# Approximation of conformal mappings by circle patterns and discrete minimal surfaces

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## Summary

To a rhombic embedding of a planar graph with quadrilateral faces and vertices colored black and white there is an associated isoradial circle pattern $\mathscr{C}_1$ with centers of circles at white vertices and radii equal to the edge length. Let $\mathscr{C}_2$ be another circle pattern such that the rhombi correspond to kites of intersecting circles with the same intersection angles. We consider the mapping $g_{\mathscr{C}}$ which maps the centers of circles and the intersection points to the corresponding points and which is an affine map on the rhombi.

Let $g$ be a locally injective holomorphic function. We specify the circle pattern $\mathscr{C}_2$ by prescribing the radii or the angles on the boundary corresponding to values of $\log g'$. We show that $g_{\mathscr{C}}$ approximates $g$ and its first derivative uniformly on compact subsets and that a suitably normalized sequence converges to $g$ if the radii of $\mathscr{C}_1$ converge to 0. In particular, we study the case that $\mathscr{C}_1$ is a quasicrystallic circle pattern, that is the number of different edge directions of the rhombic embedding is bounded by a fixed constant (for the whole sequence). For a class of such circle patterns we prove the convergence of discrete partial derivatives of arbitrary order to the corresponding continuous derivatives of $g$. For this purpose we use a discrete version of Hölder's inequality and a discrete regularity lemma for solutions of elliptic differential equations.

Furthermore, we consider the special case of regular circle patterns with the combinatorics of the square grid and two (different) intersection angles, which correspond to the two different edge directions. We show the uniqueness of the embedded infinite circle pattern (up to similarities) and prove an estimation for the quotients of radii of neighboring circles of such an (finite) circle pattern with error of order 1/combinatorial distance of the circle to the boundary. We also carry this result over to certain classes of quasicrystallic circle patterns. In addition, we study the $Z^\gamma$-circle patterns with the combinatorics of the square grid and regular intersection angles for $\gamma \in (0, 2)$. We prove the uniqueness (up to scaling) of such embedded circle patterns which cover a corresponding sector of the plane, subject to some conditions on the intersection angles and $\gamma$. Similar results are also shown for some classes of quasicrystallic $Z^\gamma$-circle patterns.

For the case of orthogonal circle patterns with the combinatorics of the square grid we consider the problem to approximate an homeomorphism of a square onto a kite which is conformal in the interior and maps the corner points of the square to the corner points of the kite. We prove uniform convergence on the square and convergence of all discrete derivatives on compact sets which do not contain any of the corner points. This result is generalized for other polygonal domains and stereographic projections of spherical polygonal domains which are bounded by arcs of great circles and contained in an open half-sphere of the unit sphere. As a consequence, we prove the convergence of $S$-isothermic discrete minimal surfaces to the corresponding smooth minimal surfaces away from nodal points. Furthermore, we construct examples of $S$-isothermic discrete minimal surfaces.

## Zusammenfassung

Zu einer rhombischen Einbettung eines planaren Graphen mit viereckigen Flächen und schwarz-weiß gefärbten Knoten gehört ein isoradiales Kreismuster $\mathscr{C}_1$ mit Mittelpunkten in den weißen Knoten und Radien gleich der Kantenlänge. Für ein weiteres Kreismuster $\mathscr{C}_2$, bei dem den Rhomben Drachen von sich schneidenden Kreisen mit denselben Schnittwinkeln entsprechen, betrachten wir die Abbildung $g_{\mathscr{C}}$, die entsprechende Mittelpunkte und Schnittpunkte der Kreismuster aufeinander abbildet und affin auf den Rhomben ist.

Für eine lokal injektive holomorphe Funktion $g$ bestimmen wir das Kreismuster $\mathscr{C}_2$ durch die Vorgabe von Radien oder Winkeln am Rand mit Hilfe von $\log g'$. Wir zeigen, dass $g_{\mathscr{C}}$ die Abbildung $g$ und ihre Ableitung gleichmäßig auf kompakten Teilmengen approximiert und eine geeignet normierte Folge solcher Abbildungen gegen $g$ konvergiert, falls die Radien von $\mathscr{C}_1$ gegen 0 konvergieren. Insbesondere untersuchen wir den Fall, dass $\mathscr{C}_1$ ein quasikristallisches Kreismuster ist, d.h. die Anzahl der verschiedenen Kantenrichtungen der rhombischen Einbettung ist durch eine feste Konstante beschränkt (für die gesamte Folge). Für eine Klasse solcher Kreismuster beweisen wir die Konvergenz diskreter partieller Ableitungen beliebiger Ordnung gegen die entsprechenden kontinuierlichen Ableitungen von $g$. Dafür verwenden wir eine diskrete Hölderungleichung und ein diskretes Regularitätslemma für Lösungen elliptischer Differentialgleichungen.

Außerdem betrachten wir den Spezialfall regelmäßiger Kreismuster mit Quadratgitterkombinatorik und zwei (verschiedenen) Schnittwinkeln, die den zwei Kantenrichtungen entprechen. Wir zeigen die Eindeutigkeit des eingebetteten unendlichen Kreismusters (bis auf Ähnlichkeitstransformationen) und beweisen eine Abschätzung für die Radienquotienten für benachbarte Kreise eines solchen (endlichen) Kreismuster mit Fehler der Ordnung 1/kombinatorischen Abstand der Kreise zum Rand. Dieses Ergebnis übertragen wir auch auf gewisse Klassen quasikristallischer Kreismuster. Ferner untersuchen wir die $Z^\gamma$-Kreismuster mit Quadratgitterkombinatorik und regelmäßigen Schnittwinkeln für $\gamma \in (0, 2)$. Wir beweisen die Eindeutigkeit (bis auf Skalierung) solcher eingebetteter Kreismuster, die einen entsprechenenden Sektor der Ebene überdecken, unter bestimmten Bedingungen an die Schnittwinkel und $\gamma$. Ähnliche Aussagen zeigen wir auch für einige Klassen quasikristallischer $Z^\gamma$-Kreismuster.

Für den Fall orthogonaler Kreismuster mit Quadratgitterkombinatorik betrachten wir das Problem, den im Inneren konformen Homeomorphismus eines Quadrates auf einen Drachen zu approximieren, der die Eckpunkte aufeinander abbildet. Wir beweisen gleichmäßige Konvergenz auf dem Quadrat und Konvergenz aller diskreter Ableitungen auf kompakten Mengen, die keinen der Eckpunkte enthalten. Dieses Ergebnis verallgemeinern wir für andere polygonale Gebiete und stereographische Projektionen sphärischer polygonaler Gebiete, die von Großkreisbögen begrenzt werden und in einer offenen Halbsphäre der Einheitssphäre liegen. Als Folgerung beweisen wir die Konvergenz von $S$-isothermen diskreten Minimalflächen außerhalb von Nabelpunkten gegen entsprechende glatte Minimalflächen. Des Weiteren konstruieren wir Beispiele von $S$-isothermen diskreten Minimalflächen.

# Contents

Chapter 1

# Introduction

## Circle packings and approximations of conformal mappings

An *embedded planar circle packing* is a configuration of closed disks with disjoint interiors in the plane $\mathbb{C}$. Connecting the centers of touching disks by straight lines, one obtains an embedded graph, the *tangency graph* of the circle packing, which has triangular faces as depicted in Figure 1.1. We only consider the case when the tangency graph triangulates a simply connected region of $\mathbb{C}$. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two circle packings whose tangency graphs are combinatorially the same. Then there is a mapping $g_{\mathscr{C}} : \mathscr{C}_1 \to \mathscr{C}_2$ which maps the centers of circles of $\mathscr{C}_1$ to the corresponding centers of circles of $\mathscr{C}_2$ and is an affine map on each triangular region corresponding to three mutually tangent circles.

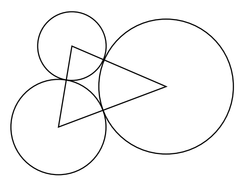

**Figure 1.1:** A triangular face of a circle packing.

Bill Thurston first introduced in his talk [73] on "The finite Riemann mapping Theorem" the idea to interprete $g_{\mathscr{C}}$ as a discrete analog and approximation of a conformal mapping. Here, a conformal map denotes a locally injective holomorphic function. In particular, take $\mathscr{C}_1$ to be a part of the regular hexagonal packing where all disks have the same radius and each disk has exactly six touching disks. Assume further that the disks of $\mathscr{C}_2$ are contained in the closed unit disk $\bar{U}$ such that all boundary disks are tangent to the boundary $\partial\bar{U}$. The existence of such a circle packing $\mathscr{C}_2$ is guaranteed by a theorem of Koebe [51]. Furthermore, normalize the packing $\mathscr{C}_2$ using suitable a Möbius transformation. Rodin and Sullivan proved Thurston's conjecture in [64]:

**Theorem 1.1** (Rodin and Sullivan)**.** *Under the above hypotheses, the mapping $g_{\mathscr{C}}$ approximates the Riemann mapping for the region covered by $\mathscr{C}_1$. The convergence is uniform on compact subsets of $U$, as the radii of the circles of the regular hexagonal circle pattern go to zero.*

This result has then been extended and modified in several directions, revealing further connections between circle packings and classical complex analysis. A beautiful introduction and surway is presented by Stephenson in [72]. As this thesis is concerned a lot with convergence, we only mention three results belonging to this topic.

In [44] He and Schramm generalized the uniform convergence to general simply connected domains and general circle packings, thereby giving a different proof of the Rodin-Sullivan-Theorem 1.1.

**Theorem 1.2** (He and Schramm)**.** *Let $D, \tilde{D} \subset \mathbb{C}$ be two simply connected bounded domains and let $p_0 \in D$. For each $n \in \mathbb{N}$ let $\mathscr{C}_1^{(n)}$ and $\mathscr{C}_2^{(n)}$ be two circle packings whose tangency graphs are combinatorially the same and such that the packings are contained in $D$ and $\tilde{D}$ respectively. Denote by $g_n = g_{\mathscr{C}^{(n)}}$ the corresponding mappings as above.*

*Let $\varepsilon_n$ be a sequence of positive numbers, $\varepsilon_n \to 0$. Assume that the radii of the circles of $\mathscr{C}_1^{(n)}$ are less than $\varepsilon_n$, that each boundary circle of $\mathscr{C}_1^{(n)}$ is within a distance $\varepsilon_n$ of $\partial D$, and that each boundary circle of $\mathscr{C}_2^{(n)}$ is within a distance $\varepsilon_n$ of $\partial\tilde{D}$. Suppose that $g_n(p_0)$ is defined for all $n$ and lies in a fixed compact subset of $\tilde{D}$.*

*Then there exists a subsequence $\{g_{n_j}\}_{j\in\mathbb{N}}$ which converges uniformly on compact subsets to a conformal homeomorphism $g : D \to \tilde{D}$.*

Using the additional structure of the regular hexagonal circle packing, He and Schramm also improved the order of convergence in [45].

**Theorem 1.3** (He and Schramm)**.** *The mappings $g_{\mathscr{C}}$ considered in the Rodin-Sullivan-Theorem 1.1 converge in $C^\infty$ uniformly on compact subsets.*

Thurston's idea as well as Theorems 1.1–1.3 are based on a rather geometric approach to prescribe the image circle packing $\mathscr{C}_2$ by some suitable boundary conditions. There is another natural analytic way to specify boundary values. To approximate a given (locally) injective holomorphic function $g$, the values of $g$ and its derivatives at boundary points (for example centers of circles or/and intersection points) can be used to assign boundary values (for example radii of boundary circles) which then determine the circle packing $\mathscr{C}_2$. Of course, the existence of $\mathscr{C}_2$ has to be proven. To this end, the definition of a circle packing is generalized to allow overlapping disks, as long as for each interior vertex the disks corresponding to this vertex and its neighbors have disjoint interiors. Carter and Rodin proved in [24] that the corresponding mapping $g_{\mathscr{C}}$ then approximates the given mapping $g$.

**Theorem 1.4** (Carter and Rodin)**.** *Let $D \subset \mathbb{C}$ be a bounded simply connected region. Let $g$ be a locally injective holomorphic function defined on an open neighborhood of $\bar{D}$. Let $\varepsilon_n$ be a sequence of positive numbers, $\varepsilon_n \to 0$. For each $n \in \mathbb{N}$ let $\mathscr{C}_1^{(n)}$ be a part of the regular hexagonal circle patterns with radius $\varepsilon_n$ such that the region covered by the triangular cells of $\mathscr{C}_1^{(n)}$ is simply connected and all boundary vertices have a distance less than $\varepsilon_n$ to the boundary $\partial D$. Define radii at boundary vertices $v$ by $r_n(v) = \varepsilon_n |g'(v)|$.*

*Then there is a corresponding circle packing $\mathscr{C}_2^{(n)}$ whose boundary circles have these radii $r_n(v)$. Suitably normalizing $\mathscr{C}_2^{(n)}$ by translation and rotation, the mappings $g_{\mathscr{C}^{(n)}}$ converge to $g$ uniformly on compacta of $D$. Furthermore, the quotient of radii of corresponding circles of $\mathscr{C}_1^{(n)}$ and $\mathscr{C}_2^{(n)}$ converges to $|g'|$ uniformly on compacta of $D$.*

Another way to generalize the Rodin-Sullivan-Theorem 1.1 is to consider circle patterns instead of circle packings. Note that for each circle packing there is an associated orthogonal circle pattern. Simply add a circle for each triangular face which passes through the three touching points.

**'Definition'.** Let $G$ be an embedded connected planar graph, possibly with boundary, and let $\alpha : E(G) \to (0, \pi)$ be a labelling on the edges $E(G)$. We always assume that for all edges incident to an interior face $f$ of $G$ we have

$$\sum_{e \text{ incident to } f} \alpha(e) = 2\pi.$$

An *immersed planar circle pattern* with adjacency graph $G$ and intersection angles $\alpha$ is a collection of circles for each vertex, such that the following conditions hold.

(1) For each edge $[u,v] \in E(G)$, the two circles associated to $u, v \in V(G)$ intersect with exterior intersection angle $\alpha([u,v])$, as in Figure 1.2.

(2) The circles corresponding to the vertices adjacent to the same face of $G$ intersect in one point.

(3) Consider a counterclockwise cyclic order of the intersection points from (2) on the circle corresponding to an interior vertex $v \in V_{int}(G)$. This order agrees with the counterclockwise cyclic order of the cycle of faces of $G$ adjacent to $v$.

A circle pattern is called *isoradial* if all its circles have the same radius. A circle pattern is called *embedded* if the interiors of the kites which are associated to intersecting circles for edges $[u,v] \in E(G)$ as in Figure 1.2 are disjoint.

Another slightly different definition of circle patterns as well as results on existence and uniqueness and further characterizations and properties are presented in Chapter 2–4.

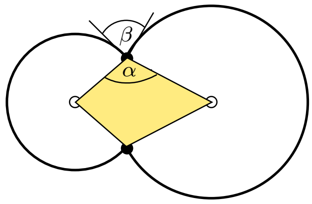


**Figure 1.2:** The exterior intersection angle $\alpha$ and the interior intersection angle $\beta = \pi - \alpha$ of two intersecting circles, and the associated kite built from centers and intersection points.

## Quasicrystallic circle patterns

Embedded isoradial circle patterns are closely related to rhombic embeddings of graphs with quadrilateral faces whose vertices are bicolored (called *b-quad-graphs*), as the kites associated to adjacent vertices as in Figure 1.2 are all rhombi. Also, given a rhombic embedding of a b-quad-graph a corresponding isoradial circle pattern is obtained by adding circles with centers at the vertices corresponding to one color and radii equal to the edge length. Thus to a rhombic embedding there are two associated isoradial circle patterns.

The notion *quasicrystallic* for a rhombic embedding, as introduced by Bobenko, Mercat, and Suris in [14], refers to the property that the number of different edge directions of the rhombic embedding is bounded. Note that this number is related to the number of different intersection angles of a corresponding isoradial circle pattern. Quasicrystallic rhombic embeddings of b-quad-graphs contain a lot of additional structure. One way to describe their regularity is to use the correspondence of such a rhombic embedding to a two-dimensional subcomplex (*combinatorial surface*) $\Omega$ of $\mathbb{Z}^d$ which is locally homeomorphic to the unit disk. The dimension $d$ depends only on the number of different edge directions of the rhombic embedding. This correspondence is presented and discussed in [14] and reveals a connection of quasicrystallic isoradial circle patterns and integrable systems. Examples for rhombic embeddings of b-quad-graphs (and associated isoradial circle patterns) are classical Penrose tilings (see Figures 2.3(c)) or generalizations of this construction (see Figure 1.5 (left) for a very symmetric example and Section 3.2 for examples of general construction schemes). Furthermore, a circle pattern is called *quasicrystallic* if there is an isoradial circle pattern with the same combinatorics and the same intersection angles such that the associated rhombic embedding is quasicrystallic.

The combinatorial surface $\Omega$ is especially important because of its connection with integrability. More precisely, assume that we have given a function on the vertices of $\Omega$ which satisfies a 3D-consistent equation on all faces of $\Omega$. In the case of quasicrystallic circle patterns, this function gives the radii and the rotations of the edge stars at intersection points when changing from an isoradial circle pattern with the same combinatorics and intersection angles to the given circle pattern. The Hirota Equation (2.16) or (3.12), which is 3D-consistent, encodes the closing condition for the kites built from centers of circles and intersection points for circles corresponding to incident vertices. *3D-consistency* means that if there are equations on the two-dimensional faces of a three dimensional cube and we have given values of a function on seven vertices such that the equations are satisfied for each of the three faces with values on its four vertices, then three values for the eighth vertex of the cube can be calculated using the equations on the remaining faces and all these values agree. This is the case for the Hirota equation and implies that we can extend the given function of radii and rotations at intersection points to the convex hull of $\Omega$ in $\mathbb{Z}^d$. In this way we obtain quasicrystallic circle patterns with (locally) different combinatorics. See also Section 3.4. A more detailed and deepened study of consistency and integrability is presented by Bobenko and Suris in [18].

In order to obtain a locally different rhombic embedding of a b-quad-graph (and thus a locally different quasicrystallic circle pattern) we can also consider *flips*. For each interior vertex of the corresponding combinatorial surface $\Omega$ which is incident to exactly three two-dimensional facets of $\Omega$ we replace these facets by the three other faces lying on the same three-dimensional cube; see Figure 1.3 for an illustration. Some further aspect of flips are studied in Section 3.5.

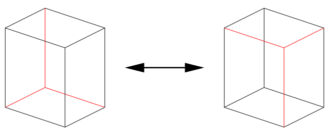

**Figure 1.3:** A flip of a three-dimensional cube. The red edges are not part of the surface in $\mathbb{Z}^d$.

(Quasicrystallic) rhombic embeddings of b-quad-graphs are also closely related to the linear theory of discrete holomorphic functions; see [40, 34, 35, 54, 14]. Some definitions are summarized in Section 5.4. Kenyon derived in [49] an asymptotic development for the discrete Green's function. In Section 3.3 we slightly improve Kenyon's result in order to show a discrete analog for Hölder's Inequality and a Regularity Lemma for solutions of discrete elliptic equations.

Note that given a rhombic embedding of a b-quad-graph, we can define two *associated graphs* $G$ *and* $G^*$ by taking all vertices of one of the two colors as vertices of $G$ and the remaining vertices for $G^*$. Two vertices are connected by an edge in $G$ (or in $G^*$) if they are incident to the same face of the b-quad-graph. Define an associated labelling on these edges to be the angle of the face of rhombic embedding at the vertices which are not incident to this edge.

The following definition of a Laplacian for functions on the vertices $V(G)$ can also be found in [14].

**Definition.** Consider a rhombic embedding of a b-quad-graph with associated graphs $G$ and $G^*$. For an edge $e = [z_1, z_2] \in E(G)$ denote by $l(e)$ the distance of the points corresponding to $z_1$ and $z_2$ in the rhombic embedding. Similarly, we define $l(e^*)$ for an edge $e^* \in E(G^*)$. Note that these are the lengths of the diagonals of the rhombi.

Let $\eta : V(G) \to \mathbb{R}$ be a function. Define a *discrete Laplacian* of $\eta$ at an interior vertex $z_0 \in V_{int}(G)$ with incident vertices $z_1, \dots, z_m$ in $G$ by

$$\Delta\eta(z) := \sum_{j=1}^{m} \frac{l(e_j^*)}{l([z_0, z_j])} (\eta(z_j) - \eta(z_0)). \tag{1.1}$$

where the edge $e_j^* \in E(G^*)$ corresponds to the same rhombus as the edge $[z_0, z_j]$.

**Theorem 1.5.** *Let $u : V(G) \to \mathbb{R}$ be a non-negative harmonic function on the vertices of the graph $G$ associated to a quasicrystallic rhombic embedding of a b-quad-graph $\mathscr{D}$. Let $x_0 \in V(G)$ be an interior vertex. There is a constant $B_1$ independent of $u$ such that*

$$|u(x_0) - u(x_1)| \le B_1 u(x_0)/\rho$$

*for all vertices $x_1 \in V(G)$ incident to $x_0$, where $\rho$ denotes the combinatorial distance of $x_0$ to the boundary of $G$.*

**Lemma 1.6.** *Let $\mathscr{D}$ be a b-quad-graph with a quasicrystallic rhombic embedding and associated graph $G$. Let $x_0 \in V(G)$ be an interior vertex. Let $W \subset V(G)$ and let $u : W \to \mathbb{R}$ be any function. Set $\|u\|_W = \max_{v\in W} |u(v)|$ and $M(u) = \max_{v\in W_{int}} |\Delta u(v)/(4F^*(v))|$, where $F^*(v)$ is half of the sum of the areas of the rhombi incident to the vertex $v$. There are constants $B_2, B_3 > 0$ independent of $W$ and $u$ such that*

$$|u(x_0) - u(x_1)|\rho \le B_2\|u\|_W + \rho^2 B_3 M(u)$$

*for all vertices $x_1 \in W$ incident to $x_0$, where $\rho$ is the combinatorial distance of $x_0$ to the boundary $W_\partial$ of $W$.*

An interesting special case of isoradial circle patterns are regular circle patterns with the combinatorics of the square grid $\mathbb{Z}^2$. There are only two different intersection angles $\psi \in (0, \pi)$ and $\pi - \psi$ such that parallel edges of the square grid $\mathbb{Z}^2$ carry the same angle. Circle patterns for $\mathbb{Z}^2$ and this regular labelling $\alpha_\psi$ are called *SG-circle patterns.* Schramm studied in [67] orthogonal $SG$-circle patterns, that is $\psi = \pi/2 = \pi - \psi$. He defined suitable Möbius invariants $\tau, \sigma \in \mathbb{R}^+$ using cross-ratios of intersection points, such that $\log\tau + i\log\sigma$ is an analog of the Schwarzian derivative $S_g = (g''/g')' - \frac{1}{2}(g''/g')^2$. Schramm proved Cauchy-Riemann-type equations for $\tau$ and $\sigma$, a nonlinear Laplace-type equation for $\tau$, and existence and uniqueness results for orthogonal $SG$-circle patterns with prescribed values of $\tau$ at the boundary. These definitions and results are generalized for $SG$-circle patterns in Section 3.6.1. In particular, we deduce the following uniqueness theorem for embedded $SG$-circle patterns.

**Theorem 1.7** (Rigidity of $SG$-circle patterns)**.** *Suppose that $\mathscr{C}$ is an embedded planar circle pattern for $\mathbb{Z}^2$ and $\alpha_\psi$. Then $\mathscr{C}$ is the image of a regular isoradial circle pattern for $\mathbb{Z}^2$ and $\alpha_\psi$ with radius 1 under a similarity.*

In Section 3.7, we generalize this result for a class of quasicrystallic circle patterns.

**Theorem 1.8** (Rigidity of quasicrystallic circle patterns)**.** *Let $\mathscr{D}$ be an infinite connected and simply connected quasicrystallic rhombic embedding which covers the whole complex plane. Assume further that for the corresponding combinatorial surface $\Omega_{\mathscr{D}}$ in $\mathbb{Z}^d$ there are at least two different indices $j_1, j_2 \in \{1, \dots, d\}$ such that $\min_{\mathbf{n}\in V(\Omega_{\mathscr{D}})} n_{j_k} = -\infty$ and $\max_{\mathbf{n}\in V(\Omega_{\mathscr{D}})} n_{j_k} = \infty$ holds for $k = 1, 2$. Let $G$ be an associated graph and let $\alpha$ be the associated labelling. Let $\mathscr{C}$ be an embedded circle pattern for $G$ and $\alpha$. Then $\mathscr{C}$ is the image of an isoradial circle pattern for $G$ and $\alpha$ under a similarity of the complex plane.*

Consider a circle of an embedded $SG$-circle pattern. Assume that this circle is surrounded by $n$ *generations* of circles for some $n \geq 3$, that is the combinatorial distance of the corresponding vertex in the adjacency graph $G$ to the boundary is at least $n$. Let $q_n$ be the maximum of the quotients of radii of this circle and its intersecting circles corresponding to an incident vertex. Denote by $1 + s_n$ the supremum of $q_n$ for all such embedded circle patterns with $n$ surrounding generations. From the uniqueness result of Theorem 1.7, we can easily deduce that $s_n$ decreases to 0. In addition, we have the following estimation on the rate of convergence.

**Theorem 1.9.** *There is a constant $A$, which depends only on $\psi$, but not on $n$, such that $s_n \leq A/n$ for all $n \geq 3$.*

Our proof given in Section 3.6.2 adapts arguments by Aharonov [5, 6] of the corresponding result for hexagonal circle packings and uses discrete potential theory, in particular Hölder's Inequality 1.5 and the Regularity Lemma 1.6, and results on the convergence of quasiconformal mappings. We also generalize Theorem 1.9 for some classes of quasicrystallic circle patterns in Corollary 3.49.

### Convergence of circle patterns

Given two circle patterns $\mathscr{C}_1$ and $\mathscr{C}_2$ with the same underlying combinatorics for the adjacency graph $G$ and the same labelling of the edges $\alpha$, a mapping $g_{\mathscr{C}} : \mathscr{C}_1 \to \mathscr{C}_2$ can be constructed in a similar way as for circle packings. Namely, take $g_{\mathscr{C}}$ to map the centers of circles and the intersection points of $\mathscr{C}_1$ corresponding to vertices and faces of $G$ to the corresponding centers of circles and intersection points of $\mathscr{C}_2$ and extend it to an affine map on each kite (see Figure 1.2). Examples are indicated in Figures 1.4 and 1.5.

As for circle packings, the natural question arises "Do the circle patterns, or more precisely, do the mappings $g_{\mathscr{C}}$ approximate a holomorphic function?" In Chapters 5 and 7 we study this approximation for some classes of isoradial circle patterns $\mathscr{C}_1$.

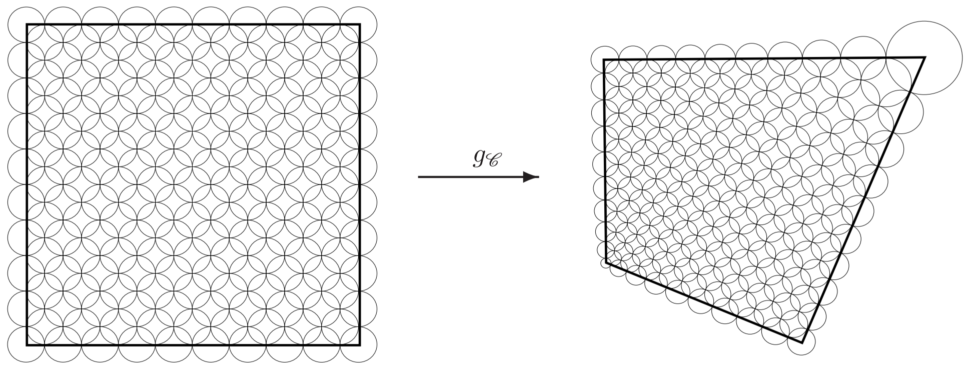


**Figure 1.4:** An example of two corresponding orthogonal circle patterns filling a square and a kite-shaped convex quadrilateral respectively.

*Convergence for orthogonal circle patterns with the combinatorics of the square grid*

First, focus on the special class of orthogonal $SG$-circle patterns, that is $\alpha \equiv \pi/2$. Figure 1.4 shows two examples.

Beginning with the more geometric viewpoint, consider the following variant of the Riemann mapping and its approximation by circle patterns. Fix a symmetric bounded convex Euclidean quadrilateral $\mathcal{Q}$ with straight boundary edges, that is, a convex kite. Let $\mathscr{C}_1$ be a part of an isoradial orthogonal $SG$-circle pattern such that the union of closed kites, corresponding to intersecting circles of $\mathscr{C}_1$, fills a square $\mathcal{R}$ with edge length one; see Figure 1.4 (left) for an example. Denote the four boundary vertices of $\mathcal{R}$ and $\mathcal{Q}$ enumerated in counterclockwise order by $C_1^{\mathcal{R}}, C_2^{\mathcal{R}}, C_3^{\mathcal{R}}, C_4^{\mathcal{R}}$ and $C_1^{\mathcal{Q}}, C_2^{\mathcal{Q}}, C_3^{\mathcal{Q}}, C_4^{\mathcal{Q}}$ respectively. Fix $p \in \{1,2,3,4\}$ and define a bijective mapping $g_{corn}$ by $g_{corn}(C_j^{\mathcal{R}}) = C^{\mathcal{Q}}_{(j+p) \pmod 4}$ for $j = 1,2,3,4$. Let $\mathscr{C}_2$ be the embedded orthogonal circle pattern with the same combinatorics as $\mathscr{C}_1$ such that the centers of the circles corresponding to the boundary vertices of $\mathscr{C}_1$ lie on the edges of $\mathcal{Q}$ and such that $g_{\mathscr{C}}$ agrees with $g_{corn}$ at the corner points of $\mathcal{R}$; see Figure 1.4 (right) for an example. The existence of $\mathscr{C}_2$ is guaranteed by a theorem of Bobenko and Springborn [16]; see also Theorem 2.25.

**Theorem 1.10.** *The mapping $g_{\mathscr{C}}$ approximates the conformal homeomorphism $g$ of the closed unit square $\mathcal{R}$ onto the convex hull of the convex kite $\mathcal{Q}$ such that $g$ agrees with $g_{corn}$ at the corner points of $\mathcal{R}$. The convergence is uniform on $\mathcal{R}$ and in $C^\infty$ on compact sets $K$ of $\mathbb{C}$ such that $K \subset \mathcal{R} \setminus \{corner\ points\}$. Furthermore, the quotients of corresponding radii converge to $|g'|$ in $C^\infty$ uniformly on compact sets $K$ of $\mathbb{C}$ such that $K \subset \mathcal{R}\setminus\{corner\ points\}$.*

This theorem can be generalized to some regions $\mathcal{Q}'$ bounded by Euclidean polygons if the unit square is substituted by a suitable polygonal region $\mathcal{R}'$ whose boundary edges are all parallel to the real or imaginary axis of $\mathbb{C}$ or to one of the two diagonals. Furthermore $\mathcal{R}'$ has to be chosen such that there is a conformal homeomorphism $g : \mathcal{R}' \to \mathcal{Q}'$ mapping the corner points of $\mathcal{R}'$ to the corresponding corner points of $\mathcal{Q}'$ (possibly, there are additional corner points on the edges of $\mathcal{Q}'$). The region $\mathcal{Q}'$ can also be the stereographic projection of a suitable spherical polygon (i.e. a polygon in $\mathbb{S}^2$ whose edges are parts of great circles) which is contained in the interior of one half-sphere.

Theorem 1.10 is motivated by its application to the convergence of discrete minimal surfaces to their smooth analogs. More precisely, we consider $S$-isothermic discrete minimal surfaces. $S$-isothermic surfaces have been discovered by Bobenko and Pinkall in [15]. The class of $S$-isothermic discrete minimal surfaces has been studied by Bobenko, Hoffmann, and Springborn in [13]. A short introduction is presented in Appendix A. Applying this concept in Section 8.2, we have constructed examples of discrete minimal analogs to triply-period minimal surfaces and to minimal surfaces spanned by polygonal boundary frames. Some aspects of the construction scheme are studied in Section 8.1. The main ingredient always consists of an orthogonal $SG$-circle pattern of a region region $\mathcal{Q}'$ which is the stereographic projection of a spherical polygon, often a spherical kite. As $S$-isothermic discrete minimal surfaces have a Weierstrass-type representation in terms of these circle patterns (cf. Theorem A.9 and [13]), the convergence result follows immediately from the generalized version of Theorem 1.10 and is presented in Section 8.3.

The proof of Theorem 1.10 is specified in Section 7.1 (and in 7.2 for the generalization) and adapts reasonings of the proofs for the similar Theorems 1.1 and 1.3 for hexagonal circle packings. First we show that the angles of the circle pattern $\mathscr{C}_2$ are uniformly bounded away from 0 and $\pi/2$, independently of the number of circles. Thus $g_{\mathscr{C}}$ is a quasiconformal mapping. Furthermore, we prove that the quotients of radii of intersecting circles converge to one. This implies the uniform convergence of $g_{\mathscr{C}}$ to $g$. Next, we use Theorem 1.9 and the Regularity Lemma 1.6 and deduce that the discrete derivatives of the logarithm of the radii of $\mathscr{C}_2$ converge to the corresponding smooth derivatives of $\log|g'|$.

In order to deal with the circle patterns in a neighborhood of the corner points, we use the $Z^\gamma$-circle patterns with square grid combinatorics as reference patterns. These have been discovered by Bobenko [9] and were studied further by Agafonov and Bobenko [4, 1, 2]. Note that similar circle patterns have also been investigated for hexagonal combinatorics by Bobenko and Hoffmann in [12]. The definition of the $Z^\gamma$-circle patterns as well as some of the known properties are resumed in Section 4.1. To meet our requirements, we summarize and extend some results concerning bounds on the radii and on the distances of the centers of circles to the origin in Section 4.2. Moreover, Theorem 1.9 and well-known results from discrete potential theory imply the uniqueness of the $Z^\gamma$-circle patterns; see Section 4.3.

**Theorem 1.11** (Rigidity of orthogonal $Z^\gamma$-circle patterns)**.** *Let $\gamma \in (0,2)$. Let $\mathscr{C}$ be an orthogonal embedded circle pattern with the same combinatorics as the $Z^\gamma$-circle pattern $\mathscr{C}_\gamma$. Assume that the kites built from centers and intersection points and corresponding to intersecting circles cover the same infinite sector bounded by two half-lines which intersect at the origin and enclose an angle $\gamma\pi/2$ for both circle patterns. Assume further that the centers of the boundary circles lie on the same boundary half lines. Then $\mathscr{C}$ is obtained from $\mathscr{C}_\gamma$ by scaling.*

In Section 4.5 we generalize Theorem 1.11 for certain classes of quasicrystallic $Z^\gamma$-circle patterns.

*Convergence for isoradial and quasicrystallic circle patterns*

As for circle packings, we may also consider the more analytic viewpoint and obtain approximating circle patterns by specifying suitable boundary values.

Using the Möbius invariants $\tau$ and $\sigma$ defined for orthogonal $SG$-circle patterns, Schramm proved in [67] the following convergence theorem, which is similar to Theorem 1.4.

**Theorem 1.12** (Schramm)**.** *Let $D \subset \mathbb{C}$ be a bounded simply connected region. Let $g$ be a locally injective meromorphic function defined on an open neighborhood of $\bar{D}$. Let $\varepsilon_n$ be a sequence of positive numbers, $\varepsilon_n \to 0$. For each $n \in \mathbb{N}$ let $\mathscr{C}_1^{(n)}$ be a part of an isoradial orthogonal $SG$-circle pattern with radius $\varepsilon_n$ such that the region covered by the orthogonal kites (in fact squares) of $\mathscr{C}_1^{(n)}$ is simply connected and all boundary vertices have a distance less than $2\varepsilon_n$ to the boundary $\partial D$. Define the Möbius invariant $\tau_n$ at boundary vertices $v$ by $\tau_n(v) = 1 + \varepsilon_n^2 \operatorname{Re}(S_g(v))$, where $S_g = (g''/g')' - \frac{1}{2}(g''/g')^2$ denotes the Schwarzian derivative of $g$.*

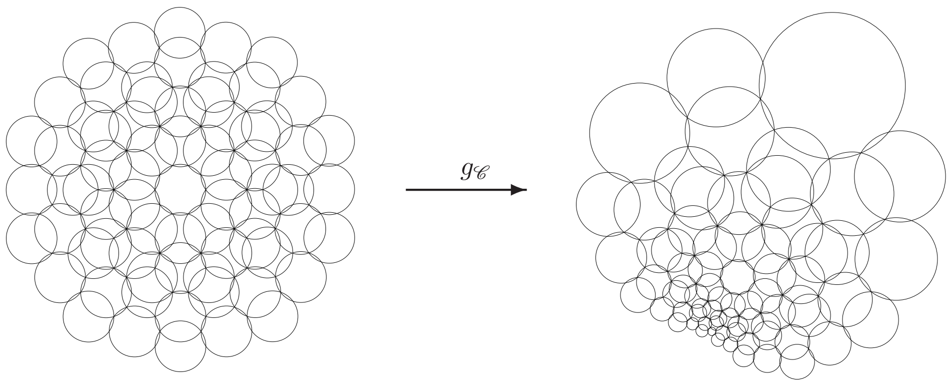


**Figure 1.5:** An example of the approximating map $g_{\mathscr{C}}$ for an isoradial circle pattern.

*Then there is a corresponding orthogonal SG-circle pattern $\mathscr{C}_2^{(n)}$ whose Möbius invariant $\tau_n$ has these boundary values. Considering a suitable normalization of $\mathscr{C}_2^{(n)}$ by a Möbius transformation, the mappings $g_{\mathscr{C}^{(n)}}$ converge to $g$ uniformly on compacta of $D$ with errors of order $\varepsilon_n$. The circle patterns also approximate $g'$ and $g''$ uniformly on compacta of $D$ with errors of order $\varepsilon_n$.*

In Chapter 5 we consider a modification of this theorem for (Euclidean) boundary conditions specifying radii or angles, which is similar to the setting of Carter and Rodin in Theorem 1.4. Furthermore, we generalize the conditions for $\mathscr{C}_1$ and admit isoradial circle patterns, whose intersection angles are uniformly bounded away from 0 and $\pi$. An example is given in Figure 1.5 (left).

**Theorem 1.13.** *Let $D \subset \mathbb{C}$ be a bounded simply connected region. Let $g$ be a locally injective holomorphic function defined on an open neighborhood of $\bar{D}$. Let $\varepsilon_n$ be a sequence of positive numbers, $\varepsilon_n \to 0$. For each $n \in \mathbb{N}$ let $\mathscr{C}_1^{(n)}$ be an embedded isoradial circle pattern with radius $\varepsilon_n$ such that the region covered by all kites (in fact rhombi) of $\mathscr{C}_1^{(n)}$ is simply connected and all boundary vertices have a distance less than $\hat{C}\varepsilon_n$ to the boundary $\partial D$, where $\hat{C}$ is some positive constant independent of $n$. Assume that the intersection angles $\alpha_n$ of $\mathscr{C}_1^{(n)}$ are uniformly bounded, that is $|\alpha_n(e) - \pi/2| < C$ for all edges $e$ and some constant $C \in (0, \pi/2)$ independent of $n$. Define radii at boundary vertices $v$ by $r_n(v) = \varepsilon_n |g'(v)|$.*

*Then there exists a corresponding circle pattern $\mathscr{C}_2^{(n)}$ whose boundary circles have radii $r_n(v)$. Suitably normalizing $\mathscr{C}_2^{(n)}$ by translations and rotations, the mappings $g_{\mathscr{C}^{(n)}}$ converge to $g$ uniformly on compacta of $D$. Furthermore, the quotient of radii of corresponding circles of $\mathscr{C}_1^{(n)}$ and $\mathscr{C}_2^{(n)}$ converges to $|g'|$ uniformly on compacta of $D$. The order of the approximation error is $1/\sqrt{-\log \varepsilon_n}$.*

There is a corresponding version specifying angles at boundary vertices according to $\arg g'$, see Theorem 5.10.

The proof of Theorem 1.13 is presented in Section 5.1 (and in 5.2 for Neumann boundary conditions) and uses ideas of Schramm's proof of Theorem 1.12. The main idea is to interpret the closing condition (condition (3) in the above definition) for circle patterns as a nonlinear discrete Laplace equation for the radius function. This equation turns out to be a (good) approximation of a linear Laplace equation and can be used in the case of isoradial circle patterns to compare discrete and smooth solutions of the corresponding elliptic problems, that is the logarithm of the radii of $\mathscr{C}_2^{(n)}$ and $\log |g'| = \mathrm{Re}(\log g')$.

Using an estimation for superharmonic functions by Saloff-Coste [65], we deduce that the rotation angles of the edge stars at intersection points, when changing from the rhombic embedding corresponding to $\mathscr{C}_1^{(n)}$ to the corresponding immersion of kites corresponding to $\mathscr{C}_2^{(n)}$, approximate $\arg g' = \operatorname{Im}(\log g')$.

As in the case of circle packings in Theorem 1.3, the convergence can be improved to $C^\infty$ on compacta if additional structural properties of the circle patterns $\mathscr{C}_1^{(n)}$ are exploited. This is for example the case for 'very regular' isoradial circle patterns, like isoradial $SG$-circle patterns, which are part of an infinite embedded isoradial circle pattern filling the whole plane and which are invariant under translation along any of the edges of the adjacency graph $G$. But the ideas of the proof of Theorem 1.3 can also be adapted for a class of quasicrystallic isoradial circle patterns. We additionally need an estimation on the set of vertices which can be reached by flips from the given combinatorial surface associated to the quasicrystallic rhombic embedding.

**Theorem 1.14.** *Suppose that the isoradial circle patterns $\mathscr{C}_1^{(n)}$ considered in Theorem 1.13 are additionally quasicrystallic with bound on the number of different edge directions of the corresponding rhombic embeddings independent of $n$. Assume further that one of the constants $C_{J_0}(\mathcal{F}(\Omega_{\mathscr{D}_n}))$ introduced in Definition 5.15 is bounded from below independently of $n$. Then the mappings $g_{\mathscr{C}^{(n)}}$ converge to $g$ in $C^\infty$ uniformly on compacta of $D$. The order of the approximation error is $\varepsilon_n$.*

The main idea of the proof is to use the additional regularity of quasicrystallic circle patterns. In particular, we use the relation of a finite embedded isoradial circle pattern $\mathscr{C}_1^{(n)}$, which is always quasicrystallic, to a rhombic embedding of a b-quad-graph and to a corresponding combinatorial surface $\Omega$ in $\mathbb{Z}^d$. Next, we benefit from the integrability to extend the radius function to a suitable neighborhood of $\Omega$ in $\mathbb{Z}^d$ whose size depends on the constant $C_{J_0}(\mathcal{F}(\Omega_{\mathscr{D}_n}))$. This enables us to define discrete derivatives of the radius function. The connection of the closing condition for the circle pattern $\mathscr{C}_2^{(n)}$ to a linear Laplacian and the Regularity Lemma 1.6 together with our assumption on the extension of this neighborhood of $\Omega$ then allow us to prove convergence by similar arguments as in the proof of Theorems 1.13 and 1.3.

For the sake of completeness, we state and prove in Chapter 6 the convergence result corresponding to Theorem 1.14 for isoradial circle packings. Note that isoradial circle packings are necessarily parts of the regular hexagonal packing.

## Open questions

The various connections between circle patterns and classical complex analysis are far from being fully understood. In particular, the general question on convergence stated above is not completely answered. Related to the topics of this thesis we note the following interesting unsolved aspects.

First, the convergence of $S$-isothermic discrete minimal surfaces has only been shown excluding a neighborhood of any umbilic point. In order to treat the convergence in a neighborhood of such a point by similar arguments as for regular points, we need uniform convergence of the first derivatives of the orthogonal $SG$-circle patterns at the corresponding point. To cope with this task, a further study of the orthogonal $Z^\gamma$-circle patterns seems to be necessary. In particular, it would be sufficient to prove an extension of Theorem 1.9 in the following sense. Let $\gamma \in (0, 2)$ and consider a finite, simply connected part of the $Z^\gamma$-circle pattern. Compare the quotients of radii of two intersecting circles of this part with the corresponding quotient of radii of an embedded orthogonal $SG$-circle pattern with the same combinatorics and the same straight lines as boundary conditions as this part of the $Z^\gamma$-circle pattern, and estimate the difference.

Furthermore, it would be interesting to generalize the rigidity results 1.11 and 1.8 to all quasicrystallic $Z^\gamma$-circle pattern and all embedded quasicrystallic circle patterns covering the whole complex plane.

Related to the analytic viewpoint, there remains the important challenge to prove convergence for a setting similar to Theorem 1.13, but for more general embedded non-isoradial initial circle patterns $\mathscr{C}_1$. As an application, a discrete holomorphic approximation to a given locally injective holomorphic map $g$ could then be defined given only a set of points in $D$ by using the the associated Delaunay circle pattern.

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Chapter 2

# Circle patterns

## 2.1 Definitions and existence

Let $\mathscr{D}$ be a finite cell decomposition of a bounded domain in $\mathbb{C}$ with boundary such that all 2-cells (faces) are embedded and counterclockwise oriented. We assume that $\mathscr{D}$ is strongly regular. A cell decomposition is called *strongly regular* if all the characteristic maps, that map closed disks onto the closed cells, are homeomorphisms and the intersection of two closed cells is empty or equal to a single closed cell. If all faces of $\mathscr{D}$ are quadrilaterals, that is if there are exactly four edges incident to each face, $\mathscr{D}$ is called a *quad-graph.* A quad-graph $\mathscr{D}$ whose 1-skeleton is a bipartite graph is called *b-quad-graph.* In particular, a quad-graph whose 1-skeleton is simply connected, is also a b-quad-graph. $\mathscr{D}$ is called *simply connected* if $\mathscr{D}$ the cell decomposition of a simply connected domain of $\mathbb{C}$ and if every closed chain of faces is null homotopic in $\mathscr{D}$.

Given a b-quad-graph $\mathscr{D}$, we will always assume that its vertices are colored black and white. From these two sets of vertices we construct two associated planar graphs $G$ and $G^*$ as follows. The vertices $V(G)$ are all white vertices of $V(\mathscr{D})$. The edges $E(G)$ correspond to faces of $\mathscr{D}$, that is two vertices of $G$ are connected by an edge if and only if the corresponding vertices of $\mathscr{D}$ are incident to the same face. The dual graph $G^*$ is constructed similarly by taking as vertices $V(G^*)$ all black vertices. The edges $E(G^*)$ correspond to faces of $\mathscr{D}$ as in the case of $E(G)$.

In the following, we will make frequent use of a labelling $\alpha : F(\mathscr{D}) \to (0, \pi)$ of the faces of $\mathscr{D}$. By abuse of notation, this labelling can also be understood as a function $\alpha : E(G) \to (0, \pi)$ or $\alpha : E(G^*) \to (0, \pi)$. We will always assume, that the labelling $\alpha$ satisfies the following condition at all interior black vertices $v \in V_{int}(G^*)$:

$$\sum_{f \text{ incident to } v} \alpha(f) = 2\pi. \tag{2.1}$$

In the sequel, such a labelling will be called *admissible.*

We now state a definition of circle patterns which is a different formulation of the definition given in the introduction.

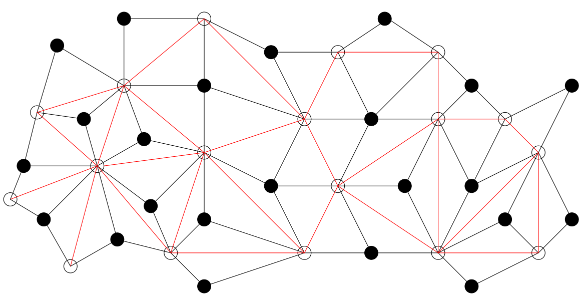


**Figure 2.1:** An example of a b-quad-graph $\mathscr{D}$ (black edges and bicolored vertices) and its associated graph $G$ (red edges and white vertices).

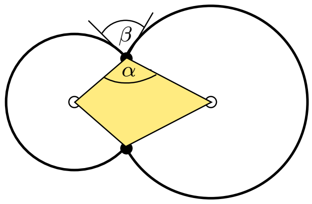



**Figure 2.2:** The exterior intersection angle $\alpha$ of two intersecting circles and the associated kite built from centers and intersection points.

**Definition 2.1.** Let $\mathscr{D}$ be a b-quad-graph and let $\alpha : E(G) \to (0,\pi)$ be an admissible labelling. An *(immersed planar) circle pattern for $\mathscr{D}$ and $\alpha$* are an indexed collection $\mathscr{C} = \{C_z : z \in V(G)\}$ of oriented circles in $\mathbb{C}$ and an indexed collection $\mathscr{K} = \{K_e : e \in E(G)\} = \{K_f : f \in F(\mathscr{D})\}$ of oriented closed kites, such that the following conditions hold.

(1) All circles and all kites carry the same orientation.

(2) If $z_1, z_2 \in V(G)$ are incident vertices in $G$, that is $[z_1, z_2] \in E(G)$, then the corresponding circles $C_{z_1}, C_{z_2}$ intersect with exterior intersection angle $\alpha([z_1, z_2])$. Furthermore, the kite $K_{[z_1,z_2]}$ is bounded by the centers of the circles $C_{z_1}, C_{z_2}$, the two intersection points, and the corresponding edges, as in Figure 2.2.

The intersection points are associated to black vertices of $V(\mathscr{D})$ or to vertices of $V(G^*)$.

(3) If two faces are incident in $\mathscr{D}$, then the corresponding kites have one edge in common. If both kites only have angles strictly less than $\pi$, then they share exactly one edge.

(4) Let $f_1, \dots, f_n \in F(\mathscr{D})$ be all the faces incident to an interior vertex $v \in V_{int}(\mathscr{D})$. Then all corresponding kites $K_{f_1}, \dots, K_{f_n}$ have mutually disjoint interiors and their boundaries all contain a point $p(v)$ corresponding to $v$. The union $U = K_{f_1} \cup \dots \cup K_{f_n}$ is homeomorphic to a closed disc and $p(v)$ is an interior point of $U$.

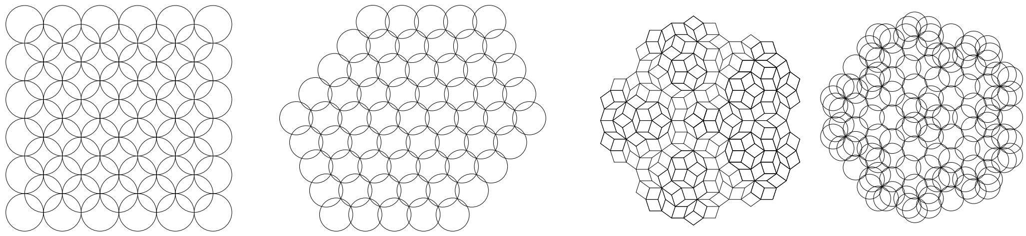
The circle pattern is called *embedded* if all kites of $\mathscr{K}$ have mutually disjoint interiors. The circle pattern is called *isoradial* if all circles of $\mathscr{C}$ have the same radius.

There are also other definitions for circle patterns, for example associated to a *Delaunay decomposition* of a domain in $\mathbb{C}$. This is a cell decomposition such that the boundary of each face is a polygon with straight edges which is inscribed in a circular disk, and these disks have no vertices in their interior. Note that the corresponding circle pattern can be associated to the graph $G^*$. The Poincaré-dual decomposition of a Delaunay decomposition with the centers of the circles as vertices and straight edges is a *Dirichlet decomposition* (or *Voronoi diagram*) and can be associated to the graph $G$.

Furthermore the definition of circle patterns can be extended allowing cone-like singularities in the vertices; see [16] and the references therein.

Note that we associate a circle pattern to an immersion of the kite pattern corresponding to $\mathscr{D}$ where the edges incident to the same white vertex are of equal length. The kites may also be non-convex if the angle at one of the white vertices is larger than $\pi$. As the kites can be constructed from a (suitable) given set of circles and from the combinatorics of $G$, we will often only consider the circles $\mathscr{C}$. Also, we call $\mathscr{C}$ an (immersed planar) circle pattern for $G$ and $\alpha$.

**Remark 2.2** (Isoradial circle patterns and rhombic embeddings)**.** An embedded isoradial circle pattern for the b-quad-graph $\mathscr{D}$ always leads to a rhombic embedding of $\mathscr{D}$, as all kites are rhombi. In particular, all kites are convex. Isoradial circle patterns contain additional structure and will be studied further in Chapter 3.



(a) A regular square grid circle pattern.

(b) A regular hexagonal circle pattern.

(c) A rhombic embedding (a part of a Penrose tiling) and a corresponding isoradial circle pattern.

**Figure 2.3:** Examples of isoradial circle patterns.

Conversely, given a rhombic embedding of a b-quad-graph $\mathscr{D}$, add circles with centers at all white points and with radii equal to the edge length of the rhombi. The black points will then be the intersection points of neighboring circles, i.e. circles corresponding to white points which are incident to the same face of $\mathscr{D}$. This gives rise to an isoradial circle pattern for $\mathscr{D}$. Figure 2.3(c) shows an example.

Given a b-quad-graph and an admissible labelling, a natural question is whether there exists any circle pattern for this data. To formulate an answer we introduce the notion of a coherent angle system.

**Definition 2.3.** Let $\mathscr{D}$ be a b-quad-graph and let $G$ be the associated graph constructed from white vertices. Let $\alpha : E(G) \to (0, \pi)$ be an admissible labelling. Denote by $\vec{E}(G)$ the oriented edges of $E(G)$, that is each edge $e = [u, v] \in E(G)$ is replaced by the two oriented edges $\vec{e} = \overrightarrow{uv}$ and $-\vec{e} = \overrightarrow{vu}$. A *coherent angle system* is a map $\phi : \vec{E}(G) \to \mathbb{R}$ which satisfies the following conditions.

(i) $\phi(\vec{e}) > 0$ for all oriented edges $\vec{e} \in \vec{E}(G)$.

(ii) $2 \sum_{\overrightarrow{uv} \in \vec{E}(G)} \phi(\overrightarrow{uv}) = 2\pi$ for all interior vertices $u \in V_{int}(G)$.

(iii) $\phi(\overrightarrow{uv}) + \phi(\overrightarrow{vu}) = \pi - \alpha([u, v])$ for all edges $[u, v] \in E(G)$.

A necessary and sufficient condition for the existence of a circle pattern is now given by the following theorem. More general versions and a proof can by found in [16].

**Theorem 2.4.** *Let $\mathscr{D}$ be a b-quad-graph with associated graph $G$ constructed from the white vertices. Suppose exterior intersection angles are prescribed by an admissible labelling $\alpha : E(G) \to (0, \pi)$. Then a planar circle pattern for $\mathscr{D}$ and $\alpha$ exists if and only if there is a coherent angle system for $G$ and $\alpha$.*

*Furthermore, there exists a isoradial planar circle pattern for $\mathscr{D}$ and $\alpha$ if and only if for every interior vertex $u \in V_{int}(G)$ the following condition holds:*

$$\sum_{[u,v] \in E(G)} (\pi - \alpha([u, v])) = 2\pi.$$

*If there exists an isoradial planar circle pattern for $G$ and $\alpha$, there is also a isoradial planar circle pattern for $G^*$ and $\alpha^* = \pi - \alpha$, which is called* conjugate circle pattern.

**Examples.** Figures 1.4, 1.5, and 2.3 show some examples of circle patterns. Examples of isoradial circle patterns include regular patterns like the regular square grid or hexagonal circle pattern as well as more general examples like circle pattens corresponding to a Penrose tiling; see Figure 2.3.

## 2.2 THE RADIUS FUNCTION

For the study of a planar circle pattern $\mathscr{C}$ we will frequently use the radius function $r_{\mathscr{C}} = r$ which assigns to every vertex $z \in V(G)$ the radius $r_{\mathscr{C}}(z) = r(z)$ of the corresponding circle $C_z$. The index $\mathscr{C}$ will be dropped whenever there is no confusion likely.

The following proposition specifies a condition for a radius function to originate from a planar circle pattern.

**Proposition 2.5.** *(1) Let $G$ be a graph constructed from a b-quad-graph $\mathscr{D}$ and let $\alpha$ be an admissible labelling. Suppose that $\mathscr{C}$ is a planar circle pattern for $\mathscr{D}$ and $\alpha$ with radius function $r = r_{\mathscr{C}}$. Then for every interior vertex $z_0 \in V_{int}(G)$ we have*

$$\left( \sum_{[z,z_0]\in E(G)} f_{\alpha([z,z_0])}(\log r(z) - \log r(z_0)) \right) - \pi = 0, \tag{2.2}$$

*where*

$$f_\theta(x) := \frac{1}{2i} \log \frac{1-e^{x-i\theta}}{1-e^{x+i\theta}},$$

*and the branch of the logarithm is chosen such that $0 < f_\theta(x) < \pi$.*

*In the special case of an orthogonal circle pattern, where $G$ is a part of the square grid, condition (2.2) can be expressed as*

$$\sum_{[z,z_0]\in E(G)} \arctan \tfrac{r(z)}{r(z_0)} = \pi \tag{2.3}$$

*or equivalently as*

$$r(z_0) = H(r(z_1), r(z_2), r(z_3), r(z_4)),$$

*where $z_1, z_2, z_3, z_4$ are incident to $z_0$ and*

$$H(r_1, r_2, r_3, r_4) = \sqrt{\frac{(r_1^{-1} + r_2^{-1} + r_3^{-1} + r_4^{-1})r_1r_2r_3r_4}{r_1 + r_2 + r_3 + r_4}}.$$

*(2) Let $\mathscr{D}$ be a simply connected b-quad-graph with associated graph $G$ and let $\alpha$ be an admissible labelling of $E(G)$.*

*Suppose that $r : V(G) \to (0, \infty)$ satisfies (2.2) for every $z \in V_{int}(G)$. Then there is a planar circle pattern for $G$ and $\alpha$ such that* radius$(C_z) = r(z)$ *for all $z \in V(G)$. This pattern is unique up to isometries of $\mathbb{C}$.*

A proof can be found in [16], and in [67] for the special case of orthogonal circle patterns with the combinatorics of a square grid, which will also be called *orthogonal SG-circle patterns.* $2f_{\alpha([z,z_0])}(\log r(z) - \log r(z_0))$ is the angle at $z_0$ of the kite with edge lengths $r(z)$ and $r(z_0)$ and angle $\alpha([z, z_0])$, as in Figure 2.2. Condition (2.2) is the closing condition for the closed chain of kites which correspond to the edges incident to $z_0$. This corresponds to condition (4) of Definition 2.1.

**Remark 2.6.** Equation (2.2) can be interpreted as a nonlinear Laplace equation for the radius function. This is motivated by several reasons.

First, there are a maximum principle and a Dirichlet principle which are discret versions of the known properties of smooth solutions of the Laplace equation. See Lemma 2.9 and Theorem 2.10 below.

Second, let $G$ be a graph and let $\alpha$ be an admissible labelling. Assume there is a one-parameter family of planar circle patterns $\mathscr{C}_\varepsilon$ for $G$ and $\alpha$ with radius function $r_\varepsilon$

for $\varepsilon \in (-1,1)$. Then for each interior vertex $z_0$ with incident vertices $z_1,\dots,z_m$ and all $\varepsilon \in (-1,1)$ Proposition 2.5 implies that

$$\sum_{j=1}^{m} 2f_{\alpha([z_j,z_0])}(\log r_\varepsilon(z_j) - \log r_\varepsilon(z_0)) = 2\pi.$$

Differentiating this equation with respect to $\varepsilon$ at $\varepsilon = 0$, we obtain

$$\sum_{j=1}^{m} 2f'_{\alpha([z_j,z_0])}(\log r_0(z_j) - \log r_0(z_0))(v(z_j) - v(z_0)) = 0, \tag{2.4}$$

where $v(z) = \frac{d}{d\varepsilon}\log r_\varepsilon(z)|_{\varepsilon=0}$. Thus the derivatives $v$ satisfy a linear discrete Laplace equation with positive weights $2f'_{\alpha([z_j,z_0])}(\log r_0(z_j) - \log r_0(z_0))$. In order to understand this weights better, consider the kite of the circle pattern $\mathscr{C}_0$ containing the points $p(z_1), p(z_2)$ corresponding to two incident vertices $z_1, z_2 \in V(G)$. Denote the two other corner points of this kite by $p(v_1), p(v_2)$ and the distance between $p(z_1), p(z_2)$ by $l([z_1,z_2])$ and between $p(v_1), p(v_2)$ by $l^*([z_1,z_2])$. Then Lemma 2.7 below and a simple calculation show that

$$2f'_{\alpha([z_1,z_2])}(\log r_0(z_1) - \log r_0(z_2)) = \frac{l^*([z_1,z_2])}{l([z_1,z_2])}.$$

Linear discrete Laplacians with such weights are common in the linear theory of discrete holomorphic functions; see Section 5.4 or [14] for more details.

Summarizing, we have shown that a tangent space to the set of planar circle patterns for $G$ and $\alpha$ at a given point $\mathscr{C}_0$ consists of harmonic functions with respect to the linear discrete Laplacian defined by

$$\Delta u(z_0) = \sum_{[z,z_0]\in E(G)} \frac{l^*([z,z_0])}{l([z,z_0])}(u(z) - u(z_0))$$

for interior vertices $z_0 \in V(G)$. The weights are calculated from the lengths of the diagonals of the kites of $\mathscr{C}_0$.

In the following chapters, we will frequently use this connection to a linear discrete Laplacian for the case when there exits an isoradial planar circle pattern for $G$ and $\alpha$, that is $r_0 \equiv 1$.

For further use we mention some properties of $f_\theta$.

**Lemma 2.7** ([70, Lemma 2.2]). *(1) The derivative of $f_\theta$ is*

$$f'_\theta(x) = \frac{\sin\theta}{2(\cosh x - \cos\theta)} > 0.$$

*So $f_\theta$ is strictly increasing. It follows that the sum in equation* (2.2) *is strictly increasing in the variables $r(z)$ and strictly decreasing in $r(z_0)$.*

*(2) The function $f_\theta$ satisfies the functional equation*

$$f_\theta(x) + f_\theta(-x) = \pi - \theta.$$

*(3) For $0 < y < \pi - \theta$ the inverse function of $f_\theta$ is $f_\theta^{-1}(y) = \log\frac{\sin y}{\sin(y+\theta)}$.*

If there exists an isoradial circle pattern, we can obtain another circle pattern from a given radius function.

**Lemma 2.8.** *Let $G$ be a graph constructed from a b-quad-graph $\mathscr{D}$ and let $\alpha$ be an admissible labelling. Suppose that there exists an isoradial circle pattern for $G$ and $\alpha$. Let $r$ be the radius function of a planar circle pattern for $\mathscr{D}$ and $\alpha$. Then there is a circle pattern $\mathscr{C}$ for $G$ and $\alpha$ with radius function $r_{\mathscr{C}} = 1/r$.*

*Proof.* By Proposition 2.5 (2) it is sufficient to show that the function $1/r$ satisfies condition (2.2) for all interior vertices $z_0 \in V_{int}(G)$. This is due to Lemma 2.7 (2). In particular,

$$\begin{aligned}\sum_{[z,z_0]\in E(G)} f_{\alpha([z,z_0])}(\log \frac{r(z_0)}{r(z)}) &= \sum_{[z,z_0]\in E(G)} f_{\alpha([z,z_0])}(\log \frac{1}{r(z)} - \log \frac{1}{r(z_0)}) \\ &= \sum_{[z,z_0]\in E(G)} (\pi - \alpha([z,z_0])) - \sum_{[z,z_0]\in E(G)} f_{\alpha([z,z_0])}(\log \frac{r(z)}{r(z_0)}) \\ &= 2\pi - \pi = \pi.\end{aligned}$$

We have also used that $\sum_{[z,z_0]\in E(G)}(\pi - \alpha([z,z_0])) = \sum_{[z,z_0]\in E(G)} f_{\alpha([z,z_0])}(0) = 2\pi$ since there is an isoradial circle pattern for $G$ and $\alpha$ and the assumption that $r$ is the radius function of a circle pattern for $G$ and $\alpha$. □

In analogy to harmonic functions, the radius function of a planar circle pattern satisfies a maximum principle and a Dirichlet principle.

**Lemma 2.9** (Maximum Principle, [43, Lemma 2.1]). *Let $G$ be a finite graph constructed from a b-quad-graph as above with some admissible labelling $\alpha$. Suppose $\mathscr{C}$ and $\mathscr{C}^*$ are two planar circle patterns for $G$ and $\alpha$ with radius functions $r_{\mathscr{C}}, r_{\mathscr{C}^*} : V(G) \to (0,\infty)$. Then the maximum and minimum of the quotient $r_{\mathscr{C}}/r_{\mathscr{C}^*}$ is attained at the boundary.*

The main idea of the proof is to use the monotonicity of $f_\theta$ in equation (2.2).

If there exists an isoradial planar circle pattern for $G$ and $\alpha$, the usual maximum principle for the radius function follows by taking $r_{\mathscr{C}^*} \equiv 1$.

**Theorem 2.10** (Dirichlet Principle). *Let $\mathscr{D}$ be a finite simply connected b-quad-graph with associated graph $G$ and let $\alpha$ be an admissible labelling.*

*Let $r : V_\partial(G) \to (0,\infty)$ be some positive function on the boundary vertices of $G$. Then $r$ can be extended to $V(G)$ in such a way that equation (2.2) holds at every interior vertex $z \in V_{int}(G)$ if and only if there exists a coherent angle system that is if and only if there exists any circle pattern for $G$ and $\alpha$. If it exists, the extension is unique.*

*Proof.* The only if part follows directly from Proposition 2.5 (2).

To show the if part, assume that there exists a circle pattern for $\mathscr{D}$ and $\alpha$ with radius function $R : V(G) \to (0,\infty)$. A function $\kappa : V(G) \to (0,\infty)$ which satisfies the inequality

$$\left(\sum_{[z,z_0]\in E(G)} f_{\alpha([z,z_0])}(\log \kappa(z) - \log \kappa(z_0))\right) - \pi \geq 0 \tag{2.5}$$

at every interior vertex $z \in V_{int}(G)$ will be called *subharmonic* in G. Let $b$ be the minimum of the quotient $r/R$ on $V_\partial(G)$ and let $\kappa_1$ be equal to $r$ on $V_\partial(G)$ and to $bR$ on $V_{int}(G)$. Then $\kappa_1$ is subharmonic which can easily be deduced from the assumption that $R$ is a radius function and from the monotonicity of $f_\theta$. Writing $\log \kappa = \log(\kappa/R) - \log R$ for any subharmonic function $\kappa$, the same arguments imply that the maximum of $\kappa/R$ is attained at the boundary, which is a simple generalization of the Maximum Principle 2.9. Let $r^*$ be the supremum of all subharmonic functions on $G$ that coincide with $r$ on $V_\partial(G)$. Thus $r^*$ is bounded from above by the maximum of $r/R$ on $V_\partial(G)$ which is finite. One easily checks that $r^*$ satisfies condition (2.2).

The uniqueness claim follows directly from the Maximum Principle 2.9. □

The Dirichlet Principle may also be proven using the Euclidean functional of [16].

As a consequence of these results, the study of planar circle patterns for finite graphs $G$ can be reduced to the study of radius functions satisfying condition (2.2).

## 2.3 Estimations on the radius function

Estimations of the radius function will be important for our proof in Chapter 7. Therefore we gather and slightly generalize some of the known results.

First note that Proposition 2.5 and the Maximum Principle 2.9 can be generalized for circle patterns immersed with cone-like singularities at the vertices.

**Proposition 2.11.** *Under the conditions of part (1) of Proposition 2.5, if the circle pattern is immersed in a cone with apex at the center of circle corresponding to the vertex $v_0 \in V(G)$, then for $v_0$ equation (2.2) has to be modified to*

$$\sum_{[z,v_0]\in E(G)} f_{\alpha([z,v_0])}(\log r(z) - \log r(v_0)) = \beta,$$

*for $\beta \in (0,\infty)$ which is one half of the cone angle.*

*Furthermore, part (2) of Proposition 2.5 also holds true for circle patterns on a cone using the above equation for the vertex corresponding to the apex.*

**Lemma 2.12.** *Lemma 2.9 also holds if the circle patterns for $G$ and $\alpha$ are immersed on the same cone with apex a center of circle corresponding to a fixed vertex $\hat{v} \in V(G)$.*

Consider the Poincaré disk model of the hyperbolic plane consisting of the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\} \subset \mathbb{C}$ together with the hyperbolic metric $|ds| = \frac{2}{1-|z|^2}|dz|$. In the following, $U$ will be called *the hyperbolic disk*.

A version of the Maximum Principle 2.9 also holds for circle patterns in the hyperbolic disk $U$. For a closed disk $D$ in $U$, we denote by $r_{hyp}(D)$ its hyperbolic radius. If $D = D_v$ is a disk filling the circle $C_v$ of the circle pattern $\mathscr{C}$ and corresponding to the vertex $v$, we denote $r_{hyp}(v) := r_{hyp}(D_v)$.

For a circle pattern on a cone constructed as above, hyperbolic radii can be defined in the following way. Consider the region on the cone with Euclidean distance from the apex strictly less than 1. Place the apex at the origin and unroll the cone on the plane. Now the hyperbolic radii in U can be taken to be the 'hyperbolic radii' of the disks on the cone.

**Lemma 2.13** (cf. [43, Lemma 2.2])**.** *Let $G$, $r$ and $r^*$ be as in Lemma 2.12, but assume that $\alpha : E(G) \to [\pi/2, \pi)$. Assume further that the circle patterns corresponding to $r$ and $r^*$ are contained in $U$ or in a corresponding region on a cone. Then:*

*(i) The maximum of $\frac{r_{hyp}(v)}{r^*_{hyp}(v)}$, if $> 1$, is never attained at an interior vertex.*

*(ii) In particular, if the inequality $r_{hyp}(v) \le r^*_{hyp}(v)$ holds for each boundary vertex, then it holds for all vertices.*

*Proof.* Let $D_0$ be a fixed closed disk in $U$ centered at 0 and let $D_1$ be a closed disk intersecting $D_0$ with exterior intersection angle $\alpha \in [\pi/2, \pi)$. Assume that the hyperbolic radii of $D_0$ and of $D_1$ are both positive. Denote by $\gamma D = \{\gamma z : z \in D\}$ the result of the Euclidean scaling of the disk $D \subset U$ about the origin by the factor $\gamma > 0$ . Using the Maximum Principle 2.9, it is sufficient to show that

$$\frac{r_{hyp}(\gamma D_1)}{r_{hyp}(D_1)} > \frac{r_{hyp}(\gamma D_0)}{r_{hyp}(D_0)}$$

for all $\gamma > 1$ such that $\gamma D_1 \subset U$.

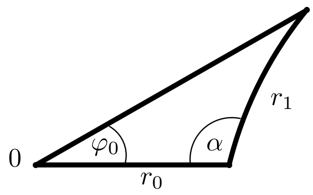



**Figure 2.4:** The angle $\varphi_0$ in the hyperbolic triangle with one vertex at the origin of $U$ and with hyperbolic edge lengths $r_0, r_1$ and angle $\alpha$.

In order to prove this inequality, consider the hyperbolic triangle built by the hyperbolic centers of the disks $D_0$ and $D_1$ and by one of the intersection points of their boundaries. Let $\varphi_0 \in (0, \pi/2)$ be the unsigned angle in this triangle at the center of $D_0$. See Figure 2.4 for an illustration. Then

$$\varphi_0 = f_\alpha(\rho_1 - \rho_0) - f_\alpha(\rho_1 + \rho_0),$$

where $\rho_j = \log \tanh(r_{hyp}(D_j)/2)$ for $j = 0, 1$. A derivation of this formula can be found in [16]. Note that the angle $\varphi_0$ does not change if the disks are scaled in an Euclidean way as above. Thus for all (suitable) $\gamma \geq 1$

$$\varphi_0 = f_\alpha(\rho_1(\gamma) - \rho_0(\gamma)) - f_\alpha(\rho_1(\gamma) + \rho_0(\gamma)),$$

where $\rho_j(\gamma) = \log \tanh(r_{hyp}(\gamma D_j)/2)$ for $j = 0, 1$. Differentiation with respect to $\gamma$ gives

$$0 = f'_\alpha(\rho_1(\gamma) - \rho_0(\gamma))(\rho'_1(\gamma) - \rho'_0(\gamma)) - f'_\alpha(\rho_1(\gamma) + \rho_0(\gamma))(\rho'_1(\gamma) + \rho'_0(\gamma)).$$

Denoting $a_j(\gamma) = \frac{r_{hyp}(\gamma D_j)}{r_{hyp}(D_j)}$ for $j = 0, 1$ and using Lemma 2.7, we obtain by a simple calculation that

$$a'_1(\gamma) = a'_0(\gamma) \frac{r_{hyp}(D_0)}{r_{hyp}(D_1)} \sinh^2(r_{hyp}(\gamma D_1))(\coth(r_{hyp}(\gamma D_0)) \coth(r_{hyp}(\gamma D_1)) - \cos\alpha). \quad (2.6)$$

For $\alpha \in [\pi/2, \pi)$, a straightforward calculation shows that

$$\frac{r_{hyp}(D_0)}{r_{hyp}(D_1)} \sinh^2(r_{hyp}(D_1))(\coth(r_{hyp}(D_0)) \coth(r_{hyp}(D_1)) - \cos\alpha) > 1$$

because

$$\begin{aligned} &r_{hyp}(D_0) \left(\coth(r_{hyp}(D_0)) \coth(r_{hyp}(D_1)) - \cos\alpha\right) \\ &\quad \geq r_{hyp}(D_0) \coth(r_{hyp}(D_0)) \coth(r_{hyp}(D_1)) > \coth(r_{hyp}(D_1)) > \frac{r_{hyp}(D_1)}{\sinh^2(r_{hyp}(D_1))}. \end{aligned}$$

Furthermore, the relation (2.6) shows that the function

$$g(\gamma) = \sinh^2(r_{hyp}(\gamma D_1))(\coth(r_{hyp}(\gamma D_0)) \coth(r_{hyp}(\gamma D_1)) - \cos\alpha)$$

is monotonically increasing. This implies $a'_1(\gamma) > a'_0(\gamma)$. Remebering that $a_1(1) = a_0(1) = 1$, we conclude that $a_1(\gamma) > a_0(\gamma)$ for all (suitable) $\gamma > 1$ which proves the claim. □

This lemma can be extended to the case, when all disks of the disk pattern have nonempty intersection with $U$. We then define $r_{hyp}(D)$ as follows: If $D$ is contained in $U$, $r_{hyp}(D)$ is the hyperbolic radius as before. If $D$ intersects $\mathbb{C} \setminus U$, let $\beta = \beta(D)$ denote the dihedral angle of the intersection. We define $r_{hyp}(D)$ to be the symbol $\infty^\beta$. We make the convention that for any angles $\beta_1 \geq \beta_2 \geq 0$ and for any real number $a$, we have: $\infty^{\beta_1} \geq \infty^{\beta_2} > a$. With the same proof, we obtain:

**Lemma 2.14** ([43, Lemma 2.3])**.** *Part (ii) of Lemma 2.13 stills holds if we assume that each closed disk of the circle patterns corresponding to $r$ and $r^*$ has nonempty intersection with $U$, instead of requiring them to be contained in $U$.*

For a disk $D \subset \mathbb{C}$ with center $c(D)$ denote the disk with the same center and half of the radius by

$$\tfrac{1}{2}D := \{z \in \mathbb{C} : 2(z - c(D)) + c(D) \in D\}.$$

The maximum principle will be used in Sections 7.1.2 and 7.1.3 in the following way.

**Lemma 2.15.** *Let $G$ be a finite planar graph with boundary and let $\alpha : E(G) \to [\pi/2, \pi)$ be an admissible labelling. Let $r$ and $r^*$ be the radius functions of two immersed circle patterns $\mathscr{C}$ and $\mathscr{C}^*$ for $G$ and $\alpha$ in the plane or on the same cone with apex in one of the centers of circles corresponding to a fixed vertex $\hat{v} \in V(G)$. Assume that there are two closed disks $D$ and $D^*$ with radii $R$ and $R^*$ such that*

*(i) the closed disk $D$ is covered by disks of $\mathscr{C}$ such that no boundary disk is entirely contained in the interior of $D$ and*

*(ii) all closed disks of $\mathscr{C}^*$ are contained in the interior of $D^*$.*

*If $\mathscr{C}$ and $\mathscr{C}^*$ are immersed circle patterns on the same cone, we further assume that $D$ and $D^*$ are centered at the apex of the cone.*

*Then there is a universal constant $C_0 > 0$ such that for each $v \in V(G)$ with $C_v \subset \frac{1}{2}D$ the following inequality holds:*

$$r^*(v) \leq \frac{2C_0 R^*}{R} r(v). \tag{2.7}$$

*Proof.* Without loss of generality we can assume that $V(G)$ consists only of vertices whose corresponding circles have nonempty intersection with the closure $\overline{U}$ of $U$. Scale the circle patterns $\mathscr{C}^*$ by $\frac{1}{2R^*}$ and $\mathscr{C}$ by $\frac{1}{R}$ and denote the scaled radii by $\rho^*(v) := \frac{r^*(v)}{2R^*}$ and $\rho(v) := \frac{r(v)}{R}$. Then $\frac{1}{2R^*}\mathscr{C}^*$ is contained in $U$ and all circles of $\frac{1}{R}\mathscr{C}$ have nonempty intersection with $U$. By assumption, all boundary circles of $\frac{1}{R}\mathscr{C}$ have nonempty intersection with $\mathbb{C}\setminus U$. Therefore we can deduce from Lemma 2.14 that $\rho^*_{hyp}(v) \leq \rho_{hyp}(v)$ holds true for all vertices.

Note that actually $\frac{1}{2R^*}\mathscr{C}^*$ is contained in $\frac{1}{2}U$. For all circles $\frac{1}{R}C_v$ which are contained in $\frac{1}{2}U$, hyperbolic and Euclidean radii are comparable. Therefore there is a universal constant $C_0$, such that $\rho^*(v) \leq C_0\rho(v)$. This implies the desired estimation. ☐

The Length-Area Lemma presented by Rodin and Sullivan in [64] provides another idea to estimate the radius function.

**Lemma 2.16** (Length-Area Lemma, cf. [64])**.** *Let $G$ be a planar graph with boundary and let $\alpha$ be an admissible labelling. Let $r$ denote the radius functions of an immersed circle pattern $\mathscr{C}$ for $G$ and $\alpha$. Assume that $\mathscr{C}$ is contained in the closed unit disk $\bar{U}$ and let $C_z$ be a circle of $\mathscr{C}$. Let $S_1, \dots, S_k$ be chains of circles of $\mathscr{C}$ such that consecutive circles of a chain intersect and each point in the open domain covered by the open disks filling the circles of $S_j$ is contained in at most two different disks. Assume that the domains covered by the disks in $S_j$ are disjoint for different chains and do not intersect the open disk bounded by $C_z$. Assume further that each chain $S_j$ is either closed and separates $C_z$ from a fixed boundary circle or the minimal distance within the domain covered by $S_j$ between the centers of the boundary circles is bigger than the diameter of $C_z$. Denote by $n_j$, $j = 1, \dots, k$, the number of circles of the chain $S_j$. Then*

$$r(z) \leq 2(n_1^{-1} + n_2^{-1} + \dots + n_k^{-1})^{-\frac{1}{2}}.$$

The main idea of the proof is to compare the squared length of the curve connecting the centers of circles of a chain to the area covered by the disks which fill the circles of the chain. The estimation follows since different chains are disjoint and contained in $\overline{U}$. Thus the sum of the areas is bounded by $\pi$.

## 2.4 Relations between radius and angle function

In this section we study relations between the radius function and a suitably chosen angle function for a given planar circle pattern. Furthermore, we compare these functions for two different circle patterns with the same combinatorics $G$ and the same labelling $\alpha$ and deduce equations, which are non-linear discrete analogs of the Cauchy-Riemann equations for holomorphic functions.

**Definition** (Angle function)**.** Let $G$ be a finite graph constructed from a b-quad-graph $\mathscr{D}$ as above and let $\alpha$ be an admissible labelling. Suppose that $\mathscr{C}$ is a planar circle pattern for $\mathscr{D}$ and $\alpha$ with radius function $r_{\mathscr{C}} = r$. We define an *angle function* on the set of oriented edges $\vec{E}(\mathscr{D})$ to be the angle of the line through the center of $C_z$ and the intersection point $p(v) \in C_z$ with the real axis.

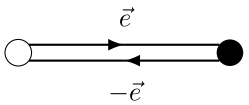



**Figure 2.5:** The orientation of directed edges of $\vec{E}(\mathscr{D})$.

In order to obtain the set of oriented edges $\vec{E}(\mathscr{D})$, each edge of $E(\mathscr{D})$ is replaced by two oriented edges of opposite orientation. As each edge of $\mathscr{D}$ connects a black and a white vertex, we denote by $\vec{e}$ the edge oriented from white to black and by $-\vec{e}$ the edge oriented from black to white, as in Figure 2.5.

Consider a circle $C_z$ with center $p(z)$ of the planar circle pattern $\mathscr{C}$ and one of the intersection points $p(v) \in C_z$. Then the complex number $b = p(v) - p(z)$ corresponds to the oriented edge $\vec{e} = \overrightarrow{zv}$. We define $\varphi(\vec{e}) = \arg b$ to be the argument of $b$, that is the angle between $b$ and the positively oriented real axis. This argument is in general only unique up to addition of multiples of $2\pi$, but $e^{i\varphi(\vec{e})}$ is well defined. So we will mostly consider $\varphi \in \mathbb{R}/(2\pi\mathbb{Z})$. Note, that we have

$$\varphi(\vec{e}) - \varphi(-\vec{e}) = \pi \pmod{2\pi}. \tag{2.8}$$

Our choice of the angle function leads to the following connections between radius and angle function for a planar circle pattern.

**Theorem 2.17.** *Let $\mathscr{D}$ be a b-quad-graph and let $\alpha : F(\mathscr{D}) \to (0, \pi)$ be an admissible labelling.*

*(1) Let $\mathscr{C}$ be a planar circle pattern for $\mathscr{D}$ and $\alpha$ with radius function $r_{\mathscr{C}} = r$ and angle function $\varphi_{\mathscr{C}} = \varphi$. Then the following equations hold on every face $f \in F(\mathscr{D})$.*

*Without loss of generality, we assume that the notation for the vertices and edges of $f$ is taken from Figure 2.6 (left). More precisely, the white vertices of $f$ are labelled $z_-$ and $z_+$ and the black vertices are labelled $v_-$ and $v_+$ such that the points $z_-, v_-, z_+, v_+$ appear in this cyclical order using the counterclockwise orientation of $f$. Furthermore the edges are labelled respecting the above convention such that $\vec{e}_1 = \overrightarrow{z_-v_+}$, $\vec{e}_2 = \overrightarrow{z_-v_-}$, $-\vec{e}_3 = \overrightarrow{v_-z_+}$, $-\vec{e}_4 = \overrightarrow{v_+z_+}$.*

$$\begin{aligned}
\varphi(\vec{e}_1) - \varphi(-\vec{e}_3) &= \alpha(f) - \pi + 2f_{\alpha(f)}(\log r(z_+) - \log r(z_-)) \pmod{2\pi} && (2.9)\\
\varphi(-\vec{e}_4) - \varphi(\vec{e}_2) &= \alpha(f) - \pi + 2f_{\alpha(f)}(\log r(z_+) - \log r(z_-)) \pmod{2\pi} && (2.10)\\
\varphi(\vec{e}_1) - \varphi(\vec{e}_2) &= 2f_{\alpha(f)}(\log r(z_+) - \log r(z_-)) \pmod{2\pi} && (2.11)\\
\varphi(-\vec{e}_4) - \varphi(-\vec{e}_3) &= -2f_{\alpha(f)}(\log r(z_-) - \log r(z_+)) \pmod{2\pi} && (2.12)\\
\varphi(-\vec{e}_3) - \varphi(\vec{e}_2) &= \pi - \alpha(f) \pmod{2\pi} && (2.13)\\
\varphi(-\vec{e}_4) - \varphi(\vec{e}_1) &= \alpha(f) - \pi \pmod{2\pi} && (2.14)
\end{aligned}$$

*If $f_0, f_1, \ldots, f_n$ is a chain of faces of $\mathscr{D}$ such that $f_{j-1}$ and $f_j$ $(j = 1, \ldots, n)$ have a common edge and $f_k$ and $f_j$ have no common edge if $k \neq j-1, j+1$, then the angle*

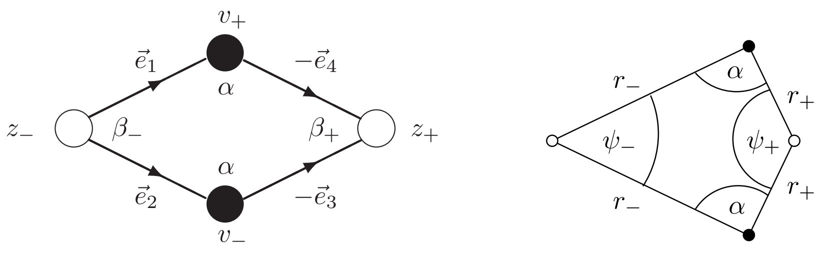



**Figure 2.6:** *Left:* A face of $\mathscr{D}$ with oriented edges. *Right:* A kite with intersection angle $\alpha$ and radii $r_-$ and $r_+$.

*function $\varphi$ can be defined with values in $\mathbb{R}$ on the oriented edges of these faces, such that equations* (2.9)–(2.14) *hold without the* (mod $2\pi$)*-term.*

*Additionally, the angle function $\varphi$ satisfies the following* monotonicity condition. *Let $z \in V(G)$ be a white vertex and let $e_1, \ldots, e_n$ be the sequence of all incident edges in $E(\mathscr{D})$ which are cyclically ordered respecting the counterclockwise orientation of $C_z$. Then the values of $\varphi \in \mathbb{R}$ can be changed by suitably adding multiples of $2\pi$ such that $\varphi(\vec{e}_j)$ is an increasing function of the index $j$ and if $z$ is an interior vertex, then also $\varphi(\vec{e}_j) - \varphi(\vec{e}_1) < 2\pi$ for all $j = 1, \ldots, n$.*

*(2) Conversely, assume that $\mathscr{D}$ is simply connected and let $r : V(G) \to (0,\infty)$ and $\varphi : \vec{E}(\mathscr{D}) \to \mathbb{R}$ be two functions which satisfy equations* (2.8), (2.9)–(2.14) *and the monotonicity condition. Then there is a corresponding planar circle pattern with radius function $r$ and angle function $\varphi$. This circle pattern is unique up to translation.*

*Proof.* (1) Consider Figure 2.6 (left) with $\alpha = \alpha(f)$,

$$\beta_- = 2f_{\alpha(f)}(\log r(z_+) - \log r(z_-)), \quad \beta_+ = 2f_{\alpha(f)}(\log r(z_-) - \log r(z_+)).$$

Then equations (2.9)–(2.14) are satisfied by the definition of the angle function $\varphi$. The monotonicity condition follows immediately by considering the definition of the angle function around one circle.

Given a sequence $f_0, f_1, \ldots, f_n$ of faces of $\mathscr{D}$ as above, the angle function $\varphi$ can be defined on the oriented edges of these faces by one initial value and by equations (2.8) and (2.9)–(2.14) (in each case considered without the (mod $2\pi$)-term). By construction, these assignments give correct values.

(2) For each face $f \in F(\mathscr{D})$, we can construct a kite $k_f \in \mathscr{K}$ as in Figure 2.6 (right), where $r_+, r_-$ are the radii of the white vertices incident to $f$ and $\alpha = \alpha(f)$. The remaining angles are $\psi_- = 2f_{\alpha(f)}(\log r_+ - \log r_-)$ and $\psi_+ = 2f_{\alpha(f)}(\log r_- - \log r_+)$. Lay out one interior kite (fixing the rotational freedom according to $\varphi$). Then successively add all other kites according to the combinatorics of $\mathscr{D}$. We have to show that all these kites fit around each vertex to result in a circle pattern (after adding circles for all white vertices). More precisely, at every vertex the angles of the kites having this vertex in common have to add up to $2\pi$. By condition (2.1), this is true for all black vertices. For all interior white vertices $z \in V_{int}(G)$ with incident vertices $z_1, \ldots, z_n \in V(G)$ condition (2.11) implies

$$\sum_{j=1}^{n} 2f_{\alpha([z_j,z])}(\log r(z_j) - \log r(z)) = 0 \pmod{2\pi}.$$

The monotonicity condition together with $2f_{\alpha([z_j,z])}(\log r(z_j) - \log r(z)) \in (0, 2\pi)$ then shows, that the value of the sum is exactly $2\pi$. So this construction leads to a circle pattern for $\mathscr{D}$ and $\alpha$ with radius function $r$ and angle function $\varphi$.

□

**Remark 2.18.** The set of equations (2.9)–(2.14) is not minimal for part (2) of Theorem 2.17. A minimal set is for example given by equations (2.11)–(2.13).

For the approximation of holomorphic functions, we will compare different circle patterns with the same combinatorics. In particular, equation (2.15) of the following theorem can be interpreted as a nonlinear discrete version of the Cauchy-Riemann equations for holomorphic functions. In Section 5.4 we also recall a connection to the linear theory of discrete holomorphic functions.

**Theorem 2.19.** *Let $\mathscr{D}$ be a b-quad-graph and let $\alpha$ be an admissible labelling.*

*(1) Let $\mathscr{C}$ and $\hat{\mathscr{C}}$ be two planar circle patterns for $\mathscr{D}$ and $\alpha$ with radius functions $r_{\mathscr{C}} = r$ and $\hat{r}_{\mathscr{C}} = \hat{r}$ and angle functions $\varphi_{\mathscr{C}} = \varphi$ and $\hat{\varphi}_{\mathscr{C}} = \hat{\varphi}$ respectively. Then the difference $\hat{\varphi} - \varphi$ gives rise to a function $\delta : V(G^*) \to \mathbb{R}$ such that the following condition holds on every face $f \in F(\mathscr{D})$.*

$$2f_{\alpha(f)}\left(\log\left(\tfrac{w(z_+)}{w(z_-)}\right) + \log\left(\tfrac{r(z_+)}{r(z_-)}\right)\right) - 2f_{\alpha(f)}\left(\log\left(\tfrac{r(z_+)}{r(z_-)}\right)\right) = \delta(v_+) - \delta(v_-) \quad (2.15)$$

*Here we have defined $w : V(G) \to \mathbb{R}$, $w(z) = \hat{r}(z)/r(z)$ and the notation is taken from Figure 2.6 (left) as in Theorem 2.17 (1).*

*(2) Conversely, assume that $\mathscr{C}$ is a planar circle pattern for $\mathscr{D}$ and $\alpha$ with radius functions $r_{\mathscr{C}} = r$ and angle function $\varphi_{\mathscr{C}} = \varphi$. Let $\delta : V(G^*) \to \mathbb{R}$ and $w : V(G) \to \mathbb{R}_+$ be two functions which satisfy equation* (2.15) *for every face $f \in F(\mathscr{D})$. Then $(rw)$ and $(\varphi+\delta)$ are the radius and angle function of a planar circle pattern for $\mathscr{D}$ and $\alpha$. This circle pattern is unique up to translation.*

*Proof.* (1) Equations (2.8), (2.13), and (2.14) imply that the difference $\hat{\varphi} - \varphi$ is constant for all edges incident to any fixed black vertex $v \in V(G^*)$. Therefore $\delta \pmod{2\pi}$ is well defined on the intersection points and encodes the relative rotation of the star of edges at $v$.

To get a function $\delta$ with values in $\mathbb{R}$, fix $\delta(v_0) \in [0, 2\pi)$ for one arbitrary vertex $v_0 \in V(G^*)$. Then the values of $\delta$ for all incident vertices in $G^*$ are given by equation (2.15). In fact, this equation is only the difference of equation (2.11) for the two circle patterns. This construction can be continued until we have assigned a value to all vertices. We will never encounter a contradiction following this proceedure, as the sum of the left hand side of equation (2.15) is zero for simple closed paths in $E(G^*)$ around a white vertex. This is due to the form of equation (2.15) and to the fact, that equation (2.2) holds for both circle patterns.

(2) As $r$ is a radius function, equation (2.15) implies that $(wr)$ fullfills equation (2.2). By assumption, $r$ and $\varphi$ satisfy equations (2.8) and (2.9)–(2.14). Equation (2.15) now implies that this is also true for $(wr)$ and $(\varphi + \delta)$. The monotonicity condition also holds for $(\varphi+\delta)$. This can be seen as follows. Take the notation of Theorem 2.17 (i) and adjust $(\varphi + \delta)(\vec{e}_1) \in [0, 2\pi)$. Denote by $z_j \in V(G)$ the white vertex which is incident to the same face as $\vec{e}_j$ but not to $\vec{e}_j$ itself. Then using equations (2.11) and (2.2) we can change the values of $(\varphi + \delta)$ such that

$$(\varphi + \delta)(\vec{e}_j) - (\varphi + \delta)(\vec{e}_1) = \sum_{k=1}^{j-1} 2f_{\alpha([z_k,z])}(\log(wr)(z_k) - \log(wr)(z)) < 2\pi$$

for all $j = 1, \ldots, n$. Now the claim follows from Theorem 2.17 (2).

$\square$

**Lemma 2.20.** *Under the assumptions of Theorem 2.19 (1) the following* closing condition *holds for all kites. For every face $f \in F(\mathscr{D})$ we have*

$$e^{i\delta(v_-)}w(z_-)(v_- - z_-) + e^{i\delta(v_-)}w(z_+)(z_+ - v_-) \\ + e^{i\delta(v_+)}w(z_+)(v_+ - z_+) + e^{i\delta(v_+)}w(z_-)(z_- - v_+) = 0. \quad (2.16)$$

*with the notation taken from Figure 2.6 (left) as in Theorem 2.17. This relation is known as* Hirota equation*, see also* (3.12)*.*

*Proof.* Equation (2.16) encodes the closing condition for the edges of the kites of $\hat{\mathscr{K}}$. $\square$

**Remark 2.21.** For the special case of orthogonal $SG$-circle patterns as considered for example in Chapter 7, we can assume without loss of generality that

$$\begin{aligned} V(\mathscr{D}) &\subset \mathbb{Z} + i\mathbb{Z}, \\ V(G) &\subset \{n + im \in \mathbb{Z} + i\mathbb{Z} : n + m = 0 \pmod 2\} \qquad \text{and} \\ V(G^*) &\subset \{n + im \in \mathbb{Z} + i\mathbb{Z} : n + m = 1 \pmod 2\}. \end{aligned}$$

The angle function may be compared to the angle function of the regular isoradial pattern and can be associated to vertices $v \in V(G^*)$ corresponding to intersection points. Therefore, this (comparison) angle function $\phi$ can be defined such that it is unique up to global addition of a constant $2\pi k$ ($k \in \mathbb{Z}$). Equation (2.15) then takes the form

$$\phi(z + i) - \phi(z + 1) = \arctan\left(\frac{r(z + 1 + i)^2 - r(z)^2}{2r(z + 1 + i)r(z)}\right), \quad (2.17)$$

$$\phi(z + i) - \phi(z - 1) = -\arctan\left(\frac{r(z - 1 + i)^2 - r(z)^2}{2r(z - 1 + i)r(z)}\right), \quad (2.18)$$

where we assume the arctan-function to take values in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

## 2.5 THE ANGLE FUNCTION

To underline the analogy of the radius and the angle function for a planar circle pattern, we prove some necessary and sufficient conditions for the angle function to originate from a circle pattern. We also give an existence and uniqueness result, which is useful when dealing with Neumann boundary conditions.

Fist, we investigate in the relation between angle functions and coherent angle systems.

**Proposition 2.22.** *Let $\mathscr{D}$ be a b-quad-graph and let $G$ and $G^*$ be the associated graphs constructed from white and black vertices respectively. Let $\alpha$ be an admissible labelling.*

*Let $\mathscr{C}$ be a planar circle pattern for $G$ and $\alpha$ with radius function $r_{\mathscr{C}} = r$ and angle function $\varphi_{\mathscr{C}} = \varphi$. Then there is a coherent angle system $\phi : \vec{E}(G) \to (0, 2\pi)$ defined by*

$$\phi(\overrightarrow{z_- z_+}) = 2f_{\alpha([z_+, z_-])}(\log r(z_+) - \log r(z_-)) = \varphi(\vec{e}_1) - \varphi(\vec{e}_2) \pmod{2\pi}$$

*for all edges $\overrightarrow{z_- z_+} \in \vec{E}(G)$, where the notation is taken from Figure 2.6 (left).*

*Conversely, suppose that $\mathscr{D}$ is simply connected and that there is a coherent angle system $\phi$. Then there is an (angle) function $\varphi : \vec{E}(\mathscr{D}) \to \mathbb{R}/(2\pi\mathbb{Z})$ such that*

$$\phi(\overrightarrow{z_- z_+}) = \varphi(\vec{e}_1) - \varphi(\vec{e}_2) \pmod{2\pi}$$

*holds for all edges* $\overrightarrow{z_-z_+} \in \vec{E}(G)$*, where the notation is again taken from Figure 2.6 (left). Furthermore,* $\varphi$ *also satisfies equations* (2.8), (2.13), (2.14) *and the monotonicity condition of Theorem 2.17 (1) at every white vertex. The map* $\varphi$ *is unique up to adding a global constant.*

*Proof.* Given a circle pattern for $G$ and $\alpha$, the properties of the radius and the angle function given in Theorems 2.5 and 2.17 imply that the mapping $\phi$ satisfies the conditions of Definition 2.3 of a coherent angle system.

Now assume that there is a coherent angle system $\phi$. Then we construct an (angle) function $\varphi : \vec{E}(\mathscr{D}) \to \mathbb{R}/(2\pi\mathbb{Z})$ in the following way. Start with any (positively oriented) edge $\vec{e}_0$ and fix $\varphi(\vec{e}_0) \in \mathbb{R}$. Consider a face $f_1 \in F(\mathscr{D})$ incident to $\vec{e}_0$. Then $\varphi$ can be determined by the following conditions for all edges which are incident to $f_1$. The notation corresponds to Figure 2.6 (left), where $\vec{e}_0$ is one of the edges $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$.

$$\begin{aligned}
\varphi(\vec{e}_2) &= \varphi(\vec{e}_1) - \phi(\overrightarrow{z_-z_+}) && \pmod{2\pi} \\
\varphi(\vec{e}_3) &= \varphi(\vec{e}_1) - \phi(\overrightarrow{z_-z_+}) - \alpha(f_1) && \pmod{2\pi} \\
\varphi(\vec{e}_4) &= \varphi(\vec{e}_1) + \alpha(f_1) && \pmod{2\pi} \\
\varphi(-\vec{e}_j) &= \varphi(\vec{e}_j) - \pi && \pmod{2\pi} \quad \text{for } j = 1, \dots, 4
\end{aligned}$$

As $\mathscr{D}$ is simply connected, this construction can be continued until $\varphi$ is defined at all edges. The values of $\varphi$ (mod $2\pi$) are independent of the particular sequence of the faces considered. To see this, consider the assignments of $\varphi$ when following the faces of a cycle around a vertex of $\mathscr{D}$. Then the value of the first edge and the assignment corresponding to the above algorithm after performing one cycle only differ by $2\pi$, as the values of the coherent angle system and the intersection angles add to $2\pi$ around each vertex. This proves the claim on existence and uniqueness. Equations (2.8), (2.13), (2.14) and the monotonicity condition of Theorem 2.17 (1) are also satisfied by construction, using the properties of $\phi$. □

The following theorem characterizes angle functions associated to planar circle patterns.

**Proposition 2.23.** *(1) Let* $\mathscr{D}$ *be a b-quad-graph with associated graphs* $G$ *and* $G^*$ *and let* $\alpha$ *be an admissible labelling.*

*Suppose that* $\mathscr{C}$ *is a planar circle pattern for* $\mathscr{D}$ *and* $\alpha$ *with angle function* $\varphi = \varphi_{\mathscr{C}}$*. Then* $\varphi$ *satisfies equations* (2.8), (2.13), (2.14)*, the monotonicity condition of Theorem 2.17 (1) at every white vertex of* $\mathscr{D}$*, and the following two conditions.*

*(i) Let* $f$ *be a face of* $\mathscr{D}$ *and let* $e_1$ *and* $e_2$ *be two edges incident to* $f$ *and to the same white vertex and assume that* $e_1$ *and* $e_2$ *are enumerated in clockwise order as in Figure 2.6 (left). Define an angle* $\beta \in (0, \pi)$ *by*

$$2\beta = \varphi(\vec{e_1}) - \varphi(\vec{e_2}) \pmod{2\pi}.$$

*Then*

$$\beta + \alpha(f) < \pi. \tag{2.19}$$

*(ii) For every interior black vertex* $v \in V_{int}(G^*)$*, denote by* $e_1, \dots, e_n, e_{n+1} = e_1$ *all incident edges of* $\mathscr{D}$ *in counterclockwise order and by* $f_j$ *the face of* $\mathscr{D}$ *incident to* $e_j$ *and* $e_{j+1}$*. Denote by* $e_j^*$ $(j = 1, \dots, n)$ *the edge incident to* $e_j$ *and* $f_j$ *which is not incident to* $v$*. For* $j = 1, \dots, n$ *define as above* $\beta_j \in (0, \pi)$ *by* $2\beta_j = \varphi(\vec{e_j}^*) - \varphi(\vec{e_j})$ (mod $2\pi$)*. Then*

$$\sum_{j=1}^{n} f^{-1}_{\alpha(f_j)}(\beta_j) = 0. \tag{2.20}$$

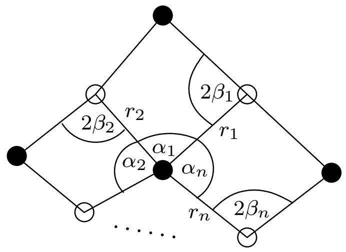



**Figure 2.7:** An interior intersection point with its neighboring faces.

*(2) Let $\mathscr{D}$ be a simply connected b-quad-graph and let $\alpha$ be an admissible labelling.*

*Suppose that $\varphi : \vec{E}(\mathscr{D}) \to \mathbb{R}/(2\pi\mathbb{Z})$ satisfies equations* (2.8), (2.13), (2.14)*, the monotonicity condition of Theorem 2.17 (1), condition* (2.19) *at every white vertex of $\mathscr{D}$, and condition* (2.20) *at every interior black vertex. Then there is a planar circle pattern for $\mathscr{D}$ and $\alpha$ with angle function $\varphi$. This pattern is unique up to scaling and translation.*

*Proof.* (1) For a given planar circle pattern, part (1) of Theorem 2.17 holds. To show (2.19), consider a kite corresponding to a face of $\mathscr{D}$ as in Figure 2.6 (left). Note that $\beta_- = 2\beta$ by equation (2.11), $\beta_- + \beta_+ + 2\alpha = 2\pi$, and $\beta_-, \beta_+, \alpha > 0$.

Using notation of Figure 2.7 we further deduce that

$$f_{\alpha_j}^{-1}(\beta_j) = \log r_{j+1} - \log r_j \tag{2.21}$$

for $j = 1, \dots, n$, where we identify $r_1 = r_{n+1}$. Now (2.20) follows immediately.

(2) Using the assumptions on the given angle function, we prove the existence of a corresponding circle pattern by constructing a radius function $r : V(G) \to (0, \infty)$ such that $r$ and $\varphi$ satisfy the assumptions of Theorem 2.17 (2).

Let $z \in V(G)$ be an interior white vertex. Set $r(z) = 1$. Consider a neighboring white vertex $z' \in V(G)$ and the face $f \in F(\mathscr{D})$ incident to $z$ and $z'$. Denote the edges of $\mathscr{D}$ incident to $z$ and $f$ by $e_1$, $e_2$ in clockwise order. Define $\psi_- \in (0, 2\pi)$ by $\psi_- = \varphi(\vec{e_1}) - \varphi(\vec{e_2}) \pmod{2\pi}$. Condition (2.19) implies $\psi_-/2 + \alpha(f) < \pi$, so $r(z') = \exp(f_{\alpha(f)}^{-1}(\psi_-/2))$ is well defined and positive. Then by construction and with the assumptions on the angle function $\varphi$, equations (2.9)–(2.14) are satisfied for the face $f$. We proceed in this way until a radius has been assigned to all white vertices. Condition (2.20) guarantees that these assignments do not lead to different values when turning around a black vertex (see Figure 2.7 with $\psi = 2\beta$). Thus $r$ is uniquely determined up to the choice of the initial radius, which corresponds to a scaling of the whole pattern. Now, the claim follows from Theorem 2.17 (2).

□

**Remark 2.24.** In the special case of an orthogonal *SG*-circle pattern, define the angle function $\phi$ at white vertices $v \in V(G^*)$ as in the Remark 2.21. Then the necessary and sufficient conditions of Proposition 2.23 can be formulated as follows.

For every interior vertex $v_0 \in V_{int}(G^*)$ we have

$$|\phi(v_0 + i^k(1+i)) - \phi(v_0)| < \frac{\pi}{2} \text{ for all } k \in \{0,1,2,3\} \text{ and}$$
$$F(\phi(v_0), \phi(v_0 - 1 - i), \phi(v_0 + 1 - i), \phi(v_0 + 1 + i), \phi(v_0 - 1 + i)) = 0,$$

where

$$F(x_0, x_1, x_2, x_3, x_4) = \tan\left(\tfrac{\pi}{4} + \tfrac{x_1 - x_0}{2}\right)\tan\left(\tfrac{\pi}{4} + \tfrac{x_3 - x_0}{2}\right) - \tan\left(\tfrac{\pi}{4} + \tfrac{x_0 - x_2}{2}\right)\tan\left(\tfrac{\pi}{4} + \tfrac{x_0 - x_4}{2}\right).$$

**Theorem 2.25** (Existence for Neumann boundary conditions)**.** *Let $\mathscr{D}$ be a simply connected b-quad-graph and let $\alpha$ be an admissible labelling.*

*Let $v_0, v_1, \dots, v_{n-1}, v_n = v_0$ be the chain of boundary vertices starting with a white vertex $v_0$ such that $v_j$ and $v_{j+1}$ are adjacent in $\mathscr{D}$ in counterclockwise order, that is the faces of $\mathscr{D}$ are to the left when following the boundary chain. Let $e_0, e_1, \dots, e_{n-1}, e_n = e_0$ be the corresponding chain of boundary edges in $\mathscr{D}$, where $e_j$ is incident to $v_j$ and $v_{j+1}$ for $j = 0, \dots, n-1$. Let $\varphi$ be an angle function on the boundary edges which satisfies* (2.8) *and*

$$\varphi(\vec{e}_j) - \varphi(-\vec{e}_{j+1}) = \sum_{f \text{ incident to } v_{j+1}} \alpha(f) \pmod{2\pi},$$

$$\varphi(\vec{e}_{j+1}) \neq \varphi(\vec{e}_{j+2}) \pmod{2\pi}$$

*for all even indices $j \in \{0, 2, \dots, n-2\}$. For every white boundary vertex $v_j \in V_\partial(G)$ (that is with an even index $j \in \{2, 4, \dots, n\}$) define $\Phi(v_j) \in (0, 2\pi)$ by*

$$\Phi(v_j) = \varphi(\vec{e}_{j-1}) - \varphi(\vec{e}_j) \pmod{2\pi}.$$

*For $v \in V_{int}(G)$ set $\Phi(v) = 2\pi$. Then a planar circle pattern for $G$ and $\alpha$ and the given boundary data exists if and only if the following condition is satisfied:*

*If $V' \subsetneqq V(G)$ is a nonempty set of vertices and $E' \subset E(G)$ the set of all edges which are incident with any vertex $v \in V'$, then*

$$\sum_{v \in V'} \Phi(v) < \sum_{e \in E'} 2(\pi - \alpha(e)). \tag{2.22}$$

*If it exists, the circle pattern leads to an extension of the angle function to $\vec{E}(\mathscr{D})$ such that equations* (2.8), (2.13), (2.14) *are satisfied, the monotonicity condition of Theorem 2.17 (1) and condition* (2.19) *hold at every white vertex of $\mathscr{D}$, and condition* (2.20) *holds at every black vertex.*

*Proof.* The proof is a direct application of Theorem 3 of [16] cited below. The equality condition follows by the construction of $\Phi(v_j)$ at the boundary. The properties of the angle functions are a consequence of Proposition 2.23. □

**Theorem 2.26** ([16, Theorem 3])**.** *Let $\Sigma$ be a cell decomposition of a compact oriented surface with or without boundary. Let $G^*$ be the graph associated to the 1-skeleton of $\Sigma$ and let $G$ be the corresponding dual graph. Suppose exterior intersection angles are prescribed by a function $\alpha \in (0, \pi)^{E_0}$ on the set $E_0$ of interior edges of $G^*$ (or of $G$). Let $\Phi \in (0, \infty)^V$ be a function on the set $V = V(G)$ of vertices of $G$ which correspond to the faces of $G^*$. $\Phi$ prescribes, for interior vertices of $G$ (faces of $G^*$), the cone angle $\Phi - 2\pi$ and, for boundary vertices of $G$ (faces of $G^*$), the Neumann boundary conditions. A planar circle pattern corresponding to these data exists if and only if the following condition is satisfied:*

*If $V' \subseteq V$ is any nonempty set of vertices and $E' \subseteq E_0$ is the set of all interior edges which are incident with any vertex $v \in V'$, then*

$$\sum_{v \in V'} \Phi_f \leq \sum_{e \in E'} 2(\pi - \alpha_e),$$

*where equality holds if and only if $V' = V$.*

Chapter 3

# Quasicrystallic circle patterns

In this chapter we focus on the special class of quasicrystallic circle patterns. These are closely connected to quasicrystallic rhombic embeddings and to combinatorial surfaces in $\mathbb{Z}^d$ for some $d \in \mathbb{N}$. Furthermore, using this additional structure we extend a known result on the asymptotics of discrete Green's function and deduce a discrete version of Hölder's inequality for harmonic functions and a regularity lemma for discrete solutions of elliptic equations. These results are then used to prove a rigidity theorem for a certain class of quasicrystallic circle patterns. Moreover, we prove an analog of the Rodin-Sullivan Conjecture for regular circle patterns with square grid combinatorics and generalize it to a class of quasicrystallic circle patterns.

Some of the results, in particular Sections 3.2–3.5 and 3.6.2, are useful for our convergence proofs in Chapters 5 and 7.

## 3.1 Quasicrystallic rhombic embeddings and $\mathbb{Z}^d$

**Definition 3.1.** Let $\mathscr{D}$ be a b-quad-graph. A *rhombic embedding of $\mathscr{D}$ in $\mathbb{C}$* is an embedding with the property that each face of $\mathscr{D}$ is mapped to a rhombus.

For a rhombic embedding of $\mathscr{D}$ in $\mathbb{C}$ there is a labelling $\beta : \vec{E}(\mathscr{D}) \to \mathbb{C}$ of the directed edges of $\mathscr{D}$ associated to this embedding: For each directed edge we take the direction of the vector of its embedding (as a complex number with length one). Then $\beta(-\vec{e}) = -\beta(\vec{e})$ for any edge $\vec{e} \in \vec{E}(\mathscr{D})$ and the values of $\beta$ on two opposite and equally directed edges of any quadrilateral face are equal.

A rhombic embedding of $\mathscr{D}$ in $\mathbb{C}$ is called *quasicrystallic* if the set of values of the corresponding labelling $\beta : \vec{E}(\mathscr{D}) \to \mathbb{C}$ is finite. Half of the number of different values, $d = |\beta(\vec{E}(\mathscr{D}))|/2$, is called the *dimension* of the quasicrystallic rhombic embedding.

A circle pattern for a b-quad-graph $\mathscr{D}$ is called a *quasicrystallic circle pattern* if there exists a quasicrystallic rhombic embedding of $\mathscr{D}$ (with a corresponding labelling).

A combinatorial criterion for the existence of a rhombic embedding of a quad-graph can be found in [50].

**Remark.** The notion "quasicrystallic" is not uniquely defined in literature. Here we adopt the definition given in [14]. Certainly, this property only makes sense for infinite graphs or infinite sequences of graphs with growing number of vertices and edges. The use of the term "dimension $d$" indicates the connection of quasicrystallic rhombic embeddings with the multi-dimensional lattice $\mathbb{Z}^d$. This structure can be described using the relation to a two-dimensional subcomplex (surface) in a multi-dimensional regular square lattice $\mathbb{Z}^d$. This representation is explained below and has been used in [14]. The idea is connected to the grid projection method for the construction of quasiperiodic tilings of the plane; see for example [29, 36, 41, 69] and Example 3.3.

In the following we will frequently identify the b-quad-graph $\mathscr{D}$ with a rhombic embedding of $\mathscr{D}$.

Consider a quasicrystallic rhombic embedding of a b-quad-graph $\mathscr{D}$ and denote the set of the different edge directions by $\mathcal{A} = \{\pm a_1, \dots, \pm a_d\} \subset \mathbb{S}^1$. We suppose that $d > 1$ and that any two non-opposite elements of $\mathcal{A}$ are linearly independent over $\mathbb{R}$. This implies in particular that all rhombi are non-degenerate. Any rhombic embedding of a b-quad-graph can be seen as a projection of a certain two-dimensional subcomplex (combinatorial surface) $\Omega_{\mathscr{D}}^{\mathcal{L}}$ of a multi-dimensional lattice which is isomorphic to $\mathbb{Z}^d$. An illustrating example is given in Figure 3.1.

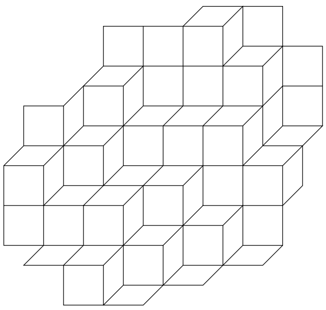


**Figure 3.1:** An example of a combinatorial surface $\Omega_{\mathscr{D}} \subset \mathbb{Z}^3$.

In order to establish this correspondence, embed the plane $\mathbb{C} \cong \mathbb{R}^2$ containing the rhombic embedding into $\mathbb{R}^d$ such that $\mathbb{R}^2$ is spanned by the first two vectors of the standard orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_d$ of $\mathbb{R}^d$ and such that 0 is a white vertex. Define a new basis by setting $\mathbf{v}_1 = a_1$, $\mathbf{v}_2 = a_2$ and $\mathbf{v}_j = a_j + \mathbf{e}_j$ for $j = 3, \ldots, d$. Then the lattice

$$\mathcal{L} = \{\sum_{j=1}^{d} n_j \mathbf{v}_j : n_j \in \mathbb{Z},\ j = 1, \ldots, d\}$$

is isomorphic to the regular square lattice $\mathbb{Z}^d$. Furthermore, $\mathbf{v}_j$ is projected onto $a_j$ by the orthogonal projection of $\mathbb{R}^d$ onto $\mathbb{R}^2$.

Now construct the combinatorial surface $\Omega_{\mathscr{D}}^{\mathcal{L}}$ of $\mathcal{L}$ (and the isomorphic combinatorial surface $\Omega_{\mathscr{D}}$ of $\mathbb{Z}^d$) in the following way. Begin with the vertex at the origin. Add the edges of $\{\pm\mathbf{v}_1, \ldots, \pm\mathbf{v}_d\}$ which correspond to the edges of $\{\pm a_1, \ldots, \pm a_d\}$ incident to 0 in $\mathscr{D}$, together with their endpoints. Then continue the construction at the new endpoints. Also add two-dimensional facets (faces) of $\mathcal{L}$ corresponding to faces of $\mathscr{D}$, spanned by incident edges. Then $\Omega_{\mathscr{D}}^{\mathcal{L}}$ is a combinatorial surface and projects onto the rhombic embedding by orthogonal projection of $\mathbb{R}^d$ onto $\mathbb{R}^2$.

A combinatorial surface $\Omega_{\mathscr{D}}$ in $\mathbb{Z}^d = \{\sum_{j=1}^{d} n_j \mathbf{e}_j : n_j \in \mathbb{Z},\ j = 1, \ldots, d\}$ corresponding to a quasicrystallic rhombic embedding can be characterized using the following monotonicity property. For a proof of this criterion see [14, Section 6].

**Lemma 3.2** (Monotonicity criterium)**.** *Any two points of $\Omega_{\mathscr{D}}$ can be connected by a path in $\Omega_{\mathscr{D}}$ with all directed edges lying in one $d$-dimensional octant, that is all directed edges of this path are elements of one of the $2^d$ subsets of $\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_d\}$ containing $d$ linearly independent vectors.*

*Notation*

To mark the correspondence of $\mathscr{D}$ and $\Omega_{\mathscr{D}}$, we will denote the points in $\mathbb{Z}^d$ corresponding to a vertex $z \in V(\mathscr{D})$ by $\hat{\mathbf{z}} \in V(\Omega_{\mathscr{D}})$. Remember that we assume the vertices of the b-quad-graph $\mathscr{D}$ to be colored white and black. By construction of $\Omega_{\mathscr{D}}$, the points of $\mathbb{Z}^d$ may be colored white and black such that the vertices of $\Omega_{\mathscr{D}}$ are colored in the same way as the corresponding vertices of $\mathscr{D}$ and incident vertices carry different colors. We denote by $V_w(\Omega_{\mathscr{D}})/V_w(\mathbb{Z}^d)$ and by $V_b(\Omega_{\mathscr{D}})/V_b(\mathbb{Z}^d)$ the white and black vertices of $\Omega_{\mathscr{D}}/\mathbb{Z}^d$ respectively.

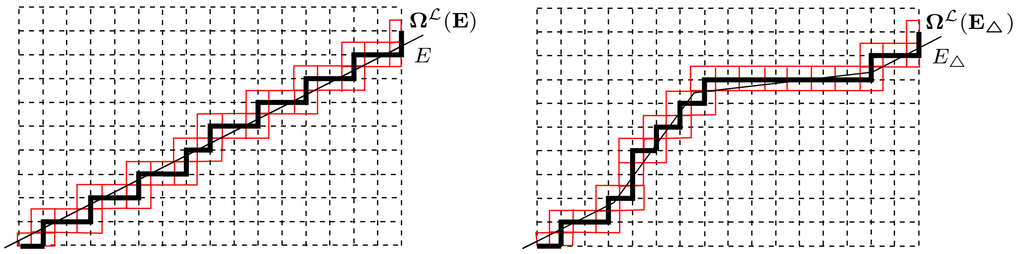
## 3.2 Examples of quasicrystallic rhombic embeddings

An important class of examples of rhombic embeddings of b-quadgraphs can be constructed using ideas of the grid projection method.

**Example 3.3** (Quasicrystallic rhombic embedding obtained from a plane in $\mathbb{Z}^d$)**.** Let

$$E = \{\mathbf{t} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 : \lambda_1, \lambda_2 \in \mathbb{R}\}$$

be a two-dimensional plane in $\mathbb{R}^d$. Denote the following segments in direction of $\mathbf{e}_1, \ldots, \mathbf{e}_d$ by $s_j = \{\mathbf{t} + \lambda\mathbf{e}_j : \lambda \in [0,1]\}$ for $j = 1, \ldots, d$. We assume that $E$ does not contain any




(a) Straight line $E$.

(b) Modified line $E_\triangle$ with a "hat".

**Figure 3.2:** Example for the usage of the grid-projection method in $\mathbb{Z}^2$. The Voronoi cells which contain $\Omega^{\mathcal{L}}$ are marked in red.

of these segments, that is $s_1, \dots, s_d \not\subset E$. If $E$ contains two different segments $s_{j_1}$ and $s_{j_2}$, the following construction only leads to the standard square grid pattern $\mathbb{Z}^2$. If $E$ contains exactly one such segment $s_j$, then the following construction may be adapted for the remaining dimensions (excluding $\mathbf{e}_j$), but we do not consider this case. We further assume that the orthogonal projections onto $E$ of the two-dimensional facets $E_{j_1,j_2} = \{\lambda_1\mathbf{e}_{j_1} + \lambda_2\mathbf{e}_{j_2} : \lambda_1, \lambda_2 \in [0,1]\}$ for $1 \leq j_1 < j_2 \leq d$ are non-degenerate parallelograms. Then we can choose positive constants $c_1, \dots, c_d$ such that the orthogonal projections $P_E(c_j\mathbf{e}_j)$ have length 1.

Consider around each vertex $\mathbf{p}$ of the lattice $\mathcal{L} = c_1\mathbb{Z} \times \cdots \times c_d\mathbb{Z}$ the hypercuboid $V = [-c_1/2, c_1/2] \times \cdots \times [-c_d/2, c_d/2]$, that is the Voronoi cell $\mathbf{p} + V$. These translations of $V$ then cover $\mathbb{R}^d$. We build an infinite monotone two-dimensional surface $\Omega^{\mathcal{L}}(E)$ in the lattice $\mathcal{L}$ which projects to a rhombic embedding on $E$ by the following construction. The basic idea is illustrated in Figure 3.2 (left) for the toy example of a line in $\mathbb{Z}^2$.

If $E$ intersects the interior of the Voronoi cell of a lattice point (i.e. $(\mathbf{p} + V)^\circ \cap E \neq \emptyset$ for $\mathbf{p} \in \mathcal{L}$), then this point belongs to $\Omega^{\mathcal{L}}(E)$. Undirected edges correspond to intersections of $E$ with the interior of a $(d-1)$-dimensional facet bounding two Voronoi cells. Thus we get a connected graph in $\mathcal{L}$. An intersection of $E$ with the interior of a translated $(d-2)$-dimensional facet of $V$ corresponds to a rectangular two-dimensional face of the lattice. By construction, the orthogonal projection of this graph onto $E$ results in a planar connected graph whose faces are all of even degree (= number of incident edges or of incident vertices). A face of degree bigger than 4 corresponds to an intersection of $E$ with the translation of a $(d-k)$-dimensional facet of $V$ for some $k \geq 3$. Consider the vertices and edges of such a face and the corresponding points and edges in the lattice $\mathcal{L}$. These points lie on a combinatorial $k$-dimensional hypercuboid contained in $\mathcal{L}$. Enumerate the edges $w_1, \dots, w_{2k}$ such that consecutive edges have a vertex in common. Then the edges are such that $w_j$ is parallel to $w_{j+k}$ for $j = 1, \dots, k$. Also the corresponding edges $\hat{w}_1, \dots, \hat{w}_k$ in $\mathcal{L}$ are linearly independent. Therefore there are two points of the $k$-dimensional hypercuboid which are each incident to $k$ of the given vertices. We choose a point with least distance from $E$ and add it to the surface. Adding edges to neighboring vertices splits the face of degree $2k$ into $k$ faces of degree 4.

Thus we obtain an infinite monotone two-dimensional combinatorial surface $\Omega^{\mathcal{L}}(E)$ which projects to an infinite rhombic embedding covering the whole plane $E$.

**Example 3.4** (Modification of the construction in Example 3.3)**.** The method used in the preceding example can be modified to result in similar but different rhombic embeddings. The basic idea of the modified construction is illustrated in Figure 3.2 (right).

Let $E$ be a two-dimensional plane in $\mathbb{R}^d$ as in Example 3.3. We also make the same assumptions, that is $s_1, \dots, s_d \not\subset E$ and the orthogonal projections onto $E$ of the two-dimensional facets $E_{j_1,j_2} = \{\lambda_1\mathbf{e}_{j_1} + \lambda_2\mathbf{e}_{j_2} : \lambda_1, \lambda_2 \in [0,1]\}$ for $1 \leq j_1 < j_2 \leq d$ are non-degenerate parallelograms. Furthermore, we choose positive constants $c_1, \ldots, c_d$ such that $\|P_E(c_j\mathbf{e}_j)\| = 1$.

Let $\mathbf{N} \neq 0$ be any vector orthogonal to $E$. Let $\triangle$ be an equilateral triangle in $E$ with vertices $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ and let $\mathbf{s}$ be the intersection point of the bisecting lines of the angles. Consider the two-dimensional facets of the three-dimensional tetrahedron $T(\triangle, \mathbf{N})$ spanned by the four vertices $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{s} + \mathbf{N}$. Exactly one of these facets is completely contained in $E$ (this is the triangle $\triangle$). We remove the triangle from $E$ and add instead the remaining facets of $T(\triangle, \mathbf{N})$. Let $E_\triangle$ be the resulting two-dimensional surface. Note that $E_\triangle$ is closed and orientable like $E$ and that $E_\triangle$ can be seen as the graph of a real-valued function over $E$. Define $\gamma \in (0, \pi/2)$ by $\gamma = \arctan(2\sqrt{3}\|\mathbf{N}\|/\|\mathbf{t}_1 - \mathbf{t}_2\|)$, where $\|\cdot\|$ denotes the Euclidean norm of vectors in $\mathbb{R}^d$. Then $\gamma$ is the acute angle between $E$ and one of the two-dimensional facets of the tetrahedron $T(\triangle, \mathbf{N})$ not contained in $E$.

By assumption, the intersection of $E$ with each of the $(d-1)$-dimensional hyperspaces

$$H_j = \text{span}\{\mathbf{e}_k : k \in \{1, \ldots, d\} \setminus \{j\}\}$$

for $j = 1, \ldots, d$ is a line $\ell_j \subset E$. Let $\ell_j = \{\mathbf{p}_j + \mathbf{v}_{\ell_j}\lambda : \lambda \in \mathbb{R}\}$ for suitable vectors $\mathbf{p}_j, \mathbf{v}_{\ell_j} \in \mathbb{R}^d$. Let $\mathbf{v}_{\ell_j}^\perp$ be orthogonal to $\mathbf{v}_{\ell_j}$ such that $\mathbf{p}_j + \mathbf{v}_{\ell_j}^\perp \in E$. Then the *intersection angle* $\theta_j \in (0, \pi/2)$ of $E$ and $H_j$ can be defined by

$$\cos\theta_j = \max\{\langle \mathbf{v}_{\ell_j}^\perp, \mathbf{w}\rangle : \mathbf{w} \in H_j, \mathbf{w} \perp \mathbf{v}_{\ell_j}, \|\mathbf{w}\| = 1\}.$$

Assume that $\gamma < \theta_{min}/2$ where $\theta_{min} = \min\{\theta_j : j = 1, \ldots, d\} > 0$ is the smallest possible intersection angle between $E$ and $H_j$.

**Lemma 3.5.** *The intersection of $E_\triangle$ with one of the translated $(d-1)$-dimensional hyperspaces $\mathbf{p}_0 + H_j$ for $j = 1, \ldots, d$ and $\mathbf{p}_0 \in \mathbb{R}^d$ is exactly one curve consisting of at most four different parts of straight lines.*

*The intersection of $E_\triangle$ with one of the translated $(d-2)$-dimensional hyperspaces $\mathbf{p}_0 + H_{j_1,j_2} = \mathbf{p}_0 + span\{\mathbf{e}_k : k \in \{0, \ldots, d\} \setminus \{j_1, j_2\}\}$ for $0 \leq j_1 < j_2 \leq d$ and $\mathbf{p}_0 \in \mathbb{R}^d$ is exactly one point.*

*Proof.* Both properties are consequences of the suitable choice of the angle $\gamma$.

The inequality $\gamma < \theta_{min}/2 < \theta_j$ implies that the intersection of the boundary of the tetrahedron $T(\triangle, \mathbf{N})$ with the hyperspaces $H_j + \mathbf{p}_0$ is either empty or exactly one point of the triangle $\triangle$ or a closed simple curve which has non-empty intersection with the triangle $\triangle$. This implies the first claim.

Note that by our assumptions on $E$ the intersection of a translated $(d-2)$-dimensional hyperspaces $H_{j_1,j_2} + \mathbf{p}_0$ with $E$ is exactly one point. The choice of the angle $\gamma$ implies that $H_{j_1,j_2} + \mathbf{p}_0$ cannot intersect $E \setminus \triangle^\circ$ and $E_\triangle \setminus E$ at the same time, more precisely

$$(H_{j_1,j_2} + \mathbf{p}_0) \cap (E \setminus \triangle^\circ) \neq \emptyset \quad \Longrightarrow \quad (H_{j_1,j_2} + \mathbf{p}_0) \cap (E_\triangle \setminus E) = \emptyset.$$

Thus if $(H_{j_1,j_2} + \mathbf{p}_0) \cap (E_\triangle \setminus E) \neq \emptyset$ we also have $(H_{j_1,j_2} + \mathbf{p}_0) \cap \triangle^\circ = \{\mathbf{q}\} \neq \emptyset$. Now if $(H_{j_1,j_2} + \mathbf{p}_0) \cap (E_\triangle \setminus E) \neq \emptyset$ contains two different points $\{\mathbf{p}_1, \mathbf{p}_2\}$, we can construct an affine line $\mathbf{q} + \lambda((\mathbf{p}_1 - \mathbf{q})\langle \mathbf{p}_2 - \mathbf{q}, \mathbf{N}\rangle - (\mathbf{p}_2 - \mathbf{q})\langle \mathbf{p}_1 - \mathbf{q}, \mathbf{N}\rangle)$ lying in $(H_{j_1,j_2} + \mathbf{p}_0) \cap E$. This is a contradiction as the intersection of $E$ and $H_{j_1,j_2} + \mathbf{p}_0$ contains exactly one point. □

Our goal is to construct a monotone two-dimensional surface in the lattice $\mathcal{L}$ in an analogous way as in the preceding example. Therefore we are interested in properties of the intersection of a Voronoi cell $\mathbf{p}_0 + V$ with $E_\triangle$.

**Corollary 3.6.** *Assume that $(V + \mathbf{p}_0)^\circ \cap E_\triangle \neq \emptyset$. Then the intersection with the boundary of the Voronoi cell, $(\partial V + \mathbf{p}_0) \cap E_\triangle$, is a simple closed curve $\kappa$ which intersects each of the $(d-2)$-dimensional facets of $V + \mathbf{p}_0$ in at most one point.*

*Furthermore, assume that the intersection of $\kappa$ with a $(d-1)$-dimensional facet $F$ of $V + \mathbf{p}_0$ is non empty. Then the intersection of $\kappa$ with the boundary of $F$ (i.e. the union of all incident $(d-2)$-dimensional facets of $V + \mathbf{p}_0$) consists of at most two different points.*

*Proof.* The first claim is a direct consequence of Lemma 3.5.

For the second claim, assume the contrary. Then there are three different points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ of $\kappa$ on the boundary of the $(d-1)$-dimensional facet $F$. By definition we have $F \subset H_j + \mathbf{p}_1$ for some $j \in \{1, \dots, d\}$. By Lemma 3.5 the intersection $E_\triangle \cap H_j + \mathbf{p}_1$ defines a curve $g$. If $g$ was a straight line, $g$ would intersect the boundary of $F$ in at most two points. Thus $g$ has the form $\_\_\diagup\diagdown\_\_$. The assumption on the intersection angles ($\gamma < \theta_{min}/2$) implies that the acute angles between different parts of straight lines are strictly smaller than $\theta_{min}$. We can assume that $\mathbf{p}_1 < \mathbf{p}_2 < \mathbf{p}_3$ using an orientation of $g$. Connect $\mathbf{p}_j$ and $\mathbf{p}_{j+1}$ by straight edges for $j = 1, 2$ to obtain a curve $\hat{\kappa}$ which is contained in the facet $F$. Moreover, the acute angle between the vectors $(\mathbf{p}_3 - \mathbf{p}_2)$ and $(\mathbf{p}_2 - \mathbf{p}_1)$ is also strictly smaller than $\theta_{min}$. Therefore $\mathbf{p}_2$ has to lie in the interior of a $(d-2)$-dimensional facet. But now a translation of this facets has two different intersection points with $\hat{\kappa}$ and by construction there is also a translation of the corresponding $(d-2)$-dimensional hyperspace which has two different intersection points with $\kappa$. This is impossible by Lemma 3.5. □

We now apply the same construction algorithm to $E_\triangle$ as for the plane $E$ in the previous example. As above, if $E_\triangle$ intersects a $(d-2)$-dimensional facet of a Voronoi cell in its interior, we get the corresponding two-dimensional face of the lattice $\mathcal{L}$. By Corollary 3.6 this face has exactly one neighboring face for each of its incident edges. If $E_\triangle$ intersects a $(d-k)$-dimensional facet of a Voronoi cell for $k \geq 3$ then we apply the same reasoning as in the previous example. Thus we obtain a closed two-dimensional surface $\Omega^{\mathcal{L}}(E_\triangle)$ in the lattice $\mathcal{L}$. As every $(d-2)$-dimensional hyperspace intersects $E_\triangle$ exactly once by Lemma 3.5, this surface is monotone.

Furthermore, $E$ and $E_\triangle$ can be oriented. We choose the orientations such that they agree on $E \setminus \triangle$. Also the monotone surface $\Omega^{\mathcal{L}}(E)$ can be oriented. We choose the orientation such that positively oriented orthogonal vector to $E$ are are positively oriented with respect to the orientation of $\Omega^{\mathcal{L}}(E)$. This is possible because of the monotonicity of $\Omega^{\mathcal{L}}(E)$. As $\Omega^{\mathcal{L}}(E_\triangle)$ is also monotone and coincides with $\Omega^{\mathcal{L}}(E)$ except for finitely many faces, we can take the corresponding orientation. Due to the monotonicity, every positively oriented orthogonal vector to $E$ can intersect $\Omega^{\mathcal{L}}(E_\triangle)$ at most once and the same is true for negatively oriented orthogonal vectors. Thus the orthogonal projection of $\Omega^{\mathcal{L}}(E_\triangle)$ onto $E$ is a rhombic embedding which coincides with the rhombic embedding from $\Omega^{\mathcal{L}}(E)$ except for a finite part.

**Remark 3.7.** The construction used for Example 3.4 can be generalized further using non-equilateral triangles and less restrictions on the angles of the tetrahedron built from this triangle. Moreover, the combinatorial surfaces $\Omega^{\mathcal{L}}(E_\triangle)$ can also be obtained using a suitable sequence of flips which is a more general proceedure explained below in Section 3.5.

**Definition 3.8.** An infinite rhombic embedding which can be obtained by the constructions and methods presented in Examples 3.3 and 3.4 or indicated in Remark 3.7 is called a *plane based quasicrystallic rhombic embedding.*

## 3.3 Properties of discrete Green's function and some consequences

Throughout this section, we assume that $\mathscr{D}$ is a (possibly infinite) simply connected quasicrystallic rhombic embedding of a b-quad-graph with edge directions $\mathcal{A} = \{\pm a_1, \dots, \pm a_d\}$ and associated graph $G$. Also, the edge lengths of $\mathscr{D}$ are supposed to be normalized to one.

Fix some interior vertex $x_0 \in V_{int}(G)$. Following Kenyon [49] and Bobenko, Mercat, and Suris [14], we define the *discrete Green's function* $\mathcal{G} : V(G) \to \mathbb{R}$ by

$$\mathcal{G}(x_0, x) = -\frac{1}{4\pi^2 i} \int_\Gamma \frac{\log(\lambda)}{2\lambda} e(x; \lambda) d\lambda \tag{3.1}$$

for all $x \in V(G)$. Here $e(x; z) = \prod_{k=1}^d \left(\frac{z+a_k}{z-a_k}\right)^{n_k}$ is the *discrete exponential function*, where $\mathbf{n} = (n_1, \dots, n_d) = \hat{\mathbf{x}} - \hat{\mathbf{x}}_0 \in \mathbb{Z}^d$ and $\hat{\mathbf{x}}, \hat{\mathbf{x}}_0 \in V(\Omega_\mathscr{D})$ correspond to $x, x_0 \in V(\mathscr{D})$. The integration path $\Gamma$ is a collection of $2d$ small loops, each one running counterclockwise around one of the points $\pm a_k$ for $k = 1, \dots, d$. The branch of $\log(\lambda)$ depends on $x$ and is chosen as follows. Without loss of generality, we assume that the circular order of the points of $\mathcal{A}$ on the positively oriented unit circle $\mathbb{S}^1$ is $a_1, \dots, a_d, -a_1, \dots, -a_d$. Set $a_{k+d} = -a_k$ for $k = 1, \dots, d$ and define $a_m$ for all $m \in \mathbb{Z}$ by $2d$-periodicity. To each $a_m = e^{i\theta_m} \in \mathbb{S}^1$ we assign a certain value of the argument $\theta_m \in \mathbb{R}$: choose $\theta_1$ arbitrarily and then use the rule

$$\theta_{m+1} - \theta_m \in (0, \pi) \qquad \text{for all } m \in \mathbb{Z}.$$

Clearly we then have $\theta_{m+d} = \theta_m + \pi$. The points $a_m$ supplied with the arguments $\theta_m$ can be considered as belonging to the Riemann surface of the logarithmic function (i.e. a branched covering of the complex $\lambda$-plane). Since $\Omega_\mathscr{D}$ is a monotone surface, there is an $m \in \{1, \dots, 2d\}$ and a directed path from $x_0$ to $x$ in $\mathscr{D}$ such that the directed edges of this path are contained in $\{a_m, \dots, a_{m+d-1}\}$ (see [14, Lemma 18]). Now, the branch of $\log(\lambda)$ in (3.1) is chosen such that

$$\log(a_l) \in [i\theta_m, i\theta_{m+d-1}], \qquad l = m, \dots, m+d-1.$$

**Definition 3.9.** Let $\mathscr{D}$ be a b-quad-graph with associated graphs $G$ and $G^*$ and let $\alpha$ be an admissible labelling. Let $\mathscr{D}$ be simply connected and assume that there is an isoradial circle pattern for $G$ and $\alpha$. Let $\eta : V(G) \to \mathbb{R}$ be a function.

Define a *discrete Laplacian* of $\eta$ at an interior vertex $z \in V_{int}(G)$ with incident vertices $z_1, \dots, z_m$ in $G$ by

$$\Delta\eta(z) := \sum_{j=1}^m c([z, z_j])(\eta(z_j) - \eta(z)). \tag{3.2}$$

where

$$c([z, z_j]) := 2f'_{\alpha([z,z_j])}(0). \tag{3.3}$$

**Remark 3.10.** Consider a kite (rhombus) of an isoradial circle pattern as in the preceeding definition with white vertices $z_1$ and $z_2$ and black vertices $v_1$ and $v_2$. Then simple calculations (see also Remark 2.6) imply, that the weight $c([z_1, z_2])$ is the quotient of the length of the diagonals of the kite. More precisely,

$$c([z_1, z_2]) = \frac{|v_2 - v_1|}{|z_2 - z_1|}.$$

This choice for the weights has already been used in [35, 14].

As $2f'_{\alpha([z,z_j])}(0) > 0$ we immediately have the following

**Lemma 3.11** (Maximum Principle). *If $\Delta\eta \geq 0$ on $V_{int}(G)$ then the maximum of $\eta$ is attained at the boundary $V_\partial(G)$.*

In the following, we will frequently use the notation $f(u_1, \dots, u_k) = \mathcal{O}(g(u_1, \dots, u_k))$. This means that there is a constant $C$, depending only on global assumptions, but not on $u_1, \dots, u_k$ such that $|f(u_1, \dots, u_k)| \leq Cg(u_1, \dots, u_k)$ holds.

**Lemma 3.12** ([49, Theorems 7.1 and 7.3]). *The discrete Green's function $\mathcal{G}(x_0, \cdot)$ defined in equation* (3.1) *has the following properties.*

*(i)* $\Delta\mathcal{G}(x_0, v) = -\delta_{x_0}(v)$, *where the Laplacian is taken with respect to the second variable.*

*(ii)* $\mathcal{G}(x_0, x_0) = 0$.

*(iii)* $\mathcal{G}(x_0, v) = \mathcal{O}(\log(|v - x_0|))$.

Note that $\mathcal{G}(x_0, \cdot)$ may also be defined by these three conditions.

### *3.3.1 Asymptotic development for discrete Green's function*

Kenyon derived an asymptotic development for the discrete Green's function using standard methods of complex analysis. His result can be slightly strengthened to the following estimation with error of order $\mathcal{O}(1/|v-x_0|^2)$. Note, that there is the summand $-\log 2/(2\pi)$ missing in Kenyon's formula (but not in his proof).

**Theorem 3.13** (cf. [49, Theorem 7.3]). *For $v \in V(G)$ there holds*

$$\mathcal{G}(x_0, v) = -\frac{1}{2\pi}\log(2|v - x_0|) - \frac{\gamma_{Euler}}{2\pi} + \mathcal{O}\left(\frac{1}{|v - x_0|^2}\right). \tag{3.4}$$

*Here $\gamma_{Euler}$ denotes the* Euler constant *which was defined in 1735 by Leonhard Euler as the limiting difference between the harmonic series and the natural logarithm:*

$$\gamma_{Euler} = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log(n)\right).$$

*Furthermore, suitable calculations show that $\gamma_{Euler}$ satisfies*

$$\gamma_{Euler} = \int_0^1 \frac{1 - e^{-t}}{t}dt - \int_0^1 \frac{e^{-t}}{t}dt,$$

*see for example [76, 22] for additional information.*

*Proof.* Consider a directed path $x_0 = w_0, \ldots, w_k = v$ in $\mathscr{D}$ from $x_0$ to $v$ such that the directed edges of this path are contained in $\{a_m, \ldots, a_{m+d-1}\}$ for some $1 \leq m \leq 2d$ as above. Note that $k$ is even since $x_0$ and $v$ are both white vertices of $\mathscr{D}$. The integration path $\Gamma$ in definition (3.1) can be deformed into a connected contour lying on a single leaf of the Riemann surface of the logarithm, in particular to a simple closed curve $\Gamma_1$ which surrounds the set $\{a_m, \ldots, a_{m+d-1}\}$ in a counterclockwise sense and has the origin and a ray $\mathscr{R} = \{se^{i\tilde{\theta}} : s > 0\}$ in its exterior. In particular, we also assume that $\Gamma_1$ is contained in the sector $\{z = re^{i\varphi} : r > 0,\ \varphi \in [\tilde{\theta} + \frac{\pi}{2} + \eta, \tilde{\theta} + \frac{3\pi}{2} - \eta]\}$ for some $\eta > 0$ independent of $v$, $x_0$, and $m$. This is possible due to the fact that $\theta_{m+d-1} - \theta_m < \pi - \delta$ for some $\delta > 0$ independent of $v$, $x_0$ and $m$.

Let $N = |v - x_0|$. Take $0 < \varrho_1 \ll 1/N^3$ and $\varrho_2 \gg N^3$, but not exponentially smaller than $1/N$ or bigger than $N$ respectively. (For example $\varrho_1 = C_1/N^4$ and $\varrho_2 = C_2N^4$ for two suitable constants $C_1, C_2 > 0$.) The curve $\Gamma_1$ is again homotopic to a curve $\Gamma_2$ which runs counterclockwise around the circle of radius $\varrho_2$ about the origin from the angle $\tilde{\theta}$ to $\tilde{\theta} + 2\pi$, then along the ray $\mathscr{R}$ from $\varrho_2 e^{i\tilde{\theta}}$ to $\varrho_1 e^{i\tilde{\theta}}$, then clockwise around the circle of radius $\varrho_1$ about the origin from the angle $\tilde{\theta} + 2\pi$ to $\tilde{\theta}$, and finally back along the ray $\mathscr{R}$ from $\varrho_1 e^{i\tilde{\theta}}$ to $\varrho_2 e^{i\tilde{\theta}}$. Without loss of generality, we assume that $\mathscr{R}$ is the negative real axis.

Kenyon showed in [49] that the integrals along the circles of radius $\varrho_1$ and $\varrho_2$ give

$$(-1)^k\frac{\log\varrho_1}{4\pi}(1 + \mathcal{O}(N\varrho_1)) - \frac{\log\varrho_2}{4\pi}(1 + \mathcal{O}(N/\varrho_2)).$$

The difference between the value of $\log z$ above and below the negative real axis is $2\pi i$. Thus the two integrals along the negative real axis can be combined into

$$-\frac{1}{4\pi}\int_{-\varrho_2}^{-\varrho_1}\frac{1}{z}\prod_{j=0}^{k-1}\frac{z+b_j}{z-b_j}dz,$$

where $b_j = w_{j+1} - w_j = \mathrm{e}^{i\beta_j} \in \mathcal{A}$ is the directed edge from $w_j$ to $w_{j+1}$ and $k = \mathcal{O}(N)$ is the number of edges of the path. This integral can be split into three parts: from $-\varrho_2$ to $-\sqrt{N}$, from $-\sqrt{N}$ to $-1/\sqrt{N}$, and from $-1/\sqrt{N}$ to $-\varrho_1$.

Kenyon proved that the integral is neglectible for the intermediate range because

$$\left|\frac{t+\mathrm{e}^{i\beta}}{t-\mathrm{e}^{i\beta}}\right| \leq \mathrm{e}^{2t\cos\beta/(t-1)^2}$$

for negative $t<0$ and due to our assumptions.

For small $|t|$ we have

$$\prod_{j=0}^{k-1}\frac{t+b_j}{t-b_j} = \prod_{j=0}^{k-1}\frac{t\bar{b}_j+1}{t\bar{b}_j-1} = (-1)^k\mathrm{e}^{2\sum_{j=0}^{k-1}\bar{b}_j t}(1+\mathcal{O}(kt^3)),$$

since

$$\frac{1+t\bar{b}_j}{1-t\bar{b}_j}\mathrm{e}^{-2\bar{b}_j t} = \left(1+\frac{2t\bar{b}_j}{1-t\bar{b}_j}\right)(1-2t\bar{b}_j+4t^2\bar{b}_j^2+\mathcal{O}(t^3)) = 1+O(t^3)$$

using the Neumann series. Thus the integral near the origin is

$$-\frac{(-1)^k}{4\pi}\left(\int_{-1/\sqrt{N}}^{-\varrho_1}\frac{\mathrm{e}^{2(\bar{v}-\bar{x}_0)t}}{t}dt + \int_{-1/\sqrt{N}}^{-\varrho_1}\mathcal{O}(kt^3)\frac{\mathrm{e}^{2(\bar{v}-\bar{x}_0)t}}{t}dt\right).$$

Applying similar reasonings and estimations as in Kenyon's proof, we obtain

$$-\frac{(-1)^k}{4\pi}\left(\log(2\varrho_1(\bar{v}-\bar{x}_0))+\gamma_{\mathrm{Euler}}\right)+\mathcal{O}\left(\frac{1}{N^2}\right).$$

Here $\gamma_{\mathrm{Euler}}$ denotes the Euler $\gamma$ constant.

For large $|t|$ the estimations are very similar. Since

$$\prod_{j=0}^{k-1}\frac{t+b_j}{t-b_j} = \prod_{j=0}^{k-1}\frac{1+b_jt^{-1}}{1-b_jt^{-1}} = \mathrm{e}^{2\sum_{j=0}^{k-1}b_jt^{-1}}(1+\mathcal{O}(kt^{-3})),$$

we get

$$-\frac{(-1)^k}{4\pi}\left(\int_{-\varrho_2}^{-\sqrt{N}}\frac{\mathrm{e}^{2(v-x_0)t^{-1}}}{t}dt + \int_{-\varrho_2}^{-\sqrt{N}}\mathcal{O}(kt^{-3})\frac{\mathrm{e}^{2(v-x_0)t^{-1}}}{t}dt\right)$$
$$= -\frac{1}{4\pi}\left(-\log(\frac{\varrho_2}{2(v-x_0)})+\gamma_{\mathrm{Euler}}\right)+\mathcal{O}\left(\frac{1}{N^2}\right).$$

Since $k$ is even, the sum of all the above integral parts is therefore

$$-\frac{1}{2\pi}\log(2|v-x_0|)-\frac{\gamma_{\mathrm{Euler}}}{2\pi}+\mathcal{O}(1/N^2).$$

□

For a bounded domain we also consider a discrete Green's function with vanishing boundary values. Let $W \subset V_{int}(G)$ be a finite subset of vertices. Denote by $W_\partial \subset W$ the set of boundary vertices which are incident to at least one vertex in $V(G) \setminus W$. Set $W_{int} = W \setminus W_\partial$ the interior vertices of $W$. Let $x_0 \in W_{int}$ be an interior vertex. The *discrete Green's function* $\mathcal{G}_W(x_0, \cdot)$ is uniquely defined by the following properties.

(i) $\Delta \mathcal{G}_W(x_0, v) = -\delta_{x_0}(v)$ for all $v \in W_{int}$, where the Laplacian is taken with respect to the second variable.

(ii) $\mathcal{G}_W(x_0, v) = 0$ for all $v \in W_\partial$.

In the following, we choose $W$ to be a special disk-like set. Let $x_0 \in V(G)$ be a vertex and let $\rho > 2$. Denote the closed disk with center $x_0$ and radius $\rho$ by $B_\rho(x_0) \subset \mathbb{C}$. Suppose that this disk is entirely covered by the rhombic embedding $\mathscr{D}$. Denote by $V(x_0, \rho) \subset V(G)$ the set of white vertices lying within $B_\rho(x_0)$. For $x_1 \in V_{int}(x_0, \rho)$ we denote

$$\mathcal{G}_{x_0,\rho}(x_1, \cdot) = \mathcal{G}_{V(x_0,\rho)}(x_1, \cdot).$$

**Proposition 3.14.** *(1) There is a constant $C_1$ independent of $\rho$ and $x_0$ such that*

$$|\mathcal{G}_{x_0,\rho}(x_0, v)| \leq C_1/\rho$$

*for all vertices $v \in V_{int}(x_0, \rho)$ which are incident to a boundary vertex in $V_\partial(x_0, \rho)$.*

*(2) Let $x_1 \in V_{int}(x_0, \rho)$ be an interior vertex incident to $x_0$. Then there is a constant $C_2$ independent of $\rho$ such that for all vertices $v \in V(x_0, \rho)$ there holds*

$$|\mathcal{G}_{x_0,\rho}(x_0, v) - \mathcal{G}_{x_0,\rho}(x_1, v)| \leq C_1/(|v - x_0| + 1).$$

*Proof.* Consider the function $h_\rho(x_0, \cdot) : V(x_0, \rho) \to \mathbb{R}$ defined by

$$h_\rho(x_0, v) = \mathcal{G}_{x_0,\rho}(x_0, v) - \mathcal{G}(x_0, v) - \tfrac{1}{2\pi}(\log(2\rho) + \gamma_{\text{Euler}}).$$

$h_\rho(x_0, \cdot)$ is harmonic on $V_{int}(x_0, \rho)$ by definition of $\mathcal{G}$ and $\mathcal{G}_{x_0,\rho}$. For boundary vertices $v \in V_\partial(x_0, \rho)$, Theorem 3.13 implies

$$h_\rho(x_0, v) = -\mathcal{G}(x_0, v) - \frac{1}{2\pi}(\log(2\rho) + \gamma_{\text{Euler}}) = \frac{1}{2\pi} \log(\frac{|v - x_0|}{\rho}) + \mathcal{O}(\frac{1}{|v - x_0|^2}) = \mathcal{O}(\frac{1}{\rho}).$$

The Maximum Principle 3.11 yields $|h_\rho(x_0, v)| \leq C/\rho$ for all $v \in V(x_0, \rho)$ and some constant $C$ independent of $\rho$ and $v$. This shows the first estimation.

To prove the second claim, we also consider the harmonic function

$$h_\rho(x_1, v) = \mathcal{G}_{x_0,\rho}(x_1, v) - \mathcal{G}(x_1, v) - \frac{1}{2\pi}(\log(2\rho) + \gamma_{\text{Euler}})$$

for an fixed interior vertex $x_1 \in V_{int}(x_0, \rho)$ incident to $x_0$. By similar reasonings as for $h_\rho(x_0, \cdot)$, we deduce that $|h_\rho(x_1, v)| \leq \tilde{C}/\rho$ for all $v \in V(x_0, \rho)$ and some constant $\tilde{C}$ independent of $\rho$ and $v$. Theorem 3.13 implies that

$$\mathcal{G}_{x_0,\rho}(x_0, v) - \mathcal{G}_{x_0,\rho}(x_1, v) = \mathcal{G}(x_0, v) - \mathcal{G}(x_1, v) + h_\rho(x_0, v) - h_\rho(x_1, v) = \mathcal{O}\left(\frac{1}{|v - x_0| + 1}\right).$$

□

### *3.3.2 Regularity of discrete solutions of elliptic equations*

Our next aim is to use the asymptotics of the discrete Green's function in order to prove Hölder's inequality for harmonic functions.

The following discrete version of Green's Identity is a straightforward generalization of Lemma 1 in [33].

**Lemma 3.15** (Green's Identity, cf. [33]). *Let $W \subset V(G)$ be a finite subset of vertices. Let $u, v : W \to \mathbb{R}$ be two functions. Then*

$$\sum_{x \in W_{int}} (v(x)\Delta u(x) - u(x)\Delta v(x)) = \sum_{[p,q] \in E_\partial(W)} c([p,q])(v(p)u(q) - u(p)v(q)), \tag{3.5}$$

*where $E_\partial(W) = \{[p,q] \in E(W) : p \in W_{int}, q \in W_\partial\}$ and $c([p,q])$ is defined in* (3.3).

**Corollary 3.16** (Representation of harmonic functions; cf. [33, Theorem 3]). *Let $u$ be a real valued harmonic function defined on $V(x_0,\rho)$. Then*

$$u(x_0) = \sum_{q \in V_\partial(x_0,\rho)} c(q)u(q),$$

*where*

$$c(q) = \sum_{\substack{p \in V_{int}(x_0,\rho) \\ \text{and } [p,q] \in E(G)}} c([p,q])\mathcal{G}_{x_0,\rho}(x_0,p) = \mathcal{O}(1/\rho). \tag{3.6}$$

The estimation in (3.6) is a consequence of Proposition 3.14 and of the boundedness of the weights $c(e)$ or equivalently of the intersection angles $\alpha(e)$.

The following theorem can be interpreted as an analog to the Theorem of Gauss in potential theory.

**Theorem 3.17** (cf. [33, Theorem 4]). *Let $u : V(x_0,\rho) \to \mathbb{R}$ be a non-negative harmonic function. There is a constant $C_3$ independent of $\rho$ and $u$ such that*

$$\left| u(x_0) - \frac{1}{\pi\rho^2} \sum_{v \in V_{int}(x_0,\rho)} F^*(v)u(v) \right| \leq \frac{C_3 u(x_0)}{\rho}, \tag{3.7}$$

*where*

$$F^*(v) = \frac{1}{4} \sum_{[z,v] \in E(G)} c([z,v])|z-v|^2 = \frac{1}{2} \sum_{[z,v] \in E(G)} \sin\alpha([z,v])$$

*is the area of the face of the dual graph $G^*$ corresponding to the vertex $v \in V_{int}(G)$.*

*Proof.* Consider the function

$$p(z) = \mathcal{G}(x_0,z) + \frac{1}{2\pi}(\log(2\rho) + \gamma_{\text{Euler}}) + \frac{|z-x_0|^2 - \rho^2}{4\pi\rho^2}.$$

In order to calculate $\Delta p$, let $z \in V_{int}(G)$ be an interior vertex. Denote by $z_1, \dots, z_m$ the incident vertices of $z$ in $G$ in counterclockwise order. Consider the chain of faces $f_j$ of $\mathscr{D}$ ($j = 1, \dots, m$) which are incident to $z$ and $z_j$. The enumeration of the vertices $z_j$ (and hence of the faces $f_j$) and of the black vertices $v_1, \dots, v_m$ incident to these faces can be chosen such that $f_j$ is incident to $v_{j-1}$ and $v_j$ for $j = 1, \dots, m$, where $v_0 = v_m$. Furthermore, using this enumeration we have

$$\frac{z_j - z}{|z_j - z|} i = \frac{v_j - v_{j-1}}{|v_j - v_{j-1}|} \quad \Longleftrightarrow \quad \frac{|v_j - v_{j-1}|}{|z_j - z|}(z_j - z) = -i(v_j - v_{j-1}); \tag{3.8}$$

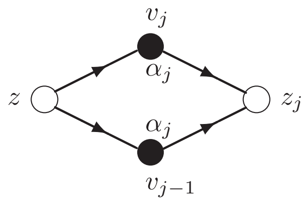



**Figure 3.3:** A rhombic face of $\mathscr{D}$ with oriented edges.

see Figure 3.3. Denoting

$$\alpha_j = \alpha([z, z_j]), \quad l_j = |z_j - z| = 2\sin(\alpha_j/2), \quad \hat{l}_j = |v_j - v_{j-1}| = 2\cos(\alpha_j/2),$$

we easily obtain by a simple calculation that

$$c([z, z_j]) = 2f'_{\alpha_j}(0) = \frac{\hat{l}_j}{l_j} = \frac{|z_j - z|}{|v_j - v_{j-1}|}.$$

Since $|z_j - x_0|^2 - |z - x_0|^2 = -2\operatorname{Re}((\overline{z - x_0})(z_j - z)) + |z_j - z|^2$, the properties of $\mathcal{G}$ and the definition of $F^*(z)$ imply that

$$\begin{aligned}
\Delta p(z) &= \Delta\mathcal{G}(x_0, v) + \frac{1}{4\pi\rho^2}\sum_{j=1}^{m} c([z, z_j])(|z_j - x_0|^2 - |z - x_0|^2) \\
&= -\delta_{x_0}(z) + \operatorname{Re}\left(\frac{2(\overline{z - x_0})}{4\pi\rho^2}\sum_{j=1}^{m}\underbrace{\frac{|v_j - v_{j-1}|}{|z_j - z|}(z_j - z)}_{=-i(v_j - v_{j-1})}\right) + \frac{1}{4\pi\rho^2}\sum_{j=1}^{m} c([z, z_j])|z_j - z|^2 \\
&= -\delta_{x_0}(z) + F^*(z)/(\pi\rho^2).
\end{aligned}$$

Let $v$ be incident to a vertex of $V_\partial(x_0, \rho)$. Theorem 3.13 implies that

$$\begin{aligned}
p(v) &= -\frac{1}{2\pi}\log\frac{|v - x_0|}{\rho} + \frac{|v - x_0|^2 - \rho^2}{4\pi\rho^2} + \mathcal{O}\left(\frac{1}{|v - x_0|^2}\right) \\
&= -\frac{1}{2\pi}\log\left(1 + \frac{|v - x_0| - \rho}{\rho}\right) + \frac{|v - x_0| - \rho}{2\pi\rho}\left(1 + \frac{|v - x_0| - \rho}{2\rho}\right) + \mathcal{O}\left(\frac{1}{|v - x_0|^2}\right) \\
&= \mathcal{O}(1/\rho^2).
\end{aligned}$$

Thus there is a constant $B_1$ independent of $\rho$ and $v$ such that $p_1(v) := p(v) + B_1/\rho^2 \geq 0$ and $|p_1(v)| \leq 2B_1/\rho^2$ for all vertices $v \in V_{int}(x_0, \rho)$ incident to a vertex of $V_\partial(x_0, \rho)$. Applying Green's Identity 3.15 to $p_1$ and the non-negative harmonic function $u$, we obtain

$$\begin{aligned}
u(x_0) - \frac{1}{\pi\rho^2}\sum_{v \in V_{int}(x_0,\rho)} F^*(v)u(v) &= \sum_{x \in V_{int}(x_0,\rho)} (p_1(x)\Delta u(x) - u(x)\Delta p_1(x)) \\
&= \sum_{[z,q] \in E_\rho} c([z, q])(p_1(z)u(q) - \underbrace{u(z)p_1(q)}_{\geq 0}), \\
&\leq \frac{2B_1}{\rho^2}4\pi\sum_{q \in V_\partial(x_0,\rho)} u(q) \\
&\leq \frac{8\pi B_1 B_2}{\rho}u(x_0),
\end{aligned}$$

Here $E_\rho = \{[p,q] \in E(G) : p \in V_{int}(x_0,\rho),\ q \in V_\partial(x_0,\rho)\}$ and we have used the estimations

$$\sum_{[p,q]\in E(G)} c([p,q]) = c(p) \le \sum_{[p,q]\in E(G)} c([p,q])|p-q|^2 = 4F^*(p) < 4\pi$$

for all vertices $p \in V_{int}(G)$ and

$$\sum_{q\in V_\partial(x_0,\rho)} u(q) \le B_2 \rho u(x_0)$$

for some constant $B_2 > 0$, which is a consequence of Corollary 3.16.

For the reverse inequality, note that there is also a constant $B_3$ independent of $\rho$ and $v$ such that $p_1(v) := p(v) - B_3/\rho^2 \le 0$ and $|p_1(v)| \le 2B_3/\rho^2$ for all vertices $v \in V_{int}(x_0,\rho)$ incident to a vertex in $V_\partial(x_0,\rho)$. Combining both estimation proves the claim. □

Using Theorem 3.17 we deduce Hölder's Inequality for non-negative harmonic function.

**Theorem 3.18** (Hölder's Inequality, cf. [33, Theorem 5])**.** *Let $u : V(x_0,\rho) \to \mathbb{R}$ be a non-negative harmonic function. There is a constant $C_4$ independent of $\rho$ and $u$ such that*

$$|u(x_0) - u(x_1)| \le C_4 u(x_0)/\rho \tag{3.9}$$

*for all vertices $x_1 \in V(x_0,\rho)$ incident to $x_0$.*

*Proof.* Our proof adapts Duffin's ideas and uses Theorem 3.17.

First note that the harmonicity of $u$ at $x_0$, that is

$$\Delta u(x_0) = 0 \iff \sum_{[v,x_0]\in E(G)} c([v,x_0])u(v) = \sum_{[v,x_0]\in E(G)} c([v,x_0])u(x_0),$$

together with the boundedness of the intersection angles $\alpha(e)$ and thus of the weights $c(e)$ implies that for all vertices $v$ incident to $x_0$ we have

$$-B_1 u(x_0) \le u(v) \le B_2 u(x_0),$$

where $B_1, B_2 > 0$ are two constants. This proves the claim for 'small' $\rho$, say $\rho < 10$.

Now assume that $\rho \ge 10$ and denote $\rho_0 = \rho - 4$, $\rho_1 = \rho - 2$. Let $x_1$ be incident to $x_0$. We apply Theorem 3.17 for $V(x_0,\rho_0)$ and $V(x_1,\rho_1)$ and obtain

$$\rho_1^2 u(x_1) = \frac{1}{\pi} \sum_{v\in V_{int}(x_1,\rho_1)} F^*(v)u(v) + u(x_1)\mathcal{O}(\rho_1),$$
$$\rho_0^2 u(x_0) = \frac{1}{\pi} \sum_{v\in V_{int}(x_0,\rho_0)} F^*(v)u(v) + u(x_0)\mathcal{O}(\rho_0).$$

Since $\rho_0^2 = \rho_1^2 + \mathcal{O}(\rho_1)$ we deduce

$$\rho_1^2 u(x_0) = \frac{1}{\pi} \sum_{v\in V_{int}(x_0,\rho_0)} F^*(v)u(v) + u(x_0)\mathcal{O}(\rho_1).$$

By definition we have $V_{int}(x_0,\rho_0) \subset V_{int}(x_1,\rho_1)$ and thus

$$\sum_{v\in V_{int}(x_1,\rho_1)} F^*(v)u(v) - \sum_{v\in V_{int}(x_0,\rho_0)} F^*(v)u(v) \ge 0.$$

This finally implies that

$$\rho_1^2(u(x_1) - u(x_0)) \ge -A\rho_1(u(x_1) + u(x_0))$$

for some constant $A > 0$. Reversing the roles of $x_0$ and $x_1$ gives the reverse estimation and proves the claim. □

As a corollary of Hölder's Inequality and of Proposition 3.14 we obtain the following result on the regularity of discrete solutions to elliptic equations. For further use, we define the norm $\|\eta\|_W := \max\{|\eta(z)| : z \in W\}$.

**Lemma 3.19** (Regularity Lemma)**.** *Let $W \subset V(G)$ and let $u : W \to \mathbb{R}$ be any function. Let $M(u) = \max_{v \in W_{int}} |\Delta u(v)/(4F^*(v))|$, where $F^*(v)$ is the area of the face dual to $v$ as in Theorem 3.17. There are constants $C_5, C_6 > 0$ independent of $W$ and $u$ such that*

$$|u(x_0) - u(x_1)|\rho \leq C_5 \|u\|_W + \rho^2 C_6 M(u) \tag{3.10}$$

*for all vertices $x_1 \in W$ incident to $x_0$, where $\rho$ is the Euclidean distance of $x_0$ to the boundary $W_\partial$.*

*Proof.* Let $x_1 \in W$ be a fixed vertex incident to $x_0$.

First we suppose that $\rho \geq 4$. Consider the auxiliary function $f(z) = M(u)|z - x_0|^2$. Since $|x_1 - x_0| < 2$, we obviously have

$$|f(x_0) - f(x_1)| = M(u)|x_1 - x_0|^2 \leq 4M(u).$$

Let $h : V(x_0, \rho) \to \mathbb{R}$ be the unique harmonic function with boundary values $h(v) = u(v) + f(v)$ for $v \in V_\partial(x_0, \rho)$. Hölder's Inequality (3.9) and the Maximum Principle 3.11 imply that

$$|h(x_0) - h(x_1)|\rho \leq B_1 \|h\|_{V(x_0,\rho)} \leq B_1(\|u\|_W + M(u)\rho^2)$$

for some constant $B_1$ independent of $h$, $\rho$, $x_0$, $x_1$.

Next consider $s = u + f - h$ on $V(x_0, \rho)$. Then

$$\begin{cases} \Delta s = \Delta u + 4F^* M(u) \geq 0 & \text{on } V_{int}(x_0, \rho) \\ s(v) = 0 & \text{for } v \in V_\partial(x_0, \rho) \end{cases}.$$

The Maximum Principle 3.11 implies that $s \leq 0$. Green's Identity 3.15 for $s$ and $\mathcal{G}_{x_0,\rho}$ gives

$$\begin{aligned} s(x_0) + \sum_{v \in V_{int}(x_0,\rho)} \mathcal{G}_{x_0,\rho}(x_0, v)\Delta s(v) &= \sum_{v \in V_{int}(x_0,\rho)} (\mathcal{G}_{x_0,\rho}(x_0, v)\Delta s(v) - s(v)\Delta \mathcal{G}_{x_0,\rho}(x_0, v)) \\ &= \sum_{[p,q] \in E_\rho} c([p, q])(\mathcal{G}_{x_0,\rho}(x_0, p)s(q) - s(p)\mathcal{G}_{x_0,\rho}(x_0, q)) \\ &= 0, \end{aligned}$$

where $E_\rho = \{[p, q] \in E(G) : p \in V_{int}(x_0, \rho), q \in V_\partial(x_0, \rho)\}$. Analogously we obtain

$$s(x_1) + \sum_{v \in V_{int}(x_0,\rho)} \mathcal{G}_{x_0,\rho}(x_1, v)\Delta s(v) = 0.$$

Using the estimation $\Delta s(v) \leq 8F^*(v)M(u)$ we deduce that

$$|s(x_0) - s(x_1)| \leq \sum_{v \in V_{int}(x_0,\rho)} |\mathcal{G}_{x_0,\rho}(x_0, v) - \mathcal{G}_{x_0,\rho}(x_1, v)| 8F^*(v)M(u).$$

Now Proposition 3.14 (2) implies that

$$|s(x_0) - s(x_1)| \leq 8B_2 M(u) \sum_{v \in V_{int}(x_0,\rho)} \frac{F^*(v)}{|v - x_0| + 1} \leq 8B_2 M(u) B_3 \rho,$$

where $B_2$ and $B_3$ are constants independent of $s$, $\rho$, $x_0$, $x_1$.

Combining the above estimations for $f$, $h$, and $s$, we finally obtain

$$\begin{aligned}|u(x_0) - u(x_1)|\rho &\le |s(x_0) - s(x_1) - (f(x_0) - f(x_1)) + h(x_0) - h(x_1)|\rho \\ &\le B_1\|u\|_W + \rho^2(4 + B_1 + 8B_2B_3)M(u).\end{aligned}$$

This implies the claim for $\rho \ge 4$. In order to prove the estimation for small $\rho$ note that

$$-4F^*(x_0)M(u) \le \Delta u(x_0) = \sum_{[x_0,v]\in E(G)} c([x_0,v])(u(v) - u(x_0)) \le 4F^*(x_0)M(u).$$

Thus

$$|u(x_0) - u(x_1)| \le 4F^*(x_0)M(u) + \sum_{[x_0,v]\in E(G)} c([x_0,v])|u(v) - u(x_0)| \le 4\pi M(u) + 8\pi\|u\|_W,$$

since $F^*(x_0) \le \pi$ and $\sum_{[x_0,v]\in E(G)} c([x_0,v]) \le 4\pi$. □

## 3.4 Quasicrystallic circle patterns and integrability

Let $\mathscr{D}$ be a quasicrystallic rhombic embedding of a b-quad-graph. The combinatorial surface $\Omega_{\mathscr{D}}$ in $\mathbb{Z}^d$ is important by its connection with integrability. See also [18] for a more detailed presentation and a deepened study of integrability and consistency. Assume we have given a function on all vertices of $\Omega_{\mathscr{D}}$ which satisfies some 3D-consistent equation on all faces of $\Omega_{\mathscr{D}}$. Then the function can be extended to the *brick*

$$\Pi(\Omega_{\mathscr{D}}) := \{\mathbf{n} = (n_1,\dots,n_d) \in \mathbb{Z}^d : \min_{\mathbf{n}\in V(\Omega_{\mathscr{D}})} n_k \le n_k \le \max_{\mathbf{n}\in V(\Omega_{\mathscr{D}})} n_k,\ k = 1,\dots,d\}.$$

Note that the brick $\Pi(\Omega_{\mathscr{D}})$ is the hull of $\Omega_{\mathscr{D}}$. A proof may be found in [14, Section 6]. This extension of a function using a 3D-consistent equation will now be applied for circle patterns.

**Definition 3.20.** Denote by $R$ and $\phi$ the (constant) radius and the angle function respectively of the isoradial circle pattern which corresponds to the quasicrystallic rhombic embedding $\mathscr{D}$. Assume that there is another circle pattern $\mathscr{C}$ with the same combinatorics and the same intersection angles. Denote its radius and angle function by $r$ and $\varphi$ respectively. Define a *comparison function* $w : V(\mathscr{D}) \to \mathbb{C}$ by

$$\begin{cases} w(y) = r(y) & \text{for } y \in V(G), \\ w(x) = \mathrm{e}^{i(\varphi(\vec{e})-\phi(\vec{e}))} \in \mathbb{S}^1 & \text{for } x \in V(G^*) \text{ and any directed edge } \vec{e} = \overrightarrow{xy}. \end{cases} \tag{3.11}$$

Note that $w(y)/R$ is the scaling factor of the circle corresponding to $y \in V(G)$ and $w(x)$ gives the rotation of the edge-star at $x \in V(G^*)$ when changing from the isoradial circle pattern to $\mathscr{C}$. Furthermore, $w$ satisfies the following *Hirota Equation* for all faces $f \in F(\mathscr{D})$, which is a version of equation (2.16).

$$w(x_0)w(y_0)a_0 - w(x_1)w(y_0)a_1 - w(x_1)w(y_1)a_0 + w(x_0)w(y_1)a_1 = 0 \tag{3.12}$$

Here $x_0, x_1 \in V(G^*)$ and $y_0, y_1 \in V(G)$ are the black and white vertices incident to $f$ and $a_0 = x_0 - y_0$ and $a_1 = x_1 - y_0$ are the directed edges. Remember that equation (3.12) is the closing condition for the kite corresponding to the face $f$, see Lemma 2.20. Furthermore, the Hirota Equation is 3D-consistent (direct computation); see Sections 10 and 11 of [14] for more details. 3D-consistency implies that $w$ considered as a function on $V(\Omega_{\mathscr{D}})$ can be extended to the brick $\Pi(\Omega_{\mathscr{D}})$ such that equation (3.12) holds on all two-dimensional facets. This extension – also denoted by $w$ – can be obtained in the following way. Assume that

$w$ is defined at some point $\mathbf{p} \in \mathbb{Z}^d$ and at three incident vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ such that these points span a three-dimensional unit cube in the lattice $\mathbb{Z}^d$. Then equation (3.12) on the two-dimensional faces defines the values of $w$ for the remaining four points of this cube. Although there are several possibilities to determine these values, this does note change the result. We never encounter any contradition by applying this method because of the 3D-consistency. Additionally, $w$ is real valued on white points $V_w(\Omega_{\mathscr{D}})$ and has value in $\mathbb{S}^1$ for black points $V_b(\Omega_{\mathscr{D}})$. The extension of $w$ also has this property for white and black vertices of the brick $\Pi(\Omega_{\mathscr{D}})$. This can easily be deduced from the Hirota Equation (3.12), which can be rewritten in the following equivalent forms:

$$\frac{w(y_1)}{w(y_0)} = \frac{a_1 w(x_1) - a_0 w(x_0)}{a_1 w(x_0) - a_0 w(x_1)} \iff \frac{w(x_1)}{w(x_0)} = \frac{a_1 w(y_1) + a_0 w(y_0)}{a_1 w(y_0) + a_0 w(y_1)}.$$

The extension of $w$ can be used to define a radius function for any rhombic embedding with the same boundary faces as $\mathscr{D}$.

**Lemma 3.21.** *Let $\mathscr{D}$ and $\mathscr{D}'$ be two simply connected finite rhombic embeddings of b-quad-graphs with the same edge directions. Assume that $\mathscr{D}$ and $\mathscr{D}'$ agree on all boundary faces. Let $\mathscr{C}$ be an (embedded) planar circle pattern for $\mathscr{D}$ and the labelling given by the rhombic embedding. Then there is an (embedded) planar circle pattern $\mathscr{C}'$ for $\mathscr{D}'$ which agrees with $\mathscr{C}$ for all boundary circles.*

*Proof.* Consider the monotone combinatorial surfaces $\Omega_{\mathscr{D}}$ and $\Omega'_{\mathscr{D}}$. Without loss of generality, we can assume that $\Omega_{\mathscr{D}}$ and $\Omega'_{\mathscr{D}}$ have the same boundary faces in $\mathbb{Z}^d$. Thus they both define the same brick $\Pi(\Omega_{\mathscr{D}}) = \Pi(\Omega'_{\mathscr{D}}) =: \Pi$. Given the circle pattern $\mathscr{C}$, define the function $w$ on $V(\Omega_{\mathscr{D}})$ by (3.11). Extend $w$ to the brick $\Pi$ such that condition (3.12) holds for all two-dimensional facets. Consider $w$ on $\Omega'_{\mathscr{D}}$ and build the corresponding pattern $\mathscr{C}'$, such that the points on the boundary agree with those of the given circle pattern $\mathscr{C}$. Equation (3.12) guarantees that all rhombi of $\Omega'_{\mathscr{D}}$ are mapped to closed kites. Due to the combinatorics, the chain of kites is closed around each vertex. Since the boundary kites of $\mathscr{C}'$ are given by $\mathscr{C}$ which is an immersed circle pattern, at every interior white point the angles of the kites sum up to $2\pi$. Thus $\mathscr{C}'$ is an immersed circle pattern.

Furthermore, assume that $\mathscr{C}$ is embedded. Consider the mapping $\mu$ from $\mathscr{D}'$ to the region covered by the kites of $\mathscr{C}'$ such that each rhombus is mapped to the corresponding kite by an affine mapping. Then there is a constant $d > 0$, depending only on the smallest radius of the circles of $\mathscr{C}'$ and on the intersection angles, such that the mapping $\mu$ restricted to a Euclidean disk with radius $d$ about any point of $\mathscr{D}'$ is injective. Thus $\mu$ is continuous, locally injective, and injective on a neighborhood of the embedded boundary curve of $\mathscr{D}'$ (as $\mathscr{C}$ and $\mathscr{C}'$ agree on boundary vertices). This implies that $\mu$ is globally injective. More precisely, consider any interior point $w_0$ of the region covered by the kites of $\mathscr{C}'$. As $\mu$ is locally injective and $\mathscr{D}'$ is compact, the set $\mu^{-1}(\{w_0\})$ is finite. Thus there exists a continuous deformation of the oriented boundary curve of $\mathscr{D}'$ to a curve which consists of small circles around the points of $\mu^{-1}(\{w_0\})$ and curves connecting these circles, such that the winding number of the points of $\mu^{-1}(\{w_0\})$ stays constant ($+1$ or $-1$). The corresponding continuous deformation of the boundary curve of $\mathscr{C}'$ with corresponding orientation does not change the winding number about $w_0$, which has absolute value 1. Thus we conclude that the set $\mu^{-1}(\{w_0\})$ only contains one point which implies that $\mu$ is injective. □

## 3.5 Local changes of rhombic embeddings

In order to specify for a given rhombic embedding of a b-quad-graph the family of embeddings for which Lemma 3.21 can be applied, we investigate (local) changes of rhombic embeddings. This also generalizes the construction of Example 3.4.

**Definition 3.22.** Let $\mathscr{D}$ be a rhombic embedding of a finite simply connected b-quad-graph and let $\Omega_{\mathscr{D}}$ be the corresponding combinatorial surface in $\mathbb{Z}^d$.

Let $\hat{\mathbf{z}} \in V_{int}(\Omega_{\mathscr{D}})$ be an interior vertex with exactly three incident two-dimensional facets of $\Omega_{\mathscr{D}}$. Consider the three-dimensional cube with these boundary facets. Replace the three given facets with the three other two-dimensional facets of this cube. This procedure is called a *flip*; see Figure 3.4 for an illustration.

A vertex $\mathbf{z} \in \mathbb{Z}^d$ *can be reached with flips from* $\Omega_{\mathscr{D}}$ if $\mathbf{z}$ is contained in a combinatorial surface obtained from $\Omega_{\mathscr{D}}$ by a suitable sequence of flips.

The set of all vertices which can be reached with flips (including $V(\Omega_{\mathscr{D}})$) will be denoted by $\mathcal{F}(\Omega_{\mathscr{D}})$.

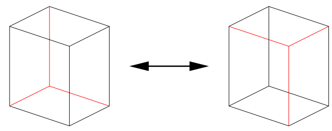


**Figure 3.4:** A flip of a three-dimensional cube. The red edges are not part of the surface in $\mathbb{Z}^d$.

Using the correspondence of rhombic embeddings and two-dimensional subcomplexes of $\mathbb{Z}^d$, a flip may also be defined in terms of the given rhombic embedding $\mathscr{D}$. Note that performing a flip always leads to a (locally different) rhombic embedding. In particular, the new combinatorial surface in $\mathbb{Z}^d$ is still monotone.

Let $\mathscr{D}$ be a rhombic embedding of a finite simply connected b-quad-graph and let $\Omega_{\mathscr{D}}$ in $\mathbb{Z}^d$ be the corresponding combinatorial surface. Assume that the boundary of $\Omega_{\mathscr{D}}$ is simple in the sense that any two-dimensional face of $\mathbb{Z}^d$ belongs to the surface $\Omega_{\mathscr{D}}$ if three of its incident edges do. Then the points of $\mathcal{F}(\Omega_{\mathscr{D}})$ may be characterized in terms of the boundary curve of $\Omega_{\mathscr{D}}$ which remains fixed.

Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be the standard orthonormal basis of $\mathbb{R}^d$ spanning the lattice $\mathbb{Z}^d$. For $1 \le i < j \le d$ denote by

$$E_{ij} := \{\lambda_1 \mathbf{e}_i + \lambda_2 \mathbf{e}_j : \lambda_1, \lambda_2 \in \mathbb{R}\}$$

the two-dimensional planes which are parallel to the two-dimensional facets of $\mathbb{Z}^d$. Consequently, these planes are parallel or orthogonal to the two-dimensional facets of any combinatorial surface $\Omega_{\mathscr{D}}$. Consider the closed curve $\Gamma(\Omega_{\mathscr{D}})$ of the boundary edges. Then the orthogonal projection $\Gamma_{ij}(\Omega_{\mathscr{D}})$ of $\Gamma(\Omega_{\mathscr{D}})$ onto one of the planes $E_{ij}$ is a Jordan curve and thus $E_{ij} \setminus \Gamma_{ij}(\Omega_{\mathscr{D}})$ contains exactly one unbounded component $U_{ij}(\Gamma_{ij}(\Omega_{\mathscr{D}})) \subset E_{ij}$.

**Lemma 3.23.** *A vertex* $\mathbf{z}$ *can be reached with flips if and only if for all* $0 \le i < j \le d$ *the orthogonal projection* $\mathbf{z}_{ij}$ *of* $\mathbf{z}$ *to* $E_{ij}$ *is contained within the projected boundary curve* $\Gamma_{ij}(\Omega_{\mathscr{D}})$, *that is* $\mathbf{z}_{ij} \in E_{ij} \setminus U_{ij}(\Gamma_{ij}(\Omega_{\mathscr{D}}))$.

*In particular,* $\mathcal{F}(\Omega_{\mathscr{D}}) \subset \Pi(\Omega_{\mathscr{D}})$ *and the inclusion is in general strict.*

*Proof.* The "only if"-part is clear from the Monotonicity Lemma 3.2.

The "if"-part is proven by induction over $n$, where $2n$ is the length of the boundary curve of $\Gamma(\Omega_{\mathscr{D}})$. For $n = 2, 3$ the claim is obvious. So assume that the claim holds for some $n \ge 3$ and let $\Omega_{\mathscr{D}}$ be a monotone combinatorial surface as above with $2(n+1)$ boundary edges. Let $\mathbf{z} \in \mathbb{Z}^d$ be such that all orthogonal projections $\mathbf{z}_{ij}$ of $\mathbf{z}$ to $E_{ij}$ are contained within the corresponding projected boundary curves $\Gamma_{ij}(\Omega_{\mathscr{D}})$. If $\mathbf{z} \in \Omega_{\mathscr{D}}$, the claim is obvious. So we assume that $\mathbf{z} \in \mathbb{Z}^d \setminus \Omega_{\mathscr{D}}$. Denote by $(n_1, \dots, n_d)$ the coordinates of $\mathbf{z}$. We want to show that $\mathbf{z}$ can be reached with flips from a part of $\Omega_{\mathscr{D}}$ (or from another monotone surface) with less boundary edges. We distinguish three possible cases.

First, assume that there is an index $j \in \{1, \dots, d\}$ such that

$$\min\{m_j \in \mathbb{Z} : \sum_{k=1}^{d} m_k \mathbf{e}_k \in V(\Omega_{\mathscr{D}})\} < n_j < \max\{m_j \in \mathbb{Z} : \sum_{k=1}^{d} m_k \mathbf{e}_k \in V(\Omega_{\mathscr{D}})\}.$$

Then neither of the half-spaces $HS_j^{\le} := \{\sum_{k=1}^{d} \lambda_k \mathbf{e}_k : \lambda_j \le n_j\}$ and $HS_j^{\ge} := \{\sum_{k=1}^{d} \lambda_k e_k : \lambda_j \ge n_j\}$ contains $\Omega_{\mathscr{D}}$. Consider the curve of edges in $E(\Omega_{\mathscr{D}}) \cap HS_j^{\le}$ which are incident to a two-dimensional facet of $F(\Omega_{\mathscr{D}})$ not contained in $HS_j^{\leq}$. Due to the Monotonicity Lemma 3.2 and to the fact that $\Omega_{\mathscr{D}}$ is a simply connected two-dimensional combinatorial surface in $\mathbb{Z}^d$, this curve is simple and contains exactly two boundary points. Using the choice of the index $j$, this curve thus divides $\Omega_{\mathscr{D}}$ in two non-empty, monotone, two-dimensional combinatorial surfaces $\Omega_{\mathscr{D}}^{\geq}$ and $\Omega_{\mathscr{D}}^{\leq} \subset HS_j^{\leq}$. Note that $\Omega_{\mathscr{D}}^{\leq}$ is also simply connected. Again due to the Monotonicity Lemma 3.2 and to the choice of the index $j$, the boundary curve of $\Omega_{\mathscr{D}}^{\leq}$ is strictly shorter than the boundary curve of $\Omega_{\mathscr{D}}$. Furthermore, the orthogonal projections of $\mathbf{z}$ to the two-dimensional planes $E_{ik}$ for $0 \leq i < k \leq d$ are contained within the projected boundary curve of $\Omega_{\mathscr{D}}^{\leq}$. Now the induction hypothesis implies that $\mathbf{z}$ can be reached by flips from $\Omega_{\mathscr{D}}^{\leq}$. As $\Omega_{\mathscr{D}}^{\leq} \subset \Omega_{\mathscr{D}}$, this proves the inductions step and the claim for the first case.

Second, suppose that the assumption of the first case is not true, but there are an index $j \in \{1, \dots, d\}$ and $l_j \in \mathbb{Z}$ such that

$$\min\{m_j \in \mathbb{Z} : \sum_{k=1}^{d} n_k \mathbf{e}_k \in V(\Omega_{\mathscr{D}})\} < l_j < \max\{m_j \in \mathbb{Z} : \sum_{k=1}^{d} n_k \mathbf{e}_k \in V(\Omega_{\mathscr{D}})\}.$$

Then $n_j$ is equal to the min- or to the max-term of this inequality. Define the half-spaces $HS_j^{\leq} := \{\sum_{k=1}^{d} \lambda_k \mathbf{e}_k : \lambda_j \leq l_j\}$ and $HS_j^{\geq} := \{\sum_{k=1}^{d} \lambda_k \mathbf{e}_k : \lambda_j \geq l_j\}$ analogously as above. Again, neither of this half-spaces contains $\Omega_{\mathscr{D}}$. Without loss of generality we assume that $\mathbf{z} \in HS_j^{\leq}$. The reasoning for $\mathbf{z} \in HS_j^{\geq}$ is analogous. Construct the two non-empty, monotone, two-dimensional combinatorial surfaces $\Omega_{\mathscr{D}}^{\geq}$ and $\Omega_{\mathscr{D}}^{\leq} \subset HS_j^{\leq}$ as in the first case. If all projection of $\mathbf{z}$ to the two-dimensional planes $E_{ik}$ for $0 \leq i < k \leq d$ are contained within the corresponding projected boundary curves $\Gamma_{ik}(\Omega_{\mathscr{D}}^{\leq})$ we can conclude as in the first case. If not, let $E_{k_1 k_2}, \dots, E_{k_{2s-1} k_{2s}}$ be the different planes where the projection of $\mathbf{z}$ is not contained within the projected boundary curve of $\Omega_{\mathscr{D}}^{\leq}$, for some $s \geq 1$. Consider the point $\mathbf{w} \in \mathbb{Z}^d$ with coordinates $w_j = l_j$, $w_{k_i} = n_{k_i}$ for $i = 1, \dots, 2s$, and such that all projection of $\mathbf{w}$ to the two-dimensional planes $E_{ik}$ are contained within the projected boundary curve of the original surface $\Omega_{\mathscr{D}}$. This is possible due to our assumptions on $\Omega_{\mathscr{D}}$. Now, the first case applies to the point $\mathbf{w}$. Using the induction hypothesis, there is a simply connected, monotone, two-dimensional combinatorial surfaces $\Omega_{\mathbf{w}}$ containing $\mathbf{w}$ which is obtained from $\Omega_{\mathscr{D}}$ by a suitable sequence of flips. Construct the two non-empty, simply connected, monotone, two-dimensional combinatorial surfaces $\Omega_{\mathbf{w}}^{\geq}$ and $\Omega_{\mathbf{w}}^{\leq} \subset HS_j^{\leq}$ as above. Then all projections of $\mathbf{z}$ to the two-dimensional planes $E_{ik}$ for $0 \leq i < k \leq d$ are contained within the corresponding projected boundary curves $\Gamma_{ik}(\Omega_{\mathbf{w}}^{\leq})$ and we can again conclude as in the first case.

If neither the first nor the second case applies, then the two-dimensional combinatorial surface $\Omega_{\mathscr{D}}$ is contained in a $d$-dimensional unit hypercube. Without loss of generality we may assume that this cube has all coordinates in the interval $[0, 1]$. First consider the case that $\Omega_{\mathscr{D}}$ contains a parallel two-dimensional facet for all $d(d-1)/2$ different two-dimensional planes $E_{ik}$ for $0 \leq i < k \leq d$. Then it is easy to see that all points of the hypercube can be reached by flips. For the remaining case, we proceed by induction on the dimension $d$. The cases $d = 2$ and $d = 3$ are obvious. Let $E_{ik}$ be a two-dimensional plane such that $\Omega_{\mathscr{D}}$ does not contain a parallel two-dimensional facet, but contains parallel edges to the vectors $\mathbf{e}_i$ and $\mathbf{e}_k$. Without loss of generality, we assume that $\Omega_{\mathscr{D}}$ actually contains the vectors $\mathbf{e}_i$ and $\mathbf{e}_k$. The other cases are very similar. Consider the projection

$$T : \mathbb{R}^d \to \mathbb{R}^{d-1}, \quad \sum_{l=1}^{d} \lambda_l \mathbf{e}_l \mapsto \sum_{\substack{l=1 \\ l \neq i,k}}^{d} \lambda_l \mathbf{e}_l + (\lambda_i - \lambda_j)\mathbf{e}_i.$$

Then $T(\Omega_{\mathscr{D}})$ is a simply connected, monotone, two-dimensional combinatorial surface in $\mathbb{Z}^{d-1}$ by our assumption on $E_{ik}$. Furthermore, all projections of $T(\mathbf{z})$ to the two-dimensional

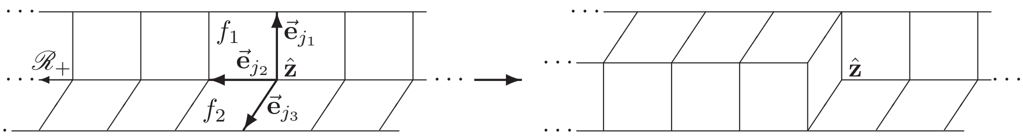



**Figure 3.5:** An example of an infinite flip.

planes of $\mathbb{Z}^{d-1}$ are contained within the corresponding projected boundary curve of $T(\Omega_{\mathscr{D}})$. Moreover, a flip can be performed for $T(\Omega_{\mathscr{D}})$ if and only if this can be done for $\Omega_{\mathscr{D}}$. Therefore the proof can be continued as in the previous cases. This completes the induction step on $d$, and thus completes also the induction step on $n$ and the proof. $\square$

For further use, we also consider the effect of flips for an (embedded) quasicrystallic circle pattern. Let $\mathscr{D}$ be a quasicrystallic rhombic embedding of a b-quad-graph with associated graph $G$ built from white vertices. Let $\mathscr{C}$ be a circle pattern for $G$ and the corresponding labelling taken from $\mathscr{D}$. Let $w$ be the comparison function defined in (3.11). As explained in Section 3.4 this function $w$ may be extended to $\mathcal{F}(\Omega_{\mathscr{D}})$. Now perform a flip to obtain a locally different combinatorial surface $\Omega'_{\mathscr{D}}$ from $\Omega_{\mathscr{D}}$. By Lemma 3.21, a circle pattern $\mathscr{C}'$ can be obtained using the values of the extension of $w$, such that $\mathscr{C}'$ agrees with $\mathscr{C}$ except for the three kites which correspond to the faces which have been flipped. Furthermore, if $\mathscr{C}$ is embedded this is also true for $\mathscr{C}'$. Therefore, we may consider a flip not only in terms of the combinatorial surface in $\mathbb{Z}^d$, but also directly for circle patterns using the kites which correspond to the two-dimensional facets of $\Omega_{\mathscr{D}}$.

**Definition 3.24.** Furthermore, we can define *flips for simply oder doubly infinite strips* of the following form. See Figure 3.5 for an illustration. Let $\Omega_{\mathscr{D}} \subset \mathbb{Z}^d$ be a simply connected monotone combinatorial surface. Let $\hat{\mathbf{z}} \in V_w(\Omega_{\mathscr{D}})$ be a white vertex. Let $\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \mathbf{e}_{j_3}$ be three different edges incident to $\hat{\mathbf{z}}$ such that there are two-dimensional faces $f_1, f_2$ of $\Omega_{\mathscr{D}}$ incident to $\mathbf{e}_{j_1}$ and $\mathbf{e}_{j_2}$, and to $\mathbf{e}_{j_2}$ and $\mathbf{e}_{j_3}$, respectively. Let $\alpha_1 = \alpha(f_1)$ and $\alpha_2 = \alpha(f_2)$ be the intersection angles associated to these faces. Let $\alpha_3$ be the intersection angles associated to the two-dimensional facet of $\mathbb{Z}^d$ incident to $\mathbf{e}_{j_1}$ and $\mathbf{e}_{j_3}$. Then $\sum_{i=1}^{3} \alpha_i = 2\pi$ or $\sum_{i=1}^{2}(\pi - \alpha_i) + \alpha_3 = 2\pi$. In the first case, consider the half-axis $\mathscr{R}_+ = \{\hat{\mathbf{z}} + \lambda \vec{\mathbf{e}}_{j_2} : \lambda \geq 0\}$, where $\vec{\mathbf{e}}_{j_2}$ is the vector corresponding to the edge $\mathbf{e}_{j_2}$ and pointing away from $\hat{\mathbf{z}}$. In the second case, consider the other half-axis $\mathscr{R}_- = \{\hat{\mathbf{z}} + \lambda \vec{\mathbf{e}}_{j_2} : \lambda \leq 0\}$. In both cases we may also consider the whole axis $\mathscr{R} = \{\hat{\mathbf{z}} + \lambda \vec{\mathbf{e}}_{j_2} : \lambda \in \mathbb{R}\}$. Assume that the translations of $f_1$ and $f_2$ along these (half-)axis, that is the faces $f_j + n\vec{\mathbf{e}}_{j_2} + \hat{\mathbf{z}}$ for $j = 1, 2$ and $n \in \mathbb{N}$, $n \in \mathbb{Z} \setminus \mathbb{N}$, or $n \in \mathbb{Z}$ respectively, are contained in $\Omega_{\mathscr{D}}$. We only consider the case of the positive half-axis $\mathscr{R}_+$ further. For $\mathscr{R}_-$ the argumentation is analogous and the case of the whole axis $\mathscr{R}$ is a simple consequence. Replace each face $f_1 + n\vec{\mathbf{e}}_{j_2} + \hat{\mathbf{z}}$ by its translate $f_1 + n\vec{\mathbf{e}}_{j_2} + \hat{\mathbf{z}} + \vec{\mathbf{e}}_{j_3}$ for $n \in \mathbb{N}_0$ and similarly $f_2 + n\vec{\mathbf{e}}_{j_2} + \hat{\mathbf{z}}$ by $f_2 + n\vec{\mathbf{e}}_{j_2} + \hat{\mathbf{z}} + \vec{\mathbf{e}}_{j_1}$ for $n \in \mathbb{N}_0$, where $\vec{\mathbf{e}}_{j_1}$ and $\vec{\mathbf{e}}_{j_3}$ are the vectors corresponding to the edges $\mathbf{e}_{j_1}$ and $\mathbf{e}_{j_3}$ respectively and pointing away from $\hat{\mathbf{z}}$. Adding the face incident to $\hat{\mathbf{z}}$, $\mathbf{e}_{j_1}$, and $\mathbf{e}_{j_3}$, we obtain a different, but still monotone simply connected combinatorial surface. The definition of an infinite flip for a black vertex $\hat{\mathbf{z}} \in V_b(\Omega_{\mathscr{D}})$ is similar.

Note that all points of $\Pi(\Omega_{\mathscr{D}})$ may be obtained performing usual flips or flips for such infinite strips.

These types of infinite flips may also be defined for corresponding circle patterns as for simple flips using the values of the extended comparison function $w$.

**Lemma 3.25.** *Let $\Omega_{\mathscr{D}} \subset \mathbb{Z}^d$ be a simply connected monotone combinatorial surface and let $\Omega'_{\mathscr{D}}$ be the simply connected monotone combinatorial surface obtained from $\Omega_{\mathscr{D}}$ after performing a flip for a simply infinite strip. Let $\mathscr{C}$ be a circle pattern for $\mathscr{D}$ and the* *corresponding labelling and let $\mathscr{C}'$ be the corresponding circle pattern after performing the corresponding infinite flip as for $\Omega'_{\mathscr{D}}$. Then the resulting circle pattern $\mathscr{C}'$ is embedded if the original one $\mathscr{C}$ is.*

*Proof.* The proof is based on similar arguments as the proof of Lemma 3.21.

Using the same notation as in Definition 3.24 for a quasicrystallic rhombic embedding $\Omega_{\mathscr{D}}$ containing an infinite strip, we assume without loss of generality that $\sum_{i=1}^{3} \alpha_i = 2\pi$. The other case is analogous. Let $\hat{\mathbf{v}}$ be a black point on $\mathscr{R}_+$ with incident white vertices $\hat{\mathbf{z}}_1 = \hat{\mathbf{v}} + \vec{\mathbf{e}}_{j_1}$, $\hat{\mathbf{z}}_2 = \hat{\mathbf{v}} + \vec{\mathbf{e}}_{j_2}$, $\hat{\mathbf{z}}_3 = \hat{\mathbf{v}} + \vec{\mathbf{e}}_{j_3}$. Then the second intersection point of the circles of $\mathscr{C}$ corresponding to $\hat{\mathbf{z}}_1$ and $\hat{\mathbf{z}}_3$ as well as the kite built by this intersection point, the intersection point corresponding to $\hat{\mathbf{v}}$, and the centers of circles corresponding to $\hat{\mathbf{z}}_1$ and $\hat{\mathbf{z}}_3$ is contained in the region covered by the four kites which contain the center of circle corresponding to $\hat{\mathbf{z}}_2$. This is due to geometric reasons and the simple combinatorics at $\hat{\mathbf{z}}_2$.

Finally, consider in $\Omega'_{\mathscr{D}}$ two white vertices $\hat{\mathbf{z}}_1 = \hat{\mathbf{v}} + \vec{\mathbf{e}}_{j_1} \in V(\Omega'_{\mathscr{D}})$ and $\hat{\mathbf{z}}_3 = \hat{\mathbf{v}} + \vec{\mathbf{e}}_{j_3} \in V(\Omega'_{\mathscr{D}})$ which are incident to the same black vertex $\hat{\mathbf{v}} \in \mathscr{R}_+$. Take any simple closed curve in $\Omega'_{\mathscr{D}}$ containing the edges at $\hat{\mathbf{z}}_1$ and $\hat{\mathbf{z}}_3$ which are parallel to $\mathbf{e}_{j_3}$ and to $\mathbf{e}_{j_1}$ respectively and containing no other points incident to $\mathscr{R}_+$. Consider the region of the new circle pattern $\mathscr{C}'$, after performing the infinite flip, which corresponds to the faces of $\Omega'_{\mathscr{D}}$ lying within this boundary curve. Then the above reasoning and the arguments in the proof of Lemma 3.21 imply that the kites of this region are embedded. As this is true for arbitrary large regions, we conclude that the whole circle pattern is embedded. □

## 3.6 Regular circle patterns with square grid combinatorics

This section contains two results for regular circle patterns with square grid combinatorics. First, we prove uniqueness of embedded circle patterns which is a generalization of the uniqueness result for orthogonal circle patterns with square grid combinatorics. Second, we consider an analog of the Rodin-Sullivan Conjecture.

To begin with, we fix some notation. Let $SGD$ be the regular square grid cell decomposition of the complex plane associated to the lattice $\mathbb{Z}^2$, that is the vertices are $V(SGD) = \mathbb{Z} + i\mathbb{Z}$ and the edges are given by pairs of vertices $[z, z']$ with $z, z' \in V(SGD)$ and $|z - z'| = 1$. The 2-cells are squares $\{z + a + ib : a, b \in [0, 1]\}$ for $z \in V(SGD)$. As $SGD$ is a b-quad-graph, we may assume the vertices

$$V(SG) = \{n + im \in \mathbb{Z} + i\mathbb{Z} : n + m = 0 \pmod 2\}$$

to be colored white. The corresponding graph $SG$ has edges connecting vertices $z_1, z_2 \in V(SG)$ with $|z_1 - z_2| = \sqrt{2}$, as illustrated in Figure 3.8. The dual graph is denoted by $SG^*$.

Furthermore, let $\psi \in (0, \pi)$ be a fixed angle. We consider the following regular labelling $\alpha_\psi$ on the edges of $SG$. Let $[z_1, z_2] \in E(SG)$ be an edge connecting the vertices $z_1, z_2 \in V(SG)$. Without loss of generality, we assume that $\mathrm{Re}(z_1) \leq \mathrm{Re}(z_2)$. Then $\alpha_\psi([z_1, z_2]) = \psi$ if $\mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$ and $\alpha_\psi([z_1, z_2]) = \pi - \psi$ if $\mathrm{Im}(z_1) \geq \mathrm{Im}(z_2)$.

If a graph $G$ is a subgraph of $SG$, a circle pattern for $G$ and $\alpha_\psi$ is called *$SG$-circle pattern.* The choice $\psi = \pi/2$ leads to orthogonal $SG$-circle patterns as considered by Schramm in [67].

### 3.6.1 *Uniqueness of regular circle patterns with square grid combinatorics*

Our main aim is to prove uniqueness of quasicrystallic isoradial circle patterns under some natural conditions in Section 3.7. As a preparation we consider $SG$-circle patterns which can be associated to the $\mathbb{Z}^2$-sublattices of the lattice $\mathbb{Z}^d$.

For orthogonal $SG$-circle patterns, Schramm considered in [67] two Möbius invariants $\tau$ and $\sigma$ such that $\log \tau + i \log \sigma$ is an analog of the Schwarzian derivative $S_g = (g''/g')' -$

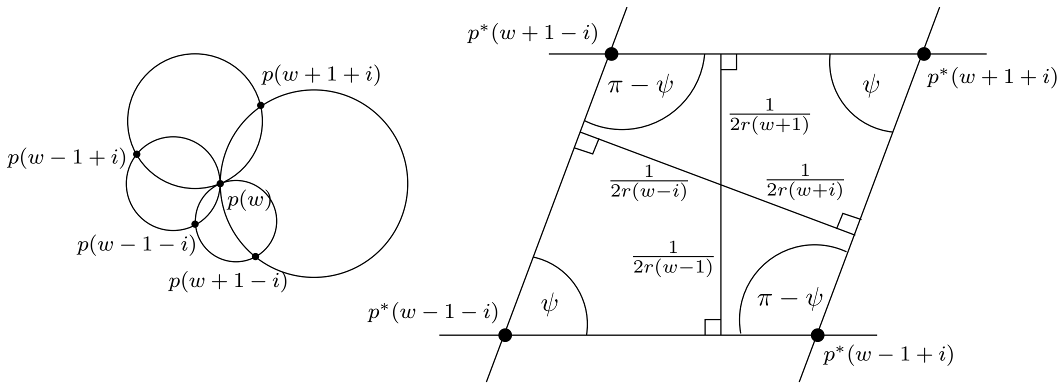



**Figure 3.6:** The configuration used for the $\sigma$-invariant.

$\frac{1}{2}(g''/g')^2$ for a holomorphic function $g$. These invariants can be defined and used in a very similar way for the more general $SG$-circle patterns with intersection angles $\alpha_\psi$.

We denote the cross-ratio of four complex points $w_1, w_2, w_3, w_4 \in \mathbb{C}$ by

$$\mathrm{cr}[w_1, w_2, w_3, w_4] := \frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)}.$$

**Definition 3.26.** Let $\mathscr{D}$ be a b-quad graph which is a simply connected part of $SGD$. Denote the associated graphs built from white and black vertices by $G \subset SG$ and $G^* \subset SG^*$ respectively. Let $\mathscr{C}$ be a circle pattern for $\mathscr{D}$ and $\alpha_\psi$. Denote by $p(z)$ the center of the circle $C_z$ corresponding to $z \in V(G)$ and by $p(w)$ the intersection point corresponding to $w \in V(G^*)$.

For $z \in V_{int}(G)$ the $\tau$-*invariant* of $\mathscr{C}$ is defined by

$$\tau_{\mathscr{C}}(z) = -\mathrm{cr}[p(z+i), p(z+1), p(z-i), p(z-1)]. \tag{3.13}$$

For $w \in V_{int}(G^*)$, the $\sigma$-*invariant* of $\mathscr{C}$ is defined by

$$\sigma_{\mathscr{C}}(w) = \mathrm{e}^{2i\psi}\mathrm{cr}[p(w+1+i), p(w+1-i), p(w-1-i), p(w-1+i)]. \tag{3.14}$$

**Remark.** Let $\mathscr{D}$, $G$, $G^*$, $\alpha_\psi$, and $p(\cdot)$ be as in Definition 3.26.

(1) $\tau$ and $\sigma$ are Möbius invariants of the circle pattern $\mathscr{C}$. This means, if $\mathscr{C}'$ is the image of $\mathscr{C}$ under a Möbius transformation, then $\mathscr{C}'$ and $\mathscr{C}$ have the same $\tau$- and $\sigma$-invariants.

(2) As the points $p(z+i)$, $p(z+1)$, $p(z-i)$, $p(z-1)$ appear in a cyclic order (clockwise) around the circle $C_z$ of $\mathscr{C}$, we have $\tau_{\mathscr{C}}(z) > 0$. Expressing $\tau_{\mathscr{C}}(z)$ in terms of the radius function $r_{\mathscr{C}}$, we obtain

$$\tau_{\mathscr{C}}(z) = |\tau_{\mathscr{C}}(z)| = \frac{\sin f_\psi(\log \frac{r_{\mathscr{C}}(z+1+i)}{r_{\mathscr{C}}(z)}) \sin f_\psi(\log \frac{r_{\mathscr{C}}(z-1-i)}{r_{\mathscr{C}}(z)})}{\sin f_{\pi-\psi}(\log \frac{r_{\mathscr{C}}(z+1-i)}{r_{\mathscr{C}}(z)}) \sin f_{\pi-\psi}(\log \frac{r_{\mathscr{C}}(z-1+i)}{r_{\mathscr{C}}(z)})}. \tag{3.15}$$

(3) Using an adaption of Schramm's argumentation in [67] we show that $\sigma_{\mathscr{C}}(w) > 0$.

Apply to $\mathscr{C}$ a Möbius transformation $\mu$ which takes $p(w)$ to $\infty$. Then the four circles $\mu(C_{w\pm1\pm i})$ are lines and form a parallelogram with corners $p^*(w+1+i)$, $p^*(w+1-i)$, $p^*(w-1-i)$, $p^*(w-1+i)$, where $p^* = \mu \circ p$. See Figure 3.6 for an illustration. Hence

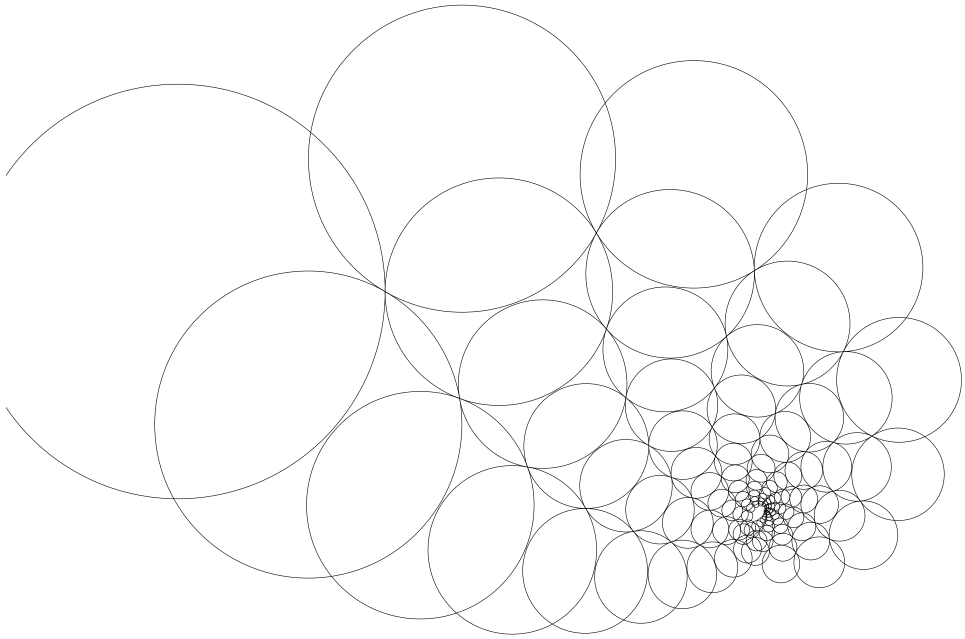


**Figure 3.7:** An example of an $SG$-Doyle spiral for $\psi = \pi/3$ and $a = 0.3$.

$$p^*(w+1-i) - p^*(w-1-i) = p^*(w+1+i) - p^*(w-1+i), \tag{3.16}$$
$$p^*(w-1+i) - p^*(w-1-i) = p^*(w+1+i) - p^*(w+1-i), \tag{3.17}$$
$$\frac{p^*(w+1+i) - p^*(w+1-i)}{p^*(w+1+i) - p^*(w-1+i)} \in \mathrm{e}^{-i\psi}\mathbb{R}_+. \tag{3.18}$$

Therefore we obtain

$$\sigma_{\mathscr{C}}(w) = \mathrm{e}^{2i\psi}\Big(\frac{p^*(w+1+i) - p^*(w+1-i)}{p^*(w+1+i) - p^*(w-1+i)}\Big)^2 \tag{3.19}$$
$$= \left|\frac{p^*(w+1+i) - p^*(w+1-i)}{p^*(w+1+i) - p^*(w-1+i)}\right|^2. \tag{3.20}$$

$\sigma_{\mathscr{C}}(w)$ may also be expressed in terms of the radius function $r_{\mathscr{C}}$. For this purpose, note that the transformation $z \to 1/z$ maps a circle with radius $r > 0$ through the origin to a line with distance $1/(2r)$ to the origin. Thus, a simple calculation gives

$$\sigma_{\mathscr{C}}(w) = \left(\frac{1/r_{\mathscr{C}}(w+i) + 1/r_{\mathscr{C}}(w-i)}{1/r_{\mathscr{C}}(w+1) + 1/r_{\mathscr{C}}(w-1)}\right). \tag{3.21}$$

**Example 3.27** (Doyle spirals)**.** An simple, but important example is the following generalization of Doyle spirals. Peter Doyle had the idea for a construction of entire immersed hexagonal circle packings which are analogous to the exponential map. These packing are known as *Doyle spirals*, see [8] or [71, Appendix C] for more details.

A generalization of Doyle spirals for hexagonal circle patterns and for $SG$-circle patterns has been considered by Bobenko and Hoffmann in [11].

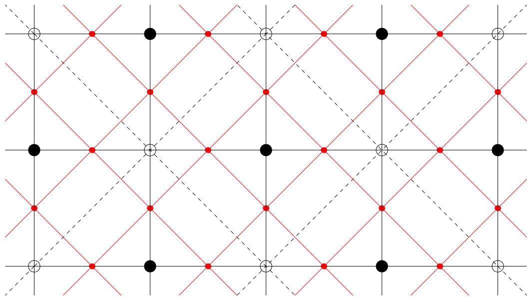


**Figure 3.8:** A part of the cell decomposition $SGD$ (black and white vertices and black edges), of the graph $SG$ (white vertices and dashed edges), and of the graph $HG$ (red vertices and edges).

Let $a \in \mathbb{C}$ be a fixed complex number. Denote $a_1 = \mathrm{Re}(a(1+i))$ and $a_2 = \mathrm{Re}(a(1-i))$. Define $r(z) = |\mathrm{e}^{az}|$ for all $z \in SG$. Lemma 2.7 (2) implies that

$$f_\psi(\log \tfrac{r(z_0+1+i)}{r(z_0)}) + f_\psi(\log \tfrac{r(z_0-1-i)}{r(z_0)}) = f_\psi(a_1) + f_\psi(-a_1) = \pi - \psi, \tag{3.22}$$

$$f_{\pi-\psi}(\log \tfrac{r(z_0+1-i)}{r(z_0)}) + f_{\pi-\psi}(\log \tfrac{r(z_0-1+i)}{r(z_0)}) = f_{\pi-\psi}(a_2) + f_{\pi-\psi}(-a_2) = \psi. \tag{3.23}$$

Thus by Theorem 2.5 (2), $r$ is the radius function of a circle pattern $\mathscr{C}_D$, which will be called *$SG$-Doyle spiral*. Figure 3.7 shows an example.

Using equations (3.15), (3.21), (3.22), and (3.23), we easily see that $\mathscr{C}_D$ has constant $\tau$- and $\sigma$-invariants:

$$\tau_{\mathscr{C}_D} = \frac{\cos(2f_\psi(a_1) - (\pi - \psi)) - \cos(\pi - \psi)}{\cos(2f_{\pi-\psi}(a_2) - \psi) - \cos\psi}, \qquad \sigma_{\mathscr{C}_D} = \left(\frac{\mathrm{e}^{a_1} + \mathrm{e}^{a_2}}{1 + \mathrm{e}^{a_1+a_2}}\right)^2.$$

Note that the $\sigma$-invariant does not depend on the parameter $\psi$.

For $a = 0$ the radius $r \equiv 1$ is constant and we obtain an isoradial circle pattern $\mathscr{C}_1$ for $SG$ and $\alpha_\psi$. In this case we have $\tau_{\mathscr{C}_1} = \cot^2(\psi/2)$ and $\sigma_{\mathscr{C}_1} = 1$.

In the following, we derive necessary and sufficient conditions for a pair of functions $\tau : V(SG) \to (0,\infty)$ and $\sigma : V(SG^*) \to (0,\infty)$ to be the invariants of an $SG$-circle pattern.

Let $\mathscr{C}$ be a circle pattern for $SG$ and $\alpha_\psi$. As we consider Möbius invariants, $\mathscr{C}$ may be a planar circle pattern as characterized in Definition 2.1 or (the stereographic projection of) a spherical circle pattern on $\mathbb{S}^2$, which can be defined in an analogous manner using circles and spherical kites in $\mathbb{S}^2$. We will determine equations relating the invariants $\tau$ and $\sigma$ of $\mathscr{C}$ in a similar way as Schramm did in [67]. Denote by $HG = (1/2)SG + 1/2$ the square grid $SG$ with edge length $\sqrt{2}/2$ such that 0 is a center of a square, as in Figure 3.8. Note that the vertices of $HG$ correspond to the edges of $SGD$ and the faces of $HG$ correspond to vertices of $SG$ and of $SG^*$ and to the faces of $SGD$, which in turn correspond to the edges of $SG$ or $SG^*$. The edges of $HG$ correspond to the angles of the kites built by the centers of two circles for incident vertices of $SG$ and by their intersection points.

For $v \in V(HG)$ let $z \in V(SG)$ be the unique vertex of $SG$ that is closest to $v$. Set

$$w_1 = z + 2(v - z), \quad w_2 = z + 2i(v - z), \quad w_3 = z - 2i(v - z).$$

Then $w_1, w_2, w_3 \in V(SG^*)$. Let $m_v$ be the Möbius transformation that takes $\infty, 0, 1$ to $p(w_1), p(w_2), p(w_3)$ respectively. For any directed edge $\overrightarrow{v_1v_2} \in \vec{E}(HG)$ we set $m_{\overrightarrow{v_1v_2}} = m_{v_1}^{-1} \circ m_{v_2}$. This Möbius transformation $m_{\overrightarrow{v_1v_2}}$ does not change if we apply any Möbius transformation to $\mathscr{C}$, therefore it is a Möbius invariant of the circle pattern. We now compute $m_{\overrightarrow{v_1v_2}}$ for the different types of edges in terms of the invariants $\tau$ and $\sigma$. Without loss of generality, we will assume that $m_{v_1}$ is the identity. This is possible because $\mathscr{C}$ may be replaced by its Möbius image $m_{v_1}^{-1}(\mathscr{C})$.

First we consider the edges of $HG$ which surround a face corresponding to $w \in V(SG^*)$. Let $v_1 = w - i/2$ and $v_2 = w + 1/2 = v_1 + (1+i)/2$. By assumption we have $p(w) = \infty$, $p(w-1-i) = 0$, $p(w+1-i) = 1$. Since $p(w) = \infty$, the four points $p(w \pm 1 \pm i)$ are the vertices of a parallelogram. Using equation (3.19) for $\sigma(w)$ and remembering the properties (3.16)–(3.18), we obtain

$$p(w+1+i) = p(w+1-i) + \mathrm{e}^{-i\psi}\sqrt{\sigma(w)}(p(w+1-i) - p(w-1-i)) = 1 + \mathrm{e}^{-i\psi}\sqrt{\sigma(w)}.$$

As $m_{\overrightarrow{v_1v_2}}$ takes $\infty, 0, 1$ to $p(w), p(w+1-i), p(w+1+i)$ respectively, we easily get

$$m_{\overrightarrow{v_1v_2}}(\zeta) = 1 + \mathrm{e}^{-i\psi}\sqrt{\sigma(w)}\zeta.$$

Similar computations can be made for $v_1 = w \pm i/2$ or $v_1 = w \pm 1/2$ and $v_2 = w + i(v_1 - w)$ and yield the following formulas which are valid for all $w \in V(SG^*)$.

$$m_{\overrightarrow{(w-\frac{i}{2})(w+\frac{1}{2})}}(\zeta) = m_{\overrightarrow{(w+\frac{i}{2})(w-\frac{1}{2})}}(\zeta) = 1 + \mathrm{e}^{-i\psi}\sqrt{\sigma(w)}\zeta, \tag{3.24}$$

$$m_{\overrightarrow{(w+\frac{1}{2})(w+\frac{i}{2})}}(\zeta) = m_{\overrightarrow{(w-\frac{1}{2})(w-\frac{i}{2})}}(\zeta) = 1 - \mathrm{e}^{i\psi}\sqrt{\sigma(w)^{-1}}\zeta. \tag{3.25}$$

Next, we consider the edges of $HG$ incident to a face which corresponds to $z \in V(SG)$. Let $v_1 = z + i/2$ and $v_2 = z - 1/2$. By assumption, we have $p(z+i) = \infty$, $p(z-1) = 0$, $p(z+1) = 1$. From the definition of $\tau(z)$ in (3.13) we obtain

$$p(z-i) = (1 + \tau(z)^{-1})^{-1}.$$

As $m_{\overrightarrow{v_1v_2}}$ takes $\infty, 0, 1$ to $p(z-1), p(z-i), p(z+i)$ respectively, we easily deduce

$$m_{\overrightarrow{v_1v_2}}(\zeta) = \frac{(1+\tau(z)^{-1})^{-1}}{1-\zeta}.$$

Similar computations can be made for $v_1 = z \pm i/2$ or $v_1 = z \pm 1/2$ and $v_2 = z + i(v_1 - z)$ and yield the following formulas which are valid for all $z \in V(SG)$.

$$m_{\overrightarrow{(z+\frac{i}{2})(z-\frac{1}{2})}}(\zeta) = m_{\overrightarrow{(z-\frac{i}{2})(z+\frac{1}{2})}}(\zeta) = \frac{(1+\tau(z)^{-1})^{-1}}{1-\zeta} \tag{3.26}$$

$$m_{\overrightarrow{(z-\frac{1}{2})(z-\frac{i}{2})}}(\zeta) = m_{\overrightarrow{(z+\frac{1}{2})(z+\frac{i}{2})}}(\zeta) = \frac{(1+\tau(z))^{-1}}{1-\zeta}. \tag{3.27}$$

From the definition of the Möbius transformations $m_{\overrightarrow{v_1v_2}}$ it is obvious that for every closed path $v_0, v_1, \ldots, v_n = v_0$ in $V(HG)$ the composition of the Möbius transformations corresponding to the edges of the path is the identity, that is

$$m_{\overrightarrow{v_0v_1}} \circ m_{\overrightarrow{v_1v_2}} \circ \cdots \circ m_{\overrightarrow{v_{n-1}v_n}} = \text{identity}. \tag{3.28}$$

In particular,

$$(m_{\overrightarrow{v_1v_2}})^{-1} = m_{\overrightarrow{v_2v_1}}. \tag{3.29}$$

We now consider the closed paths in $HG$ which surround faces of $HG$ corresponding to the edges of $SG$ or $SG^*$. Set $a = z + (-1+i)/2$ for a vertex $z \in V(SG)$ and $v_j = a + i^j/2$ for $j \in \mathbb{N}$. Then $v_0, v_1, v_2, v_3, v_4 = v_0$ is a closed path in $HG$. Consequently, we have

$$m_{\overrightarrow{v_0v_3}} \circ m_{\overrightarrow{v_3v_2}} \circ m_{\overrightarrow{v_2v_1}} \circ m_{\overrightarrow{v_1v_0}} = \text{identity}.$$

Using appropriate formulas from equations (3.24)–(3.27), this can be written as

$$1 - e^{i\psi}\sqrt{\sigma(a + \tfrac{1+i}{2})^{-1}}\left(\frac{(1+\tau(a+\frac{1-i}{2})^{-1})^{-1}}{1-\left(1+e^{i\psi}\sqrt{\sigma(a-\frac{1+i}{2})^{-1}}\left(\frac{(1+\tau(a-\frac{1-i}{2})^{-1})^{-1}}{1-\zeta}\right)\right)}\right) = \zeta.$$

Simplifications and the abbreviation $\omega = (1+i)/2$ yield

$$\frac{\sigma(a+\omega)}{\sigma(a-\omega)} = \left(\frac{1+\tau(a+i\omega)^{-1}}{1+\tau(a-i\omega)^{-1}}\right)^2. \tag{3.30}$$

Choosing $b = z + (1+i)/2$ for $z \in V(SG)$ and setting $v_j = b + i^j/2$ for $j \in \mathbb{N}$, we obtain by similar calculations

$$\frac{\sigma(b+i\omega)}{\sigma(b-i\omega)} = \left(\frac{1+\tau(b+\omega)}{1+\tau(b-\omega)}\right)^2. \tag{3.31}$$

Note that equations (3.30) and (3.31) do not depend on $\psi$ and are therefore the same as for the orthogonal case $\psi = \pi/2$ considered by Schramm in [67]. They may be interpreted as nonlinear discrete versions of the Cauchy-Riemann equations for $\tau$ and $\sigma$.

**Theorem 3.28** (cf. [67, Theorem 5.1])**.** *(1) Let $\mathscr{C}$ be a circle pattern for $SG$ and $\alpha_\psi$. Then its $\tau$- and $\sigma$-invariants satisfy equations* (3.30) *and* (3.31)*.*

*(2) Suppose that $\tau : V(SG) \to (0,\infty)$ and $\sigma : V(SG^*) \to (0,\infty)$ are positive functions which satisfy equations* (3.30) *and* (3.31)*. Then there is a circle pattern $\mathscr{C}$ for $SG$ and $\alpha_\psi$ such that $\tau_{\mathscr{C}} = \tau$ and $\sigma_{\mathscr{C}} = \sigma$.*

*(3) Suppose that $\mathscr{C}$ and $\mathscr{C}'$ are circle patterns for $SG$ and $\alpha_\psi$, and $\tau_{\mathscr{C}} = \tau_{\mathscr{C}'}$, $\sigma_{\mathscr{C}} = \sigma_{\mathscr{C}'}$. Then $\mathscr{C}'$ is the image of $\mathscr{C}$ by a Möbius transformation.*

*Proof.* Part (1) has already been proven above.

Parts (2) and (3) can be proven by the same arguments used by Schramm for the case $\psi = \pi/2$ with only minor adaptions. The main idea is to define the Möbius transformations $m_{\overrightarrow{v_1v_2}}$ by equations (3.24)–(3.27) and (3.29). Then simple calculations and the assumptions (3.30) and (3.31) show that equation (3.28) holds for all closed paths in $HG$. Now let $v_0 \in V(HG)$ be an arbitrary vertex and set $m_{v_0} =$ identity. For $v \in V(HG)$ define $m_v = m_{\overrightarrow{v_0v_1}} \circ m_{\overrightarrow{v_1v_2}} \circ \cdots \circ m_{\overrightarrow{v_{n-1}v}}$ for any path $v_0, v_1, \dots, v_{n-1}, v$ in $HG$ from $v_0$ to $v$. Then for $z \in V(SG)$ define a circle $C_z = m_v(\mathbb{R} \cup \{\infty\})$ for any $v \in V(HG)$ which is nearest to $z$. Similarly, define $p(w) = m_v(\infty)$ for any $v \in V(HG)$ which is nearest to $w \in V(SG^*)$. One easily verifies that this leads to a circle pattern $\mathscr{C}$ for $SG$ and $\alpha_\psi$ with invariants $\tau_{\mathscr{C}} = \tau$ and $\sigma_{\mathscr{C}} = \sigma$. □

As noticed by Schramm, the invariant $\sigma$ can be eliminated from equations (3.30) and (3.31) by dividing the terms for $a_\mp = z \mp (-1+i)/2$ and $b_\pm = z \pm (1+i)/2$ respectively and multiplying these quotients. This leads to an equation for $\tau$ which is a nonlinear discrete analog of the Laplace equation for harmonic functions.

**Theorem 3.29** ([67, Theorem 5.2])**.** *(1) Suppose that the positive functions $\tau : V(SG) \to (0,\infty)$ and $\sigma : V(SG^*) \to (0,\infty)$ satisfy equations* (3.30) *and* (3.31)*. Then $\tau$ satisfies the following equation for every $z \in V(SG)$:*

$$\tau(z)^2 = \frac{(\tau(z+1+i)+1)(\tau(z-1-i)+1)}{(\tau(z-1+i)^{-1}+1)(\tau(z+1-i)^{-1}+1)}. \tag{3.32}$$

*(2) Suppose that $\tau : V(SG) \to (0,\infty)$ satisfies equation (3.32) for all $z \in V(SG)$. Then there is a function $\sigma : V(SG^*) \to (0,\infty)$ such that $\tau$ and $\sigma$ together satisfy equations (3.30) and (3.31). $\sigma$ is unique, up to multiplication with a positive constant.*

For further use we introduce the function

$$\tilde{H}(\tau_1,\tau_2,\tau_3,\tau_4) = \sqrt{\frac{(\tau_1+1)(\tau_3+1)}{(\tau_2^{-1}+1)(\tau_4^{-1}+1)}}. \tag{3.33}$$

Note that equation (3.32) can be written as

$$\tau(z) = \tilde{H}(\tau(z+1+i), \tau(z-1+i), \tau(z-1-i), \tau(z+1-i)).$$

Furthermore $\tau_1 = \tilde{H}(\tau_1,\tau_1,\tau_1,\tau_1)$ for all $\tau_1 > 0$ and $\tilde{H}$ is strictly monotonically increasing in each of its arguments.

**Definition.** Let $v \in V(SG)$ be a vertex and let $k \in \mathbb{N}$. The *$k$th generation around $v$* is the set of all vertices with combinatorial distance $k$ from $v$ in $SG$. Denote by $SG(n,v)$ the subgraph of $SG$ composed of the first $n$ generations around $v$.

For further use, note the following generalization of the Ring Lemma of [64].

**Lemma 3.30.** *Let $r$ be the radius function of an embedded circle pattern for $SG(3,0)$ and $\alpha_\psi$. There is a constant $C = C(\psi) > 0$ independent of $r$ such that for $k = 0,1,2,3$*

$$\frac{r(i^k(1+i))}{r(0)} > C.$$

*Proof.* Assume the contrary. Then there is a sequence of embedded circle patterns for $SG(3,0)$ and $\alpha_\psi$ such that $r_n(0) = 1$ and $r_n(i^k(1+i)) \to 0$ as $n \to \infty$ for some $k \in \{0,1,2,3\}$. Without loss of generality we assume that $k = 0$. We also may assume that the circle $C_0$ corresponding to the vertex $0 \in V(SG)$ and the intersection point corresponding to $1 \in V(SG^*)$ are fixed for the whole sequence. Then there is a subsequence such that all the circles converge to circles or lines, that is converge in the Riemann sphere $\hat{\mathbb{C}} \cong \mathbb{S}^2$. We distinguish three possible cases for the limit.

First assume that the radii $r_{n_m}(1-i)$ and $r_{n_m}(-1+i)$ of the circles tangent to $C_{1+i}$ do both not converge to 0. Then equation (2.2) implies that the radius of at least one of the circles $C_{-2i}, C_{2-2i}, C_2, C_{2i}, C_{-2+2i}, C_{-2}$ does not converge to 0. Without loss of generality assume that this circle is $C_2$. The other cases are analogous. In the limit, this circle or line intersects the limit circle or line corresponding to the vertex $-1+i$. Consequently, there exits some kites associated to these circles which intersect in the limit in their interiors. But this is a contradition to the embeddedness of the sequence.

Second, suppose that $\lim_{m\to\infty} r_{n_m}(1-i) = 0$, but the radii $r_{n_m}(-1+i)$ and $r_{n_m}(-1-i)$ both do not converge to 0. Then equation (2.2) again implies that the radius of at least one of the circles $C_{-2i}, C_{-2-2i}, C_{-2}, C_{-2+2i}, C_{2i}$ does not converge to 0. Now similar arguments as in the previous case yield a contradition. An analogous reasoning applies for the case that $\lim_{m\to\infty} r_{n_m}(-1+i) = 0$ and $\lim_{m\to\infty} r_{n_m}(1-i) \neq 0$, $\lim_{m\to\infty} r_{n_m}(-1-i) \neq 0$.

For the last case, assume that the radii for two of the vertices $-1+i, -1-i, 1-i$ converge to 0 and the radii of one vertex does not converge to 0. Considering the limit circle or line for the incident vertices to this vertex we obtain a contradition to the embeddedness as in the previous cases. This completes the proof. $\square$

**Remark 3.31.** In the orthogonal case, that is $\psi = \pi/2 \equiv \alpha_\psi$, Lemma 3.30 also holds for $SG(2,0)$ instead of $SG(3,0)$.

**Theorem 3.32** (Rigidity of $SG$-circle patterns, cf. [67])**.** *Suppose that $\mathscr{C}$ is an embedded planar circle pattern for SG and $\alpha_\psi$. Then $\mathscr{C}$ is the image of a regular isoradial circle pattern for SG and $\alpha_\psi$ under a similarity.*

The proof given by Schramm and using the Möbius invariants $\tau$ and $\sigma$ remains valid for the more general labelling $\alpha_\psi$. First note, that Lemma 3.30 together with the representations (3.15) and (3.21) implies that the Möbius invariants $\tau$ and $\sigma$ of $\mathscr{C}$ are bounded. Let $M$ be the supremum of the values of $\tau$. The main idea of Schramm's proof is to use the Laplace equation (3.32) for $\tau$, the monotonicity of $\tilde{H}$, and the fixed combinatorics of $SG$, in order to show that the circle pattern $\mathscr{C}$ has arbitrarily large parts where $\tau$ is arbitrarily close to $M$ and $\sigma$ is almost fixed. In these parts, $\mathscr{C}$ approximates an $SG$-Doyle spiral or the isoradial $SG$-circle pattern. Thus the embeddedness implies that $M = \sup \tau = \cot^2(\psi/2)$. Similar reasonings show that $\inf \tau = \cot^2(\psi/2)$. Since $\mathscr{C}$ is embedded, we also have $\sigma \equiv 1$.

### *3.6.2 An analog of the Rodin-Sullivan Conjecture*

In this section, we establish a result for $SG$-circle patterns, which is an analog of the Rodin-Sullivan Conjecture.

We begin with an adapted version of the Hexagonal Packing Lemma considered by Rodin and Sullivan in [64].

**Lemma 3.33.** *There is a sequence $s_n = s_n(\psi)$, decreasing to $0$, with the following property. Let $G$ be a subgraph of $SG$ and let $\mathscr{C}$ be an embedded circle pattern for $G$ and $\alpha_\psi$ with radius function $r$. Let $v \in V(G)$ be a vertex and suppose that $SG(n, v) \subset G$, that is $G$ contains $n$ generations of $SG$ around $v$ for some $n \geq 3$. Then for all vertices $w$ incident to $v$ there holds*

$$1 - s_n \leq \frac{r(w)}{r(v)} \leq 1 + s_n. \tag{3.34}$$

The proof is analogous to the proof of the Hexagonal Packing Lemma of [64] and is a consequence of the Rigidity Theorem 3.32.

*Proof.* Let $v \in V(SG)$ be a fixed vertex. For each $n \geq 3$ let $G_n$ be a subgraph of $SG$ and let $\mathscr{C}_n$ be an embedded circle pattern for $G_n$ and $\alpha_\psi$ with radius function $r_n$. Suppose that $SG(n, v) \subset G_n$, that is $G_n$ contains $n$ generations of $SG$ around $v$. Furthermore, we assume that the circle patterns $\mathscr{C}_n$ are scaled such that $r_n(v) = 1$ for all $n$. Note that scaling does not change the quotient $r_n(w)/r_n(v)$. Lemma 3.30 shows that for $n \geq k+3$ the radii of the circles of generation $k$ about $v$ are uniformly bounded from above and below. Therefore we can choose a subsequence of $(\mathscr{C}_n)_{n\geq 3}$ such that all circles of generation one converge geometrically, and so on.

In this way, we obtain a limit embedded circle pattern for $SG$ and $\alpha_\psi$. The Rigidity Theorem 3.32 implies that this is the regular circle pattern. In particular the radii of the circles for the vertices which are incident to $v$ are one. □

The rest of this section is devoted to the proof of $s_n = \mathcal{O}(1/n)$ for any fixed $\psi \in (0, \pi)$. The argumentation is an adaption of the corresponding proof for the case of hexagonal circle packings by Aharonov in [5, 6].

**Lemma 3.34.** *Consider a circle pattern for $SG(1, 0)$ and $\alpha_\psi$. Denote the interior vertex by $v_0$ and the boundary vertices by $v_1, v_2, v_3, v_4$. Let $r_j = r(v_j)$ be the radii of the corresponding circles and set $c_j = c([v_0, v_j]) = 2f'_{\alpha_\psi([v_0,v_j])}(0) = \cot(\alpha_\psi([v_0, v_j])/2)$ for $j = 0, \ldots, 4$. Then*

$$\sum_{j=1}^{4} c_j r_0 \leq \sum_{j=1}^{4} c_j r_j \qquad \textit{and} \qquad \sum_{j=1}^{4} c_j \frac{1}{r_0} \leq \frac{1}{4} \sum_{j=1}^{4} c_j \frac{1}{r_j}.$$

*Proof.* Denote $x_j = r_j/r_0 \geq 0$ for $j = 1, \dots, 4$. Using condition (2.2) we have

$$g(x_1, x_2, x_3, x_4) := \sum_{j=1}^{4} f_{\alpha_\psi([v_0,v_j])}(\log x_j) = \pi. \tag{3.35}$$

Consider the function $f(x_1, x_2, x_3, x_4) := \sum_{j=1}^{4} c_j x_j$. Straightforward, but lengthy calculations yield that the minimum of $f$ on $\{x_1, x_2, x_3, x_4 \geq 0\}$ under the condition (3.35) is attained at $(1, 1, 1, 1)$, such that $f(x_1, x_2, x_3, x_4) \geq \sum_{j=1}^{4} c_j$ holds for all points of $\{x_1, x_2, x_3, x_4 \geq 0\}$. This gives the first claim.

To prove the second estimation, remember that if $r$ is a radius function for $SG(1, 0)$ and $\alpha_\psi$ then $1/r$ is also a radius function for $SG(1, 0)$ and $\alpha_\psi$ by Lemma 2.8. Thus the second claim follows from the first claim. □

Remember the definition of the *discrete Laplacian* on $SG$ in (3.2) by

$$\Delta u(x) = \sum_{k=0}^{3} c_k(u(x + \omega i^k) - u(x)), \tag{3.36}$$

where $\omega = 1 + i$ and we define for $x \in V(SG)$ and $k \in \mathbb{Z}$

$$c_k = c([x, x + \omega i^k]) = 2f'_{\alpha_\psi([x,x+\omega i^k])}(0). \tag{3.37}$$

**Lemma 3.35** (cf. [5, Lemma 3.2]). *Let $u > 0$ be a function which satisfies $\Delta u \geq 0$ and $\Delta(1/u) \geq 0$. Then*

$$|\Delta(\log u)(x)| \leq \sum_{k=0}^{3} c_k \frac{(u(x + \omega i^k) - u(x))^2}{u(x + \omega i^k)u(x)}.$$

The proof of this lemma is an obvious adaption of the corresponding proof in [5]. The main idea is to show that $\Delta(\log u) \leq \Delta u/u$ by an estimation of the logarithm and to use that $(\Delta u)/u + (\Delta \frac{1}{u})/\frac{1}{u} = \sum_{k=0}^{3} c_k \frac{(u(x+\omega i^k)-u(x))^2}{u(x+\omega i^k)u(x)}$.

**Theorem 3.36** (cf. [5, Theorem 3.1]). *There exists an absolute constant $\rho > 0$, independent of $n$, such that if the ratio $M_n/m_n < 1 + \rho$ and $n \geq 3$, then $s_n = \mathcal{O}(1/n)$. Here $M_n$ ($m_n$) is the maximum (minimum) of the radii of circles in an embedded SG-circle pattern $\mathscr{C}_n$ for $SG(n, 0)$ and $\alpha_\psi$.*

The following proof is based on the arguments of the corresponding proof in [5] and uses Green's Identity 3.15 , Hölder's Inequality 3.18, and the Regularity Lemma 3.19.

*Proof.* The proof uses induction on the number of generations $n$. For $n = 3$ the statement follows from Lemma 3.30.

Assume that the theorem has been proven up to $n - 1$, i.e. for some $A > 0$

$$s_k \leq \frac{A}{k}, \qquad \text{for } 3 \leq k \leq n - 1.$$

Without loss of generality we can assume $n - 1 \geq 5$. Let $1 \leq n_1 < n - 3$ be fixed later. Set $v = \log r$, where $r$ is the radius function of an embedded circle pattern for $SG(n, 0)$ and $\alpha_\psi$. Lemma 3.35 and the induction hypothesis imply that

$$|\Delta v(x)| \leq \frac{4A^2 C_\psi}{(n - n_1 - 1)^2} \tag{3.38}$$

for vertices $x$ in the sublattice $SG(n_1, 0)$, where $C_\psi = \max\{c_0, c_1\}$. The same estimation is true for $\hat{v} = v - \log m_n$.

The Regularity Lemma 3.19 implies that there are constants $B_1$, $B_2$ independent of $n$, $n_1$, and $\hat{v}$ such that

$$\begin{aligned} n_1|v(0)-v(\omega)| = n_1|\hat{v}(0)-\hat{v}(\omega)| &\leq B_1\|\hat{v}\|_{SG(n_1,0)} + B_2 n_1^2\|\Delta\hat{v}\|_{SG(n_1-1,0)} \\ &\leq B_1 \log\left(\frac{M_n}{m_n}\right) + n_1^2\frac{A^2B_2C_\psi}{(n-n_1-1)^2}, \end{aligned}$$

where $\omega = 1+i$. In order to finish the proof, we need an estimate for $|r(0)-r(\omega)|/r(0)$. Indeed,

$$|v(0)-v(\omega)| = \left|\log\left(\frac{r(\omega)}{r(0)}\right)\right| = \left|\log\left(1+\frac{r(\omega)-r(0)}{r(0)}\right)\right| > B_3\left|\frac{r(\omega)-r(0)}{r(0)}\right|$$

for some absolute constant $B_3$ (as $n \geq 6$ and $s_n \to 0$). Hence we get

$$\left|\frac{r(\omega)-r(0)}{r(0)}\right| < \frac{B_4\log(M_n/m_n)}{n_1} + \frac{B_5A^2n_1}{(n-n_1-1)^2},$$

where $B_4$ and $B_5$ are absolute constants depending on the previous constants $B_1, B_2, B_3, C_1$. Our aim is to show that if $n_1$ is properly chosen and if $\log(M_n/m_n)$ is small enough, then

$$\frac{B_4\log(M_n/m_n)}{n_1} + \frac{B_5A^2n_1}{(n-n_1-1)^2} \leq \frac{A}{n}.$$

But $M_n/m_n < 1+\rho$ implies $\log(M_n/m_n) < \rho$ and therefore it suffices to prove

$$\frac{B_4\rho}{n_1} + \frac{B_5A^2n_1}{(n-n_1-1)^2} \leq \frac{A}{n}$$

for a small enough $\rho$. For convenience we write $n_1 = \lambda n/A$ and choose the optimal $\lambda$ instead of the optimal $n_1$. Thus we have

$$\frac{B_4\rho A}{n\lambda} + \frac{B_5A^2\lambda n}{A(n-n\lambda/A-1)^2} \leq \frac{A}{n} \quad\Leftrightarrow\quad \frac{B_4\rho}{\lambda} + \frac{B_5\lambda}{(1-\lambda/A-1/n)^2} \leq 1.$$

If we make the obvious restriction $\lambda/A - 1/n \leq 3/4$, we get $B_4\rho\lambda^{-1} + 16B_5\lambda \leq 1$. We now take $\lambda^2 = B_4\rho/(16B_5)$ to get the desired restriction $\rho = (64B_5B_4)^{-1}$. Since $B_4$ and $B_5$ are absolute constants, $\rho$ does not depend on $n$.

The proof is complete except two additional conditions. The first minor technical detail is that $n_1$ has to be an integer. But obviously, $n_1$ has also to satisfy $n_1 \geq 1$. We have

$$\lambda^2 = \frac{B_4\rho}{16B_5} = \frac{B_4}{16B_5}(64B_5B_4)^{-1} = \frac{1}{32^2B_5^2}.$$

Hence $\lambda = (32B_5)^{-1}$ and $n_1 = n/(32AB_5)$. Thus the condition $n_1 \geq 1$ is equivalent to $A/n \leq 1/(64B_5)$. But since $s_n \to 0$ is known and $B_5$ is an absolute constant, we obviously have $s_n \leq 1/(2\cdot 64B_5) = \lambda/4$ for all $n \geq n_0$ with some $n_0 \in \mathbb{N}$. Now we can choose a constant $A$ and $N \geq n_0$ such that $s_k < A/k$ for all $k = 1,\ldots,N-1$ and $A/N \leq \lambda/2$ and use this $n \geq N$ for the induction hypothesis. $\square$

**Remark 3.37** ([6])**.** Theorem 3.36 is actually a statement on a certain class of functions, which can be formulated as follows.

There exist two numerical constants $A > 0$ and $0 < \rho \leq \min(\log 2, A)$ with the following property. Let $u > 0$ be a function on $SG(n,0)$ for $n \geq 3$ such that $u$ and $1/u$ are subharmonic. Assume that $u$ satisfies the *$\rho$-condition*

$$\exp(-3\rho) \leq u \leq \exp(3\rho) \tag{3.39}$$

on $SG(n,0)$. Then for any natural $n \geq 3$ we have

$$\max_{k=1,2,3,4} \frac{u(\omega i^k) - u(0)}{u(0)} < \frac{A}{n}. \tag{3.40}$$

Consider the function $u(z) = \mathrm{e}^{M \operatorname{Re}(z(-1+i))/n}$ for $M > 0$ on $SG(n,0)$. Then it is easy to see that both $u$ and $1/u$ are subharmonic. But the quotient $(u(\omega i^k) - u(0))/u(0) = \mathrm{e}^{M \operatorname{Re}(-2i^k)/n} - 1$ depends on $M$ for even $k$. Thus estimation 3.40 cannot be satisfied for an arbitrary large $M$. Therefore Theorem 3.36 is sharp in the sense that the $\rho$-condition cannot be removed.

*Brief review of quasiconformal mappings*

For further use, we briefly recall the notion of quasiconformality and some properties of such mappings. This can be found in standard textbooks, for example by Letho and Virtanen [52] or by Ahlfors [7].

A *topological quadrilateral* $Q(z_1, z_2, z_3, z_4)$ in the plane consists of a domain $Q$ whose boundary is a Jordan curve and of four different boundary points $z_1, z_2, z_3, z_4$. The order of the points is assumed to be the same when using the positive orientation of the boundary curve induced by the standard orientation of $Q$ and starting with $z_1$. The Riemann mapping theorem implies that each quadrilateral $Q(z_1, z_2, z_3, z_4)$ can be mapped to a quadrilateral $Q'(0, 1, 1+iM, iM) = \{z \in \mathbb{C} : \operatorname{Re}(z) \in (0,1),\ \operatorname{Im}(z) \in (0,M)\}$ such that $Q'$ is bounded and if the map is continued to the closure $\bar{Q}$ of $Q$, then the four different boundary points $z_1, z_2, z_3, z_4$ are mapped to $0, 1, 1+iM, iM$ respectively, where $M > 0$. Then $M = M(Q)$ is uniquely determined by $Q(z_1, z_2, z_3, z_4)$ and is called *(conformal) modulus* of $Q$.

Given an orientation preserving homeomorphism $f$ on a domain $G$, denote by $Q'$ the image under $f$ of the quadrilateral $Q$. Then $K_f(G) := \sup_{\bar{Q} \subset G} \frac{M(Q')}{M(Q)}$ is called *maximum dilatation* of $f$ in the domain $G$. If $K_f(G)$ is finite, that is $K_f(G) \leq K < \infty$, then $f$ is called *$K$-quasiconformal*. There are equivalent definitions of $K$-quasiconformality using annuli (cf. [52, Theorem 7.2]) or using complex dilatation (cf. [52, §5] or [7]).

Some basic facts on quasiconformal mappings are:

(i) $K$-quasiconformality is a local property [52, Theorem 9.1], and

(ii) a 1-quasiconformal mapping is conformal and conversely [52, Theorem 5.1].

We shall also use the fact that simplicial homeomorphisms are $K$-quasiconformal for $K$ depending only on the shapes of the triangles involved. To see this, note that an affine map is $K$-quasiconformal for $K$ depending only on the shapes of one triangle and its image. For a piecewise affine homeomorphism observe [52, Theorem 8.3] that a homeomorphism of a domain which is $K$-quasiconformal on the complement of an analytic arc in the domain is actually $K$-quasiconformal in the entire domain.

**Lemma 3.38** ([6, Lemma 2.1], [52, Theorem 5.1])**.** *Let $F$ be a family of $K$-quasiconformal mappings in a planar domain $G$. Let $a, b$ be two distinct points in $G$ such that $f(a) = a$ and $f(b) = b$ holds for each $f \in F$. Assume further that for each $f \in F$ a certain fixed value (say $\infty$) is omitted. Then $F$ is a normal family.*

To finish the proof of $s_n = \mathcal{O}(1/n)$, we show that the radius function satisfies the $\rho$-condition (3.39) and thus the assumptions of Theorem 3.36.

Let $q > 0$ and $m \in \mathbb{N}$, $m \geq 1$. We denote by $SG_{\psi,q}(m,0)$ the embedded isoradial circle pattern for $SG(m,0)$ and $\alpha_\psi$ with radius $r \equiv q$ and such that $0$ and $2q\sin(\psi/2)\mathrm{e}^{i(\pi-\psi)/2} = q(1 - \cos\psi + i\sin\psi)$ are the centers of the circles corresponding to the vertices $0 \in$

$V(SG(m,0))$ and $\omega = (1+i) \in V(SG(m,0))$ respectively. Let $SG'_\psi(m)$ be an embedded $SG$-circle pattern for $SG(m,0)$ and $\alpha_\psi$.

Let $l, N \geq 2$ be positive integers. Consider the mapping from $SG_{\psi,N^{-1}}(lN+3,0)$ to $SG'_\psi(lN+3)$ which maps centers and intersection points of $SG_{\psi,N^{-1}}(lN+3,0)$ to the corresponding points of $SG'_\psi(lN+3)$. This map can be continued by barycentric coordinates on the triangles formed by two centers corresponding to incident vertices and one of the intersection points on the corresponding circles. Therefore this induces a map $f_{lN}$ from $SG_{\psi,N^{-1}}(lN,0) \subset SG_{\psi,N^{-1}}(lN+3,0)$ to $SG'_\psi(lN) \subset SG'_\psi(lN+3)$ which is $K$-quasiconformal. Note that $K$ is independent of $N$ and $l$ by Lemma 3.30. Without loss of generality we may also assume that $f_{lN}(0) = 0$.

Next, consider the function $g_{lN} = f_{lN}/f_{lN}(1)$ which satisfies the normalization $g_{lN}(0) = 0$ and $g_{lN}(1) = 1$. Denote the image of $SG_{\psi,N^{-1}}(lN,0)$ under $g_{lN}$ by $SG^*_\psi(lN)$, which is $SG'_\psi(lN)$ "divided" by $f_{lN}(1)$. In fact, the original circle pattern $SG'_\psi(lN)$ is scaled by the factor $1/|f_{lN}(1)|$ and rotated about 0 by the angle $-\arg f_{lN}(1)$. The normalization of $g_{lN}$ is needed to deduce from Lemma 3.38 that $\{g_{lN}\}$ builds a normal family of $K$-quasiconformal mappings. Note the following simple but important invariance property. The scaling by $f_{lN}(1)^{-1}$ does not change the ratio between two radii.

**Lemma 3.39** ([6, Lemma 3.1]). *Let $\{g_{lN}\}$ be defined as above. Given $\varepsilon > 0$ and a compact domain $D \subset \mathbb{C}$, there are constants $N_0 = N(\varepsilon, D, \psi)$ and $l_0 = l(\varepsilon, D, \psi)$ such that*

$$|g_{lN}(z) - z| < \varepsilon \qquad \text{for all } z \in D,\ N \geq N_0,\ l \geq l_0.$$

*The constants $N_0$ and $l_0$ do not depend on the particular sequence $\{g_{lN}\}$.*

The proof given by Aharonov uses Lemma 3.38 and the fact, that if $\{g_{l_jN_j}\}_{j\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{C}$ for some subsequence with $l_j \to \infty$ and $N_j \to \infty$ as $j \to \infty$, then the limit function is an injective holomorphic mapping of $\mathbb{C}$ and thus the identity due to the normalization of $g_{lN}$.

Lemma 3.39 will be considered in a variant, that will be more convenient for our applications. For this aim, take the closed disk $\{z : |z| \leq 5\}$ as a particular $D$. Consider the function $\varphi_{lN}(z) = N g_{lN}(z/N)$ which maps $SG_{\psi,1}(lN,0)$ onto $SG^*_\psi(lN)$ "multiplied" by $N$. Denote this scaled circle pattern by $SG^{**}_\psi(lN)$. Then it is clear that

$$\varphi_{lN}(0) = 0, \qquad \varphi_{lN}(N) = N. \tag{3.41}$$

Furthermore we get as a simple consequence of Lemma 3.39.

**Lemma 3.40** ([6, Lemma 3.2]). *Let $\{\varphi_{lN}\}$ be defined as above. Then given $\varepsilon > 0$, there exist constants $N_0 = N(\varepsilon, \psi)$ and $l_0 = l(\varepsilon, \psi)$ such that for $N \geq N_0, l \geq l_0$*

$$|\varphi_{lN}(z) - z| < \varepsilon N \qquad \text{if } |z| \leq 5N.$$

*Also, $N_0$ and $l_0$ do not depend on the particular sequence $\{\varphi_{lN}\}$.*

**Corollary 3.41.** *Let $\overline{D}$ be a closed parallelogram with boundary $\partial D$. Assume that for some $N_1 \geq 2$ and for all $N \geq N_1$ we have $\overline{D} \subset \{z \in \mathbb{C} : |z| \leq 5N\}$. Then given $\varepsilon > 0$, there exist constants $N_0 = N(\varepsilon, \psi) \geq N_1$ and $l_0 = l(\varepsilon, \psi)$ such that for $N \geq N_0, l \geq l_0$*

$$|F(\varphi_{lN}(\overline{D})) - F(\overline{D})| < \varepsilon N \mathbb{L}(\partial D) + \pi(\varepsilon N)^2.$$

*Here, $F(K)$ denotes the area of the closed set $K$ and $\mathbb{L}(\gamma)$ the length of the curve $\gamma$.*

In the following, $\rho$ and $A$ denote the specific numerical constants appearing in Theorem 3.36 and Remark 3.37. We further define

$$\varepsilon_1 := \varepsilon(\rho, A, \psi) = \frac{\rho^2 \sin\psi}{16A}. \tag{3.42}$$

Set $M_1 = \max\{2, \lceil 4/\sin\psi + 1/2\rceil\}$ and let $N_1$ be any specific natural satisfying

$$N_1 > \max\{N_0(\varepsilon_1, \psi), M_1 A/\rho\}. \tag{3.43}$$

Now take $l \geq l_0(\varepsilon_1, \psi)$ sufficiently large such that in addition we also have $s_{(l-2)N_1} < A/N_1$. For convenience we add the trivial restriction $l \geq 4$. So choose a specific $l$ satisfying all these restrictions and denote it by $l_2$. Hence,

$$s_{(l-2)N_1} < A/N_1 \qquad \forall l \geq l_2 \geq \max\{l_1(\varepsilon_1), 4\}. \tag{3.44}$$

The proof of the next theorem uses ideas of the corresponding proof of Aharonov.

**Theorem 3.42** (cf. [6, Theorem 4.1])**.** *Let $\rho$, $A$, $\varepsilon_1$, $N_1$, and $l_2$ be the numbers defined above. In addition, let $N_k = 2N_{k-1} = 2^{k-1}N_1$ for any natural $k$. Consider the functions $\{f_{l_2N_k}\}$, $\{g_{l_2N_k}\}$, $\{\varphi_{l_2N_k}\}$ as defined above, so $\varphi_{l_2N_k}$ maps $SG_{\psi,1}(l_2N_k, 0) \subset SG_{\psi,1}(l_2N_k + 3, 0)$ onto $SG_\psi^{**}(l_2N_k) \subset SG_\psi^{**}(l_2N_k + 3)$ and satisfies the normalization* (3.41) *for $N = N_k$ and $l = l_2$. Then the radii of the circles in $SG_\psi^{**}(2N_k)$ satisfy the $\rho$-condition* (3.39)*.*

*Proof.* Before turning to the proof, we point out two facts. First, it is enough to prove the $\rho$-condition (3.39) only for boundary circles of the given configuration. This is due to the Maximum Principle 3.11 which holds for the subharmonic functions $r$ and $1/r$.

The second useful fact was already mentioned above. If we scale the configuration $SG'_\psi(m)$ by some constant $\lambda > 0$, then the ratio $r_j/r_k$ is transformed to $(\lambda r_j)/(\lambda r_k)$ which is the same. Thus, given an arbitrary sequence $\{f_{lN}\}$ constructed as above, we may consider instead the sequence $\{\varphi_{lN}\}$ without loss of generality to prove the claim.

We now turn to the proof, which will be by induction. First, consider the case $k = 1$. Consider the configuration $SG_\psi^{**}(l_2N_1)$ and its subconfiguration $SG_\psi^{**}(2N_1)$. By our choice of $l_2$ we have $s_{(l_2-2)N_1} < A/N_1$ by (3.44). Our aim is to show that the $\rho$-condition (3.39) is satisfied for the radii of the boundary circles of $SG_\psi^{**}(2N_1)$. Denote by $r^{**}$ the radius function of $SG_\psi^{**}(2N_1)$ and assume the contrary, namely, that for some boundary circle $C_p$ of $SG_\psi^{**}(2N_1)$, at least one of the following holds

$$\text{either} \qquad r^{**}(p) \leq \mathrm{e}^{-3\rho} \tag{3.45a}$$

$$\text{or} \qquad r^{**}(p) \geq \mathrm{e}^{3\rho}. \tag{3.45b}$$

We now show that each of the two assumptions (3.45a) and (3.45b) leads to a contradiction. For this purpose, observe that

$$r^{**}(v_j)/r^{**}(v_k) < 1 + A/N_1 \tag{3.46}$$

for any two incident vertices $v_j, v_k \in V(SG(2N_1, 0))$. Indeed, since $SG_\psi^{**}(2N_1)$ is a subconfiguration of $SG_\psi^{**}(l_2N_1)$, each circle of $SG_\psi^{**}(2N_1)$ is surrounded by at least $(l_2-2)N_1$ generations, which gives $r^{**}(v_j)/r^{**}(v_k) \leq 1 + s_{(l_2-2)N_1} < 1 + A/N_1$ by (3.44).

Next, choose a natural $m \in \mathbb{N}$ satisfying

$$\rho/(2A) \leq m/N_1 \leq \rho/A \leq 1 \qquad \text{and} \qquad m \geq M_1. \tag{3.47}$$

Such an $m$ actually exists, since $\rho/A - \rho/(2A) = \rho/(2A) > 1/N_1$ and $N_1 \geq M_1A/\rho$ by (3.43), and also $\rho \leq A$ by our assumption.

Consider any vertex $v$ with combinatorial distance less than $2m$ to $p$ in $SG(2N_1, 0)$ and assume that (3.45a) holds. Then we have the following estimation of its radius:

$$r^{**}(v) \leq (1 + A/N_1)^{2m} r^{**}(p) \overset{(3.45a)}{\leq} (1 + A/N_1)^{2m}\mathrm{e}^{-3\rho} < \mathrm{e}^{2mA/N_1 - 3\rho} \overset{(3.47)}{\leq} \mathrm{e}^{-\rho}.$$

Let $D_\psi(z, 2m) = \{z + \lambda_1 + \lambda_2\mathrm{e}^{i(\pi-\psi)} \in \mathbb{C} : \lambda_1, \lambda_2 \in [0, 2m]\}$ denote the parallelogram which is a scaled and translated version of the union of kites corresponding to the circle pattern $SG_{\psi,1}(2N_1, 0)$. As $2m \leq 2N_1$ there is a vertex $v_0 \in V(SG(2N_1, 0))$ such that for the corresponding center of circle $x_0$ we have $D_\psi(x_0, 2m) \subset SG_{\psi,1}(2N_1, 0)$. By construction, all vertices which correspond to centers of circles contained in $D_\psi(x_0, 2m)$ have a combinatorial distance less than $2m$ to $p$. Consider a rhombus $T$ of $SG_{\psi,1}(2N_1, 0)$ built from centers of circles and intersection points and corresponding to incident vertices. Suppose that $T$ is contained in $D_\psi(x_0, 2m)$. Then

$$0 \geq F(\varphi_{l_2 N_1}(T)) - F(T) \geq (\mathrm{e}^{-\rho} - 1)F(T),$$

where $F(\cdot)$ denotes the area. Therefore we deduce from Corollary 3.41 that

$$\begin{aligned} \varepsilon_1 N_1 8m + \pi\varepsilon_1^2 N_1^2 &> |F(\varphi_{l_2 N_1}(D_\psi(x_0, 2m))) - F(D_\psi(x_0, 2m))| \\ &\geq (1 - \mathrm{e}^{-\rho})F(D_\psi(x_0, 2m)) = \mathrm{e}^{-\rho}(\mathrm{e}^{\rho} - 1)4m^2 \sin\psi \geq 2\rho m^2 \sin\psi. \end{aligned}$$

The last inequality follows since $\rho \leq \log 2$. Combining this inequality with the definition of $\varepsilon_1$ in (3.42) and with estimation (3.47), we obtain

$$\varepsilon_1 > \frac{\rho \sin\psi}{8N_1/m + \varepsilon_1 \pi (N_1/m)^2} = \frac{\rho \sin\psi}{8N_1/m + \rho^2 \sin\psi \pi (N_1/m)^2/(16A)} \geq \frac{\rho^2 \sin\psi}{9A},$$

which is a contradition to the choice of $\varepsilon_1$ in (3.42).

Assuming (3.45b) instead of (3.45a) and choosing $m$ and $D_\psi(x_0, 2m)$ as above, we obtain by similar observations

$$r^{**}(v) \geq (1 + A/N_1)^{-2m} r^{**}(p) \overset{(3.45a)}{\geq} (1 + A/N_1)^{-2m} \mathrm{e}^{3\rho} > \mathrm{e}^{-2mA/N_1 + 3\rho} \overset{(3.47)}{\geq} \mathrm{e}^{\rho}.$$

Again, we deduce from Corollary 3.41 that

$$\begin{aligned} \varepsilon_1 N_1 8m + \pi\varepsilon_1^2 N_1^2 &> |F(\varphi_{l_2 N_1}(D_\psi(x_0, 2m))) - F(D_\psi(x_0, 2m))| \\ &\geq (\mathrm{e}^{\rho} - 1)F(D_\psi(x_0, 2m)) = (\mathrm{e}^{\rho} - 1)4m^2 \sin\psi \geq 4\rho m^2 \sin\psi. \end{aligned}$$

Combining this inequality with (3.47), we obtain by similar estimations $\varepsilon_1 > \frac{2\rho^2 \sin\psi}{9A}$, which is again a contradition to the choice of $\varepsilon_1$ in (3.42). Hence, the proof for the case $k = 1$ is complete.

Now we proceed with the induction proof. Assume that the statement has been established for $N_k$ for some $k \geq 1$. The induction hypothesis implies that the $\rho$-condition (3.39) is satisfied for for $SG_\psi^{**}(2N_k)$. Theorem 3.36 implies that $s_{l_2 N_k} < A/(2N_k) = A/N_{k+1}$. Note that for the subgraph $SG(2N_{k+1}, 0) \subset SG(l_2 N_{k+1}, 0)$ each boundary vertex is surrounded by $(l_2 - 2)N_{k+1} \geq l_2 N_k$ generations since $N_{k+1} = 2N_k$ and $l_2 \geq 4$. Hence we deduce that

$$r^{**}(v_j)/r^{**}(v_k) < 1 + A/N_{k+1} \tag{3.48}$$

for each two neighboring vertices of $SG(2N_{k+1}, 0)$. The remaining proof is now analogous to the previous treatment taking $N_{k+1}$ instead of $N_1$. This completes the induction step and the proof. □

The following theorem is a direct consequence of the preceeding theorem.

**Theorem 3.43** ([6, Theorem 5.1]). *Let $A$, $\varepsilon_1$, $N_1$, and $l_2$ be the numerical constants defined above. Then for all $n \geq l_2 N_1$*

$$s_n < l_2 A/n.$$

**Corollary 3.44.** *There is some absolute constant $C > 0$, depending only on $\psi$, such that for all $n \geq 3$*

$$s_n \leq C/n. \tag{3.49}$$

For the orthogonal case $\psi = \pi/2 \equiv \alpha_\psi$, we can also take $n \geq 2$.

## 3.7 Uniqueness of embedded quasicrystallic circle patterns

The uniqueness of regular circle patterns with square grid combinatorics is used in this section in order to establish the uniqueness of a certain class of embedded quasicrystallic circle patterns. Let $\mathscr{E}$ be the family of all infinite quasicrystallic rhombic embeddings of connected and simply connected b-quad-graphs $\mathscr{D}$ which cover the entire complex plane and such that the brick $\Pi(\Omega_{\mathscr{D}})$ of the corresponding combinatorial surface $\Omega_{\mathscr{D}}$ contains a $\mathbb{Z}^2$-sublattice, that is there are at least two different indices $j_1, j_2$ such that $\min_{\mathbf{n}\in V(\Omega_{\mathscr{D}})} n_{j_k} = -\infty$ and $\max_{\mathbf{n}\in V(\Omega_{\mathscr{D}})} n_{j_k} = \infty$ for $k = 1, 2$. Note that $\mathscr{E}$ contains in particular the examples of Section 3.2, like the Penrose tiling, for which $\Pi(\Omega_{\mathscr{D}}) = \mathbb{Z}^d$.

**Theorem 3.45** (Rigidity of quasicrystallic isoradial circle patterns)**.** *Let $\mathscr{D} \in \mathscr{E}$ be an infinite quasicrystallic rhombic embedding with associated graph $G$ and corresponding labelling $\alpha$. Let $\mathscr{C}$ be an embedded circle pattern for $G$ and $\alpha$. Then $\mathscr{C}$ is the image of the isoradial circle pattern corresponding to $\mathscr{D}$ under a similarity of the complex plane.*

For our proof and for further use we introduce the following notation.

**Definition 3.46.** Let $\mathscr{D}$ be a rhombic embedding of a finite simply connected b-quad-graph and let $\Omega_{\mathscr{D}}$ be the corresponding combinatorial surface in $\mathbb{Z}^d$.

Let $\hat{\mathbf{z}} \in V(\Omega_{\mathscr{D}})$. A *combinatorial $K$-environment of $\hat{\mathbf{z}}$ in $\Omega_{\mathscr{D}}$* consists of the part of $\Omega_{\mathscr{D}}$ whose vertices have combinatorial distance at most $K$ from $\hat{\mathbf{z}}$ within $\Omega_{\mathscr{D}}$. This part of $\Omega_{\mathscr{D}}$ corresponds to the combinatorial $K$-environment in $\mathscr{D}$ of the corresponding point $z \in V(\mathscr{D})$.

For $\hat{\mathbf{z}} \in V_{int}(\Omega_{\mathscr{D}})$ denote by $d(\hat{\mathbf{z}}, \partial\Omega_{\mathscr{D}})$ the combinatorial distance of $\hat{\mathbf{z}}$ to the boundary $\partial\Omega_{\mathscr{D}}$, that is the largest integer $K$ such that a combinatorial $(K-1)$-environment of $\hat{\mathbf{z}}$ in $\Omega_{\mathscr{D}}$ does not contain any boundary points of $\Omega_{\mathscr{D}}$.

Let $n \in \mathbb{N}$ and $\hat{\mathbf{z}} \in V(\Omega_{\mathscr{D}})$. Assume that $n \le d(\hat{\mathbf{z}}, \partial\Omega_{\mathscr{D}})$. The vertices of the combinatorial $n$-environment of $\hat{\mathbf{z}}$ that are not contained in the combinatorial $(n-1)$-environment of $\hat{\mathbf{z}}$ belong to the *nth generation of vertices about $\hat{\mathbf{z}}$*.

*Proof of Theorem 3.45.* Let $\mathscr{D} \in \mathscr{E}$ be an infinite quasicrystallic rhombic embeddings with associated graph $G$ and corresponding labelling $\alpha$. Let $\mathscr{C}$ be an embedded circle pattern for $G$ and $\alpha$. Consider the comparison function $w$ defined by (3.11) on $\Omega_{\mathscr{D}}$ and extend it to $\Pi(\Omega_{\mathscr{D}})$. Let $\hat{\mathbf{z}} \in V(\Omega_{\mathscr{D}})$. By assumption on $\mathscr{D}$, there is a $\mathbb{Z}^2$-sublattice $\Omega(\hat{\mathbf{z}})$ with $\hat{\mathbf{z}} \in V(\Omega(\hat{\mathbf{z}}))$ which is contained in $\Pi(\Omega_{\mathscr{D}})$. Furthermore, we can perform flips for $\Omega_{\mathscr{D}}$ and corresponding flips for the circle pattern $\mathscr{C}$ such that the resulting combinatorial surface $\Omega'$ contains an arbitrary number of generations of the lattice $\Omega(\hat{\mathbf{z}})$ about $\hat{\mathbf{z}}$. As the corresponding circle pattern $\mathscr{C}'$ is embedded and as the number of generations about $z$ can be chosen arbitrarily large, we deduce from the Rigidity Theorem 3.32 that the radius function is constant on $\Omega(\hat{\mathbf{z}})$. More precisely, the extension of $w$ is constant on white and black vertices of $\Omega(\hat{\mathbf{z}})$ respectively.

This argumentation is valid for all vertices $\hat{\mathbf{z}} \in V(\Omega_{\mathscr{D}})$ and all $\mathbb{Z}^2$-sublattice $\Omega(\hat{\mathbf{z}})$ with $\hat{\mathbf{z}} \in V(\Omega(\hat{\mathbf{z}}))$ which are contained in $\Pi(\Omega_{\mathscr{D}})$. Therefore the extension of $w$ is constant, on white and black vertices respectively, on all $\mathbb{Z}^2$-sublattices which are contained in $\Pi(\Omega_{\mathscr{D}})$. Due to the combinatorics of $\mathscr{D}$, the radius function which is $w$ restricted to white vertices has to be constant on the whole brick $\Pi(\Omega_{\mathscr{D}})$. This implies in particular that $\mathscr{C}$ is an isoradial circle pattern and thus is the image of the isoradial circle pattern corresponding to $\mathscr{D}$ under a similarity of the complex plane. $\square$

**Corollary 3.47.** *Let $\mathscr{D} \in \mathscr{E}$ be an infinite quasicrystallic rhombic embeddings with corresponding labelling $\alpha$. Let $v_0 \in V_w(\mathscr{D})$ be a white vertex. Then there are a constant $n_0 = n_0(\mathscr{D}) \in \mathbb{N}$ and a sequence $s_n(v_0, \mathscr{D})$ decreasing to $0$ for $n \to \infty$ such that the following holds.*

*For $n \in \mathbb{N}$, $n \geq n_0$, let $\mathscr{D}_{2n}(v_0)$ be the rhombic embedding corresponding to the $2n$-environment of $v_0$. Let $G_n(v_0)$ be the corresponding graph built from white vertices. Let $\mathscr{C}_n$ be an embedded circle pattern for $G_n(v_0)$ and the labelling $\alpha$ taken from $\mathscr{D}$ with radius function $r_n$. Then there holds*

$$\left|\frac{r_n(v_0)}{r_n(v_1)} - 1\right| \leq s_n(v_0, \mathscr{D}) \tag{3.50}$$

*for all vertices $v_1 \in V(G_n(v_0))$ incident to $v_0$.*

*Proof.* The proof is analogous to the proof of Lemma 3.33 by considering a sequence of embedded circle patterns $(\mathscr{C}_n)_{n \geq n_0}$ for $G_n(v_0)$ and the labelling $\alpha$ taken from $\mathscr{D}$. The role of Theorem 3.32 is substituted by Theorem 3.45. Furthermore, instead of Lemma 3.30 a generalized version has to be used which is presented in Lemma 3.48 below. □

**Lemma 3.48.** *Let $\mathscr{D}$ be a quasicrystallic rhombic embeddings with corresponding graph $G$ built from white vertices and labelling $\alpha$. Denote by $\alpha_{min} = \min\{\alpha(e) : e \in E(G)\}$ the smallest intersection angle. Let $n_0 \in \mathbb{N}$ be such that $(n_0 - 3)\alpha_{min} > \pi$. Let $v_0 \in V_w(\mathscr{D})$ be a white vertex. Assume that $\mathscr{D}$ contains a $(2n_0)$-environnement about $v_0$. Then there is a constant $C = C(\mathscr{D}) > 0$ such that the following holds.*

*Let $r$ be the radius function of an embedded circle pattern for $\mathscr{D}$ and $\alpha$ and let $v_1$ be a vertex incident to $v_0$ in $G$. Then*

$$\frac{r(v_1)}{r(v_0)} > C.$$

*Proof.* The ideas of the proof are very similar as in the proof of Lemma 3.30.

Suppose that there is a vertex $v_1$ incident to $v_0$ and a sequence of embedded circle patterns for $\mathscr{D}$ and $\alpha$ with radius functions $r_n$ such that $r_n(v_0) = 1$ and $r_n(v_1) \to 0$ as $n \to \infty$. Without loss of generality we may assume that the circle $C_0$ corresponding to the vertex $v_0$ and the intersection point corresponding to one fixed black vertex $w_0$ incident to $v_0$ and $v_1$ in $\mathscr{D}$ are fixed for the whole sequence. Then there is a subsequence such that all the circles converge to circles or lines, that is converge in the Riemann sphere $\hat{\mathbb{C}} \cong \mathbb{S}^2$.

Equation (2.2) implies that there are at least two circles corresponding to vertices incident to $v_0$ whose radii do not converge to 0. If the limit is finite, there are at least two circles whose radii do not converge to 0 corresponding to vertices incident to this vertex of the first generation and so on.

Consider the kites which contain in the limit the intersection point corresponding to $w_0$ and apply the above argument at most $n_0$ times. Then by our assumption on $n_0$ we obtain two kites whose interiors intersect in the limit. This is a contradition to the embeddedness of the sequence. □

If $\mathscr{D}$ is one of the examples constructed by the methods of Section 3.2 then we obtain the following uniform estimation of $s_n$ which is an analog of the Rodin-Sullivan Conjecture and a generalization of Corollary 3.44.

**Corollary 3.49.** *Let $\mathscr{D}$ be a plane based quasicrystallic rhombic embedding. There are some absolute constants $C = C(\mathscr{D}) > 0$ and $n_0 = n_0(\mathscr{D}) \in \mathbb{N}$, depending only on $\mathscr{D}$, such that for all white vertices $v_0 \in V_w(\mathscr{D})$ and all $n \geq n_0$ there holds*

$$s_n(v_0, \mathscr{D}) \leq s_n(\mathscr{D}) \leq C/n. \tag{3.51}$$

*Proof.* Let $v_0 \in V_w(\mathscr{D})$ be any white vertex. If $n \geq n_0(\mathscr{D})$ is big enough, for each $\mathbb{Z}^2$-sublattice $\Omega(\hat{\mathbf{v}}_0)$ the set $\mathcal{F}(\Omega_{\mathscr{D}})$ contains a $\lfloor B(\mathscr{D})n \rfloor$-environment of $\hat{\mathbf{v}}_0$ in $\Omega(\hat{\mathbf{v}}_0)$, where the constant $B(\mathscr{D})$ depends only on the construction parameters of $\mathscr{D}$. Here $\lfloor p \rfloor$ denotes the largest integer smaller than $p \in \mathbb{R}$. Therefore we can choose $s_n(\mathscr{D})$ to be the maximum of $s_{\lfloor B(\mathscr{D})n \rfloor}(\mathscr{D}(\hat{\mathbf{v}}_0))$ for all possible regular rhombic embeddings $\mathscr{D}(\hat{\mathbf{v}}_0)$ corresponding to $\mathbb{Z}^2$-sublattice $\Omega(\hat{\mathbf{v}}_0)$. Now the claim follows from Corollary 3.44 for $SG$-circle patterns. □

Chapter 4

# Some properties of the $Z^\gamma$-circle patterns

In this chapter we present two results for (orthogonal) $Z^\gamma$-circle patterns. First, we derive some estimations which are useful for our convergence proof in Chapter 7. Second, we prove uniqueness of the $Z^\gamma$-circle patterns. We also extend this result for some classes of quasicrystallic $Z^\gamma$-circle patterns.

We begin with a brief review of know results on orthogonal $Z^\gamma$-circle patterns and introduce the notation used in the following.

## 4.1 Brief review of orthogonal $Z^\gamma$-circle patterns

The standard orthogonal circle pattern with the combinatorics of the square-grid associated to the map $Z^\gamma$ was defined by Bobenko in [9]. Further development of the theory and the proofs of the following results are due to Bobenko and Agafonov and can be found in [4, 1, 2].

**Definition 4.1.** Let $D \subset \mathbb{Z}^2$. A map $f : D \to \mathbb{C}$ is called *discrete conformal* if all its elementary quadrilaterals are conformal squares, i.e. their cross-ratios are equal to -1:

$$q(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) := \frac{(f_{n,m} - f_{n+1,m})(f_{n+1,m+1} - f_{n,m+1})}{(f_{n+1,m} - f_{n+1,m+1})(f_{n,m+1} - f_{n,m})} = -1. \quad (4.1)$$

Note that this definition is Möbius invariant and is motivated by the following characterization for smooth mappings:
A smooth map $f : \mathbb{C} \subset D \to \mathbb{C}$ is called *conformal* (holomorphic or antiholomorphic) if and only if for all $z = x + iy \in D$ there holds

$$\lim_{\varepsilon \to 0} q(f(x, y), f(x + \varepsilon, y), f(x + \varepsilon, y + \varepsilon), f(x, y + \varepsilon)) = -1.$$

**Definition 4.2.** A discrete conformal map $f_{n,m}$ is called an *immersion* if the interiors of adjacent elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ are disjoint.

A discrete conformal map $f_{n,m}$ is called *embedded* if the interiors of different elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ are disjoint.

To construct an embedded discrete analog of $z^\gamma$ the following approach is used. Equation (4.1) can be supplemented with the following nonautonomous constraint:

$$\gamma f_{n,m} = 2n \frac{(f_{n+1,m} - f_{n,m})(f_{n,m} - f_{n-1,m})}{(f_{n+1,m} - f_{n-1,m})} + 2m \frac{(f_{n,m+1} - f_{n,m})(f_{n,m} - f_{n,m-1})}{(f_{n,m+1} - f_{n,m-1})}. \quad (4.2)$$

This constraint, as well as its compatibility with (4.1), is derived from some monodromy problem; see [4] for more details. We assume $0 < \gamma < 2$ and denote

$$\mathbb{Z}^2_+ = \{(n, m) \in \mathbb{Z}^2 : n, m \geq 0\}.$$

The asymptotics of the constraint (4.2) for $n, m \to \infty$ and the properties

$$z^\gamma(\mathbb{R}_+) = \mathbb{R}_+, \qquad z^\gamma(i\mathbb{R}_+) = \mathrm{e}^{\gamma\pi i/2}\mathbb{R}_+$$

of the holomorphic mapping $z^\gamma$ motivate the following definition of the discrete analog.

**Definition 4.3.** For $0 < \gamma < 2$, the discrete conformal map $Z^\gamma : \mathbb{Z}^2_+ \to \mathbb{C}$ is the solution of equations (4.1) and (4.2) with the initial conditions

$$Z^\gamma(0,0) = 0, \quad Z^\gamma(1,0) = 1, \quad Z^\gamma(0,1) = \mathrm{e}^{\gamma\pi i/2}.$$

From this definition, the properties $Z^\gamma(n,0) \in \mathbb{R}_+$ and $Z^\gamma(0,m) \in \mathrm{e}^{\gamma\pi i/2}\mathbb{R}_+$ are obvious for all $n, m \in \mathbb{N}$. Furthermore, the discrete conformal map $Z^\gamma$ from Definition 4.3 determines an $SG$-circle pattern. In fact (see [4, Proposition 1]), all edges at the vertex $f_{n,m}$ with $n+m = 0 \pmod 2$ have the same length and all angles between neighboring edges at the vertex $f_{n,m}$ with $n+m = 1 \pmod 2$ are equal to $\pi/2$. Thus, all elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ build orthogonal kites and for any $(n,m) \in \mathbb{Z}^2_+$ with $n+m = 0 \pmod 2$ the points $f_{n+1,m}, f_{n,m+1}, f_{n-1,m}, f_{n,m-1}$ lie on a circle with center $f_{n,m}$. All such circles form an orthogonal circle pattern with combinatorics of the square grid. Therefore, we consider the sublattice $\{(n,m) \in \mathbb{Z}^2_+ : n+m = 0 \pmod 2\}$ and denote by $\mathbb{V}$ the quadrant

$$\mathbb{V} = \{z = N + iM : N, M \in \mathbb{Z}, M \geq |N|\},$$

where

$$N = (n-m)/2, \quad M = (n+m)/2.$$

For this sublattice, complex labels $z = N + iM$ will be used. Denote by $C(z)$ the circle of radius

$$R(z) = |f_{n,m} - f_{n+1,m}| = |f_{n,m} - f_{n,m+1}| = |f_{n,m} - f_{n-1,m}| = |f_{n,m} - f_{n,m-1}|$$

with center at $f_{N+M,M-N} = f_{n,m}$. As the discrete conformal map $Z^\gamma$ from Definition 4.3 is an immersion (see Theorem 4.4 (ii) below), the radius function $R : \mathbb{V} \to \mathbb{R}_+$ satisfies the following equations derived in [4]:

$$\begin{aligned} R(z)R(z+1)(-2M-\gamma) + R(z+1)R(z+1+i)(2(N+1)-\gamma) \\ + R(z+1+i)R(z+i)(2(M+1)-\gamma) + R(z+i)R(z)(-2N-\gamma) = 0 \end{aligned} \tag{4.3}$$

for $z \in \mathbb{V} \cup \{-N + i(N-1) | N \in \mathbb{N}\}$,

$$\begin{aligned} (N+M)(R(z)^2 - R(z+1)R(z-i))(R(z+i) + R(z+1)) \\ + (M-N)(R(z)^2 - R(z+i)R(z+1))(R(z+1) + R(z-i)) = 0 \end{aligned} \tag{4.4}$$

for $z \in \mathbb{V} \setminus \{N + iN | N \in \mathbb{N}\}$,

$$\begin{aligned} (N+M)(R(z)^2 - R(z+i)R(z-1))(R(z-1) + R(z-i)) \\ + (M-N)(R(z)^2 - R(z-1)R(z-i))(R(z+i) + R(z-1)) = 0 \end{aligned} \tag{4.5}$$

for $z \in \mathbb{V} \setminus \{-N + iN | N \in \mathbb{N}\}$ and

$$\begin{aligned} (N+M)(R(z)^2 - R(z+i)R(z-1))(R(z+1) + R(z+i)) \\ + (N-M)(R(z)^2 - R(z+1)R(z+i))(R(z+i) + R(z-1)) = 0 \end{aligned} \tag{4.6}$$

for $z \in \mathbb{V} \setminus \{\pm N + iN | N \in \mathbb{N}\}$.

For further use, we gather some of the known results.

**Theorem 4.4** ([4, 1, 2]). *(i) If $R(z)$ denotes the radius function corresponding to the discrete conformal map $Z^\gamma$ for some $0 < \gamma < 2$, then it holds that*

$$(\gamma - 1)(R(z)^2 - R(z-i)R(z+1)) \geq 0 \tag{4.7}$$

*for all $z \in \mathbb{V} \setminus \{\pm N + iN | N \in \mathbb{N}\}$.*

*(ii)* *For* $0<\gamma<2$*, the discrete conformal maps* $Z^\gamma$ *given by Definition 4.3 are embedded.*

*(iii)* *The circle patterns corresponding to the discrete conformal maps* $Z^\gamma$ *for* $0<\gamma<2$ *are embedded.*

*(iv)* *If* $R(z)$ *denotes the radius function which corresponds to the discrete conformal map* $Z^\gamma$ *for some* $0<\gamma<2$*, then* $\tilde{R}(z)=1/R(z)$ *is the radius function corresponding to the discrete conformal map* $Z^{\tilde\gamma}$ *for* $\tilde\gamma=2-\gamma$*.*

*(v)* *Let* $R(z)$ *denote the radius function corresponding to the discrete conformal map* $Z^\gamma$ *for some* $0<\gamma<2$*. Then for all* $N\geq 0$*, the radius the boundary circles and of their neighbors can be represented in terms of the* $\Gamma$*-function:*

$$\begin{aligned} R(N+iN) &= c(\gamma)\frac{\Gamma(N+\gamma/2)}{\Gamma(N+1-\gamma/2)} = c(\gamma)N^{\gamma-1}\left(1+O\left(\frac{1}{N}\right)\right), \\ R(N+i(N+1)) &= \left(\tan\frac{\gamma\pi}{2}\right)^{(-1)^N}\left(\frac{(2(N-1)+\gamma)(2(N-3)+\gamma)(2(N-5)+\gamma)\cdots}{(2N-\gamma)(2(N-2)-\gamma)(2(N-4)-\gamma)\cdots}\right)^2 \\ &= c(\gamma)N^{\gamma-1}\left(1+O\left(\frac{1}{N}\right)\right), \end{aligned}$$

*where* $c(\gamma)=\frac{\Gamma(1-\gamma/2)}{\Gamma(1-\gamma/2)}$*. Furthermore, this implies for the asymptotics of* $Z^\gamma$

$$Z^\gamma(n,k)=\frac{2c(\gamma)}{\gamma}\left(\frac{n+ik}{2}\right)^\gamma\left(1+O\left(\frac{1}{n^2}\right)\right) \tag{4.8}$$

*for* $k=0,1$ *and* $n\to\infty$*.*

*(vi)* *If* $R(N+iM)$ *denotes the radius function corresponding to the discrete conformal map* $Z^\gamma$ *for some* $0<\gamma<2$*, then there holds*

$$R(N_0+iM)\simeq K(\gamma)M^{\gamma-1} \qquad \text{as } M\to\infty, \tag{4.9}$$

$$Z^\gamma(n_0+n,m_0+n)\simeq e^{i\gamma\pi/4}K(\gamma)n^\gamma \qquad \text{as } n\to\infty \tag{4.10}$$

*with a constant* $K(\gamma)>0$ *independent of* $N_0$*,* $n_0$*, and* $m_0$*.*

In the following two sections we continue to use the notation of this section. In particular the radius function is denoted by $R$ and we have the normalization $R(0)=1$.

## 4.2 Geometric properties of the $Z^\gamma$-circle patterns and consequences

In this section we exploit and generalize the proofs of Theorem 3 and Lemma 1 of [1].

For some fixed $n>0$, let $\Gamma_n$ be the piecewise linear curve formed by the segments $[f_{n,m},f_{n,m+1}]$ for $m\geq 0$ and let $\Gamma_n^{(l)}$ be the part of $\Gamma_n$ for $0\leq m\leq l$. By Theorem 4.4 (ii), this curve is embedded. Consider the vector $\mathbf{v}_n(m)=f_{n,m+1}-f_{n,m}$ along this curve. Due to the connection of $f_{n,m}$ with an orthogonal circle pattern, this vector rotates only in vertices with $n+m=0 \pmod 2$, which correspond to centers of circles, as $m$ increases along the curve. The rotation angle at the point $f_{n,m}$ is denoted $\theta_n(m)$ and defined by $\mathbf{v}_n(m)=\mathrm{e}^{i\theta_n(m)}\mathbf{v}_n(m-1)$ for $m\geq 1$, where $-\pi<\theta_n(m)<\pi$. For geometric reasons the sign of $\theta_n(m)$ is given by the sign of the expression $R(z)^2-R(z-i)R(z-1)$, where $z=(n-m)/2+i(n+m)/2$ is the label for the circle with center at $f_{n,m}$. Note that there is no rotation if this expression vanishes. If $n+m=1 \pmod 2$, we have $\theta_n(m)=0$. Furthermore, define $\theta_n(0)\in(-\pi/2,\pi/2)$ to be the angle between $\mathbf{v}_n(0)$ and the imaginary axes. Then $\theta_n(0)=0$ for odd $n$. Equations (4.4)–(4.6) together with Theorem 4.4 (i) imply that

$$(\gamma-1)(R(N+i(N+1))^2-R(N+iN)R(N+1+i(N+1)))\geq 0$$

holds for all $N\geq 0$. Thus $\theta_n(0)\leq 0$ for $0<\gamma<1$ and $\theta_n(0)\geq 0$ for $1<\gamma<2$.

**Lemma 4.5.** *(i) The vector $\mathbf{v}_n(m)$ rotates with increasing $m$ in the same direction for all $n$, and, namely, clockwise for $0 < \gamma < 1$ and counterclockwise for $1 < \gamma < 2$.*

*(ii) For $n, m \geq 0$, the radii $R(n - m + i(n + m))$ of the circles with centers on the curve $\Gamma_n$ are a decreasing function of $m$ for $0 < \gamma < 1$ and an increasing function of $m$ for $1 < \gamma < 2$.*

*(iii) [1, Lemma 1] For the curve $\Gamma_n$ it holds that*

$$\left|\sum_{m=0}^{n-1} \theta_n(m)\right| < \frac{\pi}{4}(1 + |1 - \gamma|). \tag{4.11}$$

*(iv) For the curve $\Gamma_n$ and all $N \geq 0$ it holds that*

$$\left|\sum_{m=0}^{n-1} \theta_n(m)\right| \leq \frac{\pi}{4}|\gamma - 1| \qquad \textit{and} \qquad \left|\sum_{m=0}^{N} \theta_n(m)\right| \leq \frac{\pi}{2}|\gamma - 1|. \tag{4.12}$$

*Proof.* (i) This is a direct consequence of the sign of $\theta_n(m)$.

(ii) This can be deduced from equations (4.4)–(4.6) and is contained in the proof of Lemma 1 in [2].

(iv) The proof is a generalization of the proof of Lemma 1 in [1]. Define the angle $\alpha_n(m)$ between the imaginary axis $i\mathbb{R}_+$ and the vector $\mathbf{v}_n(m)$ by $f_{n,m+1} - f_{n,m} = \mathrm{e}^{i(\alpha_n(m)+\pi/2)}|f_{n,m+1} - f_{n,m}|$, where $-2\pi < \alpha_n(m) \leq 0$ for $0 < \gamma < 1$ and $0 \leq \alpha_n(m) < 2\pi$ for $1 < \gamma < 2$ and $0 \leq m < n$. Then $\alpha_n(0) = \theta_n(0)$ and

$$\left|\sum_{m=0}^{l} \theta_n(m)\right| = |\alpha_n(l)| + 2\pi k_n(l)$$

for some integer $k_n(l) \geq 0$ increasing with $l$. The proof of Lemma 1 in [1] shows, that $k_n(l) = 0$ for $0 \leq l \leq n-1$. Therefore it is sufficient to prove $|\alpha_n(n-1)| \leq |\gamma - 1|\pi/4$. This follows for geometric reasons together with claim (iii).

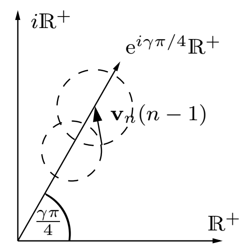



**Figure 4.1:** Geometric considerations for $\mathbf{v}_n(n-1)$.

More precisely, the angle between $i\mathbb{R}_+$ and the diagonal $\mathrm{e}^{i\gamma\pi/4}\mathbb{R}_+$ is $-\pi/2 + \gamma\pi/4 < 0$; see Figure 4.1. The radii $R(iM)$ on the diagonal are a decreasing function of $M \geq 0$ for $0 < \gamma < 1$ and an increasing function of $M \geq 0$ for $1 < \gamma < 2$. This is also a consequence of equations (4.4)–(4.6) as shown in the proof of Lemma 1 in [2]. This implies that the angle between the diagonal and $\mathbf{v}_n(n-1)$, that is between $\mathrm{e}^{i\gamma\pi/4}\mathbb{R}_+$ and $\mathrm{e}^{i(\alpha_n(n-1)+\pi/2)}\mathbb{R}_+$, is bigger than $\pi/4$ for $0 < \gamma < 1$ and smaller than $\pi/4$ for $1 < \gamma < 2$. Combing these two facts, we arrive at the desired estimations:

$$\begin{aligned} & 0 \leq -\alpha_n(n-1) \leq \frac{\pi}{2} - \gamma\frac{\pi}{4} - \frac{\pi}{4} = \frac{\pi}{4}(1 - \gamma) \quad \text{for } 0 < \gamma < 1 \\ \text{and} \qquad & 0 \leq \alpha_n(n-1) \leq \frac{\pi}{4} - \left(\frac{\pi}{2} - \gamma\frac{\pi}{4}\right) = \frac{\pi}{4}(\gamma - 1) \quad \text{for } 1 < \gamma < 2. \end{aligned}$$

This also proves the second estimation of (4.12) for $0 \leq N \leq n-1$. Now we proceed by induction on $N$. Assume that

$$\left|\sum_{m=0}^{N} \theta_n(m)\right| = |\alpha_n(N)| \leq \frac{\pi}{2}|\gamma - 1|$$

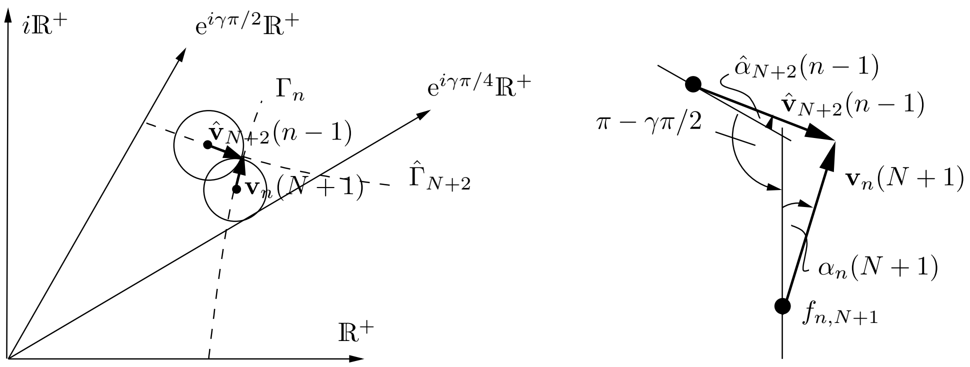



**Figure 4.2:** Geometric considerations for $\hat{\mathbf{v}}_n(N+2)$.

is true for some $N \geq n-1$. If $\theta_n(N+1)=0$, the claim is proven for $N+1$. So assume that $\theta_n(N+1)\neq 0$. We will only consider the case $0<\gamma<1$ as the proof for $1<\gamma<2$ is analogous. Thus we have $-\pi<\theta_n(N+1)<0$. Remember that the $Z^\gamma$-circle patterns are symmetric with respect to mirror reflections in the diagonal $e^{i\gamma\pi/4}\mathbb{R}_+$. Let $\hat{\Gamma}_{N+2}$ denote the curve $\Gamma_{N+2}$ reflected in the diagonal $e^{i\gamma\pi/4}\mathbb{R}_+$ and let $\hat{\alpha}_{N+2}(m)$ be the angle between the reflected imaginary axis $e^{-i(1-\gamma)\pi/2}\mathbb{R}_+$ and the reflected vector $\hat{\mathbf{v}}_{N+2}(m)=f_{m+1,N+2}-f_{m,N+2}$; see Figure 4.2. Due to the reflection and to the first claim we have $0\leq\hat{\alpha}_{N+2}(m)\leq(1-\gamma)\pi/4$ for $0\leq m\leq N-1$. As $\theta_n(N+1)\neq 0$, we deduce that $f_{n,N+1}$ is the center of a circle and $f_{n,N+2}$ is an intersection point of circles. Therefore, the vectors $\mathbf{v}_n(N+1)$ and $\hat{\mathbf{v}}_{N+2}(n-1)$ are orthogonal. As the angle between $e^{-i(1-\gamma)\pi/2}\mathbb{R}_+$ and $i\mathbb{R}_+$ is $\pi-\gamma\pi/2$ and $0\leq\hat{\alpha}_{N+2}(n-1)=\hat{\alpha}_{N+2}(n)\leq(1-\gamma)\pi/4$, we obtain

$$\alpha_n(N+1)=\alpha_n(N+2)=-(\pi-\gamma\frac{\pi}{2})+\hat{\alpha}_{N+2}(n-1)+\frac{\pi}{2}$$
$$\text{and thus}\qquad -(1-\gamma)\frac{\pi}{4}\geq\alpha_n(N+1)=\alpha_n(N+2)\geq-(1-\gamma)\frac{\pi}{2}.$$

Now using the induction hypothesis, we get

$$0\geq\sum_{m=0}^{N+1}\theta_n(m)=\alpha_n(N)+\theta_n(N+1)=\alpha_n(N+1)-2\pi k_n(N+1)$$

with some nonnegative integer $k_n(N+1)$. As

$$0\leq 2\pi k_n(N+1)=\alpha_n(N+1)-\alpha_n(N)-\theta_n(N+1)\leq(1-\gamma)\pi/2+\pi<2\pi,$$

we deduce $k_n(N+1)=0$. This proves the induction step and thus the second claim. □

The properties of the curve $\Gamma_n$ together with the asymptotics of the radius function and of $Z^\gamma$ of Theorem 4.4 (v) and (vi) imply the following estimations.

**Lemma 4.6.** *(i) There are two constants $\underline{B_\gamma},\overline{B_\gamma}>0$, depending only on $\gamma\in(0,2)$, such that for all circles with centers on the curve $\Gamma_n^{(n-1)}$ the radii are bigger than $\underline{B_\gamma}n^{\gamma-1}$ and smaller than $\overline{B_\gamma}n^{\gamma-1}$.*

*(ii) There are two constants $\underline{A_\gamma},\overline{A_\gamma}>0$, depending only on $\gamma\in(0,2)$, such that the distance of a point on the curve $\Gamma_n^{(n-1)}$ to the origin is bigger than $\underline{A_\gamma}n^\gamma$ and smaller than $\overline{A_\gamma}n^\gamma$.*

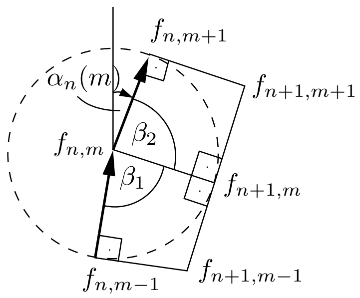



**Figure 4.3:** The angles at the point $f_{n,m}$.

*(iii) There is a constant $D_\gamma > 0$, depending only on $\gamma \in (0,2)$, such that the quotient of radii of orthogonally intersecting circles of the $Z^\gamma$-circle pattern is smaller than $D_\gamma$.*

*Proof.* (i) This is a direct consequence of Lemma 4.5 (ii) together with the estimations in Theorem 4.4 (v) and (vi).

(ii) By Lemma 4.5, the curve $\Gamma_n^{(n-1)}$ is the graph of a monotone (concave) function over an interval of $\mathbb{R}^+$ and is contained in the sector $s = \{z = \rho e^{i\beta} \in \mathbb{C} : \rho \geq 0,\ \beta \in [0, \gamma\pi/4]\}$. Furthermore, $\Gamma_n^{(n-1)}$ intersects each of the boundary half-lines of $s$ exactly once. Using Theorem 4.4 (v) and (vi), we can deduce the desired estimations.

(iii) Consider a center of circle $f_{n,m}$ on the curve $\Gamma_n^{(n)}$ for $0 \leq m < n$ and let $\beta_1$ and $\beta_2$ denote the angles at $f_{n,m}$ in the orthogonal kites with vertices $f_{n,m-1}$, $f_{n+1,m-1}$, $f_{n+1,m}$ and $f_{n,m+1}$, $f_{n+1,m+1}$, $f_{n+1,m}$ respectively, as shown in Figure 4.3. Geometric considerations and the proof of part (iv) of Lemma 4.5 imply for $0 < \gamma < 1$ that

$$\frac{\pi}{2} - \frac{\pi}{4}(1-\gamma) \leq \pi - (\frac{\pi}{2} - \alpha_{n+1}(m)) \leq \beta_1 \leq \frac{\pi}{2} - \alpha_n(m-1) \leq \frac{\pi}{2} + \frac{\pi}{4}(1-\gamma),$$
$$\frac{\pi}{2} - \frac{\pi}{4}(1-\gamma) \leq \frac{\pi}{2} + \alpha_n(m) \leq \beta_2 \leq \pi - \beta_1 \leq \frac{\pi}{2} + \frac{\pi}{4}(1-\gamma).$$

Thus for $0 < \gamma < 1$ all angles of the orthogonal kites are uniformly bounded away from 0 and $\pi$ and so the quotients of corresponding radii are uniformly bounded from above and below by a constant, which depends only on $\gamma$. Using Theorem 4.4 (iv), the claim also follows for the case $1 < \gamma < 2$.

□

For further use, we note a simple consequence of part (iii) of the preceeding lemma.

**Corollary 4.7.** *For $\gamma \in (0,2)$, all angles* $\arctan \frac{R(v_k)}{R(v_j)}$ *of the $Z^\gamma$-circle pattern for two vertices $v_k, v_j \in \mathbb{V}$ corresponding to intersecting circles are uniformly bounded away from 0 and $\pi/2$. The bound depends only on $\gamma$.*

## 4.3 Uniqueness of the $Z^\gamma$-circle patterns

This section is devoted to the proof of following uniqueness result.

**Theorem 4.8** (Rigidity of $Z^\gamma$-circle patterns)**.** *For $\gamma \in (0,2)$ the infinite orthogonal embedded circle pattern corresponding to $Z^\gamma$ is the unique embedded orthogonal circle pattern (up to global scaling) with the following two properties.*

*(i) The union of the corresponding kites of the $Z^\gamma$-circle pattern covers the infinite sector $\{z = \rho e^{i\beta} \in \mathbb{C} : \rho \geq 0,\ \beta \in [0, \gamma\pi/2]\}$ with angle $\gamma\pi/2$.*

*(ii) The centers of the boundary circles lie on the boundary half lines $\mathbb{R}_+$ and $e^{i\gamma\pi/2}\mathbb{R}_+$.*

Our proof uses results of discrete potential theory or of the theory of random walks which can be found in standard textbooks, for example by Doyle and Snell [32] or by Woess [77]. We remind some basic terminology and notation and cite adapted versions of a few theorems which will be useful for our argumentation.

*Brief review of some results of the theory of random walks*

By abuse of notation, we denote by $\mathbb{Z}^2$ the points $(a,b) \in \mathbb{R}^2 \cong \mathbb{C}$ with $a, b \in \mathbb{Z}$ as well as the graph with vertices at these points and edges $e = [z_1, z_2]$ if $|z_1 - z_2| = 1$ with the standard metric of $\mathbb{R}^2$. The meaning will be clear from the context.

Consider the network $(\mathbb{Z}^2, c)$ with *conductances* $c(e) > 0$ and *resistances* $1/c(e)$ on the undirected edges $e \in E(\mathbb{Z}^2)$. Then a *transition probability function* $p$ is given by

$$p(z_1, z_2) := \begin{cases} c([z_1, z_2])/\big(\sum_{e=[z_1,z]\in E(\mathbb{Z}^2)} c(e)\big) & \text{if } [z_1, z_2] \in E(\mathbb{Z}^2) \\ 0 & \text{otherwise} \end{cases}.$$

A stochastic process on $\mathbb{Z}^2$ given by this probability function $p$ is called a *reversible random walk* or a *reversible Markov chain* on $\mathbb{Z}^2$. The *simple random walk* on $\mathbb{Z}^2$ is given by specifying $c(e) = 1$ for all edges which leads to $p(z_1, z_2) = 1/4$ if $[z_1, z_2] \in E(\mathbb{Z}^2)$.

Denote by $p_{\text{esc}}$ the probability that a random walk starting at any point will never return to this point. The network $(\mathbb{Z}^2, c)$ is called *recurrent* if $p_{\text{esc}} = 0$ (and *transient* otherwise). Note that $p_{\text{esc}} = 1/R_{\text{eff}}$, where $R_{\text{eff}}$ denotes the effective resistance from a point to infinity.

Let $f : \mathbb{Z}^2 \to \mathbb{R}$ be a function. Then $f$ is called *superharmonic* with respect to the probability function $p$ or with respect to the conductances $c$ if for every vertex $v \in \mathbb{Z}^2$ we have $\sum_{w\in\mathbb{Z}^2} p(v,w)f(w) \leq f(v)$. Similarly, $f$ is called *subharmonic* with respect to $p$ or $c$ if $\sum_{w\in\mathbb{Z}^2} p(v,w)f(w) \geq f(v)$ for every vertex $v \in \mathbb{Z}^2$.

**Theorem 4.9.** *(i) The simple random walk on $\mathbb{Z}^2$ is recurrent.*

*(ii) Let $(\mathbb{Z}^2, c_1)$ and $(\mathbb{Z}^2, c_2)$ be two networks with conductances $c_1(e) > 0$ and $c_2(e) > 0$ on the edges. If $c_2(e) \leq c_1(e)$ for all edges $e \in E(\mathbb{Z}^2)$, then the recurrence of $(\mathbb{Z}^2, c_1)$ implies the recurrence of $(\mathbb{Z}^2, c_2)$.*

For a proof see for example [32, Chapters 5, 7, and 8] or [77, Sections 1.A, 1.B, and Corollary (2.14)].

**Theorem 4.10** ([77, Theorem (1.16)])**.** *A network is recurrent if and only if all nonnegative superharmonic functions are constant.*

The following generalization of Lemma 3.34 for the case $\alpha \equiv \pi/2$ shows that the quotient of the radius functions of two orthogonal $SG$-circle patterns is subharmonic with respect to suitably chosen conductances.

**Proposition 4.11.** *Consider two orthogonal circle patterns for $SG(1,0)$. Denote the radii by $\rho_j$ and $r_j$ respectively, where $\rho_0$ and $r_0$ denote the radii of the inner circles. Then*

$$\sum_{j=1}^{4} c_j \frac{r_j}{\rho_j} \geq \frac{r_0}{\rho_0} \sum_{j=1}^{4} c_j \qquad \text{and} \qquad \sum_{j=1}^{4} c_j \frac{\rho_j}{r_j} \geq \frac{\rho_0}{r_0} \sum_{j=1}^{4} c_j, \tag{4.13}$$

*where $c_j = 1/((\rho_j/\rho_0) + (\rho_0/\rho_j))$.*

*Proof.* The proof is based on a Taylor expansion of $\arctan(x+y)$ about $y=0$.

$$\arctan(x+y) = \arctan x + \frac{y}{1+x^2} - \frac{\xi y^2}{(1+\xi^2)^2}$$

with $\xi = x(1-t)+ty$ for some $t \in (0,1)$. Equation (2.3) for the two circle patterns implies

$$\begin{aligned}\pi &= \sum_{j=1}^{4} \arctan \frac{r_j}{r_0} = \sum_{j=1}^{4} \arctan\left(\frac{\rho_j}{\rho_0} + \left(\frac{r_j}{r_0} - \frac{\rho_j}{\rho_0}\right)\right) \\ &= \underbrace{\sum_{j=1}^{4} \arctan \frac{\rho_j}{\rho_0}}_{=\pi} + \sum_{j=1}^{4} \frac{1}{1+\left(\frac{\rho_j}{\rho_0}\right)^2}\left(\frac{r_j}{r_0} - \frac{\rho_j}{\rho_0}\right) + \sum_{j=1}^{4} -\frac{\xi_j}{(1+\xi_j^2)^2}\left(\frac{r_j}{r_0} - \frac{\rho_j}{\rho_0}\right)^2,\end{aligned}$$

where $\xi_j = (1-t_j)\frac{\rho_j}{\rho_0} + t_j(\frac{r_j}{r_0} - \frac{\rho_j}{\rho_0}) > 0$ with some suitable $t_j \in (0,1)$ for $j = 1,2,3,4$. Thus

$$\sum_{j=1}^{4} \frac{1}{\frac{\rho_j}{\rho_0} + \frac{\rho_0}{\rho_j}} \left(\frac{r_j}{r_0}\frac{\rho_0}{\rho_j} - 1\right) \geq 0.$$

This implies the first claim. The second claim follows from the fact, that $1/\rho_j$ and $1/r_j$ are also the radii of orthogonal circle patterns for $SG(1,0)$ by Lemma 2.8. Also, the coefficients $c_j$ are invariant under the transformation $\rho_j \mapsto 1/\rho_j$. □

*Proof of Theorem 4.8.* Let $\gamma \in (0,2)$ and denote by $R : \mathbb{V} \to \mathbb{R}_+$ the radius function of the embedded $Z^\gamma$-circle pattern, where $\mathbb{V} = \{z = N + iM : N, M \in \mathbb{Z}^2, M \geq |N|\}$, as in Section 4.1. By definition we have $R(0) = 1$. Let $r : \mathbb{V} \to \mathbb{R}_+$ denote the radius function of an embedded orthogonal circle pattern with the same combinatorics and the same boundary conditions (orthogonal boundary circles to the half lines $\mathbb{R}_+$ and $\mathrm{e}^{i\gamma\pi/2}\mathbb{R}_+$) with the same normalization $r(0) = 1$. In the following, we will show that the radius functions take the same values on all of $\mathbb{V}$. This implies that both circle patterns coincide.

As both circle patterns are embedded, Corollary 3.44 implies that for some constant $A > 0$ and $n \geq 2$

$$1 - \frac{A}{n} \leq \frac{r(z_j^{(n)})}{r(z_0^{(n)})} \leq 1 + \frac{A}{n} \tag{4.14}$$

holds for all radii $r(z_0^{(n)})$ for vertices $z_0^{(n)}$ of the $n$th generation away from the origin and their incident vertices $z_j^{(n)}$. Remember that vertices $z \in \mathbb{V}$ belong to the $n$th generation if their distance in $\mathbb{V}$ to the origin is $n$. For estimation (4.14) we have also used that the reflection of the circle pattern in one of the boundary lines $\mathbb{R}_+$ or $\mathrm{e}^{i\gamma\pi/2}\mathbb{R}_+$ also leads to an embedded orthogonal circle pattern. The same reasoning applies to the radii of the $Z^\gamma$-circle pattern, so

$$1 - \frac{A}{n} \leq \frac{R(z_j^{(n)})}{R(z_0^{(n)})} \leq 1 + \frac{A}{n} \tag{4.15}$$

for $n \geq 2$ with the same constant $A$. Therefore there is a constant $K > 0$ such that

$$\frac{1}{K} \leq \frac{r(z_j)}{r(z_0)} \leq K \qquad \text{and} \qquad \frac{1}{K} \leq \frac{R(z_j)}{R(z_0)} \leq K \tag{4.16}$$

for all incident vertices $z_j$ and $z_0$.

We now define two undirected networks $(\mathbb{Z}^2, C)$ and $(\mathbb{Z}^2, \tilde{C})$ as follows. Consider the vertices $z \in \mathbb{V}$ corresponding to centers of circles. Two vertices are connected by an edge if and only if the corresponding circles intersect orthogonally. On these edges we define two conductance functions $C$ and $\tilde{C}$ by

$$C(e) = C(R(z_j), R(z_k)) := \left(\frac{R(z_j)}{R(z_k)} + \frac{R(z_k)}{R(z_j)}\right)^{-1},$$
$$\tilde{C}(e) = \tilde{C}(r(z_j), r(z_k)) := \left(\frac{r(z_j)}{r(z_k)} + \frac{r(z_k)}{r(z_j)}\right)^{-1},$$

where the edge $e = [z_j, z_k]$ connects the vertices $z_j, z_k \in \mathbb{V}$. Now estimations (4.16) imply that both positive functions $C > 0$ and $\tilde{C} > 0$ are uniformly bounded away from 0 (and from infinity). These two conductance networks on $\mathbb{V} \subset \mathbb{Z}^2$ can be continued to all of $\mathbb{Z}^2$ by reflection in the lines $\{|M| = |N|\}$. From Theorem 4.9 we deduce that the two networks $(\mathbb{Z}^2, C)$ and $(\mathbb{Z}^2, \tilde{C})$ are reccurent.

Consider the following positive functions on $\mathbb{V}$

$$f_1(z) = r(z)/R(z) > 0 \qquad \text{and} \qquad f_2(z) = R(z)/r(z) = 1/f_1(z) > 0.$$

By Proposition 4.11 these functions are subharmonic, in particular

$$\sum_{j=1}^{4} p(z_0, z_j) f_1(z_j) = \sum_{j=1}^{4} \frac{C(R(z_0), R(z_j))}{\sum_{j=1}^{4} C(R(z_0), R(z_j))} f_1(z_j) \geq f_1(z_0) \qquad \text{and}$$
$$\sum_{j=1}^{4} \tilde{p}(z_0, z_j) f_2(z_j) = \sum_{j=1}^{4} \frac{\tilde{C}(r(z_0), r(z_j))}{\sum_{j=1}^{4} \tilde{C}(r(z_0), r(z_j))} f_2(z_j) \geq f_2(z_0),$$

where $z_1, z_2, z_3, z_4$ are incident to the interior vertex $z_0 \in \mathbb{V}$. Using the boundary conditions of the circle patterns, the above inequalities are also true if $f_1$ and $f_2$ are continued to all of $\mathbb{Z}^2$ using reflection. Note that $M - f_1$ and $M - f_2$ are superharmonic for all constants $M \in \mathbb{R}$. If $f_1$ or $f_2$ is bounded from above, we thus get a positive superharmonic function using the upper bound. Then Theorem 4.10 implies that both functions are constant. Thus $r \equiv R$ and consequently both circle patterns coincide.

To conclude, we prove that $f_1$ and $f_2$ are bounded from above. Denote by $M_1(n)$ and $M_2(n)$ the maximum of $f_1$ and $f_2$, respectively, for the set of vertices of the $n$th generation. As $f_1$ and $f_2$ are subharmonic, they assume their maxima on the boundary. Therefore the functions $M_1$ and $M_2$ are monotonically increasing. The estimations (4.14) and (4.15) imply that the quotients of any two radii of one circle pattern in the $n$th generation are bounded from above for $n \geq 2$, as two vertices in the $n$th generation can be connected by at most $4n$ edges using only vertices of the $n$th and $n+1$st generation. So their quotient is bounded by $\mathrm{e}^{4A}$ for both radius functions $r$ and $R$. Note that with the normalization $r(0) = 1 = R(0)$, the maxima $M_1$ and $M_2$ are bounded from below by 1. Thus their product

$$M_1(n)M_2(n) = f_1(z_{M_1}^{(n)}) f_2(z_{M_2}^{(n)}) = \frac{R(z_{M_1}^{(n)})}{r(z_{M_1}^{(n)})} \frac{r(z_{M_2}^{(n)})}{R(z_{M_2}^{(n)})} \leq \mathrm{e}^{8A}$$

is bounded from above. Here $z_{M_1}^{(n)}$ and $z_{M_2}^{(n)}$ denote the vertices of the $n$th generation where $f_1$ and $f_2$ assume their maxima, respectively. Therefore $M_1$ and $M_2$ are also bounded. This finishes the proof of uniqueness. □

## 4.4 Brief review on $Z^\gamma$-circle patterns corresponding to regular $SG$-circle patterns

Remember from Section 3.6 the definitions of the regular square grid cell decomposition $SGD$ of the complex plane associated to the lattice $\mathbb{Z}^2$ and of the graph $SG$ with vertices

$$V(SG) = \{n + im \in \mathbb{Z} + i\mathbb{Z} : n + m = 0 \pmod 2\}$$

and edges connecting vertices $z_1, z_2 \in V(SG)$ with $|z_1 - z_2| = \sqrt{2}$. Let $\psi \in (0, \pi)$ be a fixed angle. As in Section 3.6, denote by $\alpha_\psi$ the following regular labelling $\alpha_\psi$ on the edges $E(SG)$. Let $[z_1, z_2] \in E(SG)$ be an edge connecting the vertices $z_1, z_2 \in V(SG)$. Without loss of generality, we may assume that $\mathrm{Re}(z_1) \leq \mathrm{Re}(z_2)$. Then $\alpha_\psi([z_1, z_2]) = \psi$ if $\mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$ and $\alpha_\psi([z_1, z_2]) = \pi - \psi$ if $\mathrm{Im}(z_1) \geq \mathrm{Im}(z_2)$. Using this notation, we consider the following generalization of Definition 4.3.

**Definition 4.12** ([3, Definition 2]). For $0 < \gamma < 2$, the discrete map $Z^\gamma : \mathbb{Z}^2_+ \to \mathbb{C}$ is the solution of

$$q(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) := \frac{(f_{n,m} - f_{n+1,m})(f_{n+1,m+1} - f_{n,m+1})}{(f_{n+1,m} - f_{n+1,m+1})(f_{n,m+1} - f_{n,m})} = \mathrm{e}^{2i(\psi - \pi)} \tag{4.17}$$

and (4.2) with the initial conditions

$$Z^\gamma(0,0) = 0, \quad Z^\gamma(1,0) = 1, \quad Z^\gamma(0,1) = \mathrm{e}^{\gamma(\pi - \psi)i}.$$

Similar arguments as in the orthogonal case imply that one can again associate a circle pattern (for a quadrant of $SG$ corresponding to $\mathbb{Z}^2_+$ and related to $\mathbb{V}$ and $\alpha_\psi$) to the map $Z^\gamma$. Equation (4.3) is still satisfied and the corresponding version of equation (4.4) reads (see [3, Proposition 3]):

$$\begin{aligned}&(N+M)(R(z)^2 - R(z+1)R(z-i) - \cos\psi R(z)(R(z-i) - R(z+1)))(R(z+i) + R(z+1)) \\ &+(M-N)(R(z)^2 - R(z+i)R(z+1) - \cos\psi R(z)(R(z+i) - R(z+1)))(R(z+1) + R(z-i)) = 0.\end{aligned} \tag{4.18}$$

The corresponding versions of equations (4.5) and (4.6) are similar. Furthermore, the following results generalize the orthogonal case.

**Theorem 4.13** ([3]). *(i) If $R(z)$ denotes the radius function corresponding to the discrete conformal map $Z^\gamma$ for some $0 < \gamma < 2$, then it holds that*

$$(\gamma - 1)(R(z)^2 - R(z-i)R(z+1) - \cos\psi R(z)(R(z-i) - R(z+1)) \geq 0 \tag{4.19}$$

*for all $z \in \mathbb{V} \setminus \{\pm N + iN | N \in \mathbb{N}\}$.*

*(ii) For $0 < \gamma < 2$, the discrete conformal maps $Z^\gamma$ given by Definition 4.3 are embedded.*

*(iii) The circle patterns corresponding to the discrete maps $Z^\gamma$ for $0 < \gamma < 2$ are embedded.*

*(iv) If $R(z)$ denotes the radius function which corresponds to the discrete conformal map $Z^\gamma$ for some $0 < \gamma < 2$, then $\tilde{R}(z) = 1/R(z)$ is the radius function corresponding to the discrete conformal map $Z^{\tilde{\gamma}}$ for $\tilde{\gamma} = 2 - \gamma$.*

### 4.5 UNIQUENESS OF QUASICRYSTALLIC $Z^\gamma$-CIRCLE PATTERNS

As explained in Section 13 of [14], discrete analogs of the power function $z^\gamma$ can also be defined for quasicrystallic rhombic embeddings instead of $\mathbb{Z}^2_+$ (or $\mathbb{Z}^2$). In particular, let $\mathcal{A} = \{\pm a_1, \ldots, \pm a_d\} \subset \mathbb{S}^1$ be the set of edge directions. Suppose that $d > 1$ and that any two non-opposite elements of $\mathcal{A}$ are linearly independent over $\mathbb{R}$. For $0 < \gamma < 2$ define the following values of the comparison function $w$ on the coordinate semi-axis of $\mathbb{Z}^d_+$:

$$w(ne_k) = \begin{cases} 1 & \text{if } n = 0, \\ a_k^{\gamma - 1} = \mathrm{e}^{(\gamma-1)\log a_k} & \text{if } n \text{ is odd}, \\ \prod_{m=1}^{n/2} \frac{m - 1 + \frac{\gamma}{2}}{m - \frac{\gamma}{2}} & \text{if } n \geq 2 \text{ and } n \text{ is even}. \end{cases} \tag{4.20}$$

The value of the logarithm $\log a_k$ is chosen similarly as in the definition of the discrete Green's function in Section 3.3. Using the Hirota Equation (3.12), this function $w$ can be extended to the whole sector $\mathbb{Z}^d_+$. Using suitable branches of the logarithm, $w$ may also be extended to other sectors or to a branched covering of $\mathbb{Z}^d$.

Note that for $d = 2$ the above definition leads to the same circle patterns as Definition 4.12. Therefore, by Theorem 4.13 (ii), the circle pattens corresponding to the restriction of $w$ to combinatorial surfaces $\mathbb{Z}^2_+ \subset \mathbb{Z}^d_+$ which are spanned by two coordinate semi-axis are embedded. We can apply finite and infinite flips to obtain other monotone combinatorial surfaces corresponding to rhombic embeddings. In particular, we obtain restrictions to $\mathbb{Z}^d_+$ of the plane based combinatorial surfaces constructed in Examples 3.3 and 3.4. Lemmas 3.21 and 3.25 imply that these again lead to embedded circle patterns. Thus we have

**Theorem 4.14** (Embeddedness of quasicrystallic $Z^\gamma$-circle patterns)**.** *Let $\Omega \subset \mathbb{Z}^d_+$ be a simply connected monotone combinatorial surface. Then the circle pattern given by the function $w$ with initial values* (4.20) *is embedded.*

The aim of this section is to prove uniqueness of quasicrystallic $Z^\gamma$-circle patterns using the same arguments as for the orthogonal case. The following proposition is a simple generalization of Proposition 4.11 and requires a (geometric) restriction which ensures that the kites corresponding to intersecting circles are convex. Note that flips (finite or infinite) may destroy convexity, that is if all kites of a circle pattern are convex before performing a flip this is not necessarily true afterwards.

**Proposition 4.15.** *Let $\mathscr{D}$ be a quasicrystallic rhombic embedding with associated graph $G$. Let $\alpha$ be the labelling corresponding to $\mathscr{D}$. Let $v_0$ be an interior vertex of $G$ with incident vertices $v_1, \dots, v_m$. Consider two circle patterns for $G$ and $\alpha$ with radius functions $r$ and $\rho$. Denote $r_j = r(v_j)$ and $\rho_j = \rho(v_j)$ for $j = 0, 1, \dots, m$ and suppose that $r_j \geq r_0 \cos\alpha_j$ and $\rho_j \geq \rho_0 \cos\alpha_j$ for $j = 1, \dots, m$, where $\alpha_j = \alpha([v_0, v_j])$. Then*

$$\sum_{j=1}^4 c_j \frac{r_j}{\rho_j} \geq \frac{r_0}{\rho_0} \sum_{j=1}^4 c_j \qquad \text{and} \qquad \sum_{j=1}^4 c_j \frac{\rho_j}{r_j} \geq \frac{\rho_0}{r_0} \sum_{j=1}^4 c_j, \tag{4.21}$$

*where $c_j = \sin\alpha_j / ((\rho_j/\rho_0) + (\rho_0/\rho_j) - 2\cos\alpha_j)$ for $j = 1, \dots, m$.*

*Proof.* The proof is similar to the proof of Proposition 4.11 and based on a Taylor expansion of $f_\alpha(x + \log y)$ about $y = 1$.

$$f_\alpha(x + \log y) = f_\alpha(x) + f'_\alpha(x)(y-1) - \frac{\sin\alpha(\mathrm{e}^{x+\log\xi} - \cos\alpha)}{2\xi^2(\cosh(x + \log\xi) - \cos\alpha)^2}(y-1)^2$$

with $\xi = t + (1-t)y$ for some $t \in (0,1)$. Equation (2.2) for the two circle patterns implies

$$\begin{aligned}
\pi &= \sum_{j=1}^m f_{\alpha_j}(\tfrac{r_j}{r_0}) = \textstyle\sum_{j=1}^m f_{\alpha_j}(\log\frac{\rho_j}{\rho_0} + \log\frac{r_j\rho_0}{r_0\rho_j}) \\
&= \underbrace{\sum_{j=1}^m f_{\alpha_j}(\log\tfrac{\rho_j}{\rho_0})}_{=\pi} + \sum_{j=1}^m f'_{\alpha_j}(\log\tfrac{\rho_j}{\rho_0})\left(\frac{r_j\rho_0}{r_0\rho_j} - 1\right) \\
&\qquad\qquad - \sum_{j=1}^m \frac{(\frac{\rho_j}{\rho_0}\xi_j - \cos\alpha_j)\sin\alpha_j}{2\xi_j^2(\cosh(\rho_j/\rho_0 + \log\xi_j) - \cos\alpha_j)^2}\left(\frac{r_j\rho_0}{r_0\rho_j} - 1\right)^2,
\end{aligned}$$

where $\xi_j = t_j + (1-t_j)\frac{r_j\rho_0}{r_0\rho_j} > 0$ with suitable $t_j \in (0,1)$ for $j = 1, \dots, m$. Furthermore

$$\frac{\rho_j}{\rho_0}\xi_j - \cos\alpha_j = t_j\frac{\rho_j}{\rho_0} + (1-t_j)\frac{r_j}{r_0} - \cos\alpha_j \geq 0.$$

by our assumption. Thus

$$\sum_{j=1}^{m} f'_{\alpha_j}(\log \tfrac{\rho_j}{\rho_0}) \left(\frac{r_j \rho_0}{r_0 \rho_j} - 1\right) \geq 0.$$

This implies the first claim since $f'_{\alpha_j}(\log(\rho_j/\rho_0)) = \sin\alpha_j/((\rho_j/\rho_0) + (\rho_0/\rho_j) - 2\cos\alpha_j)$.

The second claim follows from the fact, that $1/\rho$ and $1/r$ are also radius function of circle patterns for $G$ and $\alpha$ by Lemma 2.8. Also, the coefficients $c_j$ are invariant under the transformation $\rho \mapsto 1/\rho$. □

In order to apply this proposition, we need the following result on the convexity of the kites of the $Z^\gamma$-circle patterns. For the cases excluded below, there exist non-convex kites, because already the kite built from the circle centered at the origin is non-convex.

**Lemma 4.16.** *If $\psi \geq \pi/2$ and $0 < \gamma < 2$, all kites in the $Z^\gamma$-circle pattern given by definition 4.12 are convex.*

*If $\psi < \pi/2$ and $(\pi - 2\psi)/(\pi - \psi) \leq \gamma \leq \pi/(\pi - \psi)$, all kites in the $Z^\gamma$-circle pattern given by definition 4.12 are convex.*

*Proof.* The claims on convexity are simple consequences of equation (4.18) and the results on embeddedness of Theorem 4.13 (i) and (ii).

In particular, for $\pi - \psi \leq \pi/2$ the kites with intersection angle $\psi \geq \pi/2$ at black vertices are always convex. From equation (4.18) and inequality (4.19) we can deduce by simple calculations that $R(z+1)\cos(\pi - \psi) \leq R(z)$ and $R(z)\cos(\pi - \psi) \leq R(z+1)$. Thus the remaining kites are also convex.

If $\psi < \pi/2$ and $(\pi - 2\psi)/(\pi - \psi) \leq \gamma \leq \pi/(\pi - \psi)$, the kites with intersection angle $\pi - \psi > \pi/2$ at black vertices are convex. In this case inequality (4.19) only implies that $R(z+i)\cos\psi \leq R(z)$ if $0 < \gamma < 1$ and $R(z)\cos\psi \leq R(z+i)$ if $1 < \gamma < 2$. This shows that for all kites with white vertices $z$ and $z+i$ and intersection angle $\psi$ the angle at the point corresponding to $z$ for $0 < \gamma < 1$ and to $z+i$ for $1 < \gamma < 2$ respectively is smaller than $\pi$. This excludes some types of non-convex kites, but not all.

For further use, note that the curves $\Gamma_n$ of the $Z^\gamma$-circle pattern corresponding to regular $SG$-circle patterns which are constructed in an analogous way as in Section 4.2 have analogous properties due to Theorem 4.13 (i) and (ii). They are embedded without self-intersections and the vector $\mathbf{v}_n(m)$ rotates clockwise for $0 < \gamma < 1$ and counterclockwise for $1 < \gamma < 2$ along these curves.

Without loss of generality, we only consider the case $1 < \gamma < 2$ further. For $0 < \gamma < 1$ the proof is very similar. First, consider a kite on the symmetry axis, that is with white vertices corresponding to $iK$ and $i(K+1)$. Then the assumption $\gamma(\pi - \psi) \leq \pi$ and the properties of the curves $\Gamma_n$ imply that the angle of this kite at the vertex corresponding to $i(K+1)$ is larger than $\pi - 2\psi$. Consequently, the angle at the vertex corresponding to $iK$ is smaller than $2\pi - (\pi - 2\psi) - 2\psi = \pi$. Thus the kites on the symmetry axis are convex.

Next, we consider the intersection angles with the half lines $\mathbb{R}^+$ and $\mathrm{e}^{i\gamma(\pi-\psi)/2}\mathbb{R}^+$ and of the lines in direction of the vector $\mathbf{v}_n(m)$ and of the vector $\hat{\mathbf{v}}_m(n)$ of the mirror reflected curve $\hat{\Gamma}_m$. We additionally assume that $n$ is odd. See Figure 4.4 for an illustration and for the notation of the angles. As the kites on the symmetry axis are convex, we deduce $\alpha \leq \psi$ and $\beta \leq \pi/2$. Furthermore using the assumption $\gamma(\pi - \psi) \leq \pi$ we obtain

$$\frac{\pi}{2} \geq \beta = \pi - \gamma\frac{\pi - \psi}{2} - \alpha \geq \pi - \frac{\pi}{2} - \psi = \frac{\pi}{2} - \psi,$$
$$\frac{\pi}{2} = \pi - \psi - \frac{\pi}{2} + \psi \geq \eta = \pi - \psi - \beta \geq \pi - \psi - \frac{\pi}{2} = \frac{\pi}{2} - \psi.$$

Finally, consider a kite with white vertex $f_{n,m}$ for odd $n$ and $m < n$. We estimate the angles $\alpha_1$ and $\alpha_2$ at this vertex of the kites containing the points $f_{n,m-1}$, $f_{n,m}$, $f_{n+1,m}$

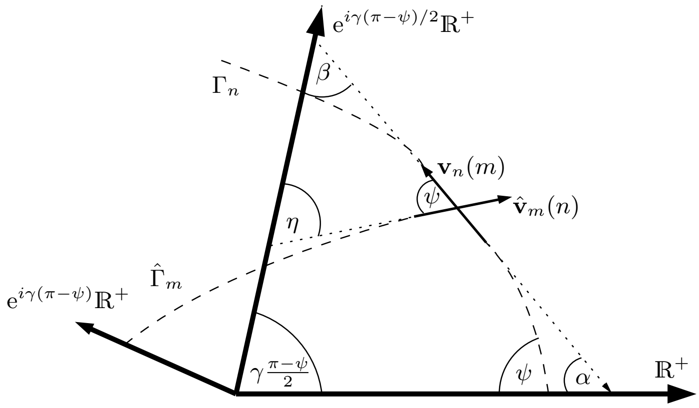



**Figure 4.4:** Geometric considerations for the vectors along the curves $\Gamma_n$ for odd $n$.

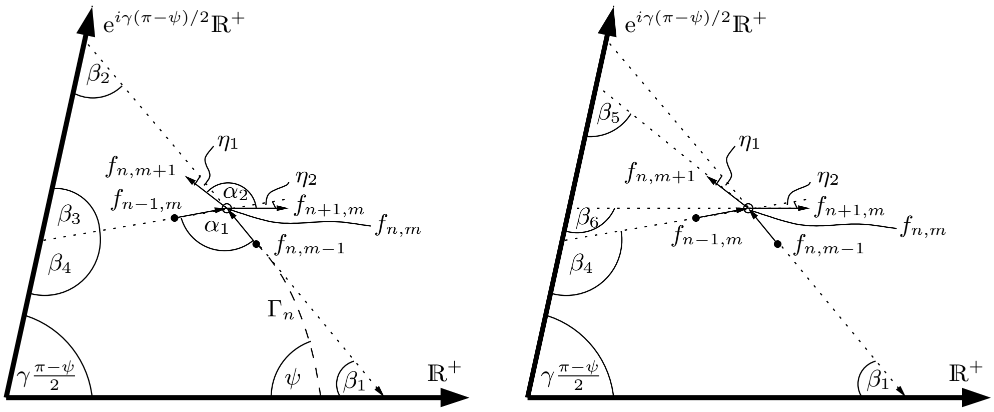



**Figure 4.5:** Geometric considerations for the angles of the kites at $f_{n,m}$ for odd $n$ and $m < n$.

and $f_{n+1,m}$, $f_{n,m}$, $f_{n,m+1}$ respectively. Note that these kites both have intersection angles $\psi$. See Figure 4.5 for an illustration and for the notation. Using the above estimations we obtain $\beta_1 \le \psi$, $\beta_4 \ge \pi/2$, $\beta_5 \le \pi/2$, and $\beta_6 \ge \pi/2$ and thus

$$\begin{aligned}
\alpha_1 &= 2\pi - \beta_1 - \beta_4 - \gamma\frac{\pi-\psi}{2} \ge 2\pi - \psi - \frac{\pi}{2} - \psi - \frac{\pi}{2} = \pi - 2\psi,\\
\eta_1 &= \pi - 2\pi + \beta_5 + \gamma\frac{\pi-\psi}{2} + \beta_1 \le -\frac{\pi}{2} + \gamma\frac{\pi-\psi}{2} + \beta_1,\\
\eta_2 &= \pi - \beta_6 - (\pi - \beta_4) \le \beta_4 - \frac{\pi}{2},\\
\alpha_2 &= \alpha_1 + \eta_1 + \eta_2 \le 2\pi - \beta_1 - \beta_4 - \gamma\frac{\pi-\psi}{2} - \frac{\pi}{2} + \gamma\frac{\pi-\psi}{2} + \beta_1 + \beta_4 - \frac{\pi}{2} = \pi.
\end{aligned}$$

Therefore the angle at $f_{n-1,m-1}$ of the kite containing the points $f_{n,m-1}$, $f_{n,m}$, $f_{n+1,m}$ is $2\pi - 2\psi - \alpha_1 \le \pi$. So this kite is convex. Furthermore, we deduce that the kite containing the points $f_{n+1,m}$, $f_{n,m}$, $f_{n,m+1}$ is also convex. Consequently all kites containing the white vertex $f_{n,m}$ are convex. This shows that the $Z^\gamma$-circle pattern with $\gamma$ and $\psi$ satisfying the above assumptions only contains convex kites. $\square$

The rigidity of regular $Z^\gamma$-circle patterns can now be proven by similar arguments as for the orthogonal case.

**Theorem 4.17** (Rigidity of regular $Z^\gamma$-circle patterns). *If $\psi \geq \pi/2$ and $0 < \gamma < 2$ or $\psi < \pi/2$ and $(\pi - 2\psi)/(\pi - \psi) \leq \gamma \leq \pi/(\pi - \psi)$ then the $Z^\gamma$-circle pattern given by Definition 4.12 is the unique embedded regular SG-circle pattern for $\mathbb{Z}^2_+$ and $\alpha_\psi$ (up to global scaling) with the following properties.*

*(i) The infinite sector $\{z = \rho e^{i\beta} \in \mathbb{C} : \rho \geq 0,\ \beta \in [0, \gamma(\pi - \psi)]\}$ with angle $\gamma(\pi - \psi)$ is covered by the union of the corresponding kites of the circle pattern.*

*(ii) The centers of the boundary circles lie on the boundary half lines.*

*(iii) All kites corresponding to intersecting circles are convex.*

The proof of Theorems 4.8 and 4.17 actually also shows the following generalization.

**Theorem 4.18.** *Let $\psi \in (0, \pi)$ and $\gamma \in (0, 2) \cap [\frac{\pi-2\psi}{\pi-\psi}, \frac{\pi}{\pi-\psi}]$. Define $Z^\gamma$-circle pattern on all four sectors $\mathbb{Z}_\pm \times \mathbb{Z}_\pm$ according to Definition 4.12 and glue these patterns such to a circle pattern $\mathscr{C}_\gamma$ on a cone with cone angle $2\pi\gamma$. Then any embedded circle pattern with the same combinatorics and intersection angles which covers the same cone with one center of circle placed at the apex and which has only convex kites coincides with $\mathscr{C}_\gamma$ (up to scaling and rotation about the apex of the cone).*

The following theorem is a direct consequence of this generalization.

**Theorem 4.19** (Rigidity of quasicrystallic $Z^\gamma$-circle patterns I). *Let $\mathscr{D}$ be a quasicrystallic rhombic embedding of a b-quad-graph covering the whole plane. Let $\mathcal{A} = \{\pm a_1, \ldots, \pm a_d\} \subset \mathbb{S}^1$ be the edge directions, where $d > 1$ and any two non-opposite elements of $\mathcal{A}$ are linearly independent over $\mathbb{R}$. Denote by $\psi_{min}$ the minimum of the undirected angles between any two elements of $\mathcal{A}$. Let $\gamma \in (0, 2)$ with $(\pi-2\psi_{min})/(\pi-\psi_{min}) \leq \gamma \leq \pi/(\pi-\psi_{min})$. Assume that the origin is a white vertex of $\mathscr{D}$. Then a quasicrystallic $Z^\gamma$-circle pattern $\mathscr{C}_\gamma$ corresponding to $\mathscr{D}$ and embedded on a cone with cone angle $2\pi\gamma$ can be defined using the definition of the comparison function $w$ on the $2d$ sectors of $\mathbb{Z}^d$ which contain the combinatorial surface $\Omega_\mathscr{D}$; see (4.20) and the remarks below. Assume further that the brick $\Pi(\Omega_\mathscr{D})$ contains the whole lattice $\mathbb{Z}^d$.*

*Let $\mathscr{C}$ be an embedded circle pattern with the same combinatorics and the same intersection angles which covers the same cone with one center of circle placed at the apex. Extend the comparison function $w$ for $\mathscr{C}$ from $\Omega_\mathscr{D}$ to $\mathbb{Z}^d$. For each $\mathbb{Z}^2$-sublattice which contains two coordinate axes suppose that the corresponding circle pattern built according to this comparison function has only convex kites. Then $\mathscr{C}$ coincides with $\mathscr{C}_\gamma$ up to scaling and rotation about the apex of the cone.*

Note that the assumption on the convexity of the kites is only a restriction for a (small) neighborhood of the origin. This is due to Lemma 3.33 (or Corollary 3.44) which implies that the ratio of the radii is almost one and thus the corresponding angles are almost as the same in the isoradial case if the combinatorial distance to the origin is big enough.

If all intersection angles of the labelling $\alpha$ taken from the rhombic embedding are larger than $\pi/2$, then the kites of any corresponding circle pattern are convex. Note that this restriction is the same as for the hyperbolic maximum principle in Lemma 2.13. Examples of such rhombic embeddings are suitable regular hexagonal patterns as shown in Figure 2.3(b). Hexagonal circle patterns and in particular analogs of the holomorphic mappings $z^\gamma$ have been studied by Bobenko and Hoffmann in [12]. In the case that all intersection angles are larger than $\pi/2$. Thus the proof of Theorem 4.8 can be directly adapted.

**Theorem 4.20** (Rigidity of quasicrystallic $Z^\gamma$-circle patterns II). *Let $\mathscr{D}$ be a quasicrystallic rhombic embedding of a b-quad-graph. Assume that the corresponding labelling $\alpha : F(\mathscr{D}) \to [\pi/2, \pi)$ only has values larger than $\pi/2$. Assume further that the origin is a white vertex. Let $\gamma \in (0, 2)$.*

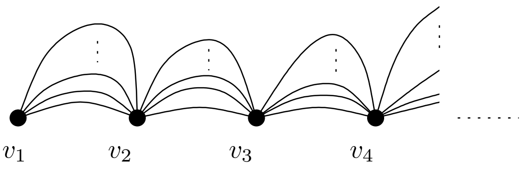



**Figure 4.6:** An illustration of the network obtained from $G$ by shortening.

*Define a quasicrystallic $Z^\gamma$-circle pattern $\mathscr{C}_\gamma$ for $\mathscr{D}$ and $\alpha$ which is embedded on a cone with cone angle $2\pi\gamma$ using the definition of the comparison function on the $2d$ sectors of $\mathbb{Z}^d$ whose union contains the combinatorial surface $\Omega_{\mathscr{D}}$; see* (4.20)*. In particular, the circle corresponding to the origin is centered at the apex of the cone.*

*Let $\mathscr{C}$ be an embedded circle pattern for $\mathscr{D}$ and $\alpha$ which covers the same cone. Suppose that the circle corresponding to the origin is centered at the apex and that $\mathscr{C}$ has only convex kites. Then $\mathscr{C}$ coincides with $\mathscr{C}_\gamma$ (up to scaling and rotation about the apex of the cone).*

In order to apply the same arguments as for the proof of Theorem 4.8, note that the simple random walk on the associated graph for an infinite rhombic embedding is recurrent.

**Lemma 4.21.** *Let $\mathscr{D}$ be a quasicrystallic rhombic embedding of a b-quad-graph which covers the whole plane $\mathbb{C}$. Let $G$ be the associated infinite graph built from white vertices. Then the simple random walk on $G$ is recurrent.*

*Proof.* Without loss of generality we assume that all edges of $\mathscr{D}$ have length one. Since $\mathscr{D}$ is a quasicrystallic rhombic embedding, there is a constant $C_1 > 0$ such that the area of any rhombus of $\mathscr{D}$ is bigger than $C_1$ (and smaller than one). Furthermore, the number of rhombi incident to a vertex of $\mathscr{D}$ is uniformly bounded from above by a constant $C_2 = C_2(d)$, where $d$ is the dimension of $\mathscr{D}$.

Without loss of generality we assume that the origin is a vertex of $G$. In the following, we construct another graph $G'$ from $G$ by shortening. Recall that for the simple random walk all edges have conductance $c(e) = 1$ and also resistance $r(e) = 1/c(e) = 1$. If two incident vertices are identified, that is the conductance of the connecting edge is increased to $\infty$ and the resistance decreased to 0, then the effective resistance of the new network is certainly smaller. More generally, the proceedure of identifying a set of vertices of $G$, which corresponds to increasing the conductances of the edges between these vertices to $\infty$ (and to decreasing the corresponding resistance to 0), is called *shortening* and reduces the effective resistance; see [32, Section 2.2.2] or [77, Theorem (2.19)]. Remember that our aim is to prove that the effective resistance $R_{\text{eff}}$ of the network $G$ with unit conductances is infinite. Therefore, it is sufficient to show that we have infinite effective resistance $R'_{\text{eff}} = \infty$ for a network $G'$ which is obtained from $G$ by shortening.

In particular, set $\varrho_k = 4k$ for $k \in \mathbb{N}$. Denote by $V_k$ the set of vertices of $G$ which are contained in the annulus $A_k = \{z \in \mathbb{C} : \varrho_{k-1} \leq |z| < \varrho_k\}$ for $k \geq 1$. Then $V(G) = \cup_{k=1}^\infty V_k$. Identify the vertices of each $V_k$ to one new vertex $v_k$. Then by construction, $v_k$ is only incident to $v_{k-1}$ for $k \geq 2$ and to $v_{k+1}$ for $k \geq 1$. An illustration of the shortened network is shown in Figure 4.6. Denote by $\#E_k$ the number of edges which are incident to $v_k$ and $v_{k+1}$ for $k \geq 1$. Then the effective resistance of the shortened network is $R'_{\text{eff}} = \sum_{k=1}^\infty 1/\#E_k$. For $k \geq 2$ we have

$$\begin{aligned}\#E_k &\leq \text{number of rhombi of } \mathscr{D} \text{ which intersect the circle } \{\{z \in \mathbb{C} : |z| = \varrho_k\} \\ &\leq F(\{z \in \mathbb{C} : \varrho_k - 2 \leq |z| < \varrho_k + 2\})/C_1 \\ &= (\pi(\varrho_k + 2)^2 - \pi(\varrho_k - 2)^2)/C_1 = k\tfrac{32\pi}{C_1},\end{aligned}$$

where $F(\cdot)$ denotes the Euclidean area. This implies that $R'_{\text{eff}} = \infty$. Thus the simple random walk on $G$ is recurrent. $\square$

Chapter 5

# Convergence for isoradial circle patterns

In this chapter we state and prove convergence theorems for isoradial circle patterns. First we prove $C^1$-convergence for a general class of isoradial circle patterns with Dirichlet or Neumann boundary conditions in Sections 5.1 and 5.2 respectively. For a certain class of quasicrystallic circle patterns, the additional regularity is then used to prove $C^\infty$-convergence in Section 5.3.

## 5.1 $C^1$-convergence for Dirichlet boundary conditions

**Theorem 5.1.** *Let $D \subset \mathbb{C}$ be a simply connected bounded domain, and let $W \subset \mathbb{C}$ be open with $\overline{D} \subset W$. Let $g : W \to \mathbb{C}$ be a locally injective holomorphic function. Assume, for convenience, that $0 \in D$.*

*For $n \in \mathbb{N}$ let $\mathscr{D}_n$ be a b-quad-graph with corresponding graphs $G_n$ and $G_n^*$ constructed as above and let $\alpha_n$ be an admissible labelling. We assume that $\mathscr{D}_n$ is simply connected and that $\alpha_n$ is uniformly bounded in the sense that for all $n \in \mathbb{N}$ and all faces $f \in F(\mathscr{D}_n)$*

$$|\alpha_n(f) - \pi/2| < C \tag{5.1}$$

*with some constant $0 < C < \pi/2$ independent of $n$.*

*Let $\varepsilon_n \in (0, \infty)$ be a sequence of positive numbers such that $\varepsilon_n \to 0$ for $n \to \infty$. For each $n \in \mathbb{N}$, assume that there is an isoradial circle pattern for $G_n$ and $\alpha_n$, that is all circles have the same radius $\varepsilon_n$. Assume further that all centers of circles lie in the domain $D$ and that any point $x \in \overline{D}$ which is not contained in any of the disks bounded by the circles of the pattern has a distance less than $\hat{C}\varepsilon_n$ to the nearest center of a circle and to the boundary $\partial D$, where $\hat{C} > 0$ is some constant independent of $n$. Denote by $R_n \equiv \varepsilon_n$ and $\phi_n$ the radius and the angle function of the above circle pattern for $G_n$ and $\alpha_n$. By abuse of notation, we do not distinguish between the realization of the circle pattern, that is the centers of circles $z_n$, the intersection points $v_n$ and the edges connecting corresponding points in $\mathscr{D}_n$ or $G_n$, and the abstract b-quad-graph $\mathscr{D}_n$ and the graphs $G_n$ and $G_n^*$. Also, the index $n$ will be dropped from the notation of the vertices and the edges.*

*Define another radius function on $G_n$ as follows. At boundary vertices $z \in V_\partial(G_n)$ set*

$$r_n(z) = R_n(z)\,|g'(z)|\,. \tag{5.2}$$

*Using Theorem 2.10, we can extend $r_n$ to a solution of the Dirichlet problem on $G_n$. Let $z_0 \in V(G_n)$ be such that the disk bounded by $C_{z_0}$ contains $0$ and let $e = [z_0, v_0] \in E(\mathscr{D}_n)$ be one of the edges incident to $z_0$ such that $\phi_n(\vec{e}) \in [0, 2\pi)$ is minimal.*

*Let $\varphi_n$ be the angle function corresponding to $r_n$ that satisfies*

$$\varphi_n(\vec{e}) = \arg(g'(v_0)) + \phi_n(\vec{e}). \tag{5.3}$$

*Let $\mathscr{C}_n$ be the planar circle pattern with radius function $r_n$ and angle function $\varphi_n$. Suppose that $\mathscr{C}_n$ is normalized by a translation such that*

$$p_n(v_0) = g(v_0), \tag{5.4}$$

*where $p_n(v)$ denotes the intersection point corresponding to $v \in V(G_n^*)$.*

*For $z \in D$ set*

$$g_n(z) = p_n(w) \quad \text{and} \quad q_n(z) = \frac{r_n(v)}{R_n(v)} e^{i(\varphi_n(\overrightarrow{vw}) - \phi_n(\overrightarrow{vw}))},$$

*where $w$ is a vertex of $V(G_n^*)$ closest to $z$ and $v$ is a vertex of $V(G_n)$ closest to $z$ such that $[v,w] \in E(\mathscr{D}_n)$.*

*Then $q_n \to g'$ and $g_n \to g$ uniformly on compact subsets in $D$ as $n \to \infty$.*

**Remark 5.2.** The proof of Theorem 5.1 actually shows the following a priori estimations for the approximating functions $q_n$ and $g_n$.

$$\|q_n - g'\|_{V(G_n)\cap K} \leq C_1(-\log_2 \varepsilon_n)^{-\frac{1}{2}} \qquad \text{and} \qquad \|g_n - g\|_{V(G_n)\cap K} \leq C_2(-\log_2 \varepsilon_n)^{-\frac{1}{2}}$$

for all compact sets $K \subset D$, where the constants $C_1, C_2$ depend on $K$, $g$, $D$ and the constants of Theorem 5.1.

We begin with an a priori estimation for the quotients of the radius functions.

**Lemma 5.3.** *For $z \in V(G_n)$ set*

$$h_n(z) = \log|g'(z)|,$$
$$t_n(z) = \log(r_n(z)/R_n(z)).$$

*Then*

$$h_n(z) - t_n(z) = \mathcal{O}(\varepsilon_n).$$

Here and below the notation $s_1 = \mathcal{O}(s_2)$ means that there is a constant $C$ which may depend on $W, D, g$ but not on $n$ and $z$, such that $|s_1| \leq Cs_2$ wherever $s_1$ is defined. A direct consequence of Lemma 5.3 is

$$r_n(z) = R_n(z)\,|g'(z)| + \mathcal{O}(\varepsilon_n^2). \tag{5.5}$$

Our proof uses ideas of Schramm's proof of the corresponding Lemma 9.2 in [67].

*Proof.* Consider the function

$$p(z) = t_n(z) - h_n(z) + \beta|z|^2,$$

where $\beta \in (0,1)$ is some function of $\varepsilon_n$. We want to choose $\beta$ such that $p$ will have no maximum in $V_{int}(G_n)$.

Suppose that $p$ has a maximum at $z \in V_{int}(G_n)$. Denote by $z_1, \dots, z_m$ the incident vertices of $z$ in $G_n$ in counterclockwise order. Then for $j = 1, \dots, m$ we have

$$t_n(z_j) - t_n(z) \leq x_j \tag{5.6}$$

where

$$x_j = h_n(z_j) - h_n(z) - \beta|z_j|^2 + \beta|z|^2. \tag{5.7}$$

First, we gather a little information about the $x_j$s. Since $z \in V_{int}(G_n)$, we have $|z| = \mathcal{O}(1)$ and by assumption $z - z_j = \mathcal{O}(\varepsilon_n)$. With $\beta \in (0,1)$ this leads to $\beta|z_j|^2 - \beta|z|^2 = \mathcal{O}(\varepsilon_n)$. Using this estimate and the smoothness of $\mathrm{Re}(\log g')$, we get $x_j = \mathcal{O}(\varepsilon_n)$.

From (5.6), the definition of $t_n(z) = \log r_n(z) - \log R_n(z)$ and the monotonicity of the sum in equation (2.2) (see Lemma 2.7), we get

$$\begin{aligned}
0 &= \left(\sum_{j=1}^{m} f_{\alpha(z,z_j)}(\log r_n(z_j) - \log r_n(z))\right) - \pi \\
&= \left(\sum_{j=1}^{m} f_{\alpha(z,z_j)}(t_n(z_j) - t_n(z) + \log(R_n(z_j)/R_n(z)))\right) - \pi \\
&\leq \left(\sum_{j=1}^{m} f_{\alpha(z,z_j)}\left(x_j + \log\left(\tfrac{R_n(z_j)}{R_n(z)}\right)\right)\right) - \pi. \qquad (5.8)
\end{aligned}$$

The preceeding reasonings also apply for general circle patterns with radius function $R_n$ (not necessarily isoradial) which approximate $D$. In the following, we exploit special properties of the given isoradial circle patterns, that is for $R_n \equiv \varepsilon_n$.

Remembering $x_j = \mathcal{O}(\varepsilon_n)$, we can consider a Taylor expansion of the right hand side of inequality (5.8) about $\log\left(\frac{R_n(z_j)}{R_n(z)}\right) = 0$ in order to make an $\mathcal{O}(\varepsilon_n^3)$-analysis.

Consider the chain of faces $f_j$ of $\mathscr{D}_n$ $(j = 1, \ldots, m)$ which are incident to $z$ and $z_j$. The enumeration of the vertices $z_j$ (and hence of the faces $f_j$) and of the black vertices $v_1, \ldots, v_m$ incident to these faces can be chosen such that $f_j$ is incident to $v_{j-1}$ and $v_j$ for $j = 1, \ldots, m$, where $v_0 = v_m$. Furthermore, using this enumeration we have

$$\frac{z_j - z}{|z_j - z|} i = \frac{v_j - v_{j-1}}{|v_j - v_{j-1}|}, \qquad (5.9)$$

see Figure 5.1. Moreover, each face $f_j$ of an isoradial circle pattern is a rhombus. So we can write, using the notations of Figure 5.1

$$z_j - z = a_{j-1} + a_j \qquad \text{and} \qquad v_j - v_{j-1} = a_j - a_{j-1}.$$

Denoting

$$\begin{aligned}
\alpha_j &= \alpha([z, z_j]), \\
l_j &= |z_j - z| = 2R_n \sin(\alpha_j/2), \\
\hat{l}_j &= |v_j - v_{j-1}| = 2R_n \cos(\alpha_j/2),
\end{aligned}$$

where $R_n \equiv \varepsilon_n$ is the constant radius function of the given isoradial circle pattern, we easily obtain by a simple calculation that

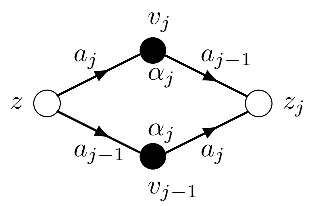



**Figure 5.1:** A rhombic face of $\mathscr{D}_n$ with oriented edges.

$$f_{\alpha_j}(0) = (\pi - \alpha_j)/2, \quad f'_{\alpha_j}(0) = \hat{l}_j/(2l_j), \quad f''_{\alpha_j}(0) = 0.$$

Taking into account that equation (2.2) holds with $R_n \equiv \varepsilon_n$ and using the uniform boundedness (5.1) of the labeling $\alpha$, inequality (5.8) yields

$$0 \leq \sum_{j=1}^{m} f'_{\alpha_j}(0) x_j + \mathcal{O}(\varepsilon_n^3). \qquad (5.10)$$

To evaluate this sum, expand

$$\log g'(z_j) - \log g'(z) = a(z_j - z) + b(z_j - z)^2 + \mathcal{O}(\varepsilon_n^3)$$

and

$$\begin{aligned} x_j &= h_n(z_j) - h_n(z) - \beta|z_j|^2 + \beta|z|^2 \\ &= \mathrm{Re}(a(z_j - z) + b(z_j - z)^2 - 2\beta\bar{z}(z_j - z)) - \beta l_j^2 + \mathcal{O}(\varepsilon_n^3). \end{aligned}$$

Noting that $f'_{\alpha_j}(0)(z_j - z) = (v_j - v_{j-1})/(2i)$ we get

$$\begin{aligned} \sum_{j=1}^m f'_{\alpha_j}(0)x_j &= \mathrm{Re}\left(\frac{a - 2\beta\bar{z}}{2i}\sum_{j=1}^m (v_j - v_{j-1}) + \frac{b}{2i}\sum_{j=1}^m (v_j - v_{j-1})(z_j - z)\right) \\ &\qquad - \beta\sum_{j=1}^m \frac{l_j\hat{l}_j}{2} + \mathcal{O}(\varepsilon_n^3) \\ &= \mathrm{Re}\left(\frac{b}{2i}\sum_{j=1}^m (a_j - a_{j-1})(a_j + a_{j-1})\right) - \beta\sum_{j=1}^m \frac{l_j\hat{l}_j}{2} + \mathcal{O}(\varepsilon_n^3) \\ &= -\beta\sum_{j=1}^m \frac{l_j\hat{l}_j}{2} + \mathcal{O}(\varepsilon_n^3) \end{aligned}$$

Thus we arrive at

$$0 \le -\beta\varepsilon_n^2\sum_{j=1}^m \sin(\alpha_j/2)\cos(\alpha_j/2) + \mathcal{O}(\varepsilon_n^3) \quad\iff\quad \beta\sum_{j=1}^m \sin(\alpha_j) \le \mathcal{O}(\varepsilon_n).$$

Note that $\varepsilon_n^2\sum_{j=1}^m \sin(\alpha_j) > \pi\varepsilon_n^2$ is the area of the rhombic faces incident to the vertex $z$. We conclude from this estimate that

$$\beta = \mathcal{O}(\varepsilon_n).$$

This means, that if we choose $\beta = C\varepsilon_n$ with $C > 0$ a sufficiently large constant and if $\varepsilon_n$ is small enough such that $C\varepsilon_n < 1$, then $p$ will have no maximum in $V_{int}(G_n)$. In that case, as we have $p(z) = \beta|z|^2 = \mathcal{O}(\varepsilon_n)$ on $V\partial(G_n)$, we deduce that $p(z) \le \mathcal{O}(\varepsilon_n)$ in $V(G_n)$ and thus

$$t_n(z) - h_n(z) \le \mathcal{O}(\varepsilon_n) \qquad \text{for } z \in V(G_n). \tag{5.11}$$

The proof for the reverse inequality is almost the same. The only modifications needed are reversing the sign of $\beta$ and a few inequalities. □

**Remark 5.4.** The statement of Lemma 5.3 can be improved to

$$h_n(z) - t_n(z) = \mathcal{O}(\varepsilon_n^2) \tag{5.12}$$

in the case of a 'very regular' isoradial circle pattern. These are isoradial circle patterns such that for each oriented edge $e_{j_1} = z_{j_1} - z \in \vec{E}(G)$ incident to an interior vertex $z \in V(G)$ there is another parallel edge $e_{j_2} = z_{j_2} - z \in \vec{E}(G)$ with opposite direction incident to $z$, that is $e_{j_2} = -e_{j_1}$. Furthermore, the corresponding intersection angles agree: $\alpha([z, z_{j_1}]) = \alpha([z, z_{j_2}])$. This additional regularity property holds for example for an orthogonal circle pattern with the combinatorics of a part of the square grid.

The proof of estimation (5.12) follows the same reasonings as above, but makes an $\mathcal{O}(\varepsilon_n^4)$-analysis. In particular, we obtain

$$\begin{aligned} &\sum_{j=1}^m f'_{\alpha_j}(0)x_j + \sum_{j=1}^m f'''_{\alpha_j}(0)x_j^3 + O(\varepsilon_n^4) \\ &\quad = -\beta\sum_{j=1}^m \frac{l_j\hat{l}_j}{2} + \mathrm{Re}\left(\sum_{j=1}^m f'_{\alpha_j}(0)c(z_j - z)^3\right) + \sum_{j=1}^m f'''_{\alpha_j}(0)\left(\mathrm{Re}(a(z_j - z))\right)^3 + O(\varepsilon_n^4), \end{aligned}$$

where

$$x_j = \operatorname{Re}(a(z_j - z) + b(z_j - z)^2 + c(z_j - z)^3 - 2\beta\bar{z}(z_j - z)) - \beta l_j^2 + \mathcal{O}(\varepsilon_n^4).$$

The additional regularity implies that all terms of order $\mathcal{O}(\varepsilon_n^3)$ vanish.

**Remark 5.5.** Remember the definition of the discrete Laplacian in (3.2) by

$$\Delta\eta(z) = \sum_{[z,z_j]\in E(G)} 2f'_{\alpha([z,z_j])}(0)(\eta(z_j) - \eta(z)).$$

The proof of Lemma 5.3 actually shows, that the difference $t_n - h_n$ is almost harmonic. More precisely, we have $\Delta(t_n - h_n) = \mathcal{O}(\varepsilon_n^3)$. Adding a suitable subharmonic function $\beta|z|^2$ with $\beta > 0$, we deduce that the resulting function $p$ is subharmonic, that is $\Delta p \geq 0$, such that $p$ attains its maximum at the boundary. This reasoning will also be important for the proof of the following lemma.

**Lemma 5.6.** *Let $t_n$ and $h_n$ be defined as in Lemma 5.3. Let $K \subset D$ be a compact subset in $D$. Then the following estimation holds for all $n \in \mathbb{N}$ and every interior vertex $z \in V_{int}(G_n) \cap K$ such that all its incident vertices $z_1, \ldots, z_l$ are also in $V_{int}(G_n) \cap K$:*

$$t_n(z_j) - h_n(z_j) - (t_n(z) - h_n(z)) = \mathcal{O}(\varepsilon_n(-\log\varepsilon_n)^{-\frac{1}{2}}). \tag{5.13}$$

*for $j = 1, \ldots, l$. The constant in $\mathcal{O}$-notation may depend on $K$, but not on $n$ or $z$.*

The proof of Lemma 5.6 uses the following estimation for superharmonic functions, which is a version of Corollary 3.1 of [65]; see also [65, Remark 3.2 and Lemma 2.1].

**Proposition 5.7** ([65])**.** *Let $G$ be an undirected connected graph without loops and let $c : E(G) \to \mathbb{R}^+$ be a positive weight function on the edges. Denote $c(e) = c(x, y)$ for an edge $e = [x, y] \in E(G)$ and assume that*

$$m = \max_{[x,y]\in E(G)} \sum_{[x,z]\in E(G)} \frac{c(x, z)}{c(x, y)} < \infty.$$

*Denote by $d(x, y)$ the combinatorial distance between two vertices $x, y \in V(G)$ in the graph $G$. Let $B_x(r) = \{y \in V(G) : d(x, y) \leq r\}$ be the combinatorial ball of radius $r > 0$ around the vertex $x \in V(G)$. Fix $x \in V(G)$ and $R \geq 4$ and set*

$$A = \sup_{1\leq r\leq R} r^{-2}W_x(r), \qquad \textit{where} \qquad W_x(r) = \sum_{\substack{z\in B_x(r),\ y\in V(G)\\ d(x,z)<d(x,y)}} c(z, y).$$

*Let $u$ be a positive superharmonic function in $B_x(R + 1)$, that is*

$$\sum_{[z,w]\in E(G)} c(z, w)(u(z) - u(w)) \leq 0$$

*for all $w \in B_x(R + 1)$. Let $y$ be incident to $x$ in $G$. Then*

$$\left|\frac{u(x)}{u(y)} - 1\right| \leq \frac{4m^2\sqrt{A}}{\sqrt{c(x, y)\log_2 R}}.$$

*Proof of Lemma 5.6.* Lemma 5.3 implies that $t_n(z_j) - t_n(z) = \mathcal{O}(\varepsilon_n)$ for all incident vertices $z, z_j \in V(G_n)$ since $h_n = \log|g'|$ is a $C^\infty$-function. Consider a Taylor expansion about 0 of

$$0 = \left(\sum_{j=1}^{m} f_{\alpha(z,z_j)}(t_n(z_j) - t_n(z))\right) - \pi.$$

Similar reasonings as in the proof of Lemma 5.3 imply that

$$\Delta t_n(z) = \sum_{j=1}^{m} 2f'_{\alpha(z,z_j)}(0)(t_n(z_j) - t_n(z)) = \mathcal{O}(\varepsilon_n^3).$$

Let $p = t_n - h_n + \beta|z|^2$ with $\beta \in (0,1)$. From the estimations in the proof of Lemma 5.3 we deduce that

$$\begin{aligned}\Delta p(z) &= \sum_{j=1}^{m} 2f'_{\alpha(z,z_j)}(0)(t_n(z_j) - t_n(z)) - \sum_{j=1}^{m} 2f'_{\alpha(z,z_j)}(0)(h_n(z_j) - h_n(z) - \beta|z_j|^2 + \beta|z|^2) \\ &= 2\beta\varepsilon_n^2 \sum_{j=1}^{m} \sin\alpha([z,z_j]) + \mathcal{O}(\varepsilon_n^3).\end{aligned}$$

Thus, if we choose $\beta = C\varepsilon_n$ with $C > 0$ a sufficiently large constant, then $\Delta p(z) \geq 0$ for all interior vertices $z \in V_{int}(G_n)$. So fix such a suitable $C$ and define the positive function

$$\hat{p} = \varepsilon_n + \|p\| - p.$$

Then $\Delta\hat{p}(z) \leq 0$ for all $z \in V_{int}(G_n)$. The proof of Lemma 5.3 shows that there is a constant $C_1$, depending only on $g$, $D$, and the labelling $\alpha$, such that $\|p\| \leq C_1\varepsilon_n$. Thus $\|\hat{p}\| \leq C_2\varepsilon_n$ with $C_2 = 2C_1 + 1$.

To finish to proof, we apply Proposition 5.7 to the superharmonic function $\hat{p}$. Remember that $G_n$ is a connected graph without loops and $c(e) := 2f'_{\alpha(e)}(0) > 0$ defines a positive weight function on the edges. The bound (5.1) on the labelling $\alpha$ implies that

$$m = \max_{[x,y]\in E(G_n)} \sum_{[x,z]\in E(G_n)} \frac{c(x,z)}{c(x,y)} < \frac{2\pi}{\pi/2 - C} \frac{\cot(\pi/4 - C/2)}{\cot(\pi/4 + C/2)} =: C_3 < \infty.$$

Hence $m$ is uniformly bounded from above for all $n \in \mathbb{N}$ and the bound $C_3$ depends only on the constant $C$ of (5.1). Let $x \in V_{int}(G_n)$. To find an upper bound for $A = \sup_{1\leq r\leq R} r^{-2}W_x(r)$, note that

$$W_x(r) = \sum_{\substack{z\in B_x(r), y\in V(G_n)\\ d(x,z)<d(x,y)}} c(z,y) \leq \max_{e\in E(G_n)} c(e)\#F_w(x,r),$$

where $F_w(x,r)$ is the set of all faces of $\mathscr{D}_n$ with one white vertex $z \in B_x(r)$ and $\#F_w(x,r)$ denotes the number of faces of $F_w(x,r)$. Now, $\max_{e\in E(G_n)} c(e) < \cot(\pi/4 - C/2)/2$ and

$$F_w(x,r) \subset D_x((r+1)2\varepsilon_n) = \{w \in \mathbb{C} : |w - x| \leq (r+1)2\varepsilon_n\},$$

as the edge lengths in $G_n$ are smaller than $2\varepsilon_n$. Remember that $F(f) = \varepsilon_n^2 \sin\alpha(f)$ is the area of the face $f \in F(\mathscr{D}_n)$. Thus $F(f) > \varepsilon_n^2 \sin(\pi/2 - C)$ and

$$\#F_w(x,r) < \frac{\pi((r+1)2\varepsilon_n)^2}{\varepsilon_n^2 \sin(\pi/2 - C)} \leq \frac{16\pi r^2}{\sin(\pi/2 - C)} =: r^2 C_4$$

for all $r \geq 1$. Therefore we obtain $A = \sup_{1\leq r\leq R} r^{-2}W_x(r) < C_4$, where the upper bound $C_4$ is independent of $R$ and $n \in \mathbb{N}$ and depends only on the constant $C$ of (5.1).

Let $K \subset D$ be compact. Denote by $\mathbb{e}$ the Euclidean distance (between a point and a compact set or between closed sets of $\mathbb{R}^2 \cong \mathbb{C}$) and let $\hat{C}$ be the same constant as in Theorem 5.1. As $\varepsilon_n \to 0$, we can choose $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\begin{aligned}K_{\varepsilon_n} &:= \{z \in \mathbb{C} : \mathbb{e}(z,K) \leq 2\varepsilon_n\} \subset \mathscr{D}_n \subset D, \\ a(\varepsilon_n) &:= \mathbb{e}(\partial D, K) - (2 + \hat{C})\varepsilon_n > 0, \\ C_5 &:= \min_{n\geq n_0} \frac{a(\varepsilon_n)}{2\cos(\pi/2 - C/2)} \geq \varepsilon_n^2, \text{ and} \\ \varepsilon_n &\leq C_5/4.\end{aligned}$$

Let $z \in V(G_n) \cap K$ and set $R+1 = d(z, V\partial(G_n))$ to be the combinatorial distance from $z$ to the boundary of $G_n$. Let $z_j \in V(G_n)$ be incident to $z$. Then $z_j \in K_{\varepsilon_n}$ and $R \geq d(z_j, V\partial(G_n))$. As the Euclidean distance $\mathbb{e}(z_1, z_2) = 2\varepsilon_n \cos(\alpha([z_1, z_2])/2)$ between two vertices of $G_n$ is smaller than $2\varepsilon_n \cos(\pi/4 - C/2)$, we obtain

$$R \geq \frac{\mathbb{e}(z_j, V\partial(G_n))}{2\varepsilon_n \cos(\pi/4 - C/2)}.$$

Now $z_j \in K_{\varepsilon_n}$ implies

$$\mathbb{e}(z_j, V\partial(G_n)) \geq \mathbb{e}(K_{\varepsilon_n}, V\partial(G_n)) \geq \mathbb{e}(K_{\varepsilon_n}, \partial D) - \hat{C}\varepsilon_n \geq \mathbb{e}(\partial D, K) - (2+\hat{C})\varepsilon_n = a(\varepsilon_n) > 0$$

for $n \geq n_0$. Thus $R \geq \varepsilon_n^{-1} a(\varepsilon_n)/(2\cos(\pi/2 - C/2)) \geq \varepsilon_n^{-1} C_5 \geq 4$ and

$$\frac{1}{\sqrt{\log_2 R}} \leq \frac{1}{\sqrt{\log_2 C_5 - \log_2 \varepsilon_n}} \leq \frac{\sqrt{2}}{\sqrt{-\log_2 \varepsilon_n}}$$

hold for all $n \geq n_0$ and all $z \in V(G_n) \cap K$ by our assumptions. Proposition 5.7 implies

$$\left| \frac{\hat{p}(z)}{\hat{p}(z_j)} - 1 \right| \leq \frac{4C_3^2 \sqrt{C_4}\sqrt{2}}{\sqrt{-c(z, z_j) \log_2 \varepsilon_n}}$$

for all incident vertices $z, z_j \in V(G_n) \cap K$ and $n \geq n_0$. As $c(z, z_j) \leq \cot(\pi/4 - C/2)/2$ and $\|\hat{p}\| \leq C_2 \varepsilon_n$ we finally arrive at the desired estimation

$$|t_n(z_j) - h_n(z_j) - (t_n(z) - h_n(z))| = |\hat{p}(z) - \hat{p}(z_j)| \leq C_6 \varepsilon_n (-\log_2 \varepsilon_n)^{-\frac{1}{2}}$$

for all incident vertices $z, z_j \in V(G_n) \cap K$ and $n \geq n_0$, where the constant $C_6$ depends on the previous constants $C_2, \ldots, C_5$, that is $C_6$ depends only on $g$, $D$, $C$, $\hat{C}$, and $K$. □

**Lemma 5.8.** *Let $\vec{e} = \overrightarrow{uv} \in \vec{E}(\mathscr{D}_n)$ be a directed edge with $u \in V(G_n)$ and $v \in V(G_n^*)$. Denote by $\delta_n(e)$ the combinatorial distance in graph $\mathscr{D}_n$ from $e = [u, v]$ to $[z_0, v_0]$, that is the least integer $k$ such that there is a sequence of edges $\{[z_0, v_0] = e_1, e_2, \ldots, e_k = e\} \subset E(\mathscr{D}_n)$ such that the edges $e_{m+1}$ and $e_m$ are incident to the same face in $\mathscr{D}_n$ for $m = 1, \ldots, k-1$. Then*

$$\varphi_n(\vec{e}) = \arg g'(v) + \phi_n(\vec{e}) + \delta_n(e)\mathcal{O}(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}}). \tag{5.14}$$

*The constant in the notation $\mathcal{O}(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}})$ may depend on the distance of $v$ to the boundary $\partial D$.*

Note that if $\partial D$ is smooth, then $\delta_n(e) = \mathcal{O}(\varepsilon_n^{-1})$. In general we have $\delta_n(e) = \mathcal{O}(\varepsilon_n^{-1})$ on compact subsets $K \subset D$, where the constant in the notation $\mathcal{O}(\varepsilon_n^{-1})$ may depend on $K$. In any case, on compact subsets of $D$ we have

$$\varphi_n(\vec{e}) = \arg g'(v) + \phi_n(\vec{e}) + \mathcal{O}((-\log_2 \varepsilon_n)^{-\frac{1}{2}}). \tag{5.15}$$

*Proof.* We use estimation (5.13) and Theorem 2.17.

Using the notation of Figure 2.6 (left), equation (5.13) implies

$$\begin{aligned}
f_\alpha(\log r_n(z_+) - \log r_n(z_-)) &= f_\alpha(\log \frac{r_n(z_+)}{R_n(z_+)} - \log \frac{r_n(z_-)}{R_n(z_-)}) \\
&= f_\alpha(\log |g'(z_+)| - \log |g'(z_-)|) + \mathcal{O}(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}}) \\
&= f_\alpha(0) + f'_\alpha(0)(\log |g'(z_+)| - \log |g'(z_-)|) \\
&\quad + \mathcal{O}(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}}).
\end{aligned}$$

As in Lemma 5.3 we may write

$$2f_\alpha'(0) = 2\frac{\cos(\alpha/2)}{2\sin(\alpha/2)} = \frac{|v_+ - v_-|}{|z_+ - z_-|} = \frac{v_+ - v_-}{i(z_+ - z_-)},$$

which yields

$$\begin{aligned} 2f_\alpha'(0)(\log|g'(z_+)| - \log|g'(z_-)|) &= \frac{v_+ - v_-}{i(z_+ - z_-)}\operatorname{Re}(a(z_+ - z_-)) + \mathcal{O}(\varepsilon_n(-\log_2\varepsilon_n)^{-\frac{1}{2}}) \\ &= \operatorname{Im}(a(v_+ - v_-)) + \mathcal{O}(\varepsilon_n(-\log_2\varepsilon_n)^{-\frac{1}{2}}) \\ &= \arg g'(v_+) - \arg g'(v_-) + \mathcal{O}(\varepsilon_n(-\log_2\varepsilon_n)^{-\frac{1}{2}}), \end{aligned}$$

where $a = \frac{g''((z_+ + z_-)/2)}{g'((z_+ + z_-)/2)} = \frac{g''((v_+ + v_-)/2)}{g'((v_+ + v_-)/2)}$.

By part (1) of Theorem 2.17 we can choose the angle functions $\phi_n$ and $\varphi_n$ on any minimal sequence of edges $\{[z_0, v_0] = e_1, e_2, \ldots, e_k = e\} \subset E(\mathscr{D}_n)$ such that equations (2.9)–(2.14) are satisfied without the (mod $2\pi$)-term. Using the above considerations of $2f_\alpha(\log r_n(z_+) - \log r_n(z_-))$ and the normalization of $\varphi_n$, we arrive at equation (5.14). □

*Proof of Theorem 5.1.* Consider a compact subset $K$ of $D$. Let $z \in V(G_n) \cap K$ and $v \in V(G_n^*) \cap K$ be vertices which are incident in $\mathscr{D}_n$, that is $[z, v] \in E(\mathscr{D}_n)$. Then Lemmas 5.3 and 5.8 imply

$$\begin{aligned} \log g'(z) &= \log|g'(z)| + i\arg g'(z) \\ &= \log(r_n(z)/R_n(z)) + i(\varphi_n(\vec{zv}) - \phi_n(\vec{zv})) + \mathcal{O}((-\log_2\varepsilon_n)^{-\frac{1}{2}}). \end{aligned}$$

As $g'$ and thus the quotient $r_n/R_n$ is uniformly bounded, we obtain

$$g'(z) = \frac{r_n(z)}{R_n(z)} e^{i(\varphi_n(\vec{zv}) - \phi_n(\vec{zv}))} + \mathcal{O}((-\log_2\varepsilon_n)^{-\frac{1}{2}}) = q_n(z) + \mathcal{O}((-\log_2\varepsilon_n)^{-\frac{1}{2}}). \quad (5.16)$$

This implies the uniform convergence on compact subsets of $D$ of $q_n$ to $g'$.

Convergence of $g_n$ is now proven by using suitable integrations of $g'$ and $q_n$.

Let $w \in V(G_n^*)$ and consider a shortest path $\gamma$ in $G_n^*$ from $v_0$ to $w$ with vertices $\{v_0 = w_1, w_2, \ldots, w_k = w\} \subset V(G_n^*)$. Then

$$\begin{aligned} g(w) = g(v_0) + \int_\gamma g'(\zeta)d\zeta &= g(v_0) + \sum_{j=1}^{k-1} g'(w_{j+1})(w_{j+1} - w_j) + \mathcal{O}(\varepsilon_n) \\ &= g(v_0) + \sum_{j=1}^{k-1} q_n(w_{j+1})(w_{j+1} - w_j) \\ &\quad + \sum_{j=1}^{k-1}(g'(w_{j+1}) - q_n(w_{j+1}))(w_{j+1} - w_j) + \mathcal{O}(\varepsilon_n) \\ &= g(v_0) + \sum_{j=1}^{k-1} q_n(w_{j+1})(w_{j+1} - w_j) + \mathcal{O}((-\log_2\varepsilon_n)^{-\frac{1}{2}}), \end{aligned}$$

because $g'(w_j) - q_n(w_j) = \mathcal{O}((-\log_2\varepsilon_n)^{-\frac{1}{2}})$, $w_{j+1} - w_j = \mathcal{O}(\varepsilon_n)$ and $k = \mathcal{O}(\varepsilon_n^{-1})$ on compact sets. The claim on uniform convergence of $g_n$ on compact sets follows, if we can show that

$$p_n(w) = g(v_0) + \sum_{j=1}^{k-1} q_n(w_{j+1})(w_{j+1} - w_j) + \mathcal{O}((-\log_2\varepsilon_n)^{-\frac{1}{2}}). \quad (5.17)$$

As we have given an isoradial circle pattern, we get for two incident vertices $v_-, v_+$ in $G_n^*$

$$\begin{aligned} v_+ - v_- &= 2R_n(z_-)\cos(\alpha/2)\mathrm{e}^{i(\phi_n(\overrightarrow{z_-v_-})+f_\alpha(0)+\pi/2)} \\ &= 2R_n(z_+)\cos(\alpha/2)\mathrm{e}^{i(\phi_n(\overrightarrow{z_+v_-})-f_\alpha(0)-\pi/2)}, \end{aligned}$$

where we have used the notation of Figure 2.6 (left). Now using $r_n(z) = \mathcal{O}(\varepsilon_n)$ and the estimations of Lemma 5.6, we deduce that

$$\begin{aligned} f_\alpha(0) &= f_\alpha(\log r_n(z_+) - \log r_n(z_-)) + \mathcal{O}((-\log_2 \varepsilon_n)^{-\frac{1}{2}}) \\ &= f_\alpha(\log r_n(z_-) - \log r_n(z_+)) + \mathcal{O}((-\log_2 \varepsilon_n)^{-\frac{1}{2}}), \\ |p_n(v_+) - p_n(v_-)| &= 2r_n(z_-)\cos(\alpha/2) + \mathcal{O}(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}}) \\ &= 2r_n(z_+)\cos(\alpha/2) + \mathcal{O}(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}}) \end{aligned}$$

with the same notation. Remembering

$$q_n(v_+) = \frac{r_n(z_+)}{R_n(z_+)}\mathrm{e}^{i(\varphi_n(\overrightarrow{w_{j+1}z_+})-\phi_n(\overrightarrow{w_{j+1}z_+}))} + \mathcal{O}((-\log_2 \varepsilon_n)^{-\frac{1}{2}}).$$

and using

$$(w_{j+1} - w_j) = 2R_n(z_+)\cos(\alpha([w_{j+1}, w_j])/2)\mathrm{e}^{i(\phi_n(\overrightarrow{w_{j+1}z_+})-(\pi/2-\alpha([w_{j+1},w_j])/2))},$$

we can conclude that

$$q_n(w_{j+1})(w_{j+1} - w_j) = p_n(w_{j+1}) - p_n(w_j) + \mathcal{O}(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}}),$$

where $z_-, z_+ \in V(G)$ are incident to $w_{j+1}$ and $w_j$ and we have used the notations are as in Figure 2.6 (left) with $w_j = v_-$ and $w_{j+1} = v_+$. As we have normalized $g(v_0) = p(v_0)$, this proves equation (5.17) and therefore the uniform convergence of $p_n$ to $g$ on compact subsets of $D$. □

**Remark 5.9.** Theorem 5.1 may easily be generalized in the following two ways. First, we may consider 'nearly isoradial' circle pattern in the sense that $R_n(z) = \mathcal{O}(\varepsilon_n)$ for all vertices $z \in V(G_n)$ and $R_n(z_1)/R_n(z_2) = 1 + \mathcal{O}(\varepsilon_n^3)$ for all edges $[z_1, z_2] \in E(G_n)$. The proof is completely analogous with some minor adaptions in the proofs of Lemmas 5.3 and 5.6.

Second, we may omit the assumption that the whole domain $D$ is approximated by the rhombic embeddings $\mathscr{D}_n$. Then the convergence claims remain true for compact subsets of any open domain $D' \subset D$ which is covered or approximated by the rhombic embeddings and contains $v_0$.

## 5.2 $C^1$-Convergence for Neumann boundary conditions

The following theorem is the analog of Theorem 5.1 for Neumann boundary conditions.

**Theorem 5.10.** *Let $D \subset \mathbb{C}$ be a simply connected bounded domain, and let $W \subset \mathbb{C}$ be open with $\bar{D} \subset W$. Let $g : W \to \mathbb{C}$ be a locally injective holomorphic function. Assume, for convenience, that $0 \in D$.*

*For $n \in \mathbb{N}$ let $\mathscr{D}_n$ be a b-quad-graph with associated graphs $G_n$ and $G_n^*$ and let $\alpha_n$ be an admissible labelling, which is uniformly bounded with constant $C$ as in Theorem 5.1. Let $\varepsilon_n \in (0, \infty)$ be a sequence of positive numbers such that $\varepsilon_n \to 0$ for $n \to \infty$. For each $n \in \mathbb{N}$, assume that there is a isoradial circle pattern for $G_n$ and $\alpha_n$. Suppose as in Theorem 5.1 that all centers lie in the domain $D$ and that any point $x \in \bar{D}$ which is not contained in any of the discs bounded by the circles of the pattern has a distance less* *than $\hat{C}\varepsilon_n$ to the nearest center of a circle and to the boundary $\partial D$, where $\hat{C} > 0$ is some constant independent of $n$. Denote by $R_n \equiv \varepsilon_n$ and $\phi_n$ the radius and the angle function of this isoradial circle pattern for $G_n$ and $\alpha_n$.*

*Further assume that $\varepsilon_n$ is sufficiently small such that for all $n \in \mathbb{N}$*

$$\sup_{v\in D} \max_{\theta\in[0,2\pi]} |\arg g'(v + 2R_n e^{i\theta}) - \arg g'(v)| < \frac{\pi}{2} - C < \min_{e\in E(G_n)} (\pi - \alpha(e)). \tag{5.18}$$

*Define an angle function on the oriented boundary edges by*

$$\varphi_n(\vec{e}) = \phi_n(\vec{e}) + \arg g'(v),$$

*where $\vec{e} = \overrightarrow{zv} \in \vec{E}(\mathscr{D}_n)$ and $v \in V(G_n^*)$. Then there is a circle pattern $\mathscr{C}_n$ for $G_n$ and $\alpha_n$ with radius function $r_n$ and angle function $\varphi_n$ with the given boundary values.*

*Suppose that this circle pattern is normalized such that*

$$r_n(z_0) = R_n(z_0)|g'(z_0)| \qquad \text{and} \qquad \varphi_n(\vec{e}) = \arg g'(v_0) + \phi_n(\vec{e}),$$

*where $z_0 \in V(G_n)$ is chosen such that the disk bounded by $C_{z_0}$ contains $0$ and $e = [z_0, v_0] \in E(\mathscr{D}_n)$ is one of the edges incident to $z_0$ such that $\phi_n(\vec{e}) \in [0, 2\pi)$ is minimal. Suppose further that $\mathscr{C}_n$ is normalized by a translation such that*

$$p_n(v_0) = g(v_0), \tag{5.19}$$

*where $p_n(v)$ denotes the intersection point corresponding to $v \in V(G_n^*)$.*

*For $z \in D$ set*

$$g_n(z) = p_n(w) \quad \text{and} \quad q_n(z) = \frac{r_n(v)}{R_n(v)} e^{i(\varphi_n(\overrightarrow{vw}) - \phi_n(\overrightarrow{vw}))},$$

*where $w$ is a vertex of $V(G_n^*)$ closest to $z$ and $v$ is a vertex of $V(G_n)$ closest to $z$ such that $[v, w] \in E(\mathscr{D}_n)$.*

*Then $q_n \to g'$ and $g_n \to g$ uniformly on compact subsets in $D$ as $n \to \infty$.*

*Proof.* First, we prove the existence claim for the circle pattern with Neumann boundary conditions using Theorem 2.25.

Define two functions $\Phi_n, \hat{\Phi}_n : V(G_n) \to (0, \infty)$ by $\Phi_n(z) = 2\pi = \hat{\Phi}_n(z)$ for interior vertices $z \in V_{int}(G_n)$ and by $\Phi_n(z) = \sum_{e=[z,u]\in E(G_n)} (\pi - \alpha(e))$ and

$$\hat{\Phi}_n(z) = \Phi_n(z) + \arg g'(v_-(z)) - \arg g'(v_+(z))$$

for boundary vertices $z \in V_\partial(G_n)$. Here and below $v_-(z)$ and $v_+(z)$ denote the two boundary vertices of $G_n^*$ incident to $z$ in $\mathscr{D}_n$. The notation is chosen such that passing from $v_-(z)$ to $z$ to $v_+(z)$ along the edges $\overrightarrow{v_-(z)z}$ and $\overrightarrow{zv_+(z)}$ means advancing on the boundary of $\mathscr{D}_n$ in counterclockwise orientation (so the circle pattern is to the left of the oriented path).

Let $V' \subset V(G_n)$ be a nonempty subset with $V' \neq V(G_n)$. We have to show that

$$\sum_{z\in V'} \hat{\Phi}_n(z) = \sum_{z\in V'} \Phi_n(z) + \sum_{z\in V'\cap\partial V(G_n)} \arg g'(v_-(z)) - \arg g'(v_+(z))$$
$$< \sum_{\substack{[z,u]\in E(G_n)\\ \text{with } z\in V'}} 2(\pi - \alpha([z, u])). \tag{5.20}$$

This is obvious if $V' \cap V_\partial(G_n) = \emptyset$ and if $V' \cap V_\partial(G_n) = V_\partial(G_n)$ as $\Phi_n$ comes from a circle pattern. So assume that $V_\partial(G_n) \cap V'$ consists of $k \geq 1$ non empty components $B_j$, $j = 1, \ldots, k$, which do not contain $V_\partial(G_n)$. Each component $B_j$ consists of a chain of boundary vertices of $V(G_n)$. Follow this chain in counterclockwise order along the boundary of $\mathscr{D}_n$ beginning with $z_-^j$ with incident vertex $v_-^j = v_-(z_-^j) \in V_\partial(G_n^*)$ and ending at $z_+^j$ with incident vertex $v_+^j = v_+(z_+^j) \in V_\partial(G_n^*)$. Then

$$\sum_{z \in B_j} \arg g'(v_-(z)) - \arg g'(v_+(z)) = \arg g'(v_-^j) - \arg g'(v_+^j).$$

Now consider the set $S = \{e = [z, u] \in E(G_n) : z \in V', u \notin V'\}$ of all edges with only one vertex in $V'$ and the corresponding dual edges $S^* = \{e^* \in E(G_n^*) : e \in S\}$ associated to the same faces of $\mathscr{D}_n$. The components of $S^*$ are chains of dual edges. It follows from the construction, that if such a chain starts at a boundary vertex of $G_n^*$, then it also ends at another boundary vertex and there are exactly $k$ such chains. Proceeding along such an oriented chain $c$ starting at $v_-^{j_1}$ and ending at $v_+^{j_2}$, we obtain using assumption (5.18)

$$\begin{aligned} \arg g'(v_-^{j_1}) - \arg g'(v_+^{j_2}) &= \sum_{e^* \in c} \arg g'(v_-(e^*)) - \arg g'(v_+(e^*)) \\ &< \sum_{e^* \in c} (\pi - \alpha(e^*)) = \sum_{e \in S \text{ with } e^* \in c} (\pi - \alpha(e)), \end{aligned}$$

where $v_-(e^*)$ and $v_+(e^*)$ denote the two vertices incident to the edge $e^*$ such that $v_-(e^*)$ and $v_+(e^*)$ occur in this order when advancing along the chain $c$ from $v_-^{j_1}$ to $v_+^{j_2}$. From this estimation we conclude that

$$\sum_{z \in V' \cap \partial V(G_n)} \arg g'(v_-(z)) - \arg g'(v_+(z)) < \sum_{e \in S} (\pi - \alpha(e)).$$

As we have given an isoradial circle pattern, we have $\Phi_n(z) = \sum_{e=[z,u] \in E(G_n)} (\pi - \alpha(e))$ for all $z \in V(G_n)$. Thus with $E' = \{e = [z, u] \in E(G_n) : z, u \in V'\}$ we get

$$\sum_{z \in V'} \Phi_n(z) = \sum_{e \in E'} 2(\pi - \alpha(e)) + \sum_{e \in S} (\pi - \alpha(e)).$$

Combining these two estimations proves inequality (5.20).

Theorem 2.19 shows that the difference $\varphi_n - \phi_n$ gives rise to a function $\delta_n : V(G_n^*) \to \mathbb{R}$. Without loss of generality, we may assume the normalization $\delta_n(v_0) = \arg g'(v_0)$.

The proof of the convergence claim is similar to the proof of Theorem 5.1. The roles of $\delta_n = \varphi_n - \phi_n$ and $\log(r_n/R_n)$ have to be interchanged in Lemmas 5.3, 5.6, and 5.8 and similarly $\arg g' = \operatorname{Im} \log g'$ has to be considered instead of $\log |g'| = \operatorname{Re} \log g'$. The role of equation (2.2) is substituted by equation (2.20). More precisely, we have the following proof.

**Lemma 5.11.** *For $v \in V(G_n^*)$ set*

$$\begin{aligned} h_n(v) &= \arg g'(v), \\ t_n(v) &= \delta_n(v). \end{aligned}$$

*Then*

$$h_n(v) - t_n(v) = \mathcal{O}(\varepsilon_n).$$

*Proof.* The proof uses the same reasoning as the proof of Lemma 5.3. Consider the function $p(v) = t_n(v) - h_n(v) + \beta|v|^2$, where $\beta \in (0, 1)$ is some function of $\varepsilon_n$. We want to choose $\beta$ such that $p$ will have no maximum in $V_{int}(G_n^*)$.

Suppose that $p$ has a maximum at $v \in V_{int}(G_n^*)$. Denote by $v_1, \dots, v_m$ the incident vertices of $v$ in $G_n^*$ in counterclockwise order. Then for $j = 1, \dots, m$ we have

$$t_n(v_j) - t_n(v) \le x_j \tag{5.21}$$

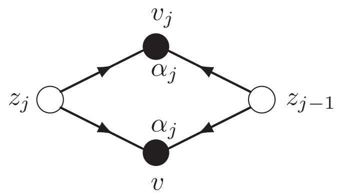



**Figure 5.2:** A rhombic face of $\mathscr{D}_n$ with oriented edges.

where

$$x_j = h_n(v_j) - h_n(v) - \beta|v_j|^2 + \beta|v|^2. \tag{5.22}$$

As in the proof of Lemma 5.3, we have $|v| = \mathcal{O}(1)$ and by assumption $v - v_j = \mathcal{O}(\varepsilon_n)$. With $\beta \in (0,1)$ this leads to $\beta|v_j|^2 - \beta|v|^2 = \mathcal{O}(\varepsilon_n)$. Using this estimate and the smoothness of $\operatorname{Im}(\log g')$, we get $x_j = \mathcal{O}(\varepsilon_n)$.

Consider the chain of faces $f_j$ of $\mathscr{D}_n$ ($j = 1, \dots, m$) which are incident to $v$ and $v_j$. The counterclockwise enumeration of the vertices $v_j$ (and hence of the faces $f_j$) and of the white vertices $z_1, \dots, z_m$ incident to these faces can be chosen such that $f_j$ is incident to $z_{j-1}$ and $z_j$ for $j = 1, \dots, m$, where $z_0 = z_m$. See Figure 5.2 for an illustration. Denote $\alpha_j = \alpha([v_j, v]) = \alpha([z_{j-1}, z_j])$.

Furthermore, we adjust the values of $\varphi$ and $\phi$ (by suitably adding multiples of $2\pi$) such that for all $j = 1, \dots, m$

$$\varphi(\overrightarrow{z_j v_j}) - \varphi(\overrightarrow{z_j v}) \in (0, 2\pi) \qquad \text{and} \qquad \phi(\overrightarrow{z_j v_j}) - \phi(\overrightarrow{z_j v}) \in (0, 2\pi).$$

From inequality (5.21), the definition of $t_n = \delta_n = \varphi_n - \phi_n$ and the monotonicity of the sum in equation (2.20) (see Lemma 2.7), we get

$$\begin{aligned} 0 &= \sum_{j=1}^{m} f_{\alpha_j}^{-1}\left(\frac{\varphi(\overrightarrow{z_j v_j}) - \varphi(\overrightarrow{z_j v})}{2}\right) \\ &= \sum_{j=1}^{m} f_{\alpha_j}^{-1}\left(\frac{t_n(v_j) - t_n(v)}{2} + \frac{\phi(\overrightarrow{z_j v_j}) - \phi(\overrightarrow{z_j v})}{2}\right) \\ &\leq \sum_{j=1}^{m} f_{\alpha_j}^{-1}\left(\frac{x_j}{2} + \frac{\phi(\overrightarrow{z_j v_j}) - \phi(\overrightarrow{z_j v})}{2}\right). \end{aligned} \tag{5.23}$$

Remembering $x_j = \mathcal{O}(\varepsilon_n)$, we can consider a Taylor expansion of the terms of the sum about $(\phi(\overrightarrow{z_j v_j}) - \phi(\overrightarrow{z_j v}))/2 = (\pi - \alpha_j)/2$ in order to make an $\mathcal{O}(\varepsilon_n^3)$-analysis.

We use the above enumeration of the vertices and the relations mentioned in the proof of Lemma 5.3 and denote

$$l_j = |z_j - z_{j-1}| = 2R_n \sin(\alpha_j/2), \qquad \hat{l}_j = |v_j - v| = 2R_n \cos(\alpha_j/2),$$

where $R_n \equiv \varepsilon_n$ is the constant radius function of the given isoradial circle pattern. Then we easily obtain by a simple calculation that

$$f_{\alpha_j}^{-1}(\tfrac{\pi-\alpha_j}{2}) = 0, \quad (f_{\alpha_j}^{-1})'(\tfrac{\pi-\alpha_j}{2}) = 1/f'_{\alpha_j}(0) = 2l_j/\hat{l}_j, \quad (f_{\alpha_j}^{-1})''(\tfrac{\pi-\alpha_j}{2}) = 0.$$

Taking into account that equation (2.20) holds with $\beta_j = (\pi - \alpha_j)/2$ and using the uniform boundedness (5.1) of the labeling $\alpha$, inequality (5.23) yields

$$0 \leq \sum_{j=1}^{m} (f_{\alpha_j}^{-1})'(\tfrac{\pi-\alpha_j}{2})\tfrac{x_j}{2} + \mathcal{O}(\varepsilon_n^3). \tag{5.24}$$

To evaluate this sum, expand

$$\log g'(v_j) - \log g'(v) = a(v_j - v) + b(v_j - v)^2 + \mathcal{O}(\varepsilon_n^3)$$

and

$$\begin{aligned} x_j &= h_n(v_j) - h_n(v) - \beta|v_j|^2 + \beta|v|^2 \\ &= \operatorname{Im}(a(v_j - v) + b(v_j - v)^2 - 2\beta i\bar{v}(v_j - v)) - \beta l_j^2 + \mathcal{O}(\varepsilon_n^3). \end{aligned}$$

Noting that $(f_{\alpha_j}^{-1})'(\frac{\pi-\alpha_j}{2})(v_j - v) = 2(z_j - z_{j-1})/i$ we get with the same reasoning as in the proof of Lemma 5.3

$$\begin{aligned} \sum_{j=1}^{m} (f_{\alpha_j}^{-1})'(\tfrac{\pi-\alpha_j}{2})\frac{x_j}{2} &= \operatorname{Im}\left(\frac{a - 2i\beta\bar{z}}{i}\sum_{j=1}^{m}(z_j - z_{j-1}) + \frac{b}{i}\sum_{j=1}^{m}(z_j - z_{j-1})(v_j - v)\right) \\ &\qquad - \beta\sum_{j=1}^{m} l_j\hat{l}_j + \mathcal{O}(\varepsilon_n^3) \\ &= -\beta\sum_{j=1}^{m} l_j\hat{l}_j + \mathcal{O}(\varepsilon_n^3) \end{aligned}$$

Thus we arrive at the same estimations for $\beta$ as in the proof of Lemma 5.3 and can conclude using the same arguments. $\square$

**Remark 5.12.** As indicated for Dirichlet boundary conditions in Remark 5.4, the statement of Lemma 5.11 can similarly be improved to

$$h_n(v) - t_n(v) = \mathcal{O}(\varepsilon_n^2) \tag{5.25}$$

in the case of a 'very regular' isoradial circle pattern. These are isoradial circle patterns such that for each oriented edge $e_{j_1} = v_{j_1} - v \in \vec{E}(G^*)$ incident to an interior vertex $v \in V(G^*)$ there is another parallel edge $e_{j_2} = v_{j_2} - v \in \vec{E}(G^*)$ with opposite direction incident to $v$, that is $e_{j_2} = -e_{j_1}$. Furthermore, the corresponding intersection angles agree: $\alpha([v, v_{j_1}]) = \alpha([v, v_{j_2}])$. This additional regularity property holds for example for an orthogonal circle pattern with the combinatorics of a part of the square grid.

**Lemma 5.13.** *Let $t_n$ and $h_n$ be defined as in Lemma 5.11. Let $K \subset D$ be a compact subset in $D$. Then the following estimation holds for all $n \in \mathbb{N}$ and every interior vertex $v \in V_{int}(G_n^*) \cap K$ such that all its incident vertices $v_1, \ldots, v_l$ are also in $V_{int}(G_n^*) \cap K$:*

$$t_n(v_j) - h_n(v_j) - (t_n(v) - h_n(v)) = \mathcal{O}(\varepsilon_n(-\log\varepsilon_n)^{-\frac{1}{2}}). \tag{5.26}$$

*for $j = 1, \ldots, l$. The constant in $\mathcal{O}$-notation may depend on $K$, but not on $n$ or $v$.*

The proof is essentially the same as for Lemma 5.6 using the definitions of $t_n$ and $h_n$ of Lemma 5.11, the weight function $c(e) = (f_{\alpha(e)}^{-1})'(\frac{\pi-\alpha(e)}{2})/2 = 1/(2f'_{\alpha(e)}(0))$, Proposition 5.7, and some obvious adaptions.

**Lemma 5.14.** *Let $z \in V(G_n)$. Denote by $\delta_n(z)$ the combinatorial distance in graph $\mathscr{D}_n$ from $z_0$ to $z$, that is the least integer $k$ such that there is a sequence of edges $\{[z_0, v_0] = e_1, e_2, \ldots, e_k = [z_k, z]\} \subset E(\mathscr{D}_n)$ such that the edges $e_{m+1}$ and $e_m$ are incident to the same face in $\mathscr{D}_n$ for $m = 1, \ldots, k-1$. Then*

$$\log(r_n(z)/R_n(z)) = \log|g'(z)| + \delta_n(z)\mathcal{O}(\varepsilon_n(-\log_2\varepsilon_n)^{-\frac{1}{2}}). \tag{5.27}$$

*The constant in the notation $\mathcal{O}(\varepsilon_n(-\log_2\varepsilon_n)^{-\frac{1}{2}})$ may depend on the distance of $z$ to the boundary $\partial D$.*

Note that if $\partial D$ is smooth, then $\delta_n(z) = \mathcal{O}(\varepsilon_n^{-1})$. In general we have $\delta_n(z) = \mathcal{O}(\varepsilon_n^{-1})$ on compact subsets $K \subset D$, where the constant in the notation $\mathcal{O}(\varepsilon_n^{-1})$ may depend on $K$. In any case, on compact subsets of $D$ we have

$$\log(r_n(z)/R_n(z)) = \log|g'(z)| + \mathcal{O}((-\log_2 \varepsilon_n)^{-\frac{1}{2}}). \tag{5.28}$$

*Proof.* The proof is similar to the proof of Lemma 5.8. We use estimation (5.26) and Theorem 2.17.

Using the notation of Figure 2.6 (left), in particular the labelling of the vertices and their cyclical order using the counterclockwise orientation of the face, equation (5.26) implies that

$$\begin{aligned}
f_\alpha^{-1}\left(\frac{\varphi(\overrightarrow{z_-v_+}) - \varphi(\overrightarrow{z_-v_-})}{2}\right) &= f_\alpha^{-1}\left(\frac{t_n(v_+) - t_n(v_-)}{2} + \frac{\phi(\overrightarrow{z_-v_+}) - \phi(\overrightarrow{z_-v_-})}{2}\right) \\
&= f_\alpha^{-1}\left(\frac{\arg g'(v_+) - \arg g'(v_-)}{2} + \frac{\pi-\alpha}{2}\right) \\
&\quad + \mathcal{O}(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}}) \\
&= f_\alpha^{-1}(\tfrac{\pi-\alpha}{2}) + (f_\alpha^{-1})'(\tfrac{\pi-\alpha}{2})(\arg g'(v_+) - \arg g'(v_-))/2 \\
&\quad + \mathcal{O}(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}}).
\end{aligned}$$

As in Lemma 5.11 (see also Lemma 5.3) we may write

$$\frac{1}{2}(f_\alpha^{-1})'(\tfrac{\pi-\alpha}{2}) = \frac{|z_+ - z_-|}{|v_+ - v_-|} = \frac{i(z_+ - z_-)}{v_+ - v_-},$$

which yields

$$\begin{aligned}
\frac{(f_\alpha^{-1})'(\frac{\pi-\alpha}{2})}{2}(\arg g'(v_+) - \arg g'(v_-)) &= \frac{i(z_+ - z_-)}{v_+ - v_-}\operatorname{Im}(a(v_+ - v_-)) + \mathcal{O}(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}}) \\
&= \operatorname{Re}(a(z_+ - z_-)) + \mathcal{O}(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}}) \\
&= \log|g'(z_+)| - \log|g'(z_-)| + \mathcal{O}(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}}),
\end{aligned}$$

where $a = \frac{g''((z_+ + z_-)/2)}{g'((z_+ + z_-)/2)} = \frac{g''((v_+ + v_-)/2)}{g'((v_+ + v_-)/2)}$.

Using

$$\begin{aligned}
\log r_n(z_+) - \log r_n(z_-) &= f_\alpha^{-1}\left(\frac{\varphi(\overrightarrow{z_-v_+}) - \varphi(\overrightarrow{z_-v_+})}{2}\right), \\
\log R_n(z_+) - \log R_n(z_-) &= f_\alpha^{-1}(\tfrac{\pi-\alpha}{2}),
\end{aligned}$$

the above considerations, and the normalization of $r_n$, we arrive at equation (5.27). ☐

The rest of the proof of Theorem 5.10 is the same as in the proof of Theorem 5.1, see pages 84–85. ☐

## 5.3 $C^\infty$-CONVERGENCE OF QUASICRYSTALLIC CIRCLE PATTERNS

For a certain class of quasicrystallic circle patterns, the order of convergence in Theorem 5.1 can be improved. Remember from Chapter 3 that the class of quasicrystallic rhombic embeddings is closely connected to combinatorial surfaces in $\mathbb{Z}^d$ for some $d$. Our proof of $C^\infty$-convergence exploits some of this additional structure, in particular the integrability of the Hirota equation which allows to extend the radius function, for example to the white vertices of the region $\mathcal{F}(\Omega_{\mathscr{D}})$ which can be reached by flips from $\Omega_{\mathscr{D}}$, and the Regularity Lemma 3.19.

Let $\mathscr{D}$ be a quasicrystallic rhombic embedding of a b-quad-graph with associated combinatorial surface $\Omega_\mathscr{D} \subset \mathbb{Z}^d$. First we enlarge the set $\mathcal{F}(\Omega_\mathscr{D})$ of vertices which can be reached by flips from $\Omega_\mathscr{D}$ in the following way. For a set of vertices $W \subset V(\mathbb{Z}^d)$ denote by $W^{[1]}$ the set $W$ together with all vertices incident to a two-dimensional facet of $\mathbb{Z}^d$ where three of its four vertices belong to $W$. Define $W^{[k+1]} = (W^{[k]})^{[1]}$ inductively for all $k \in \mathbb{N}$. In particular, we denote

$$\mathcal{F}_\kappa(\Omega_\mathscr{D}) = (\mathcal{F}(\Omega_\mathscr{D}))^{[\kappa]}$$

for some arbitrary, but fixed $\kappa \in \mathbb{N}$. Note that $\mathcal{F}_\kappa(\Omega_\mathscr{D}) \subset \Pi(\Omega_\mathscr{D})$.

Our main aim will be to study partial derivatives of the extended radius function. Let $\mathbf{e}_{j_1}, \mathbf{e}_{j_2}$ be two linearly independent directions corresponding to $a_{j_1}, a_{j_2} \in \mathcal{A}$. For any function $h$ defined on white vertices $\mathbf{z} \in \mathbb{Z}^d$ define *discrete partial derivatives* by

$$\partial_{\mathbf{e}_{j_1} \pm \mathbf{e}_{j_2}} h(\mathbf{z}) = \frac{h(\mathbf{z} + \mathbf{e}_{j_1} \pm \mathbf{e}_{j_2}) - h(\mathbf{z})}{\varepsilon_n |a_{j_1} \pm a_{j_2}|}.$$

The following constants are introduced in order to estimate the possible orders of partial derivatives for a function defined on $\mathcal{F}_\kappa(\Omega_\mathscr{D})$ and the comparison of these orders with the distance to the boundary.

**Definition 5.15.** Let $\mathscr{D}$ be a rhombic embedding of a finite simply connected b-quad-graph and let $\Omega_\mathscr{D}$ be the corresponding combinatorial surface in $\mathbb{Z}^d$. Let $J \subset \{1, \dots, d\}$ contain at least two different indices. Let $\mathbf{z} \in V(\mathbb{Z}^d)$ and $\hat{\mathbf{z}} \in V(\Omega_\mathscr{D})$.

For $B \geq 0$ define a *combinatorial ball* of radius $B$ about $\mathbf{z}$ using the directions $\{\mathbf{e}_j : j \in J\}$ by

$$U_J(\mathbf{z}, B) = \{\zeta = \mathbf{z} + \sum_{j \in J} n_j \mathbf{e}_j : \sum_{j \in J} |n_j| \leq B\}. \tag{5.29}$$

The *radius of the largest ball* about $\mathbf{z}$ using the directions $\{\mathbf{e}_j : j \in J\}$ which is contained in $\mathcal{F}_\kappa(\Omega_\mathscr{D})$ is denoted by

$$B_J(\mathbf{z}, \mathcal{F}_\kappa(\Omega_\mathscr{D})) = \max\{B \in \mathbb{N} : U_J(\mathbf{z}, B) \subset \mathcal{F}_\kappa(\Omega_\mathscr{D})\}. \tag{5.30}$$

A *combinatorial $K$-environment of $\hat{\mathbf{z}}$ in* $\Omega_\mathscr{D}$ consists of the part of $\Omega_\mathscr{D}$ whose vertices have combinatorial distance at most $K$ from $\hat{\mathbf{z}}$ within $\Omega_\mathscr{D}$. This part of $\Omega_\mathscr{D}$ corresponds to the combinatorial $K$-environment in $\mathscr{D}$ of the corresponding point $z \in V(\mathscr{D})$. Denote by $d(\hat{\mathbf{z}}, \partial\Omega_\mathscr{D})$ the *combinatorial distance* of $\hat{\mathbf{z}}$ to the boundary $\partial\Omega_\mathscr{D}$, that is the largest integer $K$ such that a combinatorial $(K-1)$-environment of $\hat{\mathbf{z}}$ in $\Omega_\mathscr{D}$ does not contain any boundary points of $\Omega_\mathscr{D}$.

For further use, we define the constant

$$C_J(\mathcal{F}_\kappa(\Omega_\mathscr{D})) = \min\left\{ \frac{B_J(\hat{\mathbf{z}}, \mathcal{F}_\kappa(\Omega_\mathscr{D})) + 1}{d(\hat{\mathbf{z}}, \partial\Omega_\mathscr{D})} : \hat{\mathbf{z}} \in V_{int}(\Omega_\mathscr{D}) \right\} > 0. \tag{5.31}$$

Note as an immediate consequence that for all $\hat{\mathbf{z}} \in V_{int}(\Omega_\mathscr{D})$

$$U_J(\hat{\mathbf{z}}, \lceil C_J(\mathcal{F}_\kappa(\Omega_\mathscr{D})) d(\hat{\mathbf{z}}, \partial\Omega_\mathscr{D}) - 1 \rceil) \subset \mathcal{F}_\kappa(\Omega_\mathscr{D}),$$

where $\lceil s \rceil$ denotes the smallest integer bigger than $s \in \mathbb{R}$.

**Theorem 5.16.** *Let $W \subset \mathbb{C}$ be open and let $g : W \to \mathbb{C}$ be a locally injective holomorphic function. Let $D \subset \mathbb{C}$ be a simply connected bounded domain with $\overline{D} \subset W$. Assume, for convenience, that $0 \in D$.*

*Let $d, \kappa \in \mathbb{N}$ be integers with $d \geq 2$, let $J_0 \subset \{1, \dots, d\}$ contain at least two indices, and let $B, C_{J_0} > 0$ and $0 < C < \pi/2$ be real constants. Let $\varepsilon_n \in (0, \infty)$ be a sequence of positive numbers such that $\varepsilon_n \to 0$ for $n \to \infty$.*

*For $n \in \mathbb{N}$ let $\mathscr{D}_n$ be a b-quad-graph with associated graphs $G_n$ and $G_n^*$ constructed from white and black vertices. We assume that $\mathscr{D}_n$ is simply connected and that there is a quasicrystallic rhombic embedding of $\mathscr{D}_n$ in $D$ with edge lengths $\varepsilon_n$ and dimension $d_n \leq d$. The directions of the edges are elements of the set $\{\pm a_1^{(n)}, \dots, \pm a_{d_n}^{(n)}\} \subset \mathbb{S}^1$ such that each two of the vectors of $\{a_1^{(n)}, \dots, a_{d_n}^{(n)}\}$ are linearly independent. The possible angles are uniformly bounded, that is for all $n \in \mathbb{N}$ the scalar product $\langle a_i^{(n)}, a_j^{(n)} \rangle$ is strictly bounded away from $\pm 1$*

$$|\langle a_i^{(n)}, a_j^{(n)} \rangle| \leq \cos(\pi/2 + C) < 1 \tag{5.32}$$

*for all $1 \leq i < j \leq d_n$. Consequently, the intersection angles $\alpha_n$ are uniformly bounded in the sense that for all $n \in \mathbb{N}$ and all faces $f \in F(\mathscr{D}_n)$*

$$|\alpha_n(f) - \pi/2| \leq C \tag{5.33}$$

*holds. Moreover, we suppose that $J_0 \subset \{1, \dots, d_n\}$ for all $n \in \mathbb{N}$ and*

$$C_{J_0}(\mathcal{F}_\kappa(\Omega_{\mathscr{D}_n})) \geq C_{J_0} > 0.$$

*Assume further that any point $x \in \overline{D}$ which is not contained in any of the faces of $\mathscr{D}_n$ has a distance less than $B\varepsilon_n$ to the nearest white vertex of $\mathscr{D}_n$ and to the boundary $\partial D$.*

*Construct the corresponding isoradial circle pattern for the rhombic embedding $\mathscr{D}_n$. Denote by $R_n = \varepsilon_n$ and $\phi_n$ the radius and the angle function of the this circle pattern. Define another radius function on $V(G_n)$ as follows. At boundary vertices $z \in V_\partial(G_n)$ set*

$$r_n(z) = R_n(z)\, |g'(z)|\,. \tag{5.34}$$

*Using Theorem 2.10, we can extend $r_n$ to a solution of the Dirichlet problem on $G_n$. Let $z_0 \in V(G_n)$ be such that the disk bounded by $C_{z_0}$ contains $0$ and let $e = [z_0, v_0] \in E(\mathscr{D}_n)$ be one of the edges incident to $z_0$ such that $\phi_n(\vec{e}) \in [0, 2\pi)$ is minimal.*

*Let $\varphi_n$ be the angle function corresponding to $r_n$ that satisfies*

$$\varphi_n(\vec{e}) = \arg(g'(v_0)) + \phi_n(\vec{e}). \tag{5.35}$$

*Let $\mathscr{C}_n$ be the planar circle pattern with radius function $r_n$ and angle function $\varphi_n$. Suppose that $\mathscr{C}_n$ is normalized by a translation such that $p_n(v_0) = g(v_0)$, where $p_n(v)$ denotes the intersection point corresponding to $v \in V(G_n^*)$. For $z \in D$ set*

$$g_n(z) = p_n(w) \quad \text{and} \quad q_n(z) = \frac{r_n(v)}{R_n(v)} e^{i(\varphi_n(\overrightarrow{vw}) - \phi_n(\overrightarrow{vw}))},$$

*where $w$ is a vertex of $V(G_n^*)$ closest to $z$ and $v$ is a vertex of $V(G_n)$ closest to $z$ such that $[v, w] \in E(\mathscr{D}_n)$.*

*Then $q_n \to g'$ and $g_n \to g$ in $C^\infty(D)$ as $n \to \infty$.*

Simple examples of sequences of quasicrystallic circle patterns for this theorem are subgraphs of suitably scaled infinite rhombic embeddings which have been considered in Section 3.5. Simply connected parts of these rhombic embeddings which are large enough satisfy the conditions $C_J(\mathcal{F}_\kappa(\Omega_{\mathscr{D}_n})) > C_0 > 0$ for all subsets $J \subset \{1, \dots, d\}$, where the constant $C_0$ only depends on the construction parameters. This is a consequence of the modified construction explained in Example 3.4. The simplest examples are subgraphs of suitably scaled infinite square grid or hexagonal circle patterns with fixed intersection angles $\alpha_1 > 0$ and $\alpha_2 = \pi - \alpha_1 > 0$ or $\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_3 = \pi - \alpha_1 - \alpha_2 > 0$ respectively; see Examples 2.3. For all these examples we can define $\mathscr{D}_n$ in the following way. Let $\mathscr{C}$ be an embedded isoradial circle pattern with radius 1 which fills the whole plane $\mathbb{C}$. Let $\mathscr{D}$ be the corresponding infinite b-quad-graph with associated graphs $G$ and $G^*$. Let $\varepsilon_n \in (0, \infty)$

be sequence of positive numbers such that $\varepsilon_n \to 0$ for $n \to \infty$. For $n \in \mathbb{N}$ choose $\mathscr{D}_n$ to be the b-quad-graph with associated graphs $G_n$ and $G_n^*$ constructed from the circles of the scaled circle pattern $\varepsilon_n \mathscr{C}$ whose centers lie in the domain $D$ and such that $\mathscr{D}_n$ is simply connected.

**Remark 5.17.** Similarly as for Theorem 5.1, the proof shows that we have in fact a priori bounds in $O(\varepsilon_n)$ on compact subset of $D$ for the difference of directional derivatives of $\log g'$ (and thus of $g'$ and $g$) and corresponding discrete partial derivatives of $\log \frac{r_n}{\varepsilon_n} + i(\varphi_n - \phi_n)$ (and thus $q_n$ and $g_n$ respectively). The uniform convergence on compact subsets of some discrete partial derivatives of all orders is called *convergence in* $C^\infty(D)$. The constant in the estimation depends on the order of the partial derivatives and on the compact subset.

Furthermore, for the 'very regular' case where the isoradial circle patterns are subpatterns of suitably scaled infinite circle patterns with square grid or hexagonal combinatorics and $\Omega_{\mathscr{D}_n} \subset \mathbb{Z}^2$ or $\Omega_{\mathscr{D}_n} \subset \mathbb{Z}^3$ respectively, the a priori bounds on the above partial derivatives have order $O(\varepsilon_n^2)$ on compact subset of $D$. This is due to the improved estimation explained in Remark 5.4.

**Remark 5.18.** There is an analogous version of Theorem 5.16 of $C^\infty$-convergence for quasicrystallic circle patterns with Neumann boundary conditions.

*Outline of the proof*

For the circle patterns $\mathscr{C}_n$ define the comparison function $w_n$ according to (3.11) and extend it to $\mathcal{F}_\kappa(\Omega_{\mathscr{D}_n})$ as explained in Section 3.4. This extension is again denoted by $w_n$. The restriction of $w_n$ to white vertices is also called *extended radius function* and denoted by $r_n$ as the original radius function for $\mathscr{C}_n$. We also extend $|g'|$ to $\mathcal{F}_\kappa(\Omega_{\mathscr{D}_n})$ by defining $|g'(\mathbf{z})|$ to be the value $|g'(z)|$ at the projection $z$ of $\mathbf{z} \in \mathbb{Z}^d$ onto the plane of the rhombic embedding $\mathscr{D}_n$. If $n$ is big enough which will be assumed in the following, then $z \in D$ or $z \in W \setminus D$ and the distance of $z$ to $D$ is bounded independently of $n$.

The main idea of our proof is to study suitable discrete partial derivatives of $\log(r_n/R_n)$ and $\log|g'|$, where $R_n \equiv \varepsilon_n$. As a first step, we show a uniform estimation of order $\mathcal{O}(\varepsilon_n)$ on the difference of the $\log(r_n/R_n)$ and $\log|g'|$. Next, we estimate the partial derivatives of all orders of this difference, but using only directions associated to the indices in $J_0$. The constant $C_{J_0}$ is useful to define regions of $\mathcal{F}_\kappa(\Omega_{\mathscr{D}_n})$ where these estimations hold. The proofs use calculations of the proof of Lemma 5.3, a scaled version of the linear Laplacian defined in (3.2), and the Regularity Lemma 3.19 which provides an estimate on the first partial derivatives using the norms of the function and of its Laplacian.

Thus all discrete partial derivatives of $r_n/R_n$ converge to the corresponding smooth directional derivatives of $|g'|$ with errors of order $\mathcal{O}(\varepsilon_n)$ uniformly on compact subsets of $D$. Using the angle function $\varphi_n$ and its connection to the radius function $r_n$, we finally obtain that $q_n$ and $g_n$ converge to $g'$ and $g$ in $C^\infty$.

In the following, we assume that the assumptions of Theorem 5.16 hold.

### 5.3.1 *Estimations on the partial derivatives of* $\log(r_n/\varepsilon_n) - \log|g'|$

Denote $t_n = \log(r_n/R_n) = \log(r_n/\varepsilon_n)$ and $h_n = \log|g'|$. Using the above extension, these functions are defined on all white vertices of $\mathcal{F}_\kappa(\Omega_{\mathscr{D}_n})$.

**Lemma 5.19.** *The estimation*

$$h_n(\mathbf{z}) - t_n(\mathbf{z}) = \mathcal{O}(\varepsilon_n).$$

*holds for all white vertices* $\mathbf{z} \in V_w(\mathcal{F}_\kappa(\Omega_{\mathscr{D}_n}))$. *The constant in the* $\mathcal{O}$*-notation may depend on* $\kappa$ *and on the dimension* $d_n$ *of* $\mathscr{D}_n$.

*Proof.* Let $\mathbf{z} \in V_w(\mathcal{F}(\Omega_{\mathscr{D}_n}))$ be a white vertex. By definition of $\mathcal{F}(\Omega_{\mathscr{D}_n})$ there is a combinatorial surface $\Omega'(\mathbf{z})$ containing $\mathbf{z}$ with the same boundary curve as $\Omega_{\mathscr{D}_n}$. Lemma 3.21 implies that we can define a circle pattern $\mathscr{C}'$ using the values of $w_n$ on $\Omega_{\mathscr{D}_n}$. Now the claim follows from Lemma 5.3.

For $\mathbf{z} \in V_w(\mathcal{F}_\kappa(\Omega_{\mathscr{D}_n})) \setminus V(\mathcal{F}(\Omega_{\mathscr{D}_n}))$ observe that equation (3.12) may be used to extend the estimation of $h_n - t_n$. Each steps adds an error of order $\varepsilon_n$, therefore the final constant depends on $\kappa$ and on $d_n$. ☐

Let $M > 0$. Denote by $\Omega_{\mathscr{D}_n}^M$ the part of $\Omega_{\mathscr{D}_n}$ with vertices of combinatorial distance bigger than $M$ to the boundary. Let $K$ be a compact subset of $D$. Consider the set of vertices $\hat{\mathbf{z}} \in V(\Omega_{\mathscr{D}_n}^M)$ whose corresponding vertices $z \in V(\mathscr{D}_n)$ lie in $K$. Let $J \subset \{1, \ldots, d\}$ contain at least two different indices. In order to define partial derivatives in the directions $\pm\mathbf{e}_{j_1} \pm \mathbf{e}_{j_2}$, where $j_1, j_2 \in J$ and $j_1 \neq j_2$, we attach a ball $U_J(\hat{\mathbf{z}}, M)$ at each of these points:

$$U_J(K, M, \Omega_{\mathscr{D}_n}) = \bigcup_{\hat{\mathbf{z}} \in V(\Omega_{\mathscr{D}_n}^M) \text{ with } z \in K} U_J(\hat{\mathbf{z}}, M). \tag{5.36}$$

Note that if $M \leq B_J(\hat{\mathbf{z}})$, for example if $M \leq C_J(\mathcal{F}_\kappa(\Omega_{\mathscr{D}_n}))d(\hat{\mathbf{z}}, \partial\Omega_{\mathscr{D}_n}) - 1$ for all $\hat{\mathbf{z}} \in V(\Omega_{\mathscr{D}_n}^M)$ with $z \in K$, then $U_J(K, M, \Omega_{\mathscr{D}_n}) \subset \mathcal{F}_\kappa(\Omega_{\mathscr{D}_n})$.

Next, we specify the partial derivatives. Consider all possible different vectors $\pm\mathbf{e}_{j_1} \pm \mathbf{e}_{j_2}$ for $0 \leq j_1, j_2 \leq d$ such that $\mathbf{e}_{j_1}$ and $\mathbf{e}_{j_2}$ are not collinear. Let $\mathbf{v}_1, \ldots, \mathbf{v}_{2d_n(d_n-1)}$ be an enumeration of these vectors and denote by $v_1, \ldots v_{2d_n(d_n-1)}$ the corresponding enumeration of the directions $\pm a_{j_1}^{(n)} \pm a_{j_2}^{(n)}$ for $0 \leq j_1, j_2 \leq d$. Denote by $\mathbf{V}_n = \{\mathbf{v}_1, \ldots, \mathbf{v}_{2d_n(d_n-1)}\}$ the set of all possible directions for partial derivatives.

**Definition 5.20.** For any function $h$ on white vertices in $\mathbb{C}$ and/or in $\mathbb{Z}^d$ define *discrete partial derivatives* in direction $\mathbf{v}_i$ or $v_i$ by

$$\partial_{\mathbf{v}_i} h(\mathbf{z}) = \frac{h(\mathbf{z} + \mathbf{v}_i) - h(\mathbf{z})}{\varepsilon_n |v_i|} \qquad \text{and} \qquad \partial_{v_i} h(\hat{z}) = \frac{h(z + \varepsilon_n v_i) - h(z)}{\varepsilon_n |v_i|}$$

respectively. Furthermore, we call a direction $\mathbf{v}_i$ or $v_i$ to be *contained in* $\Omega_{\mathscr{D}_n}$ or $\mathscr{D}_n$ at a vertex $\hat{\mathbf{z}}$ or $z$ respectively if there is a two-dimensional facet of $\Omega_{\mathscr{D}_n}$ incident to $\hat{\mathbf{z}}$ whose diagonal incident to $\hat{\mathbf{z}}$ is parallel to $\mathbf{v}_i$, that is $\{\hat{\mathbf{z}} + \lambda\mathbf{v}_i : \lambda \in [0, 1]\} \subset \Omega_{\mathscr{D}_n}$.

Corresponding to these partial derivatives we define a Laplacian which is a scaled version of (3.2). For a function $\eta$ and an interior vertex $z$ with incident vertices $z_1, \ldots, z_L$ define the *discrete Laplacian* by

$$\Delta^{\varepsilon_n}\eta(z) := \Delta^{\varepsilon_n}_{\mathbf{v}_{\mu_1}, \ldots, \mathbf{v}_{\mu_L}}\eta(z) := \frac{1}{\varepsilon_n^2}\sum_{j=1}^{L} 2f'_{\alpha_n([z, z_j])}(0)(\eta(z_j) - \eta(z)). \tag{5.37}$$

Here $v_{\mu_j} = v([z, z_j]) = (z_j - z)/\varepsilon_n$ and the notation $\Delta^{\varepsilon_n}_{\mathbf{v}_{\mu_1}, \ldots, \mathbf{v}_{\mu_L}}$ emphasizes the dependence of the Laplacian on the directions $v([z, z_j])$ of the edges $[z, z_j] \in E(G_n)$ for $j = 1, \ldots, L$.

Let $z_0 \in V_{int}(G_n)$ be an interior vertex and let $\mathbf{v}_{\mu_1}, \ldots, \mathbf{v}_{\mu_L} \in \mathbf{V}_n$ be the directions which correspond to the directions of the edges of $G_n$ incident to $z_0$. Let $\mathbf{z}_1 \in \mathcal{F}_\kappa(\Omega_{\mathscr{D}_n})$ be a vertex such that $\mathbf{z}_1 + \mathbf{v}_{\mu_i} \in \mathcal{F}_\kappa(\Omega_{\mathscr{D}_n})$ for all $i = 1, \ldots, L$. Our next aim is to study $\Delta^{\varepsilon_n}_{\mathbf{v}_{\mu_1}, \ldots, \mathbf{v}_{\mu_L}} t_n(\mathbf{z}_1)$. For this purpose we may assume that we have translated the facets of $\Omega_{\mathscr{D}_n}$ incident to $\mathbf{z}_0$ to $\mathbf{z}_1$, that is we consider the (very small) monotone surface consisting of the two-dimensional facets incident to $\mathbf{z}_1$ which contain a diagonal $\{\mathbf{z}_1 + \lambda\mathbf{v}_{\mu_i} : \lambda \in [0, 1]\}$ for $i = 1, \ldots, L$. Using the extension of the comparison function $w_n$, the closed chain of this two-dimensional facets incident to $\mathbf{z}_1$ is mapped to a closed chain of kites. Thus we have

$$\sum_{l=1}^{L} 2f_{\alpha_{\mu_l}}(t_n(\mathbf{z}_1 + \mathbf{v}_{\mu_l}) - t_n(\mathbf{z}_1)) \in 2\pi\mathbb{N},$$

where $\alpha_{\mu_l}$ denotes the labelling of the two-dimensional facet containing $\mathbf{z}_1$ and $\mathbf{z}_1 + \mathbf{v}_{\mu_l}$. By assumption $\sum_{l=1}^{L} 2f_{\alpha_{\mu_l}}(0) = 2\pi$ and we know that $t_n(\mathbf{z}_1 + \mathbf{v}_{\mu_l}) - t_n(\mathbf{z}_1) = \mathcal{O}(\varepsilon_n)$ by Lemma 5.19. Since the intersection angles and thus the maximum number of neighbors are uniformly bounded, we deduce that

$$\left( \sum_{l=1}^{L} 2f_{\alpha_{\mu_l}}(t_n(\mathbf{z}_1 + \mathbf{v}_{\mu_l}) - t_n(\mathbf{z}_1)) \right) - 2\pi = 0$$

if $\varepsilon_n$ is small enough. Using a Taylor expansion of this equation about 0, we obtain

$$\begin{aligned} \Delta^{\varepsilon_n}_{\mathbf{v}_{\mu_1},\ldots,\mathbf{v}_{\mu_L}} t_n(\mathbf{z}_1) &= \frac{2}{\varepsilon_n^2} \left( -\sum_{l=1}^{L} \sum_{m=3}^{\infty} \frac{f^{(m)}_{\alpha_{\mu_l}}(0)}{m!} (t_n(\mathbf{z}_1 + \mathbf{v}_{\mu_l}) - t_n(z))^m \right) \\ &= \varepsilon_n \left( -2 \sum_{l=1}^{L} \sum_{m=3}^{\infty} \frac{f^{(m)}_{\alpha_{\mu_l}}(0)}{m!} |v_{\mu_l}|^m (\partial_{\mathbf{v}_{\mu_l}} t_n(\mathbf{z}_1))^m \varepsilon_n^{m-3} \right) \\ &=: \varepsilon_n F_{\mathbf{v}_{\mu_1},\ldots,\mathbf{v}_{\mu_L}}(\varepsilon_n, \partial_{\mathbf{v}_{\mu_1}} t_n, \ldots, \partial_{\mathbf{v}_{\mu_L}} t_n; \mathbf{z}_1), \end{aligned} \tag{5.38}$$

Note that $F_{\mathbf{v}_{\mu_1},\ldots,\mathbf{v}_{\mu_L}}$ is a $C^\infty$-function in the variables $\varepsilon_n, \partial_{\mathbf{v}_{\mu_1}} t_n, \ldots, \partial_{\mathbf{v}_{\mu_L}} t_n$. This fact will be important for the proof of Lemma 5.21 below.

For further use we define $K + d_0$ to be a compact $d_0$-neighborhood of the compact set $K \subset \mathbb{C}$, that is

$$K + d_0 = \{z \in \mathbb{C} : \mathbb{e}(K, z) \leq d_0\},$$

where $\mathbb{e}$ denotes the Euclidean distance between a point and a compact set or between two compact subsets of $\mathbb{C}$.

**Lemma 5.21.** *Let $K \subset D$ be a compact set and let $0 < d_0 < \mathbb{e}(K, \partial\mathscr{D}_n)$.*

*Let $k, n_0 \in \mathbb{N}_0$ and let $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_k} \in \bigcap_{n \geq n_0} \mathbf{V}_n$ be $k$ not necessarily different directions. Let $J \subset \{1, \ldots, d\}$ be a minimal subset of indices such that $\{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_k}\} \subset span\{\mathbf{e}_j : j \in J\}$. Let $B_0, C_0 \geq 0$ be some constants. Assume that all discrete partial derivatives using at most $k$ of the directions $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_k}$ exist on $U_0 = U_J(K + d_0, B_0, \Omega_{\mathscr{D}_n})$ and are bounded on $U_0$ by $C_0\varepsilon_n$ for all $n \geq n_0$.*

*Let $n \geq n_0$ and let $\Omega_{\hat{\mathscr{D}}_n}$ be a two-dimensional monotone combinatorial surface. Let $\hat{\mathscr{D}}_n$ be the corresponding rhombic embedding with edge lengths $\varepsilon_n$ and such that*

$$\{\hat{\mathbf{z}} \in V(\Omega_{\hat{\mathscr{D}}_n}) : z \in V(\hat{\mathscr{D}}_n) \cap (K + d_0)\} \subset U_0.$$

*Let $z_0 \in V_w(\hat{\mathscr{D}}_n) \cap (K + d_0/2)$ be a white vertex such that $\mathbb{e}(z, \partial\hat{\mathscr{D}}_n) \geq d_0/2$. Let $\mathbf{v}_{i_{k+1}} \in \mathbf{V}$ be a direction contained in $\Omega_{\hat{\mathscr{D}}_n}$ at $\hat{\mathbf{z}}_0$. Then there is a constant $C_1$ depending on $K$, $D$, $g$, and on the constants $d_0$, $B_0$, $C_0$, $\kappa$, $C$, but not on $z_0$, $\mathbf{v}_{i_{k+1}}$ and $n$, such that*

$$|\partial_{\mathbf{v}_{i_{k+1}}} \partial_{\mathbf{v}_{i_k}} \cdots \partial_{\mathbf{v}_{i_1}} (t_n - h_n)(z_0)| \leq C_1 \varepsilon_n.$$

*Proof.* The proof is an application of the Regularity Lemma 3.19 for $u = \partial_{\mathbf{v}_{i_k}} \cdots \partial_{\mathbf{v}_{i_1}} (t_n - h_n)$ and the part of the rhombic embedding $\hat{\mathscr{D}}_n$ contained in $K + d_0$. Note that the edge lengths of $\hat{\mathscr{D}}_n$ are $\varepsilon_n$ and we suppose that $\mathbb{e}(z, \partial\hat{\mathscr{D}}_n) \geq d_0/2$.

By assumption we have $\|u\|_{V(\hat{\mathscr{D}}_n) \cap (K + d_0)} \leq C_0 \varepsilon_n$. As $h_n$ is a $C^\infty$-function, this implies that $\partial_{\mathbf{v}_i} \partial_{\mathbf{v}_{i_k}} \cdots \partial_{\mathbf{v}_{i_1}} t_n = \mathcal{O}(1)$ on $U_0$ for all possible directions $\mathbf{v}_i \in \mathbf{V}$ whenever this partial derivative is defined in $\mathcal{F}_\kappa(\Omega_{\hat{\mathscr{D}}_n})$.

Let $z_1 \in V_{int}(\hat{G}_n) \cap (K + d_0)$ be an interior white vertex of $\hat{\mathscr{D}}_n$ and let $\mathbf{v}_{\mu_1}, \ldots, \mathbf{v}_{\mu_L} \in \mathbf{V}_n$ be the directions corresponding to the directions of the edges of $\hat{G}_n$ incident to $z_1$. Note

$$\Delta^{\varepsilon_n}_{\mathbf{v}_{\mu_1},\ldots,\mathbf{v}_{\mu_L}} u = \Delta^{\varepsilon_n}_{\mathbf{v}_{\mu_1},\ldots,\mathbf{v}_{\mu_L}} \partial_{\mathbf{v}_{i_k}} \cdots \partial_{\mathbf{v}_{i_1}} (t_n - h_n) = \partial_{\mathbf{v}_{i_k}} \cdots \partial_{\mathbf{v}_{i_1}} \Delta^{\varepsilon_n}_{\mathbf{v}_{\mu_1},\ldots,\mathbf{v}_{\mu_L}} (t_n - h_n).$$

From the above consideration in (5.38) we know that $\Delta^{\varepsilon_n}_{\mathbf{v}_{\mu_1},\dots,\mathbf{v}_{\mu_L}} t_n = \varepsilon_n F_{\mathbf{v}_{\mu_1},\dots,\mathbf{v}_{\mu_L}}$, where $F_{\mathbf{v}_{\mu_1},\dots,\mathbf{v}_{\mu_L}}$ is a $C^\infty$-function in the variables $\varepsilon_n, \partial_{\mathbf{v}_{\mu_1}} t_n, \dots, \partial_{\mathbf{v}_{\mu_L}} t_n$. From our assumptions we know that all partial derivatives $\partial_{\mathbf{v}_{\mu_i}} \partial_{\mathbf{v}_{i_k}} \cdots \partial_{\mathbf{v}_{i_1}} t_n(z_1)$ and those containing less than $k+1$ of the derivatives $\partial_{\mathbf{v}_{\mu_i}}, \partial_{\mathbf{v}_{i_k}}, \dots, \partial_{\mathbf{v}_{i_1}}$ are defined and uniformly bounded by a constant independent of $z_1$ and $\varepsilon_n$. Thus we can conclude that

$$\left| \partial_{\mathbf{v}_{i_k}} \cdots \partial_{\mathbf{v}_{i_1}} \Delta^{\varepsilon_n}_{\mathbf{v}_{\mu_1},\dots,\mathbf{v}_{\mu_L}} t_n(z_1) \right| \leq C_{\mathbf{v}_{\mu_1},\dots,\mathbf{v}_{\mu_L}} \varepsilon_n,$$

where the constant $C_{\mathbf{v}_{\mu_1},\dots,\mathbf{v}_{\mu_L}}$ does not depend on $z_1$ and $\varepsilon_n$. As $h_n$ is a harmonic $C^\infty$-function, we obtain by similar reasonings as in the proof of Lemma 5.3 that

$$\partial_{\mathbf{v}_{i_k}} \cdots \partial_{\mathbf{v}_{i_1}} \Delta^{\varepsilon_n}_{\mathbf{v}_{\mu_1},\dots,\mathbf{v}_{\mu_L}} h_n = \mathcal{O}(\varepsilon_n).$$

As $\mathbf{V}_n$ is a finite set, we deduce that $\Delta^{\varepsilon_n} u$ is uniformly bounded on $V_{int}(\hat{G}_n) \cap (K + d_0)$ by $C_2 \varepsilon_n$ for some constant $C_2$ independent of $\varepsilon_n$. Now the Regularity Lemma 3.19 gives the claim. □

**Lemma 5.22.** *Let $K \subset D$ be a compact set and let $0 < d_1 < \mathrm{e}(K, \partial D)$. Let $n_0 \in \mathbb{N}$ be such that $K + d_1$ is covered by the rhombi of $\mathscr{D}_n$ for all $n \geq n_0$. Then there is a constant $C_1 = C_1(K, C_{J_0}(\mathcal{F}_\kappa(\Omega_{\mathscr{D}_n}))) > 0$ such that for all $z \in K + d_1$ we have*

$$C_1 \varepsilon_n^{-1} \leq C_{J_0}(\mathcal{F}_\kappa(\Omega_{\mathscr{D}_n})) \cdot d(z, \partial \mathscr{D}_n) - 1.$$

*Furthermore, let $k \in \mathbb{N}_0$ and let $\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_k} \in \mathbf{V}$ be $k$ (not necessarily different) directions such that $\mathbf{v}_{i_l} \in span\{e_j : j \in J_0\}$ for all $l = 1, \dots, k$. Then there are constants $n_0 \leq n_1(k, K) \in \mathbb{N}$ and $C(k, K) > 0$ which may depend on $k$, $K$, $d_1$, $D$, $g$, $\kappa$, $C_{J_0}(\mathcal{F}_\kappa(\Omega_{\mathscr{D}_n}))$, but not on $\varepsilon_n$, such that for all $n \geq n_1(k, K)$ we have*

$$\|\partial_{\mathbf{v}_{i_k}} \cdots \partial_{\mathbf{v}_{i_1}} (t_n - h_n)\|_{U_k} \leq C(k, K) \varepsilon_n, \tag{5.39}$$

*where $U_k = U_{J_0}(K + 2^{-k} d_1, 2^{-k} C_1 \varepsilon_n^{-1}, \Omega_{\mathscr{D}_n})$.*

*Proof.* The existence of the constant $C_1$ follows from the fact that $d(z, \partial \mathscr{D}_n)/\varepsilon_n$ is bounded from below for $z \in (K + d_1)$ since the distance $\mathrm{e}(K + d_1, \partial \mathscr{D}_n) > 0$ is positive and the angles of the rhombi are uniformly bounded.

The proof of the estimation (5.39) uses induction on the number of partial derivatives $k$. For $k = 0$ the claim has been shown in Lemma 5.19.

Let $k \in \mathbb{N}_0$ and assume that the claim is true for all $\nu \leq k$. Let $\mathbf{v}_{i_{k+1}} \in \mathbf{V}$ be a direction with $\mathbf{v}_{i_{k+1}} \in \{\pm \mathbf{e}_{j_1} \pm \mathbf{e}_{j_2}\} \subset \mathrm{span}\{\mathbf{e}_j : j \in J_0\}$. Using the induction hypotheses, we can apply Lemma 5.21 for $U_0 = U_k$, $d_0 = 2^{-k} d_1$, $\Omega_{\hat{\mathscr{D}}_n} = U_{\{j_1, j_2\}}(K + 2^{-k} d_1, 2^{-k} C_1 \varepsilon_n^{-1}, \Omega_{\mathscr{D}_n}) \subset U_k$ and the corresponding rhombic embedding $\hat{\mathscr{D}}_n$ obtained by projection, and $z_0 \in \hat{\mathscr{D}}_n \cap (K + 2^{-k-1} d_1)$. This completes the induction step and the proof. □

### 5.3.2 *Proof of $C^\infty$-convergence*

Identify $\mathbb{C}$ with $\mathbb{R}^2$ in the standard way and fix two orthogonal unit vectors $e_1, e_2$. Define discrete partial derivatives $\partial_{e_1}, \partial_{e_2}$ in these directions the using the discrete partial derivatives in two orthogonal directions $v_{i_1} = a^{(n)}_{j_1} + a^{(n)}_{j_2}$ and $v_{i_2} = a^{(n)}_{j_1} - a^{(n)}_{j_2}$ for $j_1, j_2 \in J_0$. This definition depends on the choice of $a^{(n)}_{j_1}, a^{(n)}_{j_2}$, which may be different for each $n$, but this does not affect the proof.

$$\partial_{e_j} h(z) := \langle \frac{v_{i_1}}{|v_{i_1}|}, e_j \rangle \partial_{\mathbf{v}_{i_1}} h(z) + \langle \frac{v_{i_2}}{|v_{i_2}|}, e_j \rangle \partial_{\mathbf{v}_{i_2}} h(z) \quad \text{for } j = 1, 2.$$

As the possible intersection angles are bounded and as $h_n$ is a $C^\infty$-function, we deduce that

$$\|\partial_{e_{j_k}} \cdots \partial_{e_{j_1}} h_n - \partial_{j_k} \cdots \partial_{j_1} h_n\|_K \leq C_1(k, K)\varepsilon_n$$

on every compact set $K$ for $j_k, \ldots, j_1 \in \{1, 2\}$. Here $\partial_1, \partial_2$ denote the standard partial derivatives associated to $e_1, e_2$ for smooth functions and $C_1(k, K)$ is a constant which depends only on $K$, $k$, and $g$. Lemma 5.22 implies that

$$\|\partial_{\hat{v}_{j_k}} \cdots \partial_{\hat{v}_{j_1}} t_n - \partial_{j_k} \cdots \partial_{j_1} h_n\|_{U_{J_0}(K+2^{-k}d_1, 2^{-k}C_1\varepsilon_n^{-1}, \Omega_{\mathscr{D}_n})} \leq C_2(k, K)\varepsilon_n$$

if $n$ is big enough. Using a version of Lemma 5.8 with error of order $\mathcal{O}(\varepsilon_n)$, we deduce that $t_n + i(\varphi_n - \phi_n)$ converges to $\log g'$ in $C^\infty(D)$. Now the convergence of $q_n$ and $g_n$ follows by similar arguments as in the proof of Theorem 5.1.

**Remark 5.23.** In addition to the observations in Remark 5.9, Theorem 5.16 may be generalized in the following ways.

(i) The estimations of Lemma 5.22 and in the proof of Theorem 5.16 hold (with possibly different constants and domains) for all directions $J \subset \{1, \ldots, d\}$, where $C_J(\mathscr{D}_n) > A_J > 0$ is uniformly bounded from below.

(ii) If $C_{J_0}(\mathscr{D}_n)$ (or any other $C_J(\mathscr{D}_n)$) is uniformly bounded from below only on a simply connected open subset $U \subset D$, consider as rhombic embedding $\mathscr{U}_n$ the largest simply connected part of $\mathscr{D}_n \cap U$. Then the estimations of Lemma 5.22 and in the proof of Theorem 5.16, and thus $C^\infty$-convergence of $g_n$, hold for compact subsets of $U$, using $\mathscr{U}_n$ instead of $\mathscr{D}_n$.

## 5.4 Connections to the linear theory of discrete holomorphic functions

The linear theory of discrete holomorphic functions is based on a linear discretization of the Cauchy-Riemann equations. The theory was first developed for functions on the regular square lattice $\mathbb{Z}^2$ in [40] and [34]. Duffin then developed a generalization to planar graphs with rhombic faces in [35] and Mercat extended this approach further to discrete Riemann surfaces in [54]. In [14] the authors present a connection of this theory with the class of integrable circle patterns. Such circle patterns admit an isoradial realization, that is there exists an isoradial circle pattern with the same combinatorics and the same intersection angles.

We shortly introduce the main definitions and then focus on the connection of discrete holomorphic functions and isoradial circle patterns, and on convergence results.

**Definition 5.24.** Let $\mathscr{D}$ be a b-quad-graph. Let $\nu : \vec{E}(G) \to \mathbb{C}$ be a weight function on the edges of the associated graph $G$. We define $\nu(e^*) = 1/\nu(e)$ for the dual edges $e^* \in E(G^*)$, where $e$ and $e^*$ correspond to the same face of $\mathscr{D}$. Then $g : V(\mathscr{D}) \to \mathbb{C}$ is called *discrete holomorphic* (with respect to the weights $\nu$) if for every face $f \in F(\mathscr{D})$ with vertices $z_-, v_-, z_+, v_+$ in positive orientation as in Figure 2.6 (left), there holds

$$\frac{g(v_+) - g(v_-)}{g(z_+) - g(z_-)} = i\nu([z_+, z_-]) = -\frac{1}{i\nu([v_+, v_-])}, \tag{5.40}$$

where $z_+, z_- \in V(G)$ and $v_+, v_- \in V(G^*)$. Equations (5.40) are called *discrete Cauchy-Riemann equations*. Furthermore, define a *Laplacian* (with respect to the weights $\nu$) by

$$(\Delta g)(x_0) = \sum_{[x, x_0] \in E(G)} \nu([x, x_0])(g(x) - g(x_0)) \tag{5.41}$$

for a function $g : V(G) \to \mathbb{R}$. For a function $g : V(G^*) \to \mathbb{R}$, a Laplacian can be defined in an analogous way. Then $g : V(G) \to \mathbb{R}$ or $g : V(G^*) \to \mathbb{R}$ is called *discrete harmonic* (with respect to the weights $\nu$) if $\Delta g = 0$.

### *5.4.1 Connections of discrete holomorphic functions and isoradial circle patterns*

Let $w_\varepsilon : V(\mathscr{D}) \to \mathbb{C}$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ be a family of solutions of the Hirota equation (3.12). Then $g = (w_\varepsilon^{-1} \frac{dw_\varepsilon}{d\varepsilon})_{\varepsilon=0}$ is discrete holomorphic, that is $g$ solves the discrete Cauchy-Riemann equations (5.40) for the weights $\nu([z_+, z_-]) = |v_+ - v_-|/|z_+ - z_-|$. This is summarized by the following theorem.

**Theorem 5.25** ([14, Theorem 39 (b)])**.** *The tangent space to the set of integrable circle patterns of a given combinatorics, at a point corresponding to an isoradial circle pattern, consists of discrete holomorphic functions on the corresponding embedding of the b-quad-graph, which take real values on $G$ and purely imaginary values on $G^*$.*

Following the spirit of this theorem, equation (2.15) is a nonlinear version of the linear Cauchy-Riemann equation (5.40). More precisely, assume that for the radius function $r : V(G) \to \mathbb{R}^+$ we have $\log r(z_+) - \log r(z_-) = \mathcal{O}(\varepsilon)$ for all edges $[z_+, z_-] \in E(G)$ as for example in the proof of Theorem 5.1 or of Theorem 5.16. Then using equation (2.15) for the corresponding function $\delta : V(G^*) \to \mathbb{R}$, we obtain

$$\begin{aligned}\delta(v_+) - \delta(v_-) &= 2f_{\alpha([z_+,z_-])}(\log r(z_+) - \log r(z_-)) - (\pi - \alpha)\\ &= 2f'_{\alpha([z_+,z_-])}(0)(\log r(z_+) - \log r(z_-)) + \mathcal{O}(\varepsilon^2).\end{aligned}$$

This implies

$$\frac{i\delta(v_+) - i\delta(v_-)}{\log \frac{r(z_+)}{\varepsilon} - \log \frac{r(z_-)}{\varepsilon}} = \frac{i\delta(v_+) - i\delta(v_-)}{\log r(z_+) - \log r(z_-)} = i2f'_{\alpha([z_+,z_-])}(0) + \mathcal{O}(\varepsilon) = i\frac{|v_+ - v_-|}{|z_+ - z_-|} + \mathcal{O}(\varepsilon).$$

Thus

$$g(z) = \begin{cases}\log \frac{r(z)}{\varepsilon} & \text{if } z \in V(G)\\ i\delta(z) & \text{if } z \in V(G^*)\end{cases}$$

is almost a discrete holomorphic function.

Our convergence proofs exploit this connection to the linear theory of discrete holomorphic functions by making extensive use of the (simpler) linear Laplacian which is (a scaled version of) (5.41).

### *5.4.2 Convergence results*

The convergence claim of Theorem 5.1 also holds for the (simpler) linear case with weights $\nu([z_+, z_-]) = 2f'_{\alpha([z_+,z_-])}(0) = |v_+ - v_-|/|z_+ - z_-|$ taken from the rhombic embedding. Equation (2.2) for the circle pattern has to be replaced by the linear Laplace equation $\Delta h_n = 0$ using the Laplacian of (5.41) with respect to the weights $\nu$. The boundary conditions are given by $\operatorname{Re} g'$ or $\operatorname{Im} g'$. The discrete harmonic function $h_n : V(G_n) \to \mathbb{R}$, which is the solution of the Dirichlet problem for the linear Laplace equation, gives rise to a conjugate discrete harmonic function $h_n^* : V(G_n^*) \to \mathbb{R}$ and a discrete holomorphic function $g_n : V(\mathscr{D}_n) \to \mathbb{C}$, $g_n(z) = h_n(z)$ if $z \in V(G_n)$ and $g_n(z) = ih_n^*(z)$ if $z \in V(G_n^*)$. The real and imaginary part of $g_n$ converge to $\operatorname{Re} g'$ and $\operatorname{Im} g'$ respectively and thus $g_n$, or more precisely a small modification to $h_n(z) + ih_n^*(v)$ for $z \in V(G_n)$, $v \in V(G_n^*)$ and $[z, v] \in E(\mathscr{D}_n)$ converges to $g'$. Integration gives the convergence to $g$. The proof is an easy adaption of the proof of Theorem 5.1. $C^\infty$-convergence for this linear case can also be proven by similar (but simpler) arguments as in Section 5.3.

There are also other types of convergence results. Mercat proved that if a sequence of discrete holomorphic functions converges (pointwise) to a limit function $g$, then $g$ is holomorphic [54, Theorem 3]. Matthes considered regular square lattices and orthogonal $SG$-circle patterns in [53], but his approach differs a lot from the convergence results of

Theorems 5.1 and 5.16. Instead of an elliptic problem which is the most natural way to deal with harmonic functions and associated holomorphic mappings, Matthes studied a discrete Cauchy-problem and prescribed analytic initial values corresponding to a given holomorphic function on a zig-zag line whose points lie on two parallel lines of distance $\varepsilon$. His results imply $C^\infty$-convergence of the discrete solution to the given function on a suitably square-shaped domain with error of order $\varepsilon^2$; see [53] for more details.

Chapter 6

# Convergence for isoradial circle packings

The convergence theorems for isoradial circle patterns presented in Chapter 5 can be adapted to the case of isoradial circle packings which are regular hexagonal circle packings.

Recall that given any circle packing there is an orthogonal circle pattern obtained by the following procedure. To each interstice formed by three mutually tangent circles of the packing add an orthogonally intersecting circle containing the three points of tangency. Note that the additional circles touch if the corresponding interstices share a common edge. In the case of a regular hexagonal circle packing, the additional circles all have the same radius ($1/\sqrt{3}$ times the radius of the original circles). But the main part of our proof is based directly on the isoradial circle packing.

Consider a regular hexagonal circle packing where all circles have radius 1 and which fills the whole plane $\mathbb{C}$. Let $G$ be the corresponding contact graph, that is the vertices of $G$ are the centers of circles and the edges connect vertices if the corresponding circles touch. $G$ can also be interpreted as the 1-skeleton of a cell decomposition $\mathcal{T}$ of the plane with triangular 2-cells, which correspond to three mutually tangent circles. By abuse of notation, we do not distinguish between the realization of the circle packing and the abstract graph $G$. Without loss of generality we also assume that $0 \in \mathbb{C}$ is a vertex of $\mathcal{T}$, i.e. $0 \in V(\mathcal{T}) = V(G)$.

## 6.1 Radius and angle function

Given any hexagonal circle packing with contact graph $\hat{G}$, where $\hat{G}$ is a part of $G$, the radius function $r : V(\hat{G}) \to (0, \infty)$ can be associated in an obvious way.

In order to determine a corresponding angle function, we use the fact that $\hat{G}$ corresponds to a part of the regular hexagonal circle packing. Given an edge $e = [v, w] \in E(\hat{G})$, let $\varphi(e)$ denote the oriented angle between the oriented line from the center $c(v)$ to $c(w)$ of the given packing and the corresponding oriented line in the part of the regular hexagonal circle packing. This angle is only unique up to addition of multiples of $2\pi$. But we can choose the angle such that neighboring angles only differ by less then $\pi$. If $t(e)$ is the tangency point corresponding to $e$, we can also associate the angle function to these points and denote $\varphi(t(e))$.

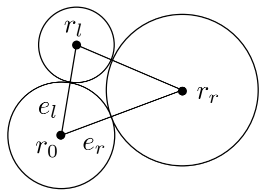



**Figure 6.1:** A triangular 2-cell of a circle packing.

The discrete version of the Cauchy-Riemann equations, corresponding to (2.11) or to (2.17) and (2.18), then reads

$$\varphi(e_l) - \varphi(e_r) = \hat{f}(\log r_l - \log r_0, \log r_r - \log r_0) - \hat{f}(0,0), \tag{6.1}$$

where

$$\hat{f}(x_1, x_2) = \arctan \sqrt{\frac{\mathrm{e}^{x_1}\mathrm{e}^{x_2}}{\mathrm{e}^{x_1} + \mathrm{e}^{x_2} + 1}}. \tag{6.2}$$

and the edges $e_l$ and $e_r$ are chosen in clockwise order around the center $c_0$ with radius $r_0$, see Figure 6.1 for the notation. Note that $\hat{f}$ is strictly increasing in both variables.

The closing condition corresponding to equation (2.2) is given by

$$\left(\sum_{k=1}^{6} \hat{f}(\log r_n(z_k) - \log r_n(z),\ \log r_n(z_{k+1}) - \log r_n(z))\right) - \pi = 0 \tag{6.3}$$

for an interior vertex $z$ and its incident vertices $z_1, \ldots, z_6$. Here and below we use the notation $z_m = z_{m \pmod 6}$ and the definition of $\hat{f}$ given in (6.2) .

## 6.2 $C^\infty$-convergence for isoradial circle packings

**Theorem 6.1.** *Let $D \subset \mathbb{C}$ be a simply connected bounded domain, and let $W \subset \mathbb{C}$ be open with $\bar{D} \subset W$. Let $g : W \to \mathbb{C}$ be a locally injective holomorphic function. Assume, for convenience, that $0 \in D$.*

*Let $\varepsilon_n \in (0, \infty)$ be a sequence of positive numbers such that $\varepsilon_n \to 0$ for $n \to \infty$. For $n \in \mathbb{N}$, scale the regular hexagonal circle packing corresponding to $G$ and $\mathcal{T}$ by $\varepsilon_n$ and consider all faces of $\varepsilon_n\mathcal{T}$ lying within the domain $D$. We choose the connected component built of theses faces which contains $0 \in V(\varepsilon_n G)$. Let $\mathcal{T}_n$ be the corresponding cell decomposition with 1-skeleton $G_n$. As above, we do not distinguish between the realization of the circle packing approximating $D$, that is the the centers of circles $z_n$ and the edges connecting corresponding points in $\mathcal{T}_n$ or $G_n$, and the abstract cell decomposition $\mathcal{T}_n$ and the graph $G_n$. Also, the index $n$ will be dropped from the notation of the vertices and edges.*

*Denote by $R_n \equiv \varepsilon_n$ the radius function of the above circle packing for $G_n$ and let $t_n(e)$ denote the touching point corresponding to the edge $e \in E(G_n)$.*

*Define another radius function as follows. At boundary vertices $z \in V_\partial(G_n)$ set*

$$r_n(z) = R_n(z)\, |g'(z)|\,. \tag{6.4}$$

*Using Theorem 2.10 for the extended orthogonal circle pattern described above, we can extend $r_n$ to a solution of the Dirichlet problem on $G_n$. Fix one of the edges $e = [0, v_0] \in E(G_n)$ incident to $0$. Let $\varphi_n$ be the angle function corresponding to $r_n$ that satisfies*

$$\varphi_n(e) = \arg\left(g'(t_n(e))\right). \tag{6.5}$$

*Let $\mathscr{C}_n$ be the planar circle pattern with radius function $r_n$ and angle function $\varphi_n$. Suppose that $\mathscr{C}_n$ is normalized by a translation such that*

$$p_n(t_n(e)) = g(t_n(e)), \tag{6.6}$$

*where $p_n(t_n(e))$ denotes the touching point which corresponds to the original touching point $t_n(e)$. For $z \in D$ set*

$$g_n(z) = p_n(w) \quad \textit{and} \quad q_n(z) = \frac{r_n(v)}{R_n(v)} e^{i\varphi_n(e)},$$

*where $w = t_n(e)$ is a touching point closest to $z$ and $v$ is a vertex of $V(G_n)$ closest to $z$ and incident to $e$.*

*Then $q_n \to g'$ and $g_n \to g$ in $C^\infty$ uniformly on compact subsets in $D$ as $n \to \infty$.*

**Remark 6.2.** As for Theorem 5.16, we prove in fact a priori bounds in $\mathcal{O}(\varepsilon_n)$ on compact subset of $D$ for the difference of partial derivatives of $g$ and corresponding discrete partial derivatives of $g_n$ and $q_n$ respectively.

The proof is very similar to the corresponding proofs of Theorem 5.1 and Theorem 5.16. There are some adaptions which are explained in the following.

**Lemma 6.3.** *For $z \in V(G_n)$ set $h_n(z) = \log|g'(z)|$ and $t_n(z) = \log(r_n(z)/R_n(z))$. Then*

$$h_n(z) - t_n(z) = \mathcal{O}(\varepsilon_n^2).$$

*Proof.* The proof is very similar to the proof of the corresponding Lemma 5.3. The first change occurs in the estimation leading to inequality (5.8). Instead of equation (2.2) we have to consider equation (6.3). Now we obtain as in Lemma 5.3

$$\begin{aligned} 0 &= \left(\sum_{j=1}^{6} \hat{f}(\log r_n(z_j) - \log r_n(z),\ \log r_n(z_{j+1}) - \log r_n(z))\right) - \pi \\ &\leq \left(\sum_{j=1}^{6} \hat{f}(x_j, x_{j+1})\right) - \pi \end{aligned} \tag{6.7}$$

Remembering $x_j = h_n(z_j) - h_n(z) - \beta|z_j|^2 + \beta|z|^2 = \mathcal{O}(\varepsilon_n)$, we can consider a Taylor expansion about 0 to make an $\mathcal{O}(\varepsilon_n^4)$-analysis. Noting that

$$\begin{aligned} &\hat{f}(0,0) = \frac{\pi}{6}, && \partial_1 \hat{f}(0,0) = \frac{1}{4\sqrt{3}} = \partial_2 \hat{f}(0,0), \\ &\partial_1^2 \hat{f}(0,0) = -\frac{1}{24\sqrt{3}} = \partial_2^2 \hat{f}(0,0), && \partial_1\partial_2 \hat{f}(0,0) = \frac{1}{12\sqrt{3}} \\ &\partial_1^3 \hat{f}(0,0) = -\frac{1}{12\sqrt{3}} = \partial_2^3 \hat{f}(0,0), && \partial_1^2\partial_2 \hat{f}(0,0) = 0 = \partial_1\partial_2^2 \hat{f}(0,0), \end{aligned}$$

we arrive at

$$\begin{aligned} 0 \leq &\frac{1}{2\sqrt{3}} \sum_{j=1}^{6} x_j - \frac{1}{24\sqrt{3}} \sum_{j=1}^{6} x_j^2 + \frac{1}{12\sqrt{3}} \sum_{j=1}^{6} x_j x_{j+1} - \frac{1}{36\sqrt{3}} \sum_{j=1}^{6} x_j^3 + \mathcal{O}(\varepsilon_n^4) \\ = &\frac{1}{2\sqrt{3}} \sum_{j=1}^{6} x_j - \frac{1}{48\sqrt{3}} \sum_{j=1}^{6} (x_j + x_{j+3})^2 + \frac{1}{12\sqrt{3}} (x_1 + x_3 + x_5)(x_2 + x_4 + x_6) \\ &- \frac{1}{36\sqrt{3}} \sum_{j=1}^{6} x_j^3 + \mathcal{O}(\varepsilon_n^4). \end{aligned}$$

We now estimate the different sums using $l = 2\varepsilon_n$ and

$$\begin{aligned} \log g'(z_j) - \log g'(z) &= a(z_j - z) + b(z_j - z)^2 + c(z_j - z)^3 + \mathcal{O}(\varepsilon_n^4), \\ x_j &= h_n(z_j) - h_n(z) - \beta|z_j|^2 + \beta|z|^2 \\ &= \mathrm{Re}((a - 2\beta\bar{z})(z_j - z) + b(z_j - z)^2 + c(z_j - z)^3) - \beta l^2 + \mathcal{O}(\varepsilon_n^4), \\ x_j^3 &= (\mathrm{Re}((a - 2\beta\bar{z})(z_j - z)))^3 + \mathcal{O}(\varepsilon_n^4). \end{aligned}$$

Using the symmetry of $G$, we may assume without loss of generality that $(z_{j+1} - z) = \mathrm{e}^{i\pi/3}(z_j - z)$ for $j = 1, \ldots, 6$. Thus we get

$$\begin{aligned} \sum_{j=1}^{6} x_j &= -4\beta\varepsilon_n^2 + \mathcal{O}(\varepsilon_n^4), \\ \sum_{j=1}^{6} (x_j + x_{j+3})^2 &= \mathcal{O}(\varepsilon_n^4) && \text{as } x_j + x_{j+3} = \mathcal{O}(\varepsilon_n^2), \\ (x_1 + x_3 + x_5)(x_2 + x_4 + x_6) &= \mathcal{O}(\varepsilon_n^4) && \text{as } x_{j-1} + x_{j+1} + x_{j+3} = \mathcal{O}(\varepsilon_n^2), \\ \sum_{j=1}^{6} x_j^3 &= \mathcal{O}(\varepsilon_n^4) && \text{as } x_j^3 + x_{j+3}^3 = \mathcal{O}(\varepsilon_n^4). \end{aligned}$$

Therefore we finally arrive at

$$0 \leq -4\beta\varepsilon_n^2 + \mathcal{O}(\varepsilon_n^4) \quad \Longleftrightarrow \quad \beta \leq \mathcal{O}(\varepsilon_n^2).$$

Now the conclusion and the rest of the proof is analogous as for Lemma 5.3. □

**Corollary 6.4.** *Let $t_n$ and $h_n$ be defined as in Lemma 6.3. Then the following estimation holds for all $n \in \mathbb{N}$ and all incident vertices $z_j$ and $z$:*

$$t_n(z_j) - h_n(z_j) - (t_n(z) - h_n(z)) = \mathcal{O}(\varepsilon_n^2).$$

**Remark 6.5.** The estimation for the radii in Lemma 6.3 can easily be extended to the orthogonal circles corresponding to faces of $\mathcal{T}_n$. The radius of such a circle is given by

$$r = \sqrt{\frac{r_1 r_2 r_3}{r_1 + r_2 + r_3}},$$

where $r_1, r_2, r_3$ are the radii of the three mutually touching circles corresponding to the chosen face. Applying the estimation of the above lemma, we get

$$\log(r_n(z)\sqrt{3}/\varepsilon_n) - \log|g'(z)| = \mathcal{O}(\varepsilon_n^2),$$

where $z$ is the center of the corresponding orthogonal circle in $\mathcal{T}_n$.

**Lemma 6.6.** *Let $e \in E(G)$ and denote by $\delta_n(e)$ the combinatorial distance in the graph $T_n$ from $e$ to $[0, v_0]$. Then*

$$\varphi_n(e) = \arg g'(t_n(e)) + \delta_n(e)\mathcal{O}(\varepsilon_n^2).$$

*Proof.* Using the Cauchy-Riemann equations (6.1) for hexagonal circle packings, the proof is analogous to the proof of the corresponding Lemma 5.8. □

Note that if $\partial D$ is smooth, then $\delta_n(e) = \mathcal{O}(\varepsilon_n^{-1})$. In general we have $\delta_n(e) = \mathcal{O}(\varepsilon_n^{-1})$ on compact subsets $K \subset D$, where the constant in the notation $\mathcal{O}(\varepsilon_n)$ may depend on $K$. But in any case, on compact subsets of $D$ we have

$$\varphi_n(e) = \arg g'(t_n(e)) + \mathcal{O}(\varepsilon_n).$$

To prove $C^\infty$-convergence, He and Schramm already made in [45] extensive use of the discrete Laplacian for the regular hexagonal circle packing

$$\Delta^\varepsilon \eta(z) := \frac{1}{\varepsilon^2}\frac{2}{3}\sum_{j=1}^{6}(\eta(z_j) - \eta(z)),$$

where $z_1, \ldots, z_6$ are incident of the interior vertex $z$. They also proved a corresponding version of the Regularity Lemma 7.19 in [45, Section 7]. Using this Regularity Lemma, the $C^\infty$-convergence of $\log(r_n/R_n)$ to $\log|g'|$ may be proved with the same methods as in the proof of Theorem 5.16. The roles of $f_\alpha$ and equation (2.2) are substituted by $\hat{f}$ and equation (6.3). Furthermore, not only $t_n$ and its discrete derivatives

$$\partial_j t_n(z) = (t_n(z + \omega^j 2\varepsilon_n) - t_n(z))/\varepsilon_n \qquad \text{for } j = 1, \ldots, 6$$

are considered, where $\omega = \mathrm{e}^{i\frac{\pi}{3}}$, but also

$$\begin{aligned} \tau_j &= (\eta_j + \eta_{j+3})/\varepsilon_n && \text{for } j = 1, 2, 3, \\ \sigma_j &= (\eta_j + \eta_{j+2} + \eta_{j+4})/\varepsilon_n && \text{for } j = 1, 2, \end{aligned}$$

which are related to second derivatives of $|g'|$. Lemma 6.3 implies that the functions $t_n$, $\partial_j t_n$, $\tau_j$, and $\sigma_j$ are uniformly bounded. Furthermore, similar calculations as in the proof of Lemma 6.3 show that

$$\begin{aligned}
\Delta^{\varepsilon_n} t_n &= \varepsilon_n^2 \hat{F}(\varepsilon_n, \partial_1 t_n, \dots, \partial_6 t_n, \tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2; z),\\
\Delta^{\varepsilon_n} \tau_j &= \hat{F}_{\tau,j}(\varepsilon_n, \partial_1 t_n, \dots, \partial_6 t_n, \tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2; z),\\
\Delta^{\varepsilon_n} \sigma_j &= \hat{F}_{\sigma,j}(\varepsilon_n, \partial_1 t_n, \dots, \partial_6 t_n, \tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2; z),
\end{aligned}$$

where $\hat{F}, \hat{F}_{\tau,j}, \hat{F}_{\sigma,j}$ are $C^\infty$-function in the variables $\varepsilon_n, \partial_1 t_n, \dots, \partial_6 t_n, \tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2$. Now a similar proof as for Lemma 5.22 shows the following estimations.

**Lemma 6.7.** *Under the assumptions of Theorem 6.1, let $K$ be a compact subset of $D$ and let $k \in \mathbb{N}$. Then there are constants $C = C(k, K) > 0$ and $n_0 = n_0(k, K) \in \mathbb{N}$ such that*

$$\|\partial_{\hat{v}_{j_k}} \cdots \partial_{\hat{v}_{j_1}} (t_n - h_n)\|_{V(G_n)\cap K} \leq C\varepsilon_n^2,$$
$$\|\partial_{\hat{v}_{j_k}} \cdots \partial_{\hat{v}_{j_1}} \tau_l\|_{V(G_n)\cap K} \leq C, \qquad \textit{and} \qquad \|\partial_{\hat{v}_{j_k}} \cdots \partial_{\hat{v}_{j_1}} \sigma_m\|_{V(G_n)\cap K} \leq C,$$

*for all $n \geq n_0$, $j_1, \dots, j_k \in \{1, \dots, 6\}$, $l \in \{1, 2, 3\}$, and $m \in \{1, 2\}$.*

The proof of Theorem 6.1 is now very similar to the proofs of Theorems 5.1 and 5.16.

Chapter 7

# Convergence of orthogonal circle patterns with square grid combinatorics for polygonal image domains

## 7.1 Convergence of orthogonal circle patterns with square grid combinatorics for kite-shaped image domains

Let $\mathcal{D}$ be a bounded convex planar domain, which is either a convex symmetric quadrilateral whose edges are straight segments, i.e. a convex kite, or a suitable stereographic projection of a symmetric convex spherical quadrilateral (i.e. a spherical convex kite whose edges are parts of great circles) lying strictly within one half-sphere.

Denote by $\mathcal{R} = \{x + iy : x, y \in [0, 1]\}$ the closed unit square in $\mathbb{R}^2 \cong \mathbb{C}$. Remember the definition of the regular orthogonal square grid embedding $SG$ in Section 3.6.1. For $n \in \mathbb{N}$, let $SG_n$ denote the embedding $SG$ scaled by the factor $1/(2n) > 0$. Denote the subgraph corresponding to all vertices of $SG_n$ which have nonempty intersection with $\mathcal{R}$ by $SG_n^{\mathcal{R}}$. By abuse of notation, we will not distinguish in the following between the abstract graph $SG_n^{\mathcal{R}}$ and its embedding into $\mathcal{R}$. In particular, we have $V(SG_n^{\mathcal{R}}) \subset \mathcal{R}$. Note that combinatorially, we may also identify $SG_n^{\mathcal{R}}$ with $SG(n, 0)$ which is a part of the regular square grid $SG$. The isoradial orthogonal circle pattern of all circles with radius $1/(2n)$ and centers in $V(SG_n^{\mathcal{R}})$ is denoted by $\mathscr{C}_n$.

The four vertices $a + ib \in V(SG_n^{\mathcal{R}})$ with $a, b \in \{0, 1\}$ at the corners of $\mathcal{R}$ will be referred to as *corner vertices* or *corner points* of $\mathcal{R}$.

Assume that for each $n \in \mathbb{N}$ there is an orthogonal circle pattern $\mathscr{C}_n^{\mathcal{D}}$ with the combinatorics of $SG_n^{\mathcal{R}}$ such that all boundary circles intersect the boundary $\partial\mathcal{D}$ orthogonally and the circles corresponding to corner circles of $SG_n^{\mathcal{R}}$ intersect two corresponding boundary lines of $\partial\mathcal{D}$ orthogonally. In the case of a convex kite, the existence of $\mathscr{C}_n^{\mathcal{D}}$ is guaranteed by Theorem 2.26. The general question on existence for the spherical case is still open, see Section 8.3 for further remarks and a special case.

Denote the radius function of $\mathscr{C}_n^{\mathcal{D}}$ by $r_n : V(SG_n^{\mathcal{R}}) \to (0, \infty)$ and the centers of circles by $c_n : V(SG_n^{\mathcal{R}}) \to \mathcal{D}_n$, where $\mathcal{D}_n \supset \mathcal{D}$ is the convex hull of the centers and the intersection points of $\mathscr{C}_n^{\mathcal{D}}$. The isomorphism of circle patterns $\mathscr{C}_n \to \mathscr{C}_n^{\mathcal{D}}$ determines an approximating mapping $g_n : \mathcal{R} \to \mathcal{D}_n$ by the following definition. Enumerate the four corner points of $\mathcal{R}$ and $\mathcal{D}$ in counterclockwise order respectively. Then fix a bijective mapping $g^{corn}$ from the corner points of $\mathcal{R}$ to the corner points of $\mathcal{D}$ respecting this order. As to each corner point of $\mathcal{D}$ there corresponds a unique center of nearest circle, denote by $g_n^{corn}$ the corresponding bijective mapping which maps the corner points of $\mathcal{R}$ to the corresponding centers of circles of $\mathscr{C}_n^{\mathcal{D}}$. Define $g_n$ to agree with $g_n^{corn}$ for the corner vertices and to map the orthogonal kites (which are in fact squares) formed by the two centers of orthogonally intersecting circles of $\mathscr{C}_n$ and by the corresponding intersection points to the corresponding orthogonal kites of $\mathscr{C}_n^{\mathcal{D}}$ using barycentric coordinates. An example of $\mathscr{C}_n$ and $\mathscr{C}_n^{\mathcal{D}}$ (and hence $g_n$) is given in Figure 1.4.

For abbreviation we denote

$$\mathcal{R}^* = \mathcal{R} \setminus \{\text{corner points}\} \qquad \text{and} \qquad \mathcal{D}^* = \mathcal{D} \setminus \{\text{corner points}\}.$$

Compact sets $K \subset \mathcal{R}^*$ or $K \subset \mathcal{D}^*$ will always mean compact subsets of $\mathbb{C}$ with respect to the standard metric which are contained in $\mathcal{R}^*$ or $\mathcal{D}^*$ respectively.

**Theorem 7.1.** *The mappings $g_n$ converge to the unique conformal homeomorphism $g : \mathcal{R} \to \mathcal{D}$ which coincides with $g^{corn}$ at the corner points. The convergence is uniform and in $C^\infty$ on compact subsets of $\mathcal{R}^*$. Furthermore, the quotients of corresponding radii, $2nr_n$, converge to $|g'|$ in $C^\infty$ uniformly on compact subsets of $\mathcal{R}^*$.*

**Remark 7.2.** If the angle of $\mathcal{D}$ at a corner point is $\pi/2$ then the convergence is also in $C^\infty$ on compact sets of $\mathcal{R}$ including the corresponding corner point (but none of those corner points where the angles of $\mathcal{D}$ at the corresponding points are different from $\pi/2$).

**Remark 7.3.** Instead of $\mathcal{R}$, we may consider the rotated region $\mathcal{R}' = \mathrm{e}^{i\pi/4}\mathcal{R}$ and the corresponding scaled regular orthogonal square grid circle patterns. Note that the corresponding combinatorics differ from $SG_n^{\mathcal{R}}$. Theorem 7.1 also holds in this case.

The proof is organized as follows. First, we derive estimations for the radius function in Sections 7.1.2 and 7.1.3 which lead to uniform convergence $g_n \to g$ in Section 7.1.4. Next, we prove pointwise convergence of the quotients of radii and then show in Section 7.1.6 that this convergence is actually $C^\infty(\mathcal{R})$. Finally, in Section 7.1.7, we deduce $C^\infty$-convergence of the sequence $(g_n)_{n\in\mathbb{N}}$.

In Section 7.2 we indicate how this theorem may be generalized for more general polygonal image domains.

In the sequel, we always assume the hypothesis of Theorem 7.1 given.

### *7.1.1 Embeddedness of the circle patterns $\mathscr{C}_n^{\mathcal{D}}$*

This section is concerned with the following simple, but crucial topological result. For lack of a reference, we include a proof.

**Lemma 7.4.** *The circle patterns $\mathscr{C}_n^{\mathcal{D}}$ are embedded for all $n$. Equivalently, the mappings $g_n$ are injective.*

*Proof.* We prove the second assertion.

By definition, $g_n$ is continuous. Using Definition 2.1 of a circle pattern and equation (2.3), we deduce that $g_n$ is an immersion. More precisely, let $D(z,(2n)^{-1})$ denote a closed disk with center $z$ and radius $(2n)^{-1}$. Then for every $z \in \mathcal{R}$ the mapping $g_n$ restricted to $D(z,(2n)^{-1})\cap\mathcal{R}$ is a homeomorphism and in particular injective. These properties imply that $g_n$ maps compact/open/connected sets to compact/open/connected sets respectively. Furthermore, $g_n$ maps each of the boundary arcs of $\mathcal{R}$ injectively to the corresponding boundary arc of $\mathcal{D}$ if this is a straight edge or to a simple polygon with edges tangent to the corresponding boundary part of $\mathcal{D}$ if this is an arc of a circle. Thus $\mathcal{D}_n \supset \mathcal{D}$ and we can easily deduce that $\mathcal{D}_n$ which is defined as the convex hull of the centers and intersection points of the circle pattern $\mathscr{C}_n^{\mathcal{D}}$ is in fact the convex hull of $g_n(\partial\mathcal{R})$.

For $\delta > 0$ define $W_\delta$ as the points of $\mathcal{R}$ with distance strictly less than $\delta$ to the boundary $\partial\mathcal{R}$. Then

$$\delta^* := \sup\{0 < \delta \le 1 : g_n|_{W_\delta} \text{ is injective}\} > 0,$$

as $g_n$ is a local homeomorphism and injective on the boundary $\partial\mathcal{R}$. To conclude, we distinguish three possible cases.

If $\delta^* > 1/2$, then $g_n$ is globally injective.

If $\delta^* = 1/2$, there is a point $z \in W_{1/2}$ such that $g_n(z) = g_n((1+i)/2)$. But this is a contradiction, as $g_n$ is a local homeomorphism and injective on $W_{1/2}$.

So assume that $\delta^* < 1/2$. Then $g_n|_{\partial W_{\delta^*}}$ is not injective and $V_{\delta^*} := \mathcal{D}_n \setminus g_n(\overline{W_{\delta^*}})$ is open and consists of at least two non-empty components (otherwise we obtain a contradiction as in the previous case). Consider a parametrization $\gamma$ of $\partial W_{\delta^*}$ which is a simple closed curve. Then the winding number of $g_n \circ \gamma$ about any point in $V_{\delta^*}$ is the same. As $g_n|_{W_{\delta^*}}$ is injective and $g_n(W_{\delta^*})$ is open and connected, this implies that the closures of different components of $V_{\delta^*}$ have at most one point in common. Now consider the open set $U_{\delta^*} := \mathcal{R} \setminus \overline{W_{\delta^*}}$. Then we deduce that $g_n(U_{\delta^*})$ is open, connected, and covers $V_{\delta^*}$. Furthermore,

$$\partial g_n(U_{\delta^*}) \subset g_n(\overline{U_{\delta^*}}) \setminus g_n(U_{\delta^*}) \subset g_n(\partial U_{\delta^*}) = g_n(\partial W_{\delta^*}),$$

and hence $g_n(U_{\delta^*}) \subset V_{\delta^*}$ which is a contradiction, since $g_n(U_{\delta^*})$ is connected and $V_{\delta^*}$ is not connected by assumption. This completes the proof. □

### *7.1.2 Estimations of $r_n$ away from the corner points*

The proof of Theorem 7.1 is based on properties of the radius function $r_n$. Therefore, we first exploit the estimations of Section 2.3 for the given circle patterns $\mathscr{C}_n$ and $\mathscr{C}_n^{\mathcal{D}}$.

**Lemma 7.5.** *As $n \to \infty$ we have $r_n \to 0$ uniformly.*

*Proof.* Due to the given combinatorics, to each disk in $\mathscr{C}_n^{\mathcal{D}}$ – except possibly the corner disks and its neighbors – there is a chain of circles of combinatorial length $\geq n$ for $n \geq 3$ which does not intersect the given disk. Thus the claim follows from the Length-Area Lemma 2.16. For the corner disks and their neighbors, Lemma 2.16 can be adapted for chains with minimal distance of the boundary centers $c$ within the chain $\geq A \cdot \text{diameter}(c)$ for some suitable constant $A$. The constant $A$ depends on the angle at the corner point. Now the claim follows again from the Length-Area Lemma 2.16 with suitable scaling. □

Using Lemma 2.15, we prove an estimation from above for the radii of all vertices except the corner vertices, which are considered in the next section. Our argumentation needs the following observation.

**Remark 7.6** (Continuation of circle patterns across boundary arcs)**.** Let $\mathcal{A}$ be one of the boundary arcs of $\mathcal{R}$ or $\mathcal{D}$ (i.e. an arc of a circle or a part of a straight line). Denote by $M_{\mathcal{A}}$ the mirror reflection in $\mathcal{A}$ and by $M_{\mathcal{A}}(\mathscr{C})$ the circle pattern obtained by the mirror reflection of the circle pattern $\mathscr{C}$ in $\mathcal{A}$. As the boundary circles of the circle patterns $\mathscr{C} = \mathscr{C}_n$ and $\mathscr{C} = \mathscr{C}_n^{\mathcal{D}}$ intersect the corresponding boundary arcs orthogonally, the union $\mathscr{C} \cup M_{\mathcal{A}}(\mathscr{C})$ is also an embedded orthogonal $SG$-circle pattern.

**Proposition 7.7.** *Let $\mathcal{K} \subset \mathcal{R}^*$ be a compact set. Then there are constants $n_0 \in \mathbb{N}$ and $C_u = C_u(\mathcal{K}) > 0$ such that*

$$r_n(v) \leq C_u/n. \tag{7.1}$$

*for all $n > n_0$ and for all vertices $v \in V(SG_n^{\mathcal{R}})$ with center $c_n(v) \in \mathcal{K}$.*

*Proof.* The proof is a direct application of Lemma 2.15 with $G = SG_n^{\mathcal{R}}$, $\alpha \equiv \pi/2$, $R$ is the distance of $\mathcal{K}$ to $\partial\mathcal{R}$, and $R^* > 0$ such that the distance between any two points of $\partial\mathcal{D}$ is smaller than $R^*/2$. Using Remark 7.6, we can also include neighborhoods of boundary vertices (except for the corner vertices). □

The analogous estimation from below of the radius function holds under the assumption that the centers $c_n(v)$ are bounded away from the corner points of $\mathcal{D}$ (uniformly on compact set of $\mathcal{R}^*$). In Corollary 7.15 we will prove that this assumption is satisfied under the hypothesis of Theorem 7.1.

**Proposition 7.8.** *Let $\mathcal{K}_{\mathcal{D}} \subset \mathcal{D}^*$ be a compact set. Then there are constants $n_0 \in \mathbb{N}$ and $C_l = C_l(\mathcal{K}_{\mathcal{D}}) > 0$ such that the following holds. Let $v \in V(SG_n^{\mathcal{R}})$ be a vertex such that the centers $c_n(v)$ of the corresponding circles of $\mathscr{C}_n^{\mathcal{D}}$ lie in $\mathcal{K}_{\mathcal{D}}$ for all $n > n_0$. Then we have for all $n > n_0$*

$$C_l/n \leq r_n(v). \tag{7.2}$$

*Proof.* The proof is analogous to the proof of Proposition 7.7 by interchanging the roles of $\mathcal{R}$ and $\mathcal{D}$. □

We will now prove one part of the assumption for Proposition 7.8.

**Proposition 7.9.** *Let $n_0 \in \mathbb{N}$ and let $v_0 \in V(SG_n^{\mathcal{R}})$ be a fixed point of $\mathcal{R}$ for all $n > n_0$. Let $\mathcal{K_D} \subset \mathcal{D}^*$ be a compact set and assume that the centers of circles $(c_n(v_0))_{n>n_0} \subset \mathcal{K_D}$ lie in $\mathcal{K_D}$. In particular, all accumulation points of $(c_n(v_0))_{n>n_0}$ have positive distance to the corner points of $\mathcal{D}$. Then there are constants $d_{\mathcal{R}} > 0$ and $C > 0$ such that for all vertices $v \in V(SG_n^{\mathcal{R}})$ with $|v - v_0| \leq d_{\mathcal{R}}$ and for all $n > n_0$ we have*

$$|c_n(v) - c_n(v_0)| \geq C|v - v_0|.$$

*Proof.* Without loss of generality, we can assume that

$$\min_{n>n_0} \operatorname{dist}(c_n(v_0), \partial\mathcal{K_D}) =: a > 0$$

by slightly enlarging the compact set $\mathcal{K_D}$ without including any of the corner points.

Let $\mathcal{K}_d(v_0)$ be a compact disk with center $v_0$ and radius $d$ such that $\mathcal{K}_d(v_0)$ does not contain any of the corner points of $\mathcal{R}$. Then by Proposition 7.7 all points of the corresponding compact neighborhood $\mathcal{K}_n^{\mathcal{D}}(c_n(v_0))$ of $c_n(v_0)$ in $\mathcal{D}_n$ (that is the closure of all disks corresponding to the disks lying in $\mathcal{K}$) have Euclidean distance less than $C_u(\mathcal{K})d$ from $c_n(v_0)$. By choosing an appropriately small radius $d_{\mathcal{R}}$ and using our assumptions on $\mathcal{K_D}$ and $c_n(v_0)$, we deduce that $\mathcal{K}_n^{\mathcal{D}}(c_n(v_0))$ is contained in $\mathcal{K_D}$. Then by Proposition 7.8 there is a constant $C_l > 0$ such that $C_l/n \leq r_n(v)$ for every vertex $v \in V(SG_n^{\mathcal{R}})$ with center $v \in \mathcal{K}_d(v_0)$ and all $n > n_0$.

Consider for fixed $n$ the center $v_0$. Successively, we can build pairwise distinct closed polygons $\mathcal{P}_m$ with vertices in the centers of the circles of $\mathscr{C}_n$ of the $m$th generation around $v_0$. More precisely, take $\mathcal{P}_0$ to be the center $v_0$. Given $\mathcal{P}_m$, construct $\mathcal{P}_{m+1}$ as follows: Consider all circles with centers in the vertices of $\mathcal{P}_m$. Then take all centers of the circles which intersect one of these circles orthogonally and are not contained in $\mathcal{P}_{m-1}$. Join two of these centers by a straight line, if the corresponding circles touch.

In the same way, we can construct corresponding polygons $\mathcal{P}_m^{\mathcal{D}}$ in $\mathcal{D}$ using $\mathcal{P}_0^{\mathcal{D}} = c_n(v_0)$. Then by construction, $\mathcal{P}_m$ (resp. $\mathcal{P}_m^{\mathcal{D}}$) separates $\mathcal{P}_{m-1}$ (resp. $\mathcal{P}_{m-1}^{\mathcal{D}}$) from $\mathcal{P}_{m+1}$ (resp. $\mathcal{P}_{m+1}^{\mathcal{D}}$). If all vertices of the polygons $\mathcal{P}_m$, $0 \leq m \leq M$, are contained in $\mathcal{K}$, then by construction and the above inequalities the distance between the polygons $\mathcal{P}_m^{\mathcal{D}}$ and $\mathcal{P}_{m+1}^{\mathcal{D}}$ is at least $C_l/n$ (as the circles intersect orthogonally).

Now consider a vertex $v \in V(SG_n^{\mathcal{R}})$ with $v \in \mathcal{P}_m$ for an $m \in \mathbb{N}$. Then by construction $m/n \leq |v - v_0| \leq \sqrt{2}m/n$. Assume we have $\mathcal{P}_m \subset \mathcal{K}$ for all $m \leq M$. To determine the distance from $c_n(v)$ to $c_n(v_0)$, we consider a line $L$ joining these points. This line intersects all polygons $\mathcal{P}_k^{\mathcal{D}}$ with $0 \leq k \leq m$ at least once. As the distance between two consecutive polygons is at least $C_l/n$, we deduce

$$|L| = |c_n(v) - c_n(v_0)| \geq C_l m/n \geq \frac{C_l}{\sqrt{2}}|v - v_0|.$$

This proves the claim. □

Using Proposition 7.7 and the same polygons as in the proof of Proposition 7.9, we immediately deduce the following estimation from above.

**Proposition 7.10.** *Let $n_0 \in \mathbb{N}$ and let $v_0 \in V(SG_n^{\mathcal{R}})$ be a fixed point of $\mathcal{R}$ for all $n > n_0$. Then there are constants $d_{\mathcal{R}} > 0$ and $C > 0$ such that for all vertices $v \in V(SG_n^{\mathcal{R}})$ with $|v - v_0| \leq d_{\mathcal{R}}$ and for all $n > n_0$ we have*

$$|c_n(v) - c_n(v_0)| \leq C|v - v_0|.$$

### 7.1.3 Estimations of $r_n$ near the corner points

In a similar manner as in the previous section, we consider the radius function $r_n$ in a neighborhood of the corner points.

Let $p$ be a corner point of $\mathcal{D}$ and denote the intersection angle of the two boundary arcs meeting in $p$ by $\alpha$. If $\alpha = \pi/2$, the circle patterns $\mathscr{C}_n$ and $\mathscr{C}_n^{\mathcal{D}}$ can be reflected in the boundary lines such that the resulting circle pattern is closed and embedded and has $\mathbb{Z}^2$ combinatorics at the corner point. Thus the reasonings of the previous section apply in this case. Therefore we will only consider the case $\alpha \neq \pi/2$. Without loss of generality we assume that the two arcs intersecting in $p$ are straight edges. This case can always be obtained by application of a suitable Möbius transformation. The inequalities for the radius function are only changed by a constant factor by such a Möbius transformation.

**Lemma 7.11.** *Let $p$ be a corner point of $\mathcal{D}$. Consider a Möbius transformation $M$, which maps $p$ to the origin and the two arcs intersecting in $p$ to straight lines. By the assumptions of Theorem 7.1 and by Lemma 7.5 the image circle pattern $M(\mathscr{C}_n^{\mathcal{D}})$ is bounded (at least for all $n > n_0$ and a suitable choice of $n_0$). Denote by $r_n^M$ the radius function of $M(\mathscr{C}_n^{\mathcal{D}})$. Then there are two constants $C_1 = C_1(M, n_0) > 0$ and $C_2 = C_2(M, n_0) > 0$, depending only on $M$ (and eventually on $n_0$), such that for all $v \in V(SG_n^{\mathcal{R}})$ the following inequality holds:*

$$C_1 r_n(v) \leq r_n^M(v) \leq C_2 r_n(v).$$

Thus we assume that $p = c_n(v_p)$, where $v_p \in V(SG_n^{\mathcal{R}})$ is one of the corner vertices of $\mathcal{R}$. In the following, we consider the circle pattern $\mathscr{C}_n^{\mathcal{D}}$ obtained by reflection in one of the straight boundary lines intersecting at $p$. Identifying corresponding boundary circles, we may assume that this circle pattern $\hat{\mathscr{C}}_n^{\mathcal{D}}$ is embedded in an appropriate cone with apex at $p$. Denote the corresponding radius function by $\hat{r}_n$.

As "standard" circle pattern for comparison, we use the regular embedded $Z^\gamma$-circle pattern $\mathscr{C}^\gamma$ with $\gamma = 2\alpha/\pi$, see Section 4.1–4.3 for more details. The radius function of $\mathscr{C}^\gamma$ will be denoted by $R$ and we use the normalization $R(0) = 1$. $\mathscr{C}^\gamma$ can be continued by reflection in the same way as $\mathscr{C}_n^{\mathcal{D}}$ to result in a circle pattern $\hat{\mathscr{C}}^\gamma$ on a cone with apex at 0. We denote by $\mathscr{C}_n^\gamma$ the part of the $Z^\gamma$-circle pattern on the lattice $\mathbb{V}_n = \{z = N + iM : N, M \in \mathbb{Z}, M \geq |N|, M + N \leq n\} \subset \mathbb{V}$ and by $\hat{\mathscr{C}}_n^\gamma$ the corresponding reflected circle pattern.

We now compare the corresponding radii of $\hat{\mathscr{C}}_n^{\mathcal{D}}$ and $\hat{\mathscr{C}}_n^\gamma$.

**Proposition 7.12.** *Assume that the disks $\hat{\mathscr{C}}_n^{\mathcal{D}}(v_p)$ and $\hat{\mathscr{C}}_n^\gamma(0)$ are both centered at the origin. Then there are disks $D_1, D_2$ resp. $D_3^{(n)}, D_4^{(n)}$, centered at the origin, such that for all $n \in \mathbb{N}$*

*(i) $D_1$ contains $\hat{\mathscr{C}}_n^{\mathcal{D}}$ and $D_3^{(n)}$ contains $\hat{\mathscr{C}}_n^\gamma$,*

*(ii) $D_2$ is entirely covered by $\hat{\mathscr{C}}_n^{\mathcal{D}}$ and $D_4^{(n)}$ is entirely covered by $\hat{\mathscr{C}}_n^\gamma$,*

*(iii) the radii $\rho_3^{(n)}$ and $\rho_4^{(n)}$ of $D_3^{(n)}$ and $D_4^{(n)}$ satisfy: $0 < k_3 n^\gamma \leq \rho_3^{(n)} \leq \rho_4^{(n)} \leq k_4 n^\gamma$, where $k_3, k_4 > 0$ are constants independent of $n$.*

*Furthermore there are two constants $C_l = C_l(\alpha, \mathcal{D}) > 0$ and $C_u = C_u(\alpha, \mathcal{D}) > 0$, depending only on the intersection angle $\alpha = \gamma\pi/2$ at $p$ and on $\mathcal{D}$, such that for all $n \in \mathbb{N}$ and for all $v$ with corresponding circles $\hat{\mathscr{C}}_n^{\mathcal{D}}(v)$ being contained in $\frac{1}{2}D_2$ and $\hat{\mathscr{C}}_n^\gamma(v)$ being contained in $\frac{1}{2}D_4^{(n)}$ the following estimations holds*

$$C_l \frac{R((v - v_p)2ne^{-i\eta})}{n^\gamma} \leq \hat{r}_n(v) \leq C_u \frac{R((v - v_p)2ne^{-i\eta})}{n^\gamma}, \tag{7.3}$$

*where $\eta$ denote the signed angle between the imaginary axis and the vector $v_1 - v_p \in \mathbb{R}^2 \cong \mathbb{C}$, where $v_1 \in V(SG_n^{\mathcal{R}})$ is the unique vertex incident to $v_p$ in $SG_n^{\mathcal{R}}$.*

*Proof.* The existence and the properties of the disks $D_1, D_2, D_3^{(n)}, D_4^{(n)}$ can be deduced from the construction of $\mathscr{C}_n^{\mathcal{D}}$ and from Lemma 4.6 (ii). Now, the proposition follows from a direct application of Lemma 2.15 for circle patterns on a cone. □

A corresponding version to Proposition 7.9 can also be proven near the corner points.

**Proposition 7.13.** *There are constants $\delta_{\mathcal{R}} > 0$ and $C > 0$ such that for all vertices $v \in V(SG_n^{\mathcal{R}})$ with $|v - v_p| \le \delta_{\mathcal{R}}$ we have*

$$|c_n(v) - c_n(v_p)| \ge C|v - v_p|^\gamma.$$

*Proof.* The proof is similar to the proof of the corresponding Proposition 7.9.

As in the proof of Proposition 7.9, for fixed $n$ we can successively build pairwise distinct polygons $\mathcal{P}_m$ with vertices in centers of the circles of $\mathscr{C}_n$ starting with $\mathcal{P}_0 = v_p$. For odd $m$ we add orthogonal segments from the center of the boundary circle to the corresponding boundary line. Then these polygons all connect the two boundary lines intersecting at p.

In the same way, we can construct corresponding polygons $\mathcal{P}_m^{\mathcal{D}}$ in $\mathcal{D}$ starting with $\mathcal{P}_0^{\mathcal{D}} = c_n(v_p)$. Then by construction, $\mathcal{P}_m$ (resp. $\mathcal{P}_m^{\mathcal{D}}$) separates $\mathcal{P}_{m-1}$ (resp. $\mathcal{P}_{m-1}^{\mathcal{D}}$) from $\mathcal{P}_{m+1}$ (resp. $\mathcal{P}_{m+1}^{\mathcal{D}}$). If all vertices of the polygons $\mathcal{P}_m$, $0 \le m \le M$, are contained in $\mathcal{K}$, then by construction, Lemma 4.6 (i) and estimations (7.3) the distance between the polygons $\mathcal{P}_m^{\mathcal{D}}$ and $\mathcal{P}_{m+1}^{\mathcal{D}}$ is at least $C_l \underline{B_\gamma}(m+1)^{\gamma-1}/n^\gamma$ for $0 < \gamma < 1$ and $C_l \underline{B_\gamma} m^{\gamma-1}/n^\gamma$ for $1 < \gamma < 2$ as the circles intersect orthogonally.

Now let $1 < \gamma < 2$ and consider a vertex $v \in V(SG_n^{\mathcal{R}})$ with $v \in \mathcal{P}_m$ for some $m \in \mathbb{N}$. Then $m/n \le |v - v_p| \le \sqrt{2}m/n$. Assume that $v \in \frac{1}{2}D_2$, where $D_2$ is a closed disk as in Proposition 7.12. As in the proof of Proposition 7.9, consider a straight line $L$ joining the two points $c_n(v)$ and $c_n(v_0)$. From the above estimation we deduce

$$\begin{aligned} |L| = |c_n(v) - c_n(v_p)| &\ge \frac{C_l \underline{B_\gamma}}{n^\gamma}(\sum_{k=1}^{m-1} k^{\gamma-1} + 1) \\ &\ge \frac{C_l \min\{\underline{B_\gamma}, 1\}}{n^\gamma}(m-1)^\gamma \ge \frac{C_l \min\{\underline{B_\gamma}, 1\}}{(2\sqrt{2})^\gamma}|v - v_p|^\gamma. \end{aligned}$$

This proves the claim for $\gamma \in (1, 2)$. For $\gamma \in (0, 1)$ the estimation is obtained analogously. □

Again, we obtain the corresponding estimation from above with a similar proof.

**Proposition 7.14.** *There are constants $\delta_{\mathcal{R}} > 0$ and $C > 0$ such that for all vertices $v \in V(SG_n^{\mathcal{R}})$ with $|v - v_p| \le \delta_{\mathcal{R}}$ we have*

$$|c_n(v) - c_n(v_p)| \le C|v - v_p|^\gamma.$$

Proposition 7.13 together with Proposition 7.9 and Lemma 7.4 imply that the hypothesis of Proposition 7.8 holds.

**Corollary 7.15.** *For all $\delta_{\mathcal{R}} > 0$ there exists a constant $\hat{d} = \hat{d}(\delta_{\mathcal{R}}) > 0$ such that*

$$\mathbb{e}(v, \partial\mathcal{R}) \ge \delta_{\mathcal{R}} \quad \Longrightarrow \quad \mathbb{e}(c_n(v), \partial\mathcal{D}) \ge \hat{d}$$

*for all $n \in \mathbb{N}$ and $v \in V(SG_n^{\mathcal{R}})$, where $\mathbb{e}$ denotes the standard Euclidean distance of a point to a piecewise differentiable curve.*

*In particular, for every interior vertex $v$ in $\mathcal{R}$, the sequence $(c_n(v))_{n>n_0}$ remains bounded away from the boundary of $\mathcal{D}$. Therefore, all accumulation points of $(c_n(v))_{n>n_0}$ lie in the interior of $\mathcal{D}$.*

Using Corollary 7.15 together with Propositions 7.7 and 7.8 we summarize

**Corollary 7.16.** *Let $\mathcal{K} \subset \mathcal{R}^*$ be a compact set. Then there are constants $C_u = C_u(\mathcal{K}) > 0$, $C_l = C_l(\mathcal{K}) > 0$, $C^u = C^u(\mathcal{K}) > 0$, $C^l = C^l(\mathcal{K}) > 0$, and $n_0 \in \mathbb{N}$ such that the following estimations hold.*

*Let $n > n_0$ and let $v, w \in V(SG_n^{\mathcal{R}})$ be two vertices with $v, w \in \mathcal{K}$. Then*

$$C_l/n \leq r_n(v) \leq C_u/n. \tag{7.4}$$

*and*

$$C^l \leq \frac{r_n(v)}{r_n(w)} \leq C^u. \tag{7.5}$$

An analogous result near the corner points can be derived combining Corollary 4.7 together with Proposition 7.12. From the existence of bounds in a neighborhood of the corner points combined with Corollary 7.16 we then obtain uniform bounds on the quotients of radii of incident vertices for the whole circle pattern $\mathscr{C}_n^{\mathcal{D}}$.

**Corollary 7.17.** *In the circle pattern $\mathscr{C}_n^{\mathcal{D}}$, all angles* $\arctan \frac{r_n(v_k)}{r_n(v_j)}$ *for incident vertices $v_k$ and $v_j$ are uniformly bounded away from 0 and $\frac{\pi}{2}$. The bounds are independent of $n$.*

### *7.1.4 Uniform convergence of $g_n$ to $g$*

In this section we prove the claim on uniform convergence of Theorem 7.1. Our reasoning is an adaption of the original proof of uniform convergence of circle packings by Rodin and Sullivan in [64].

As already mentioned above, given the circle patterns $\mathscr{C}_n$ and $\mathscr{C}_n^{\mathcal{D}}$, we can build topologically embedded (by Lemma 7.4) triangulations $T_n$ and $T_n^{\mathcal{D}}$ of $\mathcal{R}$ and $\mathcal{D}_n$ by the following construction: Take as vertices the centers of the circles and the intersection points of the circles of the patterns $\mathscr{C}_n$ and $\mathscr{C}_n^{\mathcal{D}}$ respectively. Two vertices are joint by a straight edge if the corresponding circles intersect orthogonally or if the intersection point lies on the corresponding circle. In particular, all triangles of $T_n$ and $T_n^{\mathcal{D}}$ are right-angled. Let $g_n : \mathcal{R} \to \mathcal{D}_n$ be the simplicial map determined by the correspondence of vertices and edges of $T_n$ and $T_n^{\mathcal{D}}$. We may assume, that $g_n$ is orientation preserving. Denote by $g_n^{-1} : \mathcal{D}_n \to \mathcal{R}$ the inverse mapping of $g_n$.

Corollary 7.17 implies that the quotient of the radii of intersecting circles is uniformly bounded from above and below with bounds independent of $n$. Denote the upper bound by $K$. Recall the notion of quasiconformal mappings and some of their properties. A brief review can be found on page 55. The maps $g_n : \mathcal{R} \to \mathcal{D}_n$ are $K$-quasiconformal since they map isoscele right-angled triangles to right-angled triangles of uniformly bounded distortion. (An analogous observation is true for the family $g_n^{-1}$ on $\mathcal{D}$.)

The definition of $\mathcal{D}_n$ and Lemma 7.5 imply that $\mathcal{D}_n$ converges to $\mathcal{D}$ in the sense that $\mathcal{D} = \bigcap_{n \in \mathbb{N}} \mathcal{D}_n$. Note that $g_n$ and $g_n^{-1}$ are equicontinuous using Propositions 7.9, 7.10, 7.13, and 7.14. Thus $g_n$ (resp. $g_n^{-1}$) forms a normal family and from the convergence of $\mathcal{D}_n$ to $\mathcal{D}$ it follows that every limit function $g$ of a convergent subsequence is a continuous mapping of $\mathcal{R}$ to $\mathcal{D}$ which is $K$-quasiconformal and which agrees with $g^{corn}$ on the corner vertices of $\mathcal{R}$. To see that $g$ is one-to-one onto $\mathcal{D}$, pick $w_0 \in \mathcal{D}^*$. Consider a sufficiently small compact neighborhood $\mathcal{K} \subset \mathcal{D}^*$ of $w_0$. Denote the restriction of $g_n^{-1}$ to $\mathcal{K}$ by $h_n$. Then $\mathcal{R} \supset h_n(\mathcal{K})$. Choose $n_m \to \infty$ such that $g_{n_m} \to g$ and $h_{n_m} \to h$ uniformly on compacta. Using the uniform convergence of $g_{n_m}$ near $h(w_0)$, it follows from $g_{n_m}(h_{n_m}(w_0)) = w_0$ that $g(h(w_0)) = w_0$. Thus $\mathcal{D} = g(\mathcal{R})$. Since the roles of $\mathcal{R}$ and $\mathcal{D}$ can be reversed, $g$ is one-to-one.

Lemma 3.33 shows that the simplicial mapping $g_n$ restricted to a fixed compact subset of $\mathcal{R}^*$ maps right-angled isoscele triangles to right-angled triangles which become arbitrary close to isoscele as $n \to \infty$. Therefore any limit function $g$ of the $g_n$'s will be

1-quasiconformal and thus conformal. This property is also true for boundary points except possibly for the corner points. Since the image of the corner points remains fixed for all $n$ if the intersecting arcs are straight lines or converges a corner point of $\mathcal{D}$ otherwise and as the boundary arcs of $\mathcal{R}$ are mapped onto the corresponding boundary arcs of $\mathcal{D}$, we deduce that all limit functions are equal to the unique continuous mapping of $\mathcal{R}$ onto $\mathcal{D}$, which is conformal in the interior of $\mathcal{R}$ coincides with $g^{corn}$ on the corner vertices of $\mathcal{R}$.

Note further that for a compact set in a small neighborhood of a corner point, Theorem 4.8 implies that $g_n$ maps right-angled isoscele triangles to right-angled triangles which become arbitrary close to the corresponding triangles of the $Z^\gamma$-circle pattern as $n \to \infty$, where $\gamma = 2\alpha/\pi$ and $\alpha$ is the angle of $\mathcal{D}$ at the corner point.

#### *7.1.5 Pointwise convergence of $2nr_n$ to $|g'|$*

Our next aim is to show $2nr_n \to |g'|$ pointwise in $\mathcal{R}^*$. Our argumentation follows an analogous proof by Stephenson for hexagonal circle packings in [71, Proposition 20.5].

Fix some interior point $z \in \mathcal{R}^*$ such that $z$ is a center of circle, that is $z \in V(SG_n^{\mathcal{R}})$ if $n > n_0$ is big enough. Let $1 > \delta > 0$ and denote $w = g(z)$. As $g$ is holomorphic, we can choose $t > 0$ small enough, such that the closed disk $D(z,t)$ with center $z$ and radius $t$ is contained in $\mathcal{R}$ and we have

$$D(w,(1-\delta)|g'(z)|t) \subset g(D(z,t)) \subset D(w,(1+\delta)|g'(z)|t).$$

Denote the part of the circle pattern $\mathscr{C}_n$ whose circles are contained within $D(z,t)$ by $Q_n$. Then the disks of $Q_n$ cover a disk $D(z,t-\varepsilon_n)$ where $\varepsilon_n \searrow 0$. Consider the corresponding image circle pattern $Q_n^{\mathcal{D}}$ which is a part of $\mathscr{C}_n^{\mathcal{D}}$. As the sequence $g_n$ converges uniformly on compact sets, to each $\eta > 0$ there exists an $n_1 = n_1(\eta) \in \mathbb{N}$ such that for all $n > n_1$ we have

$$D(w,(1-\delta)|g'(z)|t - \eta t) \subset Q_n^{\mathcal{D}} \subset D(w,(1+\delta)|g'(z)|t + \eta t).$$

For $n$ big enough, denote appropriately scaled radii in $Q_n$ and $Q_n^{\mathcal{D}}$ by

$$\rho_n = \frac{1}{2n(t-\varepsilon_n)}, \qquad \rho_n^* = \frac{r_n}{(1+\delta)|g'(z)|t + \eta t},$$
$$\hat\rho_n = \frac{1}{2nt}, \qquad \hat\rho_n^* = \frac{r_n}{(1-\delta)|g'(z)|t - \eta t}.$$

A careful application of Lemma 2.13 then implies

$$(\rho_n^*)_{hyp}(z) \le (\rho_n)_{hyp}(z) \quad \text{and} \quad (\hat\rho_n^*)_{hyp}(z) \ge (\hat\rho_n)_{hyp}(z). \tag{7.6}$$

As $z$ is the center of $D(z,t)$, the image $c_n(z)$ converges to $w$ and the radii $1/2n$ and $r_n(z)$ converge to 0, the hyperbolic radii in the above estimation can be calculated by $(\ln\left(\frac{1+x_1}{1-x_1}\right) - \ln\left(\frac{1+x_2}{1-x_2}\right))/2$ for some appropriate choice of $x_1, x_2 \in \mathbb{R}$, $x_1 > x_2$. Note that $x_1, x_2 \to 0$ for $n \to \infty$, so we can compare hyperbolic and Euclidean radii by

$$C_l(x_1 - x_2) \le (\ln\left(\frac{1+x_1}{1-x_1}\right) - \ln\left(\frac{1+x_2}{1-x_2}\right))/2 \le C_u(x_1 - x_2)$$

where we have $C_l, C_u \to 1$ for $x_1, x_2 \to 0$. Thus estimations (7.6) lead to

$$\frac{C_l}{C_u}\frac{(1-\delta)|g'(z)|t - \eta t}{t} \le 2nr_n(z) \le \frac{C_u}{C_l}\frac{(1+\delta)|g'(z)|t + \eta t}{t - \varepsilon_n}.$$

Corollary 7.16 implies that the quotient of radii $2nr_n(z)$ is bounded for $n \to \infty$. Choose any convergent subsequence with limit $A(z)$, then

$$(1-\delta)|g'(z)| - \eta \le A(z) \le (1+\delta)|g'(z)| + \eta.$$

As $\delta$ and $\eta$ are arbitrary, we deduce that $A(z) = |g'(z)|$. This also proves that the whole sequence $2nr_n(z)$ converges to $|g'(z)|$.

### 7.1.6 $C^\infty$-convergence of $2nr_n$ to $|g'|$

The argumentations of this section follow the proof of $C^\infty$-convergence for hexagonal circle packings by He and Schramm in [45]. Note that the reasoning as well as the notation is very similar to Section 5.3.

**Definition 7.18.** Let $n \in \mathbb{N}$. Let $\eta : V(SG_n^{\mathcal{R}}) \to \mathbb{R}$ be a function. Then the *discrete Laplacian* of $\eta$ at an interior vertex $z \in V_{int}(SG_n^{\mathcal{R}})$ with incident vertices $z_1, z_2, z_3, z_4$ in $SG_n^{\mathcal{R}}$ is defined by

$$\Delta^n \eta(z) := \frac{n^2}{2} \sum_{j=1}^{4} (\eta(z_j) - \eta(z)), \tag{7.7}$$

which is a scaled version of the Laplacian defined in (3.36) and a corresponding version of the Laplacian considered in (5.37).

Let $z \in V(SG_n^{\mathcal{R}})$ be a vertex with incident vertices $u = z + (1+i)i^k/(2n) \in V(SG_n^{\mathcal{R}})$ for $k \in \{0,1,2,3\}$. The *discrete directional derivative* of $\eta$ at $z$ in direction $(1+i)i^k$ is defined by

$$\partial_k^n \eta(z) := (\eta(z + (1+i)i^k/(2n)) - \eta(z))n\sqrt{2}. \tag{7.8}$$

Let $W \subset V(SG_n^{\mathcal{R}})$ be a subset of $V(SG_n^{\mathcal{R}})$. For further use, we also remind the definition of the norm $\|\eta\|_W := \max\{|\eta(z)| : z \in W\}$ and we define the set of interior vertices of $W$ by

$$W^{(1)} = \{z \in W : z \in V_{int}(SG_n^{\mathcal{R}}) \text{ and } z + (1+i)i^k/(2n) \in W \text{ for } k = 0,1,2,3\}.$$

For $k \in \mathbb{N}$, $k \geq 2$ define $W^{(k)} = (W^{(k-1)})^{(1)}$ inductively.

Note that for any function $\kappa : V(SG_n^{\mathcal{R}}) \to \mathbb{R}$ and for any $z \in V(SG_n^{\mathcal{R}})^{(2)}$ we have

$$\Delta^n \partial_j^n \kappa(z) = \partial_j^n \Delta^n \kappa(z).$$

Denote by $x$ and $y$ the functions, which associate the real and imaginary part to $z \in \mathbb{C}$. Then we easily see that $\Delta^n x = 0 = \Delta^n y$ and $\Delta^n x^2 = 2 = \Delta^n y^2$.

Also, we immediately have a corresponding version of the Maximum Principle 3.11 and the Regularity Lemma 3.19.

**Lemma 7.19.** *Let $W \subset V(SG_n^{\mathcal{R}})$ be a subset of $V(SG_n^{\mathcal{R}})$, let $v_0 \in W^{(1)}$ be an interior vertex of $W$, and let $\delta$ be the Euclidean distance from $v_0$ to $V(SG_n^{\mathcal{R}}) \setminus W$. Let $\eta : W \to \mathbb{R}$ be any function. Then there are two constants $C_1, C_2 > 0$ such that*

$$\delta |\partial_k^n \eta(v_0)| \leq C_1 \|\eta\|_W + C_2 \delta^2 \|\Delta^n \eta\|_{W_{int}} \tag{7.9}$$

*holds for all $k \in \{0,1,2,3\}$. The constants $C_1, C_2$ are independent of $W$, $\eta$, $v_0$, and $n$.*

**Definition 7.20.** Let $f : G \to \mathbb{C}$ be some function defined on a domain $G \subset \mathbb{C}$. For each $n \in \mathbb{N}$ let $f_n$ be defined on some set of vertices $V_n \subset SG_n$ with values in $\mathbb{C}$. Suppose that for every $z \in G$ there are $n_0 \in \mathbb{N}$ and $\delta > 0$ such that $\{v \in SG_n : |v - z| < \delta\} \subset V_n$ for all $n \geq n_0$.

If for every $z \in G$ and every $\varepsilon > 0$ there are $n_0 \in \mathbb{N}$ and $\delta > 0$ such that $|f(z) - f_n(v)| < \varepsilon$ for all $n \geq n_0$ and every $v \in SG_n$ with $|v - z| < \delta$, then we say that $f_n$ *converges to $f$ locally uniformly in $G$*.

Let $k \in \mathbb{N}$ and suppose that $f$ is $C^k$-smooth. If for every sequence $l_1, \dots, l_j \in \{0, \dots, 3\}$ with $j \leq k$ we have $\partial_{l_j}^n \cdots \partial_{l_1}^n f_n \to \partial_{l_j} \cdots \partial_{l_1} f$ locally uniformly in $G$, then we say that $f_n$ *converges to $f$ in $C^k(G)$*. If this holds for all $k \in \mathbb{N}$, then the convergence is $C^\infty(G)$.

The functions $f_n$ are said to be *uniformly bounded in $C^k(G)$* provided that for every compact $\mathcal{K} \subset G$ there is some constant $C(\mathcal{K}, k)$ such that

$$\|\partial_{l_j}^n \cdots \partial_{l_1}^n f_n\|_{\mathcal{K} \cap SG_n} < C(\mathcal{K}, k)$$

whenever $j \leq k$ and $n$ is sufficiently large. The functions $f_n$ are *uniformly bounded in* $C^\infty(G)$ if they are uniformly bounded in $C^k(G)$ for every $k \in \mathbb{N}$.

**Lemma 7.21** ([45, Lemma 2.1])**.** *Let $k \in \mathbb{N}$, $k \geq 1$. Suppose that the functions $f_n$ are uniformly bounded in $C^k(G)$. Then there is a $C^k$-function $f$ and a subsequence $n_l \to \infty$ such that $f_{n_l} \to f$ in $C^{k-1}(G)$ along that subsequence.*

In the remainder of this section, we consider the functions

$$h_n = \log(2nr_n) \qquad \text{and} \qquad p_{k,n} = \partial_k^n \log(2nr_n) \text{ for } k = 0,1,2,3 \tag{7.10}$$

and prove that they converge in $C^\infty(\mathcal{R})$.

**Proposition 7.22.** *Let $\mathcal{K} \subset \mathcal{R}^*$ be a compact set. Then there is a constant $C_0 = C_0(\mathcal{K}) > 0$ such that for all $k \in \{0,1,2,3\}$, all $n \in \mathbb{N}$ and all $v \in \mathcal{K} \cap V_{int}(SG_n^{\mathcal{R}})$ we have*

$$|p_{k,n}(v)| \leq C_0.$$

*Proof.* The claim is a direct application of Corollary 3.44. Assume that $\mathcal{K} \subset \mathcal{R}^\circ$, where $\mathcal{R}^\circ$ denotes the interior of $\mathcal{R}$. Then all vertices in $\mathcal{K}$ are surrounded by $A \cdot n$ generations in $SG_n^{\mathcal{R}}$ where $A > 0$ is some suitable constant depending on $\mathcal{K}$. Consider two incident vertices $v, w \in \mathcal{K}$. Corollary 3.44 implies that

$$1 - B/n \leq r_n(v)/r_n(w) \leq 1 + B/n$$

with some constant $B$ depending on $A$. As the quotient $r_n(v)/r_n(w)$ is uniformly bounded on $\mathcal{K}$ by Corollary 7.16, we deduce that $|\log(r_n(v)/r_n(w))| \leq D/n$ with some constant $D$. As $D$ only depends on $\mathcal{K}$, the claim follows from the definition of $p_{k,n}(v)$ in (7.10).

The above reasoning can easily be adapted for compact neighborhoods of boundary points in $\mathcal{R}^*$ using the continuation of circle patterns across boundary edges explained in Remark 7.6. $\square$

Our next aim is to show that $h_n$ is uniformly bounded in $C^\infty(\mathcal{R})$. Consider equation (2.2) at some interior vertex $z \in V_{int}(SG_n^{\mathcal{R}})$. Using a Taylor expansion of

$$\left(\sum_{j=0}^{3} f_{\frac{\pi}{2}}(\log r_n(z + \tfrac{\omega i^j}{n}) - \log r_n(z))\right) - \pi = 0$$

about 0 with $\omega = (1+i)/2$, $\log r_n(z + \omega i^j/n) - \log r_n(z) = \mathcal{O}(1/n)$ by Proposition 7.22, and $f_{\frac{\pi}{2}}(x) = \arctan \mathrm{e}^x$, we obtain

$$\begin{aligned}
0 &= \left(\sum_{j=0}^{3} \frac{\pi}{4}\right) - \pi + \sum_{j=0}^{3}\sum_{k=1}^{\infty} \frac{f_{\frac{\pi}{2}}^{(k)}(0)}{k!}(\log r_n(z + \tfrac{\omega i^j}{n}) - \log r_n(z))^k \\
&= \frac{1}{2}\sum_{j=0}^{3}(\log r_n(z + \tfrac{\omega i^j}{n}) - \log r_n(z)) \\
&\quad + \sum_{j=0}^{3}\sum_{l=1}^{\infty} \frac{f_{\frac{\pi}{2}}^{(2l+1)}(0)}{(2l+1)!}(\log r_n(z + \tfrac{\omega i^j}{n}) - \log r_n(z))^{2l+1},
\end{aligned}$$

since $f_{\frac{\pi}{2}}(x) - \frac{\pi}{4}$ is odd. Remembering the definitions of $\Delta^n$ in (7.7) and of $p_{k,n}$ in (7.10), we deduce

$$\Delta^n h_n(z) = -n^{-1}\sum_{j=0}^{3}\sum_{l=1}^{\infty} \frac{f_{\frac{\pi}{2}}^{(2l+1)}(0)}{(2l+1)!}(p_{j,n}/\sqrt{2})^{2l+1} n^{-2l+2} =: n^{-1}F(p_{0,n}, \dots, p_{3,n}, n^{-1}; z).$$

Note that $F$ is a $C^\infty$-function in the variables $p_{0,n}, \dots, p_{3,n}, n^{-1}$. This fact is important for our proof of uniform boundedness of the derivatives of $h_n$.

**Lemma 7.23.** *Let $\mathcal{K} \subset \mathcal{R}^*$ be a compact set and let $k \in \mathbb{N}$. Then there are constants $C = C(k, \mathcal{K})$ and $n_0 = n_0(k, \mathcal{K}) \in \mathbb{N}$ such that*

$$\|\partial^n_{j_k} \cdots \partial^n_{j_1} h_n\|_{V(SG^{\mathcal{R}}_n)\cap\mathcal{K}} \leq C$$

*for all $n \geq n_0$ and $j_1, \ldots, j_k \in \{0, 1, 2, 3\}$.*

*Proof.* The proof uses induction on $k$. For $k = 0$ the claim follows from Corollary 7.16 and for $k = 1$ this is Proposition 7.22. So assume that the claim is true for some $k \in \mathbb{N}$. Let $s = \partial^n_{j_k} \cdots \partial^n_{j_1} h_n$. Then

$$\Delta^n s = \Delta^n \partial^n_{j_k} \cdots \partial^n_{j_1} h_n = \partial^n_{j_k} \cdots \partial^n_{j_1} \Delta^n h_n$$

at all vertices in $V(SG^{\mathcal{R}}_n)^{(k+2)}$. From the above consideration we know that $F = n\Delta^n h_n$ is a $C^\infty$-function in the variables $n^{-1}, p_{0,n}, \ldots, p_{3,n}$. Thus we can conclude from the induction hypothesis that $s$ and $\Delta^n s = n^{-1}\partial^n_{j_k} \cdots \partial^n_{j_1} F(p_{0,n}, \ldots, p_{3,n}, n^{-1})$ are bounded on $V(SG^{\mathcal{R}}_n)\cap\mathcal{K}$ if $n$ big enough. Now, the claim follows from the Regularity Lemma 7.19. This completes the induction step and the proof. $\square$

The preceeding lemma shows that the $h_n$s are uniformly bounded in $C^\infty(\mathcal{R})$. Hence Lemma 7.21 implies the uniform convergence of $\log(2nr_n)$ in $C^\infty(\mathcal{R})$ to some smooth functions $h \in C^\infty(\mathcal{R})$ along some subsequence $n_k \to \infty$. But Section 7.1.5 shows that $h = \log|g'|$ and thus the whole sequence $\log(2nr_n)$ converges to $\log|g'|$ in $C^\infty(\mathcal{R})$. Using the reflection of the circle patterns in one of the boundary arcs as above, the generalization of this result to uniform convergence in $C^\infty$ on compact subsets of $\mathcal{R}^*$ is straightforward.

### *7.1.7 $C^\infty$-convergence of $g_n$ to $g$*

Let $\varphi_n$ be the angle function associated to the radius function $r_n$ and the circle pattern $\mathscr{C}^{\mathcal{D}}_n$, see Remark 2.21. We will assume that one of the boundary edges of $\mathcal{D}$ is a straight line and $\varphi_n(v) = \beta_0 = \arg g'|_{\mathcal{A}_0} = const.$ for all intersection points $v$ on the corresponding boundary line $\mathcal{A}_0$ of $\mathcal{R}$. Note that the combinatorial distance from any intersection point $v$ on this boundary line to any other intersection point in $SG^{\mathcal{R}}_n$ is $\mathcal{O}(n)$.

For arbitrary, but fixed $0 < \varepsilon < 1/2$ consider the compact set

$$\mathcal{K}_\varepsilon = \{z \in \mathcal{R} \ : \ |z - v_p| \geq \varepsilon \text{ for all corner points } v_p\}.$$

Then by the discrete Cauchy-Riemann equations (2.17) and (2.18), combined with Corollary 7.16 and Proposition 7.22, we obtain with a suitable constant $C = C(\mathcal{K}_\varepsilon)$

$$\begin{aligned}|\varphi_n(z + \tfrac{i}{2n}) - \varphi_n(z + \tfrac{1}{2n})| \\ = \left|\arctan\left(\frac{1}{2}\left(1 - \frac{r_n(z)}{r_n(z + \frac{1+i}{2n})}\right)\left(1 + \frac{r_n(z + \frac{1+i}{2n})}{r_n(z)}\right)\right)\right| \leq C/n\end{aligned}$$

and

$$\begin{aligned}|\varphi_n(z + \tfrac{i}{2n}) - \varphi_n(z - \tfrac{1}{2n})| \\ = \left|\arctan\left(\frac{1}{2}\left(1 - \frac{r_n(z)}{r_n(z + \frac{-1+i}{2n})}\right)\left(1 + \frac{r_n(z + \frac{-1+i}{2n})}{r_n(z)}\right)\right)\right| \leq C/n\end{aligned}$$

for all $z \in V(SG^{\mathcal{R}}_n)$ such that both sides are defined. Knowing $\varphi_n(v) = \beta_0$ on one fixed boundary line of $\mathcal{R}$, we deduce that $\varphi_n$ is uniformly bounded on $\mathcal{K}_\varepsilon$ by a constant independent of $n$.

In the previous section we have shown that $\log 2nr_n$ converges to $\log|g'|$ in $C^\infty(\mathcal{R})$. Using the discrete Cauchy-Riemann equations (2.17) and (2.18) as above, we deduce that $\varphi_n$ is uniformly bounded in $C^\infty(\mathcal{R})$. Thus $\varphi_{n_k}$ converges to some function $h$ in $C^\infty(\mathcal{R})$ for a suitable subsequence $n_k \to \infty$. As the discrete Cauchy-Riemann equations are valid for all $n$ and as $\partial_k^n \log 2nr_n = \partial_k^n \log r_n$ converges to $\partial_k \log|g'|$, where $\partial_k$ denotes the smooth derivative in direction $(1+i)i^k$, we deduce that

$$\partial_1 h = \partial_0 \log|g'| \qquad \text{and} \qquad \partial_0 h = -\partial_1 \log|g'|.$$

These are the continuous Cauchy-Riemann equations for $\log g' + c$ where $c \in \mathbb{C}$ is some constant. Therefore we have $h = \operatorname{Im}\log g'$ by the appropriate choice of $\beta_0$. Also, the whole sequence $\varphi_n$ converges to $\operatorname{Im}\log g'$ in $C^\infty(\mathcal{R})$.

Summing up, we have shown that $\log 2nr_n + i\varphi_n$ converges to $\log g'$ in $C^\infty(\mathcal{R})$ (or more precisely, converges uniformly in $C^\infty$ on compact sets of $\mathcal{R}^*$) and we can deduce that $2nr_n \mathrm{e}^{i\varphi_n}$ converges in $C^\infty(\mathcal{R})$ to $g'$. Now define discrete derivatives

$$\begin{aligned}
\partial_x^n \eta(z) &= (\eta(z+\tfrac{1}{2n}) - \eta(z))2n,\\
\partial_y^n \eta(z) &= (\eta(z+\tfrac{i}{2n}) - \eta(z))2n.
\end{aligned}$$

Then

$$\begin{aligned}
\partial_x^n g_n(z) &= (g_n(z+\tfrac{1}{2n}) - g_n(z))2n = 2nr_n(z)\mathrm{e}^{i\varphi_n(z+1/(2n))},\\
\partial_y^n g_n(z) &= (g_n(z+\tfrac{i}{2n}) - g_n(z))2n = i2nr_n(z)\mathrm{e}^{i\varphi_n(z+i/(2n))},\\
\partial_x^n g_n(v) &= (g_n(v+\tfrac{1}{2n}) - g_n(v))2n = 2nr_n(v+\tfrac{1}{2n})\mathrm{e}^{i\varphi_n(v)},\\
\partial_y^n g_n(v) &= (g_n(v+\tfrac{i}{2n}) - g_n(v))2n = i2nr_n(v+\tfrac{i}{2n})\mathrm{e}^{i\varphi_n(v)},
\end{aligned}$$

for all centers of circles $z \in V(SG_n^{\mathcal{R}})$ and all intersection points $v \in F(SG_n^{\mathcal{R}})$ corresponding to faces of $SG_n^{\mathcal{R}}$. From these equations, we easily deduce the convergence of $g_n$ to $g$ in $C^\infty(\mathcal{R})$.

## 7.2 Convergence of $SG$-circle patterns for polygonal image domains

The proof of Theorem 7.1 can be adapted for more general polygonally bounded image regions $\mathcal{D}$ like (convex) Euclidean quadrilaterals or polygons with straight boundary edges or projections of corresponding spherical polygons formed by arcs of great circles and lying strictly within one halfsphere. In order to use the same reasonings for the proof, first a conformal parametrization (i.e. a homeomorphism which is a biholomorphic map in the interior) $g : \mathcal{R} \to \mathcal{D}$ and the corresponding polygonal parameter domain $\mathcal{R}$ have to be determined. Each boundary edge of $\mathcal{R}$ has to be parallel to one of the lines $\ell_k = \{t\mathrm{e}^{i\frac{\pi}{4}k} : t \in \mathbb{R}\}$ $(k = 0,1,2,3)$ and the boundary edges and corner points of $\mathcal{R}$ are mapped by $g$ to the corresponding boundary edges and corner points of $\mathcal{D}$. Note that $\mathcal{R}$ is uniquely determined by $\mathcal{D}$ up to translation, scaling and rotation by an integer multiple of $\frac{\pi}{4}$ about the origin. Conversely, $\mathcal{D}$ is uniquely determined by $\mathcal{R}$ and the intersection angles at the corner points up to isomorphisms of $\mathbb{C}$. Therefore we can assume without loss of generality that $[0,1] \subset \mathbb{R} \subset \mathbb{C}$ is a boundary edge of $\mathcal{R}$. Then for generic domains the length of the other boundary edges of $\mathcal{R}$ are not rational, thus $\mathcal{R}$ is *not* covered *exactly* by a part of the scaled orthogonal circle pattern $SG_n$. The simplest example is a rectangle $\mathcal{R} = \mathcal{D}$ with irrational quotient of the different lengths of boundary edges. Despite the obvious simplicity of this example, the conformal parametrization $g : \mathcal{R} \to \mathcal{D}$, $g = id$ can only be approximated by $SG$-circle pattern in the sense that the approximating maps $g_n$ are not defined on the same domain as $g$. Moreover, given any part of $SG_n$ approximating $\mathcal{R}$ (for example covering a largest possible simply connected subset), the region covered by the corresponding circle pattern $\mathscr{C}_n^{\mathcal{D}}$ is in general not uniquely determined by the intersection angles at the corner points of $\mathcal{D}$. Thus, it is in general not evident that $\mathscr{C}_n^{\mathcal{D}}$ also approximates $\mathcal{D}$ (up to isomorphism of $\mathbb{C}$) and, consequently and more precisely, that the approximation mappings $g_n$ converge to the given conformal mapping $g : \mathcal{R} \to \mathcal{D}$ using an appropriate normalization of $\mathscr{C}_n^{\mathcal{D}}$.

In the following, we prove this approximation property for a convex quadrilateral $\mathcal{Q} = \mathcal{D}$ with interior angles $\alpha_1, \dots, \alpha_4 \in (0, 2\pi)$ at the corner points (and $\sum_{k=1}^{4} \alpha_k = 2\pi$). In this case

$$\mathcal{R} = \mathcal{R}_q = \{z \in \mathbb{C} : \mathrm{Re}(z) \in [0,1],\ \mathrm{Im}(z) \in [0,q]\}$$

is a rectangle for a suitable $q \in \mathbb{R}^+$. The arguments can also be used for more general polygonal domains and suitable spherical analogs.

**Lemma 7.24.** *For $n \in \mathbb{N}$ let $SG_n^{\mathcal{R}_q}$ denote the subgraph of $SG_n$ corresponding to vertices lying in $\mathcal{R}_q$. Denote by $\mathcal{Q}_n$ the domain covered by the corresponding orthogonal circle pattern $\mathscr{C}_n^{\mathcal{Q}}$. Then suitably scaled and translated versions of $\mathcal{Q}_n$ converge to $\mathcal{Q}$ with respect to the Hausdorff-metric.*

**Corollary 7.25.** *Denote by $g_n$ the mapping associated to the correspondence of the circle patterns $\mathscr{C}_n$ and $\mathscr{C}_n^{\mathcal{Q}}$ as in Theorem 7.1. Then the sequence $(g_n)_{n\in\mathbb{N}}$ converges to the given conformal parameterization $g : \mathcal{R}_q \to \mathcal{Q}$ uniformly and in $C^\infty$ on compact sets away from the corner points of $\mathcal{R}_q$.*

*Proof.* The arguments of the proof of Theorem 7.1 in Sections 7.1.1–7.1.7 can without any problem be suitably adapted for this case. ☐

*Proof of Lemma 7.24.* If $\mathcal{Q}$ is a rectangle, then the claim is obvious.

Denote the corner points of $\mathcal{Q}$ by $A, B, C, D$. Thus we may assume without loss of generality that the two affine lines $\ell_{AB}$ and $\ell_{CD}$, containing the points $A, B$ and $C, D$ respectively, intersect in a point $E \in \mathbb{C}$. Fix an orientation of $\ell_{AB}$. We may assume that the point $B$ lies between $A$ and $E$ with respect to this orientation, that is $A \le B \le E$ or $E \le B \le A$. Again, there is no loss of generality.

Note in particular, that the circle pattern $\mathscr{C}_n^{\mathcal{Q}}$ is embedded by a similar proof as for Lemma 7.4. Denote by $A_n, B_n, C_n, D_n$ the corner points of $\mathcal{Q}_n$ corresponding to $A, B, C, D$. Using a suitable affine transformation, we may assume that $A_n = A$, $D_n = D$ and $B_n \in \ell_{AB}$. Then $C_n \in \ell_{CD}$. Without loss of generality, we suppose that $d(A_n, D_n) = d(A, D) = 1$. We will prove that the Euclidean distances $d(A_n, B_n)$ and $d(D_n, C_n)$ are uniformly bounded away from 0. As they are bounded from above by $d(A, E)$ and $d(D, E)$, these distances converge at least for some subsequence to non-zero values. This implies that $\mathcal{Q}_n$ converges for this subsequence to some domain $\hat{\mathcal{Q}}$ with respect to the Hausdorff-metric. Now, the same arguments as for the proof of Theorem 7.1 apply. Thus $g_n$ converges along this subsequence to a conformal mapping $g : \mathcal{R}_q \to \hat{\mathcal{Q}}$ which maps the boundary edges of $\mathcal{R}_q$ to the corresponding boundary edges of $\hat{\mathcal{Q}}$. The normalizations of $\hat{\mathcal{Q}}$ and $\mathcal{R}_q$ and the uniqueness of $\mathcal{R}_q$ now imply that $\hat{\mathcal{Q}} = \mathcal{Q}$ and $g$ is the given conformal parametrization. As we can choose a converging subsequence from any subsequence of $(\mathcal{Q}_n)_{n\in\mathbb{N}}$, we deduce that the whole sequence converges to $\mathcal{Q}$.

Let $M \in \mathbb{N}$ and denote by $U_M$ all vertices of $V(SG_n^{\mathcal{R}_q})$ which have combinatorial distance at least $M + 1$ to any of the corner points. Note that the circle patterns $\mathscr{C}_n$ and $\mathscr{C}_n^{\mathcal{Q}}$ may both be continued across each of the straight boundary edges to embedded orthogonal circle patterns. This is due to the convexity of the domains $\mathcal{R}_q$ and $\mathcal{Q}_n$ and to the boundary conditions of the circle patterns. Thus Lemma 3.33 and Corollary 3.44 imply that there is a constant $A$ independent of $n$ and $M$ such that

$$\left| \frac{r_n(v')}{r_n(v)} - 1 \right| \le \frac{A}{M} \iff |r_n(v) - r_n(v')| \le r_n(v) \frac{A}{M}$$

for all incident vertices $v, v' \in U_M$ and all $n \in \mathbb{N}$. Using induction, we deduce that

$$|r_n(v) - r_n(\hat{v})| \leq r_n(v) \frac{A}{M} \sum_{k=0}^{d(v,\hat{v})-1} \left(1 + \frac{A}{M}\right)^k$$

for all vertices $v, \hat{v} \in U_M$, where $d(v, \hat{v})$ denotes the combinatorial distance between $v$ and $\hat{v}$. For $x \in \mathbb{R}^+$ denote by $\lfloor x \rfloor$ the largest integer with $\lfloor x \rfloor \leq x$. Let $\bar{\ell} := \max\{n, \lfloor qn \rfloor\}$ and $\underline{\ell} := \min\{n, \lfloor qn \rfloor\}$. Assume that $n$ is large enough such that $\underline{\ell} \geq 1$. Then the combinatorial distance between any two points of $V(SG_n^{\mathcal{R}_q})$ is at most $2\bar{\ell}$. Thus

$$\frac{A}{M} \sum_{k=0}^{d(v,\hat{v})-1} \left(1 + \frac{A}{M}\right)^k \leq \frac{2A\bar{\ell}}{M} \left(1 + \frac{A}{M}\right)^{2\bar{\ell}} \leq \frac{2As\bar{\ell}}{\underline{\ell}} \mathrm{e}^{2As\bar{\ell}/\underline{\ell}} =: C(s),$$

where $s := \underline{\ell}/M$. Therefore, if $s$ is uniformly bounded (with respect to $n$) then the above estimations imply

$$r_n(\hat{v}) \leq r_n(v)(1 + C(s)) \tag{7.11}$$

for all $v, \hat{v} \in U_M$. In particular, the quotients of radii are uniformly bounded on $U_M$ from above and below and the bounds are independent of $n$.

Next, we will determine $s$ and $M$. By our assumptions, there is a fixed radius $R > 0$ such that $\mathscr{C}_n^{\mathcal{Q}}$ is contained in a disk of radius $R$ about any of its corner points. Consider one of the corner points $v_{p_j} \in V(SG_n^{\mathcal{R}_\mathcal{Q}})$ and set $\gamma_j = 2\alpha_j/\pi$, where $\alpha_j$ is the angle at the corresponding corner point of $\mathcal{Q}_n$. Thus we can deduce as in Proposition 7.12 using the same notation that

$$r_n(v) \leq \bar{C}_{\gamma_j} \frac{R((v - v_p)2n\mathrm{e}^{-i\eta})}{\underline{\ell}^{\gamma}}$$

for all vertices $v \in V(SG_n^{\mathcal{R}_q})$ with combinatorial distance at most $\bar{D}_{\gamma_j}\underline{\ell}$ from $v_{p_j}$ with some constants $\bar{C}_{\gamma_j}, \bar{D}_{\gamma_j}$ independent of $n$. Furthermore, for these vertices we have

$$|c_n(v) - c_n(v_{p_j})| \leq \bar{E}_{\gamma_j} |v - v_p|^{\gamma_j} \tag{7.12}$$

by Propsition 7.14 with a constant $\bar{E}_{\gamma_j}$ independent of $n$. Without loss of generality assume that $\alpha_1$ and $\alpha_4$ are the angles at the corner points $A$ and $D$. Take

$$M = \lfloor \min\{1/(4\bar{E}_{\gamma_1})^{1/\gamma_1}, 1/(4\bar{E}_{\gamma_4})^{1/\gamma_4}, 1/4\}\underline{\ell} \rfloor \qquad \text{and} \qquad s = \underline{\ell}/M,$$

and assume that $n$ is large enough such that $M > 1$. Define $U_M$ as above and denote by $U_M(AD)$ the vertices of $U_M$ which are mapped to the straight edge connecting $A$ and $D$. By assumption, the length of this edge is 1 and by estimation (7.12) and the definition of $M$ we have $\sum_{v \in U_M(AD)} 2r_n(v) \geq 1 - 2 \cdot 1/4 = 1/2$. Using the boundary conditions on the straight edges, there is a vertex $v_1 \in U_M$ with center $c_n(v_1)$ on the line $g_{AB}$ such that $r_n(v_1) \leq d(A, B_n)/2(\underline{\ell} - 2M)$. Using estimation (7.11) we deduce

$$\frac{1}{2} \leq \sum_{v \in U_M(AD)} 2r_n(v) \leq 2r_n(v_1)(1 + C(s))(\bar{\ell} - 2M) \leq d(A, B_n)(1 + C(s)) \frac{\bar{\ell} - 2M}{\underline{\ell} - 2M}.$$

This gives a lower bound for $d(A, B_n)$ independent of $n$. As the estimation of $d(D, C_n)$ is analogous, this completes the proof. □

## 7.3 Comparison with the linear theory of discrete harmonic functions

In Section 7.1 we proved convergence of certain classes of orthogonal $SG$-circle patterns. Using these patterns we defined sequences $(2nr_n)_{n\in\mathbb{N}}$, $(\varphi_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ converging to $\text{Re}(\log g')$, $\text{Im}(\log g')$, and $g$ respectively, where $g$ is the conformal mapping of Theorem 7.1. To approximate $g$, linear discrete harmonic (and holomorphic) functions on $SG_n^{\mathcal{R}}$ in the sense of Definition 5.24 (with $\nu \equiv 1$) can also be used. See Section 5.4.2 for a brief surway of known convergence results for the linear theory. The simplest approach is to consider the (real valued) linear discrete harmonic functions $h_n$ with Dirichlet boundary conditions given by $\text{Re}(g)$ (or $\text{Im}(g)$). On compact subsets of the interior of $\mathcal{R}$, the Regularity Lemma 3.19 can be used to obtain analogous estimations as in Section 7.1.6. Thus $(h_n)_{n\in\mathbb{N}}$ converges in $C^\infty$ on any compact subset in the interior of $\mathcal{R}$ at least along a subsequence. Using the simple geometric forms of $\mathcal{R}$ and $\mathcal{D}$, $\text{Re}(g)$ is easily seen to belong to a Sobolev space $W^{1,p}(\mathcal{R})$ for some $p > 2$, that is $\text{Re}(g)$ and its first partial derivatives belong to $L^p(\mathcal{R})$. The choice of $p$ depends only on the angles of $\mathcal{D}$ at the corner points. Now a result of Ciarlet and Raviart in [27] shows that $(h_n)_{n\in\mathbb{N}}$ converges to $\text{Re}(g)$ in $L^\infty(\mathcal{R})$. Thus the whole sequence converges uniformly and in $C^\infty$ on compact subsets of the interior of $\mathcal{R}$. Now build the conjugate harmonic function $h_n^*$ on the dual lattice corresponding to $h_n$ and normalize at one fixed interior point according to the value of $\text{Im}(g)$ (resp. $\text{Re}(g)$). Hölder's Inequality 3.18 implies $|h_n(z) - h_n(z')| \leq C/n$ for neighboring vertices $z$ and $z'$ on a compact subset of the interior of $\mathcal{R}$ , where the constant $C$ only depends on $g$ and the compact subset. Using the estimations of the discrete derivatives of $h_n$ on compact subsets and the definition of the conjugate harmonic function, we deduce that $(h_n^*)_{n\in\mathbb{N}}$ converges in $C^\infty$ on any compact subset at least along a subsequence to a smooth harmonic function conjugate to $\text{Re}(g)$ (resp. $\text{Im}(g)$). Due to the normalization, this function is $\text{Im}(g)$ (resp. $\text{Re}(g)$) and thus the whole sequence $(h_n^*)_{n\in\mathbb{N}}$ converges in $C^\infty$ on any compact subset of the interior of $\mathcal{R}$. This is the corresponding convergence result to Theorem 7.1. Linear discrete harmonic functions may also be used for more general domains as considered in Section 7.2.

An important advantage of the approximation using circle patterns for the given problem is the simplicity of the boundary conditions in the case of straight edges as $\text{Im} \log g'$ then is constant on each boundary edge. Note that $g'$ is unbounded at the corner points and therefore unfavorable to use for a convergent approximation by linear discrete harmonic functions. Furthermore, the orthogonal circle patterns and the approximating functions may be continued across the boundary edges by simple reflection using the symmetry of the problem.

Chapter 8

# Discrete minimal surfaces

Minimal surfaces and solutions to Plateau's problem have been studied for a long time and constitute still interesting (model) problems. Expositions of the theory of minimal surfaces can be found in standard textbooks, as for example [30, 56]. To explicitly construct minimal surfaces is an remaining challenge. Numerical approximations of minimal surfaces have been proposed and studied by several authors.

In this chapter, we first show how to apply the theory of $S$-isothermic discrete minimal surfaces to the construction of examples, yielding in particular some triply-periodic discrete minimal surfaces and examples of solutions to Plateau's problem. See also [23] for a shortened version. In Section 8.3 we apply the convergence results of Chapter 7 and prove $C^\infty$-convergence of $S$-isothermic discrete minimal surfaces to smooth analogs. Finally, we compare this result with existing convergence results for other construction methods of discrete minimal surfaces in Section 8.4.

## 8.1 Construction of $S$-isothermic discrete minimal surfaces with special boundary conditions

In the following we explain some details of how to apply the general constructions scheme for $S$-isothermic discrete minimal surfaces of Appendix A to examples with boundary conditions specified below. We focus on the first step (finding the combinatorial parametrization and boundary conditions) of the algorithm, as this constitutes the main ingredient for the remaining steps (construction of the corresponding spherical circle pattern, the Koebe polyhedron, and finally the discrete minimal surface by dualization).

### *8.1.1 Boundary conditions*

Consider the family of all smooth minimal surfaces with boundary whose boundary curve can be divided into finitely many pieces of finite positive length. Furthermore, each of this boundary arcs

either (i) *lies within a plane which intersects the surface orthogonally.* Then this boundary arc is a curvature line and the surface may be continued smoothly across the plane by reflection in this plane. Moreover the image of this boundary arc under the Gauss map is (a part of) a great circle on the unit sphere.

or (ii) *lies on a straight line.* Then this boundary arc is an asymptotic line and the surface may be continued smoothly across this straight line by 180°-rotation about it. Again the image of this boundary arc under the Gauss map is (a part of) a great circle on the unit sphere.

The implications in (i) and (ii) are well-known properties of smooth minimal surfaces and are called *Schwarz's reflection principles*; cf. for example [30, Section 4.8].

Like a smooth minimal surface, an $S$-isothermic discrete minimal surface may also be continued by reflection in the boundary planes or by 180°-rotation about straight boundary lines. This is due to the translation of the boundary conditions for the discrete minimal surfaces into angle conditions for boundary circles. In particular, the boundary circles of the spherical circle pattern intersect the corresponding great circle orthogonally.

To construct the examples, we always assume, that we know all boundary planes and straight lines and their intersection angles (and lengths if necessary). As indicated in Appendix A, to construct a discrete minimal surface, the first step consists in determining the combinatorics of the curvature lines under the Gauss map. To cope with this task, we first try to reduce the problem by symmetry. This is especially helpful when dealing with highly symmetric triply periodic minimal surfaces.

### *8.1.2 Reduction of symmetries*

To simplify the construction of a discrete minimal surface, we only consider a *fundamental piece* of the smooth (and the discrete) minimal surface. This is a piece of the surface which is bounded by planar curvature lines and/or straight asymptotic lines (like the whole surface itself) with the following two properties.

(i) The surface is obtained from the fundamental piece by successive reflection/rotation in its boundary planes/lines and the new obtained boundary planes/lines.

(ii) There is no piece strictly contained in the fundamental piece which has property (i).

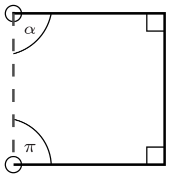
In general, the fundamental piece is not unique. In the remainder of this section, we assume that we are dealing with a fundamental piece.

### *8.1.3 Combinatorics of curvature lines*

Knowing the boundary conditions, the main step consists in finding the 'right' combinatorial data for the circle pattern. Thus we only draw general combinatorial pictures of curvature lines and indicate only sometimes intersection angles between boundary pieces. An illustrating example of the following algorithm is given in Figures 8.1 and 8.2 corresponding to the image of a fundamental piece of Schwarz's H surface under the Gauss map.

Given a fundamental piece of a smooth minimal surface, we first determine the image of the boundary pieces under the Gauss map. By our assumption explained in Section 8.1.1, each of these curves is mapped to a part of a great circle on the unit sphere $\mathbb{S}^2$. Then we extract the following ingredients for the construction (see Figure 8.1 for an example):

- a combinatorial picture of the image of the boundary pieces under the Gauss map,
- the angles between different boundary curves on $\mathbb{S}^2$,
- and eventually the lengths of the image of the boundary pieces under the Gauss map, if the angles do not uniquely determine the image of the boundary curve on $\mathbb{S}^2$ (up to rotations of the sphere). This case is more difficult; see Remark 8.1.




— curvature line
– – asymptotic line
○ umbilic point

angles of the boundary configuration on $\mathbb{S}^2$:
$\urcorner$ = right angle, $\alpha = \pi/3$

**Figure 8.1:** A combinatorial picture of the boundary conditions on $\mathbb{S}^2$ of a fundamental region of Schwarz's H surface.

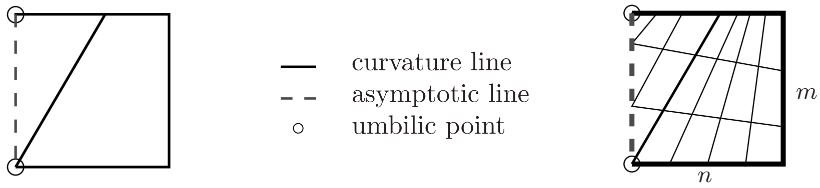



**Figure 8.2:** A combinatorial conformal parametrization of the fundamental region of Schwarz's H surface.

Given the combinatorial picture of the boundary, the remaining task consists in finding a combinatorial parametrization of the bounded domain corresponding to the curvature line parametrization of a smooth minimal surface. This implies in particular:

(1) Umbilic and singular points are taken from the smooth surface, but only their combinatorial locations matter. The smooth surface also determines the number of curvature lines meeting at these points. For interior umbilic or singular points, we additionally have to take care how many times the interior region is covered by the Gauss map. As this case does not occur in any of the examples presented in Section 8.2, we assume for simplicity that all umbilic or singular points lie on the boundary.

(2) If a boundary curvature line and a boundary asymptotic line intersect at an angle of $3\pi/4$, there is another curvature line meeting at this point. This property is a consequence of the conformality of the parametrization.

(3) The curvature lines of the combinatorial parametrization divide the domain into combinatorial squares. The only exceptions are combinatorial triangles formed by two curvature lines and by an asymptotic line on the boundary.

Hence in order to find the combinatorial curvature line parametrization, first determine all umbilic and singular points and all regular boundary points with $3\pi/4$-angles. Then continue the additional curvature line(s) meeting at these points. In this way, the combinatorial domain is divided into finitely many subdomains such that conditions (2) and (3) hold, see Figure 8.2 (left). If there are several combinatorially different possibilities to obtain such a subdivision, we have to choose the one with the same combinatorics as the corresponding curvature lines of the given smooth minimal surface. More precisely, the discrete curvature lines are chosen as an approximation of the smooth (infinite) curvature line pattern.

Given this coarse subdivision of the combinatorial domain, the parametrization of the subdomains is obvious. The only additional conditions occur at common boundaries of two subdomains, where the number of crossing curvature lines has to be equal on both sides. Hence after this step, we know the maximum number of free integer parameters corresponding to the number of different combinatorial types of curvature lines whose numbers may be chosen independently. These integer parameters of the discrete minimal surface correspond to smooth parameters of the continuous minimal surface such as scaling or quotients of lengths (for example of the boundary curves). The number of free integer parameters is greater or equal to the number of smooth parameters of the corresponding continuous minimal surface.

**Remark 8.1.** Note that the free parameters of the combinatorial curvature line parametrization take integer values and the number of (quadrilateral) subdomains obtained by the subdivision process is finite. Also, all combinatorial curvature lines are closed modulo the boundary. In general these properties do not hold for curvature lines of smooth minimal

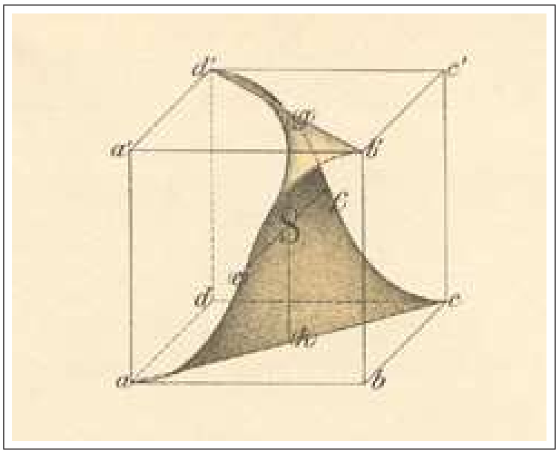


**Figure 8.3:** Gergonne's surface. Copper plate engraving from H. A. Schwarz [68].

surfaces. Furthermore, there may be dependencies between the numbers of curvature lines of different types which have to be approximated by the choice of the free integer parameters of the combinatorial curvature line parametrization. These three aspects affect the different appearances of the smooth minimal surfaces and its discrete minimal analog.

In particular, for the of a smooth minimal surface whose curvature lines are closed modulo the boundary and whose combinatorial curvature line parametrization has only one free parameter (corresponding to overall scaling), the discrete minimal analog will have exactly the same boundary planes/straight lines (up to translation and scaling). This is useful to create discrete minimal analogs to triply-periodic minimal surfaces. In general, we obtain discrete minimal approximations of such surfaces in the sense that the discrete surface does not exactly close up after suitable reflection/rotation in its boundary planes/straight edges.

## 8.2 Examples of $S$-isothermic discrete minimal surfaces

In the following, the construction of discrete minimal surfaces explained in Appendix A and in the preceeding section is applied to some examples. To create presentations of the discrete minimal surfaces resembling their smooth analogs, only the circles filled up with disks are shown in the pictures. In each case we present the boundary conditions and deduce suitable reduced conditions, a combinatorial picture of the curvature and/or the asymptotic lines and the image of the reduced boundary conditions under the Gauss map. If this image is uniquely determined by the intersection angles between arcs of great circles (up to rotation of the sphere), then the corresponding spherical circle pattern will also be unique (due to the simple combinatorics). This is the case in 8.2.1–8.2.6, for symmetric quadrilaterals and for the cubic frames considered in 8.2.7. Pictures of the corresponding smooth minimal surfaces can be found in textbooks like [30, 56] or in some of the original treatises cited below. Pictures of triply periodic minimal surfaces can for example be found in the article [48] by Karcher or on Brakke's web-page [20].

### *8.2.1 Gergonne's surface*

Gergonne's surface traces back to J. D. Gergonne [42], who posed the first geometric problem leading to minimal surfaces with free boundaries in 1816. A correct solution was only found by H. A. Schwarz in 1872; see [68, pp. 126-148].

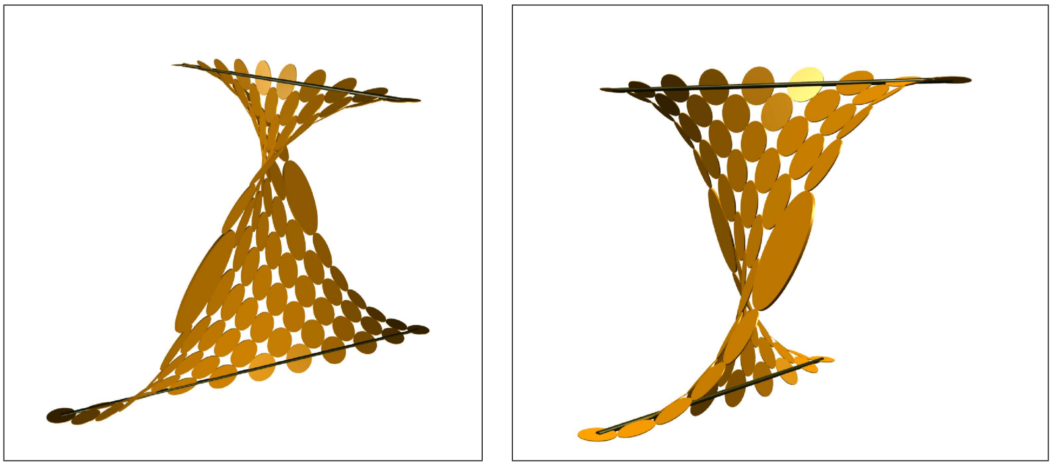
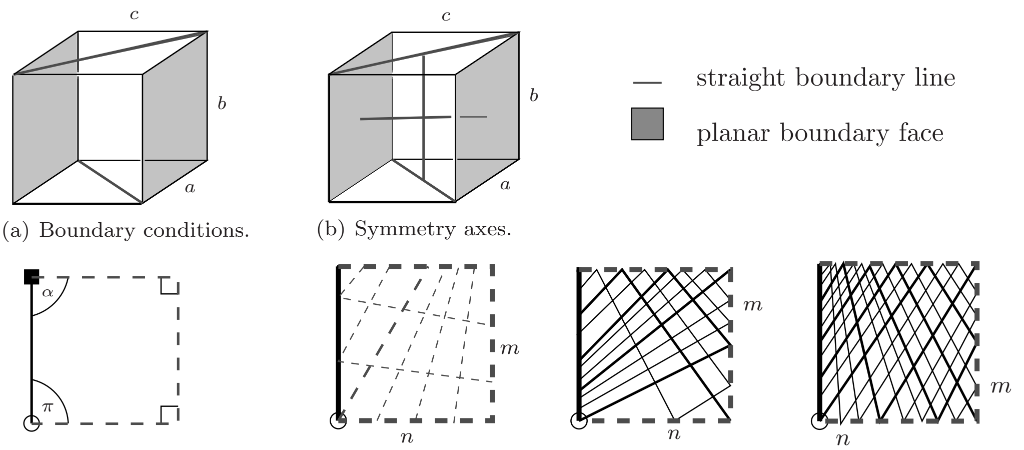


**Figure 8.4:** Discrete Gergonne's surface with $\alpha = \frac{\pi}{6}$ (left) and $\alpha = \frac{\pi}{4}$ (right).




(a) Boundary conditions. (b) Symmetry axes.

(c) Reduced boundary conditions. (d) Different types of combinatorial conformal parametrizations.

Angles of the boundary configuration on $\mathbb{S}^2$:
$\urcorner$ = right angle, $\alpha \in (0, \frac{\pi}{2})$ given angle

$n, m$ integer parameters of parameter lines

— curvature line
- - asymptotic line
○ umbilic point
■ umbilic or singular point if $\alpha \neq \frac{\pi}{4}$

**Figure 8.5:** Gergonne's surface: boundary conditions and combinatorial conformal parametrizations.

**Boundary conditions:** Given a cuboid with side lengths $a$, $b$, $c$, take two opposite faces as boundary faces and non-collinear diagonals of two other opposite faces, as in Figure 8.5(a). Then the cuboid with this boundary conditions has two axes of 180°-rotation symmetry. These axes will lie on the minimal surface and cut the surface into four congruent parts; see Figures 8.3 and 8.5(b). Therefore it is sufficient to consider reduced boundary conditions consisting of three orthogonal straight lines and one planar boundary face or equivalently three straight asymptotic lines and one planar curvature line, as in Figure 8.5(c).

Now a combinatorial picture of the asymptotic lines or of the curvature lines can be determined; see Figure 8.5(d). The two parameter numbers correspond to the free choice

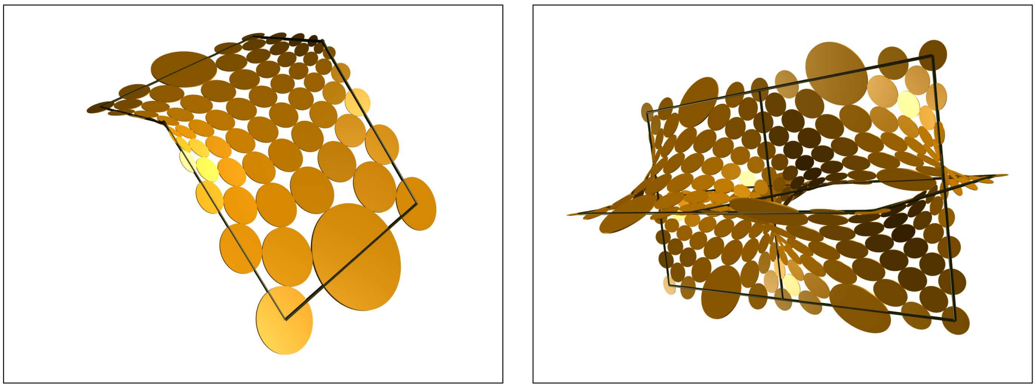


**Figure 8.6:** A generalization of Schwarz's CLP surface with $\alpha = \frac{2\pi}{3}$ (left) and Schwarz's CLP surface (right, $\alpha = \frac{\pi}{2}$).

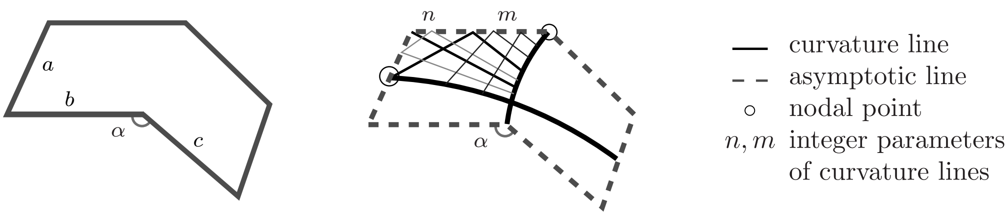



**Figure 8.7:** A generalization of Schwarz's CLP surface: boundary frame (left) and combinatorics of curvature lines (right).

of two length parameters of the cuboid, given the angle $\alpha$ between the diagonal and the planar boundary face.

If the angle $\alpha = \frac{\pi}{4}$, the minimal surface can be continued by reflection and rotation in the boundary faces/edges to result in a triply periodic (discrete) minimal surface.

The image of the reduced boundary conditions under the Gauss map is a spherical triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{2}$ and $(\frac{\pi}{2} - \alpha)$.

### *8.2.2 Schwarz's CLP surface*

Schwarz's CLP surface is one of the triply periodic minimal surfaces already constructed by H. A. Schwarz, see [68, vol. 1, pp. 92-125]. The name CLP is due to A. Schoen [66]. He considered the labyrinth formed by the periodic surface and associated the name to properties of the underlying spatial lattice.

**Boundary conditions:** Consider a frame of two pasted rectangles (with edge lengths $a, b, c$) which enclose an angle $\alpha \in (0, \pi)$, as in Figure 8.7 (left). Then there is a plane of reflection symmetry orthogonal to the sides with length $a$. Thus we get one planar curvature line. If $b = c$ there is another plane of reflection symmetry through both corners with angle $\alpha$ and yet another curvature line. This curvature line persists if $b \neq c$.

It is therefore sufficient to consider one fourth of the whole combinatorial picture bounded by two asymptotic and two curvature lines; see Figure 8.7 (right). The two integer parameters correspond to the two boundary lengths of the first rectangle. Analogously, there are two other integer parameters corresponding to the two boundary lengths of the second rectangle. The pasting procedure translates into the condition that the number of lines on the curvature line joining the pasted corners has to be equal on both sides.

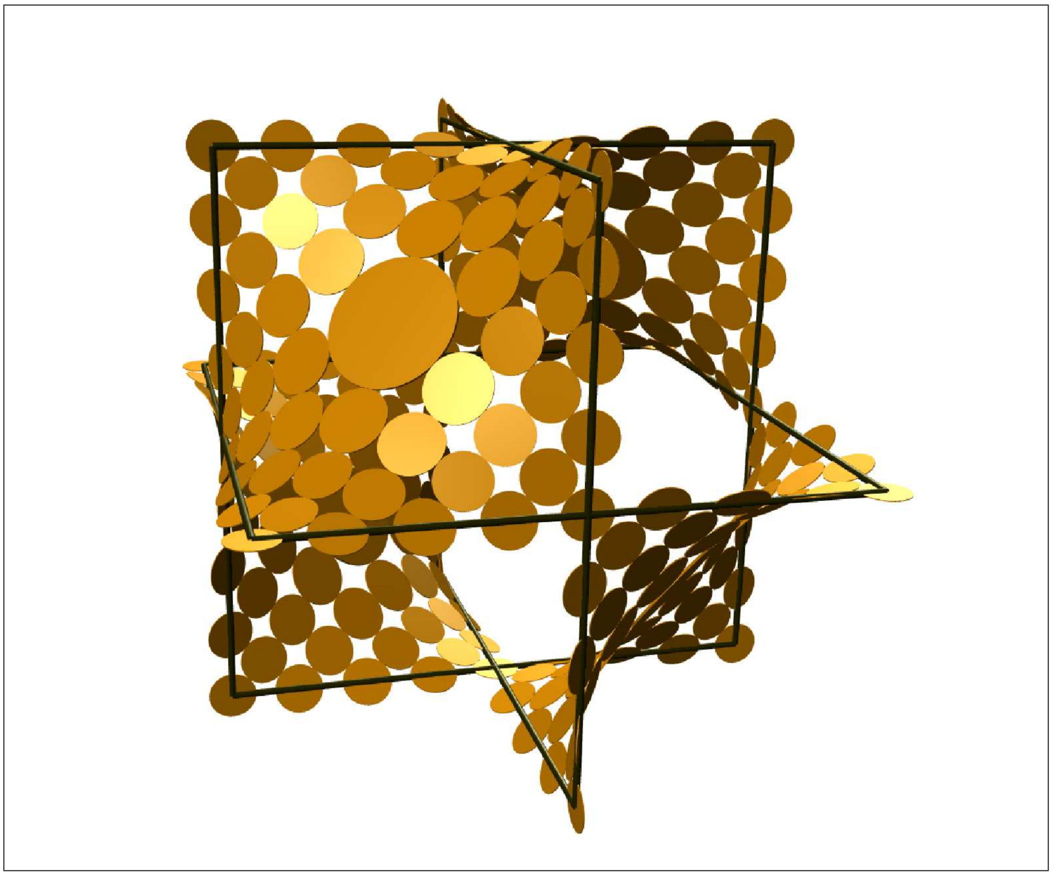


**Figure 8.8:** Schwarz's D surface.

Therefore there are three parameters left corresponding to the three lengths $a, b, c$.

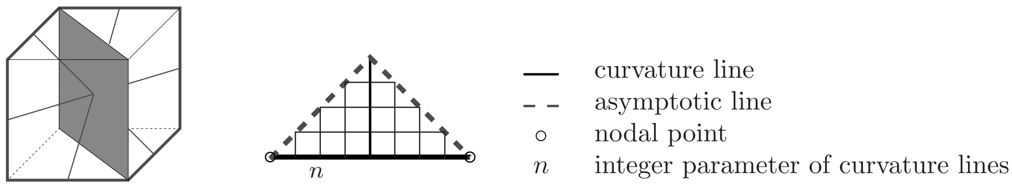
The image of the reduced boundary conditions (the plane of reflection and one half of the given boundary frame) under the Gauss map is a spherical triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{2}$ and $(\pi - \alpha)$.

If $\alpha = \frac{\pi}{2}$ and $b = c$ the surface can be continued across its boundaries to build a triply periodic discrete minimal surface. Figure 8.6 (right) shows a cubical unit cell.

### *8.2.3 Schwarz's D surface*

Schwarz's D surface is another triply periodic minimal surfaces already constructed by H. A. Schwarz, see [68, vol. 1, pp. 92-125]. The surface was named D by A. Schoen because its labyrinth graphs are 4-connected 'diamond' networks.

**Boundary conditions:** From the edges of a cuboid take a closed boundary frame consisting of two edges of each side; see Figure 8.9(a). By construction, there are three straight lines of 180°-rotation symmetry lying within the minimal surface and three planes of reflection symmetry orthogonal to the boundary frame which give planar curvature lines (see Figure 8.9(a) where only one of the three planes of reflection symmetry is shown). Therefore a fundamental piece consists of a triangle bounded by two straight asymptotic lines and one planar curvature line. The combinatorics of the curvature lines of this fundamental piece is simple and shown in Figure 8.9(b). The only parameter corresponds to overall scaling or refinement.




(a) Boundary frame with symmetries.

(b) Combinatorics of curvature lines of a fundamental piece.

**Figure 8.9:** Schwarz's D surface: boundary frame and combinatorics of curvature lines.

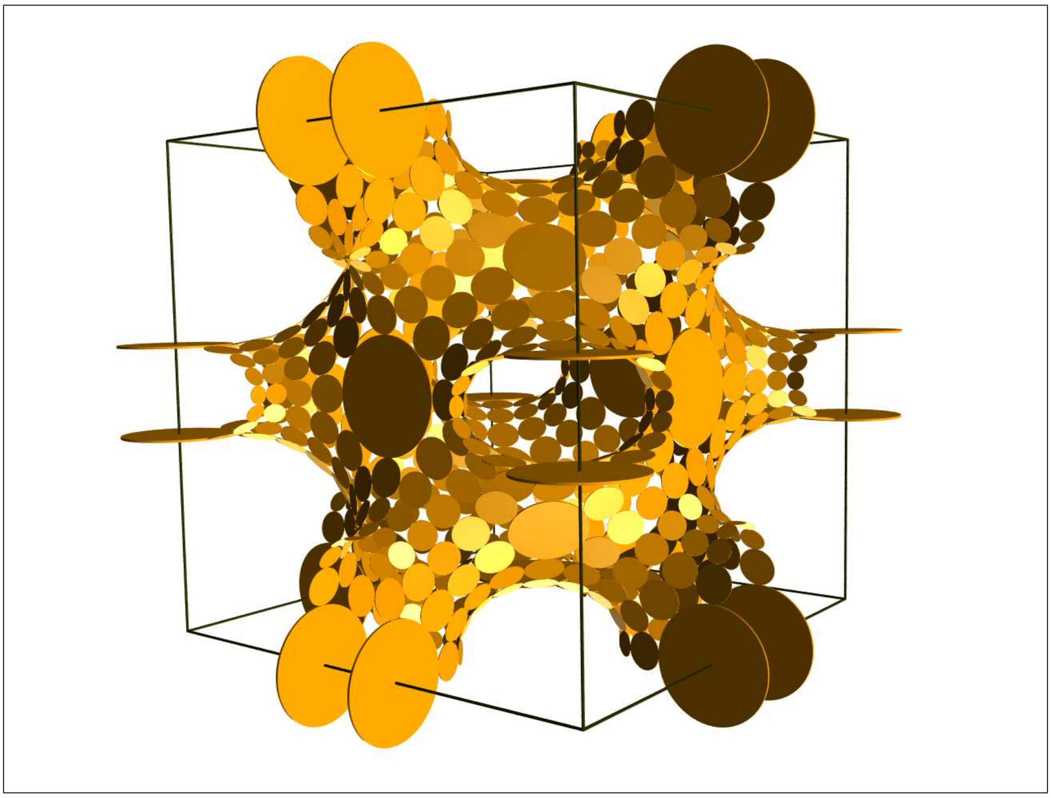


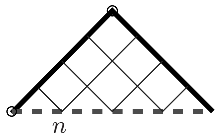
**Figure 8.10:** Neovius's surface.

The image of the reduced boundary conditions under the Gauss map is a spherical triangle with angles $\frac{\pi}{2}$ (between the two asymptotic lines) and $\frac{\pi}{4}$ and $\frac{\pi}{3}$ (between an asymptotic line and the curvature line).

The minimal surface constructed by the boundary frame in Figure 8.9(a) can be continued across the boundary to result in a triply periodic minimal surface. Figure 8.8 shows one cubical unit cell of the periodic lattice.

### *8.2.4 Neovius's surface*

H. A. Schwarz [68] began to consider minimal surfaces bounded by two straight lines and an orthogonal plane. His student E. R. Neovius continued and deepened this study and found another triply periodic surface, see [55]. This surface has the same symmetry group as Schwarz's P surface and was named C(P) by A. Schoen.




| | |
|---|---|
| — | curvature line |
| – – | asymptotic line |
| ○ | nodal point |
| $n$ | integer parameter of curvature lines |

**Figure 8.11:** Neovius's surface: combinatorics of curvature lines of a fundamental piece.

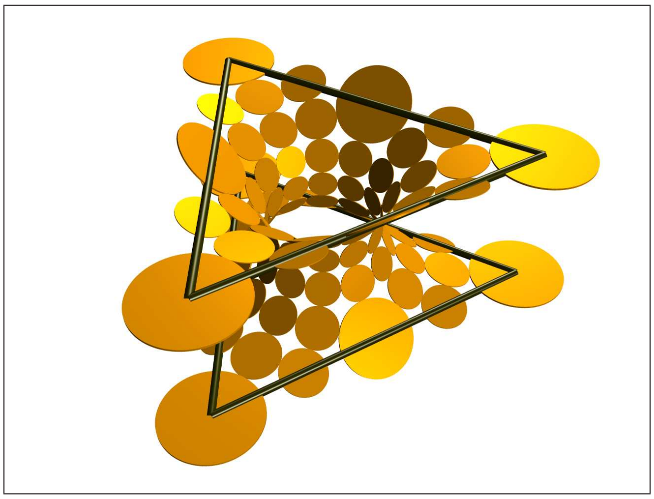


**Figure 8.12:** Schwarz's H surface.

**Boundary conditions:** One unit cell of the lattice of Neovius's surface is basically a cubical cell with one central chamber and necks out of the middle of each edge of the cube; see Figure 8.10. By symmetry, it is sufficient to consider a cuboid which is one eighth of the unit cell. All the faces of this cuboid are boundary planes. Furthermore, the surface piece has three planes of reflection symmetry and three lines of 180°-rotation symmetry (see Figure 8.9(a) where only one of the three planes of reflection symmetry is shown). Therefore a fundamental piece consists of a triangle bounded by one straight asymptotic lines and two planar curvature line.

The combinatorics of the curvature lines of this fundamental is again very simple and shown in Figure 8.11. The only parameter corresponds to overall scaling or refinement.

The image of the reduced boundary conditions under the Gauss map is a spherical triangle with angles $\frac{\pi}{4}$ and $\frac{\pi}{3}$ (between a curvature line and the asymptotic line) and $\frac{3\pi}{4}$ (between the two curvature lines).

The minimal surface constructed by the boundary conditions shown in Figure 8.11 can be continued across the boundary to result in a triply periodic minimal surface. Figure 8.10 shows one cubical unit cell of the periodic lattice.

### *8.2.5 Schwarz's H surface*

Schwarz's H surface is another triply periodic minimal surface which was already known to H. A. Schwarz, see [68, vol. 1, pp. 92-125].

**Boundary conditions:** Take two parallel copies of an equilateral triangle and consider their edges as boundary frame for a minimal surface spanned in between them, as in Fig-

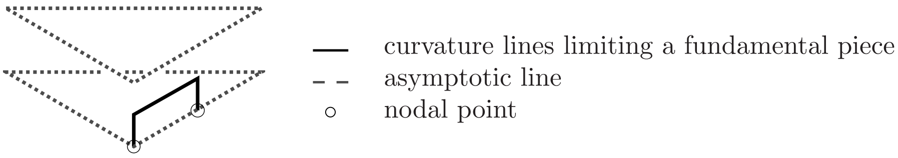



**Figure 8.13:** The boundary frame of Schwarz's H surface with a fundamental piece.

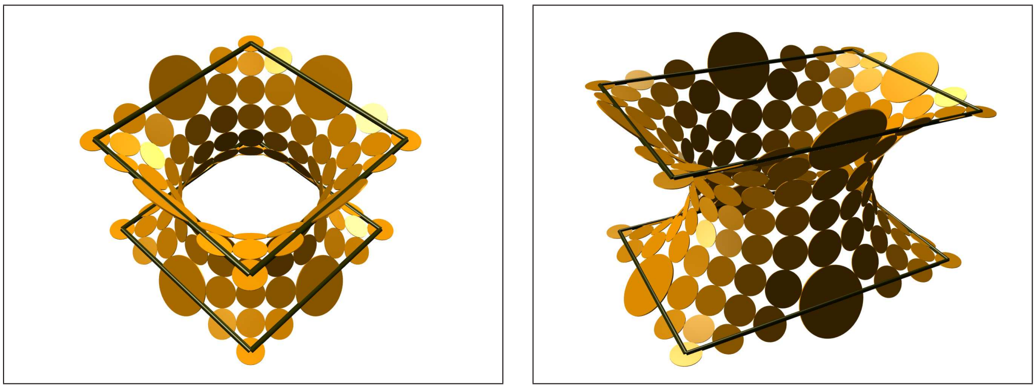


**Figure 8.14:** Schoen's I-6 surface (left) and a generalization (right).

ure 8.13. Then by construction the minimal surface has one plane of reflection symmetry parallel to the planes of the triangles and three other orthogonal planes as symmetry group of the equilateral triangle. Thus we arrive at a fundamental piece bounded by three planar curvature lines and one straight asymptotic line; see Figure 8.13.

The combinatorics of the curvature lines are shown in Figure 8.2. The two parameters correspond to the side length of the two equilateral triangles and to their distance.

The image of the reduced boundary conditions under the Gauss map is a spherical triangle with angles $\frac{\pi}{2}$ and $\frac{\pi}{2}$ (between two of the curvature lines) and $\frac{\pi}{3}$ (between the asymptotic line and a curvature line, the other angle is $\pi$).

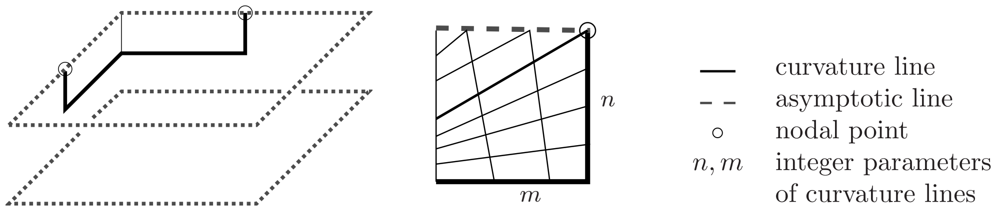
The minimal surface constructed by the boundary conditions shown in Figure 8.12 can be continued across the boundary by 180°-rotation about the straight lines to result in a triply periodic minimal surface.

### *8.2.6 Schoen's I-6 surface and generalizations*

About 1970, the physicist and crystallographer A. Schoen discovered many triply periodic minimal surfaces. His reports [66] were a bit sketchy, but H. Karcher [48] established the existence of all of Schoen's surfaces.

**Boundary conditions:** Similarly as for Schwarz's H surface, take two parallel copies of a rectangle and regard their edges as the boundary frame for a minimal surface between them. By construction the minimal surface has one plane of reflection symmetry parallel to the planes of the rectangles and two other orthogonal planes as symmetry group of the rectangle (four in case of a square). Thus we arrive at a fundamental piece bounded by three planar curvature lines and one straight asymptotic line; see Figure 8.15(a). There is one additional curvature line splitting the piece into two parts which can easily be found be reflection symmetry in the case of squares.

For both parts of this fundamental piece, the combinatorics of curvature lines are the




(a) Boundary frame with a fundamental piece.

(b) Combinatorics of the curvature lines of one half of a fundamental piece.

**Figure 8.15:** (A generalization of) Schoen's I-6 surface.

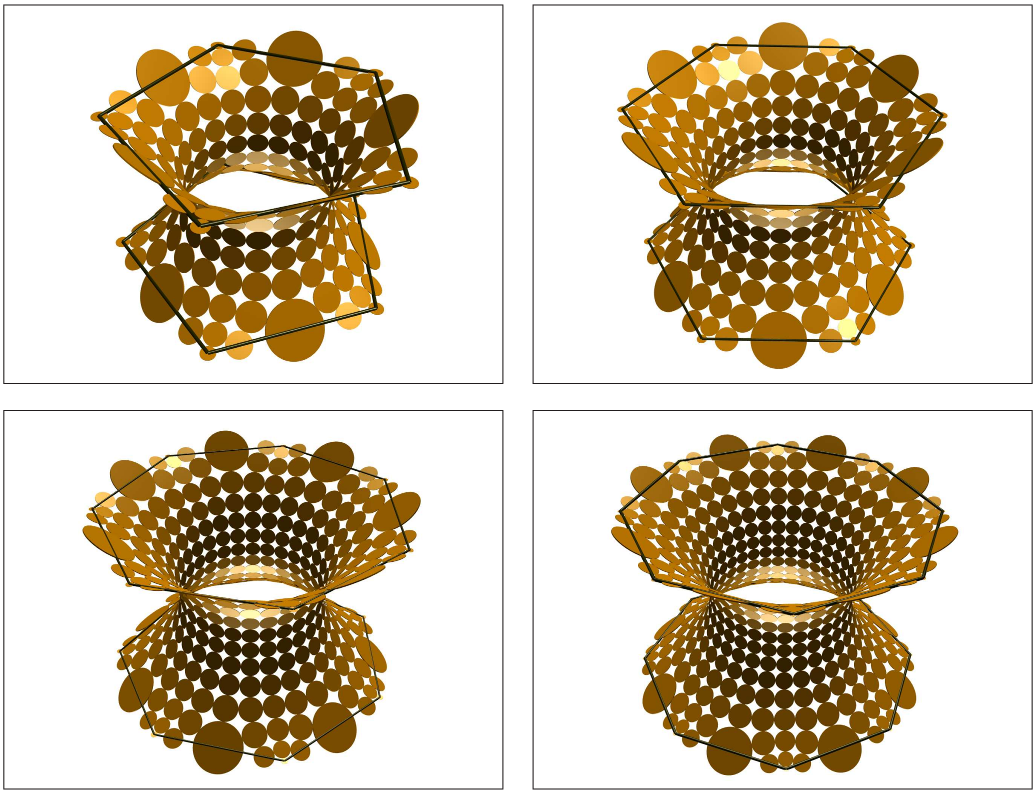


**Figure 8.16:** An approximation of a piece of a catenoid using polygonal boundary frames.

same as for the fundamental piece of Schwarz's H surface; see Figure 8.15(b) or 8.2. The four integer parameters have to be chosen such that the number of curvature lines meeting at the pasting curvature line of the two parts is the same on both sides. The remaining three parameters correspond to the side lengths of the two rectangles and to their distance.

The image of the reduced boundary conditions under the Gauss map is a spherical triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{2}$ and $\frac{\pi}{4}$.

The minimal surfaces constructed by the above boundary conditions can be continued across the boundary to result in triply periodic minimal surfaces.

In an analogous way to the construction of Schwarz's H surface and Schoen's I-6 surface, we may consider all regular symmetric planar polygons with sides of equal length. The fundamental piece is always combinatorially the same as for Schwarz's H surface. The only difference is the angle $\alpha = \frac{\pi}{n}$ for an n-gon. These minimal surfaces may also be understood

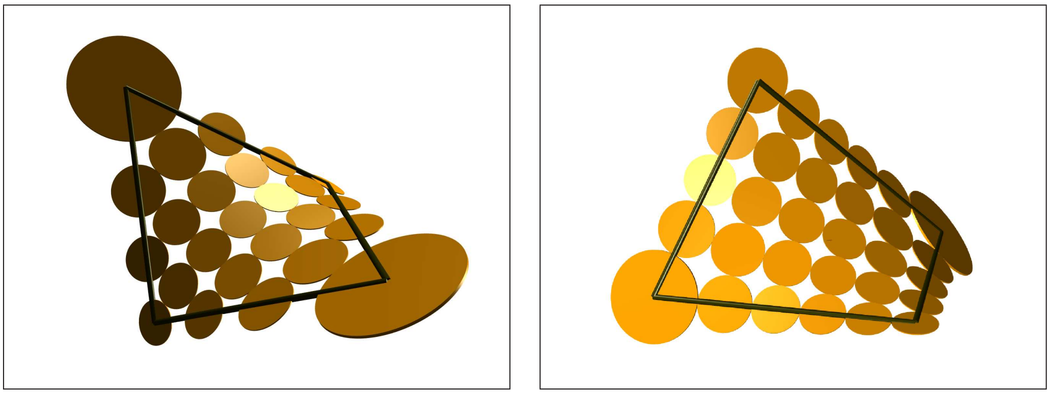
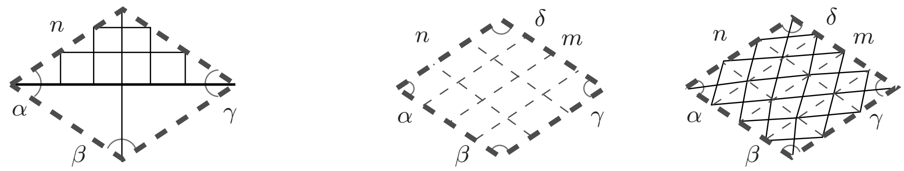


**Figure 8.17:** Two examples of discrete quadrilateral boundary frames: a symmetric and a general version.




(a) Symmetric quadrilateral. (b) General quadrilateral.

| | | | |
|---|---|---|---|
| —— | curvature line | $\alpha, \beta, \gamma, \delta$ | angles of the boundary configuration |
| – – | asymptotic line | $n, m$ | integer parameters of parameter lines |

**Figure 8.18:** Boundary conditions and combinatorial parametrizations for symmetric and general quadrilaterals.

as an approximation to a piece of the catenoid; see Figure 8.16. The boundary conditions may also be generalized to non-regular planar polygons with additional symmetry planes.

### *8.2.7 Polygonal boundary frames*

As a special class of Plateau's problems, all non-planar, simple, unknotted polygons can be considered as boundary conditions. The main task is to determine the corresponding discrete combinatorics of the conformal parameter lines. The number of different special cases increases with the number of boundary segments, so we confine ourselves to quadrilaterals, pentagons, and a cubical boundary frame as one more complicated example. Since the boundary frame consists only of straight asymptotic lines, we use the conjugate minimal surface from the associated family for the construction. A definition and some properties of the associated family can be found in the Appendix A; see Definition A.10 and Theorem A.11.

*Quadrilateral*

The minimal surface spanned by a quadrilateral with equal side lengths and equal angles of $\frac{\pi}{3}$ was the first known solution in the class of Plateau's problems. In 1865, H. A. Schwarz found the explicite solution [68, pp. 6–125]. About the same time, B. Riemann independently solved this problem [63, pp. 326–329]. His paper [62] appeared posthumously in 1867. At the same time H. A. Schwarz sent his prize-essay to the Berlin Academy. Later

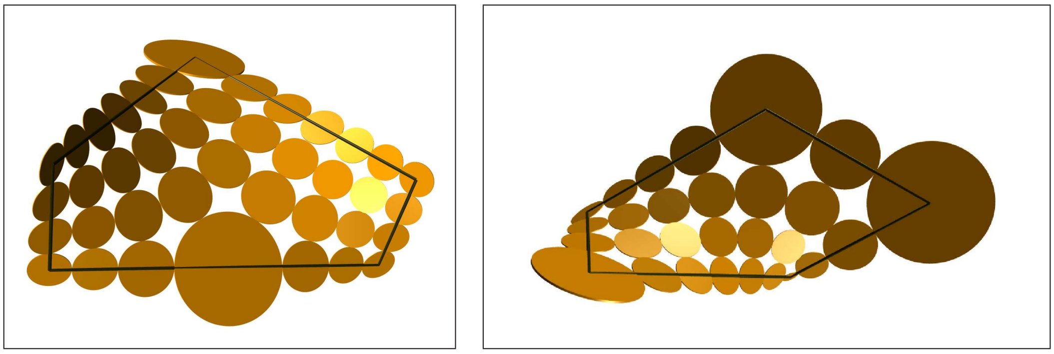
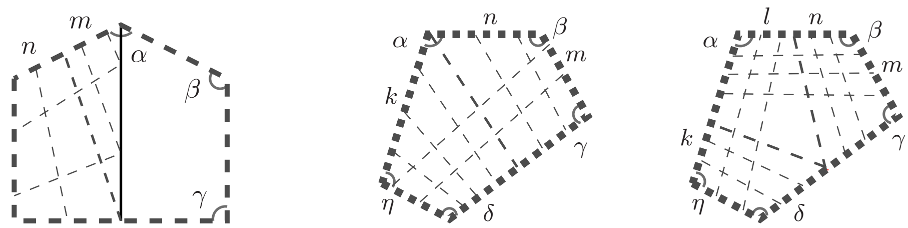


**Figure 8.19:** Two examples of discrete pentagons: a symmetric and a general boundary frame.




(a) Symmetric pentagon. (b) General pentagon.

| | | | |
|---|---|---|---|
| —— | curvature line | $\alpha, \beta, \gamma, \delta, \eta$ | angles of the boundary configuration |
| – – | asymptotic line | $n, m, k, l$ | integer parameters of asymptotic lines |

**Figure 8.20:** Boundary conditions and combinatorics of asymptotic lines for symmetric and general pentagons.

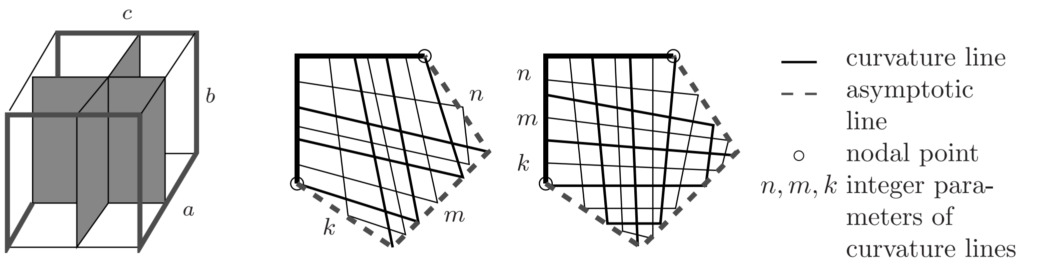
on, Plateau's problem was tackled for other polygonal boundaries, see for example the historical remarks in [30, 56].

Given a non-planar quadrilateral of straight boundary lines, the combinatorics of asymptotic lines is easily found; see Figure 8.18(b) (left). Then the corresponding curvature lines may also be determined, as in Figure 8.18(b) (right). The two parameters correspond to a global scaling and the to a ratio of lengths of the boundary segments. As explained in Remark 8.1, in general we do not obtain exactly a given quadrilateral boundary frame. Nevertheless, we get an approximation which converges for a suitable choice of parameter values.

In the case of a non-planar symmetric quadrilateral the combinatorics of curvature lines are obvious; see Figure 8.18(a). The only parameter corresponds to a global scaling or refinement. Thus we can obtain any symmetric quadrilateral boundary frame as the exact boundary of a discrete minimal surface.

*Pentagon*

First, consider a symmetric case where the boundary configuration allows a plane of mirror symmetry containing one boundary vertex and cutting the opposite edge. The remaining reduced domain is similar to the boundary conditions of Gergonne's surface and thus has the same combinatorics of asymptotic lines; see Figure 8.20(a). The corresponding curvature lines may also be determined from this parametrization.




(a) A cubic boundary frame and its symmetry planes.

(b) Different possible types for the combinatorics of curvature lines of a fundamental piece.

**Figure 8.21:** A general cubical boundary frame.

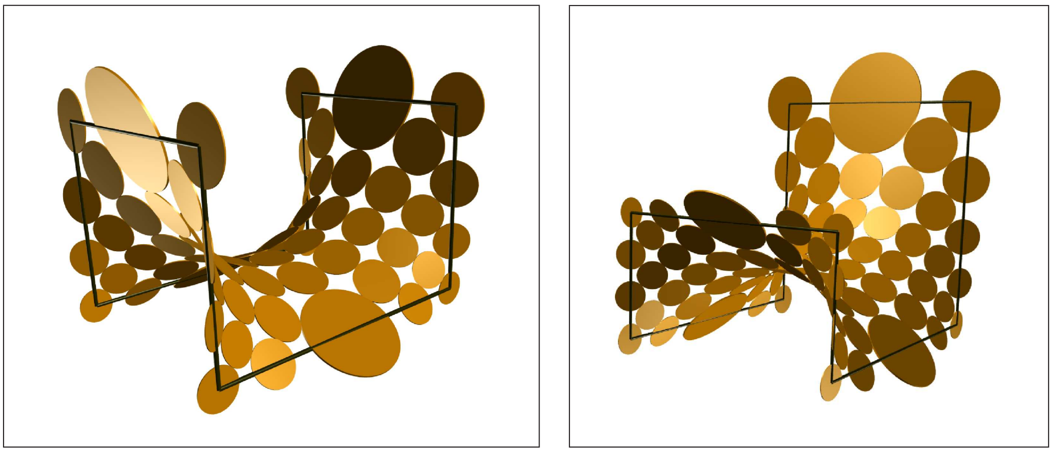


**Figure 8.22:** An example of a discrete minimal surface with cubical boundary frame (left) and a generalization (right).

The general case is more difficult. In principle, there are two possibilities for the combinatorial position of an additional umbilic point; see Figure 8.20(b). As in the case of general quadrilaterals, the integer parameters correspond to lengths of boundary segments.

*A cubical frame*

As a last example consider a more complicated polygonal boundary frame. Take a cuboid with edge lengths $a$, $b$, $c$ and select eight of its boundary edges as illustrated in Figure 8.21(a). By construction, the minimal surface spanned by this frame has two planes of reflection symmetry and yet two planar curvature lines. The fundamental piece is therefore bounded by three straight asymptotic lines and two curvature line. The possible combinatorics of curvature lines are depicted in Figure 8.21(b). The three integer parameters correspond to the three edge lengths $a$, $b$, $c$ of the cuboid.

The image of the reduced boundary conditions under the Gauss map is a spherical triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{2}$ and $\frac{\pi}{2}$.

A minimal surface constructed by this boundary conditions is shown in Figure 8.22 (left) and can be continued across the boundary to result in a triply periodic minimal surface.

Of course, this example may be generalized to related problems. Figure 8.22 (right) shows an example of such a generalization.

## 8.3 Convergence of $S$-isothermic discrete minimal surfaces

In this section we consider an application of the convergence Theorem 7.1 and its generalization discussed in Section 7.2 to $S$-isothermic discrete minimal surfaces.

We focus on the same special classes of (discrete) minimal surface as in the previous sections. These surfaces are bounded, homeomorphic to a disk, and have a boundary curve consisting of planar curvature lines or straight asymptotic lines. As an example, we take as region $\mathcal{D}$ the projection of a symmetric spherical quadrilateral bounded by parts of great circles which is entirely contained in one half of the sphere. Unfortunately, we do not know in general if a corresponding orthogonal circle patten $\mathscr{C}_n^{\mathcal{D}}$ exists, see Remark A.14 in Appendix A. Nevertheless, existence can be proven for some special cases, for example a symmetric spherical quadrilateral with angles $\alpha = 2\pi/3$, $\beta = \pi/2 = \delta$, $\gamma = \pi/2$. This choice of angles leads to (a part of) Schwarz' P surface, which has been constructed by Springborn [13].

The Weierstrass Representation Theorem A.9 together with Theorem 7.1 leads to the following convergence result.

**Theorem 8.2.** *Let $\mathcal{D}$ be the projection of a symmetric spherical quadrilateral $Q \subset S^2$ bounded by parts of great circles which is entirely contained in one half of the sphere. Assume that for all $n \in \mathbb{N}$ orthogonal circle pattens $\mathscr{C}_n^{\mathcal{D}}$ with the combinatorics of $SG_n^{\mathcal{R}}$ exist such that all boundary circles intersect the boundary $\partial\mathcal{D}$ orthogonally and the circles corresponding to corner vertices of $SG_n^{\mathcal{R}}$ intersect two corresponding boundary lines of $\partial\mathcal{D}$ orthogonally. For $n \in \mathbb{N}$, define $g_n$ as in Theorem 7.1 and $F_n$ using Theorem A.9. Then the scaled $S$-isothermic discrete minimal surfaces $M_n$ corresponding to $\frac{1}{n^2}F_n$ converge uniformly for the images of compact subsets of $\mathcal{R}^*$ to a smooth minimal surface $M$. The stereographic projection of the Gauss map, determining $M$ up to translation and scaling, is given by the conformal map $g : \mathcal{R} \to \mathcal{D}$.*

*Proof.* Bicolor the vertices of $SG$ and thus of $SG_n^{\mathcal{R}}$ using the labels ⓢ and ⓒ. Consider two vertices $x_2, x_1 \in V(SG_n^{\mathcal{R}})$, both labelled ⓢ, such that $|x_2 - x_1| = 1/n$. Then

$$\begin{aligned}(\frac{1}{n^2}F_n(x_2) - \frac{1}{n^2}F_n(x_1))n &= (F_n(x_2) - F_n(x_1))/n \\ &= \pm\mathrm{Re}\left(\frac{R_n(x_2)/n + R_n(x_1)/n}{(1+|p|^2)}\,\frac{\overline{c_n(x_2)-c_n(x_1)}}{|c_n(x_2)-c_n(x_1)|}\begin{pmatrix}1-p^2\\ i(1+p^2)\\ 2p\end{pmatrix}\right),\end{aligned}$$

where the notations are as in Theorem A.9. Theorem 7.1 implies that $g_n$ converges to $g$ uniformly and in $C^\infty$ on compact sets contained in $\mathcal{R}^*$. Thus, keeping $x_1$ fixed such that $x_2 \to x_1$ and the direction $\delta(x_2 - x_1) = const.$, the touching point $p$ converges to $f(x_1)$ uniformly and in $C^\infty(\mathcal{R})$, or more precisely, uniformly in $C^\infty$ on compact sets contained in $\mathcal{R}^*$. As the angle function $\varphi_n$ also converges to $\mathrm{Im}\log g'$ in $C^\infty(\mathcal{R})$, we obtain

$$\frac{\overline{c_n(x_2)-c_n(x_1)}}{|c_n(x_2)-c_n(x_1)|} \longrightarrow \exp(-i\arg g'(x_1) + i\delta(x_2-x_1)\pi/2)$$

in $C^\infty(\mathcal{R})$. Similarly, from $2nr_n \to |g'|$, we deduce from (A.2) that

$$\frac{R_n(x_i)}{n} = \left|\frac{1+|c(x_i)|^2 - (r_n(x_i))^2}{2nr_n(x_i)}\right| \longrightarrow \frac{1+|g(x_1)|^2}{|g'(x_1)|}$$

for $i = 1, 2$ in $C^\infty(\mathcal{R})$. Thus the difference quotient $(\frac{1}{n^2}F_n(x_2) - \frac{1}{n^2}F_n(x_1))n$ converges in $C^\infty(\mathcal{R})$ to

$$\mathrm{Re}\left(\frac{i^{\delta(x_2-x_1)}}{g'(x_1)}\begin{pmatrix}1-g(x_1)^2\\ i(1+g(x_1)^2)\\ 2g(x_1)\end{pmatrix}\right).$$

This is the partial derivative of $F$ at $x_1 \in \mathcal{R}$ in direction $x$ if $\delta(x_2 - x_1) = 0$ and $y$ otherwise, where

$$F(w) = F(x_0) + \mathrm{Re} \int_{z_0}^{w} \begin{pmatrix} 1 - g(z)^2 \\ i(1 + g(z)^2) \\ 2g(z) \end{pmatrix} \frac{1}{g'(z)} dz$$

is the smooth Weierstrass representation formula for a smooth minimal surface $M$. Now, the claim follows by suitable discrete and smooth integration. $\square$

In analogous way, convergence results for more general regions can be deduced from Section 7.2.

## 8.4 Comparison with linear theories of discrete minimal surfaces and of discrete harmonic functions

To explicitly construct minimal surfaces, numerical approximations of such surfaces have been studied by several authors, mostly using linear finite elements. Thus triangles are used to define the discrete minimal surface (and to parameterize the domain) and the class of functions under consideration is linear on each triangle and continuous at the vertices (conforming finite elements) or at the edge midpoints (non-conforming finite elements). Note that $S$-isothermic discrete minimal surfaces also belong to this class of surfaces.

### *8.4.1 Convergence results for discrete minimal surfaces*

In the following we briefly review some of the known numerical methods where convergence results are available. The estimations use convenient $L^p$-norms and a suitable variational formulation of Plateau's problem. Furthermore, there are in general regularity assumptions for the minimal surfaces, the boundary curve and the triangulations of the domain. Several investigations concern (discrete) minimizers or stationary points of the Dirichlet integral. This idea is due to Douglas [31] and Radó [60] and later reformulated by Courant [28]. We restrict ourselves to disk-type minimal surfaces.

*Minimal surfaces as graphs*

The first convergence results were obtained for minimal surfaces which are graphs of a function over a (strictly convex) domain, see for example [47, 61, 26]. Using appropriate regularity conditions for the boundary of the domain and the given function on the boundary, the difference between the finite element solution and the smooth solution is estimated in $W^{2,p}$- Sobolev norms and in $L^2$. This gives an convergence rate of order $O(h^2)$ for regular triangulations with mesh size controlled by $h$.

*Parametric approach*

To treat a broader class of solutions to Plateau's problem, (discrete) minimal surfaces are represented via a parameterization. The parameter domain is often taken to be the unit disk $D$ (up to some normalization).

Tsuchiya seems to be the first to give a complete proof in [74, 75] for the convergence of discrete minimal surfaces to a continuous solution in the $H^1(D)$-norm. The smooth and the discrete minimal surfaces have to be stable minimizers and there are also additional regularity conditions for the boundary curve and the triangulations of the domain. The ideas of Tsuchiya follow the lines of the existence proof for the Plateau problem by Douglas. His algorithm can be decomposed in two steps. First given a fixed triangulation, minimize the Dirichlet energy by varying the points in the image space. This is a linear problem.

Second, minimize the Dirichlet energy by varying (boundary) points of the triangulation of the disk. This can be interpreted as a variation of the metric to approximate conformality for the mapping.

Dziuk and Hutchinson generalize these results in [38, 39]. They prove that given any nondegenerate minimal surface $u : \overline{D} \to \mathbb{R}^3$ spanning a given regularly parameterized $C^3$-curve, there exists discrete minimal surfaces $u_h : \overline{D_h} \to \mathbb{R}^3$ such that $\|u - u_h\|_{H^1(D_h)} \leq ch$, where the constant $c$ only depends on the curve and on the non-degeneracy constant for $u$. Here $h$ controls the grid size of the quasi-uniform triangulation $D_h$ of the unit disk $D$.

*Mean curvature flow*

Using the fact that the mean curvature of a minimal surface vanishes, it is a natural approach to compute discrete minimal surfaces via mean curvature flow. In contrast to the former methods, these algorithms only work in the image space without having a two-dimensional parameter domain. Implementations of such algorithms are for example due to [37] and [19].

Pinkall and Polthier use similar ideas in [57]; see also [58, 59]. The discrete minimal surface is found by sequentially solving the Dirichlet problem with respect to the metric of the current iterate. This is equivalent to computing a harmonic mapping with respect to the Laplace-Betrami operator of the discrete surface (the so-called cot-*Laplacian*) satisfying some boundary conditions. The boundary points are also allowed to vary if they lie on straight boundary lines or on free boundary curves restricted to planes. Therefore the resulting discrete minimal surface may be extended along boundary symmetry lines as discrete minimal surfaces. Thus examples of periodic minimal surfaces can be constructed. The algorithm also allows to compute conjugate minimal surfaces using non-conforming finite elements.

Recently, Bobenko and Springborn improved this algorithm in [17]. They use the discrete Laplace-Betrami operator corresponding to the Delaunay triangulation of the given discrete surface. Then for any harmonic function there is a convex hull property (maximum principle), that is the value at any point lies in the convex hull of the values at neighboring points of the Delaunay triangulation.

There is only one convergence result for these classes of discrete minimal surfaces which is due to Hildebrandt, Polthier and Wardetzky; see [46, Theorem 7]. Given a sequence of discrete minimal surfaces whose triangles have bounded aspect ratio and which converges to a smooth surface totally normally (that is, the discrete surfaces converge in Hausdorff distance and the normals converge in $L^\infty$), they prove that the limit surface is minimal in the classical sense.

### *8.4.2 Comparison with the nonlinear theory of discrete minimal surfaces*

In [13], Bobenko, Hoffmann and Springborn constructed discrete minimal surfaces using disks and spheres as a special class of $S$-isothermic surfaces. Their results are presented in Appendix A. These surfaces can also be considered as simplicial surfaces. In particular, consider a sphere and an orthogonally intersecting disk. Take as vertices the centers of the sphere and of the disk and the two intersection points and build two orthogonal triangles. Constructing triangles in this way for all points of orthogonally intersecting spheres and disks of an $S$-isothermic discrete minimal surface, we obtain a simplicial surface which additionally is a Delaunay triangulation. Note that this surface is in general not minimal in the sense of the previous paragraph, as the identity is in general not harmonic with respect to the Laplace-Beltrami operator (except at centers of disks).

The greatest advantage of the finite element methods using simplicial surfaces is their flexibility. Boundary conditions and the combinatorics can be adapted, boundary points may be forced to lie on the given (arbitrary) boundary curve. On the other hand, several steps are necessary to actually compute the discrete minimal surface for a given combinatorics and convergence is only proved for suitably smooth boundary curves.

By contrast, $S$-isothermic minimal surfaces are computed using only one minimization step and a dualization procedure. Furthermore, the structural properties allow to obtain similar claims as in the smooth theory, for example a Weierstrass-type representation and the associated family. Away from nodal points a given smooth minimal surface can be approximated in $C^\infty$ using a $C^\infty$-approximation of the function used in the Weierstrass representation. For this purpose, Theorem 5.16 can be used or convergence results of [67, 53] using other types of boundary or initial values. Also, examples of periodic $S$-isothermic discrete minimal surfaces may be obtained. Of course, this additional structure is also a drawback, as it reduces flexibility. To compute an $S$-isothermic discrete minimal surface, the combinatorics of the curvature line parameterization of the smooth minimal surface has to be determined and the boundary conditions have to be translated into Neumann or Dirichlet boundary conditions for the circle patterns as indicated in Appendix A and in Section 8.1.1. In general, boundary points do not lie on the given boundary curve (only for simple cases). Furthermore, in order to actually construct the $S$-isothermic discrete minimal examples of Section 8.2, we computed spherical circle patterns which were suited best for the given boundary conditions. The method we used for this computation worked very well for all our examples, although existence and uniqueness are not yet proven in general for these spherical circle patterns; see also Remark A.14.

Appendix A

# $S$-isothermic discrete minimal surfaces

In this appendix we present the definition of $S$-isothermic discrete minimal surfaces and mention some of their properties. $S$-isothermic surfaces have been defined by Bobenko and Pinkall [15]. Special properties of $S$-isothermic discrete minimal surfaces were introduced and studied by Bobenko, Hoffmann, and Springborn in [13], were also more details and proofs can be found. See also [10] for a brief review.

**Definition A.1.** Let $\mathscr{D}$ be a quad-graph such that the degree of every interior vertex is even. A function $f : V(\mathscr{D}) \to \mathbb{R}^3$ is called a *discrete isothermic surface* if for every face of $\mathscr{D}$ with vertices $v_1, v_2, v_3, v_4$ in cyclic order, the points $f(v_1), f(v_2), f(v_3), f(v_4)$ form a *conformal square*, that is their cross-ratio is $-1$ and the points thus lie on a circle.

**Definition A.2.** Let $\mathscr{D}$ be a b-quad-graph with vertices colored white and black. $\mathscr{D}$ is called a *S-quad-graph* if all interior black vertices have degree 4 and if the white vertices can be labelled ⓒ and ⓢ in such a way that each quadrilateral has one white vertex labelled ⓒ and one white vertex labelled ⓢ.

**Definition A.3.** Let $\mathscr{D}$ be an S-quad-graph and let $V_b(\mathscr{D})$ be the set of all black vertices. A *discrete $S$-isothermic surface* is a map $f_b : V_b(\mathscr{D}) \to \mathbb{R}^3$ with the following properties:

(i) If $v_1, \ldots, v_{2m} \in V_b(\mathscr{D})$ are the neighbors of a ⓒ-labelled vertex in cyclic order, then $f_b(v_1), \ldots, f_b(v_{2m})$ lie on a circle in $\mathbb{R}^3$ in the same cyclic order. This defines a map from the ⓒ-labelled white vertices to the set of circles in $\mathbb{R}^3$.

(ii) If $v_1, \ldots, v_{2m} \in V_b(\mathscr{D})$ are the neighbors of a ⓢ-labelled vertex in cyclic order, then $f_b(v_1), \ldots, f_b(v_{2m})$ lie on a sphere in $\mathbb{R}^3$. This defines a map from the ⓢ-labelled white vertices to the set of spheres in $\mathbb{R}^3$.

(iii) If $v_c$ and $v_s$ are the ⓒ-labelled and the ⓢ-labelled vertices of a quadrilateral of $\mathscr{D}$, the circle corresponding to $v_c$ intersects the sphere corresponding to $v_s$ orthogonally.

If $f_b : V_b(\mathscr{D}) \to \mathbb{R}^3$ is a discrete $S$-isothermic surface, the *central extension* of $f_b$ is the discrete isothermic surface $f : V(\mathscr{D}) \to \mathbb{R}^3$ defined by

$$f(v) = f_b(v) \quad \text{if } v \in V_b(\mathscr{D}),$$

and otherwise by

$$f(v) = \text{the center of the circle or sphere corresponding to } v.$$

Note that the quadrilaterals corresponding to the faces of $\mathscr{D}$ are planar right-angled kites. Thus they are conformal squares since their cross-ratio is $-1$. Hence, this definition implies that a discrete $S$-isothermic surface is a polyhedral surface such that the faces have inscribed circles and the inscribed circles of neighboring faces touch their common edge in the same point.

**Proposition A.4** ([13, Proposition 2]). *Let $f : V(\mathscr{D}) \to \mathbb{R}^3$ be a discrete isothermic surface, where the quad-graph $\mathscr{D}$ is simply connected. Then the edges of $\mathscr{D}$ may be labelled*

*"+" and "−" such that each quadrilateral has two opposite edges labelled "+" and the other opposite edges labelled "−". The* dual discrete isothermic surface $f^*$ *is defined by the formula*

$$\Delta f^* = \pm \frac{\Delta f}{\|\Delta f\|^2},$$

*where $\Delta f$ denotes the difference of neighboring vertices and the sign is chosen according to the edge label.*

**Proposition A.5** ([13, Proposition 3])**.** *The dual of a central extension of a discrete $S$-isothermic surface is itself a central extension of a discrete $S$-isothermic surface.*

The definition and construction of $S$-isothermic discrete minimal surfaces is based on the following characterization of smooth minimal surfaces due to Christoffel [25]. A proof can also be found in [30].

**Theorem A.6** (Christoffel)**.** *Minimal surfaces are isothermic. An isothermic immersion is a minimal surface if the dual immersion defined by*

$$f_x^* = \frac{f_x}{\|f_x\|^2}, \quad f_y^* = -\frac{f_y}{\|f_y\|^2}$$

*is contained in a sphere. In that case the dual immersion is in fact the Gauss map of the minimal surface, up to scale and translation.*

As discrete versions of the Gauss map we consider *Koebe polyhedra.* These are convex polyhedra with all edges tangent to the sphere $\mathbb{S}^2$. The idea is now to define a discrete minimal surface as an $S$-isothermic surface which is dual to a (part of a) Koebe polyhedron. Together with Theorem A.8 below this leads to the following definition.

**Definition A.7.** An *$S$-isothermic discrete minimal surface* is an $S$-isothermic discrete surface $F : V(\mathscr{D}) \to \mathbb{R}^3$ which satisfies one of the equivalent conditions (i)–(iii) below. Suppose that $v_s \in V(\mathscr{D})$ is a white vertex labelled ⓢ such that $F(v_s)$ is the center of a sphere. Let $y_1, \ldots, y_{2m}$ be the vertices incident to $v_s$ in $\mathscr{D}$ in cyclic order. Let $F(y_j) = F(v_s) + b_j$. Then the following equivalent conditions hold.

(i) The points $F(v_s) + (-1)^j b_j$ lie on a circle.

(ii) There is an $N \in \mathbb{R}^3$ such that $(-1)^j (b_j, N)$ is the same for $j = 1, \ldots, 2m$.

(iii) There is a plane through $F(v_s)$ and the centers of the orthogonal circles which intersect the sphere with center $F(v_s)$ orthogonally. Then the points $\{F(y_j) : j \text{ even}\}$ and the points $\{F(y_j) : j \text{ odd}\}$ lie in planes which are parallel to it at the same distance on opposite sides.

**Theorem A.8** ([13, Theorem 5])**.** *An $S$-isothermic discrete surface is an $S$-isothermic discrete minimal surface if and only if the dual $S$-isothermic discrete surface corresponds to a Koebe polyhedron.*

Given any planar orthogonal circle pattern with the combinatorics of the square grid, a corresponding $S$-isothermic discrete minimal surface can be obtained using a formula which resembles the Weierstrass representation formula.

**Theorem A.9** (Discrete Weierstrass representation; [13, Theorem 6])**.** *Let $\mathscr{D}$ be an $S$-quad-graph and let $\mathscr{C}$ be a planar orthogonal circle pattern for $\mathscr{D}$. The $S$-isothermic discrete minimal surface*

$$F : \{x \in V(\mathscr{D}) : x \textit{ is labelled } ⓢ\} \to \mathbb{R}^3,$$
$$F(x) = \textit{the center of the sphere corresponding to } x$$

*that corresponds to this circle pattern is given by the following formula. Let* $x_1, x_2 \in V(\mathscr{D})$ *be two vertices, both labelled* ⓢ*, which correspond to touching circles of the pattern. Let* $y \in V(\mathscr{D})$ *be the black vertex between* $x_1$ *and* $x_2$*, which corresponds to the point of contact. Then the centers* $F(x_1)$ *and* $F(x_2)$ *of the corresponding touching spheres of the* $S$*-isothermic discrete minimal surface* $F$ *satisfy*

$$F(x_2) - F(x_1) = \pm\mathrm{Re}\left(\frac{R(x_2)+R(x_1)}{1+|p|^2}\,\frac{\overline{c(x_2)-c(x_1)}}{|c(x_2)-c(x_1)|}\begin{pmatrix}1-p^2\\ i(1+p^2)\\ 2p\end{pmatrix}\right), \tag{A.1}$$

*where* $p = c(y)$ *and the radii* $R(x_j)$ *of the spheres are*

$$R(x_j) = \left|\frac{1+|c(x_j)|^2 - |c(x_j)-p|^2}{2|c(x_j)-p|}\right|. \tag{A.2}$$

*The sign on the right-hand side of equation* (A.1) *depends on whether the two edges of the quad-graph connecting* $x_1$ *with* $y$ *and* $y$ *with* $x_2$ *are labelled '+' or '−'.*

Every smooth minimal surface comes with an associated family of isometric minimal surfaces with the same Gauss map. The members of this family remain conformally, but not isothermically, parameterized. This concept carries over to the discrete case, where the discrete surfaces of the associated family are not $S$-isothermic, but should be considered as discrete conformally parameterized minimal surfaces.

**Definition A.10.** The *associated family* $F_\varphi$ of an $S$-isothermic discrete minimal surface $F_0$ consists of the one-parameter family of discrete surfaces that are obtained by the following construction. Before dualizing the Koebe-polyhedron, rotate each edge by an equal angle $\varphi$ in the plane which is tangent to the unit sphere in the point where the edge touches the unit sphere.

This definition implies that the discrete associated family keeps essential properties of the smooth associated family. In particular, the surfaces are isometric, have the same Gauss map and the Weierstrass-type formula of Theorem A.9 may be extended to the associated family.

**Theorem A.11** ([13, Theorem 8])**.** *The discrete surfaces* $F_\varphi$ *of the associated family of an* $S$*-isothermic discrete minimal surface* $F_0$ *consist of touching spheres. The radii of the spheres do not depend on* $\varphi$.

*In the generic case, when the* $S$*-quad-graph is a part of* $SG$*, there are also circles through the points of contact. The normals of these circles do not depend on* $\varphi$.

**Theorem A.12** ([13, Theorem 7])**.** *With the notation of Theorem A.9, the discrete surfaces* $F_\varphi$ *of the associated family satisfy*

$$F_\varphi(x_2) - F_\varphi(x_1) = \pm\mathrm{Re}\left(e^{i\varphi}\frac{R(x_2)+R(x_1)}{1+|p|^2}\,\frac{\overline{c(x_2)-c(x_1)}}{|c(x_2)-c(x_1)|}\begin{pmatrix}1-p^2\\ i(1+p^2)\\ 2p\end{pmatrix}\right).$$

As explained in section 9 of [13], the definition and the properties of $S$-isothermic discrete minimal surfaces can be used to deduce a *construction scheme* for discrete minimal analogs of known smooth minimal surfaces. In Section 8.1, we present some more details how to construct the concrete examples presented in Section 8.2. The difficult part is finding the right circle pattern (steps 1 and 2 below).

**Step 1:** Consider a given smooth minimal surface together with its conformal curvature line parameterization. Map the curvature lines to the unit sphere by the Gauss map to obtain a qualitative picture. The goal is to *understand the combinatorics of the curvature lines.*

From the combinatorial picture of the curvature lines we choose finitely many curvature lines to obtain a finite cell decomposition of the unit sphere $\mathbb{S}^2$ (or a part or a branched covering of $\mathbb{S}^2$) with quadrilateral cells. The choice of the number of curvature lines corresponds to the level of refinement (and possibly to the choice of different length parameters). Generically, all vertices have degree 4. Exceptional vertices correspond to umbilic or singular points, ends or boundary points of the minimal surface. The vertices have to be colored black and white. This choice is usually determined by exceptional vertices, since umbilics and singular points have to be white vertices. So the cell decomposition leads to an $S$-quad-graph which provides a *combinatorial conformal parameterization.* This cell decomposition (together with additional informations about the behavior at the boundary or near ends if necessary) is the main ingredient to construct the analog discrete minimal surface.

**Step 2:** Given the combinatorics from step 1, *construct an orthogonal spherical circle pattern* where circles correspond to white vertices of the cell decomposition. If two white vertices are incident to the same face then the corresponding circles intersect orthogonally. Circles corresponding to vertices which are not incident to one another but to the same black vertex touch. At boundary vertices we use information about the smooth minimal surface to prescribe angles. Ends have also to be taken care of but we do not consider this case further.

For some comments on how to practically calculate the circle pattern, see remark A.14 below.

**Step 3:** From the circle pattern, *construct the Koebe-polyhedron.* Here a choice is made which of the white vertices are labelled ⓒ and ⓢ. The two choices lead to different discrete surfaces close to each other. Take the circles labelled ⓢ and build the spheres which intersect $\mathbb{S}^2$ orthogonally in these circles. Then build the Koebe polyhedron by joining the centers of touching spheres by an edge (tangent to $\mathbb{S}^2$).

**Step 4:** *Dualize the Koebe polyhedron* to arrive at the desired $S$-isothermic discrete minimal surface.

For some simple cases, this scheme can be used in a straightforward way. In general, the first two of the above construction steps require some care.

**Remark A.13.** The cell decomposition constructed in step 1 (together with additional information about angles and lengths at the boundary if necessary) is the only ingredient for the remaining construction of the discrete minimal surface. Therefore the main task consists in finding a suitable cell decomposition which corresponds to the given smooth minimal surface.

If there are combinatorially different types of curvature lines, the choice of different numbers of curvature lines of different types may affect the constructed discrete minimal surface; see Section 8.1.3.

Additionally to perform step 2, we need a finite cell decomposition, that is we only need curvature lines which are closed modulo the boundary. More precisely, the following property has to hold for all chosen (combinatorial) curvature lines: Construct a curve by first following one curvature line until it meets the boundary. This boundary part is either a curvature line or an intersection point between two boundary asymptotic lines (in these cases we are done) or an asymptotic line. In the latter case, follow the other curvature line emanating from the boundary point to the next boundary and so on. The curve constructed by this procedure then ends at a boundary curvature line (or in a boundary corner) or it has to be closed after only finitely many steps. This is not generally the case for curvature lines of smooth minimal surfaces. Thus, for the combinatorial conformal parametrization we cannot always take curvatures lines directly from the smooth surface. Instead, we choose approximating lines to obtain a similar discrete parametrization.

**Remark A.14.** Given the combinatorial data (and some boundary data if necessary), the goal of step 2 is to actually calculate the circle pattern corresponding to the given cell decomposition (and boundary constraints). The rest of the construction does not need additional data.

Existence and uniqueness (up to Möbius transformations) of such a spherical circle pattern was first proven by Koebe in case of a triangulation of the sphere in [51]. Generalized versions for polytopal cell decompositions of the sphere may be found in [21, 16, 70].

Many of the examples in Section 8.2 do not rely on cell decompositions of the whole sphere, but only of a part with given boundary data. Thus one has to find a circle pattern with Neumann boundary conditions. Now, the main task is to suitably specify the boundary conditions. Note that for the examples presented in Section 8.2 the boundary conditions are simplest for the *spherical* circle patterns. Therefore, we have actually used a method developed by Springborn for calculating the spherical circle patterns by a variational principle, see [13, Section 8] or [70] for more details. The solutions of the spherical circle pattern problem with given boundary angles are in one-to-one correspondence with the critical points of a functional. Since this functional is not convex and has negative index at least one, the critical point cannot be obtained by simply minimizing the functional. In order to numerically compute the spherical circle pattern, a convenient reduced functional is used instead. Existence and uniqueness of a solution are not yet proven. Nevertheless, this method has proven to be amazingly powerful, in particular to produce all the spherical circle patterns used for the examples in Section 8.2.

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