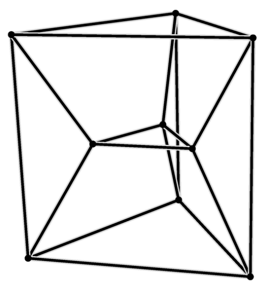
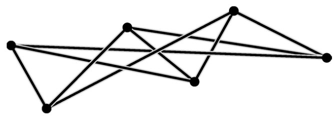
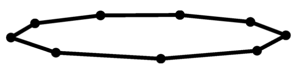
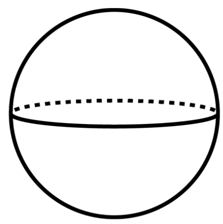
Constructions and obstructions for extremal polytopes





$$\begin{tikzcd} \Delta_2 \times \Delta_2 \arrow[d, "\pi"] & \Longleftrightarrow & \mathsf{K}_{3,3} \arrow[d, hook] \\ \mathcal{P}_9 & & S^2 \end{tikzcd}$$





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Constructions and obstructions for extremal polytopes

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aus Berlin

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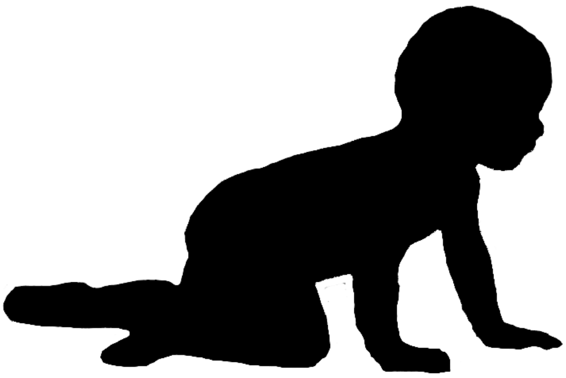


expected October 2008

Acknowledgements.

In the fall of 2001, I was finishing my computer science undergrad studies and **Martin Skutella** suggested that I take the course "Linear optimization", a course offered to beginning math grad students and taught by **Prof. Günter Ziegler**. Following his advice, my fellow student **Roman Wagner** and I took the course, and failed badly. We lacked the necessary linear algebra and so the teaching assistant had a hard time understanding our "solutions" to the exercises. However, I liked the little that I understood and after the next semester break, which I spent with a linear algebra book, I re-started the ADM-lecture-cycle, this time with "Graph and network algorithms" and more computer scientists: **Moritz Hilger** and **Oliver Wirjadi** joined the team. We had a great time spending afternoons at the "beach" solving the exercises. But the best of all was our tutor **Vanessa Kääb**: In three months from now we will get married and in two more months we will have our first child – ain't that great! The teaching assistant was **Frank Lutz** who offered me the job as a tutor for the linear optimization course – not a bad setup as I did not have to do the exercises again. He also encouraged me to attend the "Discrete Geometry" course in the 2004 summer term, also taught by Günter. In the following semester break I started working on my diploma thesis under Günter's supervision. I got a desk in the workgroup and I met lots of people to talk about math, including **Jakob Uszkoreit**, someone who can listen to more math than I'm able to tell. I did a decent job in my thesis and Günter offered me a PhD position, first on his Leibnitz grant and later in the research training group "Methods for Discrete Structures". That was in October 2005. Now it is May 2008 and the end of the journey.

I had a wonderful time. Günter taught me how to do research, how to "write stuff up," and how to give talks. I enjoyed the frequent morning coffees with him in the "Molotov Depot" (it opens up at 7:20 am, weekends a little later) and I appreciated the fact that he had always time for yet another (stupid) question. I benefited not only from what he is doing *in* but also from what he is doing *for* mathematics. It was a pleasure to receive steady support from everyone on the **Combi-Flur**. **Thilo Rörig** has been a great colleague and friend. **Axel Werner** and his whiteboard sacrificed an incredible amount of time to put some of my ideas through their paces. **Carsten Schulz** was my personal walking topology encyclopedia, best to be consulted over a beer. **Elke Pose** was my secret weapon for cutting through the red tape. **Christian Haase**, at our branch office at the FU, became my official BMS-mentor and a dear friend and we spent many hours together not only talking math.

Bernd Sturmfels put a lot of confidence in me when he nominated me for a Miller Fellowship and agreed to co-supervise this thesis. **Andreas Paffenholz**, Axel, Christian, Thilo, and Vanessa agreed to proofread (parts of) this thesis and they did a fine job (didn't they?).

Most importantly, I was blessed with the endless support from my family: my mother (and soon to be a grandmother) and my future wife Vanessa.

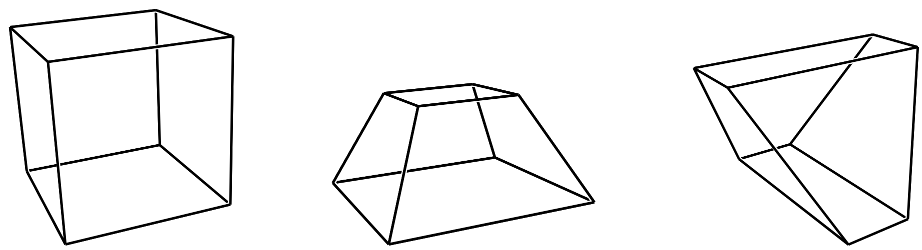
Thank you all for making this happen ... and stay tuned for more.

Contents

Introduction

A famous quote of Kalai [30] moans the scarcity of *interesting* examples in mathematics and it is true that a life time can be spend in search for an object with specific qualities. In discrete geometry, the domain in which this thesis is rooted, the objects are usually *convex polytopes* and *polyhedral complexes*, the properties are *combinatorial* in nature and the quest starts early – more often than not in dimensions 3 or 4. The aim of this work is twofold: In reverse order, we construct interesting polytopes in the second part of this thesis and we investigate limitations of a construction principle that provided us with many interesting examples.

As we will give precise definitions for all concepts in the following chapter, we will be content with motivating this work along examples. A collection of some 3-dimensional polytopes is this



Each polytope has 8 vertices (=extreme points) and every vertex is connected to exactly three other vertices along an edge (=1-dimensional face) and incident to three facets (here quadrilaterals). Although there are clear-cut differences in terms of shape, all three specimens exhibit certain common features. They all have 26 (proper, non-empty) faces: 8 vertices, 12 edges, and 6 facets. Moreover the incidences among faces of various dimensions – their face lattices – coincide and we define the combinatorial type of these polytopes as the class of all 3-dimensional polytopes with this face lattice. A coarser invariant is the incidence structure of only the vertices and edges, the 1-*skeleton*, and a natural question to ask is: Is the 1-skeleton sufficient to tell the combinatorial type?

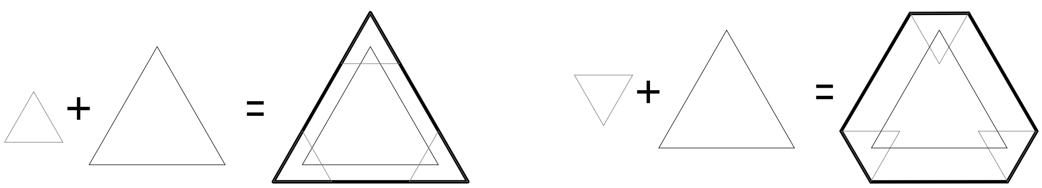
While this is true in dimensions $d \leq 3$, it miserably fails in dimensions $d \geq 4$: For every $d \geq 4$ there is a 4-polytope and a d-polytope, each on $d+1$ vertices and having a *complete* 1-skeleton, i.e. every two vertices share an edge. In a precise sense this makes the mentioned 4-polytope combinatorially *extreme*. A fruitful approach to the construction of combinatorially *extremal* polytopes was undertaken in joint work with Günter Ziegler [53]. Extending ideas from [64], we presented a construction principle that yields d-dimensional polytopes whose k-skeleta are isomorphic to those of polytopes of dimension larger than d. The idea is to construct a realization of a given polytope such that the projection to d-space *strictly preserves* all faces of dimension $\leq k$.

In **Part I** of this thesis we investigate the limitations of this construction principle: *Given a d-dimensional combinatorial type, what are the necessary conditions for the existence of a polytope of that type such that a projection to $\mathbb{R}^e$ yields a polytope with the given k-skeleton?*

In Chapter 2 we give a careful introduction to projections of polytopes and, in particular, introduce the combinatorially convenient class of *strictly preserved faces*. The concept of strictly preserved faces models essentially how faces of a polytope behave under *generic* projections: The image of a face under a generic projection is either a face affinely isomorphic to its preimage or it fails to be a face at all. We present three applications in which strictly preserved faces play a prominent role and each of which asks for the existence of polytopes with certain combinatorial and geometric qualities.

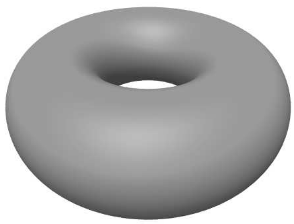
The first application sketches a main result of [53], the construction of a d-polytope having the $\lfloor \frac{d-2}{2} \rfloor$-skeleton of a product of $r \geq \frac{d}{2}$ polygons, each with an even number of vertices. The emphasis is put on "even", as the analogous construction for odd polygons fails. In the course of the first part of the thesis we will develop tools to prove that there is some non-trivial topology that *obstructs* the odd case from working out.

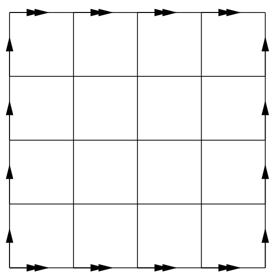
The second application treats the extremal behavior of *Minkowski sums* of polytopes: The pointwise (vector) sum of two convex polytopes in $\mathbb{R}^d$ turns out to be a convex polytope. This recycling operation of polytopes is highly non-combinatorial in nature; the combinatorial structure of the sum heavily depends on the geometry of the summands. Nevertheless, it is reasonable to wonder how extremal the sum can be with respect to the summands. Is it, for example, possible to sum two triangles in the plane to get a 9-gon? Here are two typical attempts:

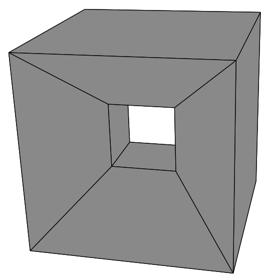


Some experimentation yields that, depending on the shape of the two summands, it is possible to obtain every polygon with three to six vertices, but no 9-gon. What makes us think that there should be a 9-gon, is that in 3-space there exist two tetrahedra such that their sum has $4 \cdot 4 = 16$ vertices and, more general, Fukuda & Weibel [18] show that in d-dimensional space it is possible to sum $\leq d-1$ polytopes with arbitrarily many vertices $n_1, n_2, \ldots, n_{d-1}$ and obtain a polytope having a total of $n_1 \cdot n_2 \cdots n_{d-1}$ vertices. What happens when we require at least d summands? Experimentation in the plane suggests that it is not possible, but how do we go about proving such a result? We will make some observations that reduces the general problem to essentially a single case per dimension. The crucial insight will be that the Minkowski sum of two polytopes is the image under projection of a certain polytope and the fact that all possible vertices show up in the sum requires the projection to strictly preserve all vertices. We show that the sum of every collection of $r \geq d$ polytopes, each having at least $d+1$ vertices, fails to have the maximal number of vertices and some counting yields a lower bound on the number of vertices that go amiss.

The third application concerns the realization of polyhedral surfaces. A polyhedral surface is a collection of (combinatorial) polygons with an instruction manual of how to glue them edge-to-edge such that at every vertex the incident polygons can be arranged cyclically with neighboring polygons sharing an edge. The following three figures show a torus, a (partial) gluing of a torus from quadrilaterals, and a realization of that gluing with flat and convex faces[1].







Topologically, every orientable surface can be embedded in 3-space but, by a geometric result of Betke & Gritzmann [6], there exist polyhedral surfaces that posses no realization with flat and convex faces in d-space for *any* d. McMullen, Schulz, and Wills [41, 42] constructed families of *equivelar* surfaces, that is, polyhedral surfaces with only one type of polygon and the same number of polygons meeting at every vertex, that are realized in $\mathbb{R}^3$

[1] Courtesy of Thilo Rörig.

and have "unusually large genus" – the number of holes is $\mathcal{O}(n \log n)$ with n being the number of vertices. A different realization of one of those families was given by Joswig & Rörig [26] via the following observation: Every surface in that family is a subcomplex of the 2-skeleton of an n-dimensional cube and there is a realization of that cube such that a projection to $\mathbb{R}^4$ strictly preserves the surface. Thus, every such surface is embedded in the boundary of a 4-dimensional polytope and may be embedded with flat and convex faces in 3-space via Schlegel diagrams, i.e. stereographic projections from the boundary sphere.

In [49], Rörig & Ziegler pursued this line of thought and generalized their construction to *wedge products*, a generalization of joins and products of polytopes. They showed that certain wedge products contain a multitude of equivelar surfaces and for a certain family they constructed realizations of the wedge product that safely carry the surfaces to 4-space under projection. In Section 2.3 we give an introduction to wedge products from a geometric and a combinatorial perspective that is different from that in [49]. Our presentation highlights the kinship of wedge products to joins and products that is reminiscent of the products that occur in the previously discussed applications. We give a short account of the surfaces considered by Rörig & Ziegler and prove that for most of their surfaces there is no realization of the corresponding wedge product that retains the surface under projection to $\mathbb{R}^4$.

Chapter 3 is devoted to the study of geometric and topological properties of projections. The advantage of working with strictly preserved faces is that they have an algebraic characterization – the Projection Lemma (cf. Lemma 3.1). Under mild conditions, such as the strict preservation of all vertices, the Projection Lemma together with Gale duality yields a polytope $\mathcal{A}$ associated to the projection of polytopes. The dimension of $\mathcal{A}$ essentially depends on the dimension of the target space of the projection. Moreover, every face that is strictly preserved under projection induces a face of $\mathcal{A}$ and the union of all induced faces yields a subcomplex of the boundary of $\mathcal{A}$, a topological sphere of prescribed dimension. The bottom line is that if a certain collection of faces is supposed to be strictly preserved under projection then the union of all induced faces yields a topological space that is embedded into a sphere of fixed dimension. Thus, in order to obstruct the existence of a realization with the required properties under projection, we show that the corresponding topological space is not embeddable into a sphere of the given dimension.

In our situation the topological sphere is a simplicial complex and therefore has a combinatorial description. We give an account of the tools from topological combinatorics that yield lower bounds on the embeddability dimension of simplicial complexes and we devise tools for treating the special types of complexes that occur.

In Chapter 4 we give the missing proofs for the results in Chapter 2. We calculate obstructions to the projectability of k-skeleta for three families of polytopes: products of polygons, products of simplices, and wedge products of polygons and simplices. Using the results of Chapter 3, this reduces to finding colorings of certain graphs and finding lower bounds on the optimal value of certain integer linear programs. In the case of products of simplices, the graphs in question are the famous *Kneser graphs*.

The results in Part I appear in [51] and [48], the latter of which is written in collaboration with Thilo Rörig.

In **Part II** we investigate the validity of three conjectures of Kalai [29] regarding centrally symmetric polytopes. A polytope $P \subset \mathbb{R}^d$ is centrally symmetric if P is symmetric with respect to the origin, i.e. $P = -P$. The most basic examples are the regular cubes and crosspolytopes.

The first conjecture, the 3^d*-conjecture*, states that every centrally symmetric d-dimensional polytope has at least 3^d non-empty faces. Although it seems that the validity or invalidity of this conjecture has no far reaching implications, it is a rather provocative conjecture in the sense that playing with examples furnishes the impression that the conjecture should be true or false for basically *trivial* reasons. However, while the conjecture is obviously true in dimension ≤ 2 and a simple exercise in dimension 3, the last 18 years saw no progress in dimensions ≥ 4. In Section 5.1, we prove the 3^d-conjecture in dimension 4. The main ingredient for the proof, which is interesting in its own right, is a tight lower bound on the flag-vector functional g_2^{tor} for the class of centrally symmetric d-polytopes. The flag-vector functional g_2^{tor} is a certain linear combination of the dimension, the number of vertices, the number of 2-dimensional faces as well as the number of vertex–2-face incidences. The proof depends on a beautiful connection made by Kalai [28] between these combinatorial numbers and the statics of *frameworks* associated to the 2-skeleton of a polytope.

The most elementary d-dimensional polytope is the *d-simplex*, which can be characterized as the d-dimensional polytope that minimizes the number of k-dimensional faces among d-polytopes for all k simultaneously. For the class of centrally symmetric polytopes there is no single such polytope: The d-dimensional crosspolytope has the minimal number of vertices while the d-dimensional cube has the minimal number of facets (=($d-1$)-dimensional faces). Whatever a set of *minimal* centrally symmetric polytopes might be, it should attain the minimum of 3^d faces. A potential candidate is given by the class of *Hanner* polytopes [24], a class of centrally symmetric polytopes that subsumes the cubes and crosspolytopes, and the claim that this is the right class is precisely the statement of Kalai's second conjecture. In dimension 4, the arguments that prove the 3^d-conjecture basically yield the validity of the second conjecture. Yet a main achievement is the construction of counterexamples in all dimensions $d \geq 5$. However, we show that the number of counterexamples is finite in fixed dimension.

The third conjecture is a generalization of the second conjecture and thus false in dimensions ≥ 5. In dimension 4 we construct an infinite class of centrally symmetric polytopes that refutes an even stronger version. We close with two examples from an interesting family of centrally symmetric polytopes that might lead the way to counterexamples for the 3^d-conjecture.

The results of Part II are based on joint work [52] with Axel Werner and Günter Ziegler.

Chapter 1

Basics

As the title suggests, *the* main objects of this thesis are polytopes. We will consider convex polytopes from a geometric, a combinatorial, and a topological perspective and we use this chapter to collect some of the necessary language and notation surrounding polytopes. We are, however, of the opinion that it pays off to introduce concepts only when they are needed and then only as succinct as possible in order not to deviate too much from the main story line. Thus, additional material not only regarding polytopes is scattered throughout with concise definitions and statements and furnished with pointers to the relevant literature.

In this spirit, here are the main characters of this play. We start with polytopes as geometric objects but quickly reduce them to their key combinatorial properties in the form of posets. We introduce the reader to simplicial complexes that, for us, bridge the gap between combinatorics and topology. Our main references for polytopes are the books by Grünbaum [23] and Ziegler [63], for partially ordered sets and relatives the book by Stanley [57, Sect. 3.2] is recommended, and for further information about simplicial complexes we refer the reader to the books of Matoušek [37] and Munkres [44] as well as the handbook article by Björner [7].

Definition 1.1 (Polytope)**.** A *polytope* $P \subset \mathbb{R}^d$ is a bounded subset that arises

($\mathcal{V}$) as the convex hull $P = \mathsf{conv}\, V$ of a finite set of points $V \subset \mathbb{R}^d$, or

($\mathcal{H}$) as the intersection of finitely many halfspaces, i.e. there are normal vectors $\mathsf{a}_1, \ldots, \mathsf{a}_m \in \mathbb{R}^d$ and displacements $b_1, \ldots, b_m \in \mathbb{R}$ such that

$$P \;=\; \bigcap_{i=1}^{m} \{\mathbf{x} \in \mathbb{R}^d : \mathsf{a}_i^{\mathsf{T}}\,\mathbf{x} \le b_i\}.$$

The former presentation is called the $\mathcal{V}$- or *interior* presentation whereas the latter is called the $\mathcal{H}$- or *exterior* presentation. The main theorem of polytope theory unites the two parts of the definition.

Theorem 1.2 (Minkowski–Weyl Theorem [63, Thm. 1.1])**.** *Let $P \subset \mathbb{R}^d$. Then P is a $\mathcal{V}$-polytope if and only if P is an $\mathcal{H}$-polytope.*

Geometric polarity, which basically proves the main theorem, can also be held responsible for the fact that every notion in polytope theory comes essentially in two different flavors – interior and exterior. The inclusion-minimal set $\mathsf{vert}\, P = V \subset \mathbb{R}^d$ such that $P = \mathsf{conv}\, V$ is called the *vertex set* of P. For an irredundant collection of halfspaces as in the above definition, we economically collect the normals a_i^T as the rows of a matrix $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$ likewise and write $P = P(A, b) = \{\mathbf{x} : A\,\mathbf{x} \leq b\}$. The bounding hyperplane of a halfspace from this irredundant collection is said to be *facet defining*.

Definition 1.3 (Face)**.** Let $P \subset \mathbb{R}^d$ be a polytope. A subset $F \subseteq P$ is a *face* of P if there is a normal vector $\mathsf{c} \in \mathbb{R}^d$ and a displacement $\delta \in \mathbb{R}$ such that $\mathsf{c}^\mathsf{T}\,\mathbf{x} \leq \delta$ for all $\mathbf{x} \in P$ and

$$F = \{\mathbf{x} \in P : \mathsf{c}^\mathsf{T}\,\mathbf{x} = \delta\}.$$

Note that the definition comprises the degenerated cases $(\mathsf{c}, \delta) = (\mathbb{0}, +1)$ and $= (\mathbb{0}, 0)$ and thus $F = \varnothing$ and $F = P$ are both faces of P. Every face $F \neq P$ is called *proper*. The *dimension* of a face F is $\mathsf{dim}\, F := \mathsf{dim}\,\mathsf{aff}\, F$, the dimension of the affine span of F. The dimension of $P \subset \mathbb{R}^d$ is the dimension of it as a face and P is called *full-dimensional* if $\mathsf{dim}\, P = d$. The faces of dimension $0, 1, d-2$, and $d-1$ are called *vertices, edges, ridges,* and *facets*, respectively. To avoid cumbersome descriptions we mostly abbreviate "d-dimensional polytope" and "k-dimensional face" by *d-polytope* and *k-face*, respectively.

Proposition 1.4 ([63, Prop. 2.3])**.** *Let $P = P(A, b)$ be a polytope with $A \in \mathbb{R}^{m \times d}, b \in \mathbb{R}^m$ and $V = \mathsf{vert}\, P$. If $F \subseteq P$ is a face then*

i) $F = \mathsf{conv}\,(F \cap V)$*, and*
ii) $F = \{\mathbf{x} \in P : \mathsf{a}_i^\mathsf{T}\,\mathbf{x} = b_i$ *for all* $i \in I(F)\}$ *with*

$$I(F) := \{i \in [m] : \mathsf{a}_i^\mathsf{T}\,\mathbf{x} = b_i \textit{ for all } \mathbf{x} \in F\} \subseteq [m].$$

For every face F we call the set $I(F)$ the *facet incidences* or the *equality set* of F.

In preparation for later chapters, we need to abstract from the geometry of P and focus on its combinatorial properties. The combinatorial notion to which we relate polytopes is that of a *partially ordered set* (or *poset*, for short) or, more precisely, a *lattice*.

Definition 1.5. Let P be a polytope. The *face lattice* $\mathcal{L}(P)$ is the collection of faces of P ordered by inclusion and graded by $\mathsf{dim}\, F - 1$. Two polytopes P and Q are *combinatorially isomorphic* if $\mathcal{L}(P) \cong \mathcal{L}(Q)$ as ranked posets. We call a graded poset $\mathcal{P}$ a *combinatorial type* of dimension d (or d*-type*, for short) if $\mathcal{P} \cong \mathcal{L}(P)$ for some d-polytope P and, conversely, P is called a *realization* of the combinatorial type $\mathcal{P}$.

We want to think about combinatorial types as polytopes stripped of their geometric realization and we will therefore stick to our geometric terminology and, for example, call $F \in \mathcal{P}$ a face of $\mathcal{P}$. The poset $\mathcal{L}(P)$ is indeed a lattice: The faces $\varnothing$ and P play the role of the minimum and the maximum in $\mathcal{L}(P)$ and every two faces F and G have a unique maximal face $F \cap G$ contained in both and a unique minimal face $\mathsf{conv}\,(F \cup G)$ containing both faces F and G. An implication of Proposition 1.4 is that we can get $\mathcal{L}(P)$ as a subposet of $2^M := \{M' : M' \subseteq M\}$ for some discrete M in two different ways.

Corollary 1.6. *Let P be a d-polytope on m facets with $V = \mathsf{vert}\, P$. Then $\mathcal{L}(P)$ is isomorphic to the following two posets*

1. $\{\mathsf{vert}\, F : F \subseteq P$ *a face*$\} \subseteq 2^V$ *ordered by inclusion, and*
2. $\{I(F) : F \subseteq P$ *a face*$\} \subseteq 2^{[m]}$ *ordered by reverse inclusion.* □

The first part of Corollary 1.6 shows that $\mathcal{L}(P)$ is *atomic*, i.e. a face is the convex hull of its vertices, whereas the second poset shows that $\mathcal{L}(P)$ is *coatomic*, every face is the intersection of facets. For a combinatorial type $\mathcal{P}$ on m facets we write $I_{\mathcal{P}}(F) \subseteq [m]$ for the incidence relation of faces and facets. When there is no possible confusion, we omit the subscript $\mathcal{P}$.

A virtue of working with combinatorial types is that certain geometric operations on polytopes can be carried out entirely on the level of posets. For each of the following three operations we give a geometric and a combinatorial description. Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be polytopes of dimensions d and e with $\mathbb{0}$ in the interior.

The join $P * Q$ is the $(d+e+1)$-dimensional polytope obtained by embedding P and Q in skew affine subspaces in $\mathbb{R}^{d+e+1}$ and taking the convex hull. If $P = P(A, \mathbb{1})$ and $Q = P(B, \mathbb{1})$ then the join is given by the points $(\mathbf{x}, \mathbf{y}, z) \in \mathbb{R}^{d+e+1}$ satisfying

$$\begin{array}{ccl} A\mathbf{x} & & +z \le \mathbb{1} \\ & B\mathbf{y} & -z \le \mathbb{1}. \end{array}$$

The face lattice of the join is given by $\mathcal{L}(P * Q) = \mathcal{L}(P) \times \mathcal{L}(Q)$, the direct product of posets. The order relation is componentwise inclusion. The dimension of $(F, G) \in \mathcal{L}(P * Q)$ is $\mathsf{dim}(F, G) = \mathsf{dim}\, F + \mathsf{dim}\, G + 1$. We write a face of the join also as $F * G$.

The product $P \times Q = \mathsf{conv}\{(p,q) : p \in P, q \in Q\}$ is a $(d+e)$-dimensional polytope. For $P = P(A, \mathbb{1})$ and $Q = P(B, \mathbb{1})$ the product is the set of solutions to the inequality system

$$\begin{pmatrix} A & \mathbb{0} \\ \mathbb{0} & B \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \leq \begin{pmatrix} \mathbb{1} \\ \mathbb{1} \end{pmatrix}.$$

The face lattice $\mathcal{L}(P \times Q)$ is the induced subposet of the direct product of the respective face lattices given by the set of elements $(F, G) \in \mathcal{L}(P) \times \mathcal{L}(Q)$ satisfying $F = \varnothing$ if and only if $G = \varnothing$. The dimension of a nonempty face $(F, G) \neq (\varnothing, \varnothing)$ is $\mathsf{dim}(F, G) = \mathsf{dim}\, F + \mathsf{dim}\, G$.

The direct sum $P \oplus Q$ is the $(d+e)$-polytope given by the convex hull of $P \times \{\mathbb{0}\} \cup \{\mathbb{0}\} \times Q$. Thus

$$P \oplus Q = \mathsf{conv}\left\{ \begin{pmatrix} u \\ \mathbb{0} \end{pmatrix}, \begin{pmatrix} \mathbb{0} \\ v \end{pmatrix} : u \in \mathsf{vert}\, P, v \in \mathsf{vert}\, Q \right\}.$$

Similar to the product, the face lattice is the induced subposet of the direct product given by $\mathcal{L}(P \oplus Q) = \{(F, G) : F = P \text{ iff } G = Q\} \subset \mathcal{L}(P) \times \mathcal{L}(Q)$ with dimension function $\mathsf{dim}(F, G) = \mathsf{dim}\, F + \mathsf{dim}\, G + 1$ for $(F, G) \neq (P, Q)$.

If $P \subset \mathbb{R}^d$ is a d-polytope with $\mathbb{0}$ in the interior then

$$P^{\triangle} = \{\ell \in (\mathbb{R}^d)^* : \ell(\mathbf{x}) \leq 1 \text{ for all } \mathbf{x} \in P\},$$

the set of linear functionals with value at most 1 over P is again a convex d-polytope called the *polar dual* to P. The combinatorics of $P^{\triangle}$ is given by $\mathcal{L}(P^{\triangle}) = \mathcal{L}(P)^{\mathsf{op}}$, where $\mathcal{L}(P)^{\mathsf{op}}$ is the lattice on the same underlying set but ordered by reverse inclusion. For joins it is easily seen that $\mathcal{L}(P^{\triangle} * Q^{\triangle}) \cong \mathcal{L}(P * Q)^{\mathsf{op}}$ and the intimate relation between the product and the direct sum is $\mathcal{L}(P^{\triangle} \oplus Q^{\triangle}) \cong \mathcal{L}(P \times Q)^{\mathsf{op}}$.

We will also be interested in certain subposets of the face lattices: For $\mathcal{P}$ a d-type and $0 \leq k \leq d$, we call the collection of faces $F \in \mathcal{P}$ with $\mathsf{dim}\, F \leq k$ the *k-skeleton* of P. For $k = 0$, this is just the collection of vertices. The 1-skeleton is also called the *graph* of $\mathcal{P}$ and the $(d-1)$-skeleton is called the *boundary* of $\mathcal{P}$ and is denoted by $\partial\mathcal{P}$.

A numeric invariant associated to a d-type $\mathcal{P}$ is $f(\mathcal{P}) = (f_{-1}, f_0, \ldots, f_d)$ the vector of *face numbers* or *f-vector* with f_i equal to the number of i-faces of $\mathcal{P}$. Clearly, the first and the last entry are both equal to 1 and, depending on the context, we omit these entries sometimes.

The *d-simplex* is the combinatorial type $\Delta_d = 2^{[d+1]}$ with the dimension function $\mathsf{dim}\, F = |F| - 1$ which is the face lattice of the convex hull of any

$d + 1$ affinely independent points. A d-type is *simplicial* if every proper face is a simplex, that is if $|\mathsf{vert}\, F| = k + 1$ for every face F of dimension $k < d$. A d-type $\mathcal{P}$ is *simple* if $\mathcal{P}^{\mathsf{op}}$ is simplicial, i.e. we have $|I(F)| = d - \mathsf{dim}\, F$ for every k-face with $k < d$. The geometric property of simplicial and simple polytopes is that the combinatorial type is stable under small perturbations applied to the vertices or the facet defining hyperplanes, respectively.

The boundary of a simplicial polytope belongs to a class of combinatorial/topological objects that will play an important role.

Definition 1.7. Let V be a finite set. A *simplicial complex* $\mathsf{K} \subseteq 2^V$ is a collection of subsets of V with the properties that

i) K is closed under taking subsets, and
ii) $\varnothing \in \mathsf{K}$.

The elements $\sigma \in \mathsf{K}$ are called *faces* and the *dimension* of σ is $\mathsf{dim}\, \sigma = |\sigma| - 1$. The dimension of K is $\mathsf{dim}\, \mathsf{K} = \mathsf{max}\, \{\mathsf{dim}\, \sigma : \sigma \in \mathsf{K}\}$ and K is called *pure* if all inclusion-maximal faces are of equal dimension. The *join* of two simplicial complexes $\mathsf{K} \subseteq 2^V$ and $\mathsf{L} \subseteq 2^W$ on the vertex sets V and W is the simplicial complex

$$\mathsf{K} * \mathsf{L} = \{\sigma \uplus \tau : \sigma \in \mathsf{K}, \tau \in \mathsf{L}\} \subseteq 2^{V \uplus W}$$

where the disjoint union is defined as $A \uplus B := A \times \{1\} \cup B \times \{2\}$. It is easy to verify that the dimension of $\mathsf{K} * \mathsf{L}$ is $\mathsf{dim}\, \mathsf{K} * \mathsf{L} = \mathsf{dim}\, \mathsf{K} + \mathsf{dim}\, \mathsf{L} + 1$.

Proposition 1.8. *Let P and Q be simplicial polytopes. The direct sum $P \oplus Q$ is a simplicial polytope and $\partial(P \oplus Q) \cong \partial P * \partial Q$.* □

The connection with topology stems from the fact that every (finite) simplicial complex yields a certain closed subset of the sphere: Let $\mathsf{K} \subseteq 2^{[n+1]}$ be a simplicial complex and let $\Delta_n = \mathsf{conv}\, \{v_1, \ldots, v_{n+1}\} \subset \mathbb{R}^n$ be an n-simplex. The collection of faces $F_\sigma = \mathsf{conv}\, \{v_i : i \in \sigma\}$ for $\sigma \in \mathsf{K}$ gives a closed subset of the (topological) boundary of Δ_n which is an $(n-1)$-sphere. We denote by

$$\|\mathsf{K}\| = \bigcup_{\sigma \in \mathsf{K}} F_\sigma \subset \mathbb{R}^n$$

the underlying space. Note that Proposition 1.8 implies that if K and L are simplicial subcomplexes of the boundaries of the simplicial polytopes P and Q, then $\mathsf{K} * \mathsf{L}$ is a subcomplex of the boundary of $P \oplus Q$.

Part I

Projections of polytopes

Chapter 2

Introduction and applications

Show that every polytope on n vertices is the image under projection of an $(n-1)$-dimensional simplex.

Virtually every course on polytope theory features the above exercise in one way or the other and, therefore, it should go without saying that "projections of polytopes" are worth studying. However, the above exercise also shows that projections of polytopes are intrinsically difficult objects. Let $P \subset \mathbb{R}^d$ be a polytope and let $\pi : \mathbb{R}^d \to \mathbb{R}^e$ be an affine projection. Considering a single k-face $F \subset P$, the image under projection can be an ℓ-face for $\ell \leq k$ or part of an ℓ-face with $\ell \geq k$. Thus the combinatorial type of $\pi(P)$ is hard to predict. As a simplifying measure, we restrict our attention to faces that behave *nicely* under projection in the sense of the following definition.

Definition 2.1 (Preserved and strictly preserved faces; cf. [51, 64])**.** Let P be a polytope, $F \subset P$ a proper face, and $\pi : P \to \pi(P)$ a projection of polytopes. The face F is *preserved* under π if

i) $G = \pi(F)$ is a proper face of $\pi(P)$ and

ii) F and G are combinatorially isomorphic.

If, in addition,

iii) $\pi^{-1}(G)$ is equal to F

then F is *strictly preserved.*

The first two conditions should trigger an agreeing nod since they model the intuition behind "preserved faces." The third condition is a little less clear. In order to talk about *distinct* faces of a projection we have to rule out that two preserved faces come to lie on top of each other and this situation is dealt with in condition iii). Figure 2 shows instances of non-preserved, preserved, and strictly preserved faces.

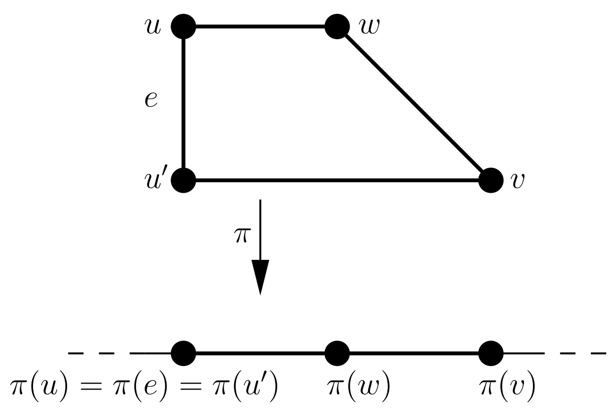


Figure 2.1: Faces under projection: u and u' are preserved but not strictly, w and e are not preserved, and v is strictly preserved.

In this chapter we discuss three problems in which the notion of strictly preserved faces plus the surrounding theory and tools play a vital role for their solution:

▶ The first problem concerns the approach to *dimensional ambiguity* taken by Sanyal & Ziegler [53]. They construct products of even polygons such that a projection to e-space strictly preserves the k-skeleton for certain k. However, their approach is restricted to *even* polygons and we show that this is no shortcoming of their construction.

In some sense, this problem serves as a *role model* as the task at hand is to construct a realization of a certain combinatorial type such that the projection to e-space retains the whole k-skeleton for $k \geq 0$.

▶ In the next section we prove non-trivial upper bounds on the number of vertices of the Minkowski sum of polytopes. The essential observation is that this bound is related to projections of products of simplices.

▶ The final problem concerns the realization of polyhedral surfaces in the boundary of 4-polytopes by Rörig & Ziegler [49]: The polyhedral surface is identified as part of the 2-skeleton of a high-dimensional polytope and a suitable realization retains the surface under projection. The surfaces considered in [49] are embedded in *wedge products*, a construction that generalizes joins. We present a different approach to wedge products that highlights their relation to joins and products. This similarity is then exploited to prove that almost all of their surfaces can not be projected to 4-space.

Convention. Let P be a d-polytope and let $\pi : P \to \pi(P) \subseteq \mathbb{R}^e$ be an affine projection. Throughout this chapter it is understood that $d \geq e$ and that $\pi(P)$ is full-dimensional.

2.1 Deformed products of odd polygons

According to Grünbaum [23, Ch. 12], a polytope $P \subset \mathbb{R}^d$ is *dimensionally k-ambiguous* if the k-skeleton of P is isomorphic to that of a polytope Q and $\mathsf{dim}\, Q \neq \mathsf{dim}\, P$. So, not only is the k-skeleton of such a polytope *not* characteristic but, even worse, it does not even give away the dimension in which to look for it. Unfortunately, there is no effective way to decide when a polytope is dimensionally ambiguous and even the list of known instances of such polytopes is rather short. Presumably the polytope most commonly known to be dimensionally $\lfloor\frac{d-3}{2}\rfloor$-ambiguous for $d \geq 3$ is the d-simplex: This follows from the existence of *neighborly simplicial polytopes* such as the cyclic polytopes (cf. [63, Corollary 0.8]). However, in recent years two more families of polytopes joined the list: The family of cubes via the existence of *neighborly cubical polytopes* [27] and the family of products of even polygons in guise of the *projected deformed products of polygons* [64]. In both cases, the construction principle (unified in [53]) is to give a special realization of the combinatorial type and to verify that a projection to lower dimensions retains the skeleton in question.

A main feature of the approach to dimensional ambiguity via projection is that the preservation of the k-skeleton can be checked *locally* by means of strictly preserved faces.

Lemma 2.2. *Let P be a polytope and let $\pi : P \to \pi(P)$ be a projection of polytopes. For $0 \leq k < \mathsf{dim}\, P$ the polytopes P and $\pi(P)$ have isomorphic k-skeleta if and only if every k-face of P is strictly preserved under π.*

Proof. Assume that P and $\pi(P)$ have isomorphic k-skeleta. If for $l \geq 1$ the $(l-1)$-skeleton is strictly preserved under projection, then the preimage of every l-face of $\pi(P)$ is an l-face. Indeed, let $\bar{F} \subset \pi(P)$ be an l-face and let $F = \pi^{-1}(\bar{F})$. Then the map $\pi|_F : F \to \pi(F) = \bar{F}$ is a projection of polytopes that strictly preserves the $(l-1)$-skeleton and thus the $(l-1)$-skeleton of F is a subcomplex of an $(l-1)$-sphere. Since $f_i(P) = f_i(\pi(P))$ for $0 \leq i \leq k$, the 0-skeleton is strictly preserved and the claim follows by induction on l.

Conversely, since every i-face for $i \leq k$ is strictly preserved we have that the k-skeleton of P is isomorphic to a subposet of the k-skeleton of $\pi(P)$. Assume that the inclusion is strict and let $H \subset P$ be a proper face of dimension greater than k and $\pi(H)$ a k-face of $\pi(P)$. As a polytope, H has a proper face F of dimension k and since "a face of a face is a face", F is a k-face of P with $\pi(F) = \pi(H)$. Thus F is not strictly preserved. $\square$

The benefit of working with strictly preserved faces is that the defining conditions can be checked prior to the projection by purely (linear) algebraic means. The key to that is the Projection Lemma. In order to convey a better feeling for the geometry we state here the *matrix version* of the Projection Lemma from [64] and [53]. However, for the purpose of this thesis we will work with (and prove) a *coordinate-free* version in Section 3.1.

The matrix version gives an algebraic way to *read off* the preserved faces from a polytope in exterior presentation. Every affine projection $\pi : \mathbb{R}^d \to \mathbb{R}^e$ factors into an affine transformation followed by $\pi_e : \mathbb{R}^{d-e} \times \mathbb{R}^e \to \mathbb{R}^e$, the linear projection that deletes the first $d-e$ coordinates, that is $\pi_e(\mathbf{x}', \mathbf{x}'') = \mathbf{x}'' \in \mathbb{R}^e$ for all $(\mathbf{x}', \mathbf{x}'') \in \mathbb{R}^{d-e} \times \mathbb{R}^e$. Therefore, we will focus on the projections π_e "to the last e coordinates". For a polytope $P = P(A, b) \subset \mathbb{R}^d$ in exterior presentation the projection map π_e naturally partitions the columns of A into $A = (\overline{A}|\overline{\overline{A}})$.

Lemma 2.3 (Projection Lemma: Matrix version [53, Lem. 2.5])**.** *For a polytope $P = P(A, b) \subset \mathbb{R}^d$ let $F \subset P$ be a proper face and $I = I(F)$ the index set of the inequalities that are tight at F. Then F is strictly preserved by the projection $\pi_e : \mathbb{R}^d \to \mathbb{R}^e$ to the last e coordinates if and only if the rows of $\overline{A}$ indexed by I are positively spanning.*

The construction scheme of Sanyal & Ziegler [53] is as follows: Let $P_1, \ldots, P_r$ be a collection of simple polytopes with $P_i = P(A_i, b_i)$ for $i = 1, \ldots, r$ and consider the product $P = P_1 \times P_2 \times \cdots \times P_r$. The exterior presentation of P is given by

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{pmatrix}.$$

As can be seen, the system of inequalities is rather sparse. The product is a simple polytope if and only if each of its factors is. Thus, the combinatorial type stays fixed if we apply small perturbations to A below the (block-)diagonal and thereby turning it into a *deformed product*. By choosing the right realizations of the polytopes P_i and the right perturbations it might be possible to retain the k-skeleton of P under projection to the last d coordinates. A major result of this construction principle is this.

Theorem 2.4 ([53, Theorem 4.1])**.** *Let P be a product of r polygons and let $2 \leq e < 2r$. If P is a product of* even *polygons, i.e. each polygon has an even number of vertices, then there is an e-polytope whose $\lfloor \frac{e-2}{2} \rfloor$-skeleton is combinatorially isomorphic to that of P.*

It is a curiosity left in connection with the deformed products of polygons that the above construction scheme fails for *odd* polygons, i.e. polygons with an odd number of vertices. On the other hand it is known, though apparently nowhere written up in detail, that there is no realization of a product of two odd polygons such that a projection to the plane strictly preserves all vertices.

As a corollary to Theorem 4.4 we will prove that the deformed-products scheme has a good reason to fail for odd polygons.

Corollary 2.5. *Let $r_o, r_e \geq 0$ and $1 \leq e \leq 2r = 2(r_o + r_e)$. Assume that*

$$\begin{array}{rcll} \lceil \frac{3e-2}{4} \rceil & < & r & \text{if } r_e < \lfloor \frac{e}{4} \rfloor, \text{ and} \\ \lfloor \frac{e}{2} \rfloor & < & r_o & \text{if } r_e \geq \lfloor \frac{e}{4} \rfloor. \end{array}$$

Then there is no realization of a product of r_e many even and r_o many odd polygons such that the image under projection to e-space retains the $\lfloor \frac{e-2}{2} \rfloor$-skeleton.

Furthermore, in Section 4.1 we will generalize the mentioned obstruction to the projectability of products of odd polygons to

Corollary 2.6. *Let $0 \leq k < 2r$. If*

$$e < r + 1 + \left\lfloor \frac{k}{2} \right\rfloor,$$

then there is no realization $P \subset \mathbb{R}^{2r}$ of a product of r odd polygons such that a projection $\mathbb{R}^{2r} \to \mathbb{R}^e$ strictly preserves the k-skeleton.

In particular, there is no realization of the r-fold product of odd polygons such that a projection to r-space retains all vertices.

2.2 Minkowski sums of polytopes

For two polytopes $P, Q \subset \mathbb{R}^d$ their *Minkowski sum* is the convex polytope

$$P + Q = \{p + q : p \in P,\, q \in Q\} \subset \mathbb{R}^d.$$

Minkowski sums have starred in applied areas, such as robot motion planning [32] and computer aided design [17], as well as in fields of pure mathematics, among them commutative algebra and tropical geometry [59]. In applications it is essential to understand the facial structure of $P+Q$. However, even with quite detailed knowledge of P and Q, it is in general difficult to determine the combinatorics of $P+Q$. Even for *special cases*, the knowledge of complete face lattices is meager. The best understood Minkowski sums are zonotopes [63] and sums of perfectly centered polytopes with their polar duals [18].

So it is natural (and vital) to investigate the combinatorial structure of Minkowski sums. From the standpoint of combinatorial geometry, a less ambitious goal one can settle for is the question of f-vector shapes. This includes different kinds of upper and lower bounds for the f-vector entries with respect to the corresponding entries of the summands. Starting with the first entry of an f-vector, the number of vertices, Fukuda & Weibel [18] studied the maximal number of vertices of a Minkowski sum. Their starting point was the following upper bound on the number of vertices.

Proposition 2.7 (Trivial Upper Bound; cf. [18])**.** *Let $P_1, \ldots, P_r \subset \mathbb{R}^d$ be polytopes. Then*

$$f_0(P_1 + \cdots + P_r) \quad \leq \quad \prod_{i=1}^{r} f_0(P_i).$$

Fukuda and Weibel gave a construction showing that the trivial upper bound can be attained independent of the dimension but with a restricted number of summands.

Theorem 2.8 (Fukuda & Weibel [18])**.** *For every $r < d$ there are d-polytopes $P_1, \ldots, P_r \subset \mathbb{R}^d$, each with an arbitrarily large number of vertices, such that the Minkowski sum $P_1 + \cdots + P_r$ attains the trivial upper bound on the number of vertices.*

This result is our point of departure: The following result asserts that the restriction to the number of summands is best possible.

Theorem 2.9. *Let $P_1, P_2, \ldots, P_r \subset \mathbb{R}^d$ be $r \geq d$ polytopes and let each polytope have $f_0(P_i) \geq d+1$ many vertices for all $i = 1, \ldots, r$. Then*

$$f_0(P_1 + \cdots + P_r) \;\leq\; \left(1 - \frac{1}{(d+1)^d}\right) \prod_{i=1}^{r} f_0(P_i).$$

However, the requirement on the number of vertices of each summand is unnecessarily strong; see the end of this section for a discussion of strengthenings of the result.

The problem, some reductions, and a reformulation

Forming Minkowski sums is not a purely combinatorial construction, i.e. in contrast to basic polytope constructions such as products, direct sums, joins, etc. the resulting face lattice is not determined by the face lattices of the polytopes involved. For a sum $P+Q$ of two polytopes P and Q it is easy to see that if $F \subseteq P+Q$ is a non-empty face, then F is of the form $F = G+H$ with $G \subseteq P$ and $H \subseteq Q$ being uniquely determined faces. This, in particular, sheds new light on the "Trivial Upper Bound" in the last section: It states that the set of vertices of a Minkowski sum is a subset of the pairwise sums of vertices of the polytopes involved.

As an example let us consider the first non-trivial case: Are there two triangles P and Q in the plane whose sum is a 9-gon? An *ad-hoc* argument for this case, which uses notions of normal fan and refinement as presented in [63, Sect. 7.1], is the following: The polytope $P+Q$ is a 9-gon if its normal fan $\mathcal{N}(P+Q)$ has nine extremal rays. The normal fan $\mathcal{N}(P+Q)$ equals $\mathcal{N}(P) \wedge \mathcal{N}(Q)$, the common refinement of the fans $\mathcal{N}(P)$ and $\mathcal{N}(Q)$. Thus, the cones in $\mathcal{N}(P+Q)$ are pairwise intersections of cones of $\mathcal{N}(P)$ and $\mathcal{N}(Q)$. It follows that the extremal rays, i.e. the 1-dimensional cones, of $\mathcal{N}(P+Q)$ are just the extremal rays of P and of Q. If P and Q are triangles, then each one has only three extremal rays. Therefore, $\mathcal{N}(P+Q)$ has at most six extremal rays and falls short of being a 9-gon. The same reasoning yields the following result.

Proposition 2.10. *Let P and Q be two polygons in the plane. Then*

$$f_0(P+Q) \le f_0(P) + f_0(Q).$$

However, this elementary geometrical reasoning fails in higher dimensions and we will employ topological machinery for the general case. But for now let us make some observations that will simplify the general case.

The first observation concerns the dimensions of the polytopes involved in the sum.

Observation 2.11 (Dimension of summands)**.** *Let $P_1, P_2, \ldots, P_r \subset \mathbb{R}^d$ be polytopes each having at least $d+1$ vertices. Then there are* full-dimensional *polytopes $P'_1, P'_2, \ldots, P'_r$ with $f_0(P'_i) = f_0(P_i)$ and*

$$f_0(P'_1 + P'_2 + \cdots + P'_r) \;\ge\; f_0(P_1 + P_2 + \cdots + P_r).$$

Clearly, if one of the summands, say P_1, is not full-dimensional, then the number of vertices prevents P_1 from being a lower dimensional simplex and thus P_1 has a vertex v_1 that is not a cone point. Let $c \in \mathbb{R}^d$ be a vector perpendicular to the affine hull of P_1 and let $P_1(\varepsilon)$ be the result of pulling v_1 ε-far in direction c. The polytope $P_1(\varepsilon)$ has $f_0(P_1(\varepsilon)) = f_0(P_1)$ vertices and dimension $\mathsf{dim}\, P_1(\varepsilon) = \mathsf{dim}\, P_1 + 1$ unless $\varepsilon = 0$. We claim that for $\varepsilon > 0$ sufficiently small $P(\varepsilon) = P_1(\varepsilon) + P_2 + \cdots + P_r$ has at least as many vertices as $P = P(0)$. To that end, choose a linear function ℓ_u for every vertex $u = u_1 + \cdots + u_r$ of P such that the unique maximum of ℓ_u over P is attained at u. The corresponding point in $P(\varepsilon)$ is a vertex if and only if ℓ_u attains its maximum over $P_1(\varepsilon)$ in the vertex corresponding to v_1. This gives a finite collection of strict linear inequalities in ε with $\varepsilon = 0$ being a feasible solution.

Observation 2.12 (Number of summands)**.** *Let $P_1, P_2, \ldots, P_r \subset \mathbb{R}^d$ be d-polytopes such that $P_1 + P_2 + \cdots + P_r$ attains the trivial upper bound, then so does every subsum $P_{i_1} + P_{i_2} + \cdots + P_{i_k}$ with $\{i_1, \ldots i_k\} \subseteq [r]$.*

Thus for the proof of Theorem 2.9 we can restrict to the situation of sums with d summands. The next observation turns out to be even more valuable. It states that we can even assume that every summand is a simplex.

Observation 2.13 (Combinatorial type of summands)**.** *Let $P_1, P_2, \ldots, P_r$ be polytopes such that $P_1 + P_2 + \cdots + P_r$ attains the trivial upper bound. For every $i \in [r]$ let $P_i' \subseteq P_i$ be a vertex induced subpolytope. Then $P_1' + P_2' + \cdots + P_r'$ attains the trivial upper bound.*

The last observation casts the problem into the realm of polytope projections.

Observation 2.14. *The Minkowski sum $P + Q$ is the projection of the product $P \times Q$ under the map $\pi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ with $\pi(x, y) = x + y$.*

Thus, the fact that the Minkowski sum $P + Q$ attains the trivial upper bound is equivalent to saying that a special projection of the standard product $P \times Q$ strictly preserves the 0-skeleton. By means of the above observations this boils down to the following result about products of simplices. We will derive the corollary from a general result (Theorem 4.10) concerning the projectability of simplex products in Section 4.2.

Corollary 2.15. *Let P be a polytope combinatorially equivalent to a d-fold product of d-simplices and let $\pi : P \to \mathbb{R}^d$ be a linear projection. Then*

$$f_0(\pi(P)) \leq f_0(P) - 1 = (d+1)^d - 1.$$

Special emphasis should be put on the phrase "combinatorially equivalent to a product." As discussed in Chapter 1, The *standard* product $P \times Q$ of two polytopes $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ is obtained by taking the Cartesian product of P and Q, that is taking the convex hull of $\mathsf{vert}\, P \times \mathsf{vert}\, Q \subset \mathbb{R}^{d+e}$. One feature of this construction is that if $P' \subset P$ and $Q' \subset Q$ are vertex induced subpolytopes, then $P \times Q$ contains $P' \times Q'$ again as a vertex induced subpolytope. This no longer holds for *combinatorial products*: The *Goldfarb cube* G_4 is a 4-polytope combinatorially equivalent to a 4-cube with the property that a projection to 2-space retains all 16 vertices (cf. [21, 53]). The 4-cube itself is combinatorially equivalent to a product of two quadrilaterals which, in the standard product, contains a vertex induced product of two triangles. The subpolytope on the corresponding vertices of G_4 is not a combinatorial product of two triangles; indeed, this would contradict Corollary 2.15 for $d = 2$.

Proof of Theorem 2.9. By Observations 2.11 and 2.13 we can assume that all polytopes P_i are full-dimensional and have their vertices in general position. Let us first consider the case $r = d$. For each $i = 1, \ldots, r$ set $k_i = \lfloor \frac{f_0(P_i)}{d+1} \rfloor$ and let $V_{i,1}, V_{i,2}, \ldots, V_{i,k_i} \subseteq V_i = \mathsf{vert}\, P_i$ be pairwise disjoint subsets each of cardinality $d + 1$. Let $T = \{t \in \mathbb{N}^r : 1 \le t_i \le k_i \text{ for all } i = 1, \ldots, r\}$. Then for every $(v_1, \ldots, v_r) \in \prod_{i=1}^r V_i$ there is at most one $t \in T$ such that $v_i \in V_{i,t_i}$. The subpolytopes $P_{i,j} = \mathsf{conv}\, V_{i,j} \subseteq P_i$ are d-simplices and for every $t \in T$ the polytope $P_t = P_{1,t_1} + P_{2,t_2} + \cdots + P_{r,t_r}$ falls short of attaining the trivial upper bound by at least one vertex due to Corollary 2.15. Thus the number of sums of vertices of $P_1 + \cdots + P_r$ is at most

$$\prod_{i=1}^{d} f_0(P_i) - |T| \;=\; \prod_{i=1}^{d} f_0(P_i) - \prod_{i=1}^{d} k_i \le \left(1 - \frac{1}{(d+1)^d}\right) \prod_{i=1}^{d} f_0(P_i).$$

Now, for $r > d$ we use the associativity of Minkowski sums to obtain

$$\begin{aligned} f_0(P_1 + \cdots + P_r) \;&\le\; f_0(P_1 + \cdots + P_d) \cdot f_0(P_{d+1} + \cdots + P_r) \\ &\le\; \left(1 - \frac{1}{(d+1)^d}\right) \cdot \prod_{i=1}^{r} f_0(P_i). \end{aligned}$$

□

The statement of Corollary 2.15 is tight with respect to the number of vertices that survive the projection: In Section 3.4 we give a realization of a product of two triangles such that a projection to the plane has 8 vertices. In comparison, Proposition 2.10 shows that the bound given in Theorem 2.9 is *not* tight for $d = 2$: The sum of two triangles in the plane has at most 6 vertices.

As a final remark concerning the Minkowski sum of polytopes let us mention that at the Oberwolfach-Workshop "Geometric and Topological Combinatorics" in January 2007 Rade Živaljević suggested a different argument, which involves Lovász' *colored Helly theorem.*

Theorem 2.16 (Colored Helly Theorem; cf. [33])**.** *Let $\mathcal{C}_1, \dots, \mathcal{C}_r$ be collections of convex sets in $\mathbb{R}^d$ with $r \geq d+1$. If $\cap_{i=1}^r C_i \neq \varnothing$ for every choice $C_i \in \mathcal{C}_i$ then there is a $j \in [r]$ such that $\cap_{C \in \mathcal{C}_j} C \neq \varnothing$.*

The following proof refines an idea of Imre Bárány (personal communication). Let $P \subset \mathbb{R}^d$ be a polytope. The *normal cone* $N_P(v) \subset (\mathbb{R}^d)^*$ of a vertex $v \in \mathsf{vert}\, P$ is the set of linear functions $\ell \in (\mathbb{R}^d)^*$ such that ℓ attains its unique maximum over P in v. The set $N_P(v)$ is an open convex cone of dimension d. We already made use of the fact that if P and Q are polytopes and $u \in \mathsf{vert}\, P$ and $v \in \mathsf{vert}\, Q$ are vertices then $u+v$ is a vertex of $P+Q$ if and only if $N_P(u) \cap N_Q(v) \neq \varnothing$. Further, denote by $S_d \subset (\mathbb{R}^d)^*$ the unit sphere and for $p \in S_d$ let $\psi_p : S_d \setminus \{p\} \to (\mathbb{R}^{d-1})^*$ be the stereographic projection from p.

Let us assume that $P_1, P_2, \dots, P_d \subset \mathbb{R}^d$ are polytopes such that $P_1 + \dots + P_d$ attains the trivial upper bound. Choose a fixed vertex $u_i \in P_i$ for every $i = 1, \dots, d$ and let $p \in \bigcap_{i=1}^d N_{P_i}(u_i)$. For every $i = 1, \dots, r$ the set $\mathcal{C}_i = \{\psi_p(N_{P_i}(v) \cap S_d) : v \in \mathsf{vert}\, P_i, v \neq u_i\}$ is a collection of convex sets in $(\mathbb{R}^{d-1})^*$ and $\mathcal{C}_1, \dots, \mathcal{C}_d$ satisfy the requirements of the Colored Helly Theorem. Since $f_0(P_i) \geq 3$ for every i, the consequence is that for at least one summand there are two distinct vertices whose normal cones intersect non-trivially. This, clearly, is impossible.

Let us emphasize that this considerably strengthens Theorem 2.9 inasmuch as in the proof the sets $V_{i,j}$ can be chosen of cardinality $|V_{i,j}| = 3$. This yields

Corollary 2.17. *Let $P_1, \dots, P_r \subset \mathbb{R}^d$ be $r \geq d$ polytopes with $f_0(P_i) \geq 3$ for all $i = 1, \dots, r$. Then*

$$f_0(P_1 + P_2 + \dots + P_r) \;\leq\; \left(1 - \frac{1}{3^d}\right) \prod_{i=1}^r f_0(P_i).$$

However, this result can also be obtained by appealing to Corollary 2.6.

Based on a preliminary version of [51], C. Weibel (personal communication) noted that the requirement on the number of vertices is unnecessarily strong.

Proposition 2.18 ([61, Thm 4.1.4])**.** *Let $P_1, \dots, P_{d-1} \subset \mathbb{R}^d$ be 1-dimensional polytopes and let $Q \subset \mathbb{R}^d$ be a triangle. Then*

$$f_0(P_1 + P_2 + \dots + P_{d-1} + Q) \;<\; 3 \cdot 2^{d-1}.$$

We give a different proof that uses the machinery above.

Proof. Let $P = \operatorname{conv}\{p, q\} \subset \mathbb{R}^d$ a segment in d-space. The normal cones $N_P(p)$ and $N_P(q)$ correspond to the two halfspaces determined by the hyperplane $H_P = \{\ell \in (\mathbb{R}^d)^* : \ell(p-q) = 0\}$. Let $L = \bigcap_{i=1}^{d-1} H_{P_i}$ be the intersection of the hyperplanes $H_{P_i} \subset (\mathbb{R}^d)^*$ determined by the segments $P_1, \dots, P_{d-1}$. The set L is a linear space of dimension $\dim L \geq 1$ and let $p \in L \cap S_d$ be an arbitrary point. For $i = 1, \dots, d-1$ let $\mathcal{C}_i = \{\psi_p(N_{P_i}(v) \cap S_d) : v \in \operatorname{vert} P_i\}$ and set $\mathcal{C}_d = \{\psi_p(N_Q(v) \cap S_d) : v \in \operatorname{vert} Q, p \notin N_Q(v)\}$. Each $\mathcal{C}_i$ is a collection of at least two convex sets in $(\mathbb{R}^d)^*$. Assuming that $P_1 + \cdots + P_{d-1} + Q$ attains the trivial upper bound and applying the Colored Helly Theorem yields the desired contradiction. $\square$

This further strengthens Theorem 2.9 and, indeed, Proposition 2.18 cannot be deduced from the machinery to be developed in Chapter 3. The reason is that the machinery would need to obstruct the existence of a polytope isomorphic to an $(d-1)$-fold prism over a triangle such that a projection to d-space preserves all vertices. However, such a polytope is easily constructed.

2.3 Wedge products and equivelar surfaces

The properties of products of polygons and products of simplices that we will exploit for proving the non-projectability of certain k-skeleta also hold for more general polytope constructions: Most notably they hold for *wedge products* of polytopes. The wedge product, introduced by Rörig & Ziegler in [49], generalizes the more familiar constructions of product and join in a combinatorial-geometric way: The wedge products correspond to certain geometric sections of joins that can be described combinatorially. We sketch an approach to the geometric construction of wedge products that differs from the presentation in [49]. However, our emphasis will lie on wedge products from the perspective of combinatorial types.

Wedge products and their duals, the *wreath products,* starred recently in connection with the construction of spheres with large symmetry groups by Joswig & Lutz [35] and as geometric embedding spaces for *polyhedral surfaces* in Rörig & Ziegler [49]. Our motivation for studying wedge products was spawned by the latter. Rörig & Ziegler showed that some families of polyhedral surfaces naturally embed into the 2-skeleton of $\mathcal{W}_{r,n-1} = \mathcal{P}_r \lhd \Delta_{n-1}$, the wedge product of an r-gon with an $(n-1)$-simplex. Furthermore, they showed that a slight variation of the idea underlying the deformed products of polygons (cf. Section 2.1) yields a realization of a particular sub-family in 3-space: For each member of that family there is a realization of the corresponding wedge product such that a projection to 4-space preserves the surface and, from there, can be realized in $\mathbb{R}^3$ via Schlegel diagrams or orthogonal projection. We briefly review the polyhedral surfaces and prove that *nearly* none of the surfaces outside this sub-family can be *realized* in that way – unfortunately, we miss one other sub-family. The proof relies on the close connection of wedge products with products of simplices.

Geometric and combinatorial wedge products

The *wedge product* $P \lhd Q$ of two polytopes P and Q is a construction that generalizes the join Q^{*r}. It may be obtained by an iteration of the *subdirect product* construction of McMullen [38] or explicitly in terms of the exterior representations of P and Q in the following way (see also [49] for an alternative description). Let $P \subset \mathbb{R}^e$ be an e-polytope on r facets given by

$$P = \{\mathbf{y} \in \mathbb{R}^e : \mathsf{a}_j^\mathsf{T}\,\mathbf{y} \ \leq\ 1 \text{ for all } j = 1, \ldots, r\}$$

and, for simplicity, let us assume that $\mathsf{a}_r = -(\mathsf{a}_1 + \mathsf{a}_2 + \cdots + \mathsf{a}_{r-1})$. Let $Q = P(B, \mathbb{1}) \subset \mathbb{R}^d$ be a d-polytope with $B \in \mathbb{R}^{m \times d}$. A realization of the r-fold join of Q is given by the points $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r, \mathbf{z}) \in (\mathbb{R}^d)^r \times \mathbb{R}^{r-1}$ satisfying

$$\begin{array}{llr} B\,\mathbf{x}_i & + \qquad\qquad z_i\mathbb{1} & \leq \mathbb{1} \text{ for } i = 1, \ldots, r-1 \text{ and} \\ B\,\mathbf{x}_r & -\ (z_1 + \cdots + z_{r-1})\mathbb{1} & \leq \mathbb{1}. \end{array} \tag{JN}$$

Let $\bar{A} \in \mathbb{R}^{(r-1)\times d}$ be the matrix with rows $\mathsf{a}_1, \ldots, \mathsf{a}_{r-1}$ and consider the linear space

$$L = (\mathbb{R}^d)^r \times \{\mathbf{z} \in \mathbb{R}^{r-1} : \mathbf{z} = \bar{A}\,\mathbf{y} \text{ for } \mathbf{y} \in \mathbb{R}^d\}.$$

Definition 2.19 (Geometric wedge product; cf. [49, Definition 2.7])**.** Let $P = P(A, \mathbb{1}) \subset \mathbb{R}^e$ and $Q = P(B, \mathbb{1}) \subset \mathbb{R}^d$ be polytopes with $A \in \mathbb{R}^{r\times e}$ and rows $\mathsf{a}_1^\mathsf{T}, \ldots, \mathsf{a}_r^\mathsf{T}$. The intersection $Q^{*r} \cap L$ is the polytope given by the points $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r, \mathbf{y}) \in (\mathbb{R}^d)^r \times \mathbb{R}^e$ satisfying

$$\begin{array}{ccccrl} B\,\mathbf{x}_1 & & & & + \mathbb{1}(\mathsf{a}_1^\mathsf{T}\,\mathbf{y}) & \leq \mathbb{1} \\ & B\,\mathbf{x}_2 & & & + \mathbb{1}(\mathsf{a}_2^\mathsf{T}\,\mathbf{y}) & \leq \mathbb{1} \\ & & \ddots & & \vdots & \\ & & & B\,\mathbf{x}_r & + \mathbb{1}(\mathsf{a}_r^\mathsf{T}\,\mathbf{y}) & \leq \mathbb{1} \end{array} \tag{WP}$$

and is called the *wedge product* $P \lhd Q$ of P and Q.

Thus, the wedge product $P \lhd Q$ is a section of Q^{*r} *along* P. Although difficult to see from the exterior presentation (WP), the wedge product has a purely combinatorial description in terms of the facial structure of P and Q. As we will primarily need the combinatorial structure of $P \lhd Q$ and its combinatorial relation to joins, we give a definition of wedge products in terms of combinatorial types. We will reconcile both definitions immediately afterwards. Recall that the face lattice of an r-fold join $\mathcal{L}(Q^{*r}) = \mathcal{L}(Q)^r$ is an r-fold direct product of the face lattice of Q.

Definition 2.20 (Combinatorial wedge product)**.** Let $\mathcal{P}$ and $\mathcal{Q}$ be combinatorial types and let $I_{\mathcal{P}} : \mathcal{P} \to 2^{[r]}$ be the face–facet incidence function of $\mathcal{P}$ with r being the number of facets of $\mathcal{P}$. The *wedge product* $\mathcal{P} \lhd \mathcal{Q} \subseteq \mathcal{Q}^r$ is the induced sub-poset of $\mathcal{Q}^{*r}$ defined by the following condition: A face $G = (G_1, G_2, \ldots, G_r) \in \mathcal{Q}^{*r}$ is a face of the wedge product if and only if

$$\{i \in [r] : G_i = \varnothing\} \ = \ I_{\mathcal{P}}(F) \tag{CWP}$$

for some face $F \in \mathcal{P}$.

We have the following relation among the geometric and the combinatorial wedge product.

Theorem 2.21. *Let P and Q be polytopes. Then*

$$\mathcal{L}(P \lhd Q) \cong \mathcal{L}(P) \lhd \mathcal{L}(Q).$$

Proof. By definition, the geometric wedge product $P \lhd Q$ is the section of the r-fold join Q^{*r} with a linear space L. Every face G' of $P \lhd Q$ is of the form $G' = G \cap L$ for $G = G_1 * G_2 * \cdots * G_r \subseteq Q^{*r}$ a face. Thus it suffices to prove that the intersection $G \cap L$ is non-empty if and only if there is a face $F \subseteq P$ with

$$\{i : G_i = \varnothing\} = I_P(F). \tag{*}$$

Note that $L_0 = (\mathbb{R}^d)^r \times \{\mathbb{0}\} \subset L$ is a subspace and $L_0 \cap Q^{*r} = Q^r$ is the r-fold Cartesian product of Q. Thus L intersects all faces of $Q^r \subset Q^{*r}$ and the faces of Q^r are precisely the faces of Q^{*r} with the left hand side of (*) being empty. This equals the right hand side for $F = P$. Thus we can restrict to those faces $G = G_1 * \cdots * G_r \subseteq Q^{*r}$ for which $J = \{i : G_i = \varnothing\}$ is not empty.

Let $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r, \mathbf{z})$ be a point in the relative interior of G. From the inequality system (JN) it is easy to see that $z_i \leq 1$ and with equality whenever $i \in J$. If $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r, \mathbf{z})$ is a point in the intersection $G \cap L$ then the definition of L yields $\mathbf{z} = \bar{A}\,\mathbf{y} \leq \mathbb{1}$ and with equality whenever $i \in J$ and so $\mathbf{y}$ is a point in the relative interior of some face $F \subseteq P$ with $I(F) = J$.

Let $F \subseteq P$ be a face, $\mathbf{y} \in \operatorname{\mathsf{relint}} F$ and set $L_{\mathbf{y}} = (\mathbb{R}^d)^r \times \{\bar{A}\,\mathbf{y}\}$. From the inequality system (WP) it is easy to see that $Q^{*r} \cap L_{\mathbf{y}}$ is a scaled product with those factors $\mathbf{x}_i$ contracted to a point for which $i \in I(F)$. Any face of that product yields a face of $P \lhd Q$. $\square$

The theorem enables us to state properties of the wedge product that are trivial to verify if we are allowed to alternate between the geometric and the combinatorial characterization. The dimension of $\mathcal{P} \lhd \mathcal{Q}$ is $r \cdot \operatorname{\mathsf{dim}} \mathcal{Q} + \operatorname{\mathsf{dim}} \mathcal{P}$. If $\mathcal{P}$ is of positive dimension, then the wedge product is simple if and only if $\mathcal{P}$ is simple and $\mathcal{Q}$ is a simplex. In the case that $\mathcal{P}$ is a point it is sufficient that $\mathcal{Q}$ is simple.

Proposition 2.22. *Let $\mathcal{P} \lhd \mathcal{Q}$ be a simple wedge product. Then the dimension of a face $G = (G_1, \ldots, G_r) \in \mathcal{P} \lhd \mathcal{Q}$ is*

$$\operatorname{\mathsf{dim}} G = \sum_{i=1}^{r} \operatorname{\mathsf{dim}} G_i + \operatorname{\mathsf{dim}} \mathcal{P}. \tag{DF}$$

For $\mathcal{P} = \Delta_{r-1}$ the condition (CWP) is vacuous and we obtain $\Delta_{r-1} \lhd \mathcal{Q} = \mathcal{Q}^r$.

THE WEDGE PRODUCTS $\mathcal{W}_{r,n-1} := \mathcal{P}_r \lhd \Delta_{n-1}$

Consider an r-gon $\mathcal{P}_r$ with edges labeled by $[r]$ in cyclic order and let Δ_{n-1} be an $(n-1)$-simplex with vertices labeled by $[n]$. The (combinatorial) wedge product $\mathcal{W}_{r,n-1} := \mathcal{P}_r \lhd \Delta_{n-1}$ is an $(r(n-1)+2)$-dimensional simple type with rn facets. Let us identify a face of Δ_{n-1} by its set of vertex labels. The faces of Δ_{n-1}^{*r} are therefore represented by $H = (H_1, H_2, \ldots, H_r)$ with $H_i \subseteq [n]$ and, according to (CWP), H corresponds to a non-empty face of the wedge product $\mathcal{W}_{r,n-1}$ if

i) the set $\{i : H_i = \varnothing\}$ is of size at most two, and

ii) in the case that $\{i : H_i = \varnothing\} = \{k, l\}$ with $k \neq l$, then the indices are cyclically adjacent, i.e. $k - \ell \equiv \pm 1 \bmod n$.

From the dimension formula (DF) for simple wedge products it follows that the vertices of $\mathcal{W}_{r,n-1}$ are given by

$$\mathcal{H}_V = \{H \in \mathcal{W}_{r,n-1} : |H_i| \leq 1 \text{ for all } i = 1, \ldots, r \text{ and } H_i = \varnothing \text{ for exactly two cyclically adjacent indices } \}.$$

In preparation for Section 2.3 where we give a description of the polyhedral surfaces constructed in [49] let us have a quick look at some of the 2-faces and edges of the wedge product. The set

$$\mathcal{H}_R = \{(H_1, \dots, H_r) \in \mathcal{W}_{r,n-1} : |H_i| = 1 \text{ for all } i = 1, \dots, r\}$$

is a collection of 2-faces each of which is isomorphic to an r-gon: The dimension of the faces in $\mathcal{H}_R$ is 2 by the dimension formula (DF) and every face contains exactly r vertices of $\mathcal{H}_V$. The collection of edges $\mathcal{H}_E$ incident to the 2-faces in $\mathcal{H}_R$ is given by those faces $H = (H_1, H_2, \dots, H_r) \in \mathcal{W}_{r,n-1}$ with $|H_i| = 1$ for all but a unique $i_0 \in [r]$ with $H_{i_0} = \varnothing$. In particular, every edge of $\mathcal{H}_E$ is incident to exactly n r-gons. Note that $\mathcal{H}_E$ does not exhaust all edges of $\mathcal{W}_{r,n-1}$.

Regular wedge product surfaces

The preceding discussion shows that the 2-skeleton of $\mathcal{W}_{r,n-1}$ is *rich* in r-gon faces. In [49] this fact is used to construct *regular (polyhedral) surfaces of type* $\{r, 2n\}$, i.e. (orientable) polyhedral 2-manifolds that are

- *equivelar*: all faces are r-gons and every vertex is incident to $2n$ faces, and
- *regular*: the automorphism group acts transitively on the flags of the surface.

Note that a regular surface is automatically equivelar. For detailed information regarding polyhedral surfaces we refer the reader to the handbook article of Brehm & Wills [11].

For $r \geq 3$ and $n \geq 2$ consider the 2-dimensional (polyhedral) subcomplex $\mathcal{S}_{r,2n}$ generated by the collection of r-gons

$$\Big\{(j_1, \dots, j_r) \in \Delta_{n-1}^{*r} : 1 \leq j_i \leq n,\ \sum_{k=1}^{r} j_k \equiv 0 \text{ or } 1 \bmod n\Big\} \subseteq \mathcal{H}_R.$$

The subcomplex $\mathcal{S}_{r,2n}$ contains all vertices $\mathcal{H}_V$ of the wedge product and all the special edges $\mathcal{H}_E$ of the wedge product given previously. Moreover, the subcomplex has the following properties.

Theorem 2.23 (see [49]). *For $r \geq 3$ and $n \geq 2$ the subcomplex $\mathcal{S}_{r,2n}$ of the wedge product $\mathcal{W}_{r,n-1} = \mathcal{P}_r \lhd \Delta_{n-1}$ is a closed connected orientable regular 2-manifold of type $\{r, 2n\}$ with f-vector $f(\mathcal{S}_{r,2n}) = n^{r-2}(r, rn, 2n)$ and genus $1 + \frac{1}{2}n^{r-2}(rn - r - 2n)$.*

The surfaces $\mathcal{S}_{3,2n}$ show up already in the book of Coxeter & Moser on discrete groups [15, p. 139]. For $n = 2$ the surface $\mathcal{S}_{3,4}$ is just the octahedron and for $n = 3$ we obtain Dyck's regular map $\mathcal{S}_{3,6}$ (cf. [8]). Embeddings of Dyck's regular map into $\mathbb{R}^3$ were constructed by Bokowski [8] and Brehm [10]. It would be very interesting to obtain embeddings in $\mathbb{R}^3$ for the surfaces $\mathcal{S}_{3,2n}$ for arbitrary $n \geq 2$ because the surfaces have $3n$ vertices and genus $1 + \frac{1}{2}n(n-3)$, so the genus is quadratic in the number of vertices. Embeddings of the surfaces $\mathcal{S}_{r,4}$ ($n = 2$) in $\mathbb{R}^3$ are constructed by McMullen, Schulz, and Wills [42]. These surfaces of "unusually large genus" [42] are the surfaces of largest genus on the given number of 2^r vertices known to be embeddable in $\mathbb{R}^3$. Via deformation and projection of the wedge products $\mathcal{W}_{r,1}$ other embeddings of the surfaces $\mathcal{S}_{r,4}$ are obtained in [49].

Theorem 2.24 (see [49])**.** *The wedge product $\mathcal{W}_{r,1}$ has a realization such that the surface $\mathcal{S}_{r,4} \subset \mathcal{W}_{r,1}$ is preserved by the projection to $\mathbb{R}^4$.*

Note that the surface is embedded in the boundary of a convex 4-polytope and, via Schlegel diagrams, is then realizable in 3-space with planar r-gons.

An open question was whether there are appropriate realizations of the wedge product $\mathcal{W}_{r,n-1}$ for $n \geq 3$ that would give analogous results for the remaining surfaces. A main result of this thesis is to answer this question to the negative.

Corollary 2.25. *There is no realization of the wedge product $\mathcal{W}_{r,n-1}$, with $n \geq 3$ and $r \geq 4$, such that the surface $\mathcal{S}_{r,2n}$ is strictly preserved by the projection to $\mathbb{R}^e$ for $e < r + 1$.*

The corollary follows from the fact that every realization of $\mathcal{W}_{r,n-1}$ that admits a projection to $\mathbb{R}^e$ which retains the surface automatically preserves the edges $\mathcal{H}_E$. As an application of the tools developed in Chapter 3 we will prove that this is impossible whenever $e < r + 1$.

Theorem 2.26. *There is no realization of the wedge product $\mathcal{W}_{r,n-1}$, with $n \geq 3$ and $r \geq 4$, such that the collection of edges $\mathcal{H}_E$ is strictly preserved by the projection to $\mathbb{R}^e$ for $e < r + 1$.*

We prove Theorem 2.26 in Section 4.2 as part of an investigation of the non-projectability of skeleta of the wedge products $\mathcal{W}_{r,n-1}$ which, in turn, relies on the kinship of $\mathcal{W}_{r,n-1}$ with $(\Delta_{n-1})^r$, the r-fold product of an $(n-1)$-simplex.

However, our tools yield no result for $r = 3$: In this case the surface $\mathcal{S}_{r,2n}$ is triangulated and the wedge product $\mathcal{W}_{3,n-1}$ is a $(3n-1)$-simplex. As we will see in Chapter 3, the obstruction to the projection is the non-embeddability of an associated simplicial complex into the boundary of a polytope of a certain dimension. Curiously, in the case $r = 3$ the associated complex is $\mathcal{S}_{3,2n}$ and the polytope is 4-dimensional. Thus, we do not find a realization and a projection that yields a 4-polytope carrying the surface if we can show that there is no 4-polytope with $\mathcal{S}_{3,2n}$ in the boundary. Topologically, the embedding into a (polyhedral) 3-sphere is possible, since $\mathcal{S}_{r,2n}$ is orientable.

By the following result of Perles (written up by Grünbaum) we get an embedding of the surface into $\mathbb{R}^5$. For the statement recall that a polyhedral complex is a collection of polytopes in some $\mathbb{R}^d$ that is closed under taking faces and every two polytopes in the collection intersect in a (possibly empty) face.

Theorem 2.27 ([23, p. 204])**.** *Let $\mathcal{K} \subset \mathbb{R}^d$ be a k-dimensional polyhedral complex. Then there is a straight-line embedding of $\mathcal{K}$ into $\mathbb{R}^{2k+1}$.*

The theorem does *not* assert the realization of $\mathcal{K}$ in the boundary of a polytope and hence does not contradict Theorem 2.25 for $r \geq 7$.

Concerning the problem of surfaces with average vertex degree and polygon size at least 5 in $\mathbb{R}^3$ we easily deduce the following corollary.

Corollary 2.28. *For $r \geq 4$ and $n \geq 3$ there exists no realization of the wedge product $\mathcal{W}_{r,n-1}$ such that the surface $\mathcal{S}_{r,2n}$ is strictly preserved by the projection to $\mathbb{R}^4$.*

This corollary does not obstruct the realization of the surfaces $\mathcal{S}_{r,2n}$ in $\mathbb{R}^3$ – it merely reveals the limitations of the construction principle presented here.

Chapter 3

The geometry and topology of projections

As exemplified in the previous chapter, our motivation was to investigate the limitations of *skeleton-preserving* projections. To be more specific: Given the combinatorial type $\mathcal{P}$ of a d-polytope and parameters $k \leq e \leq d$, what are necessary conditions for the existence of a polytope P of the prescribed type $\mathcal{P}$ and an affine projection $\pi : P \to \mathbb{R}^e$ such that P and $\pi(P)$ have isomorphic k-skeleta.

In the current chapter we devise tools that give fairly good necessary conditions on the existence of such pairs (P, π). The main observation is that if $\pi : P \to \pi(P)$ retains the k-skeleton for $k \geq 0$ then there is an associated pair of spaces $(S, \|\mathsf{K}\|)$ with $\|\mathsf{K}\| \subseteq S$ where K is a simplicial complex and S is a (polytopal) sphere. The simplicial complex K is defined in terms of the combinatorics of P whereas the dimension of the sphere S depends on $\mathsf{dim}\, \pi(P)$. Thus, the existence of (P, π) implies that K is embeddable into a sphere of a specific dimension.

We will illustrate the concepts along the following instructive example which appears in the first two sections of Chapter 2.

Guiding Example. Is there a realization of $\Delta_2 \times \Delta_2$, the product of two triangles, such that a projection to the plane strictly preserves all 9 vertices?

3.1 The geometry of projections

Let P be a d-polytope and let $\pi : P \to \pi(P)$ be a projection of polytopes. Recall the definition of a strictly preserved face (Definition 2.1): A proper face $F \subset P$ is strictly preserved if $\pi(F)$ is a face combinatorially equivalent to F and, moreover, F is the unique face that maps to $\pi(F)$. A key observation is that if π strictly preserves *sufficiently many* vertices, then this spawns the existence of an *associated* polytope that certifies the preservation of certain faces. En route to turning this into a rigorous statement, we state and prove a *coordinate-free* version of the Projection Lemma (see Lemma 2.3).

Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope on m facets with 0 in the interior and let

$$P^{\triangle} \;=\; \{\ell \in (\mathbb{R}^d)^* : \ell(x) \le 1 \;\text{ for all } x \in P\} \;=\; \mathsf{conv}\,\{\ell_1, \ldots, \ell_m\} \;\subset\; (\mathbb{R}^d)^*$$

be the *dual polytope* of P. Thus, the m facets of P correspond to the sets $F_i = \{x \in P : \ell_i(x) = 1\}$ for $i = 1, \ldots, m$. For every face $F \subseteq P$ we denote by

$$F^{\diamond} \;=\; \left\{\ell \in P^{\triangle} : \ell|_F = 1\right\} \;=\; \mathsf{conv}\;\{\ell_i : \ell_i|_F = 1 \text{ for } i \in [m]\}$$

the face of $P^{\triangle}$ dual to F and we define

$$I(F) := \{i \in [m] : \ell_i|_F = 1\} \subseteq [m].$$

Let $\pi : \mathbb{R}^d \to \mathbb{R}^e$ be a linear projection and let

$$U \;=\; \ker \pi \;\cong\; \mathbb{R}^{d-e}$$

be the kernel of the linear map π. The characterization of strictly preserved faces will be in terms of restrictions of dual faces to U.

Lemma 3.1 (Projection Lemma)**.** *Let P be a polytope and $F \subset P$ a proper face. Then F is strictly preserved if and only if*

$$0 \in \mathsf{int}\, F^{\diamond}|_U = \mathsf{int}\,\mathsf{conv}\,\{\ell_i|_U : i \in I(F)\}.$$

We first sort out the situation for (non-strictly) preserved faces.

Proposition 3.2. *Let $F \subset P$ be a proper face. Then $\pi(F)$ is a proper face of $Q = \pi(P)$ if and only if $0 \in F^{\diamond}|_U$.*

Proof. Let $H = \pi^{-1}(\pi(F)) \subseteq P$ be the preimage of $\pi(F)$. Then $\pi(F)$ is a proper face of Q iff there is an $\ell \in (\mathbb{R}^e)^*$ such that

$$\begin{aligned} \ell \circ \pi(x) < 1 \quad &\text{for all } x \in P \setminus H \text{ and} \\ \ell \circ \pi(x) = 1 \quad &\text{for all } x \in H \end{aligned}$$

This is the case iff $\pi^*(\ell) \in H^{\diamond} \subseteq F^{\diamond}$ which in turn holds if and only if $0 = \pi^*(\ell)|_U \in H^{\diamond}|_U \subseteq F^{\diamond}|_U$. □

Proposition 3.3. *Let $F \subset P$ be a proper face. Then F is preserved under π if and only if $F^{\diamond}|_U$ is full-dimensional with $0 \in F^{\diamond}|_U$.*

Proof. In light of Proposition 3.2 it is sufficient to show that F and $\pi(F)$ are isomorphic iff $F^\diamond|_U$ is of full dimension $d-e$. The affine image $\pi(F)$ is isomorphic to F iff π is injective on $\mathsf{aff}\, F$. Let $\mathsf{aff}\, F = x_0 + L$ with L being a linear space. Then $\mathsf{aff}\, F^\diamond$ is a translate of $L^0 = \{\ell \in (\mathbb{R}^n)^* : \ell|_L \equiv 0\}$. Now, π is injective on L iff $L \cap U = \{0\}$. This is the case if and only if $L^0|_U = U^*$. $\square$

For the finishing touch we need the following technical result.

Lemma 3.4. *Let P be a polytope and $\pi : P \to \pi(P)$ a projection. If $F \subset P$ is a facet such that $\mathsf{dim}\, \pi(F) < \mathsf{dim}\, \pi(P)$ then $\pi(F) \subset \partial \pi(P)$.*

Proof. Assume that $\pi(F)$ is not contained in the boundary of $\pi(P)$. Thus, for every hyperplane $H \supset \pi(F)$ there are two points $p, q \in P$ whose images are separated by H. Then $\pi^{-1}(H)$ is a hyperplane containing F and p and q lie on the same side. $\square$

Proof of the Projection Lemma. By the preceding results we can assume that F is preserved and that $0 \in F^\diamond|_U$.

Assume first that F is not strictly preserved and let H be an inclusion minimal face properly containing F and such that $\pi(F) = \pi(H)$. Thus $H^\diamond$ is a facet of $F^\diamond$ with $0 \in H^\diamond|_U$. Since H is not preserved, $H^\diamond|_U$ is of dimension strictly less than $d-e$ and Lemma 3.4 guarantees that $0 \notin \mathsf{int}\, F^\diamond|_U$.

Conversely, assume that 0 is contained in the boundary of $F^\diamond|_U$ and let $H^\diamond \subset F^\diamond$ be a facet with $0 \in H^\diamond|_U$. It follows that $H \supset F$ is a non-preserved face which implies that $\pi(F) \subseteq \pi(H)$ are both faces of $\pi(P)$ of the same dimension. $\square$

The associated polytope

We made use of the fact that $F^\diamond|_U = \mathsf{conv}\, \{\ell|_U : \ell \in \mathsf{vert}\, F^\diamond\}$ and for later reference we denote by $\mathcal{G} = \{g_i = \ell_i|_U : i = 1, \dots, m\}$ the projection of the vertices of $P^\triangle$. Note that we will not treat $\mathcal{G}$ as the set of vertices of a polytope (especially since not all would be vertices) but as a configuration of vectors. In case that all vertices survive the projection, this vector configuration has some strong properties: It is the *Gale transform* of a polytope. Gale transforms are a well-known notion from discrete geometry; we refer the reader to Matoušek [36] and Ziegler [63] for full treatments (from different perspectives) and to McMullen [39] for an extensive survey.

A set of vectors $W = \{w_1, \dots, w_k\} \subset \mathbb{R}^r$ is *positively spanning* if every point in $\mathbb{R}^r$ is a non-negative combination of the vectors w_i, that is, if $\mathsf{cone}\, W = \mathbb{R}^r$.

Equivalently, W is positively spanning if and only if $\operatorname{conv} W$ is a full dimensional polytope with 0 in its interior. We also need the weaker notion of *positively dependent* which holds if $0 \in \operatorname{relint} \operatorname{conv} W$.

Definition 3.5 (Polytopal Gale transform)**.** A spanning vector configuration $\mathcal{G} = \{g_1, \ldots, g_m\}$ is a *polytopal Gale transform* if for every $i = 1, \ldots, m$ the subconfiguration $\mathcal{G} \setminus g_i$ is positively dependent.

The main reason why polytopal Gale transforms are useful is that they are yet another way to represent polytopes. For the sake of brevity, we drop the specification "polytopal" and simply call $\mathcal{G}$ a Gale transform.

Theorem 3.6 (Gale duality [36, Cor. 5.6.3])**.** *Let $\mathcal{G} = \{g_1, \ldots, g_m\}$ be a Gale transform in r-dimensional space. Then there is an polytope Q of dimension $m - r - 1$ with vertices $\operatorname{vert} Q = \{v_1, \ldots, v_m\}$ such that for every $I \subseteq [m]$*

$$\operatorname{conv}\{v_i : i \in [m] \setminus I\} \textit{ is a face of } Q \;\Leftrightarrow\; \{g_j : j \in I\} \textit{ is positively dependent.}$$

The condition given by the Projection Lemma can be rephrased in terms of positive spans. It is an easy observation that under certain mild conditions the set $\mathcal{G}$ is actually a Gale transform if all vertices survive the projection.

Proposition 3.7. *Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope with m facets and let $\pi : \mathbb{R}^d \to \mathbb{R}^e$ be a linear projection. If for every $i \in [m]$ there is a strictly preserved vertex v with $i \notin I(v)$, then $\mathcal{G}$ is a polytopal Gale transform.*

Proof. Let $i \in [m]$ and let $v \in \operatorname{vert} P$ be the vertex strictly preserved under π with $i \notin I(v)$. By the Projection Lemma the set $\{g_j : j \in I(v)\} \subset \mathcal{G} \setminus g_i$ is positively spanning. □

This result leads to a very useful corollary.

Corollary 3.8. *Let P be a polytope and let r be the maximal number of vertices in a facet of P. If $\pi : \mathbb{R}^d \to \mathbb{R}^e$ is a linear projection that strictly preserves at least $r + 1$ vertices, then $\mathcal{G}$ is a polytopal Gale transform. In particular, this holds if π strictly preserves all vertices of P.* □

Definition 3.9 (Associated polytope)**.** Let P be a polytope and let π be a projection of polytopes such that (P, π) satisfies the conditions of Proposition 3.7. We call the polytope $\mathcal{A} = \mathcal{A}(P, \pi)$ Gale-dual to $\mathcal{G}$ the *associated polytope*.

In case that P is simple, we can even assume that $\mathcal{A}$ is simplicial: We can *wiggle* the facet defining hyperplanes of P without changing the combinatorial type. For strictly preserved faces of P, the condition as dictated by the Projection Lemma is *open*, i.e. stable under small perturbations. Thus perturbing the bounding hyperplanes of P alters neither the type nor the fact that all vertices survive the projection. The effect on the Gale transform $\mathcal{G}$ is that every positively dependent set is positively spanning. On the other hand, this is yet another characterization of the fact that $\mathcal{A}(P, \pi)$ is a polytope with vertices in general position and hence simplicial.

Guiding Example (continued)**.** Suppose $P \subset \mathbb{R}^4$ is combinatorially equivalent to the product of two triangles and that $\pi : P \to \pi(P)$ is a projection to the plane retaining the 0-skeleton. Then by Corollary 3.8 and Theorem 3.6 the associated polytope $\mathcal{A} = \mathcal{A}(P, \pi)$ is a 3-dimensional simplicial polytope on 6 vertices $\mathsf{vert}\,\mathcal{A} = \{u_1, u_2, u_3, v_1, v_2, v_3\}$.

3.2 The topology of projections

Given that certain vertices of P survive the projection we obtain an associated polytope $\mathcal{A} = \mathcal{A}(P, \pi)$. Furthermore, for every preserved vertex $v \in \mathsf{vert}\,P$ the polytope $v^{\diamond}|_U$ has the origin in its interior which, in particular, implies that its vertices are positively dependent. By Gale duality (Theorem 3.6), this induces a face in $\mathcal{A}$. The collection of all the induced faces is a polytopal complex in the boundary of $\mathcal{A}$. If P is a simple polytope, we argued that $\mathcal{A}$ can be assumed to be simplicial. However, the next proposition shows that all the induced faces will be simplex faces. In total, this gives a simplicial complex whose combinatorics is determined by the sole knowledge of the combinatorics of P. We give a rather general description of this complex since it seems that it is the first occurrence in the literature.

Proposition 3.10. *Let P be a d-dimensional polytope on m facets and let $\pi : P \to \pi(P)$ be a projection meeting the conditions of Proposition 3.7. Then the associated polytope $\mathcal{A} = \mathcal{A}(P, \pi)$ is of dimension $m - d - 1 + \mathsf{dim}\,\pi(P)$ with vertices $a_1, a_2, \ldots, a_m$ such that the following holds: If $F \subset P$ is a face strictly preserved under projection, then*

$$\mathcal{A}_F = \mathsf{conv}\,\{a_i : i \in [m] \setminus I(F)\}$$

is a simplex face of $\mathcal{A}$.

Proof. Let $e = \mathsf{dim}\,\pi(P)$ and fix a strictly preserved face F. By the Projection Lemma the set $\{g_i : i \in I(F)\}$ is positively spanning. This implies that for every superset $J \supseteq I(F)$, the set $\{g_i : i \in J\}$ will be positively spanning as well. By Theorem 3.6 the subset $\mathcal{A}_F$ is a face of $\mathcal{A}$ and every subset of the vertices of $\mathcal{A}_F$ is the vertex set of a face. This, however, characterizes $\mathcal{A}_F$ as being a simplex. □

The combinatorics of the surviving faces yields the following simplicial complex.

Definition 3.11 (Associated complex)**.** Let P be a d-polytope on m facets and let $\pi : P \to \pi(P)$ be a projection retaining all vertices of P. The *associated complex* $\mathsf{K} = \mathsf{K}(P, \pi)$ is the simplicial complex on the vertex set $[m]$ generated by the sets

$$[m] \setminus I_P(F)$$

for every strictly preserved face $F \subset P$.

We may now rephrase Proposition 3.10 as follows.

Theorem 3.12. *Let P be a d-polytope on m facets and $\pi : P \to \pi(P)$ be a projection satisfying the conditions of Proposition 3.7. Then $\mathsf{K}(P, \pi)$ is embedded in a (polyhedral) sphere of dimension $m - d - 2 + \operatorname{dim} \pi(P)$.* □

Theorem 3.12 will be the main source for necessary conditions regarding the existence of a polytope with prescribed combinatorics and the existence of a projection retaining a given k-skeleton. In particular, the results will state the non-existence of a suitable realization of a given combinatorial type. For that reason we phrase the results in terms of *combinatorial types* – cf. Chapter 1 for definitions.

As we are primarily interested in the preservation of full skeleta of a given dimension we introduce the following complex.

Definition 3.13 (k-th coskeleton complex)**.** Let $\mathcal{P}$ be a combinatorial type of dimension d on m facets $F_1, F_2, \ldots, F_m$. For $-1 \leq k \leq d$, the *k-th coskeleton complex* is the simplicial complex $\Sigma_k(\mathcal{P}) \subseteq 2^{[m]}$ given by

$$\Sigma_k(\mathcal{P}) = \{\tau \subseteq [m] : \tau \cap I_{\mathcal{P}}(G) = \varnothing \text{ for some } k\text{-face } G \in \mathcal{P}\}.$$

Thus the maximal faces of $\Sigma_k(\mathcal{P})$ are in bijection with the k-faces of $\mathcal{P}$ under the correspondence $G \mapsto [m] \setminus I_P(G)$. The connection to $\mathsf{K}(P, \pi)$ is given by the following

Observation 3.14. *If $\pi : P \to \pi(P)$ is a projection retaining the k-skeleton then*

$$\{\varnothing\} = \Sigma_{-1}(P) \subset \Sigma_0(P) \subset \Sigma_1(P) \subset \cdots \subset \Sigma_k(P) \subset \mathsf{K}(P, \pi)$$

is a tower of subcomplexes.

In case of P being a simple polytope, there is a nice description for $\Sigma_0(P)$.

Definition 3.15 (Complement complex)**.** Let $\mathsf{K} \subseteq 2^V$ be a simplicial complex on vertices V. The *complement complex* K^{c} of K is the simplicial complex defined by

$$\mathsf{K}^{\mathsf{c}} = \{\tau \subseteq V : \tau \cap \sigma = \varnothing \text{ for some facet } \sigma \in \mathsf{K}\}.$$

If P is a simple polytope, then its dual $P^{\triangle}$ is simplicial and thus the boundary $\partial P^{\triangle}$ is a simplicial sphere.

Proposition 3.16. *If P is a simple polytope, then $\Sigma_0(P) = (\partial P^{\triangle})^{\mathsf{c}}$.* □

For a non-simple polytope the above statement stays true if $\partial P^{\triangle}$ is replaced by the *crosscut complex* [7, p. 1850] of $\partial P^{\triangle}$ with respect to the vertices. The complement complex has several favorable properties some of which are summarized in the following proposition and implicitly used in the upcoming discussions.

Proposition 3.17. *Let K and L be simplicial complexes. Then the following statements hold*

i) $(\mathsf{K}^{\mathsf{c}})^{\mathsf{c}} = \mathsf{K}$
ii) *If K is pure, then K^{c} is pure of dimension $f_0(\mathsf{K}) - \mathsf{dim}\,\mathsf{K} - 2$.*
iii) $(\mathsf{K} * \mathsf{L})^{\mathsf{c}} = \mathsf{K}^{\mathsf{c}} * \mathsf{L}^{\mathsf{c}}$. □

In particular, the first property states that no information is lost in the passage from K to its complement complex. To the best of our knowledge there has been no work on this construction of simplicial complexes. A possible reason for that is the lack of topological plausibility. For a simplicial complex K there seems to be no obvious relation between K and K^{c} concerning homotopy type and/or (co)homology. For a complex being a *matroid* (see [46]), the complement complex is just the matroid dual which is well understood combinatorially and topologically. But matroids are rare among simplicial complexes.

Guiding Example (continued)**.** In terms of intersection of facets, the combinatorial type of $\Delta_2 \times \Delta_2$ is given by $\mathcal{P} = \{(A, B) : A, B \subset [3]\}$ with componentwise *reverse* inclusion. The dimension of a face $(A, B) \in \mathcal{P}$ is $(2 - |A|) + (2 - |B|)$. The 0-th coskeleton complex $\Sigma_0 = \Sigma_0(\mathcal{P})$ is therefore given by

$$\Sigma_0 = (\partial\Delta_2 * \partial\Delta_2)^{\mathsf{c}} = \binom{[3]}{\leq 1} * \binom{[3]}{\leq 1}.$$

In the language of graph theory Σ_0 is $\mathsf{K}_{3,3}$, the complete bipartite graph on 6 vertices. By Theorem 3.12, $\mathsf{K}_{3,3}$ is embedded in the 2-sphere. On the other hand, Kuratowski's Theorem (see for example [16, Theorem 4.4.6]) asserts that $\mathsf{K}_{3,3}$ is not planar. Thus, there is no realization of a product of two triangles such that a projection to the plane retains all vertices.

3.3 Interlude: Embeddability of simplicial complexes

Let $\mathsf{K} \subseteq 2^{[m]}$ be a (finite) simplicial complex. By the discussion at the end of Chapter 1, K has a canonical realization in the boundary of the $(m-1)$-simplex and therefore $\|\mathsf{K}\|$ is embeddable into a sphere of dimension $m - 2$. However, it is reasonable to ask for the smallest dimension of a sphere in which $\|\mathsf{K}\|$ can be realized.

Definition 3.18 (Embeddability dimension)**.** Let $\mathsf{K} \subseteq 2^{[m]}$ be a simplicial complex. The *embeddability dimension* $\mathsf{edim}(\mathsf{K})$ is the smallest integer d such that $\|\mathsf{K}\|$ may be embedded into the d-sphere, i.e. $\|\mathsf{K}\|$ is homeomorphic to a closed subset of the d-sphere.

The above discussion shows that $\mathsf{edim}(\mathsf{K})$ is at most the number of vertices minus one and, hence, finite. But it is a well-known result that the embeddability dimension can be bounded in terms of the dimension of K.

Proposition 3.19 ([23, Ex. 4.8.25, Thm. 11.1.8])**.** *Let* K *be a simplicial complex of dimension* $\dim \mathsf{K} = \ell$*. Then*

$$\ell \;\leq\; \mathsf{edim}(\mathsf{K}) \;\leq\; 2\ell + 1. \qquad \square$$

Putting it all together we obtain

Corollary 3.20. *Let* $\mathcal{P}$ *be a* d*-type on* m *facets and for* $0 \leq k \leq d$ *let* $\Sigma_k = \Sigma_k(\mathcal{P})$ *be the* k*-th coskeleton complex of* $\mathcal{P}$*. If*

$$e \;<\; \mathsf{edim}(\Sigma_k) + d - m + 2$$

then there is no realization of $\mathcal{P}$ *such that a projection to* $\mathbb{R}^e$ *retains the* k*-skeleton.*

Proof. By contradiction, assume that $P \subset \mathbb{R}^d$ is a realization of $\mathcal{P}$ and that $\pi : P \to \pi(P)$ is a projection retaining the k-skeleton and the dimension of the image is $\mathsf{dim}\,\pi(P) = e < \mathsf{edim}(\Sigma_k) + d - m + 2$. By Theorem 3.12, the above observation and the fact that the embeddability dimension is monotone along subcomplexes, the complex Σ_k is realized in a sphere of dimension

$$\mathsf{edim}(\Sigma_k) \leq m - d - 2 + e < \mathsf{edim}(\Sigma_k). \qquad \square$$

Let us consider the statement of Corollary 3.20 with the bounds given in Proposition 3.19. If $\mathsf{edim}(\Sigma_k)$ attains the lower bound of Proposition 3.19 then Corollary 3.20 implies that the dimension of the target space has to be at least $e \geq k + 1$. This is reasonable as the projection embeds $\Sigma_k(\mathcal{P})$ into a sphere of dimension $e - 1$. Now let us assume that $\mathsf{edim}(\Sigma_k)$ attains the upper bound. If $\mathcal{P}$ is simple, then $\mathsf{dim}\,\Sigma_k(\mathcal{P}) = m - (d - k) - 1$ and the k-skeleton is not projectable to e-space if $e < m - d + 2k + 1$. This implies the polyhedral part of the classical Van Kampen–Flores result:

Theorem 3.21. *Let $\mathcal{P}$ be a d-type and let $0 \leq k \leq \lfloor \frac{d-2}{2} \rfloor$. If*

$$e \leq 2k + 1$$

then there is no realization of $\mathcal{P}$ such that a projection to e-space retains the k-skeleton.

Proof. By a result of Grünbaum [22] the boundary complex of a d-polytope is a refinement of the boundary complex of a d-simplex. This implies that the k-skeleton of $\mathcal{P}$ contains a refinement of the k-skeleton of Δ_d. The Van Kampen–Flores result (e.g. see [37]) states that for $k \leq \lfloor \frac{d-2}{2} \rfloor$ the k-skeleton of a d-simplex is not homeomorphic to a subset of a $2k$-sphere. $\square$

In general it is hard to decide whether a complex K embeds into a sphere of a given dimension. In the following we give an executive summary of the techniques and results for treating embeddability questions as presented in the (fantastic) book [37].

The category of free $\mathbb{Z}_2$-spaces consists of topological spaces X together with a free action of the group $\mathbb{Z}_2$, i.e. a fixed point free involution on X. Morphisms in this category are continuous maps that commute with the respective $\mathbb{Z}_2$-actions. The foremost examples of $\mathbb{Z}_2$-spaces are spheres S^d with the antipodal action. For a $\mathbb{Z}_2$-space X a numerical invariant is the $\mathbb{Z}_2$-*index* $\mathsf{ind}_{\mathbb{Z}_2} X$ which is the smallest integer d such that there is a $\mathbb{Z}_2$-equivariant map $X \to_{\mathbb{Z}_2} S^d$. For example $\mathsf{ind}_{\mathbb{Z}_2} S^d = d$, which is a statement equivalent to the Borsuk–Ulam theorem. Let $Y = \|\mathsf{K}\|$ be the underlying topological space and let $f : Y \to S^d$ be an embedding of K into the d-sphere. Since f is necessarily injective, there is an induced map

$$f_\Delta^{*2} : (Y)_\Delta^{*2} \to_{\mathbb{Z}_2} (S^d)_\Delta^{*2}$$

where $(Y)_\Delta^{*2} = \{[(x,y,t)] \in Y^{*2} : x = y \Rightarrow t \neq \frac{1}{2}\} \subset Y^{*2}$ is the *deleted join* of the topological space Y (cf. [37, Sect. 5.5]). The deleted join is naturally a $\mathbb{Z}_2$-space and it can be shown that $(S^d)_\Delta^{*2} \simeq_{\mathbb{Z}_2} S^d$ as $\mathbb{Z}_2$-spaces. Thus, if K is embeddable into a d-sphere, then $\mathsf{ind}_{\mathbb{Z}_2} (Y)_\Delta^{*2} \leq d$.

We started with a finite simplicial complex which is a particularly nice topological space. For that, we are rewarded with a simple combinatorial construction that turns K into a $\mathbb{Z}_2$-space. We define the *deleted join* of K to be the complex

$$\mathsf{K}_\Delta^{*2} = \{\sigma \uplus \tau : \sigma, \tau \in \mathsf{K}, \sigma \cap \tau = \varnothing\}$$

The deleted join turns an arbitrary simplicial complex into a free $\mathbb{Z}_2$-complex by means of $\sigma \uplus \tau \mapsto \tau \uplus \sigma$. As $\|\mathsf{K}\|_\Delta^{*2}$ and $\|\mathsf{K}_\Delta^{*2}\|$ are $\mathbb{Z}_2$-homotopy equivalent, we get

Theorem 3.22 ([37, Theorem 5.5.5])**.** *Let* K *be a simplicial complex. Then*

$$\mathsf{edim}(\mathsf{K}) \geq \mathsf{ind}_{\mathbb{Z}_2} \mathsf{K}_\Delta^{*2}.$$

There is a beautiful theorem due to Karanbir Sarkaria (see [37]) that gives an effective way to obtain lower bounds on the $\mathbb{Z}_2$-index of the simplicial complex K_Δ^{*2} which is purely combinatorial in nature. We codify the main notion of that theorem in the definition of the Sarkaria index below. We need the following additional concepts.

Minimal non-faces. Let $\mathsf{K} \subseteq 2^V$ be a simplicial complex. We denote by $\mathcal{F}(\mathsf{K})$ the set of *minimal non-faces*, i.e. the inclusion-minimal sets in $2^V \setminus \mathsf{K}$.

Generalized Kneser graphs. For a collection of sets $\mathcal{F} = \{F_1, \ldots, F_k\}$ we denote by $\mathsf{KG}(\mathcal{F})$ the (abstract) graph with vertex set $\mathcal{F}$. Two vertices F_i, F_j share an edge iff $F_i \cap F_j = \varnothing$. Such a graph is called a *generalized Kneser graph.*

Chromatic number. For a graph G we denote by $\chi(G)$ its *chromatic number*, i.e. the minimal number of colors to properly color the graph.

Definition 3.23 (Sarkaria index)**.** Let K be a simplicial complex on m vertices and $\mathcal{F} = \mathcal{F}(\mathsf{K})$ the collection of minimal non-faces. The *Sarkaria index* is

$$\mathsf{ind}_{\mathsf{SK}} \mathsf{K} := m - \chi(\mathsf{KG}(\mathcal{F})) - 1.$$

Theorem 3.24 (Sarkaria's coloring/embedding theorem [37, Sect. 5.8]). *Let* K *be a simplicial complex. Then*

$$\mathsf{edim}(\mathsf{K}) \;\geq\; \mathsf{ind}_{\mathbb{Z}_2} \mathsf{K}^{*2}_{\Delta} \;\geq\; \mathsf{ind}_{\mathsf{SK}} \mathsf{K}.$$

As it will come in handy in later sections, the next proposition determines the Sarkaria index for two coskeleton complexes that depend only on the number of facets of the combinatorial type.

Proposition 3.25. *Let* $\mathcal{P}$ *be a* d*-type on* m *facets. Then* $\Sigma_d(\mathcal{P}) = \Delta_{m-1}$ *is homeomorphic to an* $(m-1)$*-ball and*

$$m - 1 = \mathsf{edim}(\Sigma_d(\mathcal{P})) = \mathsf{ind}_{\mathsf{SK}} \Sigma_d(\mathcal{P}).$$

For the $(d-1)$*-skeleton we have that* $\Sigma_{d-1}(\mathcal{P}) = \partial\Delta_{m-1} \cong S^{m-2}$ *and*

$$m - 2 = \mathsf{edim}(\Sigma_{d-1}(\mathcal{P})) = \mathsf{ind}_{\mathsf{SK}} \Sigma_{d-1}(\mathcal{P}).$$

Proof. In both cases the claim about the homeomorphism type follows from the definition of the coskeleton complex. Thus the embeddability dimensions are $m-1$ and $m-2$, respectively. For the Sarkaria index we get in the former case that the Kneser graph of the minimal non-faces of $\Sigma_d(\mathcal{P})$ has no vertices, whereas in the latter case the graph has no edges. $\square$

Although determining upper bounds on the chromatic number of graphs is easier than finding equivariant maps, it is, in general, still hard enough. The key property that enables us to calculate chromatic numbers for the Kneser graphs we will encounter is that the complexes are made up of (possibly) simpler ones, that is they are joins of complexes. The following results will show that this continues to hold if we pass from complexes to non-faces and then to Kneser graphs.

Lemma 3.26. *Let* $\mathsf{K} \subseteq 2^V$ *and* $\mathsf{L} \subseteq 2^W$ *be simplicial complexes. Then*

$$\mathcal{F}(\mathsf{K} * \mathsf{L}) = \{F \uplus \varnothing : F \in \mathcal{F}(\mathsf{K})\} \cup \{\varnothing \uplus G : G \in \mathcal{F}(\mathsf{L})\}.$$

Proof. The left hand side is easily seen to be contained in the right hand side. Let $F \uplus G \in \mathcal{F}(\mathsf{K} * \mathsf{L})$ and assume that $F, G \neq \varnothing$. Since $F \uplus G$ is a minimal non-face, it follows that $(F \setminus i) \uplus G$ and $F \uplus (G \setminus j)$ are both in $\mathsf{K} * \mathsf{L}$ for $i \in F$ and $j \in G$. This, however, implies that $F \in \mathsf{K}$ and $G \in \mathsf{L}$ and $F \uplus G \in \mathsf{K} * \mathsf{L}$. $\square$

On the level of Kneser graphs this fact results in a *bipartite sum* of the respective Kneser graphs. Let G and H be graphs with disjoint vertex sets U and V. The bipartite sum of G and H is the graph $G \bowtie H$ with vertex set $U \cup V$ and edges $E(G) \cup E(H) \cup (U \times V)$.

Proposition 3.27. *Let G and H be graphs. Then*

$$\chi(G \bowtie H) = \chi(G) + \chi(H).$$

Proof. The edges $U \times V \subset E(G \bowtie H)$ force the set of colors on U and V to be disjoint. Thus a coloring on $G \bowtie H$ is minimal iff it is minimal on the subgraphs G and H. □

The last two observations together now show that the Sarkaria index harmonizes with joins.

Proposition 3.28. *Let K and L be simplicial complexes. Then*

$$\mathsf{ind}_{\mathsf{SK}}\, \mathsf{K} * \mathsf{L} = \mathsf{ind}_{\mathsf{SK}}\, \mathsf{K} + \mathsf{ind}_{\mathsf{SK}}\, \mathsf{L} + 1.$$

□

Guiding Example (the wrap-up)**.** In Section 3.2 we inferred that the 0-th coskeleton complex $\Sigma_0 = \Sigma_0(\Delta_2 \times \Delta_2)$ of a product of two triangles is, as a 1-dimensional simplicial complex, isomorphic to $\mathsf{K}_{3,3} = \binom{[3]}{\leq 1}^{*2}$. We appealed to Kuratowski's Theorem to deduce the non-planarity of $\mathsf{K}_{3,3}$. Let us verify this using the Sarkaria index alone. By Proposition 3.28, we need to calculate the Sarkaria index for $\mathsf{D}_3 = \binom{[3]}{\leq 1}$, the simplicial complex on three isolated vertices. The minimal non-faces of D_3 are $\mathcal{F} = \mathcal{F}(\mathsf{D}_3) = \binom{[3]}{2}$ and thus the Kneser graph $\mathsf{KG}(\mathcal{F})$ has three vertices and no edges. It follows that $\mathsf{ind}_{\mathsf{SK}}\, \mathsf{D}_3 = 1$ and $\mathsf{ind}_{\mathsf{SK}}\, \mathsf{K}_{3,3} = 3$ which asserts the non-embeddability into a 2-sphere. A more direct way would have been to note that the Kneser graph of minimal non-faces of $\mathsf{K}_{3,3}$ is isomorphic to $\mathsf{K}_{3,3}$ which has chromatic number $\chi(\mathsf{K}_{3,3}) = 2$. Thus we have finally proved

Corollary 3.29. *There is no realization of $\Delta_2 \times \Delta_2$ such that a projection to the plane yields a 9-gon.*

The statement of the above corollary is sharp: There exists a polytope $P \subset \mathbb{R}^4$ combinatorially equivalent to Δ_2^2 that projects to an 8-gon. To see that, consider the polytope P given by the solution to the following set of inequalities

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{pmatrix} 1 & 1 & & \\ -1 & 1 & & \\ 0 & -1 & -\varepsilon & \\ & -\varepsilon & -1 & 0 \\ & & 1 & 1 \\ & & 1 & -1 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

The numbers to the left label the facets of P. For $\varepsilon = 0$ this is just a Cartesian product of two triangles and, since this is a simple polytope, we can choose $\varepsilon > 0$ sufficiently small without changing the combinatorial type. Taking $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ to be the projection to the first and last coordinate and using the matrix version of the Projection Lemma (Lemma 2.3) it can be verified that all but one vertex of P survive the projection. A different argument is the following: By identifying $(\mathbb{R}^4)^*$ with $\mathbb{R}^4$ via the standard inner product we get an ordered set

$$\mathcal{G} = \mathsf{vert}\, P^{\triangle}|_U = \begin{pmatrix} 1 & 1 & -1 & -\varepsilon & & \\ & & -\varepsilon & -1 & 1 & 1 \end{pmatrix}$$

For $0 < \varepsilon \leq 1$ the set $\mathcal{G}$ is the Gale transform of a polytope $\mathcal{A} = \mathcal{A}(P, \pi)$ combinatorially equivalent to an octahedron (e.g. set $\varepsilon = 1$ and observe that $\mathcal{G}$ is a Gale transform of a regular octahedron.) As intersections of facets the set of vertices is given by $S = \{[6] \setminus \{i, j\} : \{i, j\} \in \mathsf{K}\}$ where the complement complex K is the complete bipartite graph on the partition $\{1, 2, 3\}$ and $\{4, 5, 6\}$. We show that the only vertex v_0 that fails to survive the projection is given by the intersection of the facets $[6] \setminus \{1, 4\}$. By the Projection Lemma and Gale duality this is the case if and only if $\mathsf{K} - \{1, 4\}$ is a subcomplex of the 1-skeleton of $\mathcal{A}$. Figure 3.1 shows $\mathcal{A}$ and the embedding of $\mathsf{K} - \{1, 4\}$, thus finishing the proof. The missing edge between the vertices 1 and 4 shows that v_0 falls short of being a vertex of $\pi(P)$.

3.4 Compound types and cotype complexes

A feature common to the product, the wedge product, and the join of polytopes is that the faces of each are *compound* in the following sense.

Definition 3.30 (Compound type)**.** Let $\mathcal{P}_1, \ldots, \mathcal{P}_r$ be combinatorial types and let $\mathcal{P} \subseteq \mathcal{P}_1 \times \cdots \times \mathcal{P}_r$ be a combinatorial type with corresponding maximal faces. Let $F = (F_1, \ldots, F_r) \in \mathcal{P}$ be a k-face with $\lambda_i = \mathsf{dim}\, F_i$ for $i = 1, \ldots, r$.

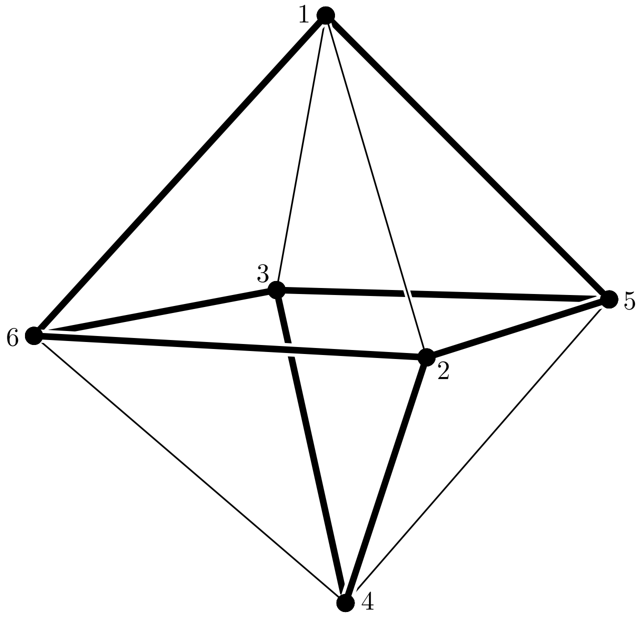


Figure 3.1: The polytope $\mathcal{A}$ associated to P. The marked edges correspond to the embedding of $\mathsf{K} - \{1, 4\}$, that is $\mathsf{K}_{3,3}$ minus an edge.

The type $\mathcal{P}$ is called a *compound type* if every face $F' \in \prod_{i=1}^{r} \mathcal{P}_i$ satisfying $\operatorname{dim} F'_i = \lambda_i$ for $i = 1, \ldots, r$ is a face of $\mathcal{P}$.

We call $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ a *face type* of $\mathcal{P}$ and, as every face arises from some type, the k-skeleton of $\mathcal{P}$ is determined by the types $\mathcal{P}_1, \ldots, \mathcal{P}_r$ and the knowledge of the *admissible face types* of dimension k

$$\Lambda_k(\mathcal{P}) = \{(\operatorname{dim} F_1, \ldots, \operatorname{dim} F_r) : F = (F_1, \ldots, F_r) \in \mathcal{P}, \operatorname{dim} F = k\} \subset \mathbb{Z}^r.$$

The product and the join are compound types: The admissible faces types of dimension $k \geq 0$ for the product $\mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_r$ are given by

$$\Lambda_k(\mathcal{P}) = \{\boldsymbol{\lambda} \in \mathbb{Z}^r : 0 \leq \lambda_i \leq \operatorname{dim} \mathcal{P}_i \text{ for } i \in [r], \lambda_1 + \cdots + \lambda_r = k\},$$

and for the join $\mathcal{P} = \mathcal{P}_1 * \cdots * \mathcal{P}_r$ they are

$$\Lambda_k(\mathcal{P}) = \{\boldsymbol{\lambda} \in \mathbb{Z}^r : -1 \leq \lambda_i \leq \operatorname{dim} \mathcal{P}_i \text{ for } i \in [r], \lambda_1 + \cdots + \lambda_r = k - r + 1\}.$$

The reason for discussing compound types is that in the case of a compound type we can define a natural cover of the coskeleton complex by subcomplexes.

Definition 3.31 (Cotype complex)**.** Let $\mathcal{P}$ be a compound type of the types $\mathcal{P}_1, \ldots, \mathcal{P}_r$. For $\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P})$ we define the *cotype complex* $\Sigma_{\boldsymbol{\lambda}}(\mathcal{P})$ as

$$\Sigma_{\boldsymbol{\lambda}}(\mathcal{P}) := \Sigma_{\lambda_1}(\mathcal{P}_1) \ * \ \Sigma_{\lambda_2}(\mathcal{P}_2) \ * \ \cdots \ * \ \Sigma_{\lambda_r}(\mathcal{P}_r).$$

The fact that the face types partition the collection of k-faces proves that the cotype complexes cover the coskeleton complex.

Proposition 3.32. *Let $\mathcal{P}$ be a compound type of the combinatorial types $\mathcal{P}_1, \ldots, \mathcal{P}_r$ and let $0 \leq k \leq \operatorname{\mathsf{dim}} \mathcal{P}$. Then*

$$\Sigma_k(\mathcal{P}) = \bigcup_{\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P})} \Sigma_{\boldsymbol{\lambda}}(\mathcal{P}).$$

□

The gain in introducing the cotype complexes is reflected in the following two results. The first states that a large embeddability dimension of a cotype complex is sufficient to obstruct the projectability of the k-skeleton. It follows from Corollary 3.20 and the fact that the embeddability dimension is monotone with respect to subcomplexes.

Corollary 3.33. *Let $\mathcal{P}$ be a compound type and $0 \leq k < \operatorname{\mathsf{dim}} \mathcal{P}$. If there is a face type $\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P})$ such that*

$$e < \operatorname{\mathsf{edim}}(\Sigma_{\boldsymbol{\lambda}}) + d - m + 2$$

then there is no realization of $\mathcal{P}$ such that a projection to $\mathbb{R}^e$ retains the k-skeleton. □

The second benefit of the cotype complex lies in its structural simplicity and the (almost) additive behavior of the Sarkaria index with respect to joins (see Proposition 3.28).

Corollary 3.34. *Let $\mathcal{P}$ be a compound type with types $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_r$ and let $\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P})$. Then*

$$\operatorname{\mathsf{ind}}_{\mathsf{SK}} \Sigma_{\boldsymbol{\lambda}}(\mathcal{P}) = \sum_{i=1}^{r} \operatorname{\mathsf{ind}}_{\mathsf{SK}} \Sigma_{\lambda_i}(\mathcal{P}_i) + r - 1.$$

□

Suppose $\mathcal{P}$ is the r-fold product of $\mathcal{Q}$. By Corollary 3.34 the Sarkaria index of a $\Sigma_{\boldsymbol{\lambda}}(\mathcal{P})$ is computed in terms of $\operatorname{\mathsf{ind}}_{\mathsf{SK}} \Sigma_{\lambda_i}(\mathcal{Q})$ for $i = 1, \ldots, r$. The largest Sarkaria index among the cotype complexes can be computed by solving the following *knapsack-type* problem.

Proposition 3.35. *Let $\mathcal{P}$ be the r-fold product of the d-type $\mathcal{Q}$ and let $0 \le k \le rd-1$. Further, let $s_i = \mathsf{ind}_{\mathsf{SK}}\,\Sigma_i(\mathcal{Q})$ for $i = 0,\dots,d$. If s^* is the optimal value of*

$$\begin{array}{rrcl} \mathsf{max} & s_0\,\mu_0 + s_1\,\mu_1 + \cdots + s_d\,\mu_d & & \\ \mathsf{s.t.} & 0\,\mu_0 + \ 1\,\mu_1 + \cdots + \ d\,\mu_d & = & k \\ & \mu_0 + \quad \mu_1 + \cdots + \quad \mu_d & = & r \end{array}$$

with $\mu_0,\dots,\mu_d \in \mathbb{Z}_{\ge 0}$. Then $\mathsf{edim}(\Sigma_k(\mathcal{P})) \ge s^* + r - 1$.

Proof. To a face type $\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P})$ we associate the non-negative numbers $\mu = (\mu_0,\mu_1,\dots,\mu_d)$ with $\mu_i = \#\{j \in [r] : \lambda_j = i\}$. Since $\boldsymbol{\lambda}$ is a partition of k in r parts, the numbers μ satisfy

$$\begin{array}{rcl} 0\,\mu_0 + 1\,\mu_1 + \cdots + d\,\mu_d & = & k \text{ and} \\ \mu_0 + \ \ \mu_1 + \cdots + \ \ \mu_d & = & r \end{array}$$

The Sarkaria index of $\Sigma_{\boldsymbol{\lambda}}(\mathcal{P})$ is given by $\sum_i s_i\,\mu_i + r - 1$. Vice versa, every such non-negative collection of numbers μ_i that satisfies the conditions of the integer program gives rise to a valid face type. $\square$

Example 3.36. To illustrate the usefulness of the cotype complex, consider the following question: Is there a realization of $\mathcal{P} = \Delta_1 \times \Delta_2$, a prism over a triangle, such that a projection to the plane preserves the three *vertical* edges (see Figure 3.2). The ad-hoc negation of the question is that by *Desargues' Theorem* (cf. [14, Sect. 14.3]) the three vertical edges in the prism meet in a common point (maybe at infinity) and a linear projection retains this property. Using the developed machinery, we see that the assumed projection satisfies the conditions of Proposition 3.7 and the vertical edges correspond to the face type $\boldsymbol{\lambda} = (1,0)$. The cotype complex is also shown in Figure 3.2; it consists of three triangles that share a common edge. Corollary 3.33 implies that such a projection does not exist as $\Sigma_{(1,0)}(\mathcal{P})$ is not planar.

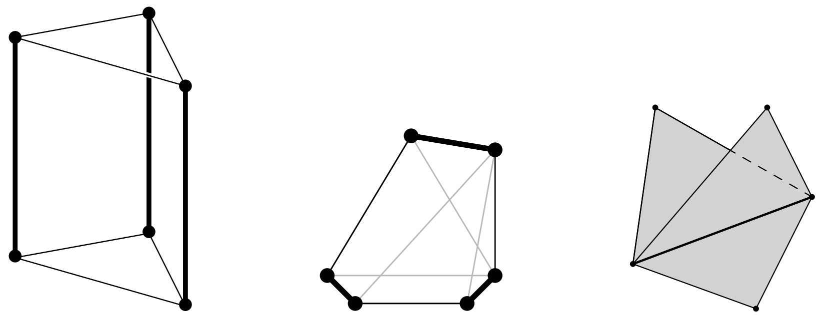


Figure 3.2: The triangular prism to the left with bold vertical edges. An alleged projection in the middle with preserved vertical edges. And the associated complex to the right.

Chapter 4

Non-projectability of polytope skeleta

After having developed the machinery to prove non-projectability results of skeleta in Chapter 3, we go about calculating the obstructions for three families of polytopes. We treat products of polygons and of simplices where we extensively use the results from Section 3.4 plus information about single polygons and simplices, respectively. The close relation of products of simplices to the wedge products $\mathcal{W}_{r,n-1} = \mathcal{P}_r \lhd \Delta_{n-1}$ will also yield non-projectability results for the latter.

4.1 Products of polygons

Denote by $\mathcal{P}_m$ the combinatorial type of an m-gon, that is a 2-dimensional combinatorial type on $m \geq 3$ facets labeled in cyclic order. We seek to find necessary conditions for the existence of a realization of the product of polygons $\mathcal{P} = \mathcal{P}_{m_1} \times \mathcal{P}_{m_2} \times \cdots \times \mathcal{P}_{m_r}$ such that a projection to $\mathbb{R}^e$ retains the k-skeleton. By the technology developed in Chapter 3 we need to bound from below the embeddability dimension of $\Sigma_k(\mathcal{P})$ for $0 \leq k < 2r = \mathsf{dim}\,\mathcal{P}$.

An interesting peculiarity of the results to come is that (bounds on) the embeddability will only depend on the m_i *modulo* 2 (cf. Lemma 4.1). For this reason, we fix the following prime example for the rest of the section. We denote by

$$\mathcal{P} \;=\; \mathcal{P}_{m_e}^{r_e} \times \mathcal{P}_{m_o}^{r_o}$$

the product of r_e *even* m_e-gons and r_o *odd* m_o-gons. Furthermore, we denote by $r = r_e + r_o$ the total number of factors and by $m = r_e m_e + r_o m_o$ the total number of facets.

Coskeleton complexes of polygons

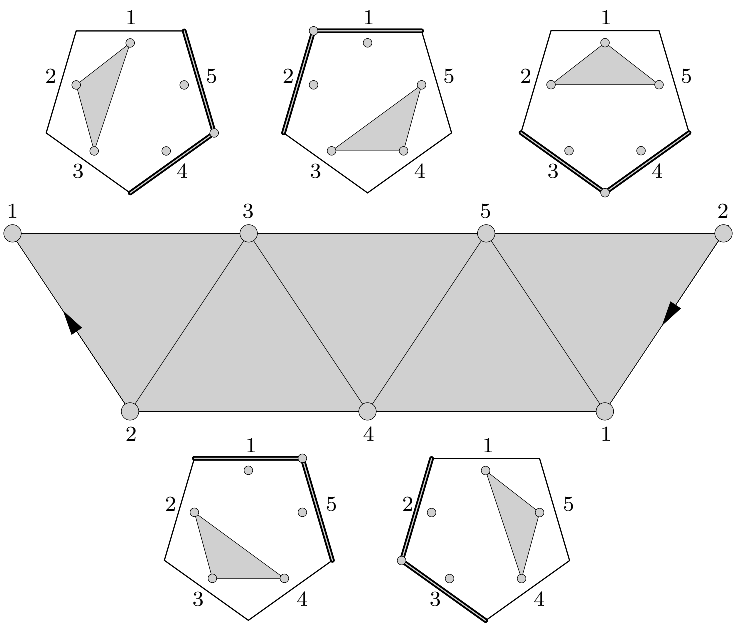
Eventually, we want to apply Proposition 3.35 for which we need to determine the Sarkaria index of $\Sigma_k(\mathcal{P}_m)$ for $k = 0, 1, 2$. In light of Proposition 3.25, we are left to determine only the Sarkaria index for the 0-th coskeleton complex of polygons.

Lemma 4.1. *Let $m \geq 3$ and $\mathcal{P}_m$ the combinatorial type of an m-gon. The Sarkaria index for the 0-th coskeleton complex is given by*

$$\mathsf{ind}_{\mathsf{SK}}\, \Sigma_0(\mathcal{P}_m) = \begin{cases} m-3, & \text{if } m \text{ is even, and} \\ m-2, & \text{if } m \text{ is odd.} \end{cases}$$

Proof. We show that the Kneser graph of minimal non-faces of $\Sigma_0(\mathcal{P}_m)$ has chromatic number 2 and 1, respectively. For that let us determine the minimal non-faces of $\Sigma_0(\mathcal{P}_m)$: A subset $\sigma \subseteq [m]$ of the facets of $\mathcal{P}_m$ is a non-face of $\Sigma_0(\mathcal{P}_m)$ if and only if every vertex of $\mathcal{P}_m$ is incident to at least one facet F_i of $\mathcal{P}_m$ with $i \in \sigma$. If a vertex of $\mathcal{P}_m$ is covered twice by σ then every other minimal non-face intersects σ and thus σ is an isolated vertex in the Kneser graph. If σ covers every vertex exactly once, then $[m] \setminus \sigma$ is again a minimal non-face. It follows that for odd m the Kneser graph consists of isolated vertices alone while for even m there is exactly one edge. $\square$

Example 4.2. As an illustration, let us consider $\Sigma_0(\mathcal{P}_5)$, the 0-th coskeleton complex of a pentagon. The facets (edges) of the pentagon are labeled by 1 to 5 in cyclic order, the facet of $\Sigma_0(\mathcal{P}_5)$ corresponding to a chosen vertex of $\mathcal{P}_5$ is shown inside the respective pentagon.



These triangles fit together to form a *Möbius strip.* Thus $\Sigma_0(\mathcal{P}_5)$ not embeddable in the 2-sphere.

COSKELETON COMPLEXES OF PRODUCTS OF POLYGONS

We are now ready to deal with the coskeleton complexes of products of polygons using the *knapsack-type* integer program introduced in Proposition 3.35.

Theorem 4.3. *Let $\mathcal{P} = \mathcal{P}_{m_e}^{r_e} \times \mathcal{P}_{m_o}^{r_o}$ be a product of $r = r_e + r_o$ polygons with a total of $m = r_e m_e + r_o m_o$ facets. For $0 \le k < 2r$ we have*

$$\mathsf{edim}(\Sigma_k(\mathcal{P})) \ge m - 1 - r + \left\lfloor \frac{k}{2} \right\rfloor + \mathsf{min}\left\{0, \left\lceil \frac{k}{2} \right\rceil - r_e\right\}.$$

Proof. In the spirit of Proposition 3.35 consider the following integer linear program

$$\begin{array}{llllllllll}
\mathsf{max} & 2\,\mu_0^{\mathsf{even}} & + & \mu_0^{\mathsf{odd}} & + & \mu_1 & & & & \\
\mathsf{s.t.} & & & & & \mu_1 & + 2\,\mu_2 & & = & k \\
& \mu_0^{\mathsf{even}} & + & \mu_0^{\mathsf{odd}} & + & \mu_1 & + & \mu_2 & = & r \\
& \mu_0^{\mathsf{even}} & & & & & & & \le & r_e \\
& & & \mu_0^{\mathsf{odd}} & & & & & \le & r_o
\end{array}$$

with $\mu_0^{\mathsf{even}}, \mu_0^{\mathsf{odd}}, \mu_1, \mu_2 \in \mathbb{Z}_{\ge 0}$. Every face type $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in \Lambda_k(\mathcal{P})$ gives rise to a feasible solution by the association

$$\begin{array}{lcll}
\mu_2 & := & \#\{i : \lambda_i = 2\} & \text{(polygons)} \\
\mu_1 & := & \#\{i : \lambda_i = 1\} & \text{(edges)} \\
\mu_0^{\mathsf{even}} & := & \#\{i : \lambda_i = 0, 1 \le i \le r_e\} & \text{(even vertices)} \\
\mu_0^{\mathsf{odd}} & := & \#\{i : \lambda_i = 0, r_e < i \le r\} & \text{(odd vertices)}
\end{array}$$

and, vice versa, every feasible solution yields a face type. The integer program reduces to a problem in essentially two variables and the optimal solution is easily calculated to be

$$s^* = r - \left\lfloor \frac{k}{2} \right\rfloor + \mathsf{max}\left\{0, r_e - \left\lceil \frac{k}{2} \right\rceil\right\}.$$

The result then follows from the fact that $\mathsf{edim}(\Sigma_k(\mathcal{P})) \ge m - 1 - s^*$. □

Combining the bounds of Theorem 4.3 with Corollary 3.20 we obtain the following obstructions to the projectability of products of polygons.

Theorem 4.4. *Let $\mathcal{P} = \mathcal{P}_{m_e}^{r_e} \times \mathcal{P}_{m_o}^{r_o}$ be a product of r_e even and r_o odd polygons and let $0 \le k < 2r = 2(r_e + r_o)$. If*

$$e < r + 1 + \left\lfloor \frac{k}{2} \right\rfloor + \mathsf{min}\left\{0, \left\lceil \frac{k}{2} \right\rceil - r_e\right\}.$$

then there is no realization of $\mathcal{P}$ such that a projection to $\mathbb{R}^e$ retains the k-skeleton. □

In closing with our treatment of products of polygons, let us put the above result in perspective by determining upper bounds on the embeddability dimension of $\Sigma_k(\mathcal{P}_{m_e}^{r_e} \times \mathcal{P}_{m_o}^{r_o})$.

Proposition 4.5. *Let $\mathcal{P} = \mathcal{P}_{m_e}^{r_e} \times \mathcal{P}_{m_o}^{r_o}$ be a product of polygons on a total of $m = r_e m_e + r_o m_o$ facets and let $0 \le k < 2r = 2(r_e + r_o)$. Then*

$$\mathsf{edim}(\Sigma_k(\mathcal{P})) \le \begin{cases} m - r - r_e - 1, & \text{if } k = 0 \\ m - r - 1, & \text{if } k = 1 \\ m - 1, & \text{otherwise.} \end{cases}$$

Proof. Let $\ell = \min\{k, 2\}$, set $\boldsymbol{\ell} = (\ell, \ell, \dots, \ell)$, and define

$$\Sigma_{\boldsymbol{\ell}} = \Sigma_{\boldsymbol{\ell}}(\mathcal{P}) = \Sigma_\ell(\mathcal{P}_{m_e})^{*r_e} * \Sigma_\ell(\mathcal{P}_{m_o})^{*r_o}.$$

For every admissible face type $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in \Lambda_k(\mathcal{P})$ we have $\lambda_i \le \ell$ for $i = 1, \dots, r$ and, by Observation 3.14 and the relation of subcomplexes among joins, this shows $\Sigma_{\boldsymbol{\lambda}}(\mathcal{P}) \subseteq \Sigma_{\boldsymbol{\ell}}$. By the natural identification of vertex sets, we obtain that Σ_k is a subcomplex of $\Sigma_{\boldsymbol{\ell}}$ and we will bound the embeddability dimension of $\Sigma_{\boldsymbol{\ell}}$ from above.

For $\ell = 2$, we have by Proposition 3.25 that $\Sigma_2(\mathcal{P}_n) = \Delta_{n-1} \hookrightarrow \partial\Delta_n$ and thus $\Sigma_{\boldsymbol{\ell}}$ embeds into the boundary of $\Delta_{m_e}^{\oplus r_e} \oplus \Delta_{m_o}^{\oplus r_o}$, a simplicial sphere of dimension $r_e m_e + r_o m_o - 1 = m - 1$.

For $\ell = 1$, we again make use of Proposition 3.25 to get $\Sigma_1(\mathcal{P}_n) = \partial\Delta_{n-1}$ and $\Sigma_{\boldsymbol{\ell}} \hookrightarrow \partial(\Delta_{m_e-1}^{\oplus r_e} \oplus \Delta_{m_o-1}^{\oplus r_o})$, which is a simplicial sphere of dimension $r_e(m_e - 1) + r_o(m_o - 1) - 1 = m - r - 1$.

For $\ell = 0$, the 0-th coskeleton complex of $\mathcal{P}_n$ may be embedded into the boundary of an $(n-1)$-simplex. However, for even $n = 2t$ we can do better: Consider the $(n-2)$-dimensional polytope $Q_t = \Delta_{t-1} \oplus \Delta_{t-1}$ and the mapping from the vertices of $\Sigma_0(\mathcal{P}_n)$ that maps the i-th vertex to the $\lfloor \frac{i}{2} \rfloor$-th vertex of the first summand if i is even and of the second otherwise. We claim that this gives an embedding. Every vertex v of $\mathcal{P}_n$ is the intersection of an odd and an even edge. Thus the corresponding facet $[n] \setminus I(v)$ is the disjoint union of $t-1$ odd and $t-1$ even vertices. These sets correspond to facets of Q_t. Thus $\Sigma_0(\mathcal{P}) = \Sigma_{\boldsymbol{\ell}}$ embeds into the boundary of $Q_t^{\oplus r_e} \oplus \Delta_{m_o-1}^{\oplus r_o}$ with $t = \frac{m_e}{2}$. $\square$

The upper and lower bounds of Proposition 4.5 and Theorem 4.3 match in the special case of the 0-skeleton of a product of polygons and of the 1-skeleton of a product of odd polygons. This entails that the corresponding bounds in Theorem 4.4 are sharp.

4.2 Products of simplices

It is known to both discrete geometers and topologists that no d-polytope is dimensionally k-ambiguous for $k > \lfloor \frac{d}{2} \rfloor$ (cf. Theorem 3.21). Essentially, the reason is that the statement is already false for the d-simplex, which serves as the *general* polytope in this situation. For the class of polytopes that are *products*, it is reasonable that products of simplices should take the place of the general polytopes. In this section we provide some evidence by proving obstructions to the projectability of skeleta of products of simplices by calculating lower bounds for the Sarkaria index of the coskeleton complex. This depends on the Sarkaria index of a single simplex which is intrinsically related to colorings of *classical* Kneser graphs.

The last section relates our results to the wedge products $\mathcal{W}_{r,n-1} = \mathcal{P}_r \triangleleft \Delta_{n-1}$.

Coskeleton complexes of simplices

The key to determining the embeddability dimension and the Sarkaria index of $\Sigma_k(\Delta_{n-1})$ will be the following.

Observation. *For $n \geq 1$ and $0 \leq k \leq n-1$ the k-th coskeleton complex $\Sigma_k(\Delta_{n-1})$ of the $(n-1)$-simplex is isomorphic to the k-skeleton of Δ_{n-1}.*

Thus $\Sigma_k(\Delta_{n-1})$ is a *well known* complex and the calculation of the Sarkaria index involves the *classical* Kneser graphs $\mathsf{KG}_{n,\ell} = \mathsf{KG}\binom{[n]}{\ell}$ for $0 \leq \ell \leq n$, the Kneser graphs on the collection of all ℓ-sets of $[n]$.

Theorem 4.6 (Lovász [34]). *For $1 \leq \ell \leq n$ the chromatic number of $\mathsf{KG}_{n,\ell}$ is given by*

$$\chi(\mathsf{KG}_{n,\ell}) = \begin{cases} n - 2\ell + 2 & \text{if } \ell \leq \frac{n+1}{2} \\ 1 & \text{otherwise.} \end{cases}$$

The Sarkaria index of the k-th coskeleton complex may easily be calculated as follows.

Lemma 4.7. *For $n \geq 2$ and $0 \leq k \leq n-1$ the Sarkaria index of the k-th coskeleton complex $\Sigma_k = \Sigma_k(\Delta_{n-1})$ of the $(n-1)$-simplex is*

$$\mathsf{ind}_{\mathsf{SK}}\, \Sigma_k = \begin{cases} 2k+1, & \text{if } 0 \leq k \leq \frac{n-3}{2}, \\ n-2, & \text{if } \frac{n-3}{2} < k \leq n-2, \\ n-1, & \text{if } k = n-1. \end{cases}$$

Proof. By the above observation, we have $\mathsf{KG}(\mathcal{F}(\Sigma_k)) = \mathsf{KG}_{n,k+2}$. The result follows from Theorem 4.6 except for the cases where $\frac{n-3}{2} \leq k \leq n-2$ and we have no disjoint non-faces and where $k = n-1$ and we have no non-faces. The latter case also follows from Proposition 3.25. □

In combination with Proposition 3.19 and Theorem 3.24 this gives

Corollary 4.8. *Let $\Sigma_k = \Sigma_k(\Delta_{n-1})$ be the k-th coskeleton complex of the $(n-1)$-simplex for $n \geq 2$. Then the embeddability dimension satisfies*

$$\mathsf{edim}(\Sigma_k) = \begin{cases} 2k+1, & \text{if } \quad 0 \leq k \leq \frac{n-3}{2}, \\ n-2, & \text{if } \frac{n-3}{2} < k \leq n-2, \\ n-1, & \text{otherwise.} \end{cases}$$ □

COSKELETON COMPLEXES OF PRODUCTS OF SIMPLICES

The Sarkaria index for a single simplex enables us to deal with products of simplices. In the following we denote by

$$\Delta_{n-1}^r = \underbrace{\Delta_{n-1} \times \Delta_{n-1} \times \cdots \times \Delta_{n-1}}_{r}$$

an r-fold product of $(n-1)$-simplices.

Theorem 4.9. *Let $n \geq 2$, $r \geq 1$ and $0 \leq k < r(n-1)$. The embeddability dimension of the k-th coskeleton complex $\Sigma_k = \Sigma_k(\Delta_{n-1}^r)$ is bounded from below by*

$$\mathsf{edim}(\Sigma_k) \geq \begin{cases} 2r+2k-1, & \text{if } \quad 0 \leq k \leq r\lfloor\frac{n-3}{2}\rfloor \\ \frac{1}{2}rn+k-1, & \text{if } r\lfloor\frac{n-3}{2}\rfloor < k \leq r\lfloor\frac{n-2}{2}\rfloor \\ r(n-1)+\alpha-1, & \text{if } r\lfloor\frac{n-2}{2}\rfloor < k < r(n-1) \end{cases}$$

and

$$\alpha = \left\lfloor \frac{k - r\lfloor\frac{n-2}{2}\rfloor}{\lfloor\frac{n+1}{2}\rfloor} \right\rfloor.$$

Proof. We use the knowledge gained from Lemma 4.7 to set up the integer linear program from Proposition 3.35. Set $c = \lfloor\frac{n-3}{2}\rfloor$ and let $0 \leq k < r(n-1)$. The program is

$$\begin{array}{lll} \mathsf{max} & \sum_{j=0}^{c}(2j+1)\,\mu_j + (n-2)\sum_{j=c+1}^{n-2}\mu_j + (n-1)\mu_{n-1} & \\ \mathsf{s.t.} & \mu_0 + \mu_1 + \cdots + \mu_{n-1} & = r \\ & 0\,\mu_0 + 1\,\mu_1 + \cdots + (n-1)\mu_{n-1} & = k \end{array}$$

and subject to the condition that the μ_i are non-negative and integral. Any feasible solution with value s gives the bound $\mathsf{edim}(\Sigma_k) \geq r - 1 + s$.

By using the two equations to rewrite the objective function, the optimal value is equal to the optimal value of

$$r + 2k - \mathsf{min} \sum_{j=c+1}^{n-1} (2j - n + 3)\mu_j - \mu_{n-1}$$

subject to the same constraints as above. Note that all coefficients are non-negative and thus the minimum is at most 0.

For $0 < k \leq r\lfloor \frac{n-2}{2} \rfloor$ set $\ell = \lceil \frac{k}{r} \rceil \leq c + 1$. Define $\mu = (\mu_0, \dots, \mu_{n-1}) \in \mathbb{Z}^n$ by

$$\begin{pmatrix} \mu_{\ell-1} \\ \mu_\ell \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \ell - 1 & \ell \end{pmatrix}^{-1} \begin{pmatrix} r \\ k \end{pmatrix} = \begin{pmatrix} r\ell - k \\ k - r(\ell - 1) \end{pmatrix}$$

and $\mu_j = 0$ otherwise. For n odd we have $\ell \leq \lfloor \frac{n-2}{2} \rfloor = c$ and μ gives a feasible solution with value 0 in the minimization above. If n is even and $\ell = c + 1$ the feasible solution yields a total value of $r + 2k - (k + r\ell) = k + \frac{1}{2}rn$. Note that the second case is vacuous for n odd.

For $r\lfloor \frac{n-2}{2} \rfloor < k$, let $h = k - r\lfloor \frac{n-2}{2} \rfloor - \alpha\lfloor \frac{n+1}{2} \rfloor$ set

$$\mu_{n-1} = \alpha \qquad \mu_c = r - \alpha - 1 \qquad \mu_{c+h} = 1$$

for n odd and

$$\mu_{n-1} = \alpha \qquad \mu_{c+1} = r - \alpha - 1 \qquad \mu_{c+h+1} = 1$$

for n even and $\mu_j = 0$ for all other j. $\square$

As can be seen in the proof, the feasible solution for $\ell \leq \lfloor \frac{n-3}{2} \rfloor$ is given by a basic solution to the linear program relaxation and it can be checked that this indeed gives the optimal solution. However, the coefficient for μ_{n-1} keeps this circumstance from being true for $\ell > \lfloor \frac{n-3}{2} \rfloor$.

In conjunction with Corollary 3.20 this gives the following definitive result concerning the non-projectability of skeleta of Δ_{n-1}^r.

Theorem 4.10. *Let $n \geq 2$ and $r \geq 1$ and set $\alpha = \left\lfloor \frac{k - r\lfloor \frac{n-2}{2} \rfloor}{\lfloor \frac{n+1}{2} \rfloor} \right\rfloor$. If*

$$e < \begin{cases} r + 2k + 1, & \text{for} \quad 0 \leq k \leq r\lfloor \frac{n-3}{2} \rfloor \\ \frac{1}{2}r(n-2) + k + 1, & \text{for} \quad r\lfloor \frac{n-3}{2} \rfloor < k \leq r\lfloor \frac{n-2}{2} \rfloor \\ r(n-2) + \alpha + 1, & \text{for} \quad r\lfloor \frac{n-2}{2} \rfloor < k < r(n-1) \end{cases}$$

then there exists no realization of the r-fold product Δ_{n-1}^r of $(n-1)$-simplices such that a projection to $\mathbb{R}^e$ retains the k-skeleton. $\square$

The result can be viewed as a generalization of the polyhedral version of the Van Kampen–Flores theorem (cf. Theorem 3.21) from simplices (the case $r = 1$) to products of simplices. Moreover, it proves Corollary 2.15 used in Section 2.2 and it gives yet another proof of the fact that a product of two triangles does not linearly map to a 9-gon.

Again, let us view the statement of Theorem 4.10 in comparison with upper bounds on the embeddability dimension of the complexes $\Sigma_k(\Delta_{n-1}^r)$.

Proposition 4.11. *Let $\Sigma_k = \Sigma_k(\Delta_{n-1}^r)$ be the k-th coskeleton complex of the r-fold product of $(n-1)$-simplices with $n \geq 2$ and $0 \leq k < r(n-1)$. Then*

$$\mathsf{edim}(\Sigma_k) \leq \mathsf{min}\,\{2k + 2r - 1, rn - 1\}.$$

Proof. We work along the same lines as in the proof of Proposition 4.5 and we use the fact that

$$\Sigma_\ell(\Delta_{n-1}) \cong \tbinom{[n]}{\leq \ell+1} \hookrightarrow \partial\Delta_n$$

for all $0 \leq \ell \leq n-1$. Therefrom it follows that $\Sigma_k \hookrightarrow \partial\Delta_n^{\oplus r} = \partial(\Delta_n^r)^\triangle$ and thus $\mathsf{edim}(\Sigma_k) \leq rn - 1$. However, since $\mathsf{dim}\,\Sigma_k = r + k - 1$ the bound given by Proposition 3.19 is better for $k \leq \frac{1}{2}r(n-2)$. $\square$

Combining the upper bounds with the lower bounds from Theorem 4.9 yields that the result of Theorem 4.10 is sharp for $k \leq r\lfloor\frac{n-3}{2}\rfloor$. See also the *guiding example* at the end of Section 3.3 for an example of a product of simplices whose projection retains all but one vertex.

GENERALIZATION TO WEDGE PRODUCTS

Let us recall the definition of a combinatorial wedge product: For combinatorial types $\mathcal{P}$ and $\mathcal{Q}$ and r the number of facets of $\mathcal{P}$, the wedge product $\mathcal{P} \lhd \mathcal{Q}$ is given by the faces $G = (G_1, G_2, \ldots, G_r) \in \mathcal{Q}^r$ such that

$$\{i \in [r] : G_i = \varnothing\} = I_\mathcal{P}(F)$$

for some face $F \in \mathcal{P}$. In particular, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{Z}^r$ is an admissible face type for $\mathcal{P} \lhd \mathcal{Q}$ if and only if $\boldsymbol{\lambda}$ is admissible for $\mathcal{Q}^r$ and

$$\{i \in [r] : \lambda_i = -1\} = I_\mathcal{P}(F).$$

In order to deduce the non-projectability of the k-skeleton of $\mathcal{P} \lhd \mathcal{Q}$ our punchline comes from the following lemmas.

Lemma 4.12. *Let $\mathcal{P}$ and $\mathcal{Q}$ be combinatorial types and let r be the number of facets of $\mathcal{P}$. For every admissible face type $\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P} \lhd \mathcal{Q})$ we have*

$$\Sigma_{\boldsymbol{\lambda}}(\mathcal{P} \lhd \mathcal{Q}) \cong \Sigma_{\boldsymbol{\lambda}}(\mathcal{Q}^r).$$

Proof. The result trivially follows from the fact that the facets of $\mathcal{P} \lhd \mathcal{Q}$ and $\mathcal{Q}^r$ are canonically isomorphic: The defining property (CWP) of wedge products is trivially fulfilled for the facets of $\mathcal{Q}^r$. Hence, $I_{\mathcal{P}\lhd\mathcal{Q}}(G) = I_{\mathcal{Q}^r}(G)$ for every face $G \in \mathcal{P} \lhd \mathcal{Q}$ and therefore $\Sigma_{\boldsymbol{\lambda}}(\mathcal{P} \lhd \mathcal{Q}) \cong \Sigma_{\boldsymbol{\lambda}}(\mathcal{Q}^r)$. □

Note that Lemma 4.12 in conjunction with Corollary 3.33 yields obstructions to the projectability of the skeleta that are independent of the combinatorics of $\mathcal{P}$. We can further relate the cotype complexes of the wedge product to that of an ℓ-fold product of $\mathcal{Q}$.

Lemma 4.13. *Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \Lambda_k(\mathcal{Q}^r)$ be a face type with $k \geq 0$ and let $\ell = \#\{i : \lambda_i \neq -1\}$ be the number of non-empty faces. For the truncated face type $\bar{\boldsymbol{\lambda}} = (\lambda_i : \lambda_i \geq 0) \in \Lambda_{k-\ell+1}(\mathcal{Q}^\ell)$ we have*

$$\Sigma_{\boldsymbol{\lambda}}(\mathcal{Q}^r) \;\cong\; \Sigma_{\bar{\boldsymbol{\lambda}}}(\mathcal{Q}^\ell)$$

and in particular

$$\mathsf{ind}_{\mathsf{SK}}\, \Sigma_{\boldsymbol{\lambda}}(\mathcal{Q}^r) \;=\; \mathsf{ind}_{\mathsf{SK}}\, \Sigma_{\bar{\boldsymbol{\lambda}}}(\mathcal{Q}^\ell).$$

Proof. For the first assertion note that $\Sigma_{-1}(\mathcal{Q}) = \{\varnothing\}$ for every combinatorial type $\mathcal{Q}$. The claim now follows from the fact that $\mathsf{K} * \{\varnothing\} \cong \mathsf{K}$ for every simplicial complex K. The second statement follows from the definition of the Sarkaria index. □

The combination of the Lemmas 4.12 and 4.13 with Theorem 4.9 yields the following result on the projectability of the k-skeleton of $\mathcal{W}_{r,n-1}$.

Theorem 4.14. *Let $r \geq 3$, $n \geq 2$. For $0 \leq k < r(n-1)+2$ let*

$$\beta = \max\{\mathsf{ind}_{\mathsf{SK}}\, \Sigma_{\boldsymbol{\lambda}}(\Delta_{n-1}^{r-\ell}) : \boldsymbol{\lambda} \in \Lambda_{k+\ell-2}(\Delta_{n-1}^{r-\ell}) \text{ and } 0 \leq \ell \leq 2\}.$$

If $e < 4 - r + \beta$ then there is no realization of $\mathcal{W}_{r,n-1} = \mathcal{P}_r \lhd \Delta_{n-1}$ such that a projection to e-space strictly preserves the k-skeleton.

Proof. The combination of Lemma 4.12 and Lemma 4.13 together with the fact that $\Sigma_k(\mathcal{P}_r \lhd \Delta_{n-1}) \supseteq \Sigma_{\boldsymbol{\lambda}}(\mathcal{P}_r \lhd \Delta_{n-1})$ for all $\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P}_r \lhd \Delta_{n-1})$ yields $\mathsf{ind}_{\mathsf{SK}}\, \Sigma_k(\mathcal{P}_r \lhd \Delta_{n-1}) \geq \beta$. The result then follows from Corollary 3.33. □

Finally, we can give a proof of Theorem 2.26 concerning the non-projectability of the polyhedral surfaces $\mathcal{S}_{r,2n} \subset \mathcal{W}_{r,n-1}$.

Proof of Theorem 2.26. We first note that the collection of edges $\mathcal{H}_E$ cover all vertices of $\mathcal{W}_{r,n-1}$. Thus, finding a projection that preserves the surface necessarily preserves the 0-skeleton and we can use Corollary 3.33.

From the definition of $\mathcal{H}_E$ it follows that the edges of type

$$\boldsymbol{\lambda} = (-1, \underbrace{0, 0, \ldots, 0}_{r-1})$$

form a proper subset of $\mathcal{H}_E$. Thus, if there is a realization of $\mathcal{W}_{r,n-1}$ such that a projection to e-space retains $\mathcal{H}_E$, then it also preserves all edges of type $\boldsymbol{\lambda}$. By using Lemma 4.12 and Lemma 4.13 it follows that

$$\Sigma_{\boldsymbol{\lambda}}(\mathcal{W}_{r,n-1}) \cong \Sigma_{\bar{\boldsymbol{\lambda}}}(\Delta_{n-1}^{r-1}) = \Sigma_0(\Delta_{n-1}^{r-1})$$

where $\bar{\boldsymbol{\lambda}} = (0, \ldots, 0)$. By Theorem 4.9, we get that $\mathsf{edim}(\Sigma_{\boldsymbol{\lambda}}(\mathcal{W}_{r,n-1})) \geq 2r-3$ and an application of Corollary 3.33 yields the result. $\square$

Part II

Centrally symmetric polytopes

Chapter 5

Kalai's conjectures concerning centrally symmetric polytopes

A convex d-polytope P is *centrally symmetric*, or *cs* for short, if $P = -P$. Concerning face numbers, this implies that for $0 \leq i \leq d-1$ the number of i-faces $f_i(P)$ is even and that $f_0(P), f_{d-1}(P) \geq 2d$. Beyond this, only very little is known for the general case. That is to say, the extra (structural) information of a central symmetry yields no substantial additional constraints for the face numbers on the restricted class of polytopes.

Not uncommon to the f-vector business, the knowledge about face numbers is concentrated on the class of centrally symmetric *simplicial*, or dually *simple*, polytopes. In 1982, Bárány and Lovász [3] proved a lower bound on the number of vertices of simple cs polytopes with prescribed number of facets, using a generalization of the Borsuk–Ulam theorem. Moreover, they conjectured lower bounds for all face numbers of this class of polytopes with respect to the number of facets. In 1987 Stanley [56] proved a conjecture of Björner concerning the h-vectors of simplicial cs polytopes which implies the one by Bárány and Lovász. The proof uses Stanley-Reisner rings and toric varieties plus a pinch of representation theory. The result of Stanley [56] for centrally symmetric polytopes was reproved in a more geometric setting by Novik [45] using "symmetric flips" in McMullen's weight algebra [40]. For general cs polytopes, lower bounds on the *toric* h-vector were recently obtained by A'Campo-Neuen [2] using combinatorial intersection cohomology. Unfortunately, the toric h-vector contains only limited information about the face numbers of general (cs) polytopes and thus the applicability of the result is limited (see Section 5.1).

In [29], Kalai stated three conjectures about the face numbers of general cs polytopes. Let P be a (cs) d-polytope with f-vector $f(P) = (f_0, f_1, \ldots, f_{d-1})$. Define the function $s(P)$ by

$$s(P) := 1 + \sum_{i=0}^{d-1} f_i(P) = f_P(1)$$

where $f_P(t) := f_{d-1}(P) + f_{d-2}(P)t + \cdots + f_0(P)t^{d-1} + t^d$ is the f-polynomial. Thus, $s(P)$ measures the total number of non-empty faces of P. Here is Kalai's first conjecture from [29], the "3^d-*conjecture*".

Conjecture A. Every centrally symmetric d-polytope has at least 3^d non-empty faces, i.e. $s(P) \geq 3^d$.

For the d-cube C_d and for its dual, the d-dimensional crosspolytope $C_d^{\triangle}$, we have

$$s(C_d) = s(C_d^{\triangle}) = 1 + \sum_{i=0}^{d-1} \binom{d}{i} 2^{d-i} = (2+1)^d$$

and thus they satisfy the 3^d-conjecture. It takes a moment's thought to see that in dimensions $d \geq 4$ these are not the only polytopes with 3^d non-empty faces. The following important class of cs polytopes motivated Conjecture **A**.

Definition 5.1 (Hanner polytopes [24]). A d-dimensional cs polytope H is a *Hanner polytope* if

i) the dimension d is at most 1, or
ii) H is the direct sum or direct product of two (lower dimensional) Hanner polytopes H' and H''.

In particular, the cube and the crosspolytope are Hanner polytopes and it is easy to see that if H is a d-dimensional Hanner polytope, then $s(H) = 3^d$. The number of Hanner polytopes grows exponentially in the dimension d, with a Catalan-type recursion. It is given by the number of two-terminal networks with d edges, $n(d) = 1, 1, 2, 4, 8, 18, 40, 94, 224, 548, 1356, \ldots,$ for $d = 1, 2, \ldots,$ as counted by Moon [43]; see also [54].

It is reasonable to ask whether the Hanner polytopes simultaneously minimize all entries of the f-vector and this is exactly the content of the second conjecture.

Conjecture B. For every cs d-polytope P there is a d-dimensional Hanner polytope H such that

$$f_i(P) \geq f_i(H)$$

for all $i = 0, \ldots, d-1$.

For a d-polytope P and $S = \{i_1, i_2, \ldots, i_k\} \subseteq [d] = \{0, 1, \ldots, d-1\}$ let $f_S(P) \in \mathbb{Z}^{2^{[d]}}$ be the number of chains of faces $F_1 \subset F_2 \subset \cdots \subset F_k \subset P$ with $\mathsf{dim}\, F_j = i_j$ for all $j = 1, \ldots, k$. We call $(f_S(P) : S \subseteq \{0, \ldots, d-1\})$ the *flag-vector* of P. Identifying $\mathbb{R}^{2^{[d]}}$ with its dual space via the standard inner product, we write $\alpha(P) := \sum_S \alpha_S f_S(P)$ for $(\alpha_S)_{S \subseteq [d]} \in \mathbb{R}^{2^{[d]}}$. The set

$$\mathcal{P}_d = \left\{ (\alpha_S)_{S \subseteq [d]} \in \mathbb{R}^{2^{[d]}} : \alpha(P) = \sum_S \alpha_S f_S(P) \geq 0 \text{ for all } d\text{-polytopes } P \right\}$$

is the polar to the set of flag-vectors of d-polytopes, that is, the cone of all linear functionals that are non-negative on all flag-vectors of (not necessarily cs) d-polytopes.

Conjecture C. For every cs d-polytope P there is a d-dimensional Hanner polytope H such that

$$\alpha(P) \geq \alpha(H)$$

for all $\alpha \in \mathcal{P}_d$.

It is easy to see that $\mathbf{C} \Rightarrow \mathbf{B} \Rightarrow \mathbf{A}$: Define $\alpha^i(P) := f_i(P)$, then $\alpha^i \in \mathcal{P}_d$ and the validity of **C** on the functionals α^i implies **B**; the remaining implication follows since $s(P)$ is a non-negative combination of the $f_i(P)$.

We investigate the validity of these three conjectures in various dimensions. The main results and the organization of the second part of this thesis are as follows.

In Section 5.1 we establish a lower bound on the flag-vector functional g_2^{tor} on the class of centrally symmetric polytopes. Specializing this functional to cs 4-polytopes and combining it with some combinatorial and geometric reasoning this leads to the following result.

Theorem 5.2. *The conjectures* **A** *and* **B** *hold for centrally symmetric polytopes of dimension $d \leq 4$.*

In Section 5.2, we present an infinite family of centrally symmetric, 2-simple, 2-simplicial polytopes that shows

Theorem 5.3. *Conjecture* **C** *is false in dimension $d = 4$.*

In Section 5.3 we consider centrally symmetric hypersimplices in odd dimensions; combined with basic properties of Hanner polytopes, this gives a proof of

Theorem 5.4. *For all $d \geq 5$ both conjectures* **B** *and* **C** *fail.*

We also show that in fixed dimension d there are only finitely many counterexamples to conjecture **B**. We close with two further interesting examples of centrally symmetric polytopes in Section 5.4.

5.1 Conjectures A and B in dimensions $\mathbf{d} \leq 4$

In this section we prove Theorem 5.2, that is, the conjectures **A** and **B** for polytopes in dimensions $d \leq 4$. The work of Stanley [56] implies **A** and **B** for simplicial and thus also for simple polytopes. Furthermore, if $f_0(P) = 2d$, then P is linearly isomorphic to a crosspolytope. *Therefore, we assume throughout this section that all cs d-polytopes P are neither simple nor simplicial, and that $f_{d-1}(P) \geq f_0(P) \geq 2d+2$.*

The main work will be in dimension 4. The claims for dimensions one, two, and three are vacuous, clear, and easy to prove, in that order. In particular, the case $d = 3$ can be obtained from an easy f-vector calculation. But, to get in the right mood, let us sketch a geometric argument. Let P be a cs 3-polytope. Since P is not simplicial, P has a non-triangle facet. Let F be a facet of P with $f_0(F) \geq 4$ vertices. Let $F_0 = P \cap H$ with H being the hyperplane parallel to the affine hulls of F and of $-F$ that contains the origin. Now, F_0 is a cs 2-polytope and it is clear that every face G of P that has a nontrivial intersection with H is neither a face of F nor of $-F$. We get

$$s(P) \quad \geq \quad s(F) + s(F_0) + s(-F) \geq 3 \cdot 3^2.$$

This type of argument fails in dimensions $d \geq 4$. Applying small (symmetric) perturbations to the vertices of a prism over an octahedron yields a cs 4-polytope with the following two types of facets: prisms over a triangle and square pyramids. Every such facet has less than 3^3 faces, which shows that less than a third of the alleged 81 faces are concentrated in every facet.

Let's come back to dimension 4. The proof of the conjectures **A** and **B** splits into a combinatorial part (*f-vector yoga*) and a geometric argument. We partition the class of cs 4-polytopes into *large* and (few) *small* polytopes, where "large" means that

$$f_0(P) + f_3(P) \quad \geq \quad 24. \tag{5.1}$$

We will reconsider an argument of Kalai [28] that proves a lower bound theorem for polytopes and, in combination with flag-vector identities, leads to a tight flag-vector inequality for cs 4-polytopes. With this new tool, we prove that (5.1) implies conjectures **A** and **B** for dimension 4.

We show that the *small* cs 4-polytopes, i.e. those not satisfying (5.1), are *twisted prisms* over 3-polytopes. We then establish basic properties of twisted prisms that imply the validity of conjectures **A** and **B** for small centrally symmetric 4-polytopes.

Equivariant rigidity and the parameter g_2^{tor}

For a general simplicial d-polytope P the *h-vector* $h(P)$ is the ordered collection of the coefficients of the polynomial $h_P(t) := f_P(t-1)$, the *h-polynomial* of P. Clearly, $h_P(t)$ encodes the same information as the f-polynomial, but additionally $h_P(t)$ is a unimodal, palindromic polynomial with non-negative, integral coefficients (see e.g. [63, Sect. 8.3]). This gives more insight in the nature of face numbers of simplicial polytopes and, in a compressed form, this numerical information is carried by its *g-vector* $g(P)$ with $g_i(P) = h_i(P) - h_{i-1}(P)$ for $i = 1, \ldots, \lfloor \frac{d}{2} \rfloor$. There are various interpretations for the h- and g-numbers and, via the g-Theorem (see e.g. [63, Sect. 8.6]), they carry a complete characterization of the f-vectors of simplicial d-polytopes.

For general d-polytopes a much weaker invariant is given by the *generalized* or *toric* h-vector $h^{\mathsf{tor}}(P)$ introduced by Stanley [55]. In contrast to the ordinary h-vector, the toric h-numbers $h_i^{\mathsf{tor}}(P)$ are not determined by the f-vector: They are linear combinations of the face numbers and of other entries of the flag-vector of P. For example,

$$g_2^{\mathsf{tor}} \;=\; h_2^{\mathsf{tor}} - h_1^{\mathsf{tor}} \;=\; f_1 + f_{02} - 3f_2 - df_0 + \tbinom{d+1}{2}.$$

The corresponding toric h-polynomial shares the same properties as its simplicial relative but, unfortunately, carries quite incomplete information about the f-vector.

For example, in the case of P being a *quasi-simplicial* polytope, i.e. every facet of P is simplicial, the toric h-vector depends only on the f-numbers $f_i(P)$ for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$ and, therefore, does not carry enough information to determine a lower bound on $s(P)$ for $d \geq 5$. However, the information gained in dimension 4 will be a major step in the direction of a proof of Theorem 5.2. To be more precise, for the class of centrally symmetric d-polytopes there is a refinement of the flag-vector inequality $g_2^{\mathsf{tor}} = h_2^{\mathsf{tor}} - h_1^{\mathsf{tor}} \geq 0$.

Theorem 5.5. *Let P be a centrally symmetric d-polytope. Then*

$$g_2^{\mathsf{tor}}(P) \;=\; f_1(P) + f_{02}(P) - 3f_2(P) - df_0(P) + \tbinom{d+1}{2} \;\geq\; \tbinom{d}{2} - d.$$

With Euler's equation and the generalized Dehn-Sommerville equations [5] it is routine to derive the following inequality for the class of cs 4-polytopes.

Corollary 5.6. *If P is a centrally symmetric 4-polytope, then*

$$f_{03}(P) \;\geq\; 3f_0(P) + 3f_3(P) - 8. \tag{5.2}$$

We will prove Theorem 5.5 using the theory of *infinitesimally rigid frameworks.* For information about rigidity beyond our needs we refer the reader to Roth [50] for a very readable introduction and to Whiteley [62] and Kalai [29] for rigidity in connection with polytopes.

Let $d \geq 1$ and let $G = (V, E)$ be an abstract simple undirected graph. The *edge function* associated to G and d is the map

$$\begin{aligned} \Phi : (\mathbb{R}^d)^V &\to \mathbb{R}^E \\ (p_v)_{v \in V} &\mapsto \left(\|p_u - p_v\|^2\right)_{uv \in E}, \end{aligned}$$

which measures the (squared) lengths of the edges of G for any choice of coordinates $\mathbf{p} = (p_v)_{v \in V} \in (\mathbb{R}^d)^V$. The pair $(G, \mathbf{p})$ is called a *framework* in $\mathbb{R}^d$ and the points of $\Phi_{\mathbf{p}} := \Phi^{-1}(\Phi(\mathbf{p}))$ give the possible frameworks in $\mathbb{R}^d$ with constant edge lengths $\Phi(\mathbf{p})$.

Let $n = |V| \geq d + 1$ and let $\mathbf{p}$ be a generic embedding. Then the set $\Phi_{\mathbf{p}} \subset (\mathbb{R}^d)^V$ is a smooth submanifold on which the group of *Euclidean/rigid motions* $E(\mathbb{R}^d)$ acts smoothly and faithfully. Therefore the dimension of $\Phi_{\mathbf{p}}$ is $\dim \Phi_{\mathbf{p}} \geq \binom{d+1}{2}$ and in case of equality the framework $(G, \mathbf{p})$ is *infinitesimally rigid*. The *rigidity matrix* $R = R(G, \mathbf{p}) \in (\mathbb{R}^d)^{E \times V}$ of $(G, \mathbf{p})$ is the Jacobian matrix of Φ evaluated at $\mathbf{p}$. Invoking the Implicit Function Theorem, it is easy to see that $(G, \mathbf{p})$ is infinitesimally rigid if and only if $\operatorname{rank} R = dn - \binom{d+1}{2}$.

A *stress* on the framework $(G, \mathbf{p})$ is an assignment $\omega = (\omega_e)_{e \in E} \in \mathbb{R}^E$ of weights $\omega_e \in \mathbb{R}$ to the edges $e \in E$ such that there is an equilibrium

$$\sum_{u : uv \in E} \omega_{uv}(p_v - p_u) = 0$$

at every vertex $v \in V$. We denote by $S(G, \mathbf{p}) = \{\omega \in \mathbb{R}^E : \omega^{\mathsf{T}} R = 0\}$ the kernel of R^{T}, called the *space of stresses* on $(G, \mathbf{p})$.

Theorem 5.7 (Whiteley [62, Thm. 8.6 with Thm. 2.9])**.** *Let $P \subset \mathbb{R}^d$ be a d-polytope. Let $G = (V, E)$ be the graph obtained from a triangulation of the 2-skeleton of P without new vertices and let $\mathbf{p} = \mathbf{p}(P)$ be the vertex coordinates. Then the resulting framework $(G, \mathbf{p})$ is infinitesimally rigid.*

The theorem does not specify the triangulation of the 2-skeleton. The important fact to note is that the graph of any triangulation of the 2-skeleton of P will have exactly $m := |E| = f_1(P) + f_{02}(P) - 3f_2(P)$ edges: In addition to the $f_1(P)$ edges of P, $k - 3$ edges are needed for every 2-face with k vertices.

For the dimension of the space of stresses $S(G, \mathbf{p})$ we get

$$\begin{aligned} 0 \leq \dim S(G, \mathbf{p}) &= m - \operatorname{rank} R \\ &= m - dn + \tbinom{d+1}{2} \\ &= f_1(P) + f_{02}(P) - 3f_2(P) - df_0(P) + \tbinom{d+1}{2} \\ &= g_2^{\mathsf{tor}}(P). \end{aligned}$$

Now let P be a centrally symmetric d-polytope, $d \ge 3$. Let $G = (V, E)$ be the graph in Theorem 5.7 obtained from a triangulation that respects the central symmetry of the 2-skeleton and let $\mathbf{p} = \mathbf{p}(P)$ be the vertex coordinates of P. The antipodal map $\mathbf{x} \mapsto -\mathbf{x}$ induces a free action of the group $\mathbb{Z}_2$ on the graph G. We denote by $\overline{V} = V/\mathbb{Z}_2$ and $\overline{E} = E/\mathbb{Z}_2$ the respective quotients and, after choosing representatives, we denote by $V = V^+ \uplus V^-$ and $E = E^+ \uplus E^-$ the decompositions of the sets of vertices and edges according to the action. Since the action is free we have $|\overline{V}| = |V^\pm| = \frac{n}{2}$ and $|\overline{E}| = |E^\pm| = \frac{m}{2}$.

Concerning the rigidity matrix, it is easy to see that

$$R \quad = \quad \begin{array}{c c} & \begin{array}{cc} V^+ & V^- \end{array} \\ \begin{array}{c} E^+ \\ E^- \end{array} & \begin{pmatrix} R_1 & R_2 \\ -R_2 & -R_1 \end{pmatrix} \end{array} \quad \in \quad (\mathbb{R}^d)^{E \times V}$$

with labels above and to the left of the matrix. The embedding $\mathbf{p} = \mathbf{p}(P)$ respects the central symmetry of G and we can augment the edge function by a second component that takes the symmetry information into account:

$$\begin{array}{rrcl} \Phi^{\mathsf{sym}} : & (\mathbb{R}^d)^{V^+} \times (\mathbb{R}^d)^{V^-} & \to & \mathbb{R}^E \times (\mathbb{R}^d)^{\overline{V}} \\ & \mathbf{p} = (\mathbf{p}_{V^+}, \mathbf{p}_{V^-}) & \mapsto & (\Phi(\mathbf{p}), \mathbf{p}_{V^+} + \mathbf{p}_{V^-}) \,. \end{array}$$

Thus Φ^{sym} additionally measures the degree of asymmetry of the embedding. By the symmetry of P, $\Phi^{\mathsf{sym}}(\mathbf{p}) = (\Phi(\mathbf{p}), 0)$ for $\mathbf{p} = \mathbf{p}(P)$. The preimage of this point under Φ^{sym} is $\Phi^{\mathsf{sym}}_{\mathbf{p}} \subset \Phi_{\mathbf{p}}$, the set of all centrally symmetric embeddings with edge lengths $\Phi(\mathbf{p})$. Any small (close to identity) rigid motion that fixes the origin takes $\mathbf{p} \in \Phi^{\mathsf{sym}}_{\mathbf{p}}$ to a *distinct* centrally symmetric realization $\mathbf{p}' \in \Phi^{\mathsf{sym}}_{\mathbf{p}}$. Thus the action of the subgroup $O(\mathbb{R}^d)$, the group of orthogonal transformations, on $\Phi^{\mathsf{sym}}_{\mathbf{p}}$ locally gives a smooth embedding of $O(\mathbb{R}^d)$ into $\Phi^{\mathsf{sym}}_{\mathbf{p}}$. It follows that $\dim \Phi^{\mathsf{sym}}_{\mathbf{p}} \ge \dim O(\mathbb{R}^d) = \binom{d}{2}$ and thus

$$\operatorname{rank} R^{\mathsf{sym}} \quad \le \quad dn - \tbinom{d}{2}, \tag{5.3}$$

where we can compute the rank of R^{sym}, the Jacobian of Φ^{sym} at $\mathbf{p}$, as

$$\operatorname{rank} R^{\mathsf{sym}} \quad = \quad \operatorname{rank} \begin{pmatrix} R_1 & R_2 \\ -R_2 & -R_1 \\ I_{V^+} & I_{V^-} \end{pmatrix} \quad = \quad \frac{dn}{2} + \operatorname{rank}\,(R_1 - R_2)\,. \tag{5.4}$$

Proof of Theorem 5.5. Consider the space of *symmetric stresses*, that is, the linear subspace

$$\begin{aligned} S^{\mathsf{sym}}(G, \mathbf{p}) &= \{\omega = (\omega_{E^+}, \omega_{E^-}) \in S(G, \mathbf{p}) : \omega_{E^+} = \omega_{E^-}\} \\ &\cong \{\overline{\omega} \in \mathbb{R}^{\overline{E}} : \overline{\omega}^{\mathsf{T}}\,(R_1 - R_2) = 0\}. \end{aligned}$$

From (5.3) and (5.4) it follows that

$$\dim S^{\mathsf{sym}}(G,\mathbf{p}) \;=\; \frac{m}{2} - \operatorname{rank}(R_1 - R_2) \;\geq\; \frac{m}{2} - \frac{dn}{2} + \binom{d}{2}.$$

The theorem follows from noting that $S^{\mathsf{sym}}(G,\mathbf{p}) \subseteq S(G,\mathbf{p})$ and therefore

$$\begin{aligned} 0 &\leq 2\,[\dim S(G,\mathbf{p}) - \dim S^{\mathsf{sym}}(G,\mathbf{p})] \\ &= (m-dn) + 2\binom{d+1}{2} - 2\binom{d}{2} \\ &= (m-dn) + \binom{d+1}{2} - \binom{d}{2} + d \\ &= g_2^{\mathsf{tor}} - \binom{d}{2} + d. \end{aligned}$$

□

Theorem 5.5 can also be deduced from the following result of A'Campo-Neuen [2]; see also [1].

Theorem 5.8 ([2, Theorem 2])**.** *Let P be a centrally symmetric d-polytope and let $h_P^{\mathsf{tor}}(t) = \sum_{i=0}^{d} h_i^{\mathsf{tor}}(P)\, t^i$ be its toric h-polynomial. Then the polynomial*

$$h_P^{\mathsf{tor}}(t) - h_{C_d^{\triangle}}^{\mathsf{tor}}(t) = h_P^{\mathsf{tor}}(t) - (1+t)^d \in \mathbb{Z}[t]$$

is palindromic and unimodal with non-negative, even coefficients. In particular,

$$g_i^{\mathsf{tor}}(P) = h_i^{\mathsf{tor}}(P) - h_{i-1}^{\mathsf{tor}}(P) \geq \tbinom{d}{i} - \tbinom{d}{i-1} \text{ for all } 1 \leq i \leq \left\lfloor \tfrac{d}{2} \right\rfloor.$$

The proof of Theorem 5.8 relies on the (heavy) machinery of combinatorial intersection cohomology for fans. Theorem 5.5 concerns the special case of the coefficient of the quadratic term. In light of McMullen's *weight algebra* [40], it would be interesting to know whether/how Theorem 5.8 can be deduced by considering (generalized) stresses. A connection between the combinatorial intersection cohomology set-up for fans and rigidity was established by Braden [9, Sect. 2.9].

LARGE CENTRALLY SYMMETRIC 4-POLYTOPES

In order to prove conjectures **A** and **B** for large polytopes, we need one more ingredient.

Proposition 5.9. *Let P be a 4-polytope. Then*

$$\begin{array}{rcl} f_{03}(P) & \leq & 4f_2(P) - 4f_3(P) \\ & = & 4f_1(P) - 4f_0(P). \end{array} \tag{5.5}$$

Equality holds if and only if P is center-boolean, *i.e. if every facet is simple.*

Proof. The inequality was first proved by Bayer [4]. Every facet F of P is a 3-polytope satisfying $3f_0(F) \leq 2f_1(F)$. By summing up over all facets of P we get

$$3f_{03}(P) = \sum_{F \text{ facet}} 3f_0(F) \leq \sum_{F \text{ facet}} 2f_1(F) = 2f_{13}(P).$$

By one of the Generalized Dehn-Sommerville Equations [5] we have

$$f_{03} - f_{13} + f_{23} = 2f_3,$$

which, together with $f_{23} = 2f_2$ immediately implies the asserted inequality. Equality holds if the above inequality for 3-polytopes holds with equality for all facets of P, which means that all facets are simple 3-polytopes. The last equality in the assertion is Euler's equation. $\square$

Combining the inequalities (5.2) and (5.5), we obtain

$$\begin{array}{rcccl} f_2 & \geq & \frac{1}{4}(3f_0 + 7f_3) - 2 & = & f_3 + \frac{3}{4}(f_0 + f_3) - 2 \\ f_1 & \geq & \frac{1}{4}(7f_0 + 3f_3) - 2 & = & f_0 + \frac{3}{4}(f_0 + f_3) - 2. \end{array} \tag{5.6}$$

In terms of f_0 and f_3 this gives

$$s(P) \geq \frac{14}{4}(f_0 + f_3) - 3 \geq 81$$

where the last inequality holds if P is large.

To prove conjecture **B** for large polytopes, we have to show that the f-vector of every large polytope is component-wise larger than the f-vector of one of the following four Hanner polytopes:

	(f_0, f_1, f_2, f_3)
C_4	$(16, 32, 24, \ 8)$
$C_4^{\triangle}$	$(\ 8, 24, 32, 16)$
bip C_3	$(10, 28, 30, 12)$
prism $C_3^{\triangle}$	$(12, 30, 28, 10)$

It suffices to treat the case $f_0 + f_3 = 24$. Indeed, for $f_0 + f_3 \geq 26$ and $f_3 \geq f_0 \geq 10$ we get from (5.6) that

$$f_1 \geq f_0 + 18 \geq 28$$
$$f_2 \geq f_3 + 18 \geq 30$$

and thus $f(\mathsf{bip}\, C_3)$ is componentwise smaller.

We claim that the same bounds hold for $f_0 + f_3 = 24$. Otherwise, if $f_1 \leq 26$ or $f_2 \leq 28$, then by using (5.5) together with $f_0 \geq 10$ and $f_3 \geq 12$ we get in both cases that $f_{03} \leq 64$. In fact, we now get $f_{03} = 64$ from (5.2), which tells us that P is *center boolean*, i.e. every facet is simple. Granted that every facet of P is simple and has at most 6 vertices, the possible facet types are the 3-simplex Δ_3 and the triangular prism $\mathsf{prism}\,\Delta_2$. Using the assumption that P is not simplicial, there is a facet $F \cong \mathsf{prism}\,\Delta_2$. The three quad faces of F give rise to three more prism facets and, due to the number of vertices, no two of them are antipodes. For the same reason, any two prism facets cannot intersect in a triangle face. In total, we note that P has exactly eight prism facets and four tetrahedra. Since every antipodal pair of prism facets give a partition of the vertices, it follows that every vertex is contained in a simplex and exactly 4 prism facets. Therefore, every vertex has degree ≥ 6 and thus $2f_1 \geq 6 \cdot 12$. By Euler's equation, the same holds for f_2.

Summing up, we have proved conjectures **A** and **B** for the class of large centrally symmetric 4-polytopes.

TWISTED PRISMS AND THE SMALL POLYTOPES

The class of small centrally symmetric polytopes consists of all cs 4-polytopes P with $12 \geq f_3(P) \geq f_0(P) = 10$. Since P is not simplicial, P has a facet F that has $5 = d + 1 = f_0(F)$ vertices, and $P = \mathsf{conv}\,(F \cup -F)$. In particular, F is a 3-polytope with $3 + 2$ vertices, which does not leave much diversity in terms of combinatorial types. The facet F is combinatorially equivalent to

- a pyramid over a quadrilateral, or
- a bipyramid over a triangle.

Definition 5.10 (Twisted prism)**.** Let $Q \subset \mathbb{R}^{d-1}$ be a $(d-1)$-polytope. The centrally symmetric d-polytope

$$P = \mathsf{tprism}\, Q = \mathsf{conv}\ (Q \times \{1\} \quad \cup \quad -Q \times \{-1\}) \subset \mathbb{R}^d$$

is called the *twisted prism* over the base Q.

The following basic properties of twisted prisms will be of good service.

Proposition 5.11. *Let $Q \subset \mathbb{R}^{d-1}$ be a $(d-1)$-polytope and* $\mathsf{tprism}\, Q$ *the twisted prism over Q.*

1. *If $\mathsf{T} : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ is a non-singular affine transformation, then* $\mathsf{tprism}\, Q$ *and* $\mathsf{tprism}\, \mathsf{T}Q$ *are affinely isomorphic.*
2. *If $Q = \mathsf{pyr}\, Q'$ is a pyramid with base Q', then* $\mathsf{tprism}\, Q$ *is combinatorially equivalent to* $\mathsf{bip}\,\mathsf{tprism}\, Q'$*, a bipyramid over the twisted prism over Q'.* □

The second statement of Proposition 5.11 actually proves the conjectures **A** and **B** for half of the small cs 4-polytopes: Let $P = \mathsf{tprism}\, Q$ and Q a pyramid over a quadrilateral. By the second statement P is combinatorially equivalent to $\mathsf{bip}\, P'$, where P' is a cs 3-polytope. In terms of f-polynomials, it is easy to show that for a bipyramid $f_{\mathsf{bip}\, Q}(t) = (2+t) f_Q(t)$. Thus

$$s(P) = f_{\mathsf{bip}\, P'}(1) = 3 f_{P'}(1) \geq 3^4.$$

Since **B** is true in dimension 3 there is a 3-dimensional Hanner polytope H such that $f_i(P') \geq f_i(H)$ for $i = 0, 1, 2$. From the above identity of f-polynomials it follows that $f_i(\mathsf{bip}\, P') \geq f_i(\mathsf{bip}\, H)$ for $1 \leq i \leq 3$, where $\mathsf{bip}\, H = I \oplus H$ is a Hanner polytope.

The next lemma shows that twisted prisms with base a bipyramid over a triangle are not small and thereby settles **A** and **B** for dimension 4.

Lemma 5.12. *Let $d \geq 4$ and let $P = \mathsf{tprism}\, F \subset \mathbb{R}^d$ be a cs d-polytope with F combinatorially equivalent to $\Delta_i \oplus \Delta_{d-i-1}$ and $1 \leq i \leq \frac{d-1}{2}$. Then*

$$f_{d-1}(P) \quad \geq \quad 2(1 + (i+1)(d-i)) \quad \geq \quad 2(2d-1).$$

Proof. The facet F has $(i+1)(d-i)$ faces of dimension $d-2$ and thus F and its neighboring facets account for $1 + (i+1)(d-i)$ facets. The result now follows by considering $-F$ as soon as we have checked that no facet G shares a ridge with F and with $-F$. This, however, is impossible, since G would have to have two vertex disjoint $(d-2)$-simplices as maximal faces and, therefore, at least $f_0(G) \geq 2d-2$ vertices. Since G and $-G$ are vertex disjoint, we have $2d+2 = f_0(P) \geq f_0(G) + f_0(-G) \geq 4d-4$. □

Corollary 5.13. *If $P = \mathsf{tprism}\, Q$ with $Q \cong \mathsf{bip}\, \Delta_2$, then P is large.* □

5.2 Conjecture **C** in dimension 4

We will refute conjecture **C** *strongly* in dimension 4: We exhibit a non-negative flag-functional $L_{2,2} \in \mathcal{P}_4$ such that $L_{2,2}(H) > 0$ for all 4-dimensional Hanner polytopes and such that $\mathcal{L}^{\mathsf{cs}}_{2,2}$, the class of 4-dimensional cs polytopes with $L_{2,2}(P) = 0$, is infinite.

Geometrically, this means that there is an oriented hyperplane in the vector space $\mathbb{R}^{2^{[d]}}$ that contains all flag vectors on its non-negative side, the flag vectors of $\mathcal{L}^{\mathsf{cs}}_{2,2}$ on the hyperplane, and the 4-dimensional Hanner polytopes strictly in the positive halfspace.

Definition 5.14 (k-simple, l-simplicial polytopes [23, Sect. 4.5])**.** A polytope P is called *l-simplicial* if every l-face is a simplex. It is called *k-simple* if its dual $P^{\triangle}$ is k-simplicial.

The class of k-simple l-simplicial polytopes is characterized in terms of the vanishing of a certain flag functional.

Proposition 5.15. *Consider the flag-functional*

$$L_{k,l}(P) = \left[f_{0k}(P^{\triangle}) - (k+1)f_k(P^{\triangle}))\right] + \left[f_{0l}(P) - (l+1)f_l(P))\right].$$

Then $L_{k,l}$ is a non-negative flag functional and $L_{k,l}(P) = 0$ if and only if P is k-simple and l-simplicial.

Proof. Every j-face of P has at least $j+1$ vertices and therefore

$$\sum_F (f_0(F) - (j+1)) = f_{0j}(P) - (j+1)f_j(P) \geq 0$$

where the sum is taken over all j-faces F of P. The sum is equal to zero if and only if P is j-simplicial. □

A first step towards the refutation of Conjecture **C** is given by the following result.

Proposition 5.16. *Let $d \geq 4$ and let $k, l \geq 2$. There is no d-dimensional Hanner polytope that is k-simple and l-simplicial, i.e. $L_{k,l}(H) > 0$ for every d-dimensional Hanner polytope H.*

Proof. The result follows from the fact that if P and Q are polytopes of positive dimension, then $P \times Q$ has quadrilateral 2-faces and thus is never l-simplicial for $l \geq 2$.

Therefore, H cannot be a product of two Hanner polytopes. On the other hand, if H is a direct sum, then $H^{\triangle}$ is not 2-simplicial. □

In the rest of the section we will prove that the class $\mathcal{L}_{2,2}^{\mathsf{cs}}$ of centrally symmetric, 2-simple 2-simplicial 4-polytopes is not empty – in fact, we show that there are infinitely many of them.

Theorem 5.17. *There are infinitely many centrally symmetric,* 2*-simple,* 2*-simplicial* 4*-polytopes.*

The construction relies on the fact that there are polytopes that can be truncated by hyperplanes in a special way.

Definition 5.18 (Truncatable Polytopes; Paffenholz & Ziegler [47, Def. 3.2])**.** Let $P \subset \mathbb{R}^d$ be a d-polytope with vertex set $V = \mathsf{vert}\, P$. The polytope P is called *truncatable* if there is a collection of hyperplanes $\{H_v \subset \mathbb{R}^d : v \in V\}$ with the properties that

i) v and $V \setminus v$ are in different connected components of $\mathbb{R}^d \setminus H_v$, and
ii) if $e = \mathsf{conv}\, \{u, v\}$ is an edge, then $e \cap H_u \cap H_v = \{d_e\}$ is a single point.

For a truncatable polytope P denote by $D_1(P) = \mathsf{conv}\, \{d_e : e \text{ edge of } P\}$ the polytope obtained by *truncating* P by the hyperplanes H_v.

Theorem 5.19 ([47, Proposition 3.3 with Corollary 2.3])**.** *If P is a simplicial truncatable* 4*-polytope, then $D_1(P)^{\triangle}$ is a* 2*-simple,* 2*-simplicial* 4*-polytope.*

Thus for a proof of Theorem 5.17 it suffices to show that there are infinitely many simplicial, truncatable 4-polytopes P such that $D_1(P)$ is centrally symmetric. This, however, turns out to be easy and the justification is essentially given in the proof of Theorem 3.5 in [47]. For this recall the operation of stacking a facet: Let $P = P(A, b) \subset \mathbb{R}^d$ be a d-polytope and let $F_i = \{\mathbf{x} \in P : \mathsf{a}_i^{\mathsf{T}}\, \mathbf{x} = b_i\}$ be a facet. The polytope resulting from *stacking* F is $P' = \mathsf{conv}\, (P \cup v_F)$ were v_F is any point satisfying

$$\begin{aligned} &\mathsf{a}_i^{\mathsf{T}}\, v_F > b_i, \text{ and} \\ &\mathsf{a}_j^{\mathsf{T}}\, v_F < b_j \text{ for all } j \neq i. \end{aligned} \tag{ST}$$

The change in combinatorics can be described by removing the relative interior of F and taking a cone over its boundary and, in particular, is independent of the choice of v_F.

Proposition 5.20. *Let P be a truncatable d-polytope and let F be a simplex facet. Then F can be stacked in a way such that the resulting polytope is truncatable.*

Proof. Let $P = P(A, b)$ and let $H_v = \{\mathbf{x} : \mathsf{c}_v^{\mathsf{T}}\,\mathbf{x} = \delta_v\}$ be a collection of truncating hyperplanes satisfying the conditions given in Definition 5.18. There exists a point v_F satisfying (ST) and $\mathsf{c}_v^{\mathsf{T}}\,v_F < \delta_v$ for all $v \in \operatorname{vert} P$. As F is a simplex, the polytope $P' = \operatorname{conv}(P \cup v_F)$ has d edges emanating from v_F. These edges intersect the corresponding hyperplanes H_v in d affinely independent points which define a truncating hyperplane H_{v_F}. The extended collection of hyperplanes proves that P' is truncatable. □

Proof of Theorem 5.17. It is easily verified that the 4-dimensional crosspolytope $C_4^{\triangle}$ is truncatable such that $D_1(C_4^{\triangle})$ is cs. The preceding proposition and its proof show that we can successively apply symmetric stacking operations to $C_4^{\triangle}$ and obtain an infinite family of centrally symmetric 2-simple 2-simplicial 4-polytopes. □

It is possible to construct infinitely many counterexamples in every dimension $d \geq 4$ by adapting the proof of Theorem 3.8 in Paffenholz & Ziegler [47]. They construct infinitely many 2-simple $(d-2)$-simplicial d-polytopes by *gluing* crosspolytopes facet-to-facet and then repairing a certain geometric defect by adding in simplices. Iterating this procedure, they obtain a (modified) *stack* of n crosspolytopes. A thorough inspection of their argument shows that if n is odd, then the resulting polytope is centrally symmetric.

5.3 The central hypersimplices $\tilde{\Delta}_k = \Delta(k, 2k)$

The (k, d)-*hypersimplex* for $d > k > 0$ is the $(d-1)$-dimensional polytope

$$\Delta(k, d) = \operatorname{conv}\left\{\mathbf{x} \in \{0,1\}^d : x_1 + x_2 + \cdots + x_d = k\right\} \subset \mathbb{R}^d.$$

Hypersimplices were considered e.g. as (semiregular) polytopes in [13, §11.8] (see also [47, Sect. 3.3.2] and [23, Exercise 4.8.16]), as well as in connection with algebraic geometry in [19], [20], and [58].

One rather simple observation is that $\Delta(k, d)$ and $\Delta(d-k, d)$ are affinely isomorphic under the map $\mathbf{x} \mapsto \mathbb{1} - \mathbf{x}$. In particular, the hypersimplex $\tilde{\Delta}_k := \Delta(k, 2k)$ is a centrally symmetric $(2k-1)$-polytope with $f_0(\tilde{\Delta}_k) = \binom{2k}{k}$ vertices.

In a different, full-dimensional realization, the central hypersimplex is given by

$$\tilde{\Delta}_k \quad \cong \quad \mathsf{conv}\left\{\mathbf{x} \in \{+1, -1\}^{2k-1} : -1 \leq x_1 + x_2 + \cdots + x_{2k-1} \leq 1\right\}.$$

From this realization it is easy to see that for $k \geq 2$ the hypersimplex $\tilde{\Delta}_k$ is a twisted prism over $\Delta(k, 2k-1)$ with $f_{2k-2}(\tilde{\Delta}_k) = 4k = 2(2k-1)+2$ facets: Since the above realization lives in an odd-dimensional space, the sum of the coordinates for any vertex is either $+1$ or -1. The points satisfying $\sum_i x_i = 1$ form a face that is affinely isomorphic to $\Delta(k, 2k-1)$. To verify the number of facets, observe that $\tilde{\Delta}_k$ is the intersection of the $2k$-cube with a hyperplane that cuts all its $4k$ facets.

We will show that in odd dimensions $d = 2k-1 \geq 5$ a d-dimensional Hanner polytope that has no more facets than $\tilde{\Delta}_k$ has way too many vertices for conjecture **B**. In even dimensions $d \geq 6$ Theorem 5.4 follows then by taking a prism over $\tilde{\Delta}_k$. The following proposition gathers the information needed about Hanner polytopes.

Proposition 5.21. *Let H be a d-dimensional Hanner polytope. Then*

(a) $f_{d-1}(H) \geq 2d$.
(b) *If $f_{d-1}(H) = 2d$, then H is a d-cube.*
(c) *If $f_{d-1}(H) = 2d+2$, then $H = C_{d-3} \times C_3^{\triangle}$.*

Proof. Since all three claims are certainly true for Hanner polytopes of dimension $d \leq 3$, let us assume that $d \geq 4$. By definition, H is the direct sum or product of two Hanner polytopes H' and H'' of dimensions i and $d-i$ with $1 \leq i \leq \frac{d}{2}$.

If $H = H' \oplus H''$, then, by induction on d, we get

$$f_{d-1}(H) = f_{i-1}(H') \cdot f_{d-i-1}(H'') \geq 4i(d-i) \geq 2d+4.$$

Therefore, we can assume that $H = H' \times H''$ is a product and the number of facets is $f_{d-1}(H) = f_{i-1}(H') + f_{d-i-1}(H'') \geq 2d$ which proves (a). The condition in (b) is satisfied if and only if it is satisfied for each of the two factors. By induction both factors are cubes and so is their product.

Similarly, the condition in (c) is satisfied iff it is satisfied for one of the two factors. By using (a) we see that the remaining factor is a cube, which proves (c). $\square$

Proof of Theorem 5.4. Let $d = 2k-1 \geq 5$ and let H be a d-dimensional Hanner polytope with $f_i(H) \leq f_i(\tilde{\Delta}_k)$ for all $i = 0, \ldots, d-1$. Since the hypersimplex $\tilde{\Delta}_k$ has $2d+2$ facets, it follows from Proposition 5.21 that H is either C_{2k-1} or $C_{2k-4} \times C_3^{\triangle}$. In either case, the Hanner polytope satisfies $f_0(H) \geq 3 \cdot 2^{2k-3} > \binom{2k}{k}$, where the last inequality holds for $k \geq 3$.

For even dimensions $d = 2k$ consider $\mathsf{prism}\,\tilde{\Delta}_k = I \times \tilde{\Delta}_k$, the prism with base $\tilde{\Delta}_k$. Then number of facets is $f_{d-1}(\mathsf{prism}\,\tilde{\Delta}_k) = 2(2k-1)+4 = 2d+2$ facets. Again by Proposition 5.21, a Hanner polytope H with componentwise smaller f-vector is of the form $I \times H'$ and the result follows from the odd case.

Recalling that conjecture **C** implies conjecture **B** finishes the proof. □

Note that in comparison to conjecture **C**, see Section 5.2, conjecture **B** is too restrictive to allow infinitely many counterexamples.

Proposition 5.22. *If $\mathcal{F}$ is a family of combinatorially distinct cs d-polytopes violating conjecture* **B**, *then $\mathcal{F}$ is finite.*

Proof. Assume that $\mathcal{F}$ is an infinite family violating conjecture **B** in fixed dimension d. For $0 \leq i \leq d-1$ let m_i be the minimal number of i-faces among all d-dimensional Hanner polytopes. As $\mathcal{F}$ is infinite, there is a j such that

$$\{P \in \mathcal{F} : f_j(P) < m_j\} \subseteq \mathcal{F}$$

is still infinite. However, bounding the number of j-faces implicitly bounds the number of facets: Every facet is uniquely determined by the collection of incident j-faces for $j \geq 0$. Thus the number of facets is bounded from above by 2^{m_j} and on finitely many facets there are only finitely many distinct types. □

5.4 Two more examples

We wish to discuss two examples of centrally symmetric polytopes that exhibit some remarkable properties, two of which are being *self-dual* and being counterexamples to conjecture **C**. Both polytopes are instances of *Hansen polytopes* [25], for which we sketch the construction.

Let $G = (V, E)$ be a simple graph on the vertices $V = \{1, \ldots, d-1\}$. Further assume that neither G nor its *complementary graph* $\bar{G} = (V, \binom{V}{2} \setminus E)$ has an odd cycle of length ≥ 5, that is G is *perfect* by the Strong Perfect Graph Theorem [12]. Let $\mathsf{Ind}(G) \subseteq 2^V$ be the *independence complex* of G. So $\mathsf{Ind}(G)$ is the simplicial complex on the vertices V defined by the relation that $S \subseteq V$ is contained in $\mathsf{Ind}(G)$ if and only if the vertex induced subgraph

$G[S]$ has no edges. To every independent set $S \in \mathsf{Ind}(G)$ associate the (characteristic) vector $\tilde{\chi}_S \in \{+1,-1\}^{d-1}$ with $(\tilde{\chi}_S)_i = +1$ if and only if $i \in S$. The collection of vectors is a subset of the vertex set of the $(d-1)$-cube. Let $P_{\mathsf{Ind}(G)} = \mathsf{conv}\{\tilde{\chi}_S : S \in \mathsf{Ind}(G)\} \subset [-1,+1]^{d-1}$ be the vertex induced subpolytope. The *Hansen polytope* $H(G)$ associated to G is the twisted prism over $P_{\mathsf{Ind}(G)}$. In particular, $H(G)$ is a centrally symmetric d-polytope with $f_0(H(G)) = 2\,|\mathsf{Ind}(G)|$ vertices. A graph $G = (V,E)$ is *self-complementary* if G is isomorphic to its complementary graph $\overline{G}$.

Proposition 5.23. *If $G = (V,E)$ is a self-complementary, perfect graph on $d-1$ vertices, then $H(G)$ is a centrally symmetric, self-dual d-polytope.*

Proof. By [25, Thm. 4], the polytope $H(G)^{\triangle}$ is isomorphic to $H(\overline{G}) = H(G)$. □

Example 5.24. Let G_4 be the path on four vertices v_1, v_2, v_3, v_4. This is a self-complementary perfect graph, so $H(G_4)$ is a 5-dimensional self-dual cs polytope. We compute its f-vector, and compare it to the f-vectors of the 5-dimensional hypersimplex $\tilde{\Delta}_3$ and of the eight 5-dimensional Hanner polytopes. This results in the following table (the four Hanner polytopes not listed are the duals of the ones given here, with the corresponding reversed f-vectors):

	$(f_0, f_1, f_2, f_3, f_4)$	$f_0 + f_4$	s
$H(G_4)$	$(16, 64, 98, 64, 16)$	32	259
$\tilde{\Delta}_3$	$(20, 90, 120, 60, 12)$	32	303
$C_5^{\triangle}$	$(10, 40, 80, 80, 32)$	42	243
$\mathsf{bip}\,\mathsf{bip}\,C_3$	$(12, 48, 86, 72, 24)$	36	243
$\mathsf{bip}\,\mathsf{prism}\,C_3^{\triangle}$	$(14, 54, 88, 66, 20)$	34	243
$\mathsf{prism}\,C_4^{\triangle}$	$(16, 56, 88, 64, 18)$	34	243

Thus $H(G_4)$ refutes conjecture **B** in dimension 5 *strongly*: its value for f_0+f_4 is smaller than for any Hanner polytope. Furthermore, $H(G_4)$ has a smaller face number sum s than the hypersimplex, so in that sense it is even a better example to look at in view of conjecture **A**.

Example 5.25. Let G_5 be the *bull*, that is the graph depicted in Figure 5.1. This is a self-complementary perfect graph, so we obtain a 6-dimensional self-dual cs polytope $H(G_5)$. Again its f-vector can be computed and compared to that of the prism over the 5-dimensional hypersimplex, $I \times \tilde{\Delta}_3$, which we had used for Theorem 5.4 as well as the eighteen Hanner polytopes in dimension 6 (again we do not list the duals explicitly):

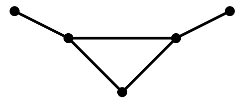


Figure 5.1: The bull graph

	$(f_0, f_1, f_2, f_3, f_4, f_5)$	$f_0 + f_5$	s
$H(G_5)$	$(24, 116, 232, 232, 116, 24)$	48	745
prism $\tilde{\Delta}_3$	$(40, 200, 330, 240, 84, 14)$	54	908
$C_6^{\triangle}$	$(12, 60, 160, 240, 192, 64)$	76	729
bip bip bip C_3	$(14, 72, 182, 244, 168, 48)$	62	729
bip bip prism $C_3^{\triangle}$	$(16, 82, 196, 242, 152, 40)$	56	729
bip prism $C_4^{\triangle}$	$(18, 88, 200, 240, 146, 36)$	54	729
bip bip C_4	$(20, 100, 216, 232, 128, 32)$	52	729
prism $C_5^{\triangle}$	$(20, 90, 200, 240, 144, 34)$	54	729
bip prism bip C_3	$(22, 106, 220, 230, 122, 28)$	50	729
prism bip bip C_3	$(24, 108, 220, 230, 120, 26)$	50	729
$C_3 \oplus C_3$	$(16, 88, 204, 240, 144, 36)$	52	729

Thus $H(G_5)$ is a self-dual cs polytope that also refutes conjecture **B** in dimension 6 *strongly.* Moreover, also looking at the pair (f_1, f_4) suffices to derive a contradiction to conjecture **B**. In these respects, $H(G_5)$ is the nicest and strongest counterexample that we currently have for conjecture **B** in dimension 6.

Note that there are no self-complementary (perfect) graphs on 6 or on 7 vertices, since $\binom{6}{2} = 15$ and $\binom{7}{2} = 21$ are odd. Thus, we cannot derive self-dual polytopes in dimensions 7 or 8 from Hansen's construction.

The Hansen polytopes, derived from perfect graphs, are subject to further research. For example, $H(G_4)$ and $H(G_5)$ are interesting examples in view of the Mahler conjecture, since they exhibit only a small deviation from the Mahler volume of the d-cube, which is conjectured to be minimal (see Kuperberg [31] and Tao [60]).

The Hansen polytopes in turn are special cases of *weak Hanner polytopes*, as defined by Hansen [25], which are twisted prisms over any of their facets. Greg Kuperberg has observed that all of these are equivalent to ± 1-polytopes.

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