On Wave Kinematics in Harsh Seaway and Freak Waves

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Summary

Ever since, harsh seaway and extreme waves have been a risk for ships and offshore structures inducing extreme loads and motions. Nevertheless, a practical mathematical model to calculate wave kinematics in the splash zone of steep, irregular seaway is still pending. The model of irregular seaway consisting of a large number of harmonic component wave proves to be efficient and accurate for the surface description and wave generation. For the calculation of near surface wave kinematics it is proven to be inadequate. Higher order wave models cover only regular waves that do not reflect the nature of irregular seaway, since front-back-asymmetry and wave breaking are not covered by theories for regular wave. They have, however, crucial impact on the wave kinematics and therefore on the loads exerted on ships and offshore-structures.

In this thesis, the Stokes-Approximation is presented, a new method to predict particle kinematics within steep irregular waves. The model is based on a perturbation solution for interacting Stokes-waves by Pierson (1993) that is extended to six component waves accounting for interaction terms up the third order. The method is adapted for the use in restricted water depth. By a Subplex search algorithm a parametric solution is fitted onto a surface elevation that may be a time registration or as a surface contour. For validation purposes measurements of dynamic pressure and particle velocities are taken in wave tanks. Three wave signals are analyzed:

1. A steep focussing wave packet generated in the small wave tank of TU-Berlin is analyzed at an early stage consisting of fifteen individual waves. Measurements are taken by wave gauges and an ultrasonic flow-meter. At the point of breaking, the surface contour is recorded by successive application of wave gauges and the velocity profile is recorded by a Laser-Doppler-Velocimeter. It is shown that the Stokes-Approximation produces results even close to wave breaking. With the determined velocity field the CFD-solver Fluent is initialized to continue the simulation revealing the breaking process.

2. The ”Yura Wave” is a wave sequence with an embedded freak that has been registered in the Japan Sea off the coast of Yura Harbor by Mori et al. (2000). It is reconstructed in the small wave tank at a scale of 1:120. Measurements of dynamic pressure and velocities are compared to results from linear, stretching and regular Stokes wave theory as well as with the new Stokes-Approximation.
3. A reproduction of the "New Year Wave" a freak wave sequence measured on 01/01/1995 in the North Sea (Haver (2000)) is generated in the large model basin of TU-Berlin at scale 1:81. In parallel the horizontal velocities within the wave crest are recorded. The same experiment is conducted virtually in the numerical wave tank WAVETUB showing good agreement in terms of wave propagation as well as the resulting velocity profile in the freak wave.

It is shown that for all cases the Stokes-Approximation gives excellent results regarding the velocity field underneath the wave crest – for the freak waves as well as for the surrounding steep irregular seaway.

On the one hand the superposition of many wave components is indispensable to model, transform, and generate irregular, deterministic surface elevations while on the other hand only high order regular theories are applicable to model the crest kinematics sufficiently well. The presented and validated method bridges the gap allowing reliable estimates of the wave crest kinematics in irregular seaway that can be utilized for the calculation of loads on offshore-structures.
**Kurzfassung** (Abstract)

Von je her stellen rauer Seegang und Monsterwellen eine Gefahr für Schiffe und meerechnische Konstruktionen dar, indem sie extreme Belastungen ausüben und starke Bewegungsausschläge bewirken. Die Formulierung eines rationalen Verfahrens für die Berechnung von Wellenlasten im steilen, irregulären Seegang im Oberflächenbereich steht bis heute aus. Die Beschreibung von natürlichem Seegang durch Überlagerung vieler harmonischer Wellen hat sich als effizientes und genaues Werkzeug für die Darstellung der Wasseroberfläche, wie auch für die Wellengenerierung und Transformation erwiesen. Für die Berechnung der Wellenkinematik in Oberflächenähe ist sie ungeeignet. Wellentheorien höherer Ordnung bilden nur reguläre Wellen exakt ab, die keine Besonderheiten des natürlichen Seegangs wie die horizon tale Asymmetrie und Wellenbrechen aufweisen. Diese Effekte haben entscheidenden Einfluss auf die Wellenkinematik und damit auf die resultierenden Seegangslasten auf Schiffe und Offshore-Plattformen.


Zur Validierung werden Messungen von Partikelgeschwindigkeiten und den dynamischen Wellendrücken in Wellenkanälen herangezogen. Folgende Wellensignale werden untersucht:

Welle simuliert.


Es wird gezeigt, dass die neue Approximationsmethode in allen Fällen exzellente Ergebnisse liefert.

Das harmonische Superpositionsmodell, das die Dispersion für jede einzelne Wellenkomponente vorsieht, ist unverzichtbar für die Modellierung, Simulation und Transformation irregulärer und deterministischer Wellenzüge. Gleichzeitig können nur Theorien höherer Ordnung für regelmäßige Wellen die Kinetik im Wellenberg richtig darstellen. Die hier präsentierte und validierte Methode stellt eine Verbindung beider Konzepte dar, mit deren Hilfe, die Kinetik steiler, unregelmäßiger Wellenzüge schnell und zuverlässig ermittelt werden kann.
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Chapter 1

Introduction

Ever since large waves and harsh seaway have been a risk for human life and goods at sea. Although the last centuries seagoing vessels are constantly improved by ship building experience, research, and evolving construction technology and materials, seaway still poses a latent risk which is difficult to manage due to the ever changing stochastic nature of seaway. The Münchener Rückversicherungs-Gesellschaft estimates an annual loss between 2500 and 10,000 containers going overboard while the value of each unit often exceeds 2.5 millions USD (Wichmann (2006)). If a passenger carrying ship encounters extreme heel angles or sinks in heavy seas the extent of the losses is tremendous since each single incident leads to a great number of casualties attracting world wide attention with a subsequent loss of confidence. Therefore, given the economic and psychological damage only a very low risk is acceptable. The management of this has been a difficult task for ship builders and navigators ever since.

Not only high single waves impose a considerable risk. The main seaway related capsize hazards for ships can be identified as

- Loss of stability: Especially in modern ship designs the bow and aft section provide a large portion of the hydrostatic stability. The emergence of the bow and stern in a hogging situation may cause the righting mo-
ment to vanish. Especially fast ships with pronounced bow flare and flat sterns are prone to this detrimental effect on the hydrostatic stability. If a following wave with a wave length in the order of the ship length travelling with a similar speed, the unstable situation may last long enough to cause spontaneous capsizing (see Blume and Hattendorf (1984) and Blume (1986)).

- Roll Resonance: Following and head seas can initiate resonance phenomena if the encounter period is close to the natural period or one half of it. Parametric rolling, a coupled motion between pitch, heave and roll motion, is excited by periodic changes in the hydrostatic restoring forces and can lead to large – sometimes fatal – roll angles (Clauss and Hennig (2004), France et al. (2001)).

- Broaching-to: If a small ship encounters a steep following wave with a wave length more that twice the ship length the stern is lifted up into the wave crest and the incident flow at the rudder and with it the rudder force may reverse. This leads to a severe course instability and may cause a rapid turn of the ship on the front slope of the approaching wave followed by immediate capsizing.

Other wave related hazards are:

- Internal structural damages on the pay load due to excessive local heel angles and accelerations, and

- Slamming Impact: Large water impact loads may occur in rough seas with large relative ship motions due to forefoot slamming as a result of vessel heave/pitch motions. From steep wave impact with the hull or bow plating it is also referred to as slapping (Xu et al. (2008)). Other forms are: bottom slamming, bow flare slamming and bow slamming. As a result, local damages in the plating as well as large scale buckling and fatigue occur.

Seaway related risks are apparent when stationary oil rigs are considered. Offshore platforms remain at fixed locations for 20 years or more and op-
operators do not have the option to avoid heavy seas by changing speed or course nor postponing trips. Direct calculations of sea loads are necessary to guarantee the safety of the crew and to sustain the investment requiring adequate wave models. Prominent failure modes affecting fixed and floating offshore structures are

- Over-tipping combined with structural overload on slender structure members in the splash zone due to wave kinematics. Fig. 1.1 shows the offshore oil production platform Eugene Island 322 struck by the Hurricane Lilli in the Gulf of Mexico, in 2002 (DeFranco et al. (2004)).

![Offshore oil production platform Eugene Island 322 struck by Hurricane Lilli](image)

**Fig. 1.1**: Offshore oil production platform EUGENE ISLAND 322 having been struck by the Hurricane Lilli in the Gulf of Mexico, in 2002 (DeFranco et al. (2004)) showing considerable deformation due to structural overload.

- Wet deck slamming is another threat to fixed and floating offshore platforms. If a wave crest exceeds the air gap slamming loads impact on the underside of the main deck of a platform which are not designed to withstand impact loads. Fig. 1.2 shows the damages to the underside of the main deck of a platform in the Gulf of Mexico.

- Structural responses such as "Whipping" or "Springing" which are global resonant hull vibration excited by the seaway leading to sharp,
elastic deformations, that on the long term lead to cracks and fatigue within the structure.

![Image](image.png)

**Fig. 1.2:** Damage to the underside of the main deck of a platform in the Gulf of Mexico (photo courtesy of Forristall [2006]).

For most failure mechanisms the combination of wave kinematics i.e. the local velocity in the wave crest region and the dynamic shape of the surface elevation play the crucial roles. To understand and avoid the risk powerful models are needed to predict the possible wave impact and the dynamic and mechanical loads.

### 1.1 Freak Waves

Freak waves are episodic single waves or wave groups with an exceptional wave height compared to the surrounding seaway ([Clauss](2002)). The encounter of Freak Waves – also referred to as rogue waves – imposes a particular threat for passengers, mariners, and payload. Their unforeseeable
occurrence and their unexpected, extrem impact on ships and structures has lead to colorful depictions even in the scientific world. While the mechanisms causing their emergence are yet under discussion their devastating consequences are obvious: On February 1982, the Canadian semi-submersible Ocean Ranger was hit by a breaking freak wave 267 km east of St. John’s, Newfoundland. The platform lost water tight integrity and capsized killing 84 crew members (Kjeldsen (1996)). More fortunate were the passengers and the crew aboard the cruise ship M/S Bremen which was hit by a rogue wave in the South Atlantic in March 2001. Despite smashing of the bridge windows, flooding of the bridge and the subsequent total blackout the crew managed to regain control and to head the ship to a safe harbor. Eye witness reports are documented by Schulz (2001).

Similar catastrophic scenarios have been reported from the cruise ships Cale-
donian Star also in March 2001, the cruise ships Norwegian Dawn in 2005 and Voyager on the way from Tunis to Barcelona ([Bertotti and Cavaleri (2008)]). The world’s merchant fleet is also heavily affected: In Fig. 1.3, 22 accidents of super carriers between 1969 and 1994 are assembled that have been associated with freak waves and their geographic locations. Although reports of freak waves hitting ships appear numerous and impressive, conclusive measurements on the incident wave height or surface elevation are sparse given the missing reference frame on a moving ship. It is possible that in some cases the vessel executes a sudden pitch motion, e.g. due to resonance excitation leading to an extremely high and steep appearance of the next encountering, yet ordinary, wave. Therefore, to extract conclusive data for analysis surface recording taken from fixed platforms are utilized in the scientific world and throughout this thesis.
1.1 Freak Waves

Since first introduced by Draper (1964) there exists no exact mathematical definition of the phenomenon "Freak Wave": Faulkner (2000) considers a single wave with the ratio of $H_{\text{max}} > 2.4H_S$ to be a freak wave whereas $H_{\text{max}} > 2.2H_S$ is proposed as criterion by Liu and MacHutchon (2006). A similar definition is given by Dysthe et al. (2008) adding a criterion for the crest height: $\zeta_c/H_s > 1.25$. Hennig (2005) treats a freak wave as a single wave with $H_{\text{max}} > 2.0H_S$ in a surrounding seaway with $H_S > 10$ m. In wave records taken at North Alwyn during a 5 day storm 21 waves within 40 minutes fulfill this criterion (Wolfram et al. (2000)). From long term measurements between 1994 to 1998 Wolfram et al. (2000) conclude that these waves are generally 50% steeper than the significant steepness, with wave heights $H_{\text{max}} > 2.3H_S$. They appear in groups named "Three Sisters" in which the preceding and succeeding waves have steepness values around half the significant values while their heights are around the significant height. Therefore, rogue waves of that kind are not necessarily to be seen as exceptional events and must be included in any off-shore design considerations.

One famous recording of a freak wave has been named ”New Year Wave” since it was recorded on January 1, 1995 on the Draupner-platform (see Fig. 1.4) in the North Sea (Fig. 1.5). Although it suffered severe damages due to this encounter, the wave data recorded during the incident are of good quality. An in depth analysis of this record is shown in Section 8.2. Other registrations of extreme single waves recorded in the Gorm field on Danish Continental shelf have shown that they possess an exceptional large steepness and asymmetry that exceed short term statistical distributions (Sand et al. (1990)). In a long series Kjeldsen (1990) investigated exceptional waves that are reported from the Norwegian Frigg field (Kjeldsen (1990)) with $H_s = 8.49$ m, $H_{\text{max}} = 19.98$ m, water depth $d = 99.4$ m observing a typically strong vertical asymmetry and skew of freak waves. Here, the ratio of front to back steepness is referred to as skew. Statistical investigation on that asymmetries and steepness distribution for wind-driven ocean waves from severe storms are published by Stansell et al. (2003). Further observations of
freak waves in the North Cormorant field in the North Sea are published by Guedes Soares et al. (2004b) and within the hurricane Camille in the Gulf of Mexico (Guedes Soares et al. (2004a)).

Closely related to the question of the definition of the freak wave is the question whether a freak wave is an extreme value of a valid probability density distribution of the wave heights or if a freak wave is caused by mechanisms that are not covered by conventional seaway statistics. Following the conventional wave statistics which obey the Rayleigh distribution the Most Probable Maximum (MPM) in 1000 waves (which corresponds to an about 3 hour storm with $T_0 = 10$ s) lies at $H_{MPM} = 1.86H_S$. The mean maximum wave height is $1.94H_S$ (see e.g. Jacobsen (2005)). By numerical integration of the joint probability distribution of 1080 waves it is shown that the 5 per cent margin (95th percentile) lies at $H_{max} > 2.23H_S$ and the one per cent margin (99th percentile) at $H_{max} > 2.41H_S$ (see Sec. 3.3.2 for details). The NORSOK (Norsk Sokkels Konkuranseposisjon – Safety standards developed by the Norwegian petroleum industry NORSOK (2004)) defines a ratio of $H_{max}/H_S = 1.9$ for the maximum wave height within a 100 year storm of 2 hours duration. For an annual exceedance probability of $10^{-4}$, the NORSOK standard suggests the accidental limit state, i.e. the maximum significant wave height, to be $H_{s,10000} = 1.25H_{s,100}$, with a ratio of $H_{max}/H_s = 2.375$ which must not cause a complete loss of the integrity of an offshore platform. ($H_{s,x}$ denotes here the highest significant wave height in $x$ years.) A probability distribution relating to extreme wave crest heights are proposed by Jahns and Wheeler (1972) (with coefficients refined by Haring and Heidemann (1978)) and Forristall (2000).

So far, statistical extrapolation methods to predict extreme events are deviating from each other. The scientific community agrees that the punctual measurements and satellite photos currently available do not provide a sufficient database to derive reliable statistics on extreme waves yet (see e.g. Bitner-Gregersen and Hagen (2004)).
1.1 Freak Waves

While interaction of wind generated seas are the main source of freak waves other confounding factors may be crucial. Lavrenov (1998) observed that the unusual nature of the Agulhas Current at the Horn of Africa interacts with the opposing south western swell intensifying it and leading to an increase of the wave energy density where development of large wave groups is prone to take place. Refraction and diffraction of waves, either due to currents or bathometry have additional influence on the local wave climate in coastal areas.

Fig. 1.5: Locations of platforms, freak wave measurements, and surface registrations in the North Sea that are mentioned in this thesis.
In December 2000 the European Union initiated a scientific project called MaxWave to confirm the widespread occurrence of rogue waves, model how they occur (Clauss et al. 2003) and consider their implications for ship and offshore structure design criteria (Olagnon and Athanassoulis 2000).

1.2 Deterministic Analysis

The term "deterministic" denotes that an exactly predefined single wave or wave train is used as a basis for wave load or response calculation. Phase relations and nonlinear interactions are key parameters to specify the relevant surface profile at the structure as well as the associated wave kinematics and dynamics. Gudmestad (1989) differentiates between a "pseudo-static design wave analysis" where the wave load is applied as a static load. The maximum load is found by systematically varying the angle of attack, phase angle, and wave heights and steepness. This approach is also referred to as design wave approach. Usually, high order Stokes Wave Theory is applied for deep water (the perturbation method and a formulation according to Skjelbreia and Hendrickson 1961 is shown in Sec. 3.2.1) Dean’s Stream Function Theory is often used in case of restricted water depth but also persists in deep water.

Alternatively, deterministic time-dependent methods are typically utilized whereby a time registration of regular or irregular free surface waves is used to calculate time dependent loads and motions. By applying deterministic wave trains Clauss and Hennig (2004) carried out investigations on modelling parametric roll motions and capsizes of ships in tailor made wave sequences (see also Clauss (2005), Clauss et al. (2006a), Clauss (2008)). The technique allows to generate reproducible results in realistic, irregular wave sequences.

The ”New Wave Theory” states the most probable shape of the extreme surface elevation to be the autocorrelation of the chosen spectrum under the assumption that the irregular seaway is a Gaussian process. The average
shape of the extreme wave is linked to the scaled autocorrelation function of the surface elevation \( \text{Tromans et al. (1991)} \). It does not allow conclusions about the point of occurrence and travelling of the wave. Extreme wave kinematics are related not only to the elevation but also to the wave steepness. \( \text{Xu and Barltrop (2005)} \) enhanced the method by including combinations of wave height and front steepness with specified exceedance probability.

The search for adequate models for near surface kinematics has brought up a number of models for linear and nonlinear waves in regular and irregular seaway. Other than for the surface description, the bandwidth and the asymmetry of the waves, i.e. nonlinear effects, play an important role. \( \text{Gudmestad and Connor (1986)} \) established second order corrections for irregular seaway to better fit wave kinematics with irregular seaway. \( \text{Zhang et al. (1996)} \) published a hybrid wave model for second order, unidirectional irregular waves which has been validated with measurements by \( \text{Sell et al. (1996)} \). Another second order model was implemented to predict wave impact loading on offshore structures by \( \text{Voogt (2004)} \) within the SAFE-FLOW-Project. \( \text{Xu and Barltrop (2008)} \) added another two alternatives to estimate water wave kinematics of steep waves based on modified linear wave theory in order to conduct investigations on slamming loads.

Each wave theory demands validation which generally is conducted by measurements in wave tanks. Given the sensitivity to disturbances especially in the wave crest region non-intrusive sampling methods to measure velocities are necessary. \( \text{Hertel et al. (1969)} \) developed and patented a method for flow visualization by photographic imaging the particle path through a rotating tilted mirror. The path lines show a trochoidal modulation with significant peaks marking subsequent revolutions. That way the path as well as the instantaneous particle velocities are photographically registered. (see Fig. 1.6). The analysis must be carried out manually also the link between instantaneous particle motion and surface elevation cannot be reproduced reliably. It gives, however, a complete impression of the entire flow field as shown in
Fig. 1.6: Photographic imaging: The particle path is photographically recorded through a rotating tilted mirror. The path lines show a trochoidal modulation with significant peaks marking subsequent revolutions. That way the path as well as the instantaneous particle velocities are photographically registered (Hertel et al. (1969)).

Fig. 1.6

Standard methods to measure 2-D full field velocity measurements are Particle Induced Velocimetry (PIV) (see e.g. Gray et al. (1987) and Skyner et al. (1989)) and Laser Doppler Velocimetry (LDV) (also referred to as Laser Doppler Anemometry – LDA). Measurements of instationary flows by laser applications are challenging since the data quality strongly depends on the fluctuating velocity and flow direction as well as on the degree of particle distribution within the fluid. Previous laboratory studies comparing LDV-measurements of wave kinematics with current wave models applied on regular waves and focussing wave packages are published by Baldock et al.
1.3 Scope of the Thesis

Knowledge of wave kinematics is crucial for the estimation of wave impact on ships and offshore structures. The linear and modified linear superposition of many wave components has proven to model the surface elevation of irregular seaway and the wave propagation well. But, it does not provide reliable predictions of wave kinematics in the surface domain – the region of largest velocities. Higher order wave theories provide good models for wave kinematics only for regular waves that do not possess characteristics of harsh irregular seaway. Numerical methods fulfill that task, but their required calculation times are too large for practical applications – especially when longer wave records are to be analyzed or if the wave modelling is part of a Monte-Carlo-Simulation within a design process. Furthermore the time step integration methods (e.g. WAVETUB: Runge-Kutta, FLUENT: Euler-explicit-scheme) requires sensible initialization.

To overcome the gap and provide a rational, fast, and reliable model for kinematics of asymmetric and irregular waves is the scope of this thesis.

Measurements of the velocity and dynamic pressure are utilized to validate the new approximation method. The analysis includes realizations of measured freak waves that are transferred at various scales into wave tanks. The method provides a reliable non-linear flow field initialization for CFD-solvers.
that allows numerical investigation of breaking waves and slamming events in short time.

1.4 Structure of this Work and General Remarks

After this introduction the work continues with the theoretical fundamentals beginning with the basic equations of fluid dynamics and the boundary value problem in chapter 2. Its solutions state the different types of wave theories.

Beginning with the series expansion leading the Airy and second order Stokes wave theory chapter 3 covers fundamentals of the Stokes-Wave Theory expansion for regular waves. Results are presented for the second and fifth order. For shallow water wave theories according to Dean and the Cnoidal wave theory are also presented. Methods to model irregular seaway are shown in Sec. 3.3. In this section selected empirical wave spectra are presented and statistical properties including statistics of extreme waves are discussed.

The problem of modelling near crest kinematics in irregular seaway is shown in Sec. 3.4. Conventional approaches such as stretching theories according to Wheeler (1970) and Rodenbusch and Forristall (1986) are shown as well as the Local-Fourier-Approximation algorithm by Sobey (1992b). The chapter on wave theory closes with an outline of an approach to model water waves using the nonlinear Schrödinger Equation by Dysthe et al. (2008).

Following the chapter on wave theory is an introduction on modelling water waves by means of Computational Fluid Dynamics (CFD). Two established codes are presented: The Pot/FE-Code WAVETUB, the RANSE-solver Fluent and an example of a combination of both.
Based on the perturbation of Pierson (1993) a new approach to model irregular water waves is presented in Chapter 5. It begins with the presentation of the perturbation scheme of three interacting Stokes waves of third order followed by the adaption for the application in limited water depth. For the practical application on external wave elevations the Subplex-Search algorithm is utilized. The configuration of the optimization problem, i.e. the objective function and the application of constraints are presented in Sec. 5.4.3.

To validate the new theory model tests are carried in two different wave tanks. In chapter 6 the wave tanks used and all measuring devices are described that found application in the wave generation and measurements of the kinematics. Experimental procedures are briefly described. In order to carry out validations three wave trains are investigated:

1. A steep focussing wave packet was studied. Surface elevation and particle velocities are recorded at three distinct positions. Special attention is given on the state where the wave packet focusses to one single wave which is about to break. The surface contour just before breaking is reconstructed by sequential surface measurements and the particle velocities are recorded up to the wave crest. To model the breaking process the velocity field of the almost breaking wave is calculated and handed over into CFD-code Fluent. The numerical code continues the calculation reproducing the observed breaking pattern well.

2. A realization of a freak wave that has been recorded off Yura harbor ($H_s = 5.09$ m, $H_{max} = 13.6$ m, $\zeta_c = 8.2$ m) by Japanese National Maritime Institute (Mori et al. (2000)) is realized in the small wave tank at scale 1:120. Dynamic pressure and the horizontal particle velocities are measured simultaneously revealing the maximum velocity and pressure below the crest of the freak wave; and

3. the New Year Wave ($H_s = 11.2$ m, $H_{max} = 25.6$ m, $\zeta_c = 18.5$ m) has been realized in the large wave tank of the Technical University Berlin (tank dimensions: length 80 m, width 4 m, water depth 1.5 m, piston
type wave generator) at a scale of 1:81. On a length of 27 m the surface elevation is recorded successively at distances of 0.2 to 0.3 m revealing the genesis of the freak wave as time registrations as well as surface contours shown in Sec. 8.2.

The thesis closes with a discussion of all results and an outlook for further research. A nomenclature and a list of abbreviations that are used throughout the text is found Sec. 10 and Sec. 10 at the end. Additional symbols that turn up only at specific subsections are described in the text. The coordinate system is assumed to be earth-fixed - positioned in the mean water line (MWL) with the wave travelling along the x-axis. The z-coordinate marks the vertical component and is pointing up, i.e. out of the water.
Chapter 2

Basic Equations

This section covers the basic equations of fluid mechanics necessary to model the seaway. In Sec. 2.1 the equations of motion regarding the fluid particles are briefly described followed by the definition of the fundamental boundary value problem in Sec. 2.2. Its solutions are based on perturbation theory and form - depending on the expansion and truncation - the different wave theories, which are covered in the next section (Sec. 3). Table 2.1 describes most important symbols used throughout this section. A complete list is printed in Sec. 10 at the end of this thesis.

2.1 Fluid Mechanics

Water waves are instationary flows that can be described according to the basic equations of fluid mechanics. Water is assumed incompressible, i.e. the density remains constant. The conservation of mass then gives the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$  (2.1)

The fluid is assumed to be ideal (non-viscous), thus, due to the absence of shear stresses the particle motion is purely translatory and irrotational.
which allows the definition of the flow potential $\Phi$ by the Laplace differential equation:

$$\Delta \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0. \quad (2.2)$$

The partial derivatives of $\Phi$ lead to the fluid particle velocity:

$$\frac{\partial \Phi}{\partial \vec{n}} = \vec{v}. \quad (2.3)$$

Since each potential $\Phi$ is a valid linear solution of a linear differential equation, it may be superimposed with other solution $\Phi_1 + \Phi_2 + \ldots$. The resulting pressure distribution $p$ is derived by considering the equilibrium of forces (Newton's law) on an infinitesimal volume element. It leads to the Euler
2.2 Water Waves as a Boundary Value Problem

differential equation

\[
\rho \frac{\partial \vec{v}}{\partial t} = \rho \left[ \frac{\partial \vec{v}}{\partial t} + \vec{v} \nabla \vec{v} \right] = -\nabla (p + \rho g z).
\] (2.4)

The integration of the Euler equation (Eq. 2.4) along a stream line leads to the Bernoulli differential equation for instationary flow (see e.g. Clauss et al. (1992)):

\[
\rho \frac{\partial \Phi}{\partial t} + \rho \frac{\partial |\vec{v}|^2}{\partial t} + p + \rho g z = p_0
\] (2.5)

where \(p_0\) is an integration constant that may be time dependent. For linear theory \(p_0\) is assumed to be the surrounding air pressure - in nonlinear wave theory it is a finite, time-dependent expression (see Eq. 3.118 in Sec. 3.4.1 (Longuet-Higgins (1975)).

In the vicinity of solid walls or phase boundaries (e.g. due to wave breaking) the assumption of irrotational, frictionless fluid and absence of shear stresses does not persist. Instead, the Euler equation (Eq. 2.4) gains a new term, that includes the dynamic viscosity \(\mu\):

\[
\rho \frac{\partial \vec{v}}{\partial t} = \rho \left[ \frac{\partial \vec{v}}{\partial t} + \vec{v} \nabla \vec{v} \right] = -\nabla (p + \rho g z) + \mu \nabla^2 \vec{v}.
\] (2.6)

Yet, analytical solutions of the Navier-Stokes-Equation exist only for simple cases, thus a numerical method must be employed: Namely the RANSE method (RANSE: Reynolds Averaged Navier Stokes Equations).

2.2 Water Waves as a Boundary Value Problem

The general objective of all wave theories is to determine a formulation of the velocity potential. In order to find a solution for the wave potential a boundary value problem (BVP) is stated as sketched in Fig. 2.1. The most
Kinematic Free Surface Boundary Condition (KFSBC) The surface, $S$, is formulated implicitly, where $\zeta$ is the surface elevation:

$$S(x, z, t) = \zeta(x, t) - z = 0.$$  \hfill (2.7)
2.2 Water Waves as a Boundary Value Problem

Since the closed surface is defined implicitly the total differential also equals zero

\[ dS = dS_x + dS_z + dS_t = 0 \]

\[ \iff \partial_x (\zeta(x,t) - z) dx + \partial_z (\zeta(x,t) - z) dz + \partial_t (\zeta(x,t) - z) dt = 0 \]

\[ \iff \zeta_x \ dx - 1 \cdot \ dz + \zeta_t \ dt = 0 \]

\[ \iff \zeta_x \frac{dx}{dt} + \zeta_t - \frac{dz}{dt} = 0 \]

\[ \iff \zeta_x \phi_x + \zeta_t - \phi_z = 0 \quad \text{at } z = \zeta(x,t). \] (2.8)

Dynamic Free Surface Boundary Condition (DFSBC)  In the free fluid surface at \( S = 0 \iff z = \zeta \) it is assumed that the pressure, \( p \), is equal to the constant ambient pressure, \( p_0 \), i.e. the integration constant. The influence of the wind on the surface is not considered

\[ p = p_0(t). \] (2.9)

Applying this equation to the Bernoulli equation, (Eq. 2.5), leads to the DFSBC:

\[ \Phi_t + \frac{1}{2} \| \vec{v} \|^2 + g \ z = 0 \quad \text{at } z = \zeta(x,t). \] (2.10)

Substituting \( \zeta \) for \( z \) and rearranging this equation produces a formula for the surface elevation:

\[ \zeta = -\frac{1}{g} \left( \phi_t + \frac{1}{2} \| \vec{v} \|^2 \right). \] (2.11)

Combined Exact Free Surface Boundary Condition (CEFSBC)  The assumption that the pressure is constant in the free fluid surface implies that the sum of all differentials of the pressure is zero, therefore the total differ-
ential, \( \frac{d\rho}{dt} \), equals zero:

\[
\frac{d\rho}{dt} = pt + p_x \Phi_x + p_z \Phi_z = 0 \quad \text{at} \quad z = \zeta.
\] (2.12)

Rearranging the Bernoulli equation, (Eq. 2.5), provides an expression for the pressure, \( p \):

\[
p = -\rho \Phi_t - \frac{\rho}{2} |\vec{v}|^2 - p + \rho \ g \ z + p_0
\]

\[
\Leftrightarrow p = -\rho \Phi_t - \frac{\rho}{2} |\vec{v}|^2 - p + \rho \ g \ \zeta + p_0.
\] (2.13)

The derivatives of the pressure are:

\[
p_t = -\rho \left( \phi_t + \frac{1}{2} \frac{\partial}{\partial t} |\vec{v}|^2 + g \ \zeta \right)
\] (2.14)

\[
p_x = -\rho \left( \phi_{tx} + \frac{1}{2} \frac{\partial}{\partial x} |\vec{v}|^2 + g \ \zeta_x \right)
\] (2.15)

\[
p_z = -\rho \left( \phi_{tz} + \frac{1}{2} \frac{\partial}{\partial z} |\vec{v}|^2 + g \cdot 0 \right).
\] (2.16)
Introducing these terms into the total differential of the pressure in equation (2.12) gives the EFSBC:

\[
\Phi_{tt} + \frac{1}{2} \frac{\partial}{\partial t} |\vec{v}|^2 + g \zeta_t + \Phi_x \left( \Phi_{tx} + \frac{1}{2} \frac{\partial}{\partial x} |\vec{v}|^2 + g \zeta_x \right) + \\
\Phi_z \left( \Phi_{tz} + \frac{1}{2} \frac{\partial}{\partial z} |\vec{v}|^2 + g \phi \right) = 0
\]

\[\Leftarrow \Phi_{tt} + g \Phi_z + \partial_t |\vec{v}|^2 + \frac{1}{2} \left( \Phi_x \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial z} \right) |\vec{v}|^2 = 0 \]

at \( z = \zeta(x,t) \).

(2.17)

This equation is also referred to as Combined Free Surface Boundary Condition (CFSBC).

**Bottom Boundary Condition** (BBC) The sea bottom is assumed to be flat, horizontal, and not penetrable. Therefore, the velocity normal to the seafloor must vanish:

\[
\frac{\partial \Phi}{\partial z} = w = 0 \text{ at } z = -d = \text{ const.}.
\]

(2.18)

If the water depth is large (i.e. \( d > L/2 \approx 0.78 \ T^2 \)) it has negligible impact on the solution since wave related kinematics decay exponentially to less than five per cent of the kinematics at the MWL (Clauss et al. (1992)). The BBC regarding the oscillating motion can then be generalized to

\[
\nabla \Phi = 0 \text{ for } z \to \infty.
\]

(2.19)
Lateral Boundary Condition  Assuming unrestricted waters with constant water depth all wave theories (with the exception of the solution wave theory) are assumed to be periodic in space and time hence

$$\Phi(x, z, t) = \Phi(x, z, t + T) = \Phi(x + L, z, t). \quad (2.20)$$

This also implies that the mean wave energy is evenly distributed over the whole area. This includes irregular seaway which is modelled as a superposition of many harmonic components with very long return periods. Local disturbances such as radiation potentials (e.g. due to wave-body-interaction, wave maker, wave breaking) have only a local impact on the wave field. The disturbing potential $\Phi_{\text{dist}}$ is decreasing with an increasing distance (in the 3-D case radius) $R$ from the point of disturbance according to the Radiation- or Sommerfeld condition:

$$\lim_{R \to \infty} \sqrt{R} \left( \frac{\partial \Phi_{\text{dist}}}{\partial R} - ik \Phi_{\text{dist}} \right) = 0. \quad (2.21)$$

Solutions of the BVP  The Boundary conditions can be classified as combined Neumann-Dirichlet-Boundary-Condition, meaning that the Boundary-Conditions apply for the derivatives and the potential function itself. An exception is the BBC for limited water depths (Eq. 2.18) which is only a pure Neumann-Condition (i.e. it applies only on derivatives). The BVP does not have a unique solution. The essential problem lies in the free surface that is itself an integral part of the solution. The different orders of the same wave theory are distinguished by the level of approximation to the free surface boundary conditions where higher orders are providing enhanced fidelity at the cost of mathematical complexity. The wave theories that are compared in the following comprise the simplifying assumption that limit the validity of the solution to ranges of the parameter combinations $H, L, T, d$. On the one hand, the quality of each formulation must be assessed analytically in terms of how well the boundary conditions are being adhered. On the other hand, the experimental validity must be shown in terms of how well observations compare to measurements, e.g. in a wave tank experiments (Sobey (1992a)).
Chapter 3

Wave Theory

Each wave theory represents a particular solution of the above described boundary value problem. The solutions are found by means of perturbation theory and differ in term of truncation order and perturbation parameters. The most important parameters are

- wave steepness $\frac{H}{L}$ of $k\zeta_a$ perturbation parameter of the Airy’s and Stokes Wave Theory (see Sec. 3.1 and 3.2),
- relative water depth $d/L$ constituting the Shallow Water Wave theories (shown in sec. 3.2.3),
- Ursell number $(HL^2)/d^3$
- the combination of wave steepness $k\zeta_a$ and band width $\Delta k$ which leads to the Nonlinear Schrödinger Equation as outlined in Sec. 3.5

The most common Linear Wave Theory (also named Airy wave Theory) (Airy 1845) is shown in Sec. 3.1. In Sec. 3.2 the basic equations of wave theories of higher order are covered namely Stokes wave theory in Sec. 3.2.1 for steep regular waves, Dean’s Stream function theory in Sec. 3.2.2 and Shallow water wave theory (Cnoidal Wave Theory) in Sec. 3.2.3. The standard model for irregular seaway –the Gaussian Spectral Random Wave Model– including empirical spectral formulations for natural seaway is presented in Sec. 3.3.1.
A general outline, which theory to use based on the limiting parameters is shown in Fig. 3.1.

Fig. 3.1: Non-dimensional diagram of the limitations of wave theories for regular waves (Clauss et al. (1992))

3.1 Airy Wave Theory

The Airy Wave Theory (also referred to as Linear Wave Theory or first order Stokes wave theory) is yielded when in the perturbation of the wave potential as well as in all surface boundary conditions nonlinear terms are omitted. The
3.1 Airy Wave Theory

KFSBC and the DFSBC then become

\[
\frac{\partial \Phi}{\partial z} = \frac{\partial \zeta}{\partial t} \tag{3.1}
\]

\[
\zeta = -\frac{1}{g} \frac{\partial \Phi}{\partial t} \tag{3.2}
\]

both valid at \( z = \zeta \approx 0 \). In the KFSBC (Eq. 3.1) the wave steepness and in Eq. 3.2 the convective velocities are neglected. All following terms down to Eq. 3.10 are harmonically oscillating with respect to time \( t \) and propagation direction \( x \). The CFSBC can therefore be formulated using the harmonic approach \( \Phi = \varphi e^{-i\omega t} \) as follows

\[
\frac{\partial \Phi}{\partial z} - \frac{\omega^2}{d} \Phi = 0 \text{ at } z = \zeta. \tag{3.3}
\]

In the following the linear wave field kinematics are listed referring to a two dimensional flow field where \( x \) denotes the horizontal coordinate and \( z \) denotes the vertical coordinate with \( z = 0 \) lying at the mean water level. All derivations follow the formulations of Clauss et al. (1992).

**Linear Wave Field Kinematics** The wave flow potential and stream functions are

\[
\Phi = \frac{\zeta g \cosh k(z + d)}{\omega \cosh kd} \sin(kx - \omega t) \tag{3.4}
\]

\[
\Psi = \frac{\zeta g \sinh k(z + d)}{\omega \cosh kd} \cos(kx - \omega t). \tag{3.5}
\]

The particle velocities are

\[
u = \frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial z} = \zeta \omega \frac{\cosh k(z + d)}{\sinh kd} \cos(kx - \omega t) \tag{3.6}
\]

\[
w = \frac{\partial \Phi}{\partial z} = \frac{\partial \Psi}{\partial x} = \zeta \omega \frac{\sinh k(z + d)}{\sinh kd} \sin(kx - \omega t). \tag{3.7}
\]
The particle accelerations are

\[
\begin{aligned}
\dot{u} &= \frac{\partial u}{\partial t} = \zeta a \omega^2 \frac{\cosh k(z + d)}{\sinh kd} \sin(kx - \omega t) \quad (3.8) \\
\dot{w} &= \frac{\partial w}{\partial t} = -\zeta a \omega^2 \frac{\sinh k(z + d)}{\sinh kd} \cos(kx - \omega t). \quad (3.9)
\end{aligned}
\]

The remaining dynamic pressure component derived from the "instationary" or "transient" part of the Bernoulli equation

\[
p_d = -\rho \frac{\partial \Phi}{\partial t} = \zeta a \rho g \cosh k(z + d) \sinh kd \sin(kx - \omega t).
\]  

(3.10)

All expressions describing the wave kinematics (Eq.3.4 through Eq.3.10) contain a harmonic phase function \( \Theta \):

\[kx - \omega t = \Theta. \]  

(3.11)

For a constant \( \Theta \) the wave phase speed (also called celerity) \( c \) can be determined:

\[c = \frac{\omega}{k} = \frac{L}{T} = \frac{g}{\omega} \tanh kd. \]  

(3.12)

Hence, other than acoustic and electromagnetic waves the propagation speed of water waves strongly depends on the wave frequency and length as well as on the water depth \( d \) if the wave length exceeds two times the water depth.

**Dispersion Relation**  The dispersion equation

\[0 = \omega^2 - g k \tanh kd \]  

(3.13)

describes the unique relationship among \( k, \omega \) and \( d \) (respectively: \( T, L \) and \( d \)). It contains the assumption that the phase function other than \( k, \omega \) remains constant as a wave propagates from deep into shallow water. Eq.3.13 cannot be solved for \( k \) explicitly. If \( k \) is not known a solution can be found
### 3.1 Airy Wave Theory

by

- iteration e.g. Newton-Raphson-method,
- graphical solution according to Fig. 3.2
- a rational approximation according to Fenton and McKee (1990):

\[
\frac{k}{\omega^2/g} = \left[ \tanh \left( \frac{\omega^2 d}{g} \right)^{3/4} \right]^{-2/3}.
\]

(3.14)

For large water depths the hyperbolic term \(\tanh(kd)\) approaches one. Thus, for wave lengths of \(0.5L < d\) the dispersion relation can be simplified to

\[
\omega^2 = kg
\]

(3.15)

i.e. the water depth looses its influence on the wave. If a co-flowing (or Eule-rian) current \(U\) is present the Doppler-frequency shift \(kU\) must be accounted for by subtracting it from the total wave frequency

\[
(\omega - kU)^2 = kg \tanh kd.
\]

(3.16)

For shallow water where \((\omega^2 d)/g \leq 0.3\) the tanh-function approaches \(kd\) i.e.

\[
c \approx \frac{g}{\omega} kd = \frac{gd}{\omega/k} \\
c = \sqrt{gd} = c(d).
\]

(3.17)

Waves of this type are characterized as non-dispersive and have the same phase speed which solely depends on the water depth. A typical example for a non-dispersive wave is a Tsunami.

At very high wave frequencies the surface tension (nominally assumed to be \(\gamma = 0.074 \text{ N/m} \) at the air/water interface) influences the wave propagation. Lighthill (1978) derived a formulation for the dispersion relation for so called
surface ripples

\[ \omega^2 = \left( g + \frac{\gamma k^2}{\rho} \right) k \tanh kd. \]  

(3.18)

The influence the surface tension can be quantified by the Weber Number with WE

\[ WE = \frac{\rho g^3}{\omega^4 \gamma}. \]  

(3.19)

At Weber numbers of order one or less (respectively a wave length of 0.17 m or less) the surface tension becomes effective. It is obvious that the surface tension does not have any influence on wave forces on offshore structures in the real sea. It constitutes, however, a limiting parameter for the model scale at which wave experiments can be conducted – especially when nonlinear effects of high order wave theories are under consideration.

**Integral Properties** The **potential energy** \( P \) of a volume element \( dV = l \ dx dz \) where \( l \) is the length along the wave crest and can be normalized by \( l \) is

\[ dP = \rho gz dV. \]  

(3.20)
3.1 Airy Wave Theory

The total potential energy is then given by

\[ P_{\text{tot}} = \int_0^L \int_0^{\zeta(x)} \rho g z \, dzdx \]
\[ = \frac{1}{4} \rho g \zeta_a^2 \ell \quad (3.21) \]
related to the area \( \ell \cdot L \). We obtain the potential energy per square meter

\[ P = \frac{1}{4} \rho g \zeta_a^2 \quad (3.22) \]

(Clauss et al. (1992)). The kinetic energy is given by

\[ dK = \frac{1}{2} \rho (u^2 + w^2) \, dV. \quad (3.23) \]

Introducing the linear velocity components from Eq. 3.6 and 3.7 and after integration over one wave length the mean kinetic energy \( K \) is

\[ K_{\text{tot}} = \ell \int_0^L \int_{-d}^{\zeta(x)} \frac{1}{2} \rho (u^2 + w^2) dzdx \]
\[ = \frac{1}{4} \rho g \zeta_a^2 \ell \quad \text{related to the area } \ell \cdot L \text{ or} \]
\[ K = \frac{1}{4} \rho g \zeta_a^2 = P \quad (3.24) \]
i.e. the mean potential and the mean kinetic energy are identical. Thus, the total energy of a linear wave can be quantified by

\[ E = P + K = \frac{1}{2} \rho g \zeta_a^2 \quad (3.25) \]
which is proportional to the variance of the surface elevation by the factor \( \rho g \). Despite apparent cyclic orbits low magnitudes of energy, momentum and mass are transported by linear progressive waves as shown in the following:

The energy flux \( F \) (i.e. the rate of doing work) is given by

\[ dF = p_d \, dAu = p_d \, dzu \quad (3.26) \]
The wave averaged energy flux per unit length of crest is then

\[
F = \frac{1}{L} \int_0^L \int_{-d}^{\zeta(x)} \left[ \frac{1}{2} \rho (u^2 + w^2) + p + \rho g z \right] u \, du \, dx
\]

\[
= \frac{1}{2} \rho g \zeta_0 \frac{1}{2} \left( 1 + \frac{2kd}{\sinh 2kd} \right) \omega \frac{\omega}{k} \tag{3.27}
\]

To shorten the further derivations the hyperbolic term will be abbreviated by

\[
\Upsilon = \frac{1}{2} \left( 1 + \frac{2kd}{\sinh 2kd} \right) \tag{3.28}
\]

The speed of propagation of wave energy or the **group speed** can then be derived by the division

\[
c_{gr} = \frac{E}{F} = \frac{1}{2} \left( 1 + \frac{2kd}{\sinh 2kd} \right) c = \Upsilon c. \tag{3.29}
\]

This expression leads to \(c_{gr} \approx 0.5c\) in deep water, and \(c \approx c_{gr}\) for shallow water.

The differential mass transport rate, the **mass flux**, is \(dI \rho u \, dA\). Integration over one wave length and the water depth reveals the wave averaged mass flux per unit length of the wave crest

\[
I = \frac{1}{L} \int_0^L \int_{-d}^{\zeta(x)} \rho u \, du \, dx = \frac{E}{c}. \tag{3.30}
\]

The **momentum flux** (also being referred to as ”radiation stress”) is defined by the momentum transport rate

\[
dS = (p + \rho u^2) \, dA. \tag{3.31}
\]
Wave-averaged over one wave length per unit length of the wave crest it is

\[ S = \frac{1}{L} \int_0^L \int_{-d}^{\zeta(x)} (p_d + \rho u^2) dz dx \]
\[ = \frac{1}{2} \rho g \zeta_a^2 \Upsilon. \] (3.32)

### 3.2 Higher Order Wave Theory

Classical wave theory is valid only for regular waves. Each wave in a wave train has identical properties without changes when the wave travels. The lateral boundary condition applies for each individual wave crest. Therefore, the application is suitable to simulate the response to individual waves or a low number of successive waves.

In the following section, Sec. 3.2.1, the perturbation solution leading to the second order Stokes Wave Theory is described. The results from the fifth order Stokes Theory are also shown with the respective coefficients listed in Appendix B. Dean’s Stream Function Theory is shown in Section 3.2.2 which often applied to model waves in restricted water depth (but is also valid for deep water), followed by an outline of the Cnoidal Wave Theory in Sec. 3.2.3 which is applied under shallow water condition.

#### 3.2.1 Stokes Wave Theory

This chapter provides the theoretical principles of Stokes’s finite amplitude wave theories. The boundary conditions are approximated with Taylor series. Solutions of the boundary conditions are derived for the first and second order to provide the basis for the Stokes second order wave potential. After the boundary value problem has been formulated for first and second order, the corresponding members of the perturbation series of the wave potential function and surface elevation is solved. Solutions are presented for
Airy’s linear wave theory and for Stokes theory of second order. Skjelbreia and Hendrickson [1961] published an elaborate procedure including partially precalculated coefficients to determine the explicit form of the wave potential function and surface elevation after Stokes’ wave theory fifth order. The results of the method are presented at the end of this section with coefficients listed in Appendix B.

Perturbation The solution of the wave potential, $\Phi$, as well as the surface elevation, $\zeta$, will be approximated by a power series in terms of a non-dimensional perturbation parameter

$$\epsilon = k \frac{H}{2},$$

(3.33)
3.2 Higher Order Wave Theory

$H$ is the wave height and $k$ the wave number, defined as

$$k = \frac{2 \pi}{L}. \quad (3.34)$$

$\Phi$ can then be expressed in a serial form, where $\epsilon^l \Phi^{(l)}$ is the $l^{th}$ order solution for $\Phi$

$$\Phi(x, z, t, \epsilon) = \sum_{l=1}^{n=\infty} \epsilon^l \Phi^{(l)}(x, z, t). \quad (3.35)$$

Similarly, the wave profile, $\zeta$, is defined as

$$\zeta(x, t, \epsilon) = \sum_{l=1}^{n=\infty} \epsilon^l \zeta^{(l)}(x, t). \quad (3.36)$$

If the series are truncated after the second summand, so that $n = 2$, the following solution is obtained

$$\Phi(x, z, t) = \epsilon \Phi^{(1)}(x, z, t) + \epsilon^2 \Phi^{(2)}(x, z, t) \quad (3.37)$$

$$\zeta(x, t) = \epsilon \zeta^{(1)}(x, t) + \epsilon^2 \zeta^{(2)}(x, t). \quad (3.38)$$

These approximations include an error of the scale $\epsilon^3$ and their derivatives can be applied on the FSBCs.

Furthermore, Taylor expansions (see e.g. Bronstein et al. (1993)) are conducted to determine approximations for the $l^{th}$ order wave potential function, $\Phi^{(l)}$, and its derivatives at still water level, $z_0 = 0$

$$\Phi^{(l)}(x, z, t) = \Phi^{(l)}(x, z_0, t) + \Phi_z^{(l)}(x, z_0, t)(z - z_0) + \frac{1}{2} \Phi_{zz}^{(l)}(x, z_0, t)(z - z_0)^2 + \ldots. \quad (3.39)$$

Note, that the Taylor expansion series for any derivative of $\Phi^{(l)}$ has the same form as the Taylor expansion for $\Phi^{(l)}$ itself. Hence, they are not obtained by taking the derivative of the Taylor expansion of $\Phi^{(l)}$. At the surface $z = \zeta$, in the Taylor series (3.39) $z$ is substituted by the second order perturbation
series of the surface elevation, (Eq. 3.38). Truncating terms of order higher than two results in

\[
\Phi^{(l)}(x, z, t) = \Phi^{(l)}(x, 0, t) + \Phi_x^{(l)}(x, 0, t) (\epsilon \zeta^{(1)}(x, t) + \epsilon^2 \zeta^{(2)}(x, t)) + \frac{1}{2} \epsilon^2 \Phi_{zz}^{(l)}(x, 0, t) \zeta^{(1)}(x, t)^2.
\] (3.40)

Introducing the perturbation series (Eq. 3.37) and (Eq. 3.38) into the KFSBC (Eq. 2.8) gives

\[
(\epsilon \zeta^{(1)}_t(x, t) + \epsilon^2 \zeta^{(2)}_t(x, t)) (\epsilon \Phi^{(1)}_x(x, z, t) + \epsilon^2 \Phi^{(2)}_x(x, z, t)) + (\epsilon \zeta^{(1)}_x(x, t) + \epsilon^2 \zeta^{(2)}_x(x, t)) (\epsilon \Phi^{(1)}_{zz}(x, 0, t) \zeta^{(1)}(x, t)) - (\epsilon \Phi^{(1)}_x(x, z, t) + \epsilon^2 \Phi^{(2)}_x(x, z, t)) = 0.
\] (3.41)

Disregarding all terms higher than the second order results in

\[
\epsilon^2 \zeta^{(1)}_x(x, t) \Phi^{(1)}_x(x, z, t) + \epsilon \zeta^{(1)}_t(x, t) + \epsilon^2 \zeta^{(2)}_t(x, t) - \epsilon \Phi^{(1)}_x(x, z, t) - \epsilon^2 \Phi^{(2)}_x(x, z, t) = 0.
\] (3.42)

The Taylor series for \(\Phi^{(l)}\) and its derivatives are now introduced into (Eq. 3.42)

\[
\epsilon^2 \zeta^{(1)}_x(x, t) \Phi^{(1)}_x(x, 0, t) + \epsilon \zeta^{(1)}_t(x, t) + \epsilon^2 \zeta^{(2)}_t(x, t) - \epsilon \Phi^{(1)}_x(x, 0, t) - \epsilon^2 \Phi^{(2)}_x(x, 0, t) = 0.
\] (3.43)

By comparison of the coefficients, the KFSBC for \(\Phi^{(1)}\) and \(\Phi^{(2)}\) follows

\[
\zeta^{(1)}(x, t) - \Phi^{(1)}_z(x, 0, t) = 0 \quad \text{at } z = 0
\] (3.44)

\[
\zeta^{(1)}(x, t) \Phi^{(1)}_x(x, 0, t) + \zeta^{(2)}_t(x, t) - \Phi^{(1)}_z(x, 0, t) - \Phi^{(2)}_z(x, 0, t) = 0 \quad \text{at } z = 0.
\] (3.45)
Introducing the perturbation series of the velocity potential (Eq. 3.37) and of the surface elevation (Eq. 3.38) into the DFSBC (Eq. 2.11) and neglecting higher order terms leads to

\[
g \left( \epsilon \zeta^{(1)}(x,t) + \epsilon^2 \zeta^{(2)}(x,t) \right) + \\
\frac{1}{2} \left( (\epsilon \Phi^{(1)}(x,z,t))_x \right)^2 + (\epsilon \Phi^{(1)}(x,z,t))_z \right)^2 + \\
\epsilon \Phi^{(1)}(x,z,t)_t + \epsilon^2 \Phi^{(2)}(x,z,t)_t = 0.
\]

(3.46)

The Taylor series are again introduced

\[
g \left( \epsilon \zeta^{(1)}(x,t) + \epsilon^2 \zeta^{(2)}(x,t) \right) + \\
\frac{1}{2} \epsilon^2 \left( \Phi^{(1)}_{xx}(x,0,t)^2 + \Phi^{(1)}_{zz}(x,0,t)^2 \right) + \\
\epsilon \left( \Phi^{(1)}_t(x,0,t) + \epsilon \Phi^{(1)}_{tz}(x,0,t) \zeta^{(1)}(x,t) + \epsilon \Phi^{(1)}_z(x,0,t) \zeta^{(1)}_t(x,t) \right) = 0.
\]

(3.47)

From comparing the coefficients, the dynamic free surface boundary conditions for \( \Phi^{(1)} \) and \( \Phi^{(2)} \) follow

\[
\zeta^{(1)}(x,t) = -\frac{1}{g} \Phi^{(1)}_t(x,0,t) \text{ at } z = 0
\]

(3.48)

\[
\Phi^{(1)}_{tz}(x,0,t) \zeta^{(1)}(x,t) + \Phi^{(2)}_t(x,0,t) + \\
\frac{1}{2} \left( \Phi^{(1)}_{xx}(x,0,t)^2 + \Phi^{(1)}_{zz}(x,0,t)^2 \right) + g \zeta^{(2)}(x,t) = 0 \text{ at } z = 0.
\]

(3.49)

The EFSBC (2.17) can be rewritten as

\[
\Phi_{tt} + g \Phi_z + 2 (\Phi_x \Phi_{xt} + \Phi_z \Phi_{zt} + \Phi_x \Phi_x \Phi_{xx}) + \Phi_x^2 \Phi_{xx} + \Phi_z^2 \Phi_{zz} = 0.
\]

(3.50)

Introducing the derivatives of the perturbation series of the potential func-
tion, (3.37), and of the surface elevation, (3.38), gives

\[
e^{\Phi(1)}(x, z, t)_{tt} + e^2\Phi(2)(x, z, t)_{tt} + g\left(e^{\Phi(1)}(x, z, t)_z + e^2\Phi(2)(x, z, t)_z\right) + \\
2\left[(e^{\Phi(1)}(x, z, t)_x + e^2\Phi(2)(x, z, t)_x)\left(e^{\Phi(1)}(x, z, t)_{xt} + e^2\Phi(2)(x, z, t)_{xt}\right) + \\
(e^{\Phi(1)}(x, z, t)_z + e^2\Phi(2)(x, z, t)_z)\left(e^{\Phi(1)}(x, z, t)_{zt} + e^2\Phi(2)(x, z, t)_{zt}\right) + \\
(e^{\Phi(1)}(x, z, t)_x + e^2\Phi(2)(x, z, t)_x)\left(e^{\Phi(1)}(x, z, t)_z + e^2\Phi(2)(x, z, t)_z\right)
\right] + \\
\left(\epsilon\Phi(1)(x, z, t)_{xx} + e^2\Phi(2)(x, z, t)_{xx}\right) + \left(\epsilon\Phi(1)(x, z, t)_x + e^2\Phi(2)(x, z, t)_x\right)^2. \\
\left(\epsilon\Phi(1)(x, z, t)_{xz} + e^2\Phi(2)(x, z, t)_{xz}\right) + \left(\epsilon\Phi(1)(x, z, t)_z + e^2\Phi(2)(x, z, t)_z\right)^2. \\
\left(\epsilon\Phi(1)(x, z, t)_{zz} + e^2\Phi(2)(x, z, t)_{zz}\right) = 0.
\]

(3.51)

All terms higher than second order are neglected

\[
e^{\Phi(1)}(x, z, t)_{tt} + e^2\Phi(2)(x, z, t)_{tt} + g\left(e^{\Phi(1)}(x, z, t)_z + e^2\Phi(2)(x, z, t)_z\right) + \\
2\epsilon^2\left[\Phi(1)(x, z, t)_x \Phi(1)(x, z, t)_{xt} + \Phi(1)(x, z, t)_z \Phi(1)(x, z, t)_{zt}\right] = 0.
\]

(3.52)

The Taylor series, (3.40), of \(\Phi^{(k)}\) and its derivatives are introduced. Higher order terms are neglected

\[
e\left[\Phi^{(1)}_{tt}(x, 0, t) + e\Phi^{(1)}_{ttz}(x, 0, t) \zeta^{(1)}(x, t)\right] + e^2\Phi^{(2)}_{tt}(x, 0, t) + \\
g\left[\epsilon \left(\zeta^{(1)}(x, 0, t) + e\zeta^{(1)}(x, 0, t) \zeta^{(1)}(x, t)\right) + e^2\zeta^{(2)}(x, 0, t)\right] + \\
2\epsilon^2\left[\Phi^{(1)}_{xz}(x, 0, t) \Phi^{(1)}_{st}(x, 0, t) + \Phi^{(1)}_{zz}(x, 0, t) \Phi^{(1)}_{zt}(x, 0, t)\right] = 0.
\]

(3.53)

By means of coefficient comparison, the linearized and generalized free surface boundary condition is obtained for the first order terms, and the conditional equation for second-order potentials is obtained for the second order terms

\[
\Phi^{(1)}_{tt}(x, 0, t) + g\Phi^{(1)}_{zz}(x, 0, t) = 0 \quad \text{at} \ z = 0.
\]

(3.54)

The equation for the velocity potential second order, \(e^2\Phi^{(2)}\), is expected to have a similar form to the velocity potential first order, ( Eq. 3.4); whereby, \(k\)
3.2 Higher Order Wave Theory

is replaced by $2k$ and the phase, $\Theta$, by $2\Theta$. Applying the boundary conditions for $\Phi^{(2)}$, (Eq. 2.10) and (Eq. 3.42), leads to the second order term

$$
\epsilon^2 \Phi^{(2)} = \frac{3H^2 \omega \cosh (2k(z + d))}{2 \sinh^4(kd)} \sin(2\Theta). \quad (3.55)
$$

The sum of the first and second order solutions gives the velocity potential of Stokes second order wave theory

$$
\Phi = \epsilon \Phi^{(1)} + \epsilon^2 \Phi^{(2)} = \frac{H \omega \cosh (k(z + d))}{2k} \sin(\Theta) + \frac{3H^2 \omega \cosh (2k(z + d))}{2 \sinh^4(kd)} \sin(2\Theta).
\quad (3.56)
$$

The surface elevation results from introducing the values of $\phi^{(1)}$ and $\Phi^{(2)}$ into the linearized generalized boundary condition, (3.54), and into the exact free surface boundary condition, (Eq. 2.17)

$$
\zeta = \frac{H}{2} \cos(\Theta) + \frac{\pi H^2 \cosh(kd)}{8L \sinh^3(kd)} (2 + \cosh(2kd)) \cos(2\Theta).
\quad (3.57)
$$

The respective list of results for second order velocities and acceleration components are printed by [Clauss et al. (1992), Chakrabarti (2005)], and others. A photo of steep regular waves with typical Stokes-Wave-Theory profile with pointed crests with long shallow troughs is shown in Fig. 3.3.

**Solutions of the Stokes Wave Theory of fifth Order**

An equally detailed derivation of the fifth order wave theory would well exceed the scope of this thesis. Therefore, a simplified formulation as suggested by Skjelbreia and Hendrickson (1961) will be presented for the wave potential function of the fifth order. The wave height is given by

$$
H = \frac{2}{k} \left[ \lambda + B_{33} \lambda^3 + (B_{35} + B_{55}) \lambda^5 \right]. \quad (3.58)
$$
The equations to determine the coefficients $\lambda, A_{xx}$ and $B_{xx}$ are banished to Appendix B. The wave elevation is

$$\zeta = \frac{1}{k} \left[ \lambda \cos(\Theta) + (\lambda^2 B_{22} + \lambda^4 B_{24}) \cos(2\Theta) + (\lambda^3 B_{33} + \lambda^5 B_{35}) \cos(3\Theta) 
+ \lambda^4 B_{44} \cos(4\Theta) + \lambda^5 B_{55} \cos(5\Theta) \right]$$

(3.59)

The wave potential

$$\Phi = \frac{c}{k} \sum_{l=1}^{5} \lambda_l \cosh(lk(z+d)) \sin(l(kx-\omega t))$$

(3.60)

and the celerity, $c$, is given by

$$c^2 = \left(\frac{\omega}{k}\right)^2 = c_0^2(1 + \lambda^2 C_1 + \lambda^4 C_2)$$

(3.61)

In Fig. 3.4 the results of the fifth order Stokes wave theory are shown in comparison with Airy wave theory.

### 3.2.2 Stream Function Theory

Following the formulation by Dean (1965) the Stream Function Theory describes a steady, regular wave form in a coordinate system travelling with the constant wave propagation speed $c$. The time dependency vanishes and the surface elevation becomes constant in time. The Laplace equation and the BBC (bottom boundary condition) remain the same

$$\Delta \Phi = \Delta \Psi = \nabla^2 \Psi = 0 \text{ and } w = 0 \text{ at } z = -d$$

(3.62)

For the KFSBC (Kinematic free surface boundary condition, (see Eq. 2.8)) remains

$$\frac{\partial \zeta}{\partial x} = \frac{w}{u-C} \text{ at } z = \zeta(x)$$

(3.63)
3.2 Higher Order Wave Theory

Fig. 3.4: Results of the fifth order Stokes wave theory in comparison with Airy Wave Theory surface elevation (top), horizontal velocities (middle row) and accelerations (bottom row) at mean water level. All graphs are extrapolated to the crest height according to their wave theory and may exceed the actual water elevation.

and the DFSBC (Dynamic free surface Boundary condition), (see Eq. 2.10)

\[ \zeta + \frac{1}{2g} \left[ (u - C)^2 + w^2 \right] - \frac{C^2}{2g} = Q \text{ at } z = \zeta(x) \]  

\[ (3.64) \]
where the Bernoulli constant in the steady frame $Q$ includes the surrounding air pressure $p_0$. The motion is periodic with a spatial periodicity of the wave length $L$. The solution features a formulation of the Stream function as a truncated Fourier series

$$\Psi(x, z) = \frac{L}{T} z + \sum_{j=1}^{n} X_j \sinh [k_j(d + z)] \cos(k_j x)$$  \hspace{1cm} (3.65)

where $n$ represents the order of representation and $X_j$ coefficients. By setting $z = \zeta$ the surface elevation is

$$\zeta(x) = \frac{T}{L} \Psi_\zeta - \frac{T}{L} \sum_{j=1}^{n} X_j \sinh [k_j(d + \zeta)] \cos(k_j x).$$ \hspace{1cm} (3.66)

$\Psi_\zeta$ represents the constant value of the on the free surface. Eq. 3.65 and 3.66 are formulations that describe a high order regular wave form that is symmetric in $x$ about the crest. No requirements are placed on the permanence of the wave form, i.e. the wave could change its form as it propagates due to relative motion and interference of components with various speeds. Consequently, it does not have predictive capabilities in terms of wave propagation nor a solution for the dispersion relation.

If a particular wave condition is given in parametric form – that is wave height, period and water depth are specified – Eq.3.65 satisfies the bottom boundary condition as well as the KFSBC exactly for all combinations of $L$, $\Psi_\zeta$ and $X_j$. The, yet unknown, parameters $L$ and $X_j$ are then determined by a nonlinear numerical perturbation procedure, e.g. as shown by Dean (1974).

The Stream function theory can also be applied to a given irregular surface elevation. The wave length, the related wave number and wave period are taken from the record, e.g. by the trough-trough method. The coefficients $X_j$ are determined by approximating the real surface elevation with Eq. 3.66 e.g. by a least square algorithm.
3.2 Higher Order Wave Theory

3.2.3 Shallow Water Wave Theory

The Cnoidal Wave Theory is applicable to finite-amplitude shallow-water waves and includes both nonlinearity and dispersion effects. Korteweg and de Vries (1895) developed this wave theory. Further extensive studies are published by Keulegan and Patterson (1940) who analyzed the irrotational translation of a solitary wave, later complemented by Keller (1948) and Laitone (1960) who developed first- through third-order approximations to the Cnoidal Wave Theory.

Cnoidal waves are periodic with sharp crests separated by wide flat troughs as shown in Fig. 3.5. The approximate range of validity of the Cnoidal Theory is \( d/L < 1/8 \) when the Ursell number is \( UR > 25 \). Wiegel (1960) summarized the principal results in a more convenient form by presenting such wave characteristics as length, celerity, and period in tabular and graphical form to facilitate their application in coastal engineering. Cnoidal Wave Theory is restricted to waves progressing uniformly in only one direction. In the following descriptions the horizontal coordinate \( X = (x - ct)/L \) is chosen, that is moving continuously with the wave.

It forms the solution of the ordinary differential equation that has the general form

\[
\frac{d^3F}{dX^3} + \alpha F \frac{dF}{dX} - \beta \frac{dF}{dA} = 0 \tag{3.67}
\]

where \( F \) is a function of \( X \). This equation has a periodic solution containing the Jacobian elliptic functions \( cn \), and \( sn \) (see Appendix D, Eq. D.2 to D.5) with the parameter of the elliptic functions \( m \). The first order results are according to Sobey (2002).

\[\text{1} \text{Jacobian elliptic integrals can alternatively be given in terms of the modulus } k \text{ (not to be confused with the wave number) with } k^2 = m.\]
The expressions \( K(m) \) and \((E)(m)\) stand for the complete elliptical integral of the first and second kind. Its mathematical definition is shown in Appendix D. Both functions \( cn^2(\alpha X; m) \) and \( scd(\alpha X; m) \) are periodic with the period argument \( 2K(m) \). For the limit \( m \rightarrow 1 \) the period becomes infinite which leads to the solitary wave theory. In the first order shallow water approximation the wave length \( L \) is initially unknown and can be approximated by

\[
\left( \frac{2K(m) \, d}{L} \right) = \frac{H}{d} \frac{3}{4m}.
\] (3.72)

The Jacobian \( m \) is used as dependent variable in the dispersion relation – similar to the wave number \( k \) in the Linear- and Stokes Wave Theory. The first order dispersion equation is given by

\[
\frac{L}{T} = \frac{2K(m)d/T}{\sqrt{\varepsilon F_1}}
\] (3.73)

where a particular dependence on the wave height \( H \) through the parameter \( \varepsilon = \frac{H}{d} \) becomes obvious, even at first order. Eq.\(3.73\) can only be solved numerically. Above the maximum steepness of \( H/d = 0.78 \) no convergence is reached (Wiegel (1960)). As the ratio of wave height to water depth becomes small, i.e. infinitesimal wave height, the wave profile approaches...
3.2 Higher Order Wave Theory

the sinusoidal profile as predicted by the linear theory. Additional terms that form the second order solutions are given by Sobey (1990) as follows

\[
\Psi_{cn2}(X, z) = \Psi_{cn}(X, z) + \sqrt{gh} \left[ \varepsilon^2 (1 + z/d) \left( A_{20} + A_{21} cn^2(\alpha X; m) + A_{22} cn^4(\alpha X; m) \right) + \frac{1}{3} \varepsilon^2 (1 + z/d)^3 A_{11} F_1 \left( -1 + m + 2(1 - 2m) \right) \right] \]

\[
\zeta_{cn2}(X) = \zeta_{cn}(X) + h \left[ \varepsilon^2 \left( B_{20} + B_{21} cn^2(\alpha m; m) + B_{22} cn^4(\alpha X; m) \right) \right] \]

\[
u_{cn2}(X, z) = \nu_{cn}(X, z) + \sqrt{gd} \left[ \varepsilon^2 \left( A_{20} + A_{21} cn^2(\alpha; m) + A_{22} cn^4(\alpha X; m) \right) + 2 \varepsilon^2 (1 + \frac{z}{d})^2 A_{11} F_1 \left( -1 + m + 2(1 - 2m) \right) \right] \]

\[
w_{cn2}(X, z) = w_{cn}(X, z) + \sqrt{gd^2 \alpha scd(\alpha X; m)} \left[ \varepsilon^2 (1 + \frac{z}{d}) \left( 2 A_{21} + 4 A_{22} cn^2(\alpha X; m) \right) + \frac{\varepsilon^2 A_{21} F_1 (1 - 2m)}{3} + 3m \right] \]

Up to the second order the dynamic pressure resulting from the wave is neglected. Hence, only the hydrostatic portion is effective relating to the instantaneous surface

\[
p_{dyn} = \rho g (z + \zeta(x)) \]
Fig. 3.5: Surface Profile according to second order Cnoidal Wave Theory (H/d = 0.5)
3.3 Irregular Seaway

Irregular or natural seaway is considered as a random process inspiring the quote "The Basic law of the seaway is the apparent lack of any law." John William Strutt, 3rd Baron Rayleigh. Despite of this perception Lord Rayleigh found the Rayleigh-distribution which is widely used as probability distribution for the wave heights in seaway statistics. The following section outlines the current concepts in seaway statistics including the probabilities of extreme events followed by a short discussion. In Sec. 3.3.3 spectral aspects are explained.

3.3.1 Statistical Properties

Irregular Seaway can only be described by statistical means. In the Gaussian Spectral Random Wave Model the irregular seaway is interpreted as a superposition of harmonic component wave with uniformly distributed phases. Independent from the probability distributions of the amplitude and phases the Central Limit Theorem assures a Gaussian distributed surface elevation with zero mean and standard deviation $\sigma$. Assuming that the seaway is narrow banded ($\varepsilon > 0.8$ see Sec. 3.3.3), and that the waves are independent of each other the probability density functions of the wave heights (or envelope curve) is given by the Rayleigh distribution

$$\phi_R(H) = \frac{H}{4\sigma^2} e^{-\frac{H^2}{8\sigma^2}}$$

(3.79)

where $\sigma$ denotes the standard deviation of the surface elevation. With the significant wave height being $H_S = 4\sigma$

$$\phi_R(H) = \frac{4H}{H_S^2} e^{-\frac{2H^2}{H_S^2}}$$

(3.80)

The probability of exceedance is accordingly

$$P(H > H_0) = e^{-\frac{2H_0^2}{H_S^2}}.$$  

(3.81)
48 Wave Theory

Tab. 3.1: Extreme wave heights to be expected during a 3 hour storm according to Rayleigh’s PDF: The 95th percentile denotes an extreme wave height that appears once in 20 storms while the 99th percentile once in 100.

<table>
<thead>
<tr>
<th>$T_0$</th>
<th>$N$</th>
<th>$H_{max}/H_S$ (p95)</th>
<th>$H_{max}/H_S$ (p99)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 s</td>
<td>2160</td>
<td>2.31</td>
<td>2.47</td>
</tr>
<tr>
<td>10 s</td>
<td>1080</td>
<td>2.23</td>
<td>2.41</td>
</tr>
<tr>
<td>12 s</td>
<td>900</td>
<td>2.21</td>
<td>2.39</td>
</tr>
<tr>
<td>15 s</td>
<td>720</td>
<td>2.19</td>
<td>2.36</td>
</tr>
</tbody>
</table>

3.3.2 Extreme Events

For a stationary short term seaway consisting of $N$ waves the particular probabilities of exceedance is accumulated to find the PDF of the maximum wave height

$$
\phi(H_{max}) = N \left(1 - e^{-\frac{2n^2}{H_S}}\right)^{N-1} \frac{4H}{H_S^2} e^{-\frac{2n^2}{H_S}}
$$

(3.82)

assuming that the waves are independent from each other. For $N = 1080$ (which represents a stationary seaway with a duration of three hours and $T_0 = 10$ s $\phi(H_{max})$ has its maximum (also referred to as ”most probable maximum”) at $1.86H_S$ and its mean at approx. $1.95H_S$. To derive (short term) design wave heights based on Eq. 3.82 the 95th or 99th percentile can be used with $H_{p95}$ resulting from the relation

$$
0.95 = \int_{0}^{H_{p95}} \phi(H_{max})dH
$$

(3.83)

($H_{p99}$-respectively). Fig. 3.6 shows the PDF of the maximum wave height (Eq. 3.82) for a three hours sea state with various zero-upcrossing periods.

For large number of waves the Gumbel distribution – a two parameter Weibull distribution – which is a generalized extreme value distribution for $x = H_{max}/H_S$
Fig. 3.6: PDFs of the maximum wave height of a point estimate over three hours (10800 s) for different zero-upcrossing periods.
may be used \( (\text{Gumbel (1958)}) \)

\[
G(x, \mu, b) = \exp \left( - \exp \left( - \frac{x - \mu}{b} \right) \right) \text{ with } \\
\mu = \sqrt{\frac{\log N}{2}} \text{ and } b = \frac{1}{4\mu}. \tag{3.84}
\]

A probability distribution of the maximum wave crest heights is proposed by \( \text{Forristall (2000)} \):

\[
F(\zeta_c) = 1 - \exp \left[ - \left( \frac{\zeta_c}{(\alpha_F h)^{\beta_F}} \right)^{\beta_F} \right] \text{ with } \\
\alpha_F = 0.3536 + 0.2892 s_1 + 0.1060 UR \\
\beta_F = 2 - 2.1597 s_1 + 0.0968 UR^2 \\
s_1 = \frac{2\pi H_s}{g(0.75TP)^2}
\]

with the Ursell parameter defined as

\[
UR = \frac{H_S}{k_P d^3}
\]

\( k_P \) is the peak-wave number that is associated with the peak period. Alternative formulations are published by \( \text{Jahns and Wheeler (1972)} \) with coefficients specified by \( \text{Haring and Heidemann (1978)} \). Another formulation is published by \( \text{Tayfun (1980)} \) which is preferred in modelling the crest height in steep irregular waves.

So far, statistics have referred to single point estimations in irregular seaways. In design considerations the probability of extreme waves or crest heights must refer to a defined area, e.g. the main deck area of a platform is relevant. Applying Piterbarg’s Theorem \( \text{(Piterbarg (1996))} \) for asymptotic distributions for Gaussian processes over large multi-dimensional spaces \( \text{Krogstad et al. (2004)} \) derived probability distributions for the maximum crest height over large sea areas. They found the expected maximum crest height over
3.3 Irregular Seaway

Fig. 3.7: PDFs of the maximum wave height. The Rayleigh probability density function "smears" into the negative domain for greater bandwidths approaching the normal distribution. \[ \epsilon = \sqrt{1 - \frac{m_2}{m_0 m_4}} \quad \text{with } [0 \leq \epsilon \leq 1] \] (see Sec.3.3.4)

\[ \zeta_{\text{max}} = 1.32 H_S. \] (3.86)

Similar derivations and numerical simulations by Forristal (2006) showed that considering an area of 60 \( m^2 \) the expected values of maximum crest heights exceed the point estimate by 20 per cent in a linear model as well as in a second order model (\( H_S = 8 \text{ m}, T_P = 10 \text{ s}; \) over a period of \( T=1024 \text{ s} \)).

Discussion The validity of wave statistics based on the Rayleigh distribution (Eq. 3.79) is under discussion since in natural seaway its fundamental assumptions are often violated. The Rayleigh distribution is restricted to strictly narrow banded seaway which is not satisfied for typical wind wave spectra. As shown in Fig. 3.7 the probability density function "smears" into the negative domain for greater bandwidth approaching the normal distribution (Rice (1944)). The probability of extreme values (in the upper tail) lowers. A very small bandwidth, on the other hand, abets the development of wave modulations and self-amplifying wave groups as shown in Sec. 3.5. In this case the assumption of each wave being independent from each other does not hold. The assumption of stationarity may be challenged if seaway in fast changing weather conditions is investigated such as in hurricanes.
Bitner-Gregersen and Hagen (2004) point out that the punctual measurements and satellite photos currently available do not provide a sufficient database to derive reliable long term statistics of extreme waves. Further analysis of reliable long term data is necessary for reliable statistics on extreme waves.

### 3.3.3 Spectral Properties of Natural Seaway

The linearity of the Airy Wave Theory opens the possibility to apply spectral analysis on irregular seaway. Irregular seaway is assumed to be a superposition of a large number harmonic elementary waves. Although an irregular seaway cannot be considered periodic the lateral boundary condition (defined in Sec. 2.2) holds up since every summation of harmonic functions with rational arguments is periodic in time and space. Its period increases to a scale which exceeds the scope of short term seaway analysis. The complex Fourier spectrum $F(\omega)$ can be generated through Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} \zeta(t)e^{-i\omega t} dt \quad (3.87)$$

or, vice versa, created from a Fourier-spectrum by inverse Fourier transform

$$\zeta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{i\omega t} d\omega \quad (3.88)$$
For finite records of surface registrations with sampling rate \( f = 1/\Delta t \) and length \( T = N\Delta t \) the Fourier transform pair (Eq.3.87 and 3.88) turns into

\[
F(n\Delta \omega) = \Delta t \sum_{n_0}^{n=N-1} \zeta(n\Delta t)e^{-i2\pi r n/N} \\
\text{with } r = 0, 1, 2, \ldots, N/2 
\]

(3.89)

\[
\zeta(n\Delta t) = \frac{\Delta \omega}{2\pi} \sum_{r_0}^{n=N/2} F(r\Delta \omega)e^{i2\pi r n/N} \\
\text{with } n = 0, 1, 2, \ldots, (N - 1) 
\]

(3.90)

(3.91)

where \( \zeta(n\Delta t) \) represents the discrete time record and \( \Delta \omega = 2\pi/(N\Delta t) \) the respective frequency resolution. An efficient and widely used routine to evaluate the sums is the Fast Fourier Transformation as presented and implemented by [Cooley and Tukey (1965)](https://www.jstor.org/stable/1991503). The Fourier spectrum and the energy spectrum (also called variance spectrum) are related through the Parseval Theorem

\[
S(r\Delta \omega) = \frac{1}{\pi T}|F(r\Delta \omega)|^2 
\]

(3.92)

During the transition to the energy spectrum all phase information is lost. Thus, a specific wave train cannot be reconstructed. The energy per square meter of each component wave \( \zeta_n(t) = \zeta_n \cos(kx - \omega_n t + \phi) \) is represented by a partial area under the spectrum

\[
\frac{1}{2}\zeta_n^2 = S(\omega)\Delta \omega
\]

(3.93)
An alternative procedure to generate a one-sided energy spectrum from the Fourier Transform of the autocorrelation function

\[
S(\omega) = \frac{1}{\pi} \int_{0}^{\infty} R(\tau) e^{-i\omega \tau} d\tau
\]

with the autocorrelation function

\[
R(\tau) = \lim_{T \to \infty} \int_{-\infty}^{\infty} \frac{1}{T} \zeta(t) \zeta(t + \tau) dt
\]

is named the Wiener-Khinchine-Theorem (Bendat and Piersol (1986)). The advantage is the averaging quality of the convolution integral leading to smoother energy spectra while the numerical effort is much higher than using the direct transform in Eq.3.87 and smoothing afterwards. The phase shift \( \varphi \) is distributed uniformly between zero and \( 2\pi \). Alternatively it is possible not to chose a fixed frequency resolution but to employ a resolution that creates harmonic wave components that have identical energy content – i.e. amplitudes (Söding (1987)). This leads to a higher resolution of high energetic regions in frequency domain while the low energetic regions – i.e. the high frequent tail of an empirical spectrum – is not represented as exact.

**Spectral Parameters** To characterize irregular seaway spectral moments \( m_n \) are used

\[
m_n = \int_{0}^{\infty} \omega^n S(\omega) d\omega
\]

where the area under the spectral curve \( m_0 \) represents the over all wave energy per square meter normalized by \( \rho g \). It is equal to the variance \( \sigma^2 \) of the surface elevation. The spectral moment \( m_0 \) and moments of higher order can be used to derive

- significant wave height: \( H_s = 4\sqrt{m_0} = 4\sigma \)

- modal frequency: \( \omega_1 = \frac{m_1}{m_0} \)
3.3 Irregular Seaway

- zero-upcrossing frequency: \( \omega_z = \sqrt{\frac{m_2}{m_0}} \)

- bandwidth parameter: \( \epsilon = \sqrt{1 - \frac{m_2^2}{m_0 m_4}} \) with \( 0 \leq \epsilon \leq 1 \)

The use of the fourth spectral moment \( m_4 \) for measured energy spectra may imply inaccuracies since high frequencies regarded at fourth power are difficult to measure and have only little physical impact. Alternatively Longuet-Higgins (1952) suggests the measure \( \nu \)

\[
\nu = \sqrt{\frac{m_0 m_2}{m_1^2}} - 1. \tag{3.97}
\]

3.3.4 Empirical Wave Spectra

The parameters for the description of the surface elevation in irregular seaways consist of statistical properties, which are

- the significant wave height \( H_S \): The mean wave height of the one third highest waves,

- the Zero-Upcrossing-Period \( T_0 \): The average time between two zero upcrossings,

- the shape of the wave energy spectrum, and

- a directional wave distribution function.

Alternatively the significant wave height can be substituted with the root mean square RMS of the wave height or the standard deviation \( \sigma \) of the elevation (providing that the mean water level lays exactly at \( z = 0 \))

\[
H_S = 1.416 H_{RMS} = 4\sigma \tag{3.98}
\]

The approximate similarity in shape of most measured wind spectra – especially under growing sea conditions – has lead to numerous attempts to
define a universal spectral form. For design purposes parametric standard spectra have been established to describe a typical distribution of the wave energy in the frequency domain.

Pierson-Moskowitz-Spectrum  The Pierson-Moskowitz-Spectrum was originally formulated in dependence of the mean wind speed measured $U_{19.5}$ at a height of 19.5 m meters above sea level

$$
S(\omega) = \frac{\alpha_p g^2 2 \pi}{\omega} e^{-0.74\left(\frac{\omega}{v_{19.5}}\right)^4}.
$$

(3.99)

It describes a fully developed seaway with unlimited fetch. The Phillips coefficient $\alpha_p$ characterizes atmospheric energy transfer and is therefore proportional with the wave steepness $\alpha \sim (k\zeta_a)^2$. [Pierson and Neumann (1963)] propose an approximation with $\alpha_p = 0.0081$. Introducing the relation

$$
H_S = \frac{0.21}{g} U_{19.5}^2 \text{ and } T_0 = 0.81 \left(\frac{2\pi}{g} U_{19.5}\right)
$$

(3.100)

a generic, parametric formulation was gathered that found wide spread application in the offshore industry [Pierson and Moskowitz (1964)]

$$
S(\omega) = 124 \frac{H_S^2}{T_0^4} \frac{1}{\omega^5} e^{\frac{496}{\gamma^2}}
= 490 \frac{H_S^2}{T_P^4} \frac{1}{\omega^5} e^{\frac{1955}{\gamma^2}} \text{ with } T_P = \frac{2\pi}{\omega_P}.
$$

(3.101)

The JONSWAP spectrum  The JONSWAP spectrum (JONSWAP: Joint North Sea Wave Project) was extracted from a large number of measurements within the North Sea by Hasselmann et al. (1973). The formulation by Houmb and Overik (1976) in dependence on $\omega$ instead of $f$

$$
S(\omega) = \frac{\alpha_p g^2}{\omega^5} e^{-\frac{\omega}{\gamma}} e^{-\frac{\omega}{v_{19.5}}}.
$$

(3.102)
3.3 Irregular Seaway

\[
\begin{array}{|c|cccccc|}
\hline
\gamma & 1 & 2 & 3 & 3.3 & 4 & 5 & 6 \\
\hline
T_p/T_1 & 1.296 & 1.240 & 1.206 & 1.198 & 1.183 & 1.165 & 1.151 \\
T_p/T_0 & 1.408 & 1.338 & 1.295 & 1.285 & 1.264 & 1.240 & 1.221 \\
T_1/T_0 & 1.086 & 1.079 & 1.073 & 1.072 & 1.069 & 1.065 & 1.061 \\
\hline
\end{array}
\]

**Tab. 3.2:** Ratios of peak period \( T_p \), zero-upcrossing period \( T_0 \), and modal period \( T_1 \) according to Wichers (1979)

with

\[
\alpha_p = 0.3395 \left( \frac{H_s \omega_p^2}{g} \right)^{2.036} (1 - 0.298 \ln \gamma) \quad \text{(3.103)}
\]

\[
B = e^{-\omega^2 \omega_p^2 / 2} \quad \text{and}
\]

\[
\sigma = \begin{cases} 
0.07 \text{ for } \omega < \omega_p \\
0.09 \text{ for } \omega > \omega_p 
\end{cases} \quad \text{(3.104)}
\]

differs from the original formulation in which the peak-enhancement-factor \( \gamma \) describes the ratio of the enhanced peak height and the original PM-spectrum. Physically \( \gamma \) represents the influence of a limited fetch. Its most probable value is \( \gamma = 3.3 \). In the present formulation the Phillips-constant \( \alpha \) is modified (Eq. 3.103) in order to keep total energy content independent from \( \gamma \). Thus, the relation \( H_S = 4 \sqrt{m_0} \) remains valid at all times. The spectral moments, and therefore the ratios of peak period, zero-upcrossing period, and modal period are related according to table 3.2 (Wichers (1979)).

**The Torsethaugen Spectrum** While the above spectra refer to pure wind seas the Torsethaugen Spectrum includes two components: Wind sea generated by the local wind, and swell seas which consist of waves entering into the location from other areas which has a greater peak period than the period of a fully developed wind sea \( T_{p,f} \): \( T_{p,\text{swell}} > T_{p,f} \). The peak period of the wind sea is associated with the significant wave height by the ratio

\[
T_{p,f} = a_f \sqrt{H_s} \quad \text{(3.105)}
\]
where $a_f$ is an empirical factor that accounts for the fetch: $a_f = 5.3 \frac{m}{m}$ for a fetch of 100 km and $a_f = 6.6 \frac{m}{m}$ for 370 km. Each of the two spectral components is represented by separate formulation derived from the JONSWAP spectrum which is then superimposed with the other. The original formulation is based upon data collected during nine years at the Statfjord field ([Torsethaugen (1993)]), Gullfaks C and other locations in the North Sea see Fig. 1.5). A later simplified formulation with a reduced number of parameters for wind sea dominated two peaked spectrum ([Torsethaugen (2004)]) is given by

$$S(\omega) = 3.26 \frac{H_s^2 T_p 1 + 1.1 [\ln \gamma]^{1.19}}{16} \frac{1558.5}{\omega_{1n}^3} e^{-\frac{1558.5}{\omega_{1n}^3} \gamma^p} + 3.26 \frac{(1 - H_s^2) T_p 1}{16} \frac{1558.5}{\omega_{2n}^5} e^{-\frac{1558.5}{\omega_{2n}^5} \gamma^p}$$

(3.106)

with $p = e^{-\left[\frac{1}{2\pi^2} (\frac{2}{\sigma} - 1)^2\right]}$.

where the first line represents the primary system driven by local wind and the second line the swell. $\sigma$ conforms with the JONSWAP spectrum (see Eq. 3.104). The particular peak frequencies of the components $\omega_{xn}$ are normalized with the peak frequency $\omega_{xn} = \frac{\omega_x}{\omega_p}$. The relation of the component wave heights of the spectra $H_{s1}$ and $H_{s2}$ is defined by the expressions

$$H_{s1} = R_w H_s$$

$$H_{s2} = \sqrt{1 - R_w^2} H_s$$

where $R_w$ is an empirical coefficient relating to the fetch and other local properties. Details are presented by [Torsethaugen (2004)] along with a similar formulation for swell dominated seaway. Given the empirical nature of the spectrum it is only applicable in the North Sea area, where it has been extracted. A more generic spectrum, the Ochi- Six-Parameter-Spectrum, was published by [Ochi and Hubble (1998)] which had been extracted from point measurements with wave buoys, and refined by vertical wave radar measure-
It should be noted that wave systems of different origins are most likely to have different propagation directions. Therefore, short crested seaway should be assumed.

**Fig. 3.8:** Spectral density functions of a wind dominated seaway with $H_s = 3$ m and $T_P = 5$ s. The Pierson-Moskowitz Spectrum and the JONSWAP spectrum ($\gamma = 3.3$) represent pure wind sea while the Torsethaugen also includes a low frequent swell sea component.
3.4 Modelling Wave Kinematics of Irregular Seaway

While the linear wave theory is consistent for the wave kinematics and wave propagation for each single wave component, their superposition shows strong local inaccuracies. Especially when steep wave groups occur the superposition does not obey the restrictions regarding the limited wave steepness resulting from the linearisation of the kinematic free surface boundary condition (KFSBC, Eq. 2.8). The hyperbolic (for deep water exponential) functions become unrealistic large for components of interacting wave components with differing wave numbers. This leads to a failure of the linear model to predict the water particle kinematics. The errors arising from the breach of the KFSBC are quantified by Sobey (2002) as

\[ K(x, t) = w(\zeta, t) - \frac{\partial \Phi(\zeta, t)}{\partial z}, \quad (3.107) \]

and of the DFSBC as

\[ D(x, t) = \frac{\partial \Phi}{\partial t} + \frac{1}{2} (u^2 + w^2) + g\zeta(x, t) - p_0. \quad (3.108) \]

As illustrated by a theoretical example in Appendix A the error due to the breach of linearized KFSBC \( K(x, t) \) becomes dominant when a superposition of many elementary waves is considered.

Stretching Theories

A simple extrapolation of kinematics of many harmonic wave components above the mean water line does not lead to valid results in steep irregular waves. As a fast and rational solution Wheeler (1970) proposed to keep the exponential term up to the wave crest and to apply a stretching term which rescales the vertical coordinate of the exponential term from the linear wave theory (or hyperbolic term if applied in restricted water depth) according to
3.4 Modelling Wave Kinematics of Irregular Seaway

\[ z' = \frac{z + d}{1 + \frac{\zeta}{d}} - d \]  

(3.109)

where \( z' \) is the new computational, vertical coordinate \((-d \leq z' \leq 0)\), \( z \) is the actual vertical coordinate \((-d \leq z \leq \zeta)\), \( d \) is the water depth, \( \zeta \) is the instantaneous surface elevation. Particle kinematics that are computed in the mean water line are "stretched" to the wave crest. Alternative, yet similar, stretching terms are introduced by [Chakrabarti (2005)] including one for second order wave profiles.

This stretching method is easy to apply and constitutes the current industry standard according to ISO 19901-1. The scientific community, however, agrees that stretching theories lead to a substantial underestimation of wave kinematics and therefore have to be used only with caution for design purposes [Gudmestad (1989), Kjeldsen et al. (2000), Baldock et al. (1996), Clauss et al. (2007)] and once more in Sec. [8.1]. The so called "Delta Stretching" is a compromise between linear extrapolation and Wheeler’s method introducing the two empirical ”tuning” coefficients \( \Delta \) and \( d_\Delta \). The latter is the depth above which the kinematics profile is stretched.

\[ z_\Delta = (z + d_\Delta) \frac{d_\Delta + \Delta \zeta(t)}{d_\Delta + \zeta(t)} \quad \text{for} \quad \begin{cases} z > -d_\Delta & z_\Delta = z \\ \zeta > 0 & z_\Delta = \end{cases} \]

(3.110)

The parameter combination \( \Delta = 0 \) and \( d_\Delta = d \) yields the stretching term according to Wheeler whereas \( \Delta = 1 \) and \( d_\Delta = d \) results in a pure extrapolation of the linear wave theory. The value of \( d_\Delta \) must be chosen sufficiently small compared to random wave heights. As the free surface dips below \( d_\Delta \) Eq. 3.110 becomes ill defined. [Rodenbusch and Forristall (1986)] propose the combination \( \Delta = 0.3 \) and \( d_\Delta = 2\sigma \) based on measurements taken from the Fulmar Platform (North Sea) and laboratory data.
3.4.1 Local Fourier Approximation

The Local Fourier Approximation proposed by Sobey (1992a) is a practical approach to model highly nonlinear wave kinematics underneath a known water surface elevation. The problem is formulated as an optimization task where the two goal functions are stated by the free surface boundary conditions as shown in Eq. 3.112 and Eq. 3.113. With $U$ being an Eulerian current, the potential function is considered similar to superposition of linear wave potentials:

$$\Phi(x, z, t) = Ux + \sum_{j=1}^{n} A_j \frac{\cosh j(k(d + z))}{\cosh jkd} \sin j(kx - \omega t) \quad (3.111)$$

with a set of parameters $\omega, j, k, kx, A_j$ that is valid only for a local window of a given surface elevation in time. The width of the considered window lies in the order of one tenth of the zero-upcrossing period. $kx$ can be interpreted as a spatial phase shift, which is independent from the wave number $k$. The
index \( j \) denotes the order of each component while the form of summation is responsible for its name. \( U \) denotes an Eulerian current. It should be noted, however, that all components are bound and have no relative phase shift which is the essential difference to the linear superposition carried out by the discrete Fourier transform.

To find the instantaneous set of parameters an optimization algorithm is proposed that uses implicit formulations of the FSBCs as goal functions. Using the index \( i \) for each point distributed within the considered window, the specific formulation of the KFSBC is

\[
f^K_i(\omega, k, kx, A_j) = w_i - \frac{\partial \zeta_i}{\partial t} - u_i \frac{\partial \zeta_i}{\partial x} \Rightarrow 0.
\]

Since spatial gradients are not available from the measured record at a single site it can be substituted with the assumption of a steady profile assumption

\[
\frac{\partial \zeta}{\partial x} = \frac{k \partial \zeta}{\omega \partial t} = \frac{1}{c} \frac{\partial \zeta}{\partial t}.
\]

Note that this assumption is not imposed beyond the considered window. The DFSBC is implicitly formulated by

\[
f^D_i(\omega, k, kx, A_j) = \frac{\partial \Phi_i}{\partial t} + \frac{1}{2} u_i^2 + \frac{1}{2} w_i^2 + g\zeta_i - B \Rightarrow 0
\]

with the derivatives

\[
u_i = \frac{\partial \Phi_i(x, \zeta_i)}{\partial x} = U + \sum_{j=1}^{n} jkA_j \frac{\cosh jk(d + \zeta_i)}{\cosh jkd} \cos j(kx - \omega t_i)
\]

\[
w_i = \frac{\partial \Phi_i(x, \zeta_i)}{\partial z} = -\sum_{j=1}^{n} jkA_j \frac{\sinh jk(d + \zeta_i)}{\cosh jkd} \sin j(kx - \omega t_i)
\]

\[
\frac{\partial \Phi_i}{\partial t} = -\sum_{j=1}^{n} j\omega A_j \frac{\cosh jk(d + \zeta_i)}{\cosh jkd} \cos j(kx - \omega t_i).
\]
In the present coordinate system, in which the mean water line defines \( z = 0 \) the nonlinear Bernoulli constant reduces to (Longuet-Higgins (1975))

\[
B = U^2 + \frac{1}{4} \sum_{j=1}^{n} \left( \frac{j k A_j}{\cosh(j k d)} \right)^2.
\] (3.118)

The given record of the surface elevation may be extracted from numerical simulations, experiments, or real sea measurements. To extract a sensible value for the first derivative \( \frac{\partial \zeta}{\partial t} \) which is part of the goal function (used in Eq. 3.112 as well as in Eq. 3.113) the investigated record of the surface elevation must be of high quality i.e. smooth. Especially for normally low frequent real sea measurements a cubic spline interpolation is proposed to increase the number of data points (Sobey (1992a)). The total number of nodes \( M \) depends on the order \( n \) to which the wave potential should be solved. Numerically an over-specification is advantageous, thus \( M \geq n + 3 \).

To overcome remaining difficulties in convergence a moving average filter is proposed to suppress unrealistic results. The filter width \( T_{FW} \) must always remain well below the period of the highest order wave component

\[
T_{FW} \ll \frac{2\pi n}{\omega}
\] (3.119)

which poses a limit for the highest reasonable order of analysis. It must be pointed out that with respect to Eq. 3.111 the "higher order" terms are determined by an optimization algorithm and are not a result of any expansion. The formulation consists of a superposition of linearly formulated wave components which in combination fulfill –locally– the free surface boundary conditions. Hence the name: Local Fourier Approximation.
3.5 The Nonlinear Schrödinger Equation Approach

Using perturbation theory applied on the nonlinear free surface boundary condition [Benjamin and Feir (1967)] showed that weakly nonlinear regular water waves are unstable, i.e. once water waves are propagating portions of the energy migrate to adjacent side band frequencies. This leads to modulations and magnification of the initially regular wave field. [Zakharov (1968)] classified this effect as negative pressure-type instability and showed that is exists for all types of dispersive waves for which the sign of the second derivative ($\partial^2 \omega / \partial k^2$) is different from the sign of the frequency shift due to the non-linearity, in this case the dependence of the wave velocity on the wave number. An expansion solution of the BVP up to the fourth order is presented by [Dysthe (1979)], in which a modulation of the finite amplitude wave train causes non-uniformity of the radiation stress leading to a slow varying Doppler shift that has a 'detuning' effect on the modulation instability.

A narrow banded wave field with a bandwidth on $\Delta k$ and characteristic amplitude $a$ is assumed. Beginning from Laplace equation

$$\nabla^2 \Phi = 0 \tag{3.120}$$

and the nonlinear kinematic and dynamic surface conditions

$$\frac{\partial \zeta}{\partial t} + \nabla \Phi \cdot \nabla \zeta = \frac{\partial \Phi}{\partial z} \quad \text{at } z = \zeta(x, y, t) \tag{3.121}$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} (\nabla \Phi)^2 + g\zeta = 0 \quad \text{at } z = \zeta(x, y, t) \tag{3.122}$$
With a phase function defined as \( \theta = (kx - \omega t) \) the expansion consists of

\[
\Phi = \Phi + \frac{1}{2} \left( Ae^{i\theta + k z} + A_2 e^{2i\theta + 2k z} + \ldots + Ae^{-i\theta + k z} + A_2 e^{-2i\theta + 2k z} + \ldots \right)
\]

\[
(3.123)
\]

\[
\zeta = \zeta + \frac{1}{2} \left( Be^{i\theta} + B_2 e^{2i\theta} + \ldots + Be^{-i\theta} + B_2 e^{-2i\theta} + \ldots \right).
\]

\[
(3.124)
\]

The perturbation is applied on a very narrow banded wave field with wave numbers grouped around the central wave number \( k \) and a typical bandwidth \( \triangle k \). This preconditions typically lead to a modulated wave train. The perturbation parameters are the wave steepness \( \epsilon = ka \) and additionally the relative bandwidth \( \mu = \triangle k / k \) ([Dysthe and Trulsen (1999)]). The slow drift term \( \Phi \) and horizontal offset term \( \zeta \) are considered to be slowly varying functions that result from the perturbation expansion (similar to the steady term \( 1 / 2 / k^2 \) that appears in Stokes’s expansion as explained in Sec. [3.2]). The order of the terms is given by \( \Phi = O(\epsilon^2) \) and \( \zeta = O(\mu \epsilon^2) \). The expressions \( A, B, A_2, B_2, \ldots \) are constant terms of the order \( O(\epsilon^n) \). The most discussed truncation combinations referring to the orders \( O(\epsilon^3) \) and \( O(\epsilon^\mu) \) is the Nonlinear Schrödinger Equation (NLS) honoring the theoretical findings of [Schrödinger (1926)].

\[
i \left( \frac{\partial B}{\partial t} + v_{gr} \frac{\partial B}{\partial x} \right) + \frac{1}{2} \left( \frac{dc_{gr}}{dk} \frac{\partial^2 B}{\partial x^2} + \frac{c_{gr}}{k} \frac{\partial^2 B}{\partial y^2} \right) - \delta \omega B = 0.
\]

\[
(3.125)
\]

It describes the evolution of modulated wave trains. It is valid for dispersive and weakly nonlinear waves in isotropic media finding application also in nonlinear optics and plasma physics. The assumption of narrow bandwidth, limits its application on real wind sea dominated sea states which initiated the formulation of an extended version – the Modified Nonlinear Schrödinger Equation (MNSE) by [Trulsen and Dysthe (1996)] for greater bandwidth. Focussing on large waves due to the modulation instability and magnification – also named the ”breather solution” of the NLS [Janssen (2003)] introduced
3.6 Discussion

In the quest of finding adequate models and prediction methods of Freak Waves the utilization of the NLS solution is a new alternative to the established superposition model based on dispersion. All solutions including those of the Modified Nonlinear Schrödinger Equation refer to a rather narrow banded and almost unidirectional seaway. It has a very peaked spectra that typically appear due to swell or in regions of limited fetch such as the North Sea which are not exclusively prone to Freak waves. Furthermore, the breather solution is still valid for weakly nonlinear waves i.e. with limited wave steepness, which challenges their candidacy for an adequate Freak wave model (Zakharov and Dyachenko (2008)). Yet, definitive cause-effect-relations of real sea events are sparse and unsatisfactory. However, supported by hindcast data and real sea measurements attempts have been made to

the Benjamin Feir Index BFI with

\[ BFI = \frac{\sqrt{2}a_0 k_0}{\Delta k/k_0} \]  

(3.126)

to characterize the stability indicating whether irregular seaway is prone to large waves magnification (BFI > 1) or not (BFI < 1). Here, \( \Delta k \) is the bandwidth distributed around the dominating wave number \( k_0 \) with dominating amplitude \( a_0 \). Simulations and experiments by [Onorato et al. (2003)] conducted with peaked JONSWAP spectra (see Sec 3.3.4) identified the Phillips constant (and with it the wave steepness given that \( \alpha_p \sim (k \zeta a)^2 \)) and the peak-enhancement-factor \( \gamma \) as governing parameters, i.e. for \( \alpha_p = 0.01 \) the BFI exceeds one if \( \gamma > 4.4 \), and for \( \alpha_p = 0.02 \) if \( \gamma > 2.2 \). Simulations as well as experiments in various testing facilities indicate that the upper tail of the cumulative Rayleigh distribution can be exceeded drastically for irregular waves with \( BFI = 0.9 \) and above. It appears that the modulation instability is a mechanism for explaining exceptionally large waves that fall out of the conventional Rayleighian statistics ([Onorato et al. (2003), Stansberg (2000)]).
link solutions of the NLS to the occurrence of possible accidents due to freak waves. Such are the analysis of a storm that hit the cruise ship Voyager (Bertotti and Cavaleri (2008)), findings by Gunson and Magnusson (2002) and Lechuga (2006).
Chapter 4

Computational Fluid Dynamics Methods

Modelling free propagating water waves with universal codes for flow dynamics is challenging since the instationary as well as the free surface demand complex models. This section describes two numerical approaches:

- The numerical wave tank WAVETUB (WAVE simulation code developed at the Technical University Berlin) which is based on potential theory combined with a finite element discretization, and

- a realization of a numerical wave tank utilizing the universal RANSE/VoF-solver Fluent.

The section closes with a discussion and a new approach to combine both methods. Providing that fluids are assumed to be incompressible the hydrodynamic state is uniquely described by the pressure and three components of the particle velocity vector. These are found by solving the field equations - namely the continuity equation and the Navier-Stokes-Equation which describes the conservation of momentum in a viscous fluid. For a numerical approximation for non breaking water waves it is possible to neglect the viscosity i.e. to apply potential theory. The numerical wave tank WAVETUB (WAVE simulation code developed at TU-Berlin) solves
Fig. 4.1: Scheme of the numerical wave tank WAVETUB with boundary conditions. $l_{\text{beach}}$ denotes the region where artificial damping terms to the kinematic and dynamic free surface boundary condition are added in order to suppress reflections. $x_b$ symbolizes the wave board motion and $\Gamma_B$, $\Gamma_S$, and $\Gamma_W$ are implicit formulations of the boundary conditions: Wave board, free surface, and tank walls.

the Laplace-Equation (see Eq. 2.2) by applying a finite element approach to a finite surface elevation. The more complex model, the RANSE/VoF-solver (Reynolds-Averaged-Navier-Stokes Equations/Volume of Fluid) Fluent, that allows to simulate wave breaking processes and wave-structure-interaction (see Sec. 4.2).

4.1 Pot-FE-Solver WAVETUB

This section presents the Pot-FE-solver WAVETUB which allows the simulation of wave propagation in a numerical wave tank based on potential theory (Steinhagen 2001). Following an approach by Wu and Eatock-Taylor (1994), the finite element approach is applied for the discretisation of the fluid do-
main. The two dimensional nonlinear free surface flow problem is solved in time domain using potential theory: The fluid is considered inviscid, incompressible, and the flow is irrotational (see Sec. [2]). Only long crested waves are considered and the surface tension is neglected. Piston type wave makers are implemented as well as flap type wave makers with one and two combined flaps. The flow field is modelled by a velocity potential which satisfies the Laplace equation. Following a combined Euler-Lagrange approach at each time step a new boundary-fitted mesh is created and the velocity potential is calculated in the entire fluid domain using the finite element method. From this solution the velocities at the free surface are determined by second-order differences. For long time simulations a “numerical beach” is implemented at the end of the wave tank by adding artificial damping terms to the kinematic and dynamic free surface boundary condition in order to suppress reflections. To develop the solution in time domain the fourth-order Runge-Kutta integration scheme is applied. The procedure is executed until the desired time step is reached, or the wave train becomes unstable due to wave breaking. The abortion of the simulation due to breaking of minor waves can be avoided by choosing a coarse grid which leads to a smoother yet more stable simulation. In this case, remeshing evens out small steep waves suppressing the initiation of wave breaking. This leads to an energy loss – similar to the energy loss that can be observed in real wave breaking (Stück (2008)). This is a purely numerical side effect which coincides with observations during experiments, but has no physical coherence. For investigations on breaking waves on a firm physical basis as well as wave structure interaction the use of the RANSE/VoF-Solver Fluent is necessary as described in the next section.

4.2 RANSE/VoF-Solver Fluent

For the simulation of the wave breaking process the general purpose CFD solver FLUENT is used. The solver describes the viscous flow by conservation equations of mass and momentum. These equations result in a system of four coupled, nonlinear partial differential equations, the Navier-Stokes Equations
in component form, which describe the entire flow field including all turbulent characteristics:

\[
\begin{align*}
\rho(u_t + uu_x + vu_y + wu_z) &= \rho f_1 - p_x + \mu(u_{xx} + u_{yy} + u_{zz}) \quad (4.1) \\
\rho(v_t + uv_x + vv_y + wv_z) &= \rho f_2 - p_y + \mu(v_{xx} + v_{yy} + v_{zz}) \quad (4.2) \\
\rho(w_t + uw_x + vw_y + ww_z) &= \rho f_3 - p_z + \mu(w_{xx} + w_{yy} + w_{zz}) \quad (4.3)
\end{align*}
\]

and the continuity equation:

\[u_x + v_y + w_z = 0.\quad (4.4)\]

Unlike in the potential theory a stress related, second order expression containing the dynamic viscosity \(\mu\) is accounted for on the right hand side of Eq. 4.1 through 4.3. \(f_1, \ldots, f_3\) denote volume forces such as gravity and the subscripts denote the partial derivative of the particular velocity component \(u, v, w\) and the pressure \(p\). Algebraic approximations have to be used to solve the differential equations numerically. The most important simplification is to decompose the pressure and velocity variables into a mean and a time
4.2 RANSE/VoF-Solver Fluent

varying portion:

\[ u = \bar{u} + u', \quad v = \bar{v} + v', \quad w = \bar{w} + w', \quad p = \bar{p} + p' \]  \hspace{2cm} (4.5)

which leads to additional terms that must be added to the right hand side of the Navier-Stokes equations (Eq. 4.1 through 4.3)

\[ -\rho(\bar{u}'u' + \bar{u}'v' + \bar{w}'w') \]  \hspace{2cm} (4.6)

\[ -\rho(\bar{v}'u' + \bar{v}'v' + \bar{v}'w') \]  \hspace{2cm} (4.7)

\[ -\rho(\bar{w}'u' + \bar{w}'v' + \bar{w}'w') \]  \hspace{2cm} (4.8)

Eq. 4.1, 4.2, and 4.3 completed with Eq. 4.6 through Eq. 4.8 form the Reynolds-Averaged Navier-Stokes Equations (RANSE). Due to the time averaging of the turbulent fluctuations in velocity and pressure the problem of closure arises i.e. the system has more variables than equations. Consequently a turbulence model is introduced, modelling the effects of turbulence by approximations and empirical constants. The k-\( \varepsilon \) and k-\( \omega \) models are eddy viscosity models for the Reynolds stress which finally leads to two new partial differential equations for the turbulent kinetic energy and the dissipation. To approximate the conservation equations the flow domain is discretized applying the finite volume method.

For describing the free surface the VoF method (VoF: Volume of Fluid) allows the simulation of wave breaking phenomena. The VoF method introduces a scalar indicator function between zero and one to model the location of the free surface (see Fig. 4.2) by solving the continuity equation for the volume fraction of the respective phase \( q \):

\[ \frac{\partial \alpha_q}{\partial t} + \vec{v} \cdot \nabla \alpha_q = 0 \]  \hspace{2cm} (4.9)

The volume elements of the fluid domain are filled with two phases: air and/or water. The VoF formulation relies on the fact that two or more fluids are not interpenetrating. For each phase a new variable "volume fraction" \( \alpha_q \) is introduced. A cell is partially filled with phase \( q \) if \( 0 < \alpha_q < 1 \). \( \alpha_q = 0 \)
denotes that the cell is empty of \( q \) whereas \( \alpha_q = 1 \) means that the cell is entirely filled (Hirt and Nichols (1981), Ubbink (1997)). The sum of all volume fractions in a control volume must be unity, i.e.

\[
\sum_{q=1}^{n} \alpha_q = 1.
\] (4.10)

The tracking of the interface is accomplished by solving a continuity equation for the volume fraction. Other than WAVETUB the VoF method is a surface-capturing method, i.e. it is applied in a fixed grid and cannot reconstruct the surface between phases exactly (see Ferziger and Peric (2002) for further details).

### 4.3 Combinations of WAVETUB and Fluent

As presented by Schmittner (2005) and Stück (2008) both numerical codes have the ability to model wave generation, propagation, wave-wave-interaction, and kinematics reasonably well with varying effort of problem-specific fine tuning. The numerical wave tank WAVETUB is specialized for wave modelling. The implemented potential theory with its adapted mesh on the free surface boundary conditions show much faster convergence than the general purpose RANSE/VoF-Solver Fluent. As a rule of thumb, one hour calculation time of WAVETUB equals one day of calculation time with a RANSE solver Fluent. The reason for that lies in the larger numbers of equations to be solved by the RANSE solver as well as the need for a higher resolution in time and space, i.e. more cells and smaller time steps. Other drawbacks of the RANSE/VoF-Solver is numerical damping affecting wave propagation and the need for high, problem-adapted grid quality. To model wave breaking and complex wave-structure interaction in detail, however, it is indispensable.

Clauss et al. (2006c) implemented coupling mechanisms between WAVETUB and the RANSE-solver – one regarding a specific instance in time, one regarding a specific location – triggered by wave breaking criteria in order to save computation time as an example. Fig. 4.3 shows a numerical wave tank.
with coupling between WAVETUB and the RANSE solver (25 m from the wave board): A wave packet advancing through the POT/FE solver domain into a RANSE solver domain where it breaks. Stück (2008) perfected the coupling between WAVETUB and Fluent.

4.4 Discussion

To retrace deterministic wave groups to their origin – the transfer of atmospheric energy into wave energy – is currently beyond the capabilities of most mathematical models. Hence, the numerical simulation of specific water wave groups for engineering purposes requires a proper starting point (or wave train) which may come from radar- or satellite scans of the ocean surface or from registrations by wave gauges. In either case the initial field kinematics are unknown.

The numerical wave tank WAVETUB begins the simulation with a still water surface at the inset of the wave board motion – like in the physical experiment. To model a specific wave group all time steps of the wave generation and wave propagation to the predefined point of interest must be simulated until the requested wave group appears. None of the numerical codes possesses the capability to transform a given wave train to the wave maker and therefore to extract a wave board motion in order to gain a predefined wave pattern. In order to automatically generate a predefined wave sequence at a specified location the numerical code can be combined with optimization routines. Starting from a signal derived by linear wave propagation the required wave board signal is synthesized by trial and error (Clauss and Steinhagen (2000), Clauss et al. (2001), Steinhagen (2001)). This method is extremely demanding in time and computer power even when only potential theory is employed since a huge number of waves and wave propagation must be simulated until the one desired wave group appears. Other obstacles are numerical instabilities due to preliminary wave breaking and numerical dif-
fusion. This can be bypassed by the coupling mechanism as mentioned above (in Sec. 4.3) at the cost of further computation time.

A much faster and more efficient method is to initialize the CFD-solver with the kinematics of a particular wave or wave group, immediately before the point of interest. This may be just before (or in front of) wave breaking or hitting a structure. This way, the whole procedure of simulating the history of the wave generation is omitted saving up to 98 per cent of the computational time. A new method to estimate the wave kinematics within steep wave groups, the Stokes-Approximation, is introduced in the next chapter. It determines a velocity field underneath a given irregular surface based on high order wave theory. The following chapters contain examples of applications on deterministic wave sequences with Freak Waves as well as an example of initialization of FLUENT to simulate a breaking wave.
Fig. 4.3: Numerical wave tank with coupling mechanism between RANSE solver and WAVETUB (25 m from the wave board): Showing a wave packet advancing the POT/FE solver domain (left hand side, color: velocity magnitude) to a RANSE solver domain (right hand side, color: volume fraction) at five instances of time.
Chapter 5

Stokes-Approximation

In this section a new rational model is introduced to estimate kinematics of steep irregular wave trains with known surface elevation. The model is based on the oscillatory third-order perturbation solutions for sums of of interacting long-crested stokes waves on deep water by [Pierson (1993)]. It is based on formulations by [Lamb (1932)]. It differs slightly from the general Stokes-III theory by [Skjelbreia (1959)] which is strictly limited to regular waves. In Sec. 5.1 the theoretical derivation of the solution of the boundary value problem is outlined and the results are summarized. Adjustments for practical application in limited water depth and are shown in Sec. 5.2 followed by a definition of the optimization problem in Sec. 5.3 and its solution scheme in Sec. 5.4 — the Subplex Search Algorithm. The validation and application on measured wave groups in the wave tanks is presented in Sec. 7 and the following sections.

5.1 Interaction of Stokes Waves

Like in the conventional wave theory (as shown in Chapter 3) the fluid motion is treated as a free boundary value problem where the free surface $\zeta(x,t)$ must be determined as part of the solution. There is no wind or atmospheric pressure acting on the surface. Therefore, no generation of waves nor dissipa-
tion mechanisms are considered. The 2-D boundary value problem is stated by the Laplace equation

$$\triangle \Phi = 0 \quad (5.1)$$

and the dynamic and kinematic boundary conditions:

$$gz - \Phi_t + (\Phi^2_x + \Phi^2_z)/2 = F(t) \quad \text{for } z = \zeta \quad (5.2)$$
$$\zeta_t + \Phi_z - \Phi_x \zeta_x = 0 \quad \text{for } z = \zeta \quad (5.3)$$

where $F(t)$ is the Bernoulli constant. The domain of definition is given by

$$-\infty < x < \infty$$
$$-\infty < t < \infty$$
$$-d < z < \zeta(x,t)$$

Within this domain a purely oscillatory solution is sought in which the resulting functions (and derivatives) of $\Phi$ and $\zeta$ cross the t- and x-axis an infinite number of times. Secular terms that become unbounded as $t$ approaches infinity (e.g. found in the Stokes III solution for finite water depth by Skjelbreia (1959)) by are not considered. This excludes also the secular solution which is referred to the Benjamin-Feir-instability (see Eq. 3.126 in Sec. 3.5) and other types of instabilities although the condition $BFI < 1$ is met. There are no initial conditions given but the premise that all waves travel in the same direction along the x-axis. Each wave is uniquely defined by its first order amplitude $a_n$, its wave number $k_n$ and a phase lag $\alpha_n$. The respective wave frequency $\omega_n$ is calculated from the deep water dispersion relation

$$\omega_n = \sqrt{k_n g}. \quad (5.4)$$

In the following the expansion is carried out for deep water waves. Adjustments for shallow water waves will be introduced later in Sec. 5.2.
The third order perturbation solution goes back to a solution for a regular wave by Lamb (1932):

\[
\zeta(x) = \zeta + 2\zeta + 3\zeta = \frac{a^2k}{2} + a\cos kx + \frac{a^2k}{2}\cos 2kx + \frac{3a^3k^2}{8}\cos 3kx \quad (5.5)
\]

\[
\Phi(z,x) = \Phi + 3\Phi = -a\sqrt{g/k}e^{kz}\sin kx + (a^3k^2\sqrt{g/k}/8)e^{kz}\sin kx \quad (5.6)
\]

\[
C = \sqrt{g/k}(1 + a^2k^2/2). \quad (5.7)
\]

The prefix subscript defines the order of the term. Note that the wave potential function does not contain sinusoidal terms corresponding to 2\(kx\) or 3\(kx\). Since the constant term \(a^2k/2\) in Eq. 5.5 affects the mean water line and has no effect on the further perturbation results it will be omitted in the following. The amplitudes (in this section denoted with the symbol \(a\)) are assumed to be small. The perturbation parameter is the wave steepness \(\delta = ka\) for one- or \(\delta_1 = k_1a_1, \delta_2 = k_2a_2, \ldots\) for more interacting waves. It the following solutions it does not appear explicitly but is factored in.

We now assume the phase function to be dependent on space and time based on the celerity \(C\), hence

\[
k[x - Ct] = k \left[ x - \sqrt{g/k}(1 + a^2k^2/2)t \right] = kx - \omega(1 + a^2k^2/2)t \quad (5.8)
\]

In the further development the exponential decay function is replaced by a truncated Maclaurin series

\[
e^{k\zeta} = 1 + k\zeta(x,t) + \frac{k^2}{2}\zeta(x,t)^2 + \ldots \quad (5.9)
\]

As an example the time derivative of the first order term of the regular wave
potential in Eq. 5.6 is then:

\[ \Phi_t = ak\omega \sqrt{g/k} \left(1 + \frac{a^2 k^2}{2}\right) \left(1 + k\zeta + \frac{k^2}{2} \zeta^2\right) \cos[kx - \omega(1 + a^2 k^2/2)t] \]

\[ = ak\omega \sqrt{g/k} \left(1 + k\zeta + \frac{k^2}{2} \zeta^2\right) \cos[kx - \omega(1 + a^2 k^2/2)t] + \left(ak\omega \sqrt{g/k} \frac{a^2 k^2}{2}\right) \cos[kx - \omega(1 + a^2 k^2/2)t] \]  (5.10)

In this example the third order term in Eq. 5.10 is essential to fulfil the DFSBC (see Eq. 5.2). Later it will be shown that the third order terms often have a much greater impact than the second order terms and bear important contributions to the final solution.

The according surface elevation in space and time is

\[ \zeta(x, t) = \frac{a^2 k}{2} + a \cos[kx - \omega(1 + \frac{a^2 k^2}{2})t] + \frac{a^2 k}{2} \cos 2[kx - \omega(1 + \frac{a^2 k^2}{2})t] + \frac{3a^3 k^2}{8} \cos 3[kx - \omega(1 + \frac{a^2 k^2}{2})t] \]  (5.11)

**Extension to more than one Stokes wave:** The defining field equation (Eq. 5.1) and the free surface boundary conditions (Eq. 5.2 and 5.3) can be
rewritten identifying the first-, second, and third order terms:

\[ \Phi_{xx} + \Phi_{zz} + \frac{1}{2} \left[ (\Phi_x + \Phi_z)^2 + (\Phi_x + \Phi_z)^2 \right] = F_1(t) \]  
\[ \kappa_t + \Phi_z + \frac{1}{2} \left[ (\Phi_x + \Phi_z)^2 + (\Phi_x + \Phi_z)^2 \right] = F_2(t) \]  
\[ \kappa_t + \Phi_x - \frac{1}{2} \left[ (\Phi_x + \Phi_z)^2 + (\Phi_x + \Phi_z)^2 \right] = F_3(t) \]

\( F_1(t), F_2(t) \) and \( F_3(t) \) are the Bernoulli constants of each respective order. The terms that contain \( a_1^2, a_1a_2, \) etc. are referred to as "second order" and terms involving initial amplitudes to the third power (\( a_1^3, a_1a_3^2, \) etc.) are denoted as third order. All terms higher than third order will be omitted in the following derivations. For lack of space and better clarity the number of initial waves will be limited to three in this text. All additional terms for more components consist of permutations of the results given in the following text. The numerical implementation handles up to six third order wave components.
Second Order Solution  The solution for three interacting Stokes waves that contain interaction terms up to the second order is given by

\[
\zeta_p(x,t) = a_1 \cos \theta_1 + \frac{a_1^2 k_1}{2} \cos 2 \theta_1 + \frac{3a_1^3 k_1^2}{8} \cos 3 \theta_1
+ a_2 \cos \theta_2 + \frac{a_2^2 k_2}{2} \cos 2 \theta_2 + \frac{3a_2^3 k_2^2}{8} \cos 3 \theta_2
+ a_3 \cos \theta_3 + \frac{a_3^2 k_3}{2} \cos 2 \theta_3 + \frac{3a_3^3 k_3^2}{8} \cos 3 \theta_3
+ \frac{a_1 a_2 (k_1 + k_2)}{2} \cos (\theta_2 + \theta_1) - \frac{a_1 a_2 (k_2 - k_1)}{2} \cos (\theta_2 - \theta_1)
+ \frac{a_1 a_3 (k_3 + k_1)}{2} \cos (\theta_3 + \theta_1) - \frac{a_1 a_3 (k_3 - k_1)}{2} \cos (\theta_3 - \theta_1)
+ \frac{a_2 a_3 (k_2 + k_3)}{2} \cos (\theta_2 + \theta_3) - \frac{a_2 a_3 (k_3 - k_2)}{2} \cos (\theta_3 - \theta_2)
\]

(5.21)

with the phase functions include a dependency on the wave steepness:

\[
\theta_n = k_n x - \omega_n (1 + \varepsilon) t + \alpha_n
\]

(5.22)

\[
\varepsilon = \frac{(a_1 k_1 + a_2 k_2 + \ldots + a_n k_n)^2}{2}
\]

(5.23)

and \(n\) being the number of the particular component Stokes wave. Note that the restriction to second order refers to the interaction terms only while the particular Stokes waves remain complete to the third order. \(\alpha_n\) is the particular phase shift. Like in Eq. 5.6 the second order potential function is
5.1 Interaction of Stokes Waves

simpler

\[
\Phi_p(x, z, t) = \left[ \frac{a_1 \omega_1}{k_1} + \frac{a_1^3 k_1 \omega_1}{8} \right] e^{k_1 z} \sin \theta_1 \\
+ \left[ \frac{a_2 \omega_2}{k_2} + \frac{a_2^3 k_2 \omega_2}{8} \right] e^{k_2 z} \sin \theta_2 \\
+ \left[ \frac{a_3 \omega_3}{k_3} + \frac{a_3^3 k_3 \omega_3}{8} \right] e^{k_3 z} \sin \theta_3 \\
+ a_1 a_2 \omega_2 e^{(k_2 - k_1)z} \sin(\theta_2 - \theta_1) + a_1 a_3 \omega_3 e^{(k_3 - k_1)z} \sin(\theta_3 - \theta_1) \\
+ a_2 a_3 \omega_3 e^{(k_3 - k_2)z} \sin(\theta_3 - \theta_2)
\] (5.24)

To derive the respective second order velocities of the interacting Stokes waves the partial derivatives of the flow potential used (note the sign):

\[ u_p = -\frac{\partial \Phi_p}{\partial x} \quad \text{and} \quad w_p = -\frac{\partial \Phi_p}{\partial x}. \] (5.25)

Hence,

\[
u_p = \left[ a_1 \omega_1 - \frac{1}{8} a_1^3 k_1 \omega_1 \right] e^{k_1 z} \cos \theta_1 \\
+ \left[ a_2 \omega_2 - \frac{1}{8} a_2^3 k_2 \omega_2 \right] e^{k_2 z} \cos \theta_2 \\
+ \left[ a_3 \omega_3 - \frac{1}{8} a_3^3 k_3 \omega_3 \right] e^{k_3 z} \cos \theta_3 \\
+ \frac{a_1 a_2 \omega_2}{g} (\omega_2^2 - \omega_1^2) e^{(k_2 - k_1)z} \cos(\theta_2 - \theta_1) \\
+ \frac{a_1 a_3 \omega_3}{g} (\omega_3^2 - \omega_1^2) e^{(k_3 - k_1)z} \cos(\theta_3 - \theta_1) \\
+ \frac{a_2 a_3 \omega_3}{g} (\omega_3^2 - \omega_2^2) e^{(k_3 - k_2)z} \cos(\theta_3 - \theta_2). \] (5.26)
The vertical velocity is respectively

\[ w_p = \left[ a_1 \omega_1 - \frac{1}{8} a_1^3 k_1^2 \omega_1 \right] e^{k_1 z} \sin \theta_1 \]
\[ + \left[ a_2 \omega_2 - \frac{1}{8} a_2^3 k_2^2 \omega_2 \right] e^{k_2 z} \sin \theta_2 \]
\[ + \left[ a_3 \omega_3 - \frac{1}{8} a_3^3 k_3^2 \omega_3 \right] e^{k_3 z} \sin \theta_3 \]
\[ + \frac{\omega_2^2 - \omega_1^2}{g} a_1 a_2 \omega_2 e^{(k_2 - k_1)z} \sin(\theta_2 - \theta_1) \]
\[ + \frac{\omega_3^2 - \omega_1^2}{g} a_1 a_3 \omega_3 e^{(k_3 - k_1)z} \sin(\theta_3 - \theta_1) \]
\[ + \frac{\omega_3^2 - \omega_2^2}{g} a_2 a_3 \omega_3 e^{(k_3 - k_2)z} \sin(\theta_3 - \theta_2) \]  \( \text{(5.27)} \)

The particle accelerations are

\[ \dot{u}_p = \frac{\partial u_p}{\partial t} = \frac{\partial}{\partial t} \left( \frac{\partial \Phi_p}{\partial x} \right) \]  \( \text{(5.28)} \)

and

\[ \dot{w}_p = \frac{\partial w_p}{\partial t} = \frac{\partial}{\partial t} \left( \frac{\partial \Phi_p}{\partial z} \right) \]  \( \text{(5.29)} \)
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i.e.:

\[
\dot{u}_p = \left[ a_1 \omega_1 - \frac{1}{8} a_1^2 k_1^2 \omega_1 \right] \omega_1 \left[ 1 + \frac{a_1 \omega_1^2 + a_2 \omega_2^2 + a_3 \omega_3^2}{2g^2} \right] e^{k_1 z \sin \theta_1} \\
+ \left[ a_2 \omega_2 - \frac{1}{8} a_2^2 k_2^2 \omega_2 \right] \omega_2 \left[ 1 + \frac{a_1 \omega_1^2 + a_2 \omega_2^2 + a_3 \omega_3^2}{2g^2} \right] e^{k_2 z \sin \theta_2} \\
+ \left[ a_3 \omega_3 - \frac{1}{8} a_3^2 k_3^2 \omega_3 \right] \omega_3 \left[ 1 + \frac{a_1 \omega_1^2 + a_2 \omega_2^2 + a_3 \omega_3^2}{2g^2} \right] e^{k_3 z \sin \theta_3} \\
+ \frac{a_1 a_2 \omega_2}{g} (\omega_2 - \omega_1) \left( \omega_1 - \omega_2 \right) \left[ 1 + \frac{a_1 \omega_1^2 + a_2 \omega_2^2 + a_3 \omega_3^2}{2g^2} \right] e^{(k_2 - k_1) z \sin \theta_2 - \theta_1} \\
+ \frac{a_1 a_3 \omega_3}{g} (\omega_3 - \omega_1) \left( \omega_1 - \omega_3 \right) \left[ 1 + \frac{a_1 \omega_1^2 + a_2 \omega_2^2 + a_3 \omega_3^2}{2g^2} \right] e^{(k_3 - k_1) z \sin \theta_3 - \theta_1} \\
+ \frac{a_2 a_3 \omega_3}{g} (\omega_3 - \omega_2) \left( \omega_2 - \omega_3 \right) \left[ 1 + \frac{a_1 \omega_1^2 + a_2 \omega_2^2 + a_3 \omega_3^2}{2g^2} \right] e^{(k_3 - k_2) z \sin \theta_3 - \theta_2}.
\]

(5.30)

and

\[
\dot{w}_p = -\left[ a_1 \omega_1 - \frac{1}{8} a_1^2 k_1^2 \omega_1 \right] \omega_1 \left[ 1 + \frac{a_1 \omega_1^2 + a_2 \omega_2^2 + a_3 \omega_3^2}{2g^2} \right] e^{k_1 z \cos \theta_1} \\
- \left[ a_2 \omega_2 - \frac{1}{8} a_2^2 k_2^2 \omega_2 \right] \omega_2 \left[ 1 + \frac{a_1 \omega_1^2 + a_2 \omega_2^2 + a_3 \omega_3^2}{2g^2} \right] e^{k_2 z \cos \theta_2} \\
- \left[ a_3 \omega_3 - \frac{1}{8} a_3^2 k_3^2 \omega_3 \right] \omega_3 \left[ 1 + \frac{a_1 \omega_1^2 + a_2 \omega_2^2 + a_3 \omega_3^2}{2g^2} \right] e^{k_3 z \cos \theta_3} \\
- \frac{a_1 a_2 \omega_2}{g} (\omega_2 - \omega_1) \left( \omega_1 - \omega_2 \right) \left[ 1 + \frac{a_1 \omega_1^2 + a_2 \omega_2^2 + a_3 \omega_3^2}{2g^2} \right] e^{(k_2 - k_1) z \cos \theta_2 - \theta_1} \\
- \frac{a_1 a_3 \omega_3}{g} (\omega_3 - \omega_1) \left( \omega_1 - \omega_3 \right) \left[ 1 + \frac{a_1 \omega_1^2 + a_2 \omega_2^2 + a_3 \omega_3^2}{2g^2} \right] e^{(k_3 - k_1) z \cos \theta_3 - \theta_1} \\
- \frac{a_2 a_3 \omega_3}{g} (\omega_3 - \omega_2) \left( \omega_2 - \omega_3 \right) \left[ 1 + \frac{a_1 \omega_1^2 + a_2 \omega_2^2 + a_3 \omega_3^2}{2g^2} \right] e^{(k_3 - k_2) z \cos \theta_3 - \theta_2}.
\]

(5.31)

Third Order Terms  The additional coupling term of third order can be of greater order than those of second order. For easier reading \(N\) and \(P\) are introduced which are summations of the lengthy expressions shown in
To determine the third order portion of the horizontal and vertical velocity elevation function is: also in the partial derivatives of $\Phi$. The third order portion of the surface are not time dependent nor a function of $x$. Hence, they remain the same also in the partial derivatives of $\Phi$. The third order portion of the surface elevation function is:

$$
\zeta_{pp}(x, t) = \frac{a^2 a_1(2k_2 + k_1)^2}{8} \cos(2\theta_2 + \theta_1) + \frac{a_2 a_1^2(2k_1 + k_2)^2}{8} \cos(2\theta_1 + \theta_2)
$$

$$
+ \frac{a_1 a_2 a_3(k_3 + k_2 + k_1)^2}{4} \cos(\theta_3 + \theta_2 + \theta_1)
$$

$$
+ N_{(2\theta_2 - \theta_1)} \cos(2\theta_2 - \theta_1) + N_{(2\theta_1 - \theta_2)} \cos(2\theta_1 - \theta_2) +
$$

$$
+ N_{(2\theta_3 - \theta_1)} \cos(2\theta_3 - \theta_1) + N_{(2\theta_1 - \theta_3)} \cos(2\theta_1 - \theta_3) +
$$

$$
+ N_{(2\theta_3 - \theta_2)} \cos(2\theta_3 - \theta_2) + N_{(2\theta_2 - \theta_3)} \cos(2\theta_2 - \theta_3) +
$$

$$
+ N_{(\theta_1 + \theta_2 - \theta_1)} \cos(\theta_3 + \theta_2 - \theta_1) + N_{(\theta_3 - \theta_2 + \theta_1)} \cos(\theta_3 - \theta_2 + \theta_1)
$$

$$
+ N_{(\theta_2 + \theta_1 - \theta_3)} \cos(\theta_2 + \theta_1 - \theta_3)
$$

$$
+ N_{\theta_1} \cos(\theta_1) + N_{\theta_2} \cos(\theta_2) + N_{\theta_3} \cos(\theta_3)
$$

$$
(5.32)
$$

The third order potential function is accordingly

$$
\Phi_{pp}(x, y, z) = P_{(2\theta_2 - \theta_1)} e^{(2k_2 - k_1)z} \sin(2\theta_2 - \theta_1) + P_{(2\theta_1 - \theta_2)} e^{(2k_1 - k_2)z} \sin(2\theta_1 - \theta_2)
$$

$$
+ P_{(2\theta_3 - \theta_1)} e^{(2k_3 - k_1)z} \sin(2\theta_3 - \theta_1) + P_{(2\theta_1 - \theta_3)} e^{(2k_1 - k_3)z} \sin(2\theta_1 - \theta_3)
$$

$$
+ P_{(2\theta_3 - \theta_2)} e^{(2k_3 - k_2)z} \sin(2\theta_3 - \theta_2) + P_{(2\theta_2 - \theta_3)} e^{(2k_2 - k_3)z} \sin(2\theta_2 - \theta_3)
$$

$$
+ P_{(\theta_1 + \theta_2 - \theta_1)} e^{(k_3 + k_2 - k_1)z} \sin(\theta_3 + \theta_2 - \theta_1) + P_{(\theta_3 - \theta_2 + \theta_1)} e^{(k_1 - k_3 - k_2)z}
$$

$$
\cdot \sin(\theta_3 - \theta_2 + \theta_1) + P_{(\theta_2 + \theta_1 - \theta_3)} e^{(k_2 + k_3 - k_1)z} \sin(\theta_2 + \theta_1 - \theta_3)
$$

$$
+ P_{\theta_1} e^{k_1z} \sin(\theta_1) + P_{\theta_2} e^{k_2z} \sin(\theta_2)
$$

$$
+ P_{\theta_3} e^{k_3z} \sin(\theta_3).
$$

$$
(5.33)
$$

To determine the third order portion of the horizontal and vertical velocity the partial derivatives are taken:

$$
u_{pp} = -\frac{\partial \Phi_{pp}}{\partial x}
$$

$$
(5.34)
$$
and
\[ w_{pp} = -\frac{\partial \Phi_{pp}}{\partial x} \] (5.35)

which leads to

\[
\begin{align*}
\mathbf{u}_{pp}(x,y,z) &= P(2\theta_2-\theta_1)e^{(2k_2-k_1)z}(2k_2-k_1)\cos(2\theta_2-\theta_1) \\
&+ P(2\theta_1-\theta_2)e^{(2k_1-k_2)z}(2k_1-k_2)\cos(2\theta_1-\theta_2) \\
&+ P(2\theta_3-\theta_1)e^{(2k_3-k_1)z}(2k_3-k_1)\cos(2\theta_3-\theta_1) \\
&+ P(2\theta_1-\theta_3)e^{(2k_1-k_3)z}(2k_1-k_3)\cos(2\theta_1-\theta_3) \\
&+ P(2\theta_3-\theta_2)e^{(2k_3-k_2)z}(2k_3-k_2)\cos(2\theta_3-\theta_2) \\
&+ P(2\theta_2-\theta_3)e^{(2k_2-k_3)z}(2k_2-k_3)\cos(2\theta_2-\theta_3) \\
&+ P(\theta_3+\theta_2-\theta_1)e^{(k_3+k_2-k_1)z}(k_3+k_2-k_1)\cos(\theta_3+\theta_2-\theta_1) \\
&+ P(\theta_1-\theta_2+\theta_1)e^{(k_3-k_2+k_1)z}(k_3-k_2+k_1)\cos(\theta_3-\theta_2+\theta_1) \\
&+ P(\theta_2+\theta_1-\theta_3)e^{(k_2+k_1-k_3)z}(k_2+k_1-k_3)\cos(\theta_2+\theta_1-\theta_3) \\
&+ P_{\theta_1}e^{k_1z}k_1\cos(\theta_1) + P_{\theta_2}e^{k_2z}k_2\cos(\theta_2) \\
&+ P_{\theta_3}e^{k_3z}k_3\cos(\theta_3). \\
\end{align*}
\] (5.36)
The respective third order terms for the vertical velocities are

\[
\begin{align*}
    w_{pp}(x, y, z) &= P(2\theta_2 - \theta_1)e^{(2k_2 - k_1)z}(2k_2 - k_1)\sin(2\theta_2 - \theta_1) \\
    &\quad + P(2\theta_1 - \theta_2)e^{(2k_1 - k_2)z}(2k_1 - k_2)\sin(2\theta_1 - \theta_2) \\
    &\quad + P(2\theta_3 - \theta_1)e^{(2k_3 - k_1)z}(2k_3 - k_1)\sin(2\theta_3 - \theta_1) \\
    &\quad + P(2\theta_1 - \theta_3)e^{(2k_1 - k_3)z}(2k_1 - k_3)\sin(2\theta_1 - \theta_3) \\
    &\quad + P(2\theta_3 - \theta_2)e^{(2k_3 - k_2)z}(2k_3 - k_2)\sin(2\theta_3 - \theta_2) \\
    &\quad + P(2\theta_2 - \theta_3)e^{(2k_2 - k_3)z}(2k_2 - k_3)\sin(2\theta_2 - \theta_3) \\
    &\quad + P(\theta_3 + \theta_2 - \theta_1)e^{(k_3 - k_2 + k_1)z}(k_3 + k_2 - k_1)\sin(\theta_3 + \theta_2 - \theta_1) \\
    &\quad + P(\theta_3 - \theta_2 + \theta_1)e^{(k_3 - k_2 + k_1)z}(k_3 - k_2 + k_1)\sin(\theta_3 - \theta_2 + \theta_1) \\
    &\quad + P(\theta_2 + \theta_1 - \theta_3)e^{(k_2 + k_1 - k_3)z}(k_2 + k_1 - k_3)\sin(\theta_2 + \theta_1 - \theta_3) \\
    &\quad + P_1 e^{k_1 z}k_1 \sin(\theta_1) + P_2 e^{k_2 z}k_2 \sin(\theta_2) \\
    &\quad + P_3 e^{k_3 z}k_3 \sin(\theta_3). \\
\end{align*}
\]

(5.37)

The third order terms of the particle accelerations are derived by

\[
\dot{u}_{pp} = \frac{\partial u_{pp}}{\partial t} = \frac{\partial}{\partial t} \left( \frac{\partial \Phi_{pp}}{\partial x} \right) 
\]

(5.38)

and

\[
\dot{w}_{pp} = \frac{\partial w_{pp}}{\partial t} = \frac{\partial}{\partial t} \left( \frac{\partial \Phi_{pp}}{\partial z} \right).
\]

(5.39)

Since the argument of the sinusoidal terms are the only time dependent properties the derivation is can be found applying the identity:

\[
\frac{\partial \sin(2\theta_a - \theta_b)}{\partial t} = \cos(2\theta_a - \theta_b)(2\omega_a - \omega_b)(1 + \varepsilon)
\]

(5.40)
or similar for each term. The coefficients $N_\theta$ and $P_\theta$ in Eqs. 5.33, 5.36 and 5.37 contain collected terms from the perturbation. They are

\begin{align*}
N_{(2\theta_2 - \theta_1)} &= \frac{(2k_2 - k_1)B_{(2\theta_2 - \theta_1)} - (2\omega_2 - \omega_1)(1 + \varepsilon)K_{(2\theta_2 - \theta_1)}}{(2\omega_2 - \omega_1)^2(1 + \varepsilon)^2g(2k_2 - k_1)} \\
N_{(2\theta_1 - \theta_2)} &= \frac{(2k_1 - k_2)B_{(2\theta_1 - \theta_2)} - (2\omega_1 - \omega_2)(1 + \varepsilon)K_{(2\theta_1 - \theta_2)}}{(2\omega_1 - \omega_2)^2(1 + \varepsilon)^2g(2k_1 - k_2)} \\
N_{(2\theta_3 - \theta_1)} &= \frac{(2k_3 - k_1)B_{(2\theta_3 - \theta_1)} - (2\omega_3 - \omega_1)(1 + \varepsilon)K_{(2\theta_3 - \theta_1)}}{(2\omega_1 - \omega_3)^2(1 + \varepsilon)^2g(2k_3 - k_1)} \\
N_{(2\theta_1 - \theta_3)} &= \frac{(2k_1 - k_3)B_{(2\theta_1 - \theta_3)} - (2\omega_1 - \omega_3)(1 + \varepsilon)K_{(2\theta_1 - \theta_3)}}{(2\omega_1 - \omega_3)^2(1 + \varepsilon)^2g(2k_1 - k_3)} \\
N_{(2\theta_3 - \theta_2)} &= \frac{(2k_3 - k_2)B_{(2\theta_3 - \theta_2)} - (2\omega_3 - \omega_2)(1 + \varepsilon)K_{(2\theta_3 - \theta_2)}}{(2\omega_3 - \omega_2)^2(1 + \varepsilon)^2g(2k_3 - k_2)} \\
N_{(2\theta_2 - \theta_3)} &= \frac{(2k_2 - k_3)B_{(2\theta_2 - \theta_3)} - (2\omega_2 - \omega_3)(1 + \varepsilon)K_{(2\theta_2 - \theta_3)}}{(2\omega_2 - \omega_3)^2(1 + \varepsilon)^2g(2k_2 - k_3)}
\end{align*}

\begin{align*}
N_{(\theta_2 + \theta_2 - \theta_1)} &= \frac{(k_3 + k_2 - k_1)B_{(\theta_2 + \theta_2 - \theta_1)} - (\omega_3 + \omega_2 - \omega_1)(1 + \varepsilon)K_{(\theta_2 + \theta_2 - \theta_1)}}{(\omega_3 + \omega_2 - \omega_1)^2(1 + \varepsilon)^2 - g(k_3 + k_2 - k_1)} \\
N_{(\theta_3 - \theta_2 + \theta_1)} &= \frac{(k_3 - k_2 + k_1)B_{(\theta_3 - \theta_2 + \theta_1)} - (\omega_3 - \omega_2 + \omega_1)(1 + \varepsilon)K_{(\theta_3 - \theta_2 + \theta_1)}}{(\omega_3 - \omega_2 + \omega_1)^2(1 + \varepsilon)^2 - g(k_3 - k_2 + k_1)} \\
N_{(\theta_2 + \theta_1 - \theta_3)} &= \frac{(k_2 + k_1 - k_3)B_{(\theta_2 + \theta_1 - \theta_3)} - (\omega_2 + \omega_1 - \omega_3)(1 + \varepsilon)K_{(\theta_2 + \theta_1 - \theta_3)}}{(\omega_2 + \omega_1 - \omega_3)^2(1 + \varepsilon)^2 - g(k_2 + k_1 - k_3)}
\end{align*}

\begin{align*}
N_{\theta_1} &= \frac{\omega_1 B_{\theta_1}}{g} - (1 + \varepsilon)K_{\theta_1} - a_1 \omega_1 \varepsilon (2 + \varepsilon) - \varepsilon (a_3^3 \omega_1^5 / 8g^2) \\
N_{\theta_2} &= \frac{\omega_2 B_{\theta_2}}{g} - (1 + \varepsilon)K_{\theta_2} - a_2 \omega_2 \varepsilon (2 + \varepsilon) - \varepsilon (a_3^3 \omega_2^5 / 8g^2) \\
N_{\theta_3} &= \frac{\omega_3 B_{\theta_3}}{g} - (1 + \varepsilon)K_{\theta_3} - a_3 \omega_3 \varepsilon (2 + \varepsilon) - \varepsilon (a_3^3 \omega_3^5 / 8g^2)
\end{align*}
\[ P_{(\theta_2-\theta_1)} = \frac{gK_{(\theta_2-\theta_1)} - (2\omega_2 - \omega_1)(1 + \varepsilon)B_{(\theta_2-\theta_1)}}{(2\omega_2 - \omega_1)^2(1 + \varepsilon)^2g(2k_2 - k_1)} \] (5.53)

\[ P_{(\theta_1-\theta_2)} = \frac{gK_{(\theta_1-\theta_2)} - (2\omega_1 - \omega_2)(1 + \varepsilon)B_{(\theta_1-\theta_2)}}{(2\omega_1 - \omega_2)^2(1 + \varepsilon)^2g(2k_1 - k_2)} \] (5.54)

\[ P_{(\theta_3-\theta_1)} = \frac{gK_{(\theta_3-\theta_1)} - (2\omega_3 - \omega_1)(1 + \varepsilon)B_{(\theta_3-\theta_1)}}{(2\omega_3 - \omega_1)^2(1 + \varepsilon)^2g(2k_3 - k_1)} \] (5.55)

\[ P_{(\theta_1-\theta_3)} = \frac{gK_{(\theta_1-\theta_3)} - (2\omega_1 - \omega_3)(1 + \varepsilon)B_{(\theta_1-\theta_3)}}{(2\omega_1 - \omega_3)^2(1 + \varepsilon)^2g(2k_1 - k_3)} \] (5.56)

\[ P_{(\theta_2-\theta_3)} = \frac{gK_{(\theta_2-\theta_3)} - (2\omega_2 - \omega_3)(1 + \varepsilon)B_{(\theta_2-\theta_3)}}{(2\omega_2 - \omega_3)^2(1 + \varepsilon)^2g(2k_2 - k_3)} \] (5.57)

\[ P_{(\theta_1+\theta_2-\theta_1)} = \frac{gK_{(\theta_1+\theta_2-\theta_1)} - (\omega_3 + \omega_2 - \omega_1)(1 + \varepsilon)B_{(\theta_1+\theta_2-\theta_1)}}{(\omega_3 + \omega_2 - \omega_1)^2(1 + \varepsilon)^2 - g(k_3 + k_2 - k_1)} \] (5.59)

\[ P_{(\theta_1-\theta_2+\theta_1)} = \frac{gK_{(\theta_1-\theta_2+\theta_1)} - (\omega_3 - \omega_2 + \omega_1)(1 + \varepsilon)B_{(\theta_1-\theta_2+\theta_1)}}{(\omega_3 - \omega_2 + \omega_1)^2(1 + \varepsilon)^2 - g(k_3 - k_2 + k_1)} \] (5.60)

\[ P_{(\theta_2+\theta_1-\theta_3)} = \frac{gK_{(\theta_2+\theta_1-\theta_3)} - (\omega_2 + \omega_1 - \omega_3)(1 + \varepsilon)B_{(\theta_2+\theta_1-\theta_3)}}{(\omega_2 + \omega_1 - \omega_3)^2(1 + \varepsilon)^2 - g(k_2 + k_1 - k_3)} \] (5.61)

\[ P_{\theta_1} = \frac{gK_{\theta_1} - (1 + \varepsilon)B_{\theta_1}\omega_1 - a_1 g \omega_1 \varepsilon (2 + \varepsilon) - \varepsilon (a_1^3 \omega_1^5 / 8g)}{\omega_1^2 \varepsilon (2 + \varepsilon)} \] (5.62)

\[ P_{\theta_2} = \frac{gK_{\theta_2} - (1 + \varepsilon)B_{\theta_2}\omega_2 - a_2 g \omega_2 \varepsilon (2 + \varepsilon) - \varepsilon (a_2^3 \omega_2^5 / 8g)}{\omega_2^2 \varepsilon (2 + \varepsilon)} \] (5.63)

The coefficients \( B \) and \( K \) refer to the dynamic free surface boundary condition (DFSBC) and the kinematic free surface boundary condition (KFSBC) are listed in Sec. C. The final results are found by the addition of first-,
second- and third order terms:

\[
\begin{align*}
\zeta &= \zeta_p + \zeta_{pp} \\
\Phi &= \Phi_p + \Phi_{pp} \\
u &= u_p + u_{pp} \\
w &= \ldots
\end{align*}
\] (5.64)

In the current formulation the wave numbers must be chosen so that

\[
\begin{align*}
k_1 &\leq k_2 \leq k_3 \leq 2k_1 \\
k_3 + k_2 + k_1 &\geq 0 \\
k_3 + k_2 - k_1 &\geq 0 \\
k_1 + k_2 &\geq k_3
\end{align*}
\] (5.65, 5.66, 5.67, 5.68)

If these conditions are not met high order interaction-terms may reverse their travelling direction leading to spurious solutions as the wave propagates. For three identical wave numbers \((k = k_1 = k_2 = k_3)\) and phase shifts all interaction terms in Eq. 5.21 and 5.24 vanish. Remaining is a regular third order Stokes wave of the form

\[
\begin{align*}
\zeta(x,t) &= (a_1 + a_2 + a_3) \cos(\theta) + \left[ (a_1 + a_2 + a_3)^2 k/2 \right] \cos(2\theta) \\
&\quad+ \left[ 3(a_1 + a_2 + a_3)^3 k^2 / 8 \right] \cos(3\theta)
\end{align*}
\] (5.69)

and the potential function

\[
\Phi(z,x) = -(a_1 + a_2 + a_3) \sqrt{g/k} e^{kz} \sin \theta + \left[ (a_1 + a_2 + a_3)^3 k^2 / 8 \right] e^{kz} \sin \theta
\] (5.70)

which complies with the formulation of a single Stokes wave of third order according Eqs. 5.5 and 5.6 by replacing \((a_1 + a_2 + a_3)\) with \(a\).
The third order dispersion relation for deep water - valid for each component wave with frequency $\omega_N$ is given with:

$$\omega_N = \sqrt{gk_N \left[ 1 + \left( \sum a_i k_i \right)^2 \right]} \quad \text{for } i = 1, \ldots, n. \quad (5.71)$$

The relations for the period $T_N = 2\pi/\omega_N$ and wave lengths $L_N = 2\pi/k_N$ persist. Hence, given the dependency on the wave steepness it is possible to have component waves of equal frequency with differing wave lengths.

Unlike in the conventional high order Stokes Wave Theory the solution consists only of periodic, sinusoidal terms which are bound by fixed phase relations. The mean water line coincides with the formal origin of the coordinate system. However, the second and third order terms must not be interpreted as a superposition of independent components. They physically cannot exist for themselves but are only existent to modify the linear solution in order to "better fit" the free surface boundary conditions and provide the pointed crests and flat troughs. The analysis by a Fourier decomposition into an amplitude and phase spectra does not account for these bound modes and therefore cannot represent the nonlinear single wave shapes nor the underlying kinematics.

In Fig. 5.1 on top a surface elevation profile of a wave is shown with $a_1 = a_2 = a_3 = \frac{L}{60}$ (i.e. wave steepness $H/L = 1/10$) and $k_1 = k_2 = k_3 = \frac{2\pi}{L}$ in comparison with a profile of three linear waves with identical parameters.

In the middle line of Fig. 5.1 the surface elevation of modulation is shown resulting from $k_1 = \frac{2\pi}{1.17}$, $k_2 = \frac{2\pi}{L}$ and $k_3 = \frac{2\pi}{0.9L}$ – again in comparison with the result from the linear wave theory. The differences therefore the resulting high order terms are shown in the third line of Fig. 5.1. The entire field for vertical and horizontal velocity- and acceleration components are shown in Fig. 5.2 plotted in space and in Fig. 5.3 plotted over time.
5.1 Interaction of Stokes Waves

Fig. 5.1: Surface profile of three interacting Stokes waves reducing to one single wave with \( a_1 = a_2 = a_3 = \frac{L}{60} \) (i.e. wave steepness \( H/L = 1/10 \)) and \( k_1 = k_2 = k_3 = k = \frac{2\pi}{L/10} \) all in phase (top). In the middle a modulation of the waves with \( k_1 = \frac{2\pi}{10L} \), \( k_2 = \frac{2\pi}{L} \) and \( k_3 = \frac{2\pi}{60L} \) is shown in comparison with results from the linear wave theory. Resulting Interaction terms are in the third line. Comparison of horizontal velocity profiles between the linear wave theory and the Stokes-Approximation for a regular wave (left) and the modulation of three waves are shown at the bottom.
**Fig. 5.2**: Comparison of potential and velocity fields between Airy Wave Theory and Stokes-Approximation for a modulation of three waves according to Fig. 5.1. The modulation of the linear waves (left column) shows a slightly higher wave potential and particle velocities in the wave crest than the interacting Stokes waves (right column).
Fig. 5.3: Comparison of potential and velocity fields between Airy Wave Theory and Stokes-Approximation for a modulation of three waves according to Fig. 5.1. The modulation of the linear waves (left column) shows a slightly higher wave potential and particle velocities in the wave crest than the interacting Stokes waves (right column).
5.2 Adjustments

To apply the, yet theoretical, derivations to the experimental and real sea measurements the following adjustments are made:

**Finite Water Depth**  The bottom boundary condition

\[ w(z = -d) = \frac{\partial \phi}{\partial z} = 0. \]  
(5.72)

causes the particle trajectories to become of elliptical nature which is mathematically expressed by the hyperbolic decay terms \( D_h, D_v \) valid for displacement, velocities, and accelerations which are for all horizontal particle dynamics:

\[ D_h = \cosh(k(z + d)) \frac{\sinh(kd)}{\sinh(kd)} \]

and for the respective vertical particle motion

\[ D_v = \sinh(k(z + d)) \frac{\sinh(kd)}{\sinh(kd)} \]

The water depth can be neglected if it exceeds one half of the wave length \( d/L = \omega^2 d/(2\pi g) < 1/2 \). At this depth the wave kinematics decay down to less than 5 per cent and are therefore negligible. The decay terms can then be approximated by the general exponential function \( e^{kz} \):

\[
\cosh k(z + d) \approx \sinh k(z + d) \approx \frac{1}{2} e^{kz+kd} \\
\cosh kd \approx \sinh kd \approx \frac{1}{2} e^{kd} \\
\Rightarrow D_h \approx D_v \approx e^{kz}
\]  
(5.73)

To apply this simplification to the irregular seaways it must be assured that also the low frequent interaction terms fulfil this deep water condition.
The Maclaurin series expansion of the hyperbolic decay terms (here truncated after third order)

$$\cosh k(z+d) = \cosh kd + (k \sinh kd)\zeta(x,t) + (k^2 \cosh kd)\frac{\zeta(x,t)^2}{2} + \ldots$$  \hspace{1cm} (5.74)

induces a large number of hyperbolic functions leading to a significant slowdown of the overall calculation. It has been found reasonable to account only for the linear decay terms. Given that the hyperbolic functions had to be evaluated in each interaction term (coming from the wave potential) this theoretical inconsistency can be seen as a compromise between accuracy and computation time. In terms of the dispersion relation the third order solution for finite water depths is accounted for:

$$\omega = \sqrt{kg \tanh(kd + (ak)^2 \frac{\cosh(4kd) + 8}{8 \sinh(kd^4)}}. \hspace{1cm} (5.75)$$

**Number of Component Waves** The number of interacting component waves is extended from three to six. The additional interaction terms are found by a modulation of the indices of the existing terms. This leads to a significant increase of interaction terms. If \(n_c\) denotes the number of component waves, the number of second order terms \(n(2)\) increases with

$$n(2) = n_c^2$$  \hspace{1cm} (5.76)

and the number of third order terms \(n(3)\) increases to

$$n(3) = n_c + 2n_c(n_c - 1) + 2n_c(n_c - 1)(n_c - 2)/3. \hspace{1cm} (5.77)$$

The according dramatic increase of the numbers is illustrated in the right column of Tab. 5.1. Still, all terms of order higher than three are omitted. Therefore, each interaction term represents a maximum of three component waves being then superimposed with others. In practical application on wave groups consisting of three to five wave crests \(n_c = 4\) is usually sufficient. The
An iteration scheme is setup to find the component wave amplitudes $a_1, \ldots a_6$ the wave numbers $k_1, \ldots k_6$ and the phase shifts $\alpha_1, \ldots \alpha_6$ by fitting the calculated surface to a given measured or computed surface using a least square algorithm. Therefore, the surface elevation is considered realistic by itself and a restraint of the amplitudes or wave steepness except from the conditions Ineq. 5.65 through 5.68 is not imposed.
A superposition of three to six Stokes waves with shifted phase relations is capable to provide third order solutions for wave potentials of asymmetric and skewed surface profiles. The surface elevation can be processed as surface contour $\zeta(x)$ as well as time registration $\zeta(t)$. The parametric perturbation solution is fitted to the measured surface elevation on a wave-by-wave basis by shifting a window along a considered surface elevation record. The window size must cover at least one wave length or one wave period if applied on a surface registration. Best convergence is reached for values between 1.2 and 1.5 times the dominant wave length (i.e. $n_s \Delta x > L; t = \text{const.}$) or between 1.3 and 1.1 times the mean zero-upcrossing period ($n \Delta t > T_0; x = \text{const.}$). The surface fitting is done by a modified least square algorithm. The objective function is the minimization of $g_f$ with

$$g_f = \sum_{i=1}^{n_s} \left( |\zeta|^p \text{sign}(\zeta) - |\zeta_{\text{approx}}|^p \text{sign}(\zeta) \right)^2.$$ \hspace{1cm} (5.78)

$n_s$ is the number of discrete sample points of the given surface record. The exponent $p$ is introduced to increase and control the gradient of the goal function and therefore increases the speed of convergence. Good results are achieved with $p = 2.0, \ldots, 2.5$ when applied on a surface contour ($t = \text{const.}$) or $p = 2.6, \ldots, 3.0$ on a time registration. Each component wave is defined by an amplitude, a wave number and a phase information. If the superposition consists of $n_w$ component waves, the set of free variables is formed by the particular amplitudes $a_1, \ldots, a_{nw}$, the wave numbers $k_1, \ldots, ka_{nw}$, and the phase shifts $\alpha_1, \ldots, \alpha_{nw}$. The choice of taking the wave numbers instead of the wave frequencies in all cases (even when time registrations are considered) allows to solve the dispersion relation without approximation or numerical iteration scheme.
5.4 The Solution Scheme

The objective function $g_f$ can be characterized as being strongly non-linear and multi-modal. Especially phase jitter between signal and approximation leads to steep gradients and discontinuities of $g_f$. The presence of high frequent disturbances induces noise which poses a challenge to the optimization algorithms. The Subplex Search Algorithm is a robust, non-linear search method and has proven to be a universal and reliable algorithm for unconstrained optimization problems. It is a systematic application of the Nelder-Mead-Simplex method which is outlined in the next two sections.

5.4.1 Nelder-Mead-Simplex

Nelder-Mead-Simplex method (also named downhill simplex method) is a function comparison method for unconstrained optimization \cite{Nelder and Mead (1965)}. A simplex is a set of $n + 1$ points in an $n$-dimensional space where $n$ denotes the number of the degrees of freedom in the given optimization problem. In a 2-D-space a simplex can be represented by a triangle, in a 3-D space by a tetrahedron, and so forth. In the Nelder-Mead simplex method, a simplex ”moves” through the $n$-dimensional domain space of the objective function according to the values of its vertices $f(x_1) \leq f(x_2) \leq \ldots \leq f(x_{n+1})$. Given an initial simplex typically formed by setting $x_0 = x$ and $x_i = sc_i e_i$ (where $e_i$ is the unit vector in each dimension and $sc_i$ is the scale which is defined by the problem). The simplex changes automatically its shape and size by successively carrying out the following iterations:

Reflection $x_r = x_o + \alpha (x_o - x_{n+1})$ where $x_o$ ist the center of gravity of all points. If $f(x_1) < f(x_2) < f(x_n)$ a new simplex is created with the reflected point $x_r$ rejecting $x_{n+1}$, $[\alpha = 1]$

Expansion $x_e = \rho x_r + (1 - \rho)x_0$, If $f(x_e) < f(x_r)$ use new simplex with $x_e$ instead of $x_r$. $[\rho = 2]$
5.4 The Solution Scheme

**Contraction**, If $f(x_n) \leq f(x_r) \leq f(x_{n+1})$ let $x_c = x_{n+1} + \gamma(x_o - x_{n+1})$.
Adopt $x_c$ if $f(x_c) \leq f(x_r)$. \[ \gamma = 0.5 \]

**Shrinkage**, set for all $n$ vertices $x_i = x_1 + \sigma(x_i - x_1)$ with $i = 2, \ldots, n + 1$,
\[ \sigma = 0.5 \]

In Fig. 5.4 the operations are depicted in a 2-D example. In the square brackets standard values coefficients $\alpha, \rho, \gamma$, and $\sigma$ (in that combination also referred to as strategy) are given. Depending on the particular problem a simplex may become stretched or flattened after many expansions or contractions. The algorithm is restarted periodically from its best point. The termination of the process is reached when the standard deviation of the simplex function values fall below a specified tolerance $f_{tol}$ with

$$ f_{tol} > \sqrt{\frac{1}{n} \sum_{i=1}^{n} (f_i - \bar{f})^2}. $$

(5.79)

The simplex search algorithm is especially suited for noisy functions since it does not relay on the gradients of the considered function. Given the large number of function evaluations it performs efficiently only if the number of dimensions is small (i.e. $n \leq 5$) (Box (1966)).

5.4.2 Subplex Search Algorithm

The Subplex Search Algorithm as presented and implemented by [Rowan 1990] is a generalization of the Nelder-Mead Simplex method. It overcomes the drawbacks regarding dimensions and constraints by decomposing the problem into low-dimensional, mutually orthogonal subspaces $n_{si}$ that can be handled efficiently by the simplex algorithm:

$$ \sum_{i=1}^{n} n_{si} = n, \text{ with } n_{smin} \leq n_{si} \leq n_{smax} $$

(5.80)
Fig. 5.4: Nelder-Mead-Simplex method (also named Downhill Simplex Method) for unconstrained optimization. A simplex is a set of $n+1$ points in an $n$-dimensional space where $n$ denotes the number of the degrees of freedom in the given optimization problem. In a 2-D-space a simplex can be represented by a triangle moving through the $n$-dimensional domain space (Rowan (1990)).
where \( n_{s\text{min}} \) and \( n_{s\text{max}} \) define the subspace dimensions,

\[
1 \leq n_{s\text{min}} \leq n_{s\text{max}} \quad \text{and} \quad n_{s\text{min}}(n/n_{s\text{min}}) \leq n
\]

(5.81)

where \( n_s \) is the total number of subspaces. The first step in determining the subspaces is to sort the components of the vector of progress \( \Delta x = (\Delta x_1, \ldots, \Delta x_n) \) by decreasing magnitude into a new vector denoted as \( \Delta \bar{x} = (\Delta x_{p1}, \ldots, \Delta x_{pn}) \), where \( |x_{pi}| \geq |x_{pi+1}| \) for \( i = 1, \ldots, n-1 \). The first subspace dimension \( n_{s1} \) is the value of \( k \) that minimizes the function defined by

\[
\frac{||\Delta x||_1}{k} - \frac{||\Delta x||_1}{n-k} \quad \text{for} \quad k < n \text{ or } n
\]

(5.82)

\[
\frac{||\Delta x||_1}{k} \quad \text{for} \quad k = n
\]

(5.83)

subject to \( n_{s\text{min}} \leq n_{si} \leq n_{s\text{max}} \) and \( n_{s\text{min}}[(n-k)/n_{s\text{max}}] \leq n-k \). The first constraint forces \( n_{s1} \) to be in proper range, and the second constraint guarantees that the remaining \((n-n_{s1})\)-vector can be partitioned. The process is repeated to determine \( n_{s2}, n_{s3} \) and so forth. At each iteration of the outer loop, the new vertices of the initial simplices for inner Nelder-Mead-minimizations are determined by

\[
x_i = x_l + \mu_i e_i \quad \text{with} \quad i_1, \ldots, n
\]

(5.84)

Here \( x_l \) is the current best point and \( \mu \) denotes the rescaled step sizes of the previous iteration \( \mu_{\text{prev}} \):

\[
\mu = \begin{cases} 
\min \left[ \max \left( \frac{||\Delta x||_1}{|p|}, \omega_s \right), \frac{1}{\omega_s} \right] \mu_{\text{prev}}, & \text{for} \quad n_s > 1 \\
\psi_s \mu_{\text{prev}}, & \text{for} \quad n_s = 1
\end{cases}
\]

(5.85)

Besides these limits two more parameters are added to the strategy also search strategy: \( \psi_s \) to control for the accuracy of the subspace searches, and \( \omega_s \) which controls the step size. Low values of \( \omega_s \) lead to a fast convergence at the cost of possibly missing a lower minimum, while for large values the convergence is slower but leads to a more thorough search. The accuracy of
the subspace searches is controlled by the coefficient $\psi_s$ by stating that the size of the inner simplex is sufficiently reduced if the condition

$$\text{dist}(x_l, x_h) \leq \psi_s \text{dist}(x_l, x_h)_{\text{initial}}$$

is satisfied. The termination condition for the outer loop is defined by

$$\frac{\max (||\Delta x||_{\infty}, \psi_s ||\mu||_{\infty})}{\max (||x||_{\infty}, 1)} \leq f_{\text{tol}}$$

### 5.4.3 Application of the Stokes-Approximation

In the present application the feasible space of the free variable is limited:

$$a_1, \ldots, a_{nw} \geq 0$$

$$k_1, \ldots, k_{nw} > 0$$

$$0 \leq \alpha_1, \ldots, \alpha_{nw} \leq 2\pi$$

Furthermore, the relation of the wave numbers must adhere to the inequalities as shown in Eq. 5.65 through 5.68. If constraints are imposed by applying penalty functions to infeasible points a simplex algorithm tends to adhere to the feasible domain borders collapsing into a subspace neglecting possible other global minima. Therefore, in order to minimize the number of explicit penalty functions the problem is transferred into a less constraint optimization problem. This is done by modifying the approximation to fulfill the conditions Eq. 5.88 and the following operations:

- Negative values for the component amplitudes and wave numbers are allowed but only their absolute values are used;
- In case of a wave number being zero a small value is added in order to avoid singularities that lead to a program crash;
- before each surface calculation the component waves are ordered with respect to the absolute value of their wave numbers wave numbers.
The conditions limiting the relative bandwidth Eq. [5.65] through [5.68] remain and are accounted for with penalty functions. The maximum number of wave components used in this work is $n_w=6$. More are possible but lead to slowdown of the calculation while the increase in accuracy is marginal. Simulations have shown that often only two or three wave components become effective. The others amplitudes often tend to be close to zero.
Chapter 6

Experimental Investigation

In this section the experimental procedures and equipment are outlined. All experimental results are presented in model scale being linked to the real world by Froude’s law. It states that the ratio between inertial forces and gravitational forces in the model scale is equal to the ratio in the large scale. With $\lambda$ being the scale it leads to the following scaling ratios model scale to fulls scale:

- length/linear displacement: $\frac{1}{\lambda}$
- linear velocity: $\frac{1}{\sqrt[3]{\lambda}}$
- linear acceleration: 1
- area: $\frac{1}{\lambda^2}$
- volume, mass, force: $\frac{1}{\lambda^3}$
- time: $\frac{1}{\sqrt[3]{\lambda}}$
- work: $\frac{1}{\lambda^5}$
- power: $\frac{1}{\lambda^{8/3}}$

Starting from an outline of the wave generation technique in Sec. 6.1 the measuring devices are briefly described in Sec. 6.2.
6.1 Wave Generation

Irregular seaway is assumed to be a random process that results from superposition and interaction of wind driven waves of various frequencies and wave lengths. Extreme events happen sporadic and seldom. A reproduction in a wave tank would demand long trial periods and is therefore very inefficient. Deterministic wave sequences, in which a "model freak wave" appears at a predefined instance of time and location at the tank makes the analysis feasible and reproducible. By a proper choice of a location and an early time of appearance detrimental influences such as residual reflections from the wave absorbers and a permanent circulation induced by the Stokes drift can be avoided.

The generation of predefined wave sequences consists of the four steps:

1. Definition of the desired target wave train, e.g. by scaling a measured wave signal as shown in Sec. 8.2 and 8.1 and transformation in frequency domain $F_{\text{target}}(\omega)$;

2. up-stream transformation of the desired wave from the requested location in the tank up to the wave maker. In this procedure the amplitude spectrum is considered constant whereas the phase spectrum is systematically varied according to the position in the wave tank (Kühnlein (1997)).

3. Determination of the electric signal by application of the hydro-electric transfer function $F_{\text{hydro-electric}}(\omega)$ function, and

4. performance of the model tests.

In mathematical terms the wave board motion is determined:

$$x_{\text{waveboard}}(t) = \text{IFFT} \left[ F_{\text{target}}(\omega) \cdot F_{\text{transformation}}(\omega) \cdot F_{\text{hydro-electric}}(\omega) \right]. \quad (6.1)$$

For extreme waves, the linear model is not sufficient due to substantial non-linear effects such as amplitude dependent-velocity and wave tank dependent
effects. For steep waves with $H/L > 0.05$ the amplitude-depending dispersion relation must be accounted for which leads to faster propagation of steep waves than as compared to of flat waves of same length:

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k} \tanh kd \left(1 + (\zeta a k)^2 \frac{\cosh 4kd + 8}{\sinh^4 kd}\right)}.$$  \hspace{1cm} (6.2)

This effect has been analyzed by Hennig (2005). An iterative scheme is designed that allows the nonlinear upstream transformation taking into account the amplitude dependent propagation. A more physical approach has been introduced by Clauss et al. (2006b). First the wave signal is transformed utilizing the linear wave propagation. The resulting approximate signal is then further varied by a local wavelet optimization scheme. The comparison of the real and the target wave train is done by experiments in the small wave tank that are conducted constantly over several days. Further details are given in Sec. 8.1.

### 6.2 Measuring Devices

The experiments modelling the Yura Wave and the breaking wave packet are carried out in a small wave tank of TU-Berlin (shown in Fig. 6.1) with a length of 15 m, a width of 0.5 m and a water depth of 0.4 m. A computer driven piston type wave maker was employed to realize each wave signal. To suppress disturbing wave reflections at the opposite end of the tank an artificial "beach" is installed. Velocity measurements are sensitive to short waves that are radiated from the breaking at the beach zone. Therefore, to suppress wave breaking additional wave absorbers are installed which consist of an arrangement of grids with 18% porosity as shown in Fig. 6.2. After each experiment a waiting period of 7 up to 10 minutes follows not only to calm down the water surface but also to suppress the permanent vortex initiated by the wave induced reverse current (also known as Stokes drift), that decays slower than residual wave reflections on the surface.
The time history of surface elevation is registered by surface piercing wave gauges. The wave gauges consist of two vertical wires 3 mm in diameter being mounted 18 mm apart. The electronic signal is generated through the varying resistance and transformed into a digital signal at a sampling rate of 200 Hz. The accuracy has been found to be better than 1 mm. The pressure measurements are realized with a pressure transducer (Bell & Howell, type BHL 4104-00-05110). By pointing the sensor upwards the pressure record above the mean water line was taken with minimum disturbances.

Flow field measurements were made by means of a two component laser Doppler system (Polytec LDV-580-2D), (Laser 70 mW, $\lambda = 821$ nm, focal length 30 mm) in combination with two real time signal analyzer units RSA 1000-L. To increase the data rate the water was seeded with aluminium particles. To ensure a data rate of more than 40 Hz at all locations each ex-
Fig. 6.2: Arrangement of wave damping grids with a porosity of 18 % in front of the "beach" to suppress wave breaking and high frequent reflections from the end of the wave tank.

The experiment was repeated three or more times. The accuracy strongly depends on the incident particle velocity. The higher the velocity the more particles pass the control volume inducing a higher data rate and therefore a higher reliability of the data.

As an alternative device an ultrasonic flow meter (DS 102 - Denshi Kogyo Co. LTD) is engaged. The particle motion is tracked by registration of the phase shift of an ultrasonic signal between an emitter and a receiver which is proportional to the delay time. The sampling volume consisting of an emitter/receiver pair for each coordinate axis has the size of $\Delta x = 37$ mm, $\Delta y = 40$ mm, and $\Delta z = 38$ mm.

Compared to the LDV the advantage lies in the output of a constant data rate and the good resolution of velocities between 0 and 1 m/s. A major
draw back is the extend of the sample volume in which local velocity gradi-
ents are averaged out. Its applicability close to the surface is limited since
the presence of the phase boundary nearby leads to echoes and scattering
which in turns causes to distorted registrations. The ultrasonic flow meter is
therefore utilized only well below the free surface.
Measurements of the **New Year Wave** (see Sec. 8.2) are taken in the large
wave tank (length: 80 m, a width of 0.4 m and a water depth of 1.5 m,
piston type wave maker). Lateral windows allow the application of the LDA
at distinct positions along the channel. Fig. 6.3 shows photos of the wave
maker (left) and the arrangement of the lateral underwater windows in the
empty tank (right). At the end of the tank an artificial beach causes the
wave to break and dissipate most of the energy in order to suppress dis-
turbing reflections. The velocity measurements are taken through the lateral
windows. Before each run the domain that is expected to drift in front of
the window is seeded with a cloud of aluminium particles. Therefore, there
is only a restricted span of time in which a sufficient data rate is achieved in
each experiment. The time to allow the disturbed surface to settle between
the experiments lies between 20 and 35 min.

![Large Wave Tank](image)

**Fig. 6.3**: Large Wave Tank (length: 80 m, width: 0.4 m, water depth:
1.5 m, piston type wave maker), vertical grids in front of the wave maker
suppress lateral standing wave systems. Left: View at the wave maker,
right: arrangement of underwater windows in the (empty) tank.
Chapter 7

Validation of the Stokes Approximation

In this section the application of the Stokes-Approximation on measurements of wave sequences is presented for the purpose of validation. Each time the surface elevation is measured by a single or an array of wave gauges. The recorded surface elevation is used to determine a valid combination of parameters of the interacting Stokes waves – the amplitudes, wave numbers, and phase lags. This is done by variation until the irregular surface elevation is reconstructed. The resulting velocity profile is compared with according velocity measurements taken by LDV or ultrasonic flow meters. The goal is to find a valid combination of the irregular surface elevation (within a given window in time or space covering one wave or more) and the underlying velocity profile. Here, the velocity profile underneath the wave crest is chosen as criterion since at that point the maximum velocities occur. Furthermore, the LDV measurements gather the highest data rate at this point providing the lowest error in measurement.

To validate the results of the Stokes-approximation three different wave signals are chosen and realized in wave tanks. All results in this and the following section show comparisons between measurements and results from the
Stokes-Approximation:

1. A steep wave packet that focuses to one single breaking wave: Generated in the small wave tank at TU-Berlin the vertical and horizontal velocities are measured at three distinct positions: At an early stage 2 m from the wave maker the wave package consists of 16 single waves where the length of each wave is a little bit increased compared to the preceding wave. Further measurements are taken at an intermediate stage 8 m from the wave maker where only seven remaining waves pass. Finally, the wave contour at the point of focus, 10.4 m from the wave maker is recorded by an array of wave gauges.

2. The ”Yura Wave” is generated in the small wave tank at a scale of 1:120. The approximation is carried out based on the surface elevation registration at the point, where the Yura Wave appears.

3. The ”New Year Wave” is realized in the large wave tank at scale 1:81. Within a range of 30 meters wave gauges register the surface elevation at distances of 0.2 to 0.3 m allowing the reconstruction of the formation process of the critical wave group. In parallel numerical calculations are conducted using the Pot/FE-solver WAVETUB, revealing numerical results on the wave propagation as well as the kinematics.

In each experiment the measurements of the velocities are used as benchmark for the validity of the numerical results and the Stokes approximation.

### 7.1 Analysis of a Steep Focussing Wave Packet

Transient wave packets have been established for the generation of large waves as well as for seakeeping model tests. According to a predefined Fourier-Spectrum (see top of Fig. [7.1]) the wave maker generates a transient wave train by beginning with high frequency components, and lowering the frequency constantly. The leading short waves are followed by longer and therefore faster waves which in turn are also followed by longer waves. The
7.1 Analysis of a Steep Focussing Wave Packet

Fig. 7.1: The Fourier spectrum used for the generation of the transient wave package

Fig. 7.2: Focussing wave package: Time record of the surface elevation at three locations $x_1 = 2\ \text{m}$, $x_2 = 8\ \text{m}$, and $x_3 = 10.42\ \text{m}$. The focussed single wave breaks immediately behind $x_3 = 10.4\ \text{m}$.

Whole wave train focusses at one point where only one large wave builds up. Transient wave packages can be "hidden" in an irregular seaway leading to the sudden appearance of large waves – like freak waves. Low transient wave packages are used to determine the transfer functions of loads and motions of floating structures in one single experiment which is a much more efficient procedure than the use of regular waves.

A steep focussing wave train is generated with its theoretical concentration
point to one single wave at $x = 11$ m behind the wave maker. In the experiment this extreme wave becomes unstable and starts breaking at $x = 10.4$ m. At three different positions, $x = 2$ m, $x = 8$ m, and $x = 10.4$ m, the vertical and horizontal velocities ($u$ and $v$) are measured up to the mean water level by an ultrasonic flow meter (DS 102 - Denshi Kogyo Co. LTD). The first row of Fig. 7.2 shows the surface elevation of the wave packet passing by wave gauges which are located at 2, 8 and 10.4 m behind the wave maker. The second and the third row show horizontal as well as vertical velocity measurements taken at water depth 0.2 m by an ultrasonic flow meter. A series of close-ups for each wave at $x = 2$ m is shown in Figs. 7.3 through 7.5 in comparison with the approximated surface by the Stokes-Approximation. For each wave the velocity profiles of the Stokes-Approximation under the wave crests are compared to measurements by an ultrasonic flow meter. On the right hand side of Figs. 7.3 through 7.5 the Stokes approximations of the profiles of horizontal velocities under each particular wave crest are compared to measurements by an ultrasonic flow meter. Deviations appearing close to the surface are attributed to echoes of the ultrasonic signal from the phase boundary leading to biased results. For the remaining waves at $x = 8$ m the according approximations are shown in the Figs. 7.6 and 7.7.
Fig. 7.3: Stokes-Approximations of velocities under the wave crests (marked with approx.) compared to measurements by an ultrasonic flow meter measured at position $x = 2$ m. The whole wave train passing this position is shown on top. The approximated region is marked.
Fig. 7.4: Stokes-Approximations of velocities under the wave crests (marked with approx.) compared to measurements by an ultrasonic flow meter measured at position $x = 2$ m. The whole wave train passing this position is shown on top.
Fig. 7.5: Stokes-Approximations of velocities under the wave crests (marked with approx.) compared to measurements by an ultrasonic flow meter measured at position $x = 2$ m. The whole wave train passing this position is shown on top.
Fig. 7.6: Stokes-Approximations of velocities under the wave crests (marked with approx.) compared to measurements by an ultrasonic flow meter measured at position $x = 8$ m. The whole wave train passing this position is shown in the top left corner of Fig. 7.2.
Fig. 7.7: Stokes-Approximations of velocities under the wave crests (marked with approx.) compared to measurements by an ultrasonic flow meter measured at position $x = 8$ m. The whole wave train passing this position is shown in the top left corner of Fig. 7.2 (part 3).
7.2 Kinematics of the breaking Wave

The wave package focuses and breaks at $x \approx 10.4$ m. Surface elevation over time with the resulting extreme single wave about to break is shown in on top of Fig. 7.8. Since the tank bottom is flat and the water depth constant at 0.4 m the breaking process is only initiated by the steepness of the single wave. To examine the breaking process the surface elevation is recorded at intervals of $\Delta x = 0.02$ m within the breaking zone to track the deformation of the wave crest just before it breaks as shown in Fig. 7.8 on top. The last time step at which a converging solution was found ($t = 30.22$ s) is used to approximate the wave kinematics in space domain as shown in the middle of Fig. 7.8. In Fig. 7.9 the resulting field velocity components in $u(x, z, t)$ and $w(x, z, t)$ is shown based on the approximation in the whole domain (Clauss et al. (2008)).

Discussion  Regarding the surface of the wave packet the approximation catches the surface elevation with its asymmetric wave crests wave quite well. Good agreement is achieved between the measurements of the ultrasonic flow meter and the theoretical results. Close to the surface the measurements become spurious due to the detrimental surface effects.

The LDV measurements at the focal point show good agreement even at the strongly asymmetric wave at the brink of breaking. The continuation according to results of the RANS/VoF calculation that has been initialized by a flowfield determined by the Stokes-Approximation shows good agreement with observations during the experiment as presented in the plot and photo series of Fig. 7.11.
Fig. 7.8: Top: Surface elevation over time at $x = 10.24$ m with the resulting extreme single wave being marked. Middle line: Evolution of the surface elevation until the wave breaks ($x = 10.42$ m). Bottom: Measured and approximated surface elevation just before breaking occurs ($t = 30.22$ s, $x = 10.42$ m). It is the last point of time where the approximation provides a converging solution.
Fig. 7.9: The resulting velocity field in components $u(x, z, t)$ and $w(x, z, t)$ based on the approximation. The flow field data are transferred to the RANS-solver Fluent, which continues the simulation of the wave breaking.
Fig. 7.10: Comparison between the LDA-measurement and velocity calculated by the third order Stokes-Approximation according the surface profile at the bottom of Fig. 7.8 ($t = 30.2$ s, $x = 10.42$ m) – just before wave breaking.
Fig. 7.11: Photo sequence of the breaking wave at $x = 10.5$ m. The right column shows results of the RANS/VoF calculation that has been initialized by a flow field determined by the Stokes-Approximation.
Chapter 8

Analysis of Freak Wave Sequences

Reliable measurements of the surface elevation of freak waves in the real sea are sparse, and there is no information on the underlying wave kinematics. Therefore, the analysis is conducted by generation and analysis of synthetic freak waves – scaled registrations of measured wave elevation in the real sea. In this section two famous, high quality recordings of real sea measurements are analyzed– the ”Yura Wave” (Sec. 8.1) and the ”New Year Wave” (Sec. 8.2). After a short introduction results regarding the wave formation process and comparisons between velocity measurements with predictions from wave theories are presented along with results of the new Stokes-Approximation. Each section closes with a short discussion.

8.1 The ”Yura Wave”-Sequence

The so called ”Yura Wave” is a rogue wave that was recorded in the Japan Sea off the coast of Yura harbor by the Japanese National Maritime Institute (Mori et al. (2000)). It has been recorded in a sea state of $H_s = 5.09m$ and $T_p = 11.7s$ with $H_{max} = 13.6m = 2.67 \cdot H_s$ and $\zeta_c = 8.2m = 0.6 \cdot H_s$. In Fig. 8.1 the registration of the wave group in its surrounding seaway is shown.
Analysis of Freak Wave Sequences

<table>
<thead>
<tr>
<th></th>
<th>prior wave</th>
<th>optimized Yura Wave</th>
<th>following wave</th>
</tr>
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<tr>
<td>$H_i$</td>
<td>5.89 m</td>
<td>15.63 m</td>
<td>5.89 m</td>
</tr>
<tr>
<td>$\zeta_{ci}$</td>
<td>2.65 m</td>
<td>11.0 m</td>
<td>3.91 m</td>
</tr>
<tr>
<td>$T_i$</td>
<td>13.06 s</td>
<td>10.53 s</td>
<td>10.84 s</td>
</tr>
<tr>
<td>$T_{ci}$</td>
<td>5.48 s</td>
<td>4.62 s</td>
<td>5.90 s</td>
</tr>
<tr>
<td>$t_{target}$</td>
<td>-</td>
<td>10 s</td>
<td>-</td>
</tr>
</tbody>
</table>

Tab. 8.1: Definition of the target parameters of the $i^{th}$ waves (full scale) embedded in a random sea state with Pierson-Moskowitz-Energy-spectrum, significant wave height $H_s = 5.1$ m, and peak period $T_p = 11.7$ s.

The non-breaking wave train is reproduced in a wave tank (tank dimensions: length 15 m, width 0.3 m, water depth 0.4 m) slightly exaggerated at a scale 1:120. Fig. 8.2 shows a snapshot of the wave passing the wave gauge 8.5 m from the wave maker. The velocity- and pressure field of the Yura Wave are measured simultaneously and the results are compared to current wave theories. To transfer the recorded wave into the wave tank an optimization approach for the experimental generation of wave sequences with
8.1 The "Yura Wave"-Sequence

Fig. 8.2: Snapshot of the Yura Wave passing the wave gauge at \( x = 8.5 \) m. The wave train is generated in the small wave tank (tank dimensions: length 15 m, width 0.3 m, water depth 0.4 m) at scale 1:120.

predefined characteristics is used according to Clauss and Schmittner (2005). The method is applied to generate scenarios with a single high wave superimposed to irregular seas. Special emphasis is laid on the slightly exaggerated reproduction of the wave height \( H_i \), crest height \( \zeta_{ci} \), zero-upcrossing wave period \( T_{i} \) and the duration between zero-upcrossing and zero-downcrossing \( T_{ci} \) of the particular \( i^{th} \) wave. \( t_{\text{target}} \) denotes the time of occurrence. The characteristics of the prior and following wave are included in the experimental optimization. Tab. 8.1 shows all target parameters used in the experimental optimization of the sea state with the modified Yura Wave. The optimization begins with a realization of an irregular seaway according to a given energy spectrum with random phase distribution. By systematic variations of the particular phase spectrum driven by an optimization-algorithm the requested wave contour in time domain is generated at the respective position in the tank. At the end of the first optimization part a preliminary control signal for the wave generator is produced.

Since the initial control signal is calculated by linear theory nonlinear effects such as wave-wave interaction and wave breaking are not taken into account. Therefore, the measured wave sequence differs from the defined target parameters significantly. From here the experimental optimization routine starts and improves iteratively the control signal aiming to satisfy the target parameters. At this stage only the wave environment close to the embedded rogue wave is modified. This is realized by applying discrete wavelet transform
Fig. 8.3: Computer controlled optimization scheme of waves: The wave sequence is automatically created, measured, evaluated and modified until convergence is achieved, i.e. the target wave train is generated.

which alters the wave signal at a hybrid stage between frequency domain (as done in the first step) and time domain (see e.g. Daubechies (1988), Valens (1999), or Meyer (1992)). The fitting of the wave train is achieved by applying the Subplex optimization method. Nonlinear free surface such as wave breaking between wave maker and target location do not inhibit the fitting process since the values of the objective function are determined from the nonlinear wave tank experiment, constituting a "self-validating" process as sketched in Fig. 8.3. At the end of this "hybrid" optimization process a control signal for a wave sequence satisfying all target parameters is obtained (Clauss and Schmittner (2005), Schmittner (2005), and Clauss et al. (2007)).

The reproduced wave train with the embedded critical wave group at scale 1:120 is shown in the top line of Figs. 8.4 and 8.5.

**Analysis and Discussion** To conduct the analysis of the wave kinematics a decomposition into 1024 harmonic elementary wave components is shown in Fig. 8.4. To calculate the surface elevation, the dynamic pressure (fourth line) and the horizontal velocities calculated
by extrapolation the linear wave theory (see Eq. 3.6 and 3.10 up to the wave crest, and
- by applying a stretching terms according to Wheeler (1970) (see Sec. 3.4) to each component wave.

for each component and then superimposed. The results are plotted in Fig. 8.5 in comparison with the measurements. When the sensors emerge from the water, the values are set to zero. The theoretical results of the extrapolated solution turn out to be unrealistically high, and are cut off at the mean water line for easier depiction. The solution indicates once more that the extrapolation solution with many components is not able to describe kinematics (see Appendix A) and the stretching theory tends to underestimate the velocities. Therefore, another solution on a wave-to-wave basis is needed. At the bottom of Fig. 8.5 the comparison of the measured velocity profiles of the critical wave group are shown. In comparison with regular third order Stokes wave theory the results for the main freak wave is represented well reasonably while the results for the skewed preceding and following wave are unsatisfactory. The application of the Stokes-approximation on the registration of the time signal takes the skewed wave profiles into account and shows good compliance with the measurements as shown in Figs. 8.6 through 8.6.

Regarding the dynamic pressure a maximum is visible at the mean water line with an almost linear drop towards the wave crest. This drop – similar to the calm water hydrostatics – affirms the dynamic free surface boundary condition (DFSBC) but is not represented by any exponential-like increasing behavior of any deep water wave theory. It does, however, support the theory of oscillating lever arms in which the fluctuating surface contour induces periodic motions on ships motion by oscillating ”restoring” forces.
Fig. 8.4: The Fourier Analysis of the Yura Wave Sequence shows the decomposition into elementary wave components with their contributions to the surface elevation (second line), dynamic pressure (fourth line) and horizontal velocities (bottom line) all calculated according to Airy wave theory (see Sec. 3.1).
8.1 The "Yura Wave"-Sequence

Fig. 8.5: Yura Wave: Recording of the surface elevation (top), horizontal velocity (right), and dynamic pressure (left) are shown. Values taken above the deflected surface are set to zero. Bottom: Pressure and velocity profiles are shown in comparison with results from conventional wave theories.
Fig. 8.6: Approximations of the surface elevation of the Yura Wave train and the resulting profiles of the horizontal velocity: Comparison between measurement and results of the Stokes-Approximation (part 1).
8.1 The "Yura Wave"-Sequence

Fig. 8.7: Approximations of the surface elevation of the Yura Wave train and the resulting profiles of the horizontal velocity: Comparison between measurement and results of the Stokes-Approximation (part. 2).
Fig. 8.8: Approximations of the surface elevation of the Yura Wave train and the resulting profiles of the horizontal velocity: Comparison between measurement and results of the Stokes-Approximation (part. 3).
8.2 The "New Year Wave"-Sequence

This famous registration of a rogue wave with a wave height of 25.6 m (significant wave height $H_s = 11.92$ m) was recorded at the Draupner platform in the North Sea on January 1, 1995 while the platform suffered severe damages (Haver (2000)). The wave is reproduced in the large wave tank of the Technical University Berlin (tank dimensions: length 80 m, width 4 m, water depth 1.5 m, piston type wave generator) at a scale of 1:81 using the wave generation algorithm as described in Sec. 6.1. Fig. 8.9 shows a comparison of the original measurement with the reproduced signal being transferred into the original scale. A Fourier decomposition is shown in Fig. 8.10. Measurements are taken with six wave gauges mounted on a towing carriage. The wave sequence is repeated 22 times while each time the carriage was positioned at other subsequent location, measuring time recordings at 132 positions that are assembled to surface contours. That way a dynamic wave profile was generated showing the wave evolution in time and space. Velocity measurements by LDA are taken through lateral windows at $x = 25.3$ m which is the point where the steep single wave reaches its highest crest height prior to wave breaking.

Fig. 8.9: Registration of the surface elevation measured from the Draupner Platform on January 1st 1995, 15:20 GMT (Haver (2000)) in comparison with its reproduction in the large wave tank at scale 1:81 (here shown in full scale)
In parallel the experiment is conducted virtually using the numerical wave tank WAVETUB. The tank is discretized over a length of 60 m into 120,000 cells (2400 cells along the tank and 50 cells in vertical direction). The time step size is set to $\Delta t = 0.005$ s. Utilizing the fourth order Runge-Kutta-Time-step integration scheme a computation time of approximately 13 hours is needed up to the point where the main wave breaks and the calculation aborts.

Figs. 8.11 and 8.12 show a series of the wave profiles – numerical and experimental – before and during the emergence of the freak wave. The full series of the evolution of the surface elevation in time and in space are shown in Appendix E in Figs. E.1 E.2 – along with the evolution of the mean envelope curves which are smoothed using a cubic spline interpolation. A close-up of instances of the surface elevation showing the emergence of the New Year Wave in comparison between measurement and numerical result is shown in Fig. 8.13 (Clauss et al. (2006c)). Wave breaking sets in at about $t = 71$ s.

The velocity field extracted from WAVETUB is shown in Fig. 8.14. Besides by the application of WAVETUB (shown in the top right of Fig. 8.17 the according velocity profile is determined with regular waves, superimposed harmonic waves and finally the Stokes-Approximation. The profiles of the regular waves are fitted to the front steepness, since it has been identified as the governing parameter for impact loads (Xu and Barltrop (2008)). The wave height is taken as the distance between wave crest and the mean of the preceding and following trough. The assumed wave period is the time span between the preceding trough and the wave crest multiplied by two. The following profiles are shown in comparison with LDA measurements taken at the large wave tank:

- Based on a Fourier analysis the wave group is decomposed into 4 up to 512 component waves. The resulting surface elevations are shown in Fig. 8.15. The according results of linear and stretching theory are shown in in Fig. 8.16.
8.2 The ”New Year Wave”-Sequence

\[ \zeta_a(t) \ [\text{m}] \]

\[ \omega \ [\text{rad s}^{-1}] \]

\[ \varepsilon \ [\text{rad}] \]

Fig. 8.10: Fourier decomposition of the New Year Wave-registration: The amplitude spectrum (top) and the phase distribution (bottom) is shown.

- Regular linear wave (Airy) is shown in the top left part of Fig. 8.18, which is extrapolated up to the wave crest, in comparison with regular linear wave (Airy), where a stretching term according to Wheeler (1970) is applied.

- Conventional regular Stokes III theory according to Skjelbreia (1959) is compared to the velocity profile extracted from the WAVETUB-calculation in Fig. 8.17 (Clauss et al. (2007)).

Finally the Stokes-Approximation shows the best agreement with the measurements as shown in the bottom of Fig. 8.17.

The respective diagram over time with a fixed location is shown in the series of diagrams in Fig. E.3. Again a full series of the whole wave group is given in Appendix E in Figs. E.3–E.4 with the evolution of the respective smoothed...
envelope curves. By comparison it shows that the wave group moves with
a group speed of \( c_{gr} = 1.16 \text{ m/s} \) somewhat increasing over time while the
individual waves prior breaking travel with \( c = 2.25 \text{ m/s} \). The approximate
period of the wave group shrinks from 7.2 s (at position \( x = 18.1 \text{ m} \)) down to
4.3 s just before the wave breaks at \( x = 25.0 \text{ m} \). The periodic characteristic
of the wave group indicated by the envelope curve (shown on the left side of
Fig. E.3 and E.4) leads to the association with beats (or breather-like) wave
systems that consist of only two ”detuned” waves. In this case the fitting
process revealed the following parameters:

<table>
<thead>
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<th>amplitude [m]</th>
<th>wave number [1/m]</th>
<th>( \omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 = 0.075 )</td>
<td>( k_1 = 2.08 )</td>
<td>( \omega_1 = 4.487 )</td>
</tr>
<tr>
<td>( a_2 = 0.075 )</td>
<td>( k_2 = 1.165 )</td>
<td>( \omega_2 = 3.255 )</td>
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</table>

Fig. 8.18 shows the surface elevation plotted over space (top) and time(bottom)
in comparison to a superposition of two detuned, interacting Stokes waves.
The characteristic not only of the freak wave but also of the surrounding wave
group is met quite well. The profile of the horizontal velocity proves to meet
the measurements quite accurately as shown in Fig. 8.17. The theoretical
group velocity

\[
c_{gr} = \frac{\Delta \omega}{\Delta k} = 1.37
\]  
(8.1)

exceeds the measurement by 18 per cent.
Fig. 8.11: Sequence of surface elevation contours (plotted over $x$) with the developing the New Year Wave at scale 1:81. The red line shows the numerical result (WAVETUB).
Fig. 8.12: Time record sequence of surface elevations (plotted over time $t$) with the developing the New Year Wave at scale 1:81. The red line shows the numerical result (WAVETUB).
8.2 The "New Year Wave"-Sequence

Fig. 8.13: Close-up of instances of the surface elevation showing the emergence of the New Year Wave in comparison between measurement and numerical result (WAVETUB). Wave breaking sets in after $t = 70.8$ s.

Fig. 8.14: New Year Wave realized in a physical and a numerical wave tank. Horizontal particle velocity field at $t = 70.5$ s is shown calculated by the Pot/FE solver WAVETUB.
Fig. 8.15: The wave train consisting the New Year ($x = 25$ m) is reconstructed out of $n$ harmonic wave components. The according velocity profile underneath the wave crest ($t = 72.1$ s) are shown in Fig. 8.16.
Fig. 8.16: According the wave elevations in Fig. 8.15 the velocity profile underneath the wave crest \( t = 72.1 \) s is determined using linear and stretching theory for irregular wave trains (Wheeler (1970)) with varying number of wave components \( n \).
Fig. 8.17: Top: The horizontal velocity profiles of the New Year Wave at $t = 70.4$ s, $x = 24.9$ m. in comparison with conventional wave theories. The profiles from the linear and the Stokes wave are based on a regular wave, that is fitted to the front steepness: The wave height is defined by the wave crest and the mean of the preceding and the following wave trough. Bottom: Velocity profile underneath the crest determined by the Stokes-Approximation (see Fig. 8.18 for surface fitting).
**8.2 The ”New Year Wave”-Sequence**

**Fig. 8.18:** New Year Wave realized in a wave tank: Surface elevation plotted over space (top) and time (bottom) in comparison to a superposition of two detuned, interacting Stokes waves.
8.3 Discussion

Once more it is demonstrated the results of the Fourier decomposition can represent the surface elevation well – even locally when only a small number of components is used. The kinematics, however, are overestimated by the strictly linear model (which has its limit at the mean water line) while the stretching term predicts lower velocities than the measurements reveal.

The comparison between the measured surface elevation with the numerical results proves how well the Pot-FE model predicts the general characteristics of wave propagation and wave-wave-interaction. As well, the kinematics also show good agreement with the measurements as shown in Fig. 8.17. Like in physical testing facilities wave groups are generated and modified by inducing and altering the –numerical– wave maker motion. That way, the simulation must cover the whole wave signal from the beginning of the wave motion (with zero energy in the tank) up to the point where the desired wave group occurs - sometimes aborting the simulation due to premature wave breaking. In case of the New Year Wave the simulation time took 13 hours until the actual wave group appeared.

By neither numerical nor experimental investigation is the influence of short-crestedness. While the original point measurement of the New-Year-Wave-record by Haver (2000) does not reveal directional information, hindcast studies taken at that time in this sea area imply the presence of a swell and wind sea spectrum. In recent experiments in the Large Offshore Basin of Marin, Netherlands, Hennig (2008) reproduced the New-Year-Wave based on a directional double peaked spectrum. The directional scattering of the wave propagation leads subsequently to scattering of the horizontal components of particle velocity and acceleration. Observations by Mitsuyasu et al. (1975) have shown that the main energy carrying components are narrowly grouped around the main wave direction. Therefore, with respect to the wave kinematics the long crested case can be assumed as worst case scenario or design
8.3 Discussion

case.
Chapter 9

Conclusions

The impact of steep seaway and freak waves impose considerable forces on ships and offshore platforms and have led to extensive damage. To take those loads into account within the design process and classification, reliable models of wave kinematics are crucial for the estimation of the wave impact.

The superposition of many linear wave components has proven to model the surface elevation of irregular seaway and the wave propagation well. With few modifications it serves as a powerful tool for the generation of extreme waves in irregular seaway. But it does not provide reliable predictions of the wave kinematics in the surface domain which is the region of highest velocities. The use of higher order wave theories provide good estimations for wave kinematics only for regular waves that do not possess characteristics of harsh irregular seaway.

In this thesis the Stokes-Approximation is presented – a new method to connect both approaches. To a window of an irregular surface consisting of one or more waves – which may be a time registration or a surface contour – a combination of up to six interacting Stokes waves is determined. The solution is found by fitting a parametric surface of the interacting Stokes waves to the given natural surface utilizing a modified least square method combined
with a Subplex search algorithm. The perturbation solution of the Stokes waves contains interaction terms up to the third order. It is shown that the instantaneous surface contour as well as the underlying particle velocities up to the wave crest is approximated well – even when the wave is about to break.

To validate the Stokes-Approximation scheme three wave sequences generated in two wave tanks are considered:

- A steep wave packet that focusses to one large single wave is generated in the small wave tank at TU-Berlin the vertical and horizontal velocities are measured at three distinct positions. The Stokes-Approximation allows good results for the single waves at the early stage of the wave package as well as at the point where all waves focus to one single high wave which breaks. Initialized by the velocity field from the Stokes-Approximation the breaking process is simulated using the RANSE-solver Fluent showing good agreement with the observations of the experiment.

- The "Yura-Wave" a freak wave sequence measured off the coast of Yura Harbor is reproduced at scale 1:120 in the small wave tank. The Stokes-Approximation is carried out based on the surface elevation registration showing good coincidence with measurements.

- The "New Year Wave" is reproduced at scale 1:81 in the large wave tank. By application of an array of wave gauges the surface elevation is recorded in time and space allowing the reconstruction of the formation process of the critical wave group. In parallel numerical calculations are conducted using the Pot/FE-solver WAVETUB showing good agreement with respect to the wave propagation as well as the crest kinematics.

These examples show that the Stokes-Approximation is in good agreement with the physics in testing facilities as well as with numerics. It relays, however, on a given surface elevation and does not provide an estimation for the previous form of the wave train nor for the further development.
Chapter 10

Nomenclature

Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
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<tr>
<td>BBC</td>
<td>Bottom Boundary Condition</td>
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<tr>
<td>BFI</td>
<td>Benjamin-Feir-Index</td>
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<td>BVP</td>
<td>Boundary Value Problem</td>
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<td>CFD</td>
<td>Computational Fluid Dynamics</td>
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<td>CEFSBC</td>
<td>Combined Exact Free Surface Boundary Condition</td>
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<td>Free Surface Boundary Condition</td>
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<td>JONSWAP</td>
<td>Joint North Sea Wave Project</td>
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<td>KFSBC</td>
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<td>LDA</td>
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<td>Laser Doppler Velocemetry</td>
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<td>Modified nonlinear Schrödinger Equation</td>
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<td>Mean Water Line</td>
</tr>
<tr>
<td>NLS</td>
<td>Nonlinear Schödinger Equation</td>
</tr>
<tr>
<td>NORSOK</td>
<td>(Norsk Sokkels Konkurranseposisjon) Safety standards</td>
</tr>
<tr>
<td>PIV</td>
<td>Particle Induced Velocemetry</td>
</tr>
<tr>
<td>POT/FE</td>
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</tr>
<tr>
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<td>Reynolds Averaged Navier-Stokes-Equation</td>
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<td>RMS</td>
<td>Root Mean Square</td>
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<tr>
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<td>Technical University of Berlin</td>
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<tr>
<td>VoF</td>
<td>Volume of Fluid</td>
</tr>
<tr>
<td>WAVETUB</td>
<td>WAVE Analysis at Technical University of Berlin</td>
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### List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Unit</th>
<th>Description</th>
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<tbody>
<tr>
<td>$A$</td>
<td>$[m^2]$</td>
<td>area</td>
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<tr>
<td>$A_{xx}$</td>
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<td>shortcut for Stokes expansion</td>
</tr>
<tr>
<td>$B$</td>
<td>$[m^2/s^2]$</td>
<td>Bernoulli const.</td>
</tr>
<tr>
<td>$B_{xx}$</td>
<td>$[-]$</td>
<td>shortcut for Stokes expansion</td>
</tr>
<tr>
<td>$B_{(...)}$</td>
<td>$[-]$</td>
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<tr>
<td>$C_{xx}$</td>
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<tr>
<td>$C$</td>
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<tr>
<td>$C$</td>
<td>$[m/s]$</td>
<td>velocity of moving reference frame</td>
</tr>
<tr>
<td>$C$</td>
<td>$[m/s]$</td>
<td>celerity</td>
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<tr>
<td>$D(x,t)$</td>
<td>$[m^2/s^2]$</td>
<td>error of the DFSBC</td>
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<tr>
<td>$D_n, D_0$</td>
<td>$[-]$</td>
<td>decay terms</td>
</tr>
<tr>
<td>$E$</td>
<td>$[N/m]$</td>
<td>wave energy per square meter</td>
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<tr>
<td>$F(\zeta_c)$</td>
<td>$[-]$</td>
<td>Probability density of wave crests</td>
</tr>
<tr>
<td>$F(\omega)$</td>
<td>$[-]$</td>
<td>Fourier transform</td>
</tr>
<tr>
<td>$F_1(t), F_2(t), \ldots$</td>
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<td>Bernoulli const. of order 1,...</td>
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<tr>
<td>$H$</td>
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<td>wave height</td>
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<td>$H_S$</td>
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<td>significant wave height</td>
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<tr>
<td>$H_{p95}, H_{p99}$</td>
<td>$[m]$</td>
<td>95th, 99th percentile wave height</td>
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<td>$[m]$</td>
<td>maximum wave height</td>
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<td>$H_{MPM}$</td>
<td>$[m]$</td>
<td>most probable maximum wave height</td>
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<tr>
<td>$H_{RMS}$</td>
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<td>root mean square of the wave height</td>
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<tr>
<td>$K$</td>
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<td>$R$</td>
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<td>$R_w$</td>
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<td>coefficient for fetch influence</td>
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<tr>
<td>$S$</td>
<td>$[-]$</td>
<td>implicit function for the water surface</td>
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<td>$S$</td>
<td>$[N/m]$</td>
<td>radiation stress</td>
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<tr>
<td>$S(\omega)$</td>
<td>$[m^2 s]$</td>
<td>wave energy spectrum</td>
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</table>
$T$ $[s]$ wave period
$T_C$ $[s]$ crest period
$T_{Fw}$ $[s]$ filter period
$T_0$ $[s]$ zero-upcrossing period
$T_P$ $[s]$ peak period period
$U$ $[m/s]$ Eulerian current
$U_{19.5}$ $[m/s]$ wind speed 19.5 m above sea level
$U_R$ $[-]$ Ursell-parameter
$V$ $[m^3]$ volume
$X$ $[m/s]$ x-coordinate (moving frame)
$X_i$ $[m^2/s]$ amplitude of $i^{th}$ order
$a_0$ $[m]$ dominant wave amplitude
$a_{1,2,...}$ $[m]$ first, second, . . . component wave
$a_f$ $[-]$ factor for fetch influence
$c, c_0$ $[m/s]$ wave celerity
$c_{gr}$ $[m/s]$ wave group speed
$d$ $[m]$ water depth
$d_{\triangle}$ $[m]$ empirical tuning coefficient
$f^K_i, f^D_i$ $[-]$ objective functions
$g$ $[m/s^2]$ gravitational constant
$g_f$ $[m^2]$ goal function of the subplex search algorithm
$i, j, l$ $[-]$ index
$k$ $[1/k]$ wave number
$k_0$ $[1/m]$ dominant wave number
$k_{1,2,...}$ $[1/m]$ wave number of the first, second, . . . component wave
$k_P$ $[1/k]$ peak wave number (associated with $\omega_P$)
$l$ $[m]$ length in y direction
$m_n$ $[(rad/s)^n]$ $n^{th}$ spectral moment
$p$ $[pa]$ pressure
$p_d, p_{dyn}$ $[pa]$ dynamic pressure
$p_0$ $[pa]$ surrounding (air) pressure
$t$ $[s]$ time
$\tilde{v}$ $[m/s]$ velocity vector
$t$ $[-]$ time
<table>
<thead>
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<th>Symbol</th>
<th>Units</th>
<th>Description</th>
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<td>[m/s]</td>
<td>particle velocity vector/components in x,y,z</td>
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<tr>
<td>$\vec{n}, x, y, z$</td>
<td>[m]</td>
<td>normal vector/ cartesian coordinates</td>
</tr>
<tr>
<td>$x_h, x_l, x_r$</td>
<td>[-]</td>
<td>points associated with simplex algorithm</td>
</tr>
<tr>
<td>$z, z'$</td>
<td>[m]</td>
<td>scaled vertical coordinate</td>
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<td>$\Phi$</td>
<td>[m$^2$/s]</td>
<td>Wave potential</td>
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<tr>
<td>$\Psi$</td>
<td>[m$^2$/s]</td>
<td>Streamline</td>
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<td>$\Psi_{cn}$</td>
<td>[m$^2$/s]</td>
<td>Stream function of the cnoidal theory</td>
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<td>$\Upsilon$</td>
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<td>$\Theta$</td>
<td>[-]</td>
<td>phase function</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>[(m,m)]</td>
<td>Fluid domain</td>
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<td>$\alpha_1, \alpha_2, \ldots$</td>
<td>[rad]</td>
<td>phase lag</td>
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<tr>
<td>$\mu$</td>
<td>[kg/(ms)]</td>
<td>dynamic viscosity</td>
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<tr>
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<td>[kg/m$^3$]</td>
<td>water density (= 1025 for salt water)</td>
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<td>[m]</td>
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<tr>
<td>$\zeta_c$</td>
<td>[m]</td>
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<td>$\sigma$</td>
<td>[m]</td>
<td>standard deviation of the surface elevation</td>
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<td>$\omega_p$</td>
<td>[rad s$^{-1}$]</td>
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<tr>
<td>$\omega_s$</td>
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<td>simplex control parameter</td>
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Appendix A

Wave Kinematics by Superposition vs Single Wave

In this section the a simple example illustrates the inability to model wave kinematics – in this case horizontal velocities – by superposition of linear wave components. Fig. A.1 shows on to two waves ($T = 10$ s, $H = 7.8$ m, $H/L = 1/20$, deep water condition) which are to be analyzed using Airy wave theory (as presented in Sec. 3.1): While one is a purely harmonic (or sinusoidal) wave the other one is a little bit skewed and has to modelled by a superposition of harmonic wave components. The respective amplitude spectra generated by a Fast Fourier Transform are printed in the bottom of Fig. A.1 – here restricted to the first ten components. The horizontal velocity in the wave crest of the harmonic wave is

$$u_{\text{crest}}(t = 0, x = 0) = \omega \frac{H}{2} e^{kz_{\text{crest}}} = 2.870, \ [\text{m/s}] \quad (A.1)$$

the horizontal particle velocity at the mean water line is accordingly

$$u_{z=0}(t = 0, x = 0) = \omega \frac{H}{2} e^0 = 2.453 \ [\text{m/s}]. \quad (A.2)$$

The results of the spectral approach are listen in Tab. A.1 clearly indicating that from the fourth component the ($\omega > 4\omega_1$) the contribution to the sur-
Fig. A.1: Comparison between a harmonic wave ($T = 10$ s, $H = 7.8$ m, $H/L = 1/20$) and a skewed wave with identical wave height and period. The top graph shows the surface elevations, in the bottom the amplitude spectra are shown.

Face elevation almost vanishes while the theoretical contribution to the crest velocity increases at an unrealistic scale. The crest velocity of the skewed wave cannot be modelled by means of the extrapolation of the Fourier transform based linear wave theory. Even at the mean water line at $z = 0$ the results from the superposition exceed the results from the single wave (shown in Eq. A.2) significantly.
$\omega_i \text{ rad/s}$ | $\zeta ai \text{ m}$ | $\sum_i u_{z=0} \text{ m/s}$ | $\sum_i \exp(k_i z_{crest})$ | $u_{crest} \text{ m/s}$
---|---|---|---|---
1 | 0.628 | 3.877 | 2.167 | 1.170 | 2.685
2 | 1.256 | 0.510 | 2.666 | 1.874 | 1.873
3 | 1.885 | 0.136 | 2.760 | 4.111 | 0.952
4 | 2.513 | 0.050 | 2.777 | 12.345 | 0.935
5 | 3.142 | 0.023 | 2.779 | 50.754 | 3.773
6 | 3.770 | 0.012 | 2.779 | 285.68 | 16.48
7 | 4.398 | 0.007 | 2.778 | 2201.5 | 53.05
8 | 5.027 | 0.005 | 2.778 | 23227.6 | 99.16
9 | 5.655 | 0.003 | 2.777 | 335524.5 | 5537.7
10 | 6.283 | 0.002 | 2.777 | $6.6 \times 10^6$ | $9.7 \times 10^4$

Tab. A.1: Theoretical results for the crest velocity of the skewed wave: The superposition of velocity components even of the first ten waves leads to unrealistic high velocities in the wave crest due to high frequency pollution.
Appendix B

Solutions of the fifth order
Stokes Wave Theory

The following list defines a set of coefficients, which are all functions of \( kd \) only:

\[
C = \cosh(kd) \quad \text{(B.1)}
\]

\[
S = \sinh(kd) \quad \text{(B.2)}
\]

\[
A_{1,1} = 1/S \quad \text{(B.3)}
\]

\[
A_{1,3} = \frac{-C^2 \cdot (5C^2 + 1)}{(8S^5)} \quad \text{(B.4)}
\]

\[
A_{1,5} = -\left(1184C^{10} - 1440C^8 - 1992C^6 + 2641C^4 - 249C^2 + 18\right) \cdot \frac{1}{1536S^{11}} \quad \text{(B.5)}
\]

\[
A_{2,2} = \frac{3}{8S^4} \quad \text{(B.6)}
\]

\[
A_{2,4} = \frac{192C^8 - 424C^6 - 312C^4 + 480C^2 - 17}{768S^{10}} \quad \text{(B.7)}
\]

\[
A_{3,3} = \frac{13 - 4C^2}{64S^7} \quad \text{(B.8)}
\]

\[
A_{3,5} = \left(512C^{12} + 4224C^40 - 6800C^8 - 12808C^6 + 16704C^4 - 3154C^2 + 107\right) \frac{1}{4096S^{13} \cdot (6C^2 - 1)} \quad \text{(B.9)}
\]
A_{4,4} = \frac{80C^6 - 816C^4 + 1338C^2 - 197}{1536S^{10} \cdot (6C^2 - 1)} \tag{B.10}

A_{5,5} = -\left(2880C^{10} - 72480C^8 + 324000C^6 - 432000C^4 + 163470C^2 - 16245\right) \cdot \frac{1}{61440S^{11} \cdot (6C^2 - 1) \cdot (8C^4 - 11C^2 + 3)} \tag{B.11}

B_{2,2} = C \cdot \frac{2C^2 + 1}{4S^3} \tag{B.12}

B_{2,4} = C \cdot \frac{272C^8 - 504C^6 - 192C^4 + 322C^2 + 21}{384S^9} \tag{B.13}

B_{3,3} = 3 \cdot \frac{8C^6 + 1}{64S^6} \tag{B.14}

B_{3,5} = \left(88128C^{14} - 208224C^{12} + 70848C^{10} + 54000C^8 - 21816C^6 + 6264C^4 - 54C^2 - 81 \right) \cdot \frac{1}{12288S^{12} \cdot (6C^2 - 1)} \tag{B.15}

B_{4,4} = C \cdot \frac{768C^{10} - 448C^8 - 48C^6 + 48C^4 + 106C^2 - 21}{384S^9 \cdot (6C^2 - 1)} \tag{B.16}

B_{5,5} = \left(192000C^{16} - 262720C^{14} + 83680C^{12} + 20160C^{10} - 7280C^8 + 7160C^6 - 1800C^4 - 1050C^2 + 255 \right) \cdot \frac{1}{12288S^{10} \cdot (6C^2 - 1) \cdot (8C^4 - 11C^2 + 3)} \tag{B.17}

C_1 = \frac{8C^4 - 8C^2 + 9}{8S^4} \tag{B.18}

C_2 = \frac{3840C^{12} - 4096C^{10} + 2592C^8 - 1008C^6 + 5944C^4 - 1830C^2 + 147}{512S^{10} \cdot (6C^2 - 1)} \tag{B.19}

C_3 = -\frac{1}{4SC} \tag{B.20}

C_4 = \frac{12C^8 + 36C^6 - 162C^4 + 141C^2 - 27}{192C^5} \tag{B.21}

c_0 = \sqrt{\frac{g}{k} \cdot \tanh(kd)} \tag{B.22}
\[ \lambda_1 = \lambda A_{11} + \lambda^3 A_{13} + \lambda^5 A_{15} \]  \hspace{1cm} (B.23)
\[ \lambda_2 = \lambda^2 A_{22} + \lambda^4 A_{24} \]  \hspace{1cm} (B.24)
\[ \lambda_3 = \lambda^3 A_{33} + \lambda^5 A_{35} \]  \hspace{1cm} (B.25)
\[ \lambda_4 = \lambda^4 A_{44} \]  \hspace{1cm} (B.26)
\[ \lambda_5 = \lambda^5 A_{55} \]  \hspace{1cm} (B.27)
Solutions of the fifth order Stokes Wave Theory
Appendix C

Third Order Interaction Terms

Listing of interaction terms of three referring to three Interacting Stokes waves of third order. The subscript indicates the involved components waves and their factor and whether the terms refer to the addition (p) or substraction (m). e.g. \(K_{(2\theta_3-\theta_1)}\) refers to a term resulting from the KFSBC with the harmonic argument \(\cos(2\theta_3 - \theta_1)\).

Terms that arise from the KFSBC are denoted with the capital letter \(K\):

\[
K_{(2\theta_2-\theta_1)} = \frac{a_3^2a_1}{8g^2} (\omega_1^5 - 2\omega_1^4\omega_2 + 4\omega_1^2\omega_2^3 - 4\omega_1\omega_2^4 - 4\omega_2^5) \quad (C.1)
\]
\[
K_{(2\theta_1-\theta_2)} = -\frac{a_1^2a_2\omega_2}{8g^2} (8\omega_1^4 - 2\omega_1^3\omega_2 + 4\omega_1^2\omega_2^2 - 2\omega_1\omega_2^3 - \omega_2^4) \quad (C.2)
\]
\[
K_{(2\theta_3-\theta_1)} = \frac{a_3^2a_1}{8g^2} (\omega_1^5 - 2\omega_1^4\omega_3 + 4\omega_1^2\omega_3^3 - 4\omega_1\omega_3^4 - 4\omega_3^5) \quad (C.3)
\]
\[
K_{(2\theta_3-\theta_1)} = -\frac{a_1^2a_3\omega_3}{8g^2} (8\omega_1^4 - 2\omega_1^3\omega_3 + 4\omega_1^2\omega_3^2 - 2\omega_1\omega_3^3 - \omega_3^4) \quad (C.4)
\]
\[
K_{(2\theta_2-\theta_3)} = \frac{a_2^2a_3}{8g^2} (\omega_2^5 - 2\omega_2^4\omega_3 + 4\omega_2^2\omega_3^3 - 4\omega_2\omega_3^4 - 4\omega_3^5) \quad (C.5)
\]
\[
K_{(2\theta_2-\theta_3)} = -\frac{a_2^2a_3\omega_3}{8g^2} (8\omega_2^4 - 2\omega_2^3\omega_3 + 4\omega_2^2\omega_3^2 - 2\omega_2\omega_3^3 - \omega_3^4) \quad (C.6)
\]
\[ K_{(\theta_3+\theta_2-\theta_1)} = \frac{a_1a_2a_3}{4g^2} (\omega_3^5 - \omega_2^5(\omega_1 + \omega_2) - 2\omega_1^2\omega_2^3 - 2\omega_1^2\omega_1\omega_2^3) \] (C.7)

\[ K_{(\theta_3-\theta_2+\theta_1)} = \frac{a_1a_2a_3}{4g^2} (\omega_3^5 - \omega_4^5(\omega_1 + \omega_2) + 2\omega_1^2\omega_2^3 - 2\omega_2^3) \] (C.8)

\[ K_{(\theta_3+\theta_1-\theta_3)} = -\frac{a_1a_2a_3}{4g^2} (3\omega_3^5 - \omega_4^5(\omega_1 + \omega_2) - 6\omega_1^2(\omega_1^2 + \omega_2^2) + 2\omega_1^2\omega_2^3 + \omega_1^3) \] (C.9)

\[ K_{\theta_1} = -\frac{a_1}{8g^2} 2a_2(\omega_1^5 + 6\omega_1^2\omega_2^3 - 2\omega_1^3) - \frac{2a_1a_3^2}{8g^2} (\omega_3^5 + 6\omega_1^2\omega_3^3 - 2\omega_3^5) \] (C.10)

\[ K_{\theta_2} = -\frac{a_2a_3}{8g^2} 5\omega_2^5\omega_3 + 2a_1^2(2\omega_1^5 + 2\omega_1^3\omega_2 - 2\omega_1^2\omega_2^3 + 2\omega_1\omega_2^3 + \omega_1^3) \] (C.11)

\[ K_{\theta_3} = -\frac{a_3a_1}{8g^2} (5\omega_3^5\omega_3^3 + 2a_1^2(2\omega_1^5 + 2\omega_1^3\omega_3^3 - 2\omega_1^2\omega_3^3 + 2\omega_1\omega_3^3 + \omega_1^3)) \] (C.12)

Terms that arise from the DFSBC are denoted with the capital letter \(B\):

\[ B_{(2\theta_2-\theta_1)} = \frac{a_2^2a_1}{8g} (2\omega_2^4 - 6\omega_2^2\omega_1^2 + 8\omega_2\omega_1^3 - \omega_1^4) \] (C.13)

\[ B_{(2\theta_1-\theta_2)} = \frac{a_2^2a_1}{8g} (3\omega_2^4 - 8\omega_2^2\omega_1 - 6\omega_2^2\omega_1^3 + 16\omega_2\omega_1^4 - 2\omega_1^4) \] (C.14)

\[ B_{(2\theta_3-\theta_1)} = \frac{a_3^2a_1}{8g} (2\omega_3^4 - 6\omega_3^2\omega_1^2 + 8\omega_3\omega_1^3 - \omega_1^4) \] (C.15)

\[ B_{(2\theta_3-\theta_1)} = \frac{a_3^2a_1}{8g} (3\omega_3^4 - 8\omega_3^2\omega_1 - 6\omega_3^2\omega_1^3 + 16\omega_3\omega_1^4 - 2\omega_1^4) \] (C.16)

\[ B_{(2\theta_2-\theta_3)} = \frac{a_2^2a_3}{8g} (2\omega_2^4 - 6\omega_2^2\omega_3^2 + 8\omega_3\omega_3^3 - \omega_2^4) \] (C.17)

\[ B_{(2\theta_2-\theta_3)} = \frac{a_2^2a_3}{8g} (3\omega_3^4 - 8\omega_3^2\omega_2 - 6\omega_3^2\omega_2^2 + 16\omega_3\omega_2^3 - 2\omega_2^4) \] (C.18)
\begin{align*}
B_{\theta_3+\theta_2-\theta_1} &= \frac{a_1 a_3 a_3}{4g} \left( \omega_3^4 - 2 \omega_3^2 (\omega_1^2 + \omega_2^2) + 4 \omega_3 \omega_1^3 - \omega_1^4 ight) + 4 \omega_3^4 \omega_2^2 - 2 \omega_3^2 \omega_2^2 + \omega_1^4 \tag{C.19} \\
B_{\theta_3-\theta_2+\theta_1} &= \frac{a_1 a_2 a_3}{4g} \left( \omega_3^4 - 4 \omega_3^2 \omega_2^2 + 4 \omega_3 \omega_2 \omega_1^2 - \omega_1^4 ight) + 4 \omega_3^4 \omega_1 - 2 \omega_2^2 \omega_1^2 + \omega_2^4 \tag{C.20} \\
B_{\theta_2+\theta_1-\theta_3} &= \frac{a_1 a_3 a_3}{4g} \left( 3 \omega_3^4 - 4 \omega_3^2 (\omega_1 + \omega_2) - 4 \omega_3^2 (\omega_1^2 + \omega_2^2) ight) + 4 \omega_3 (\omega_1^3 + \omega_1^2 \omega_2 + \omega_1 \omega_2^2 + \omega_2^3) - \omega_1^2 + 2 \omega_1 \omega_2^3 - \omega_2^4 \tag{C.21} \\
B_{\theta_1} &= \frac{a_1 a_2 a_2}{8g} \left( 3 \omega_3^3 \omega_1^4 - 2 \omega_3^2 (\omega_1^4 - 4 \omega_1^2 \omega_2 - \omega_1 \omega_2^2 - \omega_2^4) ight) - \frac{2 a_1 a_2^2}{8g} (\omega_1^4 - 4 \omega_1^2 \omega_3 - \omega_1 \omega_3^2 + \omega_3^3) \tag{C.22} \\
B_{\theta_2} &= \frac{a_2 a_2 a_3}{8g} \left( 3 \omega_3^2 \omega_1^4 + 2 \omega_1^4 (\omega_1^4 + 4 \omega_1^2 \omega_2 - 3 \omega_1 \omega_2^2 + \omega_2^4) ight) - \frac{2 a_2 a_3}{8g} (\omega_1^3 + 4 \omega_1^2 \omega_3 - \omega_1 \omega_3^2 + \omega_3^2) \tag{C.23} \\
B_{\theta_3} &= \frac{a_3 a_2 a_2}{8g} \left( 3 \omega_2^3 \omega_1^4 + 2 \omega_1^4 (\omega_1^4 + 4 \omega_1^2 \omega_3 - 3 \omega_1 \omega_3^2 + \omega_3^2) ight) + \frac{2 a_3 a_2^2}{8g} (\omega_2^4 + 4 \omega_2^3 \omega_3 - 2 \omega_2 \omega_3^2 - 3 \omega_2 \omega_3^2 + \omega_3^3) \tag{C.24}
\end{align*}
Third Order Interaction Terms
Appendix D

Jacobian Elliptical Functions

The definitions of the elliptical functions as used in the Cnoidal Wave Theory (Sec. 3.2.3) are based on the integral

\[ u = \int_0^\phi \frac{1}{\sqrt{1 - m \sin^2 \theta}} d\theta \]  

(D.1)

in which the angle \( \phi \) is the amplitude and \( m \) is the Jacobian elliptic parameter. The functions \( sn \) and \( cn \) are called the \textit{Jacobian Elliptic Sine} and \textit{Cosine} respectively and are defined as

\[ sn(u; m) = \sin(\phi) \]  

(D.2)

\[ cn(u; m) = \cos(\phi) \]  

(D.3)

\[ dn(u; m) = \sqrt{1 - m \sin^2 \phi} \]  

(D.4)

\[ scd(u; m) = sn(u; m)cn(u; m)dn(u; m) \]  

(D.5)

(D.6)
The expressions $K(m)$ and $E(m)$ stand for the complete elliptical integral of the first and second kind and are defined as

\[ K(m) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - m \sin^2 \theta}} d\theta \quad (D.7) \]

\[ E(m) = \int_0^{\pi/2} \sqrt{1 - m \sin^2 \theta} d\theta \quad (D.8) \]
Appendix E

Evolution of the New Year Wave

The following pages show the evolution of the New Year Wave group. Figs. E.1 and E.2 show a snapshot sequence depicting the spacial development. Figs. E.3 and E.4 show the temporal evolution of the same wave group at different locations in the wave tank.
**Fig. E.1:** Snapshot sequence of surface elevations prior to developing the New Year Wave at scale 1:81 (Pt. 1). The red line shows the numerical result (WAVETUB). On the right side the mean envelop curve is shown.
Fig. E.2: Snapshot sequence of surface elevations (plotted over x) with the developing the New Year wave at scale 1:81 (Pt. 2). The red line shows the numerical result (WAVETUB). On the right side the mean envelope curve is shown.
Evolution of the New Year Wave

Fig. E.3: Time record sequence of surface elevations (plotted over time $t$) with the developing the New Year wave at scale 1:81 (Pt. 1). The red line shows the numerical result (WAVETUB). On the right side the mean envelope curve is shown.
Fig. E.4: Time record sequence of surface elevations (plotted over time $t$) with the developing the New Year Wave at scale 1:81 (Pt. 2). The red line shows the numerical result (WAVETUB). On the right side the mean envelope curve is shown.
Bibliography


G. F. Clauss and U. Steinhagen. Optimization of transient design waves in random sea. In *Proceedings of 10th International Offshore and Polar En-


J. Gunson and A. K. Magnusson. Investigating conditions for rogue waves events from spectral wave observations and models. In *7th Interna-


H. Mitsuyasu, F. Tasai, T. Suhara, S. Mizuno, M. Ohkusu, T. Honda, and
K. Rikiishi. Observations of the directional spectrum of ocean waves using

K. Mittendorf, B. Nguyen, and W. Zielke. Seegang und Seegangsbelastung
II. Technical report, Institute of Fluid Mechanics, University of Hannover,
Hannover, Germany, 2005.

N. Mori, T. Yasuda, and S. Nakayama. Statistical Properties of Freak Waves
Observed in the Sea of Japan. In *Proceedings of the 10th International Off-
shore and Polar Engineering Conference (ISOPE)*, volume 3, pages 109–


NORSOK. Standard - actions and action effects. Technical Report N-003,
2004.


M. Olagnon and G. Athanassoulis, editors. *Rogue Waves 2000*, Brest, France,
November 2000. Ifremer.

M. Onorato, A. R. Osborne, M. Serio, L. Cavalieri, C. Brandini, and C. T.
Stansberg. Extreme waves and modulational instability: wave flume exper-
iments on irregular waves. *ArXiv Nonlinear Sciences e-prints*, November
2003.

W. J. Pierson. Oscillatory Third-Order Perturbation Solutions for Sums of
Interacting Long-Crested Stokes Waves on Deep Water. *Journal of Ship

W. J. Pierson and L. Moskowitz. A proposed spectral form for fully de-
veloped wind seas based on the similarity of S. A. Kitagorodskii. *Journal of


C. Valens. A really friendly guide to wavelets. c.valens@mindless.com, 1999.

J. D. Wheeler. Method of calculating forces produced by irregular waves. 


R. L. Wiegel. A presentation of cnoidal wave theory for practical application. 


