# Distribution of Real Critical Points of Logarithmic Derivatives of Real Entire Functions

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**Mikhail Tyaglov**

**Abinsk, Rußland**

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| Vorsitzender: | Prof. Dr. G. M. Ziegler |
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# Zusammenfassung


Die vorgelegte Dissertation beschäftigt sich mit der Verteilung der kritischen Punkten der logarithmischen Ableitungen ganzener reelener Funktionen endlichen Grads mit endlich vielen nicht-reellen Nullstellen. Diese Funktionen wurden 1922 von lander klassifiziert. Er zeigte, dass eine reelle ganze Funktion $f$ zu der Klasse $V_{2n}$ gehört, wenn sie in folgender Form dargestellt werden kann:

$$f(z) = cz^d e^{-\gamma z^{2n+2}+q(z)} \prod_{j=1}^{\omega} \left(1 - \frac{z}{\alpha_j}\right) e^{\frac{z}{\alpha_j}+\frac{z^2}{2\alpha_j^2}+\ldots+\frac{z^l}{l\alpha_j^l}}, \quad 0 \leqslant \omega \leqslant \infty,$$

wobei $q(z)$ ein reelles Polynom ist und $\gamma \geqslant 0$, $\deg q \leqslant 2n+1$, $l \leqslant 2n+1$, $\alpha_j \in \mathbb{R}\ \forall j$, $\sum_j |\alpha_j|^{-l-1} < \infty$, $d \in \mathbb{Z}_+$, $c \in \mathbb{R}\backslash\{0\}$.

Eine Funktion $\varphi$ gehört zu der Klasse $U_{2n}^*$, wenn $\varphi = pf$ für ein reelles Polynom $p$ ohne reelle Nullstellen und eine Funktion $f$, die zu der Klasse $U_{2n} = V_{2n}\backslash V_{2n-2}$, $n \geqslant 1$ oder zu der Klasse $U_0 = V_0$ gehört. Die letztere Klasse wird auch Laguerre-Polya-Klasse genannt und wird mit $\mathcal{L}-\mathcal{P}$ bezeichnet. Die Klasse $U_0^* = \mathcal{L}-\mathcal{P}^*$ spielt eine wichtige Rolle in dieser Dissertation.

Für eine gegebene Funktion $\varphi \in U_{2n}^*$ wird die Anzahl der reellen Nullstellen der Funktion $Q = Q[\varphi] = (\varphi'/\varphi)'$ betrachtet in Abhängigkeit von der Anzahl der nicht-reellen Nullstellen der Funktionen $\varphi$ und $\varphi'$ und der Anzahl der reellen Nullstellen der Funktion $Q_1 = Q[\varphi'] = (\varphi''/\varphi')'$ auf einigen beschränkten und einseitig unbeschränkten Intervallen, sowie auf der ganzen reellen Achse.

Die Hauptergebnisse dieser Arbeit sind folgende Ungleichungen, die für bestimmte Teilklassen der Klasse $U_{2n}^*$ gelten:

$$Z_{\mathbb{C}}(\varphi) - Z_{\mathbb{C}}(\varphi') + 2n - 2 \leqslant Z_{\mathbb{R}}(Q) \leqslant Z_{\mathbb{C}}(\varphi) - Z_{\mathbb{C}}(\varphi') + 2n + 1 + Z_{\mathbb{R}}(Q_1),$$

wobei $Z_{\mathbb{R}}(Q)$ und $Z_{\mathbb{R}}(Q_1)$ die Anzahl der reellen Nullstellen von $Q$ bzw. $Q_1$ bezeichnen und $Z_{\mathbb{C}}(\varphi)$ und $Z_{\mathbb{C}}(\varphi')$ die Anzahl der nicht-reellen Nullstellen der Funktion $\varphi$ bzw. $\varphi'$ bezeichnen.

Für die Klasse $\mathcal{L}-\mathcal{P}^*$ werden diese Ungleichungen verbessert und für schärfere Abschätzungen der Anzahl der nicht-reellen Nullstellen verwendet. Es wird gezeigt, dass für jede Funktion $\varphi$ in der Klasse $\mathcal{L}-\mathcal{P}^*$ gilt: Die Anzahl der kritischen Punkte ihrer logarithmischen Abteilung ist kleiner oder gleich der Anzahl der nicht-reellen Nullstellen der Funktion $\varphi$. Dadurch wird die 22 Jahre alte Hawaii-Vermutung bewiesen, die so genannt wird, da deren Autoren an der University of Hawaii forschen.

Die Beweise dieser Resultate verwenden den Satz von Rolle und andere Eigenschaften von Ableitungen der Funktionen in der Klasse $U_{2n}^*$.

# Notation

| | |
|---|---|
| $\mathbb{R}$ | the real axis |
| $\mathbb{C}$ | the complex plane |
| $Z_{\mathbb{C}}(\varphi)$ | the number of nonreal zeros of the entire function $\varphi$ |
| $Z_{\mathbb{R}}(f)$ | the number of real zeros of the meromorphic function $f$ |
| $Z_{\{\alpha\}}(f)$ | the number of zeros of the meromorphic function $f$ at the point $\alpha \in \mathbb{R}$ |
| $Z_X(f)$ | the number of zeros of the meromorphic function $f$ on the set $X \subset \mathbb{R}$ |
| $E(\varphi')$ | the number of extra zeros of the derivative $\varphi'$ of the entire function $\varphi$ |
| $[a]$ | maximum integer not exceeding real $a$ |

# Contents

# Chapter 1

# Introduction

We study the real critical points of logarithmic derivatives of real entire function of finite order with finitely many nonreal zeros. For a given real entire function $\varphi$, the derivative of its logarithmic derivative is as follows

$$Q[\varphi](z) \overset{def}{=} \frac{d}{dz}\left(\frac{\varphi'(z)}{\varphi(z)}\right) = \frac{\varphi(z)\varphi''(z) - (\varphi'(z))^2}{(\varphi(z))^2}, \quad \text{where} \quad \varphi'(z) = \frac{d\varphi(z)}{dz}, \tag{1.1}$$

So, we are interested in the distribution of real zeros of this function.

In [6], Csordas, Craven and Smith, via Nagy [16], attributed to Gau the enquiry about finding a relationship between the number of nonreal zeros of $\varphi$ and the number of real zeros of the function $Q[\varphi]$ in the case when $\varphi$ is a real polynomial. They proved the following result for entire functions in the class $\mathcal{L}-\mathcal{P}^*$ [6] (see below for the definition).

**Theorem 1.1 ([6, Theorem 1]).** *Let $\varphi \in \mathcal{L}-\mathcal{P}^*$. Suppose that the order of $\varphi$ is less than 2 and that $\varphi$ has exactly $2m$, $m > 0$, nonreal zeros. Let $\sigma \in \mathbb{R}$. Then the following statements are equivalent.*

(a) $\left(\frac{d}{dz} + \sigma\right)\varphi(z) \in \mathcal{L}-\mathcal{P}$;

(b) *(i) $Z_{\mathbb{R}}(Q) = 2m$, and (ii) if $\zeta_1 \leqslant \zeta_2 \leqslant \ldots \leqslant \zeta_{2m}$ are the real zeros of $Q[\varphi]$, then $\varphi'/\varphi(\zeta_{2j-1}) \leqslant -\sigma$ and $\varphi'/\varphi(\zeta_{2j}) \geqslant -\sigma$, $1 \leqslant j \leqslant m$, and $\varphi(z) \neq 0$ for $\zeta_{2j-1} \leqslant z \leqslant \zeta_{2j}$, $1 \leqslant j \leqslant m$;*

*where $Z_{\mathbb{R}}(Q)$ denotes the number of real zeros of $Q[\varphi]$, counting multiplicities.*

We recall that the function $\varphi$ is said to be in the class $\mathcal{L}-\mathcal{P}$, the Laguerre-Pólya class (see [14, 17]), if $\varphi(z)$ can be expressed in the form

$$\varphi(z) = cz^d e^{-\gamma z^2 + \beta z} \prod_{j=1}^{\omega} \left(1 - \frac{z}{\alpha_j}\right) e^{\frac{z}{\alpha_j}}, \quad 0 \leqslant \omega \leqslant \infty,$$

where $c, \beta, \alpha_j \in \mathbb{R}$, $\gamma \geqslant 0$, $d$ is nonnegative integer and $\sum \alpha_j^{-2} < \infty$. The function $\varphi$ is in the class $\mathcal{L}-\mathcal{P}^*$ if $\varphi = pf$ where $f \in \mathcal{L}-\mathcal{P}$ and $p$ is a real polynomial with no real zeros.

Thus, if for some $\sigma \in \mathbb{R}$, the polynomial $\sigma p(z) + p'(z)$ has only real zeros, then by Theorem 1.1, the function $Q[p](z)$ has exactly $2m$ real zeros.

In light of Theorem 1.1, Csordas, Craven and Smith [6] stated the following conjecture, which was recently nicknamed by Eremenko the Hawaii conjecture, as the three authors are all from the University of Hawaii (see also [7, 19]).

**The Hawaii conjecture.** *If a real polynomial* $p$ *has precisely* $2m$ *nonreal zeros, then*

$$Z_{\mathbb{R}}(Q) \leqslant 2m,$$

*where* $Z_{\mathbb{R}}(Q)$ *denotes the number of real zeros of* $Q[p]$*, counting multiplicities.*

In [7], Csordas writes that this conjecture could also have been stated for entire functions in the class $\mathcal{L} - \mathcal{P}^*$.

Thus, the Hawaii conjecture, if it is true, gives an exhaustive answer on the question of a relationship between the number of nonreal zeros of the polynomial $p$ and the number of real zeros of the function $Q[p]$.

In [19, Chapter 9], Sheil-Small suggests several appealing ideas concerning this conjecture. He writes in his preface: "As this conjecture relates closely to the topological structure formed by the level curves on which the logarithmic derivative is real, it is a problem of fundamental interest in understanding the structure of real polynomials". In particular, Sheil-Small showed that the conjecture holds when

- the polynomial $p$ has exactly 2 non-real zeros;
- the level set $\operatorname{Im}\left(\dfrac{p'(z)}{p(z)}\right) < 0$, in the upper half-plane, is connected;
- the polynomial $p$ has purely imaginary zeros and the degree of $p$ is $2, 4, 6, 8$ or 10.

In [9], Dilcher and Stolarsky investigated the relationship between the distribution of zeros of a polynomial, $p(z)$, and those of the "Wronskian of the polynomial", $Wp(z)$, where $Wp(z) \stackrel{def}{=} p(z)p''(z) - (p'(z))^2$. Dilcher and Stolarsky established several general properties of the polynomial $Wp(z)$ and its zeros. For example, they showed that if $d$ is the minimum distance between two consecutive real zeros of $p(z)$, then the imaginary part of the zeros of $Wp(z)$ cannot be less than $d\sqrt{3}/4$ [9, Lemma 2.8].

Dilcher [8] studied the geometry of the zeros of $Wp(z)$ and proved the Hawaii conjecture for polynomials whose zeros are sufficiently well spaced [8, Theorem 2.4]. Also he showed that any *real* zero of $Wp(z)$, which is not a zero of the polynomial $p(z)$, must lie on or inside the Jensen circle of some pair of complex zeros of $p(z)$.

Finally, we also mention that recently J. Borcea and B. Shapiro [4] developed a general theory of level sets, which may imply the validity of the Hawaii conjecture as a special case. But this approach, so far, has not led to a resolution of the Hawaii conjecture.

We remark that while the upper bound of the number of real zeros of $Q$ was only conjectured recently, the lower bound was known a long time ago, at least in the following special case:

**Problem 133 ([12]).** *Let the real polynomial $f(x)$ have only real zeros and suppose that the polynomial $f(x) + a$, where $a \in \mathbb{R}\backslash\{0\}$, has $2m$ nonreal zeros. Prove that the equation $(f'(x))^2 - f(x)f''(x) - af''(x) = 0$ has at least $2m$ real roots.*

If $p(z) = f(z) + a$, then it follows that $p$ has only simple zeros and exactly $2m$ nonreal zeros. Moreover, $p' = f'$ has only real zeros. By Problem 133, $Q$ associated with the polynomial $p(z) = f(z) + a$ has at least $2m$ real zeros. In fact, $Q$ has exactly $2m$ real zeros [6, Theorem 1]. The hint provided for Problem 133 in [12] suggests using Rolle's theorem. This approach ultimately leads us to a more precise result, namely, the following proposition.

**Proposition 1.1.** *Suppose that the polynomial $p$ has $2m$ nonreal zeros and its derivative $p'$ has $2m_1$ nonreal zeros, then $Q$ has at least $2m - 2m_1$ real zeros.*

Thus, the lower bound of the number of real zeros of $Q$ can be easily determined. Moreover, as one can see, this bound depends not only on the number of nonreal zeros of the polynomial $p$ but also on the number of nonreal zeros of $p'$. Unfortunately, this simple fact, Proposition 1.1, was not well known.

In this dissertation, we establish the lower bound on the number of real zeros of the function $Q$ not only for real polynomials but also for all functions in $\mathcal{L} - \mathcal{P}^*$ (Theorem 4.19), so, *ipso facto*, we prove Proposition 1.1 for functions in $\mathcal{L} - \mathcal{P}^*$. Moreover, we extend this result to an arbitrary real entire function of finite order with finitely many nonreal zeros (Theorem 5.22). But the main goal of the present work is to estimate the number of real critical points of logarithmic derivatives of real entire functions from above. Despite geometrical importance of this problem, the basic and nearly the unique instrument we use in our investigation is Rolle's theorem. Virtually, all results received in this dissertation are nontrivial consequences of that theorem.

In Chapter 2, we introduce some definitions and auxiliary (old and new) facts, which we use throughout the dissertation. We also introduce a specific property, *property A* (see Definition 2.25), of entire functions. Although not every function in $\mathcal{L} - \mathcal{P}^*$ possesses *property A*, for a given function $\varphi \in \mathcal{L} - \mathcal{P}^*$, one can always find another function $\psi_*$ in $\mathcal{L} - \mathcal{P}^*$ with *property A*, which has the same zeros and the same associated function $Q[\varphi]$ (Theorem 2.26).

For a given entire function $\varphi$, together with the function $Q[\varphi]$ defined in (1.1), we use the function $Q_1 \stackrel{def}{=} Q_1[\varphi] \stackrel{def}{=} Q[\varphi']$. In Chapter 3, we study the relationship between the number of real zeros of the functions $Q$ and $Q_1$ on finite intervals. All results in this chapter are established for real entire functions of finite order with a finite number of nonreal zeros.

Chapter 4 contains a comprehensive account on the distribution of real critical points of logarithmic derivatives of entire functions in $\mathcal{L} - \mathcal{P}^*$. In Sections 4.1 and 4.2, we obtain further particular results on the relationship between the number of real zeros of the functions $Q[\varphi]$ and $Q_1[\varphi]$. Namely, we investigate this relationship on half-infinite intervals free of poles of these functions and on the entire real axis when the first derivative of $\varphi$ has no real zeros.

In Section 4.3, we obtain our main result for functions in the class $\mathcal{L}-\mathcal{P}^*$ with *property A*. We prove the following inequalities, which provide a connection between the number of nonreal zeros of the functions $\varphi$ and $\varphi'$ and the number of real zeros of the functions $Q[\varphi]$ and $Q_1[\varphi]$:

$$2m - 2m_1 \leqslant Z_{\mathbb{R}}(Q) \leqslant 2m - 2m_1 + Z_{\mathbb{R}}(Q_1). \tag{1.2}$$

Here $2m$ and $2m_1$ are the number of nonreal zeros of $\varphi$ and $\varphi'$, respectively. These inequalities together with one technical result mentioned above, Theorem 2.26, allow us to prove the Hawaii conjecture, Theorem 4.19.

In Chapter 5, we extend the inequalities (1.2) to additional classes of real entire functions of finite order with finitely many nonreal zeros.

# Chapter 2

# Definitions and basic properties

In this chapter, we give necessary definitions and present some theorems from the theory of entire functions that will be used in the sequel. More details can be found in [2, 3, 11, 15].

## 2.1 Entire functions. Zeros and extra zeros

An *entire function* is a function of a complex variable holomorphic in the entire plane and represented by an everywhere convergent power series

$$f(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n + \ldots \tag{2.1}$$

As shown by K. Weierstraß [3, 15], every entire function $f(z)$ can be represented in the form

$$f(z) = z^d e^{g(z)} \prod_{j=1}^{\omega} \left(1 - \frac{z}{\alpha_j}\right) e^{\frac{z}{\alpha_j} + \frac{z^2}{2\alpha_j^2} + \ldots + \frac{z^{l_j}}{l_j \alpha_j^{l_j}}}, \quad 0 \leqslant \omega \leqslant \infty, \tag{2.2}$$

where $g(z)$ is an entire function, $\alpha_j$ are the nonzero roots of the entire function $f(z)$, $d$ is the order of the zero of $f(z)$ at the origin.

Let the series

$$\sum_{j=1}^{\infty} \frac{1}{|\alpha_j|^{\lambda}} \tag{2.3}$$

converge for some positive $\lambda$. In this case, let $l$ denote the smallest integer for which the series

$$\sum_{j=1}^{\infty} \frac{1}{|\alpha_j|^{l+1}}$$

converges. Then the uniformly convergent infinite product

$$\Pi(z) = \prod_{j=1}^{\omega} \left(1 - \frac{z}{\alpha_j}\right) e^{\frac{z}{\alpha_j} + \frac{z^2}{2\alpha_j^2} + \ldots + \frac{z^{l}}{l \alpha_j^{l}}}, \quad 0 \leqslant \omega \leqslant \infty,$$

is called a *canonical product*, and the number $l$ is called the *genus of the canonical product* (see, for example, [15]). For functions with the property of zeros just discussed, it is customary to choose the canonical product as the infinite product in the representation (2.2):

$$f(z) = z^d e^{g(z)} \Pi(z) = z^d e^{g(z)} \prod_{j=1}^{\omega} \left(1 - \frac{z}{\alpha_j}\right) e^{\frac{z}{\alpha_j} + \frac{z^2}{2\alpha_j^2} + \ldots + \frac{z^l}{l\alpha_j^l}}, \quad 0 \leqslant \omega \leqslant \infty, \tag{2.4}$$

This determines uniquely the function $g$.

If $g$ is a polynomial, then the function $f$ in (2.4) is said to be an *entire function of finite genus*. The number $\max\{l, \deg g\}$ is called the genus of the entire function $f$.

The number

$$\rho = \limsup_{r \to +\infty} \frac{\ln \ln M_f(r)}{\ln r},$$

where $M_f(r) = \max_{|z|=r} |f(z)|$, is called the *order* of the entire function $f$. The order of the entire function $f$ equals the greatest lower bound of the numbers $\lambda$, for which the series (2.3) converges.

In this dissertation, we address only *real* entire function of *finite* order with *finitely many* nonreal zeros.

**Definition 2.1.** An entire function $f(z)$ is called real if it is real on the real axis or, equivalently, if it has only real coefficients in its power series (2.1).

More precisely, we study the following classes of real entire functions.

**Definition 2.2** ([1], see also [11, 13])**.** The function $f$ is said to be in the class $V_{2n}$, $f \in V_{2n}$, if

$$f(z) = cz^d e^{-\gamma z^{2n+2} + q(z)} \prod_{j=1}^{\omega} \left(1 - \frac{z}{\alpha_j}\right) e^{\frac{z}{\alpha_j} + \frac{z^2}{2\alpha_j^2} + \ldots + \frac{z^l}{l\alpha_j^l}}, \quad 0 \leqslant \omega \leqslant \infty.$$

where $\gamma \geqslant 0$, $q$ is a real polynomial with $\deg q \leqslant 2n+1$, $l \leqslant 2n+1$, $\alpha_j \in \mathbb{R}\ \forall j$, $\sum_j |\alpha_j|^{-l-1} < \infty$, $d$ is a nonnegative integer, $c \neq 0 \in \mathbb{R}$.

**Definition 2.3.** The class $U_{2n}$ is defined as follows

$$U_0 \stackrel{def}{=} V_0$$

$$U_{2n} \stackrel{def}{=} V_{2n} \setminus V_{2n-2}$$

for $n \geqslant 0$.

Note that the class $U_0$ is exactly the Laguerre-Pólya class $\mathcal{L} - \mathcal{P}$ mentioned in Introduction.

**Definition 2.4.** The function $\varphi$ is in the class $U_{2n}^*$ if $\varphi = pf$ where $f \in U_{2n}$ and $p$ is a real polynomial with no real zeros.

Each class $U_{2n}^*$ is closed under differentiation (see [11, Corollary 2.12]).

**Proposition 2.5.** *If $\varphi \in U_{2n}^*$, then $\varphi' \in U_{2n}^*$.*

The class $U_0^*$ is often denoted by $\mathcal{L}-\mathcal{P}^*$ (see, for instance, [6, 7]). It plays a central role in our investigation. All real polynomials belong to this class.

**Notation 2.6.** For $\varphi \in U_{2n}^*$, by $Z_{\mathbb{C}}(\varphi)$ we denote the number of nonreal zeros of $\varphi$, counting multiplicities. If $f$ is a real meromorphic function having only a finite number of real zeros, then $Z_{\mathbb{R}}(f)$ will denote the number of real zeros of $f$, counting multiplicities. In the sequel, we also denote the number of zeros of the function $f$ in an interval $(a, b)$ and at a point $\alpha \in \mathbb{R}$ by $Z_{(a,b)}(f)$ and $Z_{\{\alpha\}}(f)$, respectively[1]. Generally, the number of zeros of $f$ on a set $X$ will be denoted by $Z_X(f)$.

The main instrument we use in this dissertation is Rolle's theorem. Principally, we use this theorem in the following form (see, for example, [18, Part V, Problem 10]).

**Theorem 2.7** (Rolle's theorem)**.** *Let the function $f$ be real analytic. Between any two consecutive real zeros, say $a$ and $b$, $a < b$, of $f$, its derivative $f'$ has an* ***odd*** *number of real zeros (and* a fortiori *at least one).*

Thus, between any two consecutive zeros $a$ and $b$ of the function $f$, one zero of the function $f'$ is guaranteed by Rolle's theorem. Other zeros, if they exist, are called *extra zeros* of the function $f'$ in the interval $(a, b)$. Counting all zeros with multiplicities, suppose that $f'$ has $2r+1$ zeros between $a$ and $b$. Then we will say that $f'$ has $2r$ *extra zeros* between $a$ and $b$. If $f$ has the largest zero $a_L$ (or the smallest zero $a_S$), then any real zero $f'$ in $(a_L, \infty)$ (and in $(-\infty, a_S)$) is also called an *extra zero* of $f'$.

**Notation 2.8.** The total number of extra zeros of $f'$ on the entire real axis, counting multiplicities, will be denoted by $E(f')$.

**Remark 2.9.** The multiple real zeros of $f$ are not counted as real extra zeros of $f'$.

It turns out that if a function $\varphi \in U_{2n}^*$ has at most finitely many zeros, then its number of extra zeros can be calculated exactly.

**Theorem 2.10.** *Let $p$ be a real polynomial and let $\varphi$ be the function defined as follows.*

$$\varphi(z) \overset{def}{=} e^{az^{2n+1}+q(z)}p(z), \quad a \neq 0 \in \mathbb{R}, \ \ n \in \mathbb{N} \cup \{0\}, \tag{2.5}$$

*where $q$ and $p$ are real polynomials and* $\deg q \leqslant 2n$. *Then*

$$E(\varphi') = \begin{cases} Z_{\mathbb{C}}(\varphi) - Z_{\mathbb{C}}(\varphi') + 2n & \text{if } \deg p = Z_{\mathbb{C}}(p), \\ Z_{\mathbb{C}}(\varphi) - Z_{\mathbb{C}}(\varphi') + 2n + 1 & \text{if } \deg p > Z_{\mathbb{C}}(p). \end{cases}$$

[1] Thus, $Z_{\mathbb{R}}(f) = Z_{(-\infty,+\infty)}(f)$.

**Proof.** From (2.5), it is easy to see that $\varphi'$ has exactly $\deg p + 2n$ zeros, counting multiplicities.

If $\varphi(z) \neq 0$ for $z \in \mathbb{R}$, then $\deg p = Z_{\mathbb{C}}(p) = Z_{\mathbb{C}}(\varphi)$ and all real zeros of $\varphi'$ are extra zeros. Therefore, $E(\varphi') = Z_{\mathbb{C}}(\varphi) + 2n - Z_{\mathbb{C}}(\varphi')$.

If $\varphi$ has at least one real zero, then $\deg p = Z_{\mathbb{C}}(\varphi) + r$, where $r\ (> 0)$ is the number of real zeros of $\varphi$, counting multiplicities. Therefore, $\varphi'$ has exactly $Z_{\mathbb{C}}(\varphi) + 2n + r - Z_{\mathbb{C}}(\varphi')$ real zeros, counting multiplicities, $r - 1$ of which are guaranteed by Rolle's theorem. Thus, $E(\varphi') = Z_{\mathbb{C}}(\varphi) + 2n + 1 - Z_{\mathbb{C}}(\varphi')$. □

In the same way, the following two theorems can be proved.

**Theorem 2.11.** *Let $p$ be a real polynomial and let $\varphi$ be the function defined as follows.*

$$\varphi(z) \overset{def}{=} e^{az^{2n}+q(z)}p(z), \quad a > 0, \ n \in \mathbb{N} \cup \{0\}, \tag{2.6}$$

*where $q$ is a real polynomial of degree at most*[2] $2n - 1$ *and $p$ is a real polynomial. Then*

$$E(\varphi') = \begin{cases} Z_{\mathbb{C}}(\varphi) - Z_{\mathbb{C}}(\varphi') + 2n - 1 & \text{if } \deg p = Z_{\mathbb{C}}(p), \\ Z_{\mathbb{C}}(\varphi) - Z_{\mathbb{C}}(\varphi') + 2n & \text{if } \deg p > Z_{\mathbb{C}}(p). \end{cases} \tag{2.7}$$

**Theorem 2.12.** *Let $p$ be a real polynomial and let $\varphi$ be the function defined as follows.*

$$\varphi(z) \overset{def}{=} e^{-az^{2n+2}+q(z)}p(z), \quad a > 0, \ n \in \mathbb{N} \cup \{0\}, \tag{2.8}$$

*where $q$ and $p$ are real polynomials and* $\deg q \leqslant 2n + 1$. *Then*

$$E(\varphi') = \begin{cases} Z_{\mathbb{C}}(\varphi) - Z_{\mathbb{C}}(\varphi') + 2n + 1 & \text{if } \deg p = Z_{\mathbb{C}}(p), \\ Z_{\mathbb{C}}(\varphi) - Z_{\mathbb{C}}(\varphi') + 2n + 2 & \text{if } \deg p > Z_{\mathbb{C}}(p). \end{cases}$$

We should note that Theorems 2.10, 2.11 and 2.12 are parts of Lemma 2.8 of the work [11]. We state these parts in terms of our definition of extra zeros. Another parts of Lemma 2.8 in [11] concerns functions with infinitely many zeros.

**Theorem 2.13** ([11])**.** *Let $\varphi$ be in $U_{2n}^*$.*

- *If $\varphi$ has infinitely many positive and negative zeros, then*

$$E(\varphi') = Z_{\mathbb{C}}(\varphi) - Z_{\mathbb{C}}(\varphi') + 2n.$$

- *If $\varphi$ has infinitely many zeros but only finitely many positive or negative zeros, then*

$$Z_{\mathbb{C}}(\varphi) - Z_{\mathbb{C}}(\varphi') + 2n \leqslant E(\varphi') \leqslant Z_{\mathbb{C}}(\varphi) - Z_{\mathbb{C}}(\varphi') + 2n + 1.$$

For functions in the class $U_0^* = \mathcal{L} - \mathcal{P}^*$, this theorem was established by T. Craven, G. Csordas, and W. Smith in [5, p. 325].

[2] If $n = 0$, then $q(z) \equiv 0$.

## 2.2 Logarithmic derivatives of entire functions in $U_{2n}$

Given a function $\varphi$, the following function

$$L(z) \stackrel{def}{=} \frac{d\ln\varphi(z)}{dz} = \frac{\varphi'(z)}{\varphi(z)}$$

is called the logarithmic derivative of $\varphi$.

The following important fact was proved in [2, Lemma 4].

**Theorem 2.14.** *Let $\varphi$ be a function of class $U_{2n}$. Then the logarithmic derivative of $\varphi$ has a representation*

$$\frac{\varphi'(z)}{\varphi(z)} = h(z)\mu(z), \tag{2.9}$$

*where $h$ is a real polynomial, $\deg h = 2n$, the leading coefficient of $h$ is negative, and $\mu \not\equiv 0$ is a function with nonnegative imaginary part in the upper half-plane of the complex plane.*

As was shown in [2], if $\varphi \in U_{2n}$ has no zeros, then there can be only three situations.

- $\varphi(z) = e^{-az^{2n+2}+q(z)}$, where $a > 0$ and $q$ is a real polynomial, $\deg q \leqslant 2n+1$. Then $\varphi'(z)/\varphi(z) = -a(2n+2)z^{2n+1} + q'(z)$ is a real polynomial of an odd degree and, therefore, it has a real zero $\beta$. So one can put

$$h(z) = \frac{-a(2n+2)z^{2n+1} + q''(z)}{z-\beta} \quad \text{and} \quad \mu(z) = z - \beta.$$

- $\varphi(z) = e^{az^{2n+1}+q(z)}$, where $a \neq 0 \in \mathbb{R}$ and $q$ is a real polynomial, $\deg q \leqslant 2n$. Then one can set $h(z) = -|a|(2n+1)z^{2n} + q'(z)$ and $\mu(z) = -\operatorname{sign} a$.

- $\varphi(z) = e^{az^{2n}+q(z)}$, where $a > 0$ and $q$ is a real polynomial, $\deg q \leqslant 2n-1$. Then we set $h(z) = -2naz^{2n} - zq'(z)$ and $\mu(z) = -1/z$.

From now on, assume that $\varphi \in U_{2n}$ has at least one real zero $\alpha_0$. If $\varphi$ has only finitely many negative zeros and $\varphi(z) \to 0$ as $z \to -\infty$, then we follow [2] to consider $-\infty$ as a zero of $\varphi$. Similarly, we consider $+\infty$ as a zero of $\varphi$ if $\varphi$ has only finitely many positive zeros and $\varphi(z) \to 0$ as $z \to +\infty$. We arrange the zeros into an increasing sequence $\{\alpha_j\}$, where each zero occurs once, disregarding multiplicity. The range of the subscript $j$ will be $M < j < N$, where $-\infty \leqslant M < 0 \leqslant N \leqslant +\infty$, with $\alpha_{M+1} = -\infty$ and $\alpha_{N-1} = +\infty$ in the cases described above.

By Rolle's theorem, each open interval contains a zero of $\varphi'$. To make a definite choice, we take for $\beta_j$ the largest zero in this interval. Each $\beta_j$ occurs in this sequence only once, and we disregard multiplicity. As was shown in [2], in this case, the function $\mu$ in Theorem 2.14 has the following form

$$\mu(z) = \frac{1}{z - \alpha_{N-1}} \prod_{M<j<N-1} \frac{1 - z/\beta_j}{1 - z/\alpha_j},$$

where the factor $z-\alpha_{N-1}$ is omitted if $\alpha_{N-1}=+\infty$ or $N=+\infty$, and the factor $1-z/\alpha_{M+1}$ is omitted if $\alpha_{M+1}=-\infty$. If for some $j\in(M,N-1)$ we have $\alpha_j=0$ or $\beta_j=0$, then the $j^{\text{th}}$ factor has to be replaced by $(z-\beta_j)/(z-\alpha_j)$.

As was shown by Chebotarev [15, p. 310], the function $\mu$ as a function mapping the upper half-plane onto itself may be represented in the form

$$\mu(z)=az+b+\sum_{j=M}^{N}A_j\left(\frac{1}{\alpha_j-z}-\frac{1}{\alpha_j}\right),\quad -\infty\leqslant M<N\leqslant+\infty,$$

where $a\geqslant 0$, $b\in\mathbb{R}$, $A_j\geqslant 0$ where the series

$$\sum_{j=M}^{N}\frac{A_j}{\alpha_j^2}$$

converges.

From this representation it follows that $\mu'(z)>0$ for every real $z\neq\alpha_j$. In particular, if $\varphi\in U_0=\mathcal{L}-\mathcal{P}$, then $h(z)\equiv c<0$ in (2.9). Therefore,

$$\left(\frac{\varphi'(z)}{\varphi(z)}\right)'=c\mu'(z)<0\quad\text{for}\quad z\in\mathbb{R},\quad z\neq\alpha_j. \tag{2.10}$$

Thus, we proved the well-known fact (see, for example, [6, 5]) that the logarithmic derivative of a function in the Laguerre-Pólya class is a decreasing function on the intervals where it has no poles.

We use these facts in the next section and in Chapter 5.

## 2.3 Derivatives of logarithmic derivatives. Entire functions with *property A*

Let $\varphi$ be in $U_{2n}^*$ and let the function $Q=Q[\varphi]$ associated with $\varphi$ be defined as

$$Q(z)\overset{def}{=}Q[\varphi](z)\overset{def}{=}\frac{d}{dz}\left(\frac{\varphi'(z)}{\varphi(z)}\right)=\frac{\varphi(z)\varphi''(z)-(\varphi'(z))^2}{(\varphi(z))^2}. \tag{2.11}$$

We note that if $\varphi(z)=Ce^{\beta z}$, where $C,\beta\in\mathbb{R}$, then $Q(z)\equiv 0$. At the same time, all functions of the form $Ce^{\beta z}$ belong to the class $U_0^*=\mathcal{L}-\mathcal{P}^*$. Hence, we adopt the following convention throughout this dissertation.

**Convention.** *If $\varphi\in\mathcal{L}-\mathcal{P}^*$, then $\varphi$ is assumed not to be of the form $\varphi(z)=Ce^{\beta z}$, $C,\beta\in\mathbb{R}$.*

Analogously to (2.11), we also introduce the related function

$$Q_1(z)\overset{def}{=}Q[\varphi'](z)\overset{def}{=}\frac{d}{dz}\left(\frac{\varphi''(z)}{\varphi'(z)}\right)=\frac{\varphi'(z)\varphi'''(z)-(\varphi''(z))^2}{(\varphi'(z))^2}. \tag{2.12}$$

Our interest is concentrated only on the number of *real* zeros of the function $Q$, $Z_{\mathbb{R}}(Q)$, and on bounding this number. Obviously, $Q[\varphi]$ has a finite number of real zeros if $\varphi$ has a finitely many zeros. But generally speaking, $Q[\varphi]$ may have infinitely many real zeros. However, for the function $\varphi$ in the class $\mathcal{L}-\mathcal{P}^*$, the function $Q[\varphi]$ has also a finite number of real zeros even if $\varphi$ has infinitely many zeros. This fact is essentially known from [6], but we still include the proof for completeness.

**Theorem 2.15.** *Let $\varphi \in \mathcal{L}-\mathcal{P}^*$. Then the function $Q[\varphi]$ has finitely many real zeros.*

**Proof.** By definition, $\varphi$ is a product $\varphi = pf$, where $f \in \mathcal{L}-\mathcal{P}$ and $p$ is a real polynomial with no real zeros. Let $\deg p = 2m \geqslant 0$.

If $m = 0$, then $p(z) \equiv$ *const*. In this case, $\varphi \in \mathcal{L}-\mathcal{P}$. As we showed in preceding section (see (2.10)), the logarithmic derivative of a function from the Laguerre-Pólya class is a decreasing function on the intervals where it has no poles. Therefore, $Q[\varphi](z) < 0$ for any real $z$, which is not a pole of this function. Thus, if $m = 0$, then $Q[\varphi]$ has no real zeros.

We assume now that $m > 0$. Observe that

$$Q[\varphi] = Q[p] + Q[f].$$

From this equality it follows that all zeros of $Q$ are roots of the equation

$$Q[p](z) = -Q[f](z). \tag{2.13}$$

It is easy to see that if $p(z) = a_0 z^{2m} + \ldots$, $a_0 \neq 0$, then

$$Q[p](z) = \frac{-2ma_0^2 z^{4m-2} + \ldots}{a_0^2 z^{4m} + \ldots}. \tag{2.14}$$

This formula shows that $Q[p](z) \to -0$ whenever $z \to \pm\infty$. Consequently, there exist two real numbers $a_1$ and $a_2$ ($a_1 < a_2$) such that $Q[p](z) < 0$ for $z \in (-\infty, a_1] \cup [a_2, +\infty)$. But since $f \in \mathcal{L}-\mathcal{P}$, we have $Q[f](z) < 0$ for $z \in \mathbb{R}$ as we mentioned above. Thus, the right hand side of the equation (2.13) is positive for all $z \in \mathbb{R}$ but its left hand side is negative for all $z \in (-\infty, a_1] \cup [a_2, +\infty)$. Therefore, all real roots of the equation (2.13) and, consequently, all real zeros of the function $Q[\varphi]$ belong to the interval $(a_1, a_2)$, and $Q[\varphi](z) < 0$ outside this interval. Since real zeros of a meromorphic function are isolated, $Q[\varphi]$ has only finitely many real zeros, as required. □

In fact, the number of real zeros of the function $Q$ associated with functions in the class $\mathcal{L}-\mathcal{P}^*$ is even (see Theorem 2.20).

For functions in the classes $U_{2n}^*$ with $n \geqslant 1$, Theorem 2.15 is not valid. That is, if $\varphi \in U_{2n}^*$ with $n \geqslant 1$ has infinitely many zeros, then its associated function $Q[\varphi]$ may have infinitely many zeros.

**Example 2.16.** Consider the function $f(z) = e^{az^2}\sin z$ with $a > 0$. It is clear that $f \in U_2^*$ and $f$ has infinitely many real zeros $\alpha_k = \pi k, k \in \mathbb{Z}$. In this case, the function $Q[f]$ is as follows:

$$Q[f](z) = -\frac{1}{\sin^2 z} + 2a.$$

Its zeros are the roots of the equation $\sin^2 z = \dfrac{1}{2a}$. For $a \geqslant \dfrac{1}{2}$, this equation has infinitely many roots:

$$\zeta_k = \pm \arcsin\frac{1}{\sqrt{2a}} + \pi k, \quad k \in \mathbb{Z}. \tag{2.15}$$

Thus, for $a \geqslant \dfrac{1}{2}$, the function $Q[f]$ associated with $f(z) = e^{az^2}\sin z \in U_2^*$ has infinitely many real zeros given by (2.15).

**Notation 2.17.** For convenience, we use $\alpha_j$ to denote the zeros of $\varphi$, $\beta_j$ to denote the zeros of $\varphi'$, and $\gamma_j$ to denote the zeros of $\varphi''$. We have already used this convention in the preceding section.

The following simple fact is very important for the sequel.

**Proposition 2.18.** *Let $\alpha \in \mathbb{R}$ be a zero of $\varphi \in U_{2n}^*$. For all sufficiently small $\varepsilon > 0$, the following inequality holds*

$$Q(\alpha \pm \varepsilon) < 0. \tag{2.16}$$

**Proof.** If $\alpha$ is a zero of $\varphi$ of multiplicity $M \geqslant 1$, then $\varphi(z) = (z-\alpha)^M\psi(z)$, where $\psi(\alpha) \neq 0$. Thus, we have

$$Q(z) = -\frac{M}{(z-\alpha)^2} + \frac{\psi(z)\psi''(z) - (\psi'(z))^2}{(\psi(z))^2}.$$

Consequently, the function $Q$ is negative in a small punctured neighbourhood of $\alpha$ as required. □

Furthermore, as was shown in [6, p. 418] (see also the proof of Theorem 2.15), the function $Q[\varphi](z)$ associated with $\varphi \in \mathcal{L}-\mathcal{P}^*$ is negative for sufficiently large $z$. This fact and Proposition 2.18 imply the following lemma which concerns the parity of the number of real zeros of $Q$ on the half-interval $[\alpha_L, +\infty)$ where $\alpha_L$ is the largest zero of the function $\varphi \in \mathbb{R}$.

**Lemma 2.19.** *Let $\varphi \in \mathcal{L}-\mathcal{P}^*$. If $\varphi$ has the largest zero $\alpha_L$ (or the smallest zero $\alpha_S$), then $Q$ has an even number of real zeros in $(\alpha_L, +\infty)$ (or in $(-\infty, \alpha_S)$), counting multiplicities.*

**Proof.** By Proposition 2.18, $Q$ is negative for $z$ sufficiently close to $\alpha_L$ (or to $\alpha_S$). But it was already proved in Theorem 2.15 (see also (3.11) in [6, p.415] and subsequent remark there) that $Q(z) < 0$ for all sufficiently large real $z$. Therefore, $Q$ has an even number of zeros in $(\alpha_L, +\infty)$ (and in $(-\infty, a_S)$ if $\varphi$ has the smallest real zero $\alpha_S$), counting multiplicities, since $Q(z)$ is negative for all real $z$ sufficiently close to the ends of the interval $(\alpha_L, +\infty)$ (or of the interval $(-\infty, \alpha_S)$). □

Using this lemma and Proposition 2.18, it is easy to establish the following fact concerning the parity of the number $Z_{\mathbb{R}}(Q)$ for the function $Q$ associated with a function in $\mathcal{L}-\mathcal{P}^*$.

**Theorem 2.20** (Craven–Csordas–Smith [6], p. 415)**.** *If $\varphi \in \mathcal{L}-\mathcal{P}^*$, then the function $Q[\varphi]$ has an even number of real zeros, counting multiplicity.*

**Proof.** In fact, if $\varphi$ has no real zeros, then $Q$ has no real poles, and the number $Z_{\mathbb{R}}(Q)$ is even, since $Q(z) < 0$ for all sufficiently large real $z$.

If $\varphi$ has only one real zero $\alpha$, then, according to Lemma 2.19, $Q$ has an even number of zeros in each of the intervals $(-\infty, \alpha)$ and $(\alpha, +\infty)$. Thus, $Z_{\mathbb{R}}(Q)$ is also even in this case.

Let $\varphi$ have at least two real zeros. If $\alpha_j$ and $\alpha_{j+1}$ are two consecutive zeros of $\varphi$, then, according to (2.16), $Q$ has an even number of zeros, counting multiplicities, in the interval $(\alpha_j, \alpha_{j+1})$. If $\varphi$ has the largest (or/and the smallest) real zero, say $\alpha_L$ ($\alpha_S$), then, by Lemma 2.19, $Q$ has an even number of real zeros, counting multiplicities, in $(a_L, +\infty)$ (and in $(-\infty, \alpha_S)$). Therefore, the number $Z_{\mathbb{R}}(Q)$ is even. □

**Remark 2.21.** Analogously, the function $Q_1$ associated with a function in the class $\mathcal{L}-\mathcal{P}^*$ has an even number of real zeros, counting multiplicities, since the class $\mathcal{L}-\mathcal{P}^*$ is closed with respect to differentiation by Proposition 2.5.

As we will see in Chapter 5, the functions $Q[\varphi]$ associated with functions $\varphi$ in the classes $U_{2n}^*$ with $n \geqslant 1$ can be positive for all sufficiently large real $z$. Moreover, $Q[\varphi]$ can have an odd number of zeros (see Chapter 5 for more details). However, in some cases, $Q[\varphi](z)$ may be negative for all sufficiently large real $z$. In Chapter 5, we use the following theorem.

**Theorem 2.22.** *Let $\varphi$ be in $U_{2n}^*$ with $n \geqslant 1$. If $\varphi$ has the largest zero $\alpha_L$ and $\varphi'$ has an odd number of zeros, counting multiplicities, in the interval $(\alpha_L, +\infty)$, then $Q[\varphi](z) < 0$ for all sufficiently large positive $z$.*

**Proof.** By definition, $\varphi$ belong to $U_{2n}^*$ if $\varphi = pf$, where $p$ is a real polynomial with no real zeros and $f \in U_{2n}$. According to Theorem 2.14 (see (2.9)), the logarithmic derivative can be represented in the form $\varphi'/\varphi = h\mu$, where $h$ is a real polynomial of degree $2n$ whose leading coefficient is negative, and $\mu$ is the meromorphic function described in Section 2.2. Thus, for the derivative of the logarithmic derivative of the function $\varphi$, we have

$$Q[\varphi](z) = \left(\frac{\varphi'(z)}{\varphi(z)}\right)' = h(z)\mu'(z) + h'(z)\mu(z) + Q[p](z). \tag{2.17}$$

Next, by assumption, the function $\varphi'$ has an odd number of zeros in the interval $(\alpha_L, +\infty)$. Therefore, by construction (see Section 2.2), the function $\mu$ has a unique simple zero $\beta$ in the interval $(\alpha_L, +\infty)$ that coincides with one of the zeros of $\varphi'$. Moreover, as we noted in Section 2.2, $\mu'(z) > 0$ in any interval between its poles, therefore, $\mu(z)$ must be negative in the interval $(\alpha_L, \beta)$ and positive in the interval $(\beta, +\infty)$. Furthermore, the function $Q[p](z)$ is negative for all sufficiently large real $z$ as was shown in the proof of Theorem 2.15 (see (2.14)). At last, by Theorem 2.14, the polynomial $h(z)$ and its derivative $h'(z)$ are negative for all sufficiently large positive $z$, since $\deg h = 2n > 0$ and the leading coefficient of $h$ is negative.

Thus, we have shown that all the summands are negative at $+\infty$ in (2.17), that is, $Q[\varphi](z) < 0$ for all sufficiently large positive $z$, as required. $\square$

Theorem 2.22 is valid with respective modification in the case when $\varphi$ has the smallest zero $a_S$ and $\varphi'$ has an odd number of zeros in the interval $(-\infty, \alpha_S)$.

**Theorem 2.23.** *Let $\varphi$ be in $U_{2n}^*$ with $n \geqslant 1$. If $\varphi$ has the smallest zero $\alpha_S$ and $\varphi'$ has an odd number of zeros, counting multiplicities, in the interval $(-\infty, \alpha_S)$, then $Q[\varphi](z) < 0$ for all sufficiently large negative $z$.*

The following definition plays a crucial role in our investigation and in the proof of the Hawaii conjecture.

**Definition 2.24.** Let $\varphi \in U_{2n}^*$ and let $\alpha$ be a real zero of $\varphi$. Suppose that $\beta_1$ and $\beta_2$, $\beta_1 < \alpha < \beta_2$, are real zeros of $\varphi'$ such that $\varphi'(z) \neq 0$ for $z \in (\beta_1, \alpha) \cup (\alpha, \beta_2)$. The function $\varphi$ is said to possess **property A at its real zero** $\alpha$ if $Q$ has no real zeros in at least one of the intervals $(\beta_1, \alpha)$ and $(\alpha, \beta_2)$. If $\alpha$ is the smallest zero of $\varphi$, then set $\beta_1 = -\infty$, and if $\alpha$ is the largest zero of $\varphi$, then set $\beta_2 = +\infty$.

**Definition 2.25.** A function $\varphi \in U_{2n}^*$ is said to possess **property A** if $\varphi$ possesses property A at each of its real zeros. In particular, $\varphi$ without real zeros possesses property A.

To illustrate *property A*, we use a function $\varphi \in \mathcal{L} - \mathcal{P}^*$ of the form $\varphi(z) = e^{\lambda z} p(z)$, where $\lambda > 0$ and $p$ is a real polynomial. On Figure 2.1, there is the graphic of the logarithmic derivative of such a function $\varphi$ with *property A*.

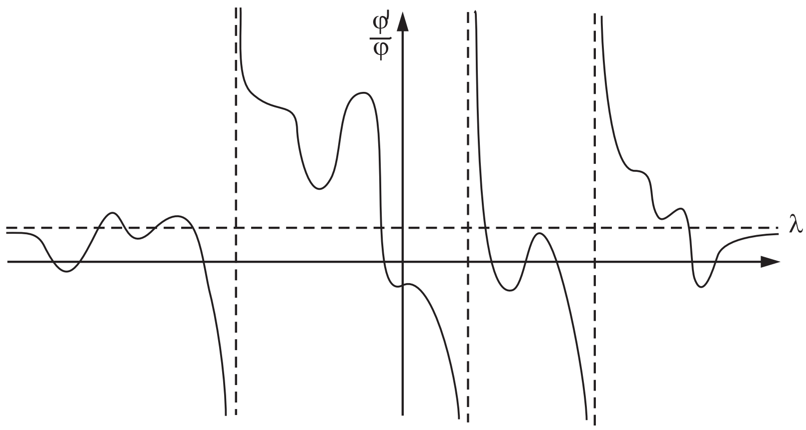


Figure 2.1:

*Property A* is a very special property of entire functions. It cannot be verified easily. However, for a given function $\varphi$ in $\mathcal{L} - \mathcal{P}^*$, we can always find another function $\psi_*$

in $\mathcal{L}-\mathcal{P}^*$ with *property A*, which has the same zeros and the same associated function $Q$. The following theorem establishes the existence of such a function.

**Theorem 2.26.** *Let $\varphi \in \mathcal{L}-\mathcal{P}^*$. Then there exists a real number $\sigma_*$ such that the function $\psi_*(z) \stackrel{def}{=} e^{-\sigma_* z}\varphi(z)$ possesses property A. Moreover, if $Z_{\mathbb{R}}(Q) \neq 0$, then $Z_{\mathbb{C}}(\psi_*') < Z_{\mathbb{C}}(\psi_*)$.*

Before proving the theorem, we notice that, for any real $\sigma$, the function $\psi(z) = e^{-\sigma z}\varphi(z)$ has the same set of zeros as the function $\varphi(z)$. At the same time, we have the following relations:

$$\frac{\psi'(z)}{\psi(z)} = \frac{\varphi'(z)}{\varphi(z)} - \sigma, \qquad Q[\psi] = Q[\varphi]. \tag{2.18}$$

Using these relations, one can easily give the main idea of the proof of Theorem 2.26. In fact, let again, for the sake of simplicity, $\varphi(z) = e^{\lambda z}p(z)$, where $\lambda > 0$ and $p$ is a real polynomial. Suppose that $\varphi$ does not possess *property A*. This means that its logarithmic derivative $\varphi'/\varphi$ has critical points on each interval between a pole of $\varphi'/\varphi$ and its zero closest to this pole (see Figure 2.2). From Figure 2.2 one can see that, for sufficiently large

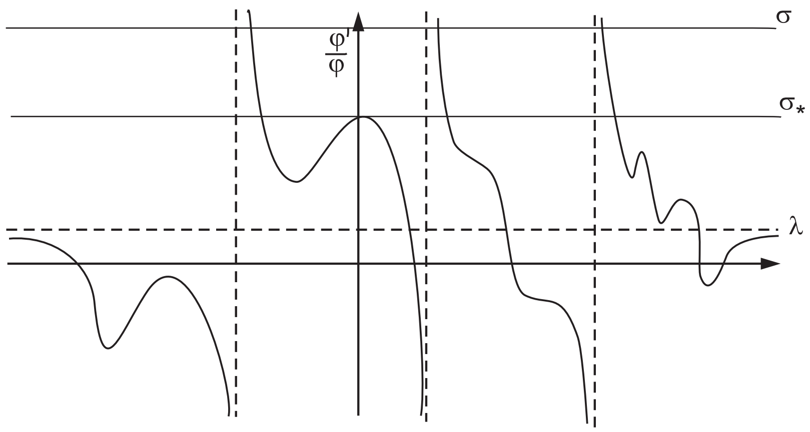


Figure 2.2:

positive $\sigma$, the function $\psi(z) = e^{-\sigma z}\varphi(z)$ possesses *property A*. Now we just have to reduce $\sigma$ until the line $y = \sigma$ meets the first critical point[3] of $\varphi'/\varphi$.

**Proof.** By Theorem 2.15, the function $Q[\varphi]$ has finitely many real zeros. If $Q[\varphi]$ has no real zeros, then, for any real $\sigma$, $\psi(z) = e^{-\sigma z}\varphi(z)$ automatically possesses *property A*.

Let now $Q[\varphi]$ have at least one real zero and let $\zeta_1 < \zeta_2 < \ldots < \zeta_n$, $n \geqslant 1$, be the *distinct* real zeros of $Q[\varphi]$. We set

$$\sigma_* \stackrel{def}{=} \max_{1\leqslant i\leqslant n}\left(\frac{\varphi'(\zeta_i)}{\varphi(\zeta_i)}\right) \quad \text{and} \quad \psi_*(z) \stackrel{def}{=} e^{-\sigma_* z}\varphi(z). \tag{2.19}$$

[3]That is, the critical point of $\varphi'/\varphi$ with maximal value in this point (see Figure 2.2).

Then from (2.18) applied to $\psi_*$ it follows that

$$\frac{\psi_*'(\zeta_i)}{\psi_*(\zeta_i)} \leqslant 0, \quad i = 1, \dots, n. \tag{2.20}$$

In this theorem, we denote by $\zeta_*$ any (fixed) zero of $Q[\varphi]$ where the maximum (2.19) is attained. Thus, we have

$$\frac{\psi_*'(\zeta_*)}{\psi_*(\zeta_*)} = 0. \tag{2.21}$$

We also denote by $\alpha_j$ ($j \in \mathbb{Z}$) the *real* zeros of $\varphi$ (and of $\psi_*$) and $I_j = (\alpha_j, \alpha_{j+1})$. If $\varphi$ has the largest zero $\alpha_L$ and the smallest zero $\alpha_S$, then we also consider the intervals

$$I_{+\infty} = (\alpha_L, +\infty) \quad \text{and} \quad I_{-\infty} = (-\infty, \alpha_S).$$

We now show that $\psi_*$ possesses *property A*. If $\varphi$ has no real zeros, then $\psi_*$ also has no real zeros and, therefore, it possesses *property A*.

Suppose that $\varphi$ has at least one real zero. Consider the interval $I_j$ for a fixed $j$. Let $\beta$ be the leftmost zero of $\psi_*'/\psi_*$ in $I_j$. The existence of this zero is guaranteed by Rolle's theorem. Since $\psi_*'(z)/\psi_*(z) \to +\infty$ whenever $z \searrow \alpha_j$ (see (2.16)), we have $\psi_*'(z)/\psi_*(z) > 0$ for $z \in (\alpha_j, \beta)$. Now from (2.20) it follows that $Q[\psi_*]$ has no zeros in $(\alpha_j, \beta)$. Consequently, $\psi_*$ possesses *property A* at $\alpha_j$. In particular, if $\psi_*$ has the smallest zero $\alpha_S = \alpha_1$, then considering the interval $I_1$, we obtain that $\psi_*$ possesses *property A* at $\alpha_S$. Let $\psi_*$ have the largest zero $\alpha_L$. If $\psi_*'/\psi_*$ has no zeros in the interval $I_{+\infty}$, then $\psi_*'(z)/\psi_*(z) > 0$ for $z \in I_{+\infty}$, since $\psi_*'(z)/\psi_*(z) \to +\infty$ whenever $z \searrow \alpha_L$ (see (2.16)). Therefore, by (2.20), we have $Q[\psi_*](z) \neq 0$ in the interval $(\alpha_L, +\infty)$, and $\psi_*$ also possesses *property A* at $\alpha_L$. If $\psi_*'/\psi_*$ has at least one zero in $I_{+\infty}$ and $\beta$ is the leftmost one, then $\psi_*'(z)/\psi_*(z) > 0$ for $z \in (\alpha_L, \beta)$. So, $Q[\psi_*]$ can not have zeros in $z \in (\alpha_L, \beta)$ by the same reasoning as above. Thus, we obtain that $\psi_*$ possesses *property A* at each of its real zero (if any). Consequently, $\psi_*$ possesses *property A*.

We now prove that $Z_{\mathbb{C}}(\psi_*') < Z_{\mathbb{C}}(\psi_*)$. There are only the following two cases.

Case I. The function $\psi_*$ have no real zeros. Then all real zeros of $\psi_*'$ are extra zeros.

If $\zeta_*$ is a zero of $Q[\psi_*]$ of multiplicity $M \geqslant 1$, then $\zeta_*$ is a zero of $\psi_*'$ of multiplicity $M+1$ according to (2.21). Thus, we have $E(\psi_*') \geqslant M+1$. Since by Theorems 2.10–2.12 (with $n = 0$)

$$Z_{\mathbb{C}}(\psi_*) - Z_{\mathbb{C}}(\psi_*') - 1 \leqslant E(\psi_*') \leqslant Z_{\mathbb{C}}(\psi_*) - Z_{\mathbb{C}}(\psi_*') + 1, \tag{2.22}$$

we have $1 \leqslant M \leqslant Z_{\mathbb{C}}(\psi_*) - Z_{\mathbb{C}}(\psi_*')$, that is, $Z_{\mathbb{C}}(\psi_*') < Z_{\mathbb{C}}(\psi_*)$.

Case II. The function $\psi_*$ have at least one real zero. Recall that all zeros of $\psi_*'$ in the every interval $I_j$ are extra zeros of $\psi_*'$ except one, counting multiplicity. At the same time, all the zeros of $\psi_*'$ in the intervals $I_{-\infty}$ and $I_{+\infty}$ are extra zeros of $\psi_*'$.

Let $\psi_*$ have at least two real zeros and let $\zeta_* \in I_j$ for some $j$. If $\zeta_*$ is a zero of $Q[\psi_*]$ of *even* multiplicity, then $\zeta_*$ is a zero of $\psi_*'$ of odd multiplicity (at least three). Since one zero of $\psi_*'$ in $I_j$ is guaranteed by Rolle's theorem, we have

$$E(\psi_*') \geqslant 2. \tag{2.23}$$

If $\psi_*'$ is not of the form (2.8) (with $n = 0$), then by Theorems 2.10, 2.11 and 2.13, $\psi_*$ satisfies the inequalities

$$Z_{\mathbb{C}}(\psi_*) - Z_{\mathbb{C}}(\psi_*') \leqslant E(\psi_*') \leqslant Z_{\mathbb{C}}(\psi_*) - Z_{\mathbb{C}}(\psi_*') + 1, \tag{2.24}$$

from which it follows that $Z_{\mathbb{C}}(\psi_*') < Z_{\mathbb{C}}(\psi_*)$ by (2.23).

If $\psi_*$ has the form (2.8) (with $n = 0$), then we have

$$\frac{\psi_*'(z)}{\psi_*(z)} = \frac{p'(z)}{p(z)} - 2az + b, \quad a > 0, \tag{2.25}$$

where $p$ is a real polynomial and $b \in \mathbb{R}$. Hence the interval $I_{-\infty}$ exists and $\psi_*'$ has an odd number of extra zeros (at least one) in $I_{-\infty}$. In fact, $\psi_*'(z)/\psi_*(z) \to +\infty$ whenever $z \to -\infty$ and $\psi_*'(z)/\psi_*(z) \to -\infty$ whenever $z \nearrow \alpha_S$ by (2.16) and (2.25). Thus, in this case, the inequality (2.23) can be improved to the following one.

$$E(\psi_*') \geqslant 3. \tag{2.26}$$

But by Theorem 2.12, $E(\psi_*') = Z_{\mathbb{C}}(\psi_*) - Z_{\mathbb{C}}(\psi_*') + 2$. Now from (2.26) it follows that $Z_{\mathbb{C}}(\psi_*') < Z_{\mathbb{C}}(\psi_*)$.

If $\zeta_*$ is a zero of $Q[\psi_*]$ of *odd* multiplicity, then $\zeta_*$ is a zero of $\psi_*'$ of even multiplicity (at least two). But by Rolle's theorem, $\psi_*'$ has an odd number of zeros in $I_j$. Therefore, if $\psi_*$ is not of the form (2.8) (with $n = 0$), then (2.23) is valid. If $\psi_*$ is of the form (2.8) (with $n = 0$), then, in this case, (2.26) can be proved by the same method as above. Thus, we also have $Z_{\mathbb{C}}(\psi_*') < Z_{\mathbb{C}}(\psi_*)$ by (2.24) and (2.22).

Let $\psi_*$ have at least one real zero and let $\zeta_* \in I_{-\infty}$ or $\zeta_* \in I_{+\infty}$. By the same reasoning as above, one can show that the inequality (2.26) holds in this case. So, we again obtain the inequality $Z_{\mathbb{C}}(\psi_*') < Z_{\mathbb{C}}(\psi_*)$ by Theorems 2.10–2.13.

Thus, we have shown that for a given $\varphi \in \mathcal{L} - \mathcal{P}^*$, there exists a real $\sigma_*$ such that the function $\psi_*(z) = e^{-\sigma_* z}\varphi(z)$ possesses *property A*. Additionally, if $Z_{\mathbb{R}}(Q) > 0$, then $Z_{\mathbb{C}}(\psi_*') < Z_{\mathbb{C}}(\psi_*)$, as required. □

**Remark 2.27.** For a given function $\varphi \in \mathcal{L} - \mathcal{P}^*$ the number $\sigma_*$ guaranteed by Theorem 2.26 is not unique. For example, one can find another number applying Theorem 2.26 to the function $\varphi(-z)$.

**Remark 2.28.** In the same way as in Theorem 2.26, one can prove that if, for a function $\varphi \in U_{2n}^*$ with $n \geqslant 1$, its associated function $Q[\varphi]$ has only finitely many real zeros, then there exists a real number $\sigma_*$ such that $\psi_*(z) = e^{-\sigma_* z}\varphi(z)$ possesses *property A*. This number is not unique. Generally speaking, Theorem 2.26 is not true for the classes $U_{2n}^*$ with $n \geqslant 1$.

We use the functions with *property A* and Theorem 2.26 in Sections 3.2 and 4.4

# Chapter 3

# Bounds on the number of real critical points of the logarithmic derivative on finite intervals

Let $\varphi$ be in $U_{2n}^*$ and let the function $Q$ be defined as in (2.11). In this chapter, we establish bounds on the number of zeros of the function $Q$ on finite intervals.

Section 3.1 is devoted to estimates of the number of zeros of $Q$ on intervals free of zeros of the functions $\varphi$, $\varphi'$ and $\varphi''$. In this section, we also prove a basic fact, Theorem 3.5, establishing an interrelation between the numbers $Z_{(a,b)}(Q)$ and $Z_{(a,b)}(Q_1)$, where $(a,b)$ is an interval such that $\varphi(z) \neq 0$, $\varphi'(z) \neq 0$ and $\varphi''(z) \neq 0$ for $z \in (a,b)$.

In Section 3.2, we establish bounds on the number of zeros of the function $Q$ on intervals free of zeros of $\varphi'$ and $\varphi$, on intervals free of zeros of $\varphi$ and on intervals with a unique zero of $\varphi$.

## 3.1 Intervals free of zeros of $\varphi$, $\varphi'$ and $\varphi''$

For a given function $\varphi \in U_{2n}^*$, by $F$ and $F_1$ we will denote the following functions

$$F(z) \stackrel{def}{=} \varphi(z)\varphi''(z) - (\varphi'(z))^2, \quad F_1(z) \stackrel{def}{=} \varphi'(z)\varphi'''(z) - (\varphi''(z))^2. \tag{3.1}$$

This denotation will be used throughout this section.

Our first result is about the number of zeros of $Q$ on a finite interval free of zeros of the functions $\varphi$, $\varphi'$, $\varphi''$ and $Q_1$.

**Lemma 3.1.** *Let $\varphi \in U_{2n}^*$ and let $a$ and $b$ be real and let $\varphi(z) \neq 0$, $\varphi'(z) \neq 0$, $\varphi''(z) \neq 0$, $Q_1(z) \neq 0$ in the interval $(a,b)$.*

I. *If, for all sufficiently small $\delta > 0$,*

$$\varphi'(a+\delta)\varphi''(a+\delta)Q(a+\delta)Q_1(a+\delta) > 0, \tag{3.2}$$

*then $Q$ has no zeros in $(a,b]$.*

II. *If, for all sufficiently small $\delta > 0$,*

$$\varphi'(a+\delta)\varphi''(a+\delta)Q(a+\delta)Q_1(a+\delta) < 0, \tag{3.3}$$

*then $Q$ has at most one zero in $(a, b)$, counting multiplicities. Moreover, if $Q(\zeta) = 0$ for some $\zeta \in (a, b)$, then $Q(b) \neq 0$ (if $Q$ is finite at $b$).*

**Proof.** The condition $\varphi(z) \neq 0$ for $z \in (a, b)$ means that $Q$ is finite at every point of $(a, b)$.
If $\zeta \in (a, b)$ and $Q(\zeta) = 0$, then $F(\zeta) = 0$ and (3.1) implies

$$\varphi'(\zeta) = \frac{\varphi(\zeta)\varphi''(\zeta)}{\varphi'(\zeta)}. \tag{3.4}$$

Now we consider $F_1$. From (3.1) and (3.4) it is easy to derive that

$$\begin{aligned} F_1(\zeta) = \varphi'(\zeta)\varphi'''(\zeta) - (\varphi''(\zeta))^2 &= \frac{\varphi(\zeta)\varphi''(\zeta)\varphi'''(\zeta)}{\varphi'(\zeta)} - (\varphi''(\zeta))^2 = \\ = \frac{\varphi''(\zeta)}{\varphi'(\zeta)}\left[\varphi(\zeta)\varphi'''(\zeta) - \varphi'(\zeta)\varphi''(\zeta)\right] &= \frac{\varphi''(\zeta)}{\varphi'(\zeta)}F'(\zeta). \end{aligned} \tag{3.5}$$

Since $\varphi'(z) \neq 0, \varphi''(z) \neq 0, Q_1(z) \neq 0$ (and therefore $F_1(z) \neq 0$) in $(a, b)$ by assumption, from (3.5) it follows that $F'(\zeta) \neq 0$. That is, all zeros of $Q$ in $(a, b)$ are simple.

I. Let the inequality (3.2) hold. Assume that, for all sufficiently small $\delta > 0$,

$$\varphi'(a+\delta)\varphi''(a+\delta)Q_1(a+\delta) > 0, \tag{3.6}$$

then $Q(a+\delta) > 0$, that is, $F(a+\delta) > 0$. Therefore, if $\zeta$ is the leftmost zero of $Q$ in $(a, b)$, then $F'(\zeta) < 0$. This inequality contradicts (3.5), since

$$\varphi'(z)\varphi''(z)Q_1(z) > 0$$

for $z \in (a, b)$, which follows from (3.6) and from the assumption of the lemma. Consequently, $Q$ cannot have zeros in the interval $(a, b)$ if the inequalities (3.2) and (3.6) hold. In the same way, one can prove that if $\varphi'(a+\delta)\varphi''(a+\delta)Q_1(a+\delta) < 0$ for all sufficiently small $\delta > 0$ and if the inequality (3.2) hold, then $Q(z) \neq 0$ for $z \in (a, b)$.

Thus, $Q$ has no zeros in the interval $(a, b)$ if the inequality (3.2) holds. Moreover, it is easy to show that $Q(b) \neq 0$ as well. In fact, if $F(b) = 0$, then, for all sufficiently small $\varepsilon > 0$,

$$\operatorname{sign}\left(\frac{\varphi'(b-\varepsilon)}{\varphi''(b-\varepsilon)}F_1(b-\varepsilon)\right) = \operatorname{sign}(F'(b-\varepsilon)) \tag{3.7}$$

according to (3.5). But if the inequality (3.2) holds, then

$$\operatorname{sign}\left(\frac{\varphi'(b-\varepsilon)}{\varphi''(b-\varepsilon)}F_1(b-\varepsilon)\right) = \operatorname{sign}(F(b-\varepsilon)) \tag{3.8}$$

for all sufficiently small $\varepsilon > 0$, since $\varphi'(z) \neq 0$, $\varphi''(z) \neq 0$, $Q_1(z) \neq 0$ in the interval $(a, b)$ by assumption and since $Q(z) \neq 0$ in $(a, b)$, which was proved above. So, if the inequality (3.2) holds and if $F(b) = 0$, then from (3.7) and (3.8) we obtain that

$$F(b-\varepsilon)F'(b-\varepsilon) > 0$$

for all sufficiently small $\varepsilon > 0$. This inequality contradicts the analyticity[1] of the function $F$. Therefore, if the inequality (3.2) holds and if $Q$ is finite at the point $b$, then $Q(b) \neq 0$. Thus, the first part of the lemma is proved.

II. Let the inequality (3.3) hold, then $Q$ may have zeros in $(a, b)$. But it cannot have more than one zero. In fact, if $\zeta$ is the leftmost zero of $Q$ in $(a, b)$, then this zero is simple as we proved above. Therefore the following inequality holds for all sufficiently small $\varepsilon > 0$

$$\varphi'(\zeta+\varepsilon)\varphi''(\zeta+\varepsilon)Q(\zeta+\varepsilon)Q_1(\zeta+\varepsilon) > 0.$$

Consequently, $Q$ has no zeros in $(\zeta, b]$ according to Case I of the lemma. □

**Remark 3.2.** Lemma 3.1 is also true if $(a, b)$ is a half-infinite interval, that is, $(a, +\infty)$ or $(-\infty, b)$.

Thus, we have found out that $Q$ has at most one real zero, counting multiplicity, in an interval where the functions $\varphi$, $\varphi'$, $\varphi''$ and $Q_1$ have no real zeros. Now we study the multiple zeros of $Q$ and its zeros common with one of the above-mentioned functions. From (2.11) it follows that all zeros of $\varphi'$ of multiplicity at least 2 are also zeros of $Q$ and all zeros of $\varphi'$ of multiplicity at least 3 are multiple zeros of $Q$. The following lemma provides information about common zeros of $Q$ and $Q_1$.

**Lemma 3.3.** *Let $\varphi \in U^*_{2n}$ and let $a$ and $b$ be real and let $\varphi(z) \neq 0$, $\varphi'(z) \neq 0$, $\varphi''(z) \neq 0$ in the interval $(a, b)$. Suppose that $Q_1$ has a unique zero $\xi \in (a, b)$ of multiplicity $M$ in $(a, b)$. If $Q(\xi) = 0$, then $\xi$ is a zero of $Q$ of multiplicity $M+1$, and $Q(z) \neq 0$ for $z \in (a, \xi) \cup (\xi, b]$.*

**Proof.** The condition $\varphi(z) \neq 0$ for $z \in (a, b)$ means that $Q$ is finite at every point of $(a, b)$.

By assumption, $\xi$ is a zero of $F_1$ of multiplicity $M$ and $F(\xi) = 0$. First, we prove that $\xi$ is a zero of $F$ of multiplicity $M + 1$.

Since $\varphi'(z) \neq 0$ for $z \in (a, b)$ by assumption, from (3.1) it follows that

$$\varphi'(z) = \frac{\varphi(z)\varphi''(z)}{\varphi'(z)} - \frac{F(z)}{\varphi'(z)}.$$

Substituting this expression into the formula (3.1), we obtain

$$\begin{aligned} F_1(z) = \frac{\varphi(z)\varphi''(z)\varphi'''(z)}{\varphi'(z)} - \frac{\varphi'(z)\varphi''(z)\varphi''(z)}{\varphi'(z)} - \frac{F(z)\varphi'''(z)}{\varphi'(z)} = \\ = \frac{\varphi''(z)}{\varphi'(z)} \cdot F'(z) - \frac{F(z)\varphi'''(z)}{\varphi'(z)}, \end{aligned}$$

[1] If a function $f$ is analytic at some neighbourhood of a real point $a$ and equals zero at this point, then, for all sufficiently small $\varepsilon > 0$,

$$f(a-\varepsilon)f'(a-\varepsilon) < 0.$$

or, equivalently,

$$\frac{\varphi'(z)}{[\varphi''(z)]^2}F_1(z)=\left(\frac{F(z)}{\varphi''(z)}\right)',$$

since $\varphi''(z)\neq 0$ for $z\in(a,b)$ by assumption. Differentiating this equality $j$ times with respect to $z$, we get

$$\left(\frac{\varphi'(z)}{[\varphi''(z)]^2}F_1(z)\right)^{(j)}=\left(\frac{F(z)}{\varphi''(z)}\right)^{(j+1)}. \tag{3.9}$$

From (3.9) it follows that $F^{(j+1)}(\xi)=0$ if $\varphi'(\xi)\neq 0$, $\varphi''(\xi)\neq 0$, $F_1^{(i)}(\xi)=0$ and $F^{(i)}(\xi)=0$, $i=0,1,\dots,j$. Consequently, $\xi$ is a zero of $F$ of multiplicity at least $M+1$. But by assumptions, the formula (3.9) gives the following equality

$$\varphi'(\xi)F_1^{(M+1)}(\xi)=\varphi''(\xi)F^{(M+2)}(\xi)\neq 0.$$

Hence, $\xi$ is a zero of $F$ of multiplicity exactly $M+1$. But $\varphi(\xi)\neq 0$ by assumption, therefore, $\xi$ is a zero of $Q$ of multiplicity $M+1$.

It remains to prove that $Q$ has no zeros in $(a,b]$ except $\xi$. In fact, consider the interval $(a,\xi)$. According to Lemma 3.1, $Q$ can have a zero at $\xi$ only if the inequality (3.3) holds and $Q(z)\neq 0$ for $z\in(a,\xi)$. Furthermore, the function $\varphi'\varphi''$ does not change its sign at $\xi$ but the function $QQ_1$ does, since $\xi$ is a zero of $QQ_1$ of multiplicity $2M+1$. Thus, for all sufficiently small $\delta>0$,

$$\varphi'(\xi+\delta)\varphi''(\xi+\delta)Q(\xi+\delta)Q_1(\xi+\delta)>0, \tag{3.10}$$

since the inequality (3.3) must hold in the interval $(a,\xi)$ by Lemma 3.1. From (3.10) it follows that Case I of Lemma 3.1 holds in the interval $(\xi,b)$, so $Q(z)\neq 0$ for $z\in(\xi,b]$. □

**Remark 3.4.** Lemma 3.3 remains valid if $(a,b)$ is a half-infinite interval, that is, $(a,+\infty)$ or $(-\infty,b)$.

Now combining the previous two lemmata, we provide a general bound on the number of zeros of $Q$ in terms of the number of zeros of $Q_1$ in a given interval.

**Theorem 3.5.** *Let $\varphi\in U_{2n}^*$ and let $a$ and $b$ be real. If $\varphi(z)\neq 0$, $\varphi'(z)\neq 0$ and $\varphi''(z)\neq 0$ for $z\in(a,b)$, then*

$$Z_{(a,b)}(Q)\leqslant 1+Z_{(a,b)}(Q_1). \tag{3.11}$$

**Proof.** If $\varphi(z)\varphi''(z)<0$ in $(a,b)$, then $Q(z)<0$ for $z\in(a,b)$ by (2.11), i.e., $Z_{(a,b)}(Q)=0$. Therefore, the inequality (3.11) holds automatically in this case.

Let now $\varphi(z)\varphi''(z)>0$ for $z\in(a,b)$. If $Q_1(z)\neq 0$ in $(a,b)$, then, by Lemma 3.1, $Q$ has at most one zero, counting multiplicity, in $(a,b)$. Therefore, (3.11) also holds in this case.

If $Q_1$ has a unique zero $\xi$ in $(a,b)$ and $Q(\xi)\neq 0$, then, by Lemma 3.1, $Q$ has at most one zero in each of the intervals $(a,\xi)$ and $(\xi,b)$:

$$Z_{(a,\xi)}(Q)\leqslant 1+Z_{(a,\xi)}(Q_1), \tag{3.12}$$

where $Z_{(a,\xi)}(Q_1) = 0$, and

$$Z_{(\xi,b)}(Q) \leqslant 1 + Z_{(\xi,b)}(Q_1), \tag{3.13}$$

where $Z_{(\xi,b)}(Q_1) = 0$. But since $Q(\xi) \neq 0$ and $Q_1(\xi) = 0$, we have

$$0 = Z_{\{\xi\}}(Q) \leqslant -1 + Z_{\{\xi\}}(Q_1). \tag{3.14}$$

Thus, summing the inequalities (3.12)–(3.14), we obtain (3.11).

If $Q_1$ has a unique zero $\xi$ in $(a, b)$ and $Q(\xi) = 0$, then, by Lemma 3.3, we have

$$Z_{\{\xi\}}(Q) = 1 + Z_{\{\xi\}}(Q_1),$$

and $Q(z) \neq 0$ for $z \in (a, \xi) \cup (\xi, b)$. Therefore, the inequality (3.11) is also true in this case.

Finally, let $Q_1$ have exactly $r \geqslant 2$ *distinct* zeros, say $\xi_1 < \xi_2 < \ldots < \xi_r$, in the interval $(a, b)$. These zeros divide $(a, b)$ into $r + 1$ subintervals. If, for some number $i$, $1 \leqslant i \leqslant r$, $Q(\xi_i) \neq 0$ and $Q(\xi_{i-1}) \neq 0$ (if $i \neq 1$), then, by Lemma 3.1, $Q$ has *at most* one zero, counting multiplicity, in $(\xi_{i-1}, \xi_i]$ ($\xi_0 \overset{def}{=} a$). But $Q_1$ has *at least* one zero in $(\xi_{i-1}, \xi_i]$, counting multiplicities (at the point $\xi_i$). Consequently,

$$Z_{(\xi_{i-1},\xi_i]}(Q) \leqslant Z_{(\xi_{i-1},\xi_i]}(Q_1) \tag{3.15}$$

If, for some number $i$, $1 \leqslant i \leqslant r - 1$, $Q(\xi_i) = 0$ and $\xi_i$ is a zero of $Q_1$ of multiplicity $M$, then, by Lemma 3.3, $Q$ has only one zero $\xi_i$ of multiplicity $M + 1$ in $(\xi_{i-1}, \xi_{i+1}]$. But in the interval $(\xi_{i-1}, \xi_{i+1}]$, $Q_1$ has *at least* $M + 1$ zeros, counting multiplicities (namely, $\xi_i$ which is a zero of multiplicity $M$, and $\xi_{i+1}$). Therefore, in this case, the following inequality holds

$$Z_{(\xi_{i-1},\xi_{i+1}]}(Q) \leqslant Z_{(\xi_{i-1},\xi_{i+1}]}(Q_1) \tag{3.16}$$

Thus, if $Q(\xi_r) \neq 0$, then from (3.15)–(3.16) it follows that

$$Z_{(a,\xi_r]}(Q) \leqslant Z_{(a,\xi_r]}(Q_1). \tag{3.17}$$

But by Lemma 3.1, $Q$ has *at most* one real zero, counting multiplicity, in the interval $(\xi_r, b)$. Consequently, if $Q(\xi_r) \neq 0$, then the inequality (3.11) is valid.

If $Q(\xi_r) = 0$, then, by Lemma 3.3, $Q(\xi_{r-1}) \neq 0$ (otherwise, $\xi_r$ cannot be a zero of $Q$) and from (3.15)–(3.16) it follows that

$$Z_{(a,\xi_{r-1}]}(Q) \leqslant Z_{(\xi_a,\xi_{r-1}]}(Q_1). \tag{3.18}$$

Now Lemma 3.3 implies

$$Z_{(\xi_{r-1},b)}(Q) = 1 + Z_{(\xi_{r-1},b)}(Q_1), \tag{3.19}$$

therefore, the inequality (3.11) follows from (3.18)–(3.19). □

**Remark 3.6.** Theorem 3.5 is also true if $(a, b)$ is a half-infinite interval, that is, $(a, +\infty)$ or $(-\infty, b)$ (see Remarks 3.2 and 3.4).

The inequality (3.11) is a very important result. Per se, all subsequent inequalities are consequences of this inequality and Rolle's theorem.

## 3.2 Intervals between zeros of $\varphi$, $\varphi'$, and $\varphi''$

At first, we estimate the parities of the number of real zeros of $Q$ in certain intervals.

**Lemma 3.7.** *Let $\varphi \in U_{2n}^*$ and let $\beta_1$ and $\beta_2$ be two real zeros of $\varphi'$.*

I. *If $\beta_1$ and $\beta_2$ are consecutive zeros of $\varphi'$, and $\varphi(z) \neq 0$ for $z \in (\beta_1, \beta_2)$, then $Q$ has an odd number of zeros in $(\beta_1, \beta_2)$, counting multiplicities.*

II. *If $\beta_1$ and $\beta_2$ are two zeros of $\varphi'$ such that $\varphi$ has a unique zero $\alpha$ in $(\beta_1, \beta_2)$ and $\varphi'(z) \neq 0$ for $z \in (\beta_1, \alpha) \cup (\alpha, \beta_2)$, then $Q$ has an even number of zeros, counting multiplicities, in each of the intervals $(\beta_1, \alpha)$ and $(\alpha, \beta_2)$.*

**Proof.** I. In fact, since $\varphi'/\varphi$ equals zero at the points $\beta_1$ and $\beta_2$, its derivative, the function $Q$, has an odd number of zeros in $(\beta_1, \beta_2)$ by Rolle's theorem.

II. According to Proposition 2.18 (see (2.16)), $\varphi'/\varphi$ is decreasing in a small vicinity of $\alpha$, and, therefore, $\varphi'(z)/\varphi(z) \to -\infty$ whenever $z \nearrow \alpha$. Since $\varphi'/\varphi$ equals zero at $\beta_1$, its derivative, the function $Q$, must have an even number of zeros in $(\beta_1, \alpha)$ by Rolle's theorem.

By the same argumentation, $Q$ has an even number of zeros in $(\alpha, \beta_2)$. $\square$

We now embark on a more detailed analysis of the zeros of $\varphi$, $\varphi'$, and $\varphi''$. In the following lemma, we consider an arbitrary pair of zeros of $\varphi''$. According to Notation 2.17, we denote them by $\gamma^{(1)}$, $\gamma^{(2)}$.

**Lemma 3.8.** *Let $\varphi \in U_{2n}^*$ and let $\gamma^{(1)}$ and $\gamma^{(2)}$, $\gamma^{(1)} < \gamma^{(2)}$, be real zeros of $\varphi''$ such that $\varphi(z) \neq 0$ and $\varphi'(z) \neq 0$ for $z \in [\gamma^{(1)}, \gamma^{(2)}]$ and suppose that $\varphi''$ has exactly $q \geqslant 2$ zeros, counting multiplicities, in the interval $[\gamma^{(1)}, \gamma^{(2)}]$. Then $Q$ has an even number of zeros in $[\gamma^{(1)}, \gamma^{(2)}]$ and one of the following hold:*

I. *If $q$ is an odd number, then*

$$0 \leqslant Z_{[\gamma^{(1)},\gamma^{(2)}]}(Q) \leqslant Z_{[\gamma^{(1)},\gamma^{(2)}]}(Q_1). \tag{3.20}$$

II. *If $q$ is an even number and $\varphi(\gamma^{(1)} - \varepsilon)\varphi''(\gamma^{(1)} - \varepsilon) > 0$ for all sufficiently small $\varepsilon > 0$, then*

$$0 \leqslant Z_{[\gamma^{(1)},\gamma^{(2)}]}(Q) \leqslant -1 + Z_{[\gamma^{(1)},\gamma^{(2)}]}(Q_1). \tag{3.21}$$

III. *If $q$ is an even number and $\varphi(\gamma^{(1)} - \varepsilon)\varphi''(\gamma^{(1)} - \varepsilon) < 0$ for all sufficiently small $\varepsilon > 0$, then*

$$0 \leqslant Z_{[\gamma^{(1)},\gamma^{(2)}]}(Q) \leqslant 1 + Z_{[\gamma^{(1)},\gamma^{(2)}]}(Q_1). \tag{3.22}$$

**Proof.** We assume that $\varphi''$ has exactly $r \leqslant q$ *distinct* zeros in the interval $[\gamma^{(1)}, \gamma^{(2)}]$, say $\gamma^{(1)} = \gamma_1 < \gamma_2 < \ldots < \gamma_{r-1} < \gamma_r = \gamma^{(2)}$. Below in the proof, we denote the interval $[\gamma^{(1)}, \gamma^{(2)}]$ by $[\gamma_1, \gamma_r]$.

From (2.11) it follows that

$$Q(\gamma_i) = -\frac{\left(\varphi'(\gamma_i)\right)^2}{\left(\varphi(\gamma_i)\right)^2} < 0, \tag{3.23}$$

since $\varphi(z) \neq 0$ and $\varphi'(z) \neq 0$ in $[\gamma_1, \gamma_r]$ by assumption. Thus, $Q$ has an even number of zeros in the interval $[\gamma_1, \gamma_r]$.

I. Let $q$ be an odd number.

If $\varphi(z)\varphi''(z) \leqslant 0$ for $z \in [\gamma_1, \gamma_r]$, then, by (2.11), we have the inequalities

$$0 = Z_{[\gamma_1,\gamma_r]}(Q) \leqslant Z_{[\gamma_1,\gamma_r]}(Q_1),$$

which are exactly (3.20).

If, for some $i$, $2 \leqslant i \leqslant r$, $\varphi(z)\varphi''(z) > 0$ in the interval $(\gamma_{i-1}, \gamma_i)$, then, by Theorem 3.5, we have

$$Z_{(\gamma_{i-1},\gamma_i)}(Q) \leqslant 1 + Z_{(\gamma_{i-1},\gamma_i)}(Q_1). \tag{3.24}$$

Moreover, there are only two possibilities:

I.1. The number $\gamma_i$ is a zero of $\varphi''$ of *even* multiplicity (at least two). Then according to (2.12), $\gamma_i$ is a zero of $Q_1$ of multiplicity *at least* one, but $Q(\gamma_i) \neq 0$ by (3.23). From these facts and from the inequality (3.24) we obtain

$$Z_{(\gamma_{i-1},\gamma_i]}(Q) \leqslant Z_{(\gamma_{i-1},\gamma_i]}(Q_1). \tag{3.25}$$

For the sequel, we notice that, in this case, $\varphi''$ has an *even* number of zeros, counting multiplicities, in the interval $(\gamma_{i-1}, \gamma_i]$.

I.2. The number $\gamma_i$, $i < r$, is a zero of $\varphi''$ of *odd* multiplicity (at least one). Then $\varphi(z)\varphi''(z)$ is nonpositive in $[\gamma_i, \gamma_{i+1}]$, so $Q(z) \neq 0$ in this interval by (2.11). But $Q_1$ has an odd number (at least one) of real zeros in $(\gamma_i, \gamma_{i+1})$ according to Lemma 3.7 applied to $Q_1$, therefore,

$$0 = Z_{(\gamma_i,\gamma_{i+1}]}(Q) \leqslant Z_{(\gamma_i,\gamma_{i+1}]}(Q_1) - 1. \tag{3.26}$$

This inequality and the inequality (3.24) imply

$$Z_{(\gamma_{i-1},\gamma_{i+1}]}(Q) \leqslant Z_{(\gamma_{i-1},\gamma_{i+1}]}(Q_1). \tag{3.27}$$

Suppose that $\gamma_j$, $i + 1 \leqslant j \leqslant r$, is the next zero of $\varphi''$ of odd multiplicity, i.e., we assume that the multiplicity of each of $\gamma_{i+1}, \ldots, \gamma_{j-1}$ is even, while the multiplicity of $\gamma_j$ is odd. Then we have $\varphi(z)\varphi''(z) \leqslant 0$ and, therefore, $Q(z) < 0$ for $z \in (\gamma_{i+1}, \gamma_j]$ according to (2.11). But by Lemma 3.7 applied to $Q_1$ on the intervals $(\gamma_{i+1}, \gamma_{i+2}), \ldots, (\gamma_{j-1}, \gamma_j)$, $Q_1$ has *at least* $j - i - 1$ zeros in $(\gamma_{i+1}, \gamma_j]$. This fact and the inequality (3.27) imply

$$Z_{(\gamma_{i-1},\gamma_j]}(Q) \leqslant Z_{(\gamma_{i-1},\gamma_j]}(Q_1). \tag{3.28}$$

As well as in Case I.1, we notice that $\varphi''$ has an *even* number of zeros, counting multiplicities, in the interval $(\gamma_{i-1}, \gamma_j]$. It follows from the fact that $\gamma_i$ and $\gamma_j$ are zeros of $\varphi''$ of odd multiplicities and $\gamma_{i+1}, \dots, \gamma_{j-1}$ are zeros of $\varphi''$ of even multiplicities.

Let $l$, $1 \leqslant l \leqslant r-1$, be an integer such that $\varphi(z)\varphi''(z)$ is nonpositive in the interval $[\gamma_1, \gamma_l]$ and positive in the interval $(\gamma_l, \gamma_{l+1})$. Then by (2.11), $Q(z) \neq 0$ for $z \in [\gamma_1, \gamma_l]$, and we have the following inequality

$$0 = Z_{[\gamma_1,\gamma_l]}(Q) \leqslant Z_{[\gamma_1,\gamma_l]}(Q_1), \tag{3.29}$$

since $Q_1$ has at least $l-1\,(\geqslant 0)$ zeros in $[\gamma_1, \gamma_l]$ by Lemma 3.7 applied to $Q_1$ on the intervals $(\gamma_1, \gamma_2), \dots, (\gamma_{l-1}, \gamma_l)$.

If $\gamma_r$ is a zero of $\varphi''$ of even multiplicity or if $\varphi(z)\varphi''(z) < 0$ in $(\gamma_{r-1}, \gamma_r)$, then the interval $(\gamma_l, \gamma_r]$ consists only of the subintervals described in Cases I.1 and I.2. Consequently, from the inequalities (3.25), (3.28), (3.29) we obtain

$$Z_{[\gamma_1,\gamma_r]}(Q) \leqslant Z_{[\gamma_1,\gamma_r]}(Q_1). \tag{3.30}$$

Let $\gamma_r$ be a zero of $\varphi''$ of odd multiplicity and let $\varphi(z)\varphi''(z) > 0$ in $(\gamma_{r-1}, \gamma_r)$ and let $l \geqslant 2$. Then the interval $(\gamma_l, \gamma_{r-1}]$ consists only of subintervals described in Cases I.1 and I.2 and, therefore,

$$Z_{(\gamma_l,\gamma_{r-1}]}(Q) \leqslant Z_{(\gamma_l,\gamma_{r-1}]}(Q_1). \tag{3.31}$$

Moreover, since $l \geqslant 2$ by assumption, we have $Q(z) \neq 0$ for $z \in [\gamma_1, \gamma_l]$, but $Q_1$ has *at least* one zero in $[\gamma_1, \gamma_l]$ as we mentioned above. Consequently, in this case, we can improve the inequality (3.29) to the following one:

$$0 = Z_{[\gamma_1,\gamma_l]}(Q) \leqslant Z_{[\gamma_1,\gamma_l]}(Q_1) - 1. \tag{3.32}$$

Furthermore, we have

$$Z_{(\gamma_{r-1},\gamma_r]}(Q) \leqslant 1 + Z_{(\gamma_{r-1},\gamma_r]}(Q_1) \tag{3.33}$$

by Theorem 3.5 and by the fact that $Q(\gamma_r) \neq 0$ (see (3.23)). Summing the inequalities (3.31)–(3.33), we again obtain (3.30).

At last, let $\gamma_r$ also be a zero of $\varphi''$ of odd multiplicity and let $\varphi(z)\varphi''(z) > 0$ in $(\gamma_{r-1}, \gamma_r)$, but let $l = 1$. In this case, by the same reasoning as above, the inequalities (3.31) (for $l = 1$) and (3.33) hold. Recalling the final remarks in Cases I.1 and I.2, we conclude that $\varphi''$ has an *even* number of zeros in the interval $(\gamma_1, \gamma_{r-1}]$. Since $\varphi''$ has an odd number $q$ of zeros in $[\gamma_1, \gamma_r]$ and $\gamma_r$ is a zero of $\varphi''$ of odd multiplicity by assumption, $\gamma_1$ must be a zero of $\varphi''$ of even multiplicity. But (2.12) shows that every zero of $\varphi''$ of even multiplicity is a zero of $Q_1$ of odd multiplicity. Consequently, $\gamma_1$ is a zero of $Q_1$ of multiplicity *at least* one, and we have

$$0 = Z_{\{\gamma_1\}}(Q) \leqslant Z_{\{\gamma_1\}}(Q_1) - 1. \tag{3.34}$$

Summing the inequalities (3.31), (3.33) and (3.34), we also obtain the inequality (3.30).

Thus, the number of zeros of $Q$ in the interval $[\gamma_1, \gamma_r]$ does not exceed the number of zeros of $Q_1$ in this interval. This implies the validity of the inequalities (3.20) where the lower bound cannot be improved, since the number of zeros of $Q$ in the interval $[\gamma_1, \gamma_r]$ is even as we showed above (see (3.23)).

II. Now let $q = 2M$ and $\varphi(\gamma_1 - \varepsilon)\varphi''(\gamma_1 - \varepsilon) > 0$ for all sufficiently small $\varepsilon > 0$.

At first, we assume that one of the $\gamma_i$, $i = 1, \dots, r$, is a zero of $\varphi''$ of odd multiplicity. Let $\gamma_j$, $1 \leqslant j \leqslant r-1$, be the closest to $\gamma_1$ (possibly $\gamma_1$ itself) zero of $\varphi''$ of odd multiplicity[2]. Since $\varphi''$ has an odd number of zeros in the interval $[\gamma_1, \gamma_j]$, $\varphi''$ has different signs in the interval $(\gamma_j, \gamma_{j+1})$ and in the left-sided neighbourhood of $\gamma_1$, where $\varphi(z)\varphi''(z)$ is positive by assumption. Consequently, $\varphi(z)\varphi''(z)$ is negative in $(\gamma_j, \gamma_{j+1})$, because $\varphi(z) \neq 0$ for $z \in [\gamma_1, \gamma_r]$ by assumption. Hence, according to (2.11), $Q(z) \neq 0$ in the interval $(\gamma_j, \gamma_{j+1})$, and we have

$$0 = Z_{(\gamma_j, \gamma_{j+1})}(Q) \leqslant -1 + Z_{(\gamma_j, \gamma_{j+1})}(Q_1), \tag{3.35}$$

since $Q_1$ has an odd number (at least one) of zeros, counting multiplicities, in $(\gamma_j, \gamma_{j+1})$ by Lemma 3.7 applied to $Q_1$ on the interval $(\gamma_j, \gamma_{j+1})$. On the other hand, inasmuch as $\varphi''$ has an odd number of zeros, counting multiplicities, in both the intervals $[\gamma_1, \gamma_j]$ and $[\gamma_{j+1}, \gamma_r]$, it follows from Case I that[3]

$$0 \leqslant Z_{[\gamma_1, \gamma_j]}(Q) \leqslant Z_{[\gamma_1, \gamma_j]}(Q_1), \tag{3.36}$$

$$0 \leqslant Z_{[\gamma_{j+1}, \gamma_r]}(Q) \leqslant Z_{[\gamma_{j+1}, \gamma_r]}(Q_1). \tag{3.37}$$

The inequalities (3.35)–(3.37) together give the inequality (3.21).

Now we assume that all $\gamma_i$, $i = 1, \dots, r$, are zeros of $\varphi''$ of even multiplicities. Therefore,

$$1 \leqslant r \leqslant M \tag{3.38}$$

and $\varphi''$ has equal signs in all intervals $(\gamma_i, \gamma_{i+1})$, $i = 1, \dots, r-1$. Since $\varphi(z)\varphi''(z)$ is positive in a small left-sided neighbourhood of $\gamma_1$ by assumption, it is positive in each of the intervals $(\gamma_i, \gamma_{i+1})$, $i = 1, \dots, r-1$. Thus, we have

$$0 \leqslant Z_{X_1}(Q) \leqslant r - 1 + Z_{X_1}(Q_1) \leqslant M - 1 + Z_{X_1}(Q_1), \tag{3.39}$$

where $X_1 = \bigcup_{i=1}^{r-1}(\gamma_i, \gamma_{i+1})$. Indeed, since $Q(\gamma_i) < 0$ for $i = 1, \dots, r$ (see (3.23)), $Q$ has an even number of zeros in each of the interval $(\gamma_i, \gamma_{i+1})$. This fact implies the lower estimate in (3.39). The upper bound follows from Theorem 3.5 applied to the intervals $(\gamma_i, \gamma_{i+1})$

[2]Since $\varphi''$ has an *even* number of real zeros in $[\gamma_1, \gamma_r]$ by assumption, there is always *even* number of zeros of $\varphi''$ of odd multiplicities in $[\gamma_1, \gamma_r]$. Thus, $\gamma_r$ cannot be the closest to $\gamma_1$ (and thereby the only) zero of $\varphi''$ of odd multiplicity.

[3]If $\gamma_j = \gamma_1$ and $\gamma_1$ is a zero of $\varphi''$ of multiplicity $K > 1$, then $\gamma_1$ is a zero of $Q_1$ of multiplicity $K - 1$ according to (2.12). If $\gamma_1$ is a simple zero of $\varphi''$, then $\gamma_1$ is not a zero of $Q_1$. In both cases, (3.36) reduces to the following inequality

$$0 = Z_{\{\gamma_1\}}(Q) \leqslant Z_{\{\gamma_1\}}(Q_1).$$

and from (3.38). Further, $Q_1$ has exactly $2M - r$ zeros, counting multiplicities, among the points $X_2 = \bigcup_{i=1}^{r}\{\gamma_i\}$. Consequently, by (3.38), we have

$$Z_{X_2}(Q_1) = 2M - r \geqslant M. \tag{3.40}$$

Now we note that $[\gamma_1, \gamma_r] = X_1 \bigcup X_2$ and $Z_{X_2}(Q) = 0$. Therefore, from (3.39)–(3.40) it follows that

$$0 \leqslant Z_{[\gamma_1,\gamma_r]}(Q) = Z_{X_1}(Q) \leqslant M - 1 + Z_{X_1}(Q_1) =$$

$$= M - 1 + Z_{[\gamma_1,\gamma_r]}(Q_1) - Z_{X_2}(Q_1) \leqslant -1 + Z_{[\gamma_1,\gamma_r]}(Q_1),$$

where the lower bound cannot be improved, since $Q$ has an even number of zeros in the interval $[\gamma_1, \gamma_r]$ as we showed above (see (3.23)). So, the inequalities (3.21) are also valid in this case.

III. At last, let $q = 2M$ and let

$$\varphi(\gamma_1 - \varepsilon)\varphi''(\gamma_1 - \varepsilon) < 0 \tag{3.41}$$

for sufficiently small $\varepsilon > 0$. If $\varphi''$ has zeros of odd multiplicities in the interval $[\gamma_1, \gamma_r]$, then this case differs from Case II only by the sign of the function $\varphi\varphi''$ in the interval $(\gamma_j, \gamma_{j+1})$, where $\gamma_j$ is the closest to $\gamma_1$ (possibly $\gamma_1$ itself) zero of $\varphi''$ of odd multiplicity. Thus, now we have $\varphi(z)\varphi''(z) > 0$ in $(\gamma_j, \gamma_{j+1})$, since $\varphi''$ has an odd number of zeros in $[\gamma_1, \gamma_j]$ (see (3.41)). Therefore, instead of the inequalities (3.35) of Case II, we have

$$0 \leqslant Z_{(\gamma_j,\gamma_{j+1})}(Q) \leqslant 1 + Z_{(\gamma_j,\gamma_{j+1})}(Q_1) \tag{3.42}$$

by Theorem 3.5. As in Cases I and II, the lower bound follows from (3.23). The inequalities (3.36), (3.37) and (3.42) imply (3.22).

If all $\gamma_i$, $i = 1, \dots, r$, are zeros of $\varphi''$ of even multiplicities, then the inequality (3.41) implies $\varphi(z)\varphi''(z) \leqslant 0$ in $[\gamma_1, \gamma_r]$, so we have

$$0 = Z_{[\gamma_1,\gamma_r]}(Q) \leqslant Z_{[\gamma_1,\gamma_r]}(Q_1), \tag{3.43}$$

which also gives (3.22). □

Further, we derive an important bound on the number of real zeros of $Q$ in terms of the number of real zeros of $Q_1$ between two consecutive zeros of $\varphi'$.

**Theorem 3.9.** *Let $\varphi \in U_{2n}^*$. If $\beta_1$ and $\beta_2$, $\beta_1 < \beta_2$, are consecutive real zeros of $\varphi'$ and $\varphi(z) \neq 0$ for $z \in (\beta_1, \beta_2)$, then*

$$1 \leqslant Z_{(\beta_1,\beta_2)}(Q) \leqslant 1 + Z_{(\beta_1,\beta_2)}(Q_1). \tag{3.44}$$

**Proof.** By Lemma 3.7, $Q$ has an odd number of zeros in the interval $(\beta_1, \beta_2)$, therefore,

$$1 \leqslant Z_{(\beta_1,\beta_2)}(Q). \tag{3.45}$$

According to Rolle's theorem, $\varphi''$ has an odd number of real zeros, counting multiplicities, in the interval $(\beta_1, \beta_2)$. Let $\gamma_S$ and $\gamma_L$ be the smallest and the largest zeros of $\varphi''$ in the $(\beta_1, \beta_2)$. If $\gamma_S < \gamma_L$, then from Case I of Lemma 3.8 it follows that

$$0 \leqslant Z_{[\gamma_S,\gamma_L]}(Q) \leqslant Z_{[\gamma_S,\gamma_L]}(Q_1). \tag{3.46}$$

If $\gamma_S = \gamma_L$, then (see (3.23))

$$0 = Z_{\{\gamma_S\}}(Q) \leqslant Z_{\{\gamma_S\}}(Q_1). \tag{3.47}$$

Without loss of generality, we may assume that the function $\varphi(z)\varphi''(z)$ is positive in the interval $(\beta_1, \gamma_S)$. Then $\varphi(z)\varphi''(z) < 0$ for $z \in (\gamma_L, \beta_2)$, since $\varphi''$ has an odd number of zeros in $[\gamma_S, \gamma_L]$ and $\varphi(z) \neq 0$ in $(\beta_1, \beta_2)$ by assumption. Thus, by (2.11), we have

$$0 = Z_{(\gamma_L,\beta_2)}(Q) \leqslant Z_{(\gamma_L,\beta_2)}(Q_1). \tag{3.48}$$

We cannot improve this inequality, since $Q_1$ has an even number of zeros in $(\gamma_L, \beta_2)$ by Lemma 3.7 applied to $Q_1$ on the interval $(\gamma_L, \beta_2)$. By Theorem 3.5,

$$Z_{(\beta_1,\gamma_S)}(Q) \leqslant 1 + Z_{(\beta_1,\gamma_S)}(Q_1). \tag{3.49}$$

The inequalities (3.45)–(3.49) imply (3.44). $\square$

The following corollary is a very useful generalization of Theorem 3.9.

**Corollary 3.10.** *Let $\varphi \in U_{2n}^*$ and let $\beta^{(1)}$ and $\beta^{(2)}$, $\beta^{(1)} \leqslant \beta^{(2)}$, be zeros of $\varphi'$ and let $\varphi(z) \neq 0$ for $z \in [\beta^{(1)}, \beta^{(2)}]$. If $\varphi'$ has exactly $q \geqslant 2$ real zeros, counting multiplicities, in the interval $[\beta^{(1)}, \beta^{(2)}]$, then*

$$q - 1 \leqslant Z_{[\beta^{(1)},\beta^{(2)}]}(Q) \leqslant q - 1 + Z_{[\beta^{(1)},\beta^{(2)}]}(Q_1). \tag{3.50}$$

**Proof.** In fact, if some real number $\beta$ is a zero of $\varphi'$ of multiplicity $M$, then $\beta$ is a zero of $Q$ of multiplicity $M - 1$ according to (2.11). Therefore, the following inequalities hold:

$$M - 1 \leqslant Z_{\{\beta\}}(Q) \leqslant M - 1 + Z_{\{\beta\}}(Q_1), \tag{3.51}$$

since $Z_{\{\beta\}}(Q_1) = 0$ by (2.12). Thus, if $\varphi'$ has exactly $l$, $1 \leqslant l \leqslant q$, *distinct* real zeros, say $\beta^{(1)} = \beta_1 < \beta_2 < \ldots < \beta_l = \beta^{(2)}$, in the interval $[\beta^{(1)}, \beta^{(2)}]$, then $Q$ has exactly $q - l$ real zeros at the points $\beta_i$, $i = 1, 2, \ldots, l$, so from (3.51) it follows

$$q - l \leqslant Z_{X_2}(Q) \leqslant q - l + Z_{X_2}(Q_1), \tag{3.52}$$

where $X_2 = \bigcup_{i=1}^{l}\{\beta_i\}$ and $Z_{X_2}(Q_1) = 0$ by (2.12).

If $l = 1$, then $\beta^{(1)} = \beta^{(2)}$ is a zero of $\varphi'$ of multiplicity $q$. By (3.52), we have the following inequalities

$$q - 1 \leqslant Z_{\{\beta^{(1)}\}}(Q) \leqslant q - 1 + Z_{\{\beta^{(1)}\}}(Q_1), \tag{3.53}$$

which are equivalent to (3.50), since $Z_{\{\beta^{(1)}\}}(Q) = Z_{[\beta^{(1)},\beta^{(2)}]}(Q)$ and $Z_{\{\beta^{(1)}\}}(Q_1) = Z_{[\beta^{(1)},\beta^{(2)}]}(Q_1)$ in this case.

Now let $l > 1$. Then the points $\beta_i$, $i = 1, 2, \dots, l$, divide the interval $(\beta_1, \beta_l) = (\beta^{(1)}, \beta^{(2)})$ into $l-1$ subintervals $(\beta_i, \beta_{i+1})$, $i = 1, 2, \dots, l-1$. Applying Theorem 3.9 to each of these subintervals, we get

$$l - 1 \leqslant Z_{X_1}(Q) \leqslant l - 1 + Z_{X_1}(Q_1), \tag{3.54}$$

where $X_1 = \bigcup_{i=1}^{l-1}(\beta_i, \beta_{i+1})$.

Since $X_1 \cup X_2 = [\beta^{(1)}, \beta^{(2)}]$, we obtain (3.50) by summing the inequalities (3.52) and (3.54) or, if $\beta^{(1)} = \beta^{(2)}$, directly from (3.53). □

We now analyze the relation between the number of real zeros of $Q$ and the number of real zeros of $Q_1$ in an interval adjacent to a zero of $\varphi$. This analysis justifies the introduction of *property A* (Definition 2.24).

**Lemma 3.11.** *Let $\varphi \in U_{2n}^*$ and let $\beta$ be a real zero of $\varphi'$ and let $\alpha > \beta$ be a real zero of $\varphi$ such that $\varphi'(z) \neq 0$ and $\varphi(z) \neq 0$ for $z \in (\beta, \alpha)$.*

I. *If $Q_1$ has an even number of zeros in the interval $(\beta, \alpha)$, then*

$$0 \leqslant Z_{(\beta,\alpha)}(Q) \leqslant Z_{(\beta,\alpha)}(Q_1). \tag{3.55}$$

II. *If $Q_1$ has an odd number of zeros in the interval $(\beta, \alpha)$, then*

$$0 \leqslant Z_{(\beta,\alpha)}(Q) \leqslant 1 + Z_{(\beta,\alpha)}(Q_1). \tag{3.56}$$

**Proof.** According to Lemma 3.7, $Q$ has an even number of zeros in the interval $(\beta, \alpha)$. In particular, $Q$ may have no zeros in this interval. Thus, the lower bounds for $Z_{(\beta_1,\alpha)}(Q)$ in (3.55) and in (3.56) are valid and cannot be improved.

I. Let $Q_1$ have an even number of zeros in $(\beta_1, \alpha)$ (or have no zeros at all in this interval).

I.1. At first, we assume that $\varphi''$ has an even number of zeros, counting multiplicities, in $(\beta, \alpha)$, say $\beta < \gamma_1 \leqslant \ldots \leqslant \gamma_{2M} < \alpha$, where $M \geqslant 0$. It is easy to see that, for sufficiently small $\varepsilon > 0$, we have the following inequality

$$\varphi(\beta + \varepsilon)\varphi''(\beta + \varepsilon) < 0. \tag{3.57}$$

In fact, if $\varphi(z)$ is positive (negative) in $(\beta, \alpha)$, then it is decreasing (increasing), since $\varphi(\alpha) = 0$. Consequently, $\varphi(z)\varphi'(z) < 0$ in the interval $(\beta, \alpha)$. By the similar reason, we have $\varphi'(\beta+\varepsilon)\varphi''(\beta+\varepsilon)>0$ for all sufficiently small $\varepsilon>0$. Thus, the inequality (3.57) is true.

Let $\varphi''$ have at least one zero in the interval $(\beta,\alpha)$, that is, let $M>0$, then from (3.57) it follows that $Q(z)\neq 0$ for $z\in(\beta,\gamma_1)$ (see (2.11)), so we have

$$0 = Z_{(\beta,\gamma_1)}(Q) \leqslant Z_{(\beta,\gamma_1)}(Q_1). \tag{3.58}$$

If $\gamma_1<\gamma_{2M}$, then by (3.57) and Case III of Lemma 3.8, we obtain

$$0 \leqslant Z_{[\gamma_1,\gamma_{2M}]}(Q) \leqslant 1 + Z_{[\gamma_1,\gamma_{2M}]}(Q_1). \tag{3.59}$$

Moreover, from (3.57) it also follows that $\varphi(z)\varphi''(z)<0$ in $(\gamma_{2M},\alpha)$, since $\varphi''$ has equal signs in the intervals $(\beta,\gamma_1)$ and $(\gamma_{2M},\alpha)$. Therefore,

$$0 = Z_{(\gamma_{2M},\alpha)}(Q) \leqslant Z_{(\gamma_{2M},\alpha)}(Q_1). \tag{3.60}$$

Let $\gamma_1=\gamma_{2M}$, that is, let $\gamma_1$ be a unique zero of $\varphi''$ of multiplicity $2M$ in $(\beta,\alpha)$. Since $Z_{\{\gamma_1\}}(Q)=0$ by (3.23) and $Q_1$ has a zero of multiplicity $2M-1$ at the point $\gamma_1$ (see (2.12)), we obtain the following inequality

$$0 = Z_{\{\gamma_1\}}(Q) \leqslant 1 + Z_{\{\gamma_1\}}(Q_1) = 2M. \tag{3.61}$$

Likewise, the inequality (3.60) holds for the same reasoning as in the case $\gamma_1<\gamma_{2M}$.

Thus, the inequalities (3.58)–(3.61) imply

$$0 \leqslant Z_{(\beta,\alpha)}(Q) \leqslant 1 + Z_{(\beta,\alpha)}(Q_1), \tag{3.62}$$

which is equivalent to (3.55), since the number $Z_{(\beta,\alpha)}(Q)$ is even by Lemma 3.7 and $Z_{(\beta,\alpha)}(Q_1)$ is an even number by assumption.

At last, let $\varphi''(z)\neq 0$ for $z\in(\beta,\alpha)$. Then from (3.57) and (2.11) it follows that

$$0 = Z_{(\beta,\alpha)}(Q) \leqslant Z_{(\beta,\alpha)}(Q_1), \tag{3.63}$$

which is exactly (3.55).

Since $Q_1$ has an even number of zeros in the interval $(\beta,\gamma_1)$ by Lemma 3.7 and $Q_1$ also has an even number of zeros in $(\beta,\alpha)$ by assumption, $Q_1$ may have no zeros in these intervals[4]. Consequently, the inequalities (3.58) and (3.63) cannot be improved.

[4] The function $Q_1$ can have no zeros in $(\beta,\alpha)$ only if $\varphi''$ has no zeros in this interval. Otherwise, $Q_1$ has at least one zero in $(\beta,\alpha)$ by Lemma 3.7 applied to $Q_1$, since $\varphi''$ has an even number of zeros in this interval by assumption.

I.2. Now we assume that $\varphi''$ has an odd number of zeros, counting multiplicities, in the interval $(\beta,\alpha)$, say $\beta<\gamma_1\leqslant\ldots\leqslant\gamma_{2M+1}<\alpha$, where $M\geqslant 0$. As in Case I.1, the inequalities (3.57) and (3.58) hold.

If $\gamma_1<\gamma_{2M+1}$, then from Case I of Lemma 3.8 it follows that

$$0\leqslant Z_{[\gamma_1,\gamma_{2M+1}]}(Q)\leqslant Z_{[\gamma_1,\gamma_{2M+1}]}(Q_1), \tag{3.64}$$

If $\gamma_1=\gamma_{2M+1}$, that is, $\gamma_1$ is a unique zero of $\varphi''$ of multiplicity $2M+1$ in the interval $(\beta,\alpha)$, then $Z_{\{\gamma_1\}}(Q)=0$ (see (3.23)) and $Q_1$ has a zero of multiplicity $2M$ at the point $\gamma_1$ (see (2.12)). Therefore, we have

$$0=Z_{\{\gamma_1\}}(Q)\leqslant Z_{\{\gamma_1\}}(Q_1)=2M. \tag{3.65}$$

We note that $\varphi(z)\varphi''(z)>0$ in $(\gamma_{2M+1},\alpha)$, since $\varphi''$ has different signs in intervals $(\beta,\gamma_1)$ and $(\gamma_{2M+1},\alpha)$ and the inequality (3.57) holds. Therefore, by Theorem 3.5, we have

$$0\leqslant Z_{(\gamma_{2M+1},\alpha)}(Q)\leqslant 1+Z_{(\gamma_{2M+1},\alpha)}(Q_1). \tag{3.66}$$

Since $Q$ has an even number of zeros, counting multiplicities, in each of the intervals $(\beta,\alpha)$ and $(\beta,\gamma_1)$ (the second interval contains no zeros at all), as well as in $(\gamma_1,\gamma_{2M+1})$ by Lemmata 3.7 and 3.8, $Q$ has an even number of zeros in $(\gamma_{2M+1},\alpha)$. Therefore, the number $Z_{(\gamma_{2M+1},\alpha)}(Q)$ can be equal to zero, and we cannot improve the lower bound in (3.66).

Thus, from (3.58), (3.64)–(3.66) we obtain the inequalities (3.62) again. As above, these inequalities are equivalent to (3.55), since the number $Z_{(\beta_1,\alpha)}(Q)$ is even by Lemma 3.7 and $Z_{(\beta_1,\alpha)}(Q_1)$ is an even number by assumption.

II. If $Q_1$ has an odd number of zeros in $(\beta,\alpha)$, then, by the same argument as in Case I, we obtain the inequalities (3.62), which coincide with the inequalities (3.56). But unlike in Case I, these inequalities cannot be improved, since $Z_{(\beta,\alpha)}(Q_1)$ is odd by assumption. $\square$

**Remark 3.12.** Lemma 3.11 is true if $\beta=-\infty$ and $\varphi(z)\neq 0$, $\varphi'(z)\neq 0$ for $z\in(-\infty,\alpha)$ (see Remark 3.6). Lemma 3.11 is also valid in the case when $\beta>\alpha$ or in the interval $(\alpha,+\infty)$.

Note that the upper bound in (3.56) cannot be improved. This is confirmed by the following example constructed recently by S. Edwards [10].

**Example 3.13.** The polynomial

$$p(z)=z(z^2-1/4)(z^2+1)^{25} \tag{3.67}$$

has 3 real zeros: $\pm\dfrac{1}{2}$ and 0. Its derivative $p'(z)$ has only two real zeros: $\pm\beta$, where

$$\beta=\frac{\sqrt{4134+106\sqrt{2369}}}{212}\approx 0.4547246059.$$

In the interval $(-\beta, \beta)$, the polynomial $p$ has only one simple zero $\alpha = 0$. Straightforward computation shows that, in this case, we have $Z_{(-\beta,\alpha)}(Q(z)) = Z_{(\alpha,\beta)}(Q(z)) = 2$ and $Z_{(-\beta,\alpha)}(Q_1(z)) = Z_{(\alpha,\beta)}(Q_1(z)) = 1$, where $Q(z) = (p'(z)/p(z))'$ and $Q_1(z) = (p''(z)/p'(z))'$. Therefore, for the polynomial $p$, the inequalities (3.56) have the form

$$Z_{(-\beta,\alpha)}(Q(z)) = Z_{(-\beta,\alpha)}(Q_1(z)) + 1,$$

$$Z_{(\alpha,\beta)}(Q_1(z)) = Z_{(\alpha,\beta)}(Q_1(z)) + 1.$$

For functions with *property A*, Lemma 3.11 has the following form.

**Theorem 3.14.** *Let $\varphi \in U_{2n}^*$ and let $\beta_1$ and $\beta_2$, $\beta_1 < \beta_2$, be real zeros of $\varphi'$ and let $\varphi$ have a unique real zero $\alpha$ in the interval $(\beta_1, \beta_2)$ such that $\varphi'(z) \neq 0$ for all $z \in (\beta_1, \alpha) \cup (\alpha, \beta_2)$. If $\varphi$ possesses property A at its zero $\alpha$, then*

$$0 \leqslant Z_{(\beta_1,\beta_2)}(Q) \leqslant Z_{(\beta_1,\beta_2)}(Q_1). \tag{3.68}$$

**Proof.** We consider two situations: the number $\alpha$ is a zero of $Q_1$ of an even multiplicity (possibly not a zero of $Q_1$) and $\alpha$ is a zero of $Q_1$ of an odd multiplicity.

Case I. Let the number $\alpha$ be a zero of $Q_1$ of an even multiplicity, including the situation $Q_1(\alpha) \neq 0$. By the same method as in the proof of Lemma 3.7, one can show that $Q_1$ is negative in a small vicinity of its pole. Since the poles of $Q_1$ are the zeros of $\varphi'$, the function $Q_1$ has an even number of zeros in the interval $(\beta_1, \beta_2)$ by Rolle's theorem. Thus, there are only two possibilities:

I.1. The function $Q_1$ has an even number of zeros in each of the intervals $(\beta_1, \alpha)$ and $(\alpha, \beta_2)$. Then the inequalities (3.68) follow[5] from the fact that $Z_{\{\alpha\}}(Q) \neq 0$ and from the inequalities (3.55), which hold in this case according to Lemma 3.11 (see also Remark 3.12).

I.2. The function $Q_1$ has an odd number of zeros in each of the intervals $(\beta_1, \alpha)$ and $(\alpha, \beta_2)$. Then Lemma 3.11 and Remark 3.12 provide the validity of the inequalities (3.56) in each of the intervals $(\beta_1, \alpha)$ and $(\alpha, \beta_2)$:

$$0 \leqslant Z_{(\beta_1,\alpha)}(Q) \leqslant 1 + Z_{(\beta_1,\alpha)}(Q_1), \tag{3.69}$$

$$0 \leqslant Z_{(\alpha,\beta_2)}(Q) \leqslant 1 + Z_{(\alpha,\beta_2)}(Q_1). \tag{3.70}$$

Since $\varphi$ possesses *property A* at $\alpha$, we have $Q(z) \neq 0$ in at least one of the intervals $(\beta_1, \alpha)$ and $(\alpha, \beta_2)$.

If $Q(z) \neq 0$ for $z \in (\beta_1, \alpha)$, then $Z_{(\beta_1,\alpha)}(Q) = 0$. Since $Q_1$ has an odd number of zeros in the interval $(\beta_1, \alpha)$ (hence at least one) by assumption, the inequality (3.69) can be improved to yield

$$0 = Z_{(\beta_1,\alpha)}(Q) \leqslant Z_{(\beta_1,\alpha)}(Q_1) - 1. \tag{3.71}$$

[5]In this case, we do not use the fact that $\varphi$ possesses property A at its zero $\alpha$.

Together with (3.70) and the fact that $Z_{\{\alpha\}}(Q) \neq 0$, this inequality gives (3.68).

If $Q(z) \neq 0$ in the interval $(\alpha, \beta_2)$, then (3.68) can be proved analogously.

Case II. Let the number $\alpha$ be a zero of $Q_1$ of odd multiplicity (hence at least one).

By assumption, $\varphi$ possesses *property A*. At first, we assume that $Q(z) \neq 0$ in the interval $(\beta_1, \alpha)$. As we mentioned in Case I, $Q_1$ has an even number of zeros in the interval $(\beta_1, \beta_2)$, so there are only the following two situations:

II.1. The function $Q_1$ has an odd number of zeros in the interval $(\beta_1, \alpha)$ and it has an even number of zeros in $(\alpha, \beta_2)$. Then as above, we have the inequality (3.71) in the interval $(\beta_1, \alpha)$ and the inequality

$$0 \leqslant Z_{(\alpha,\beta_2)}(Q) \leqslant Z_{(\alpha,\beta_2)}(Q_1),$$

which follows from Lemma 3.11 and Remark 3.12. Together with (3.71) and the fact that $Z_{\{\alpha\}}(Q) \neq 0$, this inequality gives (3.68).

II.2. The function $Q_1$ has an even number of zeros in the interval $(\beta_1, \alpha)$ and $Q_1$ has an odd number of zeros in $(\alpha, \beta_2)$. Then the number $Z_{(\beta_1,\alpha)}(Q_1)$ may also be zero and we have the following inequality

$$0 = Z_{(\beta_1,\alpha)}(Q) \leqslant Z_{(\beta_1,\alpha)}(Q_1). \tag{3.72}$$

By Lemma 3.11 and Remark 3.12, we obtain the inequality (3.70). But since $\alpha$ is a zero of $Q_1$ by assumption and $Q(\alpha) \neq 0$ by (2.11), we have the inequality

$$0 = Z_{\{\alpha\}}(Q) \leqslant -1 + Z_{\{\alpha\}}(Q_1),$$

which, together with the inequality (3.70), implies

$$0 \leqslant Z_{[\alpha,\beta_2)}(Q) \leqslant Z_{[\alpha,\beta_2)}(Q_1).$$

From this inequality and the inequality (3.72) we again obtain (3.68).

If $Q(z) \neq 0$ in the interval $(\alpha, \beta_2)$, then the inequality (3.68) can be proved analogously. □

**Remark 3.15.** Theorem 3.14 remains valid if one of $\beta_1$ and $\beta_2$ is infinite rather than a zero of $\varphi'$, that is, if $(\beta_1, \beta_2) = (\beta_1, +\infty)$ or $(\beta_1, \beta_2) = (-\infty, \beta_2)$ and the numbers $Z_{(\beta_1,\beta_2)}(Q)$ and $Z_{(\beta_1,\beta_2)}(Q_1)$ are of equal parities. More general, if, say, $\beta_2 = +\infty$, then the inequalities (3.68) must turn to the following ones

$$0 \leqslant Z_{(\beta_1,+\infty)}(Q) \leqslant 1 + Z_{(\beta_1,+\infty)}(Q_1). \tag{3.73}$$

However, for functions in $U_0^* = \mathcal{L} - \mathcal{P}^*$ the inequalities (3.68) are valid for half-infinite intervals, since both functions $Q(z)$ and $Q_1(z)$ are negative for all sufficiently large real $z$ (see the proof of Theorem 2.15 and Proposition 2.5), and, therefore, the numbers $Z_{(\beta_1,\beta_2)}(Q)$ and $Z_{(\beta_1,\beta_2)}(Q_1)$ are of equal parities whether $\beta_i$'s are finite or infinite (see also Chapter 5).

**Remark 3.16.** If $\alpha$ is a multiple zero of $\varphi \in U_{2n}^*$ and $\beta_1$ and $\beta_2$, $\beta_1 < \beta_2$, are zeros of $\varphi'$ such that $\varphi(z) \neq 0$ and $\varphi'(z) \neq 0$ for all $z \in (\beta_1, \alpha) \cup (\alpha, \beta_2)$, then the inequalities (3.68) hold, since we have Case I of Lemma 3.11 in both intervals $(\beta_1, \alpha)$ and $(\alpha, \beta_2)$ in this case.

# Chapter 4

# Bounds on the number of real critical points of the logarithmic derivatives of functions in $\mathcal{L}-\mathcal{P}^*$

In this chapter, we study bounds on the number of real zeros of the function $Q[\varphi]$ associated with a function $\varphi \in \mathcal{L}-\mathcal{P}^*$ in detail.

In Section 4.1, we investigate bounds on the number of zeros of the function $Q$ in a half-infinite intervals free of zeros of the function $\varphi$. Such intervals may appear if $\varphi$ has the smallest and/or the largest zero.

Section 4.2 is devoted to the study of the number of real zeros of $Q$ associated with functions in $\mathcal{L}-\mathcal{P}^*$ whose derivatives have no zeros on the real axis. We establish some auxiliary statements and then prove Theorems 4.9 and 4.10, which we use in the next section.

In Section 4.3, we establish our main result, Theorem 4.16, providing bounds on the number of real zeros of the function $Q$ in terms of the number of real zeros of $Q_1$ and on the numbers of non-real zeros of the functions $\varphi$ and $\varphi'$.

The last section, Section 4.4, is dedicated to the proof of the Hawaii conjecture mentioned in Introduction.

## 4.1 Half-infinite intervals free of zeros of $\varphi$

In Section 2.3, we already established that if $\alpha_L$ is the largest zero of the function $\varphi$ in $\mathcal{L}-\mathcal{P}^*$, then the function $Q[\varphi]$ has an even number of zeros in the interval $(\alpha_L,+\infty)$. But if additional information on the number of real zeros of $\varphi'$ is available, then Lemma 3.7 and Lemma 2.19 can be used to derive the following sharper result.

**Lemma 4.1.** *Let $\varphi \in \mathcal{L}-\mathcal{P}^*$ and suppose that $\varphi$ has the largest zero $\alpha_L$ and $\varphi'$ has exactly $r$ extra zeros, counting multiplicities, in the interval $(\alpha_L,+\infty)$. If $\beta_L$ is the largest zero of $\varphi'$ in $(\alpha_L,+\infty)$, then $Q$ has an odd $($even$)$ number of real zeros in $(\beta_L,+\infty)$ whenever $r$ is an even $($odd$)$ number.*

**Proof.** Let $\varphi'$ have $l \leqslant r$ *distinct* zeros, say $\beta_1 < \beta_2 < \ldots < \beta_l = \beta_L$, in the interval $(\alpha_L, +\infty)$. According to Lemma 2.19, $Q$ has an even number of zeros in $(\alpha_L, +\infty)$, counting multiplicities. But from Lemma 3.7 it follows that $Q$ has an even number of zeros in $(\alpha_L, \beta_1)$, counting multiplicities, and an odd number of zeros, say $2M_i + 1$, in each of the intervals $(\beta_i, \beta_{i+1})$ $(i = 1, 2, \ldots, l-1)$. Hence, $Q$ has exactly $\sum_{i=1}^{l-1}(2M_i + 1)$ real zeros, counting multiplicities, in $\bigcup_{i=1}^{l-1}(\beta_i, \beta_{i+1})$.

Moreover, from (2.11) it follows that $\beta$ is a zero of $Q$ of multiplicity $M - 1$ whenever $\beta$ is a zero of $\varphi'$ of multiplicity $M$. Consequently, in our case, $Q$ has exactly $r - l$ zeros, counting multiplicities, at the points $\beta_i$ that are multiple zeros of $\varphi'$. Thus, the function $Q$ has exactly $r - l + \sum_{i=1}^{l-1}(2M_i + 1) = r - 1 + \sum_{i=1}^{l-1} 2M_i$ real zeros, counting multiplicities, in the interval $[\beta_1, \beta_L]$. Therefore, if $r$ is an even (odd) number, then $Q$ has an odd (even) number of zeros in $(\alpha_L, \beta_L]$. Recall that $Q$ has an even number of zeros in $(\alpha_L, \beta_1)$ by Lemma 3.7. Consequently, $Q$ has an odd (even) number of zeros in $(\beta_L, +\infty)$, since $Q$ has an even number of zeros in $(\alpha_L, +\infty)$ according to Lemma 2.19. □

**Remark 4.2.** Lemma 4.1 is valid with respective modification in the case when $\varphi$ has the smallest zero $a_S$.

The Lemmata 2.19 and 4.2 together with results of Chapter 3 imply the following theorem, which concerns half-infinite intervals adjacent to the largest zero of $\varphi$.

**Theorem 4.3.** *Let $\varphi \in \mathcal{L} - \mathcal{P}^*$ and let $\alpha_L$ be the largest zero of $\varphi$. If $\varphi'$ has exactly $r \geqslant 1$ zeros in the interval $(\alpha_L, +\infty)$, counting multiplicities, and $\beta_S$ is the minimal one, then*

$$2\left[\frac{r}{2}\right] \leqslant Z_{[\beta_S,+\infty)}(Q) \leqslant 2\left[\frac{r}{2}\right] + Z_{[\beta_S,+\infty)}(Q_1). \tag{4.1}$$

**Proof.** Let $\beta_S = \beta_1 \leqslant \beta_2 \leqslant \ldots \leqslant \beta_r$ be the zeros of $\varphi'$ in the interval $(\alpha_L, +\infty)$. Corollary 3.10 gives

$$r - 1 \leqslant Z_{[\beta_1,\beta_r]}(Q) \leqslant r - 1 + Z_{[\beta_1,\beta_r]}(Q_1). \tag{4.2}$$

We consider the following two cases.

Case I. The number $r$ is odd. There are only the following two situations:

I.1. The function $\varphi''$ has an odd number of zeros, counting multiplicities, in the interval $(\beta_r, +\infty)$, say $\gamma_1 \leqslant \gamma_1 \leqslant \gamma_2 \leqslant \ldots \leqslant \gamma_{2M+1}$, $M \geqslant 0$.

From the analyticity of $\varphi$ and $\varphi'$ it follows that

$$\varphi(\alpha_L + \varepsilon)\varphi'(\alpha_L + \varepsilon) > 0 \tag{4.3}$$

for all sufficiently small $\varepsilon > 0$, and

$$\varphi'(\beta_r + \delta)\varphi''(\beta_r + \delta) > 0 \tag{4.4}$$

for all sufficiently small $\delta > 0$. Since $r$ is an odd number, $\varphi'$ has different signs in the intervals $(\alpha_L, \beta_1)$ and $(\beta_r, +\infty)$. Therefore, from (4.3) we obtain

$$\varphi(\beta_r + \delta)\varphi'(\beta_r + \delta) < 0 \tag{4.5}$$

for sufficiently small $\delta > 0$, since $\varphi(z) \neq 0$ for $z \in (\alpha_L, +\infty)$. Hence, from (4.4) and (4.5) it follows that

$$\varphi(\beta_r + \varepsilon)\varphi''(\beta_r + \varepsilon) < 0 \tag{4.6}$$

for sufficiently small $\varepsilon > 0$. This means that $Q(z) < 0$ for $z \in (\beta_r, \gamma_1)$ by (2.11), so

$$0 = Z_{(\beta_1,\gamma_1)}(Q) \leqslant Z_{(\beta_1,\gamma_1)}(Q_1). \tag{4.7}$$

Here the upper bound cannot be improved, since $Q_1$ has an even number of zeros in the interval $(\beta_1, \gamma_1)$ according to Lemma 3.7 applied to $Q_1$. By Case I of Lemma 3.8, we have

$$0 \leqslant Z_{[\gamma_1,\gamma_{2M+1}]}(Q) \leqslant Z_{[\gamma_1,\gamma_{2M+1}]}(Q_1). \tag{4.8}$$

But $\varphi''$ has different signs in the intervals $(\beta_1, \gamma_1)$ and $(\gamma_{2M+1}, +\infty)$, therefore, by (4.6),

$$\varphi(\gamma_{2M+1} + \delta)\varphi''(\gamma_{2M+1} + \delta) > 0$$

for sufficiently small $\delta > 0$, so the number $Z_{(\gamma_{2M+1},+\infty)}(Q)$ may be positive (see (2.11)). Now we note that $Q$ has an even number of zeros in $(\beta_r, +\infty)$ and in $[\gamma_1, \gamma_{2M+1}]$ by Lemmata 4.1 and 3.8, respectively. Since $Q$ has no zeros in $(\beta_r, \gamma_1)$, it has an even number of zeros in the interval $(\gamma_{2M+1}, +\infty)$. Theorem 3.5 and Remark 3.6 imply the validity of the inequality (3.11) for $(\gamma_{2M+1}, +\infty)$:

$$Z_{(\gamma_{2M+1},+\infty)}(Q) \leqslant 1 + Z_{(\gamma_{2M+1},+\infty)}(Q_1). \tag{4.9}$$

Applying Lemma 4.1 to $Q_1$ on the interval $(\gamma_{2M+1}, +\infty)$, we obtain that $Q_1$ has an even number of zeros in $(\gamma_{2M+1}, +\infty)$ too. Consequently, the inequality (4.9) can be improved to the following one

$$0 \leqslant Z_{(\gamma_{2M+1},+\infty)}(Q) \leqslant Z_{(\gamma_{2M+1},+\infty)}(Q_1). \tag{4.10}$$

Here the lower bound follows from the fact, proved above, that $Q$ has an even number of zeros in $(\gamma_{2M+1}, +\infty)$. Now the inequalities (4.2), (4.7), (4.8) and (4.10) imply

$$r - 1 \leqslant Z_{[\beta_s,+\infty)}(Q) \leqslant r - 1 + Z_{[\beta_s,+\infty)}(Q_1), \tag{4.11}$$

which is equivalent to (4.1), since $r$ is odd, so $2\left[\dfrac{r}{2}\right] = r - 1$.

I.2. The function $\varphi''$ has an even number of zeros, counting multiplicities, in the interval $(\beta_r, +\infty)$, say $\gamma_1 \leqslant \gamma_1 \leqslant \gamma_2 \leqslant \ldots \leqslant \gamma_{2M}$, $M \geqslant 0$.

Let $M = 0$, that is, $\varphi''(z) \neq 0$ for $z \in (\beta_r, +\infty)$. Since $r$ is an odd number, the inequality (4.6) holds as we proved above. Therefore,

$$0 = Z_{(\beta_r,+\infty)}(Q) \leqslant Z_{(\beta_r,+\infty)}(Q_1).$$

Combined with (4.2), this inequality implies (4.11), which is equivalent to (4.1) as we explained above.

Let $M > 0$, then (4.6) and Case III of Lemma 3.8 imply

$$0 \leqslant Z_{[\gamma_1,\gamma_{2M}]}(Q) \leqslant 1 + Z_{[\gamma_1,\gamma_{2M}]}(Q_1). \tag{4.12}$$

As above, the inequality (4.6) implies (4.7). Moreover, from the fact that $\varphi''$ has an even number of zeros in $(\beta_r, +\infty)$ it follows that $\varphi''$ has equal signs in the intervals $(\beta_r, \gamma_1)$ and $(\gamma_{2M}, +\infty)$. Then by (4.6), we have the following inequality

$$\varphi(\gamma_{2M} + \varepsilon)\varphi''(\gamma_{2M} + \varepsilon) < 0 \tag{4.13}$$

for sufficiently small $\varepsilon$, so $Z_{(\gamma_{2M},+\infty)}(Q) = 0$ by (2.11). Applying Lemma 4.1 to $Q_1$, we obtain that $Q_1$ has an odd number of zeros in $(\gamma_{2M}, +\infty)$ (at least one). Therefore, we have

$$0 = Z_{(\gamma_{2M},+\infty)}(Q) \leqslant -1 + Z_{(\gamma_{2M},+\infty)}(Q_1). \tag{4.14}$$

Now from (4.2), (4.7), (4.12) and (4.14) we again obtain the inequalities (4.11), which are equivalent to (4.1) as above.

Case II. The number $r > 0$ is even. There are also only two situations:

II.1. The function $\varphi''$ has an even number of zeros, counting multiplicities, in the interval $(\beta_r, +\infty)$, say $\gamma_1 \leqslant \gamma_1 \leqslant \gamma_2 \leqslant \ldots \leqslant \gamma_{2M}$, $M \geqslant 0$.

Let $M > 0$. According to Lemma 4.1, $Q$ has an odd number of zeros in $(\beta_r, +\infty)$. But $Q(z)$ is negative for sufficiently large real $z$ as was shown in the proof of Theorem 2.15, therefore,

$$Q(\beta_r + \varepsilon) > 0 \tag{4.15}$$

for sufficiently small $\varepsilon > 0$. From (3.23) it follows that $Q(\gamma_1) < 0$, consequently, $Q$ has an odd number of zeros (at least one) in the interval $(\beta_r, \gamma_1)$. This fact and Theorem 3.5 imply the following inequalities

$$1 \leqslant Z_{(\beta_r,\gamma_1)}(Q) \leqslant 1 + Z_{(\beta_r,\gamma_1)}(Q_1). \tag{4.16}$$

By Lemma 3.7 applied to $Q_1$, the latter has an even number of zeros in the interval $(\beta_r, \gamma_1)$. Hence, the inequalities (4.16) cannot be improved.

Since $\varphi''$ has an even number of zeros, counting multiplicities, in the interval $(\beta_r, +\infty)$, $\varphi''$ has equal signs in the intervals $(\beta_r, \gamma_1)$ and $(\gamma_{2M}, +\infty)$. But $Q$ has at least one zero in $(\beta_r, \gamma_1)$ (see (4.16)), therefore from (2.11) it follows that $\varphi(z)\varphi''(z)$ is positive in $(\beta_r, \gamma_1)$. Consequently, $\varphi(z)\varphi''(z)$ is also positive in $(\gamma_{2M}, +\infty)$. Thus, by (2.11), the number $Z_{(\gamma_{2M},+\infty)}(Q)$ may be positive. Moreover, since $Q(z)$ is negative for $z$ sufficiently close to $\gamma_{2M}$ (see (3.23)) and for sufficiently large real $z$ (see the proof of Theorem 2.15), $Q$ has an even number of zeros in $(\gamma_{2M}, +\infty)$. Theorem 3.5 and Remark 3.6 imply

$$0 \leqslant Z_{(\gamma_{2M},+\infty)}(Q) \leqslant 1 + Z_{(\gamma_{2M},+\infty)}(Q_1). \tag{4.17}$$

Here the upper bound cannot be improved, since $Q_1$ has an odd number of zeros in the interval $(\gamma_{2M}, +\infty)$ by Lemma 4.1 applied to $Q_1$.

Since $\varphi(z)\varphi''(z) > 0$ in $(\beta_1, \gamma_1)$ (see (4.16) and (2.11)), by Case II of Lemma 3.8, we have

$$0 \leqslant Z_{[\gamma_1,\gamma_{2M}]}(Q) \leqslant -1 + Z_{[\gamma_1,\gamma_{2M}]}(Q_1). \tag{4.18}$$

Now from (4.2) and (4.16)–(4.18) we obtain the following inequalities

$$r \leqslant Z_{(\beta_s,+\infty)}(Q) \leqslant r + Z_{(\beta_s,+\infty)}(Q_1), \tag{4.19}$$

which are equivalent to (4.1), since $r$ is even, so $2\left[\dfrac{r}{2}\right] = r$.

If $M = 0$, that is, $\varphi''(z) \neq 0$ in the interval $(\beta_r, +\infty)$, then we have the inequalities

$$1 \leqslant Z_{(\beta_r,+\infty)}(Q) \leqslant 1 + Z_{(\beta_r,+\infty)}(Q_1), \tag{4.20}$$

where the upper bound follows from Theorem 3.5 and Remark 3.6, and the lower bound follows from (4.15) and from the fact that $Q(z) < 0$ for all sufficiently large real $z$.

By (4.2) and (4.20), we again obtain (4.19), which is equivalent to (4.1) as we showed above.

II.2. The function $\varphi''$ has an odd number of zeros, counting multiplicities, in the interval $(\beta_r, +\infty)$, say $\gamma_1 \leqslant \gamma_1 \leqslant \gamma_2 \leqslant \ldots \leqslant \gamma_{2M+1}$, $M \geqslant 0$.

As in Case II.1, Lemma 4.1 implies the inequality (4.15), from which the inequalities (4.16) follow by Theorem 3.5. Furthermore, from Case I of Lemma 3.8 we obtain the inequalities (4.8). Since $Q$ has at least one zero in the interval $(\beta_r, \gamma_1)$ (see (4.16)), $\varphi(z)\varphi''(z) > 0$ in $(\beta_r, \gamma_1)$ by (2.11). But $\varphi''$ has an odd number of zeros in $(\beta_r, +\infty)$ by assumption, and $\varphi(z) \neq 0$ in this interval, therefore, $\varphi(z)\varphi''(z) < 0$ in $(\gamma_{2M+1}, +\infty)$ and $Q$ has no zeros in the interval $(\gamma_{2M+1}, +\infty)$. This fact implies the following inequality

$$0 = Z_{(\gamma_{2M+1},+\infty)}(Q) \leqslant Z_{(\gamma_{2M+1},+\infty)}(Q_1),$$

which, together with (4.8), and (4.16) implies (4.20). Now by (4.2) and (4.20), we obtain (4.19), which is equivalent to (4.1) as we showed in Case II.1.

□

**Remark 4.4.** Theorem 4.3 remains valid with respective modification in the case when $\varphi$ has the smallest zero $a_S$.

**Remark 4.5.** The number $r$ in Theorem 4.3 is the number of extra zeros of $\varphi'$ in $(\alpha_L, +\infty)$.

## 4.2 Functions in $\mathcal{L}-\mathcal{P}^*$ whose first derivatives have no real zeros

So far, we have considered semi-infinite intervals. Our statements in this section addresses the entire real axis.

**Lemma 4.6.** *Let $\varphi \in \mathcal{L}-\mathcal{P}^*$. If $\varphi'(z) \neq 0$, $\varphi''(z) \neq 0$, $Q_1(z) \neq 0$ for $z \in \mathbb{R}$, then $Z_{\mathbb{R}}(Q) = 0$, i.e. $Q$ has no real zeros.*

**Proof.** The function $\varphi$ may have real zeros. But by Rolle's theorem, $\varphi$ has at most one real zero, counting multiplicity, since $\varphi'(z) \neq 0$ for $z \in \mathbb{R}$ by assumption.

If $\varphi(z) \neq 0$ for $z \in \mathbb{R}$, then, by Lemma 3.1 applied on the real axis, $Q$ has at most one real zero. But by Theorem 2.20, the number of real zeros of $Q$ is even. Consequently, $Z_{\mathbb{R}}(Q) = 0$.

If $\varphi$ has one real zero, counting multiplicity, say $\alpha$, then $\alpha$ is a unique pole of $Q$. Moreover, since $\varphi''(z) \neq 0$ for $z \in \mathbb{R}$ by assumption, the function $\varphi\varphi''$ changes its sign at $\alpha$ and, therefore, $\varphi(z)\varphi''(z) < 0$ in one the intervals $(-\infty, \alpha)$ and $(\alpha, +\infty)$. Consequently, by (2.11), $Q(z) \neq 0$ in one of the intervals $(-\infty, \alpha)$ and $(\alpha, +\infty)$. Without loss of generality, we may suppose that $Z_{(-\infty,\alpha]}(Q) = 0$. Then according to Lemma 3.1 and Remark 3.2, we obtain that $Q$ has at most one zero in the interval $(\alpha, +\infty)$. But the number of real zeros of $Q$ is even by Theorem 2.20, and $Q$ has no zeros in the interval $(-\infty, \alpha]$. Therefore, $Q$ cannot have zeros in the interval $(\alpha, +\infty)$, so $Z_{\mathbb{R}}(Q) = 0$, as required. □

**Lemma 4.7.** *Let $\varphi \in \mathcal{L}-\mathcal{P}^*$ and let $\varphi'(z) \neq 0$, $\varphi''(z) \neq 0$ for $z \in \mathbb{R}$. If $Q_1$ has only one real zero, then $Q$ has no real zeros, i.e. $Z_{\mathbb{R}}(Q) = 0$.*

**Proof.** As in Lemma 4.6, we note that $\varphi$ has at most one real zero by Rolle's theorem.

Case I. Let $\varphi(z) \neq 0$ for $z \in \mathbb{R}$ and let $\xi$ be a unique real zero of $Q_1$. By Theorem 2.20 (see Remark 2.21), $\xi$ is a zero of $Q_1$ of even multiplicity $2M$. In this case, the number $\xi$ cannot be zero of $Q$. In fact, if $Q(\xi) = 0$, then, by Lemma 3.3 applied on the real axis, $\xi$ is a unique real zero of $Q$ of multiplicity $2M + 1$, that is, $Z_{\{\xi\}}(Q) = Z_{\mathbb{R}}(Q) = 2M + 1$. This contradicts Theorem 2.20. So, $Q(\xi) \neq 0$.

Since $\xi$ is a unique real zero of $Q_1$ of even multiplicity, $Q_1$ has equal signs in the intervals $(-\infty, \xi)$ and $(\xi, +\infty)$. By Lemma 3.1 applied to these intervals, $Q$ can have at most one zero in each of the intervals $(-\infty, \xi)$ and $(\xi, +\infty)$. But if $Q$ has a zero in the interval $(-\infty, \xi)$, then $Q(z) \neq 0$ for $z \in (\xi, +\infty)$. In fact, if $\zeta \in (-\infty, \xi)$ is a zero of $Q$, then $\zeta$ is simple zero of $Q$ by Lemma 3.1. Moreover, $Q(z) \neq 0$ for $z \in (\zeta, \xi]$, since (see the proof of Lemma 3.1), for all sufficiently small $\delta > 0$, we have

$$\varphi'(\zeta + \delta)\varphi''(\zeta + \delta)Q(\zeta + \delta)Q_1(\zeta + \delta) > 0. \tag{4.21}$$

But the functions $\varphi'$, $\varphi''$, $Q$ and $Q_1$ do not change their signs at $\xi$, therefore, we have the inequality (4.21) in a small right-sided neighbourhood of $\xi$. Consequently, $Q(z) \neq 0$ in $(\xi, +\infty)$ by Lemma 3.1.

Thus, if $Q$ has real zeros, then it has at most one zero in one of the intervals $(-\infty,\xi)$ and $(\xi,+\infty)$, that is, $Z_{\mathbb{R}}(Q)\leqslant 1$. Now Theorem 2.20 implies $Z_{\mathbb{R}}(Q)=0$.

Case II. If $\varphi$ has one real zero, counting multiplicity, say $\alpha$, then $\alpha$ is a unique pole of $Q$. In this case, as in the proof of Lemma 4.6, one can show that $Q(z)\neq 0$ in one of the intervals $(-\infty,\alpha)$ and $(\alpha,+\infty)$. Without loss of generality, we may assume that $Q(z)\neq 0$ for $z\in(-\infty,\alpha]$, that is, $Z_{(-\infty,\alpha]}(Q)=0$. Then by Theorem 2.20, the number $Z_{(\alpha,+\infty)}(Q)$ is even.

Let $\xi$ be a unique real zero of $Q_1$ and $\xi\in(-\infty,\alpha]$, then, by Lemma 3.1 (see Remark 3.2), $Q$ has at most one zero in $(\alpha,+\infty)$. Since $Z_{(\alpha,+\infty)}(Q)$ is an even number, $Z_{\mathbb{R}}(Q)=0$. If $\xi\in(\alpha,+\infty)$, then, by the same argument as in Case I, one can show that $Z_{(\alpha,+\infty)}(Q)=0$. Therefore, $Z_{\mathbb{R}}(Q)=0$, since $Z_{(-\infty,\alpha]}(Q)=0$ by assumption. □

We are now in a position to establish a relation between the number of real zeros of $Q$ and $Q_1$ on the real line analogous to (3.11).

**Theorem 4.8.** *Let $\varphi\in\mathcal{L}-\mathcal{P}^*$. If $\varphi'(z)\neq 0$, $\varphi''(z)\neq 0$ for $z\in\mathbb{R}$, then*

$$0\leqslant Z_{\mathbb{R}}(Q)\leqslant Z_{\mathbb{R}}(Q_1). \tag{4.22}$$

**Proof.** As in Lemma 4.6, we note that $\varphi$ has at most one real zero, counting multiplicity, by Rolle's theorem.

Case I. Let $\varphi(z)\neq 0$ for $z\in\mathbb{R}$. Then we can apply Theorem 3.5 (see also Lemmata 4.6 and 4.7) on the real axis to obtain

$$0\leqslant Z_{\mathbb{R}}(Q)\leqslant 1+Z_{\mathbb{R}}(Q_1), \tag{4.23}$$

that is equivalent to (4.22), since the numbers $Z_{\mathbb{R}}(Q)$ and $Z_{\mathbb{R}}(Q_1)$ are both even by Theorem 2.20 and Remark 2.21.

Case II. If $\varphi$ has one real zero, counting multiplicity, say $\alpha$, then $\alpha$ is a unique pole of $Q$. In this case, as in the proof of Lemma 4.6, one can show that $Q(z)\neq 0$ in one of the intervals $(-\infty,\alpha)$ and $(\alpha,+\infty)$. Without loss of generality, we assume that $Q(z)\neq 0$ for $z\in(-\infty,\alpha]$. Then by Theorem 2.20, the number $Z_{(\alpha,+\infty)}(Q)$ is even and the following inequality holds

$$0=Z_{(-\infty,\alpha]}(Q)\leqslant Z_{(-\infty,\alpha]}(Q_1). \tag{4.24}$$

By Theorem 3.5 (see Remarks 3.2, 3.4 and 3.6), we obtain

$$0\leqslant Z_{(\alpha,+\infty)}(Q)\leqslant 1+Z_{(\alpha,+\infty)}(Q_1). \tag{4.25}$$

The inequalities (4.24)–(4.25) imply (4.23), which is equivalent to (4.22) as we showed above. □

We now address the case when $\varphi$ and $\varphi'$ have no real zeros.

**Theorem 4.9.** *Let $\varphi\in\mathcal{L}-\mathcal{P}^*$ and let $\varphi(z)\neq 0$, $\varphi'(z)\neq 0$ for $z\in\mathbb{R}$. Then the inequalities* (4.22) *hold for $\varphi$.*

**Proof.** At first, we note that $\varphi''$ cannot have an odd number of real zeros, counting multiplicities, if $\varphi(z) \neq 0$ for $z \in \mathbb{R}$. In fact, since $\varphi \in \mathcal{L}-\mathcal{P}^*$ has no real zeros by assumption, $\varphi(z) = e^{q(z)}p(z)$, where $q$ is a real polynomial of degree at most two and $p$ is a real polynomial with no real zeros. Therefore, $\deg p$ is even. Since $\varphi''$ has exactly $\deg p + 2\deg q - 2$ zeros, $\varphi''$ has an even number of real zeros, counting multiplicities, or has no real zeros.

If $\varphi''(z) \neq 0$ for $z \in \mathbb{R}$, then the inequalities (4.22) follow from Theorem 4.8.

Let $\varphi''$ have an even number of real zeros, counting multiplicities, say $\gamma_1 \leqslant \gamma_2 \leqslant \ldots \leqslant \gamma_{2r}$, $r \geqslant 1$. If $\gamma_1 = \gamma_{2r}$, then $\gamma_1$ is a unique real zero of $\varphi''$ and its multiplicity is even. Consequently, $\gamma_1$ is a zero of $Q_1$, according to (2.12). Therefore, the inequality (3.23) implies

$$0 = Z_{\{\gamma_1\}}(Q) \leqslant -1 + Z_{\{\gamma_1\}}(Q_1). \tag{4.26}$$

Since $Q(z)$ is negative when $z = \gamma_1$ (see (3.23)) and when $z \to \pm\infty$, $Q$ has an even number of zeros in each of the intervals $(-\infty,\gamma_1)$ and $(\gamma_1,+\infty)$. Then by Theorem 3.5 and Remark 3.6, the following inequalities hold

$$0 \leqslant Z_{(-\infty,\gamma_1)}(Q) \leqslant 1 + Z_{(-\infty,\gamma_1)}(Q_1), \tag{4.27}$$

$$0 \leqslant Z_{(\gamma_1,+\infty)}(Q) \leqslant 1 + Z_{(\gamma_1,+\infty)}(Q_1).$$

Together with (4.26), these inequalities imply (4.23), which is equivalent to (4.22), since the numbers $Z_{\mathbb{R}}(Q)$ and $Z_{\mathbb{R}}(Q_1)$ are both even by Theorem 2.20 and Remark 2.21.

Let $\gamma_1 < \gamma_{2r}$ and let $\varphi(z)\varphi''(z) > 0$ for $z \in (-\infty,\gamma_1)$. Then $\varphi(z)\varphi''(z) > 0$ for $z \in (\gamma_{2r},+\infty)$, since $\varphi''$ has equal signs in the intervals $(-\infty,\gamma_1)$ and $(\gamma_{2r},+\infty)$ and $\varphi(z) \neq 0$ for $z \in \mathbb{R}$ by assumption. By Theorem 3.5 and Remark 3.6, the inequalities (4.27) hold and

$$0 \leqslant Z_{(\gamma_{2r},+\infty)}(Q) \leqslant 1 + Z_{(\gamma_{2r},+\infty)}(Q_1). \tag{4.28}$$

Case II of Lemma 3.8 implies

$$0 \leqslant Z_{[\gamma_1,\gamma_{2r}]}(Q) \leqslant -1 + Z_{[\gamma_1,\gamma_{2r}]}(Q_1). \tag{4.29}$$

Now from (4.27)–(4.29) we obtain the inequalities (4.23), which are equivalent to (4.22) as we mentioned above.

If $\gamma_1 < \gamma_{2r}$ and $\varphi(z)\varphi''(z) < 0$ for $z \in (-\infty,\gamma_1)$, then $\varphi(z)\varphi''(z) < 0$ for $z \in (\gamma_{2r},+\infty)$. Therefore, by (2.11), the following inequalities hold

$$0 = Z_{(-\infty,\gamma_1)}(Q) \leqslant Z_{(-\infty,\gamma_1)}(Q_1), \tag{4.30}$$

$$0 = Z_{(\gamma_{2r},+\infty)}(Q) \leqslant Z_{(\gamma_{2r},+\infty)}(Q_1). \tag{4.31}$$

From Case III of Lemma 3.8 it follows that the inequalities (3.22) hold in the interval $[\gamma_1,\gamma_{2r}]$ and together with (4.30)–(4.31), they imply (4.23), which is equivalent to (4.22). □

At last, we examine the case when $\varphi$ has a unique real zero $\alpha$ and possesses *property A* at $\alpha$.

**Theorem 4.10.** *Let $\varphi \in \mathcal{L}-\mathcal{P}^*$ have a unique real zero $\alpha$ such that $\varphi'(z) \neq 0$ for $z \in \mathbb{R}\backslash\{\alpha\}$. If $\varphi$ possesses property A at $\alpha$, then the inequalities* (4.22) *hold.*

**Proof.** Since $\varphi$ possesses *property A*, $Q$ associated with $\varphi$ has no zeros in at least one of the intervals $(-\infty, \alpha)$ and $(\alpha, +\infty)$. Without loss of generality, we may assume that $Q(z) \neq 0$ for $z \in (-\infty, \alpha)$. Then we have

$$0 = Z_{(-\infty,\alpha]}(Q) \leqslant Z_{(-\infty,\alpha]}(Q_1), \tag{4.32}$$

since $Q(\alpha) \neq 0$. From Lemma 3.11 and Remark 3.12 it follows that

$$0 \leqslant Z_{(\alpha,+\infty)}(Q) \leqslant 1 + Z_{(\alpha,+\infty)}(Q_1). \tag{4.33}$$

The inequalities (4.32)–(4.33) imply (4.23), which is equivalent to (4.22), since $Z_{\mathbb{R}}(Q)$ and $Z_{\mathbb{R}}(Q_1)$ are even by Theorem 2.20 and Remark 2.21. □

## 4.3 The number of real zeros of the function $Q$

In this section, we establish bounds on the number of real zeros of $Q$ associated with a function $\varphi$ in $\mathcal{L}-\mathcal{P}^*$. We consider separately three types of functions in $\mathcal{L}-\mathcal{P}^*$ with finitely many real zeros and also the functions with infinitely many real zeros. The main idea to establish the mentioned bounds is to consider local estimates. That is, the real zeros $\alpha_j$ of a given function $\varphi \in \mathcal{L}-\mathcal{P}^*$ and the real zeros $\beta_j$ of its derivative $\varphi'$ split the real axis into a few types of finite and half-infinite intervals, so we can use the results from Sections 3.2 and 4.1 on those intervals. In the case when $\varphi$ has no real zeros, we use the results from Section 4.2.

So, if the function $\varphi$ has infinitely many positive and negative zeros, then its real zeros and the real zeros of its derivative split the real axis into only two type intervals appeared in Corollary 3.10 and Theorem 3.14. If $\varphi$ has infinitely many negative zeros but finitely many positive ones, then we additionally estimate the number of zeros of $Q[\varphi]$ on the half-infinite interval Theorem 4.3 deals with.

Likewise, if the function $\varphi$ has finitely many real zeros, then its real zeros (if any) and the real zeros of its derivative split the real axis into three types of intervals appeared in Corollary 3.10 and Theorems 3.14 and 4.3.

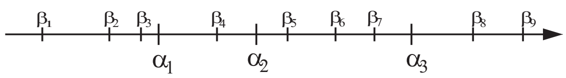


Figure 4.1:

**Example 4.11.** Let the function $\varphi$ have the form $\varphi(z) = e^{\lambda z}p(z)$, where $\lambda > 0$ and $p$ is a real polynomial. Suppose that the real zeros of $\varphi$ and zeros of its derivative $\varphi'$ are located as on Figure 4.1. Then, if $\varphi$ possesses *Property A*, we can apply Theorem 3.14 to the intervals $(\beta_3, \beta_4)$, $(\beta_4, \beta_5)$ and $(\beta_7, \beta_8)$. Applying Theorem 4.3 to the intervals $(-\infty, \beta_1)$ and $(\beta_8, +\infty)$ and applying Corollary 3.10 to the interval $[\beta_5, \beta_7]$, we obtain

$$6 \leqslant Z_{\mathbb{R}}(Q) \leqslant 6 + Z_{\mathbb{R}}(Q_1). \tag{4.34}$$

Here we take into account that $Z_{\{\beta_4\}}(Q) = Z_{\{\beta_4\}}(Q) = 0$. Since $E(\varphi') = 7$ in this case, we have $6 = 2[E(\varphi')/2]$. But by Theorem 2.10, we have

$$E(\varphi') = Z_{\mathbb{C}}(\varphi) - Z_{\mathbb{C}}(\varphi'), \tag{4.35}$$

so from (4.34)–(4.35) we obtain

$$Z_{\mathbb{C}}(\varphi) - Z_{\mathbb{C}}(\varphi') \leqslant Z_{\mathbb{R}}(Q) \leqslant Z_{\mathbb{C}}(\varphi) - Z_{\mathbb{C}}(\varphi') + Z_{\mathbb{R}}(Q_1). \tag{4.36}$$

Thus, the main idea to estimate the number $Z_R(Q)$ is described briefly. Now we give the full proof of the inequalities (4.36) for all functions in $\mathcal{L}-\mathcal{P}^*$.

At first, we establish bounds on the number of real zeros of $Q$ associated with a function $\varphi$ in $\mathcal{L}-\mathcal{P}^*$ with finitely many real zeros. At first, we express those bounds in terms of the number of extra zeros of $\varphi'$.

The next theorem concerns real polynomials multiplied by exponentials.

**Theorem 4.12.** *Let $p$ be a real polynomial and let $\varphi$ be the function*

$$\varphi(z) = e^{bz}p(z), \tag{4.37}$$

*where $b \neq 0$ is a real number. If $\varphi$ possesses property A, then*

$$2\left[\frac{E(\varphi')}{2}\right] \leqslant Z_{\mathbb{R}}(Q) \leqslant 2\left[\frac{E(\varphi')}{2}\right] + Z_{\mathbb{R}}(Q_1). \tag{4.38}$$

*Here $E(\varphi')$ is the number of extra zeros of $\varphi'$.*

**Proof.** Without loss of generality, we assume that $b > 0$ and the leading coefficient of $p$ is also positive. Then $\varphi(z) \to +\infty$ whenever $z \to +\infty$ and $\varphi(z) \to 0$ whenever $z \to -\infty$.

We consider the following two cases: I. $p$ has no real zeros, and II. $p$ has at least one real zero.

Case I. Let $p(z) \neq 0$ for $z \in \mathbb{R}$, then $\deg p$ is even and $\varphi'(z) = e^{bz}[p'(z) + bp(z)]$. Thus, $\varphi'$ has no real zeros or it has an even number of real zeros, all of which are extra zeros of $\varphi'$. (This also follows from Theorem 2.10 with $n = 0$.)

If $\varphi'(z) \neq 0$ for $z \in \mathbb{R}$, then $E(\varphi') = 0$, and, by Theorem 4.9, we obtain the validity of (4.22), which is equivalent to (4.38) in this case.

Let $\varphi'$ have an even number of real zeros, say $\beta_1 \leqslant \beta_2 \leqslant \ldots \leqslant \beta_{2r}$, where $r > 0$, then $E(\varphi') = 2r$. Since $b > 0$ and $\varphi(z) > 0$ for $z \in \mathbb{R}$ by assumption, $\varphi'(z) \to +\infty$ whenever $z \to +\infty$ and $\varphi'(z) \to +0$ whenever $z \to -\infty$. In particular, we have

$$\varphi'(z) > 0 \quad \text{for} \quad z \in (-\infty, \beta_1) \cup (\beta_{2r}, +\infty). \tag{4.39}$$

Also $\varphi''(z) \to +\infty$ whenever $z \to +\infty$ and $\varphi''(z) \to +0$ whenever $z \to -\infty$. By the analyticity of $\varphi'$, the following inequalities hold

$$\varphi'(\beta_1 - \varepsilon)\varphi''(\beta_1 - \varepsilon) < 0, \tag{4.40}$$

$$\varphi'(\beta_{2r} + \delta)\varphi''(\beta_{2r} + \delta) > 0 \tag{4.41}$$

for all sufficiently small $\varepsilon > 0$ and $\delta > 0$ (see (4.4)). These inequalities with (4.39) imply $\varphi''(\beta_1 - \varepsilon) < 0$ and $\varphi''(\beta_{2r} + \delta) > 0$ for all sufficiently small positive $\varepsilon$ and $\delta$. Therefore, $\varphi''$ has an odd number of zeros in the interval $(-\infty, \beta_1)$ and an even number of zeros (or no zeros) in the interval $(\beta_{2r}, +\infty)$. Moreover, $Q(\beta_1 - \varepsilon) < 0$ and $Q(\beta_{2r} + \delta) > 0$ for all sufficiently small $\delta > 0$ and $\varepsilon > 0$, since the sign of $Q$ in a small vicinity of a real zero of $\varphi'$ equals the sign of $\varphi\varphi''$ in that vicinity (see (2.11)) and since $\varphi(z) > 0$ for $z \in \mathbb{R}$ by assumption. As was mentioned in Theorem 2.15, $Q(z) < 0$ for all sufficiently large real $z$, consequently, $Q$ has an even number of zeros (possibly no zeros) in the interval $(-\infty, \beta_1)$ and an odd number of zeros in $(\beta_{2r}, +\infty)$. Thus,

$$Z_{(\beta_{2r}, +\infty)}(Q) \geqslant 1. \tag{4.42}$$

From Corollary 3.10 it follows that

$$2r - 1 \leqslant Z_{[\beta_1, \beta_{2r}]}(Q) \leqslant 2r - 1 + Z_{[\beta_1, \beta_{2r}]}(Q_1). \tag{4.43}$$

If $\varphi''$ has zeros in $(\beta_{2r}, +\infty)$, say $\gamma_1^{(+)} \leqslant \gamma_2^{(+)} \leqslant \ldots \leqslant \gamma_{2M}^{(+)}$, $M > 0$, then, by (4.41) and by Case II of Lemma 3.8, we have

$$0 \leqslant Z_{[\gamma_1^{(+)}, \gamma_{2M}^{(+)}]}(Q) \leqslant -1 + Z_{[\gamma_1^{(+)}, \gamma_{2M}^{(+)}]}(Q_1). \tag{4.44}$$

Theorem 3.5 and Remark 3.6 yield

$$Z_{(\gamma_{2M}^{(+)}, +\infty)}(Q) \leqslant 1 + Z_{(\gamma_{2M}^{(+)}, +\infty)}(Q_1), \tag{4.45}$$

$$Z_{(\beta_{2r}, \gamma_1^{(+)})}(Q) \leqslant 1 + Z_{(\beta_{2r}, \gamma_1^{(+)})}(Q_1). \tag{4.46}$$

Then from (4.42) and (4.44)–(4.46) it follows that

$$1 \leqslant Z_{(\beta_{2r}, +\infty)}(Q) \leqslant 1 + Z_{(\beta_{2r}, +\infty)}(Q_1). \tag{4.47}$$

These inequalities cannot be improved, since $Z_{(\beta_{2r}, +\infty)}(Q_1)$ is even by Lemma 2.19 applied to $Q_1$.

If $\varphi''(z) \neq 0$ for $z \in (\beta_{2r}, +\infty)$, then, by Theorem 3.5 with Remark 3.6 and by (4.42), we obtain the inequalities (4.47) again.

Since $\varphi''$ has an odd number of zeros in $(-\infty, \beta_1)$, say $\gamma_1^{(-)} \leqslant \gamma_2^{(-)} \leqslant \ldots \leqslant \gamma_{2N+1}^{(-)}$, $N \geqslant 0$, and $\varphi''(z) < 0$ in $(\gamma_{2N+1}^{(-)}, \beta_1)$ as we showed above, $\varphi''(z) > 0$ in $(-\infty, \gamma_1^{(-)})$. Then we have

$$0 \leqslant Z_{(-\infty, \gamma_1^{(-)})}(Q) \leqslant Z_{(-\infty, \gamma_1^{(-)})}(Q_1), \tag{4.48}$$

$$0 \leqslant Z_{[\gamma_1^{(-)},\gamma_{2N+1}^{(-)}]}(Q) \leqslant Z_{[\gamma_1^{(-)},\gamma_{2N+1}^{(-)}]}(Q_1), \tag{4.49}$$

$$0 = Z_{(\gamma_{2N+1}^{(-)},\beta_1)}(Q) \leqslant Z_{(\gamma_{2N+1}^{(-)},\beta_1)}(Q_1). \tag{4.50}$$

Indeed, the inequalities (4.48) follow from Theorem 3.5 with Remark 3.6 and the fact that $Q_1$ has an even number of zeros in $(-\infty,\gamma_1^{(-)})$ by Lemma 4.1 applied to $Q_1$. At the same time, the lower bound of the inequalities (4.48) cannot be improved, since $Q$ has an even number of zeros in the interval $(-\infty,\gamma_1^{(-)})$ by (3.23) and by the fact that $Q(z)<0$ for all sufficiently large real $z$. Further, the inequalities (4.49) are exactly (3.20). At last, since $\varphi''(z)<0$ in $(\gamma_{2N+1}^{(-)},\beta_1)$ and $\varphi(z)>0$ for $z\in\mathbb{R}$, $Q(z)$ has no zeros in $(\gamma_{2N+1}^{(-)},\beta_1)$ by (2.11). This fact we use in (4.50).

Now (4.43), (4.47) and (4.48)–(4.50) imply

$$2r \leqslant Z_{\mathbb{R}}(Q) \leqslant 2r + Z_{\mathbb{R}}(Q_1).$$

These inequalities are equivalent to (4.38), since $E(\varphi')=2r$.

Case II. Let $\varphi$ have at least one real zero.

Let $\alpha_S$ and $\alpha_L$ be the smallest and the largest zeros of $\varphi$. It is easy to see that, for all sufficiently small $\varepsilon>0$,

$$\varphi(\alpha_S-\varepsilon)\varphi'(\alpha_S-\varepsilon)<0. \tag{4.51}$$

Also since $\varphi(z)\to 0$ whenever $z\to-\infty$, we have, for all sufficiently large negative $C$,

$$\varphi(C)\varphi'(C)>0. \tag{4.52}$$

From (4.51)–(4.52) it follows that $\varphi'$ has an odd number of zeros, counting multiplicities, say $2r_-+1, r_-\geqslant 0$, in $(-\infty,\alpha_S)$, since $\varphi(z)\neq 0$ in $(-\infty,\alpha_S)$. Let $\beta_L^-$ be the largest zero of $\varphi'$ in the interval $(-\infty,\alpha_S)$. Then by Theorem 4.3 and by Remark 4.4, we have

$$2r_- \leqslant Z_{(-\infty,\beta_L^-]}(Q) \leqslant 2r_- + Z_{(-\infty,\beta_L^-]}(Q_1), \tag{4.53}$$

since $2r_- = 2\left[\dfrac{2r_-+1}{2}\right]$.

Further, by assumption, $\varphi(z)$ is positive for $z\in(\alpha_L,+\infty)$ and tends to $+\infty$ whenever $z$ tends to $+\infty$. Therefore, $\varphi'(z)\to+\infty$ as $z\to+\infty$ and, by (4.3), we have $\varphi'(z)>0$ in a small right-sided neighbourhood of the point $\alpha_L$. Consequently, $\varphi'$ has an even number of zeros, counting multiplicities, say $2r_+\geqslant 0$, in the interval $(\alpha_L,+\infty)$. If $\varphi'$ has at least one zero in $(\alpha_L,+\infty)$ and $\beta_S^+$ is the smallest one, then, by Theorem 4.3, we have

$$2r_+ \leqslant Z_{[\beta_S^+,+\infty)}(Q) \leqslant 2r_+ + Z_{[\beta_S^+,+\infty)}(Q_1) \tag{4.54}$$

since $2r_+ = 2\left[\dfrac{2r_+}{2}\right]$.

II.1 Let $\varphi$ have a unique real zero $\alpha$. Then $\alpha = \alpha_S = \alpha_L$.

If $\varphi'$ has no zeros in the interval $(\alpha, +\infty)$ and has exactly $2r_- + 1$ zeros, counting multiplicities, in $(-\infty, \alpha)$, then from Theorem 3.14 and Remark 3.15 it follows that

$$0 \leqslant Z_{(\beta_L^-, +\infty)}(Q) \leqslant Z_{(\beta_L^-, +\infty)}(Q_1), \tag{4.55}$$

where $\beta_L^-$ is the largest zero of $\varphi'$ in the interval $(-\infty, \alpha)$. Now the inequalities (4.53) and (4.55) imply (4.38), since $E(\varphi') = 2r_- + 1$ in this case.

Let $\varphi'$ have exactly $2r_+ > 0$ zeros, counting multiplicities, in the interval $(\alpha, +\infty)$ and let $\beta_S^+$ be the smallest one. By Theorem 3.14, we have

$$0 \leqslant Z_{(\beta_L^-, \beta_S^+)}(Q) \leqslant Z_{(\beta_L^-, \beta_S^+)}(Q_1). \tag{4.56}$$

This inequality, together with (4.53) and (4.54), gives (4.38), since in this case, we have $E(\varphi') = 2r_+ + 2r_- + 1$.

II.2 Let $\varphi$ have exactly $l \geqslant 2$ *distinct* real zeros, say $\alpha_S = \alpha_1 < \alpha_2 < \ldots < \alpha_l = \alpha_L$. By Rolle's theorem, $\varphi'$ has an odd number of zeros, say $2M_i + 1$, $M_i \geqslant 0$, counting multiplicities, in each of the intervals $(\alpha_i, \alpha_{i+1})$, $i = 1, 2, \ldots, l-1$. If we denote by $\beta_S^{(1)}$ and $\beta_L^{(l-1)}$ the smallest zero of $\varphi'$ in the interval $(\alpha_1, \alpha_2)$ and the largest zero of $\varphi'$ in the interval $(\alpha_{l-1}, \alpha_l)$, then, by Corollary 3.10 and Theorem 3.14, we have

$$\sum_{i=1}^{l-1} 2M_i \leqslant Z_{[\beta_S^{(1)}, \beta_L^{(l-1)}]}(Q) \leqslant \sum_{i=1}^{l-1} 2M_i + Z_{[\beta_S^{(1)}, \beta_L^{(l-1)}]}(Q_1). \tag{4.57}$$

In the interval $(\beta_L^-, \beta_S^{(1)})$, we use Theorem 3.14 to yield

$$0 \leqslant Z_{(\beta_L^-, \beta_S^{(1)})}(Q) \leqslant Z_{(\beta_L^-, \beta_S^{(1)})}(Q_1), \tag{4.58}$$

where $\beta_L^-$ is the largest zero of $\varphi'$ in the interval $(-\infty, \alpha_1)$. The existence of this zero was proved above.

If $\varphi'$ has no zeros in the interval $(\alpha_l, +\infty)$, then, by Theorem 3.14 and by Remark 3.15, we have

$$0 \leqslant Z_{(\beta_L^{(l-1)}, +\infty)}(Q) \leqslant Z_{(\beta_L^{(l-1)}, +\infty)}(Q_1). \tag{4.59}$$

Thus, if we suppose that $\varphi'$ has $2r_- + 1$ zeros, counting multiplicities, in $(-\infty, \alpha_1)$, then $E(\varphi') = 2r_- + 1 + \sum_{i=1}^{l-1} 2M_i$, so the inequalities (4.53), (4.58), (4.57) and (4.59) imply (4.38).

If $\varphi'$ has at least one zero in the interval $(\alpha_l, +\infty)$ and $\beta_S^+$ is the smallest one, then, by Theorem 3.14, we have

$$0 \leqslant Z_{(\beta_L^{(l-1)}, \beta_S^+)}(Q) \leqslant Z_{(\beta_L^{(l-1)}, \beta_S^+)}(Q_1). \tag{4.60}$$

If we suppose that $\varphi'$ has $2r_+ > 0$ zeros, counting multiplicities, in the interval $(\alpha_l, +\infty)$ and it has $2r_- + 1$ zeros, counting multiplicities, in $(-\infty, \alpha_1)$, then we have $E(\varphi') = 2r_- + 2r_+ + 1 + \sum_{i=1}^{l-1} 2M_i$, so (4.38) follows from (4.53), (4.58), (4.57), (4.60) and (4.54).

□

Now we derive a bound on the number of real zeros of $Q$ associated with a real polynomial.

**Theorem 4.13.** *Let $\varphi$ be a real polynomial possessing property A.*

I. *If $\varphi(z) \neq 0$ for $z \in \mathbb{R}$, then*

$$2\left[\frac{E(\varphi')}{2}\right] + 2 \leqslant Z_{\mathbb{R}}(Q) \leqslant 2\left[\frac{E(\varphi')}{2}\right] + 2 + Z_{\mathbb{R}}(Q_1), \tag{4.61}$$

*where $E(\varphi')$ is the number of extra zeros of $\varphi'$.*

II. *If $\varphi$ has at least one real zero, then the inequalities* (4.38) *hold.*

**Proof.** Without loss of generality, we may assume that the leading coefficient of $\varphi$ is positive. Then $\varphi(z) \to +\infty$ whenever $z \to +\infty$ and $\varphi(z) \to \infty$ whenever $z \to -\infty$.

I. Let $\varphi(z) \neq 0$ for $z \in \mathbb{R}$. Then $\deg \varphi$ is even and $\deg \varphi' = \deg \varphi - 1$. Consequently, $\varphi'$ has an odd number of zeros, all of which are extra zeros of $\varphi'$ (see (2.6)–(2.7) with $n = 0$).

We denote the real zeros of $\varphi'$ by $\beta_1 \leqslant \beta_2 \leqslant \ldots \leqslant \beta_{2r+1}$, so $E(\varphi') = 2r + 1$. By assumption, $\varphi(z) > 0$ for $z \in \mathbb{R}$, therefore, $\varphi'(z) \to +\infty$ as $z \to +\infty$ and $\varphi'(z) \to -\infty$ as $z \to -\infty$. Consequently, $\varphi'(z) < 0$ for $z \in (-\infty, \beta_1)$ and $\varphi'(z) > 0$ for $z \in (\beta_{2r+1}, +\infty)$. Also $\varphi''(z) \to +\infty$ whenever $z \to \pm\infty$. As in the proof of Theorem 4.12, one can show that the inequality (4.40) holds and implies $\varphi''(\beta_1 - \varepsilon) > 0$ for all sufficiently small $\varepsilon > 0$. Analogously to (4.41), we have

$$\varphi'(\beta_{2r+1} + \delta)\varphi''(\beta_{2r+1} + \delta) > 0 \tag{4.62}$$

for all sufficiently small $\delta > 0$, that is, $\varphi''(\beta_{2r+1} + \delta) > 0$. Therefore, $\varphi''$ has an even number of zeros in each of the intervals $(-\infty, \beta_1)$ and $(\beta_{2r+1}, +\infty)$. Moreover, since $\varphi(z) > 0$ for $z \in \mathbb{R}$ by assumption, we have $Q(\beta_1 - \varepsilon) > 0$ and $Q(\beta_{2r+1} + \delta) > 0$ for all sufficiently small $\delta > 0$ and $\varepsilon > 0$ according to (2.11). But $Q(z) < 0$ for all sufficiently large real $z$ (see the proof of Theorem 2.15), consequently, $Q$ has an odd number of zeros in each of the intervals $(-\infty, \beta_1)$ and $(\beta_{2r+1}, +\infty)$. Thus,

$$Z_{(-\infty,\beta_1)}(Q) \geqslant 1, \tag{4.63}$$

$$Z_{(\beta_{2r+1},+\infty)}(Q) \geqslant 1. \tag{4.64}$$

By Corollary 3.10, we have

$$2r \leqslant Z_{[\beta_1,\beta_{2r+1}]}(Q) \leqslant 2r + Z_{[\beta_1,\beta_{2r+1}]}(Q_1). \tag{4.65}$$

Since $\varphi''$ has an even number of zeros in each of the intervals $(-\infty, \beta_1)$ and $(\beta_{2r+1}, +\infty)$, in the same way as in Case I of Theorem 4.12 (see (4.47)), one can show that

$$1 \leqslant Z_{(-\infty,\beta_1)}(Q) \leqslant 1 + Z_{(-\infty,\beta_1)}(Q_1), \tag{4.66}$$

$$1 \leqslant Z_{(\beta_{2r+1},+\infty)}(Q) \leqslant 1 + Z_{(\beta_{2r+1},+\infty)}(Q_1). \tag{4.67}$$

Now (4.65)–(4.67) imply

$$2r + 2 \leqslant Z_{\mathbb{R}}(Q) \leqslant 2r + 2 + Z_{\mathbb{R}}(Q_1),$$

which is equivalent to (4.61), since $E(\varphi') = 2r + 1$ and, therefore, $2\left[\dfrac{E(\varphi')}{2}\right] = 2r$.

II. Let $\varphi$ have at least one real zero and let $\alpha_S$ and $\alpha_L$ be the smallest and the largest zeros of $\varphi$, respectively. It is easy to see that the inequality (4.51) holds in this case. Moreover, since $\varphi(z) \to \infty$ whenever $z \to -\infty$, we have, for sufficiently large negative $C$, $\varphi(C)\varphi'(C) < 0$. Consequently, $\varphi'$ has an even number of zeros (or has no zeros) in $(-\infty, \alpha_S)$, since $\varphi(z) \neq 0$ for $z \in \mathbb{R}$. Analogously, $\varphi'$ has an even number of zeros (or has no zeros) in $(\alpha_L, +\infty)$. Further, we consider the following two cases.

II.1 Let $\varphi$ has a unique real zero $\alpha = \alpha_S = \alpha_L$. If $\varphi'(z) \neq 0$ for $z \in \mathbb{R}\backslash\{\alpha\}$, then by Theorem 4.10, the inequality (4.22) holds. The latter is equivalent to (4.38), since $E(\varphi') = 0$ in this case.

Now suppose that $\varphi'$ has at least one zero in one of the intervals $(-\infty, \alpha)$ and $(\alpha, +\infty)$. Without loss of generality, we may assume that $\varphi'(z) \neq 0$ for $z \in (-\infty, \alpha)$ and $\varphi'$ has an even number of zeros, counting multiplicities, say $2r_+ > 0$, in $(\alpha, +\infty)$. Let $\beta_S^+$ be the smallest zero of $\varphi'$ in this interval. Then by the same argument as in Case II of Theorem 4.12, one can prove that the inequalities (4.54) hold. Moreover, by Theorem 3.14 and Remark 3.15, we have

$$0 \leqslant Z_{(-\infty,\beta_S^+)}(Q) \leqslant Z_{(-\infty,\beta_S^+)}(Q_1).$$

Combined with (4.54), these inequalities give (4.38), since $E(\varphi') = 2r_+$ in this case.

At last, let $\varphi$ have a unique real zero $\alpha$ and let $\varphi'$ has at least one zero in each of the intervals $(-\infty, \alpha)$ and $(\alpha, +\infty)$. Let $\varphi'$ have exactly $2r_- > 0$ zeros, counting multiplicities, in the interval $(-\infty, \alpha)$ and exactly $2r_+ > 0$ zeros, counting multiplicities, in $(\alpha, +\infty)$. Let $\beta_L^-$ be the largest zero of $\varphi'$ in $(-\infty, \alpha)$ and let $\beta_S^+$ be the smallest zero of $\varphi'$ in $(\alpha, +\infty)$. From Theorems 4.3 and 3.14 it follows that the inequalities (4.53), (4.54) and (4.56) hold. These inequalities imply (4.38), since $E(\varphi') = 2r_- + 2r_+$ in this case.

II.2 Let now $\varphi'$ have $l \geqslant 2$ *distinct* real zeros, counting multiplicities, say $\alpha_1 < \ldots < \alpha_l$. Then by Rolle's theorem, $\varphi'$ has an odd number of zeros, say $2M_i + 1$, $M_i \geqslant 0$, counting multiplicities, in each of the intervals $(\alpha_i, \alpha_{i+1})$, $i = 1, 2, \ldots, l-1$. If $\beta_S^{(1)}$

is the smallest zero of $\varphi'$ in the interval $(\alpha_1, \alpha_2)$ and $\beta_L^{(l-1)}$ is the largest zero of $\varphi'$ in the interval $(\alpha_{l-1}, \alpha_l)$, then, as in the proof of Case II of Theorem 4.12, the inequalities (4.57) hold by Corollary 3.10 and Theorem 3.14. Moreover, if $\varphi'$ has $2r_- \geqslant 0$ zeros, counting multiplicities, in the interval $(-\infty, \alpha_1)$ and $2r_+ \geqslant 0$ zeros, counting multiplicities, in the interval $(\alpha_l, +\infty)$, then Theorems 4.3 and 3.14 with Remark 3.15 imply

$$2r_- \leqslant Z_{(-\infty,\beta_S^{(1)})}(Q) \leqslant 2r_- + Z_{(-\infty,\beta_S^{(1)})}(Q_1) \tag{4.68}$$

and

$$2r_+ \leqslant Z_{(\beta_L^{(l-1)},+\infty)}(Q) \leqslant 2r_+ + Z_{(\beta_L^{(l-1)},+\infty)}(Q_1), \tag{4.69}$$

since $2r_- = 2\left[\dfrac{2r_-}{2}\right]$ and $2r_+ = 2\left[\dfrac{2r_+}{2}\right]$. Obviously, $E(\varphi') = 2r_+ + 2r_- + \sum_{i=1}^{l-1} 2M_i$ in this case. Therefore, the inequalities (4.38) follow from (4.57), (4.68) and (4.69).

□

Our next theorem concerns the last subclass of entire functions in $\mathcal{L}-\mathcal{P}^*$ with finitely many zeros, namely, polynomials multiplied by exponentials of the form $e^{-az^2+bz}$.

**Theorem 4.14.** *Let $p$ be a real polynomial and let $\varphi$ be the function*

$$\varphi(z) = e^{-az^2+bz}p(z), \tag{4.70}$$

*where $a > 0$ and $b \in \mathbb{R}$. Let $\varphi$ possess property A.*

I. *If $p$ has no real zeros, then the inequalities* (4.38) *hold.*

II. *If $p$ has at least one real zero, then*

$$2\left[\frac{E(\varphi')}{2}\right] - 2 \leqslant Z_{\mathbb{R}}(Q) \leqslant 2\left[\frac{E(\varphi')}{2}\right] - 2 + Z_{\mathbb{R}}(Q_1), \tag{4.71}$$

*where $E(\varphi')$ is the number of extra zeros of $\varphi'$.*

**Proof.** Without loss of generality, we may assume that the leading coefficient of $p$ is positive. Then $\varphi(z) \to +0$ whenever $z \to +\infty$ and $\varphi(z) \to 0$ whenever $z \to -\infty$.

I. Let $p(z) \neq 0$ for $z \in \mathbb{R}$. Then $\deg p$ is even and $\varphi'(z) = e^{-az^2+bz}g(z)$, where $g$ is a real polynomial of odd degree, which is equal to $\deg p + 1$. Thus, $\varphi'$ has an odd number of zeros, counting multiplicities, say $\beta_1 \leqslant \beta_2 \leqslant \ldots \leqslant \beta_{2r+1}$, all of which are extra zeros of $\varphi'$ (see also Theorem 2.12 with $n = 0$.). Therefore, the inequalities (4.65) hold in this case.

Since $p(z) \neq 0$ for $z \in \mathbb{R}$, $\varphi(z) > 0$ on the real axis and $\varphi(z) \to +0$ whenever $z \to \pm\infty$ by assumption. Then $\varphi'(z) \to -0$ as $z \to +\infty$ and $\varphi'(z) \to +0$ as $z \to -\infty$. In particular, $\varphi'(z) > 0$ for $z \in (-\infty, \beta_1)$ and $\varphi'(z) < 0$ for $z \in (\beta_{2r+1}, +\infty)$. Also $\varphi''(z) \to +0$ whenever $z \to \pm\infty$. From the inequality (4.40) it follows that $\varphi''(\beta_1 - \varepsilon) < 0$ for all sufficiently small

$\varepsilon > 0$. By the similar reasoning, one can show that $\varphi''(\beta_{2r+1}+\delta) < 0$ for all sufficiently small $\delta > 0$. Consequently, $\varphi''$ has an odd number of zeros in each of the intervals $(-\infty, \beta_1)$ and $(\beta_{2r+1}, +\infty)$. Moreover, since $\varphi(z)$ is positive on the real line by assumption, $Q(\beta_1-\varepsilon) < 0$ and $Q(\beta_{2r+1}-\delta) < 0$ for all sufficiently small $\varepsilon > 0$ and $\delta > 0$ (see (2.11)). Consequently, $Q$ has an even number of zeros in each of the intervals $(-\infty, \beta_1)$ and $(\beta_{2r+1}, +\infty)$, because $Q(z) < 0$ for all sufficiently large real $z$ (see the proof of Theorem 2.15). Thus, in the same way as in the proof of Case I of Theorem 4.12 (see (4.48)–(4.50)), it is easy to show that

$$0 \leqslant Z_{(-\infty,\beta_1)}(Q) \leqslant Z_{(-\infty,\beta_1)}(Q_1). \tag{4.72}$$

And analogously,

$$0 \leqslant Z_{(\beta_{2r+1},+\infty)}(Q) \leqslant Z_{(\beta_{2r+1},+\infty)}(Q_1). \tag{4.73}$$

Since $E(\varphi') = 2r+1$ in this case, the inequalities (4.65) and (4.72)–(4.73) imply (4.38).

II. Let $p$ have at least one real zero and let $\alpha_S$ and $\alpha_L$ be the smallest and the largest zeros of $\varphi$. It is easy to see that the inequalities (4.51) and (4.52) hold in this case. From these inequalities it follows that $\varphi'$ has an odd number of zeros, counting multiplicities, say $2r_-+1, r_- \geqslant 0$, in $(-\infty, \alpha_S)$, since $\varphi(z) \neq 0$ in this interval. Analogously, one can show that $\varphi'$ has an odd number of zeros, counting multiplicities, say $2r_+ + 1, r_+ \geqslant 0$, in the interval $(\alpha_L, +\infty)$. Let $\beta_L^-$ be the largest zero of $\varphi'$ in the interval $(-\infty, \alpha_S)$ and let $\beta_S^+$ be the smallest zero of $\varphi'$ in $(\alpha_L, +\infty)$. Then by Theorem 4.3, we have

$$2r_- \leqslant Z_{(-\infty,\beta_L^-]}(Q) \leqslant 2r_- + Z_{(-\infty,\beta_L^-]}(Q_1), \tag{4.74}$$

$$2r_+ \leqslant Z_{[\beta_S^+,+\infty)}(Q) \leqslant 2r_+ + Z_{[\beta_S^+,+\infty)}(Q_1), \tag{4.75}$$

since $2r_- = 2\left[\dfrac{2r_-+1}{2}\right]$ and $2r_+ = 2\left[\dfrac{2r_++1}{2}\right]$.

If $\varphi$ has a unique real zero $\alpha = \alpha_S = \alpha_L$, then by Theorem 3.14, we have

$$0 \leqslant Z_{(\beta_L^-,\beta_S^+)}(Q) \leqslant Z_{(\beta_L^-,\beta_S^+)}(Q_1). \tag{4.76}$$

Summing the inequalities (4.74)–(4.76), we obtain (4.71), since $E(\varphi') = 2r_-+1+2r_++1$ in this case.

If $\varphi$ has exactly $l \geqslant 2$ *distinct* zeros, say $\alpha_1 < \alpha_2 < \ldots < \alpha_l$, then by Rolle's theorem, $\varphi'$ has an odd number of zeros, say $2M_i+1$, $M_i \geqslant 0$, counting multiplicities, in each of the intervals $(\alpha_i, \alpha_{i+1})$, $i = 1, 2, \ldots, l-1$. As above, from Corollary 3.10 and Theorem 3.14 it follows that

$$\sum_{i=1}^{l-1} 2M_i \leqslant Z_{(\beta_L^-,\beta_S^+)}(Q) \leqslant \sum_{i=1}^{l-1} 2M_i + Z_{(\beta_L^-,\beta_S^+)}(Q_1). \tag{4.77}$$

Now the inequalities (4.74)–(4.75) and (4.77) imply

$$2r_- + 2r_+ + \sum_{i=1}^{l-1} 2M_i \leqslant Z_{\mathbb{R}}(Q) \leqslant 2r_- + 2r_+ + \sum_{i=1}^{l-1} 2M_i + Z_{\mathbb{R}}(Q_1),$$

which is equivalent to (4.71), since $E(\varphi') = 2r_-+1+2r_++1+\sum_{i=1}^{l-1} 2M_i$. □

Theorems 4.12–4.14 describe all functions in $\mathcal{L}-\mathcal{P}^*$ with *finitely* many zeros. The next theorem provides a bound on the number of real zeros of $Q$ associated with a function in $\mathcal{L}-\mathcal{P}^*$ with *infinitely* many real zeros.

**Theorem 4.15.** *Let $\varphi \in \mathcal{L}-\mathcal{P}^*$ and suppose that $\varphi$ has infinitely many real zeros. If $\varphi$ possesses property A, then the inequalities* (4.38) *hold.*

**Proof.** If $\varphi$ has infinitely many positive and negative zeros, then $\varphi'$ can have extra zeros only between two consecutive zeros of $\varphi$. Therefore, $E(\varphi')$ is an even number and the inequalities (4.38) follow from Corollary 3.10 and Theorem 3.14.

Let $\varphi$ have infinitely many real zeros but only finitely many positive or negative zeros. Without loss of generality, we may assume that $\varphi$ has the largest real zero, say $\alpha_L$. If $\beta < \alpha_L$ is a zero of $\varphi'$ such that $\varphi(z) \neq 0$ and $\varphi'(z) \neq 0$ for $z \in (\beta, \alpha_L)$, then applying Corollary 3.10 and Theorem 3.14 to the interval $(-\infty, \beta]$ and Theorems 3.14 and 4.3 (see also Remark 3.15) to the interval $(\beta, +\infty)$, we again obtain (4.38). □

Now we are able to prove the following general theorem.

**Theorem 4.16.** *Let $\varphi \in \mathcal{L}-\mathcal{P}^*$ and suppose that $\varphi$ has exactly $2m$ nonreal zeros, counting multiplicities. If $\varphi$ possesses property A, then*

$$2m - 2m_1 \leqslant Z_{\mathbb{R}}(Q) \leqslant 2m - 2m_1 + Z_{\mathbb{R}}(Q_1), \tag{4.78}$$

*where $2m_1 = Z_{\mathbb{C}}(\varphi')$.*

**Proof.** The assertion of the theorem follows from Theorems 2.10–2.13 combined with Theorems 4.12–4.15. □

**Remark 4.17.** Example 3.13 constructed by S. Edwards [10] can be used to show that Theorem 4.16 is not valid for every function in $\mathcal{L}-\mathcal{P}^*$. Indeed, for the polynomial (3.67), we have $Z_{\mathbb{C}}(p) = 50$, $Z_{\mathbb{C}}(p') = 50$, $Z_{\mathbb{R}}(Q) = 4$ and $Z_{\mathbb{R}}(Q_1) = 2$. So, that polynomial does not enjoy the inequalities (4.78) of Theorem 4.16, so necessarily does not possess *property A*.

**Remark 4.18.** If a function $\varphi \in \mathcal{L}-\mathcal{P}^*$ has only multiple real zeros, then, in the proofs of Theorems 4.12, 4.13 and 4.14, we can use Remark 3.16 instead of Theorem 3.14. Thus, the inequalities (4.78) also hold for functions in $\mathcal{L}-\mathcal{P}^*$ with only multiple real zeros.

## 4.4 Hawaii conjecture

Now we are in a position to prove the Hawaii conjecture for functions in the class $\mathcal{L}-\mathcal{P}^*$ by combining Theorems 2.26 and 4.16. We also prove Proposition 1.1 from Introduction not only for real polynomials but for all functions in $\mathcal{L}-\mathcal{P}^*$.

**Theorem 4.19.** *Let $\varphi \in \mathcal{L}-\mathcal{P}^*$ and $Q$ be a meromorphic function defined by (2.11). If $\varphi$ has exactly $2m$ nonreal zeros, counting multiplicities, then*

$$2m - 2m_1 \leqslant Z_{\mathbb{R}}(Q) \leqslant 2m, \tag{4.79}$$

*where $2m_1$ is the number of nonreal zeros of $\varphi'$.*

**Proof.** Indeed, by Theorem 4.16, the inequality

$$2m - 2m_1 \leqslant Z_{\mathbb{R}}(Q) \tag{4.80}$$

is valid for functions in $\mathcal{L}-\mathcal{P}^*$ possessing *property A*. However, in the proofs of Theorems 4.12, 4.13, 4.14 and 4.15, we used the fact that $\varphi$ possesses *property A* only when we applied Theorem 3.14. Moreover, in the proof of Theorem 3.14, we used *property A* only for the upper bound but not for the lower one. Therefore, the inequality (4.80) is valid for all functions in $\mathcal{L}-\mathcal{P}^*$.

If $Z_{\mathbb{R}}(Q) = 0$, then the logarithmic derivative $\varphi'/\varphi$ is decreasing on the real axis except its poles. Therefore, $2m = 2m_1$ in this case, so the theorem is true.

Let $Z_{\mathbb{R}}(Q) \neq 0$. Then according to Theorem 2.26, there exists a real $\sigma_0$ such that the function $\psi_0(z) = e^{-\sigma_0 z}\varphi(z)$ possesses property A and $Z_{\mathbb{C}}(\psi_0') < Z_{\mathbb{C}}(\psi_0)$.

If $Z_{\mathbb{R}}(Q_1[\psi_0]) \neq 0$, then we can apply Theorem 2.26 to $\psi_0'$ to get a real $\sigma_1$ such that $\psi_1(z) = e^{-\sigma_1 z}\psi_0'(z)$ possesses *property A* and $Z_{\mathbb{C}}(\psi_1') < Z_{\mathbb{C}}(\psi_1)$. If $Z_{\mathbb{R}}(Q_1[\psi_1]) \neq 0$, then we can apply Theorem 2.26 to $\psi_1'$ and so on. Thus, we obtain a sequence of the functions $\psi_0, \psi_1, \psi_2, \ldots$ with *property A* satisfying the inequalities $Z_{\mathbb{C}}(\psi_j') < Z_{\mathbb{C}}(\psi_j)$, where $\psi_j(z) = e^{-\sigma_j z}\psi_{j-1}'(z)$, $j = 1, 2, \ldots$

Since $\varphi$ has finitely many nonreal zeros by assumption, the sequence of the functions $\psi_j$ is finite. That is, there exists a nonnegative integer $l\,(\leqslant m-1)$ such that[1]

$$Z_{\mathbb{R}}(Q_1[\psi_l]) = 0, \tag{4.81}$$

while $Z_{\mathbb{R}}(Q[\psi_l]) > 0$.

By construction, all the functions $\psi_j$ possess *property A*. Consequently, we can apply Theorem 4.16 to each of them to obtain

$$2m^{(j)} - 2m_1^{(j)} \leqslant Z_{\mathbb{R}}(Q[\psi_j]) \leqslant 2m^{(j)} - 2m_1^{(j)} + Z_{\mathbb{R}}(Q_1[\psi_j]), \quad j = 0, 1, \ldots, l, \tag{4.82}$$

where we use notation $2m^{(j)} = Z_{\mathbb{C}}(\psi_j)$, $2m_1^{(j)} = Z_{\mathbb{C}}(\psi_j')$.

From (4.81) and (4.82) it follows that

$$Z_{\mathbb{R}}(Q[\psi_l]) = 2m^{(l)} - 2m_1^{(l)}. \tag{4.83}$$

Further, the inequalities (4.82) imply

$$Z_{\mathbb{R}}(Q[\psi_j]) - Z_{\mathbb{R}}(Q_1[\psi_j]) \leqslant 2m^{(j)} - 2m_1^{(j)}, \quad j = 0, 1, \ldots, l-1. \tag{4.84}$$

[1]We notice that a necessary condition for the equality (4.81) is $Z_{\mathbb{C}}(\psi_l) = Z_{\mathbb{C}}(\psi_l')$.

By construction of the functions $\psi_j$ (see (2.18)), we have

$$Z_{\mathbb{R}}\left(Q_1[\psi_j]\right) = Z_{\mathbb{R}}\left(Q[\psi_{j+1}]\right), \quad 2m_1^{(j)} = 2m^{(j+1)}, \quad j = 0, 1, \dots, l-1.$$

Therefore, (4.84) can be rewritten in the form

$$Z_{\mathbb{R}}\left(Q[\psi_j]\right) - Z_{\mathbb{R}}\left(Q[\psi_{j+1}]\right) \leqslant 2m^{(j)} - 2m^{(j+1)}, \quad j = 0, 1, \dots, l-1. \tag{4.85}$$

Summing the inequalities (4.85) for $j = 0, 1, \dots, l-1$, we obtain

$$Z_{\mathbb{R}}\left(Q[\psi_0]\right) - Z_{\mathbb{R}}\left(Q[\psi_l]\right) = Z_{\mathbb{R}}\left(Q[\psi_0]\right) - Z_{\mathbb{R}}\left(Q[\psi_1]\right) + Z_{\mathbb{R}}\left(Q[\psi_1]\right) - Z_{\mathbb{R}}\left(Q[\psi_2]\right) +$$

$$+ Z_{\mathbb{R}}\left(Q[\psi_2]\right) - Z_{\mathbb{R}}\left(Q[\psi_3]\right) + \ldots + Z_{\mathbb{R}}\left(Q[\psi_{l-1}]\right) - Z_{\mathbb{R}}\left(Q[\psi_l]\right) \leqslant$$

$$\leqslant (2m^{(0)} - 2m^{(1)}) + (2m^{(1)} - 2m^{(2)}) + \ldots + (2m^{(l-1)} - 2m^{(l)}) = 2m^{(0)} - 2m^{(l)}.$$

This inequality and the equality (4.83) yield

$$Z_{\mathbb{R}}\left(Q[\psi_0]\right) \leqslant 2m^{(0)} - 2m^{(l)} + Z_{\mathbb{R}}\left(Q[\psi_l]\right) = 2m^{(0)} - 2m_1^{(l)} \leqslant 2m^{(0)}. \tag{4.86}$$

But by construction of $\psi_0$, we have $Q = Q[\psi_0]$ (see (2.18)) and $2m = 2m^{(0)}$. Therefore, the inequality (4.86) is exactly

$$Z_{\mathbb{R}}\left(Q\right) \leqslant 2m.$$

Together with (4.80), this inequality implies (4.79). □

# Chapter 5

# Bounds on the number of real critical points of the logarithmic derivatives of functions in $U_{2n}^*$ with $n \geqslant 1$

In this chapter, we extend Theorem 4.16 to the classes $U_{2n}^*$ with $n \geqslant 1$.

As we discussed in Section 2.3, the function $Q[\varphi](z)$ associated with a function $\varphi$ in $U_0^* = \mathcal{L} - \mathcal{P}^*$ is negative for all sufficiently large $z$. But functions $Q[\varphi]$ associated with functions $\varphi$ in the classes $U_{2n}^*$ with $n \geqslant 1$ can be positive for all sufficiently large $z$, as we will see below. Therefore, before we establish an extended version of Theorem 4.16, we should prove corresponding inequalities on the number of real zeros of $Q[\varphi]$ on half-intervals free of zeros of $\varphi$ when $Q[\varphi]$ is positive at infinity.

First, we revise Theorem 4.3 in terms that are more convenient for us in this chapter and with respective modification.

**Theorem 5.1.** *Let $\varphi \in U_{2n}^*$ with $n \geqslant 1$ and let $\varphi$ have the largest zero $\alpha_L$ and let the function $Q[\varphi](z)$ is negative for all sufficiently large positive $z$. If $\varphi'$ has exactly $r \geqslant 1$ zeros in the interval $(\alpha_L, +\infty)$, counting multiplicities, and $\beta_S$ is the minimal one, then*

$$2\left[\frac{r}{2}\right] \leqslant Z_{[\beta_S,+\infty)}(Q) \leqslant 2\left[\frac{r}{2}\right] + 1 + Z_{[\beta_S,+\infty)}(Q_1). \tag{5.1}$$

The modification we made in this theorem concerns the behavior of the function $Q_1(z)$ for large real $z$. Namely, in the assumption of Theorem 5.1, the function $Q_1(z)$ may potentially be either positive or negative for all large positive $z$, therefore, the numbers $Z_{[\beta_S,+\infty)}(Q)$ and $Z_{[\beta_S,+\infty)}(Q_1)$ may be of different parities, and we have to add the unity at the upper bound.

By Theorem 2.22, if a functions $\varphi \in U_{2n}^*$ has the largest zero $\alpha_L$ and $Q[\varphi](z) > 0$ for all sufficiently large positive $z$, then in this case, $\varphi'$ has an even number of zeros in the interval $(\alpha_L, +\infty)$. We use this fact in the following theorem.

**Theorem 5.2.** *Let $\varphi \in U_{2n}^*$ with $n \geqslant 1$ and let $\varphi$ have the largest zero $\alpha_L$ and let the function $Q[\varphi](z)$ is positive for all sufficiently large positive $z$. If $\varphi'$ has exactly $r \geqslant 1$ zeros in the interval $(\alpha_L, +\infty)$, counting multiplicities, and $\beta_S$ is the minimal one, then*

$$\max\{1, 2r-1\} \leqslant Z_{[\beta_S,+\infty)}(Q) \leqslant 2r + 1 + Z_{[\beta_S,+\infty)}(Q_1). \tag{5.2}$$

This theorem can be proved in the same way as Theorem 4.3 was. The only differences we must take into account are the behavior of the function $Q(z)$ as $z \to +\infty$ and the fact that, generally speaking, the numbers $Z_{[\beta_S,+\infty)}(Q)$ and $Z_{[\beta_S,+\infty)}(Q_1)$ may be of different parities.

**Remark 5.3.** Theorems 5.1 and 5.2 remain valid with respective modification in the case when $\varphi$ has the smallest zero $a_S$ (see Remark 4.4 and Theorem 2.23).

Consider the function $\varphi$ of the following form:

$$\varphi(z) = e^{az^{2n+1}+q(z)}p(z), \quad a \neq 0 \in \mathbb{R}, \;\; n \in \mathbb{N}, \tag{5.3}$$

where $q$ is a real polynomial of degree at most $2n$ and $p$ is a real polynomial. The derivative of the logarithmic derivative of the function $\varphi$ has the form

$$Q[\varphi](z) = Q[p](z) + 2na(2n+1)z^{2n-1} + q''(z).$$

It is clear that

$$\operatorname{sign} a \; Q[\varphi](z) \to \pm\infty \quad \text{as} \quad z \to \pm\infty.$$

Analogously to Lemma 2.19 and 4.1, one can prove the following simple facts.

**Lemma 5.4.** *Let $\varphi$ be of the form* (5.3) *with $a > 0$ ($a < 0$). If $\alpha_L$ is the largest zero of $\varphi$, then $Q$ has an odd* (*even*) *number of real zeros in $(\alpha_L, +\infty)$, counting multiplicities.*

**Lemma 5.5.** *Let $\varphi$ be of the form* (5.3) *with $a > 0$ ($a < 0$). If $\alpha_S$ is the smallest zero of $\varphi$, then $Q$ has an even* (*odd*) *number of real zeros in $(-\infty, \alpha_S)$, counting multiplicities.*

**Lemma 5.6.** *Let $\varphi$ be of the form* (5.3) *with $a > 0$ ($a < 0$) and suppose that $\varphi$ has the largest zero $\alpha_L$ and $\varphi'$ has exactly $2r$ ($2r+1$) extra zeros, counting multiplicities, in the interval $(\alpha_L, +\infty)$. If $\beta_L$ is the largest zero of $\varphi'$ in $(\alpha_L, +\infty)$, then $Q$ has an even number of real zeros in $(\beta_L, +\infty)$.*

**Lemma 5.7.** *Let $\varphi$ be of the form* (5.3) *with $a > 0$ ($a < 0$) and suppose that $\varphi$ has the largest zero $\alpha_L$ and $\varphi'$ has exactly $2r+1$ ($2r$) extra zeros, counting multiplicities, in the interval $(\alpha_L, +\infty)$. If $\beta_L$ is the largest zero of $\varphi'$ in $(\alpha_L, +\infty)$, then $Q$ has an odd number of real zeros in $(\beta_L, +\infty)$.*

These lemmata can be proved in the same way as Lemmata 2.19 and 4.1 were.

From Lemmata 5.4 and 5.5 and Proposition 2.18 it is easy to determine the parity of the number of real zeros of the function $Q$ associated with functions of the form (5.3).

**Theorem 5.8.** *If $\varphi$ is of the form (5.3), then the function $Q$ associated with $\varphi$ has an odd number of real zeros, counting multiplicity.*

The next lemma and theorems are exact analogues of Theorems 4.8, 4.9 and 4.10 and can be established in the same way with respective modification taking into account that $\operatorname{sign} a\, Q(z) \to +\infty$ as $z \to +\infty$ for functions of the form(5.3).

**Lemma 5.9.** *Let $\varphi$ be of the form (5.3). If $\varphi'(z) \neq 0$, $\varphi''(z) \neq 0$ for $z \in \mathbb{R}$, then*

$$1 \leqslant Z_{\mathbb{R}}(Q) \leqslant Z_{\mathbb{R}}(Q_1). \tag{5.4}$$

**Theorem 5.10.** *Let $\varphi$ be of the form (5.3) and let $\varphi(z) \neq 0$, $\varphi'(z) \neq 0$ for $z \in \mathbb{R}$. Then the inequalities (5.4) hold for $Q[\varphi]$.*

**Theorem 5.11.** *Let $\varphi$ be of the form (5.3) and have a unique real zero $\alpha$ such that $\varphi'(z) \neq 0$ for $z \in \mathbb{R} \backslash \{\alpha\}$. If $\varphi$ possesses property $A$ at $\alpha$, then the inequalities (5.4) hold.*

Thus, for functions of the form (5.3) one can prove the following general theorem about bounds on the number of real zeros of the function $Q$.

**Theorem 5.12.** *Let $\varphi$ be of the form (5.3) and suppose that $\varphi$ has exactly $2m$ nonreal zeros. If $\varphi$ possesses property $A$, then*

$$\max\{1, 2m - 2m_1 + 2n - 1\} \leqslant Z_{\mathbb{R}}(Q) \leqslant 2m - 2m_1 + 2n + Z_{\mathbb{R}}(Q_1), \tag{5.5}$$

*where $2m_1 = Z_{\mathbb{C}}(\varphi')$.*

We omit the proof of this theorem, since it is analogous to the proof of Theorem 4.12. We only have to take into account a different behavior of the function $Q(z)$ for sufficiently large positive $z$ and then use Theorem 2.10.

Let now the function $\varphi$ is of the form

$$\varphi(z) = e^{az^{2n} + q(z)} p(z), \quad a > 0, \ \ n \in \mathbb{N}, \tag{5.6}$$

where $q$ is a real polynomial of degree at most $2n - 1$ and $p$ is a real polynomial.

The function $Q$ associated with $\varphi$ is as follows

$$Q[\varphi](z) = Q[p](z) + 2na(2n-1)z^{2n-2} + q''(z).$$

It is easy to see that

$$Q[\varphi](z) \to +\infty \quad \text{as} \quad z \to \pm\infty.$$

This immediately implies the following simple results.

**Lemma 5.13.** *Let $\varphi$ be of the form (5.6). If $\alpha_L$ is the largest zero of $\varphi$, then $Q$ has an odd number of real zeros in $(\alpha_L, +\infty)$, counting multiplicities.*

**Lemma 5.14.** *Let $\varphi$ be of the form* (5.6). *If $\alpha_S$ is the smallest zero of $\varphi$, then $Q$ has an odd number of real zeros in $(-\infty, \alpha_S)$, counting multiplicities.*

**Theorem 5.15.** *If $\varphi$ is of the form* (5.6), *then the function $Q$ associated with $\varphi$ has an even number of real zeros, counting multiplicity.*

Using these lemmata and theorem, one can establish the following theorem.

**Theorem 5.16.** *Let $\varphi$ be of the form* (5.6) *and suppose that $\varphi$ has exactly $2m$ nonreal zeros. If $\varphi$ possesses property A, then*

$$2m - 2m_1 + 2n - 2 \leqslant Z_{\mathbb{R}}(Q) \leqslant 2m - 2m_1 + 2n + Z_{\mathbb{R}}(Q_1), \tag{5.7}$$

*where $2m_1 = Z_{\mathbb{C}}(\varphi')$.*

As above, we omit the proof of this theorem, since it is analogous to the proof of Theorem 4.13. We only take into account that $Q(z) > 0$ for all sufficiently large real $z$ and then use Theorem 2.11.

If the function $\varphi$ has the form

$$\varphi(z) = e^{-az^{2n+2}+q(z)}p(z), \quad a > 0, \;\; n \in \mathbb{N}, \tag{5.8}$$

where $q$ is a real polynomial of degree at most $2n+1$ and $p$ is a real polynomial, then the function $Q[\varphi]$ is as follows

$$Q[\varphi](z) = Q[p](z) - a(2n+1)(2n+2)z^{2n-1} + q''(z).$$

It is clear that

$$Q[\varphi](z) \to -\infty \quad \text{as} \quad z \to \pm\infty.$$

Therefore, $Q(z) < 0$ for all sufficiently large real $z$ and all results of Sections 4.1–4.3 are true in this case. Consequently, for functions of the form (5.8) Theorem 4.14 is valid and can be proved in the same way. From Theorems 4.14 and 2.12 we have the following result.

**Theorem 5.17.** *Let $\varphi$ be of the form* (5.8) *and suppose that $\varphi$ has exactly $2m$ nonreal zeros. If $\varphi$ possesses property A, then*

$$2m - 2m_1 + 2n \leqslant Z_{\mathbb{R}}(Q) \leqslant 2m - 2m_1 + 2n + Z_{\mathbb{R}}(Q_1), \tag{5.9}$$

*where $2m_1 = Z_{\mathbb{C}}(\varphi')$.*

Next, if a function $\varphi \in U_{2n}^{*}$ with $n \geqslant 1$ has an infinite number of positive and negative zeros, then we use results of Chapter 3 to prove the following theorem.

**Theorem 5.18.** *Let $\varphi \in U_{2n}^{*}$ with $n \geqslant 1$ have infinitely many positive and negative zeros. If $\varphi$ possesses property A, then*

$$2m - 2m_1 + 2n \leqslant Z_{\mathbb{R}}(Q) \leqslant 2m - 2m_1 + 2n + Z_{\mathbb{R}}(Q_1), \tag{5.10}$$

*where $2m_1 = Z_{\mathbb{C}}(\varphi')$.*

This theorem follows from Theorem 2.13 and the inequality (4.38) which can be proved in the same way as in Theorem 4.15.

At last, consider the functions in $U_{2n}^*$ with $n \geqslant 1$ that have infinitely many zeros but only finitely many positive or negative zeros.

**Theorem 5.19.** *Let $\varphi \in U_{2n}^*$ with $n \geqslant 1$ have infinitely many zeros but only finitely many positive or negative zeros and suppose that $\varphi$ has exactly $2m$ nonreal zeros. If $\varphi$ possesses property A, then*

$$\max\{1, 2m - 2m_1 + 2n - 1\} \leqslant Z_{\mathbb{R}}(Q) \leqslant 2m - 2m_1 + 2n + 1 + Z_{\mathbb{R}}(Q_1), \tag{5.11}$$

**Proof.** Without loss of generality, we may assume that $\varphi$ has the largest real zero, say $\alpha_L$. If $\beta < \alpha_L$ is a zero of $\varphi'$ such that $\varphi(z) \neq 0$ and $\varphi'(z) \neq 0$ for $z \in (\beta, \alpha_L)$, then applying Corollary 3.10 and Theorem 3.14 to the interval $(-\infty, \beta]$, we have

$$2r \leqslant Z_{(-\infty,\beta]}(Q) \leqslant 2r + Z_{(-\infty,\beta]}(Q_1), \tag{5.12}$$

where $2r \geqslant 0$ is the number of extra zeros of $\varphi'$ in the interval $(-\infty, \beta]$.

If $\varphi'$ has at least one zero in the interval $(\alpha_L, +\infty)$, then by (5.12) and by Theorems 3.14 and 5.1–5.2, we obtain

$$2\left[\frac{E(\varphi')}{2}\right] - 1 \leqslant Z_{\mathbb{R}}(Q) \leqslant 2\left[\frac{E(\varphi')}{2}\right] + 1 + Z_{\mathbb{R}}(Q_1). \tag{5.13}$$

Let now $\varphi'$ has no zeros in the interval $(\alpha_L, +\infty)$, then by (3.73) (see Remark 3.15) and by (5.12) we again obtain (5.13). The inequalities (5.13) with Theorem 2.13 imply (5.11), as required. □

Now combining Theorems 5.12 and 5.16–5.19 we obtain the general result for functions in $U_{2n}^*$ with $n \geqslant 1$.

**Theorem 5.20.** *Let $\varphi$ be in $U_{2n}^*$ with $n \geqslant 1$ and suppose that $\varphi$ has exactly $2m$ nonreal zeros. If $\varphi$ possesses property A, then*

$$2m - 2m_1 + 2n - 2 \leqslant Z_{\mathbb{R}}(Q) \leqslant 2m - 2m_1 + 2n + 1 + Z_{\mathbb{R}}(Q_1), \tag{5.14}$$

*where $2m_1 = Z_{\mathbb{C}}(\varphi')$.*

As in Section 4.3, we make the following remarks.

**Remark 5.21.** If a function $\varphi \in U_{2n}^*$ with $n \geqslant 1$ has only multiple real zeros, then, in the proof of Theorem 5.14, we can use Remark 3.16 instead of Theorem 3.14. Thus, the inequalities (5.14) also hold for functions in $U_{2n}^*$, $n \geqslant 1$, with only multiple real zeros.

As in the proofs of Theorems 4.12, 4.13, 4.14 and 4.15, in the proof of Theorem 5.14, we do not use the fact that $\varphi$ possesses *property A* getting the lower bound. Therefore, in Theorem 5.14, the lower bound of the number $Z_{\mathbb{R}}(Q)$ does not depend on *property A* and holds for every function in $U_{2n}^*$ with $n \geqslant 1$, and we have our final theorem.

**Theorem 5.22.** *Let $\varphi$ be in $U_{2n}^*$ with $n \geqslant 1$ and suppose that $\varphi$ has exactly $2m$ nonreal zeros. Then*

$$Z_{\mathbb{R}}(Q) \geqslant 2m - 2m_1 + 2n - 2, \tag{5.15}$$

*where $2m_1 = Z_{\mathbb{C}}(\varphi')$.*

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# LEBENSLAUF

**Mikhail Tyaglov** Berlin, 14.04.2009

| | |
|---|---|
| Geburtsdatum: | 30. November, 1977 |
| Geburtsort: | Abinsk, Russland |
| Staatsangehörigkeit: | Russisch |
| Familienstand: | verheiratet |

## SCHULBILDUNG und STUDIUM

| | | |
|---|---|---|
| Schulbildung: | Sekundarschule, Abinsk, Russland | Sept. 1985 - Juni 1995 |
| Studium: | Fakultät für Mechanik und Mathematik, Staatliche Universität Rostov-am-Don, Russland | Sept. 1995 - Juni 1999 |
| Diplomarbeit: | *Numerical investigation of convection in the horizontal layer of isothermally incompressible fluid with elastic cover.* | |
| | Diplom-Mathematiker | Juni 1999 |
| Magisterstudium: | Fakultät für Mechanik und Mathematik, Staatliche Universität Rostov-am-Don, Russland | Juli 1999 - Juni 2001 |
| Magisterarbeit: | *On the onset of convection in the horizontal layer of isothermally incompressible fluid with elastic cover.* | |
| Magistergrad: | Magister der Mathematik | Juni 2001 |
| Promotionsstudium: | | |
| | Staatliche Universität Rostov-am-Don, Russland | Juli 2001 - Juni 2004 |
| | Technische Universität Berlin, Deutschland | Mai 2007 - Apr. 2009 |

## BERUFSTÄTIGKEIT

| | | |
|---|---|---|
| Staatliche Universität Rostov | Wissenschaftlicher Mitarbeiter | Sept. 2003 - Dec. 2006 |
| Südliche Bundesuniversität Rostov | Wissenschaftlicher Mitarbeiter | Jan. 2007 - Apr. 2007 |
| Technische Universität Berlin | Wissenschaftlicher Mitarbeiter | seit Mai 2007 |

Mikhail Tyaglov