Optimal control
of the unsteady Navier-Stokes equations

vorgelegt von

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Preface

Outline

My thesis is concerned with the study of optimal control problems for the non-stationary Navier-Stokes equations. These equations are a mathematical model to describe the behaviour of fluid flows. Simply speaking, we are considering the minimization of an objective function $J(y, u)$ depending on control u and velocity field y subject to the state equation condensed to $NS(y, u) = 0$.

This work is not the first on that particular subject. Many other authors have contributed their results. As a pioneering work stands the article of Abergel and Temam [1], where the first optimal control problem for fluid flows was considered, and where also the first existence and optimality conditions can be found[1]. My thesis follows these classical attempt and will repeat known results, but also adds some new insight to this subject.

Chapter 1 tries to give an overview of the governing equations. It provides some functional analytic background material such as function spaces and weak solution concepts. Each control action requires a reaction of the flow. Though in real life this reaction should be unique, the mathematical theory does not yet provide such a result for three-dimensional problems for low regular data. In the two dimensional case this reaction is unique, and the mapping control $\mapsto$ velocity field can be studied. It turns out that this mapping is even twice Fréchet differentiable, which enables us to use well-known Banach space programming techniques later on. The contribution of the first chapter is the study of the linearized and adjoint state equations in L^p-spaces, a topic that was not recognized in optimal control of the instationary Navier-Stokes equations. These results are the basis for arguments in the following chapters, which would be not possible without them.

The optimal control problem is formulated exactly in Chapter 2. Here also the question of existence of solutions is answered positively. The characterization of such a solution by necessary conditions then follows in Chapter 3. The first-order conditions imply a representation of optimal controls by projections, a fact which leads to new regularity results. In particular, an optimal control is a continuous function in space and time under some regularity assumptions.

Chapter 4 is devoted to the study of sufficient second-order conditions. Such conditions are a-priori assumptions to prove local optimality of a reference control, as the name 'sufficient' suggests. A refined analysis allows to prove local optimality of reference controls in weaker than L^∞-norms, which are usually encountered in optimal control problems for semilinear partial differential equations.

[1] It is also the number one in the bibliography.

This sufficient condition is an important assumption to prove stability of optimal controls under small perturbations in Chapter 5. With the help of the L^p-estimates of Chapter 1, we prove stability of optimal controls with respect to the L^∞-norm, a result which is stronger and was proven under weaker assumptions than other results of that type. This stability result directly gives the local quadratic convergence of the SQP-method for our optimal control problem in Chapter 6.

Chapter 7 is devoted to the study of more general control constraints. The control in our problem is vector-valued, thus box-constraints are not the only possible choice. The material in this chapter especially regularity of optimal controls and sufficient optimality conditions is original. Infinite-dimensional optimization problems with such contraints are rarely investigated in the literature. Some questions remain open in this chapter and may initiate further research in this area.

The thesis is completed by numerical test in Chapter 8. They confirm the convergence theory of the SQP-method. Furthermore, a new active-set method to solve problems with convex control constraints is presented.

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Chapter 1

The unsteady Navier-Stokes equations

The Navier-Stokes equations are a mathematical model to describe the behaviour of fluid flows. Navier and Stokes were two scientists of the 19th century, which were the first who tried to derive equations of motion for fluids.

In this work, we are dealing with optimal control of the non-stationary and incompressible Navier-Stokes equations. In their dimensionless form they are given by

$$\begin{aligned} y_t - \nu\Delta y + (y \cdot \nabla)y + \nabla p &= f \\ \operatorname{div} y &= 0 \\ y(0) &= y_0. \end{aligned} \tag{1.1}$$

together with no-slip boundary conditions $y = 0$ on the boundary Γ of the domain.

Here, y denotes the state of the system — the velocity field [2]. The pressure is denoted by p. The right hand side f is an given source term. Later, the control will be brought in the system in this position. The initial profile of the flow is a given function y_0.

In this chapter, we provide some functional analytic background to study the Navier-Stokes equations. The first mathematical work to understand those equation is due to Leray and Hopf. Here, we rely on the books by Constantin and Foias [21], Ladyshenzkaya [51], and Temam [70].

The existence, uniqueness and regularity of solutions to the equations (1.1) is completely understood in the two-dimensional case. However, in the three-dimensional case there are many open problems connected with smoothness and uniqueness of solutions. One of the seven millenium problems published by the Clay institute[3] concerns the Navier-Stokes equations in $\mathbb{R}^3$. Hence, it is natural to restrict our attention to the two-dimensional case.

1 Function spaces

In the context of optimal control it is useful to look for weak solutions to the state equation (1.1). To this aim, we have to introduce some function spaces. For a

[2] Unlike to the context of engineering or analysis of partial differential equations, where the velocity is denoted by u. However, since we are doing optimal control, the letter u is reserved for the control.

[3] `http://www.claymath.org/millennium/Navier-Stokes_Equations`

more detailed introduction, we refer to the textbooks [2, 70]. In the latter one, the concept of solenoidal functions is developed.

1.1 The domain Ω

All functions involved in the problem are living on some domain. It has to obey certain restrictions, so we will begin this preliminary section with their specification.

Let Ω be a domain in $\mathbb{R}^n$ that is an open and bounded set. Its boundary is denoted by $\Gamma := \partial\Omega$. The outward normal vector of the boundary is denoted by $\mathfrak{n}$. In general, we will need some smoothness properties of Ω respectively Γ. In many situations, it suffices to assume

$$\Omega \text{ is locally Lipschitz.} \tag{1.2}$$

That means that for each point $x \in \Gamma$ there is a neighborhood $\mathcal{U} \ni x$, such that $\Gamma \cap \mathcal{U}$ is the graph of a Lipschitz continuous function, see e.g. [2, 70].

However, this smoothness might be insufficient in some other cases. Sometimes, we will need that

$$\Omega \text{ is of class } C^m \tag{1.3}$$

for some integer $m \geq 1$, i.e. the boundary Γ is a $n-1$-dimensional manifold of class C^m. Obviously, the latter property implies the local Lipschitz property. Both smoothness assumption imply the so-called cone property, which is a pre-requisite for continuous imbeddings of Sobolev spaces, see Theorem 1.1 below.

1.2 L^p and Sobolev spaces

We denote by $L^p(\Omega)$, $1 \leq p \leq \infty$, the space of real functions defined on Ω whose p-th power is integrable for the Lebesgue measure $\mathrm{d}x$, respectively that are essentially bounded in the case $p = \infty$. It is a Banach space endowed with the norm

$$\begin{aligned} |u|_p &:= \|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \,\mathrm{d}x \right)^{1/p}, \; p < \infty, \\ |u|_\infty &:= \|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_\Omega |u(x)|. \end{aligned}$$

For $p = 2$, $L^2(\Omega)$ is a Hilbert space with the scalar product

$$(u, v)_2 = \int_\Omega u(x)v(x)\,\mathrm{d}x.$$

For a bounded domain Ω, each function in $L^p(\Omega)$ is also in $L^q(\Omega)$ for q smaller than p, i.e. $1 \leq q \leq p$. The respective norms can be estimated as

$$|u|_q \leq (\operatorname{meas}\Omega)^{1/q-1/p}\,|u|_p, \tag{1.4}$$

where $\operatorname{meas}\Omega = \int_\Omega 1\,\mathrm{d}x$ is the measure of the domain.

The dual space to L^p, $1 < p < \infty$ is itself a space of integrable functions. More precisely, the dual can be indentified with the space $L^{p'}$, where $p' := p/(p-1)$ is the conjugate exponent of p satisfying $1/p + 1/p' = 1$. The dual pairing is defined by

$$\langle u,\, v\rangle_{p',p} = \int_\Omega u(x)v(x)\,\mathrm{d}x.$$

We will provide some basic inequalities to deal with Lebesgue integrable functions. For $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$, we have by *Hölder's inequality*

$$\int_\Omega u(x)v(x)\,\mathrm{d}x \le \left(\int_\Omega |u(x)|^p\,\mathrm{d}x\right)^{1/p} \left(\int_\Omega |v(x)|^{p'}\,\mathrm{d}x\right)^{1/p'}. \tag{1.5}$$

It directly yields the estimate of the dual product $\langle u,\, v\rangle_{p',p} \le |u|_{p'}|v|_p$. A conclusion is the interpolation inequality for $u \in L^p(\Omega)$, $q < p$, $\theta \in (0,1)$, $1/p_\theta = \theta/p + (1-\theta)/q$,

$$|u|_{p_\theta} \le |u|_p^\theta |u|_q^{1-\theta}.$$

It can be proven by substituting $u := |u|^s$, $v := |u|^{p_\theta - s}$, $s := p(p_\theta - q)/(p-q)$ in (1.5).

The Sobolev space $W^{m,p}(\Omega)$ is the space of functions whose weak derivatives up to order m are functions in $L^p(\Omega)$. Equipped with the norm

$$|u|_{m,p} := \left(\sum_{|j|\le m} |D^j u|_p^p\right)^{1/p}$$

it is a Banach space. Again, in the case $p = 2$, the space $H^m(\Omega) := W^{m,2}(\Omega)$ is a Hilbert space with scalar product

$$(u,v)_{H^m} = \sum_{|j|\le m} (D^j u,\, D^j v)_2.$$

For a comprehensive introduction of Sobolev spaces we refer to [2].

Let $\mathcal{D}(\Omega)$ be the space of all C^∞-functions with compact support in Ω. The closure of $\mathcal{D}(\Omega)$ in the norm of $W^{m,p}(\Omega)$ is denoted by $W_0^{m,p}(\Omega)$ respectively $H_0^m(\Omega)$ when $p = 2$.

A very useful tool when dealing with Sobolev spaces is the following theorem. We refer to [2] for the proof and further details.

Theorem 1.1 (Sobolev imbedding theorem). *Let Ω be an domain in $\mathbb{R}^n$ having the cone property. Suppose $mp < n$. Then the imbedding*

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is continuous for $q < \frac{np}{n-mp}$. If $mp > n$, then we have continuity of the imbedding

$$W^{m,p}(\Omega) \hookrightarrow C(\bar{\Omega}).$$

In the sequel, we will work with two-dimensional domains. Here, we have that the imbeddings $H^1(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \le q < \infty$ and $W^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$ for $p > 2$ are continuous.

Furthermore, we will make use of an interpolation inequality proven in [70].

Lemma 1.2. *For $u \in H^1(\Omega)$, $\Omega \in \mathbb{R}^n$, $n \le 4$, it holds*

$$|u|_4 \le 2^{1/4}\,|u|_2^{1/2}\,|u|_{1,2}^{1/2}. \tag{1.6}$$

In the sequel, we will be concerned with n-dimensional vector functions. There, we will make use of the notation

$$L^p(\Omega)^n := L^p(\Omega) \times \cdots \times L^p(\Omega),$$

which is analogously employed for all other kinds of spaces. These product spaces will be equipped with the usual product norm or an equivalent norm, except the case $\mathcal{D}(\Omega)^n$ that is not a normed space at all.

1.3 Solenoidal function spaces

It is convenient to work with functions that satisfy intrinsicly the constraint $\operatorname{div} y = 0$. Let $\mathcal{V}$ be the space

$$\mathcal{V} = \{y \in \mathcal{D}(\Omega)^n : \operatorname{div} y = 0\}.$$

The closures of $\mathcal{V}$ in L^2 and H_0^1 are the basic spaces used in the theory of incompressible Navier-Stokes equations,

$$H := \text{closure of } \mathcal{V} \text{ in } L^2(\Omega)^n,$$
$$V := \text{closure of } \mathcal{V} \text{ in } H_0^1(\Omega)^n.$$

Their norms are the usual $L^2(\Omega)^n$ and $H_0^1(\Omega)^n$-norms and will be denoted by $|\cdot|_H$ and $|\cdot|_V$ respectively. Although these spaces have the same scalar product as L^2 and H_0^1, we will use $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_V$ instead. They enjoy the following useful characterizations.

Theorem 1.3. [70, Theorems I.1.4, I.1.6] *Let Ω be an open Lipschitz domain in $\mathbb{R}^n$. Then it holds*

$$\begin{aligned} H &= \{u \in L^2(\Omega)^n : \operatorname{div} u = 0,\, \gamma_{\mathfrak{n}} u = 0\}, \\ H^\perp &= \{u \in L^2(\Omega)^n : u = \nabla p,\, p \in H^1(\Omega)\}, \\ V &= \{u \in H_0^1(\Omega)^n : \operatorname{div} u = 0\}. \end{aligned}$$

Here, $\gamma_{\mathfrak{n}}$ is the normal trace operator, $\gamma_{\mathfrak{n}} : u \mapsto \mathfrak{n} \cdot u|_\Gamma$. For a very detailed discussion, we refer to the textbook [70], see also [21, 28].

We define a norm in H by

$$|y|_* := \sup_{v \in V \setminus \{0\}} \frac{(y,v)_H}{|v|_V}.$$

The closure of H with respect to the $|\cdot|_*$-norm is equal to V', which is the usual dual space of V, see [89, p. 258]. In V' we will use the above defined norm: $|\cdot|_{V'} := |\cdot|_*$. The spaces V, H, and V' are a Gelfand triple, which is characterized by

$$V \subset H = H' \subset V'$$

with continuous and dense imbeddings. The duality pairing of V' and V is then the continuation of the H-scalar product to $V' \times V$. It is denoted by $\langle \cdot, \cdot \rangle_{V',V}$. For $u \in H$ and $v \in V$, the following property is an immediate conclusion

$$\langle u, v \rangle_{V',V} = \int_\Omega u(x) v(x) \, \mathrm{d}x.$$

Setting $v := u$, we can estimate the H-norm of a function $u \in V$ by

$$|u|_H^2 = \int_\Omega |u(x)|^2 \,\mathrm{d}x = \langle u,\, u\rangle_{V,V'} \le |u|_V |u|_{V'}.$$

We will use once in a while the following spaces, which are the L^p-counterparts of H and V,

$$H_p := \text{closure of } \mathcal{V} \text{ in } L^p(\Omega)^n,$$
$$V_p := \text{closure of } \mathcal{V} \text{ in } W_0^{1,p}(\Omega)^n.$$

1.4 Spaces of abstract functions

We denote by $Q := \Omega \times (0,T)$ the space-time cylinder. Here, $T < \infty$ is a given final time. Further, we set $\Sigma := \Gamma \times (0,T)$.

We shall work in the standard space of abstract functions from $[0,T]$ to a real Banach space X, $L^p(0,T;X)$, endowed with its natural norm,

$$\begin{aligned}
\|y\|_{L^p(X)} &:= \|y\|_{L^p(0,T;X)} = \left(\int_0^T |y(t)|_X^p \,\mathrm{d}t\right)^{1/p} \qquad 1 \le p < \infty,\\
\|y\|_{L^\infty(X)} &:= \operatorname*{vrai\,max}_{t\in(0,T)} |y(t)|_X.
\end{aligned}$$

In the sequel, we will identify the spaces $L^p(0,T;L^p(\Omega)^2)$ and $L^p(Q)^2$ for $1 < p < \infty$, and denote their norm by $\|u\|_p := |u|_{L^p(Q)^2}$. However, for the case $p = \infty$, there is only an imbedding $L^\infty(0,T;L^\infty(Q)^2) \hookrightarrow L^\infty(Q)^2$. A counterexample, see [27], shows, that a function in $L^\infty(Q)$ needs not to be measurable as function of $(0,T)$ with values in $L^\infty(\Omega)$. [4]

We denote by $(\cdot,\cdot)_Q$ the usual $L^2(Q)^2$-scalar product to avoid ambiguity. In all what follows, $\|\cdot\|$ stands for norms of abstract functions, while $|\cdot|$ denotes norms of "stationary" spaces like H and V.

To deal with the time derivative in the state equation, we introduce the common spaces of functions y whose time derivatives y_t exist as abstract functions,

$$\begin{aligned}
W^\alpha(0,T;V) &:= \{y \in L^2(0,T;V) :\; y_t \in L^\alpha(0,T;V')\},\\
W(0,T) &:= W^2(0,T;V),
\end{aligned}$$

where $1 \le \alpha \le 2$. Endowed with the norm

$$\begin{aligned}
\|y\|_{W^\alpha} &:= \|y\|_{W^\alpha(0,T;V)} = \|y\|_{L^2(V)} + \|y_t\|_{L^\alpha(V')},\\
\|y\|_W &:= \|y\|_{W^2},
\end{aligned}$$

these spaces are Banach spaces. Every function of $W(0,T)$ is, up to changes on sets of zero measure, equivalent to a function of $C([0,T],H)$, and the imbedding $W(0,T) \hookrightarrow C([0,T],H)$ is continuous, cf. [2, 54]. In $W(0,T)$, we have the rule of integration by parts with $v,w \in W(0,T)$

$$\int_0^T \langle v_t(t),\, w(t)\rangle_{V',V} \,\mathrm{d}t = (v(T),w(T))_2 - (v(0),w(0))_2 - \int_0^T \langle w_t(t),\, v(t)\rangle_{V',V} \,\mathrm{d}t. \tag{1.7}$$

[4] Take $\Omega = (0,1)$, $T = 1$, $g(x,t) = 1$ for $x < t$, $g(x,t) = 0$ otherwise. Then $g \in L^\infty(Q)$, but $g(\cdot,t)$ is not a measurable function from $(0,T)$ to $L^\infty(\Omega)$.

For $v \in W(0,T)$, we can conclude

$$\int_0^T \langle v_t(t),\, v(t)\rangle_{V',V}\,\mathrm{d}t = \frac{1}{2}\left(|v(T)|_H^2 - |v(0)|_H^2\right) = \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}|v(t)|_H^2\right)\bigg|_{t=s}\,\mathrm{d}s. \quad (1.8)$$

This relation is often used to prove a-priori estimates of weak solutions in the $L^\infty(0,T;H)$-norm.

The space $W(0,T)$ is continuously imbedded in $L^4(Q)^2$ as the next lemma shows.

Lemma 1.4. *Let $y \in W(0,T)$ be given. Then it holds $y \in L^4(Q)^2$ with $\|y\|_4 \le c\|y\|_W$.*

Proof. Using inequality (1.6), the claim with $c = \sqrt[4]{2}$ follows from a simple calculation,

$$\|y\|_4^4 = \int_0^T \int_\Omega |y(x,t)|^4\,\mathrm{d}x\,\mathrm{d}t \le 2\int_0^T |y(t)|_H^2 |y(t)|_V^2\,\mathrm{d}t \le 2\|y\|_{L^\infty(H)}^2 \|y\|_{L^2(V)}^2.$$

Observe, that the claim remains true for $y \in L^2(0,T;V) \cap L^\infty(0,T;H)$, since we did not use any regularity of the time derivative. □

Besides $W(0,T)$, we will use the space $W^{4/3}(0,T)$. It should be mentioned that this space is not continuously imbeddable in $C([0,T],H)$, [65]. However, there is an imbedding $W^{4/3}(0,T) \hookrightarrow L^p(Q)^2$ for $p < 7/2$, [78].

Furthermore, we introduce the following space of abstract functions in the L^p-context:

$$W_p^{2,1} := \{y \in L^p(0,T;W^{2,p}(\Omega)^2 \cap V_p) :\; y_t \in L^p(0,T;L^p(\Omega)^2)\}. \quad (1.9)$$

We abbreviate $H^{2,1} = W_2^{2,1}$ for $p = 2$. The space $H^{2,1}$ is continuously imbedded in $C([0,T],V)$, see Lions and Magenes [54].

Let us define $W_0^{2-2/p,\,p}(\Omega)^2$ as the space of solenoidal $W^{2-2/p,\,p}$-functions where zero boundary values are prescribed if $p \ge 4/3$. It turns out that this is the right trace space of $W_p^{2,1}$, where the initial values of $W_p^{2,1}$-functions are in. By [52, Lemma II.3.4, p. 83], the initial value $y(0)$ for $y \in W_p^{2,1}$ belongs to $W_0^{2-2/p,\,p}(\Omega)^2$.

Corollary 1.5. *Every function of $W_p^{2,1}$ is up to changes on sets of zero measure a continous function with values in $W_0^{2-2/p,\,p}(\Omega)^2$, i.e. the space $W_p^{2,1}$ is continuously imbedded in the space $C([0,T],W_0^{2-2/p,\,p}(\Omega)^2)$.*

Proof. In the case $p = 2$ we have $W_0^{2-2/2,\,2}(\Omega)^2 = V$. Then the claim is equivalent to the imbedding $H^{2,1} \hookrightarrow C([0,T],V)$. For $p > 2$ the imbedding $W_p^{2,1} \hookrightarrow C([0,T],W_0^{2-2/p,\,p}(\Omega)^2)$ is even compact, see Amann [4]. □

For $p > 2$ we have even the following.

Corollary 1.6. *The space $W_p^{2,1}$ is continuously imbedded in $C(\bar{Q})^2$ for $p > 2$.*

Proof. The space $W_p^{2,1}$ is compactly imbedded in $C([0,T],W_0^{2-2/p,\,p}(\Omega)^2)$ for $p > 2$, see [52]. Moreover, the imbedding $W_0^{2-2/p,\,p}(\Omega)^2 \hookrightarrow C(\bar{\Omega})^2$ is continuous by

Theorem 1.1 if $p > 2$. Altogether, there is a continuous imbedding of $W_p^{2,1}$ in $C([0,T], C(\bar{\Omega})^2)$. The latter space can be identified with $C(\bar{Q})^2$. Thus, the claim is proven. $\square$

2 The nonlinear equation

Let us introduce for convenience a trilinear form $b : V \times V \times V \mapsto \mathbb{R}$ as the variational formulation of the Navier-Stokes nonlinearity $(u \cdot \nabla)u$ by

$$b(u, v, w) = ((u \cdot \nabla)v, w)_2 = \int_\Omega \sum_{i,j=1}^n u_i \frac{\partial v_j}{\partial x_i}\; w_j \,\mathrm{d}x.$$

It is continuous, as the following estimate shows.

Lemma 1.7. *Let $u, v, w \in V$ be given. Then it holds*

$$|b(u, v, w)| \le \sqrt{n}\, |u|_4 |v|_V |w|_4. \tag{1.10}$$

In particular, b is continuous from $V \times V \times V$ to $\mathbb{R}$.

Proof. By Hölders inequality, we find

$$\left| \int_\Omega u_i \frac{\partial v_j}{\partial x_i}\; w_j \,\mathrm{d}x \right| \le |u_i|_4 \left| \frac{\partial v_j}{\partial x_i} \right|_2 |w_j|_4.$$

Adding up, (1.10) is shown. Since the imbedding $H_0^1 \hookrightarrow L^4$ is continuous in spatial dimensions $2, 3, 4$, the continuity of b is evident. $\square$

In the two-dimensional case we estimate $|b|$ in terms of H- and V-norms. Applying inequality (1.6), we obtain for $u, v, w \in V$

$$|b(u, v, w)| \le 2\, |u|_H^{1/2} |u|_V^{1/2} |v|_V |w|_H^{1/2} |w|_V^{1/2}. \tag{1.11}$$

Following the ideas of the proof of the previous lemma, one can establish similar estimates involving stronger norms such as H^2. See for instance in [21, 70, 88].

Another very useful property of b is the following:

Lemma 1.8. *For all $u, v, w \in V$ it holds*

$$b(u, v, w) = -b(u, w, v).$$

A proof can be found for instance in [70]. As a simple conclusion of the previous lemma, we get $b(u, v, v) = 0$ for all functions $u, v \in V$.

Now, we are ready to investigate the instationary Navier-Stokes equations. Given $f \in L^2(Q)^2$ and $y_0 \in H$, we are looking for solutions of the system

$$\begin{aligned} y_t - \nu \Delta y + (y \cdot \nabla) y + \nabla p &= f && \text{on } Q, \\ \operatorname{div} y &= 0 && \text{on } Q, \\ y &= 0 && \text{on } \Sigma, \\ y(0) &= y_0 && \text{on } \Omega. \end{aligned} \tag{1.12}$$

Let us assume there is a classical solution, say $y \in C^2(\bar{Q})^2$ and $p \in C^1(\bar{Q})$ satisfying (1.12). If v is an element of V, one finds that

$$(y_t, v)_2 + \nu(y, v)_V + b(y, y, v) = \langle f, v\rangle_{V',V} \tag{1.13}$$

holds for all $t \in (0, T)$. Since y is divergence-free, the pressure disappears in the weak formulation due to $(y(t), \nabla p(t)) = 0$. Equation (1.13) suggests the following formulation of the problem (1.12), which is due to Leray.

For given $f \in L^2(0, T; V')$ and $y_0 \in H$, find a solution $y \in L^2(0, T; V)$ with $y_t \in L^2(0, T; V')$ that fulfills

$$\langle y_t(t),\, v\rangle_{V',V} + \nu(y(t), v)_V + b(y(t), y(t), v) = \langle f(t), v\rangle_{V',V} \quad \forall v \in V,\ a.e.\ on\ (0, T), \tag{1.14a}$$

$$y(0) = y_0. \tag{1.14b}$$

By the imbedding $W(0, T) \hookrightarrow C([0, T]; H)$, the initial condition in (1.14) is meaningful. We want to give an equivalent formulation as an equation in function space. To this aim, we introduce a linear, continuous operator $A : L^2(0, T; V) \mapsto L^2(0, T; V')$ for $y, v \in L^2(0, T; V)$ by

$$\begin{aligned}\langle Ay,\, v\rangle_{L^2(V'),L^2(V)} &= \int_0^T \langle (Ay)(t),\, v(t)\rangle_{V',V}\, \mathrm{d}t \\ &:= \int_0^T (y(t), v(t))_V\, \mathrm{d}t = \int_0^T \nabla y(t) \cdot \nabla v(t)\, \mathrm{d}t\end{aligned}$$

and a nonlinear operator $B : W(0, T) \mapsto L^2(0, T; V')$ for $y \in W(0, T)$, $w \in L^2(0, T; V)$ by

$$\langle B(y),\, v\rangle_{L^2(V'),L^2(V)} = \int_0^T \big\langle \big(B(y)\big)(t),\, w(t)\big\rangle_{V',V}\, \mathrm{d}t := \int_0^T b(y(t), y(t), w(t))\, \mathrm{d}t.$$

The operator B is a bounded mapping from $W(0, T)$ to $L^2(0, T; V')$, i.e. it holds $\|B(y)\|_{L^2(V')} \le c\|y\|_W^2$.

Lemma 1.9. *For all $y \in L^2(0, T; V) \cap L^\infty(0, T; H)$ it holds $B(y) \in L^2(0, T; V')$.*

Proof. Let us introduce a norm in $\tilde{W} := L^2(0, T; V) \cap L^\infty(0, T; H)$ by $\|y\|_{\tilde{W}} := \max\big(||y\|_{L^2(V)}, ||y\|_{L^\infty(H)}\big)$. A closer inspection of Lemma 1.4 reveals that the L^4-norm can be estimated in terms of the $\tilde{W}$-norm by $\|y\|_4 \le \sqrt[4]{2}\|y\|_{\tilde{W}}$. Let $v \in L^2(0, T; V)$ be given. Applying Lemma 1.8 and Hölders inequality we find

$$\begin{aligned}\big|\langle B(y),\, v\rangle_{L^2(V'),L^2(V)}\big| &= \left|\int_0^T b(y(t), y(t), v(t))\, \mathrm{d}t\right| = \left|\int_0^T b(y(t), v(t), y(t))\, \mathrm{d}t\right| \\ &\le \sqrt{2}\int_0^T |y(t)|_4 |v(t)|_V |y(t)|_4\, \mathrm{d}t \le \sqrt{2}\|y\|_4^2\|v\|_{L^2(V)} \\ &\le 2\|y\|_{\tilde{W}}^2\|v\|_{L^2(V)},\end{aligned}$$

and the claim is proven. □

Now, the system (1.14) can be transformed to an operator equation.

Definition 1.10 (Weak solution). *Let $f \in L^2(0,T;V')$ and $y_0 \in H$ be given. A function $y \in L^2(0,T;V)$ with $y_t \in L^2(0,T;V')$, i.e. $y \in W(0,T)$, is called weak solution of* (1.12) *if it fulfills*

$$y_t + \nu A y + B(y) = f \quad \text{in } L^2(0,T;V'), \tag{1.15a}$$

$$y(0) = y_0 \quad \text{in } H. \tag{1.15b}$$

Lemma 1.11. *The weak formulations* (1.14) *and* (1.15) *are equivalent.*

Proof. Since the initial value condition is the same in both formulations, we have to investigate only the equations on $(0,T)$.

At first, let $y \in W(0,T)$ be a solution in the sense of (1.14). We have to show that the equation (1.15a) holds in $L^2(0,T;V')$. Thus, it suffices to prove $\langle y_t + Ay + B(y) - f,\, w\rangle_{L^2(V'),L^2(V)} = 0$ for all test functions $w \in L^2(0,T;V)$. Take $w \in L^2(0,T;V)$ arbitrary. Then, there exists a set $M \subset (0,T)$ of full measure, such that $w(t) \in V$ for all $t \in M$. Consequently, equation (1.14a) holds with $v := w(t)$ and $t \in M$. Now, we test (1.15a) with w. Taking into account that $(0,T) \setminus M$ has zero measure, we obtain

$$\begin{aligned}
&\langle y_t + Ay + B(y) - f,\, w\rangle_{L^2(V'),L^2(V)} = \\
&= \int_0^T \langle y_t(t),\, w(t)\rangle_{V',V} + \nu(y(t), w(t))_V + b(y(t), y(t), w(t)) - \langle f(t),\, w(t)\rangle_{V',V} \\
&= \int_M \langle y_t(t),\, w(t)\rangle_{V',V} + \nu(y(t), w(t))_V + b(y(t), y(t), w(t)) - \langle f(t),\, w(t)\rangle_{V',V} \\
&= 0.
\end{aligned}$$

Hence, we found that any solution satifying (1.14) is also a solution in the sense of (1.15).

Secondly, let $y \in W(0,T)$ be a solution of (1.15). There exist a set $M \subset (0,T)$ of full measure such that $y_t(t) \in V'$, $y(t) \in V$, and $f(t) \in V'$ holds for all $t \in M$. This implies $y_t(t) + (Ay)(t) + (B(y))(t) - f(t) \in V'$ and even $y_t(t) + (Ay)(t) + (B(y))(t) - f(t) = 0$ for all $t \in M$. Therefore, every function $v \in V$ satisfies $\langle (y_t + Ay + B(y) - f)(t),\, v\rangle_{V',V} = 0$ on M. Since M has full measure and v is arbitrary, the system (1.14) is fulfilled, and y is a weak solution with respect to this definition. $\square$

In the sequel, we will work with the second and more handy definition of a weak solution (1.15). As already mentioned, we will restrict our considerations to the spatial two-dimensional case. Here, results concerning the solvability of (1.15) are standard, cf. [21, 70] for proofs and further details. In the three-dimensional case, there arise some difficulties in the analysis of (1.15). It is known that a weak solution exists, which need not be unique. Under an additional a-priori regularity assumption, the solution is unique, see for instance [70]. Therefore, the following results are only valid for two-dimensional domains.

Theorem 1.12 (Existence and uniqueness of solutions). *Let Ω be a bounded and locally Lipschitz domain in $\mathbb{R}^2$. Then for every $f \in L^2(0,T;V')$ and $y_0 \in H$,* *the equation* (1.15) *has a unique solution* $y \in W(0,T)$. *Moreover, the mapping* $(y_0, f) \mapsto y$ *is locally Lipschitz from* $L^2(0,T;V') \times H$ *to* $W(0,T)$.

Proof. For the proof of existence, which is carried out using a Galerkin approximation, we refer to Temam [70]. We will derive an a-priori bound of the solution. The proof of the Lipschitz estimate has to follow the same steps in principle. Since a similar proof will be carried out in Section 3, p. 14, we do not present it here for the sake of brevity and refer to [70].

Now, let $y \in W(0,T)$ be a solution of (1.15). Since y is in $L^2(0,T;V)$, we can test (1.15a) with $\chi_{(0,t)}y$ and get

$$\int_0^t \left(\langle y_t(s),\, y(s)\rangle_{V',V} + \nu|y(s)|_V^2\right) \mathrm{d}s = \int_0^t \langle f(s),\, y(s)\rangle_{V',V}\, \mathrm{d}s.$$

Here, we used the identity $b(y(t), y(t), y(t)) = 0$. The right-hand side can be estimated by $|\langle f,\, y\rangle_{V',V}| \le \frac{1}{2\nu}|f|_{V'}^2 + \frac{\nu}{2}|y|_V^2$. By partial integration (1.8), we derive for the left-hand side

$$\int_0^t \langle y_t(s),\, y(s)\rangle_{V',V}\, \mathrm{d}s = \frac{1}{2}\left(|y(t)|_H^2 - |y_0|_H^2\right)$$

Altogether, we arrive at

$$|y(t)|_H^2 + \nu\int_0^t |y(s)|_V^2\, \mathrm{d}s \le \int_0^t \frac{1}{\nu}|f(s)|_{V'}^2\, \mathrm{d}s + |y_0|_H^2 \le \int_0^T \frac{1}{\nu}|f(s)|_{V'}^2\, \mathrm{d}s + |y_0|_H^2.$$

Hence, $|y(t)|_H^2$ is uniformly bounded on $(0,T)$. Setting $t = T$, we obtain the estimate

$$\|y\|_{L^\infty(H)}^2 + \nu\|y\|_{L^2(V)}^2 \le \frac{1}{\nu}\|f\|_{L^2(V')}^2 + |y_0|_H^2.$$

Now, it remains to investigate $y_t = f - Ay - B(y)$. Since y is in $L^2(0,T;V) \cap L^\infty(0,T;H)$, the functional Ay is in $L^2(0,T;V')$ by definition, whereas $B(y) \in L^2(0,T;V')$ is ensured by Lemma 1.9. Hence, we find

$$\begin{aligned}\|y_t\|_{L^2(V')} &= \|f - Ay - B(y)\|_{L^2(V')}\\ &\le \|f\|_{L^2(V')} + \|y\|_{L^2(V)} + c\|y\|_{L^2(V)}\|y\|_{L^\infty(H)} < \infty,\end{aligned}$$

which implies $y \in W(0,T)$, and the a-priori bound is proven. The uniqueness of solutions is a consequence of the Lipschitz estimate. ☐

2.1 More regular solutions

For more regular data, one expects more regular solutions. The next theorem states some well-known facts, see for instance [70] for the details. Further regularity results can be found in [67, 71, 88].

Theorem 1.13 (Regularity). *For the higher regularity of the weak solutions of* (1.15) *the following holds.*

(i) Let $y_0 \in V$ and $f \in L^2(Q)^2$ be given. Then the weak solution of (1.15) *fulfills*

$$y \in L^2(0,T;H^2(\Omega)^2) \cap L^\infty(0,T;V),$$
$$y_t \in L^2(0,T;H).$$

The solution mapping $(f, y_0) \mapsto (y, y_t)$ is locally Lipschitz continuous between the mentioned spaces.

(ii) Let additionally, $y_0 \in H^2(\Omega)^2 \cap V$ and f, $f_t \in L^2(0,T;V')$ and $f(0) \in L^2(\Omega)^2$ be given. Then the weak solution y of (1.15) *satisfies*

$$y_t \in L^2(0,T;V) \cap L^\infty(0,T;H).$$

(iii) If moreover Ω is of class C^2 and f is in $L^\infty(0,T;H)$ then it holds

$$y \in L^\infty(0,T;H^2(\Omega)^2).$$

Proof. For the proofs of the three statements we refer to Temam [70, Theorems III.3.10, 3.5, 3.6]. The Lipschitz continuity of the solution mapping $(f, y_0) \mapsto y$ in case *(i)* can be obtained using a similar result for the linearized equation of Hinze [44] and estimating the nonlinear term in appropriate norms. □

The regularity of case *(ii)* implies that y is in $C(\bar{Q})^2$, which is a quite strong result. However, from the optimal control point of view this results are less satisfying. The right-hand side f will be formed by the control. One can prove that a locally optimal control is regular enough to fulfill the requirements of *(i)-(iii)*. But, stability estimates can be expected for control constrained problems only in L^p-norms. This means, stability of the optimal state can be only achieved for case *(i)*, which does not yield stability in $C(\bar{Q})^2$. See also Chapter 5, where the stability problem is adressed.

Up to now, we looked for weak solutions in Hilbert spaces. We will extend the concept of weak solutions to Banach spaces, more precisely we will use spaces with L^p-norms instead of L^2.

Definition 1.14 (Strong solution in L^p). *Let be given the source term $f \in L^p(Q)^2$ and the initial value $y_0 \in W_0^{2-2/p,\,p}(\Omega)^2$. A function $y \in L^p(0,T;V_p)$ with $y_t \in L^p(0,T;L^p(\Omega)^2)$ is called* strong solution *of* (1.12) *to the exponent p, $2 < p < \infty$, if*

$$\int_0^T (y_t, \phi)\,\mathrm{d}t + \nu \int_0^T (y, \phi)_V\,\mathrm{d}t + \int_0^T b(y, y, \phi)\,\mathrm{d}t = \int_0^T (f, \phi)\,\mathrm{d}t \tag{1.16}$$

and $y(0) = y_0$ hold for all test functions $\phi \in L^q(0,T;V_q)$, where q is the dual exponent to p, $1/q + 1/p = 1$.

Here the space $W_0^{2-2/p,\,p}(\Omega)^2$ is the natural trace space of $W_p^{2,1}$: Every abstract function of $L^p(0,T;W^{2,p}(\Omega)^2)$ with time derivative in $L^p(0,T;L^p(\Omega)^2)$ is - after changes on a zero measure set - continuous with values in $W_0^{2-2/p,\,p}(\Omega)^2$. Obviously, every strong L^p-solution is a weak solution. For existence of L^p-solutions we have the following theorem.

Theorem 1.15 (L^p-solutions). *Let Ω be of class C^3. Further, let $f \in L^p(Q)^2$ and $y_0 \in W_0^{2-2/p,\,p}(\Omega)^2$ be given with $p \geq 2$. Then the weak solution y of (1.12) in the sense of Definition 1.10 is a strong solution and satisfies*

$$\begin{aligned} y &\in L^p(0,T; W^{2,p}(\Omega)^2 \cap V_p), \\ y_t &\in L^p(0,T; L^p(\Omega)^2). \end{aligned}$$

Moreover, the mapping $(f, y_0) \mapsto y$ is locally Lipschitz continuous, hence the strong solution y is unique.

For $p = 2$ this result reduces to Theorem 1.13(i). The result in non-Hilbert space case $p > 2$ is proven in [86, 87]. It relies on a similar regularity result for instationary Stokes equations of [68]. Theorem 1.15 ensures the existence of strong solutions. Observe, that the regularity of y is more than claimed in the definition. Locally Lipschitz continuity of the solution mapping is proven in Corollary 1.34 below. Analogously as above, we can transform the equation (1.16) into an operator equation.

Corollary 1.16. *Let the assumptions of Theorem 1.15 be fulfilled. Then a function $y \in W_p^{2,1}$ is a strong solution if and only if it satisfies*

$$\begin{aligned} y_t + \nu A y + B(y) &= f \qquad \text{in } L^q(0,T; V_q) \\ y(0) &= y_0. \end{aligned} \tag{1.17}$$

2.2 Linearized equations

We will need in the following some results about linearized equations. Given a state $\bar{y} \in W(0,T)$, we consider the system

$$y_t + \nu A y + B'(\bar{y}) y = f \quad \text{in } L^2(0,T;V'), \tag{1.18a}$$

$$y(0) = y_0 \quad \text{in } H. \tag{1.18b}$$

Here, $B'(\bar{y})y$ denotes the Fréchet derivative of B with respect to the state $\bar{y}$. It is itself a functional of $L^2(0,T;V')$, which for $v \in L^2(0,T;V)$ is given by

$$\langle B'(\bar{y})y,\, v\rangle_{L^2(V'),L^2(V)} = \int_0^T \big(b(\bar{y}(t), y(t), v(t)) + b(y(t), \bar{y}(t), v(t))\big)\,\mathrm{d}t. \tag{1.19}$$

Since B is of quadratic nature, its differentiability is a simple conclusion of the above considerations.

Lemma 1.17. *The operator $B : W(0,T) \mapsto L^2(0,T;V')$ is twice Fréchet differentiable. All derivatives of third or higher order vanish. The first derivative is given by (1.19). It can be estimated as*

$$\|B'(\bar{y})y\|_{L^2(V')} \leq c\|\bar{y}\|_W \|y\|_W. \tag{1.20}$$

As for quadratic functions, the second derivative is independent of $\bar{y}$:

$$\langle B''(\bar{y})[y_1, y_2],\, v\rangle_{L^2(V'),L^2(V)} = \int_0^T b(y_1(t), y_2(t), v(t)) + b(y_2(t), y_1(t), v(t))\,\mathrm{d}t. \tag{1.21}$$

Thus it holds, compare (1.19),

$$\langle B''(\bar{y})[y_1, y_2],\, v\rangle_{L^2(V'),L^2(V)} = \langle B'(y_1)y_2,\, v\rangle_{L^2(V'),L^2(V)}. \tag{1.22}$$

Moreover, for $y = y_1 = y_2$ we obtain

$$\frac{1}{2}\langle B''(\bar{y})[y, y],\, v\rangle_{L^2(V'),L^2(V)} = \langle B(y),\, v\rangle_{L^2(V'),L^2(V)}. \tag{1.23}$$

The proofs of existence and regularity of solutions of the linearized system (1.18) follow the lines of similar proofs regarding the nonlinear equation. Summarizing, we have

Theorem 1.18 (Solutions of the linearized system).

(i) *Let $y_0 \in H$, $f \in L^2(0,T;V')$, and $\bar{y} \in W(0,T)$ be given. Then there exists a unique weak solution $y \in W(0,T)$ of* (1.18)*.*

(ii) *Let $y_0 \in V$, $f \in L^2(Q)^2$, and $\bar{y} \in H^{2,1}$ be given. Then the weak solution of* (1.18) *satisfies also $y \in H^{2,1}$.*

(iii) *Let Ω be of class C^3. Further, let $f \in L^p(Q)^2$ and $y_0 \in W_0^{2-2/p,\,p}(\Omega)^2$ be given with $p \geq 2$. Assume $\bar{y} \in W_p^{2,1}$. Then the weak solution y of* (1.18) *is a solution in the stronger sense of Definition 1.10 and satisfies $y \in W_p^{2,1}$.*

The solution mapping $(f, y_0) \mapsto y$ is linear and continuous between the mentioned spaces. Its norm depends on $\bar{y}$.

Statements *(i)* and *(ii)* are proven in [44, 46]. For the proof of the third statement we refer to Section 4. Using this theorem, we are able to prove Fréchet-differentiability of the solution mapping, see the next section. One can also prove the regularity analogously to Theorem 1.13(ii) and 1.13(iii).

3 The control-to-state mapping

Now, we will take one step towards optimal control of the state equations. We will study the mapping: right-hand side $\mapsto$ solution, the so-called *control-to-state mapping*. At first, we have to specify, how the control is brought into the system. Let $u \in L^2(Q)^2$ denote the control. Then we will use u as a source term in (1.12), e.g. the right-hand side f of (1.15) will be formed by

$$\langle f,\, v\rangle_{L^2(V'),L^2(V)} := \int_Q u(x,t)v(x,t)\,\mathrm{d}x\,\mathrm{d}t. \tag{1.24}$$

Definition 1.19 (Solution mapping). *Consider the system* (1.15)*. The mapping $u \mapsto y$, where y is the weak solution of* (1.15) *with the control right-hand side* (1.24) *and fixed initial value y_0, is denoted by S, i.e. $y = S(u)$.*

Actually, the solution mapping depends on the initial value, $S(u) = S_{y_0}(u)$. Since we are not interested in the properties of $y_0 \mapsto y$, we will treat this dependency implicitly.

3.1 Continuity and Differentiability

Lemma 1.20. *The control-to-state mapping is locally Lipschitz continuous from $L^q(Q)^2$, $4/3 \leq q \leq \infty$, to $L^2(0,T;V) \cap L^\infty(0,T;H)$ in the following sense: For $u_1, u_2 \in L^q(Q)^2 \cap L^2(0,T;V')$, there is a constant $c = c(u_2, y_0)$ such that for $y_i := S(u_i)$ the estimate*

$$\|y_1 - y_2\|_{L^\infty(H)} + \|y_1 - y_2\|_{L^2(V)} \leq c\|u_1 - u_2\|_q$$

is valid.

Remark 1.21. *The lemma states Lipschitz continuity of the solution mapping with respect to some L^q-norms. The reason, why that procedure is succesful, is the fact that the continuity of the imbedding $W(0,T) \hookrightarrow L^4(Q)^2$ implies the imbedding $(L^4(Q)^2)^* = L^{4/3}(Q)^2 \hookrightarrow W(0,T)^*$. However, we need the regularity $u \in L^2(0,T;V')$ to ensure the existence of a solution.*

Proof. [of Lemma 1.20] Let y_1, y_2 be two solutions of (1.15) with the same initial value y_0 and associated with the control functions u_1, u_2, $y_i = S(u_i)$. Theorem 1.12 ensures the regularity $y_i \in W(0,T)$. Denote by y and u the difference of them, $y = y_1 - y_2$ and $u = u_1 - u_2$. We subtract the corresponding operator equations and test with $v = \chi_{(0,t)}y$. Partial integration of the time-derivative term yields

$$\frac{1}{2}|y(t)|_H^2 + \nu \int_0^t |y(s)|_V^2 \,\mathrm{d}s = \int_0^t (u(s), y(s))_{q,q'} \,\mathrm{d}s - \int_0^t b(y(s), y_2(s), y(s)) \,\mathrm{d}s. \tag{1.25}$$

Here, we used $y(0) = y_1(0) - y_2(0) \equiv 0$. Further, the difference of the nonlinear terms was transformed to

$$\begin{aligned} b(y_1, y_1, y_1 - y_2) - b(y_2, y_2, y_1 - y_2) &= b(y_1, y_1 - y_2, y_1 - y_2) + b(y_1 - y_2, y_2, y_1 - y_2) \\ &= b(y_1 - y_2, y_2, y_1 - y_2). \end{aligned}$$

Using Hölders inequality, the inequalities (1.6), and (1.4) we derive

$$\begin{aligned} \int_0^t (u, y)_{q,q'} \,\mathrm{d}s &\leq \int_0^t |u(s)|_q |y(s)|_{q'} \,\mathrm{d}s \leq \left(\int_0^t |u(s)|_q^q \,\mathrm{d}s\right)^{1/q} \left(\int_0^t |y(s)|_{q'}^{q'} \,\mathrm{d}s\right)^{1/q'} \\ &\leq c_a \|u\|_q \left(\int_0^t |y(s)|_H^{q'/2} |y(s)|_V^{q'/2} \,\mathrm{d}s\right)^{1/q'} \\ &\leq c_a \|u\|_q \|y\|_{L^\infty(0,T;H)}^{1/2} \|y\|_{L^{q'/2}(0,t;V)}^{1/2}. \end{aligned}$$

The constants $c_a = 2^{1/4}(\operatorname{meas} Q)^\mu$ and $\mu = 1/q' - 1/4$ are given by (1.6) and (1.4). Notice, that $q' \leq 4$ implies $q'/2 \leq 2$. We can apply (1.4) with respect to the time interval $[0,t]$ to proceed

$$\begin{aligned} \int_0^t (u, y)_{q,q'} \,\mathrm{d}s &\leq c_a T^{2\mu} \|u\|_q \|y\|_{L^\infty(0,T;H)}^{1/2} \|y\|_{L^2(0,t;V)}^{1/2} \\ &\leq c_b \|u\|_q^2 + \frac{1}{\mathcal{N}} \|y\|_{L^\infty(0,T;H)}^2 + \frac{\nu}{2} \|y\|_{L^2(0,t;V)}^2, \end{aligned} \tag{1.26}$$

where $c_b = \frac{1}{4}(\operatorname{meas} Q)^{2\mu} T^{4\mu} \mathcal{N}^{1/2} \nu^{-1/2}$ and a constant $\mathcal{N} > 0$ to be specified later. The nonlinear term is estimated by (1.11),

$$\begin{aligned}\left|\int_0^t b(y(s), y_2(s), y(s))\,\mathrm{d}s\right| &\le 2\int_0^t |y(s)|_H |y(s)|_V |y_2(s)|_V\,\mathrm{d}s \\ &\le \frac{\nu}{4}\int_0^t |y(s)|_V^2\,\mathrm{d}s + \frac{4}{\nu}\int_0^t |y(s)|_H^2 |y_2(s)|_V^2\,\mathrm{d}s.\end{aligned}$$

Inserting these estimates in (1.25), we obtain

$$\frac{1}{2}|y(t)|_H^2 + \frac{\nu}{4}\|y\|_{L^2(0,t;V)}^2 \le c_b\|u\|_q^2 + \frac{1}{\mathcal{N}}\|y\|_{L^\infty(0,T;H)}^2 + \frac{4}{\nu}\int_0^t |y(s)|_H^2 |y_2(s)|_V^2\,\mathrm{d}s. \tag{1.27}$$

Since $y_2 \in L^2(0,T;V)$, the norm square $|y_2(\cdot)|_V^2$ is integrable and Gronwall's lemma applies to get

$$|y(t)|_H^2 \le \exp\left(\frac{8}{\nu}\|y_2\|_{L^2(V)}^2\right)\left(2c_b\|u\|_q^2 + \frac{2}{\mathcal{N}}\|y\|_{L^\infty(0,T;H)}^2\right).$$

Choosing $\mathcal{N} := 8\exp\left(\frac{8}{\nu}\|y_2\|_{L^2(V)}\right)$ yields that the following inequality holds for all $t \in [0,T]$:

$$\frac{1}{2}|y(t)|_H^2 \le \frac{1}{4}\|y\|_{L^\infty(0,T;H)}^2 + c\|u\|_q^2.$$

Here, the $L^\infty(0,T;H)$-norm of y appears on the right-hand side to bound $|y(t)|_H$. Since $y \in L^\infty(0,T;H)$ is given by Theorem 1.12, we can take the maximum for $t \in [0,T]$ on the left-hand side

$$\frac{1}{4}\|y\|_{L^\infty(0,T;H)}^2 \le c\|u\|_q^2.$$

The Lipschitz estimate for the $L^2(V)$-norm of y follows from inequality (1.27), which finishes the proof ☐

Now, are going to prove Fréchet differentiability of the solution mapping.

Lemma 1.22. *The control-to-state mapping is Fréchet differentiable as mapping from $L^2(0,T;V')$ to $W(0,T)$. The derivative at $\bar u \in L^2(0,T;V')$ in direction $h \in L^2(0,T;V')$ is given by $S'(\bar u)u = y$, where y is the weak solution of*

$$y_t + \nu Ay + B'(\bar y)y = h \quad \text{in } L^2(0,T;V'), \tag{1.28a}$$
$$y(0) = 0 \quad \text{in } H. \tag{1.28b}$$

with $\bar y = S(\bar u)$.

Proof. Define $y = S(\bar u + h)$. Using Lemma 1.17 to cope with the occurrences of B, we find that the difference $d := y - \bar y$ is the weak solution of

$$\begin{aligned} d_t + \nu Ad + B'(\bar y)d &= h - B(d) \quad \text{in } L^2(0,T;V'),\\ d(0) &= 0 \quad \text{in } H.\end{aligned}$$

We split d into $d = z + r$, where z and r are the weak solutions of the respective systems

$$\begin{aligned} z_t + \nu A z + B'(\bar{y})z &= h, & r_t + \nu A r + B'(\bar{y})r &= -B(d) && \text{in } L^2(0,T;V'); \\ z(0) &= 0, & r(0) &= 0 && \text{in } H. \end{aligned}$$

By Theorem 1.18(i), we find that the mapping $h \mapsto z$ is continuous from $L^2(0,T;V')$ to $W(0,T)$,

$$\|z\|_W \le c\|h\|_{L^2(V')}.$$

If we show

$$\frac{\|y - \bar{y} - z\|_W}{\|h\|_{L^2(V')}} \to 0 \quad \text{as } \|h\|^2_{L^2(V')} \to 0, \tag{1.29}$$

then the function z will be the Fréchet derivative of S at $\bar{u}$ in direction h and can be denoted by $z = S'(\bar{u})u$. We obtain by subsequent applications of Theorem 1.18(i) and Lemma 1.9

$$\|y - \bar{y} - z\|_W = \|r\|_W \le c\|B(d)\|_{L^2(V')} \le c\|d\|^2_W.$$

By Lipschitz continuity of the solution mapping, see Theorem 1.12, we get

$$\|d\|^2_W = \|y - \bar{y}\|^2_W = \|S(\bar{u} + h) - S(\bar{u})\|^2_W \le c\|h\|^2_{L^2(V')}.$$

Thus (1.29) is fulfilled, and S is Fréchet differentiable with derivative $S'(\bar{u})h = z$. ☐

Combining the results of the previous two Lemmata, we can conclude

Corollary 1.23. *With the notation of the previous theorem, we get for $\bar{u}, h \in L^2(0,T;V') \cap L^q(Q)^2$, $q \ge 4/3$,*

$$\|S(\bar{u} + h) - S(\bar{u}) - S'(\bar{u})h\|_W \le c\|h\|^2_q$$

and

$$\|S'(\bar{u})h\|_W \le c\|h\|^2_q.$$

The solution mapping is even twice continuously differentiable. We will need this property in connection with second-order necessary optimality conditions, which are treated in Section 3.3.

Lemma 1.24. *The control-to-state mapping is twice continuously differentiable as mapping from $L^2(0,T;V')$ to $W(0,T)$. The derivative at $\bar{u} \in L^2(0,T;V')$ in directions $h_1, h_2 \in L^2(0,T;V')$ is given by $S''(\bar{u})[h_1, h_2] = y$, where y is the weak solution of*

$$y_t + \nu A y + B'(\bar{y})y = -B''(\bar{y})[y_1, y_2] \quad \text{in } L^2(0,T;V'), \tag{1.30a}$$

$$y(0) = 0 \quad \text{in } H. \tag{1.30b}$$

with $\bar{y} = S(\bar{u})$ and $y_i = S'(\bar{u})h_i$, $i = 1, 2$.

Proof. Let be given $h_1, h_2 \in L^2(0,T;V')$. Define $y_1 = S'(\bar u)h_1$ and $\tilde y = S'(\bar u + h_2)h_1$. By definition, $\tilde y$ is the weak solution of

$$\begin{aligned} \tilde y_t + \nu A\tilde y + B'(S(\bar u + h_2))\tilde y &= h_1 \quad \text{in } L^2(0,T;V'), \\ \tilde y(0) &= 0 \quad \text{in } H. \end{aligned}$$

We want to write it as $\tilde y = S'(\bar u)f$ with some function f. Using Fréchet differentiability of S, we find $S(\bar u + h_2) = S(\bar u) + S'(\bar u)h_2 + r_a^1(h_2)$ with some remainder term r_a^1 satisfying

$$\frac{\|r_a^1(h_2)\|_W}{\|h_2\|_{L^2(V')}} \to 0 \quad \text{as } \|h_2\|_{L^2(V')} \to 0.$$

Since $B'(\bar y)$ depends linearly on $\bar y$, cf. (1.19), we can write

$$B'(S(\bar u+h_2)) = B'(S(\bar u)) + B'(S'(\bar u)h_2) + B'(r_a^1(h_2)) = B'(\bar y) + B'(y_2) + B'(r_a^1(h_2)).$$

Here, we employed $y_2 = S'(\bar u)h_2$. Thus, $\tilde y$ is also the weak solution of

$$\begin{aligned} \tilde y_t + \nu A\tilde y + B'(\bar y)\tilde y &= h_1 - B'(y_2)\tilde y - B'(r_a^1(h_2))\tilde y \quad \text{in } L^2(0,T;V'), \\ \tilde y(0) &= 0 \quad \text{in } H. \end{aligned}$$

Consequently, it holds

$$\tilde y = S'(\bar u)[h_1 - B'(y_2)\tilde y - B'(r_a^1(h_2))\tilde y] = y_1 - S'(\bar u)[B'(y_2)\tilde y + B'(r_a^1(h_2))\tilde y]. \tag{1.31}$$

The last term will be denoted by $r_b^1(h_2)$, i.e. $r_b^1(h_2) := -S'(\bar u)B'(r_a^1(h_2))\tilde y$. It vanishes if $\|h_2\|_{L^2(V')}$ goes to zero:

$$\|r_b^1\|_W = \|-S'(\bar u)B'(r_a^1(h_2))\tilde y\|_W \le c\|r_a^1(h_2)\|_W\|\tilde y\|_W \le c\|r_a^1(h_2)\|_W\|h_1\|_{L^2(V')}.$$

Here, we used the estimate of B' in (1.20). We will investigate the second term of (1.31). The occurrence of $\tilde y$ is replaced by the right-hand side of (1.31). By (1.22), we can use B'' instead of B' and obtain

$$\begin{aligned} B'(y_2)\tilde y &= B'(y_2)\left\{y_1 - S'(\bar u)[B'(y_2)\tilde y + B'(r_a^1(h_2))\tilde y]\right\} \\ &= B''(\bar y)[y_1, y_2] - B'(y_2)\left\{S'(\bar u)[B'(y_2)\tilde y + B'(r_a^1(h_2))\tilde y]\right\}. \end{aligned}$$

Let us denote the second addend by $\hat f$, e.g. $\hat f = -B'(y_2)\{\dots\}$. It can be estimated as

$$\begin{aligned} \|\hat f\|_{L^2(V')} &= \left\|B'(y_2)\left\{S'(\bar u)[B'(y_2)\tilde y + B'(r_a^1(h_2))\tilde y]\right\}\right\|_{L^2(V')} \\ &\le c\|y_2\|_W\left\{\|y_2\|_W\|\tilde y\|_W + \|r_a^1(h_2)\|_W\|\tilde y\|_W\right\} \\ &\le c\|h_2\|_{L^2(V')}\left\{\|h_2\|_{L^2(V')} + \|r_a^1(h_2)\|_W\right\}\|h_1\|_{L^2(V')}. \end{aligned}$$

For the relevant estimates of S' and B' we refer to Sections 2.2 and 3.1. Now, we can write $\tilde y$ as

$$\begin{aligned} \tilde y &= y_1 - S'(\bar u)[B'(y_2)\tilde y] + r_b^1(h_2) \\ &= y_1 - S'(\bar u)[B''(\bar y)[y_1, y_2] + \hat f] + r_b^1(h_2). \end{aligned} \tag{1.32}$$

So we find for the difference $d = \tilde y - y$

$$d = \tilde y - y = -S'(\bar u)[B''(\bar y)[y_1, y_2] + \hat f] + r_b^1(h_2).$$

By the previous estimates, we get for the remainder term

$$\frac{\|\tilde{y} - y - (-S'(\bar{u})B''(\bar{y})[y_1, y_2])\|_W}{\|h_2\|_{L^2(V')}} = \frac{\|S'(\bar{u})\hat{f}\|_W + \|r_b^1(h_2)\|_W}{\|h_2\|_{L^2(V')}},$$

which converges to 0 as $\|h_2\|_{L^2(V')} \to 0$. This implies that the second derivative of S is given as $S''(\bar{u})[h_1, h_2] = -S'(\bar{u})B''(\bar{y})[y_1, y_2]$ or equivalently as the weak solution of (1.30).

It remains to prove continuity of $S''(u)$ with respect to u. We derived a representation of S'' as the solution of a linear system. Using the definitions of $\bar{y}$, y_1, y_2, we obtain

$$\begin{aligned} S''(\bar{u})[h_1, h_2] &= -S'(\bar{u})B''(\bar{y})[y_1, y_2] \\ &= -S'(\bar{u})B''(S(\bar{u}))[S'(\bar{u})h_1, S'(\bar{u})h_2]. \end{aligned}$$

The solution mappings $S(\bar{u})$ and $S'(\bar{u})$ depend continuously on the paramter $\bar{u}$, while the operator B'' is in fact independent of its argument. Altogether the mapping $\bar{u} \mapsto S''(\bar{u})[h_1, h_2]$ is continuous. $\square$

3.2 Representation of the adjoint mapping

In order to establish first-order optimality conditions, we will need the adjoint operator of $S'(\bar{u})$ denoted by $S'(\bar{u})^*$. This adjoint mapping was investigated by Hinze [44] and Hinze and Kunisch [46]. In contrast to the results there, we can allow inhomogeneous initial conditions for the adjoint state, which in turn enables us to incorporate the terminal value of the state in the objective functional. The operator $S'(\bar{u})$ is the solution mapping of a linearized system with homogeneous initial value. By Lemma 1.22, we can regard $S'(\bar{u})$ as linear operator from $L^2(0, T; V')$ to W_0, where W_0 is defined as a closed linear subspace of $W(0, T)$ by

$$W_0 := \{y \in W(0, T) :\ y(0) = 0\}. \tag{1.33}$$

Hence, the adjoint will be a mapping from W_0^* to $L^2(0, T; V)$.

Lemma 1.25. *Let be $\bar{u} \in L^2(Q)^2$. Then the operator $S'(\bar{u})^*$ is linear and continuous from W_0^* to $L^2(0, T; V)$. Its action is defined as follows. Take g in W_0^*. Then $\lambda = S'(\bar{u})^* g$ holds if and only if*

$$\langle w_t + \nu A w + B'(\bar{y})w,\ \lambda\rangle_{L^2(V'), L^2(V)} = \langle g,\ w\rangle_{W_0^*, W_0} \tag{1.34}$$

for all $w \in W_0$. Moreover, we have $\|\lambda\|_{L^2(V)} \le c\|g\|_{W^*}$.

Proof. Observe at first, that the system (1.28) defining $S'(\bar{u})$ can be written equivalently as follows: find $y \in W_0$ such that for all $v \in L^2(0, T; V)$ the equation

$$\langle y_t + \nu A y + B'(\bar{y})y - h,\ v\rangle_{L^2(V'), L^2(V)} = 0$$

holds. Defining an operator $T : W_0 \to L^2(0, T; V')$ by

$$Ty = y_t + \nu A y + B'(\bar{y})y,$$

this equation can be written as

$$Ty = h.$$

By construction, we have that $S'(\bar{u})$ and T are inverse to each other, i.e. $S'(\bar{u})^{-1} = T$. By Theorem 1.18(i), the inverse T^{-1} is continuous from $L^2(0,T;V')$ to $W_0(0,T)$. Now, we investigate the adjoint T^*, which is linear and continuous from $L^2(0,T;V)$ to $W_0(0,T)^*$. Its action is given by

$$\langle T^*v,\, y\rangle_{W_0^*,W_0} = \langle y_t + \nu Ay + B'(\bar{y})y,\, v\rangle_{L^2(V'),L^2(V)}$$

for $v \in L^2(0,T;V)$, $y \in W_0$. Its inverse exists by $T^{-*} = (T^*)^{-1} = (T^{-1})^*$ and is a linear and continuous operator from W_0^* to $L^2(0,T;V)$. More specificly, given an functional $g \in W_0^*$ there exists a unique solution $\lambda \in L^2(0,T;V)$ satisfying for all $w \in W_0$

$$\langle T^*\lambda,\, w\rangle_{W_0^*,W_0} = \langle w_t + \nu Aw + B'(\bar{y})w,\, \lambda\rangle_{L^2(V'),L^2(V)} = \langle g,\, w\rangle_{W_0^*,W_0},$$

which is the claim. $\square$

Here, we get the representation of the adjoint solution operator. Since we do not know at this moment whether the time derivative of λ exists, we are not allowed to write (1.34) as an evolutionary equation. Fortunately, we get the existence of the time derivative if the right-hand side is more regular than W_0^*.

Lemma 1.26. *Let be $\bar{u} \in L^2(Q)^2$ given. Suppose the right-hand side of* (1.34) *g is in $W_0^* \cap L^{4/3}(0,T;V')$. Then $\lambda = S'(\bar{u})^*g$ is given equivalently as the weak solution of*

$$-\lambda_t + \nu A\lambda + B'(\bar{y})^*\lambda = g \quad \text{in } L^{4/3}(0,T;V'), \tag{1.35a}$$

$$\lambda(T) = 0. \tag{1.35b}$$

Moreover, the time derivative λ_t is in $L^{4/3}(0,T;V')$, and there is a constant c depending on $\bar{y}$ but not on λ, g such that the estimate

$$\|\lambda\|_{L^2(V)} + \|\lambda_t\|_{L^{4/3}(V')} \le c\left(\|g\|_{L^{4/3}(V')} + \|g\|_{W_0^*}\right)$$

is fulfilled.

Proof. At first, we prove that the solution λ of (1.34) is also a solution of (1.35) and has a time derivative in $L^{4/3}(0,T;V')$.

We will show that the weak time-derivative of λ can be identified with $\nu A\lambda + B'(\bar{y})^*\lambda - g$. To this aim, we have to do some partial integration.

The operator A is self-adjoint from $L^2(0,T;V)$ to $L^2(0,T;V')$. Since $\lambda \in L^2(0,T;V)$ holds, we can write

$$\langle \nu Aw,\, \lambda\rangle_{L^2(V'),L^2(V)} = \langle \nu A\lambda,\, w\rangle_{L^2(V'),L^2(V)} \quad \forall w \in L^2(0,T;V).$$

The adjoint of $B'(\bar{y})$, called $B'(\bar{y})^*$, is a linear and continuous operator from $L^2(0,T;V)$ to $W^* \subset W_0^*$ by definition, compare Lemma 1.17. It can be written as

$$\begin{aligned}\langle B'(\bar{y})^*\lambda,\, w\rangle_{W_0^*,W_0} &= \int_0^T b(\bar{y}(t), w(t), \lambda(t)) + b(w(t), \bar{y}(t), \lambda(t))\,\mathrm{d}t \\ &= -\int_0^T b(\bar{y}(t), \lambda(t), w(t)) + b(w(t), \lambda(t), \bar{y}(t))\,\mathrm{d}t.\end{aligned}$$

Using the same argumentation as in Lemma 1.9, we can estimate

$$|\langle B'(\bar{y})^*\lambda,\, w\rangle_{W_0^*,W_0}| \le c\|\bar{y}\|_4\|\lambda\|_{L^2(V)}\|w\|_4 \le c\|\bar{y}\|_W\|\lambda\|_{L^2(V)}\|w\|_4. \tag{1.36}$$

This proves that for $\lambda \in L^2(0,T;V)$ we have $B'(\bar{y})^*\lambda \in (L^4(Q)^2)^* = L^{4/3}(Q)^2 \subset L^{4/3}(0,T;V')$. By the imbedding $W(0,T) \hookrightarrow L^4(Q)^2$ it is also an element of W_0^*. Hence, we find that

$$\langle w_t,\, \lambda\rangle_{L^2(V'),L^2(V)} = \langle g - \nu A\lambda - B'(\bar{y})^*\lambda,\, w\rangle_{L^{4/3}(V'),L^4(V)} \tag{1.37}$$

holds for all $w \in W_0 \cap L^4(0,T;V)$. Thus, we have the existence and representation of the derivative $\lambda_t = -(g - \nu A\lambda - B'(\bar{y})^*\lambda)$ in the sense of vector-valued distributions. In the previous considerations we found $g - \nu A\lambda - B'(\bar{y})^*\lambda \in L^{4/3}(0,T;V')$, which allows us to conclude $\lambda_t \in L^{4/3}(0,T;V')$. We can estimate its norm using (1.36) and the continuity estimate in Lemma 1.26

$$\begin{aligned}\|\lambda_t\|_{L^{4/3}(V')} &= \|g - \nu A\lambda - B'(\bar{y})^*\lambda\|_{L^{4/3}(V')}\\ &\le \|g\|_{L^{4/3}(V')} + \nu\|\lambda\|_{L^2(V)} + c\|\bar{y}\|_W\|\lambda\|_{L^2(V)}\\ &\le c(1+\|\bar{y}\|_W)(\|g\|_{L^{4/3}(V')} + \|g\|_{W_0^*}).\end{aligned}$$

Since in (1.37) the value of $w(T)$ can be arbitrary, we get even $\lambda(T) = 0$, otherwise there would be an extra term $\langle\lambda(T),\, w(T)\rangle$ on the right-hand side of (1.37). Summarizing, λ is the solution of

$$-\lambda_t + \nu A\lambda + B'(\bar{y})^*\lambda = g \quad \text{in } \left(W_0 \cap L^4(0,T;V)\right)^*, \tag{1.38a}$$

$$\lambda(T) = 0. \tag{1.38b}$$

We know already that $-\lambda_t + \nu A\lambda + B'(\bar{y})^*\lambda - g$ is an element of $L^{4/3}(0,T;V')$. Density arguments justify that equation (1.38a) is fulfilled also in the larger space $L^{4/3}(0,T;V')$.

To argue that every solution of (1.35) is also a solution of (1.34), one can carry out the same steps backwards using smooth test functions. □

If the optimal control problem involves the final value $y(T)$ in the objective, then the corresponding inhomogeneity g of (1.34) has only the regularity W_0^*. But we can decompose g into an addend of class $L^{4/3}(0,T;V')$ and one addend related to the final value. And we can show that the solution λ of (1.34) is the weak solution of (1.38) but now with a non-zero terminal condition.

Corollary 1.27. *Let be $\bar{u} \in L^2(Q)^2$ given. Suppose the right-hand side g of (1.34) is in the form $g = g_1 + g_2$ with functionals $g_1 \in L^{4/3}(0,T;V') \cap W_0^*$ and $g_2 \in W_0^*$ defined by $g_2(w) = (g_T,\, w(T))_2$, $g_T \in H$. Then $\lambda = S'(\bar{u})^*g$ is the weak solution of*

$$-\lambda_t + \nu A\lambda + B'(\bar{y})^*\lambda = g_1 \quad \textit{in } L^{4/3}(0,T;V'), \tag{1.39a}$$

$$\lambda(T) = g_T. \tag{1.39b}$$

Furthermore, it holds $\lambda \in W^{4/3}(0,T)$.

Proof. We can follow the proof of the previous lemma. Instead of equation (1.37) we obtain for $w \in W_0(0,T)$

$$\langle w_t,\, \lambda\rangle_{L^2(V'),L^2(V)} = \langle g_1 - \nu A\lambda - B'(\bar{y})^*\lambda,\, w\rangle_{L^{4/3}(V'),L^4(V)} + (g_T, w(T))_2. \tag{1.40}$$

Using test functions with $w(0) = w(T) = 0$, we get the representation and the same regularity of the time derivative λ_t as above. The terminal condition (1.39b) is a consequence of the continuous — even compact — imbedding of $W^{4/3}(0,T)$ in $C([0,T];V')$, cf. [4], and the density of $C^\infty(0,T;V)$ in $W^{4/3}(0,T)$. Thus, the claim can be proven by partial integration. □

We can not give an estimation of the norm of λ in dependence of g_T in a direct way as a result of the previous proof. The common method to achieve such an estimate is to test the respective equation by the solution itself. However, this is not permitted in our case due to lack of regularity of λ. The equation (1.39a) is formulated in $L^{4/3}(0,T;V')$ but the state λ is only in $L^2(0,T;V)$ and cannot be taken as test function. Here, we have to use another technique.

Corollary 1.28. *Let the conditions of Corollary 1.27 be satisfied. Then the solution λ of* (1.39) *satisfies*

$$\|\lambda\|_{W^{4/3}} \leq c\left(\|g_1\|_{L^{4/3}(V')} + \|g_1\|_{W_0^*} + |g_T|_H\right)$$

with a constant c depending on $\bar{y}$ but not on g_1, g_T.

Proof. At first, denote by ζ the weak solution of

$$\begin{aligned} -\zeta_t + \nu A\zeta &= 0, \\ \zeta(T) &= g_T. \end{aligned}$$

Its existence and regularity $\zeta \in W(0,T)$ follows from solvability of the instationary Stokes-equation, cf. [70]. Moreover, we get the continuity estimate

$$\|\zeta\|_W \leq c\,|g_T|_H. \tag{1.41}$$

Further, let z be the weak solution of

$$\begin{aligned} -z_t + \nu Az + B'(\bar{y})^*z &= g_1 - B'(\bar{y})^*\zeta \\ z(T) &= 0. \end{aligned}$$

By Lemma 1.26, a unique solution exists, and we have the estimate

$$\|z\|_{W^{4/3}} \leq c\left(\|g_1\|_{L^{4/3}(V')} + \|g_1\|_{W_0^*} + \|B'(\bar{y})^*\zeta\|_{W_0^*} + \|B'(\bar{y})^*\zeta\|_{L^{4/3}(V')}\right) \tag{1.42}$$

Similar to the estimate (1.36), we find

$$\|B'(\bar{y})^*\zeta\|_{W_0^*} + \|B'(\bar{y})^*\zeta\|_{L^{4/3}(V')} \leq c\|\bar{y}\|_W\|\zeta\|_{L^2(V)} \leq c\|\bar{y}\|_W|g_T|_H. \tag{1.43}$$

We construct a solution of the inhomogeneous adjoint equation (1.39) by $\lambda = z + \zeta$. Using (1.41) – (1.43), we get

$$\|\lambda\|_{W^{4/3}} \leq \|z\|_{W^{4/3}} + \|\zeta\|_W \leq c\,(1 + \|\bar{y}\|_W)\left\{\|g_1\|_{L^{4/3}(V')} + \|g_1\|_{W_0^*} + |g_T|_H\right\},$$

and the claim is proven. □

We will summarize the results about the system defining the adjoint state in the next theorem.

Theorem 1.29. *Let be given $\bar{y} \in W(0,T)$, $g_1 \in L^{4/3}(0,T;V') \cap W_0^*$, and $g_T \in H$.*

Then there exists a unique weak solution λ of the system

$$-\lambda_t + \nu A\lambda + B'(\bar{y})^*\lambda = g_1 \qquad \text{in } L^{4/3}(0,T;V'), \tag{1.44a}$$
$$\lambda(T) = g_T. \tag{1.44b}$$

There is a constant c depending on $\bar{y}$ such the estimate

$$\|\lambda\|_{W^{4/3}} \leq c\,(1 + \|\bar{y}\|_W)\left\{\|g_1\|_{L^{4/3}(V')} + \|g_1\|_{W_0^*} + |g_T|_H\right\}$$

holds.

Observe, that the conditions of the previous theorem require the initial value to be in H, whereas the regularity $\lambda \in W^{4/3}(0,T)$ does not guarantee $\lambda(t) \to g_T$ in H for $t \to T$. Since λ_t is integrable and $\lambda(T)$ is in H, it is evident that λ is continuous in $[0,T]$ with values in V'.

Let us comment on available imbedding results. Amann [4] proves compactness of the imbedding $W^{4/3}(0,T) \hookrightarrow C([0,T];H^{-s})$ for $s > 1/3$. The counterexamples in [65] show that there is no continuous imbedding of $W^{4/3}(0,T)$ into $C([0,T];H^{-s})$ for $s < 1/3$. Here, H^{-s} denote the Sobolev-Slobodecki space. Hence, we can conclude that $\lambda(t)$ converges to g_T in H^{-s} for $s > 1/3$ but not in a stronger norm, e.g. not in spaces of integrable functions.

If the right-hand side g_1 is in $L^2(0,T;V') \subset L^{4/3}(0,T;V') \cap W_0^*$ then we get the following estimate, but the regularity of λ remains unchanged.

Corollary 1.30. *Let be given $\bar{y} \in W(0,T)$, $g_1 \in L^2(0,T;V')$, and $g_T \in H$. Then there exists a unique weak solution λ of the system* (1.44)*, and there is a constant c depending on $\bar{y}$ such that the estimate*

$$\|\lambda\|_{W^{4/3}} \leq c\,(1 + \|\bar{y}\|_W)\left\{\|g_1\|_{L^2(V')} + |g_T|_H\right\}$$

holds.

3.3 More regular solutions of the adjoint system

In the previous section, we derived the system defining the adjoint state λ. Due to a lack of regularity of the time derivative λ_t, the argumentation was quite complicated and technical. If the data is more regular then things are much easier. For convenience, we repeat the system of equations under consideration. Given data g_T and g_1, we are looking for a weak solution λ of

$$\begin{aligned} -\lambda_t + \nu A\lambda + B'(\bar{y})^*\lambda &= g_1, \\ \lambda(T) &= g_T. \end{aligned} \tag{1.45}$$

If $\bar{y}$, g_1, g_T are more regular than required in the previous section, then there is no gap of regularity between the adjoint state λ and the state $\bar{y}$, both enjoy the same regularity under similar assumptions on the data.

Theorem 1.31.

(i) Let $g_T \in V$, $g_1 \in L^2(Q)^2$, and $\bar{y} \in H^{2,1}$ be given. Then the weak solution λ of (1.45) *is also in $H^{2,1}$.*

(ii) *Let Ω be of class C^3. Further, let $f \in L^p(Q)^2$ and $y_0 \in W_0^{2-2/p,\,p}(\Omega)^2$ be given with $p \geq 2$. Assume $\bar{y} \in W_p^{2,1}$. Then the weak solution λ of* (1.45) *is a solution in the stronger sense of Definition 1.10 and satisfies $\lambda \in W_p^{2,1}$.*

The solution mapping $(g_1, g_T) \mapsto \lambda$ is linear and continuous between the mentioned spaces. Its norm depends on $\bar{y}$.

Statement *(i)* is proven in [44, 46] following the lines of similar proofs of [70]. The proof of the second statement is carried out in the following section. One can also prove regularity analogously to Theorem 1.13(ii) and 1.13(iii). But such results are not necessary, the regularity provided by case (ii) is enough for our concerns.

4 L^p-solutions of the linearized and adjoint equations

Here, we will provide the proofs for the existence and L^p-regularity of the linearized and the adjoint equations, since in the literature one can only find estimates for the nonlinear equation itself. Throughout the section, we assume that the domain Ω is of class C^3.

Let a function $\bar{y} \in L^p(0,T;W^{2,p}(\Omega)^2) \cap L^\infty(0,T;W_0^{2-2/p,\,p}(\Omega)^2)$ be given. Then we are looking for solutions of the linearized system

$$\begin{aligned} y_t + \nu A y + B'(\bar{y})y &= f, \\ y(0) &= y_0. \end{aligned} \tag{1.46}$$

We will show that this system admits a unique solution, which belongs to the space $W_p^{2,1}$ and depends continuously on the data.

To this end, we will rely on a regularity result for the instationary Stokes equations,

$$\begin{aligned} y_t + \nu A y &= f, \\ y(0) &= y_0. \end{aligned} \tag{1.47}$$

Concerning L^p-solutions, the following result is due to Solonnikov [68] for the two- and three-dimensional case. In v. Wahl [86] it was generalized to arbitrary spatial dimensions. A similar result can be found in [69], where equations with the linear term $(\bar{y} \cdot \nabla)y$ instead of $B'(\bar{y})y = (\bar{y} \cdot \nabla)y + (y \cdot \nabla)\bar{y}$ are studied.

Theorem 1.32. [68] *Let $p > 1$, $p \neq 3/2$, $y_0 \in W_0^{2-2/p,\,p}(\Omega)^2$, $f \in L^p(Q)^2$. Then there exists a unique weak solution y of* (1.47) *satisfying $y \in W_p^{2,1}$. Furthermore, there exists a constant $c > 0$ such that the estimate*

$$\|y_t\|_p + \|y\|_{L^p(W^{2,p})} \leq c\left\{\|f\|_p + |y_0|_{W^{2-2/p,\,p}}\right\}$$

is satisfied.

Now, we can prove the regularity result for the linearized system stated in Theorem 1.18(iii).

Theorem 1.33. *Let $\bar{y} \in L^p(0,T;W^{2,p}(\Omega)^2) \cap L^\infty(0,T;W_0^{2-2/p,\,p}(\Omega)^2)$, $f \in L^p(Q)^2$ and $y_0 \in W_0^{2-2/p,\,p}(\Omega)^2$ be given with $2 \leq p < \infty$. Then the system* (1.46) *has a unique solution $y \in W_p^{2,1}$. Moreover, there is a constant $c > 0$ independent of f and y_0 such that the following estimate holds*

$$\|y\|_{W_p^{2,1}} \leq c\left\{\|f\|_p + |y_0|_{W^{2-2/p,\,p}}\right\}. \tag{1.48}$$

Proof. STEP 1: $p = 2$. The existence of a unique weak solution was proven in [46] together with the estimate in the case $p = 2$. We write the system (1.46) in a slightly modified form

$$\begin{aligned} y_t + \nu A y &= f - B'(\bar{y})y, \\ y(0) &= y_0 \end{aligned} \tag{1.49}$$

to estimate y in terms of f, $B'(\bar{y})y$, and y_0. Here, we want to apply Theorem 1.32. To this end, we have to estimate L^p-norm of the right-hand side of (1.49) for different values of p. The proof is then carried out using bootstrapping arguments.

STEP 2: $2 < p < 4$. From the previous step, we know the existence of a unique weak solution $y \in W_2^{2,1}$ of (1.46). Let us investigate $B'(\bar{y})y = (y \cdot \nabla)\bar{y} + (\bar{y} \cdot \nabla)y$.

By assumption, we have the regularity $\bar{y} \in L^\infty(0,T; W_0^{2-2/p,\,p}(\Omega)^2)$. The trace space $W_0^{2-2/p,\,p}(\Omega)^2$ is continuously imbedded in $W^{1,q}(\Omega)^2$ for $q = \frac{2p}{4-p}$, $2 \le p < 4$, cf. [2, 72]. Furthermore, the space V is continuously imbedded in $L^{q'}(\Omega)^2$ for all $q' < \infty$. Applying Hölders inequality with $\frac{1}{p} = \frac{1}{q} + \frac{1}{q'}$, we obtain

$$\|(y \cdot \nabla)\bar{y}\|_p \le c\|y\|_{L^\infty(L^{q'})}\|\bar{y}\|_{L^\infty(W^{1,q})} \le c\|y\|_{L^\infty(V)}\|\bar{y}\|_{L^\infty(W^{2-2/p,\,p})}. \tag{1.50}$$

The estimation of the second addend of $B'(\bar{y})y$ needs a bit more effort. Using the interpolation identity [9, Theorem 6.4.5]

$$[W^{2,2}(\Omega)^2, W^{1,2}(\Omega)^2]_\theta = W^{2-2/p,\,2}(\Omega)^2, \quad \theta = 1 - 2/p,$$

and the imbedding $W^{2-2/p,\,2}(\Omega)^2 \hookrightarrow W^{1,p}(\Omega)^2$, which is valid for $2 \le p < 4$, we find for a.a. $t \in [0,T]$

$$|y(t)|^p_{W^{1,p}} \le c|y(t)|^p_{W^{2-2/p,\,2}} \le c|y(t)|^{p-2}_{W^{2,2}}|y(t)|^2_{W^{1,2}}.$$

Integrating with respect to the time variable yields

$$\|y\|^p_{L^p(W^{1,p})} \le c\|y\|^{p-2}_{L^{p-2}(W^{2,2})}\|y\|^2_{L^\infty(V)} \le c\|y\|^{p-2}_{L^2(W^{2,2})}\|y\|^2_{L^\infty(V)} \le c\|y\|^p_{W_2^{2,1}} \tag{1.51}$$

provided that $p \le 4$ holds. And, we can derive

$$\|(\bar{y} \cdot \nabla)y\|_p \le c\|\bar{y}\|_\infty\|y\|_{L^p(W^{1,p})}. \tag{1.52}$$

Collecting (1.50)–(1.52), we find

$$\|B'(\bar{y})y\|_p \le c\|\bar{y}\|_{W_p^{2,1}}\|y\|_{W_2^{2,1}} \le c\|\bar{y}\|_{W_p^{2,1}}\left\{\|f\|_2 + |y_0|_V\right\}.$$

Now, we can utilize Theorem 1.32 to obtain the solution estimate

$$\begin{aligned} \|y\|_{W_p^{2,1}} &\le c\left\{\|f\|_p + |y_0|_{W^{2-2/p,\,p}} + \|B'(\bar{y})y\|_p\right\} \\ &\le c\left\{\|f\|_p + |y_0|_{W^{2-2/p,\,p}}\right\} + c\|\bar{y}\|_{W_p^{2,1}}\left\{\|f\|_2 + |y_0|_V\right\} \\ &\le c(1 + \|\bar{y}\|_{W_p^{2,1}})\left\{\|f\|_p + |y_0|_{W^{2-2/p,\,p}}\right\}. \end{aligned}$$

Thus, the (weak) solution y is of class $W_p^{2,1}$. The test functions in the definition of weak solutions were in $L^2(0,T;V)$, which is dense in the function space $L_q(0,T;V_q)$, which was used in the definition of strong solutions. Since the weak solution y is regular enough, it is easy to verify that y is also a strong solution. Since every strong solution is a weak solution, and weak solutions are unique, it follows that y is the unique strong solution of the linearized system. This completes the proof for exponents $p \in (2,4)$.

STEP 3: $4 \le p < \infty$. By Step 2, the solution y of (1.46) is in $W^{2,1}_{4-\epsilon}$, $0 < \epsilon \le 2$. It is — after changes on a set of zero measure — continuous with values in the space $W_0^{2-2/(4-\epsilon),4-\epsilon}(\Omega)^2$, which is itself continuously imbedded in $L^\infty(\Omega)^2$. Hence, the imbedding of $W^{2,1}_{4-\epsilon}$ in $L^\infty(Q)^2$ is continuous.

Again, we have to estimate the L^p-norm of $B'(\bar y)y$. We begin with its first addend, which can be treated by

$$\|(y\cdot\nabla)\bar y\|_p \le c\|y\|_\infty \|\nabla \bar y\|_p \le c\|y\|_{W^{2,1}_{4-\epsilon}} \|\bar y\|_{W^{2,1}_p}. \tag{1.53}$$

To estimate the second addend of $B'(\bar y)y$, we observe that for $\epsilon = \frac{8}{p+2}$ the imbedding

$$W_0^{2-\frac{2}{4-\epsilon},4-\epsilon}(\Omega)^2 = W_0^{\frac{3}{2}-\frac{1}{p},\frac{4p}{p+2}}(\Omega)^2 \hookrightarrow W_0^{1,p}(\Omega)^2$$

is continuous. Consequently, we obtain for this choice of ϵ

$$y \in W^{2,1}_{4-\epsilon} \hookrightarrow L^\infty(0,T;W_0^{1,p}(\Omega)^2).$$

Hence, we arrive at

$$\|(\bar y\cdot\nabla)y\|_p \le c\|\bar y\|_\infty \|y\|_{L^\infty(W^{1,p})} \le c\|\bar y\|_{W^{2,1}_p}\|y\|_{W^{2,1}_{4-\epsilon}}, \tag{1.54}$$

which allows as to conclude by Theorem 1.32

$$\|y\|_{W^{2,1}_p} \le c(1+\|\bar y\|_{W^{2,1}_p})\left\{\|f\|_p + |y_0|_{W^{2-2/p,p}}\right\},$$

and the claim is proven for all p in $[2,\infty)$. □

The Lipschitz continuity of the solution mapping of the instationary Navier-Stokes equations with respect to L^p-norms can be proven using the previous Lemma. Let data $f_i \in L^p(Q)^2$ and $y_{0,i} \in W_0^{2-2/p,p}(\Omega)^2$ be given, $i = 1,2$. Denote the associated strong solutions by y_i, $i = 1,2$. Then the difference $y := y_1 - y_2$ satisfies

$$\begin{aligned} y_t + \nu A y + (y_1\cdot\nabla)y + (y\cdot\nabla)y_2 &= f_1 - f_2, \\ y(0) &= y_{0,1} - y_{0,2}. \end{aligned} \tag{1.55}$$

Corollary 1.34. *Let Ω be of class C^3. Further, let $f_i \in L^p(Q)^2$ and $y_{0,i} \in W_0^{2-2/p,p}(\Omega)^2$ be given with $p \ge 2$ for $i = 1,2$. Then there is a constant c independent of $(y_i, f_i, y_{0,i})$, $i = 1,2$, such that the estimate*

$$\|y_1 - y_2\|_{W^{2,1}_p} \le c(1+\|y_1\|_{W^{2,1}_p} + \|y_2\|_{W^{2,1}_p})\left\{\|f_1 - f_2\|_p + |y_{0,1} - y_{0,2}|_{W^{2-2/p,p}}\right\},$$

holds for the associated solutions y_1 and y_2.

Proof. At first we get by Theorem 1.15 that the solutions y_i are in $W^{2,1}_p$.

The function $y = y_1 - y_2$ fulfills equation (1.55). The difference to equation (1.46), which is studied in the previous theorem, is the appearance of $(y_1 \cdot$

$\nabla)y + (y \cdot \nabla)y_2$ instead of the linearized nonlinearity $B'(\bar{y})y$. However, both addends of $B'(\bar{y})y = (\bar{y} \cdot \nabla)y + (y \cdot \nabla)\bar{y}$ were estimated independent of each other, compare the estimates (1.50), (1.52)–(1.54).

Since the solutions y_1, y_2, and consequently $y = y_1 - y_2$ are regular enough, we can apply the technique of the proof of the previuos theorem. We only have to replace $\|\bar{y}\|_{W_p^{1,2}}$ by $\|y_1\|_{W_p^{1,2}} + \|y_2\|_{W_p^{1,2}}$. $\square$

Now, we want to prove a similar result for the adjoint equation. For convenience let us recall the system under consideration

$$\begin{aligned} -\lambda_T + \nu A\lambda + B'(\bar{y})^*\lambda &= g_1, \\ \lambda(T) &= g_T. \end{aligned} \tag{1.56}$$

It differs only in the term $B'(\bar{y})^*$ from the linearized equations studied above. Hence, the analysis is shorter due to the use of similar arguments.

Theorem 1.35. *Let $g_1 \in L^p(Q)^2$ and $g_T \in W_0^{2-2/p,\,p}(\Omega)^2$ be given with $p \geq 2$. If $\bar{y} \in L^p(0,T;W^{2,p}(\Omega)^2) \cap L^\infty(0,T;W_0^{2-2/p,\,p}(\Omega)^2)$, then the weak solution λ of (1.56) is a strong solution and satisfies $\lambda \in W_p^{2,1}$. Moreover, the adjoint state λ depends continuously on the given data g_1, g_T, $\bar{y}$.*

Proof. The result in the case $p = 2$ is equivalent to Theorem 1.31(i). Let us sketch the proof for the case $p > 2$. Via the transformations $w(t) = \lambda(T-t)$, $\hat{y}(t) = \bar{y}(T-t)$, $\hat{g}(t) = g_1(T-t)$, $w_0 = g_T$, this system is carried over to the forward-in-time equation

$$\begin{aligned} w_t + \nu Aw + B'(\hat{y})^*w &= \hat{g} \\ w(0) &= w_0. \end{aligned} \tag{1.57}$$

Obviously, $\hat{y}, \hat{g}, w_0$ inherits their regularity from $\bar{y}, g_1, g_T$. Hence, the adjoint state λ has the same regularity as the auxiliary state w. The proof is finished if the next Lemma is verified. $\square$

Lemma 1.36. *Let $\hat{y} \in L^p(0,T;W^{2,p}(\Omega)^2) \cap L^\infty(0,T;W_0^{2-2/p,\,p}(\Omega)^2)$, $\hat{g} \in L^p(Q)^2$ and $w_0 \in W_0^{2-2/p,\,p}(\Omega)^2$ be given with $2 \leq p < \infty$. Then the system (1.57) has a unique solution $w \in W_p^{2,1}$. Moreover, there is a constant $c > 0$ independently of $\hat{g}$ and w_0 such that the following estimate is true*

$$\|w\|_{W_p^{2,1}} \leq c\left\{\|\hat{g}\|_p + |w_0|_{W^{2-2/p,\,p}}\right\}.$$

Proof. The proof is very similar to the proof of Theorem 1.33. Therefore, we will briefly repeat its steps. At first, let us investigate the action of $B'(\hat{y})^*w$ on a test function $v \in W(0,T)$:

$$\begin{aligned} [B'(\hat{y})^*w]v &= b_Q(\hat{y},v,w) + b_Q(v,\hat{y},w) = -b_Q(\hat{y},w,v) + b_Q(v,\hat{y},w) \\ &= \sum_{i,j=1}^{2} \int_Q \left(-\hat{y}_i(x,t)\frac{\partial w_j(x,t)}{\partial x_i} v_j(x,t) + v_i(x,t)\frac{\partial \hat{y}_j(x,t)}{\partial x_i} w_j(x,t) \right) dxdt \\ &= \int_0^T [-(\hat{y} \cdot \nabla)w + (\nabla \hat{y})^T w] \cdot v \, dt. \end{aligned}$$

Here, we used the identity $b_Q(y,v,w) = -b_Q(y,w,v)$, which holds for functions $y,v,w \in L^2(0,T;V)$ [70]. Consequently, we are allowed to identify the functional $B(\hat{y})^*w$ with the function $-(\hat{y}\cdot\nabla)w+(\nabla\hat{y})^T w$. For $\hat{y}, w \in H^{2,1}$ we find $-(\hat{y}\cdot\nabla)w+(\nabla\hat{y})^T w \in L^2(Q)^2$.

STEP 1: $p=2$. The result for $p=2$ was proven, for instance, in [46]. It yields the existence of a unique weak solution $w \in W_2^{2,1} = H^{2,1}$ and the existence of a constant $c>0$ such that

$$\|w\|_{H^{2,1}} \leq c\,\{\|\hat{g}\|_2 + |w_0|_V\}$$

is satisfied.

STEP 2: $2<p<4$. With the help of (1.50)-(1.52), we conclude

$$\begin{aligned}\|B'(\hat{y})^*w\|_p &= \|-(\hat{y}\cdot\nabla)w+(\nabla\hat{y})^T w\|_p \\ &\leq c\left(\|\hat{y}\|_\infty\|w\|_{W_2^{2,1}} + \|\hat{y}\|_{L^\infty(W^{2-2/p,p})}\|w\|_{L^\infty(V)}\right).\end{aligned}$$

Then Theorem 1.32 gives us the boundedness of the solution w in $W_p^{2,1}$.

STEP 3: $4\leq p<\infty$. Let $w \in W_{4-\epsilon}^{2,1}$ be the strong solution of Step 2, $0<\epsilon<2$. Analogously as in (1.53) and (1.54) we find

$$\|B'(\hat{y})^*w\|_p = \|-(\hat{y}\cdot\nabla)w+(\nabla\hat{y})^T w\|_p \leq c\|\hat{y}\|_{W_p^{2,1}}\|w\|_{W_{4-\epsilon}^{2,1}},$$

and the claim follows immediately. $\square$

Chapter 2

The optimal control problem

Now, let us introduce the optimization problem, which we will investigate in the sequel. We are considering optimal control of the instationary Navier-Stokes equations. The minimization of the following quadratic objective functional serves as model problem:

$$
\begin{aligned}
J(y,u) = \frac{\alpha_T}{2}\int_\Omega |y(x,T) - y_T(x)|^2\,\mathrm{d}x + \frac{\alpha_Q}{2}\int_Q |y(x,t) - y_Q(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \\
+ \frac{\alpha_R}{2}\int_Q |\operatorname{curl} y(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{2}\int_Q |u(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t. \quad (2.1)
\end{aligned}
$$

Several optimization goals are involved in the functional J. They are weighted by the coefficients α_T, α_Q, α_R, and γ. The curl operator appearing in the objective is defined by

$$
\operatorname{curl} y = \nabla \times y = \frac{\partial y_1}{\partial x_2} - \frac{\partial y_2}{\partial x_1}.
$$

It is a linear and continuous mapping from $L^2(0,T;V)$ to $L^2(Q)^2$.

The free variables - state y and control u - have to fulfill the instationary Navier-Stokes equations

$$
\begin{aligned}
y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p &= u && \text{in } Q,\\
\operatorname{div} y &= 0 && \text{in } Q,\\
y &= 0 && \text{on } \Sigma,\\
y(0) &= y_0. && \text{in } \Omega.
\end{aligned}
$$

We are not allowed to apply arbitrarily large controls. The control has to satisfy inequality constraints

$$
u_{a,i}(x,t) \le u_i(x,t) \le u_{b,i}(x,t) \text{ a.e. on } Q,\ i = 1,2.
$$

To keep all considerations as simple as possible, we are dealing with quadratic objective functionals. Some extensions and generalizations are commented in Section 4. More complicated control constraints are the subject of Chapter 7.

1 Setting of the problem

Let us specify the problem setting. Unless other conditions are imposed, we assume that the ingredients of the optimal control problem satisfy the following:

- The domain Ω is supposed to be an open bounded subset of $\mathbb{R}^2$ with Lipschitz boundary Γ. We denote the time-space cylinder by $Q = \Omega \times (0,T)$, its boundary by $\Sigma = \Gamma \times (0,T)$.
- The initial value y_0 is a given function in H. The desired states have to satisfy $y_T \in H$ and $y_Q \in L^2(Q)^2$.
- The parameter ν is a positive real number. The coefficients α_T, α_Q, α_R are non-negative real numbers, where at least one of them is positive to get a non-trivial objective functional. The regularization parameter γ, which measures the cost of the control, is also a positive number.
- The control constraints $u_a, u_b \in L^2(Q)^2$ have to satisfy $u_{a,i}(x,t) \le u_{b,i}(x,t)$ a.e. on Q for $i = 1, 2$, such that there exists admissible functions.

We define the set of admissible controls U_{ad} by

$$U_{ad} = \{u \in L^2(Q)^2 : \ u_{a,i}(x,t) \le u_i(x,t) \le u_{b,i}(x,t) \text{ a.e. on } Q,\ i = 1,2\}. \tag{2.2}$$

U_{ad} is non-empty, convex and closed in $L^2(Q)^2$.

So we end up with the optimization problem in function space

$$\min J(y,u) \tag{2.3a}$$

subject to the state equation

$$y_t + \nu A y + B(y) = u \quad \text{ in } L^2(0,T;V'), \tag{2.3b}$$

$$y(0) = y_0 \tag{2.3c}$$

and the control constraint

$$u \in U_{ad}. \tag{2.3d}$$

Sometimes, it is convenient to work with the reduced objective functional ϕ that is defined by

$$\phi(u) = J\big(S(u),\, u\big). \tag{2.4}$$

Here, S is the nonlinear solution operator associated to (2.3b) and (2.3c). We get an optimal control problem, where the equality constraints (2.3b) and (2.3c) are eliminated.

2 Existence of solutions

We call a couple (y,u) of state and control *admissible* if it satisfies the constraints (2.3b)–(2.3d) of the optimal control problem. We will denote in the sequel pairs of control and state by v, e.g. $v = (y,u)$, $\bar v = (\bar y, \bar u)$, and so on.

At first, we will show that the optimal control problem has a solution.

Theorem 2.1. *The optimal control problem admits a - globally optimal - solution $\bar u \in U_{ad}$ with associated state $\bar y \in W(0,T)$.*

Proof. The proof follows a technique that is standard for optimal control problems. We will repeat it for convenience of the reader.

The set of admissible controls is non-empty and bounded in $L^2(Q)^2$. For every control in $L^2(Q)^2$, there exists ea unique solution of the state equation (2.3b) and

(2.3c), see Theorem 1.12. Furthermore, the functional J is bounded from below, $J(y,u) \geq 0$. Hence, there exists the infimum of J over all admissible controls and states

$$0 \leq \bar{J} := \inf_{(y,u) \text{ admissible}} J(y,u) \leq \infty.$$

Moreover, there is a minimizing sequence (y_n, u_n) of admissible pairs such that $J(y_n, u_n) \to \bar{J}$ for $n \to \infty$. The set $\{u_n\}$ is bounded in $L^2(Q)^2$. By Theorem 1.12, the set of states is bounded in $W(0,T)$ as well. Therefore, we can extract a subsequence $(y_{n'}, u_{n'})$ converging weakly in $W(0,T) \times L^2(Q)^2$ to some limit $(\bar{y}, \bar{u})$. The set of admissible controls is convex and closed in $L^2(Q)^2$, hence it is weakly closed in $L^2(Q)^2$. Thus, the control $\bar{u}$ is admissible, $\bar{u} \in U_{ad}$.

Now, we have to show that the pair $(\bar{y}, \bar{u})$ satisfies the state equation (2.3b). We find for $v \in L^2(0,T;V)$ the convergences

$$\begin{aligned}
\langle y_{n',t},\, v\rangle_{L^2(V'),L^2(V)} &\to \langle \bar{y}_t,\, v\rangle_{L^2(V'),L^2(V)}, \\
\langle Ay_{n'},\, v\rangle_{L^2(V'),L^2(V)} &\to \langle A\bar{y},\, v\rangle_{L^2(V'),L^2(V)}, \\
\langle u_{n'},\, v\rangle_{L^2(V'),L^2(V)} &\to \langle \bar{u},\, v\rangle_{L^2(V'),L^2(V)} \qquad \text{for } n' \to \infty.
\end{aligned}$$

The mapping $w \mapsto w(0)$ is linear and continuous from $W(0,T) \to H$. Thus, $y_n(0)$ converges weakly to $\bar{y}(0)$. By construction, we have $y_0 = y_n(0)$ for all n, hence it holds $\bar{y}(0) = y_0$.

The space $W(0,T)$ is compactly imbedded in $L^2(Q)^2$, see [4], which implies the strong convergence $y_{n'} \to \bar{y}$ in $L^2(Q)^2$. By [70, Lemma III.3.2], we obtain the convergence of the nonlinear term

$$\langle B(y_{n'}),\, v\rangle_{L^2(V'),L^2(V)} \to \langle B(\bar{y}),\, v\rangle_{L^2(V'),L^2(V)} \qquad \text{for } n' \to \infty.$$

Consequently, all addends in the weak formulation of the state equation converge, and

$$\langle \bar{y}_t + A\bar{y} + B(\bar{y}) - \bar{u},\, v\rangle_{L^2(V'),L^2(V)} = 0$$

is fulfilled for all $v \in L^2(0,T;V)$. Furthermore, the initial condition $\bar{y}(0) = y_0$ is satisfied. Hence, $\bar{y}$ is itself the weak solution of the state equation with right-hand side $\bar{u}$, i.e. $\bar{y} = S(\bar{u})$.

Finally, it remains to show $\bar{J} = J(\bar{y}, \bar{u})$. The objective functional consists of several norm squares, thus it is weakly lower semicontinuous which implies

$$J(\bar{y}, \bar{u}) \leq \liminf J(y_{n'}, u_{n'}) = \bar{J}.$$

Since $(\bar{y}, \bar{u})$ is admissible, and $\bar{J}$ is the infimum over all admissible pairs, it follows $\bar{J} = J(\bar{y}, \bar{u})$. That is the claim. □

The theorem ensures the existence of a global minimizer. In the sequel, we will always deal with locally optimal solutions, since there are no criteria to distingiush local and global optimizers except the obvious one: $J(\bar{u}) \leq J(u)$ for all $u \in U_{ad}$.

3 Lagrange functional

We want to define the Lagrange functional $\mathcal{L} : W(0,T) \times L^2(Q)^2 \times L^2(0,T;V) \mapsto \mathbb{R}$ for the optimal control problem as follows:

$$\mathcal{L}(y,u,\lambda) = J(y,u) - \langle y_t + Ay + B(y) - u,\, \lambda\rangle_{L^2(V'),L^2(V)}\,. \tag{2.5}$$

Observe that we did not introduce multipliers for the initial value and the control constraint. If we would do so, we get additional terms which are affine linear. Thus they would only affect the representation of first-order conditions in terms of the Lagrangian. The second-order analysis, where the main focus of the thesis is laid on, is not influenced.

In either case, the Lagrangian function $\mathcal{L}$ is, for given $\lambda \in L^2(0,T;V)$, twice Fréchet-differentiable with respect to $(y,u) \in W(0,T) \times L^2(Q)^2$. The first-order derivatives of $\mathcal{L}$ with respect to y and u in directions $w \in W(0,T)$ and $h \in L^2(Q)^2$ respectively are

$$\begin{aligned}\mathcal{L}_y(y,u,\lambda)w &= \alpha_T(y(T)-y_T, w(T))_H + \alpha_Q(y-y_Q,w)_Q + \alpha_R(\operatorname{curl} y, \operatorname{curl} w)_Q \\ &\quad - \langle w_t + Aw + B'(y)w,\, \lambda\rangle_{L^2(V'),L^2(V)}, \qquad (2.6a)\\ \mathcal{L}_u(y,u,\lambda)h &= \gamma(u,h)_Q + (h,\lambda)_Q. \qquad (2.6b)\end{aligned}$$

The second-order derivative of $\mathcal{L}$ with respect to $v=(y,u)$ in directions (w_1,h_1), $(w_2,h_2) \in W(0,T)\times L^2(Q)^2$ is

$$\mathcal{L}_{vv}(y,u,\lambda)[(w_1,h_1),(w_2,h_2)] = \mathcal{L}_{yy}(y,u,\lambda)[w_1,w_2] + \mathcal{L}_{uu}(y,u,\lambda)[h_1,h_2]$$

with

$$\begin{aligned}\mathcal{L}_{yy}(y,u,\lambda)[w_1,w_2] &= \alpha_T(w_1(T),w_2(T))_H + \alpha_R(\operatorname{curl} w_1, \operatorname{curl} w_2)_Q \\ &\quad + \alpha_Q(w_1,w_2)_Q - \langle B''(y)[w_1,w_2],\, \lambda\rangle_{L^2(V'),L^2(V)}\end{aligned} \qquad (2.7)$$

and

$$\mathcal{L}_{uu}(y,u,\lambda)[h_1,h_2] = \gamma(h_1,h_2)_Q. \qquad (2.8)$$

Please note that mixed derivatives do not appear.

Theorem 2.2. *Let $\lambda \in L^2(0,T;V)$ be given. Then the Lagrangian $\mathcal{L}$ is twice Fréchet-differentiable with respect to $v=(y,u)$ from $W(0,T)\times L^2(Q)^2$ to $\mathbb{R}$. The second-order derivative at (y,u) fulfills, together with the Lagrange multiplier λ, the estimate*

$$|\mathcal{L}_{yy}(y,u,\lambda)[w_1,w_2]| \le c_{\mathcal{L}}\left(1+\|\lambda\|_{L^2(V)}\right)\|w_1\|_W\|w_2\|_W \qquad (2.9)$$

for all $w_1,w_2 \in W(0,T)$ with some constant $c_{\mathcal{L}}>0$ that does not depend on $y,u,\lambda,w_1,$ and w_2.

Proof. The mappings $y \mapsto \mathcal{L}_y(y,u,\lambda)$ and $u \mapsto \mathcal{L}_u(y,u,\lambda)$ are affine linear. Their linear parts are bounded, hence continuous. Therefore, both mappings are Fréchet-differentiable. This shows that $\mathcal{L}$ is twice Fréchet-differentiable as well. Then we can estimate

$$\begin{aligned}|\mathcal{L}_{yy}(y,u,\lambda)[w_1,w_2]| &\le c\big(\|w_1\|_{L^\infty(H)}\|w_2\|_{L^\infty(H)} + \|w_1\|_{L^2(V)}\|w_2\|_{L^2(V)} \\ &\quad + \|w_1\|_W\|w_2\|_W\|\lambda\|_{L^2(V)}\big) \\ &\le c_{\mathcal{L}}(1+\|\lambda\|_{L^2(V)})\|w_1\|_W\|w_2\|_W.\end{aligned}$$

Here we used the representation of B'' in (1.21), which we estimated similarly to the procedure of Lemma 1.9. ☐

Corollary 2.3. *The objective functional J is twice continuously differentiable from $W(0,T) \times L^2(Q)^2$ to $\mathbb{R}$. The reduced objective ϕ continuously is also twice continuous differentiable from $L^2(Q)^2$ to $\mathbb{R}$.*

Proof. The differentiability of J is a consequence of the form of J as a sum of norm squares. The differentiability of ϕ follows from the differentiability of J and the continous second-order differentiability of the solution mapping S, see Lemma 1.24. $\square$

4 Generalizations

One can think of several generalizations. The simplest one is to integrate not over the whole domains Ω and Q but over measurable subsets of them. If for instance the control is defined only on a subset Ω_c of Ω, then the right-hand side of (2.3) would be $\mathcal{B}u$ instead of u, where, $\mathcal{B}$ is the extension operator $\mathcal{B} : L^2(\Omega_c)^2 \mapsto L^2(\Omega)^2$. This change will not complicate the analysis. One has to take care of the occurrences of $\mathcal{B}$ and its adjoint $\mathcal{B}^*$. Obviously, all regularity results for an optimal control has to be adapted, i.e. instead of $\bar{u} \in C(\bar{Q})^2$ one would get $\bar{u} \in C([0,T] \times \bar{\Omega}_c)^2$.

In the literature of optimal control problems for semilinear equations, the following general objective is used:

$$J_1(y,u) = \int_\Omega F(x, y(x,T))\,\mathrm{d}x + \int_Q f(x,t,y(x,t),u(x,t))\,\mathrm{d}x\,\mathrm{d}t.$$

Here one has to impose certain measurability, continuity, and differentiability assumptions on the functions f and F. For the exact statement of them we refer to Raymond and Tröltzsch [61], where associated assumptionswere posed in a different context.

Another interesting objective functional was presented by Hintermüller, Kunisch, Spasov, and Volkwein [43],

$$J_2 = \int_Q h(\det(\nabla y))\,\mathrm{d}x\,\mathrm{d}t$$

with a C^2 function h that is positive whenever its argument is positive. It was motivated by a dynamical system approach to suppress vortices.

The control constraints (2.2) are a very specific choice. In Chapter 7, we will carry over all results to constraints of the form

$$u(x,t) \in U(x,t) \text{ a.e. on } Q,$$

where U is a set-valued mapping whose images are subsets of $\mathbb{R}^2$.

Chapter 3

Necessary optimality conditions

In this chapter, we will prove first-order necesary optimality conditions. These conditions will be necessary for *local* optimality. They are important in many respects. At first, one can prove that local optimal controls enjoy more regularity than $L^2(Q)^2$, which is the regularity needed to set up the optimal control problem properly. Furthermore, first-order conditions are used to compute candidates for optimal controls numerically in the way that numerical schemes try to solve the first-order optimality system at a discrete level.

1 Convex sets and cones

At first, we will recall some definitions from the theory of convex sets. For a convex set $C \in \mathbb{R}^n$ and an element $u \in C$, we denote by $\mathcal{N}_C(u)$ and $\mathcal{T}_C(u)$ the normal cone and the polar cone of tangents of C at the point u respectively, which are defined by

$$\begin{aligned}\mathcal{N}_C(u) &= \{z \in \mathbb{R}^n : z^T(v-u) \le 0 \quad \forall v \in C\},\\ \mathcal{T}_C(u) &= \{z \in \mathbb{R}^n : z^T v \le 0 \quad \forall v \in \mathcal{N}_C(u)\}.\end{aligned}$$

Further, we will need the linear subspaces

$$N_C(u) = \operatorname{cl}\operatorname{span}\mathcal{N}_C(u), \qquad T_C(u) = N_C(u)^\perp.$$

We want to apply these notations to the case $C = U_{ad}$. For convenience, we repeat the definition of U_{ad}:

$$\begin{aligned}U_{ad} &= \{u \in L^2(Q)^2 : u_i(x,t) \in U_i(x,t) \text{ for } i = 1,2, \text{ a.e. on } Q\},\\ U_i(x,t) &= [u_{a,i}(x,t),\, u_{b,i}(x,t)].\end{aligned}$$

The set of admissible controls is a subset of $L^2(Q)^2$. Hence, we will consider the normal cone and the polar cone of tangents of U_{ad} with respect to $L^2(Q)^2$-scalar product. Let u be an admissible control, i.e. $u \in U_{ad}$. Then, we will use the following definitions

$$\begin{aligned}\mathcal{N}_{U_{ad}}(u) &= \{z \in L^2(Q)^2 : (z,\, v-u)_2 \le 0 \quad \forall v \in U_{ad}\},\\ \mathcal{T}_{U_{ad}}(u) &= \{z \in L^2(Q)^2 : (z,\, v)_2 \le 0 \quad \forall v \in \mathcal{N}_{U_{ad}}(u)\}.\end{aligned} \tag{3.1}$$

Analogously as above, we define

$$N_{U_{ad}}(u) = \operatorname{cl\,span} \mathcal{N}_{U_{ad}}(u), \quad T_{U_{ad}}(u) = N_{U_{ad}}(u)^{\perp}.$$

Here, $\operatorname{cl\,span} \mathcal{N}_{U_{ad}}(u)$ means the closure in $L^2(Q)^2$ of all finite linear combinations of elements of $\mathcal{N}_{U_{ad}}(u)$. The orthogonal complement $N^{\perp}$ is built with respect to the L^2-scalar product. By construction, we have $T_{U_{ad}}(u) \subset \mathcal{T}_{U_{ad}}(u)$. Roughly speaking, the space $T_{U_{ad}}(u)$ is the smallest linear subspace contained in $\mathcal{T}_{U_{ad}}(u)$.

It is well-known, that the sets $\mathcal{N}_{U_{ad}}(u)$, $\mathcal{T}_{U_{ad}}(u)$, $N_{U_{ad}}(u)$, and $T_{U_{ad}}(u)$ admit a pointwise representation as U_{ad} itself, cf. [7, Section 8.5] and [63]. For instance, the set $\mathcal{N}_{U_{ad}}(u)$ is given by

$$\mathcal{N}_{U_{ad}}(u) = \left\{ v \in L^2(Q)^2 : \, v_i(x,t) \in \mathcal{N}_{U_i(x,t)}(u_i(x,t)) \text{ a.e. on } Q, \, i = 1,2 \right\}.$$

In the presence of box constraints, things are even easier. Let be given a function $u \in U_{ad}$. Define sets of active constraints by

$$\begin{aligned} Q_{a,i}(u) &= \{(x,t) \in Q : \, u(x,t) = u_{a,i}(x,t)\}, \\ Q_{b,i}(u) &= \{(x,t) \in Q : \, u(x,t) = u_{b,i}(x,t)\} \quad \text{for } i = 1,2. \end{aligned}$$

Then we have the following characterization of the normal cone $\mathcal{N}_{U_{ad}}(u)$:

$$\begin{aligned} \mathcal{N}_{U_{ad}}(u) = \{ w \in L^2(Q)^2 : \, & w_i(x,t) \le 0 \text{ if } (x,t) \in Q_{a,i}(u), \\ & w_i(x,t) \ge 0 \text{ if } (x,t) \in Q_{b,i}(u), \\ & w_i(x,t) = 0 \text{ otherwise}, \; i = 1,2 \}. \end{aligned} \tag{3.2}$$

Consequently, the polar cone of tangents is

$$\begin{aligned} \mathcal{T}_{U_{ad}}(u) = \{ w \in L^2(Q)^2 : \, & w_i(x,t) \ge 0 \text{ if } (x,t) \in Q_{a,i}(u), \\ & w_i(x,t) \le 0 \text{ if } (x,t) \in Q_{b,i}(u), \; i = 1,2 \}. \end{aligned} \tag{3.3}$$

It implies the representation of the spaces of normal directions $N_{U_{ad}}(u)$ and its complement $T_{U_{ad}}(u)$ as

$$\begin{aligned} N_{U_{ad}}(u) &= \{ w \in L^2(Q)^2 : \, w_i(x,t) = 0 \text{ if } (x,t) \notin Q_{a,i}(u) \cup Q_{b,i}(u), \; i = 1,2 \}, \\ T_{U_{ad}}(u) &= \{ w \in L^2(Q)^2 : \, w_i(x,t) = 0 \text{ if } (x,t) \in Q_{a,i}(u) \cup Q_{b,i}(u), \; i = 1,2 \}. \end{aligned} \tag{3.4}$$

We did not assume that the bounds u_a and u_b are separated by some distance $\tau > 0$, i.e. $u_a + \tau \le u_b$. It implies that in the definition of the sets above the case $(x,t) \in Q_{a,i}(u) \cap Q_{b,i}(u)$ can occur. Then both of the relations that are imposed for $(x,t) \in Q_{a,i}(u)$ and $(x,t) \in Q_{b,i}(u)$ have to be fulfilled.

2 First-order necessary optimality conditions

Definition 3.1. *A control $\bar u \in U_{ad}$ is said to be* locally optimal in $L^p(Q)^2$, *if there exists a constant $\rho > 0$ such that*

$$J(\bar y, \bar u) \le J(y,u)$$

holds for all $u \in U_{ad}$ with $\|\bar u - u\|_p \le \rho$. Here, $\bar y$ and y denote the states associated with $\bar u$ and u, e.g. $\bar y = S(\bar u)$ and $y = S(u)$.

First-order necessary optimality conditions can be found in many literature references. Conditions for optimal control problems for partial differential equations can be derived from general programming principles in Banach spaces. They were investigated years ago, see for instance Ben-Tal and Zowe [8], Maurer [56], Tröltzsch [73], and the references therein.

The first proof of necessary conditions for optimal control of Navier-Stokes equations was given in the pioneering work [1] by Abergel and Temam. Other proofs can be found in Gunzburger and Manservisi [32, 33], and Hinze und Kunisch [46]. Regularity problems are adressed in Hinze [44], Hinze and Kunisch [46], and Tröltzsch and Wachsmuth [77]. Necessary optimality conditions for the case of three-dimensional flow are established by Casas in [16].

Now, we will state and prove the first-order optimality condition for convenience of the reader.

Theorem 3.2 (Necessary condition). *Let $\bar u$ be locally optimal in $L^2(Q)^2$ with associated state $\bar y = S(\bar u)$. Then there exists a unique Lagrange multiplier $\bar\lambda \in W^{4/3}(0,T;V)$, which is the weak solution of the adjoint equation*

$$\begin{aligned} -\bar\lambda_t + \nu A\bar\lambda + B'(\bar y)^*\bar\lambda &= \alpha_Q(\bar y - y_Q) + \alpha_R \operatorname{curl}^*\operatorname{curl}\bar y \\ \bar\lambda(T) &= \alpha_T(\bar y(T) - y_T). \end{aligned} \tag{3.5}$$

Moreover, the variational inequality

$$(\gamma\bar u + \bar\lambda, u - \bar u)_Q \geq 0 \quad \forall u \in U_{ad} \tag{3.6}$$

is satisfied. The curl**-operator is defined for a scalar function $v \in L^2(0,T;H^1(\Omega))$ by*

$$\operatorname{curl}^* v = \left(-\frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_1}\right).$$

Proof. We will work with the reduced objective functional ϕ already defined in (2.4) by $\phi(u) = J(S(u),u)$. The functional ϕ is twice Fréchet-differentiable from $L^2(Q)^2$ to $\mathbb{R}$, cf. Corollary 2.3. By Banach space optimization principles, we know that the variational inequality

$$\phi'(\bar u)(u - \bar u) \geq 0 \quad \forall u \in U_{ad} \tag{3.7}$$

is necessary for local optimality of $\bar u$. It remains to compute ϕ' and to derive the adjoint system. Let us write ϕ in the following way:

$$\phi(u) = \frac{\alpha_T}{2}|FS(u) - y_T|_2^2 + \frac{\alpha_Q}{2}\|S(u) - y_Q\|_2^2 + \frac{\alpha_R}{2}\|\operatorname{curl} S(u)\|_2^2 + \frac{\gamma}{2}\|u\|_2^2.$$

Here, F defined by $F(y) = y(T)$ is a linear and continuous operator from $W(0,T)$ to H. The first derivative ϕ' at $\bar u$ in direction $h \in L^2(Q)^2$ is characterized by

$$\begin{aligned} \phi'(\bar u)h = {} & \alpha_T(FS(\bar u) - y_T,\, FS'(\bar u)h)_2 \\ & + \alpha_Q(S(\bar u) - y_Q,\, S'(\bar u)h)_Q \\ & + \alpha_R(\operatorname{curl} S(\bar u),\, \operatorname{curl} S'(\bar u)h)_Q \\ & + \gamma(u,h)_Q. \end{aligned}$$

Now, we want to compute the adjoint operators of F and curl.

At first, we have by definition $\langle F^*v,\, w\rangle = (v,\, w(T))_2$ for $w \in W(0,T)$ and $v \in H$. This implies with $v := Fy$ that $\langle F^*Fy,\, w\rangle = (y(T),\, w(T))_2$ holds for all $y, w \in W(0,T)$.

The adjoint operator of curl can be obtained by partial integration. One gets for $v \in L^2(0,T;H_0^1(\Omega))$ and $y \in L^2(0,T;V)$

$$\begin{aligned}\int_Q v(x,t)\,\mathrm{curl}\, y(x,t)\,\mathrm{d}x\,\mathrm{d}t &= \int_Q v(x,t)\left(\frac{\partial y_1}{\partial x_2}(x,t) - \frac{\partial y_2}{\partial x_1}(x,t)\right)\mathrm{d}x\,\mathrm{d}t\\ &= \int_Q -y_1(x,t)\frac{\partial v}{\partial x_2}(x,t) + y_2(x,t)\frac{\partial v}{\partial x_1}(x,t))\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

which gives the vector representation

$$\mathrm{curl}^*\, v = \left(-\frac{\partial v}{\partial x_2},\, \frac{\partial v}{\partial x_1}\right)^T. \tag{3.8}$$

With the help of these adjoints, we transform $\phi'(\bar u)h$ to

$$\begin{aligned}\phi'(\bar u)h = &\left\langle \alpha_T F^*(FS(\bar u) - y_T) + \alpha_Q(S(\bar u) - y_Q) + \alpha_R\,\mathrm{curl}^*\mathrm{curl}\, S(\bar u),\; S'(\bar u)h\right\rangle_{W^*,W}\\ &+ \gamma(u,h)_Q.\end{aligned}$$

Substituting back $S(\bar u) = \bar y$, the expression

$$\begin{aligned}\phi'(\bar u)h = &\left\langle \alpha_T F^*(\bar y(T) - y_T) + \alpha_Q(\bar y - y_Q) + \alpha_R\,\mathrm{curl}^*\mathrm{curl}\,\bar y,\; S'(\bar u)h\right\rangle_{W^*,W}\\ &+ \gamma(u,h)_Q.\end{aligned} \tag{3.9}$$

is found. The state $\bar y$ is solution of the nonlinear state equation, therefore it belongs to $W(0,T)$. Altogether, the addends defining $\phi'(\bar u)$ have the following regularities:

$$\begin{aligned}F^*(\bar y(T) - y_T) &\in W(0,T)^*,\\ \bar y - y_Q &\in L^2(Q)^2 \subset L^{4/3}(0,T;V') \cap W(0,T)^*,\\ \mathrm{curl}^*\mathrm{curl}\,\bar y &\in L^2(0,T;V') \subset L^{4/3}(0,T;V') \cap W(0,T)^*.\end{aligned}$$

Let us define

$$\bar\lambda := S'(\bar u)^*\left\{\alpha_T F^*(\bar y(T) - y_T) + \alpha_Q(\bar y - y_Q) + \alpha_R\,\mathrm{curl}^*\mathrm{curl}\,\bar y\right\}. \tag{3.10}$$

The term in brackets fits in the assumption of Corollary 1.27. Thus, we can interprete $\bar\lambda$ as the weak solution of

$$\begin{aligned}-\lambda_t + \nu A\lambda + B'(\bar y)^*\lambda &= \alpha_Q(\bar y - y_Q) + \alpha_R\,\mathrm{curl}^*\mathrm{curl}\,\bar y \quad \text{in } L^{4/3}(0,T;V'),\\ \lambda(T) &= \alpha_T(\bar y(T) - y_T).\end{aligned}$$

Furthermore, we get the regularity $\lambda \in W^{4/3}(0,T)$. The space $W^{4/3}(0,T)$ is continuously imbedded in $L^p(Q)^2$ for $p < 7/2$, cf. [78]. Combining the definition of $\bar\lambda$ (3.10) with (3.7) and (3.9) yields

$$0 \le \phi'(\bar u)(u - \bar u) = (\gamma\bar u + \bar\lambda,\, u - \bar u)_Q \quad \forall u \in U_{ad},$$

which is the variational inequality (3.6). □

We want to remark that the so-called method of transposition does not work for our problem. There, we would have to test the state equation by the adjoint variable and vice-versa, but this is prohibited due to the following lack of regularity: The state $\bar{y}$ is not allowed as test-function in the weak solution of the adjoint equation. We have $\bar{y} \in L^2(0,T;V)$ but the test space is $L^4(0,T;V)$.

The necessary optimality conditions can be formulated equivalently using the Lagrangian.

Theorem 3.3. *Under the conditions of Theorem 3.2, it is necessary for local optimality of $\bar{u}$ that there exists $\lambda \in W^{4/3}(0,T)$ such that*

$$\mathcal{L}_y(\bar{y},\bar{u},\bar{\lambda})w = 0 \quad \forall w \in W_0, \tag{3.11a}$$
$$\mathcal{L}_u(\bar{y},\bar{u},\bar{\lambda})(u-\bar{u}) \geq 0 \quad \forall u \in U_{ad} \tag{3.11b}$$

is fulfilled.

Proof. Let $\bar{\lambda}$ be the solution of (3.5), which is given by (3.10). According to (2.6a), we can write $\mathcal{L}_y(\bar{y},\bar{u},\bar{\lambda})w$ as

$$\begin{aligned}\mathcal{L}_y(\bar{y},\bar{u},\bar{\lambda})w = \alpha_T(\bar{y}(T)-y_T, w(T))_H + \alpha_Q(\bar{y}-y_Q, w)_Q + \alpha_R(\operatorname{curl}\bar{y}, \operatorname{curl} w)_Q \\ - \left\langle w_t + Aw + B'(y)w,\ \bar{\lambda}\right\rangle_{L^2(V'),L^2(V)},\end{aligned}$$

By construction of $\bar{\lambda}$, the right-hand side vanishes for $w \in W_0$, cf. Lemma 1.25. Hence, we obtain $\mathcal{L}_y(\bar{y},u,\lambda)w = 0$ for all test functions $w \in W_0$.

The derivative of the Lagrangian with respect to the control u is given as

$$\mathcal{L}_u(\bar{y},\bar{u},\bar{\lambda})(u-\bar{u}) = (\gamma\bar{u}+\bar{\lambda},\ u-\bar{u})_Q,$$

cf. (2.6b). Consequently, this expression must be greater or equal to zero for all admissible controls $u \in U_{ad}$. □

The variational inequality (3.6) can be reformulated equivalently in different ways. At first, a pointwise discussion yields the projection representation of the optimal control

$$u_i(x,t) = \operatorname{Proj}_{[u_{a,i}(x,t),u_{b,i}(x,t)]}\left(-\frac{1}{\gamma}\bar{\lambda}_i(x,t)\right) \qquad \text{a.e. on } Q,\ i=1,2. \tag{3.12}$$

Here, we can see that the optimal control inherits some regularity from the adjoint state, which will be exploited in Section 4. This formula is also used in connection with Lipschitz stability of optimal controls, see Chapter 5.

Secondly, we give another formulation of the variational inequality using the normal cone $\mathcal{N}_{U_{ad}}(\bar{u})$, see its definition (3.1). Then, the inequality (3.6) can be written equivalently as the inclusion

$$\nu\bar{u} + \bar{\lambda} + \mathcal{N}_{U_{ad}}(\bar{u}) \ni 0. \tag{3.13}$$

This representation fits in the context of generalized equations, which will be employed in Chapters 5 and 6.

Remark 3.4. *The adjoint equation* (3.5) *can be seen as the weak formulation of the* *following system, where the function μ might be interpreted as the adjoint pressure,*

$$\begin{aligned} -\bar{\lambda}_t - \nu\Delta\lambda - (\bar{y}\cdot\nabla)\lambda + (\nabla\bar{y})^T\lambda + \nabla\mu &= \alpha_Q(\bar{y}-y_Q) + \alpha_R\,\mathrm{curl}^*\mathrm{curl}\,\bar{y} \\ \mathrm{div}\,\lambda &= 0 \\ \lambda(T) &= \alpha_T(\bar{y}(T)-y_T) \\ \lambda &= 0. \end{aligned}$$

3 Second-order necessary optimality conditions

For the sake of completeness, we investigate necessary optimality conditions of second order. Thereby, we follow the presentation of Casas and Tröltzsch [18]. Second-order conditions for optimization problems in infinite-dimensional spaces are well-known in the literatur, we refer for instance to Ben-Tal and Zowe [8], Ioffe [47], and Maurer [56]. They can be applied to derive optimality conditions for optimal control problems. These conditions are carried over to the case of polyhedral control constraints by Bonnans[10] and Dunn [26], .

Let $\bar{u}$ be an L^2-locally optimal solution of our optimal control problem. It will be shown that the second derivative of the Lagrangian is positive on the cone of critical directions $C_0(\bar{u})$ given by

$$C_0(\bar{u}) = \{w \in \mathcal{T}_{U_{ad}}(\bar{u}) : w_i(x,t) = 0 \text{ on } Q_{0,i},\ i=1,2\}, \tag{3.14}$$

where $Q_{0,i}$ are the sets of weakly active constraints

$$Q_{0,i} = \{(x,t)\in Q : |\gamma\bar{u}_i(x,t) + \bar{\lambda}_i(x,t)| > 0\} \quad \text{for } i=1,2.$$

Let us refer to the pointwise representation of $\mathcal{T}_{U_{ad}}(\bar{u})$ in (3.3).

The course of the proof demands to work at first with L^∞-functions, namely functions of the cone

$$\tilde{C}_0(\bar{u}) = C_0(\bar{u}) \cap L^\infty(Q)^2.$$

Moreover, we encounter another difficulty. In fact, the cones $C_0(\bar{u})$ as well as $\tilde{C}_0(\bar{u})$ do not belong to the cone of feasible directions. To cope with this difficulty, we will approximate $\tilde{C}_0(\bar{u})$ by a family of cones $\tilde{C}_\sigma(\bar{u})$ that is contained in the space of feasible directions.

We define subsets of Q indicating where the control constraints are inactive but nearly active. For $\sigma > 0$ and $i = 1, 2$ we set

$$I_{\sigma,i} = \{(x,t) : 0 < |\bar{u}_i(x,t) - u_{a,i}(x,t)| < \sigma \text{ or } 0 < |u_{b,i}(x,t) - \bar{u}_i(x,t)| < \sigma\}.$$

By construction, it holds $\cap_{\sigma>0} I_{\sigma,i} = \emptyset$. Now, let us define the cones $\tilde{C}_\sigma(\bar{u})$ for $\sigma > 0$ as

$$\tilde{C}_\sigma(\bar{u}) = \left\{ w \in \tilde{C}_0(\bar{u}) : w_i(x,t) = 0 \text{ if } (x,t)\in I_{\sigma,i},\ i=1,2 \right\}.$$

The set $\tilde{C}_\sigma(\bar{u})$ is non-void, since it contains the zero function.

Lemma 3.5. *Let be given $\bar{u} \in U_{ad}$. Then, for all $\sigma > 0$ the subspace $\tilde{C}_\sigma(\bar{u})$ lies in the cone of feasible directions at $\bar{u}$. Moreover, it holds $\cup_{n=1}^\infty \tilde{C}_{1/n}(\bar{u}) \subset \tilde{C}_0(\bar{u}) \subset C_0(\bar{u})$ with dense inclusions with respect to the L^2-norm.*

Proof. Fix $\bar u \in U_{ad}$ and $\sigma > 0$. Take some $\tilde w \in \tilde C_\sigma(\bar u)$. If $\tilde w = 0$ then it is trivially a feasible direction. Suppose $\tilde w \neq 0$, and set $s \in [0, \sigma/\|\tilde w\|_\infty]$. Then it holds by construction

$$u_{a,i}(x,t) \le \bar u_i + s\tilde w_i(x,t) \le u_{b,i}(x,t)$$

for all $i = 1,2$ almost everywhere on Q, which implies $\bar u + s\tilde w \in U_{ad}$. Hence, $\tilde w$ is a feasible direction. And the first claim is proven.

Now, let be given $\tilde w \in \tilde C_0(\bar u)$. Since the involved functions are measurable, the sets $I_{\sigma,i}$ are measurable as well. Using these indicator sets, we construct an approximation of $\tilde w$ by functions w_n defined as

$$w_{n,i}(x,t) = \begin{cases} 0 & \text{if } (x,t) \in I_{1/n,\,i} \\ \tilde w_i(x,t) & \text{otherwise.} \end{cases}$$

It follows that w_n belongs to $\tilde C_{1/n}(\bar u)$. We obtain for the difference $\tilde w - w_n$

$$\|\tilde w - w_n\|_2^2 = \sum_{i=1}^2 \int_{I_{1/n,i}} (\tilde w_i(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \le \|\tilde w\|_\infty^2 \sum_{i=1}^2 \operatorname{meas}(I_{1/n,i}).$$

The measures of $I_{1/n,i}$, $i = 1,2$, tend to zero with $n \to \infty$. Hence, we have the convergence $w_n \to \tilde w$ in $L^2(Q)^2$ for $n \to \infty$. It follows that $\cup_{n=1}^\infty \tilde C_{1/n}(\bar u)$ is dense in $C_0(\bar u)$.

Finally, take $w \in C_0(\bar u)$ and $\epsilon > 0$. The space of continuous functions $C(\bar Q)^2$ is dense in $L^2(Q)^2$. Thus, we can find $\hat w \in C(\bar Q)^2$ with $\|w - \hat w\|_2 \le \epsilon$. The sets $Q_{a,i}$ and $Q_{b,i}$ are measurable. This implies that the function $\tilde w$ defined by

$$\tilde w_i(x,t) = \begin{cases} 0 & \text{if } (x,t) \in Q_{a,i} \text{ and } w_i(x,t) \le 0 \\ 0 & \text{if } (x,t) \in Q_{b,i} \text{ and } w_i(x,t) \ge 0 \\ 0 & \text{if } (x,t) \in Q_{0,i} \\ \hat w_i(x,t) & \text{otherwise} \end{cases}$$

is in $\tilde C_0(\bar u)$ with $\|\tilde w\|_\infty \le \|\hat w\|_\infty$. Since $\tilde w$ is the pointwise projection of $\hat w$ on $\tilde C_0(\bar u)$, it holds $\|w - \tilde w\|_2 \le \epsilon$. Thus, the dense inclusion $\tilde C_0(\bar u) \subset C_0(\bar u)$ is proven. □

Using these density properties, we prove positiveness of the second derivative of the Lagrangian in all critical directions. It will be shown that each direction $w \in C_0(\bar u)$ is the L^2-limit of a sequence $w_n \in \tilde C_{1/n}(\bar u)$ satisfying additionally $\phi'(\bar u)w_n = 0$ and $\phi''(\bar u)[w_n, w_n] \ge 0$. Here, we recall that ϕ is the reduced objective functional, see Section 2.3. In contrast to the proof in [18], we gain coercivity with respect to L^2-functions rather than L^∞-functions.

Theorem 3.6. *Let $\bar u$ be a locally optimal solution of the optimal control problem* (2.3)*. Denote by $\bar y$ and $\bar\lambda$ the associated state and adjoint state.*

Then for every $w \in C_0(\bar u)$ we have

$$\mathcal{L}_{vv}(\bar y, \bar u, \bar\lambda)[(y,w)^2] \ge 0, \tag{3.15}$$

where $y = S'(\bar u)w$ is the solution of the linearized equation

$$\begin{aligned} y_t + \nu A y + B'(\bar y)y &= w \quad \text{in } L^2(0,T;V'), \\ y(0) &= 0. \end{aligned}$$

Proof. Take $w \in C_0(\bar{u})$. Then by Lemma 3.5, there is a sequence $w_n \in \tilde{C}_{1/n}$ converging to w in L^2 for $n \to \infty$. The functions w_n are feasible directions at $\bar{u}$. Since $\bar{u}$ is locally optimal, we have for the reduced objective

$$\phi(\bar{u}) \leq \phi(\bar{u} + t_n w_n)$$

for all t_n with $|t_n|$ sufficiently small. The reduced objective is twice continuously differentiable from $L^2(Q)^2$ to $\mathbb{R}$, cf. Corollary 2.3, hence the function $\tilde{\phi}(t) = \phi(\bar{u} + tw_n)$ is twice continuously differentiable in a neighborhood of the origin. Thus, it follows from principles of one-dimensional programming that $\tilde{\phi}'(0) = \phi'(\bar{u})w_n = 0$ and

$$0 \leq \tilde{\phi}''(0) = \phi''(\bar{u})[w_n, w_n] \quad \forall n. \tag{3.16}$$

Using continuity of $\phi''(\bar{u})[w, w]$ with respect to w, we obtain

$$\phi''(\bar{u})[w, w] = \phi''(\bar{u})[w - w_n, w + w_n] + \phi''(\bar{u})[w_n, w_n],$$

which gives in the limit $n \to \infty$ together with (3.16)

$$\phi''(\bar{u})[w, w] \geq 0.$$

Taking into account the identities

$$\phi(u) = J(S(u), u) = \mathcal{L}(S(u), u, \lambda) \quad \forall u \in L^2(Q)^2,\, \lambda \in W^{4/3}(0, T),$$

the claimed coercivity of the second derivative of the Lagrangian (3.15) follows. □

4 Regularity of optimal controls

In this section, we are going to prove that under certain conditions any extremal control is continuous in space and time. Here, extremal controls are controls satisfying the first-order necessary optimality conditions. The key tool in our analysis is the well-known projection formula (3.12).

Theorem 3.7. *Let $(\bar{y}, \bar{u})$ be an admissible pair satisfying the first-order necessary conditions* (3.5) *and* (3.6) *together with the associated adjoint state $\bar{\lambda}$. We assume that the following conditions are satisfied for some p with $2 < p < \infty$:*

(i) *Ω is of class C^3,*

(ii) *$y_0 \in W_0^{2-2/p,\,p}(\Omega)^2$,*

(iii) *either $\alpha_T = 0$ or $y_T \in W_0^{2-2/p,\,p}(\Omega)^2$,*

(iv) *either $\alpha_Q = 0$ or $y_Q \in L^p(Q)^2$.*

(v) *$\gamma > 0$, $\alpha_R \geq 0$,*

(vi) *u_a, $u_b \in C(\bar{Q})^2$.*

Then the control $\bar{u}$ belongs to $C(\bar{Q})^2$, the corresponding state and adjoint — $\bar{y}$ and $\bar{\lambda}$ — are of class $W_p^{2,1}$.

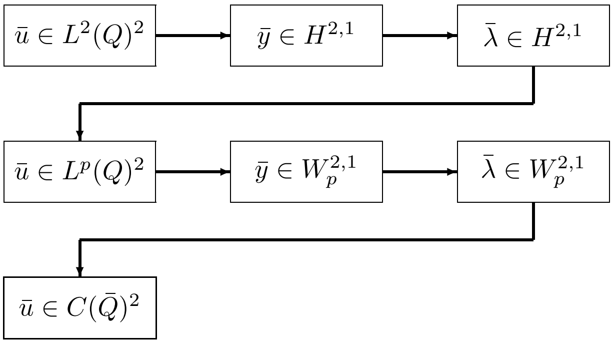

Figure 3.1. *Regularity of optimal controls*

Proof. The method of proof is sketched in Figure 3.1.

By definition of U_{ad}, we have $\bar u \in L^2(Q)^2$. Together with assumption *(ii)*, we get from Theorem 1.13(i) that $\bar y$ is in $H^{2,1}$.

Now, we have to investigate the regularity of the right-hand side of the adjoint system (3.5) in order to apply Theorem 1.31. At first, if the factor α_Q is greater than zero we have $y - y_Q \in L^r(Q)^2$ for every $2 \le r < \infty$, because the space $H^{2,1}$ is continuously imbedded in $L^r(Q)^2$ for every $r < \infty$. Since $\mathrm{curl}^*\mathrm{curl}$ is a differential operator of second order, cf. (3.8), we find for the case $\alpha_R > 0$ that $\mathrm{curl}^*\mathrm{curl}\,\bar y$ belongs to $L^2(Q)^2$. The terminal value $\bar y(T)$ is in V, thus the terminal data for the adjoint system is in V, too. Altogether, Theorem 1.31(i) implies the regularity $\bar\lambda \in H^{2,1} \hookrightarrow L^r(Q)^2$, $r < \infty$.

At this point, the projection formula comes into play. The pointwise projection is continuous between L^p-spaces. Consequently, it holds $\bar u \in L^p(Q)^2$.

This allows us to repeat the argumentation line with higher regularities of the involved quantities. Theorem 1.15 guarantees the regularity $\bar y \in W_p^{2,1}$. Hence, the right-hand side of the adjoint system (3.5) is at least in $L^p(Q)^2$. Furthermore, the terminal value in (3.5) is now of class $W_0^{2-2/p,\,p}(\Omega)^2$. Thus, by Theorem 1.31(ii) we get $\bar\lambda \in W_p^{2,1}$. The imbedding $W_p^{2,1} \hookrightarrow C(\bar Q)^2$ — see Corollary 1.6 — yields $\bar\lambda \in C(\bar Q)^2$.

A second and last application of the projection representation (3.12) gives together with the pre-requisite *(vi)* that the control $\bar u$ is continuous with respect to space and time, i.e. $\bar u \in C(\bar Q)^2$. ◻

Please note, that this result is independent of the behaviour of the velocity field, e.g. even independent of the Reynolds number $Re = 1/\nu$. It does not matter whether the flow is in laminar or turbulent regime. Only the regularity of the data is important.

Corollary 3.8. *If the conditions of the previous theorem are met for $p = 2$ then the control $\bar u$ is in $L^r(Q)^2$ for all $r < \infty$ only.*

It is known that the projection is continuous between $W^{1,p}$ spaces, see Kinderlehrer and Stampacchia [49]. If the bounds u_a and u_b are in $W^{1,p}(Q)^2$ then the optimal $\bar{u}$ is in the same space.

Corollary 3.9. *Let the conditions (i)-(v) of the previous theorem be satisfied together with*

(vi') $u_a, u_b \in W^{1,p}(Q)^2$.

Then the control $\bar{u}$ *is in* $W^{1,p}(Q)^2$ *too.*

Chapter 4

Second-order sufficient optimality conditions

We will investigate second-order sufficient optimality conditions for the optimal control problem under consideration. They are important in many respects. At first, as the name suggests, their fulfillment is sufficient for local optimality of a reference control. Secondly, they are the main pre-requisite for further results: proving stability of optimal controls and convergence of numerical algorithms and discretization schemes.

1 Sufficient optimality condition

A strong sufficient condition is to require positive definiteness of the second derivative of the Lagrangian with respect to all controls. Since such a strong condition can be applied easily to prove stability, they attracted much interest, see Hinze [44] and Hinze and Kunisch [46].

The sufficient condition that we will present here requires coercivity of the second derivative of the Lagrangian only for a subspace of the space of all possible directions. Using strongly active constraints, we are able to shrink the subspace in which the coercivity must hold. This concept is widely used in connection with control-constrained optimal control problems, see for instance Bonnans, Shapiro [13], Casas, Tröltzsch [18], Casas, Tröltzsch, Unger [19], Goldberg, Tröltzsch [29] , Tröltzsch, Wachsmuth [77].

Now, let us specify the notations of strongly active sets. Compare also the definition and use of Q_0, see Section 3.3.

Definition 4.1 (Strongly active sets). *Let $\epsilon > 0$ be given. Define sets $Q_{\epsilon,i} \subseteq Q = \Omega \times [0,T]$ for $i \in \{1,2\}$ by*

$$Q_{\epsilon,i} = \{(x,t) \in Q : |\gamma \bar{u}_i(x,t) + \bar{\lambda}_i(x,t)| > \epsilon\}.$$

For $u \in L^p(Q)^2$ and $1 \leq p < \infty$ we define the L^p-norm with respect to the sets of strongly active control constraints

$$\|u\|_{L^p(Q_\epsilon)} := \left(\sum_{i=1}^{2} \|u_i\|^p_{L^p(Q_{\epsilon,i})} \right)^{1/p}.$$

Remark 4.2. *Note that the variational inequality* (3.6) *uniquely determines $\bar{u}_i$ on*

$Q_{\epsilon,i}$. If $\gamma\bar{u}_i(x,t) + \bar{\lambda}_i(x,t) \geq \epsilon$ then $\bar{u}_i(x,t) = u_{a,i}(x,t)$ must hold. On the other hand, it follows $\bar{u}_i(x,t) = u_{b,i}(x,t)$, if $\gamma\bar{u}_i(x,t) + \bar{\lambda}_i(x,t) \leq -\epsilon$ is satisfied.

In what follows we fix $\bar{v} := (\bar{y}, \bar{u})$ to be an admissible reference pair. We suppose that $\bar{v}$ satisfies together with the adjoint state $\bar{\lambda}$ the first-order necessary optimality conditions, e.g. equations (3.5)–(3.6). Furthermore, we assume that the reference pair $\bar{v} = (\bar{y}, \bar{u})$ satisfies for some $q \geq 4/3$ the following coercivity assumption on $\mathcal{L}''(\bar{v}, \bar{\lambda})$, in the sequel called second-order sufficient condition:

$$
\textbf{(SSC)}\begin{cases}
\textit{There exist } \epsilon > 0 \textit{ and } \delta > 0 \textit{ such that} \\
\qquad \mathcal{L}_{vv}(\bar{v}, \bar{\lambda})[(z,h)]^2 \geq \delta \, \|h\|_q^2 \\
\textit{holds for all pairs } (z,h) \in W(0,T) \times L^2(Q)^2 \textit{ with} \\
\qquad h = u - \bar{u},\ u \in U_{ad},\ h_i = 0 \textit{ on } Q_{\epsilon,i} \textit{ for } i = 1,2, \\
\textit{and } z \in W(0,T) \textit{ being the weak solution of the linearized equation} \\
\qquad \begin{aligned} z_t + Az + B'(\bar{y})z &= h \\ z(0) &= 0. \end{aligned}
\end{cases}
$$

The following theorem states the sufficiency of (SSC).

Theorem 4.3. *Let $\bar{v} = (\bar{y}, \bar{u})$ be admissible for the optimal control problem and suppose that $\bar{v}$ fulfills the first order necessary optimality conditions with associated adjoint state $\bar{\lambda}$. Assume further that (SSC) is satisfied at $\bar{v}$ with $q \geq 4/3$. Then $\bar{u}$ is locally optimal in L^s. Moreover, there exist $\alpha > 0$ and $\rho > 0$ such that*

$$
J(v) \geq J(\bar{v}) + \alpha \, \|u - \bar{u}\|_q^2
$$

holds for all admissible pairs $v = (y,u)$ with $\|u - \bar{u}\|_s \leq \rho$, where the exponent s is given by $1/2 = 1/2s + 1/2q$.

Here, we can observe the two-norm discrepancy which occurs often in optimal control problems for partial differential equations: We achieve quadratical growth of the cost functional with respect to the L^q-norm in a L^s-neighborhood of the reference control. The exponents q and s are connected by the relation $1/q = 1/2s + 1/2$ or $s = q/(2-q)$, which allows us to use the interpolation estimate

$$
\|u\|_q^2 \leq \|u\|_1 \|u\|_s. \tag{4.1}
$$

Comparing with Section 3.3, we see a gap between second-order necessary and sufficient optimality conditions. It is necessary for local optimality that the second derivative of the Lagrangian is non-negative, which is far less than needed for the sufficient condition. One can close the gap with the help of Legendre forms, see Section 2. Since the unit-sphere is not compact in infinite-dimensional spaces, other compactness arguments have to be used.

The proof of the Theorem is given after an auxiliary result, which gives positivity in directions where coercivity of $\mathcal{L}_{vv}$ is not assured.

1.1 Positivity on strongly active sets

The sufficient optimality condition requires coercivity of the second derivative of the Langrangian only with respect to test functions h, whose components are zero on the strongly active sets. However, by the following lemma, we gain an additional positive term that we will need in the proof of sufficiency, see Section 1.2.

Lemma 4.4. *For all $u \in U_{ad}$ it holds*

$$(\gamma\bar{u} + \bar{\lambda},\, u - \bar{u})_Q \geq \epsilon\, \|u - \bar{u}\|_{L^1(Q_\epsilon)}.$$

Proof. The variational inequality (3.6) implies that the pointwise condition

$$(\gamma\, \bar{u}_i(x,t) + \bar{\lambda}_i(x,t)) \cdot (u_i(x,t) - \bar{u}_i(x,t)) \geq 0 \tag{4.2}$$

has to hold for almost all $(x,t) \in Q,\ i = 1,2$. Integrating over Q yields together with the inequality (4.2) and the definition of $Q_{\epsilon,i}$

$$\begin{aligned}
&\int_Q (\gamma\, \bar{u}_i(x,t) + \bar{\lambda}_i(x,t))(u_i(x,t) - \bar{u}_i(x,t))\, \mathrm{d}x\, \mathrm{d}t \\
&\qquad \geq \int_{Q_{\epsilon,i}} (\gamma\, \bar{u}_i(x,t) + \bar{\lambda}_i(x,t))(u_i(x,t) - \bar{u}_i(x,t))\, \mathrm{d}x\, \mathrm{d}t \\
&\qquad = \int_{Q_{\epsilon,i}} |\gamma\, \bar{u}_i(x,t) + \bar{\lambda}_i(x,t)| \cdot |u_i(x,t) - \bar{u}_i(x,t)|\, \mathrm{d}x\, \mathrm{d}t \\
&\qquad \geq \epsilon \int_{Q_{\epsilon,i}} |u_i(x,t) - \bar{u}_i(x,t)|\, \mathrm{d}x\, \mathrm{d}t \\
&\qquad = \epsilon\, \|u - \bar{u}\|_{L^1(Q_\epsilon)}.
\end{aligned}$$

The claim follows by summing up this expression over $i = 1,2$. ◻

1.2 Proof of Theorem 4.3

Proof. Throughout the proof, c is used as a generic constant. Suppose that $\bar{v} = (\bar{y}, \bar{u})$ fulfills the assumptions of the theorem. Let (y,u) be another admissible pair. We have

$$J(\bar{v}) = \mathcal{L}(\bar{v}, \bar{\lambda}) \quad \text{and} \quad J(v) = \mathcal{L}(v, \bar{\lambda}),$$

since $\bar{v}$ and v are admissible. Taylor-expansion of the Lagrange-function yields

$$\mathcal{L}(v,\bar{\lambda}) = \mathcal{L}(\bar{v},\bar{\lambda}) + \mathcal{L}_y(\bar{v},\bar{\lambda})(y-\bar{y}) + \mathcal{L}_u(\bar{v},\bar{\lambda})(u-\bar{u}) + \frac{1}{2}\mathcal{L}_{vv}(\bar{v},\bar{\lambda})[v-\bar{v}, v-\bar{v}]. \tag{4.3}$$

Notice that there is no remainder term due to the quadratic nature of all nonlinearities. Moreover, the necessary condition (3.11a) is satisfied at $\bar{v}$ with adjoint state $\bar{\lambda}$. Therefore, the second term vanishes. The third term is nonnegative due to the variational inequality (3.11b). However, we get even more by Lemma 4.4,

$$\mathcal{L}_u(\bar{v},\bar{\lambda})(u-\bar{u}) = \int_Q (\gamma\bar{u} + \bar{\lambda})(u - \bar{u})\, \mathrm{d}x\, \mathrm{d}t \geq \epsilon \|u - \bar{u}\|_{L^1(Q_\epsilon)}.$$

So we arrive at

$$\begin{aligned} J(v) &= J(\bar{v}) + \mathcal{L}_y(\bar{v},\bar{\lambda})(y-\bar{y}) + \mathcal{L}_u(\bar{v},\bar{\lambda})(u-\bar{u}) + \frac{1}{2}\mathcal{L}_{vv}(\bar{v},\bar{\lambda})[v-\bar{v}]^2 \\ &\geq J(\bar{v}) + \frac{1}{2}\mathcal{L}_{vv}(\bar{v},\bar{\lambda})[v-\bar{v}]^2 + \epsilon\|u-\bar{u}\|_{L^1(Q_\epsilon)}. \end{aligned} \tag{4.4}$$

We set $\delta u := u-\bar{u}$. Let us define δy to be the weak solution of the linearized system

$$\begin{aligned} \delta y_t + \nu A\delta y + B'(\bar{y})\delta y &= \delta u, \\ \delta y(0) &= 0. \end{aligned}$$

When we use δy instead of $y-\bar{y}$, we make the small error

$$r_1 := (y-\bar{y}) - \delta y = S(u) - S(\bar{u}) - S'(\bar{u})(u-\bar{u}).$$

We know that the control-to-state mapping is Fréchet differentiable, see Section 1.3. By Corollary 1.23, the remainder term satisfies

$$\frac{\|r_1\|_W}{\|\delta u\|_q} \to 0 \quad \text{as } \|\delta u\|_q \to 0.$$

Substituting $y-\bar{y} = \delta y + r_1$, we obtain

$$\begin{aligned} \mathcal{L}_{yy}(\bar{v},\bar{\lambda})[y-\bar{y}]^2 &= \mathcal{L}_{yy}(\bar{v},\bar{\lambda})[\delta y]^2 + 2\mathcal{L}_{yy}(\bar{v},\bar{\lambda})[\delta y, r_1] + \mathcal{L}_{yy}(\bar{v},\bar{\lambda})[r_1]^2 \\ &= \mathcal{L}_{yy}(\bar{v},\bar{\lambda})[\delta y]^2 + r_2. \end{aligned}$$

The remainder term can be estimated by

$$|r_2| \leq c(\|\delta y\|_W + \|r_1\|_W)\|r_1\|_W \leq c(\|\delta u\|_q + \|r_1\|_W)\|r_1\|_W,$$

and it follow that r_2 satisfies

$$\frac{|r_2|}{\|\delta u\|_q^2} \to 0 \quad \text{as } \|\delta u\|_q \to 0.$$

Let us abbreviate $\delta v = (\delta y, \delta u)$. So far, we achieved the following estimate for the difference of the objective values

$$J(v) - J(\bar{v}) \geq \frac{1}{2}\mathcal{L}_{vv}(\bar{v},\bar{\lambda})[\delta v]^2 + \epsilon\|u-\bar{u}\|_{L^1(Q_\epsilon)} + r_2.$$

In the next step, we want to apply the coercivity assumption (SSC). To this aim, we split δu in two components as follows:

$$\delta u = h_u + r_u,$$

where h_u and r_u are defined by

$$h_{u,i} = \begin{cases} 0 & \text{on } Q_{\epsilon,i} \\ \delta u_i & \text{on } Q\setminus Q_{\epsilon,i} \end{cases}, \qquad r_{u,i} = \begin{cases} \delta u_i & \text{on } Q_{\epsilon,i} \\ 0 & \text{on } Q\setminus Q_{\epsilon,i} \end{cases} \text{for } i=1,2.$$

Observe, that h_u and r_u are orthogonal, i.e. $(h_u, r_u)_Q = 0$. Moreover, it follows from the definition that the identity

$$\|r_u\|_p = \|\delta u\|_{L^p(Q_\epsilon)} = \|u-\bar{u}\|_{L^p(Q_\epsilon)} \tag{4.5}$$

holds. Analogously, we split $\delta y = h_y + r_y$, where h_y and r_y are solutions of the respective linearized systems with right-hand sides h_u and r_u. Further, we set $h_v := (h_y, h_u)$ and $r_v := (r_y, r_u)$. We continue the investigation of the Lagrangian,

$$\mathcal{L}_{vv}(\bar{v},\bar{\lambda})[\delta v]^2 = \mathcal{L}_{vv}(\bar{v},\bar{\lambda})[h_v]^2 + 2\mathcal{L}_{vv}(\bar{v},\bar{\lambda})[h_v, r_v] + \mathcal{L}_{vv}(\bar{v},\bar{\lambda})[r_v]^2. \tag{4.6}$$

Now, we can use (SSC) to obtain

$$\mathcal{L}_{vv}(\bar{v},\bar{\lambda})[h_v]^2 \geq \delta \|h_u\|_q^2. \tag{4.7}$$

The derivative $\mathcal{L}_{vv}$ can be splitted according to Theorem (2.2) into two addends, $\mathcal{L}_{uu}$ and $\mathcal{L}_{yy}$, and no mixed derivatives appear. At first, we find

$$2\mathcal{L}_{uu}(\bar{v},\bar{\lambda})[h_u, r_u] + \mathcal{L}_{uu}(\bar{v},\bar{\lambda})[r_u]^2 = 2\gamma(h_u, r_u)_Q + \gamma\|r_u\|_2^2 \geq 0. \tag{4.8}$$

Secondly, we investigate $2\mathcal{L}_{yy}(\bar{v},\bar{\lambda})[h_y, r_y] + \mathcal{L}_{yy}(\bar{v},\bar{\lambda})[r_y]^2$. The following estimate is a conclusion of the inequality (2.9) and the Lipschitz continuity of the solution mapping of the linearized system

$$\begin{aligned} \left|2\mathcal{L}_{yy}(\bar{v},\bar{\lambda})[h_y, r_y] + \mathcal{L}_{yy}(\bar{v},\bar{\lambda})[r_y]^2\right| &\geq -c\|r_y\|_W \left(\|h_y\|_W + \|r_y\|_W\right) \\ &\geq -c\|r_u\|_q \left(\|h_u\|_q + \|r_u\|_q\right) \\ &\geq -\frac{\delta}{2}\|h_u\|_q^2 - c\|r_u\|_q^2. \end{aligned} \tag{4.9}$$

Using the relation $\|h_u\|_q^2 \geq 1/2\|\delta u\|_q^2 - \|r_u\|_q^2$, we get by (4.6)–(4.9)

$$\begin{aligned} \mathcal{L}_{vv}(\bar{v},\bar{\lambda})[\delta v]^2 &\geq \frac{\delta}{2}\|h_u\|_q^2 - c\|r_u\|_q^2 \geq \frac{\delta}{4}\|\delta u\|_q^2 - c\|r_u\|_q^2 \\ &= \frac{\delta}{4}\|u-\bar{u}\|_q^2 - c\|u-\bar{u}\|_{L^q(Q_\epsilon)}^2. \end{aligned}$$

So far, we proved the following estimate

$$J(v) - J(\bar{v}) \geq \frac{\delta}{8}\|u-\bar{u}\|_q^2 + \epsilon\|u-\bar{u}\|_{L^1(Q_\epsilon)} - c\|u-\bar{u}\|_{L^q(Q_\epsilon)}^2 + r_2.$$

By the interpolation argument (4.1), we get

$$J(v) - J(\bar{v}) \geq \frac{\delta}{8}\|u-\bar{u}\|_q^2 + \left(\epsilon - c\|u-\bar{u}\|_{L^s(Q_\epsilon)}\right)\|u-\bar{u}\|_{L^1(Q_\epsilon)} + r_2.$$

We can choose ρ small enough, $\|u-\bar{u}\|_s \leq \rho$, such that it holds

$$J(v) - J(\bar{v}) \geq \frac{\delta}{16}\|u-\bar{u}\|_q^2.$$

Thus, we proved quadratic growth of the objective functional in a L^s-neighborhood of the reference control. It implies the local optimality of the pair $(\bar{y}, \bar{u})$. □

2 An equivalent formulation

Here, we comment on other formulations of second-order suficient conditions known from literature, see Bonnans [10], Bonnans, Zidani [14], and Casas, Mateos [17]. Let us consider the sufficient optimality condition (SSC) with parameter $q = 2$. Let us recall (SSC) for $q = 2$ for convenience:

$$\textbf{(SSC)}\begin{cases} \textit{There exist } \epsilon > 0 \textit{ and } \delta > 0 \textit{ such that} \\ \qquad\qquad \mathcal{L}_{vv}(\bar{v},\bar{\lambda})[(S'(\bar{u})h,h)]^2 \geq \delta\,\|h\|_2^2 \\ \textit{holds for all } h \in L^2(Q)^2 \textit{ with} \\ \qquad\qquad h = u - \bar{u},\, u \in U_{ad},\ h_i = 0 \textit{ on } Q_{\epsilon,i} \textit{ for } i = 1,2. \end{cases}$$

We will prove that (SSC) is equivalent to another formulation, introduced first by Bonnans [10, 14], which is quite close to the second-order necessary optimality condition, compare Section 3.3 and especially Theorem 3.6.

$$\textbf{(SSC}_0\textbf{)}\begin{cases} \textit{It holds} \\ \qquad\qquad \mathcal{L}_{vv}(\bar{v},\bar{\lambda})[(S'(\bar{u})h,h)]^2 > 0 \\ \textit{for all } h \in C_0(\bar{u}). \end{cases} \tag{4.10}$$

Here, $C_0(\bar{u})$ is the cone of critical directions defined in (3.14) as

$$C_0(\bar{u}) = \{w \in \mathcal{T}_{U_{ad}}(\bar{u}) :\, w_i(x,t) = 0 \text{ on } Q_{0,i},\, i = 1,2\},$$

Furthermore, we will use an equivalent characterization of the tangent cone, cf. [13]. The tangent cone on U_{ad} at $\bar{u}$ can be written as

$$\mathcal{T}_{U_{ad}}(\bar{u}) = \left\{ h \in L^2(Q)^2 \,\middle|\, h = \lim_{k\to\infty} \frac{u_k - \bar{u}}{t_k},\ u_k \in U_{ad},\ t_k \downarrow 0 \right\}.$$

Here, we see that $T(\bar{u})$ is convex, non-empty and closed in $L^2(\Omega)^n$, hence also weakly closed.

Despite the fact, that (SSC$_0$) looks weaker than (SSC), we will prove that both conditions are equivalent, see also [17, 77]. Moreover, condition (SSC$_0$) implies quadratic growth of the objective functional with respect to the L^2-norm. For finite-dimensional optimization problems it is easy to show the equivalence of (SSC) and (SSC$_0$). The proof relies on the compactness of the unit sphere. However, in infinite-dimensional problems the situations changes, since the unit sphere is not compact. Here, we have to use another compactness result. Let us denote by $q(y)$ the part of the objective that depends on y, e.g. $J(y,u) = q(y) + \frac{\gamma}{2}\|u\|_2^2$. Under some additional regularity assumptions, the mapping

$$u \mapsto q(y) = q(S(u))$$

is compact from $L^2(Q)^2$ to $\mathbb{R}$. The proof of equivalence of both sufficient conditions relies heavily on this fact.

Theorem 4.5. *Let us assume that either $\alpha_T = \alpha_R = 0$ or $y_0 \in V$ holds. Then, conditions (SSC) and (SSC$_0$) are equivalent.*

Proof. It is easy to see that (SSC) implies (SSC$_0$): Let $h \in C_0(\bar{u})$ be given. Since $Q_{\epsilon,i} \subset Q_{0,i}$, it holds $h_i = 0$ on $Q_{\epsilon,i}$. Further, there exists a sequence $h_k = (u_k - \bar{u})/t_k$ converging to h in $L^2(Q)^2$. After extracting a subsequence if necessary, we find that $(u_{k,i}(x,t) - \bar{u}_i(x,t))/t_k \to 0$ a.e. on $Q_{0,i}$. Hence, we can choose u_k such that $u_{k,i}(x,t) = \bar{u}_i(x,t)$ on $Q_{0,i}$. This implies $h_{k,i} = 0$ on $Q_{0,i}$, and h_k can be used as test function in (SSC).

Let us denote $v := (S'(\bar{u})h, h)$, $v_k := (S'(\bar{u})h_k, h_k)$. We rearrange $\mathcal{L}_{vv}$ to

$$\begin{aligned}\mathcal{L}_{vv}(\bar{v},\bar{\lambda})[v]^2 &= \mathcal{L}_{vv}(\bar{v},\bar{\lambda})[v - v_k + v_k]^2 \\ &= \mathcal{L}_{vv}(\bar{v},\bar{\lambda})[v_k]^2 + 2\mathcal{L}_{vv}(\bar{v},\bar{\lambda})[v - v_k, v_k] + L_{vv}(\bar{v},\bar{\lambda})[v - v_k]^2.\end{aligned}$$

Using the coercivity assumption of (SSC) and estimates of $\mathcal{L}_{vv}$ and S', we obtain

$$\mathcal{L}_{vv}(\bar{v},\bar{\lambda})[v]^2 \geq \delta\|h_k\|_2^2 - c_1\|h - h_k\|_2\|h_k\|_2 - c_2\|h - h_k\|_2^2 \geq \frac{\delta}{2}\|h_k\|_2^2 - c\|h - h_k\|_2^2,$$

which gives in the limit $k \to \infty$

$$\mathcal{L}_{vv}(\bar{v},\bar{\lambda})[v]^2 \geq \frac{\delta}{2}|h|_2^2 > 0,$$

and (SSC$_0$) is satisfied.

Let us prove the converse direction. Assume, that (SSC$_0$) holds true but not (SSC). Then for all $\epsilon > 0$ and $\delta > 0$ there exists $h_{\delta,\epsilon} \in L^2(Q)^2$ such that $(h_{\delta,\epsilon})_i = 0$ on $Q_{\epsilon,i}$, $h_{\delta,\epsilon} = u_{\delta,\epsilon} - \bar{u}$, $u_{\delta,\epsilon} \in U_{ad}$, and

$$\mathcal{L}_{vv}(\bar{v},\bar{\lambda})[(z_{\delta,\epsilon}, h_{\delta,\epsilon})]^2 < \delta\,\|h_{\delta,\epsilon}\|_2^2$$

is fulfilled with associated $z_{\delta,\epsilon} = S'(\bar{u})h_{\delta,\epsilon}$. Multiplying $h_{\delta,\epsilon}$ by some positive constant, we can assume $\|h_{\delta,\epsilon}\|_2 = 1$ and $h_{\delta,\epsilon} \in \mathcal{T}_{U_{ad}}(\bar{u})$. Choosing $\delta_k = \epsilon_k = 1/k$, $h_k := h_{\delta_k,\epsilon_k}$, we find

$$\mathcal{L}_{vv}(\bar{v},\bar{\lambda})[(z_k, h_k)]^2 < \frac{1}{k},$$

with $z_k = S'(\bar{u})h_k$, hence

$$\limsup_k \mathcal{L}_{vv}(\bar{v},\bar{\lambda})[(z_k, h_k)]^2 \leq 0. \tag{4.11}$$

Since the set $\{h_k\}_{k=1}^\infty$ is bounded in $L^2(Q)^2$, there exists an element $\tilde{h} \in L^2(Q)^2$, such that, after extracting a subsequence if necessary, the h_k converge weakly in $L^2(Q)^2$ to $\tilde{h}$. The tangent cone $\mathcal{T}_{U_{ad}}(\bar{u})$ is weakly closed, therefore $\tilde{h} \in \mathcal{T}_{U_{ad}}(\bar{u})$.

Next, we want to show $h_{k,i}(x,t) \to 0$ a.e. pointwise on $Q_{0,i}$. Let $(x_0,t_0) \in Q_{0,i}$ be given. Then it holds $0 < |\gamma\bar{u}_i(x_0,t_0) + \bar{\lambda}_i(x_0,t_0)| = \tau'$, which implies by definition $(x_0,t_0) \in Q_{\tau,i}$ for all $0 \leq \tau < \tau'$. Hence, there exists an index $k_i(x_0,t_0)$ such that $(x_0,t_0) \in Q_{\epsilon_k,i} = Q_{1/k,i}$ for all $k > k_i(x_0,t_0)$. By construction of h_k we conclude $h_{k,i}(x_0,t_0) = 0$ for all $k > k_i(x_0,t_0)$. It follows $\tilde{h}(x) = 0$ on Q_0 almost everywhere.

Let us denote by q the part of the $\mathcal{L}_{vv}$ that depends on the state, e.g. $\mathcal{L}_{vv}(\bar{v},\bar{\lambda})[(z,h)]^2 = q(z) + \gamma\|h\|_2^2$ with

$$q(z) = \alpha_Q\|z\|_2^2 + \alpha_T|z(T)|_2^2 + \alpha_R\|\operatorname{curl} z\|_2^2 + 2\langle B(z),\,\bar{\lambda}\rangle_{L^2(V'),L^2(V)}.$$

For abbreviation, let us define by $\tilde{q}$ the part of q that orginates from the objective functional

$$\tilde{q}(z) = \alpha_Q\|z\|_2^2 + \alpha_T|z(T)|_2^2 + \alpha_R\|\operatorname{curl} z\|_2^2.$$

We decompose $\mathcal{L}_{vv}$ and use $|h_k|_2 = 1$ to get

$$\begin{aligned}\mathcal{L}_{vv}(\bar{v},\bar{\lambda})[(z_k,h_k)]^2 &= \tilde{q}(z_k) + \gamma|h_k|_2^2 + \langle B''(\bar{y})[z_k,z_k],\,\bar{\lambda}\rangle_{L^2(V'),L^2(V)} \\ &= \tilde{q}(z_k) + \gamma + 2\langle B(z_k),\,\bar{\lambda}\rangle_{L^2(V'),L^2(V)}.\end{aligned} \tag{4.12}$$

The solution mapping $S'(\bar{u}) : h \mapsto z$ is linear and continuous from $L^2(Q)^2$ to $W(0,T)$. Thus, we obtain $z_k \rightharpoonup \tilde{z}$ in $W(0,T)$ and $z_k \to \tilde{z}$ in $L^2(Q)^2$, since $W(0,T)$ is compactly imbedded in $L^2(Q)^2$, cf. [4]. A well-known result of Temam [70, Lemma III.3.2] yields $\langle B(z_k), \bar{\lambda}\rangle_{L^2(V'),L^2(V)} \to \langle B(\tilde{z}), \bar{\lambda}\rangle_{L^2(V'),L^2(V)}$. The convergence $\tilde{q}(z_k) \to \tilde{q}(\tilde{z})$ will be proven below in Lemma 4.6.

Hence we obtain $\lim_{k\to\infty} q(z_k) = q(\tilde{z})$. Passing to the limit in (4.12), we get applying (4.11)

$$q(\tilde{z}) \leq \lim_k \mathcal{L}_{vv}(\bar{v}, \bar{\lambda})[(z_k, h_k)]^2 - \gamma \leq -\gamma < 0,$$

which proves that $\tilde{h}$ cannot vanish, remember $q(0) = 0$. Finally,

$$\mathcal{L}_{vv}(\bar{v}, \bar{\lambda})[(\tilde{z}, \tilde{h})]^2 = \gamma \|\tilde{h}\|_2^2 + q(\tilde{z}) \leq \gamma - \gamma \leq 0$$

is obtained, which contradicts (SSC_0). □

The proof is complete after we have verified the following Lemma.

Lemma 4.6. *Let the assumptions of the previous Theorem 4.5 be fullfilled. Let h_k be a sequence in $L^2(Q)^2$ that converges weakly in $L^2(Q)^2$ to $\tilde{h}$. Then it holds*

$$\tilde{q}(S'(\bar{u})h_k) \to q(S'(\bar{u})h) \text{ for } k \to \infty.$$

Proof. The state $\bar{y} = S(\bar{u})$ belongs at least to $W(0,T)$. Then the operator $S'(\bar{u})$ is a linear and continuous — hence weakly continuous — mapping from $L^2(Q)^2$ to $W(0,T)$. This implies the weak convergence $S'(\bar{u})h_k \rightharpoonup S'(\bar{u})\tilde{h}$ in $W(0,T)$. The space $W(0,T)$ is compactly imbedded in $L^2(Q)^2$, see [4], thus $S'(\bar{u})h_k$ converges in $L^2(Q)^2$ to $S'(\bar{u})\tilde{h}$. And it follows

$$\frac{\alpha_Q}{2}\|S'(\bar{u})h_k\|_2^2 \;\to\; \frac{\alpha_Q}{2}\|S'(\bar{u})\tilde{h}\|_2^2 \text{ for } k \to \infty.$$

If $\alpha_T = \alpha_R = 0$, then the claim is proven.

In the cases $\alpha_T \neq 0$ or $\alpha_R \neq 0$, we have by assumption $y_0 \in V$, which implies together with $\bar{u} \in L^2(Q)^2$ that $\bar{y}$ belongs to $H^{2,1}$, see Theorem 1.13. Then the solutions of the linearized systems are also more regular, and $S'(\bar{u})$ is a linear operator from $L^2(Q)^2$ to $H^{2,1}$. Now, let us denote for brevity $z_k := S'(\bar{u})h_k$ and $\tilde{z} = S'(\bar{u})\tilde{h}$. We find $z_k \rightharpoonup \tilde{z}$ in $H^{2,1}$, which implies $z_k(T) \rightharpoonup \tilde{z}(T)$ in V and $\operatorname{curl} z_k \rightharpoonup \operatorname{curl} \tilde{z}$ in $W(0,T)$. By the compact imbeddings $V \hookrightarrow H$ and $W(0,T) \hookrightarrow L^2(Q)^2$, we obtain the strong convergences $z_k(T) \to \tilde{z}(T)$ in H and $\operatorname{curl} z_k \to \operatorname{curl} \tilde{z}$ in $L^2(Q)^2$. Hence, we have convergence of the norms and

$$\frac{\alpha_T}{2}|z_k(T)|_2^2 + \frac{\alpha_R}{2}\|\operatorname{curl} z_k\|_2^2 \;\to\; \frac{\alpha_T}{2}|\tilde{z}(T)|_2^2 + \frac{\alpha_R}{2}\|\operatorname{curl} \tilde{z}\|_2^2 \tag{4.13}$$

for $k \to \infty$. Altogether, it implies the convergence of the quadratic form q, and the claim is proven. □

Without the assumption $y_0 \in V$, we cannot prove the convergence (4.13), since for instance the imbedding $W(0,T) \hookrightarrow C([0,T];H)$ is *not* compact, see [4].

Following the lines of Bonnans [10], we can prove that (SSC_0) even implies a L^2-growth condition in a L^2-neighborhood around $(\bar{y}, \bar{u})$.

Theorem 4.7. *Let $\bar{v} = (\bar{y}, \bar{u})$ be admissible for the optimal control problem and suppose that $\bar{v}$ fulfills the first-order necessary optimality condition with associated adjoint state $\bar{\lambda}$. Assume further that (SSC$_0$) is satisfied at $\bar{v}$, and that the regularity pre-requisites of Theorem 4.5 are met. Then there exist $\alpha > 0$ and $\rho > 0$ such that*

$$J(v) \geq J(\bar{v}) + \alpha \, \|u - \bar{u}\|_2^2 \tag{4.14}$$

holds for all admissible pairs $v = (y, u)$ with $\|u - \bar{u}\|_2 \leq \rho$.

Proof. Let us suppose that (SSC$_0$) is satisfied, whereas (4.14) does not hold. Then for all $\alpha > 0$ and $\rho > 0$ there exists $u_{\alpha,\rho} \in U_{ad}$ with $\|u_{\alpha,\rho} - \bar{u}\|_2 \leq \rho$ and

$$J(v_{\alpha,\rho}) < J(\bar{v}) + \alpha \, \|u_{\alpha,\rho} - \bar{u}\|_2^2, \tag{4.15}$$

where we used the notations $v_{\alpha,\rho} = (u_{\alpha,\rho}, y_{\alpha,\rho})$ and $y_{\alpha,\rho} := S(u_{\alpha,\rho})$ Let us choose $\alpha_k = \rho_k = 1/k$ and $u_k = u_{\alpha_k,\rho_k}$, $y_k = y_{\alpha_k,\rho_k}$.

By construction, it follows $u_k \to \bar{u}$ in $L^2(Q)^2$ as $k \to \infty$. Hence, we can write $u_k = \bar{u} + t_k h_k$, $\|h_k\|_2 = 1$ and $t_k \to 0$ as $k \to \infty$. Because of the boundedness of the set of these h_k in $L^2(Q)^2$, we can extract a subsequence denoted again by (h_k) converging weakly to $\tilde{h} \in \mathcal{T}_{U_{ad}}(\bar{u}) \subset L^2(Q)^2$. In the following, let us denote $z_k := S'(\bar{u})h_k$.

Since $(\bar{u}, \bar{y})$ and (u_k, y_k) satisfy the state equation, it holds $\mathcal{L}(\bar{v}, \bar{\lambda}) = J(\bar{v})$ and $\mathcal{L}(v_k, \bar{\lambda}) = J(v_k)$. Then we obtain

$$J(v_k) = \mathcal{L}(v_k, \bar{\lambda}) = \mathcal{L}(\bar{v}, \bar{\lambda}) + t_k \, \mathcal{L}_u(\bar{v}, \bar{\lambda})h_k + \mathcal{L}_y(\bar{v}, \bar{\lambda})(y_k - \bar{y}) + \mathcal{L}_{vv}[(y_k - \bar{y}, \, t_k h_k)^2]. \tag{4.16}$$

The first-order necessary conditions (3.11a)–(3.11b) are fulfilled, hence we find $\mathcal{L}_y(\bar{v}, \bar{\lambda})z_k = 0$ and $\mathcal{L}_u(\bar{v}, \bar{\lambda})h_k \geq 0$. If we replace $y_k - \bar{y}$, we make a small error $r_1 = (y_k - \bar{y}) - z_k$. Its norm can be estimated by $\|r_1\|_W \leq c\, t_k^2$. Then, we can write the rightmost addend of (4.16) as

$$\mathcal{L}_{vv}[(y_k - \bar{y}, \, t_k h_k)^2] = t_k^2 \, \mathcal{L}_{vv}[(z_k, h_k)^2] + r_2,$$

with a second remainder term r_2, satisfying $|r_2| \leq c\, t_k^3$ with a constant c indenpendent of t_k, h_k, and z_k. For more detailed discussion of those remainder terms we refer to the proof of Theorem 4.3 in Section 1.2.

Now, we will show $\tilde{h} = 0$ a.e. on Q_0. We derive from (4.15) and (4.16)

$$\begin{aligned} 0 \leq \mathcal{L}_u(\bar{v}, \bar{\lambda})h_k &= \frac{1}{t_k} \left(J(v_k) - J(\bar{v})\right) - t_k \, \mathcal{L}_{vv}[(z_k, h_k)^2] - \frac{r_2}{t_k} \\ &< t_k \left\{ \frac{1}{k} + c\, t_k - \mathcal{L}_{vv}[(z_k, h_k)^2] \right\}, \end{aligned} \tag{4.17}$$

which gives $\mathcal{L}_u(\bar{v}, \bar{\lambda})\tilde{h} = 0$, since $\mathcal{L}_{vv}[(z_k, h_k)^2]$ is bounded. The variational inequality

$$(\gamma \bar{u}_i(x,t) + \bar{\lambda}_i(x,t))h_{k,i}(x,t) \geq 0$$

holds a.e. on Q, $i = 1, 2$, so the weak limit $\tilde{h}_i(x,t)$ satisfies

$$(\gamma \bar{u}_i(x,t) + \bar{\lambda}_i(x,t))\tilde{h}(x,t) \geq 0$$

as well. This, together with $\mathcal{L}_u(\bar{v}, \bar{\lambda})\tilde{h} = 0$, yields $\tilde{h}(x) = 0$ on Q_0, cf. the definition of Q_0. And it follows that $\tilde{h}$ belongs to the cone of critical directions $\tilde{h} \in C_0(\bar{u})$.

Finally, we show that (4.15) contradicts (SSC_0). Obviously (4.17) implies

$$\mathcal{L}_{vv}[(z_k, h_k)^2] < \frac{1}{k} + c\, t_k.$$

Arguing as in the proof of the previous Theorem 4.5, we find that $\tilde{h}$ satisfies

$$\mathcal{L}_{vv}(\bar{v}, \bar{\lambda})[(\tilde{z}, \tilde{h})]^2 \leq 0,$$

with $h \not\equiv 0$. Since $\tilde{h}$ is admissible as test function in (SSC_0), this shows that the positivity assumption of (SSC_0) is violated. ∎

Observe that this theorem does not have to deal with the two-norm discrepancy typically appearing in optimal control of semilinear equations. This is due to the very special form of the quadratic cost functional (2.1), the linear appearance of the control u in the state equation, and the differentiability of the nonlinearity of the Navier-Stokes equations and the associated solution operator S in weaker than L^∞-norms.

Another second-order sufficient optimality condition was introduced by Casas and Mateos [17] for elliptic problems. It involves the Hamiltonian of the optimal control problem. In the case of quadratic functionals with regularization term $\frac{\gamma}{2}\|u\|_2^2$ this formulation is equivalent to (SSC_0).

3 Verification of (SSC)

The second-order sufficient conditions are essential assumptions for many results connected with approximation and convergence. In view of this importance, the verification of (SSC) is a natural desire.

3.1 Numerical check

Unfortunately, the numerical verification of (SSC) is a delicate question that has not yet been solved satisfactorily till now. This concerns optimal control problems for ODEs as well as PDEs: Any numerical check is connected with some discretization so that second-order conditions are verified in some finite-dimensional setting.

For instance, the eigenvalues of the reduced Hessian matrix can be computed for the finite-dimensional model to verify its definiteness, see Mittelmann and Tröltzsch [58]. However, it seems to be impossible to deduce from the numerical result the coercivity for the infinite-dimensional setting. In the control of Navier-Stokes equations, it is known for tracking type functionals that small residuals are sufficient for second-order conditions to hold, cf. Hinze and Kunisch [46]. Here, the problem is open whether the optimal residual is really small enough in function space. The numerical method determines the discrete solution. If its associated remainder is small, the same should hold for the continuous solution — provided that it is close to the discrete one. But this can only be expected, if second-order conditions are satisfied for the solution of the infinite-dimensional problem.

In the case of ODEs, the solvability of certain Riccati-equations can be investigated to check second-order conditions. Again, the numerical implementation of this concept needs discretization. It is not clear if the result contains reliable information on the continuous case.

Eigenvalues of the reduced Hessian, small remainders, and in the case of ODEs, the solution of Riccati equations, are useful numerical techniques to check second-order conditions. However, their numerical results will never give complete evidence for the original problem, since the computed discrete solution will be close to the exact one only under the assumption of a second-order sufficient condition. In this way, we encounter a circular reasoning.

In view of these remarks, one should handle second-order sufficient conditions similarly as constraint-qualifications in nonlinear programming: They cannot be checked in general, but they should be assumed to perform a satisfactory analysis.

3.2 A sufficient condition for (SSC)

In some references, see for instance [44, 46], sufficient conditions for the fulfillment of (SSC) are developed. The second-derivative of the Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{vv}(y,u,\lambda)[(w,h)^2] = \alpha_T |w(T)|_H^2 + \alpha_R \|\operatorname{curl} w\|_2^2 + \alpha_Q \|w\|_2^2 \\ - \langle B''(y)[w,w],\, \lambda\rangle_{L^2(V'),L^2(V)} + \gamma \|h\|_2^2\end{aligned}$$

The only addend, which can disturb coercivity is

$$\langle B''(y)[w,w],\, \lambda\rangle_{L^2(V'),L^2(V)} = 2b_Q(w,w,\lambda).$$

Hence, $\mathcal{L}_{vv}$ is positive definit if the norm of the adjoint λ is small enough. The adjoint λ depends on $\alpha_R \operatorname{curl} y$, $\alpha_Q(y - y_Q)$, and $\alpha_T(y(T) - y_T)$. Therefore, the sufficient condition is fulfilled if all those quantities are small enough. But one does not know what 'small enough' means in this context, and a numerical verfication of (SSC) based on that smallness is impossible.

Chapter 5

Stability of optimal controls

In this chapter we are dealing with stability of a locally optimal reference tripel $(\bar{y}, \bar{u}, \bar{\lambda})$ of the original problem (2.3). To be more specific, consider the perturbed optimal control problem with perturbation vector $z = (z_y, z_0, z_Q, z_T, z_u)$

$$\min J(y,u,z) = \frac{\alpha_T}{2}|y(\cdot,T) - y_T|_2^2 + (z_T, y(T))_\Omega + \frac{\alpha_Q}{2}\|y - y_Q\|_2^2 + (z_Q, y)_Q \\ + \frac{\alpha_R}{2}\|\operatorname{curl} y\|_2^2 + \frac{\gamma}{2}\|u\|_2^2 - (z_u, u)_Q \quad (5.1)$$

subject to the perturbed Navier-Stokes equations

$$\begin{array}{rcll} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p &=& u + z_y & \text{in } Q, \\ \operatorname{div} y &=& 0 & \text{in } Q, \\ y(0) &=& y_0 + z_0 & \text{in } \Omega, \end{array} \quad (5.2)$$

and the constraint

$$u \in U_{ad}.$$

The perturbations z_y and z_0 influence the state equations. They can be an unknown source term in the case of z_y respectively an uncertainty due to inexact measurements of the initial state. The perturbations in the cost functional, z_Q and z_T, represent uncertainties of the desired states.

Here arises the natural question: How depends the optimal tripel (y, u, λ) on the perturbation z? This question well be answered in the sequel.

The plan of this chapter is as follows: At first, we will introduce the concept of generalized equations, where we rely on an abstract stability result due to Robinson. Secondly, the optimality system is written as a generalized equation in function spaces. Finally, we prove stability of optimal controls provided a second-order sufficient optimality condition holds. Under suitable assumptions, we get even stability of optimal controls with respect to the L^∞-norm.

Here, we follow a well-known approach. It was used in Tröltzsch [74, 76] to prove stability of optimal controls for linear-quadratic parabolic control problems. This method of proof was extended to optimal control of semilinear elliptic as well as parabolic equations by Malanowski and Tröltzsch [55] and Unger [80]. A stability analysis for optimal control of the stationary Navier-Stokes system can be found in the paper by Roubíček and Tröltzsch [66].

1 Generalized equations

Later on, we will apply a result on generalized equations due to Robinson [62]. First, we recall some basic notations. We consider the generalized equation

$$0 \in F(x) + N(x), \tag{5.3}$$

where F is a C^1-mapping between to Banach spaces X and Z, while $N : X \mapsto 2^Z$ is a set-valued mapping. Note, that we do not require any other properties of the mapping N. In Robinson's original paper [62], it was assumed that N has closed graph. However, Dontchev [24] showed that this assumption is not needed.

Let $\bar{x}$ be a solution of (5.3). The generalized equation is said to be *strongly regular* at the point $\bar{x}$, if there are open balls $B_X(\bar{x}, \rho_x)$ and $B_Z(0, \rho_z)$ such that for all $z \in B_Z(0, \rho_z)$ the linearized and perturbed equation

$$z \in F(\bar{x}) + F'(\bar{x})(x - \bar{x}) + N(x) \tag{5.4}$$

admits a unique solution $x = x(z)$ in $B_X(\bar{x}, \rho_x)$, and the mapping $z \mapsto x$ is Lipschitz continuous $B_Z(0, \rho_z)$ from to $B_X(\bar{x}, \rho_x)$. The following theorem allows to get stability results for the perturbed nonlinear problem from stability results for the perturbed linearized equation.

Theorem 5.1. *Let $\bar{x}$ be a solution of* (5.3) *and assume that* (5.3) *is strongly regular at $\bar{x}$. Then there exist open balls $B_X(\bar{x}, \rho'_x)$ and $B_Z(0, \rho'_z)$ such that for all $z \in B_Z(0, \rho'_z)$ the perturbed equation*

$$z \in F(x) + N(x)$$

has a unique solution in $x = x(z) \in B_X(\bar{x}, \rho'_x)$, and the solution mapping $z \mapsto x(z)$ is Lipschitz continuous from $B_Z(0, \rho'_z)$ to $B_X(\bar{x}, \rho'_x)$.

In the sequel, we will apply this result to our optimal control problem.

2 Perturbed optimal control problem

Let $(\bar{y}, \bar{u}, \bar{\lambda})$ satisfy the first-order necessary optimality conditions, see Theorem 3.2, together with the second-order sufficient optimality conditions (SSC). The optimality system consisting of state equation, adjoint equation (3.5) and variational inequality (3.6), can be written in the condensed form

$$F(\bar{y}, \bar{u}, \bar{\lambda}) + \big(0, 0, 0, 0, \mathcal{N}_{U_{ad}}(\bar{u})\big)^T \ni 0, \tag{5.5}$$

where the function F,

$$F : H^{2,1} \times L^2(Q)^2 \times H^{2,1} \to L^2(Q)^2 \times V \times L^2(Q)^2 \times V \times L^2(Q)^2 \tag{5.6}$$

is given by

$$F(y, u, \lambda) = \begin{pmatrix} y_t + \nu A y + B(y) \\ y(0) \\ -\lambda_t + \nu A\lambda + B'(y)^*\lambda \\ \lambda(T) \\ \gamma u + \lambda \end{pmatrix} - \begin{pmatrix} u \\ y_0 \\ \alpha_Q(y - y_Q) + \alpha_R \vec{\operatorname{curl}} \operatorname{curl} y \\ \alpha_T(y(T) - y_T) \\ 0 \end{pmatrix}. \tag{5.7}$$

We will apply Theorem 5.1 to the generalized equation (5.5). To do so, we have to show strong regularity of this equation at the reference tripel $(\bar{y}, \bar{u}, \bar{\lambda})$. At first, we investigate the mapping F.

Corollary 5.2. *The function F defined by* (5.7) *is continuously differentiable in the setting* (5.6).

Proof. The components of F are affine linear and continuous functions except F_1, which contains the nonlinear part $B(y)$. We derive for $y, h \in H^{2,1}$, $v \in L^2(Q)^2$

$$\begin{aligned} B(y+h)v - B(y)v &= \int_0^T b(y+h, y+h, v) - b(y, y, v)\, \mathrm{d}t \\ &= \int_0^T b(y, h, v) + b(h, y, v) + b(h, h, v)\, \mathrm{d}t. \end{aligned}$$

This gives immediately the directional derivative of B in direction h as $B'(y)h = \int_0^T b(y, h, v) + b(h, y, v)\, \mathrm{d}t$. We proceed with

$$\begin{aligned} \|B(y+h) - B(y) - B'(y)h\|_2 &= \sup_{v \in L^2(Q)^2 \setminus \{0\}} \|v\|_2^{-1} \int_0^T b(h, h, v)\, \mathrm{d}t \\ &\leq \sup_{v \in L^2(Q)^2 \setminus \{0\}} \|v\|_2^{-1} c\, \|h\|_{L^4(W^{1,4})} \|h\|_4 \|v\|_2 \leq c\, \|h\|_{H^{2,1}}^2, \end{aligned}$$

which proves Frechét-differentiability of $B(y)$. To prove continuous differentiability we take $y_1, y_2 \in H^{2,1}$. Then for any direction $h \in H^{2,1}$ and element $v \in L^2(Q)^2$ we obtain

$$\begin{aligned} \big|(B'(y_1)h - B'(y_2)h)v\big| &= \left| \int_0^T b(y_1 - y_2, h, v) + b(h, y_1 - y_2, v)\, \mathrm{d}t \right| \\ &\leq c \|y_1 - y_2\|_{H^{2,1}} \|h\|_{H^{2,1}} \|v\|_2, \end{aligned}$$

which shows that the mapping $y \mapsto B'(y)$ is even Lipschitz continuous from $H^{2,1}$ in the space $\mathcal{L}(H^{2,1}, L^2(Q)^2)$. $\square$

For convenience, we introduce the space of perturbation vectors Z as

$$Z := L^2(Q)^2 \times V \times L^2(Q)^2 \times V \times L^2(Q)^2 \tag{5.8}$$

equipped with the norm $\|z\|_Z = \|z_y\|_2 + |z_0|_V + \|z_Q\|_2 + |z_T|_V + \|z_u\|_2$.

The optimality system of the perturbed problem (5.1) is equivalent to the generalized equation

$$F(y, u, \lambda) + \big(0, 0, 0, 0, \mathcal{N}_{U_{ad}}(u)\big)^T \ni z, \tag{5.9}$$

where $z = (z_y, z_0, z_Q, z_T, z_u) \in Z$. The components one to four of this inclusion are in fact equations.

The next step in proving strong regularity of (5.5) is the investigation of the linearized version of the inclusion (5.9)

$$F(\bar{y}, \bar{u}, \bar{\lambda}) + F'(\bar{y}, \bar{u}, \bar{\lambda})(y - \bar{y}, u - \bar{u}, \lambda - \bar{\lambda}) + \big(0, 0, 0, 0, \mathcal{N}_{U_{ad}}(u)\big) \ni z. \tag{5.10}$$

This generalized equation corresponds to the following system. It consists of the state equations

$$\begin{aligned} y_t + \nu A y + B'(\bar{y})y &= u + B(\bar{y}) + z_y \\ y(0) &= y_0 + z_0, \end{aligned} \tag{5.11}$$

the adjoint equations

$$\begin{aligned} -\lambda_t + \nu A\lambda + B'(\bar{y})^*\lambda &= -B'(y-\bar{y})^*\bar{\lambda} + \alpha_Q(y - y_Q) + \alpha_R \vec{\operatorname{curl}} \operatorname{curl} y + z_Q \\ \lambda(T) &= \alpha_T(y(T) - y_T) + z_T, \end{aligned} \tag{5.12}$$

and the variational inequality

$$\gamma u + \lambda + \mathcal{N}_{U_{ad}}(u) \ni z_u.$$

Here, we used the identity $B'(\bar{y})\bar{y} = 2B(\bar{y})$, which gives

$$B(\bar{y}) + B'(\bar{y})(y - \bar{y}) = B'(\bar{y})y - B(\bar{y})$$

and, consequently, $+B(\bar{y})$ on the right-hand side of (5.11). These equations build up the optimality system of the perturbed linear-quadratic optimization problem given by

$$\begin{aligned} \min J^{(z)}(y,u) = \frac{\alpha_T}{2}|y(T) - y_d|_H^2 + \frac{\alpha_Q}{2}\|y - y_Q\|_2^2 + \frac{\alpha_R}{2}\|\operatorname{curl} y\|_2^2 + \frac{\gamma}{2}\|u\|_2^2 \\ + (z_Q, y)_Q + (z_T, y(T))_\Omega - (z_u, u)_Q - b_Q(y - \bar{y}, y - \bar{y}, \bar{\lambda}) \end{aligned} \tag{5.13a}$$

subject to the linearized state equation

$$\begin{aligned} y_t + \nu A y + B'(\bar{y})y &= u + B(\bar{y}) + z_y \\ y(0) &= y_0 + z_0 \end{aligned} \tag{5.13b}$$

and the control constraint

$$u \in U_{ad}. \tag{5.13c}$$

Henceforth, we will denote the perturbed linear-quadratic problem (5.13a)–(5.13c) by (P_z).

3 An auxiliary problem

Using the second-order condition (SSC), we have no convexity of the objective functional $J^{(z)}$ in directions that are not included in (SSC). We even do not know whether the functional $J^{(z)}$ is bounded from below. To overcome these difficulties, we have to restrict the optimization problem (5.13) to a modified admissible set

$$u \in \widetilde{U_{ad}} = \{\tilde{u} \in U_{ad} : \tilde{u}_i(x,t) = \bar{u}_i(x,t) \text{ on } Q_{\epsilon,i},\ i = 1,2\}. \tag{5.13d}$$

This means, we do not allow changes of the control on the strongly active set. Let us denote the problem of minimizing the functional (5.13a) under the constraints (5.13b) and (5.13d) by $(\widetilde{P_z})$. The problem $(\widetilde{P_z})$ is equivalent to the solution of the linearized and perturbed generalized equation, compare (5.10),

$$F(\bar{y},\bar{u},\bar{\lambda}) + F'(\bar{y},\bar{u},\bar{\lambda})(y - \bar{y}, u - \bar{u}, \lambda - \bar{\lambda}) + \big(0,0,0,0,\mathcal{N}_{\widetilde{U_{ad}}}(u)\big) \ni z. \tag{5.14}$$

In this section, we investigate the optimization problem $(\widetilde{P_z})$. At first, we will prove existence of solutions. Secondly, we show that the optimal control of the perturbed problem $(\widetilde{P_z})$ depends Lipschitz continuously on the perturbation z. So we will investigate for a while strong regularity of the generalized equation

$$F(y, u, \lambda) + \left(0, 0, 0, 0, \mathcal{N}_{\widetilde{U_{ad}}}(u)\right)^T \ni z. \tag{5.15}$$

Then, we will come back to the original problem in Section 4, and prove the same stability result.

3.1 Existence of solutions

Due to the constraint $u \in \widetilde{U_{ad}}$, there exists an unique minimizer of perturbed optimization problem $(\widetilde{P_z})$.

Theorem 5.3. *Assume that $(\bar{y}, \bar{u}, \bar{\lambda})$ satisfy the first-order necessary optimality condition and the coercivity condition (SSC). Then the perturbed linear-quadratic optimal control problem $(\widetilde{P_z})$ admits a unique optimal control u_z.*

Proof. Let us denote the Lagrangian associated to $(\widetilde{P_z})$ by $\mathcal{L}^{(z)}$. Then it holds for all y, u, λ

$$\mathcal{L}^{(z)}_{vv}(y, u, \lambda) = \mathcal{L}_{vv}(\bar{y}, \bar{u}, \bar{\lambda}). \tag{5.16}$$

Now take two controls $u_1, u_2 \in \widetilde{U_{ad}}$ with associated solutions y_1, y_2 of (5.13b). Then the pair $(y_1 - y_2, u_1 - u_2)$ fits in the assumption of (SSC), and we find

$$\mathcal{L}^{(z)}_{vv}(y, u, \lambda)[(y_1 - y_2, u_1 - u_2)]^2 = \mathcal{L}_{vv}(\bar{y}, \bar{u}, \bar{\lambda})[(y_1 - y_2, u_1 - u_2)]^2 \geq \delta \|u_1 - u_2\|_q^2.$$

Thus, the problem $(\widetilde{P_z})$ is convex on the space of admissible controls $\widetilde{U_{ad}}$. Thus, the problem $(\widetilde{P_z})$ as a linear-quadratic optimization problem with strongly convex objective functional is uniquely solvable. $\square$

For a more detailed discussion of those aspects we refer to [66], where the stability analysis is made for the stationary Navier-Stokes system. We denote the unique solution of $(\widetilde{P_z})$ by $u_z = u(z)$ with associated state y_z and adjoint state λ_z.

3.2 Stability of optimal controls in L^2

Now, we are going to prove stability of optimal controls in the setting given above. To verify strong regularity we have to prove Lipschitz continuity of the solution mapping $z \mapsto (y_z, u_z, \lambda_z)$ of the perturbed linearized problem $(\widetilde{P_z})$.

Theorem 5.4. *Let (SSC) be satisfied for the reference solution $\bar{v}$ with adjoint state $\bar{\lambda}$. Let additionally y_0, $y_T \in V$, $y_Q \in L^2(Q)^2$ be given. Then the mapping $z \to (y_z, u_z, \lambda_z)$ is Lipschitz continuous from Z to $H^{2,1} \times L^2(Q)^2 \times H^{2,1}$.*

Proof. Let $z_1, z_2 \in Z$ be given. Let us denote the optimal controls of the perturbed problem $(\widetilde{P_z})$ by $u_i := u_{z_i}$ with associated states $y_i := y_{z_i}$ and adjoints $\lambda_i := \lambda_{z_i}$, $i := 1, 2$. We define their differences by $z := z_1 - z_2$, $u := u_1 - u_2$, $y := y_1 - y_2$, and $\lambda := \lambda_1 - \lambda_2$.

We know by the regularity results of Chapter 1, that the states y_i and the adjoints λ_i are in the space $H^{2,1}$. However, we will estimate their differences in quite weaker norms. This is due to the fact that the condition (SSC) gives coercivity only with respect to $L^q(Q)^2$-norms, $q \leq 2$.

Throughout the proof, we will abbreviate the scalar product in $L^2(Q)^2$ by $(\cdot,\cdot) := (\cdot,\cdot)_Q$ and the duality pairing between the function spaces $L^2(0,T;V')$ and $L^2(0,T;V)$ by $\langle \cdot,\, \cdot\rangle := \langle \cdot,\, \cdot\rangle_{L^2(V'),L^2(V)}$.

At first, we consider the variational inequality connected with the constraint $u_i \in \widetilde{U_{ad}}$,

$$(\gamma u_i + \lambda_i - z_{u,i},\, u - u_i) \geq 0 \qquad \forall u \in \widetilde{U_{ad}}.$$

Testing the inequality for u_i, $i = 1, 2$ with u_j, $j = 2 - i$, and adding them, we find

$$-(\lambda, u) + (u, z_u) \geq \gamma \|u\|_2^2. \tag{5.17}$$

Secondly, we consider the state equation. The difference y is the weak solution of

$$\begin{aligned} y_t + \nu A y + B'(\bar{y}) y &= u + z_y \\ y(0) &= z_0. \end{aligned} \tag{5.18}$$

We test this equation by $\lambda = \lambda_1 - \lambda_2$ to obtain

$$\langle y_t,\, \lambda\rangle + \nu (y, \lambda)_{L^2(V)} + \langle B'(\bar{y}) y,\, \lambda\rangle = (u, \lambda) + (z_y, \lambda). \tag{5.19}$$

And third, we investigate the adjoint equations. The difference λ of the adjoint states satisfies

$$\begin{aligned} -\lambda_t + \nu A \lambda + B'(\bar{y})^* \lambda &= -B'(y)^* \bar{\lambda} + \alpha_Q y + \alpha_R \vec{\operatorname{curl}} \operatorname{curl} y + z_Q \\ \lambda(T) &= \alpha_T y(T) + z_T. \end{aligned} \tag{5.20}$$

Testing this equation by $y = y_1 - y_2$ yields

$$\begin{aligned} &-\langle \lambda_t,\, y\rangle + \nu (\lambda, y)_{L^2(V)} + \langle B'(\bar{y}) y,\, \lambda\rangle = \\ &\qquad -\left\langle B'(y)^* \bar{\lambda},\, y\right\rangle + \alpha_Q \|y\|_2^2 + \alpha_R \|\operatorname{curl} y\|_2^2 + (z_Q, y). \end{aligned} \tag{5.21}$$

By integration by parts we find

$$\begin{aligned} -\langle \lambda_t,\, y\rangle &= \langle y_t,\, \lambda\rangle - (\lambda(T), y(T))_H + (\lambda(0), y(0))_H \\ &= \langle y_t,\, \lambda\rangle - \alpha_T |y(T)|_H^2 - (z_T, y(T))_H + (\lambda(0), z_0)_H. \end{aligned} \tag{5.22}$$

Combining (5.19), (5.21), and (5.22), the equation

$$\begin{aligned} (u, \lambda) + (z_y, \lambda) = \alpha_T &|y(T)|_H^2 + (z_T, y(T))_H - (\lambda(0), z_0)_H \\ &- \left\langle B'(y)^* \bar{\lambda},\, y\right\rangle + \alpha_Q \|y\|_2^2 + \alpha_R \|\operatorname{curl} y\|_2^2 + (z_Q, y) \end{aligned} \tag{5.23}$$

is found.

Let us introduce the auxiliary function $\tilde{y}$ as the weak solution of (5.18) with $u = 0$. Now, the coercivity assumption of $\mathcal{L}_{vv}$ comes into play. The tupel $(y - \tilde{y}, u)$ fits in the assumptions of (SSC): We have $u_i = u_{1,i} - u_{2,i} = \bar{u}_i - \bar{u}_i = 0$ on $Q_{\epsilon,i}$ by fulfillment of the constraint $u_1, u_2 \in \widetilde{U_{ad}}$. Further, the difference $y - \tilde{y}$ is equal to $S'(\bar{u})u$. With $\mathcal{L}_{vv}$ given by Theorem 2.2, we get

$$\begin{aligned} \delta \|u\|_q^2 &\leq \mathcal{L}_{vv}(\bar{v}, \bar{\lambda})[(y - \tilde{y}, u)]^2 \\ &= \mathcal{L}_{uu}(\bar{v}, \bar{\lambda})[u]^2 + \mathcal{L}_{yy}(\bar{v}, \bar{\lambda})[y]^2 - 2\mathcal{L}_{yy}(\bar{v}, \bar{\lambda})[y, \tilde{y}]^2 + \mathcal{L}_{yy}(\bar{v}, \bar{\lambda})[\tilde{y}]^2. \end{aligned} \tag{5.24}$$

The first and second addend we write according to (2.7), (2.8) as

$$\begin{aligned}
&\mathcal{L}_{uu}(\bar v,\bar\lambda)[u]^2+\mathcal{L}_{yy}(\bar v,\bar\lambda)[y]^2\\
&\qquad\qquad=\alpha_T|y(T)|_H^2+\alpha_Q\|y\|_2^2+\alpha_R\|\operatorname{curl} y\|_2^2-\left\langle B''(\bar y)[y]^2,\,\bar\lambda\right\rangle+\gamma\|u\|_2^2.
\end{aligned}$$

By Section 1.2.2, we have the identity $\left\langle B'(y)^*\bar\lambda,\,y\right\rangle=\left\langle B''(\bar y)[y]^2,\,\bar\lambda\right\rangle$. Using (5.23), we continue

$$\begin{aligned}
&\mathcal{L}_{uu}(\bar v,\bar\lambda)[u]^2+\mathcal{L}_{yy}(\bar v,\bar\lambda)[y]^2\\
&\qquad=\alpha_T|y(T)|_H^2+\alpha_Q\|y\|_2^2+\alpha_R\|\operatorname{curl} y\|_2^2-\left\langle B''(y)[y]^2,\,\bar\lambda\right\rangle+\gamma\|u\|_2^2\\
&\qquad=(u,\lambda)+(z_y,\lambda)-(z_T,y(T))_H+(\lambda(0),z_0)_H-(z_Q,y)+\gamma\|u\|_2^2.
\end{aligned}\tag{5.25}$$

We apply inequality (5.17) to obtain

$$\mathcal{L}_{uu}(\bar v,\bar\lambda)[u]^2+\mathcal{L}_{yy}(\bar v,\bar\lambda)[y]^2\le(z_u,u)+(z_y,\lambda)-(z_T,y(T))_H+(\lambda(0),z_0)_H-(z_Q,y).$$

Now, we want to estimate the right-hand side. The only addends that need caution are (z_u,u) and $(\lambda(0),z_0)_H$. The imbedding $W^{4/3}(0,T;V)\hookrightarrow C(0,T;V')$ is continuous, and we find

$$(\lambda(0),z_0)_H=\langle\lambda(0),\,z_0\rangle_{V',V}\le|\lambda(0)|_{V'}|z_0|_V\le\|\lambda\|_{W^{4/3}}|z_0|_V.$$

The other one is estimated simply by $(z_u,u)\le\|z_u\|_2\|u\|_2$. We will do some extra work later in the proof to get an estimate with respect to $\|u\|_q$. Thus, we estimate

$$\begin{aligned}
&\mathcal{L}_{uu}(\bar v,\bar\lambda)[u]^2+\mathcal{L}_{yy}(\bar v,\bar\lambda)[y]^2\\
&\qquad\le(z_u,u)+(z_y,\lambda)-(z_T,y(T))_H+(\lambda(0),z_0)_H-(z_Q,y)\\
&\qquad\le\|z_u\|_2\|u\|_2+\|z_y\|_2\|\lambda\|_2+|z_T|_H|y(T)|_H+|z_0|_V|\lambda(0)|_{V'}+\|z_Q\|_2\|y\|_2\\
&\qquad\le\|z_u\|_2\|u\|_2+\|z\|_Z\left(\|y\|_W+\|\lambda\|_{W^{4/3}}\right).
\end{aligned}$$

Inequality (5.17) yields

$$\|u\|_2\le\frac{1}{\gamma}(\|z_u\|_2+\|\lambda\|_2).\tag{5.26}$$

Hence, we can conclude

$$\mathcal{L}_{uu}(\bar v,\bar\lambda)[u]^2+\mathcal{L}_{yy}(\bar v,\bar\lambda)[y]^2\le c\|z\|_Z\left(\|z\|_Z+\|y\|_W+\|\lambda\|_{W^{4/3}}\right).$$

Now, we have to investigate the third and fourth addend of (5.24). Since $\tilde y$ is the weak solution of a linearized equation, its norm can be expressed in terms of the data z_y and z_0

$$\|\tilde y\|_W\le c\left\{\|z_y\|_2+|z_0|_V\right\}\le c\|z\|_Z.$$

Applying the bound of $\mathcal{L}_{vv}$ in (2.9), we can estimate the third and fourth addend in (5.24) by

$$\begin{aligned}
\left|2\mathcal{L}_{yy}(\bar v,\bar\lambda)[y,\tilde y]^2\right|+\left|\mathcal{L}_{yy}(\bar v,\bar\lambda)[\tilde y]^2\right|&\le c\big\{\|y\|_W\|\tilde y\|_W+\|\tilde y\|_W^2\big\}\\
&\le c\big\{\|y\|_W\|z\|_Z+\|z\|_Z^2\big\}.
\end{aligned}\tag{5.27}$$

Collecting (5.24)–(5.27), we find

$$\delta\|u\|_q^2\le c\|z\|_Z\left(\|z\|_Z+\|y\|_W+\|\lambda\|_{W^{4/3}}\right).\tag{5.28}$$

Using results of Section 1.3, we derive bounds of the differences of the states and adjoints as weak solutions of (5.18) and (5.20) by

$$\begin{aligned}\|y\|_W &\le c\left(\|u\|_q + \|z_y\|_2 + |z_0|_h\right) \le c\left(\|u\|_q + \|z\|_Z\right),\\ \|\lambda\|_{W^{4/3}} &\le c\left(\|y\|_{L^2(V)} + \|z_Q\|_2 + \|y\|_{L^\infty(H)} + |z_T|_H\right) \le c\left(\|u\|_q + \|z\|_Z\right).\end{aligned}$$

Hence, we obtain from (5.28)

$$\begin{aligned}\delta\|u\|_q^2 &\le c\|z\|_Z\left(\|z\|_Z + \|y\|_W + \|\lambda\|_{W^{4/3}}\right)\\ &\le c\|z\|_Z\left(\|z\|_Z + \|u\|_q\right)\\ &\le c\|z\|_Z^2 + \frac{\delta}{2}\|u\|_q^2,\end{aligned}$$

and Lipschitz continuity of $z \mapsto u_z$ from Z to $L^q(Q)^2$ is proven. The estimate (5.26) provides a Lipschitz estimate with respect to the L^2-norm,

$$\|u\|_2 \le c\left(\|z_u\|_2 + \|\lambda\|_2\right) \le c\left(\|z\|_Z + \|\lambda\|_{W^{4/3}}\right) \le c\left(\|z\|_Z + \|u\|_q\right) \le c\|z\|_Z.$$

By Theorems 1.1.18(ii) and 1.1.31(i) we can estimate the state and adjoint state differences in stronger norms

$$\begin{aligned}\|y\|_{H^{2,1}} &\le c\left(\|u\|_2 + \|z_y\|_2 + |z_0|_V\right),\\ \|\lambda\|_{H^{2,1}} &\le c\left(\|y\|_{L^\infty(H)} + \|y\|_{L^2(H^2)} + \|z_Q\|_2 + |z_T|_V\right).\end{aligned}$$

Altogether, this proves Lipschitz continuity of $z \mapsto (y_z, u_z, \lambda_z)$ from Z to $H^{2,1} \times L^2(Q)^2 \times H^{2,1}$ □

So far, we provided all pre-requisites to prove the L^2-stability theorem.

Theorem 5.5. *Let (SSC) be satisfied for the reference solution $\bar{v}$ with adjoint state $\bar{\lambda}$. Let additionally y_0, $y_T \in V$, $y_Q \in L^2(Q)^2$ be given. Then there exists $\rho > 0$, such that for all $z \in Z$ with $\|z\|_Z \le \rho$, the perturbed optimal control problem* (5.1) *subject to the modified constraint $u \in \widetilde{U_{ad}}$ admits a unique solution (y_z, u_z, λ_z). Moreover, the mapping $z \mapsto (y_z, u_z, \lambda_z)$ is Lipschitz continuous from Z to $H^{2,1} \times L^2(Q)^2 \times H^{2,1}$.*

Proof. Theorem 1.18(iii) yields strong regularity of the equation (5.15) at the point $(\bar{y}, \bar{u}, \bar{\lambda})$. So we can apply Theorem 5.1 which finishes the proof. □

If the vector of perturbations z is slightly more regular than stated in (5.8), say

$$z \in \tilde{Z} = L^2(Q)^2 \times V \times L^2(Q)^2 \times V \times L^s(Q)^2$$

for some $2 < s < \infty$ equipped with the norm $\|z\|_{\tilde{Z}} = \|z_y\|_2 + |z_0|_V + \|z_Q\|_2 + |z_T|_V + \|z_u\|_s$, then one can show the following

Theorem 5.6. *Let (SSC) be satisfied for the reference solution $\bar{v}$ with adjoint state $\bar{\lambda}$. Let additionally $y_0, y_T \in V$, $y_Q \in L^2(Q)^2$ be given. Then there exists $\rho > 0$, such that for all $z \in \tilde{Z}$ with $\|z\|_{\tilde{Z}} \le \rho$, the perturbed optimal control problem* (5.1) *admits a unique solution (y_z, u_z, λ_z). Moreover, the mapping $z \mapsto (y_z, u_z, \lambda_z)$ is Lipschitz continuous from $\tilde{Z}$ to $H^{2,1} \times L^s(Q)^2 \times H^{2,1}$.*

Proof. The proof is very similar to the proof of Theorem 5.11, see below. It uses the stability result of the previous Theorem 5.5, the projection formula (3.12), and the imbedding $H^{2,1} \hookrightarrow L^s(Q)^2$ for $s < \infty$. □

However, this result is maximal in the following sense. Stability of optimal controls in $C(\bar{Q})^2$ can not be achieved using Hilbert space results in the analysis of the state equation. This is due to the fact that it is not possible to derive a Lipschitz estimate for the time derivatives of the controls that would be necessary to employ Theorem 1.13(ii). To this end consider the following example.

Example 5.7. *Let $\lambda_1, \lambda_2 \in C^1[0,T]$ be given by $\lambda_1(t) = \sin(nt) + 2$ and $\lambda_2(t) = \sin(nt) - 2$. Then it holds $\lambda_1(t) - \lambda_2(t) = 4$ and $\frac{\mathrm{d}}{\mathrm{d}t}(\lambda_1(t) - \lambda_2(t)) = 0$. With $u_a(t) = 0$, $u_b(t) = +\infty$, and $\gamma = 1$ we get*

$$\mathrm{Proj}_{[0,+\infty)}(-\lambda_1(t)) - \mathrm{Proj}_{[0,+\infty)}(-\lambda_2(t)) = 0 - (2 - \sin(nt)) = \sin(nt) - 2,$$

hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{Proj}_{[0,+\infty)}(-\lambda_1(t)) - \mathrm{Proj}_{[0,+\infty)}(-\lambda_2(t))\Big) = n\cos(nt) \neq 0,$$

which proves that we cannot show Lipschitz dependence of the time derivatives of the projected adjoint states λ_i.

At this point, we have to use L^p-methods to derive a stability result in the $C(\bar{Q})^2$-norm.

Remark 5.8. *Obviously, this difficulties do not appear for the unconstrained problem $U_{ad} = L^2(Q)^2$, where the variational inequality is equivalent to $u = -\frac{1}{\gamma}\lambda$. Then, any extremal control $\bar{u}$ is as smooth as the associated adjoint $\bar{\lambda}$ and admits the same stability properties, i.e. $z \mapsto u_z$ is Lipschitz from Z to $H^{2,1}$.*

3.3 Stability of optimal controls in L^∞

Here, we give the stability result of optimal controls in norms adequate to the regularity achieved in Section 3.4. Since we want ot use the L^p-estimates of Section 1.4, we require in the sequel that Ω is of class C^3. Again, we are considering the inclusion (5.15) and the linearized and perturbed problem. Now, we regard F to be a function in the setting

$$\begin{aligned} F : W_p^{2,1} \times L^\infty(Q)^2 \times W_p^{2,1} \to \\ L^p(Q)^2 \times W_0^{2-2/p,\,p}(\Omega)^2 \times L^p(Q)^2 \times W_0^{2-2/p,\,p}(\Omega)^2 \times L^\infty(Q)^2, \end{aligned} \tag{5.29}$$

since we want to apply Theorem 5.1 to get stability in stronger norms such as L^∞ for the control.

Corollary 5.9. *The function F is continuously differentiable with respect to the spaces given by* (5.29).

Proof. The proof follows the lines of the proof of Corollary 5.2. We have to check only the continuous differentiability of F_1 respectively $B(y)$. Let us denote the adjoint exponent of p by p', i.e. $1/p + 1/p' = 1$. Take $y, h \in W_p^{2,1}$. Due to the continuous imbedding of $W_p^{2,1}$ into $L^\infty(Q)^2$, we can estimate

$$\begin{aligned}\|B(y+h) - B(y) - B'(y)h\|_p &= \sup_{v \in L^{p'}(Q)^2 \setminus \{0\}} \|v\|_{p'}^{-1} \int_0^T b(h,h,v)\,\mathrm{d}t \\ &\le \sup_{v \in L^2(Q)^2 \setminus \{0\}} \|v\|_{p'}^{-1} c\,\|h\|_\infty \|h\|_{L^p(W^{1,p})} \|v\|_{p'} \le c\,\|h\|^2_{W_p^{2,1}},\end{aligned}$$

which proves that B is Fréchet differentiable from $W_p^{2,1}$ to $L^p(Q)^2$. The continuity of $y \mapsto B'(y)$ can be proven similarly. $\square$

Accordingly, the perturbation vector z has to be in the more regular — thus smaller — space of perturbations Z_p,

$$Z_p := L^p(Q)^2 \times W_0^{2-2/p,\,p}(\Omega)^2 \times L^p(Q)^2 \times W_0^{2-2/p,\,p}(\Omega)^2 \times L^\infty(Q)^2 \tag{5.30}$$

which we endow with the norm

$$\begin{aligned}\|z\|_{Z_p} = \|(z_y, z_0, z_Q, z_T, z_u)\|_{Z_p} &:= \\ &\|z_y\|_p + |z_0|_{W^{2-2/p,\,p}} + \|z_Q\|_p + |z_T|_{W^{2-2/p,\,p}} + \|z_u\|_\infty.\end{aligned}$$

For $p = 2$ we get the identity $Z_2 = Z$, where Z is the space of perturbation already used in the previous sections. Finally, we have to modify the definition of the normal cone $\mathcal{N}_{\widetilde{U_{ad}}}$. Here, this set has to be a subset of $L^\infty(Q)^2$,

$$\tilde{\mathcal{N}}_{\widetilde{U_{ad}}}(\bar u) := \begin{cases} \left\{ z \in L^\infty(Q)^2 : (z, u - \bar u)_2 \le 0 \ \forall u \in \widetilde{U_{ad}} \right\} & \text{if } \bar u \in \widetilde{U_{ad}} \\ \emptyset & \text{otherwise.} \end{cases} \tag{5.31}$$

Observe that $\tilde{\mathcal{N}}_{\widetilde{U_{ad}}}(u)$ is a non-empty, closed and convex subset of $L^\infty(Q)^2$. By construction, we have

$$\tilde{\mathcal{N}}_{\widetilde{U_{ad}}}(u) = \mathcal{N}_{\widetilde{U_{ad}}}(u) \cap L^\infty(Q)^2 \subset \mathcal{N}_{\widetilde{U_{ad}}}(u).$$

Now, we will investigate the linearized and perturbed equation

$$F(\bar y, \bar u, \bar\lambda) + F'(\bar y, \bar u, \bar\lambda)(y - \bar y, u - \bar u, \lambda - \bar\lambda) + \big(0,0,0,0, \tilde{\mathcal{N}}_{\widetilde{U_{ad}}}(u)\big) \ni z, \tag{5.32}$$

which is the same as (5.14) except that we replaced $\mathcal{N}_{\widetilde{U_{ad}}}$ by $\tilde{\mathcal{N}}_{\widetilde{U_{ad}}}$. With additional regularity assumptions we can show that the solution of (5.14) is also a solution of (5.32).

Corollary 5.10. *Let (SSC) be satisfied for the reference solution $\bar v = (\bar y, \bar u)$ with adjoint state $\bar\lambda$. Suppose the domain Ω is of class C^3. Moreover, assume that it holds $y_0, y_T \in W_0^{2-2/p,\,p}(\Omega)^2$, $y_Q \in L^p(Q)^2$ for some p satisfying $2 < p < \infty$, and $u_a, u_b \in L^\infty(Q)^2$.*

Then there exists a solution of the generalized equation (5.32).

Proof. Let (y_z, u_z, λ_z) be the solution of the generalized equation (5.14), whose existence is guaranteed by Theorem 5.3. The generalized equations (5.14) and

(5.32) differ only in the fifth component, which contains the two different normal cones. That means we have to show that a solution to (5.14) also satisfies the fifth component of the equation (5.32). We know already that it holds

$$\gamma u_z + \lambda_z + \mathcal{N}_{\widetilde{U_{ad}}}(u_z) \ni z_u$$

or equivalently

$$-(\gamma u_z + \lambda_z - z_u) \in \mathcal{N}_{\widetilde{U_{ad}}}(u_z)$$

If we prove the regularity $\gamma u_z + \lambda_z - z_u \in L^\infty(Q)^2$ then we would obtain

$$-(\gamma u_z + \lambda_z - z_u) \in \mathcal{N}_{\widetilde{U_{ad}}}(u_z) \cap L^\infty(Q)^2 = \tilde{\mathcal{N}}_{\widetilde{U_{ad}}}(u),$$

which finishes the proof. Here, we can follow the argumentation of Theorem 3.7 that uses bootstrapping arguments to gain the desired regularities. In the case considered here, y_z is the solution of the state equation linearized at $\bar{y}$. But this is no limitation, since for the solutions of the linearized state equation there are the same regularity results as for the nonlinear ones available, see Theorems 1.15 and 1.18(iii). Altogether, we get the regularities $u_z \in L^\infty(Q)^2$ and $\lambda_z \in W_p^{2,1} \hookrightarrow L^\infty(Q)^2$, which yields the claim. $\square$

This result ensures the existence of solutions of the perturbed equation (5.32). At next, we prove Lipschitz continuity of the associated solution mapping in stronger norms than in the previous section.

Theorem 5.11. *Let (SSC) be satisfied for the reference solution $\bar{v} = (\bar{y}, \bar{u})$ with adjoint state $\bar{\lambda}$. Suppose the domain Ω is of class C^3. Moreover, assume that it holds $y_0, y_T \in W_0^{2-2/p,\,p}(\Omega)^2$, $y_Q \in L^p(Q)^2$ for some p satisfying $2 < p < \infty$, and $u_a, u_b \in L^\infty(Q)^2$.*

Then the solution mapping $z \to (y_z, u_z, \lambda_z)$ associated to $(\widetilde{P_z})$ is Lipschitz continuous from Z_p to $W_p^{2,1} \times L^\infty(Q)^2 \times W_p^{2,1}$.

Proof. To begin with, notice that the assumptions imply $\bar{u} \in L^p(Q)^2$. Thus $\bar{y}$ as well as $\bar{\lambda}$ are strong solutions of the respective equations, i.e. $\bar{y}$, $\bar{\lambda} \in W_p^{2,1}$, see Theorems 1.15 and 1.31(ii).

Let $z_1, z_2 \in Z_p$ be given. Denote the optimal controls of the perturbed problem by $u_i := u_{z_i}$ with associated states y_i and adjoints λ_i, $i = 1, 2$.

At first, Theorem 5.4 yields stability of control, state, and adjoint in $L^2(Q)^2 \times H^{2,1} \times H^{2,1}$,

$$\|u_1 - u_2\|_2 + \|y_1 - y_2\|_{H^{2,1}} + \|\lambda_1 - \lambda_2\|_{H^{2,1}} \le c\,\|z_1 - z_2\|_{Z_2}.$$

By imbedding arguments, we have

$$\|\lambda_1 - \lambda_2\|_p \le c\|\lambda_1 - \lambda_2\|_{L^\infty(V)} \le c\|z_1 - z_2\|_{Z_2}.$$

The projection formula (3.12) yields

$$\|u_1 - u_2\|_p \le c\,\{\|\lambda_1 - \lambda_2\|_p + \|z_{u,1} - z_{u,2}\|_p\} \le c\,\|z_1 - z_2\|_{Z_p}.$$

By Theorem 1.18(iii) the weak solution $y_1 - y_2$ of (5.18) is also a strong solution and satisfies

$$\begin{aligned}
&\|y_1 - y_2\|_{L^p(W^{2,p})} + \|y_1 - y_2\|_{L^\infty(W^{2-2/p,\,p})} + \|y_{1,t} - y_{2,t}\|_p \\
&\quad \le c\,\{|z_{0,1} - z_{0,2}|_{W^{2-2/p,\,p}} + \|z_{y,1} - z_{y,2}\|_p + \|u_1 - u_2\|_p\} \le c\,\|z_1 - z_2\|_{Z_p}.
\end{aligned}$$

A similar estimate is valid also for the adjoint states, cf. Theorem 1.31(ii),

$$\begin{aligned}\|\lambda_1 - \lambda_2\|_{L^p(W^{2,p})} + \|\lambda_1 - \lambda_2\|_{L^\infty(W^{2-2/p,\,p})} + \|\lambda_{1,t} - \lambda_{2,t}\|_p \\ \leq c\left\{\|z_1 - z_2\|_{Z_p} + \|y_1 - y_2\|_{L^p(W^{2,p})} + \|y_1 - y_2\|_{L^\infty(W^{2-2/p,\,p})}\right\} \\ \leq c\,\|z_1 - z_2\|_{Z_p}.\end{aligned}$$

This actually means that the mapping $z \mapsto \lambda$ is Lipschitz form Z_p to $W_p^{2,1}$. The space $W_p^{2,1}$ is continuously imbedded in $L^\infty(Q)^2$. Hence, it follows using the projection formula a last time

$$\|u_1 - u_2\|_\infty \leq c\left\{\|\lambda_1 - \lambda_2\|_{L^\infty(Q)^2} + \|z_{u,1} - z_{u,2}\|_\infty\right\} \leq c\,\|z_1 - z_2\|_{Z_p},$$

which completes the proof. □

Thus, we proved strong regularity of the equation (5.15) in the stronger setting (5.29), and Theorem 5.1 is applicable.

Theorem 5.12. *Let (SSC) be satisfied for the reference solution $\bar v$ with adjoint state $\bar\lambda$. Suppose the domain Ω is of class C^3. Additionally, assume that the regularity conditions $y_0, y_T \in W_0^{2-2/p,\,p}(\Omega)^2$, $y_Q \in L^p(Q)^2$ for some p satisfying $2 < p < \infty$, and $u_a, u_b \in L^\infty(Q)^2$ are satisfied.*

Then there exist $\rho_z > 0$ and $\rho_u > 0$ such that for all $z \in B_{Z_p}(0, \rho_z)$ it holds: the perturbed inclusion (5.15) has in $B_{L^\infty}(\bar u, \rho_u)$ a unique solution u_z. Moreover, the mapping $z \mapsto (y_z, u_z, \lambda_z)$ is Lipschitz continuous from Z_p to $W_p^{2,1} \times L^\infty(Q)^2 \times W_p^{2,1}$.

Proof. Theorem 5.11 yields strong regularity of the equation (5.15) at the point $(\bar y, \bar u, \bar\lambda)$. So we can apply Theorem 5.1. Since the state and adjoint equations are uniquely solvable in general, we need not to restrict y_z and λ_z to neighborhoods of $\bar y$ and $\bar\lambda$, and the claim follows immediately. □

As already mentioned in Example 5.7, it is not possible to derive stability results for *bounded* optimal controls in W_p^1-norms, $1 \leq p \leq \infty$. So the result of Theorem 5.12 cannot be improved in this direction.

4 Stability of solutions of the perturbed optimal control problem

Let us return to the original stability problem. We want to show strong regularity of the generalized equation (5.5). To this aim, we have to investigate the perturbed linear-quadratic optimal control problem (P_z). Here, we emphasize that we are now looking for solutions $u(z)$ in the original admissible set U_{ad}. The natural candidate is the control u_z, which solves the auxiliary problem $(\widetilde{P_z})$, e.g. it satisfies the more restrictive constraint $u_z \in \widetilde{U_{ad}}$. However, we prove in the sequel that u_z is indeed a solution of (P_z) provided the perturbation z is sufficiently small.

Let us study the behaviour of u_z on the active set Q_ϵ. Here, we rely on the L^∞-stability result of the previous section. As already mentioned, using Hilbert-space methods, i.e. looking for weak solutions in Hilbert spaces, it is not possible to derive such a result for the constrained optimal control problem of instationary

Navier-Stokes equations, cf. [84] where this issue is addressed.

Corollary 5.13. *Let the assumptions of Theorem 5.11 be fulfilled. Then there exist $\rho_z > 0$ such that for all $z \in Z_p$ with $\|z\|_{Z_p} < \rho_z$ the optimal control of $(\widetilde{P_z})$, u_z, is strongly active a.e. on $Q_{\epsilon,i}$, i.e. it holds*

$$|\gamma u_{z,i}(x,t) + \lambda_{z,i}(x,t) - z_{u,i}(x,t)| > \frac{\epsilon}{2},$$

and the signs of $(\gamma \bar{u}_i(x,t) + \bar{\lambda}_i(x,t))$ and $(\gamma u_{z,i}(x,t) + \lambda_{z,i}(x,t) - z_{u,i}(x,t))$ coincide a.e. on $Q_{\epsilon,i}$ for $i = 1, 2$.

Proof. By Theorem 5.11, the mapping $z \mapsto (y_z, u_z, \lambda_z)$ is Lipschitz continuous from Z_p to $W_p^{2,1} \times L^\infty(Q)^2 \times W_p^{2,1}$. By imbedding arguments, we find that $z \mapsto \gamma u_z + \lambda_z - z_u$ is Lipschitz as mapping form Z_p to $L^\infty(Q)^2$.
Take $(x,t) \in Q_{\epsilon,i}$ such that $\gamma \bar{u}_i(x,t) + \bar{\lambda}_i(x,t) > \epsilon$. For such (x,t) it holds

$$\begin{aligned}
\epsilon &< \gamma \bar{u}_i(x,t) + \bar{\lambda}_i(x,t) \\
&= \gamma \bar{u}_i(x,t) + \bar{\lambda}_i(x,t) - (\gamma u_{z,i}(x,t) + \lambda_{z,i}(x,t) - z_{u,i}(x,t)) \\
&\qquad + (\gamma u_{z,i}(x,t) + \lambda_{z,i}(x,t) - z_{u,i}(x,t)) \\
&\le c_L \|z\|_{Z_p} + \gamma u_{z,i}(x,t) + \lambda_{z,i}(x,t) - z_{u,i}(x,t).
\end{aligned}$$

Therefore, the choice $\rho_z := c_L^{-1}\epsilon/2$ yields $\gamma u_{z,i}(x,t) + \lambda_{z,i}(x,t) - z_{u,i}(x,t) > \epsilon/2$.

Analogously, if for $(x,t) \in Q_{\epsilon,i}$ we have $\gamma \bar{u}_i(x,t) + \bar{\lambda}_i(x,t) < -\epsilon$, then the same value of ρ_z gives $\gamma u_{z,i}(x,t) + \lambda_{z,i}(x,t) - z_{u,i}(x,t) < -\epsilon/2$. The consistency of the signs of $(\gamma \bar{u}_i + \bar{\lambda}_i)$ and $(\gamma u_{z,i} + \lambda_{z,i} - z_{u,i})$ on $Q_{\epsilon,i}$ follows from the above estimations. □

An immediate conclusion is that u_z fulfills a variational inequality for all admissible controls $u \in U_{ad}$. This is far more than the necessary optimality condition for $(\widetilde{P_z})$, where only test functions from $\widetilde{U_{ad}} \subset U_{ad}$ are allowed.

Corollary 5.14. *Let the assumptions of Theorem 5.11 be fulfilled. Then the control u_z associated to a perturbation $z \in Z_p$ with $\|z\|_{Z_p} < \rho_z$, ρ_z given by the Corollary 5.13, fulfills the variational inequality*

$$(\gamma u_z + \lambda_z - z_u,\, u - u_z) \ge 0 \quad \forall u \in U_{ad}, \tag{5.33}$$

i.e. it satisfies the first-order necessary optimality condition of the optimization problem (P_z).

Proof. Let $u \in U_{ad}$ be given. We begin with

$$\begin{aligned}
\int_Q (\gamma u_{z,i} + \lambda_{z,i} - z_{u,i})(u_i - u_{z,i}) = &\int_{Q\setminus Q_{\epsilon,i}} (\gamma u_{z,i} + \lambda_{z,i} - z_{u,i})(u_i - u_{z,i}) \\
&+ \int_{Q_{\epsilon,i}} (\gamma u_{z,i} + \lambda_{z,i} - z_{u,i})(u_i - \bar{u}_i),
\end{aligned} \tag{5.34}$$

since $u_z \in \widetilde{U_{ad}}$ means $u_{z,i}(x,t) = \bar{u}_i(x,t)$ a.e. on $Q_{\epsilon,i}$. The first integral is part of the first-order necessary optimality conditions of $(\widetilde{P_z})$. Therefore, it is nonnegative.

By Corollary 5.13, $(\gamma\bar u_i + \bar\lambda_i)$ and $(\gamma u_{z,i} + \lambda_{z,i} - z_{u,i})$ have the same sign a.e. on $Q_{\epsilon,i}$. Furthermore, $\bar u_i$ is active on this set, so that $u_i - \bar u_i$ has always the same sign for all possible choices of u_i. Since $\gamma\bar u + \bar\lambda$ satisfies $\int_{Q_{\epsilon,i}}(\gamma\bar u_i + \bar\lambda_i)(u_i - \bar u_i) \geq 0$, the same is true for $\gamma u_z + \lambda_z - z_u$, i.e.

$$\int_{Q_{\epsilon,i}} (\gamma u_{z,i} + \lambda_{z,i} - z_{u,i})(u_i - \bar u_i) \geq 0$$

is satisfied. Thus, we proved that both integrals in (5.34) are nonnegative. Adding them, we derived the claim (5.33). □

So far, we showed that (y_z, u_z, λ_z) fulfills the optimality system of the perturbed problem (P_z) or equivalently the linearized and perturbed generalized equation (5.10). We have to ask whether it might be a local minimizer of (P_z). With the previous corollary and the identity (5.16) we have all ingredients at hand to prove that (y_z, u_z, λ_z) satisfies a second-order sufficient optimality condition for the problem (P_z), i.e. it is indeed a locally optimal solution.

Theorem 5.15. *Let the assumptions of Theorem 5.11 be fulfilled. Then, there are $\rho_z, \rho_u > 0$ such that the control u_z associated to a perturbation $z \in Z_p$ with $\|z\|_{Z_p} < \rho_z$ is a locally optimal solution of (P_z), and it satisfies*

$$J^{(z)}(y_z, u_z) \leq J^{(z)}(y, u)$$

for all $u \in U_{ad}$ with $\|u - u_z\|_\infty \leq \rho_u$. Here y_z and y are the weak solutions of linerarized and perturbed state equations (5.11) associated to the controls u_z and u.

Proof. We denote by y_z and λ_z the solutions of the state respectively adjoint equations (5.11) and (5.12). By Corollary 5.14, the triple (y_z, u_z, λ_z) satisfies not only the first-order necessary optimality conditions of $(\widetilde{P_z})$ but also the necessary optimality conditions of (P_z) if the norm of the perturbation z is smaller than ρ_z, $\|z\|_{Z_p} \leq \rho_z$. This is due to the fact that the control constraints are strongly active in u_z at Q_ϵ.

The control problems (P_z) and $(\widetilde{P_z})$ differ only in the choice of the admissible control set. Since we do not include this control constraint in the Lagrangian functional, the Lagrangians of both optimization problems coincide. As already mentioned above, cf. Theorem 5.3, the second derivatives of the Lagrangians $\mathcal{L}^{(z)}$ and $\mathcal{L}$ — associated with (P_z) and $(\widetilde{P_z})$ respectively the unperturbed nonlinear problem (2.3) — are identical,

$$\mathcal{L}^{(z)}_{vv}(y, u, \lambda) = \mathcal{L}_{vv}(\bar y, \bar u, \bar\lambda).$$

Let $u \in U_{ad}$ with $u_i = \bar u_i$ a.e. on $Q_{\epsilon,i}$ be given. Denote by y the associated solution of (5.11). Set $h = u - u_z$ and $w = y - y_z$. This implies $h = 0$ a.e. on $Q_{\epsilon,i}$. Therefore, h fits in the assumptions of (SSC). The triple $(\bar y, \bar u, \bar\lambda)$ satisfies the second-order sufficient optimality condition (SSC), which means

$$\mathcal{L}^{(z)}_{vv}(y, u, \lambda)[(w, h)]^2 = \mathcal{L}_{vv}(\bar y, \bar u, \bar\lambda)[(w, h)]^2 \geq \delta \|h\|_q^2. \tag{5.35}$$

Both, Corollary 5.13 and the coercivity relation (5.35) build up the second-order sufficient optimality condition connected with $(\widetilde{P_z})$. Following the lines of the proof of Theorem 4.3, we conclude that u_z is locally optimal: there exists a constant $\rho_u > 0$ such $J^{(z)}(y_z, u_z) \leq J^{(z)}(y, u)$ for all $u \in U_{ad}$ with $\|u - u_z\|_\infty \leq \rho_u$. $\square$

The previous theorem states the existence of a solution of the linearized perturbed equation (5.10). Hence, the strong regularity of the generalized equation (5.5) follows immediately.

Corollary 5.16. *Let the assumptions of Theorem 5.11 be fulfilled. Then the generalized equation* (5.5) *is strongly regular at* $(\bar{y}, \bar{u}, \bar{\lambda})$.

Proof. By Corollary 5.13, the function F is a $C^{1,1}$-mapping. Theorem 5.15 states that u_z is the unique optimal solution of the perturbed linearized optimization problem (P_z) in the ball $B_{L^\infty}(\bar{u}, \rho_u)$ for perturbations from $B_{Z_p}(0, \rho_z)$. By Theorem 5.11, the associated state y_z lies in the ball $B_{W_p^{2,1}}(\bar{y},\, c_y\rho_z)$, whereas the adjoint state λ_z is in $B_{W_p^{2,1}}(\bar{\lambda}, c_\lambda\, \rho_z)$. Here, c_y and c_λ are the Lipschitz constants given by Theorem 5.11. This altogether yields the unique solvability of the perturbed linearized generalized equation (5.10) in $B_{W_p^{2,1}}(\bar{y}, c_y\, \rho_z) \times B_{L^\infty}(\bar{u}, \rho_u) \times B_{W_p^{2,1}}(\bar{\lambda}, c_\lambda\, \rho_z)$ for perturbations from $B_{Z_p}(0, \rho_z)$. As already mentioned, the solution mapping $z \mapsto (y_z, u_z, \lambda_z)$ is Lipschitz. Therefore, all requirements for strong regularity are fulfilled. $\square$

Theorem 5.17. *Let (SSC) be satisfied for the reference solution* $\bar{v} = (\bar{y}, \bar{u})$ *with adjoint state* $\bar{\lambda}$. *Suppose the domain* Ω *is of class* C^3. *Moreover, assume that it holds* $y_0, y_T \in W_0^{2-2/p,\,p}(\Omega)^2$, $y_Q \in L^p(Q)^2$ *for some* p *satisfying* $2 < p < \infty$, *and* $u_a, u_b \in L^\infty(Q)^2$.

Then there exist positive constants ρ_z *and* ρ_u *such that for all* $z \in Z_p$ *with* $\|z\|_{Z_p} \leq \rho_z$ *the perturbed generalized equation* (5.9) *admits a unique solution in the ball* $B_{L^\infty}(\bar{u}, \rho_u)$. *In addition, the associated solution mapping* $z \to (y_z, u_z, \lambda_z)$ *is Lipschitz continuous from* Z_p *to* $W_p^{2,1} \times B_{L^\infty}(\bar{u}, \rho_u) \times W_p^{2,1}$.

Proof. In Corollary 5.16, we proved strong regularity of (5.16). Hence, we can apply Theorem 5.1, which finishes the proof. $\square$

The previous theorem states that the optimality system connected with the perturbed optimization problem (5.1)–(5.2) is solvable. However, it is not clear whether this solution is a local optimum or not.

Theorem 5.18. *Under the assumptions of the previous Theorem 5.17 we have the following. There exist positive constants* ρ_z' *and* ρ_u *such that for all* $z \in Z_p$ *with* $\|z\|_{Z_p} \leq \rho_z'$ *the perturbed nonlinear optimal control problem* (5.1)–(5.2) *admits a unique solution* u_z *in the ball open* $B_{L^\infty}(\bar{u}, \rho_u)$, *which means especially that* u_z *is a locally optimal control.*

Proof. Observe that the triple $(\bar{y}, \bar{u}, \bar{\lambda})$ is a solution of the problem (5.1)–(5.2) with no perturbation, i.e. $z \equiv 0$. Let $z \in Z_p$ be given. By the previous Theorem 5.17 there exists a unique solution (y_z, u_z, λ_z) of the generalized equation (5.9), which is the first-order optimality system connected with (5.1)–(5.2). Moreover, there is a constant c_L such that

$$\|y_z - \bar{y}\|_{W_p^{2,1}} + \|u_z - \bar{u}\|_\infty + \|\lambda_z - \bar{\lambda}\|_{W_p^{2,1}} \leq c_L \|z\|_{Z_p}$$

hold. Additionally, the control belongs to the neighborhood $B_{L^\infty}(\bar{u}, \rho_u)$. It remains to prove that u_z is a local minimizer of (5.1)–(5.2).

At first, we note that the control constraints are strongly active for $u_{z,i}$ on the set $Q_{\epsilon,i}$. Here, the argumentation of Corollary 5.13 applies in the way that we have $|\gamma u_z + \lambda_z + z_u| > \epsilon/2$ for $\|z\|_{Z_p} \leq c_L^{-1}\epsilon/2$.

It remains to show coercivity of the second derivative of the respective Lagrangian. Since the perturbation z appears linearly in the perturbed problem, the second derivative of the Lagrangian of this problem coincides with the one of the unperturbed problem.

Now, take a function $h \in L^2(Q)^2$ with $h_i = 0$ on $Q_{\epsilon,i}$, such that it can be used as test function in (SSC). We want to prove the existence of a positive $\tilde{\delta}$ such that $\mathcal{L}_{vv}(y_z, u_z, \lambda_z)[(S'(y_z)h, h)]^2 \geq \tilde{\delta}\|h\|_q^2$ holds. To this aim, we have to investigate the difference

$$\mathcal{L}_{vv}(y_z, u_z, \lambda_z)[(S'(y_z)h, h)]^2 - \mathcal{L}_{vv}(\bar{y}, \bar{u}, \bar{\lambda})[(S'(\bar{y})h, h)]^2 =: D(h).$$

We have to ensure that it can be compensated by $\delta\|h\|_q^2$, since (SSC) implies $\mathcal{L}_{vv}(\bar{y}, \bar{u}, \bar{\lambda})[(S'(\bar{y})h, h)]^2 \geq \delta\|h\|_q^2$. Let us denote $w_z := S'(y_z)h$ and $w := S'(\bar{y})h$. Their difference will be denoted by d, $d := w_z - w$. It is the weak solution of

$$\begin{aligned} d_t + \nu A d + B'(\bar{y})d &= -B'(y_z - \bar{y})w_z \\ d(0) &= 0. \end{aligned}$$

Using results of Chapter 1, we can estimate

$$\|d\|_W \leq c\|B'(y_z - \bar{y})w_z\|_{L^2(V')} \leq c\|y_z - \bar{y}\|_W \|w_z\|_{L^2(V)} \leq c\|z\|_{Z_p}\|h\|_q.$$

We will also need bounds of the norm of $w_z + w$. Applying the estimate of S' in Corollary 1.23, we find

$$\|w_z + w\|_W \leq \|w_z\|_W + \|w\|_W = \|S'(y_z)h\|_W + \|S'(\bar{y})h\|_W \leq c\|h\|_q.$$

With the representation of $\mathcal{L}_{vv}$ given in Theorem 2.2, the difference $D(h)$ of the second derivatives of the Lagrangians can be written as

$$\begin{aligned} D(h) = {} & \alpha_T(w_z + w(T), w_z - w(T))_H + \alpha_R(\operatorname{curl}(w_z + w), \operatorname{curl}(w_z - w))_Q \\ & + \alpha_Q(w_z + w, w_z - w)_Q \\ & - \langle B''(y_z)[w_z, w_z],\, \lambda_z\rangle_{L^2(V'),L^2(V)} + \langle B''(\bar{y})[w, w],\, \bar{\lambda}\rangle_{L^2(V'),L^2(V)}. \end{aligned}$$

We can rewrite the two terms, which contain B'', applying the identity (1.23). Since $B(y)$ is Lipschitz in y, we can estimate

$$\begin{aligned} \left|\langle B''(y_z)[w_z, w_z],\, \lambda_z\rangle - \langle B''(\bar{y})[w, w],\, \bar{\lambda}\rangle\right| &= \frac{1}{2}\left|\langle B(w_z),\, \lambda_z\rangle - \langle B(w),\, \bar{\lambda}\rangle\right| \\ &= \frac{1}{2}\left|\langle B(w_z) - B(w),\, \lambda_z\rangle + \langle B(w),\, \lambda_z - \bar{\lambda}\rangle\right| \\ &\leq c\left(\|w\|_W\|w_z - w\|_W\|\lambda_z\|_{L^2(V)} + \|w_z\|_W\|w_z - w\|_W\|\lambda_z\|_{L^2(V)} \right. \\ &\qquad \left. + \|w\|_W^2\|\lambda_z - \bar{\lambda}\|_{L^2(V)}\right) \\ &\leq c\left(\|h\|_q^2\|z\|_{Z_p}\|\lambda_z\|_{L^2(V)} + \|h\|_q^2\|z\|_{Z_p}\|\lambda_z\|_{L^2(V)} + \|h\|_q^2\|z\|_{Z_p}\right) \\ &\leq c\,\|h\|_q^2\|z\|_{Z_p}\left(\|\lambda_z\|_{L^2(V)} + 1\right). \end{aligned}$$

Hence, we can conclude

$$\begin{aligned}|D(h)| &\le c\left\{\|w_z + w\|_W \|w_z - w\|_W + \|h\|_q^2 \|z\|_{Z_p}\left(\|\lambda_z\|_{L^2(V)} + 1\right)\right\}\\ &\le c\left\{\|z\|_{Z_p}\|h\|_q^2 + \|h\|_q^2\|z\|_{Z_p}\left(\|\lambda_z - \bar\lambda + \bar\lambda\|_{L^2(V)} + 1\right)\right\}\\ &\le c\,\|h\|_q^2\|z\|_{Z_p}\left\{\|z\|_{Z_p} + \|\bar\lambda\|_{L^2(V)} + 1\right\}.\end{aligned}$$

And we obtain for the difference of the second derivatives of the Lagrangians

$$\begin{aligned}\mathcal{L}_{vv}(y_z, u_z, \lambda_z)[(S'(y_z)h, h)]^2 - \mathcal{L}_{vv}(\bar y, \bar u, \bar\lambda)[(S'(\bar y)h, h)]^2&\\ \ge -c\,\|h\|_q^2\|z\|_{Z_p}\left\{\|z\|_{Z_p} + \|\bar\lambda\|_{L^2(V)} + 1\right\},&\end{aligned}$$

which gives together with (SSC)

$$\begin{aligned}\mathcal{L}_{vv}(y_z, u_z, \lambda_z)[(S'(y_z)h, h)]^2 &\ge \delta\|h\|_q^2 - c\,\|h\|_q^2\|z\|_{Z_p}\left\{\|z\|_{Z_p} + \|\bar\lambda\|_{L^2(V)} + 1\right\}\\ &= \|h\|_q^2\left(\delta - c\,\|z\|_{Z_p}\left\{\|z\|_{Z_p} + \|\bar\lambda\|_{L^2(V)} + 1\right\}\right).\end{aligned}$$

With $\|z\|_{Z_p} \le \rho_z' := \min\left(\frac{\delta}{4c\left\{\|\bar\lambda\|_{L^2(V)}+1\right\}}, \sqrt{\frac{\delta}{4c}}\right)$ we get

$$\mathcal{L}_{vv}(y_z, u_z, \lambda_z)[(S'(y_z)h, h)]^2 \ge \frac{\delta}{2}\|h\|_q^2,$$

which proves coercivity of $\mathcal{L}_{vv}(y_z, u_z, \lambda_z)$. Altogether, the triple (y_z, u_z, λ_z) fulfills a second-order sufficient optimality condition, and u_z is indeed a local minimum of the perturbed problem (5.1)–(5.2), provided the perturbation is small enough. $\square$

Chapter 6

Convergence of the SQP-method

Here, we investigate the sequential quadratic programming (SQP) method. This method is widely applied to solve finite and infinite dimensional optimization problems.

The SQP-method solves in every step a linear-quadratic optimal control problem, which is a quadratic approximation of the original nonlinear problem. Given starting values y_n, u_n, λ_n, it computes the next iterates $y_{n+1}, u_{n+1}, \lambda_{n+1}$ as the solution of

$$\begin{aligned} \min J^n(y,u) = \; & \nabla J(y_n,u_n)(y-y_n,u-u_n) \\ & + \frac{1}{2}\mathcal{L}_{vv}(y_n,u_n,\lambda_n)[(y-y_n,u-u_n)]^2 \end{aligned} \qquad (P^n)$$

subject to the linearized state equation

$$\begin{aligned} y_t + \nu Ay + B'(y_n)(y-y_n) &= u - B(y_n), \\ y(0) &= y_0, \end{aligned}$$

and the control constraint

$$u \in U_{ad}.$$

We write the functional to be minimized in each solution step of (P^n) for convenience

$$\begin{aligned} J_n(y,u) = \frac{\alpha_T}{2}\int_\Omega |y(x,T)-y_T(x)|^2\,\mathrm{d}x + \frac{\alpha_Q}{2}\int_Q |y(x,t)-y_Q(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \\ + \frac{\alpha_R}{2}\int_Q |\operatorname{curl} y(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{2}\int_Q |u(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t - b_Q(y-y_n,y-y_n,\lambda_n). \end{aligned}$$

It differs from the original J only in the additional term b_Q. However, this addend can destroy the convexity of the functional.

In the sequel, we investigate local convergence of this method. Here, the sufficient condition (SSC) plays an essential role. As one expects, we get quadratic convergence as soon as the iterates lie in neighborhood of a local solution. To obtain global convergence results one has to employ a globalization method, see for instance [41].

The local convergence of the SQP-method for optimal control of instationary Navier-Stokes equations was proven by Hintermüller and Hinze [40] using a strong second-order sufficient condition. Here we will prove local convergence under much weaker conditions, which complicates the analysis considerably. For the analysis of other local methods in connection with instationary Navier-Stokes equations we refer to Heinkenschloß [35], Hinze and Kunisch [46], where Newton- and quasi Newton techniques were utilized, and to Ulbrich [78], where the optimization problem is formulated as a non-smooth equation and solved by a semismooth Newton method.

1 Generalized Newtons method

In this section, we show, that the SQP-method can be interpreted as a Newton-method for the generalized equation

$$0 \in F(x) + N(x). \tag{6.1}$$

Here, we refer to Section 5.1, where the notation of generalized equations were introduced in connection with stability issues.

Let us write down the generalized Newton-method as follows: given iterate x^n, compute the next iterate x^{n+1} by solving

$$0 \in F(x_n) + F'(x_n)(x - x_n) + N(x). \tag{6.2}$$

In this context, we will need again the assumption of strong regularity. Under the assumption, that there exists a strongly regular solution of (6.1), the following convergence theorem holds. Proofs can be found in articles by Alt [3] and Dontchev [25]. It generalizes results from the finite-dimensional case, e.g. Josephy [48].

Theorem 6.1. *Let $\bar{x}$ be a solution of* (6.1) *and assume that* (6.1) *is strongly regular at $\bar{x}$. Then there exists an open ball $B_X(\bar{x}, \rho_x')$ such that for every starting element $x_1 \in B_X(\bar{x}, \rho_x')$ the generalized Newton method generates a unique sequence $\{x_n\}_{n=1}^\infty$. The iterates x_n remain in $B_X(\bar{x}, \rho_x')$, and it holds*

$$\|x_{n+1} - \bar{x}\|_X \le c_N \, \|x_n - \bar{x}\|_X^2 \quad \forall n \in \mathbb{N}, \tag{6.3}$$

where c_N is independent of n.

Let us now investigate the linearized generalized equation (6.2). In Section 5.2, we already have investigated a similar problem: namely the linearized and perturbed equation $z \in F(\bar{x}) + F'(\bar{x})(x - \bar{x}) + N(x)$ — compare with (5.4) — which differs from (6.2) in two respects. Here, we have no perturbations, and the nonlinear equation is linearized at an iterate x_n instead of $\bar{x}$. There, we found that the linearized generalized equation is the optimality system of an optimal control problem. Setting $z = 0$ and substituting $\bar{x}$ by x_n in (5.13), this optimization problem can be written as

$$\begin{aligned} \min J_n(y,u) = \frac{\alpha_T}{2}|y(T) - y_d|_H^2 + \frac{\alpha_Q}{2}\|y - y_Q\|_2^2 + \frac{\alpha_R}{2}\|\operatorname{curl} y\|_2^2 + \frac{\gamma}{2}\|u\|_2^2 \\ - b_Q(y - y_n, y - y_n, \lambda_n) \end{aligned} \tag{6.4a}$$

subject to the linearized state equation

$$\begin{aligned} y_t + \nu A y + B'(y_n)y &= u + B(y_n) \\ y(0) &= y_0 \end{aligned} \tag{6.4b}$$

and the control constraint

$$u \in U_{ad}. \tag{6.4c}$$

This motivates the interpretation of the SQP-method as a generalized Newton method. Furthermore, it gives the possibility to prove the convergence of the SQP-method with the help of the abstract theory, especially Theorem 6.1. In order to prove local convergence of the SQP-method, we have to verify the conditions of this theorem. Please observe, that the pre-requisites of this abstract theorem are equal to the assumptions of the abstract stability result in Theorem 5.1.

The equivalence of generalized Newton method and SQP method was first used in the context of optimal control of ODEs in Alt [3]. Tröltzsch [75] proved convergence of the SQP-method applied to optimal control problems for semilinear partial differential equations. Other convergence results for optimal control problems of different kinds of partial differential equations can be found in [30, 40, 80].

2 Local convergence of the SQP algorithm

Here, we want to apply the abstract convergence result of Theorem 6.1. We have to ensure only the strong regularity of the generalized equation. However, this problem was solved already in Section 5.4, see Corollary 5.16.

The SQP-method as stated in the beginning of the present chapter requires to find the *global* minimizer of the linear-quadratic subproblems (6.4). The analysis done so far only guarantees the existence of a *local* solution of those subproblems in the neighborhood of the reference control. In other words, the linearized and perturbed subproblems are not uniquely solvable in general but only in a neighborhood of the reference solution. Consequently, we have to modify the SQP-method to enforce the solutions of the subproblems to remain near the reference solution in the following way:

Given iterates y_n, u_n, λ_n, compute the next iterates $y_{n+1}, u_{n+1}, \lambda_{n+1}$ as the solution of (6.4) subject to the control constraint

$$u \in U_{ad}^{\rho} := U_{ad} \cap \{v \in L^\infty(Q)^2 : \|v - \bar{u}\|_\infty \le \rho\}. \tag{6.5}$$

See also [75], where those aspects are discussed more detailed.

Then Theorem 6.1 yields quadratic convergence in a neighborhood of the solution.

Theorem 6.2. *Let (SSC) be satisfied for the reference solution $\bar{v} = (\bar{y}, \bar{u})$ with adjoint state $\bar{\lambda}$. Moreover, assume that it holds $y_0, y_T \in W_0^{2-2/p,\,p}(\Omega)^2$, $y_Q \in L^p(Q)^2$ for some p satisfying $2 < p < \infty$, and $u_a, u_b \in L^\infty(Q)^2$.*

Then there is a constant $\rho_s > 0$, such that for every starting value (y_1, u_1, λ_1) with $u_1 \in U_{ad}^{\rho_s}$ the SQP-method with control constraint (6.5) generates a uniquely determined sequence (y_n, u_n, λ_n) with $u_n \in U_{ad}^{\rho_s}$, and it holds

$$\begin{aligned} \|y_{n+1} - \bar{y}\|_{W_p^{2,1}} + \|u_{n+1} - \bar{u}\|_\infty + \|\lambda_{n+1} - \bar{\lambda}\|_{W_p^{2,1}} \\ \le c_s \left(\|y_n - \bar{y}\|_{W_p^{2,1}}^2 + \|u_n - \bar{u}\|_\infty^2 + \|\lambda_n - \bar{\lambda}\|_{W_p^{2,1}}^2 \right) \end{aligned}$$

with a constant c_s independent of n. Here, y_n and λ_n are the states and adjoints associated to the control u_n.

Proof. By Corollary 5.16, the generalized equation (6.1) is strongly regular. Or equivalently, the perturbed and linearized problem (6.4) admits a unique solution in a neighborhood of the reference solution for small perturbations. Now, the claim of the Theorem follows with the abstract result of Theorem 6.1. □

The a-priori unknown solution $\bar{u}$ appears in the definition of U^{ρ}_{ad}, which is necessary to establish the convergence theory. To overcome this difficulty, one has to use globalization techniques. For an application of a globalized SQP-method to compute optimal controls of instationary Navier-Stokes equations, we refer to [41]. However, in the numerical computations it was not necessary to enforce the method to stay in a neighborhood of the last iterate. This coincide also with the experience of other authors, see [30, 40].

Chapter 7

Optimal control problems with convex control constraints

In this work, we investigate the distributed optimal control problem of the non-stationary Navier-Stokes equations. The control acts in the domain. It can be realized for instance as a force induced by an outer magnetic field in a conducting fluid, see e.g. Griesse, Kunisch [31], Gunzburger, Trenchea [34]. Mathematically, the control u is a function of class $L^2(Q)^2 = L^2(Q;\mathbb{R}^2)$. This illustrates that the control $u(x,t)$ is a directed quantity: it consists of a direction and an absolute value. In other words, the control u at a point (x,t) is a vector in $\mathbb{R}^2$. The same situation appears for the case of boundary control, which is not studied here. Then the control is a prescribed velocity vector on the boundary.

The optimization has to take into account that one is not able to realize arbitrarily large controls. To this end, control constraints were introduced to the model problem in Chapter 2. If the control $u(x,t)$ is only a scalar variable such as heating or cooling then there is only one choice of a convex pointwise control constraint: the so-called box constraints

$$u_a(x,t) \le u(x,t) \le u_b(x,t). \tag{7.1a}$$

In the previous chapters, we presented the analysis of optimal control of non-stationary Navier-Stokes equations using this particular type of control constraint. But these box constraints are not the only choice for vector-valued controls. For instance, if one wants to bound the $\mathbb{R}^2$-norm of the control, one gets a nonlinear constraint

$$|u(x,t)| = \sqrt{u_1(x,t)^2 + u_2(x,t)^2} \le \rho(x,t). \tag{7.1b}$$

What happens if the control is not allowed to act in all possible directions but only in directions of a segment with an angle less than π? Using polar coordinates $u_r(x,t)$ and $u_\phi(x,t)$ for the control vector $u(x,t)$, this can be formulated as

$$0 \le u_r(x,t) \le \psi(u_\phi(x,t),x,t), \tag{7.1c}$$

where the function ψ models the shape of the set of allowed control actions.

Here, we will use another more natural representation of the constraints. Let us denote by U the set of admissible control vectors. Then we can write the control constraints (7.1a)–(7.1c) as an inclusion

$$u(x,t) \in U.$$

The advantage of this approach is that the analysis is based on rather elementary say geometrical arguments, hence there is no need of any constraint qualification. We will impose assumptions on U that allow to apply the common theory of existence and optimality condition: non-emptyness, convexity, and closedness, but no boundedness or further regularity of the boundary. We have to admit that the assumption of convexity gives some inherent regularity, the boundary of convex sets is locally Lipschitz. However, even in the convex case, there can be very irregular situations: one can construct convex sets in $\mathbb{R}^2$ with countably many corners, which lie dense on the boundary, see Bonnesen and Fenchel [15].

The formulation of the control constraint as an inclusion has a further benefit: the set of admissible control vectors can vary over time and space by simply writing

$$u(x,t) \in U(x,t),$$

without causing any additional problems. The main difficulty appears already in the non-varying case. We will comment on that later.

Optimal control problems with such control constraints are rarely investigated in literature. Second-order necessary conditions for problems with the control constraint $u(\xi) \in U(\xi)$ were proven by Páles and Zeidan [59] involving second-order admissible variations. Second-order necessary as well as sufficient conditions were established in Bonnans [10], Bonnans and Shapiro [13], and Dunn [26]. However, the set of admissible controls has to be polygonal and independent of ξ, i.e. $U(\xi) \equiv U$. These results were extended by Bonnans and Zidani [14] to the case of finitely many convex contraints $g_i(u(\xi)) = 0$, $i = 1, \dots, l$. As already mentioned, we will follow another approach and treat the control constraint as an inclusion $u(x,t) \in U(x,t)$.

We emphasize that the restriction to two dimensions, i.e. $u \in L^2(Q)^2$, is only due to the limitation of the analysis of instationary Navier-Stokes equations. As long as there exists an applicable theory of a state equation in $\mathbb{R}^n$, all results regarding convex control constraints are ready for an extension to the n-dimensional case.

1 Set-valued functions

Before we begin with the formulation of the optimal control problem with inclusion constraints, we will provide some background material. Here, we will specify the notation and assumptions for the admissible set $U(\cdot)$. It is itself a mapping from the control domain Q to the set of subsets of $\mathbb{R}^2$, it is a so-called *set-valued mapping*. We will use the notation $U : Q \rightsquigarrow \mathbb{R}^2$.

The optimal control problem is the minimization of the objective functional subject to the state equations and to the control constraint

$$u(x,t) \in U(x,t). \tag{7.2}$$

The controls are taken from the space $L^2(Q)^2$, so it is natural to require the fulfillment of (7.2) for (only) almost all $(x,t) \in Q$. And we have to impose at least some measurability conditions on the mapping U. In the sequel, we will work with measurable set-valued mappings. For an excellent — and for our purposes complete — introduction we refer to the textbook by Aubin and Frankowska [7].

Definition 7.1. *A set-valued mapping $F : Q \rightsquigarrow X$ with closed images is called* measurable*, if the inverse of each open set is measurable. In other words, for every* *open subset $\mathcal{O} \subset X$ the inverse image*

$$F^{-1}(\mathcal{O}) = \{\omega \in Q : F(\omega) \cap \mathcal{O} \neq \emptyset\}$$

is measurable.

Observe, that for a single-valued function f the definition of measurability coincides with the definition of measurability for the set-valued function $\tilde{f}$ given by

$$\tilde{f}(\omega) = \{f(\omega)\}.$$

However, this definition does not imply the existence of a measurable selection, which is a single-valued and measurable function f satisfying $f(x,t) \in F(x,t)$ almost everywhere on Q. The existence is guaranteed under additional assumptions on F.

Theorem 7.2. *[7, Th. 8.1.4] Let $F : Q \rightsquigarrow \mathbb{R}^2$ be a set-valued mapping with non-empty closed images. Then the following three statements are equivalent:*

(i) F is measurable

(ii) There exists a sequence of measurable selections $\{f_n\}_{n=1}^{\infty}$ of F such that for all $(x,t) \in Q$ it holds

$$F(x,t) = \overline{\bigcup_{n \geq 1} \{f_n(x,t)\}}.$$

(iii) For all $v \in \mathbb{R}^2$ the function $d_v(x,t) = \operatorname{dist}(v,\, F(x,t))$ is measurable.

Here, the distance of a point $u \in \mathbb{R}^n$ to a set $C \subset \mathbb{R}^n$ is defined by

$$\operatorname{dist}(u, C) = \inf_{x \in C} |u - x|.$$

The theorem does not only give the existence of a measurable selection but also tools to prove measurability of set-valued mappings based on countable approximations and on measurability of the distance function.

In the previous chapters, we made heavily use of the projection representation of an optimal control, which we derived at the end of Section 3.2. Such a characterization is also valid in the set-valued constraint case. But as a first step, we have to make sure that the pointwise projection on the set-valued mapping U preserves measurability.

Theorem 7.3. *[7, Cor. 8.2.13] Let $F : Q \rightsquigarrow \mathbb{R}^2$ be a set-valued measurable mapping with closed, non-empty, and convex images, and $f : Q \mapsto \mathbb{R}^2$ a measurable (single-valued) mapping. Then the projection*

$$g(x,t) = \operatorname{Proj}_{F(x,t)}(f(x,t))$$

is a single-valued measurable function too.

2 The optimal control problem

Here, we will investigate the optimal control problem of Chapter 2 with the constraint (7.2). At first, we have to specify the assumptions to ensure existence of solutions.

2.1 Set of admissible controls

In this section, we want to investigate the convex control constraint, which has to hold pointwise
$$u(x,t) \in U(x,t) \text{ a.e. on } Q.$$
We recall the definition of the set of admissible controls U_{ad},
$$U_{ad} = \{u \in L^2(Q)^2 : \ u(x,t) \in U(x,t) \text{ a.e. on } Q\}.$$
Once and for all, we specify the requirements for the function U, which defines the control constraints.

(AU) *The set-valued function $U : Q \rightsquigarrow \mathbb{R}^2$ satisfies:*

- *(i) U is a measurable set-valued function.*
- *(ii) The images of U are non-empty, closed, and convex a.e. on Q. That is, the sets $U(x,t)$ are non-empty, closed and convex for almost all $(x,t) \in Q$.*
- *(iii) There exists a function $f_U \in L^2(Q)^2$ with $f_U(x,t) \in U(x,t)$ a.e. on Q.*

Please note, we did not impose any conditions on the sets $U(x,t)$ that are beyond convexity such as boundedness or regularity of the boundaries $\partial U(x,t)$. Assumptions *(i)* and *(ii)* guarantee that there exists a measurable selection of U, i.e. a measurable single-valued function f_M with $f_M(x,t) \in U(x,t)$ a.e. on Q. However, no measurable selection needs to be square-integrable as the following example shows. The existence of a square integrable, admissible function is then ensured by the third assumption. This implies that the set of admissible control is non-empty.

Example 7.4. *Set $U(t) = [t^{-1/2}, 1 + t^{-1/2}]$, $0 < t \le 1$. Assumptions (i) and (ii) are fulfilled. But every function f with $f(t) \in U(t)$ for almost all $0 < t \le 1$ cannot be in $L^2(0,1)$, since the function $g(t) = t^{-1/2}$ is not square integrable on $[0,1]$.*

We will investigate under which conditions the constraints (7.1a)–(7.1c) satisfy the assumption (AU) at the end of this section, see page 83. Now, we proceed with easy but important implications of the assumption (AU).

Corollary 7.5. *The set of admissible controls U_{ad} defined by*
$$U_{ad} = \{u \in L^2(Q)^2 : \ u(x,t) \in U(x,t) \text{ a.e. on } Q\}$$
is non-empty, convex and closed in $L^2(Q)^2$.

Proof. By assumption (AU), we have $f_U \in U_{ad}$. It is obvious that U_{ad} is convex, since $U(x,t)$ is convex for almost all $(x,t) \in Q$. Let a sequence $\{f_n\}_{n=1}^{\infty} \subset U_{ad}$ converging in L^2 to f be given. Then, we can find a subsequence f_{n_k} which converges to f pointwise almost everywhere. Since $U(x,t)$ is closed, it follows $f(x,t) \in U(x,t)$ a.e. on Q. Hence, it holds $f \in U_{ad}$. □

The assumption (AU) is as general as the analysis of the second-order condition allows it. In the case that the set-valued function U is a constant function, i.e. $U(x,t) \equiv U_0$, we can give a simpler characterization.

Corollary 7.6. *Let the set-valued function U be a constant function, i.e. $U(x,t) = U_0$ a.e. on Q for some $U_0 \subset \mathbb{R}^2$. Then the assumption (AU) is fulfilled if the set U_0 is non-empty, closed, and convex.*

Proof. At first, take an open set $\mathcal{O} \in \mathbb{R}^2$. Then we have $U^{-1}(\mathcal{O}) = Q$ if $U_0 \cap \mathcal{O} \neq \emptyset$ and $U^{-1}(\mathcal{O}) = \emptyset$ if $U_0 \cap \mathcal{O} = \emptyset$. Hence, the set-valued mapping U is measurable, which proves *(i)*. Condition *(ii)* of (AU) is satisfied, since $U(x,t) = U_0$, and U_0 is supposed to fulfill the requirements of this condition. Now, take an arbitrary element $u_0 \in U_0$ and set $f_U(x,t) = u_0$ for $(x,t) \in Q$. The so-constructed function f_U is constant over Q, consequently, it is measurable and bounded, say $f_U \in L^\infty(Q)^2$. And the claim is proven. $\square$

Assuming (AU) we can derive another interesting result. Conditon *(iii)* allows us to prove that the pointwise projection on U_{ad} of a L^2-function is itself a L^2-function.

Corollary 7.7. *Let a function $u \in L^2(Q)^2$ be given. Then the function v defined pointwise a.e. by*

$$v(x,t) = \mathrm{Proj}_{U(x,t)}(u(x,t))$$

is also in $L^2(Q)^2$. Further, if for some $p \geq 2$ the functions u and f_U are in $L^p(Q)^2$, then the projection v is in $L^p(Q)^2$ as well.

Proof. By assumption (AU), the set-valued function U is measurable with closed and convex images, and u is a measurable single-valued function. Then by Theorem 7.3 the function v is measurable as well. By Lipschitz continuity of the pointwise projection, it holds

$$\begin{aligned} |v(x,t) - f_U(x,t)| &= |\operatorname{Proj}_{U(x,t)}(u(x,t)) - \operatorname{Proj}_{U(x,t)}(f_U(x,t))| \\ &\leq |u(x,t) - f_U(x,t)| \end{aligned}$$

almost everywhere on Q. Thus, squaring and integrating gives

$$\|v - f_U\|_2^2 \leq \|u - f_U\|_2^2 < \infty,$$

which implies $v \in L^2(Q)^2$. If in addition, u and f_U are in $L^p(Q)^2$ for some $p > 2$, then we can prove analogously that the projection is also in L^p, i.e. $v \in L^p(Q)^2$. $\square$

Discussion of examples

Let us investigate under which conditions the constraints (7.1a)–(7.1c) satisfy the assumption (AU).

Box constraint. At first, we will consider box-constraints (7.1a) defined by

$$U_{\mathrm{box}}(x,t) = [u_{a,1}(x,t), u_{b,1}(x,t)] \times [u_{a,2}(x,t), u_{b,2}(x,t)]$$

with two functions $u_a, u_b \in L^2(Q)^2$ satisfying $u_a \leq u_b$ a.e. on Q. Let us briefly show that U_{box} fulfills (AU). Obviously, conditions *(ii)* and *(iii)* are fulfilled. For any vector $v \in \mathbb{R}^2$ the distance to $U_{\mathrm{box}}(x,t)$ is given by

$$d_v(x,t) = \mathrm{dist}(v, U_{\mathrm{box}}(x,t)) = |v - \mathrm{Proj}_{U_{\mathrm{box}}(x,t)}(v)| = |v - \max(u_a(x,t), \min(u_b, v))|,$$

which is measurable. Hence by Theorem 7.2, U_{box} is measurable as well, and therefore, satisfies (AU).

Norm bound constraint. Now, let us study the constraint (7.1b). For given nonnegative function $\rho \in L^2(Q)^2$ we define

$$U_{\mathrm{rad}}(x,t) := \{u \in \mathbb{R}^2 : |u| \le \rho(x,t)\}.$$

Here again, *(ii)* and *(iii)* are fulfilled by construction. The distance function is now given by

$$d_v(x,t) = \mathrm{dist}(v, U_{\mathrm{rad}}(x,t)) = |v - \mathrm{Proj}_{U_{\mathrm{rad}}(x,t)}(v)| = \max(|v| - \rho(x,t),\, 0).$$

The assumption (AU) is fulfilled in the case of U_{rad} too, since the distance function is measurable.

Constraint in polar coordinates. Finally, we do the same for the constraint (7.1c). Let us define the admissible set using polar coordinates by

$$U_{\mathrm{polar}}(x,t) := \{u = r(\cos\phi,\, \sin\phi)^T \in \mathbb{R}^2 : |r| \le \psi(\phi,x,t)\}.$$

We assume that the function ψ defining the constraint satisfies the following:

- For all $\phi \in \mathbb{R}$ the function $\psi(\phi,x,t)$ is measurable in (x,t).
- For all $(x,t) \in Q$ the function $\psi(\phi,x,t)$ is continuous in ϕ.
- For all $(x,t) \in Q$ the periodic condition $\psi(0,x,t) = \psi(2\pi,x,t)$ holds.

Furthermore, we have to impose an assumption to ensure convexity of the admissible set. One possibility is the following. A closed set is convex if and only if its associated distance function is nonnegative, positively homogeneous, and subadditive, [53, Sätze 11.1, 11.2]. In our case, this condition is equivalent to suppose

- The function $g(u,x,t) = \frac{|u|}{\psi\left(\arctan\left(\frac{u_2}{u_1}\right),x,t\right)}$ is subadditive for all $(x,t) \in Q$, i.e. $g(u^1+u^2,x,t) \le g(u^1,x,t) + g(u^2,x,t)$ holds for all $u^1,u^2 \in \mathbb{R}^2$ and all $(x,t) \in Q$.

By assumption, *(ii)* and *(iii)* are fulfilled. If we assume some additional smoothness of ψ with respect to ϕ one can find other conditions that ensure convexity. However, we will not go into detail here, since that would only complicate things too much.

It remains to prove measurability of the set-valued mapping U_{polar}. To this end, we transform the set U_{polar} to polar coordinates

$$\tilde{U}(x,t) := \{(r,\phi)^T \in \mathbb{R}^2 : 0 \le r \le \psi(\phi,x,t) \text{ and } 0 \le \phi \le 2\pi\}.$$

Here, we can use the same representation of the distance function as in the box-constrained case to prove that $\tilde{U}$ is a measurable set-valued mapping. Let us denote by p the transformation from polar to Cartesian coordinates, i.e. $p(r,\phi) = r(\cos\phi,\, \sin\phi)^T$. Then, we have $U_{\mathrm{polar}}(x,t) = p(\tilde{U}(x,t))$. Since p is continuous, U_{polar} is measurable, see [7, Theorem 8.2.8].

2.2 Existence of optimal controls

Before we can think about existence of solution, we have to specify which problem we want to solve. We will assume that all conditions of Section 1 in Chapter 2 are satified that do not concern control constraints. Moreover, we assume that $U(\cdot)$ fulfills the pre-requisite (AU). So we end up with the following optimization problem:

Minimize the funtional J given by

$$J(y,u) = \frac{\alpha_T}{2}\int_\Omega |y(x,T)-y_T(x)|^2\,\mathrm{d}x + \frac{\alpha_Q}{2}\int_Q |y(x,t)-y_Q(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \\ + \frac{\alpha_R}{2}\int_Q |\operatorname{curl} y(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{2}\int_Q |u(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{7.3a}$$

subject to the state equation

$$y_t + \nu A y + B(y) = u \quad \text{in } L^2(0,T;V'), \tag{7.3b}$$

$$y(0) = y_0 \quad \text{in } H. \tag{7.3c}$$

and the control constraint

$$u \in U_{ad}, \tag{7.3d}$$

where U_{ad} is given by (7.2).

Under the assumptions above, the optimal control problem (7.3) is solvable. We recall that in Section 2.2 the regularization parameter γ is supposed to be greater than zero. We can prove existence even with $\gamma = 0$ but then we have to require boundedness of U_{ad} in L^2.

Theorem 7.8. *The optimal control problem admits a - global optimal - solution $\bar u \in U_{ad}$ with associated state $\bar y \in W(0,T)$.*

The proof is similar to the proof of Theorem 2.1.

3 First-order necessary conditions

The necessary optimality conditions for the optimal control problem discussed in the present chapter differ slightly from the conditions derived in Section 2 in Chapter 3. However, we will repeat the exact statement for convenience of the reader.

Theorem 7.9 (Necessary condition). *Let $\bar u$ be locally optimal in $L^2(Q)^2$ with associated state $\bar y = S(\bar u)$. Then there exists a unique Lagrange multiplier $\bar\lambda \in W^{4/3}(0,T;V)$, which is the weak solution of the adjoint equation*

$$\begin{aligned} -\bar\lambda_t + \nu A\bar\lambda + B'(\bar y)^*\bar\lambda &= \alpha_Q(\bar y - y_Q) + \alpha_R \operatorname{curl}^*\operatorname{curl}\bar y \\ \bar\lambda(T) &= \alpha_T(\bar y(T) - y_T). \end{aligned} \tag{7.4}$$

Moreover, the variational inequality

$$(\gamma\bar u + \bar\lambda, u - \bar u)_Q \ge 0 \quad \forall u \in U_{ad} \tag{7.5}$$

is satisfied.

The proof is identical with the proof of Theorem 3.2.

Similar as in the box-constrained case, we can reformulate the variational inequality (7.5). The projection representation of the optimal control is now realized using the admissible sets $U(\cdot)$

$$\bar{u}(x,t) = \mathrm{Proj}_{U(x,t)}\left(-\frac{1}{\gamma}\bar{\lambda}(x,t)\right) \qquad \text{a.e. on } Q. \tag{7.6}$$

Here, it will be a little bit more difficult to prove regularity results for the optimal control using the regularity of the adjoint state.

Secondly, another formulation of the variational inequality uses the normal cone $\mathcal{N}_{U_{ad}}(\bar{u})$, see its definition (3.1). The inequality (7.5) can be written equivalently as the inclusion

$$\nu\bar{u} + \bar{\lambda} + \mathcal{N}_{U_{ad}}(\bar{u}) \ni 0. \tag{7.7}$$

We will give an explanation, how the normal cone in the set-valued case considered here may look like, in Section 5.1 below.

Second-order necessary optimality conditions for optimal control problems with set-valued constraints were developed in [59]. It involves the usage of the concept of second-order tangent sets, see e.g. [20].

4 Regularity of optimal controls

Let us comment on the regularity of a locally optimal control $\bar{u}$. By (7.6), it inherits some regularity from the associated adjoint state $\bar{\lambda}$. Here, we will show, how the regularities $\bar{\lambda} \in L^p(Q)^2$ and $\bar{\lambda} \in C(\bar{Q})^2$ can be carried over to the control $\bar{u}$. However, it is not clear whether and how it is possible to prove $\bar{u} \in W^{1,p}(Q)^2$ if $\bar{\lambda} \in W^{1,p}(Q)^2$, and what assumptions on U are needed.

4.1 Optimal controls in L^p

Corollary 7.7 gives a hint, how we can prove the regularity $\bar{u} \in L^p(Q)^2$ provided that $\bar{\lambda} \in L^p(Q)^2$ holds. We have to assume only the existence of an admissible L^p-function.

Theorem 7.10. *Let $\bar{u}$ be a locally optimal control of the optimal control problem* (7.3) *with associated adjoint state $\lambda \in L^p(Q)^2$, $p \le \infty$. If there is an admissible function $f_p \in L^p(Q)^2 \cap U_{ad}$ for some $p \le \infty$ then the optimal control $\bar{u}$ is in $L^p(Q)^2$, too.*

Proof. The proof follows immediately from the projection representation (7.6) and Corollary 7.7. □

We will complete this short section with the following corollary that states the precise regularity assumptions on the problem data, such that the pre-requisites of the previous theorem are fulfilled.

Corollary 7.11. *Let be given $y_0, y_T \in V$, $y_Q \in L^2(Q)^2$. Let the set-valued mapping U satisfy the assumption (AU). Further, we assume the existence of an admissible L^p-function $f_p \in L^p(Q)^2 \cap U_{ad}$ for some $2 \le p < \infty$.*

Then every locally optimal control of problem (7.3) *is in $L^p(Q)^2$.*

The method of proof applied here does not work to obtain continuity of an optimal control. This is investigated in the next section.

4.2 Continuity of optimal controls

Now, we are going to prove continuity of locally optimal controls. We will rely in our considerations on the projection formula (7.6), which says that the optimal control is the pointwise projection of a continuous function onto the admissible sets. Hence, this admissible sets $U(x,t)$ vary over space and time. Here, we have to impose some continuity assumptions on the set-valued mapping U.

Continuous set-valued mappings

There are at least two equivalent characterizations of continuity of a *single*-valued function f:

(i) for all x in the domain of f all sequences converging to x are mapped to sequences converging to $f(x)$.

(ii) the preimages of open sets are open sets.

In the set-valued case, however both definitions of a continuous function are no longer equivalent. They define two independent kinds of semicontinuity, see also [7, Section 1.4].

Definition 7.12. *A set-valued mapping $F : D \subset X \rightsquigarrow Y$ is called* lower semicontinuous*, if for all $x \in D$, $y \in F(x)$, and any sequence $\{x_n\} \subset D$ converging to x there is a sequence of elements $y_n \in F(x_n)$ converging to y.*

Definition 7.13. *A set-valued mapping $F : D \subset X \rightsquigarrow Y$ is called* upper semicontinuous*, if for all $x \in D$ and all open sets $O \supset F(x)$ there exists $\delta = \delta(O)$ such that $F(x') \subset O$ for all x' with $|x - x'| \leq \delta$.*

Both definition are not equivalent and are independent. There are set-valued mappings, which are lower semicontinuous but not upper and vice-versa. It is natural to define a continuous mapping to have both semicontinuous properties.

Definition 7.14. *A set-valued mapping $F : D \subset X \rightsquigarrow Y$ is called* continuous*, if U is both lower and upper semicontinuous.*

The assumption (AU) on the set-valued mapping U contains the condition that $U(x,t)$ is non-empty, closed and convex almost everywhere on Q. Do these properties of the images hold everywhere provided U is continuous? At first, we want to show the improvement from 'non-empty almost everywhere' to 'non-empty everywhere' in the continuous case.

Lemma 7.15. *Let U fulfill (AU). Further let $U : \bar{Q} \rightsquigarrow \mathbb{R}^2$ be upper semicontinuous. Then $U(x,t)$ is non-empty for all $(x,t) \in \bar{Q}$.*

Proof. We will prove it by contradiction. Let $\xi = (x,t) \in \bar{Q}$ such that $U(\xi)$ is empty. Then we take sequences of points $\xi_n = (x_n, t_n) \in Q$ with $U(\xi_n) \neq \emptyset$ converging to ξ and $u_n \in U(\xi_n)$.

At first, we consider the case that the sequence $\{u_n\}$ admits a cluster point $\tilde{u}$. Then there is a subsequence $\{u_{n_k}\}$ converging to $\tilde{u}$. Let us define an open set by $O = \mathbb{R}^2 \setminus \overline{B_\rho(\tilde{u})}$ for some $\rho > 0$. Then, we have $\emptyset = U(\xi) \subset O$ and $U(\xi_{n_k}) \not\subset O$ for all k large enough, which is a contradiction to upper semicontinuity.

Now, let the sequence $\{u_n\}$ have no cluster point. Here, we take an arbitrary $\hat{u}$. Then for $\epsilon > 0$ there is n_ϵ such that $|\hat{u} - u_n| > \epsilon$ for all $n > n_\epsilon$, otherwise there would exist a cluster point in $B_\epsilon(\hat{u})$. Let us set $O = B_\epsilon(\hat{u})$. By construction, we have $\emptyset = U(\xi) \subset O$ and $U(\xi_n) \not\subset O$ for $n > n_\epsilon$, which is a contradiction to upper semicontinuity. And the proof is complete. □

Unfortunately, the property 'closedness of the images' cannot be transferred from 'almost everywhere' to 'everywhere' for continuous U as the following counterexample shows.

Example 7.16. *Define* $F : [0,1] \rightsquigarrow \mathbb{R}$ *by*

$$F(t) = \begin{cases} (0,1] & \textit{if } t = 0 \\ [t, 1+t] & \textit{otherwise.} \end{cases}$$

Clearly, F is lower semicontinuous. It is also upper semicontinuous: every open set that contains $F(0)$ contains also $F(\epsilon)$ for sufficiently small ϵ. Hence F is continuous. It has closed images almost everywhere but not everywhere.

Now, let us prove an lemma, which will help us later on.

Lemma 7.17. *Let U fulfill (AU). In addition, we assume that U is upper semicontinuous on $\bar{Q}$ with closed images $U(x,t)$ for all $(x,t) \in \bar{Q}$. Then for given sequences (x_n, t_n) converging to $(x,t) \in \bar{Q}$ and $y_n \in U(x_n, t_n)$ converging to y the limit y lies in $U(x,t)$, $y \in U(x,t)$.*

Proof. We will use again the notation $\xi = (x,t)$ and $\xi_n = (x_n, t_n)$. Let us assume $y \notin U(\xi)$. Set $\epsilon = \operatorname{dist}(y, U(\xi))$, which is positive since $U(\xi)$ is closed. Then there exists N such that for all $n > N$ it holds $y_n \notin U(\xi)$ and $\operatorname{dist}(y_n, U(\xi)) \geq \frac{2}{3}\epsilon$. Now, we construct an open set by $O := \{v : \operatorname{dist}(v, U(\xi)) < \frac{1}{3}\epsilon\}$. It implies $y_n \notin O$ and $U(\xi_n) \not\subset O$ for $n \geq N$. This yields a contradiction to upper semicontinuity, since we have $O \supset U(\xi)$. Hence it holds $y \in U(\xi)$. □

Furthermore, it turns out that the assumption of closed images is essential to prove the convexity of the images of U.

Lemma 7.18. *Let U fulfill (AU). In addition, let U be continuous on $\bar{Q}$ with closed images $U(x,t)$ for all $(x,t) \in \bar{Q}$. Then $U(x,t)$ is convex for all $(x,t) \in \bar{Q}$.*

Proof. Let $\xi = (x,t) \in \bar{Q}$ be given with $y_1, y_2 \in U(\xi)$, $\lambda \in (0,1)$. We have to show that $\lambda y_1 + (1-\lambda) y_2$ is in $U(\xi)$. We take a sequence of points $\xi_n = (x_n, t_n) \in Q$, for which $U(\xi_n)$ is non-empty and convex, converging to ξ.

By lower semicontinuity there exist sequences of points $y_1^n, y_2^n \in U(\xi_n)$ converging to y_1 respectively y_2. The points $y^n := \lambda y_1^n + (1-\lambda) y_2^n$ are in $U(\xi_n)$ and converge to $y := \lambda y_1 + (1-\lambda) y_2$ for $n \to \infty$. The previous Lemma 7.17 implies that the limit $y = \lambda y_1 + (1-\lambda) y_2$ is in $U(\xi)$. Hence $U(\xi)$ is convex. □

Assuming the continuity of the set-valued mapping U and the adjoint state λ we can prove continuity of locally optimal controls.

Theorem 7.19. *Let U satisfy the assumption (AU). Furthermore, let $U : \bar{Q} \rightsquigarrow \mathbb{R}^n$ be continuous with closed images everywhere. Suppose $\bar{u}$ satisfies the first-order necessary optimality conditions of Theorem 7.9 together with the state $\bar{y}$ and adjoint $\bar{\lambda}$. If the adjoint state is continuous, $\bar{\lambda} \in C(\bar{Q})$, so is the control as well, $\bar{u} \in C(\bar{Q})$.*

Proof. We will show that the projection

$$\mathrm{Proj}_{U(x,t)}\left(-\frac{1}{\gamma}\bar{\lambda}(x,t)\right) = \bar{u}(x,t)$$

gives a continuous function. We abbreviate $v(x,t) := -\bar{\lambda}(x,t)/\gamma$, which is a continuous function by assumption.

Let $\xi = (x,t) \in \bar{Q}$ be given. Take a sequence $\xi_n = (x_n, t_n) \in Q$ that converges to ξ. We have to show the convergence $\bar{u}(\xi_n) \to \bar{u}(\xi)$. We will give the proof in several steps.

Step 1: $U(x,t)$ is non-empty, closed and convex everywhere on $\bar{Q}$. This follows by the preceding Lemmata 7.15 and 7.18.

Step 2: U_{ad} contains a continuous function. Define the function $m : \bar{Q} \to \mathbb{R}^n$ as

$$m(x,t) = \arg\min\left\{|v| : v \in U(x,t)\right\},$$

which gives the elements of $U(x,t)$ with the smallest norm. It is called the minimal selection of U. Since $U(x,t)$ is non-empty, closed and convex, the function m is well-defined. By [6, Chapt. 3, Sect. 1, Prop. 23, p. 120], the minimal selection m is continuous.

Step 3: Boundedness of $\{\bar{u}(\xi_n)\}$. Using Lipschitz continuity of the projection, we can estimate

$$\begin{aligned} |\bar{u}(\xi_n) - m(\xi_n)| &= |\,\mathrm{Proj}_{U(\xi_n)}(v(\xi_n)) - \mathrm{Proj}_{U(\xi_n)}(m(\xi_n))| \\ &\le |v(\xi_n) - m(\xi_n)| \le \|v - m\|_{C(\bar{Q})} < \infty, \end{aligned}$$

which proves boundedness of the set $\{\bar{u}(\xi_n)\}$.

Step 4: Every accumulation point of $\{\bar{u}(\xi_n)\}$ is in $U(\xi)$. Since $\{\bar{u}(\xi_n)\}$ is bounded in $\mathbb{R}^n$, we can select a subsequence $\{\bar{u}(\xi_{n'})\}$ converging to some element $\tilde{u}$. By Lemma 7.17, we find that $\tilde{u}$ is in $U(\xi)$.

Step 5: There is exactly one accumulation point of $\{\bar{u}(\xi_n)\}$. Take an arbitrary element $z \in U(\xi)$. By lower semicontinuity, there is a sequence of elements $z_{n'} \in F(\xi_{n'})$ converging to $z \in U(\xi)$. Since $u(\xi_{n'}) = \mathrm{Proj}_{U(\xi_{n'})} v(\xi_{n'})$, we find

$$\big(u(\xi_{n'}) - v(\xi_{n'}),\, z_{n'} - u(\xi_{n'})\big) \ge 0 \quad \forall n'.$$

Hence

$$\big(u(\xi_{n'}) - v(\xi_{n'}),\, z_{n'} - z\big) + \big(u(\xi_{n'}) - v(\xi_{n'}),\, z - u(\xi_{n'})\big) \ge 0 \quad \forall n'.$$

Passing to the limit $n' \to \infty$, we find

$$\big(\tilde{u} - v(\xi),\, z - \tilde{u}\big) \ge 0.$$

Since $z \in U(x)$ was arbitrary, it holds

$$\tilde{u} = \operatorname{Proj}_{U(\xi)} v(\xi)$$

for every accumulation point of $\{\bar{u}(\xi_n)\}$. The projection is unique hence the set $\{u(\xi_n)\}$ has exactly one accumulation point, which is $\tilde{u} = \bar{u}(\xi) = \operatorname{Proj}_{U(\xi)} v(\xi)$.

Conclusion. By the previous steps, we find that $\{\bar{u}(\xi_n)\}$ is a bounded sequence with exactly one accumulation point. Hence, the limit $\lim_{n\to\infty} \bar{u}(\xi_n)$ exists and is equal to $\bar{u}(\xi)$. Thus, the proof is complete, and $\bar{u}$ is a continuous function on $\bar{Q}$. □

Let us specify what regularity of the data is enough to satisfy the requirements of the previous theorem.

Corollary 7.20. *Assume that the conditions (i)–(v) of Theorem 3.7 are satisfied for some $p > 2$. Suppose further that $U : \bar{Q} \rightsquigarrow \mathbb{R}^n$ satisfies the assumption (AU) and is continuous with closed images everywhere.*

If $\bar{u}$ satisfies the first-order necessary optimality conditions of Theorem 7.9 together with the state $\bar{y}$ and adjoint $\bar{\lambda}$, then it is continuous, i.e. $\bar{u} \in C(\bar{Q})$.

Remark 7.21. *If $U(x,t) = U_0$ is constant over Q then the previous theorem can be proven in a significantly shorter way.*

Let $\xi = (x,t) \in \bar{Q}$ be given. Take a sequence $\xi_n = (x_n, t_n) \in Q$ that converges to ξ. Here, we can employ Lipschitz continuity of the projection, since we project on the same set. We then obtain

$$|\bar{u}(\xi) - \bar{u}(\xi_n)| = \left| \operatorname{Proj}_{U_0}\left(-\frac{1}{\gamma}\bar{\lambda}(\xi)\right) - \operatorname{Proj}_{U_0}\left(-\frac{1}{\gamma}\bar{\lambda}(\xi_n)\right)\right| \le \frac{1}{\gamma}|\bar{\lambda}(\xi) - \bar{\lambda}(\xi_n)|.$$

The right-hand side tends to zero for $n \to \infty$, because $\bar{\lambda}$ is a continuous function. Thus the left-hand side has to tend to zero, too, which implies continuity of $\bar{u}$.

Remark 7.22. *The projection formula* (7.6) *remains true if one replaces $U(x,t)$ by its closure $\overline{U(x,t)}$, provided $U(\cdot)$ is closed almost everywhere on Q. Furthermore, one can show that for continuous $U : Q \rightsquigarrow \mathbb{R}^2$ the closure $\bar{U} : Q \rightsquigarrow \mathbb{R}^2$ is also a continuous set-valued mapping. In this way, we can construct a continuous representation of a locally optimal control $\bar{u}$ without the assumption of closedness of the images of U.*

5 Second-order sufficient optimality conditions

5.1 Normal directions

Before we start with the formulation of the sufficient optimality conditions, let us recall some notation already introduced in Section 1 of Chapter 3. Let be given a convex set C. Then $\mathcal{N}_C(u)$ and $\mathcal{T}_C(u)$ are the normal and tangent cones of C at some point u, respectively. The space of normal directions is written $N_C(u) = \operatorname{span} \mathcal{N}_C(u)$ with its orthogonal complement $T_C(u) = N_C(u)^\perp$.

Now, we want to use these notations with $C = U_{ad}$. Let be given an admissible control $u \in U_{ad}$, where U_{ad} is now given by the set-valued constraint U. It is well-known, that the sets $\mathcal{N}_{U_{ad}}(u)$, $\mathcal{T}_{U_{ad}}(u)$, $N_{U_{ad}}(u)$, and $T_{U_{ad}}(u)$ admit a pointwise representation as U_{ad} itself, cf. [7, 63]. For instance, for $u \in L^2(Q)^2$ the set $\mathcal{N}_{U_{ad}}(u)$ is given by

$$\mathcal{N}_{U_{ad}}(u) = \left\{ v \in L^2(Q)^2 : \, v(x,t) \in \mathcal{N}_{U(x,t)}(u(x,t)) \text{ a.e. on } Q \right\}.$$

In a while, we will need the projection of a test function w on the space of normal directions and its complement. We will denote the resulting functions by w_N and w_T respectively. They are defined pointwise by

$$w_N(x,t) = \operatorname{Proj}_{\left[N_{U(x,t)}(u(x,t))\right]}(w(x,t)) \tag{7.8}$$

and

$$w_T(x,t) = \operatorname{Proj}_{\left[T_{U(x,t)}(u(x,t))\right]}(w(x,t)). \tag{7.9}$$

It is not easy to prove that the functions w_N and w_T are measurable. At this point, the method of Dunn [26] requires that the admissible set U is polyhedric and independent of (x,t). However, these restrictions can be overcome using the results for set-valued mappings. Let us sketch the method of the measurability proof. Behind the projections there are the following mappings:

(i) $Q \ni (x,t) \mapsto \mathcal{N}_{U(x,t)}(u(x,t)) =: \mathcal{N}(x,t)$

(ii) $Q \ni (x,t) \mapsto \operatorname{span}\{\mathcal{N}(x,t)\} = \operatorname{span}\left\{\mathcal{N}_{U(x,t)}(u(x,t))\right\} =: N(x,t)$

(iii) $Q \ni (x,t) \mapsto \operatorname{Proj}_{N(x,t)}(w(x,t)) =: w_N(x,t)$.

Here, one can see, what happens if $U(x,t)$ is constant over Q: the mapping $\mathcal{N}$ is even in this case a set-valued mapping that is *not* constant. Even the dimension of $\mathcal{N}(x,t)$ may vary. So we would not have any advantage if we assume constant admissible sets $U(x,t) = U_0$.

Theorem 7.23. *Let U fulfill (AU). Then for all measurable functions $u, w : Q \to \mathbb{R}^2$ the functions w_N and w_T defined by* (7.8) *and* (7.9)*, respectively, are measurable.*

We will give a proof of the Theorem after providing some auxiliary results. At first, we will prove that the dual cone of a measurable set-valued map, whose images are cones, is measurable as well.

Lemma 7.24. *Let us assume that the set-valued map $F : Q \rightsquigarrow \mathbb{R}^2$ is measurable, and that its images are closed and convex cones. Then the set-valued map*

$$Q \ni (x,t) \rightsquigarrow F(x,t)^*$$

is measurable. Here, K^* denotes the dual or polar cone to a cone K given by

$$K^* = \{x : \quad \langle f, x\rangle \le 0 \quad \forall f \in K\}.$$

Proof. We sketch the proof for convenience. By Theorem 7.2, there exists a sequence of measurable selections f_n, such that $\cup_{n\ge1}\{f_n(x,t)\}$ is dense in $F(x,t)$. We have

$$F(x,t)^* = \left(\overline{\bigcup_{n\ge1}\{f_n(x,t)\}}\right)^* = \left(\bigcup_{n\ge1}\{f_n(x,t)\}\right)^* = \bigcap_{n\ge1}\{f_n(x,t)\}^*$$
$$= \bigcap_{n\ge1}\{z \in \mathbb{R}^2 : \, \langle f_n(x,t), z\rangle \le 0\}.$$

The set valued function $(x,t) \rightsquigarrow \{z \in \mathbb{R}^2 : \langle f_n(x,t), z\rangle \leq 0\}$ is measurable by [7, Th. 8.2.9]. Hence, the mapping $(x,t) \rightsquigarrow F(x,t)^*$ is measurable as the intersection of countably many measurable set-valued maps, see [7, Th. 8.2.4]. $\square$

Secondly, we investigate the span operation, which is used in the definition of the subspaces N and T.

Lemma 7.25. *Let us assume that the set-valued map $F : Q \rightsquigarrow \mathbb{R}^2$ is measurable. Then the set-valued map*

$$Q \ni (x,t) \rightsquigarrow \operatorname{span}(F(x,t))$$

is measurable.

Proof. Since F is measurable, there exists a sequence of measurable selections f_n that is dense in $F(x,t)$ by Theorem 7.2. Then, the set of finite and rational linear combinations of $f_n(x,t)$, $n \in \mathbb{N}$, is countable and dense in $\operatorname{span}(F(x,t))$ for all $(x,t) \in Q$. By Theorem 7.2, we can conclude that the set-valued mapping $(x,t) \rightsquigarrow \operatorname{span}(F(x,t))$ is measurable. $\square$

Proof of Theorem 7.23. Now, the measurability of w_N and w_T can be proven as follows. The mapping $(x,t) \rightsquigarrow \mathcal{T}_{U(x,t)}(u(x,t))$ is measurable if U has closed and convex images, cf. [7, Cor. 8.5.2]. The normal cone $\mathcal{N}_{U(x,t)}(u(x,t))$ is then the dual of $\mathcal{T}_{U(x,t)}(u(x,t))$, and, by Lemma 7.24, $\mathcal{N}_U(u)$ is a measurable set-valued mapping as well. With Lemma 7.25, we can conclude that the space of normal directions $N_U(u)$ is a measurable set-valued mapping. Its orthogonal complement $T_U(u)$ can be written as $T_U(u) = N_U(u)^\perp = (N_U(u))^* \cap (-N_U(u))^*$. By Lemma 7.24, it follows that $(-N_U(u))^*$ is a measurable set-valued mapping. Hence, $T_U(u)$ as the intersection of two measurable mappings is measurable as well. Now, the claim follows with Theorem 7.3, which states that the projection on measurable set-valued mappings of measurable functions results in a measurable function, too. $\square$

As a last point here, let us define some more notations in connection to convex sets. The relative interior of a convex set is defined by

$$\operatorname{ri} C = \{x \in \operatorname{aff} C : \exists \epsilon > 0, B_\epsilon(x) \cap \operatorname{aff} C \subset C\},$$

its complement in C is called the relative boundary

$$\operatorname{rb} C = C \setminus \operatorname{ri} C.$$

5.2 Sufficiency

Let us come back to optimization with convex constraints. To motivate the following, we will investigate the finite-dimensional problem $\min_{x \in C} f(x)$ with $C \subset \mathbb{R}^n$ first. It is known that a local minimizer x of the twice differentiable function f over a convex set C fulfills

$$-\nabla f(x) \in \mathcal{N}_C(x),$$

compare also the necessary condition in Section 3 especially inclusion (7.7). Now, let us have a look on corresponding sufficient optimality conditions. For unconstrained problems it suffices to assume positive definiteness of the Hessian f''. However, in the presence of constraints there is no need for positive definiteness of f'' for directions that are not admissible directions. Here, a weaker sufficient condition is then given by

$$-\nabla f(x) \in \operatorname{ri} \mathcal{N}_C(x) \tag{7.10}$$

and

$$f''(x)[y, y] > 0 \qquad \forall y \in T_C(x). \tag{7.11}$$

It consists of a first-order part: strict complementarity and a second-order part: coercivity. Now we want to adapt this formulation to the optimal control problem considered here. Condition (7.10) would become

$$-(\gamma \bar{u}(x,t) + \bar{\lambda}(x,t)) \in \operatorname{ri} \mathcal{N}_{U(x,t)}(\bar{u}(x,t)) \qquad \text{a.e. on } Q. \tag{7.12}$$

However, this is not enough for optimal control problems, we need the satisfaction of this condition in a uniform sense. We have to assume not only that $-(\gamma\bar{u}(x,t) + \bar{\lambda}(x,t))$ lies in the relative interior of the normal cone, we need moreover that $-(\gamma\bar{u}(x,t) + \bar{\lambda}(x,t))$ has a positive distance to the relative boundary of the normal cone $\mathcal{N}_{U(x,t)}(\bar{u}(x,t))$. But this cannot be assumed for all (x,t): if $\bar{u}(x,t)$ is in the interior of the admissible set $U(x,t)$ then the normal cone consists only of the origin, and this uniform condition cannot be fulfilled. Therefore, we introduce the set of strongly active constraints as the set of points, where this condition is fulfilled,

$$Q_\epsilon = \left\{(x,t) \in Q : \operatorname{dist}\left(-(\gamma\bar{u}(x,t) + \bar{\lambda}(x,t)), \operatorname{rb} \mathcal{N}_{U(x,t)}(\bar{u}(x,t))\right) > \epsilon\right\}. \tag{7.13}$$

That is, we assume that (7.12) is only fulfilled on a subset of the domain Q. Consequently, we have to require the coercivity assumption for more directions than included in $T_{U_{ad}}(\bar{u})$. Furthermore, the inequality > 0 in (7.11) has to be replaced by a norm-square, since the proof in finite dimensions that '> 0' suffices is tied to compactness of the unit sphere, which does not hold in the infinite dimensional case.

Let us comment on the definition of the strongly active set in (7.13) when U is formed by box constraints. In Chapter 4, we introduced already a definition of strongly active sets, see Definition 4.1. Although we used there active sets $Q_{\epsilon,i}$ for each component, both concepts are almost identical. The only difference is the following: suppose we have that for some $(x,t) \in Q$ the relations $\bar{u}(x,t) = (u_{b,1}(x,t), u_{b,2}(x,t))$ and $-(\gamma\bar{u}(x,t) + \bar{\lambda}(x,t)) = (v_1, 0)$ with $v_1 > 0$ hold. That is, $\bar{u}(x,t)$ is in a vertex of the box, and the negative gradient is in the relative boundary of the normal cone. Then (x,t) cannot lie in Q_ϵ as introduced above, but it lies in the set $Q_{\epsilon,1}$ since the box-constraint is strongly active in the first component. Hence, the active sets used in Chapter 4 contain a little bit more points than the active set introduced here. This is due to the fact, that the box-constraint can be decomposed in two independent inequality constraints, whereas the convex constraint cannot be decomposed in general. It is possible to refine the definition of the strongly active set here if we assume more regularity of the sets $U(x,t)$ such as a representation of $U(x,t)$ by finitely many inequalities. However, we will proceed in the way we followed already, namely to treat the control constraint as inclusion rather than as inequality.

Altogether, we require that the following is fulfilled. We assume that the reference pair $(\bar{y}, \bar{u})$ satisfies the coercivity assumption on $\mathcal{L}''(\bar{y}, \bar{u}, \bar{\lambda})$, in the sequel called second-order sufficient condition:

$$
\textbf{(SSC)} \begin{cases}
\textit{There exist } \epsilon > 0 \textit{ and } \delta > 0 \textit{ such that} \\
\qquad\qquad \mathcal{L}''(\bar{y}, \bar{u}, \bar{\lambda})[(z,h)]^2 \geq \delta \, \|h\|_2^2 & (7.14\text{a}) \\
\textit{holds for all pairs } (z,h) \in W(0,T) \times L^2(Q)^2 \textit{ with} \\
\qquad\qquad h \in \mathcal{T}_{U_{ad}}(\bar{u}), \;\; h_N = 0 \textit{ on } Q_\epsilon, & (7.14\text{b}) \\
\textit{and } z \in W(0,T) \textit{ being the weak solution of the linearized equation} \\
\qquad\qquad \begin{aligned} z_t + Az + B'(\bar{y})z &= h \\ z(0) &= 0. \end{aligned} & (7.14\text{c})
\end{cases}
$$

In (7.14b) h_N denotes the pointwise projection of $h(x,t)$ on the subspaces $N(x,t) \subset \mathbb{R}^2$, compare (7.8). We required in (SSC) the coercivity of $\mathcal{L}''$ for more test functions than in (7.11). The set $T_{U_{ad}}(\bar{u})$, which was used there, is only a subset of $\mathcal{T}_{U_{ad}}(\bar{u})$. However, the space of test functions in (SSC) can be reformulated as: $h \in \mathcal{T}_{U_{ad}}(\bar{u})$ with $h(x,t) \in T_{U(x,t)}(\bar{u}(x,t))$ on Q_ϵ. We can use test functions with values in the spaces T only on the strongly active set due to the strong complementarity, which holds only there. On the rest of the domain, the values of the test function has to lie in the tangent cones $\mathcal{T}$.

Now, the next theorem states the sufficiency of (SSC).

Theorem 7.26. *Let $(\bar{y}, \bar{u})$ be admissible for the optimal control problem and suppose that $(\bar{y}, \bar{u})$ fulfills the first order necessary optimality conditions with associated adjoint state $\bar{\lambda}$. Assume further that (SSC) is satisfied at $(\bar{y}, \bar{u})$. Then there exist $\alpha > 0$ and $\rho > 0$ such that*

$$J(y,u) \geq J(\bar{y}, \bar{u}) + \alpha \, \|u - \bar{u}\|_2^2$$

holds for all admissible pairs (y,u) with $\|u - \bar{u}\|_\infty \leq \rho$.

We will give the proof in Section 5.4 after a series of auxiliary results. There we consider at first the set of strongly active constraints. We prove its measurability, which is a non-trivial result obtained using set-valued analysis. Secondly, we derive some positiveness in directions of test functions that are not included in (SSC) from the strongly active contraints. It is possible to work with L^q and L^s norms rather than L^2 and L^∞. Since the arguments would be the same as in Chapter 4, we discuss only the latter case for the sake of simplicity.

There are a number of sufficient second-order optimality conditions for finite-dimensional optimization problems with convex constraints, see for instance [12, 13, 60]. They all use second-order tangent sets, and it is not clear how those results are related to the condition presented here. Also, the extension of the finite-dimensional results to optimal control problems is not a trivial exercise and requires further research.

5.3 Strongly active constraints

Before we turn to the discussion of measurability, we give some interpretation of the set of strongly active constraints. To keep the illustration as simple as possible, the following considerations are only valid for two-dimensional controls, i.e. $U(x,t) \subset \mathbb{R}^2$. We will distinguish some cases whether $\bar{u}(x,t)$ lies in the interior, on an edge or in a corner of the admissible set $U(x,t)$.

At first, consider the case that $\bar{u}(x,t)$ lies in the interior of $U(x,t)$. Then it holds $\mathcal{N}_{U(x,t)}(\bar{u}(x,t)) = N_{U(x,t)}(\bar{u}(x,t)) = \{0\}$. Thus, the first-order necessary optimality conditions imply $\gamma\bar{u}(x,t)+\bar{\lambda}(x,t) = 0$, which is equivalent to $-(\gamma\bar{u}(x,t)+\bar{\lambda}(x,t)) \in \operatorname{ri}\mathcal{N}_{U(x,t)}(\bar{u}(x,t))$. Hence by condition (7.13), the set of strongly active constraints can not contain points where $\bar{u}(x,t)$ lies in the interior of $U(x,t)$. This is what one expects, since no constraint is active.

Now, assume that $\bar{u}(x,t)$ lies on a smooth part of $\partial U(x,t)$, i.e. the normal cone $\mathcal{N}_{U(x,t)}(\bar{u}(x,t))$ is one-dimensional. Then, its relative boundary is the origin, $\operatorname{rb}\mathcal{N}_{U(x,t)}(\bar{u}(x,t)) = \{0\}$. Consequently, (7.13) means $|\gamma\bar{u}(x,t) + \bar{\lambda}(x,t)| > \epsilon$ on Q_ϵ. The latter relation is often used to define strongly active constraints for box-constrained optimal control problems, cf. [19, 77].

If $\bar{u}(x,t)$ is a corner of $U(x,t)$ then the dimension of $\mathcal{N}_{U(x,t)}(\bar{u}(x,t))$ is equal to the space dimension two. Here, $\mathcal{N}_{U(x,t)}(\bar{u}(x,t))$ is the convex and conical hull of two extremal vectors n_1 and n_2. We can assume that $|n_1| = |n_2| = 1$ holds. The relative boundary of the normal cone admits the representation

$$\operatorname{rb}\mathcal{N}_{U(x,t)}(\bar{u}(x,t)) = \{a_1\, n_1 |\, a_1 \geq 0\} \cup \{a_2\, n_2 |\, a_2 \geq 0\}.$$

The condition (7.13) is equivalent to the fact that $-(\gamma\bar{u} + \bar{\lambda})$ lies in a cone that is the result of a shifting of the normal cone by $\sigma(n_1 + n_2)$, i.e.

$$-(\gamma\bar{u}(x,t) + \bar{\lambda}(x,t)) \in \sigma(n_1 + n_2) + \mathcal{N}_{U(x,t)}(\bar{u}(x,t)),$$

see Figure 7.1. Here, σ is given by $\sigma = \frac{\epsilon}{\sqrt{1-(n_1,\, n_2)^2}}$.

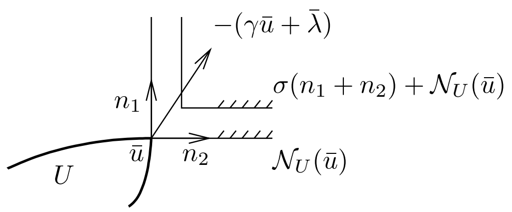

Figure 7.1.

Now, we want to prove the measurability of the set of strongly active constraints.

Lemma 7.27. *Suppose $\bar{u}$ and $\bar{\lambda}$ fulfill the first-order necessary optimality conditions, and $U : Q \rightsquigarrow \mathbb{R}^2$ is measurable. Then the set Q_ϵ defined in* (7.13) *is measurable as well.*

Proof. At first, one finds for a convex set $\mathcal{N} \subset \mathbb{R}^n$ with $0 \in \mathcal{N}$ and a vector $u \in \mathcal{N} \subset \mathbb{R}^n$ that the following identity

$$\operatorname{dist}(u,\, \operatorname{rb}\mathcal{N}) = \operatorname{dist}(u,\, \operatorname{span}\mathcal{N} \setminus \mathcal{N}) \tag{7.15}$$

holds. As already mentioned, the set-valued mapping $(x,t) \rightsquigarrow \mathcal{N}_{U(x,t)}(\bar{u}(x,t))$ is measurable. Using proving techniques of [7], one can check measurability of

$(x,t) \rightsquigarrow \operatorname{span}\mathcal{N}_{U(x,t)}(\bar{u}(x,t))$. By [7, Cor. 8.2.13], the distance between a measurable function u and a measurable set-valued function U, which is a function defined by

$$[\operatorname{dist}(u,U)](x,t) := \operatorname{dist}(u(x,t),\, U(x,t)),$$

is also measurable. This implies that the function $d_{\mathcal{N}}$ given by

$$d_{\mathcal{N}}(x,t) = \operatorname{dist}\big(-(\gamma\bar{u}(x,t)+\bar{\lambda}(x,t)),\, \operatorname{span}\mathcal{N}_{U(x,t)}(\bar{u}(x,t)) \setminus \mathcal{N}_{U(x,t)}(\bar{u}(x,t))\big)$$

is measurable. By assumption, $\bar{u}$, $\bar{\lambda}$ fulfill the first-order necessary optimality conditions especially relation (7.7). Therefore, we can apply (7.15) and obtain that

$$\operatorname{dist}\big(-(\gamma\bar{u}(x,t)+\bar{\lambda}(x,t)),\, \operatorname{rb}\mathcal{N}_{U(x,t)}(\bar{u}(x,t))\big) = d_{\mathcal{N}}(x,t)$$

is a measurable function from Q to $\mathbb{R}$. Using the representation

$$Q_\epsilon = d_{\mathcal{N}}^{-1}\left((\epsilon,+\infty)\right),$$

we finally find that Q_ϵ is a measurable set. $\square$

The condition (SSC) requires coercivity of the second derivative of the Langrangian only with respect to test functions h, whose normal components are zero on the strongly active set, i.e. $h_N = 0$ on Q_ϵ. However, by the following Lemma, we gain an additional positive term that we will need in the proof of sufficiency, see Section 5.4 below. To this aim, we denote the L^p-norm with respect to the set of positivity for $u \in L^p(Q)^2$ and $1 \le p < \infty$ by

$$\|u\|_{L^p(Q_\epsilon)} := \left(\int_{Q_\epsilon} |u(x,t)|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{1/p}.$$

The positiveness result then reads as follows.

Lemma 7.28. *For all $u \in U_{ad}$ with $\|u-\bar{u}\|_\infty < \rho$ it holds*

$$(\gamma\bar{u}+\bar{\lambda},\, u-\bar{u})_Q \ge \frac{\epsilon}{\rho}\|(u-\bar{u})_N\|^2_{L^2(Q_\epsilon)},$$

where $(\cdot)_N$ denotes the pointwise projection on $N_{U(x,t)}(\bar{u}(x,t))$, which is the space of normal directions of $U(x,t)$ at $\bar{u}(x,t)$.

Proof. Let $u \in U_{ad}$ be given. Since $(\bar{u},\bar{\lambda})$ fulfills the first-order necessary optimality conditions, it holds

$$\int_{Q\setminus Q_\epsilon} (\gamma\bar{u}(x,t)+\bar{\lambda}(x,t))\cdot(u(x,t)-\bar{u}(x,t))\,\mathrm{d}x\,\mathrm{d}t \ge 0.$$

Hence, we only need to investigate the difference $u-\bar{u}$ on the set of strongly active constraints Q_ϵ. Now, take $(x,t) \in Q_\epsilon$. We split the difference of both controls into parts belonging to the space of normal directions $N(x,t) = N_{U(x,t)}(\bar{u}(x,t))$ and its orthogonal complement $T(x,t) = T_{U(x,t)}(\bar{u}(x,t)) = N(x,t)^\perp$,

$$u(x,t)-\bar{u}(x,t) = (u(x,t)-\bar{u}(x,t))_N + (u(x,t)-\bar{u}(x,t))_T.$$

The necessary optimality conditions imply

$$-(\gamma\bar{u}(x,t) + \bar{\lambda}(x,t)) \in \mathcal{N}_{U(x,t)}(\bar{u}(x,t)) \subset N_{U(x,t)}(\bar{u}(x,t)) = N(x,t),$$

which allows us to conclude

$$(\gamma\bar{u}(x,t) + \bar{\lambda}(x,t)) \cdot (u(x,t) - \bar{u}(x,t))_T = 0 \qquad \text{a.e. on } Q_\epsilon. \tag{7.16}$$

Now, we have to distinguish two cases: whether the normal component $(u(x,t) - \bar{u}(x,t))_N$ vanishes or not. If it is zero, we have trivially

$$0 = (\gamma\bar{u}(x,t) + \bar{\lambda}(x,t)) \cdot (u(x,t) - \bar{u}(x,t))_N \geq \epsilon|(u(x,t) - \bar{u}(x,t))_N| = 0.$$

On the other hand, suppose $(u(x,t) - \bar{u}(x,t))_N \neq 0$. By definition, the gradient $-(\gamma\bar{u}(x,t) + \bar{\lambda}(x,t))$ belongs to the relative interior of $\mathcal{N}_{U(x,t)}(\bar{u}(x,t))$. Thus, there exists $\tau > 0$, such that

$$-(\gamma\bar{u}(x,t) + \bar{\lambda}(x,t)) + \tau(u(x,t) - \bar{u}(x,t))_N \in \operatorname{rb}\mathcal{N}_{U(x,t)}(\bar{u}(x,t))$$

is satisfied, which is equivalent to

$$\big(\gamma\bar{u}(x,t) + \bar{\lambda}(x,t) - \tau(u(x,t) - \bar{u}(x,t))_N\big) \cdot (u(x,t) - \bar{u}(x,t)) \geq 0. \tag{7.17}$$

But we know even more, we can estimate the norm of the correction $\tau(u - \bar{u})_N$ using (7.13) by

$$\tau|(u(x,t) - \bar{u}(x,t))_N| > \epsilon.$$

Combining (7.16), (7.17), and the previous estimate, we obtain for $(x,t) \in Q_\epsilon$

$$\begin{aligned}(\gamma\bar{u}(x,t) + \bar{\lambda}(x,t)) \cdot (u(x,t) - \bar{u}(x,t)) &\geq \tau(u(x,t) - \bar{u}(x,t))_N \cdot (u(x,t) - \bar{u}(x,t))_N \\ &= \tau|(u(x,t) - \bar{u}(x,t))_N|^2 \\ &\geq \epsilon|(u(x,t) - \bar{u}(x,t))_N|.\end{aligned} \tag{7.18}$$

Now, we integrate over Q and take (7.16), (7.18) into account to get

$$\begin{aligned}&\int_Q (\gamma\bar{u}(x,t) + \bar{\lambda}(x,t)) \cdot (u(x,t) - \bar{u}(x,t)) \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad\geq \int_{Q_\epsilon} (\gamma\bar{u}(x,t) + \bar{\lambda}(x,t)) \cdot (u(x,t) - \bar{u}(x,t)) \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad\geq \epsilon \int_{Q_\epsilon} |(u(x,t) - \bar{u}(x,t))_N| \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad= \epsilon\|(u - \bar{u})_N\|_{L^1(Q_\epsilon)}.\end{aligned}$$

An interpolation argument together with the pre-requisite $\|u - \bar{u}\|_\infty \leq \rho$ yields

$$\begin{aligned}&\int_Q (\gamma\bar{u}(x,t) + \bar{\lambda}(x,t)) \cdot (u(x,t) - \bar{u}(x,t)) \,\mathrm{d}x\,\mathrm{d}t \geq \epsilon\|(u - \bar{u})_N\|_{L^1(Q_\epsilon)} \\ &\qquad\geq \frac{\epsilon}{\rho}\|(u - \bar{u})_N\|_{L^1(Q_\epsilon)}\|(u - \bar{u})_N\|_{L^\infty(Q_\epsilon)} \geq \frac{\epsilon}{\rho}\|(u - \bar{u})_N\|^2_{L^2(Q_\epsilon)},\end{aligned}$$

which is the desired result. □

5.4 Proof of Theorem 7.26

This proof differs less from the proof of Theorem 4.3, see page 47, which concerns the sufficient condition in the box-constrained case.

Proof. Suppose that $(\bar{y}, \bar{u})$ fulfills the assumptions of the theorem. Let (y, u) be another admissible pair. We have

$$J(\bar{y}, \bar{u}) = \mathcal{L}(\bar{y}, \bar{u}, \bar{\lambda}) \quad \text{and} \quad J(y, u) = \mathcal{L}(y, u, \bar{\lambda}),$$

since $(\bar{y}, \bar{u})$ and (y, u) are admissible. Taylor-expansion of the Lagrange-function yields

$$\mathcal{L}(y, u, \bar{\lambda}) = \mathcal{L}(\bar{y}, \bar{u}, \bar{\lambda}) + \frac{\partial \mathcal{L}}{\partial y}(\bar{y}, \bar{u}, \bar{\lambda})(y - \bar{y}) + \frac{\partial \mathcal{L}}{\partial u}(\bar{y}, \bar{u}, \bar{\lambda})(u - \bar{u}) + \frac{1}{2}\mathcal{L}''(\bar{y}, \bar{u}, \bar{\lambda})[(y - \bar{y}, u - \bar{u})]^2. \quad (7.19)$$

Notice that there is no remainder term due to the quadratic nature of all nonlinearities. Moreover, the necessary conditions (7.4) are satisfied at $(\bar{y}, \bar{u})$ with adjoint state $\bar{\lambda}$. Therefore, the second term vanishes. The third term can be estimated by Lemma 7.28,

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{y}, \bar{u}, \bar{\lambda})(u - \bar{u}) = \int_Q (\gamma \bar{u} + \bar{\lambda})(u - \bar{u}) \, \mathrm{d}x \, \mathrm{d}t \geq \frac{\epsilon}{\rho} \|(u - \bar{u})_N\|^2_{L^2(Q_\epsilon)},$$

provided $\|u - \bar{u}\|_\infty \leq \rho$ holds, where ρ will be chosen sufficiently small in the course of the proof. So we arrive at

$$J(y, u) - J(\bar{y}, \bar{u}) \geq \frac{1}{2}\mathcal{L}''(\bar{y}, \bar{u}, \bar{\lambda})[(y - \bar{y}, u - \bar{u})]^2 + \frac{\epsilon}{\rho}\|(u - \bar{u})_N\|^2_{L^2(Q_\epsilon)}. \quad (7.20)$$

Let us set $\delta u = u - \bar{u}$ and define δy to be the weak solution of the linearized equation with right-hand side δu. Similar as in the proof of Theorem 4.3, we obtain the estimate

$$J(y, u) - J(\bar{y}, \bar{u}) \geq \frac{1}{2}\mathcal{L}_{vv}(\bar{v}, \bar{\lambda})[(\delta y, \delta u)]^2 + \frac{\epsilon}{\rho}\|u - \bar{u}\|^2_{L^2, Q_\epsilon} + r_2$$

with a remainder term satisfying

$$\frac{|r_2|}{\|\delta u\|_2^2} \to 0 \quad \text{as } \|\delta u\|_2 \to 0.$$

In the next step, we want to apply the coercivity assumption (SSC). To this aim, we split δu in two components as follows:

$$\delta u = h_u + r_u,$$

where h_u and r_u are defined by

$$h_u = \begin{cases} \delta u_T & \text{on } Q_\epsilon \\ \delta u & \text{on } Q \setminus Q_\epsilon \end{cases}, \qquad r_u = \begin{cases} \delta u_N & \text{on } Q_\epsilon \\ 0 & \text{on } Q \setminus Q_\epsilon \end{cases}.$$

The functions h_u and r_u are measurable, see the discussion in Section 5.1 and the result of Theorem 7.23. Observe, that h_u and r_u are orthogonal, i.e. $(h_u, r_u)_Q = 0$. Moreover, it follows from the definition that the identity

$$\|r_u\|_p = \|(u - \bar{u})_N\|_{L^p(Q_\epsilon)} \tag{7.21}$$

holds. Similarly, we split $\delta y = h_y + r_y$, where h_y and r_y are solutions of the linearized systems with right-hand sides h_u and r_u respectively. We continue the investigation of the Lagrangian,

$$\mathcal{L}''(\bar{y}, \bar{u}, \bar{\lambda})[(\delta y, \delta u)]^2 = \mathcal{L}''(\bar{y}, \bar{u}, \bar{\lambda})[(h_y, h_u)]^2 + 2\,\mathcal{L}''(\bar{y}, \bar{u}, \bar{\lambda})[(h_y, h_u), (r_y, r_u)] \\ + \mathcal{L}''(\bar{y}, \bar{u}, \bar{\lambda})[(r_y, r_u)]^2. \tag{7.22}$$

Now, we can use (SSC) to obtain

$$\mathcal{L}''(\bar{y}, \bar{u}, \bar{\lambda})[(h_y, h_u)]^2 \geq \delta \|h_u\|_2^2. \tag{7.23}$$

Analogously to the box-constrained case, we get the following estimate

$$J(y, u) - J(\bar{y}, \bar{u}) \geq \frac{\delta}{8}\|\delta u\|_2^2 + \left(\frac{\epsilon}{\rho} - c\right)\|(u - \bar{u})_N\|^2_{L^2(Q_\epsilon)} + r_2.$$

Choosing ρ small enough, we finally find

$$J(y, u) - J(\bar{y}, \bar{u}) \geq \frac{\delta}{16}\|\delta u\|_2^2 = \frac{\delta}{16}\|u - \bar{u}\|_2^2.$$

Thus, we proved quadratic growth of the objective functional in a L^∞-neighborhood of the reference control. It implies the local optimality of the pair $(\bar{y}, \bar{u})$. $\square$

6 Stability of optimal controls

Usually, the fulfillment of a second-order sufficient condition implies stability of locally optimal controls under small perturbations. We demonstrated this connection for the box-constrained optimal control problem in Chapter 5. However, in the case of general convex control constraints the sufficient condition (SSC) seems to be too weak to get stability of optimal controls. This is due to the fact, that tangent variations of the control are not necessarily admissible directions, which was an essential ingredient in the proofs in Section 5.3.3.

In finite-dimensional optimization, there are a few publications concerning stability of solutions to convex constrained optimization problems, see [11, 13]. They use the assumption of second-order regular sets. The extension of that conditions to the infinite-dimensional case considered here is not obvious, since the proofs argue by contradiction and rely on the finite-dimensionality i.e. on compactness of the unit sphere. That means, one has to use methods which differ from the indirect methods of [11] as well as from the direct proof in Chapter 5.

Obviously, if we assume coercivity of $\mathcal{L}''$ for all test directions, we can prove such stability results. Since this would be only a technical exercise, we do not proceed in this direction. Hence, the question: 'Which conditions are sufficient for (Lipschitz) stability of optimal controls?' will remain unanswered in this thesis. And an answer would require a lot more research.

Chapter 8

Numerical results

Here, we will provide numerical results. They confirm the convergence theory of the SQP-method. However, this chapter contains more conjectures than certainties. There are almost no error estimates in the literature for constrained optimal control of the instationary Navier-Stokes equations. Furthermore, there is no rigorous analysis of the employed first-order time discretization scheme in connection with the Navier-Stokes equations. Not to mention the absence of any result for the adjoint system. So every claim about approximation rates is a result of conclusions by analogy.

1 Numerical solution of the Navier-Stokes equations

At first, let us say something about the approximate solution of the Navier-Stokes equations. In contrast to the analytic considerations, we are not working with divergence free trial functions for the velocity field. So, we have to approximate the pressure p, too. We denote by $L^2_0(\Omega)$ the space of L^2-functions p with $\int_\Omega p(x)\,\mathrm{d}x = 0$. The velocity field fulfills an equation in $L^2(V')$. If its right-hand side u is in the smaller space $L^2(0,T;H^1_0(\Omega)^2)^*$, then there is a pressure $p \in W^{-1,\infty}(0,T;H^1_0(\Omega)^*)$ such that the state equation $y_t + \nu\Delta y + (y\cdot\nabla)y + \nabla p = f$ is satisfied in the sense of $W^{1,1}(0,T;H^1_0(\Omega)^2)^*$, see Temam [70, p. 307]. The pressure is unique up to an additive constant. Moreover, if the right-hand side belongs to $L^2(Q)^2$, as it is the case if $y \in H^{2,1}$, then the pressure belongs to $L^2(0,T;H^1(\Omega)\cap L^2_0(\Omega))$.

Now, let us introduce the spatial discretization. To keep the presentation as simple as possible, we assume that Ω is a polyhedral and convex domain in $\mathbb{R}^2$. Let $\mathcal{T}_h$ be a regular triangulation of the domain Ω with the maximum mesh size $h = \max_{T\in\mathcal{T}_h} \mathrm{diam}(T)$.

The discretization in space is done using the Taylor-Hood element. That is, velocity and pressure are approximated by piecewise quadratic and piecewise linear functions, respectively. Here the spaces $H^1_0(\Omega)^2$ and $L^2_0(\Omega)$ are approximated by

$$V_h = \left\{ v_h \in C(\bar\Omega)^2 : v_h|_T \in P_2(T)^2 \quad \forall T \in \mathcal{T}_h, \quad v_h|_{\partial\Omega} = 0 \right\}$$

and

$$P_h = \left\{ p_h \in C(\bar\Omega) : p_h|_T \in P_1(T)^2 \quad \forall T \in \mathcal{T}_h, \quad \int_\Omega p(x)\,\mathrm{d}x = 0 \right\}.$$

Let us denote by y_h the solution of the semidiscrete i.e. spatially discretized and continuous-in-time state equation. It is known, that this scheme gives a convergence rate of h^2 in the $L^2(Q)^2$-norm of the error, see e.g. Girault and Raviart [28] and Heywood and Rannacher [36],

$$|y_h(t) - y(t)|_2 \leq C(t)h^2,$$

with $C(t)$ being bounded on the bounded time intervall $[0, T]$. The estimate is improved by Heywood and Rannacher [37] to

$$|y_h(t) - y(t)|_2 \leq Ch^3 t^{-1/2},$$

which shows an singular behaviour for $t \to 0$ due to the lack of a compability condition at $t = 0$. Altogether, we expect [5] an error estimate of the form

$$\|y_h - y\|_{L^2(Q)^2} \leq Ch^2.$$

Now, let us introduce the time-discretization scheme that we used for our computations. The time horizon is divided into equidistant intervals of length $\tau = T/N_t$. All quantities are approximated by piecewise constant functions over these subintervals. For time integration we use an semi-implicit scheme. Semi-implicit means, the nonlinearity is taken explicitly, whereas the linear terms are treated implicitly. In every time step, we have to solve the following problem

$$\left(\frac{1}{\tau} + \nu A\right) y^{k+1} + \nabla p^{k+1} = \frac{1}{\tau} y^k - (y^k \cdot \nabla) y^k + u^{k+1}$$
$$\operatorname{div} y^{k+1} = 0.$$

The advantage of this method is that in every step a symmetric problem has to be solved. If the nonlinear term is treated implicitly we have to solve nonlinear systems. Moreover, in Newtons method unsymmetric linear problems appear, since the coefficients depend on the actual iterate. On the other hand, this is only a first-order accurate scheme, hence we have to use small time steps to approximate the solution.

The solution to the fully discretized system we denote again by y_h. It is constructed as a piecewise constant function with values in V_h. We expect for the difference to the continuous solution an error estimate

$$\|y_h - y\|_{L^2(Q)^2} \leq C(\tau + h^2).$$

Let us remark, that the linearized as well as the adjoint equations are discretized in the same way: semi-implicit in time, piecewise quadratic in space.

Discretization of the control

Now, we will introduce the discretization of the control. Similar to the states and adjoints, the control is approximated by piecewise continuous functions in time. For the spatial discretization, there are several concepts at hand.

At first, we can use piecewise constant functions, which are constant over each triangle. Here, one can expect an approximation of the optimal control with order h in the L^2 as well as L^∞-norm, see Arada, Casas and Tröltzsch [5].

[5] Here and in the sequel 'expect' means: It is likely that the claim is true, and maybe someday someone will find a proof for it...

Secondly, we implemented piecewise linear and discontinuous controls. This approach gives three degrees of freedom for each triangle in the control domain. In our experiments, there were more control variables than state variables. Recently, an approximation order of $h^{3/2}$ was proven for an linear-quadratic elliptic control problem, see Rösch and Simon [64].

These two approaches have the following advantage: If the control constraint is fulfilled for the value in the barycentre respectively in the vertices of the element, then the constraint is fulfilled on the whole interior of the triangle.

Third, we tried also piecewise quadratic and continuous control functions. Here, it is not easy to check whether the control constraint is satisfied on the whole element or not. Therefore, we treated the control constraint as a constraint only in the values at the vertices and the midpoints of the edges. Deckelnick and Hinze [22] proved an approximation of order $\tau + h^2$ for unconstrained optimal control of the instationary Stokes equations. The same authors obtain a convergence rate of h^2 for the semidiscretization of the unconstrained control of the Navier-Stokes system in [23].

Besides that, we also investigated the effects of a post-processing step due to Meyer and Rösch [57]: the obtained adjoint state $\bar{\lambda}_h$ is used in a projection step to get an better approximation of the optimal control by $w_h = \mathrm{Proj}_{U_{ad}}(-1/\gamma\bar{\lambda}_h)$. As a rule of thumb, one obtains an approximation of the optimal control by this post-processed one as for the adjoint equation. In our case, we expect to observe $\tau + h^2$.

We did not use an implicit discretization of the control enforced by the discretization of the adjoint as proposed by Hinze [45]. There, the adjoint equation is discretized in the same way as the state equation, that means by piecewise quadratic functions in space. However, it is quite complicated to evaluate whether a quadratic function fulfills the control constraint or not. This is due to the fact, that the box constraint imposed on a quadratic function cannot be reduced to a box constraint on certain values at reference points.

2 An example

Here, we provide an example, where we know a solution of the optimality system. The computational domain was chosen to be the unit square $\Omega = (0,1)^2$ with final time $T = 1$. We want to minimize the functional

$$\frac{1}{2}\int_0^1 \int_\Omega |y(x,t) - y_d(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{2}\int_0^1 \int_\Omega |u(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t$$

subject to the instationary Navier-Stokes equations on $\Omega \times (0,1)$ with distributed control

$$\begin{array}{rcll} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p &=& u + f & \text{in } Q, \\ \operatorname{div} y &=& 0 & \text{in } Q, \\ y &=& 0 & \text{on } \Sigma, \\ y(0) &=& y_0 & \text{in } \Omega. \end{array}$$

and subject to some control constraints to be specified later on.

Let us construct a triple of state, control and adjoint, that satisfies the first-order optimality system. We chose as state and adjoint state

$$\bar{y}(x,t) = e^{-\nu t}\begin{pmatrix} \sin^2(\pi x_1)\sin(\pi x_2)\cos(\pi x_2) \\ -\sin^2(\pi x_2)\sin(\pi x_1)\cos(\pi x_1) \end{pmatrix}$$

and

$$\bar{\lambda}(x,t) = \left(e^{-\nu t} - e^{-\nu}\right) \begin{pmatrix} \sin^2(\pi x_1)\sin(\pi x_2)\cos(\pi x_2) \\ -\sin^2(\pi x_2)\sin(\pi x_1)\cos(\pi x_1) \end{pmatrix}.$$

Regardless of the choice of U_{ad}, the control is computed using the projection formula as

$$\bar{u} = \mathrm{Proj}_{U_{ad}}\left(-\frac{1}{\gamma}\bar{\lambda}(x,t)\right).$$

All other quantities are now chosen in such a way that $\bar{y}$ and $\bar{\lambda}$ are the solutions of the state and adjoint equations respectively:

$$\begin{aligned} f &= \bar{y}_t - \nu\Delta\bar{y} + (\bar{y}\cdot\nabla)\bar{y} - \bar{u}, \\ y_0 &= \bar{y}(0), \\ y_d &= \bar{y} - \left(-\bar{\lambda}_t - \nu\Delta\bar{\lambda} - (\bar{y}\cdot\nabla)\lambda + (\nabla\bar{y})^T\lambda\right). \end{aligned}$$

The parameters γ and ν are varied in two examples. At first, we set $\gamma = 1$ and $\nu = 1$. These are very nice values from a computational point of view: all constants involved in approximation as well as convergence estimates grow with negative powers of γ and ν. Secondly, we set $\gamma = 0.01$ and $\nu = 0.1$. That means, the Reynolds number $Re = 1/\nu$ is higher and the control cost γ is smaller, which should result in a worse behaviour of numerical methods: slower convergence and worse approximation.

The continuous problem was discretized using Taylor-Hood finite elements with different mesh sizes. The coarsest grid consists of 256 triangles with 545 velocity and 145 pressure nodes. Further, we use the semi-implicit Euler scheme for time integration with a equidistant time discretization also with different step lengths, where in the coarsest discretization we set $\tau_0 = 0.01$. The computations are based on a finite element code of M. Hinze, TU Dresden, see [44].

We computed solutions of the optimal control problem for different spatial and time discretizations. The coarse grid was refined uniformly, i.e. each triangle was divided into four congruent triangles. It leads to a reduction of the mesh size from h_0 to $h_1 := h_0/2$, and $h_2 := h_0/4$. The time step was shortened from $\tau_0 = 0.01$ to $\tau_1 := \tau_0/4 = 0.0025$, and $\tau_2 := \tau_0/16 = 0.000625$ to get a uniform reduction of the approximation errors connected with the spatial and time discretization. See also Table 8.1, where all these values are summarized.

	triangles	velocity nodes	pressure nodes	Mesh size h	Time step τ
Coarse	256	545	145	0.125	0.01
	1024	2113	545	0.0625	0.0025
Fine	4096	8321	2113	0.03125	0.000625

Table 8.1. *Discretization parameters*

The number of unknown state, adjoint and control variables can be found in Table 8.2. Each unknown corresponds to a vector in $\mathbb{R}^2$, so one has to double these numbers to get the total number of unknown reals. Each of that floating point numbers needs 8 bytes to store on a computer. One can imagine, that a solution of such an large optimization problem requires a lot of computing resources. Note, that the fine discretizations cause a large number of unknowns such that a further uniform refinement is almost impossible to handle. During the computation, one has to store 5 data arrays of the size of the entire velocity field, and also 5 arrays of the dimension of the discretized control. That means, for the solution of the problem on the fine mesh, we need about 1.5 Gigabyte to store all the necessary variables.

In Table 8.2, one can also see the effect of the three different discretization approaches for the control. The discontinuous linear and the continuous quadratic approximation gives a number of unknown control variables similar to the number of state variables.

	velocity field	control constant	linear	quadratic
Coarse	54.500	25.600	76.800	54.500
	845.200	409.600	1.228.800	845.200
Fine	13.313.600	6.553.600	19.660.800	13.313.600

Table 8.2. *Number of unknowns*

The arising discrete control problems are solved by the SQP-method without any globalization. The constrained SQP-subproblems (P^n), p. 75, were solved by a primal-dual active-set method, see for instance Kunisch and Rösch [50] using the method of conjugate gradients (CG) for the inner loop. Since those subproblems are linear-quadratic optimization problems, this active-set strategy can be interpreted as a semi-smooth Newton method, see Hintermüller, Ito, and Kunisch [42], to solve the non-smooth equation

$$u = \mathrm{Proj}_{U_{ad}}\left(-\frac{1}{\gamma}\lambda(u)\right),$$

compare (3.12). Here, $\lambda(u)$ denotes the adjoint state for a given control u of the SQP-subproblem (P^n). This method is known to converge locally with super-linear convergence rate [42, 78, 79] if the quadratic form $\mathcal{L}''$ is coercive i.e. a sufficient optimality condition holds. Under some strong assumptions it converges even globally [50].

In all examples, the stopping criteria of the nested methods are balanced in the following way as proposed by Hintermüller and Hinze [40]: The outer SQP-loop was terminated if two successive iterates are close enough,

$$\|y^n - y^{n-1}\|_\infty + \|u^n - u^{n-1}\|_\infty + \|\lambda^n - \lambda^{n-1}\|_\infty \le \varepsilon_{\mathrm{SQP}}.$$

In the example, where the solution is known, we chose another stopping criteria: the method was stopped if the norm of the difference of two successive iterates was much smaller than the norm of the difference of the actual iterate to the solution, i.e. if

$$\begin{aligned}\|y^n - y^{n-1}\|_\infty + \|u^n - u^{n-1}\|_\infty + \|\lambda^n - \lambda^{n-1}\|_\infty& \\ \le \varepsilon_{\mathrm{EX}}\left(\|y^n - \bar{y}\|_\infty + \|u^n - \bar{u}\|_\infty + \|\lambda^n - \bar{\lambda}\|_\infty\right)&\end{aligned}$$

was satisfied. We used $\varepsilon_{\mathrm{SQP}} = 10^{-6}$ and $\varepsilon_{\mathrm{EX}} = 10^{-1}$ in the computations.

The primal-dual active set method was stopped if either the active sets of two successive control iterates coincide or the error in the variational inequality given by

$$\phi(u) = \left\| u - \mathrm{Proj}_{U_{ad}}\left(-\frac{1}{\gamma}\lambda\right)\right\|_2$$

is reduced by a factor of 0.1. The innermost iteration procedure — the CG method — was stopped if the norm of the residual was reduced by a factor of 0.01. The initial guesses u^0 and y^0 for control and state were set to zero in all computations.

2.1 Comparison of the control discretization approaches

At first, we will show the results for the approximation of the optimal control by the different discretization approaches. These tests were done with $\nu = 1$ and $\gamma = 1$. The optimization problem was subject to the box constraints

$$|u_i(x,t)| \leq 0.05 \quad \text{a.e. on } Q,\ i = 1, 2.$$

In the following, we will see some plots of error norms for the different discretizations. There, the abscissae show the number of the discretizations, where '1' corresponds to the coarse and '3' to the finest discretization. The axis of ordinates corresponds to the logarithm of the value of the error norms. Auxiliary lines are plotted that visualizes the rates $h + \tau^r$ for $r = 1, 3/2, 2$.

Figure 8.1 shows the errors $\|\bar{u}_h - \bar{u}\|_2$ ($\triangle$) and $\|\bar{u}_h - \bar{u}\|_\infty$ ($\bigtriangledown$) for the three different discretization variants. We see a reduction of the error in both norms for constant trial functions (left) of the order of $\tau + h$. The control discretized by piecewise linear functions (middle) gives an approximation order $\tau + h^{3/2}$ in the L^2-norm and an order $\tau + h$ in the L^∞-norm. The discretization with quadratic polynomials yields an order of $\tau + h^{3/2}$ in the L^2-norm, which is less than the convergence order $\tau + h^2$ for the unconstrained problem for the Stokes equation from [22].

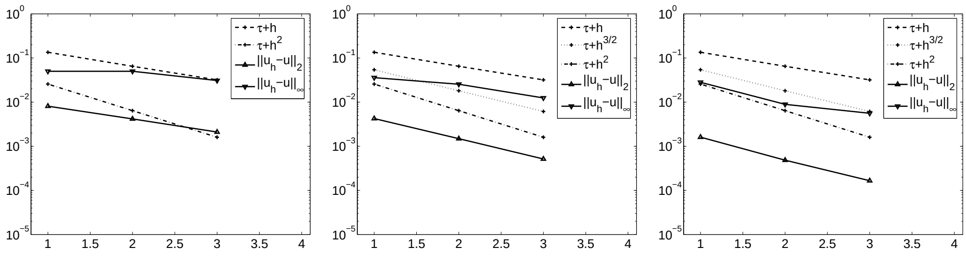

Figure 8.1. *Difference to solution: piecewise constant, linear, quadratic control*

In Figure 8.2 we see the same plot now for the post-processed control $w_h = \mathrm{Proj}_{U_{ad}}(-1/\gamma\bar{\lambda}_h)$. The behaviour of that control is pretty much the same for all three types of control discretization approaches, since the adjoint state $\bar{\lambda}$ is in all three cases approximated by $\bar{\lambda}_h$ with similar accuracy. It gives an error of order $\tau + h^2$ in both — the L^2 and the L^∞ — norms.

So we can give the following recommendation: use the piecewise constant discretization for the control, since it needs less unknown variables. After the com-

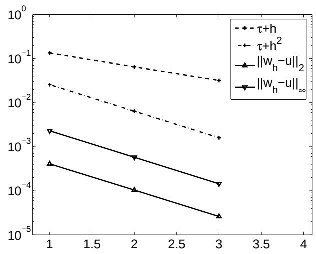

Figure 8.2. *Difference to solution: post-processed control*

putation of the discrete optimal control do a post-processing step, i.e. project the computed adjoint variable onto the admissible set.

2.2 SQP convergence

Now, let us have a look on the performance of the SQP method. Here, we use the same results as in the previous section. But we will now report about the single SQP-steps rather than the approximation of the final solution.

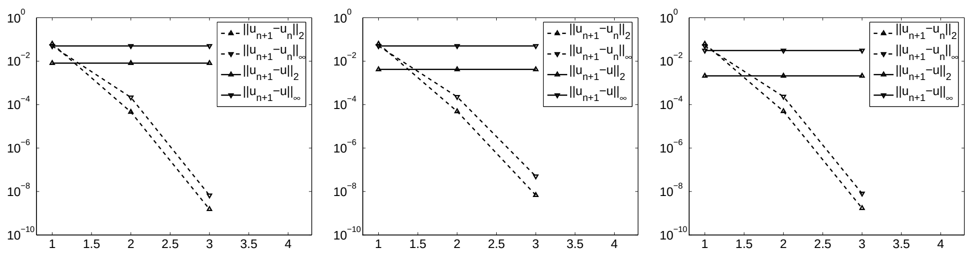

Figure 8.3. *SQP convergence: piecewise constant control; coarse, medium, fine discretization*

In Figure 8.3, we see the convergence history of the SQP-method for the three different discretizations. The solid line represents the L^2 and L^∞-norms of $u_n - \bar{u}$, where u_n is the actual SQP iterate. The dashed line plots the norm of $u_{n+1} - u_n$. As one can see, the method converges very fast towards the solution of the discrete problem. The convergence was almost quadratically, as expected. However, the overall approximation of the exact solution by the computed one did not improve after the first step! That means, it was not necessary to solve the discrete problem very accurately. One SQP-iteration was sufficient. This behaviour was observed also for tests with different parameters regardless of the used control trial space, see also Figure 8.4.

Here, the question is: What can be done to maintain a fast convergence to the exact solution? Maybe, one has to refine the discretization after each SQP-step. Since we observed a one-step convergence even for the finest discretization, it is not clear how the refinement should be done. These results underline the fast convergence of the SQP-method.

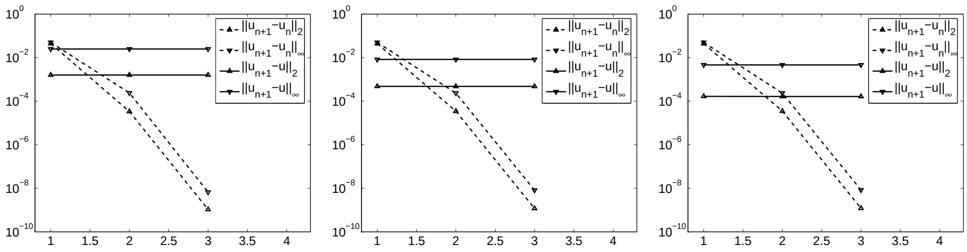

Figure 8.4. *SQP convergence: piecewise quadratic control; coarse, medium, fine discretization*

2.3 Convex control constraints

Here, we provide results for a convex constrained control problem. At first, let us explain, how we solve a convex constrained problem, i.e. an optimal control problem with the constraint $u(x,t) \in U(x,t)$ a.e. on Q. In every step of the SQP-method, an optimal control problem with linear state equation and convex control constraints has to be solved. The method was motivated in Chapter 6 by the generalized Newton step

$$0 \in F(x_n) + F'(x_n)(x - x_n) + N(x). \tag{8.1}$$

Only $N(x)$ represents the control constraint in that generalized equation. And it is the only term that was *not* linearized. Hence, the SQP-subproblem is subject to the same control constraint $u(x,t) \in U(x,t)$ as the original non-linear problem. That means, the control constraint is not linearized, even if it is written as an inequality like $u_1(x,t)^2 + u_2(x,t)^2 \le \rho(x,t)^2$.

Active-set algorithm

As for the box-constrained case we will use an active set algorithm. It is very similar to the primal-dual active-set strategy. The proposed algorithm tries to solve the projection representation of optimal controls.

Algorithm 8.1.

(i) Given u^n, y^n, λ^n. Determine the active set $\mathcal{A}^{n+1} = \mathcal{A}^{n+1}(\lambda^n)$:

$$\mathcal{A}^{n+1} := \left\{ (x,t) : -\frac{1}{\gamma}\lambda^n(x,t) \notin U(x,t) \right\}.$$

(ii) Minimize the functional J of the SQP-subproblem subject to the linearized state equation and to the control constraints

$$\left.\begin{array}{lcl} u|_{\mathcal{A}^{n+1}} & = & \mathrm{Proj}_{U(x,t)}\left(-\frac{1}{\gamma}\lambda^n(x,t)\right) \\ u|_{Q\setminus\mathcal{A}^{n+1}} & & \textit{free}. \end{array}\right\} \quad \Rightarrow \tilde{u},\ \tilde{y},\ \lambda^{n+1}$$

(iii) Project $u^{n+1} := \mathrm{Proj}_{U(x,t)}\left(-\frac{1}{\gamma}\lambda^{n+1}(x,t)\right)$, compute y^{n+1}

(iv) If $\left\| u^{n+1} - \mathrm{Proj}_{U_{ad}}\left(-\frac{1}{\gamma}\lambda^{n+1}\right)\right\|_2 < \epsilon \rightarrow$ *ready,*
else set $n := n+1$ *and go back to 1.*

In the first step of the algorithm, the active set is determined. The control constraint at a particular point (x,t) is considered active if $-1/\gamma\,\lambda^n(x,t)$ does not belong to $U(x,t)$. In the second step, a linear-quadratic optimization problem is solved. It involves no inequality constraints, since the control is fixed on the active set and the control is free on the inactive set. After that, the optimal adjoint state of that problem is projected on the admissible set to get the new control. The algorithm stops if the residual in the projection representation is small enough.

The algorithm is very similar to the primal-dual method for box-constrained problems. But there are some fine differences. Our algorithm uses only one active set $\mathcal{A}$. The primal-dual method exploits the fact that the box-constraints are formed by independent inequalities and therefore it uses active sets $\mathcal{A}_i$, $i = 1,2$ for each inequality.

Let us explain the behaviour of both algorithms in the detection of the active sets for the simple box constraint $|u_i| \le 1$, $i = 1,2$. Let us assume that $-1/\gamma\,\lambda^n(x_0,t_0) = (2,0)$, i.e. $-1/\gamma\,\lambda^n(x_0,t_0)$ is not admissible. Then in our method the point (x_0,t_0) will belong to the active set $\mathcal{A}^{n+1}$ in the next step. And at this point the value $(1,0)$ is prescribed for u^{n+1}: $u^{n+1}(x_0,t_0) = (1,0)$. In the primal-dual method, the point (x_0,t_0) will be added to the active set $\mathcal{A}_1^{n+1}$ associated to the inequality $|u_1| \le 1$. It will not belong to the active set $\mathcal{A}_2^{n+1}$ for the second inequality $|u_2| \le 1$ since this inequality was not violated. And for the inner problem we will get the constraints $u_1^{n+1}(x_0,t_0) = 1$ and $u_2^{n+1}(x_0,t_0) = free$, that is the control is allowed to vary in tangential directions on the right side of the box!

Furthermore, in the primal-dual active set method the new control iterate u^{n+1} was taken as the solution of the inner problem without projecting it. Then the lower constraint was taken to be active, if $u^n(x,t) + \frac{1}{c}m^n(x,t) < u_a(x,t)$ with the multiplier of the control constraint $m^n(x,t) = -(\gamma u^n(x,t) + \lambda^n(x,t))$. This works well, since in the box-constrained case there are only two possibilities of an active control constraint: upper and lower. Now, for convex constraints this is completely different. If we know that the constraint is active, what value should we prescribe on the active set? Here, we cannot take simply u_a or u_b. We have to compute a point on the boundary of the active set $U(x,t)$. To this end, the projection of $-1/\gamma\lambda$ is taken as the value of u on the active set.

The primal-dual method for linear-quadratic optimal control problems with box constraints is known to be equivalent the semi-smooth Newton method [42]. If we apply the semi-smooth Newton method to the non-smooth equation $u = \mathrm{Proj}_{U_{ad}}\left(-\frac{1}{\gamma}\lambda\right)$, we would end up with a different algorithm. As mentioned above, we have to allow tangential variations of the control on the active sets.

Example

Now, let us report about the results for convex control constraints compared to the box-constraints considered above. The parameters of the example are set to $\gamma = 0.01$ and $\nu = 0.1$. Thus, we use smaller values than in the previous sections. We computed solutions for the box constraint

$$|u_i(x,t)| \le 1.0$$

ϕ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$
$\psi(\phi)$	1.0	1.0	0.64	0.44	0.4	0.44	0.64	1.0

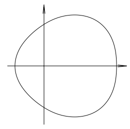

Table 8.3. *Convex constraint*

and for the convex constraint in polar coordinates

$$0 \leq u_r(x,t) \leq \psi(u_\phi(x,t)).$$

The function $\psi(\phi)$ is given as the spline interpolation with cubic splines and periodic boundary conditons of the function values in Table 8.3. The admissible control set is drawn on the right of Table 8.3. It is chosen such that the admissible set is comparable to the box constrained case.

The effect of the different shapes of the admissible set can be seen in Figure 8.5. There is a large part of the control constraints active. The convex constraint (right) gives a smoother control although the constraint is active on a large part of the domain. This is an advantage of using convex and smoother admissible sets than the box.

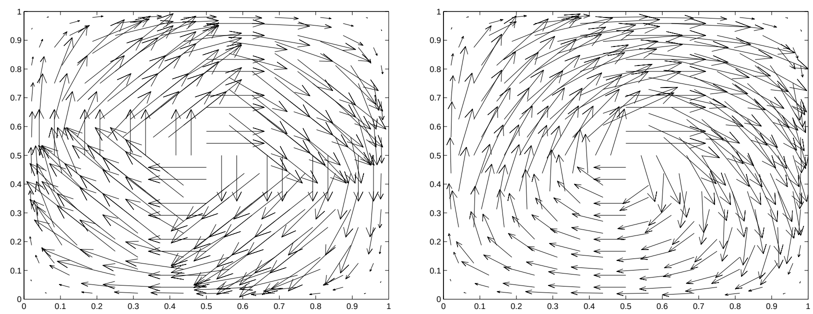

Figure 8.5. *Control snapshots at* $t = 0.05$

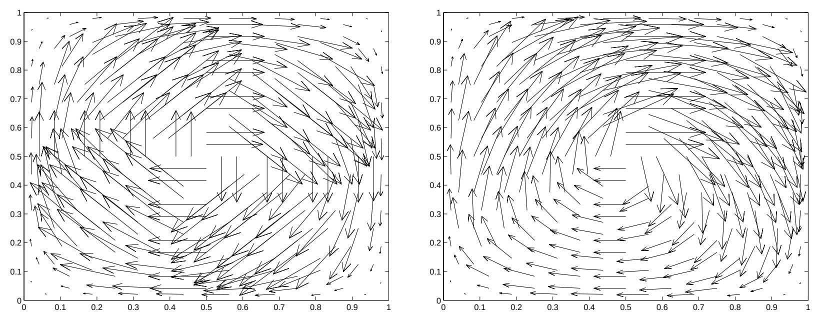

Figure 8.6. *Control snapshots at* $t = 0.2$

Figure 8.7 shows the approximation of the optimal control for the same optimization problem with box constraints and convex constraints respectively. Here, the control was approximated by piecewise constant functions. As one can see, the approximation order is not affected by the type of the constraint. The same holds for the post-processed control, see Figure 8.8.

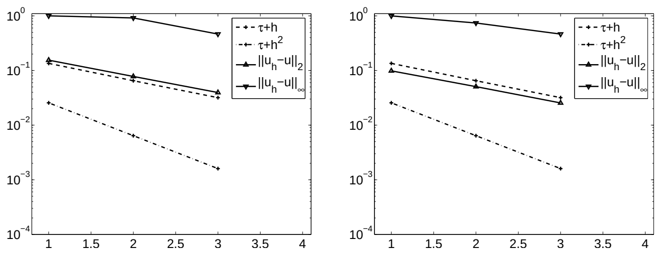

Figure 8.7. *Difference to solution: box constraints, convex constraints*

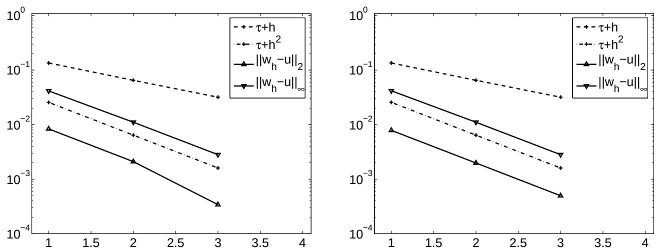

Figure 8.8. *Postprocessed control: box constraints, convex constraints*

Let us compare briefly the convergence of the active set algorithms: the primal-dual active set method for the box constrained problem and the active set method proposed above to solve the convex constrained problem. In all our computations both methods showed a similar behaviour. Although they solved optimal control problems with different control constraints, they needed almost the same number of outer and inner iterations, also the residual $u - \mathrm{Proj}(-1/\gamma\ \lambda)$ decreased in the same way. So, we can say that our algorithm is as efficient as the well-known primal dual active set method. There is also hope to give convergence proofs for our method analogously to the proofs in [39, 50].

3 Optimal control of the backward facing step

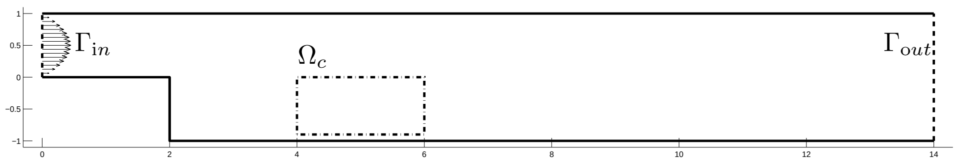

Figure 8.9. *Flow configuration*

We tested the SQP-algorithm for the tracking problem also for another flow configuration: Here the optimization goal is to reduce the recirculation bubble after the backward-facing step. We try this by minimizing the objective functional

$$J(y,u) = \frac{1}{2}\int_{Q_c} |y(x,t) - y_Q(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{2}\int_{Q_c} |u(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t$$

with $\gamma = 0.3$. The computational domain Ω is the backward-facing step. The time horizon is set to $T = 1$. Here, observation and control take place in the same part of the domain $Q_c = \Omega_c \times (0,T)$, compare Figure 8.9.

We chose the Stokes flow as desired flow y_Q, see Figure 8.10, which is the solution of the stationary Stokes equation with the same boundary conditions as used for the non-stationary simulation.

At the inflow boundary Γ_{in} a parabolic velocity profile is prescribed, whereas at the boundary Γ_{out} we use the 'do-nothing' boundary condition, cf. Heywood, Rannacher and Turek [38]:

$$\nu\frac{\partial y}{\partial n} - pn = 0 \quad \text{a.e. on } \Gamma_{out}.$$

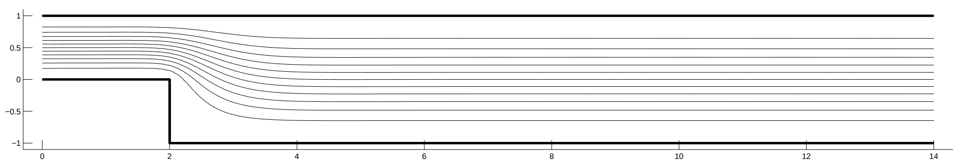

Figure 8.10. *Desired profile is the Stokes flow*

At the remaining part of the boundary we use homogeneous Dirichlet conditions. All computations were done with Reynolds number $Re = 400$, which yields a viscosity parameter $\nu = 1/400$. The initial velocity profile was chosen as the stationary limit of the uncontrolled Navier-Stokes equations, cf. Figure 8.11.

The control has to satisfy box constraints given by

$$|u_i(x,t)| \le 0.3 \quad \text{a.e. on } Q,\ i = 1,2.$$

We used the same FEM-implementation as in the previous cavity examples. The discretization parameters can be found in Table 8.4. We present results computed with three different spatial and temporal discretizations.

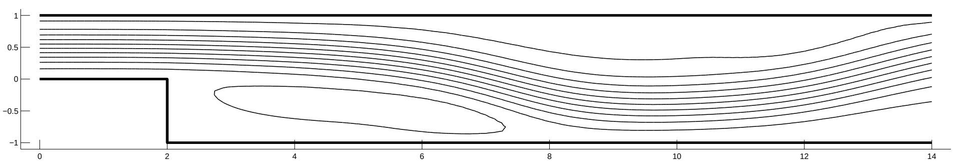

Figure 8.11. *Initial flow profile y_0*

	triangles	velocity nodes	pressure nodes	Mesh size h	Time step τ
Coarse	416	905	245	0.5	0.01
	1664	3473	905	0.25	0.0025
Fine	6656	13601	3473	0.125	0.000625

Table 8.4. *Discretization parameters*

The number of unknown control variables is $1,388,800$ for the finest discretization, whereas the number of state and adjoint variables is each about 21 millions. A further refinement will result in an optimization problem that is very expensive to solve. The computation on the finest discretization level lasted about 16 hours, after a uniform refinement the computation time will be several days. Some snapshots of the control can be seen in Figure 8.12.

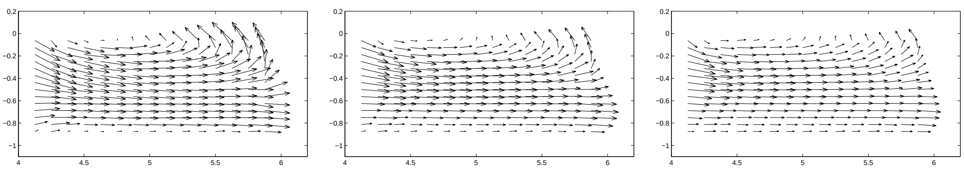

Figure 8.12. *Control at time instances $t_1 = 0.025$, $t_2 = 0.2$, $t_3 = 0.4$*

The streamlines of the terminal velocity $\bar{y}(T)$ field are depicted in Figure 8.13. The recirculation bubble after the step was reduced a bit compared to the initial profile in Figure 8.11. However, the recirculation is not supressed completely. This is due to the high control costs and to the box constraints, which were chosen such that the constraint is active on a significant part of the control domain.

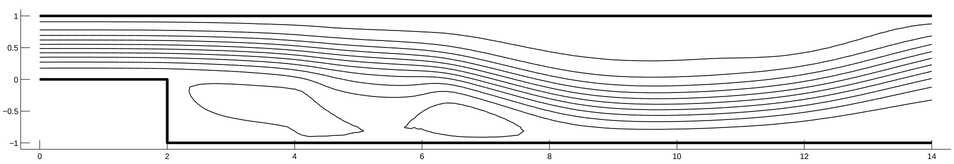

Figure 8.13. *Flow profile $\bar{y}(T)$*

Now, let us have a look on the convergence history of the SQP-method for the three discretizations. It can be found in Table 8.5. We give an estimation of the convergence speed of the method with respect to the L^2- and L^∞-norms by

$$q_2^n = \frac{\|u^n - u^{n-1}\|_2}{\|u^{n-1} - u^{n-2}\|_2^2}, \quad q_\infty^n = \frac{\|u^n - u^{n-1}\|_\infty}{\|u^{n-1} - u^{n-2}\|_\infty^2}.$$

The results for the two finest discretizations are close, so that one can see a mesh-independence behavior. The differences to the iterations on the coarsest are due to the fact that the coarsest spatial grid was too coarse, and the accuracy of solving the state equation was too low. Mesh-independence results state that the iteration rates do not depend on the mesh for discretizations that are sufficiently fine.

Grid	Iteration	$\|u^n - u^{n-1}\|_\infty$	q_2^n	q_∞^n
Coarse	1	$3.00 \cdot 10^{-1}$		
	2	$1.92 \cdot 10^{-1}$	$5.87 \cdot 10^{-1}$	$2.14 \cdot 10^{0}$
	3	$2.24 \cdot 10^{-2}$	$3.73 \cdot 10^{0}$	$6.06 \cdot 10^{-1}$
	4	$1.24 \cdot 10^{-3}$	$1.32 \cdot 10^{1}$	$2.47 \cdot 10^{0}$
	5	$4.24 \cdot 10^{-5}$	$8.98 \cdot 10^{2}$	$2.76 \cdot 10^{1}$
	1	$3.00 \cdot 10^{-1}$		
	2	$1.97 \cdot 10^{-1}$	$6.24 \cdot 10^{-1}$	$2.19 \cdot 10^{0}$
	3	$3.71 \cdot 10^{-2}$	$4.70 \cdot 10^{0}$	$9.54 \cdot 10^{-1}$
	4	$1.49 \cdot 10^{-3}$	$4.74 \cdot 10^{0}$	$1.08 \cdot 10^{0}$
	5	$5.67 \cdot 10^{-5}$	$2.76 \cdot 10^{2}$	$2.49 \cdot 10^{1}$
Fine	1	$3.00 \cdot 10^{-1}$		
	2	$2.31 \cdot 10^{-1}$	$5.90 \cdot 10^{-1}$	$2.56 \cdot 10^{0}$
	3	$2.54 \cdot 10^{-2}$	$2.61 \cdot 10^{0}$	$4.76 \cdot 10^{-1}$
	4	$1.24 \cdot 10^{-3}$	$1.01 \cdot 10^{1}$	$1.91 \cdot 10^{0}$
	5	$5.89 \cdot 10^{-5}$	$2.12 \cdot 10^{2}$	$3.84 \cdot 10^{1}$

Table 8.5. *SQP convergence*

Chapter 9

Conclusions

In the thesis we studied optimal control problems for instationary Navier-Stokes equations. Contributions have been made to various subjects of the problem. The following list summarize the main results and point to references, where some of the results has been published:

- We provided an L^p-theory for the linearized state and adjoint equations in Section 1.4, this result is contained in [81].
- In Chapter 4, we introduced a weak sufficient optimality conditon and proved its sufficiency, published in [77].
- The stability results in Chapter 5 are proven under that weak sufficient condition. They can be found in [81, 84].
- The convergence of the SQP-method was proven with the help of those stability results, see also [84].
- Chapter 7 is devoted to the study of pointwise convex control constraints, where the control constraints are described by a set-valued measurable mapping. There we proved regularity of optimal controls and gave a second-order sufficient optimality condition, which is a generalization of the condition used in the previous chapters. Some of the material in Chapter 7 can be found in [85, 82, 83].
- In the last Chapter 8, we provided numerical results for some selected problems. They confirm the convergence theory of the SQP method. Finally, we suggested a new active-set algorithm to solve convex constrained control problems and demonstrated its efficiency.

Chapter 10

Zusammenfassung

In dieser Arbeit untersuchten wir ein Optimalsteuerungsproblem für die instationäre Navier-Stokes-Gleichung. Die Schwerpunkte der Arbeit sind Optimalitätsbedingungen zweiter Ordnung und deren Anwendung sowie die Behandlung allgemeiner konvexer Steuerungsbeschränkungen.

Zu verschiedenen Aspekten des betrachteten Optimalsteuerungsproblems wurden neue Beiträge geleistet. So wurde in Abschnitt 1.4 eine Lösungstheorie für die linearisierten und die adjungierten Gleichungen in L^p-Räumen entwickelt. Diese wird benötigt, um Stabilität optimaler Steuerungen bezüglich der L^∞-Norm zu beweisen, siehe Kapitel 5. Eine weitere wichtige Voraussetzung dafür ist die schwächere hinreichende Optimalitätsbedingung aus Kapitel 4.

Aus diesen Ergebnissen folgt dann die lokale, quadratische Konvergenz des SQP-Verfahrens, siehe Kapitel 6. Dieses Verhalten wurde durch umfangreiche, numerische Tests bestätigt. In Kapitel 8 berichteten wir darüber.

In Kapitel 7 schließlich untersuchten wir allgemeine konvexe Beschränkungen an die Steuerung. Notwendige und hinreichende Optimalitätsbedingungen wurden dort bewiesen. Ein Algorithmus zur Lösung von Optimierungsproblemen mit solchen Beschränkungen wurde in Kapitel 8 vorgeschlagen und getestet.

Function spaces and norms

Space	Norm	Scalar product	Definition
$\mathbb{R}^n$	$\lvert \cdot \rvert$		
$L^p(\Omega)$	$\lvert \cdot \rvert_p$		2
$L^2(\Omega)$	$\lvert \cdot \rvert_2$	$(\cdot,\cdot)_2$	2
$W^{m,p}(\Omega)$	$\lvert \cdot \rvert_{m,p}$		3
$H^m(\Omega)$	$\lvert \cdot \rvert_{m,2}$	$(\cdot,\cdot)_{H^m}$	3
$W_0^{m,p}(\Omega)$			3
$H_0^m(\Omega)$			3
$\mathcal{D}(\Omega)$			3
$\mathcal{V}$			4
H	$\lvert \cdot \rvert_H$	$(\cdot,\cdot)_H$	4
V	$\lvert \cdot \rvert_V$	$(\cdot,\cdot)_V$	4
V'	$\lvert \cdot \rvert_{V'}$		4
H_p			5
V_p			5
$L^p(0,T;X)$	$\lVert \cdot \rVert_{L^p(X)}$		5
$W^\alpha(0,T;V)$	$\lVert \cdot \rVert_{W^\alpha}$		5
$W(0,T)$	$\lVert \cdot \rVert_W$		5
$W_p^{2,1}$			6
$W_0^{2-2/p,\,p}(\Omega)^2$	$\lvert \cdot \rvert_{W^{2-2/p,\,p}}$		6
$H^{2,1}$			6
W_0			18
Z	$\lVert \cdot \rVert_Z$		59
Z_p	$\lVert \cdot \rVert_{Z_p}$		66

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