Loop Spaces of Riemannian Manifolds

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Abstract

The present thesis approaches the loop space of a Riemannian 3-manifold $(M, \langle , \rangle)$ from a *geometric* point of view. Loops are immersed circles represented by immersions $\gamma : S^1 \to M$ modulo reparametrizations. In this setup, the loop space $\mathfrak{M}$ appears as the base of a principal bundle

$$\pi : \mathsf{Imm}(S^1, M) \to \mathsf{Imm}(S^1, M)/\mathsf{Diff}(S^1) =: \mathfrak{M}.$$

The tangent space at γ may be identified with the space $\Gamma(\perp\gamma)$ of smooth sections of the loop's normal bundle. Affine connections on $\mathfrak{M}$ are constructed. Firstly, the Levi-Civita connection ∇^{LC} belonging to the Kähler structure $(J, \langle\langle , \rangle\rangle)$, where $\langle\langle , \rangle\rangle$ denotes the L^2 product of normal fields and the almost complex structure J is given by 90° left rotation in the normal bundle. Its curvature and topological properties of the distance function induced by $\langle\langle , \rangle\rangle$ are analyzed.

Secondly, a previously unknown complex linear connection ∇^C on $\mathfrak{M}$ is described, which depends only on the conformal class of $(M, \langle , \rangle)$. The introduction of the conformally invariant harmonic mean

$$L(X) = \left(\int_\gamma \frac{1}{\|X\|} \right)^{-1}, \qquad X \in \Gamma(\perp\gamma),$$

permits the characterization of the geodesis of ∇^C as critical points of the corresponding length functional. It is shown that immersed cylinders are geodesics of the conformal connection if and only if they are – as surfaces – isothermic and their curvature lines enclose an angle of 45° with the individual loops. The whole construction is then applied to the space of immersed hypersurfaces. Here, geodesics of the conformal connection correspond to critical points of the length functional of an adapted harmonic mean. Moreover, an immersive variation is geodesic if and only if it consists – up to a conformal change of the metric on the ambient space – of parallel minimal hypersurfaces.

As an example of a special class of loops, conformal circles are discussed. In the case $\dim(M) = 3$, it is shown that they are precisely the critical points of the parallel transport in the normal bundle.

Zusammenfassung

Die vorliegende Arbeit nähert sich dem Schleifenraum $\mathfrak{M}$ einer Riemannschen 3-Mannigfaltigkeit $(M, \langle , \rangle)$ vom *geometrischen* Standpunkt aus: Schleifen sind immersierte Kreise, die durch Immersionen $\gamma : S^1 \to M$ modulo Umparametrisierung dargestellt werden. Der Schleifenraum $\mathfrak{M}$ ist die Basis des Hauptfaserbündels

$$\pi : \mathsf{Imm}(S^1, M) \to \mathsf{Imm}(S^1, M)/\mathsf{Diff}(S^1) =: \mathfrak{M}.$$

Der Tangentialraum am Punkt γ wird mit dem Raum $\Gamma(\perp\gamma)$ der glatten Schnitte des Normalenbündels identifiziert. Es werden affine Zusammenhänge auf $\mathfrak{M}$ konstruiert. Einerseits der Levi-Civita-Zusammenhang ∇^{LC}, welcher zur Kählerstruktur $(J, \langle\langle , \rangle\rangle)$ gehört, wo $\langle\langle , \rangle\rangle$ das L^2-Produkt von Normalenfeldern bezeichnet und die fast komplexe Struktur J durch $90°$-Linksdrehung im Normalenbündel von γ gegeben ist. Die Krümmung von ∇^{LC} sowie topologische Eigenschaften der zu $\langle\langle , \rangle\rangle$ gehörenden Abstandsfunktion werden analysiert.

Weiterhin wird ein bisher unbekannter komplex linearer Zusammenhang ∇^{C} auf $\mathfrak{M}$ beschrieben, welcher nur von der konformen Klasse von $(M, \langle , \rangle)$ abhängt. Die Einführung eines konform invarianten harmonischen Mittels

$$L(X) = \left(\int_\gamma \frac{1}{\|X\|} \right)^{-1}, \qquad X \in \Gamma(\perp\gamma),$$

ermöglicht die Charakterisierung der Geodätischen von ∇^{C} als kritische Punkte des zugehörigen Längenfunktionals. Es wird gezeigt, daß immersierte Zylinder genau dann Geodätische des konformen Zusammenhangs sind, wenn sie – als Fläche betrachtet – isotherm sind und ihre Krümmungslinien mit den einzelnen Kreisen einen Winkel von $45°$ einschließen. Die gesamte Konstruktion wird dann auf den Raum der immersierten Hyperflächen angewandt. Die Geodätischen des konformen Zusammenhangs entsprechen hier den kritischen Punkten des Längenfunktionals eines angepaßten harmonischen Mittels. Eine immersive Variation ist genau dann geodätisch, wenn sie – bis auf eine konforme Änderung der Metrik des umgebenden Raumes – aus parallelen Minimalflächen besteht.

Als Beispiel einer speziellen Klasse von Schleifen werden konforme Kreise betrachtet. Im Fall $\dim(M) = 3$ wird gezeigt, daß sie genau die kritischen Punkte des Paralleltransports im Normalenbündel sind.

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Contents

Introduction

The present thesis approaches the loop space $\mathfrak{M}$ of a Riemannian manifold $(M, \langle , \rangle)$ from a *geometric* point of view. Loops are immersed circles. As such, they are represented by equivalence classes of immersions $\gamma : S^1 \to M$, where two immersions are identified if they differ only by reparametrization. In this setup, the loop space appears as the base of an infinite dimensional principal bundle

$$\pi : \mathsf{Imm}(S^1, M) \to \mathsf{Imm}(S^1, M)/\mathsf{Diff}(S^1) =: \mathfrak{M}.$$

Its construction is discussed in chapter one, however, we immediately replace the circle by a compact manifold S in order to obtain the space of immersed submanifolds of diffeomorphism type S in M. The topology of the model spaces is addressed, and the tangent space of $\mathfrak{M}$ is identified with the space $\Gamma(\perp\gamma)$ of smooth sections of the normal bundle.

Chapter two introduces the canonical almost complex structure J on the loop space $\mathfrak{M}$ of a Riemannian 3-manifold given by 90° left rotation in the normal bundle of γ. The construction of affine connections on $\mathfrak{M}$ then leads to a description of the Kähler geometry of $(\mathfrak{M}, J, \langle\langle , \rangle\rangle)$ in chapter three. Here, $\langle\langle , \rangle\rangle$ denotes the L^2 product of normal fields.

In chapter four, the previous investigations lead to the discovery of a complex linear connection ∇^C on $\mathfrak{M}$ which is invariant under conformal changes of the Riemannian metric $\langle , \rangle$ on M. The introduction of the conformally invariant harmonic mean

$$L : T\mathfrak{M} \to \mathbb{R}, \quad L(X) = \left(\int_\gamma \frac{1}{\|X\|} \right)^{-1}$$

permits the characterization of the geodesics of ∇^C as critical points of the corresponding length functional on variations of loops. We show that immersed cylinders are geodedics of the conformal connection if and only if they are isothermic surfaces and their curvature lines enclose an angle of 45° with the individual loops.

Chapter five carries these findings over to spaces of immersed hypersurfaces. The curvature of the Levi-Civita connection belonging to the L^2 product is discussed. In analogy to the case of loop spaces, we derive a conformal connection. A suitable adaption of the harmonic mean allows for an identification of geodesics with the critical points of the corresponding length functional. Moreover, we show that an immersive variation is a geodesic if and only if it consists – up to a conformal change of the metric on the ambient space – of parallel minimal hypersurfaces.

A special class of loops, namely conformal circles, are discussed in chapter six. In the case $\dim(M) = 3$, the total torsion of loops can be considered as a smooth function on loop space. We show that its critical points are precisely the conformal circles.

At many points in the text, the conformal invariance of objects defined through Riemannian metrics has to be checked. Therefore, the formulas describing the conformal change of the most important curvature quantities have been collected in an appendix.

CHAPTER 1

Manifolds of Mappings

The geometric approach to the theory of loop spaces developed in this chapter serves as a foundation upon which we rely in subsequent parts of this thesis. Concerning the functional analytic depth we tried to strike a balance between tiring out the reader whose primary interest is assumed to be of differential geometric nature and leaving the discussion on the level of "formal considerations". We will focus on a description of the model spaces used to equip spaces of smooth mappings with a manifold structure. In particular, we do not repeat the explicit definitions, theorems, and proofs of the basic differential geometric language in infinite dimensions. For a comprehensive treatment of global analysis in the non-Banach setting – which is far beyond the scope of this thesis – we refer to the interesting monograph (KM97).

1. Nuclear Fréchet Manifolds

As a prerequisite for any consideration of spaces of mappings we need to decide in which category of differentiability we prefer to work. Finite differentiability, say of order $k \in \mathbb{N}$, would lead us to Banach spaces thus avoiding many analytic intricacies. Unfortunately, the coarseness of the C^k-topology causes differentiation to be discontinuous. For our purposes, this is a severe defect: The concatenation operator $\circ$ would not be continuous as its derivative involves the derivatives of the concatenated mappings. In turn, the analysis of the almost complex structure on loop space worked out in later chapters would break down. As most of the constructions we are interested in would be met by a similar fate we choose to take the burden of working with smooth maps and hence with a finer topology.

1.1 Fundamental assumptions. To begin with, we first state our basic assumptions:

(1) All finite dimensional manifolds are assumed to be second-countable, Hausdorff, without boundary, and smooth.
(2) Let S denote a compact manifold (usually S^1).
(3) Further, let M be an n-dimensional manifold.
(4) Most of the time, we will work with oriented immersions; in this case we require S and M to be orientable and choose orientations on them.

(5) Frequently, we will choose a Riemannian metric $\langle,\rangle$ on M. In each case, it will be made clear if the construction depends on the particular metric, its conformal class, or if it is completely independent of $[\langle,\rangle]$.

If we now consider a smooth immersion $f : S \to M$ we can parametrize immersions $g : S \to M$ "close to" f using sections $X \in \Gamma(f^*TM)$ of the pullback of the tangent bundle of M. One possibility is to write the nearby immersion g using the Riemannian exponential map of M:

$$g : S \to M, \quad s \mapsto g(s) = \exp_{f(s)}(X(s)).$$

This construction works for any Riemannian metric on M and for all immersions g that can be represented by sections X with $\sup \|X\|$ small enough. In the sequel, we will make precise how the above mentioned parametrizations can be used to obtain a manifold structure on the space of immersions of S into M.

1.2 The smooth topology. As indicated above, the space of smooth sections $\Gamma := \Gamma(f^*TM)$ will be used to model a neighborhood of a given immersion $f : S \to M$. On Γ, we consider the canonical C^∞ topology induced by the sequence of seminorms $(\|.\|)_{k\in\mathbb{N}}$ with

$$\|X\|_k^2 = \sum_{j=0}^{k} \int_S \left\|\nabla^j X\right\|^2$$

for $X \in \Gamma$. On the right hand side, we use the Levi-Civita connection ∇ as well as the norm associated to the Riemmanian metric on the target manifold $(M, \langle,\rangle)$. Therefore, the individual seminorms depend on the choice of metric. However, the induced topology does not (Roe98, p. 75).

1.3 Nuclearity. As a topological vector space, $\Gamma = \Gamma(f^*TM)$ is locally convex, complete, and metrizable, hence a Fréchet space. An example of a metric that yields the smooth topology is

$$d(X,Y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\|X-Y\|_k}{1+\|X-Y\|_k}.$$

Yet, d is of limited use: since every subset is d-bounded, the metric does not reflect *boundedness* in the sense of topological vector spaces.[1]

It is worthwhile to take a closer look at the topology of the space Γ. For each $k \in \mathbb{N}$, denote by

$$\Gamma_k := \left(\overline{\Gamma/\left\|.\right\|_k^{-1}(0)}, \left\|.\right\|_k\right)$$

[1] A subset is *bounded* if and only if it is bounded in *each* seminorm. The implied accuracy of discrimination between boundedness and infinity is at the root of major differences to the category of normed spaces. See also 1.4.

the completion[2] of Γ with respect to $\|.\|_k$. Moreover, write

$$\varphi_{kl} : \Gamma_l \to \Gamma_k, \quad k \leq l,$$

for the natural inclusion of Γ_l into Γ_k. Then the family (Γ_k, φ_{kl}) forms an inverse directed system. The linear subspace

$$\varprojlim \Gamma_k := \left\{ (X_k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} \Gamma_k \mid \varphi_{kl}(X_l) = X_k \text{ for all } k \leq l \right\}$$

equipped with the trace topology induced by the product is called the projective limit of the given inverse directed system (FW68). This representation summarizes the properties of the underlying convex space Γ in a convenient manner. In general, each convex space is isomorphic to a dense subspace of the projective limit of an inverse directed system of *Banach* spaces. Since Γ is metrizable, the system is countable. The completeness of Γ amounts to the equality $\Gamma = \varprojlim \Gamma_k$.

Spaces like Γ are prototypes[3] of socalled *nuclear spaces* (Gro55), which can be characterized (Sch71) by the fact that the canonical inclusions $\varphi_k : \Gamma \to \Gamma_k$ into the individual completions Γ_k are *nuclear*, that is, they can be written as

$$\varphi_k(X) = \sum_{j=0}^{\infty} \lambda_j \psi_j(X) Y_j,$$

where $\sum |\lambda_j| < \infty$, $(\psi_j) \subset \Gamma'$ is equicontinuous, and $(Y_j) \subset \Gamma^k$ is contained in a convex bounded neighborhood of zero.[4]

As a concrete example, for $C^\infty(S^1, \mathbb{C})$, one can obtain the required representation from the standard Fourier basis.

In contrast to the case of a generic Fréchet space, each normed completion Γ_k of a nuclear space is a *Hilbert space* (FW68). By taking linear subspaces, quotients with respect to closed linear subspaces, topological duals, and products of nuclear Fréchet spaces, nuclearity

[2]In the case of $(\Gamma(f^*TM), \|.\|_k)$, it is not strictly necessary to consider the quotient spaces $\Gamma/\|.\|_k$, since the individual seminorms $\|.\|_k$ are in fact norms. Nevertheless, we prefer this notation in order to emphasize the general setting.

[3] Spaces of smooth sections (resp. smooth functions) like Γ were the starting point for A. Grothendieck's theory of *nuclear spaces.* He writes (Gro55, p. 3): "Ces recherches avaient pour origine d'éclaircir et de généraliser les propriétés trés spéciales que semblaient posséder certain espaces de fonctions indéfiniment différentiables en vertu du 'théorème des noyaux' de L. Schwartz." Other treatments of nuclear spaces can be found in (Tre67), (FW68), (Sch71), (Pie72), (Jar81).

[4]In this context, Γ' denotes the strong dual of the space of smooth sections. Nuclear maps in the context of locally convex spaces are a generalization of operators of trace class on Hilbert spaces. A. Grothendieck defines a space V to be nuclear if the projective and equicontinous topologies on the tensor product of V with any other locally convex space W coincide (Gro55, chap. II, p. 34). Our definition is, of course, equivalent to his.

will always be preserved (Pie72, 5) while completeness and metrizability might be lost. Each bounded subset of a nuclear Fréchet space is precompact (Pie72, 4.4.7) – which emphasizes once more the complementary relationship of nuclear spaces to those with norm topology. Figure 1 provides an overview of the functional analytic setting.

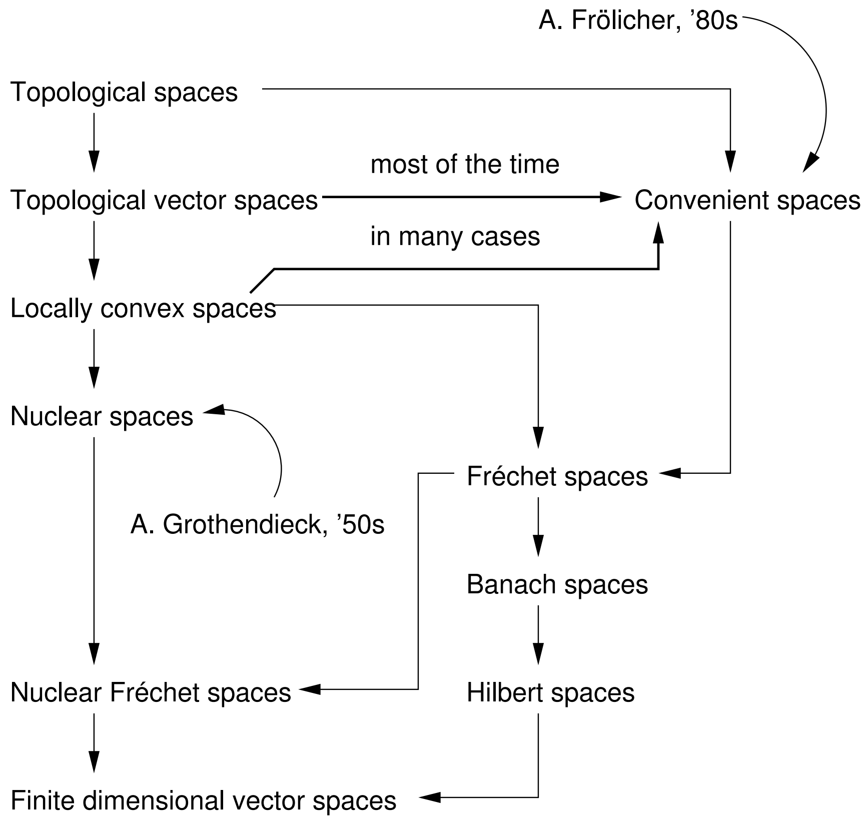

FIGURE 1. This diagram is meant to guide the reader through the functional analytic background discussed in this chapter. Arrows indicate specialization. Infinite dimensional nuclear spaces are not normable. In practice, the nuclear Fréchet spaces on the left side of the diagram can be considered complementary to the Banach and Hilbert spaces on the right. According to A. Pietsch (Pie72, p. VI), nuclear spaces "are more closely related to finite dimensional spaces than are normed spaces." Moreover, nuclear Fréchet spaces are special examples of convenient spaces. This allows us to use the results of A. Kriegl's and P. W. Michor's convenient analysis (KM97). However, in general, convenient spaces need not even be topological vector spaces (KM97, 4.20).

1.4 Smooth curves. A curve $\gamma : \mathbb{R} \to V$ in a locally convex space V is called *differentiable* if its derivative

$$\dot{\gamma}(t) := \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}$$

exists for all $t \in \mathbb{R}$. It is called C^k if its iterated derivatives up to order k exist and are continuous. Moreover, γ is said to be *smooth* if all its iterated derivatives exist. If we replace the given topology τ of V by another locally convex topology $\widetilde{\tau}$ that has the same bounded sets, then a curve γ is smooth with respect to τ if and only if it is smooth with respect to $\widetilde{\tau}$ (KM97, 1.8). Hence, smoothness of curves depends only on the bounded sets of V, the socalled *bornology*.

1.5 The c^∞-topology. Let V be a locally convex space. The *c^∞-topology* is defined to be the final topology with respect to all smooth curves (KM97, 2.12). This is to say, $U \subset V$ is open in the c^∞-topology if and only if $\gamma^{-1}(U) \subset \mathbb{R}$ is open for any smooth curve $\gamma : \mathbb{R} \to V$.

It follows that the c^∞-topology is not coarser than the given locally convex vector space topology. Generally, the c^∞-topology does *not* describe a topological vector space (KM97, 4.20). The reason why it is still considered as a key ingredient in infinite dimensional analysis is the possibility to test openness of subsets by examining preimages under smooth curves. On Fréchet spaces, the c^∞-topology coincides with the given locally convex topology.

1.6 Differentiation of maps. There are at least two successful approaches to differentiation in locally convex spaces. One is the traditional C^∞_c-analysis[5] as applied by R. S. Hamilton in his study (Ham82) of the inverse function theorem of J. F. Nash and J. Moser. Another notable work based on this concept of smoothness are J. Milnor's notes (Mil84) on infinite dimensional Lie groups. Here, a map

$$f : V \supset U \to W$$

from an open subset U of a locally convex space V to another locally convex space W is called *continuously differentiable* (C^1) if its *directitonal derivative*

$$df(p)(v) := \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t}$$

exists for all $p \in U$ and $v \in V$ and induces a continuous mapping

$$df : U \times V \to W.$$

The map f is called C^k, $k > 1$, if f is C^1 and df is C^{k-1}. It is called *smooth* if it is C^k for all $k \in \mathbb{N}$.

Another possible approach to smoothness in locally convex spaces is the socalled Convenient Analysis as developed in (KM97) by A. Kriegl

[5]This terminology was introduced by H. H. Keller in his survey (Kel74) of differential calculus on locally convex spaces.

and P. W. Michor. In this theory, the map f is said to be *smooth* if it maps smooth curves in V to smooth curves in W.

In the case of Fréchet spaces, both concepts of smoothness are equivalent (KM97, 12.8). We note that this equivalence does not hold for maps of finite differentiability.

1.7 Functional analytic summary. We summarize our basic functional analytic strategy:

(1) A subset is open if its preimages under smooth curves are open.
(2) A map is smooth if and only if it maps smooth curves to smooth curves.
(3) We do not consider finite differentiability.

1.8 Nuclear Fréchet manifolds. A *nuclear Fréchet manifold* is a set $\mathfrak{M}$ together with a smooth structure represented by an atlas[6] $(U_\alpha, u_\alpha)_{\alpha\in A}$ such that the canonical topology on $\mathfrak{M}$ with respect to this structure is Hausdorff.

A map $f : \mathfrak{M} \to \mathfrak{N}$ of nuclear Féchet manifolds is said to be *smooth in* $p \in \mathfrak{M}$ if it is smooth in one – hence all pair(s) of charts around p and $f(p)$. The map is *smooth* if it is smooth at all points of $\mathfrak{M}$. Particularly, f is smooth if and only if $f \circ \gamma$ is smooth for every smooth curve $\gamma : \mathbb{R} \to \mathfrak{M}$ (KM97, 27.2).

From now on, we require nuclear Fréchet manifolds to be *smoothly Hausdorff* (smooth functions separate points).

One can prove (KM97, 16.10) that each nuclear Fréchet manifold is *smoothly paracompact*, that is, each open cover admits a smooth partition of unity subordinated to it.

1.9 Tangent bundles. Let $p \in V$ be a point in a nuclear Fréchet space V. The *tangent space* T_pV of V at p is the set of all pairs (p, X) with $X \in V$. Equivalently, T_pV is the set of equivalence classes of smooth curves γ through p, where $\gamma_1 \sim \gamma_2$ if both have the same derivative at p.

Each tangent vector $X \in T_pV$ yields a continuous (hence bounded) derivation

$$X : C^\infty(V \supset \{p\}, \mathbb{R}) \to \mathbb{R}$$

on the germs of smooth functions at p. In the general locally convex case, it is not true that each such derivation comes from a tangent vector. However, if V is a nuclear Fréchet space (KM97, 28.7) T_pV does coincide with the set derivations on the stalk $C^\infty(V \supset \{p\}, \mathbb{R})$.

[6] As usual, *charts* are bijections from open subsets of $\mathfrak{M}$ to open subsets of a Fréchet space, whose isomorphism type has to be the same for the whole manifold. An *atlas* is a cover of $\mathfrak{M}$ by charts, where all chart changings are defined on open subsets and are required to be smooth. An equivalence class of atlas with respect to union is called a *smooth structure*.

We define the *tangent bundle* $T\mathfrak{M}$ of a nuclear Fréchet manifold to be the quotient of the disjoint union

$$\bigcup_{\alpha \in A} \{\alpha\} \times U_\alpha \times V_\alpha,$$

by the equivalence relation

$$(p, X, \alpha) \sim (q, Y, \beta) \Leftrightarrow p = q, \quad d(u_{\alpha\beta})(u_\beta(p))(Y) = X,$$

where the $u_\alpha : \mathfrak{M} \supset U_\alpha \to V_\alpha$ denote the charts, and $u_{\alpha\beta} = u_\alpha \circ u_\beta$ the chart changings of the manifold.

The strong dual of a nuclear Fréchet space need not be metrizable. In particular, this is true for spaces of smooth sections in finite dimensional vector bundles. If we would consider the strong dual of $T_p\mathfrak{M}$ as cotangent space we would drop out of the nuclear Fréchet category. In order to avoid this we will consider tensors not as sections of a certain bundle, but simply as smooth, fiberwise multilinear maps

$$A : T\mathfrak{M} \times_{\mathfrak{M}} \ldots \times_{\mathfrak{M}} T\mathfrak{M} \to E$$

with $\pi \circ A = \mathrm{id}$ and $\pi : E \to \mathfrak{M}$ a vector bundle over $\mathfrak{M}$.

2. Manifolds of Mappings

After having introduced the necessary functional analytic background we are now in position to discuss the manifold structure of spaces of smooth mappings and spaces of (oriented) immersed submanifolds.

1.10 The manifold of smooth mappings. The space of smooth mappings $C^\infty(S, M)$, where S is a compact manifold, and M an n-dimensional manifold can now be given the structure of a nuclear Fréchet manifold in the following way (see (KM97, 42.1) for a full proof).

Firstly, we choose a Riemannian metric $\langle , \rangle$ on M. On a suitable open neighborhood $U \subset TM$ of the zero section, the combination of the Riemannian exponential with the bundle projection $\pi_M : TM \to M$ yields a diffeomorphism

$$(\pi_M, \exp) : TM \supset U \to V \subset M \times M, \quad X_p \mapsto (p, \exp_p(X_p)),$$

onto an open neighborhood V of the diagonal in $M \times M$.

For $f \in C^\infty(S, M)$, each smooth section $X \in \Gamma(f^*TM)$ can be viewed as a smooth map $X : S \to TM$ with $\pi_M \circ X = f$. A smooth atlas can now be constructed from charts around f with domain

$$U_f = \{g \in C^\infty(S, M) \mid \forall s \in S : \; (f(x), g(x)) \in V\},$$

an open subset

$$\widehat{U_f} = \{X \in \Gamma(f^*TM) \mid X(S) \subset f^*U\},$$

and a bijection

$$u_f : C^\infty(S, M) \supset U_f \to \widehat{U_f} \subset \Gamma(f^*TM)$$

given by

$$u_f(g) = \Big(s \mapsto \big(s, \exp^{-1}_{f(s)}(g(s))\big) = \big(s, (\pi_M, \exp)^{-1} \circ (f, g)(x)\big)\Big).$$

Thus, the tangent space $T_f C^\infty(S, M)$ can be identified with the space $\Gamma(f^*TM)$ of smooth sections of the pullback of the tangent bundle of M. The above manifold structure on $C^\infty(S, M)$ is Hausdorff, and does not depend on the Riemannian metric $\langle , \rangle$ on M.

1.11 The space of immersions. The space

$$\mathsf{Imm}(S, M) = \{f \in C^\infty(S, M) \mid f \text{ is an immersion}\}$$

is an open subset of the manifold $C^\infty(S, M)$ of smooth mappings, and hence a nuclear Fréchet manifold itself. In particular, the diffeomorphism group

$$\mathsf{Diff}(S) = \{f \in C^\infty(S, S) \mid f \text{ is a smooth diffeomorphism}\}$$

is open in $C^\infty(S, S)$. One of its two components,

$$\mathsf{Diff}(S) = \mathsf{Diff}^+(S) \cup \mathsf{Diff}^-(S),$$

is the subgroup $\mathsf{Diff}^+(S)$ of orientation preserving diffeomorphisms. We are mostly concerned with immersed submanifolds, that is, equivalence classes $[f]$ whose elements $f_1, f_2 \in [f]$ differ only by an diffeomorphism $\varphi : S \to S$:

$$f_1 = f_2 \circ \varphi.$$

Our aim is now to introduce a manifold structure on the set

$$\mathsf{Imm}(S, M)/\mathsf{Diff}(S)$$

of immersed submanifolds $[f]$. Therefore, we need to describe the diffeomorphism group $\mathsf{Diff}(S)$ of S as well as its action on the space $\mathsf{Imm}(S, M)$ of immersions.

1.12 Definition (Mil84, 7.6, KM97, 38). A *nuclear Fréchet Lie group* G is a nuclear Fréchet manifold with a group structure such that multiplication and inversion are smooth. It is called *regular* if for every smooth path $X : \mathbb{R} \supset I \to \mathfrak{g}$ in the Lie algebra $\mathfrak{g}$ of G there is a path $\gamma : I \to G$ which solves the differential equation

$$\dot\gamma(t) = (L_{\gamma(t)})_* X(t)$$

and furthermore the correspondence

$$\mathfrak{g} \ni X \mapsto \gamma(0)^{-1}\gamma(1)$$

defines a smooth map from the space $C^\infty(I, \mathfrak{g})$ to the Lie group G.

1.13 Milnor's regularity[7] as explained in the above definition is a strengthened form of the requirement that the exponential map $\exp : \mathfrak{g} \to G$ should be defined and smooth. For a compact manifold S, the diffeomorphism group $\mathsf{Diff}(S)$ as well as $\mathsf{Diff}^+(S)$ are regular Lie groups (KM97, 43.1). The derivative of its exponential map $\exp : \Gamma(TS) \to \mathsf{Diff}(S)$ at $0 \in \Gamma(TS)$ is the identity mapping of $\Gamma(TS)$. Nevertheless, it is *not* locally surjective near zero (Gra93). Even worse, for $\dim(S) > 1$ as well as for the unit circle S^1, one can find a vector field X arbitrarily close to the zero section, such that $d\exp$ is not injective at X. This implies that $GL(\Gamma(S))$ is not open[8] in the space of endomorphisms $L(\Gamma(TS))$.

1.14 The right action of the diffeomorphism group $\mathsf{Diff}(S)$ on the space $\mathsf{Imm}(S, M)$ of smooth immersions is *not* free. A simple example for this phenomenon is the double cover

$$\gamma : S^1 \to \mathbb{C}, \quad \gamma(s) = e^{i2s}.$$

If $\varphi \in \mathsf{Diff}(S^1)$ denotes left rotation by 180 degrees, it follows that $\gamma \circ \varphi = \gamma$. For general S and M, it is not obvious if all immersions $f : S \to M$ on which $\mathsf{Diff}(S)$ does not act freely are of the type mentioned in the example. The next theorem gives a precise description of the situation.

1.15 Theorem (CMM91, 3.1,3.2).
For any immersion $f \in \mathsf{Imm}(S, M)$, the isotropy group $\mathsf{Diff}(S)_f = \{\varphi \in \mathsf{Diff}(S) \mid f \circ \varphi = f\}$ is a finite group which acts as a group of covering transformations for a finite covering $c : S \to \bar{S}$ such that f factors over c to an immersion $\bar{f} : \bar{S} \to M$ with $\bar{f} \circ c = f$ and trivial isotropy group $\mathsf{Diff}(\bar{S})_{\bar{f}} = \{\mathrm{id}_{\bar{S}}\}$.

1.16 A deeper analysis (CMM91; MM05) of the singular points (projections of immersions with non-trivial isotropy group) of the space $\mathsf{Imm}(S, M)/\mathsf{Diff}(S)$ shows that the latter space admits the structure of an "infinite dimensional orbifold". Nevertheless, we will not delve into this theory and rather content ourselves with the following workaround:

From this point on, we will only consider immersions $f : S \to M$ with trivial isotropy group $\mathsf{Diff}(S)_f$ and exclude all others from $\mathsf{Imm}(S, M)$.

1.17 Theorem (CMM91, 1.5, KM97, 39.1, 43.1). *Let S be a compact manifold and M an n-dimensional manifold equipped with a*

[7]According to (Mil84) and (KM97), all known Lie groups are regular.

[8]This shows that one cannot hope to find a classical implicit function theorem for spaces of smooth mappings.

conformal structure $[\langle,\rangle]$. *The space* $\mathsf{Imm}(S,M)$ *of free immersions is the total space of a smooth principal fiber bundle*

$$\pi : \mathsf{Imm}(S,M) \to \mathsf{Imm}(S,M)/\mathsf{Diff}(S)$$

with structure group $\mathsf{Diff}(S)$, *whose base, the space of immersed submanifolds of type* S *in* M, *is a smooth nuclear Fréchet manifold in the sense of 1.8. This bundle admits a smooth principal connection described by the horizontal bundle whose fiber over an immersion* $f \in \mathsf{Imm}(S,M)$ *is the space* $\Gamma(\perp f)$ *of smooth normal vector fields. The parallel transport for this connection exists and is smooth.*

1.18 For our purposes, the usefulness of the principal connection lies in the guaranteed existence of horizontal paths[9] in $\mathsf{Imm}(S,M)$. Moreover, $\Gamma(\perp f)$ is the horizontal lift of the tangent space $T_{[f]}(\mathsf{Imm}(S,M)/\mathsf{Diff}(S))$, and this is the representation we will work with.

We do not give a full proof of the above theorem, but include a description of saturated neighborhoods for the $\mathsf{Diff}(S)$ action which split smoothly into product of a submanifold of $\mathsf{Imm}(S,M)$ and $\mathsf{Diff}(S)$.

Step 1: Any immersion $f \in \mathsf{Imm}(S,M)$ induces a fiberwise injective bundle homomorphism $\overline{f}$:

$$\begin{array}{ccc} \perp f & \xrightarrow{\overline{f}} & TM \\ \pi_S \downarrow & & \downarrow \pi_M \\ S & \underset{f}{\rightarrow} & M \end{array}$$

Step 2: Now we choose an open cover (W_α) of S such that each W_α is connected and each compact $\overline{W_\alpha}$ is contained in a connected, open subset U_α on which $f|_{U_\alpha}$ is an embedding. The family (U_α) is chosen to be an open locally finite cover of S.

Step 3: Next, we select a Riemannian metric $\langle,\rangle$ from the given conformal class, and an open neighborhood $U_f \subset \perp f$ of the zero section in the normal bundle small enough such that for each α the map $\exp\circ\overline{f}|_{U_{f\alpha}}$ with $U_{f\alpha} = U_f|_{U_\alpha}$ is an embedding. Altogether, we get an immersion

$$\tau_f : \perp f \supset U_f \to M, \quad \tau_f = \exp\circ\overline{f},$$

of an open tube around of $0 \in \Gamma(\perp f)$ into M.

Step 4: We define an open neighborhood $I_f \subset \mathsf{Imm}(S,M)$ of immersions which respect our partition $\cup_\alpha U_\alpha = S$ from step 2:

$$I_f = \{g \in \mathsf{Imm}(S,M) \mid g(\overline{W_\alpha}) \subset \tau_f(U_{f\alpha})\ \forall\alpha\}.$$

The immersions contained in I_f will be identified with *functions* in the open subset $F_f \subset C^\infty(S,\perp f)$:

$$F_f = \{h \in C^\infty(S,\perp f) \mid h(\overline{W_\alpha}) \subset U_{f\alpha}\ \forall\alpha\}.$$

[9]If one feels uncomfortable with the assumption of trivial isotropy groups $\mathsf{Diff}(S)_f$ above, one could alternatively accept singularities in the base manifold $\mathfrak{M}(S,M)$ and then prove the existence of horizontal paths directly (MM05, 2.5).

The identification is achieved by the smooth diffeomorphism

$$\varphi_f : \mathsf{Imm}(S, M) \supset I_f \to F_f \subset C^\infty(S, \perp f)$$

with

$$\varphi_f(g) : S \to U_f, \quad s \mapsto \tau_f^{-1}(g(s)).$$

Its smooth inverse will be denoted by $\psi = \varphi_f^{-1} : F_f \to I_f$, $\psi(h) = \tau_f \circ h$. For diffeomorphisms $\sigma \in \mathsf{Diff}(S)$ close enough to $\mathrm{id} \in \mathsf{Diff}(S)$ such that $h \circ \sigma \in F_f$ we have $\psi(h \circ \sigma) = \psi(h) \circ \sigma$.

Step 5: A smooth straightening map defined on an open subset is given by

$$C^\infty(S, U_f) \supset \{\hat{h} = h \circ \sigma \mid h \in F_f, \sigma \in \mathsf{Diff}(S)\} \to \Gamma(U_f) \times \mathsf{Diff}(S),$$
$$\hat{h} \mapsto \big(\hat{h} \circ (\pi_S \circ \hat{h})^{-1}, \pi_S \circ \hat{h}\big).$$

By putting

$$\mathfrak{U}_f = \psi_f\big(\Gamma(U_f) \cap F_f\big) \subset \mathsf{Imm}(S, M)$$

we get, since the action of $\mathsf{Diff}(S)$ on f is free,

$$I_f \circ \mathsf{Diff}(S) \cong \mathfrak{U}_f \times \mathsf{Diff}(S).$$

Moreover,

$$\pi|_{\mathfrak{U}_f} : \mathfrak{U}_f \to \mathsf{Imm}(S, M)/\mathsf{Diff}(S)$$

is a bijection onto an open subset of $\mathsf{Imm}(S, M)/\mathsf{Diff}(S)$, and

$$\varphi_f \circ (\pi|_{\mathfrak{U}_f})^{-1} : \pi|_{\mathfrak{U}_f}(\mathfrak{U}_f) \to \Gamma(U_f)$$

provides a chart for the quotient space. The subset $I_f \circ \mathsf{Diff}(S)$ is an open neighborhood of $f \in \mathsf{Imm}(S, M)$ which is saturated for the action of the diffeomorphism group. $\mathfrak{U}_f$ becomes a smooth splitting submanifold of the space of immersions, diffeomorphic to an open neighborhood of the zero section in the space $\Gamma(\perp f)$.

1.19 Clearly, the above procedure remains the same in the oriented case. That is, for a compact oriented manifold S, an n-dimensional oriented manifold M with conformal structure $[\langle,\rangle]$ we can consider the space of orientation preserving immersions, mod out the orientation preserving diffeomorphisms, and arrive at the space of oriented immersed submanifolds of type S in M. We fix the following notation.

1.20 Definition. For a compact orientable manifold S and an oriented n-dimensional Riemannian manifold $(M, \langle,\rangle)$, the *space of oriented immersed submanifolds of type S in M* is denoted by

$$\mathfrak{M}(S, M) = \mathsf{Imm}^+(S, M)/\mathsf{Diff}^+(S),$$

where $\mathsf{Imm}^+(S, M)$ contains the orientation preserving immersions $S \to M$.

1.21 Definition. Let $f : S \to (M, \langle , \rangle)$ be an isometric immersion representing the immersed submanifold $[f] \in \mathfrak{M}(S, M)$. Let $X, Y \in \Gamma(\perp f)$ be normal fields representing tangent vectors of $\mathfrak{M}(S, M)$. Then the formula

$$\langle\langle X, Y \rangle\rangle := \int_S \langle X, Y \rangle \omega_f$$

for the L^2 product of the two normal fields defines a (weak) Riemannian metric on $\mathfrak{M}$. Above, ω_f denotes the induced volume form.

1.22 Aspects of the Riemannian geometry of $(\mathfrak{M}(S, M), \langle\langle , \rangle\rangle)$ – sometimes restricted to the case of embeddings – have been studied by many authors including (Bin80), (Kai84), (Bry93), (KM97), and (MM05). Alternative Riemannian metrics on spaces of immersions and embeddings have been considered in (Bin80), (MM05), (Sha08).

We note that for any chosen Riemannian metric $\langle , \rangle$ on M the projection

$$\pi : \mathsf{Imm}^+(S, M) \to \mathfrak{M}(S, M)$$

now becomes a Riemannian submersion with respect to the L^2 product on $\mathsf{Imm}^+(S, M)$ and $\mathfrak{M}(S, M)$.

CHAPTER 2

Complex Structures and Affine Connections

This chapter introduces the basic building blocks for the study of the geometry loop spaces. In particular, we discuss the canonical almost complex structure as well as affine connections on loop space.

1. The Almost Complex Structure

While the L^2 product $\langle\langle , \rangle\rangle$ induces a Riemmanian structure on spaces of immersed submanifolds of arbitrary dimension this is in particular true for loop spaces. In this case, the normal bundle has rank two. It turns out that 90 degree left rotation of normal fields along loops induces an almost complex structure J which fits together with $\langle\langle , \rangle\rangle$ such that loop spaces $(\mathfrak{M}(S^1, M), J, \langle\langle , \rangle\rangle)$ of Riemannian 3-manifolds become Hermitean. The question of integrability of J brings up surprising results due to the infinite dimensionality of mapping spaces.

2.1 In this section, we consider the space

$$\mathfrak{M} = \mathfrak{M}(S^1, M) = \mathsf{Imm}^+(S^1, M)/\mathsf{Diff}^+(S^1)$$

of oriented loops in a Riemannian 3-manifold $(M, \langle , \rangle)$. For a loop represented by

$$\gamma : S^1 \to M, \quad s \mapsto \gamma(s),$$

we write γ_s for its tangent with respect to an *arbitrary*[1] (yet regular) parametrization and let

$$T = \frac{\gamma_s}{\|\gamma_s\|}$$

denote its unit tangent with respect to the chosen metric $\langle , \rangle$.

2.2 Definition. Let $X \in \Gamma(\perp\gamma)$ be a normal field along the loop γ. Denote by $\times$ the vector product in TM induced by the Riemannian metric $\langle , \rangle$. Then the mapping

$$X \mapsto T \times X$$

induces an almost complex structure

$$J : T_{[\gamma]}\mathfrak{M} \to T_{[\gamma]}\mathfrak{M}$$

on the loop space $\mathfrak{M}$.

[1] In the literature, s is often used to denote a parametrization by arc length. We do not follow this convention, since we never assume any special parametrizations unless explicitly stated. Nevertheless, we apologize to the reader for not following the tradition.

2.3 The almost complex structure defined above has been considered by H. Hasimoto (Has72) for loops in $\mathbb{R}^3$, on the space of geodesics by N. J. Hitchin (Hit82), and on loop spaces of general 3-manifolds by J.-L. Brylinsky (Bry93). It is straightforward to see that J

(1) is smooth in the C^∞ category,[2]
(2) is not affected by the action of $\mathsf{Diff}^+(S^1)$ on the space of immersed loops,
(3) and is invariant under conformal changes $\langle,\rangle \mapsto e^{2u}\langle,\rangle$ of the Riemannian metric on the 3-manifold M.

Therefore, J is well-defined, and $(\mathfrak{M}, J)$ is an almost complex manifold. Moreover, we note that together with the L^2 product $(\mathfrak{M}, J, \langle\langle,\rangle\rangle)$ becomes a Hermitean manifold. This aspect will be further explored in chapter 3.

2.4 Definition. As in the finite dimensional case, the *Nijenhuis tensor* N of J is defined by

$$N(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y].$$

In the above formula, X and Y denote vector fields on $\mathfrak{M}$.

2.5 Theorem (Bry93, 3.4.3). *The Nijenhuis tensor N of the almost complex structure J on the loop space $\mathfrak{M}$ vanishes identically.*

2.6 *Proof.* The proof given in (Bry93, 3.4.3) is rather complicated. An alternative is to show that the almost complex structure J is parallel with respect to the Levi-Civita connection ∇^{LC} belonging to the L^2 product $\langle\langle,\rangle\rangle$ on $\mathfrak{M}$. This is done in corollary 3.3. Using this result we see that

$$\begin{aligned} N(X,Y) &= \nabla^{LC}_{JX} JY - \nabla^{LC}_{JY} JX \\ &\quad - J(\nabla^{LC}_{JX} Y - \nabla^{LC}_{Y} JX) \\ &\quad - J(\nabla^{LC}_{X} JY - \nabla^{LC}_{JY} X) \\ &\quad - \nabla^{LC}_{X} Y + \nabla^{LC}_{Y} X. \\ &= 0. \end{aligned}$$

□

2.7 In finite dimensions, the well-known theorem of Newlander-Nirenberg (NN57) states that an almost complex structure is a complex structure if and only if N vanishes identically. In light of the above theorem, one might expect to find a holomorphic atlas. Unfortunately, the next theorem shows that $(\mathfrak{M}, J)$ is a counter example to the Newlander-Nirenberg theorem in infinite dimensions.

2.8 Theorem (Lem93, 10.5). *No open subset of $\mathfrak{M}$ is smoothly biholomorphic to an open subset of a complex Fréchet space.*

[2] The definition of J works precisely in the smooth setting, which is one of our motivations not to work with maps of finite differentiability.

2.9 L. Lempert's analysis of the integrability of $(\mathfrak{M}, J)$ leads him to the following concept of weak integrability adhered by the space of smooth loops:

2.10 Definition (Lem93, 4.1). An almost complex manifold $(\mathfrak{M}, J)$ is called *weakly integrable* if for each real tangent vector $X \in T\mathfrak{M}$, $X \neq 0$, there is a J-holomorphic function

$$f : \mathfrak{M} \supset U \to \mathbb{C},$$

defined on an open neighborhood U of the base point of X such that

$$df(X) \neq 0.$$

2.11 Theorem (Lem93, 9.6). *If the Riemannian 3-manifold* $(M, \langle , \rangle)$ *is real analytic, then the corresponding loop space* $(\mathfrak{M}, J)$ *is weakly integrable.*

2.12 In order to construct holomorphic functions on $\mathfrak{M}$ L. Lempert applies C. LeBrun's theory of Twistor CR manifolds to the case of the loop space M. We refer to (LeB84) for this interesting construction.

2. Affine Connections on Loop Space

For spaces $\mathfrak{M}(S, M)$ of embedded submanifolds, the Levi-Civita connection ∇^{LC} belonging to the L^2 product $\langle\langle , \rangle\rangle$ has been known for more than two decades (Kai84). Nevertheless, a direct computation of ∇^{LC} yields very complex formulas. We will take a different route. Firstly, a certain "basic" connection $\nabla^{\perp}$ on $\mathfrak{M}(S, M)$ will be introduced. Secondly, a tensor $\mathfrak{H}$ is used to encode geometric properties of the loops. Finally, the Levi-Civita connection ∇^{LC} can be defined as a linear combination of $\nabla^{\perp}$ and $\mathfrak{H}$ (see chapter 3). This approach leads directly to an important new discovery discussed in chapter 4: an affine connection ∇^{C} which is invariant under conformal changes of the Riemannian metric on M.

2.13 Vector fields and Lie brackets. At first sight, the concept of "vector fields on loop space" might seem a bit awkward. To clarify this, we give a detailed description.

Firstly, we know from 1.9 that a tangent vector of a nuclear Fréchet manifold may be regarded as an equivalence class of smooth paths through its foot point. Hence, it may be represented by one parameter variation of a loop.

Secondly, any vector field is determined completely by its values along paths. Therefore, we may legitimately restrict our attention to vector fields along variations of loops.

Concerning Lie brackets on loop spaces, we note the following. Given a two parameter variation

$$\mathbb{R} \times \mathbb{R} \times S^1 \to M, \quad (r, t, s) \mapsto \gamma(r, t, s),$$

the three vector fields

$$Y = \gamma_r = d\gamma\left(\frac{\partial}{\partial r}\right), \quad X = \gamma_t = d\gamma\left(\frac{\partial}{\partial t}\right), \quad \gamma_s = d\gamma\left(\frac{\partial}{\partial s}\right)$$

are γ-related to the coordinate fields on $\mathbb{R} \times \mathbb{R} \times S^1$. Hence,

$$[X, Y] = d\gamma\left(\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial r}\right]\right) = 0,$$

and equally well $[X, \gamma_s] = [Y, \gamma_s] = 0$.

Now consider two vector fields X, Y on the space $\mathfrak{M}$ of (oriented) unparametrized loops. Since this space is the base of the principal bundle

$$\pi : \mathsf{Imm}^+(S^1, M) \to \mathsf{Imm}^+(S^1, M)/\mathsf{Diff}^+(S^1) = \mathfrak{M},$$

which, at the same time, is a Riemannian submersion, we have

$$\widetilde{[X, Y]} = [\widetilde{X}, \widetilde{Y}]^h = 0,$$

where "$\widetilde{\ }$" denotes the horizontal lift of vector fields, and the superscript "h" indicates the horizontal part within the tangent space to the total space.

2.14 The connection in the normal bundle. If $X \in \Gamma(\perp\gamma)$ is a normal vector field which represents a tangent vector in $T_{[\gamma]}\mathfrak{M}$ we use the Levi-Civita connection ∇ of $(M, \langle, \rangle)$ to write

$$X' := \nabla_T X + \langle X, H \rangle T$$

for the *connection in the normal bundle* of the loop γ. As a convention, T will always denote the unit tangent of the loop. Moreover, $H = \nabla_T T$ denotes the "mean" curvature vector of γ. Obviously, the connection in the normal bundle depends on the Riemannian metric of M. Its behavior under conformal changes is summarized in section 2 of the appendix.

2.15 Pullbacks of the Levi-Civita connection. We would like to apply the Levi-Civita connection ∇ of $(M, \langle, \rangle)$ to vector fields on $\mathfrak{M}$ in a certain way. To this end, consider a vector field along a path in loop space, represented as a normal field $Y \in \Gamma(\gamma^* TM)$, $Y \perp T$, where

$$\gamma : \mathbb{R} \times S^1 \to M, (t, s) \mapsto \gamma(t, s),$$

is a variation with

$$X := d\gamma\left(\frac{\partial}{\partial t}|_{t=0}\right) \perp T.$$

Using γ, we can pull back the Levi-Civita connection ∇ of M to $\mathbb{R} \times S^1$ and compute

$$\widehat{\nabla}_X Y := (\gamma^* \nabla)_{\frac{\partial}{\partial t}} Y \in \Gamma(\gamma^* TM).$$

Since this is the only possibility to differentiate (representations of) vector fields on $\mathfrak{M}$ using the Levi-Civita connection of M, it should not lead to confusion if we reuse the symbol ∇ for it: $\nabla_X Y := \widehat{\nabla}_X Y$.

2.16 Definition. Given $[X] \in T_{[\gamma]}\mathfrak{M}$ with its horizontal lift $X \in \Gamma(\perp\gamma)$ let $(t,s) \mapsto \gamma(t,s)$ be some variation which satisfies $\gamma_t = X$ for $t = 0$. Lifting a vector field $[Y] \in \Gamma(T\mathfrak{M})$ to the variation we may require $Y \perp T$ for $t = 0$. Now we define

$$\nabla^\perp_X Y := Y_t^\perp - \langle \gamma_t, T\rangle Y',$$

where $Y_t^\perp$ abbreviates the normal part

$$(\nabla_{\gamma_t} Y)^\perp = \nabla_{\gamma_t} Y - \langle \nabla_{\gamma_t} Y, T\rangle T$$

of the covariant derivative of Y along $t \mapsto \gamma(t,s)$. The projection to $T\mathfrak{M}$ of $\nabla^\perp_X Y$ defines the socalled *basic connection* on loop space, again denoted by $\nabla^\perp$.

2.17 Lemma. *The connection $\nabla^\perp$ enjoys the following properties:*

(1) *The connection $\nabla^\perp$ is well defined.*
(2) *$\nabla^\perp$ is torsion free.*
(3) *The almost complex structure J on $\mathfrak{M}$ is parallel with respect to $\nabla^\perp$.*
(4) *A conformal change of the Riemannian metric on M by $\langle , \rangle \mapsto e^{2u}\langle , \rangle$ results in a change of the basic connection according to*

$$\widetilde{\nabla^\perp} = \nabla^\perp + B^\perp$$

with

$$B^\perp_X Y = (B_X Y)^\perp = \langle X, U\rangle Y + \langle Y, U\rangle X - \langle X, Y\rangle U^\perp,$$

where $U = \operatorname{grad} u$ and $X, Y \in \Gamma(\perp\gamma)$.

2.18 *Proof.* The definition shows that $\nabla^\perp$ does not depend on the chosen parametrizations of the loops and vector fields. Moreover, one checks easily that it complies to the axioms of an affine connection. If X, Y are two (representations of) vector fields on loop space which agree along a path, that is,

$$X \circ \gamma = Y \circ \gamma$$

with

$$\gamma : \mathbb{R} \times S^1 \to M, \quad (t,s) \mapsto \gamma(t,s),$$

then, by definition of $\nabla^\perp$,

$$\nabla^\perp_{\gamma_t} X = \nabla^\perp_{\gamma_t} Y.$$

Hence, our definition of a connection along paths does indeed fix a well defined affine connection on $\mathfrak{M}$.

In order to compute the torsion tensor $T^\perp$ of $\nabla^\perp$, we take some arbitrary 2-parameter variation

$$\mathbb{R} \times \mathbb{R} \times S^1 \to M, \quad (r,t,s) \mapsto \gamma(r,t,s),$$

such that at a given loop $s \mapsto \gamma(0,0,s)$ the variational vector fields γ_r and γ_t are normal to the tangent direction γ_s, and hence represent tangent vectors X, Y of the loop space $\mathfrak{M}$. Using the definition of $\nabla^\perp$, we may now compute its torsion tensor $T^\perp$ directly:

$$\begin{aligned} T^\perp(X,Y) =& \nabla^\perp_X Y - \nabla^\perp_Y X - [X,Y] \\ =& -(\langle X,T\rangle Y' - \langle Y,T\rangle X') \\ =& 0. \end{aligned}$$

Complex linearity of $\nabla^\perp$ follows from

$$\begin{aligned} \nabla^\perp_X(JY) =& \big(\nabla_{\gamma_t}(T\times Y)\big)^\perp = \big(\gamma_t'\times Y + J\nabla_{\gamma_t}Y\big)^\perp \\ =& J\big(\nabla_{\gamma_t}Y\big)^\perp = J\nabla^\perp_X Y \end{aligned}$$

The Levi-Civita connection $\widetilde{\nabla}$ of $(M, e^{2u}\langle,\rangle)$ is given by the well known formula

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle X,U\rangle Y + \langle Y,U\rangle X - \langle X,Y\rangle U,$$

where $U = \mathrm{grad} u$. The change induced in $\nabla^\perp$ is a direct consequence of this. $\square$

2.19 Theorem. *Let $X, Y, Z \in \Gamma(\perp\gamma)$ be horizontal lifts of tangent vectors from $T_{[\gamma]}\mathfrak{M}$. Then the horizontal lift of the curvature tensor $R^\perp$ of $\nabla^\perp$ is given by*

$$R^\perp(X,Y)Z = \Big(\langle X,Y'\rangle - \langle Y,X'\rangle\Big)Z' - \Big(\langle JX',Y'\rangle + K(T)\langle JX,Y\rangle\Big)JZ,$$

where $K(T)$ denotes the sectional curvature of the plane $\{T\}^\perp$.

2.20 *Proof.* Since the value of the normal field $R^\perp(X,Y)Z \in \Gamma(\perp\gamma)$ at a given parameter $s \in S^1$ depends only on the behavior of X, Y, Z near s we may assume γ to be an embedding. In order to compute the curvature tensor we consider a two-parameter variation

$$(x,y,s) \mapsto \gamma(x,y,s)$$

of $s \mapsto \gamma(s)$ with

$$\gamma_x = X, \quad \gamma_y = Y.$$

For $x, y, z = 0$, we assume the variational fields to represent elements of $T_{[\gamma]}\mathfrak{M}$, that is $X, Y \in \Gamma(\perp\gamma)$. Moreover, we extend the normal field $Z \in \Gamma(\perp\gamma)$ along the variation and require $Z \perp T$ everywhere. Applying the definition of $\nabla^\perp$ we may compute

$$\nabla^\perp_X \nabla^\perp_Y Z = (\nabla_X\nabla_Y Z)^\perp - \langle \nabla_Y Z, T\rangle(\nabla_X T)^\perp - \langle \gamma_y, T\rangle_x Z',$$

where subscripts denote differentiation. All vector fields involved are coordinate fields, so their Lie brackets vanish. Hence, the above formula evaluates to

$$\nabla^\perp_X \nabla^\perp_Y Z = (\nabla_X\nabla_Y Z)^\perp + \langle Z, Y'\rangle X' - (\langle \nabla_{\gamma_x}\gamma_y, T\rangle + \langle Y, X'\rangle)Z'.$$

Denoting by R the curvature tensor of ∇ and by $R^{\perp}$ that of $\nabla^{\perp}$ we arrive at

$$\begin{aligned} R^{\perp}(X,Y)Z &= (R(X,Y)Z)^{\perp} \\ &\qquad + \big(\langle X, Y'\rangle - \langle Y, X'\rangle\big)Z' + \langle Z, Y'\rangle X' - \langle Z, X'\rangle Y'. \end{aligned}$$

Since $\{T\}^{\perp}$ is two-dimensional, the curvature tensor R of the Levi-Civita connection ∇ on $(M, \langle,\rangle)$ satisfies

$$(R(X,Y)Z)^{\perp} = -K(T)\langle JX, Y\rangle JZ,$$

where $\perp$ is the orthogonal projection onto $\{T\}^{\perp}$. Apart from that, Graßmann's identity yields

$$\langle Z, Y'\rangle X' - \langle Z, X'\rangle Y' = Z \times (X' \times Y') = -\langle JX', Y'\rangle JZ.$$

This completes the proof. $\square$

2.21 The affine connection $\nabla^{\perp}$ on $\mathfrak{M}$ is a good starting point for the description of the Levi-Civita connection ∇^{LC} of the Riemannian metric $(\mathfrak{M}, \langle\langle,\rangle\rangle)$. In addition to that, it will be used in the construction of a "conformal connection", too. In between, we need to come up with suitable "correction tensors" to be added to $\nabla^{\perp}$.

2.22 For the moment, assume we had already discovered the Levi-Civita connection ∇^{LC} of $(\mathfrak{M}, \langle\langle,\rangle\rangle)$ and write l for the Riemannian length functional

$$l : \mathfrak{M} \to \mathbb{R}, \quad l(\gamma) = \int_{S^1} \|\gamma_s\|\, ds,$$

on $\mathfrak{M}$. The gradient of l with respect to $\langle\langle,\rangle\rangle$ is given by

$$\mathrm{grad} l = -H.$$

Above, we view H as a vector field on $\mathfrak{M}$. A conformal change of the L^2 metric $\langle\langle,\rangle\rangle \mapsto e^{-2l}\langle\langle,\rangle\rangle$ *on the loop space* $\mathfrak{M}$ would result in a transformation of the Levi-Civita connection ∇^{LC} according to

$$\nabla^{LC} \mapsto \nabla^{LC} + \mathfrak{F},$$

where the symmetric bilinear form $\mathfrak{F}$ is given by

$$\mathfrak{F}(X,Y) = \langle\langle X, H\rangle\rangle Y + \langle\langle Y, H\rangle\rangle X - \langle\langle X, Y\rangle\rangle H.$$

Since our primary focus lies on geometric properties invariant under conformal changes of the Riemannian metric $\langle,\rangle$ on M rather than the L^2 product on $\mathfrak{M}$, a keen guess would be to investigate a pointwise version of $\mathfrak{F}$ as given in the next definition.

2.23 Definition. For normal vector fields $X, Y \in \Gamma(\perp\gamma)$, define the *mean curvature form* of γ by

$$\mathfrak{H}_X Y = \mathfrak{H}(X,Y) = \langle X, H\rangle Y + \langle Y, H\rangle X - \langle X, Y\rangle H,$$

where H denotes the mean curvature vector of the loop.

2.24 Since $\mathfrak{H}$ does not depend on the parametrization of the curve and the normal fields it defines a symmetric bilinear form on $T_{[\gamma]}\mathfrak{M}$. The next lemma summarizes the basic properties of the tensor $\mathfrak{H}$.

2.25 Lemma. *The mean curvature form $\mathfrak{H}$ is symmetric and complex linear:*

$$\mathfrak{H}_{JX}Y = \mathfrak{H}_X(JY) = J\mathfrak{H}_X Y.$$

Moreover, we have

$$\mathfrak{H}_X Y = \langle X, H\rangle Y + \langle X, JH\rangle JY$$

and

$$\mathfrak{H}_X \mathfrak{H}_Y Z = \mathfrak{H}_Y \mathfrak{H}_X Z.$$

2.26 *Proof.* Symmetry, complex linearity as well as the third property follow directly from the definition of $\mathfrak{H}$. With respect to the last equation note that due to 7.3

$$\mathfrak{H}_X \mathfrak{H}_Y Z - \mathfrak{H}_Y \mathfrak{H}_X Z = \langle X \times Y, H\rangle H \times Z = 0,$$

because X, Y are normal vectors to γ. $\square$

2.27 In preparation for the coming curvature computations, we need to know how the mean curvature vector field H of a loop and the tensor $\mathfrak{H}$ behave under normal variations. The next two lemmas provide the necessary information.

2.28 Lemma. *The mean curvature vector H, viewed as a vector field on loop space, satisfies*

$$\nabla^{\perp}_X H = X'' + R(X,T)T + \langle X, H\rangle H,$$

where $X \in \Gamma(\perp\gamma)$ represents a tangent vector of $\mathfrak{M}$, R is the Riemannian curvature tensor of $(M, \langle,\rangle)$, and prime stands for the connection in the normal bundle.

2.29 *Proof.* In order to prove the claim we consider a variation

$$\gamma : \mathbb{R} \times S^1 \to M, \quad (t,s) \mapsto \gamma(t,s),$$

with

$$X = \gamma_t|_{t=0} \perp \gamma_s,$$

and set $v := \|\gamma_s\|$. Now we compute

$$\begin{aligned}
\nabla^{\perp}_X H =& (\nabla_X H)^{\perp} = \Big(\nabla_X(\frac{1}{v}\nabla_{\gamma_s}T)\Big)^{\perp} \\
=& \Big(\frac{1}{v}\langle X, H\rangle \nabla_{\gamma_s}T + \frac{1}{v}\nabla_X\nabla_{\gamma_s}T\Big)^{\perp} \\
=& \langle X, H\rangle H + R(X,T)T + (\nabla_T\nabla_X T)^{\perp}.
\end{aligned}$$

Since

$$\nabla_X T = \nabla_X(\frac{1}{v}\gamma_s) = \langle X, H\rangle T + \nabla_T X = X'$$

we get

$$(\nabla_T \nabla_X T)^\perp = X''.$$

This completes the proof. $\square$

2.30 Lemma. *The mean curvature form $\mathfrak{H}$ of γ satisfies*

$$\begin{aligned}(d^{\nabla^\perp}\mathfrak{H})(X,Y,Z) = \big(\langle Y, X''\rangle - \langle X, Y''\rangle\big)Z \\ + \Big(\langle Y'', JX\rangle - \langle X'', JY\rangle \\ + (\|H\|^2 + \mathrm{ric}(T))\langle JX, Y\rangle\Big)JZ,\end{aligned}$$

where X, Y, Z are normal fields along γ, T and H denote the unit tangent and mean curvature vectors of the loop, and $\mathrm{ric}(T) = \mathrm{ric}(T,T)$ *is the Ricci curvature of the underlying Riemannian 3-manifold* $(M, \langle,\rangle)$.

2.31 *Proof.* We start by splitting $(d^{\nabla^\perp}\mathfrak{H})(X,Y,Z) = (\nabla^\perp_X\mathfrak{H})(Y,Z) - (\nabla^\perp_Y\mathfrak{H})(X,Z)$ into two parts:

$$(d^{\nabla^\perp}\mathfrak{H})(X,Y,Z) = \alpha Z + A,$$

where

$$\alpha := \big(\langle \nabla^\perp_X H, Y\rangle - \langle \nabla^\perp_Y H, X\rangle\big)$$

and

$$A := \langle Z, \nabla^\perp_X H\rangle Y - \langle Z, \nabla^\perp_Y H\rangle X + \langle X, Z\rangle \nabla^\perp_Y H - \langle Y, Z\rangle \nabla^\perp_X H.$$

Inserting the formula 2.28 for the derivative of H, we get for the component α in direction of Z:

$$\alpha = \langle Y, X''\rangle - \langle X, Y''\rangle.$$

We may assume $Z \neq 0$ in order to set $E := \frac{Z}{\|Z\|}$. Then we have

$$\begin{aligned}A =&\langle A, JE\rangle JE \\ =&\Big(\langle E, \nabla^\perp_X H\rangle\langle Y, JE\rangle - \langle E, \nabla^\perp_Y H\rangle\langle X, JE\rangle \\ &+ \langle X, E\rangle\langle \nabla^\perp_Y H, JE\rangle - \langle Y, E\rangle\langle \nabla^\perp_X H, JE\rangle\Big)JZ \\ =&\Big(\langle E\times JE, \nabla^\perp_X H \times Y\rangle + \langle E \times JE, X \times \nabla^\perp_Y H\rangle\Big)JZ \\ =&\Big(\langle \nabla^\perp_Y H, JX\rangle - \langle \nabla^\perp_X H, JY\rangle\Big)JZ.\end{aligned}$$

Applying 2.28 twice, we get for A

$$\begin{aligned} A =& \Big(\langle Y'', JX\rangle + \langle Y, H\rangle\langle H, JX\rangle + \langle R(Y,T)T, JX\rangle \\ &\qquad - \langle X'', JY\rangle - \langle X, H\rangle\langle H, JY\rangle - \langle R(X,T)T, JY\rangle \Big) JZ \\ =& \Big(\langle Y'', JX\rangle - \langle X'', JY\rangle + \|H\|^2 \langle JX, Y\rangle + \mathrm{ric}(T)\langle JX, Y\rangle \Big) JZ. \end{aligned}$$

This completes the proof. $\square$

CHAPTER 3

Kähler Geometry of Loop Space

In this chapter, we will apply the tools developed so far to the investigation of the Kähler geometry of the loop space $\mathfrak{M} = \mathfrak{M}(S^1, M)$ of a Riemannian 3-manifold $(M, \langle , \rangle)$.

1. The Levi-Civita Connection

The Levi-Civita connection (in the form of its connector) of $\mathfrak{M}$ was computed by (Kai84) for the case of immersions $S \to M$ of compact manifolds. Nevertheless, the formulas and proofs given below for loop spaces as well as those for spaces of hypersurfaces contained in chapter 5 may be easier to digest.

3.1 Theorem. *The Levi-Civita connection* ∇^{LC} *of* $(\mathfrak{M}, \langle\langle , \rangle\rangle)$ *is given by*

$$\nabla^{LC} = \nabla^{\perp} - \frac{1}{2}\mathfrak{H}.$$

3.2 *Proof.* Given a Riemannian metric on an infinite dimensional nuclear Fréchet manifold, one can apply the standard proof for the uniqueness of the corresponding Levi-Civita connection. So we only need to check that ∇^{LC} is a torsion free metric connection. Since ∇^{LC} is defined as the sum of the torsion free connection $\nabla^{\perp}$ and the symmetric tensor $\mathfrak{H}$ its torsion tensor vanishes.

It remains to check that ∇^{LC} is metric, too. To this end, consider a three-parameter variation $(x, y, z, s) \mapsto \gamma(x, y, z, s)$ of a given parametrization $s \mapsto \gamma(s)$ of a loop $[\gamma] \in \mathfrak{M}$. Let

$$X = \gamma_x^{\perp}, \quad Y = \gamma_y^{\perp}, \quad Z = \gamma_z^{\perp}$$

be parametrizations of tangent fields to $\mathfrak{M}$ given as normal parts of the variational vector fields. We may require that for $x, y, z = 0$

$$\gamma_x, \gamma_y, \gamma_z \in \Gamma(\perp\gamma).$$

Using this setup we compute

$$X\langle\langle Y, Z\rangle\rangle = \int_{S^1} \Big(\langle \nabla_X^{\perp} Y, Z\rangle + \langle Y, \nabla_X^{\perp} Z\rangle - \langle X, H\rangle\langle Y, Z\rangle \Big) \|\gamma_s\| \, ds.$$

The third summand in the integrand enjoys

$$\langle X, H\rangle\langle Y, Z\rangle = \frac{\langle \mathfrak{H}_X Y, Z\rangle + \langle Y, \mathfrak{H}_X Z\rangle}{2}.$$

Therefore, we may conclude

$$X\langle\langle Y, Z\rangle\rangle = \langle\langle \nabla^{LC}_X Y, Z\rangle\rangle + \langle\langle Y, \nabla^{LC}_X Z\rangle\rangle.$$

Hence, ∇^{LC} is indeed torsion free and metric. □

3.3 Corollary. *The complex structure J is parallel with respect to the Levi-Civita connection ∇^{LC}. Thus, $(\mathfrak{M}, J, \langle\langle , \rangle\rangle)$ is a pseudo-Kähler manifold.*[1]

3.4 *Proof.* The almost complex structure J on $\mathfrak{M}$ is parallel with respect to $\nabla^{\perp}$. Since $\mathfrak{H}$ is complex linear we also have $\nabla^{LC} J = 0$. □

3.5 The Kähler form of $(\mathfrak{M}, J, \langle\langle , \rangle\rangle)$ is given by

$$\omega(X, Y) = \langle\langle JX, Y\rangle\rangle = \int_{S^1} \langle JX, Y\rangle \, \|\gamma_s\| \, ds \int_{S^1} \det(\gamma_s, X, Y) ds.$$

In (LeB93), C. LeBrun explains how one can generalize the Kähler structure of the loop space $\mathfrak{M}$ to spaces of codimension two submanifolds. We will not consider this generalization, but rather compute the curvature of the Levi-Civita connection on loop space. The formula for the (holomorphic) sectional curvature given in theorem 3.8 may be compared to the one contained in (MM05).

3.6 Theorem. *The curvature tensor R^{LC} of the Levi-Civita connection on loop space is given by*

$$\begin{aligned} R^{LC}(X,Y)Z =& \big(\langle X, Y'\rangle - \langle Y, X'\rangle\big) Z' - \frac{1}{2}\big(\langle Y, X''\rangle - \langle X, Y''\rangle\big) Z \\ &+ \frac{1}{2}\bigg(\langle Y'', JX\rangle - \langle X'', JY\rangle - 2\langle JX', Y'\rangle \\ &\quad + \big(\|H\|^2 + \operatorname{ric}(T) - 2K(T)\big)\langle JX, Y\rangle \bigg) JZ, \end{aligned}$$

where H is the loop's mean curvature, $K(T)$ denotes the sectional curvature of the plane $\{T\}^{\perp}$, and $\operatorname{ric}(T) = \operatorname{ric}(T,T)$ *is the Ricci curvature of the underlying Riemannian 3-manifold $(M, \langle , \rangle)$.*

3.7 *Proof.* The definition of ∇^{LC} implies

$$R^{LC}(X,Y)Z = R^{\perp}(X,Y)Z - \frac{1}{2}\big((d^{\nabla^{\perp}}\mathfrak{H})(X,Y,Z) + \mathfrak{H}_X\mathfrak{H}_Y Z - \mathfrak{H}_Y\mathfrak{H}_X Z\big).$$

Due to lemma 2.25, $\mathfrak{H}_X\mathfrak{H}_Y Z - \mathfrak{H}_Y\mathfrak{H}_X Z = 0$. Now the asserted formula follows from theorem 2.19 and lemma 2.30. □

[1] According to (KN69, p. 149), the terminus *pseudo-Kähler* used to describe an almost Hermitian manifold with closed Kähler form and vanishing Nijenhuis tensor. After the discovery of the Newlander-Nirenberg theorem, the attribute "pseudo" could be dropped, so the old terminology sunk into oblivion. We may put it back to use for the theory of loop spaces.

3.8 Theorem. *The holomorphic sectional curvature of* $(\mathfrak{M}, \langle\langle , \rangle\rangle)$ *is given by*

$$K^{LC}(X\wedge JX) = \int_\gamma \left(2\langle X', JX\rangle^2 + \left(\|X'\|^2 - \langle X'', X\rangle - \frac{\kappa}{2}\|X\|^2 \right) \|X\|^2 \right),$$

where $X \in \Gamma(\perp\gamma)$ *is a normal field along the loop* γ *of* L^2 *length* $\langle\langle X, X\rangle\rangle = 1$. *Moreover,*

$$\kappa = \|H\|^2 + \mathrm{ric}(T) - 2K(T),$$

$\mathrm{ric}(T) = \mathrm{ric}(T, T)$ *denotes the Ricci curvature of* $(M, \langle , \rangle)$, *and* $K(T) = K(\{T\}^\perp)$ *the sectional curvature of* $(M, \langle , \rangle)$, *respectively.*

3.9 *Proof.* Since X has unit length with respect to the L^2 product we have have

$$K^{LC}(X \wedge JX) = \langle\langle R^{LC}(X, JX)JX, X\rangle\rangle.$$

In order to evaluate the right hand side, we use the formula for the curvature tensor of the Levi-Civita connection ∇^{LC} given in theorem 3.6:

$$\begin{aligned}\langle R^{LC}(X, JX)JX, X\rangle =& 2\langle X, JX'\rangle^2 \\ &+ \left(\|X'\|^2 - \langle X, X''\rangle - \frac{\kappa}{2}\|X\|^2 \right) \|X\|^2\end{aligned}$$

But this is precisely the integrand given in the theorem. □

3.10 Corollary. *For a field of constant length* $\|X\| = \frac{1}{\sqrt{l}}$, l *the length of* γ, *and constant rotation speed* τ, *we get*

$$K^{LC}(X \wedge JX) = \frac{4\tau^2}{l} - \frac{1}{2l^2}\int_\gamma \kappa.$$

If γ *is a great circle on standard* S^3, *this simplifies to*

$$K^{LC}(X \wedge JX) = \frac{2\tau^2}{\pi} + \frac{1}{4\pi}.$$

3.11 *Proof.* Let $E := \sqrt{l}X$. With $\tau = \langle E', JE\rangle$ we may compute

$$K^{LC}(X \wedge JX) = \int_\gamma \left(2\frac{\tau^2}{l^2} + \left(\frac{\tau^2}{l} + \frac{\tau^2}{l} - \frac{\kappa}{2l}\right)\frac{1}{l}\right) = \frac{4\tau^2}{l} - \frac{1}{2l^2}\int_\gamma \kappa.$$

In the special case of a great circle on the round 3-sphere, we have

$$\kappa = -1.$$

This implies the last statement of the corollary. □

3.12 Theorem. *Let*

$$\gamma : (-\varepsilon, +\varepsilon) \times S^1 \to M, \qquad (t, s) \mapsto \gamma(t, s),$$

be a parametrization of a given curve $(-\varepsilon, +\varepsilon) \ni t \mapsto \gamma(t) \in \mathfrak{M}$ *in the loop space* $\mathfrak{M}$. *As usual, denote by* $\perp$ *the orthogonal projection onto the normal bundle of a loop, and assume*

$$t = 0 \Rightarrow \forall s \in S^1 : \ \langle \gamma_t, \gamma_s\rangle = 0.$$

Then the geodesic equation of the Levi-Civita connection ∇^{LC} in $t = 0$ takes the form (subscripts denote derivatives with respect to the Levi-Civita connection of $(M, \langle,\rangle)$)

$$0 = (\|\gamma_s\|^2 \gamma_{tt} + \frac{1}{2}\|\gamma_t\|^2 \gamma_{ss})^\perp - \langle\gamma_{ss}, \gamma_t\rangle\gamma_t.$$

3.13 *Proof.* For $t = 0$, we compute

$$\begin{aligned}\nabla^{LC}_{\gamma_t}(\gamma_t^\perp) =& \nabla^\perp_{\gamma_t}(\gamma_t^\perp) - \frac{1}{2}\mathfrak{H}_{\gamma_t}(\gamma_t^\perp)\\ =& (\gamma_{tt})^\perp - \langle\gamma_t, H\rangle\gamma_t + \frac{1}{2}\|\gamma_t\|^2 H\\ =& \left(\gamma_{tt} + \frac{1}{2}\frac{\|\gamma_t\|^2}{\|\gamma_s\|^2}\gamma_{ss}\right)^\perp - \frac{1}{\|\gamma_s\|^2}\langle\gamma_{ss}, \gamma_t\rangle\gamma_t.\end{aligned}$$

Multiplication with $\|\gamma_s\|^2$ now yields the claimed formula. □

2. Topological Consequences

The formula for the holomorphic sectional curvature K^{LC} of a field of normal planes with constant rotation speed given in corollary 3.10 contains two summands. One, namely the integral of κ, incorporates information about the geometry of the loop as well as that of the underlying 3-manifold. The other is determined by the winding number of the given normal field – and diverges to $+\infty$ for increasing winding numbers. Since this effect occurs at each given loop one might ask what the consequences are for the inner metric induced by the L^2 product.

3.14 Definition. The length functional[2] F associated to the L^2 product $\langle\langle,\rangle\rangle$ on the loop space $\mathfrak{M}$ is given by

$$F(\gamma) := \int_a^b \sqrt{\langle\langle\gamma_t^\perp, \gamma_t^\perp\rangle\rangle}dt = \int_a^b \left(\int_{S^1} \|\gamma_t\|^2 \|\gamma_s\| \, ds\right)^{\frac{1}{2}} dt,$$

where the variation

$$\gamma : [a, b] \times S^1 \to M, \quad (t, s) \mapsto \gamma(t, s),$$

represents a curve in loop space connecting $\gamma(a, .)$ with $\gamma(b, .)$. The associated inner metric d^{L^2} is then defined by

$$d^{L^2}(\gamma_a, \gamma_b) := \inf\{F(\gamma)\},$$

where the infimum is taken over all variations connecting γ_a with γ_b.

3.15 The loop space $\mathfrak{M}$ is modeled on nuclear Fréchet spaces, which are, in particular, metrizable. Via its atlas, $\mathfrak{M}$ is equipped with a canonical manifold topology. This topology is metrizable if and only if it is paracompact and locally metrizable. One can prove that $\mathfrak{M}$

[2]We use F to denote the L^2 length functional, because we would like to reserve the letter L for a different concept of length (see chapter 4).

enjoys both of these properties (KM97, 42.2). Nevertheless, there is reasonable doubt that metrization can be achieved by the inner metric d^{L^2}. The next theorem shows that the L^2 product $\langle\langle,\rangle\rangle$ on $\mathfrak{M}$ is indeed a nightmare from a distance-geometric point of view.

3.16 Theorem (MM05). *The inner metric d^{L^2} induced by the L^2 product $\langle\langle,\rangle\rangle$ on the loop space $\mathfrak{M}$ vanishes identically.*

3.17 In (MM05), P. W. Michor and D. Mumford prove a more general version of this result for spaces of immersed submanifolds of any dimension and codimension. We will not repeat the proof here, but rather demonstrate the crucial idea behind it in its simplest incarnation, that is, for loops in $\mathbb{R}^2$. Hence, we consider a variation

$$\gamma : [0,1] \times [0,1] \to \mathbb{R}^2 = \mathbb{C}, \quad (t,s) \mapsto \gamma(t,s),$$

where we assume the circle to be parametrized by the second factor $[0,1]$. Moreover, we may assume $\langle \gamma_t, \gamma_s \rangle = 0$. Now we focus on modified variations

$$\widetilde{\gamma}(t,s) = \gamma(\varphi(t,s), s),$$

where

$$\varphi : [0,1] \times [0,1] \to \mathbb{R}$$

represents a reparametrization of the square. In order to construct a variation whose L^2 length is arbitrarily small, the idea is to use a reparametrization φ which scales down $\|\gamma_t^\perp\|$ while scaling up $\|\gamma_s\|$ *by the same order of magnitude.* This will result in a decrease of the L^2 length, because the normal component of the variational vector fields is squared while the length of the tangent is not:

$$F(\widetilde{\gamma}) = \int_0^1 \left(\int_0^1 \|\widetilde{\gamma}_t^\perp\|^2 \, \|\widetilde{\gamma}_s\| \, ds \right)^{\frac{1}{2}} dt.$$

Such a scaling can be achieved by a piecewise linear "saw tooth" variation.

Clearly, the more teeth the variation has the higher the sectional curvature of the regions of $\mathfrak{M}$ it passes through.

3.18 The above degeneracy of the L^2 product can be fixed by introducing a penalty on the curvature of a given loop. In (MM05), this is done by using the modified scalar product

$$\langle\langle X, Y \rangle\rangle_A = \int_{S^1} (1 + A\,\|H\|^2) \langle X, Y \rangle \, \|\gamma_s\| \, ds, \quad X, Y \in \Gamma(\perp\gamma),$$

where $A \in [0,\infty)$ is a weighting factor. The corresponding inner metric d^A can then be shown to separate points of the space of *embedded* loops or, more generally, embedded submanifolds. Another possibility is to replace the L^2 product by a conformal metric $e^{2u}\langle\langle,\rangle\rangle$, where $u \in C^\infty(\mathfrak{M})$ is meant to incorporate the length and/or curvature of the loop (Sha08).

Riemannian metrics on spaces of curves have many applications in image segmentation and more general image processing tasks. However, non of the distance measures obtained by the above constructions yields the canonical manifold topology of $\mathfrak{M}$, which is certainly much finer.

CHAPTER 4

Conformal Geometry of Loop Space

This chapter contains the central topic of this thesis, the investigation of the *conformal geometry of the loop space*, that is, the geometry of the loop space $\mathfrak{M}$ of a conformal 3-manifold $(M, [\langle,\rangle])$. The starting point of this analysis is the Levi-Civita connection ∇^{LC} which will be modified to obtain a conformally invariant connection on the loop space $\mathfrak{M}$. Moreover, we will come up with a conformally invariant replacement for the L^2 metric – a variant of the *harmonic mean*, and finally relate conformal geodesics in loop space to isothermic surfaces.

1. The Conformal Connection

The formula for the Levi-Civita connection ∇^{LC} of $(\mathfrak{M}, \langle\langle,\rangle\rangle)$ derived in theorem 3.1 suggests that it should be possible to use $\nabla^{\perp}$ and $\mathfrak{H}$ to build a connection on loop space which is invariant under conformal changes of the Riemannian metric $\langle,\rangle$ on M. This is what we will do next.

4.1 Theorem. *The connection ∇^C on $\mathfrak{M}$ given by*

$$\nabla^C = \nabla^{\perp} + \mathfrak{H}$$

is torsion free and invariant under conformal changes of $\langle,\rangle$ on M. Therefore, it is called the conformal connection on loop space.

4.2 *Proof.* The vanishing of the accompanying torsion tensor follows from the symmetry of $\mathfrak{H}$. The verification of the claimed conformal invariance is straightforward: We change the Riemannian metric $\langle,\rangle$ to $\widetilde{\langle,\rangle} = e^{2u}\langle,\rangle$ on M using some function $u \in C^{\infty}(M, \mathbb{R})$ and denote by $U = \mathrm{grad} u$ the gradient of u with respect to the old metric $\langle,\rangle$. Moreover, we decorate objects constructed from the new metric with "$\sim$". Since the new Levi-Civita connection on $(M, \widetilde{\langle,\rangle})$ is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + B_X Y$$

with

$$B_X Y = \langle X, U\rangle Y + \langle Y, U\rangle X - \langle X, Y\rangle U.$$

It follows that $\nabla^{\perp}$ on $\mathfrak{M}$ changes according to

$$\widetilde{\nabla^{\perp}}_X Y = \nabla^{\perp}_X Y + (B_X Y)^{\perp}$$

with

$$(B_X Y)^{\perp} = B_X Y - \langle B_X Y, T\rangle T$$

denoting the component of $B_X Y$ orthogonal to the tangent of the loop. Further, the mean curvature vector of the loop changes to

$$\widetilde{H} = e^{-2u}(H - U^\perp),$$

where we again use "$\perp$" to indicate the part perpendicular to the loop. This implies the following change in the tensor $\mathfrak{H}$:

$$\widetilde{\mathfrak{H}}_X Y = \mathfrak{H}_X Y - (B_X Y)^\perp.$$

Finally, the above observations lead to

$$\widetilde{\nabla^C} = \widetilde{\nabla^\perp} + \widetilde{\mathfrak{H}} = (\nabla^\perp + B^\perp) + (\mathfrak{H} - B^\perp) = \nabla^C.$$

□

4.3 Corollary. *The complex structure J is parallel with respect to the conformal connection ∇^C.*

4.4 *Proof.* The almost complex structure J on $\mathfrak{M}$ is parallel with respect to $\nabla^\perp$. Since $\mathfrak{H}$ is complex linear we have also $\nabla^C J = 0$. □

4.5 Theorem. *The curvature tensor R^C of the conformal connection on loop space is given by*

$$\begin{aligned} R^C(X,Y)Z =& \big(\langle Y, X''\rangle - \langle X, Y''\rangle\big)Z + \big(\langle X, Y'\rangle - \langle Y, X'\rangle\big)Z' \\ &+ \Big(\langle Y'', JX\rangle - \langle X'', JY\rangle - \langle JX', Y'\rangle \\ &\quad + \big(\|H\|^2 + \mathrm{ric}(T) - K(T)\big)\langle JX, Y\rangle\Big)JZ, \end{aligned}$$

where H is the loop's mean curvature, $K(T)$ denotes the sectional curvature of the plane $\{T\}^\perp$, and $\mathrm{ric}(T) = \mathrm{ric}(T,T)$ *is the Ricci curvature of the underlying Riemannian 3-manifold $(M, \langle,\rangle)$.*

4.6 *Proof.* The definition of ∇^C implies

$$R^C(X,Y)Z = R^\perp(X,Y)Z + (d^{\nabla^\perp}\mathfrak{H})(X,Y,Z) + \mathfrak{H}_X\mathfrak{H}_Y Z - \mathfrak{H}_Y\mathfrak{H}_X Z.$$

Due to lemma 2.25, $\mathfrak{H}_X\mathfrak{H}_Y Z - \mathfrak{H}_Y\mathfrak{H}_X Z = 0$. Now the asserted formula follows from theorem 2.19 and lemma 2.30. □

4.7 Theorem. *The curvature tensor R^C of the conformal connection ∇^C can be decomposed in two parts, both of which are conformally invariant:*

$$R^C(X,Y)Z = R_1(X,Y)Z + R_2(X,Y)Z$$

with

$$R_1(X,Y)Z = \Big(\langle Y, X''\rangle - \langle X, Y''\rangle\Big)Z + \Big(\langle X, Y'\rangle - \langle Y, X'\rangle\Big)Z'$$

and

$$
\begin{aligned}
R_2(X,Y)Z = \Big(& \langle Y'', JX\rangle - \langle X'', JY\rangle - \langle JX', Y'\rangle \\
& + \langle JX, Y\rangle \big(\|H\|^2 + \mathrm{ric}(T) - K(T)\big)\Big) JZ.
\end{aligned}
$$

4.8 *Proof.* We start with a given Riemannian metric $\langle,\rangle$ on M. Decorate all objects arising from $\widetilde{\langle,\rangle} = e^{2u}\langle,\rangle$ with a tilde "$\sim$" and use U to denote the gradient of u with respect to $\langle,\rangle$. Then we can compute the change induced in the first summand of R_1:

$$
\begin{aligned}
e^{2u}\big(\langle Y, \widetilde{X''}\rangle - \langle X, \widetilde{Y''}\rangle\big)Z = & \big(\langle Y, X''\rangle - \langle X, Y''\rangle\big)Z \\
& + \langle U, T\rangle\big(\langle Y, X'\rangle - \langle X, Y'\rangle\big)Z.
\end{aligned}
$$

For the second summand of R_1, we get

$$
\begin{aligned}
e^{2u}\big(\langle X, \widetilde{Y'}\rangle - \langle Y, \widetilde{X'}\rangle\big)\widetilde{Z'} = & \big(\langle X, Y'\rangle - \langle Y, X'\rangle\big)Z' \\
& + \langle U, T\rangle\big(\langle X, Y'\rangle - \langle Y, X'\rangle\big)Z.
\end{aligned}
$$

Thus, the sum R_1 of the two is not affected by the conformal change of metric. Since we know already that the curvature tensor R^C of the conformal connection is invariant, too, the equation $R_2 = R^C - R_1$ implies the claimed invariance of R_2. $\square$

4.9 The curvature tensor in the Fourier basis. Choose a Riemannian metric $\langle,\rangle$ from the conformal class of M, and consider the corresponding constant speed parametrization γ of some loop. For the sake of simplicity, we assume γ to be of length 2π. Hence, we have a torus worth of unit normals, from which we pick some $V_0 \in \perp\gamma(0)$. Denote by $V(s)$ the parallel translate of V_0 to $\perp\gamma(s)$. At 2π, $V(2\pi)$ will differ from V_0 by an angle τ, which is equal to the total torsion (see also chapter 6) of the underlying loop.

We now define Fourier normal fields E_k along γ by

$$
E_k(s) = \frac{1}{\sqrt{2\pi}}\Big(\cos(\alpha_k s)V(s) + \sin(\alpha_k s)JV(s)\Big)
$$

with

$$
\alpha_k = k - \frac{\tau}{2\pi}.
$$

Clearly, $\{E_k, JE_k\}_{k\in\mathbb{Z}}$ constitutes an orthonormal basis of (the horizontal lift of) the tangent space $T_{[\gamma]}\mathfrak{M}$ with respect to the L^2 product $\langle\langle,\rangle\rangle$. This Fourier basis induces a direct sum decomposition

$$
T_\gamma\mathfrak{M} = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}_k \text{ with } \mathbb{C}_k = \mathrm{span}(E_k, JE_k).
$$

of the tangent space which is orthogonal with respect to $\langle\langle,\rangle\rangle$. For $X, Y \in \Gamma(\perp\gamma)$ and $k \in \mathbb{Z}$,

$$
R^C(X,Y)\mathbb{C}_k \subset \mathbb{C}_k,
$$

that is,

$$R^C(X,Y) = \sum_{k\in\mathbb{Z}} R^C(X,Y) \circ \mathrm{pr}_{\mathbb{C}_k} \in \mathrm{End}_{\mathbb{C}}\big(\oplus_{k\in\mathbb{Z}} \mathbb{C}_k\big)$$

respects the decomposition into complex Fourier summands:

$$R(X,Y)|_{\mathbb{C}_k} \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}_k).$$

Hence, in the Fourier basis, $R(X,Y)$ is represented by a diagonal matrix whose elements are complex valued functions of the circle. At position $k \in \mathbb{Z}$, we have the function

$$R(X,Y)|_{\mathbb{C}_k} = \Big(\langle Y, X''\rangle - \langle X, Y''\rangle\Big) + (k - \frac{\tau}{2\pi})\Big(\langle X, Y'\rangle - \langle Y, X'\rangle + \langle Y'', JX\rangle - \langle X'', JY\rangle - \langle JX', Y'\rangle + \kappa\langle JX, Y\rangle\Big)\sqrt{-1}.$$

This shows:

- The real part is independent of k.
- The absolute value of the imaginary part tends to infinity as $|k| \to \infty$. This makes taking traces difficult.

After haven analyzed the curvature tensor R^C of ∇^C we will now turn our attention to conformal geodesics in $\mathfrak{M}$.

4.10 Theorem. *Let*

$$\gamma : (-\varepsilon, +\varepsilon) \times S^1 \to M, \qquad (t,s) \mapsto \gamma(t,s),$$

be a parametrization of a given curve $(-\varepsilon, +\varepsilon) \ni t \mapsto \gamma(t) \in \mathfrak{M}$ in the loop space $\mathfrak{M}$. As usual, denote by $\perp$ the orthogonal projection onto the normal bundle of a loop, and assume

$$t = 0 \Rightarrow \forall s \in S^1 : \ \langle \gamma_t, \gamma_s\rangle = 0.$$

Then the geodesic equation of the conformal connection ∇^C in $t = 0$ takes the form (subscripts denote derivatives with respect to the Levi-Civita connection of $(M, \langle,\rangle)$)

$$0 = (\|\gamma_s\|^2 \gamma_{tt} - \|\gamma_t\|^2 \gamma_{ss})^\perp + 2\langle \gamma_{ss}, \gamma_t\rangle\gamma_t.$$

4.11 *Proof.* For $t = 0$, we compute

$$\begin{aligned}\nabla^C_{\gamma_t}(\gamma_t^\perp) &= \nabla^\perp_{\gamma_t}(\gamma_t^\perp) + \mathfrak{H}_{\gamma_t}(\gamma_t^\perp)\\ &= (\gamma_{tt})^\perp + 2\langle \gamma_t, H\rangle\gamma_t - \|\gamma_t\|^2 H\\ &= \left(\gamma_{tt} - \frac{\|\gamma_t\|^2}{\|\gamma_s\|^2}\gamma_{ss}\right)^\perp + \frac{2}{\|\gamma_s\|^2}\langle \gamma_{ss}, \gamma_t\rangle\gamma_t.\end{aligned}$$

Multiplication with $\|\gamma_s\|^2$ now yields

$$0 = (\|\gamma_s\|^2 \gamma_{tt} - \|\gamma_t\|^2 \gamma_{ss})^\perp + 2\langle \gamma_{ss}, \gamma_t\rangle\gamma_t.$$

□

4.12 Corollary. *Let γ be a geodesic of ∇^C as described in theorem 4.10. Then*

$$\left(\frac{\|\gamma_s\|}{\|\gamma_t\|}\right)_t = 0$$

for all t, where $\gamma_t \perp \gamma_s$.

4.13 *Proof.* Scalar multiplication of the geodesic equation with γ_t yields

$$0 = \|\gamma_s\|^2 \langle \gamma_{tt}, \gamma_t \rangle + \|\gamma_t\|^2 \langle \gamma_{ss}, \gamma_t \rangle.$$

Comparing this with the derivative

$$\left(\frac{\|\gamma_t\|}{\|\gamma_s\|}\right)_t = \frac{-\big(\|\gamma_s\|^2 \langle \gamma_{tt}, \gamma_t \rangle + \|\gamma_t\|^2 \langle \gamma_{ss}, \gamma_t \rangle\big)}{\|\gamma_t\|^3 \|\gamma_s\|}$$

of $\|\gamma_t\| / \|\gamma_s\|$ yields the claim. □

2. Harmonic Mean and Isothermic Surfaces

In this section, we will introduce a smooth function on the tangent bundle of $\mathfrak{M}(S^1, M)$, the *harmonic mean.* Depending only on the conformal structure of the 3-manifold M, it will be used to define a conformal length functional on loop space. Further investigations will then yield equivalent characterizations of the geodesics of ∇^C as critical points of the harmonic mean, and as isothermic cylinders in M.

4.14 Definition. The harmonic mean is a function

$$L : T\mathfrak{M} \to \mathbb{R}$$

defined as follows. For a tangent vector of $\mathfrak{M}$ represented by a normal field $X \in \Gamma(\perp\gamma)$ without zeroes the harmonic mean is given by

$$L(X) = \left(\int_{S^1} \frac{1}{\|X\|} \|\gamma_s\| \, ds\right)^{-1}.$$

If X does have a zero we set $L(X) = 0$.

4.15 Lemma. *The harmonic mean L is a well defined function on $T\mathfrak{M}$ which is homogeneous of degree one and invariant under conformal changes of the Riemannian metric $\langle , \rangle$ on M.*

4.16 *Proof.* By definition, L is independent of the chosen parametrization γ of the loop. Clearly, $L(\lambda X) = |\lambda| L(X)$ for $\lambda \in \mathbb{R}$. Moreover, under a change of metric $\langle , \rangle \mapsto e^{2u} \langle , \rangle$ on M the integrand in the definition of L remains the same. □

4.17 The function L describes a harmonic mean of the length of a given normal field. As such, it plays together nicely with the ordinary harmonic mean

$$\mathfrak{h}(a, b) = \frac{2ab}{a+b}, \quad a, b > 0.$$

In particular, for $X, Y \in \Gamma(\perp\gamma)$, a simple computation shows

$$\mathfrak{h}(L(X), L(Y)) = L(Z) \leq \frac{L(X) + L(Y)}{2},$$

where $Z \in \Gamma(\perp\gamma)$ is some normal field with

$$\|Z\| = \mathfrak{h}(\|X\|, \|Y\|).$$

Moreover, if $\Phi : M \to M$ is a conformal diffeomorphism of $(M, [\langle,\rangle])$, then

$$\mathfrak{M} \to \mathfrak{M}, \quad [\gamma] \mapsto [\Phi \circ \gamma],$$

defines an isometry of the loop space $\mathfrak{M}$ with respect to the harmonic mean L.

4.18 Theorem. *The harmonic mean L is invariant under parallel translation with respect to ∇^C.*

4.19 *Proof.* Let $(t, s) \mapsto \gamma(t, s)$ be a variation of some loop $\gamma(0, .)$ along which we consider a ∇^C-parallel normal field Y. For $t = 0$, we may assume $\gamma_t \perp \gamma_s$. Then the parallelism of Y can be expressed as

$$0 = \nabla^C_{\gamma_t} Y = \nabla^\perp_{\gamma_t} Y + \mathfrak{H}_{\gamma_t} Y = Y_t^\perp - \langle \gamma_t, H \rangle Y - \langle \gamma_t, JH \rangle JY$$

for $t = 0$. Furthermore, we may assume that Y has no zeroes. We have

$$\frac{d}{dt}|_0 L(Y) = -L^2(Y) \int_{S^1} \left(\frac{\|\gamma_s\|}{\|Y\|} \right)_t ds$$

with

$$\begin{aligned}
\left(\frac{\|\gamma_s\|}{\|Y\|} \right)_t =& \frac{1}{\|Y\|} \left(\frac{\langle \gamma_{st}, \gamma_s \rangle}{\|\gamma_s\|} - \|\gamma_s\| \frac{\langle Y_t, Y \rangle}{\|Y\|^2} \right) \\
=& \frac{1}{\|Y\|} \left(\frac{\langle \gamma_{st}, \gamma_s \rangle}{\|\gamma_s\|} + \|\gamma_s\| \langle \gamma_t, H \rangle \right) \\
=& \frac{\langle \gamma_{st}, \gamma_s \rangle + \langle \gamma_t, \gamma_{ss} \rangle}{\|Y\| \, \|\gamma_s\|} \\
=& 0,
\end{aligned}$$

since $\gamma_t \perp \gamma_s$ for $t = 0$. □

4.20 Definition. The conformal length of a curve $\gamma : [a, b] \to \mathfrak{M}$ in loop space is defined by

$$\mathfrak{L}(\gamma) = \int_a^b L(\gamma_t) dt.$$

Since L is homogeneous of degree one $\mathfrak{L}(\gamma)$ is invariant under reparametizations.

4.21 Theorem. *Consider an immersed cylinder*

$$\gamma : [a, b] \times S^1 \to M, \quad (t, s) \mapsto \gamma(t, s).$$

representing a curve $t \mapsto \gamma(t)$ in $\mathfrak{M}$. Then the following statements are equivalent:

(1) *γ is a geodesic of the conformal connection ∇^C.*
(2) *γ is a critical point of the conformal length $\mathfrak{L}$.*
(3) *As a surface, γ is isothermic and its curvature lines make an angle of $\frac{\pi}{4}$ with the individual loops $\gamma(t,.)$.*

4.22 *Proof.* In preparation for the proof of the equivalence of (1) and (2) we consider a variation

$$(-\varepsilon,\varepsilon)\times[a,b]\times S^1 \to M, \quad (u,t,s)\mapsto \gamma(u,t,s),$$

with fixed "end points", that is, each cylinder $\gamma(u,.,.)$ has the same boundary. For $u=0$, we may assume $\gamma_t\perp\gamma_s$, otherwise we can reparametrize in s to achieve this. We put

$$\lambda = \frac{\|\gamma_s\|}{\|\gamma_t\|},$$

such that

$$\mathfrak{L}(\gamma) = \int_a^b \left(\int_{S^1} \lambda ds\right)^{-1} dt.$$

Since $\mathfrak{L}$ is invariant under reparametrization in t we may choose, for $u=0$, a reparametrization in t to *conformal arc length* which gives us

$$\forall t: \int_{S^1} \lambda ds = 1,$$

and

$$\mathfrak{L}_{u|_{u=0}} = -\int_0^l \int_{S^1} \lambda_u ds dt,$$

where

$$\lambda_u = \frac{\|\gamma_t\|^2\langle\gamma_{us},\gamma_s\rangle - \|\gamma_s\|^2\langle\gamma_{ut},\gamma_t\rangle}{\|\gamma_t\|^3\|\gamma_s\|}.$$

For $u=0$, we decompose the vector field γ_u into

$$\gamma_u = X + \alpha\gamma_t + \beta\gamma_s,$$

where X is the orthogonal projection of γ_u onto the normal space of the cylinder. With this decomposition, we compute

$$\begin{aligned}\lambda_u =& \frac{\langle X, \|\gamma_s\|^2\gamma_{tt} - \|\gamma_t\|^2\gamma_{ss}\rangle}{\|\gamma_t\|^3\|\gamma_s\|}\\ &+ \frac{\|\gamma_t\|^2\langle(\alpha\gamma_t)_s,\gamma_s\rangle - \|\gamma_s\|^2\langle(\alpha\gamma_t)_t,\gamma_t\rangle}{\|\gamma_t\|^3\|\gamma_s\|}\\ &+ \frac{\|\gamma_t\|^2\langle(\beta\gamma_s)_s,\gamma_s\rangle - \|\gamma_s\|^2\langle(\beta\gamma_s)_t,\gamma_t\rangle}{\|\gamma_t\|^3\|\gamma_s\|}\end{aligned} \tag{4.1}$$

After these preparations, we are in position to show the equivalence of the statements (1) and (2).

"(1) $\Rightarrow$ (2)": Assume $\gamma(0,.,.)$ is a geodesic of the conformal connection ∇^C. In the light of

$$0 = (\|\gamma_s\|^2 \gamma_{tt} - \|\gamma_t\|^2 \gamma_{ss})^\perp + 2\langle\gamma_{ss}, \gamma_t\rangle\gamma_t$$

the equation 4.1 for λ_u simplifies to

$$\lambda_u = (\beta\lambda)_s - (\alpha\lambda)_t.$$

Now, integration over s eleminates the first summand, integration over t the second, since $\alpha = 0$ along the boundary of the cylinder. This shows that the geodesic under consideration is indeed a critical point of the conformal length functional.

"(2) $\Rightarrow$ (1)": Under the assumption of $\gamma(0,.,.)$ being a critical point of $\mathfrak{L}$, we want to show that $\nabla^C_{\gamma_t}\gamma_t|_{t_0}$ vanishes for t_0 arbitrarily chosen from $(0,l)$. This may be done in two steps. Firstly, for variations with γ_u orthogonal to the whole cylinder, we get

$$\lambda_u = \frac{\langle\gamma_u, \|\gamma_s\|^2 \gamma_{tt} - \|\gamma_t\|^2 \gamma_{ss}\rangle}{\|\gamma_t\|^3 \|\gamma_s\|}.$$

Therefore, the component $\nabla^C_{\gamma_t}\gamma_t$ orthogonal to the cylinder has to vanish. Secondly, we may consider variations with $\gamma_u = \gamma_t$ for $u = 0$ and t close to t_0. In this case,

$$\lambda_u = -\frac{\|\gamma_s\|^2 \langle\gamma_{tt}, \gamma_t\rangle + \|\gamma_t\|^2 \langle\gamma_{ss}, \gamma_t\rangle\cdot}{\|\gamma_t\|^3 \|\gamma_s\|}.$$

Hence, it follows that

$$0 = \|\gamma_s\|^2 \langle\gamma_{tt}, \gamma_t\rangle + \|\gamma_t\|^2 \langle\gamma_{ss}, \gamma_t\rangle,$$

so $\nabla^C_{\gamma_t}\gamma_t|_{t_0} = 0$.

For the proof of the equivalence of (1) and (3) we may assume $\gamma_t \perp \gamma_s$ and put

$$E := \frac{1}{\|\gamma_t\|}\gamma_t \quad \text{and} \quad \widehat{H} := \nabla_E E,$$

in order to write the geodesic equation of the conformal connection ∇^C in the form

$$\begin{cases} \langle H, E\rangle = -\langle \frac{\gamma_{tt}}{\|\gamma_t\|^2}, E\rangle \\ \\ \langle H, JE\rangle = \langle \widehat{H}, JE\rangle \end{cases}, \tag{4.2}$$

where H and $\widehat{H}$ are the mean curvature vectors of the individual loops and variational curves, respectively. Now we observe that, using the frame (T, E, JE), the shape operator of the given cylinder with respect to the normal field JE is represented by

$$A = \begin{pmatrix} \langle H, JE\rangle & \langle\nabla_T E, JE\rangle \\ \langle\nabla_T E, JE\rangle & \langle\widehat{H}, JE\rangle \end{pmatrix}.$$

"(1) $\Rightarrow$ (3)": Under the assumption of γ being a geodesic of ∇^C corollary 4.12 implies that there is a constant $c > 0$ such that $\|\gamma_s\| = c\,\|\gamma_t\|$. Hence, we may rescale the variational parameter t such that $\|\gamma_s\| = \|\gamma_t\|$ for all t, s. To simplify the situation further we divide the metric on M by $\|\gamma_s\|^2$, such that both coordinate fields have now length one. If the cylinder is a geodesic, the eigenvalues $\kappa_{1,2}$ of A are

$$\kappa_{1,2} = \langle H, JE\rangle \pm \langle \nabla_T E, JE\rangle.$$

It follows that the corresponding eigenspaces are precisely the angle bisectors of the (T, E)-plane. If we denote by $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ the left rotation by 45 degrees $\widehat{\gamma}(x, y) = (\gamma \circ \varphi)(x, y)$ gives a new local conformal parametrization of the cylinder. Now the geodesic equation yields $\langle \widehat{\gamma}_{xy}, JE\rangle = 0$, and $\widehat{\gamma}_x$, $\widehat{\gamma}_y$ become eigenvectors of the shape operator.

"(3) $\Rightarrow$ (1)": Assuming γ to be an isothermic surface whose curvature lines make an angle of 45 degrees with the γ_s direction we see that

$$\langle H, JE\rangle = \langle \widehat{H}, JE\rangle.$$

Moreover, the cylinder fulfills $\|\gamma_s\| = \|\gamma_t\|$, hence

$$\left(\frac{\|\gamma_s\|}{\|\gamma_t\|}\right)_t = 0,$$

which is equivalent (see corollary 4.12 and its proof) to the "tangential part" of the geodesic equation 4.2. Thus, γ is indeed a geodesic of ∇^C.

$\square$

CHAPTER 5

The Space of Hypersurfaces

We now generalize the results obtained in chapters 3 and 4 to spaces of higher dimensional submanifolds. Although most of the constructions could be carried out for submanifolds of arbitrary codimension our focus will lie on hypersurfaces. We provide an interpretation of geodesics with the help of a suitable variant of the harmonic mean discussed in the previous chapter.

1. The Conformal Connection

Similar to the strategy of chapters 3 and 4 we will firstly define a "basic connection" $\nabla^\perp$ as well as a mean curvature form $\mathfrak{H}$ and then show how to obtain a conformal connection ∇^C as linear combination of $\nabla^\perp$ and $\mathfrak{H}$.

5.1 Let S be an oriented closed n-manifold, $(M, \langle,\rangle)$ an $(n+k)$-dimensional oriented Riemannian manifold. The group $\mathsf{Diff}^+(S)$ of orientation preserving diffeomorphisms of M acts on the space $\mathsf{Imm}(S, M)$ of (free) immersions $S \to M$. As discussed in chapter 1, we consider the immersed submanifolds of diffeomorphism type S in M to be given as an equivalence class $[\gamma]$ of immersions, where $\gamma_1 \sim \gamma_2$ if and only if there exists some $\varphi \in \mathsf{Diff}^+(S)$ such that $\gamma_2 = \gamma_1 \circ \varphi$. The space of all such submanifolds of M is itself a nuclear Fréchet manifold which we denote by $\mathfrak{M} = \mathfrak{M}(S, M)$.

5.2 Definition. Consider a normal variation

$$\gamma : (-\varepsilon, +\varepsilon) \times S \to (M, \langle,\rangle), \quad (t,p) \mapsto \gamma(t,p),$$

with variational vector field $\gamma_t|_{t=0} = X \in \Gamma(\perp\gamma(0))$, and a normal field $Y \in \Gamma(\perp\gamma)$ along that variation.

For $t = 0$, we may choose a local orthonormal frame $(T_1, \dots, T_n)$ of TS with respect to the induced metric $\gamma(0)^*\langle,\rangle$ on S.

Then the torsion free *basic connection* $\nabla^\perp$ on $\mathfrak{M}$ is defined by

$$\nabla^\perp_X Y := (\nabla_X Y)^\perp - \sum_i \langle X, T_i\rangle (\nabla_{T_i} Y)^\perp.$$

As a prerequisite of further investigations we compute its curvature tensor.

5.3 Theorem. *For tangent vectors of $\mathfrak{M}$ represented by normal fields $X, Y, Z \in \Gamma(\perp\gamma)$ along an immersion γ the curvature tensor $R^\perp$ of the basic connection is given by*

$$\begin{aligned} R^\perp(X,Y)Z =& (R(X,Y)Z)^\perp \\ &+ \sum_i \Big(\langle Z, \nabla_{T_i} Y\rangle (\nabla_{T_i} X)^\perp - \langle Z, \nabla_{T_i} X\rangle (\nabla_{T_i} Y)^\perp \\ &- \big(\langle Y, \nabla_{T_i} X\rangle - \langle X, \nabla_{T_i} Y\rangle\big)(\nabla_{T_i} Z)^\perp \Big). \end{aligned}$$

In the above formula, $(T_1, \dots, T_n)$ denotes a local orthonormal frame of TS.

5.4 *Proof.* We extend $X, Y, Z \in \Gamma(\perp\gamma)$ to vector fields on a neighborhood of γ and compute

$$\begin{aligned} \nabla^\perp_X \nabla^\perp_Y Z =& \Big(\nabla_X (\nabla_Y Z)^\perp\Big)^\perp - \sum_i \Big(\nabla_X\big(\langle Y, T_i\rangle (\nabla_{T_i} Z)^\perp\big)\Big)^\perp \\ =& \Big(\nabla_X \nabla_Y Z - \sum_j \langle \nabla_Y Z, T_j\rangle \nabla_X T_j\Big)^\perp \\ &- \sum_i (\langle \nabla_X Y, T_i\rangle + \langle Y, \nabla_{T_i} X\rangle)(\nabla_{T_i} Z)^\perp \\ =& (\nabla_X \nabla_Y Z)^\perp + \sum_i \Big(\langle Z, \nabla_{T_i} Y\rangle (\nabla_{T_i} X)^\perp \\ &- (\langle \nabla_X Y, T_i\rangle + \langle Y, \nabla_{T_i} X\rangle)(\nabla_{T_i} Z)^\perp \Big). \end{aligned}$$

Inserting this into

$$R^\perp(X,Y)Z = \nabla^\perp_X \nabla^\perp_Y Z - \nabla^\perp_Y \nabla^\perp_X Z$$

yields the claim. □

5.5 Conformal connection on spaces of submanifolds. Using the basic connection $\nabla^\perp$ we can generalize the *conformal connection* ∇^C to the space $\mathfrak{M}(S, M)$ of immersed submanifolds in a straightforward manner:

$$\nabla^C = \nabla^\perp + \mathfrak{H},$$

where, for normal fields $X, Y \in \Gamma(\perp\gamma)$ along an immersion γ,

$$\mathfrak{H}_X Y = \langle X, H\rangle Y + \langle Y, H\rangle X - \langle X, Y\rangle H,$$

and – writing B for the immersion's second fundamental form –

$$H = \frac{1}{n}\mathrm{Trace}(B).$$

Again, it is straightforward to show that ∇^C is invariant under conformal changes of the Riemannian metric $\langle , \rangle$ on the ambient manifold M.

For the rest of this chapter, we will restrict our attention to the space of hypersurfaces.

From this point on, we assume $\dim(S) = n$ and $\dim(M) = n+1$.

We will now compute the curvature tensor R^C of the conformal connection. In preparation for the proof of theorem 5.12 we collect some information about $R^\perp$ and $\mathfrak{H}$ in the following corollary and lemmata.

5.6 Corollary. *If γ is an immersed hypersurface with unit normal field $N \in \Gamma(\perp\gamma)$, and $X = xN$, $Y = yN$, $Z = zN$ with $x, y, z \in C^\infty(S)$,*

$$R^\perp(X, Y, Z) = \big(x\langle \mathrm{grad} y, \mathrm{grad} z\rangle - y\langle \mathrm{grad} x, \mathrm{grad} z\rangle\big)N$$

5.7 *Proof.* This is a direct consequence of the general formula for $R^\perp$ given in theorem 5.3, since, in the case of hypersurfaces,

$$(\nabla_{T_i} X)^\perp = \langle \mathrm{grad} x, T_i\rangle N,$$

where $(T_1, \ldots, T_n)$ is a local orthonormal frame. □

5.8 Lemma. *Let*

$$\gamma : (-\varepsilon, \varepsilon) \times S \to (M, \langle , \rangle), \quad (t, p) \mapsto \gamma(t, p),$$

be a normal variation of $\gamma(0)$ with variational vector field $X = xN = \gamma_t|_{t=0} \in \Gamma(\perp\gamma(0))$, $x = \langle X, N\rangle \in C^\infty(S)$, and $N \in \Gamma(\perp\gamma(0))$ a unit normal field. Then we have

$$\nabla^\perp_X H = \frac{1}{n}\big(x(\|B\|^2 + \mathrm{ric}(N, N)) + \Delta x\big)N,$$

where B, ric, and Δ denote the second fundamental form of S, the Ricci tensor of M, and the Laplace operator on $C^\infty(S)$, respectively.

5.9 *Proof.* Denote by $g(t) = \gamma(t)^*\langle , \rangle$ the induced Riemannian metric on S at time t, and write ∇ for the Levi-Civita connection of $(M, \langle , \rangle)$. For $t = 0$, $p \in S$, we choose an orthonormal frame $(T_1, \ldots, T_n)$ of T_pS, and use parallel translation along geodesic arcs emanating from p to obtain a local orthonormal frame around p with $\nabla_{T_i} T_j|_p \in \perp\gamma(0, p)$. The pushforward $\gamma(t)_* T_i$ will again be denoted by T_i in order to avoid clumsy notation. Locally, we have

$$nH = \sum_{i,j=1}^n g^{ij}(t)\langle \nabla_{T_i} T_j, N\rangle N,$$

where $g^{ij}(t)$ are the components of $g(t)^{-1}$. Hence,

$$n\Big(\frac{d}{dt}H\Big)^{\perp}|_{t=0} = \sum_{ij}\Big(\underbrace{\frac{d}{dt}g^{ij}|_{t=0}}_{(1)}\langle\nabla_{T_i}T_j, N\rangle N + \delta_{ij}\underbrace{\frac{d}{dt}\langle\nabla_{T_i}T_j, N\rangle|_{t=0}}_{(2)}N\Big).$$

(1) By definition,

$$\delta_{ij} = \sum_j g^{ij}(t)g_{jk}(t) \Rightarrow \frac{d}{dt}g^{ij}|_{t=0} = -\frac{d}{dt}\langle T_i, T_j\rangle|_{t=0}.$$

Observing $0 = [T_i, T_j] = [T_i, N]$, we deduce

$$\begin{aligned}\frac{d}{dt}g^{ij}|_{t=0} &= -(\langle\nabla_{T_i}X, T_j\rangle + \langle T_i, \nabla_{T_j}X\rangle)\\ &= 2\langle\nabla_{T_i}T_j, X\rangle = 2\langle B(T_i, T_j), X\rangle.\end{aligned}$$

(2) Moreover,

$$\begin{aligned}\frac{d}{dt}\langle\nabla_{T_i}T_i, N\rangle|_{t=0} &= \langle\nabla_X\nabla_{T_i}T_i, N\rangle\\ &= \langle R(X, T_i)T_i, N\rangle + \langle\nabla_{T_i}\nabla_{T_i}X, N\rangle\end{aligned}$$

In the light of $X = xN$,

$$\begin{aligned}\langle\nabla_{T_i}\nabla_{T_i}X, N\rangle &= \langle\nabla_{T_i}\mathrm{grad}x, T_i\rangle - \sum_j\langle X, \nabla_{T_i}T_j\rangle\langle\nabla_{T_i}T_j, N\rangle\\ &= \langle\nabla_{T_i}\mathrm{grad}x, T_i\rangle - x\sum_j\langle\nabla_{T_i}T_j, N\rangle^2.\end{aligned}$$

Therefore,

$$\sum_i\frac{d}{dt}\langle\nabla_{T_i}T_i, N\rangle|_{t=0} = x(\mathrm{ric}(N, N) - \|B\|^2) + \Delta x.$$

Combining the results of steps (1) and (2), we arrive at the claimed formula. □

5.10 Lemma. *Let $X = xN, Y = yN, Z = zN$ be normal fields along an immersed hypersurface $\gamma : S \to (M, \langle,\rangle)$ with $x, y, z \in C^\infty(S)$, and $N \in \Gamma(\perp\gamma)$ a unit normal field. Then we have*

$$(d^{\nabla^\perp}\mathfrak{H})(X, Y, Z) = \frac{1}{n}(y\Delta x - x\Delta y)Z.$$

5.11 *Proof.* By definition of $\mathfrak{H}$,

$$(\nabla^\perp_X\mathfrak{H})(Y, Z) = \langle Y, \nabla^\perp_X H\rangle Z + \langle Z, \nabla^\perp_X H\rangle Y - \langle Y, Z\rangle\nabla^\perp_X H.$$

Hence,

$$\begin{aligned}(d^{\nabla^\perp}\mathfrak{H})(X, Y, Z) &= \big(\langle Y, \nabla^\perp_X H\rangle - \langle X, \nabla^\perp_Y H\rangle\big)Z\\ &\quad + \langle Z, \nabla^\perp_X H\rangle Y - \langle Z, \nabla^\perp_Y H\rangle X\\ &\quad + \langle X, Z\rangle\nabla^\perp_Y H - \langle Y, Z\rangle\nabla^\perp_X H.\end{aligned}$$

Inserting the formula for $\nabla^\perp H$ obtained in lemma 5.8, we get the result. $\square$

5.12 Theorem. *Let $X = xN, Y = yN, Z = zN$ be normal fields along an immersed hypersurface $\gamma : S \to (M, \langle , \rangle)$ with $x, y, z \in C^\infty(S)$, and $N \in \Gamma(\perp\gamma)$ a unit normal field. Then the curvature tensor R^C of the conformal connection ∇^C on $\mathfrak{M}(S, M)$ is given by*

$$R^C(X,Y)Z = \big(x\langle \mathrm{grad} y, \mathrm{grad} z\rangle - y\langle \mathrm{grad} x, \mathrm{grad} z\rangle + \frac{1}{n} z(y\Delta x - x\Delta y)\big)N,$$

where Δ is the Laplace operator on $C^\infty(S)$.

5.13 *Proof.* Since $\nabla^C = \nabla^\perp + \mathfrak{H}$,

$$R^C(X,Y)Z = R^\perp(X,Y)Z + (d^{\nabla^\perp}\mathfrak{H})(X,Y,Z) + \mathfrak{H}_X\mathfrak{H}_Y Z - \mathfrak{H}_Y\mathfrak{H}_X Z.$$

Inserting the results for $R^\perp$ and $d^{\nabla^\perp}\mathfrak{H}$ from corollary 5.6 and lemma 5.10 yields the claim. $\square$

5.14 Next, we will derive the geodesic equation of the conformal connection ∇^C on the space of hypersurfaces. Afterwards, a characterization of geodesics will be obtained with the help of a suitable notion of harmonic mean for the codimension-one case in section 2 of this chapter.

5.15 Theorem. *Let*

$$\gamma : (-\varepsilon, +\varepsilon) \times S \to M, \qquad (t, s) \mapsto \gamma(t, s),$$

be a parametrization of a given curve $(-\varepsilon, +\varepsilon) \ni t \mapsto \gamma(t) \in \mathfrak{M}(S, M)$ in the space of hypersurfaces of type S in M. As usual, denote by $\perp$ the orthogonal projection onto the normal bundle of an immersion, and assume

$$t = 0 \Rightarrow \forall s \in S : \ \gamma_t = fN,$$

where $N : (-\varepsilon, +\varepsilon) \times S \to \perp\gamma$ is, for each t, an oriented unit normal field of the hypersurface $\gamma(t)$. Then, the geodesic equation of the conformal connection ∇^C in $t = 0$ takes the form

$$0 = f_t + f^2 h,$$

with h denoting the mean curvature function of $\gamma(t)$.

5.16 *Proof.* Since γ_t is orthogonal to the immersion, the geodesic equation of ∇^C amounts to

$$0 = \nabla^C_{\gamma_t}\gamma_t = (\gamma_{tt})^\perp + 2\langle\gamma_t, H\rangle\gamma_t - \|\gamma_t\|^2 H,$$

where we have used subscripts to denote differentiation with respect to the Levi-Civita connection of M. Using the unit normal field N, we have $h = \langle H, N\rangle$, $\gamma_t = fN$, and $(\gamma_{tt})^\perp = f_t N$. Therefore, γ is a geodesic of ∇^C if and only if

$$0 = (f_t + 2f^2h - f^2h)N = (f_t + f^2h)N.$$

□

2. Harmonic Mean for Hypersurfaces

With only minor adaptions, the idea to introduce a harmonic mean as a conformally invariant replacement for a Riemannian metric can be carried over from the case of loops (see chapter 4) to the case of hypersurfaces. This will provide for a characterization of conformal geodesics as critical points of the corresponding length functional.

5.17 Definition. Let $\gamma : S \to (M, \langle,\rangle)$ represent an immersed hypersurface. For a normal field $X \in \Gamma(\perp\gamma)$ without zeroes, the *harmonic mean* is given by

$$L(X) = \left(\int_S \frac{1}{\|X\|^n} \mathrm{dVol} \right)^{-1/n},$$

where $n = \dim(S)$. If X does have a zero we set $L(X) = 0$.

The *conformal length* of a curve $\gamma : [a, b] \to \mathfrak{M}(S, M)$ in the space of hypersurfaces is defined by

$$\mathfrak{L}(\gamma) = \int_a^b L(\gamma_t) dt.$$

5.18 The above definition[1] generalizes the harmonic mean from loops (see definition 4.14) to hypersurfaces. The exponents in the formula are chosen so that the following basic properties are maintained:

(1) invariance under the action of the diffeomorphism group $\mathsf{Diff}^+(S)$ on the space of immersions,
(2) invariance under conformal changes of the Riemannian metric $\langle,\rangle$ on the ambient manifold M,
(3) and homogeneity of degree one.

Now we are in position to prove the following characterization of conformal geodesics.

5.19 Theorem. *Consider an immersion*

$$\gamma : [a, b] \times S \to M, \quad (t, s) \mapsto \gamma(t, s).$$

representing a curve $t \mapsto \gamma(t)$ in $\mathfrak{M}(S, M)$ with oriented unit normal field N and $(\gamma_t)^\perp \neq 0$ for all t, s. Then the following statements are equivalent:

(1) *γ is a geodesic of the conformal connection ∇^C.*
(2) *γ is a critical point of the conformal length $\mathfrak{L}$.*

[1] We note that precisely the same definition would yield a conformally invariant harmonic mean for immersed submanifolds of arbitrary codimension.

(3) *Up to a conformal change of the pullback*[2] *of the Riemannian metric to* $\gamma^*\langle,\rangle / \left\|(\gamma_t)^\perp\right\|^2$ *on* γ^*TM*, the variation consists of parallel minimal hypersurfaces.*

5.20 *Proof.* Without loss of generality, we may assume, that γ_t is orthogonal to the immersion.

"(1) $\Leftrightarrow$ (3)": Assume γ is a geodesic of the conformal connection ∇^C. Using the notation from theorem 5.15 this means

$$h = -\frac{f_t}{f^2} = \left(\frac{1}{f}\right)_t$$

where $f = \langle\gamma_t, N\rangle$, $h = \langle H, N\rangle$, and N is a unit normal.

Now we change the Riemannian metric on M according to

$$\langle,\rangle \mapsto e^{2u}\langle,\rangle = \frac{1}{\|\gamma_t\|^2}\langle,\rangle,$$

and write U for the gradient of u with respect to $\langle,\rangle$. We decorate all objects that arise from the new metric with "$\,\widetilde{}\,$". This results in

$$\widetilde{N} = \gamma_t, \quad \widetilde{f} = 1, \quad \widetilde{f_t} = 0.$$

Clearly, the immersed hypersurfaces $\gamma(t, .)$ are parallel with respect to $e^{2u}\langle,\rangle$. For the new mean curvature, we get

$$\begin{aligned}\widetilde{h} &= \frac{1}{n}\sum_i e^{2u}\langle\widetilde{\nabla}_{\widetilde{E_i}}\widetilde{E_i}, \widetilde{N}\rangle = \frac{1}{n}e^{-u}\sum_i\langle\nabla_{E_i}E_i - U, N\rangle \\ &= e^{-u}(h - du(N)).\end{aligned}$$

Since the scaling function u is given by

$$u = -\frac{1}{2}\ln\langle\gamma_t, \gamma_t\rangle$$

it follows from the geodesic equation that

$$du(N) = \frac{u_t}{\|\gamma_t\|} = -\frac{\langle\gamma_{tt}, N\rangle}{\|\gamma_t\|^2} = h.$$

Therefore, $\widetilde{h} = 0$.

On the other hand, if we start with a variation consisting of parallel minimal hypersurfaces we have already

$$f_t = 0, \quad h = 0.$$

Hence, this variation fulfills $0 = f_t + f^2h$ and thus has to be a geodesic of the conformal connection ∇^C.

[2] The pullback is a technical necessity, since we consider immersions. If the whole variation is an embedded submanifold with boundary the conformal change can be applied directly to the Riemannian metric on M.

"(2) $\Leftrightarrow$ (3)": We consider a variation

$$\gamma : (-\varepsilon, \varepsilon) \times [a,b] \times S \to M, \quad (u,t,s) \mapsto \gamma(u,t,s),$$

of the given path $\gamma(0,.,.)$ in the space of hypersurfaces. The conformal length functional is given by

$$\mathfrak{L}(\gamma) = \int_a^b I(t)^{-1/n} dt$$

with

$$I(t) = \int_S \frac{1}{\|\gamma_t\|^n} \mathrm{dVol}.$$

The induced Riemannian volume form dVol on S fulfills (see, for instance, (Law80))

$$\mathrm{dVol}_u = -n\langle \gamma_u, H\rangle \mathrm{dVol}.$$

Consequently, for $u = 0$, we get

$$I_u = -n \int_S \frac{\langle \gamma_{tu}, \gamma_t\rangle + \|\gamma_t\|^2 \langle \gamma_u, H\rangle}{\|\gamma_t\|^{n+2}} \mathrm{dVol}.$$

Since the conformal length functional $\mathfrak{L}$ is invariant under reparametrizations in t, we may choose an arc length parametrization to achieve $I(t) = 1$ for all $t \in [0,l]$ with $\mathfrak{L}(\gamma) = l$. This gives us

$$\mathfrak{L}_u|_{u=0} = -\frac{1}{n}\int_0^l I_u(t) dt.$$

Moreover, we apply a conformal change to the pullback of the Riemannian metric on the ambient space M such that – in the new metric – $\|\gamma_t\| = 1$ for all t and $u = 0$. This results in

$$\mathfrak{L}_u|_{u=0} = \int_0^l \int_S \langle \gamma_u, H\rangle \ \mathrm{dVol}\ dt.$$

Therefore, the given path is a critical point of the conformal length if and only if it consists – with respect to the changed metric – of a family of minimal hypersurfaces. □

3. The Levi-Civita Connection

We will now turn to the Levi-Civita connection ∇^{LC} belonging to the L^2 product $\langle\langle , \rangle\rangle$ on the space of hypersurfaces. It will turn out that $(\mathfrak{M}(S,M)), \langle\langle , \rangle\rangle)$ considered as a Riemannian manifold has non-negative sectional curvature.

5.21 Theorem. *The Levi-Civita connection belonging to the L^2 product $\langle\langle , \rangle\rangle$ on the space of immersed hypersurfaces is given by*

$$\nabla^{LC} = \nabla^{\perp} - \frac{n}{2}\mathfrak{H}.$$

5.22 *Proof.* The connection ∇^{LC} is torsion free by definition. It remains to show that it is metric. To this end, we differentiate the L^2 product of two normal fields X, Y along a variation

$$\gamma : (-\varepsilon, +\varepsilon) \times S \to (M, \langle , \rangle), \quad (t, p) \mapsto \gamma(t, p),$$

with variational vector field $\gamma_t|_{t=0} = Z \in \Gamma(\perp\gamma(0))$. This yields

$$Z\langle\langle X, Y\rangle\rangle = \int_S \Big(\langle \nabla_Z^{\perp} X, Y\rangle + \langle X, \nabla_Z^{\perp} Y\rangle - n\langle Z, H\rangle\langle X, Y\rangle \Big) \mathrm{dVol}.$$

Now, the last summand in the integral fulfills

$$n\langle Z, H\rangle\langle X, Y\rangle = \frac{n}{2}\big(\langle \mathfrak{H}_Z X, Y\rangle - \langle X, \mathfrak{H}_Z Y\rangle\big).$$

Therefore, ∇^{LC} is a metric connection. □

5.23 Theorem. *Let $X = xN, Y = yN, Z = zN$ be normal fields along an immersed hypersurface $\gamma : S \to (M, \langle , \rangle)$ with $x, y, z \in C^\infty(S)$, and $N \in \Gamma(\perp\gamma)$ a unit normal field. Then the curvature tensor R^{LC} of the Levi-Civita connection associated to the L^2 product $\langle\langle , \rangle\rangle$ on $\mathfrak{M}(S, M)$ is given by*

$$\begin{aligned} R^{LC}(X, Y)Z = \big(&x\langle \mathrm{grad}y, \mathrm{grad}z\rangle - y\langle \mathrm{grad}x, \mathrm{grad}z\rangle \\ &+ \frac{1}{2}z(x\Delta y - y\Delta x)\big)N, \end{aligned}$$

where Δ denotes the Laplace operator on $C^\infty(S)$. Moreover,

$$\langle\langle R^{LC}(X, Y)Y, X\rangle\rangle = \frac{1}{2}\int_S \|x\mathrm{grad}y - y\mathrm{grad}x\|^2 \,\mathrm{dVol},$$

such that the sectional curvature of the space of hypersurfaces is nonnegative.

5.24 *Proof.* The formula for R^{LC} follows directly from $\nabla^{LC} = \nabla^{\perp} - \frac{n}{2}\mathfrak{H}$. Concerning the sectional curvature we note that for any pair of smooth functions $f_1, f_2 \in C^\infty(S)$,

$$\int_S f_1 \Delta f_2 \mathrm{dVol} = -\int_S \langle \mathrm{grad}f_1, \mathrm{grad}f_2\rangle \mathrm{dVol},$$

since S is compact. This implies

$$\begin{aligned} \int_S xy(x\Delta y - y\Delta x)\mathrm{dVol} &= \int_S \big(\langle \mathrm{grad}(y^2 x), \mathrm{grad}x\rangle \\ &\qquad - \langle \mathrm{grad}(x^2 y), \mathrm{grad}y\rangle\big)\mathrm{dVol} \\ &= \int_S (y^2 \|\mathrm{grad}x\|^2 - x^2 \|\mathrm{grad}y\|^2)\mathrm{dVol}. \end{aligned}$$

Finally, we get

$$\begin{aligned}\langle\langle R^{LC}(xN,yN)xN,yN\rangle\rangle &= \int_S \Big(xy\langle \mathrm{grad} x, \mathrm{grad} y\rangle - y^2 \left\|\mathrm{grad} x\right\|^2 \\ &\qquad + \frac{1}{2}xy(x\Delta y - y\Delta x)\Big)\mathrm{dVol} \\ &= -\frac{1}{2}\int_S \left\|x\mathrm{grad} y - y\mathrm{grad} x\right\|^2 \mathrm{dVol}.\end{aligned}$$

□

5.25 With the notation from theorem 5.15, the geodesic equation of ∇^{LC} takes the form

$$0 = f_t - \frac{n}{2}f^2 h.$$

Concerning the inner metric $d^{\langle\langle,\rangle\rangle}$ induced by the L^2 product $\langle\langle,\rangle\rangle$ on the space of hypersurfaces $\mathfrak{M}(S,M)$ the effects described in section 2 of chapter 3 remain the same: $d^{\langle\langle,\rangle\rangle}$ vanishes identically. We refer to (MM05) for more information on the distance metric point of view.

CHAPTER 6

Conformal Circles

Conformal circles have been studied by many authors including (Fia39), (Fer82), (BE90). Depending on the viewpoint one wishes to assume the terminology ranges from *conformal geodesics* and *conformal null curves* to *conformal circles.* It turns out that the case of circles in 3-manifolds is much easier to handle than the general situation of circles in n-dimensional manifolds. For this case, we exhibit a variational characterization of conformal circles.

1. Basic Properties

Before we come to the variational characterization of conformal circles we collect the basic definitions and elementary properties.

6.1 Definition. A loop $\gamma : S^1 \to M$ in a Riemannian n-manifold $(M, \langle,\rangle)$, $n \geq 3$, is called a *conformal circle*[1] if it satisfies the equation

$$(n-2)H' = (\mathrm{Ric}T)^{\perp},$$

where H denotes the loop's mean curvature vector, T its unit tangent, and

$$(\mathrm{Ric}T)^{\perp} = \mathrm{Ric}T - \langle \mathrm{Ric}T, T\rangle T$$

the normal part of the Ricci tensor of $(M, \langle,\rangle)$ applied to T.

6.2 Lemma. *The condition of being a conformal circle is invariant under conformal changes of the Riemannian metric $\langle,\rangle$ on M.*

6.3 *Proof.* We consider a conformal change of the Riemannian metric to $\langle,\rangle \mapsto e^{2u}\langle,\rangle = \widetilde{\langle,\rangle}$ using some smooth function $u \in C^\infty(M, \mathbb{R})$ and decorate objects arising from the new metric with "$\,\widetilde{}\,$". Unwinding the definitions, we see that the normal vector field $H' \in \Gamma(\perp\gamma)$ changes to

$$\widetilde{\widetilde{H}'} = e^{-3u}\Big(H' + \langle U, T\rangle U^{\perp} - (\nabla_T U)^{\perp}\Big),$$

where $U = \mathrm{grad} u$. On the other hand, the Ricci tensor changes to

$$(\widetilde{\mathrm{Ric}}\widetilde{T})^{\perp} = e^{-3u}\Big((\mathrm{Ric}T)^{\perp} + (n-2)\big(\langle U, T\rangle U^{\perp} - (\nabla_T U)^{\perp}\big)\Big).$$

[1]Our definition differs from that given by T. N. Bailey and M. G. Eastwood (BE90, 3.1) in that we do not demand conformal circles to be parametrized in a special way.

Hence, it does not matter which Riemannian metric we choose from a conformal class to check the circle condition. $\square$

6.4 Lemma. *If $\gamma : S^1 \to (M, \langle , \rangle)$ is an embedded loop, we can change the Riemannian metric to $e^{2u}\langle , \rangle$ on an open neighborhood V of γ such that $u|_\gamma = 1$, $\mathrm{grad} u \perp T$, and γ becomes a geodesic of the new metric $e^{2u}\langle , \rangle$. If γ is not embedded we can still apply the same strategy to the pullback $\gamma^*\langle , \rangle$ of the metric to $\gamma^* TM$.*

6.5 *Proof.* We choose some smooth $u : V \to \mathbb{R}$ with $u|_\gamma = 1$ and $\mathrm{grad} u = H$. For the changed Riemannian metric, the mean curvature vector $\widetilde{H}$ of the loop is now given by

$$\widetilde{H} = e^{-2u}(H - (\mathrm{grad} u)^\perp) = 0,$$

so the loop is a geodesic with respect to the changed metric. $\square$

6.6 Circles in Einstein manifolds. Following (NY74), we call a loop $\gamma : S^1 \to (M, \langle , \rangle)$ with parallel mean curvature vector, $H' = 0$, a *Riemannian circle.* In particular, closed geodesics are Riemannian circles. Clearly, a Riemannian circle is a conformal circle if and only if, at each point,

$$(\mathrm{Ric} T)^\perp = 0,$$

that is, its unit tangent T is an eigenvector of the Ricci tensor considered as an endomorphism $\mathrm{Ric} : T_pM \to T_pM$. Thus, if $(M, \langle , \rangle)$ is conformally Einstein the conformal circles of $(M, [\langle , \rangle])$ are exactly the Riemannian circles of the Einstein metric.

We note that in dimensions greater than three the "gap" between conformally flat and conformally non-flat Einstein manifolds is quite large: The interior of the zero set of the Weyl tensor (see 6.14) of a conformally non-flat Einstein manifold is empty (Bes87, 5.26).

Standard examples of Einstein manifolds all of whose geodesics are closed – and hence are conformal circles – include (S^n, can), $(\mathbb{C}P^n, \mathrm{can})$, and $(\mathbb{H}P^n, \mathrm{can})$ (Bes76, 3.31). One may consider the notion of conformal circles as "new" only in the case where M is *not* conformal to an Einstein manifold.

2. Conformal Circles and Total Torsion

In the case of conformal circles in 3-manifolds, we can take advantage of the fact that the codimension of a loop is always two in order to find an interesting characterization of conformal circles – regardless of any special properties of the conformal structure on the ambient space.

6.7 Definition. Let $\mathfrak{M}$ denote the loop space of a Riemannian 3-manifold $(M, \langle , \rangle)$. Then the (S^1-valued) total torsion is the function given by

$$\tau : \mathfrak{M} \to S^1 \subset \mathbb{C}, \qquad \tau([\gamma]) = \exp\left(i \int_{S^1} \langle \nabla_T E, JE \rangle \, \|\gamma_s\| \, ds \right),$$

where $E \in \Gamma(\perp\gamma)$ is an arbitrary normal vector field of length $\|E\| = 1$.

6.8 Lemma. *The total torsion $\tau : \mathfrak{M} \to S^1$ is a conformally invariant function on loop space. Precisely, τ is independent of*

(1) *the parametrization of the loop,*
(2) *the chosen unit normal field E, and*
(3) *the particular choice of Riemannian metric $\langle , \rangle$ on M representing a given conformal class.*

Moreover, τ represents the parallel transport around the loop $[\gamma]$ in its normal bundle $\perp\gamma$.

6.9 *Proof.* The definition shows that τ does not depend on the chosen parametrization of the loop. Changes introduced to the integrand by switching from one unit normal E_1 to another one E_2 are annihilated by the periodicity of exp. Finally, if we multiply the metric with e^{2u}, we need to replace E by, say, $\widetilde{E} = e^{-u}E$. Hence, the integrand does not change:

$$e^{2u}\langle \nabla_{e^{-u}T}(e^{-u}E), J(e^{-u}E)\rangle e^{u}\, \|\gamma_s\|\, ds \ = \langle \nabla_T E, JE\rangle\, \|\gamma_s\|\, ds.$$

The statement concerning the parallel transport follows from the fact that (E, JE) is an orthonormal frame for $\perp\gamma$. $\square$

6.10 The integral

$$\int_{S^1} \langle \nabla_T E, JE\rangle\, \|\gamma_s\|\, ds$$

is sometimes called the *total twist* of the unit normal field E (see (BW75)). Apart from being an example of a smooth function on the loop space of a conformal 3-manifold, the derivative $d\tau \in \Omega^1(\mathfrak{M})$ of the total torsion is intimately connected with the notion of conformal circles as the next theorem shows.

6.11 Theorem. *Let $X \in \Gamma(\perp\gamma)$ be a normal field representing a tangent vector $[X] \in T_{[\gamma]}\mathfrak{M}$ at some loop $[\gamma] \in \mathfrak{M}$. Then, the derivative in direction $[X]$ of the total torsion is given by*

$$d\tau([X]) = i\tau([\gamma]) \int_{S^1} \langle X, J\big((\mathrm{Ric}T)^{\perp} - H'\big)\rangle\, \|\gamma_s\|\, ds.$$

6.12 *Proof.* Consider a variation $(t, s) \mapsto \gamma(t, s)$ of the original loop with $\gamma_t = X\perp T$ for $t = 0$. Along the variation, choose some unit normal field E. Now we compute

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\langle \nabla_T E, JE\rangle\, \|\gamma_s\| &= \langle \nabla_{\gamma_t}\nabla_{\gamma_s} E, JE\rangle \\
&\quad + \langle \nabla_{\gamma_s} E, \nabla_{\gamma_t} T \times E\rangle + \underbrace{\langle \nabla_{\gamma_s} E, T \times \nabla_{\gamma_t} E\rangle}_{=0} \\
&= \langle \nabla_{\gamma_s}\nabla_{\gamma_t} E, JE\rangle \\
&\quad - \langle R(\gamma_s, \gamma_t)E, JE\rangle + \langle \nabla_{\gamma_s} E, \nabla_{\gamma_t} T \times E\rangle
\end{aligned}$$

From
$$\langle \nabla_{\gamma_s} E, \nabla_{\gamma_t} T \times E\rangle = \langle \gamma_t', JE\rangle\langle E, H\rangle\, \|\gamma_s\|$$
we deduce
$$\begin{aligned}\frac{d}{dt}|_{t=0}\langle \nabla_T E, JE\rangle\, \|\gamma_s\| = {}& \langle \nabla_{\gamma_s}\nabla_{\gamma_t} E, JE\rangle \\ &- \langle R(\gamma_s,\gamma_t)E, JE\rangle + \langle \gamma_t', JE\rangle\langle E, H\rangle\, \|\gamma_s\|\,.\end{aligned}$$
Concerning the first summand on the right hand side,
$$\langle \nabla_{\gamma_t} E, JE\rangle_s = \langle \nabla_{\gamma_s}\nabla_{\gamma_t} E, JE\rangle - \langle \gamma_t', E\rangle\langle E, JH\rangle\, \|\gamma_s\|\,.$$
By partial integration, we arrive at
$$\begin{aligned}\int_{S^1} \frac{d}{dt}|_{t=0}\langle \nabla_T E, JE\rangle\, \|\gamma_s\|\, ds = \int_{S^1} \Big(&\langle \gamma_t', \langle H, E\rangle JE - \langle H, JE\rangle E\rangle \\ &- \langle \gamma_t, R(E, JE)T\rangle\Big)\, \|\gamma_s\|\, ds.\end{aligned}$$
Using
$$\langle H, E\rangle JE - \langle H, JE\rangle E = JH$$
and
$$(\mathrm{Ric}\, T)^\perp = JR(E, JE)T$$
another application of partial integration finally yields
$$\int_{S^1} \frac{d}{dt}|_{t=0}\langle \nabla_T E, JE\rangle\, \|\gamma_s\|\, ds = \int_{S^1} \langle \gamma_t, J((\mathrm{Ric}\, T)^\perp - H')\rangle\, \|\gamma_s\|\, ds.$$
□

6.13 Corollary. *A loop $[\gamma]$ is a conformal circle if and only if it is a critial point of the total torsion τ, hence critical for the parallel transport in the normal bundle $\perp\gamma$.*

6.14 Conformal circles in higher dimensions. It is well known (Bes87, 1.116) that the Riemannian curvature tensor R of a manifold $(M, \langle,\rangle)$ with $\dim(M) = n \geq 4$ splits into a conformally invariant part, the Weyl tensor W, and a second part that is affected by conformal changes of the metric. One possible way to state this splitting is the following.
$$\begin{aligned}R(X,Y)Z = {}& W(X,Y)Z \\ &+ \frac{1}{n-2}\Big(\langle Z, \mathrm{Ric}X\rangle Y - \langle Z, \mathrm{Ric}Y\rangle X \\ &\qquad + \langle Z, X\rangle \mathrm{Ric}Y - \langle Z, Y\rangle \mathrm{Ric}X \\ &\qquad + \frac{\mathrm{scal}}{n-1}\big(\langle Z, Y\rangle X - \langle Z, X\rangle Y\big)\Big).\end{aligned}$$
In the above formula, Ric and scal denote the Ricci and scalar curvature of $(M, \langle,\rangle)$, respectively. In contrast to the above formula, the Riemannian curvature tensor of a 3-manifold is completely determined

by the manifold's Ricci tensor. If T is some unit vector and $X, Y \perp T$ we have

$$
\begin{aligned}
(R(X,Y)T)^{\perp} = &\ (W(X,Y)T)^{\perp} \\
&\underbrace{+\frac{1}{n-2}\Big(\langle(\mathrm{Ric}T)^{\perp}, X\rangle Y - \langle(\mathrm{Ric}T)^{\perp}, Y\rangle X\Big)}_{=: r_T(X,Y)}.
\end{aligned}
$$

Assume now that $\gamma : S^1 \to M$ is a conformal circle with unit tangent T which, at the same time, is a geodesic of $\langle , \rangle$ (we can always achieve this by lemma 6.4). We see that the property of being a conformal circle or not affects only the r_T-part, but not the Weyl tensor.

This shows that the situation in higher dimensions is much more complicated.

CHAPTER 7

Appendix

This appendix summarizes notational conventions as well as a couple of standard formulas from conformal geometry which are needed in the main text.

1. Symbols and Conventions

We list those symbols which appear most frequently in the main text and have a predefined meaning.

$(M, \langle , \rangle)$	smooth Riemannian manifold with Riemannian metric $\langle , \rangle$
∇	Levi-Civita connection of $(M, \langle ., . \rangle)$
R	Riemannian curvature tensor of $(M, \langle ., . \rangle)$
$\times$	cross product on TM for $\dim(M) = 3$
γ	smooth immersed loop in M
T	unit tangent vector of γ
H	mean curvature vector of a loop or hypersurface
$K(T)$	sectional curvature of the plane orthogonal to T for $\dim(M) = 3$
Ein	Einstein tensor considered as an endomorphism
Sch	Schouten tensor considered as an endomorphism
Ric	Ricci tensor considered as an endomorphism
ric$(., .)$	Ricci tensor considered as a bilinear form
scal	Scalar curvature
$\perp\gamma$	normal bundle of the loop γ, $\perp\gamma(s) = \{T(s)\}^\perp$ for $s \in S^1$
$\mathfrak{M}(S, M)$	space of immersed submanifolds of type S in M
$\langle\langle ., . \rangle\rangle$	L^2 scalar product on $\mathfrak{M}(S, M)$
$\nabla^\perp$	basic connection on $(\mathfrak{M}, \langle\langle ., . \rangle\rangle)$
∇^{LC}	Levi-Civita connection on $(\mathfrak{M}, \langle\langle ., . \rangle\rangle)$
∇^C	conformal connection on $(\mathfrak{M}, [\langle\langle ., . \rangle\rangle])$

Fraktur ($\mathfrak{M}$, $\mathfrak{N}$, ...) is used to denote infinite dimensional manifolds while roman type appears where the dimension is less than infinity.

2. Tools from Conformal Geometry

This section is meant to summarize some formulas from Riemannian and conformal geometry in a manner adapted to the application in the theory of loop spaces. A comprehensive introduction to the subject is (KP88).

7.1 Concerning the Riemannian curvature tensor R we employ the sign convention exhibited by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

If we change the metric according to $\langle , \rangle \mapsto \widetilde{\langle , \rangle} = e^{2u} \langle , \rangle$, where u is a smooth function with gradient $\mathrm{grad} u = U$, the Levi-Civita connection $\widetilde{\nabla}$ belonging to $\widetilde{\langle , \rangle}$ is given by

$$\widetilde{\nabla} = \nabla + B$$

with a symmetric bilinear form

$$B_X Y = \langle X, U \rangle Y + \langle Y, U \rangle X - \langle X, Y \rangle U.$$

This, in turn, yields a change in the Riemannian curvature tensor:

$$\begin{aligned}
\widetilde{R}(X,Y)Z = \; & R(X,Y)Z \\
& + \langle \nabla_X U, Z \rangle Y - \langle \nabla_Y U, Z \rangle X \\
& + \langle X, Z \rangle \nabla_Y U - \langle Y, Z \rangle \nabla_X U \\
& + \Big(\langle Y, U \rangle \langle Z, U \rangle - \langle Y, Z \rangle \|U\|^2 \Big) X \\
& + \Big(\langle X, U \rangle \langle Z, U \rangle - \langle X, Z \rangle \|U\|^2 \Big) Y \\
& + \Big(\langle X, U \rangle \langle Y, Z \rangle - \langle Y, U \rangle \langle X, Z \rangle \Big) U.
\end{aligned} \tag{7.1}$$

7.2 Curvature tensors on 3-manifolds. The main focus of this work is the investigation of loop spaces of three-dimensional Riemannian manifolds $(M, \langle , \rangle)$. Therefore, we will make explicit some simplifications special to dimension three.

In many situations, the cross product

$$\times : T_pM \times T_pM \to T_pM,$$

defined with respect to $\langle , \rangle$, turns out to be useful. With its help we can write down formulas for the Riemannian curvature, Einstein, and Schouten tensors of $(M, \langle , \rangle)$:

$$\begin{aligned}
R(X,Y)Z &= \Big(\mathrm{Ein}(X \times Y) \Big) \times Z \\
&= \Big(\big(\mathrm{Ric} - \frac{\mathrm{scal}}{2}\big)(X \times Y) \Big) \times Z \\
&= \Big(\big(\mathrm{Sch} - \frac{\mathrm{scal}}{4}\big)(X \times Y) \Big) \times Z.
\end{aligned}$$

7.3 Lemma. *Let* $\dim(M) = 3$. *For* $U \in \Gamma(TM)$, *define*

$$B_X Y = \langle X, U\rangle Y + \langle Y, U\rangle X - \langle X, Y\rangle U.$$

Then

$$B_X B_Y Z - B_Y B_X Z = \langle X \times Y, U\rangle U \times Z.$$

7.4 *Proof.* The definition of B implies

$$\begin{aligned} B_X B_Y Z - B_Y B_X Z = & \Big(\langle Z, U\rangle\langle Y, U\rangle - \langle Y, Z\rangle \|U\|^2 \Big) X \\ & - \Big(\langle Z, U\rangle\langle X, U\rangle - \langle X, Z\rangle \|U\|^2 \Big) Y \\ & + \Big(\langle Y, Z\rangle\langle X, U\rangle - \langle X, Z\rangle\langle Y, U\rangle \Big) U \\ = & - \langle U \times Y, U \times Z\rangle X + \langle U \times X, U \times Z\rangle Y \\ & + \langle X \times Y, U \times Z\rangle U. \end{aligned}$$

Assume $U \neq 0$ and set $Z^\perp = Z - \langle Z, \frac{U}{\|U\|}\rangle \frac{U}{\|U\|}$ in order to write

$$\begin{aligned} \langle U \times X, U \times Z\rangle Y - \langle U \times Y, U \times Z\rangle X &= \|U\|^2 \left(\langle Z^\perp, X\rangle Y - \langle Z^\perp, Y\rangle X \right) \\ &= \|U\|^2 Z^\perp \times (Y \times X) \\ &= \big(U \times (U \times Z)\big) \times (X \times Y) \\ &= \langle X \times Y, U\rangle U \times Z \\ &\quad - \langle X \times Y, U \times Z\rangle U. \end{aligned}$$

This proves the claim. □

7.5 Lemma. *Using a function* $u \in C^\infty(M)$ *with* $U = \operatorname{grad} u$ *change the Riemannian metric of the 3-manifold* $(M, \langle,\rangle)$ *to* $\widetilde{\langle,\rangle} = e^{2u}\langle,\rangle$ *and decorate all changed objects with "$\sim$". Then the new Riemannian curvature tensor* $\widetilde{R}$ *satisfies*

$$\widetilde{R}(X, Y)Z = R(X, Y)Z + \Big(X \times \nabla_Y U - Y \times \nabla_X U + \langle X \times Y, U\rangle U \Big) \times Z.$$

7.6 *Proof.* This formula follows by applying lemma 7.3 to the n-dimensional version given in equation 7.1. □

7.7 Let $(M, \langle,\rangle)$ be a Riemannian 3-manifold, $E \in TM$ some unit vector. Using a function $u \in C^\infty(M)$ with $U = \operatorname{grad} u$ change the Riemannian metric to $\widetilde{\langle,\rangle} = e^{2u}\langle,\rangle$ and decorate all changed objects with "$\sim$". Given a curve with unit tangent T we denote by $X^\perp$ the orthonormal projection of a vector X onto $\{T\}^\perp$, and by $Y' = (\nabla_T Y)^\perp$ the connection in the normal bundle. Then, using lemma 7.5, one can derive the following equations describing the behavior of several objects under conformal changes of the Riemannian metric on the given manifold.

$$
\begin{aligned}
\widetilde{\times} &= e^{u}\times \\
X^{\widetilde{\prime}} &= e^{-u}\Big(X' + \langle U,T\rangle X\Big) \\
X^{\widetilde{\prime\prime}} &= e^{-2u}\Big(X'' + \langle U,T\rangle X' + \big(\langle U,H\rangle + \langle \nabla_T U,T\rangle\big)X\Big) \\
\widetilde{H} &= e^{-2u}\Big(H - U^{\perp}\Big) \\
\widetilde{\mathrm{Ein}} &= e^{-2u}\Big(\mathrm{Ein} - \mathrm{Hess}u + \Delta u + \langle .,U\rangle U\Big) \\
\widetilde{K}(\widetilde{E}) &= e^{-2u}\Big(K(E) + \langle \nabla_E U,E\rangle - \Delta u - \langle E,U\rangle^2\Big) \\
\widetilde{\mathrm{scal}} &= e^{-2u}\Big(\mathrm{scal} - 4\Delta u - 2\left\|U\right\|^2\Big) \\
\widetilde{\mathrm{Ric}} &= e^{-2u}\Big(\mathrm{Ric} - \mathrm{Hess}u - \Delta u + \langle .,U\rangle U - \left\|U\right\|^2\Big) \\
\widetilde{\mathrm{ric}}(\widetilde{E},\widetilde{E}) &= e^{-2u}\Big(\mathrm{ric}(E,E) - \langle \nabla_E U,E\rangle - \Delta u + \langle E,U\rangle^2 - \left\|U\right\|^2\Big) \\
\widetilde{\mathrm{Sch}} &= e^{-2u}\Big(\mathrm{Sch} - \mathrm{Hess}u + \langle .,U\rangle U - \tfrac{1}{2}\left\|U\right\|^2\Big)
\end{aligned}
$$

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