inductive argument. One might hope that this gap can be filled in a similar way for the topological Tverberg theorem.

Recent progress is due to Schöneborn and Ziegler [62]. They have reduced the topological Tverberg theorem to the following winding number conjecture on maps of the $(d-1)$-skeleton of $\sigma^{(d+1)(q-1)}$. The smallest unresolved case $d = 2$ is thus turned into a problem on drawings of complete graphs K_{3q-2}.

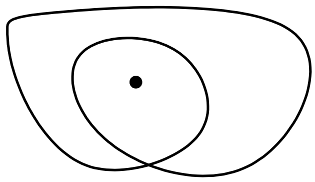

Figure 1.18: A closed curved winding twice around a point.

The concept of a closed curve winding around a point as in Figure 1.18 is very intuitive. As always in mathematics we have to formalize this concept. For this, we use what is known as the degree of a continuous map of spheres which can be found in many textbook on algebraic topology, e. g. [63]. We obtain on our way a general definition that holds not only for closed curves in the plane.

Definition 1.53 (Winding number with respect to a point)**.** Let $f : S^{d-1} \to \mathbb{R}^d$ be a continuous map, and let p be a point in $\mathbb{R}^d \setminus f(S^{d-1})$. Then f induces a homomorphism $f_* : \tilde{H}_{d-1}(S^{d-1}; \mathbb{Z}) \to \tilde{H}_{d-1}(\mathbb{R}^d \setminus \{p\}; \mathbb{Z})$ in homology. Both $(d-1)$-dimensional homology groups are isomorphic to $\mathbb{Z}$ so that our homomorphism becomes $f_* : \mathbb{Z} \to \mathbb{Z}$. Every group automorphism of $\mathbb{Z}$ is uniquely determined by the image of one of its two generators. We define the *winding number of f with respect to p* to be one of the two integers number which differ only by a sign.

The winding number $W(f, p)$ is thus defined up to sign, but the expression

$$\text{“}W(f, p) = 0\text{”}$$

is independent of this. For any d-simplex σ^d, we have for its boundary $\partial\sigma^d \cong S^{d-1}$; the winding number for continuous maps $f : \partial\sigma^d \to \mathbb{R}^d$ and points $p \notin f(\partial\sigma^d)$ is defined the same way. Again it is defined up to sign, so that the condition “$W(f, p) = 0$” makes sense.

Schöneborn [61] originally defined winding numbers using homotopy groups $[f] \in \pi_{d-1}(S^{d-1}) \cong \mathbb{Z}$ which have formal similarities to homology groups, but which are much harder to compute.

Schöneborn and Ziegler [62] show that the topological Tverberg theorem is equivalent to:

Conjecture 1.54 (Winding number conjecture)**.** Let $q \geq 2$, $d \geq 1$, and put $N = (d+1)(q-1)$. For every continuous map

$$f : (\sigma^N)^{\leq d-1} \to \mathbb{R}^d$$

there are q disjoint faces $F_1, F_2, \ldots, F_q$ of the d-skeleton of the N-simplex σ^N, and a point $p \in \mathbb{R}^d$ such that for each i, one of the following holds:
- $\dim(F_i) \leq d-1$ and $p \in f(F_i)$,
- $\dim(F_i) = d$, and either $p \in f(\partial F_i)$, or $p \notin f(\partial F_i)$ and $W(f|_{\partial F_i}, p) \neq 0$.

A set of q pairwise disjoint faces and a point as in the winding number conjecture is a *winding partition* resp. *winding point*.

Remark 1.55. In the smallest unresolved case $d = 2$, a continuous map $f : (\sigma^{3(q-1)})^{\leq 1} \to \mathbb{R}^2$ is nothing else but a drawing of the complete graph on $3(q-1)+1 = 3q-2$ vertices. If our map is a "nice" graph drawing the winding number conjecture says that
- either $q-1$ triangles (drawings of K_3 subgraphs) wind around one vertex,
- or $q-2$ triangles wind around the intersection of two edges,

where the triangles, the edges, and the vertex are disjoint in K_{3q-2}.

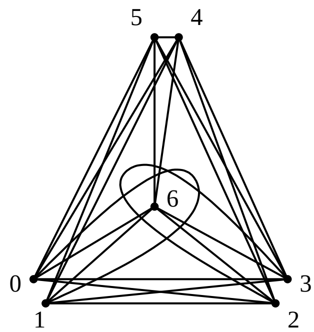

Figure 1.19: Drawing of K_7 with 6 winding partitions for $q = 3$.

Example 1.56. In Figure 1.19 we have modified the drawing 1.17 coming from Sierksma's configuration in the edges $(0,1)$ and $(2,3)$. Before the modification, we had $((q-1)!)^d = 4$ winding partitions consisting of the winding point 6 and two winding triangles for each winding partition:

$$\{(0,2,4),(1,3,5)\}, \{(0,2,5),(1,3,4)\}, \{(0,3,4),(1,2,5)\}, \{(0,3,5),(1,2,4)\}$$

The modified edges do not occur in them, so that these 4 partitions survive the modification. Moreover, two new winding partitions come up: The triangle $(2,3,4)$ winds around the intersection of $(0,1)$ and $(5,6)$, the same applies to the triangle $(2,3,5)$, and edges $(0,1)$ and $(4,6)$.

One direction of the equivalence is based on the following generalization of the well-known intermediate value theorem for continuous maps.

Theorem 1.57 (Generalized intermediate value theorem)**.** *Let $f : B^d \to \mathbb{R}^d$ be a continuous map, and $y \in \mathbb{R}^d \setminus f(S^{d-1})$ such that $W(f|_{S^{d-1}}, y) \neq 0$. Then there is a point $x \in B^d$ that is mapped to y under f.*

The case $d = 1$ implies the elementary intermediate value theorem for continuous maps. Furthermore, for a continuous map $f : \sigma^N \to \mathbb{R}^d$, every winding partition of the restriction $f|_{(\sigma^N)^{\leq d-1}}$ is a Tverberg partition of f. Hence the winding number conjecture for d and q implies the topological Tverberg theorem for d and q.

Proof. (of Theorem 1.57) Suppose $y \notin f(B^d)$, then f and $f|_{S^{d-1}}$ are homotopic to a constant map in $\mathbb{R}^d \setminus \{p\}$. Every constant map has degree resp. winding number zero. □

The other direction of the equivalence is based on the following idea:

> Extend $f : (\sigma^N)^{\leq d-1} \to \mathbb{R}^d$ to a continuous map $F : \sigma^N \to \mathbb{R}^d$ such that every Tverberg partition of F is a winding partition of f.

Unfortunately, this idea needs some adjustment: Their proof only works for $d \geq 3$. Hence the topological Tverberg theorem for $d \geq 3$ and q implies the winding number conjecture for $d \geq 3$ and q. It is not known whether this implication also holds for $d = 2$. The case $d = 2$ of the winding number conjecture follows then from the following proposition.

Proposition 1.58. *If the winding number conjecture holds for q and $d \geq 2$, then it also holds for q and $d - 1$.*

The proof is based on a modification of de Longueville's [25] proof of the same behavior for the topological Tverberg theorem. The underlying idea goes back to Sarkaria.

Another important step in their proof of the equivalence is to reduce the topological Tverberg theorem to the d-skeleton.

Conjecture 1.59 (d-skeleton conjecture)**.** Let $q \geq 2$, $d \geq 1$, and put $N = (d+1)(q-1)$. For every continuous map

$$f : (\sigma^N)^{\leq d} \to \mathbb{R}^d$$

there is a Tverberg partition in the d-skeleton of the N-simplex σ^N.

Let's come up with a suitable definition of general position for Tverberg's theorem.

Definition 1.60 (General position for affine maps)**.** Let K be a simplicial complex. A map $f : \mathsf{K} \to \mathbb{R}^d$ is *affine* if the image of every face is canonically determined by the images of its vertices. Such an affine map f is *in general position* if for every set of disjoint faces $\{F_1, F_2, \dots, F_k\}$ of K the inequality

$$\operatorname{codim}(\bigcap_{i=1}^{k} f(F_i)) \geq \sum_{i=1}^{k} \operatorname{codim}(f(F_i))$$

holds, where $\operatorname{codim}(X) = d - \dim(X)$ if $X \subset \mathbb{R}^d$. We use the convention that $\dim(\emptyset) = -\infty$ and thus $\operatorname{codim}(\emptyset) = \infty$. Thus in the case $\bigcap_{i=1}^{k} f(F_i) = \emptyset$ the general position condition holds independently from the right hand side.

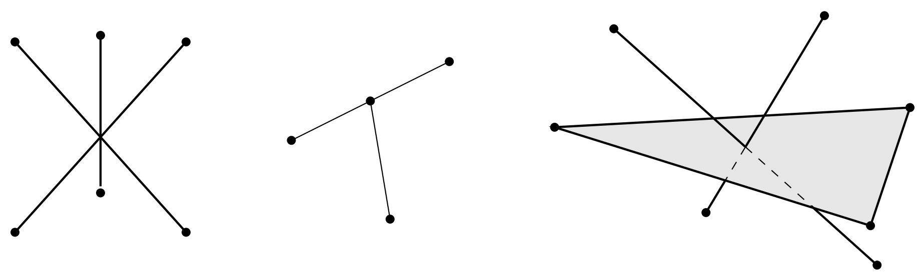

Figure 1.20: Images of maps that are not in general position in $\mathbb{R}^2$ and $\mathbb{R}^3$.

General position forces all Tverberg partition to be of the following form.

Observation 1.61. Let $f : \sigma^N \to \mathbb{R}^d$ be an affine map in general position. Then a Tverberg partition consists of:

- Type I: One vertex v, and $(q-1)$ many d-simplices containing v.
- Type II: k intersecting faces of dimension less than d, and $(q-k)$ d-simplices containing the intersection point for some $1 < k \leq \min\{d, q\}$.

For $d = 2$, type II is equal to: Two edges intersecting, and $q - 2$ many 2-simplices containing this intersection point. For arbitrary d, type II has two extremal cases: Either an edge intersects a $(d-1)$-face and $(q-2)$ many d-simplices contain the intersection point, or d many $(d-1)$-simplices intersect and $(q-d)$ many d-faces contain this intersection point. By looking at the codimension, it is easy to see that there are $P_d - 1$ many possibilities for type II. Here P_d is the number of number partition of the integer d disregarding the order of the summands, e. g. $P_1 = 1$, $P_2 = 2$, $P_3 = 3$, and $P_4 = 5$.

Schöneborn and Ziegler [62] extend general position to piecewise affine maps such that Tverberg partitions are again as in Observation 1.61 in the d-skeleton.

Remark 1.62 (General position for continuous maps)**.** General position also extends naturally to continuous maps $\mathsf{K} \to \mathbb{R}^d$ such that Tverberg partitions are as in Observation 1.61. For considering Tverberg partitions, it is sufficient to consider piecewise affine maps in general position due to Lemma 1.63.

The proof of the equivalence of the topological Tverberg theorem and the d-skeleton conjecture is based on the above analysis of piecewise linear maps which are in general position, and on the following lemma. It states that one can perturb a continuous map a little such that no new Tverberg partitions arise. The property that a given partition is not a Tverberg partition, is thus robust in this sense. The non-trivial part of the equivalence between the d-skeleton conjecture and the topological Tverberg theorem follows from the fact that continuous maps can be approximated by (general position) piecewise linear maps.

Lemma 1.63. *For every continuous map $f : (\sigma^{(d+1)(q-1)})^{\leq d} \to \mathbb{R}^d$ there is an $\epsilon_f > 0$ such that the following holds:*

> *If $\tilde{f} : (\sigma^{(d+1)(q-1)})^{\leq d} \to \mathbb{R}^d$ is a continuous map with $\|f - \tilde{f}\|_\infty < \epsilon_f$, then every Tverberg partition for $\tilde{f}$ is also Tverberg partition for f.*

Schöneborn and Ziegler also show that the number of winding partitions and the number of Tverberg partitions are closely related; see Chapter 2 for a more detailed discussion of these results.

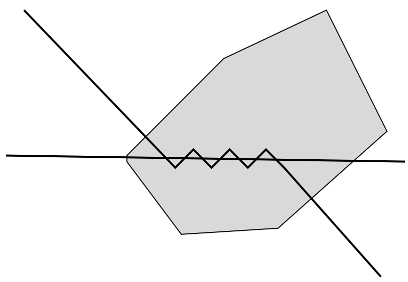

Figure 1.21: A Tverberg partition with seven Tverberg points.

Remark 1.64. In its topological version, the Tverberg point is not unique for a given Tverberg partition, even for general position maps. In Figure 1.21 the intersection of $q-2$ many 2-simplices is shaded in grey. Two disjoint edges can intersect finitely many times in the grey area.

Tverberg's conjecture. Tverberg proposed in 1989 a general conjecture which implies many classical theorems as special cases: Tverberg's theorem, Rado's theorem, the Ham sandwich theorem, non-embeddability results; see [70] for references. In its topological version due to Tverberg and Vrećica, it says the following.

Conjecture 1.65 (Tverberg's conjecture)**.** Let $0 \leq k \leq d-1$ and let $f_i : \sigma_i^{N_i} \to \mathbb{R}^d$ be continuous maps for $N_i = (q_i - 1)(d - k + 1)$, and for $i = 0, 1, 2, \ldots, k$. Then there is an affine k-dimensional subspace which intersects the images of q_i pairwise disjoint faces of the simplex σ^{N_i} for each $i = 0, 1, 2, \ldots, k$.

See Vrećica [70] for the most recent discussion of its status.

The colored Tverberg theorem. In the remaining part of this section, we discuss the colored version of Tverberg's theorem which was introduces in order to obtain upper bounds for the *k-set problem*; see [46], and [49] for a discussion of this problem from computational geometry.

The colored version of Tverberg's theorem asks the following:

Problem 1.66 (Colored Tverberg problem)**.** Let $q \geq 2$ and $d \geq 1$ be integers. Is there a minimal integer $CT(d, q)$ such that for any $d + 1$ disjoint $CT(d, q)$-element sets $C_1, C_2, \ldots, C_{d+1}$ in $\mathbb{R}^d$, there are q pairwise disjoint sets $F_1, F_2, \ldots, F_q$ such that:

- $|F_i \cap C_j| = 1$ for all i and j,
- $\bigcap_{i=1}^{q} \operatorname{conv}(F_i) \neq \emptyset$.

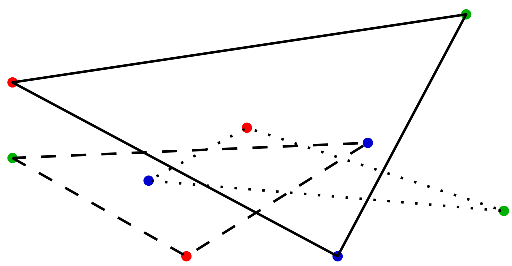

Figure 1.22: Three colored sets with a rainbow Tverberg partition.

If we imagine that the points in C_i are of color i, then we look for a Tverberg partition $F_1, F_2, \ldots, F_q$ of $\bigcup_{j=1}^{d+1} C_j$ such that every F_i is a rainbow simplex using every color once.

Theorem 1.67 (Colored Tverberg Theorem)**.** *Let $q \geq 2$ and $d \geq 1$ be integers. Then the following holds for $CT(d, q)$:*

1. $CT(d,q) = q$, *for* $d = 1,2$, *or* $q = 2$, (Bárány et al. [6]).

2. $CT(d,q) \le 2q-1$, *for prime powers* q (Vrećica and Živaljević [71], and Živaljević [74]).

3. $CT(d,q) \le 4q-3$, *for arbitrary* q (Vrećica and Živaljević [71]).

Bárány, Füredi, and Lovász [5] first showed the existence in the case $d = 2$ and $q = 2$, and conjectured that $CT(d,q) = q$. This conjecture has been verified in the case $d = 1,2$, and for $q = 2$ by Bárány, Larman, and Lovász [6]. The bound in the general case can be derived from the prime case using "Bertrand's postulate" which says for any $q > 1$ there is a prime p with $q \le p < 2q$. Using the prime power case this can be improved to $CT(d,q) \le 2q' - 1$, where q' is the smallest prime power greater than q.

The proof of the prime (power) case is based on the equivariant method used for the topological Tverberg theorem. Vrećica and Živaljević [71], [74] obtained the following topological version.

Theorem 1.68 (Topological colored Tverberg theorem)**.** *Let* $d \ge 1$*, and* q *be a prime power. Let* $C_1, C_2, \dots, C_{d+1}$ *be* $d+1$ *disjoint* $(2q-1)$*-element sets in* $\mathbb{R}^d$*, and let* K *be the abstract simplicial complex with vertex set* $C_1 \cup C_2 \cup \ldots \cup C_{d+1}$*, whose simplices are all subsets using at most one point from each* C_i*. Then for any continuous map* $f : \mathsf{K} \to \mathbb{R}^d$*, there are* q *pairwise disjoint faces* $F_1, F_2, \dots, F_q$ *of* K *such that their images intersect* $\bigcap_{i=1}^q f(F_i) \ne \emptyset$*.*

Proof. The simplicial complex K equals $[2q-1]^{*(d+1)}$, and its q-fold pairwise deleted join $(\mathsf{K})^{*q}_{\Delta(2)}$ models all possibilities to partition our points in the desired way. As for the topological Tverberg theorem, the problem can be reduced to the equivariant problem:

$$f^{*q} : (\mathsf{K})^{*q}_{\Delta(2)} \overset{S_q}{\nrightarrow} (\mathbb{R}^d)^{*q}_{\Delta}.$$

Using Lemma 1.17, we obtain

$$(\mathsf{K})^{*q}_{\Delta(2)} = ([2q-1]^{*(d+1)})^{*q}_{\Delta(2)} \cong ([2q-1]^{*q}_{\Delta(2)})^{*(d+1)}.$$

Here the space $[2q-1]^{*q}_{\Delta(2)}$ is the chessboard complex $\Delta_{2q-1,q}$ from Section 1.2. Combining our knowledge on the connectivity of the chessboard complex from Theorem 1.20, and inequality (1.1) for the connectivity of the join leads to

$$\mathrm{conn}((\mathsf{K})^{*q}_{\Delta(2)}) \ge (d+1)(q-2) + 2d = (d+1)(q-1) + d - 1.$$

In the prime case, this implies a lower bound of the $\mathbb{Z}_q$-index of $(\mathsf{K})^{*q}_{\Delta(2)}$ as desired:

$$\operatorname{ind}_{\mathbb{Z}_q}((\mathsf{K})^{*q}_{\Delta(2)}) > (d+1)(q-1)+d-1 > (d+1)(q-1)-1 \overset{(1.32)}{=} \operatorname{ind}_{\mathbb{Z}_q}((\mathbb{R}^d)^{*q}_{\Delta}).$$

The prime power case needs a little more effort, but is proved in a similar way using e. g. Volovikov's Lemma 1.33. $\square$

Remark 1.69. In the proof of Theorem 1.68, the $\mathbb{Z}_q$-index of $(\mathsf{K})^{*q}_{\Delta(2)}$ turned out to be greater than necessary. This gap has been used by Živaljević [75] to obtain another version of the colored Tverberg theorem. In its simplest case, this version implies the well-known fact that the complete bipartite graph $K_{3,3}$ is not planar, already known to Euler.

Chapter 2

Tverberg-type theorems and the equivariant method

This chapter is devoted to the application of the equivariant method in topological combinatorics. In Sections 2.2 and 2.4, we extend the lower bounds of Vućić and Živaljević [72] for primes q to the prime power case:

- for the number of Tverberg partition of continuous maps,
- for the number of splittings of a generic necklace.

The lower bound for the number of Tverberg partitions also implies a lower bound for the number of winding partitions.

Schöneborn and Ziegler initiated in [62] the concept of constraint graphs for the topological Tverberg theorem based on the question: Which graphs G in the 1-skeleton of $\sigma^{(d+1)(q-1)}$ can be can be forbidden such that any continuous map $f : \sigma^{(d+1)(q-1)} \to \mathbb{R}^d$ still admits a Tverberg partition that does not use any edge of G? In Section 2.3, we extend their result about matchings G to a wider class of graphs:

- Complete graphs K_l for $l \leq \frac{q+1}{2}$,
- complete bipartite graphs $K_{1,l}$ for $l < q-1$,
- paths P_l on l vertices for $q > 3$,
- cycles C_l on l vertices for $q > 4$,
- disjoint unions of graphs from above,

for prime powers q. This topological Tverberg theorem with constraints is a generalization of the topological Tverberg, and it confirms that $K_{1,q-1}$ is a minimal graph that is not a constraint graph. It enables us to force points to be in different blocks of the Tverberg partition, and it serves us in Chapter 3 to obtain a lower bound for the number of Tverberg points. The proof is based on connectivity results of chessboard-type complexes. We start this chapter with a short introduction to the equivariant tools in Section 2.1.

2.1 PRELIMINARIES AND TOOLS

Let's start with a summary of equivariant tools needed for this chapter. This is based on Volovikov's Lemma 1.33 from Section 1.2 which is the key result for proving the topological (colored) Tverberg theorem in the prime power case $q = p^r$ where p is a prime. Instead of looking at the symmetric group S_q, we use an elementary abelian p-group as in Volovikov's Lemma.

A group action of $(\mathbb{Z}^p)^r$ on q-fold joins and products. Let $q = p^r$ be a prime power. The additive group $(\mathbb{Z}_p)^r$ can be seen as a subgroup of the symmetric group S_q in the following way. Remember S_q is the group of permutations of the q-element set $[q]$. The group $(\mathbb{Z}_p)^r$ is of order q. Ordering the q elements of $(\mathbb{Z}_p)^r$ lexicographically leads to a bijection $[q] \to (\mathbb{Z}_p)^r$. The graph of this bijection looks in the case $p = 3$ and $r = 2$ as follows:

$$\{(1,(0,0)),(2,(0,1)),(3,(0,2)),(4,(1,0)),\dots,(9,(2,2))\}.$$

Every element $g \in (\mathbb{Z}^p)^r$ defines a permutation of the q elements of $(\mathbb{Z}_p)^r$ by translation: $h \mapsto g+h$ for $h \in (\mathbb{Z}^p)^r$. For example, the element $(1,1) \in (\mathbb{Z}_3)^2$ defines a permutation

$$(1,1) : (\mathbb{Z}_3)^2 \to (\mathbb{Z}_3)^2, \;\; (i,j) \mapsto (i+1, j+1).$$

Using the above bijection, the graph of the permutation of $(1,1)$ equals

$$\{(1,5),(2,6),(3,4),(4,8),(5,9),(6,7),(7,2),(8,3),(9,1)\}.$$

The neutral element $(0,0,\dots,0)$ of $(\mathbb{Z}_p)^r$ corresponds to the neutral element of S_q, and one can check that we obtain a monomorphism $G \hookrightarrow S_g$.

Having interpreted $(\mathbb{Z}_p)^r$ as a subgroup of S_q, we can speak of a $(\mathbb{Z}_p)^r$-action on q-fold joins and products. The symmetric group acts on a q-fold join X^{*q} (or a q-fold product X^q) by permuting its q coordinates as introduced in Example 1.21.6. The $(\mathbb{Z}_p)^r$-action on X^{*q} is the restriction of the S_q-action as in Example 1.21.2. The above element $(1,1) \in (\mathbb{Z}_3)^2$ acts on X^{*9}:

$$t_1x_1 \oplus \cdots \oplus t_9x_9 \mapsto t_9x_9 \oplus t_7x_7 \oplus t_8x_8 \oplus t_3x_3 \oplus t_1x_1 \oplus t_2x_2 \oplus t_6x_6 \oplus t_4x_4 \oplus t_5x_5.$$

Proving the topological Tverberg theorem in the prime power case. To apply Volovikov's Lemma 1.33 to the setting of the topological Tverberg theorem, we have to prove two properties for the $(\mathbb{Z}_p)^r$-space $(\mathbb{R}^d)^{*q}_{\Delta}$:

- The restriction of the S_q-action on $(\mathbb{R}^d)^{*q}_{\Delta}$ to $(\mathbb{Z}_p)^r$ is fixed point free.
- $(\mathbb{R}^d)^{*q}_{\Delta}$ is a cohomological $((d+1)(q-1)-1)$-sphere.

See the following two lemmas which verify each one of the properties.

Lemma 2.1. *Let X_Δ^{*q} be the q-fold q-wise deleted join of some space X equipped with the $(\mathbb{Z}_p)^r$-action defined as above. Then X_Δ^{*q} is a fixed point free $(\mathbb{Z}_p)^r$-space.*

Note that the $(\mathbb{Z}_p)^r$-action on X_Δ^{*q} is not free, e. g. the element $(1,1) \in (\mathbb{Z}_p)^r$ has the element

$$t_1x_1 \oplus t_2x_2 \oplus t_3x_3 \oplus t_3x_3 \oplus t_1x_1 \oplus t_2x_2 \oplus t_2x_2 \oplus t_3x_3 \oplus t_1x_1 \in X_\Delta^{*q}$$

as fixed point.

Proof. Let $x = t_1x_1 \oplus t_2x_2 \oplus \cdots \oplus t_qx_q \in X_\Delta^{*q}$, then by definition there are indices i and j such that $t_i \neq t_j$ or $x_i \neq x_j$. Using our identification of $(\mathbb{Z}_p)^r$ and $[q]$, the indices i and j correspond to elements a resp. b of $(\mathbb{Z}_p)^r$. Setting $g = b - a$, we get $x \neq g\,x$ hence $|O_x| > 1$. □

The following lemma is standard in topological combinatorics, see e. g. [46, Section 6].

Lemma 2.2. *Let $q \geq 2$ and $d \geq 1$ be integers. Then $(\mathbb{R}^d)_\Delta^{*q} \simeq S^{(d+1)(q-1)-1}$.*

Proof. Using the geometric version of the join, we get an embedding $(\mathbb{R}^d)_\Delta^{*q} \subset \mathbb{R}^{q(d+1)-1}$. More precisely, we can identify it with the subset $\{(t_1x_1, t_2x_2, t_2, \ldots, t_qx_q, t_q) \,|\, x_i \in \mathbb{R}^d, t_i \geq 0, \sum_1^q t_i = 1\}$. The diagonal of $(\mathbb{R}^d)^{*q}$ is embedded as $A = \{(x, x, \frac{1}{q}, \ldots, x, \frac{1}{q}) \,|\, x \in \mathbb{R}^d\}$, a d-dimensional affine subspace of $\mathbb{R}^{q(d+1)-1}$. Its orthogonal complement $A^\perp$ has dimension $(d+1)(q-1)$. The restriction of the orthogonal projection $p_{A^\perp}$ onto the complement maps $(\mathbb{R}^d)_\Delta^{*q}$ on $\mathbb{R}^{(d+1)(q-1)} \setminus \{\text{pt.}\}$. This map is a homotopy equivalence. □

Remark 2.3. Lemma 2.1 and Lemma 2.2 allow us to replace the index argument from the proof of the prime case of the topological Tverberg theorem in Section 1.3 with Volovikov's Lemma. The proof of the topological Tverberg theorem in the prime power case is thus complete.

Deleted products. In some applications, it is more convenient to work with the deleted product $(\mathbb{R}^d)_\Delta^q$ instead of the deleted join. Both Lemma 2.1 and Lemma 2.2 from above also hold for deleted products. The proofs are similar so that we omit them; see also [46, Section 6] for details for Lemma 2.5.

Lemma 2.4. *Let X_Δ^q be the q-fold q-wise deleted product of some space X equipped with the $(\mathbb{Z}_p)^r$-action defined as above. Then X_Δ^q is a fixed point free $(\mathbb{Z}_p)^r$-space.*

Lemma 2.5. *Let $q \geq 2$ and d be integers. Then we have $(\mathbb{R}^d)_\Delta^q \simeq S^{d(q-1)-1}$.*

2.2 ON THE NUMBER OF TVERBERG PARTITIONS IN THE PRIME POWER CASE

This is the first section of this thesis on the number of Tverberg partitions. Hence we start with some background. We obtain a new lower bound on the number of Tverberg partitions for prime powers q using the equivariant method. This result will be published in [35].

Tverberg's Theorem 1.41 establishes the existence of at least one Tverberg partition into q subsets for arbitrary sets of $(d+1)(q-1)+1$ points in $\mathbb{R}^d$. Another natural question 1.44 is to ask for a lower bound:

> How many Tverberg partitions into q subsets are there for any set of $(d+1)(q-1)+1$ points in $\mathbb{R}^d$? More generally, how many Tverberg partitions into q subsets are there for any continuous map $f : \sigma^{(d+1)(q-1)} \to \mathbb{R}^d$?

Sierksma's Conjecture 1.45 states that there are at least $((q-1)!)^d$ for any set of $(d+1)(q-1)+1$ points in $\mathbb{R}^d$. We have seen in Section 1.3 that the case $d = 1$ and arbitrary q can be proved in its topological version 1.50 using the intermediate value theorem. For $q = 2$ and arbitrary d, the conjecture reduces to the topological Radon theorem 1.34. The conjecture is unresolved in all other cases.

On the number of winding partitions. Recently, Schöneborn and Ziegler [62] have reduced the Topological Tverberg Theorem 1.50 to the Winding Number Conjecture 1.54. They show that the number of Tverberg partitions, and the number of winding partitions correspond to each other in the following way.

Proposition 2.6. *For each $q \geq 2$ and $d \geq 3$, the following three numbers are equal:*

- *The minimal number of Tverberg partitions for continuous maps $f : \sigma^{(d+1)(q-1)} \to \mathbb{R}^d$.*
- *The minimal number of Tverberg partitions in the d-skeleton for continuous maps $f : (\sigma^{(d+1)(q-1)})^{\leq d} \to \mathbb{R}^d$.*
- *The minimal number of winding partitions for continuous maps $f : (\sigma^{(d+1)(q-1)})^{\leq d-1} \to \mathbb{R}^d$.*

Moreover, the first two numbers are equal for $d = 2$.

The proof is based on the Approximation Lemma 1.63. Hence the minimal numbers are achieved for general position maps. Their proof of the reduction shows that all three numbers coincide for $d \geq 3$. The case $d = 2$ needs extra considerations: Tverberg partitions and winding partitions are not in one-to-one correspondence. Every winding partition is a Tverberg partition, but the converse does not hold for $d = 2$.

Moreover, Schöneborn and Ziegler show the equivalence of Sierksma's Conjecture 1.45 for continuous maps, and the corresponding lower bounds.

Theorem 2.7. *For each $q \geq 2$, the following three statements are equivalent:*
- *Sierksma's Conjecture 1.45 for continuous maps.*
- *For every continuous map $f : (\sigma^{(d+1)(q-1)})^{\leq d} \longrightarrow \mathbb{R}^d$ there are at least $((q-1)!)^d$ Tverberg partitions.*
- *For every continuous map $f : (\sigma^{(d+1)(q-1)})^{\leq d-1} \longrightarrow \mathbb{R}^d$ there are at least $((q-1)!)^d$ winding partitions.*

For $d \neq 1$, the proof is similar to that of Proposition 2.6. For the case $d = 2$, Schöneborn and Ziegler implicitly prove the following lemma.

Lemma 2.8. *Let $l(q)$ be a lower bound for the number of Tverberg partitions for continuous maps $f : \sigma^{4(q-1)} \longrightarrow \mathbb{R}^3$. Then $\frac{l(q)}{(q-1)!}$ is a lower bound for the number of winding partitions for continuous maps $f : K_{3q-2} \longrightarrow \mathbb{R}^2$.*

This implies that $\frac{l(q)}{(q-1)!}$ is a lower bound for the number of Tverberg partitions for continuous maps $f : \sigma^{3(q-1)} \longrightarrow \mathbb{R}^2$.

Lower bound. Up to now, the only non-trivial lower bound for the number of Tverberg partition was established by Vućić and Živaljević [72] when q is prime using a Borsuk-Ulam type argument. Hence the lower bound also holds in the general setting of continuous maps. The following result extends their result to the prime power case.

Theorem 2.9. *Let $q = p^r$ be a prime power and $d \geq 1$. For any continuous map $f : \|\sigma^N\| \to \mathbb{R}^d$, where $N = (d+1)(q-1)$, the number of unordered q-tuples $\{F_1, F_2, \ldots, F_q\}$ of disjoint faces of the N-simplex with $\bigcap_{i=1}^{q} f(\|F_i\|) \neq \emptyset$ is at least*

$$\frac{1}{(q-1)!} \cdot \left(\frac{q}{r+1}\right)^{\lceil \frac{N}{2} \rceil}.$$

The lower bound translates directly to a lower bound for the number of winding partitions for $d \geq 3$. In the case $d = 2$, we get an extra factor $\frac{1}{(q-1)!}$ due to Lemma 2.8.

In the prime case, the following proof reduces to the Vućić-Živaljević proof, in the version of Matoušek [46, Section 6.6].

Proof. (of **Theorem 2.9**) Let K be the simplicial complex $(\sigma^N)^{*q}_{\Delta(2)}$. The vertex set of K is $[N+1] \times [q]$. A maximal simplex of K is of the form $F_1 \uplus F_2 \uplus \cdots \uplus F_q$, where the F_i are pairwise disjoint subsets of the vertex set $[N+1]$ of σ^N and $\bigcup_1^q F_i = [N+1]$. In other words, there is a one-to-one correspondence between the maximal simplices K and the ordered partitions $(F_1, F_2, \ldots, F_q)$ of the vertex set $[N+1]$. Another way of looking at K: The set of all maximal simplices can be identified with the complete $(N+1)$-partite hypergraph on the vertex set $[N+1] \times [q]$. For example, a maximal simplex in the case $d = 2$ and $q = 4$ encoding a Tverberg partition for $N+1 = 10$ points in $\mathbb{R}^2$ is shown in Figure 2.1. A Tverberg partition is represented by a hyperedge consisting of 10 vertices.

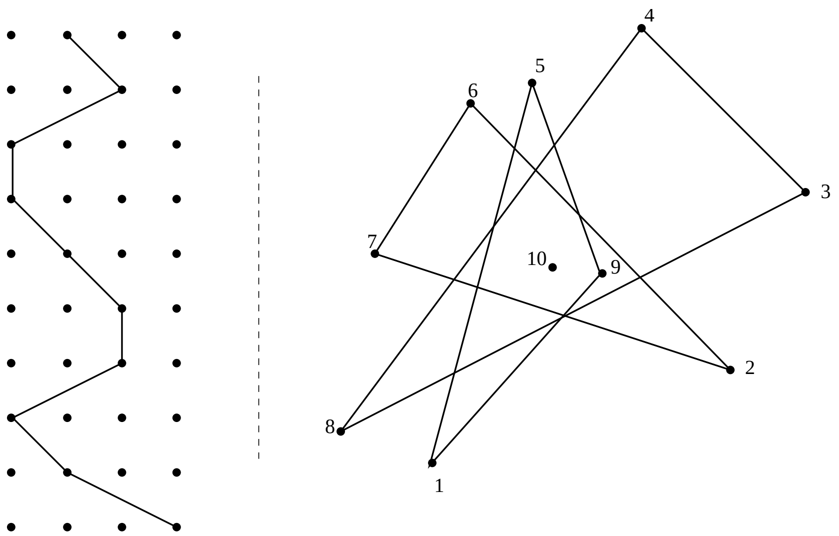

Figure 2.1: Maximal simplex of $(\sigma^N)^{*q}_{\Delta(2)}$ encoding a Tverberg partition.

The induced G-action permutes the q columns of vertices. We call a maximal face *good* if it encodes a Tverberg partition of the map f. Let $f^{*q} : \|\mathsf{K}\| \to (\mathbb{R}^d)^{*q}$ be the q-fold join of f restricted to $\|\mathsf{K}\|$, then it is a G-map. A maximal simplex S of K is good if its image $f^{*q}(\|S\|)$ intersects the diagonal of $(\mathbb{R}^d)^{*q}$. Proving a lower bound for the number of good simplices in K gives then a lower bound for the number of Tverberg partitions of f. If there are at least M good simplices we have at least $M/q!$ unordered Tverberg partitions.

In the next paragraph, we define a family $\mathcal{L}$ of subcomplexes $\mathsf{L} \subset \mathsf{K}$ having the properties: (i) L is closed under the G-action, and (ii) $\operatorname{conn}(\mathsf{L}) \geq N-1$. Then L is again a fixed point free G-space by (i) and Lemma 2.1. The reduced cohomology groups of L vanish in dimensions 0 to $N-1$ due to (ii). Now with

Lemma 2.2 we get as a direct corollary of Volovikov's Lemma that L contains one good maximal simplex S; in fact, the entire orbit of S is good and we get q good simplices in L. Suppose Q is the number of $\mathsf{L} \in \mathcal{L}$ containing any given maximal simplex of K, then we obtain the lower bound

$$M \geq q \cdot |\mathcal{L}|/Q. \tag{2.1}$$

We define the family $\mathcal{L}$ and distinguish two cases: (i) N even, that is, p or d is odd, and (ii) N odd, that is, $p = 2$ and even d. First we divide the $N+1$ rows into pairs such that we get $\frac{N}{2}$ pairs and one remaining row in the first case, and $\frac{N+1}{2}$ pairs in the second. Now we focus on the two rows of one pair; the simplices of K living on these two rows form bipartite graphs $K_{q,q}$. Suppose that we have chosen a connected G-invariant subgraph C_i of $K_{q,q}$, $i \in [\frac{N}{2}]$ resp. $i \in [\frac{N+1}{2}]$, for every pair. The maximal simplices of L to a given choice of row pairing and of the C_i, $i \in [\frac{N}{2}]$ resp. $i \in [\frac{N+1}{2}]$, are the maximal simplices of K that contain an edge of each C_i. L is G-invariant by construction. Topologically, we get in the first case

$$\mathsf{L} = C_1 * C_2 * \cdots * C_{N/2} * [q],$$

and in the second

$$\mathsf{L} = C_1 * C_2 * \cdots * C_{(N+1)/2}.$$

Here $[q]$ is the discrete space on q elements; in both cases inequality (1.1) on the connectivity of the join implies that:

$$\operatorname{conn}(\mathsf{L}) \geq N - 1.$$

In the next paragraph, we will construct distinct G-invariant connected subgraphs C of the graph $G * G$ formed by two rows. Our aim is to get as msny as possible subgraphs C so that $|\mathcal{L}|/Q$ – and at the same time our lower bound – gets as large as possible. The G-invariance implies that our subgraphs are regular, the connectivity implies that every vertex has at least degree $r+1$ (r is the smallest number of generators of the group G). We will construct

$$q(q-p^0)(q-p^1)(q-p^2)\cdots(q-p^{r-1})/(r+1)!$$

distinct G-invariant, connected subgraphs C having the smallest possible number of edges, that is $q(r+1)$.

To obtain a G-invariant subgraph choose edges and take their orbits, see Figure 2.2 for orbits in the case $q = 3^2$. The vertices are elements of $(\mathbb{Z}_p)^r$ having order p as group elements. To make sure that we count an orbit without multiplicities choose its representative edge as the edge that is incident to the upper left vertex $O := (0, 0, \ldots, 0)$.

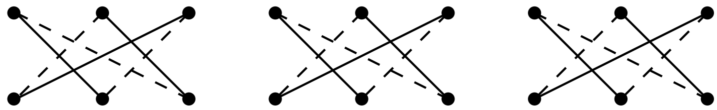

Figure 2.2: G-orbits of the edges $((0,0),(0,1))$ and $((0,0),(0,2))$.

To prove the connectivity of the graph C we show that the component K_O of the vertex O is the whole graph C. Choosing $r+1$ representative edges consecutively such that in each step a new component is connected to the component K_O leads to a connected subgraph.

More precisely, we will show inductively that after $1 \le k \le r+1$ steps: (i) there are $2p^{k-1}$ vertices in each component, p^{k-1} in each shore, and (ii) in total there are $p^{r-(k-1)}$ components. For $k=1$, the orbit of an edge consists of p^r vertex-disjoint edges, see Figure 2.2. For $k=2$, the graph of two orbits is equal to the disjoint union of p^{r-1} cycles of length $2p$, see Figure 2.2. Assume that for $1 \le k \le r$ edges the statement is true. Let the $(k+1)$-st edge be an edge, connecting K_O with one of the other remaining $p^{r-(k-1)}-1$ components. There are $q-p^{k-1}$ many representative edges to do so. The graph of the $(k+1)$-st orbit and any of the k first orbits is again a union of cycles of length $2p$, hence each p components of the graph of the first k orbits get connected. Therefore the number of components decreases by a factor p, and the number of vertices increases by the factor p in each shore.

As the order in the $r+1$ steps of our construction does not play any role, this process leads to the desired number of graphs C. Every given edge determines an orbit, hence there are

$$(q-p^0)(q-p^1)(q-p^2)\cdots(q-p^{r-1})/r!$$

connected, G-invariant graphs C containing this edge.

Finally, let π be the number of possibilities to do the row pairing in case (i) or (ii) (π cancels out in the end). Then in case (i) we get:

$$|\mathcal{L}| = \pi \cdot \left(q \cdot \prod_{i=0}^{r-1}(q-p^i)/(r+1)!\right)^{N/2},$$
$$Q = \pi \cdot \left(\prod_{i=0}^{r-1}(q-p^i)/r!\right)^{N/2},$$

and in case (ii):

$$|\mathcal{L}| = \pi \cdot \left(q \cdot \prod_{i=0}^{r-1}(q-p^i)/(r+1)!\right)^{(N+1)/2},$$
$$Q = \pi \cdot \left(\prod_{i=0}^{r-1}(q-p^i)/r!\right)^{(N+1)/2}.$$

Plugging these numbers into inequality (2.1) completes the proof. $\square$

Remark 2.10. In the proof, we really count unordered Tverberg partitions. The seven Tverberg points from the example in Figure 1.21 are counted as one Tverberg partition as they all correspond to one good simplex.

Remark 2.11. The lower bound in the prime power case due to Theorem 2.9 equals for large d and q roughly the square root of the bound conjectured by Sierksma; see Table 2.1 where both bounds are compared for small d and q. The column with $q = 6$ – the smallest integer that is not a prime power – contains the lower bound in the affine case. In general, is not known whether the lower bounds for affine maps and for continuous maps differ. All our attempts to construct an example failed, see also Observation 2.12. For a counter-example to Sierksma's Conjecture 1.45, $d \geq 2$ and $q \geq 3$ is needed. This turns counting all Tverberg partitions of a given map into a problem which can hardly be done by hand. In Chapter 4, we report about a computer project counting Tverberg partitions for $d = 2$, and $d = 3$. See also the Example 2.13 below for a continuous map f satisfying Sierksma's conjecture. In Section 3.2, we obtain new lower bounds for the number of Tverberg partitions which hold for affine maps and arbitrary q. In Section 3.3, we show that this result resp. its proof can not be carried over to the continuous case.

$d\backslash q$	2		3		4		5		6	
2	1	1	2	4	1	36	11	576	1	14400
3	1	1	3	8	1	216	64	13824	1	1728000
4	1	1	4	16	2	1296	398	331776	1	$2.07 \cdot 10^8$
5	1	1	6	32	3	7776	2484	7962624	1	$2.48 \cdot 10^{10}$

Table 2.1: Proved versus conjectured lower bounds.

Observation 2.12. A natural candidate for a counter-example to the continuous version of Sierksma's conjecture can be found in Example 2.13 below. However: Whenever Tverberg partitions vanish, new ones come up instantly. Up to now, we haven't been able to formalize this behavior. This is not without reason: The same idea could be turned into a proof of the topological Tverberg theorem for arbitrary q.

Example 2.13. Start with the configuration of points attaining Sierksma's conjectured lower bound from Figure 1.17 on page 34. The $(d+1)(q-1)+1$ points in $\mathbb{R}^d$ determine an affine map $f : \sigma^{(d+1)(q-1)+1} \to \mathbb{R}^d$ uniquely. We have seen in Remark 1.46 that all Tverberg partitions consist of the point 0 in the center and $(q-1)$ many d-faces. There are $((q-1)!)^d$ ways to obtain such Tverberg partitions. Now alter f in one $(d-1)$-face F as in Figure 2.3. There we cut out of F an small disc of radius $\epsilon > 0$, and replace it by a ϵ-tube around the point 0. Then all $q-1$ d-faces using F do not contain the point 0 any more. Hence $(q-1)((q-2)!)^d$ Tverberg partitions vanish. At the same time, the modified face F intersects all $(q-1)$ edges starting at any vertex from the unused cluster, and ending at the point 0. Every intersection point is contained in all $((q-2)!)^d$ ways to partition the remaining points as before. Altogether $(q-1)((q-2)!)^d$ new Tverberg partitions come up.

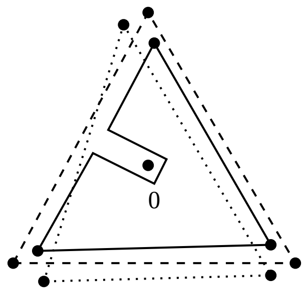

Figure 2.3: A continuous example with 24 Tverberg 4-partitions.

Let's end this section with two open problems which are related to the question whether it is possible to extend the equivariant method to arbitrary q for the Topological Tverberg Theorem 1.50.

Problem 2.14. Prove lower bounds for the number of Tverberg partition under the assumption that there exists at least one. How often does f^{*q} meet the diagonal Δ of $(\mathbb{R}^d)^{*q}$?

Problem 2.15. Prove the Affine Tverberg Theorem 1.49 using the equivariant method for arbitrary q. For this, identify new properties of f^{*q}.

2.3 Tverberg partitions with constraints

Schöneborn and Ziegler [62] showed that for p prime every continuous drawing of K_{3p-2} has a winding partition object to the following type of constraints: Certain pairs of points end up in different partition sets. In other words, there is a winding partition that does not "use" the edge connecting this pair of points. Formally, this reads as follows.

Definition 2.16. Let G be a subgraph of the 1-skeleton of $\sigma^{(d+1)(q-1)}$, and $f : \sigma^{(d+1)(q-1)} \to \mathbb{R}^d$ be a continuous map. Let $E(G)$ the set of edges of G.

A Tverberg partition resp. winding partition $F_1, F_2, \dots F_q \subset \sigma^{(d+1)(q-1)}$ of f is a *Tverberg resp. winding partition of f not using any edge of G* if

$$|F_i \cap e| \leq 1 \text{ for all } i \in [q] \text{ and all edges } e \in E(G).$$

Theorem 2.17. *Let $p \geq 3$ be a prime and M a maximal matching in K_{3p-2}. Then K_{3p-2} has a winding partition $F_1, F_2, \dots, F_p$ not using any edge from M.*

The proof of Theorem 2.17 uses subcomplexes constructed in the proof of Theorem 2.9. The proof can easily be carried over to arbitrary dimension $d \geq 1$, and to prime powers p. All results in this section also hold for winding partitions. For the sake of simplicity, we state and prove our results for Tverberg partitions.

Theorem 2.18. *Let $q = p^r > 2$ be a prime power, and M a matching in the graph of $\sigma^{(d+1)(q-1)}$. Then every continuous map $f : \|\sigma^{(d+1)(q-1)}\| \to \mathbb{R}^d$ has a Tverberg partition $F_1, F_2, \dots, F_q$ not using any edge from M.*

Theorems 2.17 and 2.18 are important steps for better understanding winding resp. Tverberg partitions: One can force pairs of points to be in different blocks of a Tverberg partition. Choose disjoint pairs of vertices of $\sigma^{(d+1)(q-1)}$, then this choice corresponds to a matching M in the 1-skeleton of $\sigma^{(d+1)(q-1)}$. For any map f, the endpoints of any edge in M end up in different partition sets. See also Figure 2.4 for an example of an affine configuration, there the pairs of points are shown as broken edges. Theorem 2.18 implies that there is a Tverberg partition such that for each pair, both points end up in different partition sets. In our example, the partition $\{2\}, \{3, 4, 7\}, \{4, 5, 6\}$ is such a Tverberg partition.

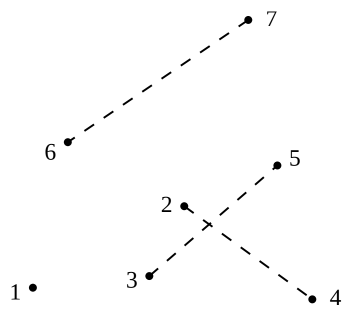

Figure 2.4: An example of a planar set for $q = 3$ with forbidden pairs $(2, 4), (3, 5), (6, 7)$.

In this section, we extend Theorem 2.18 to a wider class of graphs. This is based on the following approach.

Problem 2.19. Identify *constraint graphs* C in $\sigma^{(d+1)(q-1)}$ such that every continuous map $f : \|\sigma^{(d+1)(q-1)}\| \to \mathbb{R}^d$ has a Tverberg partition of disjoint faces $F_1, F_2, \ldots, F_q \subset \sigma^{(d+1)(q-1)}$ not using any edge from C.

Theorem 2.18 implies that any matching in $\sigma^{(d+1)(q-1)}$ is a constraint graph for prime powers q. Schöneborn and Ziegler also come up with Example 2.21 showing that the bipartite graph $K_{1,q-1}$ is not a constraint graph.

Observation 2.20. The family of constraint graphs is closed under deleting edges. It is thus a monotone graph property.

Example 2.21. Schöneborn and Ziegler [62] come up with the alternating drawing of K_{3q-2} shown in Figure 2.5 for $q = 4$. If one deletes the first $q - 1$ edges incident to the right-most vertex, then one can check that there is no winding partition resp. Tverberg partition. In Figure 2.5, the deleted edges are drawn in broken lines. Numbering the vertices from right to left with the natural numbers in $[3q - 2]$, the edges of the form $(1, 3q - 2 - 2i)$, for $0 \leq i \leq q - 2$, are deleted.

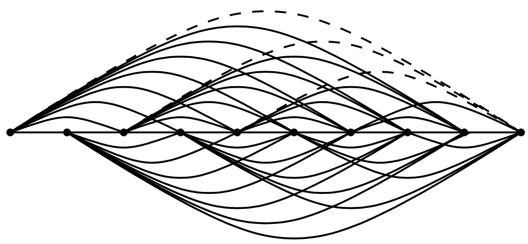

Figure 2.5: K_{10} minus three edges with no winding partition.

The following theorem generalizes the topological Tverberg theorem and Theorem 2.18. Moreover, it implies that $K_{1,q-1}$ is a minimal example: All subgraphs of $K_{1,q-1}$ are constraint graphs.

Theorem 2.22. *Let $q > 2$ be prime power then the following subgraphs of $\sigma^{(d+1)(q-1)}$ are constraint graphs:*

i) *Complete graphs K_l on l vertices for $l \leq \frac{q+1}{2}$,*

ii) *complete bipartite graphs $K_{1,l}$ for $l < q - 1$,*

iii) *paths P_l on l vertices for $q > 3$,*

iv) *cycles C_l on l vertices for $q > 4$,*

v) *and arbitrary disjoint unions of graphs from* (i)–(iv).

Theorem 2.22 serves in Section 3.2 to estimate the number of Tverberg points in the prime power case. Theorem 2.22 also holds for winding partitions.

Remark 2.23.
- Figure 2.6 shows an example of a configuration of 13 points in the plane together with a constraint graph. Theorem 2.22 implies that there is a Tverberg partition into 5 blocks that does not use any of the broken edges. In Figure 2.6, there is for example the Tverberg partition $\{6, 10\}$, $\{9, 11\}$, $\{0, 2, 8\}$, $\{1, 5, 12\}$, $\{3, 4, 7\}$ that does not use any of the broken edges. Figure 1.14 on Radon partitions shows that K_2 is not a constraint graph for $q = 2$.
- The constraint graph K_l guarantees that all l points end up in l pairwise disjoint partition sets. The constraint graph $K_{1,l}$ forces that the singular point in one shore of $K_{1,l}$ ends up in a different partition set than all l points of the other shore.
- The condition $q > 3$ for paths is necessary as $K_{1,2} = P_3$ is not a constraint graph for $q = 3$. The condition $q > 4$ for cycles is necessary as $K_3 = C_3$ is not a constraint graph for $q = 4$. The fact that K_3 is not a constraint graph for $q = 4$ follows from examples which were obtained using the algorithm given in Chapter 4.

Proof. Set $N := (d + 1)(q - 1)$, and let $q > 2$ be of the form p^r for some prime number p. As in the proof of the lower bound, we construct a good subcomplex L of $\mathsf{K} := (\sigma^N)^{*q}_{\Delta(2)}$ such that:

i) L is invariant under the $(\mathbb{Z}_p)^r$-action, and
ii) $\operatorname{conn}(\mathsf{L}) > N - 1$.

Here *good* means that L does not contain any Tverberg partitions using an edge of our constraint graph. Conditions i) and ii) together imply the existence of at least one Tverberg partition in L as noted in the proof of Theorem 2.9.

A maximal simplex of K encodes a Tverberg partition as shown in Figure 2.1, and it can be represented as a hyperedge using one point from each row of K. Our proof is based in its simplest case – for K_2 – on the following observation:

> If two points i and j end up in the same block, then the maximal face representing the partition uses one of the vertical edges between the corresponding rows i and j in K.

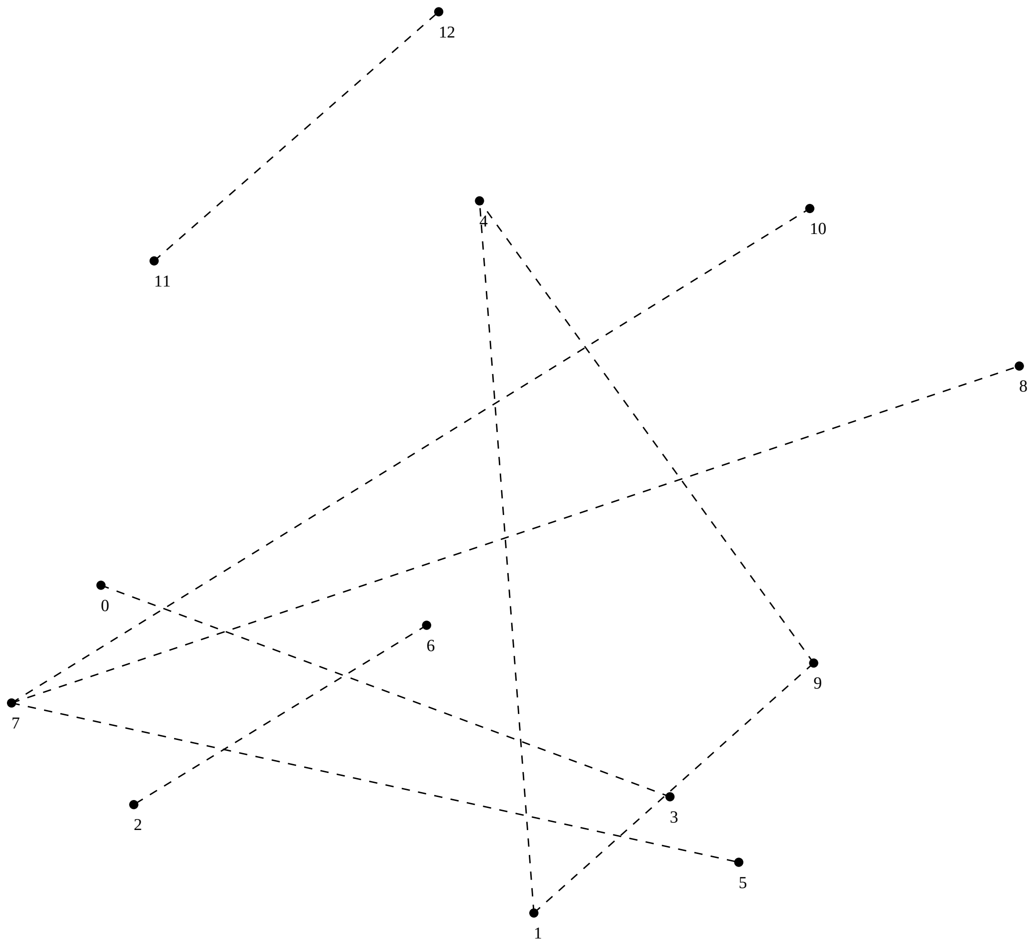

Figure 2.6: A planar configuration together with a constraint graph for $q = 5$.

To prove the K_2 case, we have to come up with a subcomplex L that does not contain maximal simplices using vertical edges between rows i and j. Let L be the join of the chessboard complex $\Delta_{2,q}$ on rows i and j, and the remaining rows. Figure 2.7 shows this construction of L for $q = 3$ and $d = 2$. The chessboard complex $\Delta_{2,q}$ does not contain any vertical edges. Moreover, L is $(\mathbb{Z}_p)^r$-invariant as only the orbit of the vertical edges is missing. For the connectivity of L see the next paragraph.

i) Construction of L for complete graphs K_l. Let L be the join of the chessboard complex $\Delta_{l,q}$ on the corresponding l rows, and the remaining rows:

$$\mathsf{L} = \Delta_{l,q} * ([q])^{*(N+1-l)}.$$

By construction L does not contain any vertical edges between any two of the l corresponding rows, so that L is good. The subcomplex L is closed under the $(\mathbb{Z}_p)^r$-action. Using Theorem 1.20 on the connectivity of the chessboard complex, and inequality (1.1) on the connectivity of the join, we obtain:

$$\begin{aligned} \operatorname{conn}(\mathsf{L}) &\geq \operatorname{conn}(\Delta_{l,q}) + \operatorname{conn}(([q])^{*(N+1-l)}) + 2 \\ &\geq \operatorname{conn}(\Delta_{l,q}) + N - l + 1 \\ &\geq N - 1. \end{aligned}$$

In the last step, we use that $\Delta_{l,q}$ is $(l-2)$-connected for $q \geq 2l - 1$.

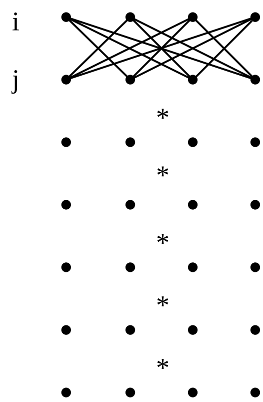

Figure 2.7: The construction of L for K_2.

ii) Construction of L for complete bipartite graphs $K_{1,l}$. We first construct an $(\mathbb{Z}_p)^r$-invariant subcomplex $C_{l,q}$ on the corresponding $l+1$ rows. For this, let i be the row that corresponds to the vertex of degree l, and $j_1, j_2, \ldots j_l$ be the corresponding rows to the l vertices of degree 1. Let $C_{l,q}$ be the maximal induced subcomplex of K on the rows $i, j_1, j_2, \ldots, j_l$ that does not contain any vertical edges starting at a vertex of row i. Then $C_{l,q}$ is the union of q many complexes $L_1, L_2, \ldots, L_q$, which are all the form $\operatorname{cone}([q-1]^{*l})$. Here the apex of L_m is the mth vertex of row i for every $m = 1, 2, \ldots, q$. In Figure 2.8 the complex L_3 is shown for $q = 4$, and $l = 2$.
Let L be the join of the complex $C_{l,q}$ and the remaining rows of K:

$$\mathsf{L} = C_{l,q} * ([q])^{*(N-l)}.$$

Now L is good and $(\mathbb{Z}_p)^r$-invariant by construction. Let's assume

$$\operatorname{conn}(C_{l,q}) \geq l - 1 \tag{2.2}$$

for $1 < l < q - 1$. The connectivity of L is then shown as above:

$$\begin{aligned} \operatorname{conn}(\mathsf{L}) &\geq \operatorname{conn}(C_{l,q}) + \operatorname{conn}(([q])^{*(N-l)}) + 2 \\ &\geq \operatorname{conn}(\Delta_{l,q}) + N - l \\ &\geq N - 1. \end{aligned}$$

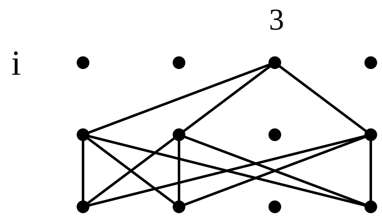

Figure 2.8: The complex L_3 for $q = 4$ and $l = 2$.

We prove assumption (2.2) in Lemma 2.24 below.

iii) Construction of L for paths P_l on $l+1$ vertices. We construct recursively a suitable good subcomplex L on $l+1$ rows such that $\operatorname{conn}(\mathsf{L}) \geq l-1$. The case $l = 1$ is covered in the proof of i) so that we can choose L to be the complex $D_{2,q} := \Delta_{2,q}$. For $l > 1$, choose L to be the complex $D_{l,q}$ which is obtained from $D_{l-1,q}$ in the following way: Order the corresponding rows $i_1, i_2, \ldots, i_{l+1}$ in the order they occur on the path. Take $D_{l-1,q}$ on the first l rows. A maximal face F of $D_{l-1,q}$ uses a point in the last row i_l in column j, for some $j \in [q]$. We want $D_{l,q}$ to be good, so that we can not choose any vertical edges between row i_l and i_{l+1}. Let $D_{l,q}$ be defined through its maximal faces: All faces of the form $F \uplus \{k\}$ for $k \neq j$. Let be the subcomplex $D^k_{l,q}$ of all faces $D_{l,q}$ ending with k. Then $D_{l,q} = \bigcup_{k=1}^q D^k_{l,q}$. In Figure 2.9 the recursive definition of the complex $D^2_{l,5}$ is shown. The complex is

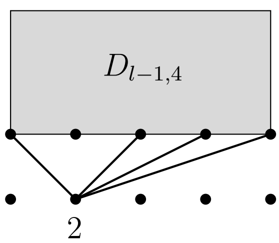

Figure 2.9: Recursive definition of $D^2_{l,4}$.

$(\mathbb{Z}_p)^r$-invariant, and the connectivity of $D_{l,q}$

$$\operatorname{conn}(D_{l,q}) \geq l-1$$

is shown in Lemma 2.25 below using the decomposition $\bigcup_{k=1}^q D^k_{l,q}$.

iv) Construction of L for cycles C_l on l vertices. Choose L to be the complex $E_{l,q}$ obtained from $D_{l-1,q}$ on l rows by removing all maximal simplices that use a vertical edge between first and last row. The following result on the connectivity of $E_{l,q}$ is shown in Lemma 2.26 below:

$$\operatorname{conn}(E_{l,q}) \geq l-2.$$

v) Construction of L for disjoint unions of constraint graphs. For every graph component construct a complex on the corresponding rows as above. Let L be the join of these subcomplexes, and of the remaining rows. Then L is a good $(\mathbb{Z}_p)^r$-invariant subcomplex by the similar arguments as above. The connectivity of L follows analogously from inequality (1.1) on the connectivity of the join. $\square$

The following three lemmas provide the connectivity results needed in the proof of Theorem 2.22. Their proofs are similar: Inductive on l, and the nerve theorem 1.39 is applied to the decompositions of the corresponding complexes that were introduced in the proof of Theorem 2.22.

Lemma 2.24. *Let $q > 2$, $d \geq 1$, and set $N = (d+1)(q-1)$. Let $C_{l,q}$ be the above defined subcomplex of $(\sigma^N)^{*q}_{\Delta(2)}$ for $1 \leq l < q-1$. Then*

$$\operatorname{conn}(C_{l,q}) \geq l-1.$$

Proof. In our proof, we use the decomposition of $C_{l,q}$ into subcomplexes $L_1, L_2, \dots, L_q$ from above.

The nerve $\mathcal{N}$ of the family $L_1, L_2, \dots, L_q$ is a simplicial complex on the vertex set $[q]$. The intersection of t many $L_{m_1}, L_{m_2}, \dots, L_{m_t}$ is $[q-t]^{*l}$ for $t > 1$ so that the nerve $\mathcal{N}$ is the boundary of the $(q-1)$-simplex. Hence $\mathcal{N}$ is $(q-3)$-connected.

Let's look at the connectivity of the non-empty intersections $\bigcap_{j=1}^{t} L_{m_j}$. For $t = 1$, every L_m is contractible as it is a cone. For $1 < t < q-1$, the space $[q-t]^{*l}$ is $(l-2)$-connected, and for $t = q-1$ the intersection is non-empty, hence its connectivity is -1. All non-empty intersections $\bigcap_{j=1}^{t} L_{m_j}$ are thus $(l-t)$-connected. The $(l-1)$-connectivity of $C_{l,q}$ immediately follows from the nerve theorem 1.39 using $q > 2$, and $l < q-1$. $\square$

Lemma 2.25. *Let $q > 3$, $d \geq 1$, and set $N = (d+1)(q-1)$. Let $D_{l,q}$ be the above defined subcomplex of $(\sigma^N)^{*q}_{\Delta(2)}$. Then*

$$\operatorname{conn}(D_{l,q}) \geq l-1.$$

Proof. In our proof, we use the decomposition of $D_{l,q}$ into subcomplexes $D^1_{l,q}, D^2_{l,q}, \dots, D^q_{l,q}$ from above. We prove the following connectivity result by an induction on $l \geq 1$:

$$\operatorname{conn}\Big(\bigcup_{j\in S} D^j_{l,q}\Big) \geq l-1, \quad \text{for any } \emptyset \neq S \subset [q]. \tag{2.3}$$

Let $l = 1$, then $D_{1,q} = \bigcup_{j\in[q]} D^j_{1,q}$ is the chessboard complex $\Delta_{2,q}$ which is connected for $q > 2$. For $l \geq 2$, look at the intersection of $t > 1$ many complexes $D^i_{l,q}$. Let $T \subset [q]$ be the corresponding index set of size $1 < t < q$, and $\bar{T}$ its complement in $[q]$. Then their intersections are

$$\bigcap_{j \in T} D^j_{l,q} = \bigcup_{j \in \bar{T}} D^j_{l-1,q}, \quad \text{and} \tag{2.4}$$

$$\bigcap_{j \in [q]} D^j_{l,q} = \bigcup_{j \in [q]} D^j_{l-2,q}. \tag{2.5}$$

The nerve $\mathcal{N}$ of the family $D^1_{l,q}, D^2_{l,q}, \ldots, D^q_{l,q}$ is a simplicial complex on the vertex set $[q]$. The nerve is the $(q-1)$-simplex, which is contractible.
For $l = 2$, let's apply the nerve theorem 1.39. For this, we have to check that the non-empty intersection of any $t \geq 1$ complexes is $(2-t)$-connected. Every $D^j_{2,q}$ is 1-connected as it is a cone. The intersection of $1 < t < q$ many complexes is 0-connected by equality (2.4). The intersection of q many complexes is $[q]$ which is -1-connected. Note that for $q = 3$ and $t = 2$ the intersection $D^1_{l,q} \cap D^2_{l,q}$ is disconnected.
Let now $l > 2$, we apply again the nerve theorem to obtain inequality (2.3). It remains to check that the non-empty intersection of any $t \geq 1$ complexes is $(l-t)$-connected. The complex $D^j_{l,q}$ is $(l-1)$-connected as it is a cone for every $j \in [q]$. The intersection of any $1 < t < q$ complexes is $(l-2)$-connected by equality (2.4) and by assumption. The intersection of q many complexes is $(l-3)$-connected by equality (2.5) and by assumption. $\square$

Lemma 2.26. *Let $q > 4$, $d \geq 1$, and set $N = (d+1)(q-1)$. Let $E_{l,q}$ be the above defined subcomplex of $(\sigma^N)^{*q}_{\Delta(2)}$. Then*

$$\operatorname{conn}(E_{l,q}) \geq l - 2.$$

Proof. The proof is similar to the proof of Lemma 2.25. The case for $l = 3$ has already been settled in the proof of case i) of Theorem 2.22. Observe that the inductive argument in the proof of Lemma 2.25 also works for $E_{l,q}$, which was obtained from $D_{l-1,q}$ by removing some maximal faces. $\square$

Remark 2.27. Figure 2.10 shows the list of constraint graphs for $q = 5$ according to Theorem 2.22. Other constraint graphs can be obtained via disjoint unions, or deleting edges. For $q = 2$, there are no constraint graphs. For $q = 3$, a single edge K_2 is the only connected constraint graph.

Let's end this section with a list of problems on possible extensions of our results. The first problem aims in the direction of finding similar good subcomplexes. The second problem asks whether it is possible to show the Tverberg theorem with constraints for affine maps, independent of the fact that q is a prime power. Moreover, we assume that this method can be adapted to the setting of the Colorful Tverberg Theorem 1.68.

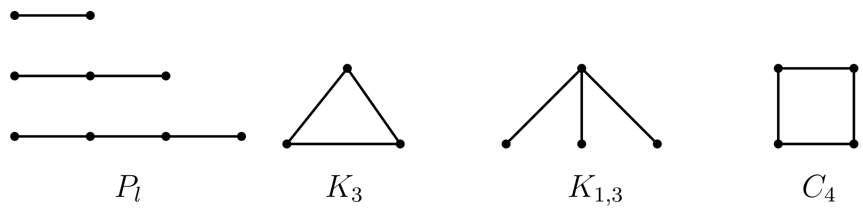

Figure 2.10: List of constraint graphs for $q = 5$.

Problem 2.28. Determine the class $\mathcal{CG}_{q,d}$ of constraint graphs. Find graphs that are not constraint graphs. Which of the constraint graphs are maximal?

Problem 2.29. Identify constraint graphs for arbitrary $q \geq 2$, especially for affine maps.

Problem 2.30. Find good subcomplexes in the configuration space $(\Delta_{2q-1,q})^{*d+1}$ of the colored Tverberg theorem to obtain a lower bound for the number of colored Tverberg partitions, and a colored Tverberg theorem with constraints.

Here a *good* subcomplex $(\Delta_{2q-1,q})^{*d+1}$ is again $(\mathbb{Z}_p)^r$-invariant, and at least $((d+1)(q-1)-1)$-connected. Constructing good subcomplexes in this setting requires more care than for the topological Tverberg theorem. One possibility to construct good subcomplexes is to identify $d+1$ many $(\mathbb{Z}_p)^r$-invariant subcomplexes L_i in the chessboard complex $\Delta_{2q-1,q}$ such that

$$\sum_{i=1}^{d+1} \operatorname{conn}(\mathsf{L}_i) \geq (d+1)(q-3)+1.$$

The join of the L_i's is then a good subcomplex in $(\Delta_{2q-1,q})^{*d+1}$. Looking at the proof for the connectivity of the chessboard complex, and studying $\Delta_{2q-1,q}$ for small q via the mathematical software system polymake [33], suggests that one obtains subcomplexes L_i by removing a non-trivial number of orbits of maximal faces. A step in this direction is formulated in the following problem.

Problem 2.31. For $q > 2$ let K be the subcomplex of $\Delta_{2q-1,q}$ which is obtained as follows: Choose $q-1$ disjoint pairs of rows in $\Delta_{2q-1,q}$. For every pair of rows choose the orbit of an edge connecting them, and remove all maximal faces of $\Delta_{2q-1,q}$ that meet this orbit. Show that

$$\operatorname{conn}(\mathsf{K}) = \operatorname{conn}(\Delta_{2q-1,q}).$$

2.4 On the number of necklace splittings

It is known that the methods introduced for the topological Tverberg theorem can also be applied to the splitting problem for necklaces for many thieves, see [46, Section 6.4]. We will extend the lower bound for the number of necklace splittings from [72] to the prime power case; this result will be published in [35].

A *necklace* is modeled in the following way: Given d continuous probability measures on $[0,1]$ and $q \geq 2$ thieves. A *fair splitting* of the necklace consists of a partition of $[0,1]$ into a number n of subintervals $I_1, I_2, \ldots, I_n$ and a partition of $[n]$ into q subsets $T_1, T_2, \ldots, T_q$ such that every thief gets an equal amount of all d materials:

$$\sum\nolimits_{j \in T_k} \mu_i(I_j) = \frac{1}{q}\text{ , for all } 1 \leq i \leq d \text{ and } 1 \leq k \leq q.$$

Noga Alon proved in 1987 that in general $d(q-1)$ is the smallest number of cuts for q thieves. A necklace is called *generic* if there is no fair splitting with less than $d(q-1)$ cuts. The splitting problem for necklaces is different from the Tverberg theorem: If a necklace is splittable for q_1 and for q_2 thieves than it can also be splitted for $q_1 q_2$ many thieves. Proving the prime case solves the problem in its generality.

The following result extends the lower bound of [72] for the number of fair splittings to the prime power case.

Theorem 2.32. *Let $q = p^r$ be a prime power. For generic necklaces made out of d continuously distributed materials the number of fair splittings with $d(q-1)$ cuts for q thieves is at least:*

$$q \cdot \left(\frac{q}{r+1}\right)^{\lceil \frac{d(q-1)}{2} \rceil}.$$

In the proof, we will again face deleted joins, but also the deleted product $(\mathbb{R}^d)^q_\Delta$ that is the q-fold Cartesian product of $\mathbb{R}^d$ without its diagonal.

Proof. Put $N = d(q-1)$. We use the one-to-one correspondence between the set of splittings of a generic necklace for q thieves and the simplicial complex $\mathsf{K} = (\sigma^{N+1})^{*q}_{\Delta(2)}$, see also the proof of Theorem 6.4.1 of [46]. For this, let $I_1, I_2, \ldots, I_{N+1}$ be a partition of the interval $[0,1]$ into $N+1$ intervals numbered from left to right. Let $T_1, T_2, \ldots, T_q$ be a partition of $[N+1]$. We encode such a splitting by a point of $\|\mathsf{K}\|$. Remember, a point z of $\|\mathsf{K}\|$ is of the form $t_1 x_1 \oplus \cdots \oplus t_q x_k$. Starting with splitting, we set t_k to be the total length of intervals assigned to the kth thief, or short $t_k = \sum_{j \in T_k} \text{length}(I_j)$.

Next, we define x_k. If $t_k = 0$, then x_k does not contribute in the join. For $t_k > 0$, we define x_k coordinate-wise

$$(x_k)_j = \begin{cases} \frac{1}{t_k}\mathrm{length}(I_j) & \text{for } j \in T_k, \\ 0 & \text{for } j \notin T_k. \end{cases}$$

We thus obtain a point z of $\|\mathsf{K}\|$ for every splitting. Conversely, a splitting is determined through a given point z.

The map $f : \|\mathsf{K}\| \to (\mathbb{R}^d)^q$, $z \mapsto f(z)_{i,k} := \sum_{j \in T_k} \mu_i(I_j)$ expressing the gains of the thieves, is a G-map. If there is no fair splitting, f would miss the diagonal of $(\mathbb{R}^d)^q$. Now let $\mathcal{L}$ be a family of subcomplexes L of K satisfying: (i) L is closed under the G-action, and (ii) $\mathrm{conn}(\mathsf{L}) \geq d(q-1)-1$. Again with Volovikov's Lemma every L contains at least one fair splitting, but as above the whole orbit of size q is good. In conclusion, the whole construction for $\mathcal{L}$ and the counting as in the proof of Theorem 2.9 can be carried over. $\square$

Chapter 3

On Birch points and Tverberg partitions

We have seen in Chapter 2 that up to now all lower bounds for the number of Tverberg partitions were based on the equivariant method. In this chapter, we go back to the roots of Tverberg's theorem. Birch proved in 1959 Tverberg's theorem in dimension $d = 2$. Starting with this original approach, we obtain in Section 3.1 properties of the number $B_0(X)$ of Birch partitions for arbitrary dimension d. Birch partitions and Tverberg partitions are closely related so that we obtain the first non-trivial lower bound for the number of Tverberg partitions which hold for arbitrary q in Section 3.2. Combining the new lower bound and the topological Tverberg Theorem 2.22 with constraints from Chapter 2, we improve this lower bound for prime powers q. This settles Sierkma's conjecture for a wide class of planar sets for $q = 3$.

In Section 3.3, we show that the properties of $B_0(X)$ do not carry over to continuous maps. As for Tverberg partitions, it is natural to consider winding Birch partitions. In the plane, this reduces to a problem on drawings of complete graphs K_{3k}. The properties of $B_0(X)$ do not hold for winding Birch partitions. A computer project came up with many counter-examples: piecewise linear drawings of K_6. This led to counter-examples for arbitrary dimension. This is in some sense a breakthrough for Tverberg's problem: Not many Tverberg-type properties of affine maps are known that do not carry over to the continuous case.

In Section 3.4, we discuss two conjectures that both imply the topological Tverberg theorem for arbitrary q. Using the approach of Roudneff [56], we settle both of them in the case of pseudoconfigurations of points in the plane. Moreover, we obtain a geodesical Tverberg theorem on the 2-sphere.

3.1 On Birch points

The two-dimensional case of Tverberg's theorem was established in 1959. Its proof was based on the following theorem by Birch [17].

Theorem 3.1. *Given $3N$ points in $\mathbb{R}^2$, we can divide them into N triads such that their convex hulls contain a common point.*

The proof of 3.1 is based on a lemma on partitioning a general measure which is due to Richard Rado [55]. This key result is nowadays known as the center point theorem. See Matoušek's textbook [49], or Tverberg and Vrećica [68] for more details.

Theorem 3.2 (Center point theorem)**.** *Let X be a finite point of sets in $\mathbb{R}^d$. Then there is a center point $x \in \mathbb{R}^d$ such that every closed half-space containing x contains at least $\frac{|X|}{d+1}$ points of X.*

Center points are also called Rado points in the literature. The relation of mass partition results, Tverberg's theorem, and non-embeddability results was first established by Vrećica and Živaljević in 1990. This led to Tverberg's Conjecture 1.65; see [70] for a recent discussion. Tverberg and Vrećica [68] came up with the following definition.

Definition 3.3 (Birch points)**.** Let X be a set of $k(d+1)$ points in $\mathbb{R}^d$. A point $p \in \mathbb{R}^d$ is a *Birch point* of X if there is a partition of X into k subsets of size $d+1$, each containing p in its convex hull. The partition of X is a *Birch partition for p*. For fixed $p \in \mathbb{R}^d$, let $B_p(X)$ be the number of unordered Birch partitions for p.

See Figure 3.1 for an example of a Birch partition for the origin $0 \in \mathbb{R}^d$ denoted by $+$. There is a another way to obtain a Birch partition for the origin in this example.

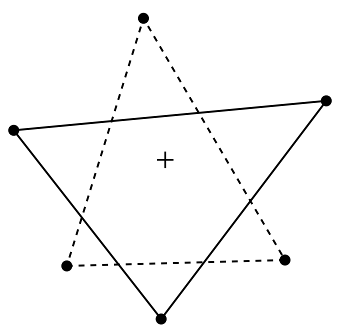

Figure 3.1: A Birch partition for 6 points in the plane.

From now on, we fix the point p to be the origin, and we write *Birch partition* instead of Birch partition for the origin for short. Tverberg and Vrećica [68] showed that the set of Birch points equals the set of center points for $d = 2$.

The main result of this section is the following theorem on the number of Birch partitions.

Theorem 3.4. *Let $d, k \geq 2$ be integers, and X be a set of $k(d+1)$ points in $\mathbb{R}^d$ in general position with respect to the origin 0. Then the following properties hold for $B_0(X)$:*

i) *$B_0(X)$ is even.*

ii) $B_0(X) > 0 \implies B_0(X) \geq k!$

Theorem 3.4 is the key step for obtaining new lower bounds for the number of Tverberg partitions in Section 3.2. Theorem 3.4 also holds for $d = 1$. The results are motivated by a computer experiment with random points which will be discussed in Chapter 4. The lower bound ii) is tight. Birch partitions and Tverberg partitions are closely related.

We prove Theorem 3.4 in two steps:

a) Property i).

b) Property i) implies Property ii).

The following lemma is a consequence of Lemma 1.9. It is essential for proving Step a).

Lemma 3.5. *Let X be a set of $d+2$ points in $\mathbb{R}^d$ that is in general position with respect to the origin. Then the number of d-simplices with vertices in X that contain the origin is even. In fact, this number is either 0, or 2.*

See Figure 3.2 for a configuration of four points in dimension $d = 2$ such that two triangles contain the origin $+$.

Proof. Suppose there is a d-simplex S spanned by $d+1$ points in X that contains the origin. Choosing any d-element subset T of S leads to a cone(T). As S contains the origin, all thus obtained cones partition $\mathbb{R}^d$. The antipode of the remaining point $p \in X \setminus S$ is in exactly one of these cones, say cone(T'). Lemma 1.9 implies that $0 \in \text{conv}(\{p\} \cup T')$. ∎

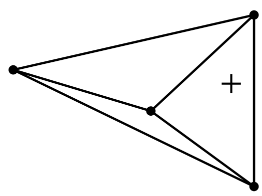

Figure 3.2: Four points that form two triangles containing the origin.

Proof. (of Theorem 3.4, Step a)) We prove i) for arbitrary $d \geq 2$ by induction on $k \geq 2$. The base case $k = 2$ is the key part.

$k = 2$: The set of normed points in X is in general position with respect to the origin, as noted in the proof of 1.12. Hence we can assume $X \subset S^{d-1}$. If all $2d + 2$ are clustered around the north pole of S^{d-1} then $B_0(X) = 0$, as $0 \notin \mathrm{conv}(X)$. We move one point p of X at a time while all other points remain fixed. Instead of following p, we look at its antipode $-p$ as for any d-element subset S of $X \setminus \{p\}$ one has due to Lemma 1.9:

$$0 \in \mathrm{conv}(S \cup \{p\}) \quad \text{iff} \quad -p \in \mathrm{cone}(S).$$

Every d-element subset of $X \setminus \{p\}$ defines a cone, and they altogether define a decomposition of the sphere S^{d-1} into cells. The boundary of a cell is defined through hyperplanes spanned by $(d-1)$-element subsets of $X \setminus \{p\}$ and the origin. At some point we are forced to move $-p$ transversally from one side of a boundary hyperplane defined by a $(d-1)$-element subset T to the other side. When $-p$ crosses such a hyperplane then $B_0(X)$ might change. We show in the case distinction below that for every change the parity of $B_0(X)$ does not change. The number $B_0(X)$ is thus even as we can move every point of X to its position while fixing all other points. As in the proof of Theorem 1.12, the cell decomposition during this process is nice: We can move $-p$ to every position on the sphere while crossing hyperplanes in a transversal way.

If $-p$ crosses the hyperplane through T transversally, the property of a d-simplex S spanned by $d + 1$ points from X containing the origin, changes in the following way. Set $\tilde{T} = T \cup \{p\}$. For all simplices that do not contain $\tilde{T}$ as a face nothing changes. If S is of the form $\tilde{T} \cup \{x\}$ for some $x \in X \setminus \tilde{T}$, then this property switches:

$$0 \in \mathrm{conv}(S) \text{ before the crossing iff } 0 \notin \mathrm{conv}(S) \text{ afterwards.}$$

A Birch partition consists of a d-simplex S and its complement $\bar{S}$ in X, which is again a d-simplex such that both contain the origin. The change of $B_0(X)$ coming from the crossing of $-p$ can thus only be affected by partitions that contain $\tilde{T}$ as a face of S, or of $\bar{S}$.

Case 1: The complements of all simplices using $\tilde{T}$ do not contain the origin. $B_0(X)$ does not change as the set of all Birch partition remains the same.

Case 2: Assume that $\tilde{T}$ is not part of a d-simplex S such that $\{S, \bar{S}\}$ is a Birch partition, and that after the crossing of $-p$ a Birch partition comes up. We show that Birch partitions come up in pairs.

See also the left side of Figure 3.3: There the point p is labeled with 2, five other points on S^1 are given through five rays starting at the origin. All six

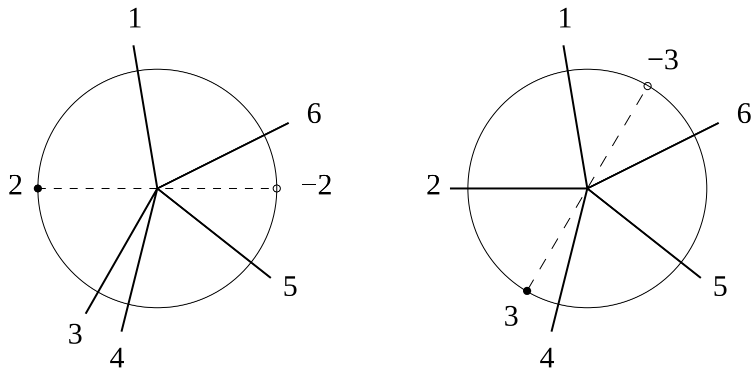

Figure 3.3: Initial configurations of six points for Cases 2 and 4.

points are labeled with the natural numbers $1, 2, 3, 4, 5, 6$ counter-clockwise. In this initial configuration, we have two Birch partitions: $\{\{1, 3, 5\}, \{2, 4, 6\}\}$, and $\{\{1, 4, 5\}, \{2, 3, 6\}\}$. The edge $\{2, 5\}$ is not part of a Birch partition. If -2 crosses the ray through 5, then two new Birch partitions come up: $\{\{1, 4, 6\}, \{2, 3, 5\}\}$, and $\{\{1, 3, 6\}, \{2, 4, 5\}\}$.
In general, the two new Birch partitions are of the form $\tilde{T}\cup\{x_1\}$ resp. $\tilde{T}\cup\{x_2\}$, with $x_1, x_2 \in X \setminus \tilde{T}$, plus their complements. Suppose there is a new one of the form $S = \tilde{T} \cup \{x_1\}$ together with its complement $\bar{S}$. Due to Lemma 3.5 there is exactly two d-simplices in $\bar{S} \cup \{x_1\}$ such that both contain the origin. One of them is $\bar{S}$, let S^* be the other. By assumption $0 \notin \bar{S^*}$ before the crossing of $-p$. In fact, $\bar{S^*} = \tilde{T} \cup \{x_2\}$ for some x_2. The set $\{\bar{S^*}, S^*\}$ is thus our second Birch partition as $0 \in \operatorname{conv}(\bar{S^*})$ afterwards.
Suppose there are three Birch partitions of the form $S_1 = \tilde{T} \cup \{x_1\}$, $S_2 = \tilde{T} \cup \{x_2\}$, and $S_3 = \tilde{T} \cup \{x_3\}$, with $x_1, x_2, x_3 \in X \setminus \tilde{T}$, together with their complements. This can not happen: One has $0 \in \bar{S_i}$ for $i = 1, 2, 3$, and $|\bigcup_{i=1}^{3} \bar{S_i}| = d + 2$. This contradicts Lemma 3.5.

Case 3: This is the inverse case of Case 2. Assume that there are exactly two Birch partitions of the form $\tilde{T} \cup \{x_1\}$ resp. $\tilde{T} \cup \{x_2\}$, with $x_1, x_2 \in X \setminus \tilde{T}$, plus their complements before the crossing. Both of them vanish after crossing of $-p$. New Birch partitions do not come up as for this we needed another $\tilde{T} \cup \{x_3\}$ such that its complement contains the origin. This cannot exist due to Lemma 3.5.

Case 4: Assume there is exactly one Birch partition for the form $S = \tilde{T} \cup \{x\}$, with $x \in X \setminus \tilde{T}$, together with its complement before the crossing. This Birch partition vanishes, and a new one comes up.

See also the right side of Figure 3.3: The initial configuration consists of the same six points labeled with the natural numbers $1, 2, 3, 4, 5, 6$ counter-clockwise as in Case 2. Let the point labeled with 3 be the moving point p.

The edge $\{3,6\}$ is part of one Birch partition. If -3 crosses the ray through 6, then $\{\{1,4,5\},\{2,3,6\}\}$ vanishes and $\{\{1,2,5\},\{3,4,6\}\}$ comes up.

In general, one has $0 \notin S$ after the crossing of $-p$ so that $\{S, \bar{S}\}$ vanishes. As in Case 2, there are exactly two d-simplices in $\bar{S} \cup \{x\}$ such that each contains the origin. One of them is $\bar{S}$, let S^* be the other. By assumption $0 \notin \bar{S^*}$ before the crossing of $-p$. In fact, $\bar{S^*} = \tilde{T} \cup \{x'\}$ for some x'. The set $\{\bar{S^*}, S^*\}$ is thus the new Birch partition as $0 \in \mathrm{conv}(\bar{S^*})$ afterwards.

Let now $k \geq 3$, and p be a point in X. Let $F_1, F_2, \ldots, F_l$ be all d-simplices using p that can be completed to a Birch partition of the origin into k subsets. For every F_i, omitting F_i leads to a Birch partition into $k-1$ subsets. By induction hypothesis, there are an even number of Birch partitions into $k-1$ subsets for the restriction of every F_i. $\square$

Remark 3.6. With some effort, the proof of the base case $k = 2$ of i) in Step a) can be extended to a proof for arbitrary $k \geq 2$.

Proof. (Theorem 3.4, Step b)) Assume Property i). We prove Property ii) by induction on $k \geq 2$. The case $k = 2$ is due to i): $B_0(X)$ is even, such that

$$B_0(X) > 0 \implies B_0(X) \geq 2 = k!$$

Let $k \geq 3$ and $B_0(X) > 0$. Then there is a Birch partition $F_1, F_2, \ldots, F_k$. If we take any $k-1$ of the F_i, they form again a Birch partition. By induction hypothesis, the union of $k-1$ many F_i has at least $(k-1)!$ Birch partitions. In particular, there are $(k-1)!$ many Birch partitions of X into k subsets that start with F_1. Let p be an element of F_1.

For every pair F_1, F_i, for $i \in \{2, 3, \ldots k\}$, one has again $B_0(F_1 \cup F_i) > 0$ such that there is a second Birch partition $\tilde{F}_1^i, \tilde{F}_i$ of $F_1 \cup F_i$. Assume without loss of generality $p \in \tilde{F}_1^i$. The k sets $F_1, \tilde{F}_1^2, \tilde{F}_1^3, \ldots, \tilde{F}_1^k$ are pairwise distinct by construction. Every one of them contributes $(k-1)!$ many Birch partitions of X by induction hypothesis. $\square$

Remark 3.7. In the proof of the induction of Step b), we didn't make use of convexity. The key is the base case $k = 2$:

$$B_0(X) > 0 \implies B_0(X) \geq 2.$$

The case $d = 2$ of Property i) of $B_0(X)$ from Theorem 3.4 also admits a simpler proof.

Proof. (of Property i) for $d = 2$) Let X be a set of $3k$ points in the plane in general position with respect to the origin. Recall Lemma 1.9:

Three points $p_1, p_2, p_3 \in X$ contain the origin iff $-p_1 \in \operatorname{cone}(p_2, p_3)$.

Choose a line through the origin. This line hits at most one point from X, and it divides the plane into two half-spaces. Choose one of the two half-spaces. Then sweep a line through the origin over the chosen half-space counter-clockwise. The ray hits all points exactly once, and the sweeping leads to a linear order on the points in X. This determines a word w_X of length $3k$ on the alphabet $\{+, -\}$ in the following way: Write for every point of X the letter $+$ when the line hits a point in the chosen half-space, and the $-$ in the other case. Choosing in Figure 3.1 the horizontal line and the upper half-space, this process outputs the word $+--+-+$.

The Birch partitions of X are encoded in w_X. Every possibility of partitioning w_X into k substrings of the form $+-+$ or $-+-$ corresponds to a Birch partition. The partition in Figure 3.1 thus corresponds to the partition $w_X[1,3,6], w_X[2,4,5]$ of w_X. A necessary condition for w_X to encode a Birch partition is therefore: There are at least k many $+$, and at least k many $-$. If X has a Birch partition there are two extremal cases. Either w_X consists of $2k$ many $+$ (resp. $-$) and k many $-$ (resp. $+$), or w_X consists of as many $+$ as $-$, plus one extra letter for odd k.

Suppose w_X has a consecutive subsequence of the letter $+$ (or $-$) of length $l > 1$. If X has a Birch partition, the l letters $+$ of the consecutive subsequence end up in l pairwise different partition sets. To each of l letters $+$ there is a substring of the $-+$ or $+-$. Every one of the $l!$ possibilities to map the l letters $+$ to the l substrings $-+$ or $+-$, leads to a new Birch partition. If w_X contains a consecutive subsequence of the letter $+$ of length $l_1 > 1$, and a consecutive subsequence of the letter $-$ of length $l_2 > 1$ one has analogously

$$B_0(X) > 0 \implies B_0(X) \geq l_1! \cdot l_2!$$

To prove i), it is thus sufficient to prove that any word of length $3k$ on the alphabet $\{+, -\}$ contains the letter $+$ (resp. $-$) twice in a row. This is true except for the alternating word of length $3k$, which clearly encodes a Birch partition. We prove that the alternating word of length $3k$ for encodes an even number of Birch partition by induction on $k \geq 2$.

Assume without loss of generality that the alternating word w_X starts with the letter $+$. For $k = 2$, the word $+-+-+-$ encodes two Birch partitions: $w_X[1,2,3], w_X[4,5,6]$ and $w_X[1,4,5], w_X[2,3,6]$. For $k \geq 3$ the word w_X is of the form $+-+w_{\tilde{X}}$ where $w_{\tilde{X}}$ is the alternating word of length $3(k-1)$ starting with $-$. The word w_X has an even number of Birch partitions with a partition set $w_X[1,2,3]$ by the induction hypothesis. The first letter $+$ of w_X has to be in partition set $+-+$ of the form $w_X[1,i,i+1]$ as otherwise the remaining word of length $3(k-1)$ would not encode a Birch partition any more. The remaining word after deleting $w_X[1, i, i+1]$ is again an alternating word of length $3(k-1)$. This remaining word encodes an even number of Birch partitions by induction hypothesis. □

Remark 3.8.
1. In general there are less point configurations than $\{+, -\}$-words, e. g. both words consisting of one of the letters correspond to the same configuration. More precisely, turning the line through the origin counter-clockwise until hitting a point of X changes w_X: The first letter is shifted to the end with a different sign.

2. A similar approach was used by Pach and Szegedy in [53] for studying the number of simplices containing the origin.

3. The $\{+, -\}$-words can be seen as an oriented matroid. For $d > 2$, it might be interesting to use concepts known from matroid theory.

Moreover, we conjecture an upper bound for the number $B_0(X)$.

Problem 3.9. Let X be a set of $k(d+1)$ points in general position in $\mathbb{R}^d$. Show that:

$$B_0(X) \leq (k!)^d.$$

Remark 3.10. Sierkma's configuration from Figure 1.17 attains the upper bound of Problem 3.9. Hence it would be maximal for the number of Birch partitions. At the same time, Sierksma conjectured it to be minimal for the number of Tverberg partitions.

Let's end this section with two problems. Both are promising starting points for future research.

Problem 3.11. Relate the properties on the number $B_p(X)$ of Birch partitions to polytope theory. Birch partitions show up while studying Gale diagrams; see Ziegler's textbook [76] for an introduction to Gale diagrams. In fact, a set X of $k(d+1)$ points in $\mathbb{R}^d$ with $B_0(X) > 0$ corresponds to a Gale diagram of a k-neighborly $(k-1)(d+1)$-dimensional simplicial polytope on $k(d+1)$ vertices.

Problem 3.12. We have seen in Chapter 1 that Radon's, Helly's, and Carathéodory's theorem are closely related. Do the results on the number of Birch partitions imply new Helly-type, or Carathéodory-type results?

3.2 New lower bounds for the number of Tverberg partitions

Section 2.2 started with a discussion on the number of Tverberg partitions. In Theorem 2.9, we obtained a lower bound. The proof of Theorem 2.9 is based on the equivariant method. The lower bound thus holds in the more general framework of continuous maps, but only for q being a prime power.

In this section, we prove the first lower bound for the number of Tverberg partitions which holds for arbitrary q. Recall Remark 1.62 on general position for Tverberg's theorem: There are Tverberg partitions of type I and of type II.

We improve this new lower bound for prime powers q. Using Theorem 2.22, we obtain a non-trivial lower bound for the number of Tverberg partitions in the case of point sets that do not have a Tverberg partition of type I. Combining this, we confirm Sierksma's conjecture 1.45 for $q = 3$ in the case of planar point sets that do not have a Tverberg partition of type I.

Theorem 3.13. *Let X be a set of $(d+1)(q-1)+1$ points in general position in $\mathbb{R}^d$, $d \geq 2$. Then the following properties hold for the number $T(X)$ of Tverberg partitions:*

i) *$T(X)$ is even for $q > d+1$.*

ii) *$T(X) \geq (q-d)!$*

The proof of Theorem 3.13 is based on the fact that Birch partitions come up in the study of Tverberg partitions. Then Properties i) and ii) of $B_0(X)$ on the number of Birch partitions immediately imply Properties i) and ii) for $T(X)$.

Remark 3.14. For $d = 2$, the lower bound ii) for $T(X)$ improves the bound of Theorem 2.9.

Using Theorem 2.22 on Tverberg partitions with constraints in the prime power case we can improve this new lower bound.

Theorem 3.15. *Let $d \geq 2$, and $q > 2$ a prime power. Then there is an integer constant $c_{d,q} \geq 2$ such that every set X of $(d+1)(q-1)+1$ points in general position in $\mathbb{R}^d$ has at least*

$$\min\{(q-1)!, c_{d,q}(q-d)!\}$$

many Tverberg partitions. Moreover, the constant $c_{d,q}$ is monotonely increasing in q. For $d = 2$, the constant $c_{2,q}$ is bounded from above by $(q-1)^2(q-2)!$, and $c_{2,3} = 4$.

Theorem 3.15 confirms Sierksma's conjecture 1.45 for point sets in the plane that do not have a Tverberg partition of type I for $q = 3$.

Proof. (of Theorem 3.13) The Tverberg Theorem 1.41 implies the existence of a Tverberg partition together with a Tverberg point p. The set X is in general position such that the partition is either of type I, or of type II as noted in Observation 1.61.
For type I, $q-1$ disjoint d-simplices contain a point p of X. The $q-1$ disjoint d-simplices make up a Birch partition for p. Theorem 3.4 implies that there are at least $(q-1)!$ many Birch partitions of p. Hence there are at least $(q-1)!$ many Tverberg partitions.
For type II, the Tverberg point p is the intersection of the convex hull of $k \leq d$ many sets of cardinality at most d. The remaining points are partitioned into $q-k$ many d-simplices containing p. For $q > d+1$, this makes up a Birch partition for p into $q-k \geq 2$ sets. Again by Theorem 3.4 there are at least $(q-k)!$ Tverberg partitions.
Properties i) and ii) follow from the corresponding results on the number of Birch partitions from Theorem 3.4. For $q > d+1$, both types of Tverberg partitions correspond bijectively to Birch partitions so that the number of Tverberg partitions is even. As we can not predict the type of the Tverberg partition, the lower bound is equal to $(q-d)!$. □

Remark 3.16. Our proof shows a bit more than a lower bound of $(q-d)!$. If we knew what type of Tverberg partition shows up then we would obtain $(q-k)!$ for some $k \in \{1, 2, \ldots, d\}$. If there is a Tverberg partition of type I then the lower bound equals $(q-1)!$.

Theorem 3.15 is based on the following observation.

Observation 3.17. Each Tverberg point contributes at least $(q-k)!$ Tverberg partitions where $k \in \{1, 2, \ldots, d\}$ depends on the type of the Tverberg point.

Observation 3.17 gives rise to the following question:

> Is there a non-trivial lower bound for the number of Tverberg points?

In general, the answer is no. Sierksma's configuration 1.17 has exactly one Tverberg point which is of type I. This leads to the term $(q-1)!$ in the lower bound of Theorem 3.15. But under the assumption that there are no Tverberg points of type I, we obtain a non-trivial lower bound for the number of Tverberg points.

Theorem 3.18. *Let $d \geq 2$, and $q > 2$ a prime power. Then there is an integer constant $c_{d,q} \geq 2$ such that every set X of $(d+1)(q-1)+1$ points in in general position in $\mathbb{R}^d$ that does not admit a Tverberg point of type I, has at least $c_{d,q}$ many Tverberg points.*
Moreover, the constant $c_{d,q} \geq 2$ is weakly growing with q. For $d = 2$, the constant $c_{2,q}$ is bounded from above by $(q-1)(q-1)!$, and $c_{2,3} = 4$.

Proof. The proof is based on Theorem 2.22 on Tverberg partitions with constraints. Let X be a set of $(d+1)(q-1)+1$ points in $\mathbb{R}^d$, and p_1 is a Tverberg point which is not of type I. The Tverberg point p_1 is the intersection point of $\bigcap_{i=1}^{k} \operatorname{conv}(F_i^1)$, where $k \in \{2, 3, \ldots, d\}$. Choose an edge e_1 in some F_i, and apply Theorem 2.22 with constraint graph $G_1 = \{e_1\}$. Then there is a Tverberg partition that does not use the edge e_1 so that there has to be second Tverberg point p_2. Now add another edge e_2 from the corresponding F_i^2 to the constraint graph G_1, and apply again Theorem 2.22 with constraint graph $G_2 = \{e_1, e_2\}$. Hence there is another Tverberg point p_3 and so on. This procedure depends on the choices of the edges, and whether G_i is still a constraint graph.
Figure 3.4 shows an example for $d = 2$ and $q = 3$: A set of seven points in $\mathbb{R}^2$. There are exactly four Tverberg points – highlighted by small circles – in this example. A constraint graph – drawn in broken lines – can remove only three among them.

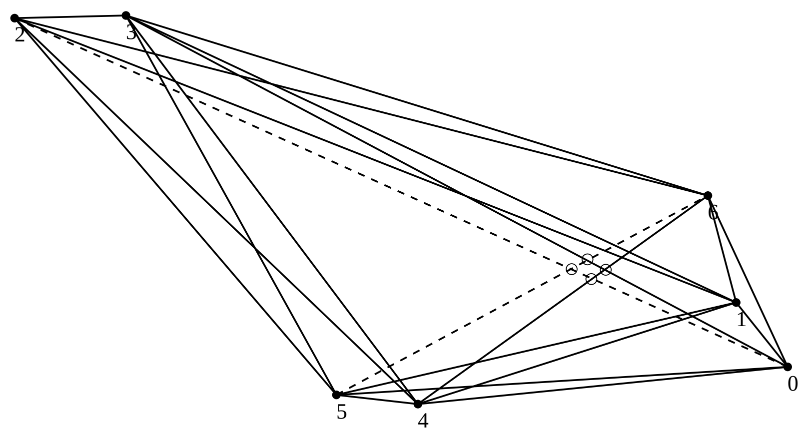

Figure 3.4: A set of 7 points in the plane together with a constraint graph.

Constraint graphs for q are also constraint graphs for $q+1$ so that our constant $c_{d,q}$ is weakly increasing in q. The constant $c_{d,q}$ also depends on d as the simplex $\sigma^{(d+1)(q-1)}$ grows in d.
For $d = 2$, the constant $c_{2,q}$ is bounded from above since there are sets with no Tverberg point of type I that attain the bound $((q-1)!)^2$, see for example

Chapter 4. The lower bound for $B_0(X)$ from Theorem 3.4 implies the upper bound for $c_{2,q}$. A lengthy case distinction shows $c_{2,3} > 3$. □

Up to now, we have not been able to determine the exact value of $c_{d,q}$ for $d > 2$, or $q > 3$ as there are just too many configurations to look at. This leads to the following problem.

Remark 3.19. While preparing the final version of this thesis, we have noted that one can show Sierksma's conjecture entirely for $q = 3$ and $d = 2$ using the method given in the proof of Theorem 3.18. The method might even work for the topological version of Sierksma's conjecture for $q = 3$ and $d = 2$.

Problem 3.20. Find configurations such that choosing a constraint graph does not remove all Tverberg points. Determine from this (bounds for) the constant $c_{d,q}$.

Remark 3.21. The proof of Theorem 3.18 translates to the setting of Tverberg points for continuous maps $f : \sigma^{(d+1)(q-1)} \to \mathbb{R}^d$. However, the constant $c_{d,q}$ turns out to be smaller in this more general setting. For $d = 2$, pairs of edges can intersect more than once, and there is in general more than one pair of intersecting edges among four points.

Remark 3.22. In the configuration from Figure 1.17 which led to Sierksma's conjecture, the point in the center is the only Tverberg point. All $((q-1)!)^d$ Tverberg partitions are of type I. Our new lower bound implies a lower bound of $(q-1)!$ for this configuration. Hence another ingredient is needed to prove Sierksma's conjecture.

3.3 ON WINDING BIRCH PARTITIONS

We have seen new lower bounds for the number of Tverberg partitions in Section 3.2. These results were based on a lower bound for the number of Birch partitions. In this section, we discuss the following questions which come up naturally in topological combinatorics.

Question 3.23. Can the lower bounds for the number of Tverberg partitions resp. their proofs be carried over to the topological Tverberg theorem? To the the number of winding partitions? Is there a topological version of Theorem 3.4 on Birch partitions? Is there a winding version of Theorem 3.4 on Birch partitions?

Our answer to all of these questions is NO. However, these negative results are important on their own. There are not many properties known that are valid for affine Tverberg partitions, and that are false in the setting of the topological Tverberg theorem. The negative results for counting winding number partitions can be seen from a graph-theoretic point of view: The lower bound for (winding) Birch partitions holds for rectilinear drawings, but the result does not hold for arbitrary graph drawings. Finally, our discussion gives rise to the following extension of Sierksma's conjecture.

Conjecture 3.24. Sierksma's Conjecture 1.45 on the number of Tverberg partitions holds not only for affine maps $f : \sigma^{(q-1)(d+1)} \to \mathbb{R}^d$, but also for continuous maps f.

Similar to Tverberg's theorem, Theorem 3.4 can be rephrased equivalently in terms of affine maps instead of convexity. An affine map $f : \mathsf{K} \to \mathbb{R}^d$ is *in general position* if the set of vertices is mapped to an $f_0(\mathsf{K})$-element set in general position with respect to the origin in $\mathbb{R}^d$. Now Theorem 3.4 can be stated equivalently as follows.

Theorem 3.25. *Let $d, k \geq 2$ be integers, and $f : (\sigma^{(d+1)k-1})^{\leq d} \to \mathbb{R}^d$ an affine map in general position. Then the following holds for the number $B_0(f)$ of partitions of $(\sigma^{(d+1)k-1})^{\leq d}$ into k disjoint d-faces $F_1, F_2, \ldots, F_k$ such that $0 \in \bigcap_{i=1}^{k} f(F_i)$:*

i) *$B_0(f)$ is even.*

ii) $B_0(f) > 0 \implies B_0(f) \geq k!$

Proof. There is a one-to-one correspondence between $(d+1)k$-element sets as in Theorem 3.4 and affine maps as above. Every Birch partition corresponds to one partition of the d-skeleton of $\sigma^{(d+1)k-1}$ as in Theorem 3.25. □

It is a natural question in topological combinatorics:

What happens if we replace affine with continuous maps?

Both Properties do in general not hold for $B_0(f)$ in the continuous case. For Properties i) and ii), this can be seen from the following example.

Example 3.26. We construct a continuous map $(\sigma^{(d+1)k-1})^{\leq d} \to \mathbb{R}^d$ for $k = 2$ which has one Birch partition. Hence Properties i) and ii) do not hold for $B_0(f)$. Start with an affine map $f : (\sigma^{2d+1})^{\leq d} \to \mathbb{R}^d$ by choosing $2d+2$ points in $\mathbb{R}^d$ such that there is no Birch partition, but at least one d-face F that contains the origin. This can be done as shown in Figure 3.5 for $d = 2$, the boundary of F is drawn in broken lines. Now f can be altered in the complementary face $\bar{F}$ of F – the boundary of $\bar{F}$ is drawn in dotted lines – such that the interior of $\bar{F}$ hits the origin. All other faces remain unchanged, and $\{F, \bar{F}\}$ is the only Birch partition. □

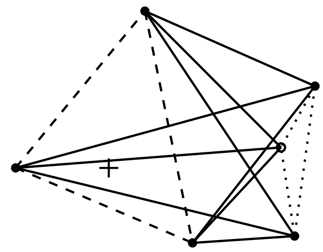

Figure 3.5: A continuous map with one Birch partition.

Tverberg partitions and Birch partitions are closely related. Property ii) of $B_0(f)$ is the key result for new lower bounds in the affine case. We have seen in Sections 1.3 and 2.2 that Tverberg partitions and winding partitions correspond to each other. Hence it is again natural to study winding Birch partitions.

Definition 3.27 (Winding Birch partition)**.** Let $f : (\sigma^{(d+1)k-1})^{\leq d-1} \to \mathbb{R}^d$ be a continuous map. A *winding Birch partition* is a partition of $(\sigma^{(d+1)k-1})^{\leq d}$ into k disjoint d-faces $F_1, F_2, \ldots, F_k$ such that the winding number $W(f|_{\partial F_i}, 0)$ is different from zero for each i. Let $WB_0(f)$ be the number of all winding Birch partitions.

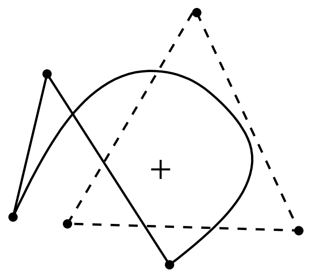

Figure 3.6: A winding Birch partition in a drawing of K_6.

For $d = 2$, see Figure 3.6 for an example of a winding Birch partition for $k = 2$. Only the edges of the winding Birch partition are drawn. In this case, we study drawings of K_6 in the plane. A winding Birch partition $F_1, F_2, \ldots, F_k$ corresponds to a C_3-factor in a drawing of the complete graph K_{3k} which comes with the extra condition that each cycle F_i winds around the origin.

Remark 3.28. Every winding Birch partition of a continuous map $f : (\sigma^{(d+1)k-1})^{\leq d} \to \mathbb{R}^d$ is a Birch partition due to the generalized intermediate value theorem 1.57:

$$W(f|_{\partial F_i}, 0) \neq 0 \implies 0 \in f(F_i).$$

The following lemma is a topological generalization of Lemma 3.5.

Lemma 3.29. *Let $f : (\partial\sigma^{d+1})^{\leq d-1} \to \mathbb{R}^d$ be a continuous map from the boundary of the $(d+1)$-simplex σ^{d+1} such that $0 \notin f((\partial\sigma^{d+1})^{\leq d-1})$. Let $WF_0(f)$ be the number of d-faces F of $\partial\sigma^{d+1}$ such that $W(f|_{\partial F}, 0) \neq 0$. Then the following implication holds:*

$$WF_0(f) > 0 \implies WF_0(f) \geq 2.$$

For $d = 2$, Lemma 3.29 implies: If in a drawing of the complete graph K_4 in $\mathbb{R}^2 \setminus \{0\}$ a C_3-subgraph winds around the origin, then there is a second C_3-subgraph winding around the origin.

Proof. We can orient the d-faces of $\partial\sigma^{d+1}$ such that any two d-faces induce different orientations on their common $(d-1)$-face. Summing up the boundary of all d-faces of $\partial\sigma^{d+1}$ gives zero as every $(d-1)$-face appears exactly twice. This sum of $(d-1)$-faces is thus a trivial element in the simplicial chain complex of $(\partial\sigma^{d+1})^{\leq d-1}$. The map $f : (\partial\sigma^{d+1})^{\leq d-1} \to \mathbb{R}^d \setminus \{0\}$ induces a map in homology $f_* : H_{d-1}((\partial\sigma^{d+1})^{\leq d-1}) \to H_{d-1}(\mathbb{R}^d \setminus \{0\})$. The above sum of $(d-1)$-faces is mapped to zero as it is trivial. By assumption, f_* maps one of the summands non-trivially, so that there has to be another one which is mapped non-trivially. □

The results from Theorem 3.4 also hold for winding Birch partitions in the case of affine maps. However, all properties do not hold in the case of continuous maps.

Theorem 3.30. *Let $k \geq 2$ be an integer, and $f : (\sigma^{(d+1)k-1})^{\leq d-1} \to \mathbb{R}^d$ an affine map in general position. Then the following properties hold for the number $WB_0(f)$*

i) *$WB_0(f)$ is even.*

ii) *$WB_0(f) > 0 \implies WB_0(f) \geq k!$*

Both properties of $WB_0(f)$ do in general not hold for continuous maps f.

The case $d = 2$ immediately implies the following statement on drawings of complete graphs K_{3k}.

Corollary 3.31. *Let f be a rectilinear drawing of the complete graph K_{3k} on $3k$ vertices in the plane, $k \geq 2$. Then the following properties hold:* i) *The number of winding C_3-factors is even.* ii) *If there is one winding C_3-factor, then there are $k!$ many of them.* iii) *Statements* i)-ii) *do in general not hold for arbitrary graph drawings.*

Proof. (of Theorem 3.30) For affine maps, the numbers $WB_0(f)$ and $B_0(f)$ coincide so that properties i) and ii) immediately follow from Theorem 3.25. Now Examples 3.32 and 3.33 below show that these properties do not hold for $WB_0(f)$ in the continuous case. □

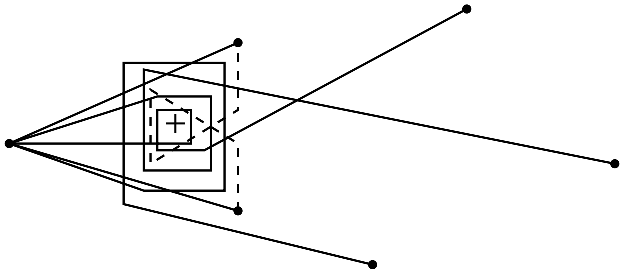

Figure 3.7: An example with $WB_0(f)$ odd, for $d = 2$ and $k = 2$.

Example 3.32. We construct an example of a continuous map f with an odd number $WB_0(f)$ of winding Birch partitions for $k = 2$. See also Figure 3.7 for the construction; all missing edges are straight. Let the dimension d be even. Choose a set X of $2(d+1)-1 = 2d+1$ points in general position in $\mathbb{R}^d$ such that $0 \notin \text{conv}(X)$. Mapping the vertices of $(\sigma^{2d})^{\leq d-1}$ to X determines an affine map $h : (\sigma^{2d})^{\leq d-1} \to \mathbb{R}^d$ uniquely. No d-simplex in X winds around the origin as $0 \notin \text{conv}(X)$. Note that there is one free vertex v of $(\sigma^{(2d+1})^{\leq d-1}$ left. Alter h in one $(d-1)$-face F as follows: Cut out of F a small disc of radius $\epsilon \geq 0$, and replace it by a tube that winds around the origin. See also Figure 3.8 for the case $d = 2$, there F is a the solid segment, and the tubular modification is drawn in a broken line that winds around the origin +. Call this altered map $\tilde{h}$.

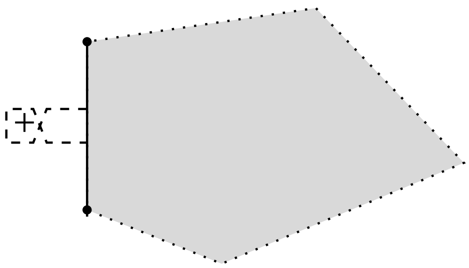

Figure 3.8: Modify h in one $(d-1)$-face to obtain $\tilde{h}$.

For our modified map $\tilde{h}$, there are exactly $d+1$ many d-faces G_1,$G_2, \ldots,$ G_{d+1} in X that wind around the origin: All d-faces of the form $F \cup \{w\}$ for $w \notin V(F)$. We extend $\tilde{h}$ to a continuous map $f : (\sigma^{2d+1})^{\leq d-1} \to \mathbb{R}^d$ such that all new $(d-1)$-faces using v wind around the origin. For this add every new $(d-1)$-face such that it winds around the origin as shown in Figure 3.9. In fact, it is sufficient that only the new $(d-1)$-faces that do not contain any vertex from $V(F)$ wind around the origin. The number of Birch partitions is by construction $d+1$, which is odd.

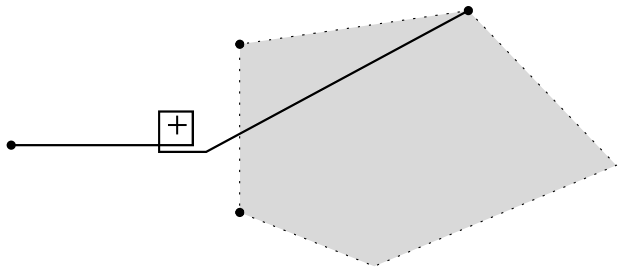

Figure 3.9: Add a $(d-1)$-face that winds around the origin.

For odd dimension d, alter h in two $(d-1)$-faces incident to a given vertex w as shown in Figure 3.8. The number of d-faces for $\tilde{h}$ that wind around the origin equals $d+1+d = 2d+1$, which is odd. Extend $\tilde{h}$ as above for even dimension. This final extension to v leads to $2d+1$ many Birch partitions.

$\square$

Example 3.33. Our computer project exposed in Chapter 4 led to many examples of piecewise linear maps that have exactly one winding Birch partition for $k=2$; one of them led to the example shown in Figure 3.10. There the only winding Birch partition is $\{1,2,3\}$ and $\{4,5,6\}$ with winding numbers ± 1 resp. ± 2.
For arbitrary dimension $d \geq 2$, we rely on the following observation: Any example for dimension d can be extended to an example in dimension $d+1$. This construction is an adapted version of Schöneborn and Ziegler's construction that led to Proposition 1.58 on winding number partitions.

Start with a map $f : (\sigma^{2d-1})^{\leq d} \longrightarrow \mathbb{R}^d$ having exactly one winding Birch partition: The boundary of one $(d+1)$-faces F, and that of its complement $\bar{F}$ wind around the origin. Embed $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$ by identifying it with the set $\mathbb{R}^d \times \{0\} = \{(x,0) \in \mathbb{R}^{d+1} \,|\, x \in \mathbb{R}^d\} \subset \mathbb{R}^{d+1}$. Choose two points p_1, p_2 in $\mathbb{R}^d \times \mathbb{R}^+$, and three points q_1, q_2, q_3 in $\mathbb{R}^d \times \mathbb{R}^-$. Now extend f to a map $\tilde{f} : (\sigma^{2d+1})^{\leq d+1} \longrightarrow \mathbb{R}^{d+1}$ as follows: (i) Map the two new vertices $2d+1$ and $2d+2$ of $(\sigma^{2d+1})^{\leq d+1}$ to p_1 resp. q_1. (ii) Perform a barycentric subdivision for the $(d+1)$-face F resp. $\bar{F}$, and map its center to q_2 resp. p_2, and then extend f canonically to F resp. $\bar{F}$. (iii) For all remaining $(d+1)$-faces G of the original complex $(\sigma^{2d-1})^{\leq d+1}$ perform a barycentric subdivision, map

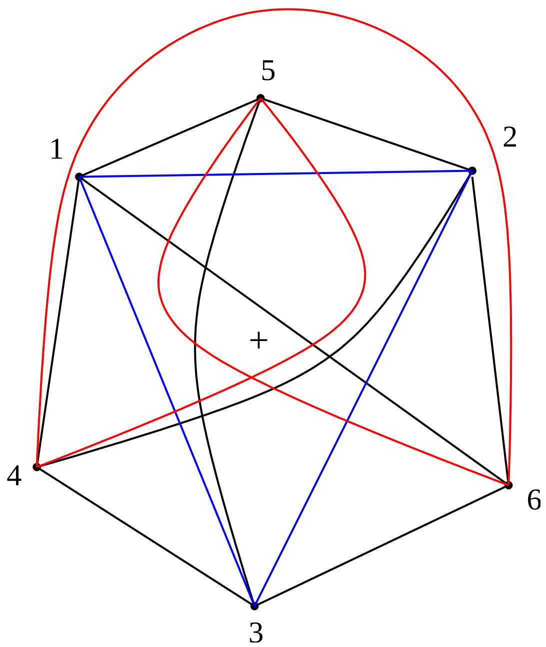

Figure 3.10: K_6 with exactly one winding Birch partition.

their centers to q_3, and then extend f canonically to G. (iv) Extend f to all $(d+1)$-faces that contain one of the two new vertices affinely.

It remains to check that $\tilde{f}$ has exactly one winding Birch partition. By construction step (ii), $F \cup \{2d+1\}$ together with its complement $\bar{F} \cup \{2d+2\}$ is a winding Birch partition, and $F \cup \{2d+2\}$ together with its complement $\bar{F} \cup \{2d+1\}$ is not. Any partition of the form $G \cup \{2d+1\}$ together with its complement $\bar{G} \cup \{2d+2\}$ is not a winding Birch partition for all $(d+1)$-faces G of the original complex $(\sigma^{2d-1})^{\leq d+1}$, as G together with $\bar{G}$ is not a winding Birch partition of f. Any partition such that the vertices $2d+1$ and $2d+2$ are in the same partition set is not a winding Birch partition, as its complement does not wind around the origin by construction step (iii): The complement lies essentially in $\mathbb{R}^d \times \mathbb{R}^-$. $\square$

Observation 3.34. Example 3.33 is a break-through concerning the numbers of Tverberg resp. winding partitions. It shows for the first time that there might be a difference between affine and continuous maps. A given Tverberg point contributes in general less to these numbers in the continuous case than in the affine case. However, all attempts to come up with a counter-example to Sierksma's conjecture in the continuous case failed, starting with one of the minimal examples mentioned in Example 3.33. This leads to the extended Sierksma's Conjecture 3.24.

The computer program that led to Example 3.33 of a piecewise linear map $f : K_{3k} \to \mathbb{R}^2$ for $k = 2$ with only one winding Birch partition, did not find any examples for $k > 2$ that contradict Property ii) of $WB_0(f)$. Up to now, we have also not been able to construct such an example by hand.

All examples that violate Property ii) of $WB_0(f)$ contain a simplex that winds around the origin an even number of times. This suggests the following problem.

Problem 3.35. Find an example of a continuous map $f : (\sigma^{(d+1)k-1})^{\leq d-1} \to \mathbb{R}^d$ such that all simplices wind around the origin an odd number of times.
Alternatively, show that Property ii) holds for continuous maps $f : (\sigma^{(d+1)k-1})^{\leq d-1} \to \mathbb{R}^d$ and winding Birch partitions $F_1, F_2, \ldots, F_k$ such that every F_i winds an odd number of times around the origin:

$$W(f|_{\partial F_i}, 0) \not\equiv 0 \mod 2 \qquad \text{for each } i \in [k].$$

The example below shows that the upper bound for $B_0(X)$ conjectured in Problem 3.9 does not hold for $WB_0(f)$ of continuous maps.

Example 3.36. We construct an example such that all potential partitions are winding Birch partitions. There are $\frac{(k(d+1))!}{((d+1)!)^k k!} > (k!)^d$ many ways to obtain unordered partitions of $k(d+1)$ points in k subsets of size $d+1$. Start with $k(d+1)$ points in $\mathbb{R}^d \setminus \{0\}$, and add every $(d-1)$-face such that it winds around the origin as shown in Figure 3.9. Now every d-face winds around the origin. $\square$

3.4 Towards the topological Tverberg theorem

In this section, we collect evidence that the Topological Tverberg Theorem 1.50 holds for arbitrary q in dimension $d = 2$. Instead of looking at the topological Tverberg theorem, we study a stronger statement: the Winding Number Conjecture 1.54. The two-dimensional version of the winding number conjecture is a problem on drawings of complete graphs on $3q-2$ vertices in the plane. It has been confirmed for prime powers q so that it is valid for drawings of K_4, K_7, K_{10}, K_{13}, K_{19}, K_{22}, ... What about K_{16}? It would be rather surprising if this problem relied on the fact that q is a prime power.

Let's discuss the following two approaches. A positive answer to each of them would imply the winding number conjecture for $d = 2$.

Conjecture 3.37 (Perturbing an edge)**.** Let $f : K_{3q-2} \to \mathbb{R}^2$ be a drawing of the complete graph on $3q-2$ vertices, and let e be an edge of K_{3q-2}. Assume that f has a winding partition. Then any perturbation of f which modifies f in e has a winding partition.

For prime powers q, Conjecture 3.37 is confirmed via the winding number conjecture. We confirm it below for arbitrary q in the case of pseudoline drawings of K_{3q-2}. Tverberg-type theorems for pseudoconfigurations of points in the plane go back to Roudneff [56]. Moreover, we obtain a geodesical Tverberg theorem as a corollary.

The following conjecture reduces for prime powers q to a special case of Theorem 2.17. An edge e can be seen as the simplest case of a matching in K_{3q-2}.

Conjecture 3.38 (Deleting an edge)**.** Let $f : K_{3q-2} \to \mathbb{R}^2$ be a drawing of the complete graph on $3q-2$ vertices, and let e be an edge of K_{3q-2}. Assume that f has a winding partition. Then there is a winding partition after deleting the edge e from the drawing f. In other words: The restriction $f|_{K_{3q-2}\setminus\{e\}}$ has a winding partition.

The computer project discussed in Chapter 4 suggests that Conjecture 3.38 should be true for arbitrary q. Let's now prove that both conjectures imply the winding number conjecture.

Proof. We have to show that any drawing $f : K_{3q-2} \to \mathbb{R}^2$ has a winding partition. The set of vertices of K_{3q-2} is mapped to $\mathbb{R}^2$ via f. Their images determine an affine drawing of K_{3q-2} uniquely. There is a winding partition for this affine drawing as the winding number conjecture holds in the case of affine drawings for arbitrary q. Both conjectures allow us to modify one edge K_{3q-2} after the other while fixing all other edges. Hence any drawing f has a winding partition as in every modification step the existence of a winding partition is guaranteed. □

Observation 3.39. A given winding partition involves $3q-3$ resp. $3q-4$ edges of K_{3q-2} for type I resp. type II. At the same time, K_{3q-2} has $\binom{3q-2}{2}$ many edges. For Conjectures 3.37 and 3.38, this means that "most" edges can be perturbed resp. deleted.

Let's introduce pseudoconfigurations of points in the plane. We look at them from a geometric point of view. They generalize affine point configurations in a natural way. In fact, they are in one-to-one correspondence with acyclic oriented matroids of rank 3. See the textbook of Björner et al. [21], or Roudneff [56] for more details on their relation to oriented matroids.

Definition 3.40 (Arrangements of pseudolines)**.** A *pseudoline* is the image of a line under a self-homeomorphism of the real projective plane $\mathbb{R}P^2$. A finite collection $\mathcal{A}$ of pseudolines is an *arrangement* if every two pseudolines of $\mathcal{A}$ have exactly one point in common in which they cross.

Any arrangement of $n \geq 2$ pseudolines decomposes the projective plane into a two-dimensional cell complex defining *faces, edges,* and *vertices.* Two arrangements are *isomorphic* if the cell complexes are isomorphic.
In every arrangement $\mathcal{A}$ we distinguish a pseudoline L_∞ called the *pseudoline at infinity.* With this convention, we can represent $\mathcal{A}$ in the "affine plane" $\mathbb{R}P^2 \setminus \{L_\infty\}$. Let $L \in \mathcal{A} \setminus \{L_\infty\}$ be a pseudoline, and $x, y \in L \setminus L_\infty$ be two distinct points. The space $L \setminus \{x, y\}$ contains two components, and the "bounded" one is the *open interval* $]x, y[$. Adding the endpoints leads to the *segment* $[x, y]$. An arrangement $\mathcal{A}$ is *simple* if no point of $\mathbb{R}P^2$ belongs to more than two pseudolines.

Definition 3.41 (Pseudoconfiguration of points)**.** Let $\mathcal{A}$ be an arrangement of pseudolines, and $V \subset \mathbb{R}P^2 \setminus L_\infty$ be an n-point set with $n \geq 3$. $\mathcal{A}(V)$ is a *pseudoconfiguration of n points in the plane* if the following properties hold:

a) Each pseudoline $L \neq L_\infty$ of $\mathcal{A}$ contains at least two points of V.

b) Each pair of points of V belongs to a pseudoline of $\mathcal{A}$.

c) The points of V are not all on the same pseudoline of $\mathcal{A}$.

We denote the pseudoline that contains a pair $\{x, y\}$ of points in V by xy. The points of V are *in general position* with respect to the pseudoconfiguration $\mathcal{A}(V)$ if the following two conditions hold:

d) Each pseudoline $L \neq L_\infty$ of $\mathcal{A}$ contains exactly two points of V.

e) If three pseudolines L_1, L_2, L_3 of $\mathcal{A}(V)$ cross at some vertex x, then $x \in V$.

This version of general position is very natural: Condition d) guarantees that no three points lie on a common pseudoline. Condition e) guarantees that no three lines meet outside of V. Let's now extend our usual concept of convexity.

Definition 3.42 (Convexity for pseudoconfigurations)**.** Given a pseudoconfiguration $\mathcal{A}(V)$, and a subset $W \subset V$. A point $x \in \mathbb{R}P^2$ is in the convex hull $\langle W \rangle$ of W if any path of x to L_∞ meets some segment $[y, z]$ for $y, z \in W$.

The following result due to Roudneff [56] extends Ringel's theorem for simple arrangements of pseudolines.

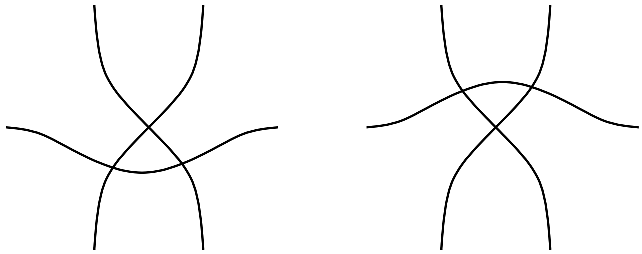

Figure 3.11: Switching of a triangle.

Theorem 3.43 (Roudneff)**.** *Any two pseudoconfigurations of n points in the plane can be transformed into each other by a finite sequence of elementary transformations: The switching of a triangle, and the crossing of a pseudoline by a point.*

See e. g. [21] also for a proof. See Figures 3.11 resp. 3.12 for an illustration of the two elementary transformations. In all our drawings of pseudoconfigurations modification occur only in the drawn parts, and outside this local neighborhood they remain unchanged.

Figure 3.12: Crossing of a pseudoline by a point.

The Winding Number Conjecture 1.54 holds for pseudoconfigurations of $3q-2$ points in the plane for arbitrary q. Tverberg's theorem in this setting has been proved by Roudneff [56].

Theorem 3.44 (Winding number theorem for pseudoconfigurations)**.** *For $q \geq 2$, any pseudoconfiguration $\mathcal{A}(V)$ of $3q-2$ points in the plane has a winding partition into q subsets.*

Theorem 3.44 immediately implies a geodesical version of the Tverberg theorem. For this, let N be the north pole of the standard 2-sphere S^2. Choosing $n+1$ points in general position on the two-dimensional sphere $S^2 \setminus \{N\}$ uniquely determines a geodesical map $f : (\sigma^n)^{\leq 2} \to S^2 \setminus \{N\}$ as follows: Map the $n+1$ vertices of the simplex σ^n to the $n+1$ points, map any edge (v_1, v_2) to its geodesic – the shortest path between x and y on S^2. Finally extend this map canonically for every 2-face to the component not containing N.

Theorem 3.45 (Geodesical Tverberg theorem)**.** *For $q \geq 2$, any geodesical map $f : (\sigma^{3(q-1)})^{\leq 2} \to S^2 \setminus \{N\}$ defined as above has a Tverberg partition F_1, F_2, ..., F_q in the 2-skeleton of $\sigma^{3(q-1)}$ such that*

$$\bigcap_{i=1}^{q} f(F_i) \neq \emptyset.$$

Theorem 3.45 extends the Conic Tverberg Theorem 1.43. Geodesical maps $f : (\sigma^{3(q-1)})^{\leq 2} \to S^2 \setminus \{N\}$ can be represented as maps to the plane $\mathbb{R}^2$ via stereographic projection, see Figure 4.2 of Section 4.1 for an example with $q = 3$.

Proof. (of Theorem 3.44) We can assume without loss of generality that the points of V are in general position. If they are not in general position, then we can slightly perturb them. Lemma 1.63 guarantees that the property of not having a winding partition is robust under small perturbations. For points in general position there are only two types of winding partitions F_1, F_2, ..., F_q due to Observation 1.61:

$$\begin{array}{ll} \text{Type I:} & |F_1| = 1,\ |F_i| = 3 \ \text{ for } i > 1, \\ \text{Type II:} & |F_1| = 2,\ |F_2| = 2, \ \text{ and } |F_i| = 3 \ \text{ for } i > 2. \end{array}$$

The winding point p is unique for both types. The existence of a winding partition for an affine point configuration follows from the Tverberg theorem 1.41. With Theorem 3.43 in mind, it remains to check that if $\mathcal{A}(V)$ has a winding partition, and $\mathcal{A}(V')$ is derived from $\mathcal{A}(V)$ by an elementary transformation T, then $\mathcal{A}(V')$ also has a winding partition. This is proved in the case distinction below. For any elementary transformation which does not involve an edge of the F_i's, the winding partition of $\mathcal{A}(V)$ is again a winding partition of $\mathcal{A}(V')$.

Case 1: Let T be the switching of a triangle. The winding point p of $\mathcal{A}(V)$ is one of the vertices of the triangle, and the opposite edge is part of F_3. The winding partition of $\mathcal{A}(V)$ is thus of type II, and after the switching it is not a winding partition of $\mathcal{A}(V')$ any more.
Let's fix some notation: $F_1 = \{a, b\}$, $F_2 = \{c, d\}$, and $F_3 = \{e, f, g\}$, see also Figure 3.13. We can assume that there is no other edge in a small neighborhood of the switched triangle such that the winding number for all F_i, with $i > 3$, is non-zero with respect to any point of the triangle. In $\mathcal{A}(V')$, the points c and g lie on different sides of the pseudoline ef such that the segment $[c, g]$ meets ef in some vertex x. The segment $[e, f]$ meets $[a, b]$ resp. $[c, d]$ in some vertex p resp. q. By the definition of switching, we have

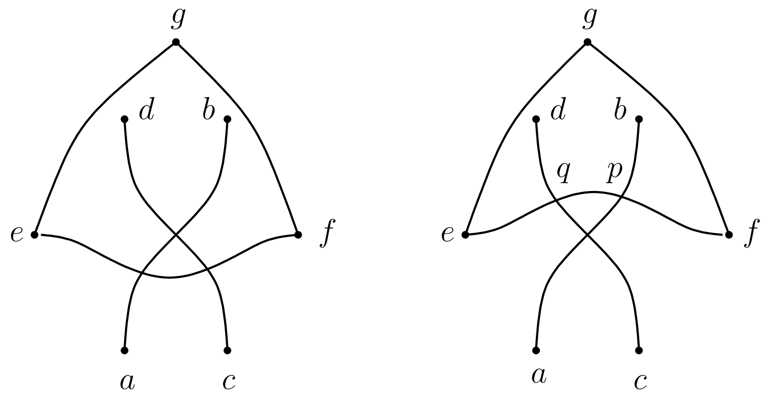

Figure 3.13: Winding partition and the switching of a triangle.

$x \notin [p, q]$. If x is right of p then our new winding partition for $\mathcal{A}(V')$ is

$$F_1' = \{a, b\},\ F_2' = \{e, f\},\ F_3' = \{c, d, g\},\ \text{ and } F_i' = F_i \text{ for } i > 3,$$

and if x is left of q then our new winding partition is

$$F_1' = \{c, d\},\ F_2' = \{e, f\},\ F_3' = \{a, b, g\},\ \text{ and } F_i' = F_i \text{ for } i > 3,.$$

Case 2: Let T be the crossing of a pseudoline ab by a point c, see also Figure 3.14. If c is the winding point of $\mathcal{A}(V)$, then the winding partition

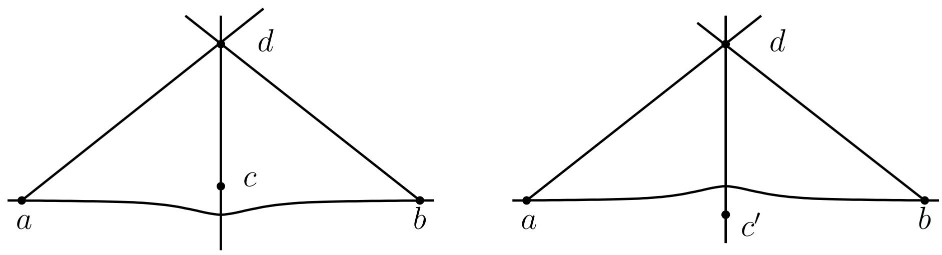

Figure 3.14: Winding partition and the crossing of a pseudoline by a point.

is of type I with $F_1 = \{c\}$, and $F_2 = \{a, b, d\}$. By definition of the crossing of a pseudoline there is no other edge with endpoint different from c in this local neighborhood. Let c' be the point c after the transformation, then the segments $[a, b]$ and $[c', d]$ meet in some vertex. Hence the winding partition of $\mathcal{A}(V')$ is

$$F_1' = \{a, b\},\ F_2' = \{c', d\},\ \text{ and } F_i' = F_i \text{ for } i > 2,$$

which is of type II. The inverse case which starts with a winding partition of type II follows from the same argument, but inversed. $\square$

Remark 3.46. We have not been able to generalize the proof of Theorem 3.44 to continuous drawings of K_{3q-2}. Figure 3.15 shows an example of Ziegler of a drawing of K_4 which does not come from a pseudoconfiguration. A winding point is indicated by a circle. If one of the edges crosses the point v as shown in Figure 3.15, then this winding partition vanishes completely.

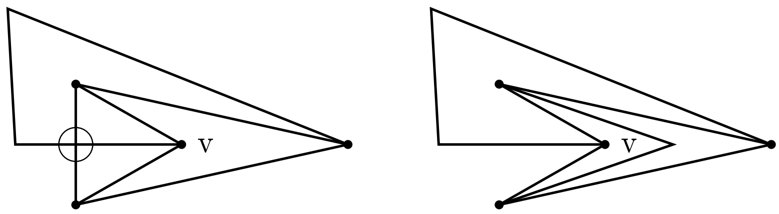

Figure 3.15: Crossing of an edge for which a winding partition vanishes.

Chapter 4

Computational data

Considering examples is essential in mathematics. In the case of Tverberg partitions, it is not obvious how to count them by hand. We are mainly interested in the number of Tverberg partitions. The smallest unresolved case of Sierksma's Conjecture 1.45 is for $d = 2$ and $q = 3$ so that one has to study configurations of seven points in the plane and their Tverberg partitions. We started a computer project which first generates random point sets, and then counts the Tverberg partitions using a brute force approach.

The results of Chapter 3 came up while staring at the computational results. This section contains a discussion of the outcome of this project.

4.1 The algorithm

In the following, we present our approach for counting Tverberg partitions, winding partitions, and Birch partitions in four different settings:

1. Sets of $3q - 2$ points in the plane.

2. Sets of $3q - 2$ points on the 2-sphere.

3. Sets of $3q - 2$ points in the plane with piecewise linear edges.

4. Sets of $4q - 3$ points in $\mathbb{R}^3$.

Setting (1): Random planar sets. We study Tverberg partitions of random point sets in $[-1, 1]^2$. See Figure 4.1 for an example of random planar set for $q = 3$. For sets X of $3q - 2$ points in general position in the plane, Tverberg partitions look as follows:

1. One point p of X, and $q - 1$ triangles containing p.

2. Two edges intersecting in $\tilde{p}$, and $q - 2$ triangles containing $\tilde{p}$.

Here all vertices, edges, and triangles are pairwise disjoint. In other words, a Tverberg partition is either a Birch partition of $X \setminus \{p\}$ with Birch point $p \in X$, or a Birch partition of $X \setminus \{p_1, p_2, p_3, p_4\}$ with Birch point $\tilde{p}$, where

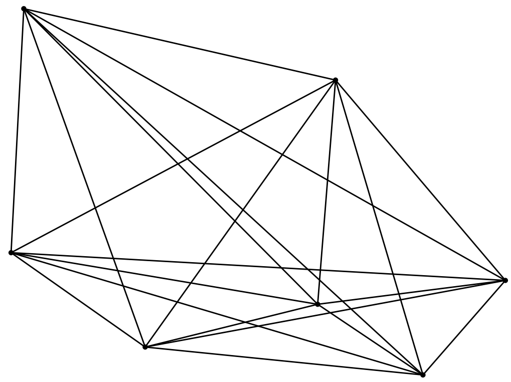

Figure 4.1: A random set of 7 points in the plane.

$\tilde{p}$ is the intersection of two edges (p_1, p_2) and (p_3, p_4). Our algorithm for counting all Tverberg partitions is then:

Generate a set X of $3q-2$ points in $[-1,1]^2$ uniformly at random
number_of_tp=0
for $p \in X$ **do**
  number_of_tp += **Count_Birch_partitions**($X \setminus \{p\}$, p)
end for
for $\tilde{p} = (p_1, p_2) \cap (p_3, p_4)$ **do**
  number_of_tp += **Count_Birch_partitions**($X \setminus \{p_1, p_2, p_3, p_4\}$, $\tilde{p}$)
end for

The subroutine **Count_Birch_partitions(Y, p)** counts the number of Birch partitions of a k-element set Y with Birch point p; here k is a multiple of 3. Implementing **Count_Birch_partitions(Y, p)** efficiently is the essential step. In our current version, it determines in a first step all triangles in Y that do not contain p. Then it counts recursively all partitions of Y into triangles that do not involve any triangle from the first step.

Setting (2): Geodesical maps on S^2. We study the number of Tverberg partitions for continuous maps defined through random points on the 2-sphere S^2. For this generate a set X of $3q-2$ points on S^2 uniformly at random. For sets X that do not contain the north pole $N \in S^2$, and no antipodal pair, define a map $f : (\sigma^{3(q-1)})^{\leq 2} \to S^2 \setminus \{N\}$ up to the 2-skeleton of the simplex $\sigma^{3(q-1)}$. Map the vertices of $\sigma^{3(q-1)}$ to X, any edge (v_1, v_2) to the geodesic connecting $f(v_1)$ and $f(v_2)$. For each 2-face F extend f as follows: The boundary of F is mapped under f to a geodesic triangle; this triangle partitions S^2 into two 2-balls. Extend f canonically to the 2-ball not containing the north pole N; see also Figure 4.2 for a projected example with $q = 3$. Composing f with the stereographic projection $S^2 \setminus \{N\} \to \mathbb{R}^2$

leads to a continuous map $\tilde{f} : (\sigma^{3(q-1)})^{\leq 2} \to \mathbb{R}^2$ such that f and $\tilde{f}$ have the same Tverberg partitions. The above extension to 2-faces corresponds in the projection to a canonical extension to the bounded component which is determined by the projected geodesic triangle. For general position maps the Tverberg partitions are as in Setting (1). Hence the algorithm is the same as above.

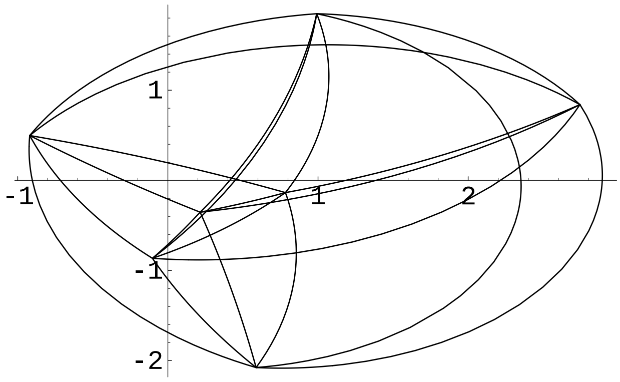

Figure 4.2: A spherical map of K_7 projected to $\mathbb{R}^2$.

Setting (3): Piecewise linear planar maps. We study the number of winding partitions for a special class of randomly generated piecewise linear drawings of K_{3q-2}. The vertices are mapped to random points in $[-1,1]^2$. The edges (v_1, v_2) are determined as follows: The segment $[v_1, v_2]$ is part of exactly two squares in the plane; choose one of them at random, and take the three other sides of the square connecting v_1 and v_2 as edge. See also Figure 4.3 for an example with $q = 3$. The maps in this setting differ from the previous ones: A pair of edges can intersect more than once. Two Tverberg points can lead to the same Tverberg partition as mentioned in Observation 1.64. The algorithm is the same as for Settings (1) and (2) with minor modifications as we face again general position maps.

Remark 4.1.
- The maps in the Settings (2) and (3) can in general not be straightened to affine maps. Even so, we see below that all examples are consistent with Sierksma's conjecture.
- The piecewise linear drawings of K_{3q-2} in Setting (3) can be extended to continuous maps $f : (\sigma^{3q-3})^{\leq 2} \to \mathbb{R}^2$ which have the same number of Tverberg partitions. To see this, start with the affine map $g : (\sigma^{3q-3})^{\leq 2} \to \mathbb{R}^2$ that is determined by the images of the vertices. There is a homotopy starting with g that ends with a map f as desired: It extends the graph drawing, and f maps every triangle to the

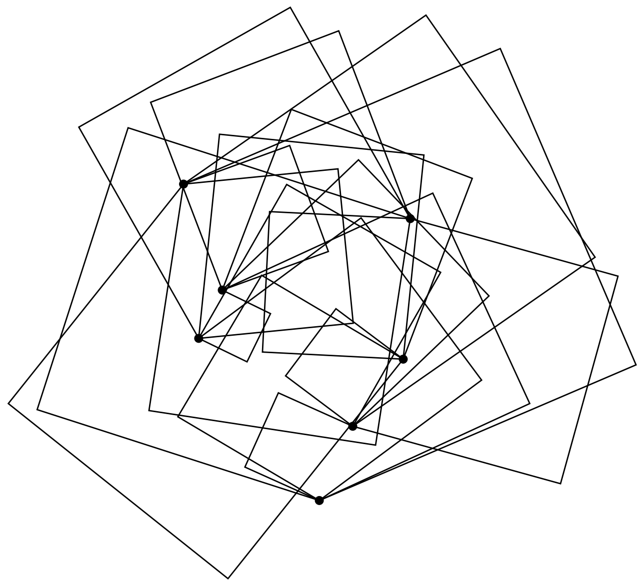

Figure 4.3: A piecewise linear drawing of K_7.

set of points having a non-zero winding number of its boundary; see also Figure 4.4 for how to do the extension step for each triangle.

- In our algorithm, we determine the intersection points of all pairs of disjoint edges. The number of all these equals the crossing number for the underlying drawing of the complete graph K_{3q-2} in Settings (1) and (2).

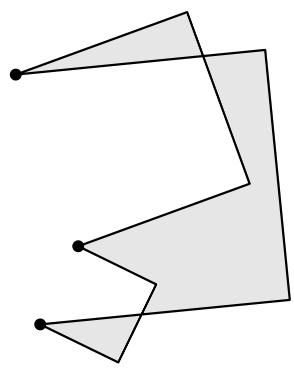

Figure 4.4: How to extend our piecewise linear drawings.

Setting (4): Random sets in $\mathbb{R}^3$. We study Tverberg partitions of random point sets in $[-1,1]^3$. For sets X of $4q-3$ points in general position in $\mathbb{R}^3$, a Tverberg partition is of one of the followings forms:

1. A point p of X, and $(q-1)$ tetrahedra containing p.
2. An edge and a triangle intersecting in p', and $(q-2)$ tetrahedra containing p'.
3. Three triangles intersecting in p'', and $(q-3)$ tetrahedra containing p''.

Here all vertices, edges, triangles and tetrahedra are pairwise disjoint. Our brute force algorithm can easily be adapted: We determine for all three cases the number of Birch partitions of all points p, p', and p''.

Implementation and running time. It was not our aim to find the fastest algorithm for finding one Tverberg point, but to implement an algorithm that finds all Tverberg partitions. See for instance Agarwal, Sharir, and Welzl [1] for the most recent results on polynomial-time algorithms for finding a (colored) Tverberg point in dimensions 2 and 3.

Initially, we investigated the non-affine case of Tverberg's theorem via Settings (2) and (3). This was motivated by the following question:

> Does Sierksma's conjecture hold for non-affine maps?

In our brute force approach, two tasks determine the running time of our algorithm: Finding all intersections, and counting all Birch partitions of a given point. The algorithm, implemented in Java, terminates in less than one day using a Intel Pentium 4 with 2.8 GHz for Settings (1)-(3) up to $q = 8$, and for Setting (4) up to $q = 5$. A special focus was on the case $q = 6$ being the smallest non-prime power. In this case the running time varies from one second up to several minutes for Settings (1)-(3).

Visualization. It is essential to have a visual representation of our mathematical objects. For Settings (1) and (3), we use `xfig`, a drawing tool which is available for most Unix platforms. A Java program converts the data of our random sets into xfig-format. The spherical maps are visualized after a stereographic projection using `Mathematica`, a commercial computer algebra system.

4.2 Computational results on the number of Tverberg partitions

In this section, we report and comment the outcome of our algorithm for counting Tverberg partitions introduced in Section 4.1. This discussion adresses the following points in all four settings from Section 4.1:

- How many examples contribute to our sample?

- How many examples attain the lower bound of Sierksma's Conjecture 1.45? Of what type are they? How do they look like?
- What about the number of Birch partitions?

(1) Results for random planar sets. Table 4.1 below summarizes the results obtained while studying random planar sets of size $3q-2$ for $3 \leq q \leq 6$. Again we write #TP for the number of Tverberg partitions. The sample size equals 20000 for every q. The second column shows the number of *minimal examples*, that is random sets attaining the conjectured lower bound $((q-1)!)^2$. In column the last two columns the minimum, resp. maximum, for the number of Tverberg partitions in our sample is given. The last column reports whether the number of Tverberg partitions is even in all examples or not.

q	# min. examples	Min for #TP	Max for #TP	#TP even?
3	13806	4	7	No.
4	2901	36	62	Yes.
5	436	576	998	Yes.
6	57	14400	26632	Yes.

Table 4.1: The number of Tverberg partitions for random planar sets.

Let's focus on minimal examples for $q = 6$. The five smallest numbers showing up are $14400, 14448, 14460, 14472, 14496$. Among the 57 examples attaining the conjectured lower bound $14400 = ((q-1)!)^2$, there are three different types: a) 13 examples have exactly one Tverberg point and all 14400 Tverberg partitions of type I, one point and 5 triangles containing this point, like in Sierksma's Example 1.17.
b) There are 10 examples having only Tverberg partitions of type II so that the Tverberg points are intersection points of two segments. One of these examples is not only minimal for the number of Tverberg partitions, but it is also extremal in the following way: It has $25 = (q-1)^2$ Tverberg points which is minimal as every Tverberg point contributes to exactly $576 = ((q-2)!)^2$ Tverberg partitions, which conjectured to be maximal in Conjecture 3.9. Figure 4.5 shows a minimal example of type b) for $q = 3$ which is also extremal in the just described sense. There are $4 = (q-1)^2$ Tverberg points of type II, and every Tverberg point contributes to $1 = ((q-2)!)^2$ Tverberg partition. The Tverberg points are the intersection points of segments $[0,2] \cap [4,6], [0,3] \cap [4,6], [0,2] \cap [5,6]$, and $[0,3] \cap [5,6]$; in Figure 4.5 they are highlighted by small circles.

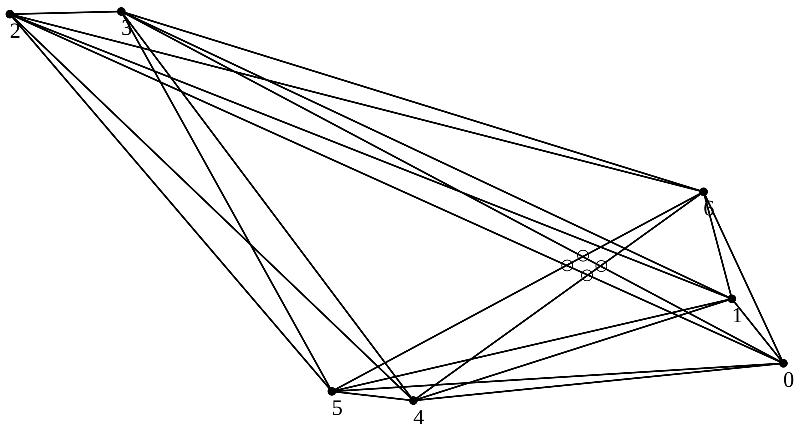

Figure 4.5: A minimal example of type b).

c) The remaining 34 examples have both Tverberg partitions of type I and of type II.

Birch partitions arise while counting Tverberg partitions. The results on the number of Birch partitions obtained in Section 3.1 show up: Evenness, lower bound, and conjectured upper bound. Both lower and upper bound are attained in many examples.

(2) Results for random spherical maps. Table 4.2 gives an overview of the results on the number of Tverberg partitions for spherical maps determined by $3q-2$ points on the 2-sphere S^2. The sample size varies with q. The minimal examples of column two attain the lower bound $((q-1)!)^2$ conjectured for affine maps as in Setting (1). The outcome for spherical maps did

q	Sample size	# min. examples	Max for #TP	#TP even?
3	5000	2326	22	No.
4	5000	431	216	Yes.
5	20067	265	3090	Yes.
6	52077	101	118044	Yes.

Table 4.2: The number of Tverberg partitions for random spherical maps.

not differ essentially from what we observed for random planar sets. There are examples attaining the conjectured lower bound, but the ratio for those is smaller than for random planar sets. Moreover, there are spherical examples with a much larger number of Tverberg partitions. All other observations for random planar sets carry over: The minimal examples are of type a), b), and c). Remarkably, the numbers predicted by the results on the number of Birch partition from Section 3.4 also show up for spherical maps.

3) Results for random piecewise linear maps. Table 4.3 shows the outcome for random piecewise linear maps in the plane. The sample size is 18000 for every $3 \leq q \leq 5$. This time we omit the case $q = 6$ in our report as it did not show any special behavior. This setting produced examples

q	# min. examples	Min for #TP	Max for #TP	#TP even?
3	41	4	78	No.
4	0	90	1250	No.
5	0	4046	42577	No.

Table 4.3: The number of winding partitions for piecewise linear maps.

that are far from being minimal for $q \geq 4$. However, this shows that there are differences concerning the number of winding resp. Tverberg partitions between affine and continuous maps. The parity result from Theorem 3.13 for $T(f)$ does not hold for piecewise linear maps as in many examples this number is not even.
Example 3.33 of a continuous drawing $f : K_6 \to \mathbb{R}^2$ with exactly one winding Birch partition is an outcome of this setting. Due to Remark 4.1, this can be extended to a continuous map $f : (\sigma^5)^{\leq 2} \to \mathbb{R}^2$ with exactly one Birch partition. As already mentioned in Section 3.3, we did not find any examples for $k > 2$ that contradict Property ii) of $WB_0(f)$.

Crossing numbers versus the number of Tverberg partitions. While counting Tverberg partitions for Settings (1), (2), and (3), we also obtained crossing numbers for random drawings of complete graphs K_{3q-2}. There are different possibilities of defining crossing numbers for graphs, see for instance Pach and Tóth's survey article [54]. In our settings, the rectilinear crossing number, the spherical crossing number, and the independent odd crossing number arise naturally. We did not find any evidence for a relation between having a small number of Tverberg partitions, and having a particularly small (or large) crossing number.

4) Results for random sets in $\mathbb{R}^3$. We conclude with Table 4.4 which highlights the important numbers from our experiment on the number of Tverberg partitions of $(4q-3)$-element sets of random points in $\mathbb{R}^3$. Our sample size is again 20000 for every $3 \leq q \leq 5$. The second column shows the number of examples attaining the conjectured lower bound $((q-1)!)^3$. Our sample set does not contain any minimal examples for $q \geq 5$. Our smallest examples for $q = 5$ has $15198 > 13824 = (4!)^3$ many Tverberg partitions. Surprisingly, all numbers are even for $q = 4$. This is based on the fact that the number of Tverberg partitions involving three intersecting triangles plus one 3-simplex turned out to be even in all examples. The results on the

q	# min. examples	Min for #TP	Max for #TP	#TP even?
3	4799	8	17	No.
4	6	216	462	Yes.
5	0	15198	36050	Yes.

Table 4.4: The number of Tverberg partitions for random sets in $\mathbb{R}^3$.

number of Birch partitions obtained in Section 3.1 appear again.
The six minimal examples for $q = 4$ are of three different types which are all different from Sierksma's Example 1.17. A Tverberg partitions consists of a point plus three tetrahedra containing this point (type I), the intersection point of a segment and a triangle plus two tetrahedra (type II), or the intersection point of three triangles plus one tetrahedron (type II'). Sierkma's example has only Tverberg partitions of type I. One of the six examples has Tverberg partitions of all three types, two of them have only Tverberg partitions of type II and II', and the remaining three have only Tverberg partitions of type I and II.

4.3 Ideas and open problems

This section comes with a list of problems suggested by our computer project.

On the number of Tverberg points. Obtaining bounds on the number of Tverberg points is an important task. We saw in Section 3.2 that every Tverberg point contributes $(q-1)!$ Birch partitions to the total number of Tverberg partitions if the Tverberg point is of type I. All random examples from Settings (1) and (4) satisfied the following upper bound on the number of Tverberg points of type I.

Problem 4.2. Let X be a set of $(d+1)(q-1)+1$ points in general position in $\mathbb{R}^d$. Show that the number of Tverberg points of type I is at most $d+1$.

Figure 4.6 shows an example which exceeds this upper bound in the continuous case: The points $3, 4, 5, 6$ are Tverberg points of type I.

Birch partitions and geodesical maps. The properties for the number $B_0(X)$ of Birch partitions from Theorem 3.4 seem to carry over to the spherical maps of Setting (2).

Problem 4.3. Show Theorem 3.4 in the case of geodesical maps. Does this carry over to pseudoconfigurations of points in the plane?

Parity in dimension 3. Theorem 3.13 implies that the number of Tverberg partitions is even for $q > d+1$. All examples of random sets in $\mathbb{R}^3$

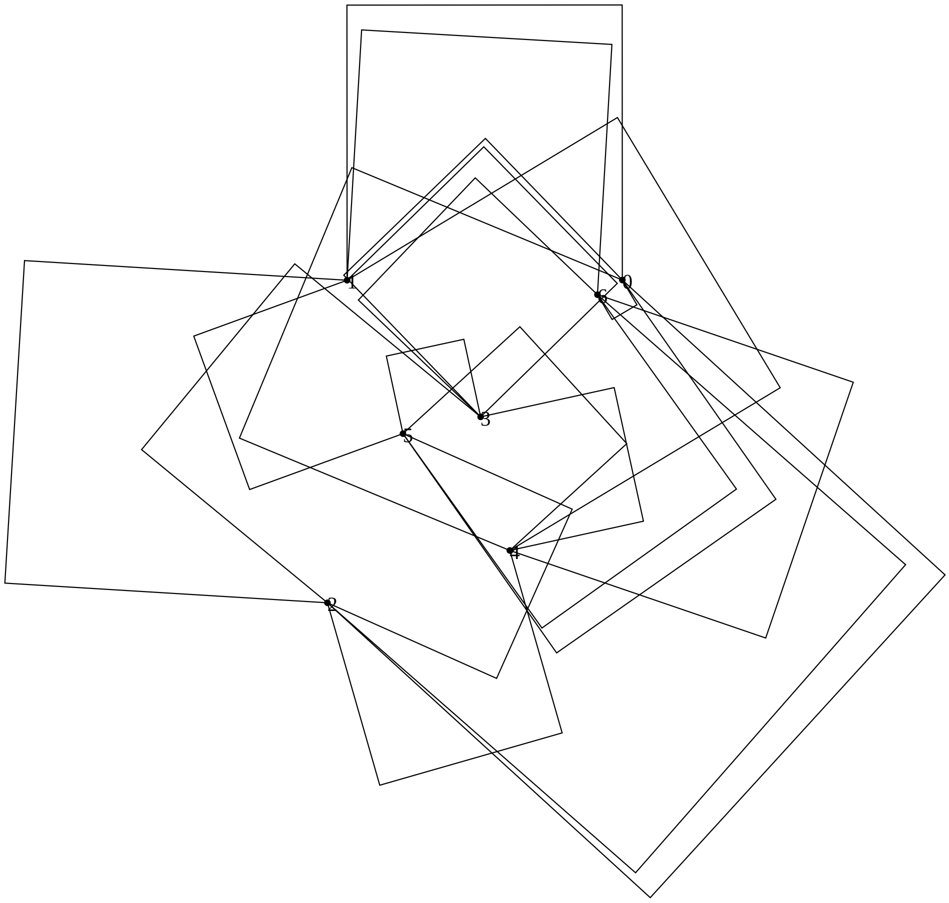

Figure 4.6: A continuous example with four Tverberg points of type I.

of Setting (4) already have an even number of Tverberg partitions for $q = 4$. This was due to the fact that the number of Tverberg partitions which involve three intersecting triangles plus one 3-simplex turned out to be even.

Problem 4.4. Let X be a set of $4q - 3$ points in general position in $\mathbb{R}^3$. Show that the number of Tverberg partitions which involve three intersecting triangles plus one 3-simplex is even.

On the colorful topological Tverberg problem. The colored Tverberg Theorem 1.68 states that $CT(2, q) = q$. In other words: Let C_1, C_2, C_3 be colored sets of points in $\mathbb{R}^2$ such that there are q points in every color class C_i. Then there is a rainbow Tverberg partition F_1, F_2, $\ldots$, F_q. In its topological version, Theorem 1.68, one needs color classes of at least $2q - 1$ points. Are there examples of non-affine maps confirming this gap?

Problem 4.5. Find examples of continuous maps with color classes of size q that do not have a rainbow Tverberg partition. A starting point would be the above program for counting Tverberg partitions of piecewise linear maps.

Chapter 5

On the Fractional Helly property

Helly's Theorem 1.3 is a classical theorem in convex geometry. For finite families $\mathcal{F}$ of convex sets in $\mathbb{R}^d$:

> If the intersection of *all* $(d+1)$-*subsets* of $\mathcal{F}$ is non-empty, then the intersection of $\mathcal{F}$ is non-empty.

In 1979, Katchalski and Liu [44] extended this result to the Fractional Helly theorem for convex sets, based on the following observation:

> If the intersection of *a fraction* α *of all* $(d+1)$-*subsets* of $\mathcal{F}$ is non-empty, then the intersection of a *fraction* β *of* $\mathcal{F}$ is non-empty.

Here the second fraction $\beta = \beta(\alpha) > 0$ depends on the first fraction α, but not of $|\mathcal{F}|$. Let us formalize this: A finite or infinite family $\mathcal{F}$ of sets has *Fractional Helly number* k if for each $\alpha \in (0, 1]$ there is a $\beta = \beta(\alpha) > 0$ such that the following implication holds:

> For all $F_1, F_2, \ldots, F_n \in \mathcal{F}$ such that $\bigcap_{i \in I} F_i \neq \emptyset$ for at least $\lfloor \alpha \binom{n}{k} \rfloor$ many index sets $I \in \binom{[n]}{k}$, there exists a point which is in at least $\lfloor \beta n \rfloor$ of the sets F_i.

The family $\mathcal{K}^d$ of convex sets in $\mathbb{R}^d$ has Fractional Helly number $d+1$. There are two main tasks concerning the Fractional Helly property:

- For a family $\mathcal{F}$ find $\beta(\alpha) > 0$ optimal, as large as possible.
- Determine new families of sets that admit a Fractional Helly property. What is their Fractional Helly number?

In this chapter, we focus on the second problem. Recently, new Fractional Helly theorems have been derived using different approaches; see Alon et al. [2], Bárány and Matoušek [8], and Matoušek [47], [50]. In Section 5.3, we show a topological Fractional Helly theorem extending the result of Alon et al. [2]. Homological conditions imply the Fractional Helly property, and the proof is based on a spectral sequence argument. This topological Fractional

Helly theorem implies a topological (p, q)-theorem. Using the same approach, we obtain in Section 5.4 a short proof for the homology version of the Nerve Theorem 1.39 due to Björner. Section 5.1 comes with a crash course on spectral sequences.

Helly-type results can also be regarded as statements on intersection patterns of finite families of sets. More precisely, the intersection pattern for a finite family $\mathcal{F}$ is measured by its nerve complex $N(\mathcal{F})$. The classification of all possible f-vectors of nerve complexes of finite families of convex sets in $\mathbb{R}^d$, due to Eckhoff and Kalai, can be seen as a generalization of Helly's theorem. In Section 5.2, we review one of the key results: The upper bound theorem for d-Leray complexes. This result stands at the basis of Fractional Helly theorem for finite good covers in Alon et al. [2]. Its proof uses algebraic shifting – a technique from exterior algebra invented by Kalai for this purpose –, Alexander duality, and the characterization of f-vectors for Cohen-Macaulay complexes discovered by Stanley in 1975. In Section 5.5, we give partial answers to a question of Kalai and Matoušek towards a general result of the form:

> Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$ so that all non-empty intersections of at most q members have *topological complexity* bounded in k, then the Fractional Helly number of $\mathcal{F}$ is bounded k and d; here q depends on k and d.

We study the relation of the homological VC-dimension introduced by Kalai in [41], and the Fractional Helly property. Our topological Fractional Helly theorem from Section 5.3 fits well in this setting. Finally, we obtain partial results towards an upper bound theorem for the family of simplicial complexes of bounded homological VC-dimension.

5.1 SPECTRAL SEQUENCES

This section summarizes some facts on what is needed in the proofs of Sections 5.3 and 5.4. We start with some remarks on singular homology, and close this section with a crash course on spectral sequences.

On singular homology. A topological space X is *connected* if it is not the disjoint union of two non-empty open subsets. For *nice* topological spaces this definition coincides with concept of 0-connectedness, defined in Section 1.2. If X is a connected space, then one has $H_0(X, G) = G$ for singular homology with coefficients in any group G. The reduced singular homology of a contractible space vanishes in all dimensions.

For a family $\mathcal{F} = \{F_i \,|\, i \in I\}$ of subspaces we define the group of singular n-chains:

$$S_n\{\mathcal{F}\} = \mathbb{Z}^{\{\tau:\sigma^p \to \bigcup \mathcal{F} \,|\, \mathrm{im}(\tau) \subset F_i \text{ for some } i\}}$$

For finite families of open sets (or of subcomplexes of a cell complex) the inclusion of the singular chain groups $S_*\{\mathcal{F}\} \hookrightarrow S_*(\bigcup \mathcal{F})$ induces an isomorphism in homology. In the this section, we write $H_*(X) := H_*(X, \mathbb{Z})$ for the singular homology with integer coefficients of a topological space X. For simplicity we use for non-empty spaces X also the reduced singular homology groups $\tilde{H}_n(X)$, thus avoiding extra considerations for the case $n = 0$. Most of our work also holds for arbitrary (co)homology theories and arbitrary coefficients.

Anomalies of singular homology. Singular homology however shows anomalies first noticed by Barratt and Milnor [12]. Subspaces $A \subset \mathbb{R}^d$ which are *not nice* can have non-vanishing homology in infinitely many dimensions. For an example, let A be the union of countable many spheres of fixed dimension $r > 1$ all having one point in common with their diameter going to zero, then A is such a subspace which is not nice. For $r = 1$, this example is known as the *Hawaiian earring*, see also Figure 5.1. To exclude such not-nice phenomena we consider only families $\mathcal{F}$ of open sets in $\mathbb{R}^d$, and of subcomplexes of CW-complexes (cell complexes) in $\mathbb{R}^d$. In both cases, one has $H_n(\bigcup \mathcal{F}) = 0$ for all $n \geq d$.

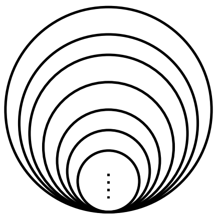

Figure 5.1: The Hawaiian earring.

Spectral sequence of a double complex. Spectral sequences are not standard tools in combinatorics, so we repeat some definitions from McCleary's textbook [51] on *spectral sequences for homology*; see also Basu [14], or Chow [23] for a short introduction. Let $C_{*,*}$ be a *double complex* with two differentials $\partial^I : C_{p,q} \to C_{p-1,q}$, and $\partial^{II} : C_{p,q} \to C_{p,q-1}$ such that $\partial^I \partial^{II} + \partial^{II} \partial^I = 0$. We associate to a double complex its *total complex* $\mathrm{Tot}(C)_n = \bigoplus_{p+q=n} C_{p,q}$ with differential $d := \partial^I + \partial^{II}$. The above relation on ∂^I and ∂^{II} implies $d \circ d = 0$. In this chapter, we are interested in first quadrant sequences so that $C_{p,q} = 0$ for $p < 0$ or $q < 0$. Before going into more details, we state the following key result on the homology $H_*(\mathrm{Tot}(C), d)$.

Theorem 5.1 (Spectral sequence of a double complex; [51, Theorem 2.15])**.** *Given a double complex $(C_{*,*}, \partial, \tilde{\partial})$ there are two spectral sequences $(E^r_{*,*}, d^r)$ and $(\tilde{E}^r_{*,*}, \tilde{d}^r)$ with*

$$E^2_{*,*} \cong H^{\partial}_{*,*} H^{\tilde{\partial}}(C) \quad \textit{and} \quad \tilde{E}^2_{*,*} \cong H^{\tilde{\partial}}_{*,*} H^{\partial}(C).$$

If $C_{p,q} = 0$ for $p < 0$ and $q < 0$ then both spectral sequences converge to $H_*(\mathrm{Tot}(C), d)$.

Here $H^{\partial}_*(C)$ stands for the homology of $C_{*,*}$ with respect to the differential ∂. The differential $\tilde{\partial}$ induces a differential map on $H^{\partial}_*(C)$ so that $H^{\tilde{\partial}}_{*,*} H^{\partial}(C)$ is well-defined. We now repeat in detail the construction of the spectral sequences to a given double complex.

Definition 5.2 (Spectral sequence of homological type)**.** A *spectral sequence* is a collection of differential bigraded modules, that is, for $r = 1, 2, 3, \ldots$, and $p, q \geq 0$ we have a module $E^r_{p,q}$, and furthermore differentials $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$. Finally, $E^{r+1}_{*,*} \cong H(E^r_{*,*}, d_r)$ for all $r \geq 1$.

Figure 5.2 shows the differentials d^r; for r big enough d^r hits the row 0. Let

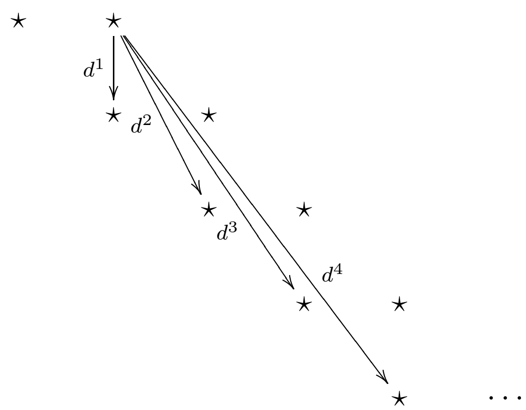

Figure 5.2: The differentials $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$.

$(C_{*,*}, \partial, \tilde{\partial})$ be a double complex with $\partial : C_{p,q} \to C_{p-1,q}$, $\tilde{\partial} : C_{p,q} \to C_{p,q-1}$, and total differential $d = \partial + \tilde{\partial}$. We define two filtrations of the total complex $(\mathrm{Tot}(C)_n, d)$ as follows:

$$F_m(\mathrm{Tot}(C)_n) = \bigoplus_{p \leq m} C_{p,n-p} \quad \text{and} \quad \tilde{F}_m(\mathrm{Tot}(C_n)) = \bigoplus_{q \leq m} C_{n-q,q}.$$

From now on, we outline the construction for the first filtration. This goes over to the construction for the second filtration by reindexing the double complex as its transpose $C^T_{p,q} = C_{q,p}$, $\partial^T = \tilde{\partial}$, $\tilde{\partial}^T = \partial$. The filtration is increasing, and it respects the total differential d:

$$0 \subset \cdots \subset F_{m-1}(\mathrm{Tot}(C)_n) \subset F_m(\mathrm{Tot}(C)_n) \subset \cdots \subset \mathrm{Tot}(C)_n$$

and,

$$d(F_m(\mathrm{Tot}(C)_n)) \subset F_m(\mathrm{Tot}(C)_n).$$

Since d respects the filtration, the homology of $(\mathrm{Tot}(C), d)$ inherits a filtration

$$0 \subset \cdots \subset F_{m-1}(H(\mathrm{Tot}(C), d)) \subset F_m(H(\mathrm{Tot}(C), d)) \subset \cdots \subset H(\mathrm{Tot}(C), d),$$

where $F_m(H(\mathrm{Tot}(C), d)) := \iota^m_*(H(F_m(\mathrm{Tot}(C)), d)) \subset H(\mathrm{Tot}(C), d)$, and $\iota^m : F_m(\mathrm{Tot}(C)) \hookrightarrow \mathrm{Tot}(C)$ is the inclusion. We know from Theorem 5.1 that the associated spectral sequence converges to $H(\mathrm{Tot}(C), d)$; more precisely, there is a $r > 0$ such that $d_r = 0$ is trivial. This is called *the sequence collapses at term* r. Therefore we have

$$E^{r+1}_{p,q} = E^\infty_{p,q} \cong F^p(H_{p+q}(\mathrm{Tot}(C), d))/F^{p-1}(H_{p+q}(\mathrm{Tot}(C), d)).$$

Hence, one gets $H_n(C, d) \cong \bigoplus_{p+q=n} E^\infty_{p,q}$ in the case of homology with field coefficients. In the case of integer coefficients this leads to an extension problem. In this paper the extension problem is trivial as only one of the groups $E^\infty_{p,q}$ with $n = p+q$ is different from zero, namely either $E^\infty_{n,0}$, or $E^\infty_{0,n}$. Therefore $H_n(\mathrm{Tot}(C), d)$ equals either $E^\infty_{n,0}$ or $E^\infty_{0,n}$.

Consider the following definitions for $r \geq 0$

$$\begin{aligned}
Z^r_{p,q} &= F_p(\mathrm{Tot}(C)_{p+q}) \cap d^{-1}(F_{p-r}(\mathrm{Tot}(C)_{p+q-1})) \\
B^r_{p,q} &= F_p(\mathrm{Tot}(C)_{p+q}) \cap d(F_{p+r}(\mathrm{Tot}(C)_{p+q+1})) \\
Z^\infty_{p,q} &= F_p(\mathrm{Tot}(C)_{p+q}) \cap \ker(d) \\
B^\infty_{p,q} &= F_p(\mathrm{Tot}(C)_{p+q}) \cap \mathrm{im}(d)
\end{aligned}$$

Then this leads to a tower of submodules

$$B^0_{p,q} \subset B^1_{p,q} \subset \cdots \subset B^\infty_{p,q} \subset Z^\infty_{p,q} \subset \cdots \subset Z^1_{p,q} \subset Z^0_{p,q}.$$

For a first quadrant sequence we get that for $r > \max\{p, q\}$

$$Z^r_{p,q} = Z^\infty_{p,q} \ \text{, and} \quad B^r_{p,q} = B^\infty_{p,q}$$

hold. This guarantees the convergence of our spectral sequence. Define

$$E^r_{p,q} = Z^r_{p,q}/(Z^{r-1}_{p-1,q} + B^{r-1}_{p,q}).$$

It can be checked that the differential $d|_{Z^r_{p,q}} : Z^r_{p,q} \to Z^r_{p-r,q+r-1}$ induces a unique homomorphism d^r such that the following diagram commutes:

$$\begin{tikzcd}
Z^r_{p,q} \arrow[r, "d"] \arrow[d, "\eta^r_{p,q}"'] & Z^r_{p-r,q+r-1} \arrow[d, "\eta^r_{p-r,q+r-1}"] \\
E^r_{p,q} \arrow[r, "d^r"] & E^r_{p-r,q+r-1}
\end{tikzcd}$$

where the maps $\eta^r_{p,q} : Z^r_{p,q} \to E^r_{p,q}$ are the canonical projections. The same constructions leads to the spectral sequence $(\tilde{E}^r, \tilde{d}^r)$, $\tilde{d}^r : \tilde{E}^r_{p,q} \to \tilde{E}^r_{p+r-1,q-r}$, for the second filtration.

5.2 d-Leray complexes, algebraic shifting, and an upper bound theorem

This section is devoted to the upper bound theorem for d-Leray complexes which is the key result for the Fractional Helly theorem for d^*-Leray families in Section 5.3. We give in fact both proofs suggested in [2] as we like each of them. On our way we introduce the necessary tools, e. g. algebraic shifting. This also serves us as a warm-up for Section 5.5. Apart from the very first definition, the content of this section is not needed for understanding Sections 5.3, and 5.4.

Definition 5.3. A simplicial complex K is *d-Leray* if $H_i(\mathsf{L}) = 0$ for all $i \geq d$, and for all induced subcomplexes $\mathsf{L} \subset \mathsf{K}$.

From now on, $H_i(\mathsf{L})$, and $\tilde{H}_i(\mathsf{L})$ denote (simplicial) homology with field coefficients. Nerve complexes of finite families of convex sets in $\mathbb{R}^d$ are d-Leray. More generally, nerve complexes of finite families of open sets (,or of subcomplexes of a cell complex) in $\mathbb{R}^d$ such that any non-empty intersection of members of the family is contractible, are d-Leray. This follows from the Nerve Theorem 1.38, and the fact that induced subcomplexes of the nerve of a family $\mathcal{F}$ of sets are just nerve complexes of subfamilies of $\mathcal{F}$. It is common to introduce the d-Leray property in a different equivalent form.

Lemma 5.4. *For any simplicial complex K the following equivalence holds:*

$$\mathsf{K} \text{ is } d\text{-Leray.} \iff \tilde{H}_i(\operatorname{lk}(F, \mathsf{K})) = 0 \text{ for all } i \geq d, \text{ and all faces } F \in \mathsf{K}.$$

Taking the link of a face $F = \{S \subset [n] \mid \bigcap_{j \in S} C_j\}$ in the nerve of a finite family $\mathcal{F} = \{C_1, C_2, \ldots, C_n\}$ of convex sets in $\mathbb{R}^d$ corresponds to looking at the nerve of the finite family

$$\{C_i \cap \bigcap_{j \in S} C_j \mid i \in [n] \setminus S \neq \emptyset\}$$

of convex sets in $\mathbb{R}^d$. A proof of Lemma 5.4 has recently been published by Kalai and Meshulam [42]. The proof uses (implicitly) the result of Bayer et al. [16] below, and induction on $|V|$.

Theorem 5.5. *Let K be a simplicial complex on the vertex set V. For each subset $W \subset V$ and $v \in V \setminus W$ there is a long exact sequence in homology*

$$\cdots \to \tilde{H}_i(\mathsf{K}[W]) \to \tilde{H}_i(\mathsf{K}[W \cup \{v\}]) \to \tilde{H}_{i-1}(\mathrm{lk}(v, \mathsf{K}[W \cup \{v\}])) \to \cdots$$

For faces $F \in \mathsf{K}$, such that $F \cap W = \emptyset$ and $v \notin F$, there is a long exact sequence in homology

$$\cdots \to \tilde{H}_i(\mathrm{lk}(F, K[W \cup F])) \to \tilde{H}_i(\mathrm{lk}(F, \mathsf{K}[W \cup F \cup \{v\}])) \to$$
$$\tilde{H}_{i-1}(\mathrm{lk}(F \cup \{v\}, \mathsf{K}[W \cup F \cup \{v\}])) \to \cdots$$

These long exact sequences can be interpreted as Mayer-Vietoris sequences of suitable pairs. The second sequence follows from the first by using the simplicial complex $\mathrm{lk}(F, \mathsf{K})[W \cup F]$ instead of $\mathsf{K}[W]$.

The upper bound theorem for families of convex sets was conjectured by Perles and Katchalski, and it was settled independently by Kalai [38] and by Eckhoff [28]. Later, Kalai generalized it as follows.

Theorem 5.6 (Upper bound theorem for d-Leray complexes)**.** *Let L a be d-Leray complex on n vertices, and $r \geq 1$. Then the following holds:*

$$f_d(\mathsf{L}) > \binom{n}{d+1} - \binom{n-r}{d+1} \Rightarrow f_{d+r}(\mathsf{L}) > 0. \tag{5.1}$$

Kalai's proof uses the technique of *algebraic shifting* which also works in this more general setting. More recently, Kalai [40] discovered another proof of Theorem 5.6 which is based on Alexander duality, and on the characterization of face numbers for Cohen-Macaulay complexes due to Stanley in 1975.

Let's start with the more recent proof of Theorem 5.6. One our way, we recall definitions and results needed in the proof.

Proof 1. (of Theorem 5.6) We can assume without loss of generality that the d-Leray complex L contains the complete $(d-1)$-skeleton. For this let $\tilde{\mathsf{L}}$ be the simplicial complex that is obtained from L by adding the complete $(d-1)$-skeleton. Then one immediately has:

- $f_i(\mathsf{L}) = f_i(\tilde{\mathsf{L}})$ for $i > d-1$, and
- $\tilde{\mathsf{L}}$ is d-Leray, as L and $\tilde{\mathsf{L}}$ coincide in dimensions greater than $d-1$.

It is thus sufficient to show Statement (5.1) for d-Leray complexes with the complete $(d-1)$-skeleton. Statement (5.1) holds if L is the simplex on n vertices. We prove Theorem 5.6 in two steps.

Step 1. In a first step, we apply the (combinatorial) Alexander duality to L. Its dual complex L^* is Cohen-Macaulay, and Statement (5.1) translates into:

$$f_{D-r}(\mathsf{L}^*) = \binom{n}{D-r+1} \Rightarrow f_D(\mathsf{L}^*) \geq \binom{n-r}{D-r+1} \tag{5.2}$$

for some $D \leq \dim(\mathsf{L}^*)$. Here a simplicial complex K is *Cohen-Macaulay* if the following condition holds for all faces $F \in \mathsf{K}$:

$$\tilde{H}_i(\mathrm{lk}(F, \mathsf{K})) = 0 \;\text{ for all } i < \dim(\mathrm{lk}(F, \mathsf{K})). \tag{5.3}$$

Definition 5.7 (Blocker dual)**.** Let K be a simplicial complex with vertex set $[n]$. The simplicial complex $\mathsf{K}^* = \{S \subset [n] \,|\, [n] \setminus S \notin \mathsf{K}\}$ is the *blocker dual of* K.

In our proof, we need some properties of the blocker dual.

Proposition 5.8 (Properties of the dual)**.** *Let K be a simplicial complex with vertex set $[n]$. Then K^* satisfies the following properties:*

i) $(\mathsf{K}^*)^* = \mathsf{K}$.

ii) $f_j(\mathsf{K}) + f_{n-2-j}(\mathsf{K}^*) = \binom{n}{j+1}$.

iii) *Alexander duality: If $[n] \notin \mathsf{K}$. Then*

$$\tilde{H}_i(\mathsf{K}^*) \cong \tilde{H}_{n-i-3}(\mathsf{K}) \;\textit{ for all }\; -1 \leq i \leq n-2.$$

iv) *If $[n] \notin \mathsf{K}$, and $S \subset [n]$ with $S \notin \mathsf{K}^*$, then*

$$\tilde{H}_i(\mathsf{K}^*[S]) \cong \tilde{H}_{|S|-i-3}(\mathrm{lk}(\bar{S}, \mathsf{K})) \;\textit{ for all }\; -1 \leq i \leq |S|-2.$$

Properties i) and ii) immediately follow from Definition 5.7. Property iii) is a combinatorial version of Alexander duality for simplicial complexes, see Meshulam and Kalai [43]. Property iv) is a consequence of Property iii).

In this paragraph, we apply Alexander duality to show that L^* is Cohen-Macaulay. We show that Property (5.3) holds for all faces $F \in \mathsf{L}^*$. For this let $\bar{F}$ be the complement of F in $[n]$. Using that L is d-Leray and Property iv) of the blocker dual, one obtains

$$\tilde{H}_j(\mathrm{lk}(F, \mathsf{L}^*)) = 0 \;\text{ for all }\; -1 \leq j \leq |\bar{F}| - d - 3.$$

The fact that L contains the complete $(d-1)$-skeleton implies for the dimension of L^*:

$$\dim(\mathsf{L}^*) \leq n - d - 2.$$

The missing part $\dim(\mathrm{lk}(F, \mathsf{L}^*)) \leq |\bar{F}| - d - 4$ follows from the inequality

$$\dim(F) + \dim(\mathrm{lk}(F, \mathsf{L}^*)) \leq \dim(\mathsf{L}^*) - 1,$$

which holds for arbitrary simplicial complexes.

Finally, Statement (5.2) is derived from Statement (5.1) using Property ii) of the blocker dual, and by setting $D = n - d - 2$.

Step 2. We use the classification for face numbers of Cohen-Macaulay complexes due to Stanley, see e. g. Stanley's textbook [64, Theorem 3.3]. Let's first introduce the h-vector of a simplicial complex, see also Ziegler's textbook [76].

Definition 5.9 (h-vector of a simplicial complex)**.** Let K be a simplicial complex of dimension $d-1$. The *h-vector of* K is

$$h(\mathsf{K}) = (h_0, h_1, \ldots, h_d) \in \mathbb{Z}^{d+1},$$

given by the formula

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}(\mathsf{K}),$$

where $f_i(\mathsf{K})$ is the number of i-faces of K, and $f_{-1}(\mathsf{K}) = 1$ by convention.

The h-vector of K can also be computed by a difference table, a variant of Pascal's triangle, known as Stanley's trick. For this write the numbers f_i (instead of $1 = \binom{d+1}{d+1}$) to the last entries of the first d rows of Pascal's triangle, and 1 into the first entry of each row. Then we compute as usual in Pascal's triangle:

$$\text{upper right neighbor} - \text{upper left neighbor}. \tag{5.4}$$

This process outputs in row $d+1$ the h-vector, an integer vector of length $d+1$.

In our proof, we need the following result on h-vectors of Cohen-Macaulay complexes.

Theorem 5.10. *If K is a Cohen-Macaulay simplex, then the h-vector $h(\mathsf{K})$ has non-negative entries.*

By assumption of Statement (5.2), the Cohen-Macaulay complex L^* has the complete $(D-r)$-skeleton. Let's look at the first $D+2$ rows for the computation of the h-vector of L^* via Stanley's trick:

$$\begin{array}{ccccccc}
1 & & & & & & \\
1 & f_0 & & & & & \\
1 & n-1 & f_1 & & & & \\
\vdots & \vdots & \vdots & \ddots & & & \\
1 & n-D-r & \binom{n-D-r+1}{2} & \cdots & f_{D-r} & & \\
\vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \\
1 & n-D & \binom{n-D+1}{2} & \cdots & \binom{n-r}{D-r+1} & \cdots & f_D
\end{array}$$

where $f_i = f_i(\mathsf{L}^*) = \binom{n}{i+1}$ for $i \leq D-r$. The entry of the $(D-r+2)$-nd column in the last row is equal to $\binom{n-r}{D-r+1}$. This follows from the usual computation rule (5.4), and the binomial identity:

$$\binom{n-1}{i-1} + \binom{n-1}{i} = \binom{n}{i}$$

The case $D = \dim(\mathsf{L}^*)$ of Statement (5.2) holds as the next rows contains the non-negative h-vector of L^*. The entries of the $(D+3)$-rd row, which are on the left of $\binom{n-r}{D-r+1}$, are thus weakly increasing.

The general case $D \leq \dim(\mathsf{L}^*)$ follows similarly: If the entries on the left of $\binom{n-r}{D-r+1}$ are not weakly increasing, then negative entries show up in the $(D+3)$-rd row. The left-most negative entry propagates until the last row which contains the non-negative h-vector. $\square$

The second proof uses the technique of *algebraic shifting*. This technique has first been applied by Kalai [37], [39] to prove Eckhoff's far-reaching h-Vermutung[1]. The h-Vermutung implies the Upper Bound Theorem 5.6 for d-Leray complexes. Since then algebraic shifting has been applied successfully in other settings. Björner and Kalai [20] have used algebraic shifting to prove their extended Euler-Poincaré theorem.

Proof 2. (of Theorem 5.6) As in Proof 1, we start with some notation and background following Eckhoff [27]. Let's define the h-vector of a d-Leray complex.

[1] The naming "h-Vermutung" is due to its formal similarity to McMullen's g-conjecture; see Ziegler's textbook [76] for the g-conjecture, and its consequences.

Definition 5.11 (h-vector of a d-Leray complex)**.** Let L be a d-Leray complex. Then define its *h-vector* $h(\mathsf{L}) = (h_0, h_1, h_2, \ldots)$:

$$h_k(\mathsf{L}) = \begin{cases} f_k(\mathsf{L}) & , k = 0, 1, \ldots, d-1, \\ \sum_{j \geq 0} (-1)^j \binom{k+j-d}{j} f_{k+j}(\mathsf{L}) & , k \geq d. \end{cases}$$

This definition of h-vectors differs from Definition 5.9 in Proof 1 as it depends on d, and $h_k(\mathsf{L}) = 0$ for all $k \geq \dim(\mathsf{L})$. However, both objects have formal similarities. The f-vector entries f_k can inversely be expressed in terms of the h-vector:

$$f_k(\mathsf{L}) = \begin{cases} h_k(\mathsf{L}) & , k = 0, 1, \ldots, d-1, \\ \sum_{j \geq 0} \binom{k+j-d}{j} h_{k+j}(\mathsf{L}) & , k \geq d. \end{cases} \tag{5.5}$$

For every non-negative integer k, every natural number n has a unique representation of the form:

$$n = \binom{a_{k+1}}{k+1} + \binom{a_k}{k} + \cdots + \binom{a_i}{i},$$

such that $a_{k+1} > a_k > \cdots > a_i \geq i \geq 1$. Let's define

$$n^{(k+1)} = \binom{a_{k+1}}{k} + \binom{a_k}{k-1} + \cdots + \binom{a_i}{i-1},$$

and put $0^{(k+1)} = 0$. Let's state Eckhoff's h-Vermutung from [27].

Theorem 5.12 (Eckhoff's h-Vermutung)**.** *An integer vector $(h_0, h_1, \ldots)$ is the h-vector of a d-Leray complex if and only if the following inequalities hold:*

$$h_k \geq 0 \quad , \textit{for } k \geq 0. \tag{5.6}$$

$$h_k^{(k+1)} \leq h_{k-1} \quad , \textit{for } k \in [d-1]. \tag{5.7}$$

$$h_k^{(d)} \leq h_{k-1} - h_k \quad , \textit{for } k \geq d. \tag{5.8}$$

In the case of nerve complexes of families of convex sets in $\mathbb{R}^d$, this theorem is due to Kalai [37], [39]. Simplicial complexes that can be obtained as nerve complexes of families of convex sets in $\mathbb{R}^d$ are called *d-representable complexes*. In its general version for d-Leray complexes, it was proved by Kalai using algebraic shifting. Before we sketch a proof of the "only if"-part, let's verify that it implies the Upper Bound Theorem 5.6 for d-Leray complexes.

Inequality (5.8) and induction on k lead to

$$h_k \leq \binom{n-k+d-1}{d}, \text{ for } k \geq d. \tag{5.9}$$

Hence $h_k = 0$ for $k \geq h_0 = n$. Instead of Statement (5.2) we show for a d-Leray complex that

$$f_{d+r}(\mathsf{L}) = 0 \Rightarrow f_d(\mathsf{L}) \leq \binom{n}{d+1} - \binom{n-r}{d+1}.$$

Assume $f_{d+r}(\mathsf{L}) = 0$ so that $h_k(\mathsf{L}) = 0$ for all $k \geq d + r$.

$$f_d(\mathsf{L}) \overset{(5.5)}{=} \sum_{j=d}^{d+r-1} h_j(\mathsf{L}) \overset{(5.9)}{\leq} \sum_{j=d}^{d+r-1} \binom{n-j+d-1}{d} = \binom{n}{d+1} - \binom{n-r}{d+1}$$

Here the last equation is obtained via the binomial identity:

$$\binom{n-j+d-1}{d} = \binom{n-j+d}{d+1} - \binom{n-j+d-1}{d+1}.$$

□

Algebraic Shifting. The remaining part of this section is devoted to algebraic shifting, see also Kalai [40] for an introduction. As a highlight, we outline the proof of the "only if"-part of the h-Vermutung 5.12. Another good starting point is Kalai [38] where exterior algebraic shifting is used for the first time to classify the f-vector of d-representable complexes.

Algebraic shifting associates to a given simplicial complex K a shifted simplicial complex $\Delta(\mathsf{K})$. The construction of $\Delta(\mathsf{K})$ is algebraic. We refer to [40] for the explicit construction based on exterior (or symmetric) algebra. We will see below that algebraic shifting preserves various topological properties of K, e. g. Betti numbers, Cohen-Macaulay, d-Leray.

Definition 5.13 (Shiftedness)**.** A family $\mathcal{A}$ of k-element sets of positive integers is *shifted* if whenever $S \in \mathcal{A}$ holds and R is obtained from S by replacing an element with a smaller element, then $R \in \mathcal{A}$. A simplicial complex with vertex set $[n]$ is *shifted* if all sets of k-faces are shifted for all k.

If a shifted family $\mathcal{A}$ contains $\{2, 4\}$ then it also contains $\{1, 4\}$, $\{2, 3\}$, $\{1, 3\}$, and $\{1, 2\}$. Let's summarize some properties of algebraic shifting in a theorem.

Theorem 5.14 (Properties of algebraic shifting)**.** *Let* K *be a simplicial complex, and let* $\Delta(\mathsf{K})$ *be the algebraic shifting of* K. *Then the following properties hold:*

i) $\Delta(\mathsf{K})$ *is a shifted simplicial complex.*

ii) f*-vector:* $f_i(\mathsf{K}) = f_i(\Delta(\mathsf{K}))$ *for all* i.

iii) *Betti numbers:*

$$\beta_i(\mathsf{K}) = \beta_i(\Delta(\mathsf{K})) = |\{S \in \Delta(\mathsf{K}) \,:\, |S| = i+1,\ S \cup \{1\} \notin \Delta(\mathsf{K})\}| \text{ for all } i.$$

iv) K *is Cohen-Macaulay* $\Rightarrow$ $\Delta(\mathsf{K})$ *is Cohen-Macaulay.*

v) K *is* d*-Leray* $\Rightarrow$ $\Delta(\mathsf{K})$ *is* d*-Leray.*

The proof of Theorem 5.14 is again algebraic, see Kalai [40] for references. Both Properties iv) and v) follow from the following theorem by Bayer et. al [16] for symmetric shifting, and by Aramova and Herzog [36] for exterior shifting.

Theorem 5.15. *Let* $i, j \geq 0$, K *be a simplicial complex. Assume that*

$$\beta_k(\mathrm{lk}(F, \mathsf{K})) = 0,$$

for all $|F| = t < j$, *and* $i \leq k \leq i + j - t$. *Then also*

$$\beta_k(\mathrm{lk}(F, \Delta(\mathsf{K}))) = 0,$$

for all $|F| = t < j$, *and* $i \leq k \leq i + j - t$, *and the quantity*

$$\sum_{|G|=j} \beta_i(\mathrm{lk}(G, \mathsf{K}))$$

is preserved under shifting.

The proof of Theorem 5.15 is based on ring-theoretic properties of classical face rings. Let's prove Property v) of algebraic shifting.

Proof. (of Theorem 5.14.v)) Let K be a d-Leray complex, then Lemma 5.4 implies that

$$\beta_k(\mathrm{lk}(F, \mathsf{K})) = 0,$$

for all $F \in \mathsf{K}$, and all $k \geq d$. Now Theorem 5.14.v) follows from Theorem 5.15 by choosing $i = d$, and j sufficiently large. □

The proof of the "only if"-part of Eckhoff's Conjecture is based on the following two results due to Kalai from [37], and [40].

Theorem 5.16. *Let $d \geq 1$, and K be a simplicial complex satisfying*

$$H^{k+1}_{d+k}(\mathsf{K}) \textit{ for all } k \geq 0,$$

then its h-vector $h(\mathsf{K})$ satisfies the inequalities (5.6)– (5.8).

Here we use Definition 5.11 of the h-vector. Moreover, $H^r_k(\mathsf{K})$ denotes the (algebraically defined) r-th iterated cohomology of a simplicial complex K. We use the following theorem to compute the r-th iterated cohomology.

Theorem 5.17. *Let K be a simplicial complex, and $r, k \geq 0$. Then*

$$H^r_k(\mathsf{K}) = |\{S \in \Delta(\mathsf{K}) \,:\, |S| = k+1,\, S \cap [r] = \emptyset,\, S \cup [r] \notin \Delta(\mathsf{K})\}|.$$

We refer again to Kalai [40] for a proof.

Proof. (of the "only if"-part of Eckhoff's h-Conjecture 5.12) Let L be a d-Leray on the vertex set $[n]$, then the algebraic shifting $\Delta(\mathsf{L})$ is also d-Leray. Using Theorems 5.16 and 5.17, it remains to show for all $k \geq 0$ that

$$|\{S \in \Delta(\mathsf{L}) \,:\, |S| = d+k+1,\, S \cap [k+1] = \emptyset,\, S \cup [k+1] \notin \Delta(\mathsf{L})\}| = 0.$$

Let $k = 0$. If there is an $S \in \Delta(\mathsf{L})$ such that $|S| = d+1$, $S \cap [1] = \emptyset$, and $S \cup [1] \notin \Delta(\mathsf{L})$, then $S \cup [1] \notin \Delta(\mathsf{L})$ implies via Property 5.14.iii) that $0 \neq \beta_d(\Delta(\mathsf{L}))$. This contradicts that $\Delta(\mathsf{L})$ is d-Leray, hence $H^1_d(\mathsf{L}) = 0$.

Let $k = 1$, and assume that there is an $S \in \Delta(\mathsf{L})$ such that $|S| = d+2$, $S \cap [2] = \emptyset$, and $S \cup [2] \notin \Delta(\mathsf{L})$. Shiftedness of $\Delta(\mathsf{L})$ implies that all proper faces of $S \cup \{2\}$ are in $\Delta(\mathsf{L})$. The induced subcomplex $\Delta(\mathsf{L})[S \cup \{2\}]$ coincides thus with the $(d+2)$-dimensional simplex with vertex set $S \cup \{2\}$ up to dimension $d+1$. It is therefore sufficient to show $S \cup \{2\} \notin \Delta(\mathsf{L})$ as $\Delta(\mathsf{L})$ is d-Leray. If $S \cup \{2\} \in \Delta(\mathsf{L})$ holds, then $S \cup \{1\} \in \Delta(\mathsf{L})$ via shiftedness. Hence all proper faces of $S \cup [2]$ are in $\Delta(\mathsf{L})[S \cup [2]]$ so that $\Delta(\mathsf{L})[S \cup [2]]$ is the boundary of the $(d+3)$-dimensional simplex. This contradicts the fact that $\Delta(\mathsf{L})$ is d-Leray.

Let $k \geq 1$ be arbitrary. Assume that there is an S such that $|S| = d+k+1$, $S \cap [k+1] = \emptyset$, and $S \cup [k+1] \notin \Delta(\mathsf{L})$. Then the shiftedness of $\Delta(\mathsf{L})$ implies – as for $k = 1$ – the existence of a subset $W \subset [n]$ such that $|W| \geq d+3$ and $\Delta(\mathsf{L})[W]$ is equal to the boundary of the simplex with vertex set W – again a contradiction. $\square$

5.3 On a topological Fractional Helly theorem

The original Fractional Helly theorem for convex sets in $\mathbb{R}^d$ by Katchalski and Liu [44] can now be stated in the following way.

Theorem 5.18 (Fractional Helly theorem for convex sets)**.** *For every $d \geq 1$, the family of convex sets in $\mathbb{R}^d$ has Fractional Helly number $d+1$.*

This result was generalized by Alon et al. [2] to *good covers* in $\mathbb{R}^d$ where a finite family $\mathcal{F}$ of sets in whose members are either all open or all closed, is a *good cover* if $\bigcap \mathcal{G}$ is either empty or contractible for all subfamilies $\mathcal{G} \subset \mathcal{F}$. Moreover, it was shown by Kalai [38] that the optimal $\beta(\alpha)$ equals $1-(1-\alpha)^{1/(d+1)}$, so that the case $\alpha = 1$ implies Helly's Theorem 1.3. Their proof of the Fractional Helly theorem for good covers uses the nerve theorem, and the following proposition for d^*-Leray families. Here a family $\mathcal{F}$ is called *d^*-Leray* if its nerve complex $N(\mathcal{F})$ is d-Leray.

Theorem 5.19 (Fractional Helly theorem for d^*-Leray families)**.** *Let $\mathcal{F}$ be a finite d^*-Leray family, and let $\mathcal{F}^{\cap}$ the family of all intersections of the sets of $\mathcal{F}$. Then $\mathcal{F}^{\cap}$ has Fractional Helly number $d+1$. Moreover, one can choose $\beta(\alpha) = 1-(1-\alpha)^{1/(d+1)}$.*

As a special case, the Fractional Helly number of $\mathcal{F}$, and its Helly number are equal to d. The Fractional Helly theorem for good covers plays a key role in proving one of the main results of Alon et al. [2], a (p,q)-theorem for finite good covers.

Here we identify topological/homological conditions – that depend on k – for families of sets which imply that their nerve complex is k-Leray. Having Proposition 5.19 in mind, we extend the Fractional Helly theorem for good covers to families with higher topological intersection complexity; see also Figure 5.4 for the relations between the results.

Theorem 5.20 (Topological Fractional Helly theorem)**.** *Let $\mathcal{F}$ be a finite family of open sets (or of subcomplexes of a cell complex) in $\mathbb{R}^d$, and $k \geq d$ such that for all subfamilies $\mathcal{G} \subset \mathcal{F}$ one of the following conditions holds:*

i) *$\bigcap \mathcal{G}$ is empty, or*

ii) *the reduced homology groups of $\bigcap \mathcal{G}$ vanish in dimension at least $k-|G|$, that is*
$$\tilde{H}_n\left(\bigcap \mathcal{G}\right) = 0 \text{ for all } n \geq k - |\mathcal{G}|.$$

Then $\mathcal{F}^{\cap}$ has Fractional Helly number $k+1$. Moreover, we can choose
$$\beta(\alpha) = 1-(1-\alpha)^{1/(k+1)}.$$

As in the case of good covers this implies Fractional Helly number $k+1$ for the family $\mathcal{F}$. The case $k = d$ implies the Fractional Helly theorem for good covers. We call a family of sets as in Theorem 5.20 satisfying conditions i) and ii) a *$k_{\mathcal{G}}$-acyclic family.*

The proof of Theorem 5.20 uses a spectral sequence argument. Table 5.3 compares the condition for good covers, and $d_{\mathcal{G}}$-acyclic families that imply Fractional Helly number $d+1$. The first column shows the conditions in the good cover case: All non-empty intersections are contractible, so their homology vanishes in all dimensions. In the second column the conditions in the $d_{\mathcal{G}}$-acyclic case are shown: Non-empty intersections of $i < d$ sets can have arbitrary homology groups in dimension less or equal than $d-i-1$.

	good cover	$d_{\mathcal{G}}$-acyclic
$\tilde{H}_n(F_i)$	0 for all $n \geq 0$	$\begin{cases} 0 & \text{for all } n \geq d-1, \\ \text{arbit.} & \text{for } n = 0, 1, \ldots, d-2. \end{cases}$
$\tilde{H}_n(F_{i_1} \cap F_{i_2})$	0 for all $n \geq 0$	$\begin{cases} 0 & \text{for all } n \geq d-2, \\ \text{arbit.} & \text{for } n = 0, 1, \ldots, d-3. \end{cases}$
$\vdots$	$\vdots$	$\vdots$
$\tilde{H}_n(F_{i_1} \cap F_{i_{d-1}})$	0 for all $n \geq 0$	$\begin{cases} 0 & \text{for all } n \geq 1, \\ \text{arbit.} & \text{for } n = 0. \end{cases}$
$\tilde{H}_n(F_{i_1} \cap \cdots \cap F_{i_t})$	0 for all $n \geq 0$	0 for all $n \geq 0$ and $t \geq d$.

Figure 5.3: The homological conditions for Fractional Helly number $d+1$.

The case $k > d$ admits even more general families of sets in $\mathbb{R}^d$. The price one has to pay for increasing the intersection complexity of the F_i is a higher Fractional Helly number. See Figure 5.5 for an example of the intersection pattern of $k = 3$ sets of a $3_{\mathcal{G}}$-acyclic family in $\mathbb{R}^2$. There the F_{i_j} can have an arbitrary number of 0- and of 1-dimensional holes. The intersection of two elements $F_{i_j} \cap F_{i_k}$ of our family still can have an arbitrary number of 0-dimensional holes.

Matoušek [50] showed a Fractional Helly Theorem 5.27 for families with bounded VC-dimension, see also Section 5.5. Matoušek's results is not a special case of our result. Bounded VC-dimension does not guarantee any Helly property, e. g. the family $\{[n] \setminus \{i\} \mid i \in [n]\}$ has bounded VC-dimension, but no Helly property. Another important example of families with bounded VC-dimension is the family of all semialgebraic subsets in $\mathbb{R}^d$ of bounded description complexity. Let's look at a concrete example: Define a semialgebraic set $F_i = \{x \in \mathbb{R}^d \mid x_1^2 + x_2^2 - i \geq 0\}$, then the family $\{F_1, F_2, \ldots, F_n\}$ has Fractional Helly number $d+1$. However, we have $\tilde{H}_1(\bigcap_{i \in I} F_i) = \mathbb{Z} \neq 0$ for all index sets $\emptyset \neq I \subset [n]$.

Bárány and Matoušek [8] showed that the family of convex lattice sets in $\mathbb{Z}^d$ has Fractional Helly number $d+1$, using a Ramsey-type argument. We can not hope to obtain directly the same Fractional Helly number for families of convex lattice sets in $\mathbb{Z}^d$ using the topological Fractional Helly Theorem 5.20 for $k_{\mathcal{G}}$-acyclic families as its Helly number of the is known to be 2^d.

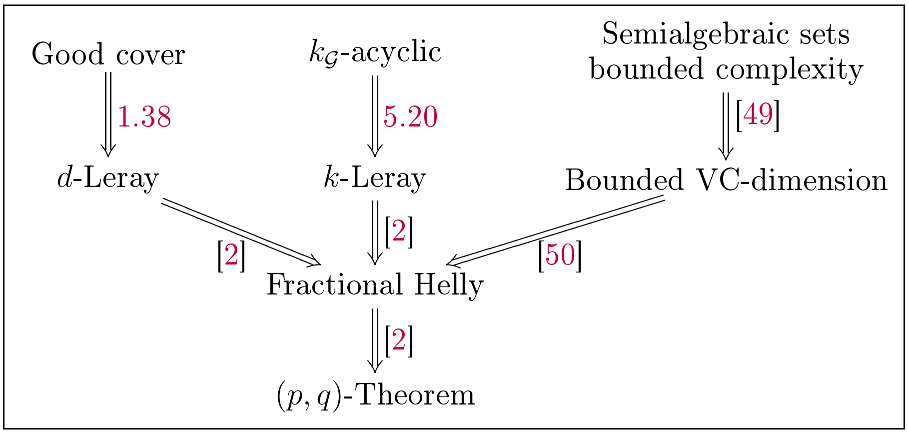

Figure 5.4: Diagram of the abstract machinery.

The (p,q)-theorem for convex sets was conjectured by Hadwiger and Debrunner, and proved by Alon and Kleitman [3], see also Eckhoff [30] for a recent survey. For this let p, q, d be integers with $p \geq q \geq d+1 \geq 2$. Then there exists a number $\mathrm{HD}(p,q,d)$ such that the following holds: Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ satisfying the (p,q)-condition; that is, among any p sets of $\mathcal{F}$, there are q sets with a non-empty intersection. Then $\tau(\mathcal{F}) \leq \mathrm{HD}(p,q,d)$, where $\tau(\mathcal{F})$ denotes the *transversal number* of $\mathcal{F}$, i. e. the smallest cardinality of a set $X \subset \bigcup \mathcal{F}$ such that $F \cap X \neq \emptyset$ for all $F \in \mathcal{F}$. It was observed by Alon et al. [2] that the crucial ingredient in the proof is a Fractional Helly theorem for the family $\mathcal{F}^{\cap}$. Therefore Theorem 5.20 immediately implies a new (p,q)-theorem using the general tools developed in Alon et al. [2].

Theorem 5.21 ((p,q)-theorem for $k_{\mathcal{G}}$-acyclic families)**.** *The assertions of the (p,q)-theorem also hold for finite $k_{\mathcal{G}}$-acyclic families of open sets (or of subcomplexes of a cell complex) in $\mathbb{R}^d$ where $p \geq q \geq k \geq d+1 \geq 2$.*

Theorem 5.21 implies the (p, q)-theorem for good covers of Alon et al. [2].

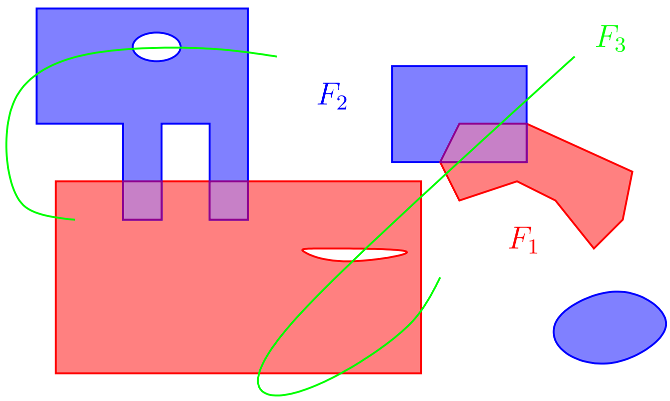

Figure 5.5: Example of $k = 3$ sets of a $3_{\mathcal{G}}$-acyclic family in $\mathbb{R}^2$.

The following lemma is the key argument in the proof of the topological Fractional Helly theorem for $k_{\mathcal{G}}$-acyclic families.

Lemma 5.22. *For $k \geq 0$, let $\mathcal{F}$ be a finite $k_{\mathcal{G}}$-acyclic family of open sets (or of subcomplexes of a cell complex) in $\mathbb{R}^d$. Then*

$$H_n(\bigcup \mathcal{F}) \cong H_n(N(\mathcal{F})) \text{ for all } n \geq k.$$

To prove Lemma 5.22 we first define a suitable double complex $(C_{*,*}, \partial, \tilde{\partial})$. Then we compute its $E^2_{*,*}$- and $\tilde{E}^2_{*,*}$-term which are shown in Figures 5.6 and 5.7. Finally, we apply Theorem 5.1 to get the conclusion.

Proof. For $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$ define a double complex

$$C_{p,q} := \bigoplus_{J \subset [n],\, |J| = q+1} S_p(\bigcap_{j \in J} F_j) \quad \text{for } p, q \geq 0.$$

Let $\partial := \oplus \partial : C_{p,q} \to C_{p-1,q}$ be the usual singular boundary operator. For index sets $J' \subset J \subset [n]$ let $i^{J,J'} : \bigcap_{j \in J} F_j \to \bigcap_{j \in J'} F_j$ be the inclusion. The group $S_p(\bigcap_{j \in J} F_j)$ is freely generated by the set of singular p-simplices $\sigma : \Delta^p \to \bigcap_{j \in J} F_j$. Define $\tilde{\partial}$ component-wise on the elements $c = \sum r_\sigma \sigma$ of $S_p(\bigcap_{j \in J} F_j)$:

$$c \mapsto \tilde{\partial}(\sigma) := (-1)^p \sum_{\sigma} \sum_{i=0}^{q} (-1)^i r_\sigma i_*^{J,J_i}(\sigma) \;\in \bigoplus_{J \subset [n], |J| = (q-1)+1} S_p(\bigcap_{j \in J} F_j),$$

where $J_i = \{j_0 < j_1 < \ldots < \hat{j}_i < \cdots < j_q\}$ is the set obtained from J by deleting the element j_i, and the factor $(-1)^p$ is added to guarantee $\partial\tilde{\partial} + \tilde{\partial}\partial = 0$.

We show that (i)

$$E^2_{p,q} = H^{\partial}_{p,q} H^{\tilde{\partial}}(C) = \begin{cases} H_p(\bigcup \mathcal{F}) & \text{for } q = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and that (ii)

$$\tilde{E}^2_{p,q} = H^{\tilde{\partial}}_{p,q} H^{\partial}(C) = \begin{cases} H_q(N(\mathcal{F})) & \text{for } p = 0,\ q \geq k \\ 0 & \text{for } p = i,\ q = n - i, \text{ for all} \\ & 1 \leq i \leq n \text{ and } n \geq k - 1. \end{cases}$$

The E^2-term is equal to zero except for the zeroth column where we get the

$$\begin{array}{ccccc} H_n(\bigcup \mathcal{F}) & 0 & 0 & 0 & \cdots \\ \vdots & 0 & 0 & 0 & \cdots \\ H_1(\bigcup \mathcal{F}) & 0 & 0 & 0 & \cdots \\ H_0(\bigcup \mathcal{F}) & 0 & 0 & 0 & \cdots \end{array}$$

Figure 5.6: E^2-term in the proof in the proof of Lemma 5.22.

homology of $\bigcup \mathcal{F}$, see Figure 5.6. The $\tilde{E}^2$-term contains in the zeroth row the homology of $N(\mathcal{F})$ for dimensions greater or equal than k, at the same time vanish all terms above the $(k-1)$-th anti-diagonal, see also Figure 5.7. Hence the $\tilde{E}^r$-sequence collapses at term 2 above this anti-diagonal. Using Theorem 5.1 we obtain

$$H_n(\bigcup \mathcal{F}) \cong H_n(\mathrm{Tot}(C)) \cong H_n(N(\mathcal{F})) \text{ for all } n \geq k.$$

To obtain (i) we first compute $H^{\tilde{\partial}}(C_{p,*})$. Using the definition of singular homology one gets:

$$C_{p,q} = \bigoplus_{J \subset [n], |J| = q+1} \quad \bigoplus_{\sigma : \Delta^p \to X, \mathrm{im}(\sigma) \subset \bigcap_{j \in J} F_j} \mathbb{Z}$$

For $\sigma : \Delta^p \to X$ let J_σ be the maximal subset $J \subset [n]$ with $\mathrm{im}(\sigma) \subset \bigcap_{j\in J} F_j$. Then this leads to:

$$C_{p,q} = \bigoplus_{\sigma:\Delta^p\to X} \quad \bigoplus_{J\subset[n],\, J\subset J_\sigma,\, |J|=q+1} \mathbb{Z}$$

Notice that for $J_\sigma = \emptyset$ a (innocent) zero is added to the direct sum. The boundary $\tilde{\partial}$ has no effect on σ so that we can look at every component

$$\bigoplus_{J\subset[n],\, J\subset J_\sigma,\, |J|=q+1} \mathbb{Z}$$

separately. For $J_\sigma \neq \emptyset$ one can check that this equals the simplicial chain complex of a $(|J_\sigma| - 1)$-dimensional simplex. The simplicial homology of the simplex vanishes in all dimensions except dimension 0 where it equals $\mathbb{Z}$. Using that

$$\bigoplus_{\sigma:\Delta^p\to X} \quad \bigoplus_{J\subset[n],\, J\subset J_\sigma,\, |J|=1} \mathbb{Z} = S_p(\mathcal{F}),$$

we obtain:

$$H_q^{\tilde{\partial}}(C_{p,*}) = \begin{cases} S_p(\mathcal{F}) & \text{for } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The induced boundary ∂ can be identified with the usual boundary on $S_*(\mathcal{F})$. For families of open sets or of subcomplexes of a cell complex the inclusion $S_*(\mathcal{F}) \to S_*(X)$ induces an isomorphism $H_*(\mathcal{F}) \to H_*(X)$ in homology.

To prove (ii) we start with computing $H^\partial(C_{*,q})$, we make use of the $k_{\mathcal{G}}$-acyclicity of our family. By the definition of ∂ we know that

$$H_p^\partial(C_{*,q}) = \bigoplus_{J\subset[n],\, |J|=q+1} H_p(\bigcap_{j\in J} F_j).$$

If $\bigcap_{j\in J} F_j = \emptyset$ then the contribution to this sum equals 0. In the case $\bigcap_{j\in J} F_j \neq \emptyset$ the summand depends on our assumption $\tilde{H}_n(\bigcap_{j\in J} F_j) = 0$ for $n \geq k - |J|$. E. g. for $p = 0$ and $|J| \geq k$ the summand equals $\mathbb{Z}$, and in general this leads to

$$H_p^\partial(C_{*,q}) = \begin{cases} \displaystyle\bigoplus_{J\subset[n],\, |J|=q+1,\, \bigcap_{j\in J} F_j\neq\emptyset} \mathbb{Z} & \text{for } p = 0,\ q \geq k-1 \\ 0 & \text{for } p = i,\ q = n-i, \text{ for all } 1 \leq i \leq n \text{ and } n \geq k-1. \end{cases}$$

It is easy to check that the induced chain complex $(H_0^\partial(C_{*,*}), \tilde{\partial})$ and the simplicial chain complex $C_*(N(\mathcal{F}))$ are isomorphic in dimension $\geq k$. The

$$\begin{array}{ccccc}
0 & 0 & 0 & 0 & \cdots \\
\star & 0 & 0 & 0 & \cdots \\
\star & \star & 0 & 0 & \cdots \\
\star & \star & \star & H_k(N(\mathcal{G})) & \cdots
\end{array}$$

Figure 5.7: The $\tilde{E}^2$-term in the proof of Lemma 5.22.

chain groups are even isomorphic in dimensions $\geq k-1$, but the differential differ in general in dimension $k-1$. To see that differentials coincide up to a sign in dimensions $\geq k$ remember that $H_0(\bigcap_{j\in J} F_j)$ is freely generated by the class of any 0-simplex, and i_*^{J,J_i} maps the class of a 0-simplex on a class of a 0-simplex. Hence row 0 of the E^2-term contains the homology $H_n(N(\mathcal{G}))$ for $n \geq k$. $\square$

After these preparations the proof of Theorem 5.20 is short. In the next section we make use of the same double complex $(C_{*,*}, \partial, \tilde{\partial})$. Similar computations lead to a nice and short proof of the homological version of a nerve theorem of Björner.

Proof. (of **Theorem 5.20**) We prove that our family $\mathcal{F}$ is $k^\star$-Leray, hence Theorem 5.20 immediately follows from Theorem 5.19. For this let $\mathsf{L} \subset N(\mathcal{F})$ be an induced subcomplex. Then L is of the form $N(\mathcal{G})$ for some $\mathcal{G} \subset \mathcal{F}$. The family $\mathcal{G}$ is again $k_{\mathcal{G}}$-acyclic so that Lemma 5.22 implies

$$H_n(\bigcup \mathcal{G}) \cong H_n(N(\mathcal{G})) \text{ for all } n \geq k.$$

Finally we have $H_n(\bigcup \mathcal{G}) = 0$ for all $n \geq d$ as $\bigcup \mathcal{G}$ is a *nice* subset of $\mathbb{R}^d$. $\square$

Matoušek observed in [50] that the Fractional Helly number is in general stable under finite unions.

Observation 5.23. If a family $\mathcal{F}$ has Fractional Helly number k, then the family $\mathcal{F}(2) = \{F_1 \cup F_2 \,|\, F_1, F_2 \in \mathcal{F}\}$ has Fractional Helly number k. For $m \geq 2$, let $\mathcal{F}(m)$ be the family of sets that are unions of m sets in $\mathcal{F}$.

This observation leads to the following problem.

Problem 5.24. Let $m \geq 2$, and $\mathcal{F}$ be a $k_{\mathcal{G}}$-acyclic family. Identify subsets of $\mathbb{R}^d$ that are in $\mathcal{F}(m)$. Is their a relation between $\mathcal{F}(m)$ and the family of semialgebraic sets of bounded description complexity?

The example from page 121 of a finite family of semialgebraic sets of the form $F_i = \{x \in \mathbb{R}^d \,|\, x_1^2 + x_2^2 - i \geq 0\}$ can also be obtained as $\mathcal{F}(3)$ of some good cover family $\mathcal{F}$. For this cut the plane into three equal pieces of cake so that the cuts are rays starting at the origin. Every F_i can be obtained as a union of three pieces of cake where the portion i has already been bitten of. The intersection of pieces of cake is either empty, or contractible.

5.4 On nerve theorems

The nerve theorem has been used in Chapter 2 to obtain new connectivity results for chessboard-type complexes. It is a standard tool in topological combinatorics, and it was first obtained by Leray [45]; see Björner [18] for a survey on nerve theorems. Here we prove a homological version of the Nerve Theorem 1.39 due to Björner.

Theorem 5.25 (Nerve theorem, homology version)**.** *For $k \geq 0$ let $\mathcal{F}$ be a family of open sets (or a finite family of subcomplexes of a cell complex) such that every $\bigcap \mathcal{G}$ is empty or $(k - |\mathcal{G}| + 1)$-connected for all non-empty subfamilies $\mathcal{G} \subset \mathcal{F}$. Then*

$$H_n(X) \cong H_n(N(\mathcal{F}))\ ,\ \textit{for } X = \bigcup \mathcal{F} \textit{ and all } n \leq k.$$

Proof. As in the proof of Lemma 5.22 we have that $H_*(X) \cong H_*(\mathrm{Tot}(C), d)$. For a k-connected space X we know from a famous theorem of Hurewicz that $\tilde{H}_n(X) = 0$ for all $n \leq k$. Using the conditions on the connectivity of $\mathcal{G}$ and analogous arguments as in the proof of Lemma 5.22 the $\tilde{E}^2$-term looks as in Figure 5.8. Hence we see that $H_n(N(\mathcal{F})) \cong H_n(C, d)$ for all $n \leq k$. ☐

5.5 Towards a homological VC-dimension

This section discusses the Fractional Helly property in a quite general context. Our starting point of this discussion is the Oberwolfach Report [41] of Kalai where he outlines further directions of research. The concept of *homological VC-dimension*, and its relation to the Fractional Helly property are studied. The naming *homological VC-dimension* suggests that this is an extension of VC-dimension – a well-known concept from computational

$$\begin{array}{ccccc}
\star & & & & \\
0 & \star & & & \\
0 & 0 & \star & & \\
0 & 0 & 0 & \star & \\
0 & 0 & 0 & 0 & \star \\
& & & & \\
H_0(N(\mathcal{F})) & H_1(N(\mathcal{F})) & \cdots & H_{k-1}(N(\mathcal{F})) & H_k(N(\mathcal{F}))
\end{array}$$

Figure 5.8: The $\tilde{E}^2$-term in the proof of Theorem 5.25.

geometry. As far as I understand, we are heading for a Fractional Helly theorem that lies between the Fractional Helly theorem for families with bounded VC-dimension, and the Fractional Helly Theorem 5.19 for d^*-Leray families.

We first recall the definition of VC-dimension, and the Fractional Helly theorem for families with bounded VC-dimension. Then homological VC-dimension is introduced, and its relation to the Fractional Helly property based on three conjectures of Kalai [41] is discussed. We will see that all three conjectures extend known results for d^*-Leray families. The relation between homological VC-dimension and the topological Fractional Helly Theorem 5.20 is shown. Finally, we obtain new results for the classification of simplicial complexes with bounded homological VC-dimension.

VC-dimension. The notion of VC-dimension was introduced by Vapnik and Chervonenkis in 1971. It plays an important role in several areas of mathematics and theoretical computer science, e. g. statistics and computational geometry. See also Matoušek's textbook [48] for more details.

Definition 5.26 (VC-dimension)**.** For a set X, let $\mathcal{F}$ be a family of subsets of X. For a subset of $Y \subset X$, we define the *restriction* of $\mathcal{F}$ to A as

$$\mathcal{F}|_Y = \{S \cap Y \mid S \in \mathcal{F}\}.$$

We define the *VC-dimension* of $\mathcal{F}$ as the supremum of the sizes of finite subsets $A \subset X$ such that
$$\mathcal{F}|_A = 2^A.$$
Subsets $A \subset X$ such that the restriction $\mathcal{F}|_A$ is the power set 2^A are *shattered.* The *shatter function* $\pi_{\mathcal{F}} : \mathbb{N} \to \mathbb{N}$ of $\mathcal{F}$ is defined as

$$\pi_{\mathcal{F}}(m) = \max_{Y \subset X, |Y|=m} |\mathcal{F}|_Y|.$$

If arbitrary large subsets of X can be shattered, then the VC-dimension of $\mathcal{F}$ is ∞. The family of all closed halfplanes in the plane is an example of VC-dimension less than 4 as no set of four points in the plane can be shattered. The family of all convex sets in $\mathbb{R}^d$ has VC-dimension ∞ as the vertex set of any n-gon can be shattered. If the VC-dimension of $\mathcal{F}$ is at most d, then it is known that the shatter function is

$$\pi_{\mathcal{F}} = O(m^d).$$

Here we use the Landau notation $f = O(g)$ for functions $f, g : \mathbb{N} \to \mathbb{N}$, if there is a constant $C > 0$ such that the quotient $|f(n)/g(n)| < C$ for all $n \in \mathbb{N}$. We write $f = o(g)$ if the limit of this quotient is 0.

The *dual VC-dimension* of $\mathcal{F}$ is the VC-dimension of its dual family with ground set $\{y_S \mid S \in \mathcal{F}\}$. The *dual shatter function* $\pi^*_{\mathcal{F}}$ is the shatter function of the dual family. Another way to understand the dual shatter function is the following: $\pi^*_{\mathcal{F}}(m)$ is the maximum number of non-empty cells in the Venn diagram of m sets of $\mathcal{F}$. A Venn diagram is the schematic diagram used in set theory to depict families of sets and represent their relationships – also used in elementary school. This approach is more intuitive, e. g. the dual shatter function of the family of all closed halfspaces in $\mathbb{R}^2$ is the maximum number of regions, into which m halfplanes partition the plane.

A primary example of geometric families of sets are semialgebraic sets in $\mathbb{R}^d$ of bounded description complexity. Here sets $A \subset \mathbb{R}^d$ are *semialgebraic sets* if they can be defined by a Boolean combination of polynomial inequalities. The *description complexity of* A is the maximum of the number of polynomials involved, and of the maximal degree of the polynomials. A classic result in real algebraic geometry due to Oleinik and Petrovskii, Thom, and Milnor implies that for the family $\mathcal{F}$ of all semialgebraic sets in $\mathbb{R}^d$ of description complexity at most B the dual shatter function is bounded

$$\pi^*_{\mathcal{F}}(m) \leq Cm^d \text{ for all } m \in \mathbb{N},$$

where $C > 0$ is a constant depending on B, see Basu et al. [15] for references.

Matoušek showed in [50] the following Fractional Helly theorem for families of bounded VC-dimension.

Theorem 5.27. *Let k be a positive integer, and let $\mathcal{F}$ be a family of sets whose dual shatter function satisfies $\pi^*_{\mathcal{F}} = o(m^k)$. Then $\mathcal{F}$ has Fractional Helly number $d+1$.*

Theorem 5.27 implies the case of families of sets with dual VC-dimension at most $k-1$, and the following corollary for the family of semialgebraic sets in $\mathbb{R}^d$.

Corollary 5.28. *Let B be a positive integer. The family of all semialgebraic sets in $\mathbb{R}^d$ of description complexity at most B has Fractional Helly number $d+1$.*

Matoušek shows Theorem 5.28 also for families of intersections of semialgebraic sets in $\mathbb{R}^d$ with an k-dimensional variety – based on results of Basu et al. [15]. Basu et al. bound the number of cells defined by a family of polynomials on a k-dimensional variety. Using the machinery shown in Figure 5.4 Theorems 5.27 and 5.28 also imply (p, q)-theorems.

Homological VC-dimension. Let K be a simplicial complex, and let $\beta(K) = \sum_{i\geq 0} \beta_i(\mathsf{K})$ denote the sum of the Betti numbers of K. A family of simplicial complexes is *strongly hereditary* if it contains with a simplicial complexes all possible links and all induced subcomplexes.

Definition 5.29 (Homological VC-dimension)**.** Let $A > 0$ be a positive constant. A strongly hereditary family $\mathcal{F}$ has *homological VC-dimension* d if for every simplicial complex $\mathsf{K} \in \mathcal{F}$ the following condition holds:

$$\beta(\mathsf{K}) < A \cdot f_0(\mathsf{K})^d. \tag{5.10}$$

We write short HVC-dimension for homological VC-dimension. The next observation gives an example of a hereditary family of HVC-dimension d. Up to now, we have not been able to show that nerves of families of bounded VC-dimension are of bounded HVC-dimension. The definition of HVC-dimension d depends on the constant A.

Observation 5.30. The family $\mathcal{L}^d$ of all d-Leray complexes is a strongly hereditary family of HVC-dimension d.

Proof. The family $\mathcal{L}^d$ is closed under taking induced subcomplexes as the induced subcomplex of an induced subcomplex is an induced subcomplex of the original complex. $\mathcal{L}^d$ is closed under taking links as for simplicial complexes K:

$$\mathrm{lk}(G, \mathrm{lk}(F, \mathsf{K})) = \mathrm{lk}(G \cup F, \mathsf{K}).$$

Let K be a d-Leray complex on n. The Betti numbers of K are zero in dimension greater or equal than d. For n large enough, one has

$$\beta(\mathsf{K}) \leq \beta((\sigma^{n-1})^{\leq d-1}) = \beta_0((\sigma^{n-1})^{\leq d-1}) + \beta_{d-1}((\sigma^{n-1})^{\leq d-1}) = 1 + \binom{n-1}{d},$$

for the complete $(d-1)$-skeleton of the simplex on n vertices. The complete $(d-1)$-skeleton of the simplex on n vertices has the maximal number of $(d-1)$-dimensional holes among d-Leray complexes. Moreover, we use that the $(d-1)$-th Betti number of the complete $(d-1)$-skeleton of the simplex on n vertices is $\binom{n-1}{d}$, and that $\beta_i(\mathsf{K}) \leq f_i(\mathsf{K}) \leq \binom{n}{i+1}$ for all i. This implies $\beta(\mathsf{K}) < An^d$ for $A = 1$ for n large enough. $\square$

This observation, and the fact that algebraic shifting maintains the property of being d-Leray, motivates the following conjecture.

Conjecture 5.31. If K is a simplicial complex from a strongly hereditary family $\mathcal{F}$ of HVC-dimension d, then so is its algebraic shifting $\Delta(\mathsf{K}) \in \mathcal{F}$.

This conjecture extends the result from Theorem 5.14 for d-Leray complexes. An approach for a proof of this conjecture is via the algebraic work of Bayer et al. [16], and Herzog [36] on generic initial ideals and graded Betti numbers. The algebraic methods to prove this conjecture seem currently to be out of my reach.

The example below shows that algebraic shifting does not maintain the property of being strongly hereditary. To determine the HVC-dimension of the family of shifted complexes, it is therefore necessary to add all induced subcomplexes, and all links.

Observation 5.32. Let $\mathcal{F}$ be a strongly hereditary family, then the family $\{\Delta(\mathsf{K}))\,|\,\mathsf{K} \in \mathcal{F}\}$ of shifted complexes is in general not strongly hereditary. Let's look at an example, also shown in Figure 5.9. There we start with the simplicial complex with maximal faces $\{1, 2\}$ and $\{3, 4\}$. This leads to a strongly hereditary family $\mathcal{F}$ by adding all links, and all induced subcomplexes which are shown on the left side of Figure 5.9. Shifting all complexes of $\mathcal{F}$ leads to the family shown on the right side. However, the family of shifted complexes is not strongly hereditary as it does not contain the induced subcomplex with maximal faces $\{1, 2\}$ and $\{1, 3\}$.

The algebraic shifting of these simple complexes can be obtained without using the algebraic machinery. It suffices to exploit the properties of algebraic shifting summarized in Theorem 5.14.

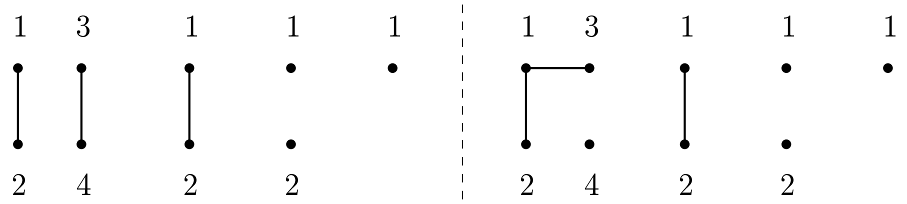

Figure 5.9: A family, and the family of shifted complexes.

Conjecture 5.33. Let $A > 0$ be a constant. Let $\mathcal{F}$ be the family of sets in $\mathbb{R}^d$ such that if S is the intersection of m sets in $\mathcal{F}$, then $\beta(S) \leq Am^d$. Then the nerve $N(\mathcal{F})$ is in the family of all simplicial complexes of HVC-dimension d.

For families of good covers, the condition on the topological complexity of any non-empty intersection M is equal to

$$\beta(S) = 1.$$

The nerves of good covers are then d-Leray complexes. In Conjecture 5.33, we relax this condition. We have seen in the topological Fractional Helly Theorem 5.20 for $k_{\mathcal{G}}$-acyclic families that is possible to relax condition even more to obtain d-Leray nerve complexes: The Betti numbers of intersections of m sets can be arbitrary in dimension less or equal than $k - m - 1$.

Recently, Basu [13], [14], and Gabrielov and Vorobjov [32] have shown bounds for the Betti numbers of semialgebraic sets. Basu uses a spectral sequence argument to show for the Betti numbers of a semialgebraic set S defined through n polynomial inequalities of degree at most d, that is contained in a k-dimensional variety:

$$\beta_i(S) \leq \binom{n}{k-i} O(d) \ \text{ for } 1 \leq i \leq n.$$

The following example shows that one can construct families of subsets in $\mathbb{R}$ such that the nerve complexes are not $(n-2)$-Leray. The sets in $\mathcal{F}$ union of two half intervals. However, the topological complexity of the non-empty intersections grows with n.

Example 5.34. For any $n \geq 2$, we construct an example of a family $\mathcal{F}$ of n subsets F_1, F_2, ..., F_n of $\mathbb{R}$ such that its nerve its nerve will be the boundary of the simplex σ^{n-1} on n vertices. Define first a family of n closed intervals G_1, G_2, ..., G_n: $G_i = [i-1, i]$ for $i \in [n]$. Now define subsets $F_i = [0, n] \setminus G_i$ for all $i \in [n]$. see also Figure 5.10. Every F_i is the union of two half-open intervals. Choose a subset M of $[n]$, then the intersection $\bigcap_{i \in M} F_i$ is the union of all open intervals $(j-1, j)$ such that $j \in [n] \setminus M$. Thus the intersection is empty for $M = [n]$, and the number of components of non-empty intersections grows with n.

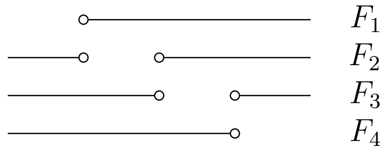

Figure 5.10: A family of sets in $\mathbb{R}$ that is not 2^*-Leray.

Let's translate the Fractional Helly property from the introduction to the setting of families of simplicial complexes.

Definition 5.35 (Fractional Helly number)**.** Let $\mathcal{F}$ be a family of simplicial complexes. The family $\mathcal{F}$ has *Fractional Helly number* k if for every $\alpha \in (0, 1]$ there is a $\beta = \beta(\alpha) >$ such that the following condition holds for all $\mathsf{K} \in \mathcal{F}$:

$$f_{k-1}(\mathsf{K}) \geq \alpha \binom{n}{k} \Rightarrow f_{\lfloor \beta n \rfloor}(\mathsf{K}) > 0, \quad \text{where } n = f_0(\mathsf{K}). \tag{5.11}$$

The Upper Bound Theorem 5.6 for d-Leray complexes implies that the family of d-Leray complexes has Fractional Helly number $d+1$, and that $\beta(\alpha) = 1 - (1-\alpha)^{1/d+1}$.

Observation 5.36. Let $\mathcal{F}$ be family with Fractional Helly number k. Then every subfamily of $\mathcal{F}$ has Fractional Helly number k, as the implication also works for any subclass.

Conjecture 5.37. A hereditary family of simplicial complexes of HVC-dimension d, for some constant $A > 0$, has Fractional Helly number $d+1$.

In analogy to the d-Leray case, this conjecture gives rise to the following question for families $\mathcal{F}$ of HVC-dimension d: Let $\mathsf{K} \in \mathcal{F}$ be of dimension $d + r$, for $r \geq 0$. What is the maximal number of d-faces of the simplicial complex K?
We will below obtain partial answers to this question for the family $\mathcal{H}^d_A$ of all simplicial complexes of HVC-dimension d.

For $r \geq 0$, let $\mathcal{H}_A^{d,r}$ be the subfamily of $\mathcal{H}_A^d$ that contains all simplicial complexes of dimension less or equal than $d+r$. The family $\mathcal{H}_A^{d,r}$ is of HVC-dimension d, for the same constant A. Let $M(n,d,r,A)$ be the maximal number of d-faces for a simplicial complex K on n vertices in $\mathcal{H}_A^{d,r}$:

$$M(n,d,r,A) = \max_{\mathsf{K} \in \mathcal{H}_A^{d,r},\, f_0(\mathsf{K})=n} f_d(\mathsf{K})$$

Clearly, $M(n,d,r,A) \leq \binom{n}{d+1}$. The next proposition shows a possible approach to disprove Conjecture 5.37. For this, we fix d, r, and A, so that $M(n,d,r,A)$ is a function $\mathbb{N} \to \mathbb{N}$.

Proposition 5.38. *Suppose that*

$$n^{d+1} = O(M(n,d,r,A)),$$

then Conjecture 5.37 does not hold for $\mathcal{H}_A^d$.

Proof. Due to Observation 5.36, it is sufficient to prove that Conjecture 5.37 does not hold for the subfamily $\mathcal{H}_A^{d,r}$. By assumption, there is a constant $C > 0$ such that $n^{d+1} \leq C \cdot M(n,d,r,A)$ for all $n \in \mathbb{N}$. Let $\alpha \in (0,1]$ such that $\alpha < \frac{d+1!}{C}$. Then there is a natural number $n_0 = n_0(C)$ such that for all $n \geq n_0$

$$M(n,d,r,A) \geq \alpha \binom{n}{d+1}.$$

Hence there is for any $n \geq n_0$ a simplicial complex $\mathsf{K} \in \mathcal{H}_A^{d,r}$ on n vertices of dimension less or equal than $d+r$ that satisfies the assumption of (5.11). For any real $\beta > 0$ the product βn exceeds $d+r$, as it tends to infinity. The family $\mathcal{H}_A^{d,r}$ has thus not Fractional Helly number $d+1$. □

Using the same arguments, which we omit here, this leads conversely to the following proposition.

Proposition 5.39. *Suppose that*

$$M(n,d,r,A) = o(n^{d+1}), \tag{5.12}$$

then Conjecture 5.37 holds for the class $\mathcal{H}_A^{d,r}$.

In the remaining part of this section, we show condition (5.12) for arbitrary $d \geq 1$, and $r = 0$. Moreover, we obtain partial results for $r \geq 1$ via algebraic shifting.

Proposition 5.40. *For $d = 1$, $M(n,d,0,A) \leq n - 3 + \lceil An \rceil$. In general,*

$$M(n,d,0,A) = O(n^d).$$

Proof. We first show the general case. Using Euler's formula, we obtain for any simplicial complex from $\mathcal{H}_A^{d,0}$:

$$(-1)^d f_d(\mathsf{K}) = \sum_{i=0}^{d}(-1)^i\beta_i(\mathsf{K}) - \sum_{i=0}^{d-1}(-1)^i f_i(\mathsf{K})$$

The absolute value of the first term on the right hand side is bounded from above by An^d. The absolute value of the second term is bounded from above by $\sum_{i=0}^{d-1}\binom{n}{i}$. Hence the maximal number $M(n,d,0,A)$ is of order at most d.

For $d=1$, bounding $b(\mathsf{K})$ by An implies via the Euler formula a bound for f_1. A short computation leads to the above stated result. □

Recall that the family $\mathcal{H}_A^{d,r}$ is strongly hereditary. This means that a simplicial complex K is in $\mathcal{H}_A^{d,r}$, if K and all its links and induced subcomplexes satisfy the homology condition (5.10).

Observation 5.41. There are 1-dimensional simplicial complexes on n vertices satisfying $f_1 \leq n-3+\lceil An\rceil$ that are not in $\mathcal{H}_A^{1,0}$. The family $\mathcal{H}_A^{1,0}$ is closed under taking induced subcomplexes. It is not hard to construct examples of simplicial complexes, in other words graphs, which contain an induced subcomplex violating $f_1 \leq n-3+\lceil An\rceil$, e. g. graphs that contain small complete graph as induced subgraph.

In the following, we construct examples of 2-dimensional simplicial complexes K such that $\beta(K) \leq An$ and $n^2 = O(f_1(\mathsf{K}))$, but $\mathsf{K} \notin \mathcal{H}_A^{1,1}$. This confirms that

$$M(n,d,r,A) = o(n^{d+1}).$$

These constructions easily generalize to the $\mathcal{H}_A^{d,1}$ case. The homology condition (5.10) for K implies that the difference of $f_d(\mathsf{K})$ and $f_{d+1}(\mathsf{K})$ is $O(n^d)$.

Example 5.42. Let K contain the full 1-skeleton on n vertices, and add $\binom{n-1}{2}$ many 2-faces to obtain an acyclic complex. Then $f_1(\mathsf{K}) = \binom{n}{2}$, and $\beta(K) = 1 \leq An$, but $\mathsf{K} \notin \mathcal{H}_A^{1,1}$. Suppose we add the $\binom{n-1}{2}$ many 2-faces uniformly at random to K. The expected number of 2-faces in an induced subcomplex on $\frac{n}{2}$ vertices is

$$\frac{1}{8}\binom{n-1}{2} = \frac{1}{16}n^2 + \cdots .$$

On the other hand, the full 1-skeleton on $\frac{n}{2}$ vertices has

$$\binom{\frac{n}{2}-1}{2} = \frac{1}{8}n^2 + \cdots$$

many 1-dimensional holes. Hence there is an induced subcomplex L on $\frac{n}{2}$ vertices in K such that

$$\beta(\mathsf{L}) \geq \beta_1(\mathsf{L}) = \frac{1}{16}n^2 + \cdots > An.$$

Example 5.43. Start with any graph G on $n-1$ vertices that has $c\binom{n}{2}-n+1$ edges, for a real constant $c > 0$. Let K be the simplicial complex that is obtained from the cone(G) by adding n many 2-faces to G, seen as a 1-dimensional simplicial complex. By construction $f_1(\mathsf{K}) = f_2(\mathsf{K}) = c\binom{n}{2}$ and $\beta(\mathsf{K}) \leq An$. However $\mathsf{K} \notin \mathcal{H}^{1,1}$, as the induced complex L on the first $n-1$ vertices has quadratically many 1-dimensional holes.

Suppose that Conjecture 5.31 holds: Algebraic shifting maintains the HVC-dimension. Then we can show the following.

Proposition 5.44. *$M(n, d, 1, A) = O(n^d)$.*

This implies together with Proposition 5.38 that $\mathcal{H}_A^{d,1}$ has Fractional Helly number $d+1$. With some more effort, one verifies similarly that $M(n, d, r, A) = O(n^d)$ for $r = 2$. Up to now, we have not been able to formalize this proof for arbitrary $r \geq 1$.

Proof. As algebraic shifting maintains the HVC-dimension, it suffices to consider only simplicial complexes K on n vertices which are algebraically shifted. Suppose that $n^d = o(f_d(\mathsf{K}))$, then we will show that $\mathsf{K} \notin \mathcal{H}_A^{d,1}$. This says for $d = 1$, that any K with a superlinear number of edges is not in $\mathcal{H}_A^{d,1}$. Using Property iii) of algebraically shifted complexes from Theorem 5.14, one obtains for any i:

$$\beta(\mathsf{K}) \geq \beta_i(\mathsf{K}) = \{S \in \mathsf{K} : |S| = i+1, S \cap \{1\} = \emptyset, S \cup \{1\} \notin \mathsf{K}\}. \quad (5.13)$$

Assume that the homology condition $\beta(\mathsf{K}) < An^d$ holds for K. It is sufficient to show that the induced complex L of K on the vertex set $[n] \setminus \{1\}$ violates the homology condition.

One has $n^d = o(f_d(\mathsf{L}))$ as there is at most $\binom{n-1}{d}$ many d-faces on the vertex set $[n]$ that use the vertex 1. On the other hand, combining the homology condition for K with inequality (5.13) for $i = d+1$ leads to

$$f_{d+1}(\mathsf{L}) \leq An^d.$$

Finally, the Euler formula implies $\beta(\mathsf{L}) > A(n-1)^d$, as there is just too many d-dimensional holes. $\square$

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