Surfaces associated to a space curve: A new proof of Fabricius-Bjerre's Formula

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Abstract

In 1962 Fabricius-Bjerre [16] found a formula relating certain geometric features of generic closed plane curves. Among bitangent lines, i.e., lines that are tangent to a curve at two points, distinguish two types: external - if the arcs of tangency lie on the same side of the line - and internal otherwise. Then, there is the following equality: the number of external bitangent lines equals the sum of the number of internal bitangent lines, the number of crossings and half of the number of inflection points of the plane curve.

Two different proofs of the formula followed, by Halpern [20] in 1970 and Banchoff [6] in 1974, as well as many generalizations. Halpern's approach is used here to provide a generalization from bitangent lines to parallel tangents pairs.

The main result of this thesis is a new proof of Fabricius-Bjerre's Theorem, which uses new methods. The idea is to view a plane curve as a projection of a space curve. The proof establishes a connection between the generic plane curves and the projections of a certain class of generic closed space curves. For a closed space curve three surfaces are constructed with maps to the sphere of projection directions. These maps encode the information of the variables appearing in the Fabricius-Bjerre formula for each projection. The surfaces are glued together to form a closed surface with a naturally defined continuous map to the sphere. The degree of that map turns out to be equal to the expression arising when all variables of the Fabricius-Bjerre formula (for a plane curve which is a projection of the space curve) are put on one side of the equality. Using methods similar to those of V.I. Arnold (used to define $J^{\pm}$ and St invariants), it is proved that the degree of the constructed map is zero for any generic space curve. An equivalence between the latter and Fabricius-Bjerre's Theorem is established. This gives a new proof of the theorem.

The methods developed in this thesis have numerous possible applications. First, the integral version of the Fabricius-Bjerre formula is obtained which states that the average bitangency number equals the sum of the average crossing number and the average absolute torsion of a generic space curve. Another application is a new proof

of the formula of Banchoff [5] and Aicardi [2] that expresses the self-linking number of a closed space curve as a sum of the writhe of a diagram and half of the signs of torsion at points of the space curve that project to inflection points in the diagram. It seems that using the methods of this work, nearly any formula relating some geometric features of a plane curve, as for example, the generalization to parallel tangents pairs, can be proved. Also, the generalization of the formula by Fabricius-Bjerre [17] to curves with cusps finds a new geometric interpretation with the new approach.

To Krzyś and Marysia

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Introduction

Let α be a closed plane curve that is generic (in a sense to be made precise later). The generic curve has a finite number of bitangent lines, i.e., lines that are tangent to α at two points. Among all bitangent lines distinguish between two types: *external* and *internal*, i.e., those where the parts of α close to the points of tangency lie on the

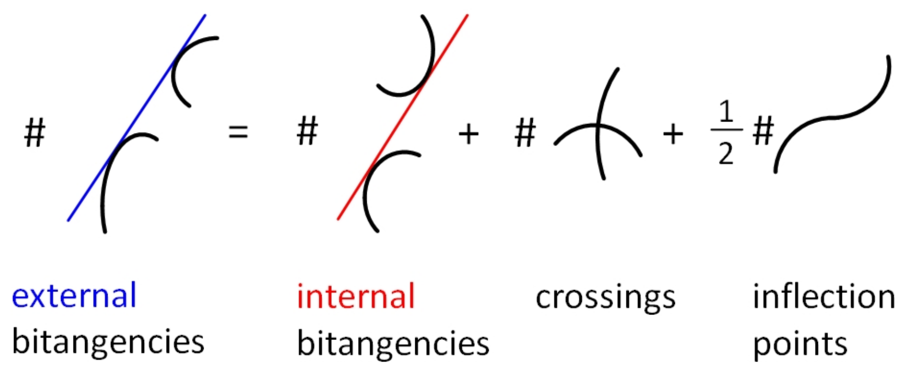


Figure 1: The Fabricius Bjerre formula.

same side or on the opposite sides of the bitangent line, respectively. Denote by ext and int their numbers respectively. Moreover, denote by cr the number of crossings and infl the number of inflection points of α. Then the formula of Fabricius-Bjerre is

$$\mathrm{ext} = \mathrm{int} + \mathrm{cr} + \frac{1}{2}\,\mathrm{infl}\,.$$

This can be graphically represented as in Figure 1. Figure 2 shows an example.

There are three known proofs of the formula each of which uses different methods. Indisputably, the credit goes to Fabricius-Bjerre [16] who as first introduced the relation in 1962. He obtained the equality by counting intersections between the curve and rays tangent to the curve. In 1970 Halpern [20] gave another much more rigorous proof of the theorem. In his approach some of the geometric features counted by the formula appear as zeros of a certain vector field associated to the torus of pairs of

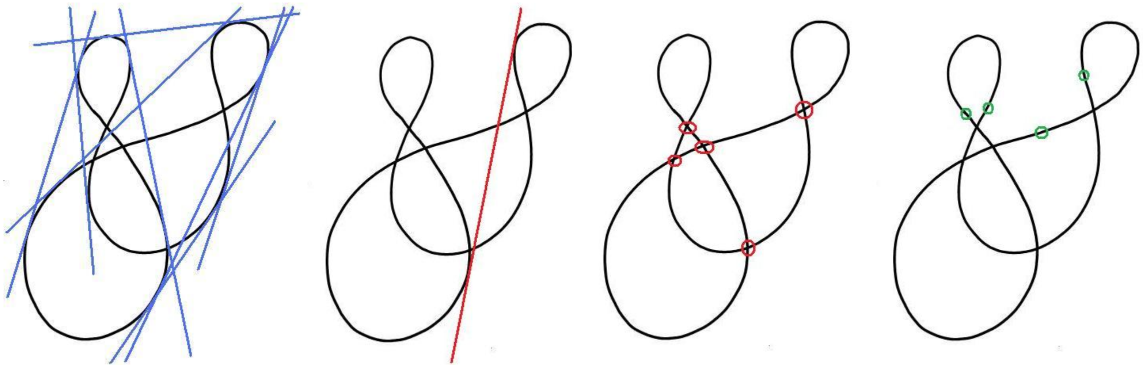


Figure 2: A curve with 8 external and 1 internal bitangent pairs, 5 crossings and 4 inflection points.

points on the curve. Integration of the vector field along suitable paths enclosing its zeros gives the formula. In 1974 Banchoff [6, 7] proved a polygonal version of the theorem and suggested a proof for the smooth case using similar methods. He uses a deformation argument. He first shows that the Fabricius-Bjerre formula holds for one curve. Then he shows that any curve can be obtained from the first one through certain deformations, which preserve the formula.

A number of generalizations followed. Fabricius-Bjerre [17] extended his own theorem to curves with cusps. Weiner [39] gave a generalization to spherical curves and applied it to the tangent indicatrix of a space curve. As a result he obtained a relation between geometric features of a space curve including its torsion-vanishing points, pairs of parallel osculating planes and pairs of parallel or anti-parallel tangent vectors. Pignoni [29] generalized the cusp version formula to curves in the real projective plane. His formula however requires a base point on a curve. Thompson [36] provided a generalization of the cusp version formula to the real projective plane without using a base point. Using duality in the real projective plane, she obtained a new formula relating further geometric features of a closed curve in the real projective plane. Ferrand [18] established a relation between Fabricius-Bjerre formula and Arnold's J^+ and J^- invariants.

There are several results in this work:

Result 1 A generalization of Fabricius-Bjerre's Theorem to parallel tangents pairs (Theorem 2.3).

Result 2 A construction (for each generic closed space curve) of the *bitangency* surface with the *bitangency* map to the unit sphere in Euclidean 3-space. The map carries information about bitangent lines of all planar projections of the space curve (Chapter 6.3).

Result 3 A proof of the fact that there are three surfaces associated to a generic closed space curve, which glued together give a closed surface (Theorem 7.4). Moreover, there is a natural map from the closed surface to the sphere, whose degree written in terms of a generic plane curve (obtained through a projection of the space curve) is equal to $-\operatorname{ext} + \operatorname{int} + \operatorname{cr} + \frac{1}{2}\operatorname{infl}$ (Theorem 7.9).

Result 4 An integral Fabricius-Bjerre formula which says that the average bitangency number equals the average crossing number plus the average of the absolute torsion of a generic space curve (Corollary 7.10).

Result 5 A new proof of a formula of Banchoff [5] and Aicardi [2] for the self-linking number of a closed space curve expressed in terms of a knot diagram. The formula is: the self-linking number equals the sum of the writhe of the diagram and half of the sum of the signs of torsion at the points on the space curve that are projected to inflection points (Chapter 8).

Result 6 A proof of the fact that the degree of the map from Result 3 is equal to zero for any generic space curve and the equivalence between this fact and Fabricius-Bjerre's Theorem (Theorem 11.4).

This work is divided into three parts. Part I is dedicated to closed plane curves. In Chapter 1 the formulation of Fabricius-Bjerre's Theorem by Halpern [20] is presented. Section 1.2 describes the three known proofs due to Fabricius-Bjerre [16], Halpern [20] and Banchoff [6]. In Section 1.3 three selected generalizations: the cusp version due to Fabricius-Bjerre [17], a spherical version of the formula due to Weiner [39] and a formula of Thompson [36] are sketched. In Chapter 2 a generalization of the Fabricius-Bjerre formula to parallel tangents pairs (Theorem 2.3, Result 1) is presented, which resulted from joint work with Jason Cantarella. Chapter 3 introduces Arnold invariants and a connection with the Fabricius-Bjerre formula established by Ferrand [18].

Part II is dedicated to closed space curves and develops the idea of viewing all orthogonal projections of a space curve as a 2-parameter family of plane curves. In particular, to each point on the unit sphere of projection directions, a projected plane curve is to be associated. Chapter 4 provides the reader with necessary preliminary material from theory of closed space curves and defines the class of generic space curves that project to generic plane curves. Extending the idea of Adams et al. [1] in the polygonal case, the author defines in Chapter 5 three curves on the unit sphere (in 3-space) for each generic space curve. These three curves are related to the three

Reidemeister moves in the sense that they capture singular projections (of the space curve) whose singularities are those observed in the Reidemeister moves. The most attention is dedicated to the II curve as there is very little known about it. Further, in Chapter 6 for each generic space curve three surfaces are constructed: the crossing surface, the inflection surface and the bitangency surface and to each of the surfaces a map to the sphere is associated. Each of the three surfaces, as their names suggest, capture the geometric features (of the 2-parameter family of projected plane curves) that are counted in the Fabricius-Bjerre formula (Result 2). Chapter 7 describes how to cut and glue the surfaces to obtain a closed surface (Theorem 7.4) with a map, whose degree is related to the Fabricius-Bjerre formula (Theorem 7.9, Result 3) and provides the integral version the Fabricius-Bjerre formula (Corollary 7.10, Result 4). Chapter 8 shows how to use the methods developed in this thesis to prove the formula of Banchoff [5] and Aicardi [2] for the self-linking number (Result 5).

Finally Part III is devoted to analysis in the infinite dimensional space of smooth curves $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$. Chapter 9 provides the necessary preliminaries: Whitney C^∞-topology and transversality theory in Section 9.1 as well as the theory of Fréchet spaces in Section 9.2. Chapter 10 analyzes the two spaces: the set of generic closed space curves $\mathcal{G}$ and its complement in $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ - the space of non-generic closed space curves $\mathcal{NG}$. Section 10.1 proves that $\mathcal{G}$ is an open and dense subset of $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$. Section 10.2 shows that the space $\mathcal{NG}$ forms a singular hypersurface in $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ and provides description of codimension 1 strata. Finally, the new proof of Fabricius-Bjerre's Theorem is presented in Chapter 11. First, Section 11.1 shows that any generic plane curve can be obtained through a projection of a generic space curve. This, together with results of Chapter 7, establishes equivalence between Fabricius-Bjerre's Theorem and the fact that the degree of the map associated to the closed surface (obtained from gluing) is zero for any generic space curve (Section 11.2). The latter is proved (Theorem 11.4, Result 6) using a method similar to that of Arnold. The strategy is the following. First, a fixed generic space curve is found, for which the degree can be explicitly computed to be zero. Then, for any curve on the codimension 1 stratum of the singular hypersurface, a path (i.e. a 1-parameter family of curves) is constructed that joins two components of $\mathcal{G}$ separated by the hypersurface around this point. Along this path the degree is the same for each generic member of the family. Hence, since any element of $\mathcal{G}$ can be connected with the fixed curve via a path that preserves the degree, the degree is equal to zero on the whole space $\mathcal{G}$. By the established equivalence, this gives a new proof of Fabricius-Bjerre's Theorem.

Some observations and final remarks are collected in Chapter 12.

Part I

Closed plane curves

Chapter 1

The Fabricius-Bjerre formula

This chapter is devoted to Fabricius-Bjerre's Theorem. Section 1.1 presents Halpern's formulation of the Fabricius-Bjerre formula. Sketches of the various proofs are presented in Section 1.2. Selected further versions and generalizations are summarized in Section 1.3.

1.1 Formulation by Halpern

Denote by $\mathbb{S}^1$ the one dimensional sphere obtained as $\mathbb{R}/L\mathbb{Z}$, where L is some positive integer. Let $\alpha : \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth immersion of a circle into the plane $\mathbb{R}^2$. Let $T = \frac{\alpha'}{\|\alpha'\|}$, $N = JT$ and $\kappa = \langle N, T' \rangle$ be the unit tangent vector, the principal normal vector and the signed curvature of α, respectively, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the rotation matrix corresponding to the rotation by $\frac{\pi}{2}$. Let square brackets denote the determinant, i.e.,

$$[u, v] := Det(u\ v) = \langle u, Jv \rangle \quad \text{for } u, v \in \mathbb{R}^2 \text{ or}$$
$$[u, v, w] := Det(u\ v\ w) = \langle u, v \times w \rangle \quad \text{for } u, v, w \in \mathbb{R}^3.$$

Definition 1.1. (i) An unordered pair $\{s, t\} \subset \mathbb{S}^1$ with $s \neq t$ is called a *double point* or a *crossing* if $\alpha(s) = \alpha(t)$. A double point is called *regular* if the intersection is transverse, i.e., if $[\alpha'(s), \alpha'(t)] \neq 0$. Denote by $\mathrm{cr} = \mathrm{cr}(\alpha)$ the number of regular double points of α.

(ii) An unordered pair $\{s, t\} \subset \mathbb{S}^1$ with $s \neq t$ is called a *bitangent pair* if it is not a double point and the tangent lines at $\alpha(s)$ and $\alpha(t)$ coincide, i.e.,

$$\alpha(t) \neq \alpha(s) \text{ and } [\alpha'(s), \alpha'(t)] = [\alpha'(s), \alpha(s) - \alpha(t)] = 0.$$

A bitangent pair is called *regular* if $\kappa(s)\kappa(t) \neq 0$ or equivalently if

$$[\alpha'(s), \alpha''(s)] \neq 0 \text{ and } [\alpha'(t), \alpha''(t)] \neq 0.$$

A regular bitangent pair is called *external* if $\kappa(s)N(s)$ and $\kappa(t)N(t)$ point in the same direction and *internal* otherwise. Denote by $\mathrm{ext} = \mathrm{ext}(\alpha)$ and $\mathrm{int} = \mathrm{int}(\alpha)$ the numbers of external, respectively internal, regular bitangent pairs.

(iii) A point $\{s\}$ is called an *inflection point* of α if $\kappa(s) = 0$. An inflection point is called *regular* if $\kappa'(s) \neq 0$ or equivalently if $[\alpha'(s), \alpha'''(s)] \neq 0$. The number of regular inflection points is denoted by $\mathrm{infl} = \mathrm{infl}(\alpha)$.

Definition 1.2. Let $\alpha : \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth immersion. The closed plane curve α is called *generic in the sense of the Fabricius-Bjerre* or *FB-generic* for short if all double points, bitangent pairs and inflection points are regular.

Theorem 1.3 (Fabricius-Bjerre [16], Halpern [20])**.** *Let $\alpha : \mathbb{S}^1 \to \mathbb{R}^2$ be a FB-generic curve. Then* cr*,* ext*,* int *and* infl *are finite and the following relation holds*

$$\mathrm{ext} = \mathrm{int} + \mathrm{cr} + \frac{1}{2}\,\mathrm{infl}\,. \tag{1.1}$$

Throughout this thesis the above theorem will be referred to as *Fabricius-Bjerre's Theorem* and the relation (1.1) will be called the *Fabricius-Bjerre formula.*

1.2 Proofs

In this section the three known proofs of the Fabricius-Bjerre formula are sketched: the original proof of Fabricius-Bjerre as well as proofs due to Halpern and Banchoff. The fact that the variables involved in the formula are finite will be presented only once in the context of Halpern's proof, in Lemma 1.6.

1.2.1 Fabricius-Bjerre's proof

Sketch of Fabricius-Bjerre's proof of Theorem 1.3. Orient the curve α. Partition the set of external and internal bitangent lines into three subsets according to how the tangent vectors at the points of tangency are directed. They may point in the same direction (type 1) or in the opposite directions: either towards (type 2) or away from each other (type 3). According to these three possibilities, distinguish three types of external and internal bitangent lines and denote their numbers by e_1, e_2, e_3 and i_1, i_2, i_3, respectively. If a bitangent line is tangent to the curve at more than two points, it is to be counted accordingly with multiplicity.

Let a point p traverse the curve $\alpha(\mathbb{S}^1)$ according to its orientation and consider the positive tangent ray based at p. Consider the points where this ray intersects the curve. This number may change precisely when p passes through one of the occurrences involved in the formula, i.e., at a crossing, at a bitangent line or at an inflection point. For example when the ray passes through an external bitangent line of type 2 (see Figure 1.1) the number of intersection points increases by 2 and

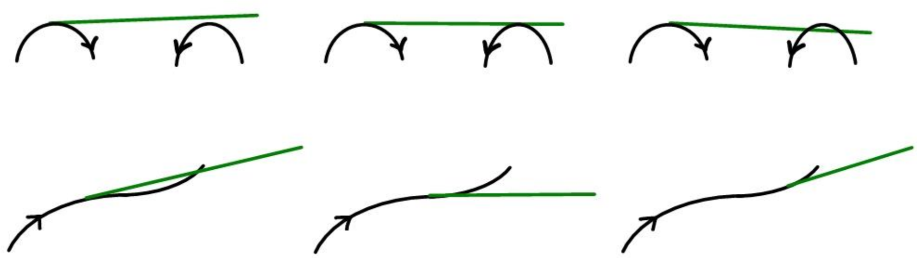


Figure 1.1: The positive tangent ray passing through an external bitangent line of the second type (top) and through an inflection point (bottom).

when the point p passes through an inflection point the number of intersection points decreases by 1. When the tangent ray makes a full turn around the curve, the total losses in the number of points common to the curve and the ray should compensate the total gains. Carefully keeping track of all the changes, one arrives at the formula

$$2e_1 + 4e_2 = 2i_1 + 4i_2 + 2\,\mathrm{cr} + \mathrm{infl}\,. \tag{1.2}$$

The same procedure applied to the negative tangent ray leads to the following equation

$$2e_1 + 4e_3 = 2i_1 + 4i_3 + 2\,\mathrm{cr} + \mathrm{infl}\,. \tag{1.3}$$

Adding the two equations gives

$$4(e_1 + e_2 + e_3) = 4(i_1 + i_2 + i_3) + 2\,\mathrm{cr} + \mathrm{infl}\,.$$

$\square$

Instead of adding (1.2) and (1.3) one can subtract them and obtain a formula that can graphically be represented by Figure 1.2.

Corollary 1.4. *Let $\alpha : \mathbb{S}^1 \to \mathbb{R}^2$ be FB-generic. Then the following relations hold*

$$e_2 + i_3 = i_2 + e_3. \tag{1.4}$$

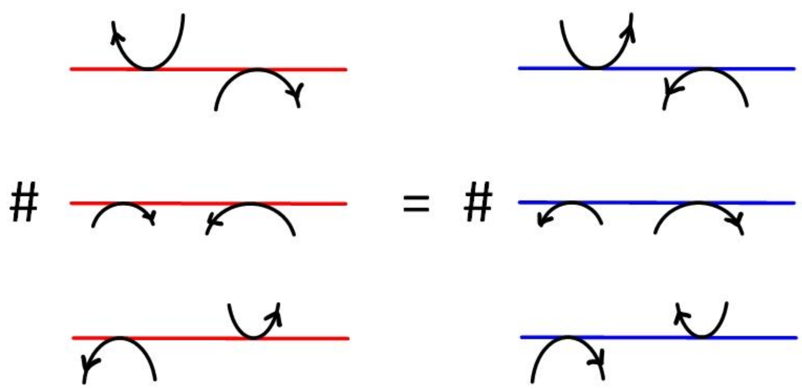


Figure 1.2: The bitangency formula (1.4).

1.2.2 Halpern's proof

Denote by Δ the diagonal of $\mathbb{S}^1 \times \mathbb{S}^1$ and let $W : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth vector field with finitely many zeros inside $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$. Then there exists an $\varepsilon > 0$ such that the closed strip around the diagonal Δ enclosed by the loops

$$\gamma^{\pm} : \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta, \quad s \mapsto \pm(s, s + \varepsilon)$$

does not contain zeros of W other than at Δ. Let $S := \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta \setminus W^{-1}(0)$ and consider the map $f : S \to \mathbb{S}^1$, given by $s \mapsto \frac{W(s)}{\|W(s)\|}$. Let γ_p be a small circle (oriented counterclockwise) around $p \in W^{-1}(0) \setminus \Delta$ that does not enclose any other zero of W. Define the *index* of W at p denoted by $\operatorname{ind}_p W$ as the degree of $f \circ \gamma_p$. Denote by $J_p W$ the Jacobian matrix of W at p.

Lemma 1.5. *Let $W : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth vector field with only finitely many zeros in $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$ and with $\det(J_p W) \neq 0$ for each $p \in W^{-1}(0) \setminus \Delta$. Let $\varepsilon > 0$,* *curves* $\gamma^{\pm}$ *and the map* f *be chosen as above. Then*

$$\deg(f\circ\gamma^{+})+\deg(f\circ\gamma^{-})=\sum_{p\in W^{-1}(0)\setminus\Delta}\operatorname{ind}_p W, \tag{1.5}$$

where $\operatorname{ind}_p W=\operatorname{sign}(\det(J_pW))$.

Proof. Let N denote the number of zeros of W in $\mathbb{S}^1\times\mathbb{S}^1\setminus\Delta$. Then, topologically, $S=\mathbb{S}^1\times\mathbb{S}^1\setminus\Delta\setminus W^{-1}(0)$ is a sphere with $N+2$ punctures. Any loop in S maps via f to a loop in $\mathbb{S}^1$ with a certain degree. The map assigning to a loop this degree is the induced map on homology groups $f^*: H_1(S)\to\mathbb{Z}\approx H_1(\mathbb{S}^1)$. Hence, homologous cycles have the same degree. For each $p\in W^{-1}(0)\setminus\Delta$ any loop around p, that does not enclose any other zero of W, is homologous in S to γ_p. In particular,

$$\gamma^{+}+\gamma^{-}-\sum_{p\in W^{-1}(0)\setminus\Delta}\gamma_p$$

is null homologous. Hence,

$$\deg(f\circ\gamma^{+})+\deg(f\circ\gamma^{-})=\sum_{p\in W^{-1}(0)\setminus\Delta}\operatorname{ind}_p W.$$

It remains to check that under the assumption $\det(J_pW)\neq 0$

$$\operatorname{ind}_p W=\operatorname{sign}(\det(J_pW))=\pm 1.$$

The latter is a standard result and can be found for example in [8, 7.7.8 Proposition]. More precisely, let $p\in W^{-1}(0)\setminus\Delta$ satisfy $J_pW\neq 0$. Then, by the Inverse Mapping Theorem, W is locally a diffeomorphism on a neighborhood U of p. Hence, there is an isotopy F between W and id_U-p if W preserves orientation, i.e., if $J_pW>0$. Each F_t can be considered as a vector field having an isolated singularity at p. By the invariance of the degree under homotopy

$$\operatorname{ind}_p W=\operatorname{ind}_p F_0=\operatorname{ind}_p F_1=\operatorname{ind}_p(\mathrm{id}_U-p)=1.$$

Analogously, if W reverses orientation, i.e., when $J_pW<0$, there is an isotopy between W and $-\mathrm{id}_U-p$ and $\operatorname{ind}_p W=-1$. $\square$

The actual proof will now reduce to finding a suitable vector field and applying Lemma 1.5.

Sketch of Halpern's proof of Theorem 1.3. Consider the vector field $W : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^2$, given by

$$(s,t) \mapsto \begin{pmatrix} [\alpha'(s), \alpha(s) - \alpha(t)] \\ [\alpha'(t), \alpha(t) - \alpha(s)] \end{pmatrix}. \tag{1.6}$$

The vector field W has zeros precisely at bitangent pairs, double points and along the diagonal Δ. By Lemma 1.6 below, the variables ext, int, infl and cr are finite. In particular, the vector field W satisfies the assumptions of Lemma 1.5. The left-hand side of (1.5) turns out to be $-\,\mathrm{infl}$. The zeros have the following indices: $+1$ for a double point and internal bitangent pair, and -1 for an external bitangent pair. Taking into account that each bitangent pair and a double point occurs as an ordered pair twice in $\mathbb{S}^1 \times \mathbb{S}^1$ the equation (1.5) reads as

$$-\,\mathrm{infl} = 2(-\,\mathrm{ext} + \mathrm{int} + \mathrm{cr}).$$

The result follows from Lemma 1.5. $\square$

Lemma 1.6. *The vector field W defined by (1.6) has finitely many zeros inside $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$. In particular, the variables* ext, int, infl *and* cr *are finite.*

Proof. The curve α defining W has only regular inflection points which are isolated zeros of the curvature function and hence, infl is finite. By an elementary computation (see [8]), genericity of α implies that the Jacobian of W at any $(s,t) \in W^{-1}(0) \setminus \Delta$ is non-zero. Hence, the zeros of W, not in the diagonal, are isolated. Suppose that there are infinitely many zeros in $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$. Then, there is a convergent subsequence in $W^{-1}(0) \setminus \Delta$ with a limit point lying on the diagonal. But this cannot happen by regularity conditions of α. Each point on the diagonal corresponds to a point s on the curve α either with $\kappa(s) \neq 0$ or with $\kappa(s) = 0 \neq \kappa'(s)$, which clearly are not limit points of a sequence of regular crossings or regular bitangent pairs. $\square$

Remark. Berger and Gastiaux [8] give a wrong argument for the proof of Lemma 1.6. They reason that W has finitely many zeros, since all zeros are isolated on the compact set $\mathbb{S}^1 \times \mathbb{S}^1$. In fact only the zeros on the open set $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$ are isolated.

The method of Halpern turns out to be very useful in obtaining different formulas.

Example 1.7. The formula (1.4) is obtained from Lemma 1.5 when applied to the vector field $W : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^2$ given by

$$(s,t) \mapsto \begin{pmatrix} [\alpha'(s), \alpha(s) - \alpha(t)] \\ [\alpha'(s), \alpha'(t)] \end{pmatrix}.$$

In this case, the left-hand side of (1.5) vanishes and the indices of the zeros of W give the result.

Example 1.8. Halpern [20] also found a formula that involves two plane curves instead of one (see also [7]). Suppose $\alpha_1, \alpha_2 : \mathbb{S}^1 \to \mathbb{R}^2$ are two plane curves satisfying some genericity conditions (to be determined as an exercise). Then Halpern's method can be applied to the vector field $W : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^2$ given by

$$(s,t) \mapsto \begin{pmatrix} [\alpha_1'(s), \alpha_1(s) - \alpha_2(t)] \\ [\alpha_2'(t), \alpha_2(t) - \alpha_1(s)] \end{pmatrix}.$$

As a result one obtains the following equivalence

$$\operatorname{ext}(\alpha_1, \alpha_2) = \operatorname{cr}(\alpha_1, \alpha_2) + \operatorname{int}(\alpha_1, \alpha_2),$$

where $\operatorname{ext}(\alpha_1, \alpha_2)$ and $\operatorname{int}(\alpha_1, \alpha_2)$ is the number of external and internal bitangent pairs respectively and $\operatorname{cr}(\alpha_1, \alpha_2)$ is the number of double points such that one point belongs to α_1 and the other to α_2.

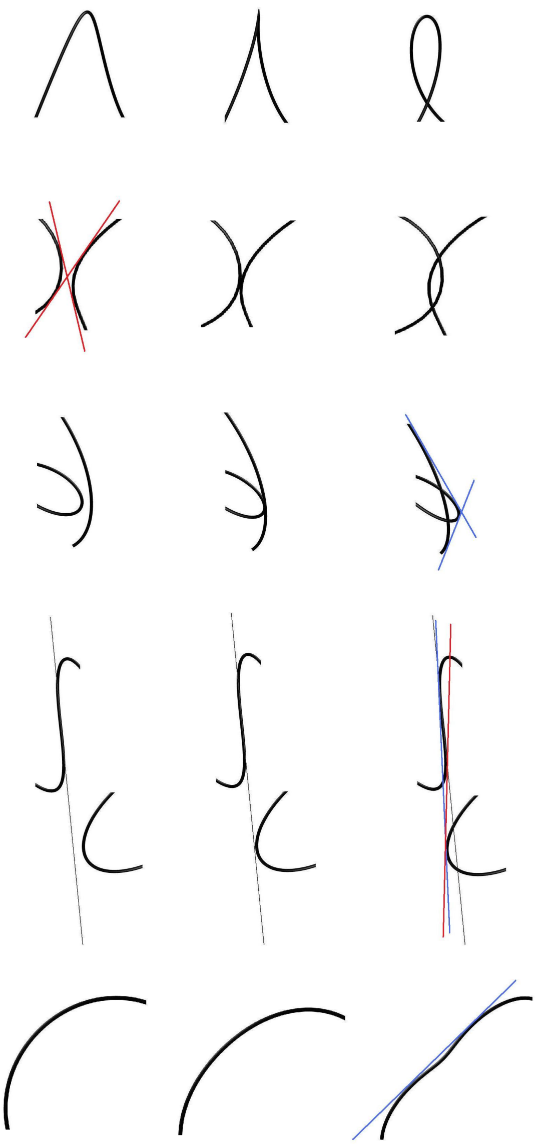
1.2.3 Banchoff's proof

Thomas Banchoff proved in [6] the polygonal version of the Fabricius-Bjerre formula using a deformation argument. In [7] he remarks that using similar methods the smooth version of the theorem can be proved. In fact, in the smooth case a lot more work is required using the deformation method since the space of smooth curves is infinite dimensional in contrast to the finite-dimensional space of polygonal closed curves. Below an outline of the smooth analog of the Banchoff's proof is given.

Sketch of the proof of Theorem 1.3 using Banchoff's method. Consider an arbitrary FB-generic curve α and a parametrization c of some convex FB-generic curve, e.g., a circle. Trivially, the Fabricius-Bjerre formula holds for c since all the variables ext, int, cr and infl are zero. Deform c into α via so called *generic deformation*, i.e., such that at each intermediate state of the homotopy the curve is FB-generic except of a finite number of times, when the curve experience one of the singularities given by Figure 1.3. In each of the five singular cases the expression

$$\operatorname{ext} - \operatorname{int} - \operatorname{cr} - \frac{1}{2} \operatorname{infl} \tag{1.7}$$

is compared before and after the move and it can be checked (see the table in Fig-



	Δ ext	Δ int	Δ cr	Δ infl
a)	0	0	1	-2
b)	0	-2	2	0
c)	2	0	2	0
d)	1	1	0	0
e)	1	0	0	2

Figure 1.3: Possible moves that change the variables in the Fabricius-Bjerre formula.

ure 1.3) that (1.7) indeed does not change. Summarizing, the formula (1.7) holds for one curve and it is preserved during any generic deformation. Hence, it holds for all smooth FB-generic closed plane curves. □

Further discussion and more details of this idea of the proof can be found in the chapter on Arnold's invariants in Section 1.2.3.

1.3 Further versions and generalizations

Below selected versions of the Fabricius-Bjerre formula are presented. Each of the formulas may have different genericity conditions imposed on the curve. These conditions will generally not be described. For a more detailed treatment the reader is asked to consult the references. Note for example the difference between a bitangent line and a bitangent pair. If a line is tangent to a curve at many points then this line will be counted with multiplicity.

The cusps version of Fabricius-Bjerre

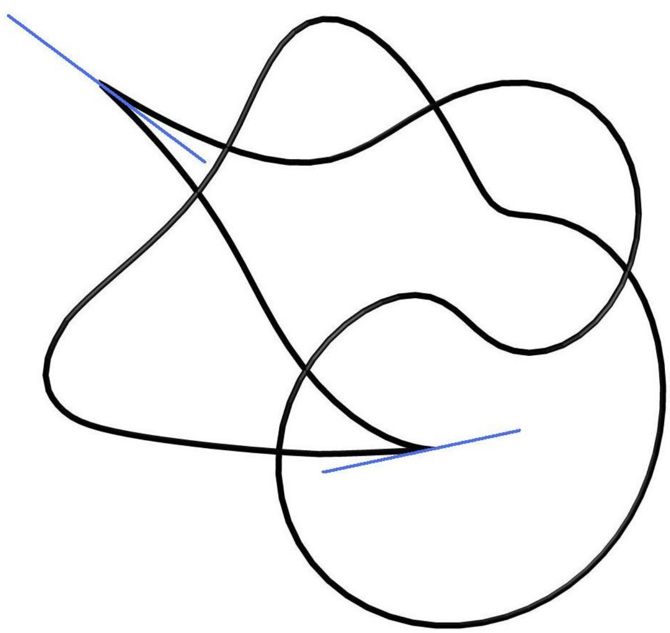


Figure 1.4: An example of a curve with cusps.

Roughly speaking, let α be a generic closed plane curve smooth everywhere except at a finite number of points, where it has cusps of the two types shown in Figure 1.5.

An example of such a curve can be found in Figure 1.4. Let c_1 be the number of cusps of the first type and c_2 the number of cusps of the second type. Let cr, infl, ext and int have the same meaning as previously. Consider additionally lines that are tangent to α and pass through a cusp and lines through two cusps. Distinguish between external and internal such lines in the obvious way. Let ext* denote the number of external and int* of internal lines of that new kind. Then the more general formula of Fabricius-Bjerre [17] is

$$\operatorname{ext} + \operatorname{ext}^* = \operatorname{int} + \operatorname{int}^* + \operatorname{cr} + c_1 + \frac{1}{2}(\operatorname{infl} + c_2). \tag{1.8}$$

The proof uses the same idea of counting the intersection points of tangent rays with the curve itself. More details, in particular the genericity requirements for α, can be found in [17]. Some remarks on this formula can be found in Chapter 12.

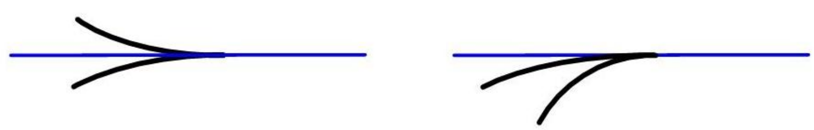


Figure 1.5: Cusp of type 1 (left) and of type 2 (right).

Weiner's generalization to spherical curves and a formula for space curves

Let $c : \mathbb{S}^1 \to \mathbb{S}^2$ be a generic smooth immersion of a circle $\mathbb{S}^1$ into the unit 2-sphere $\mathbb{S}^2$ (for genericity conditions see [39]). Let cr be the number of double points of c. An inflection point of the spherical curve c is understood as a point where the geodesic curvature of c vanishes. Let infl denote the number of such points. A bitangent line can be generalized to a great circle tangent to c at two points. The meaning of external and internal carries over in the obvious way. Let ext and int denote the numbers of external (respectively internal) bitangent great circles. Additionally, let a be the number of so called antipodal pairs of c, i.e., pairs $\{s, t\} \in \mathbb{S}^1$ with $c(s) = -c(t)$. The genericity conditions guarantee that all five variables are finite and the relation of Weiner [39] is

$$\operatorname{ext} - \operatorname{int} = \operatorname{cr} - a + \frac{1}{2}\operatorname{infl}. \tag{1.9}$$

Remark. An observation of the author is that one can obtain Weiner's formula (1.9) using Halpern's method. In particular, one applies Lemma 1.5 to the vector field

$W : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^2$ given by

$$(s,t) \mapsto \begin{pmatrix} [c'(s), c(s), c(t)] \\ [c'(t), c(t), c(s)] \end{pmatrix}.$$

This way it is easy to recover the genericity conditions needed for a curve to satisfy Weiner's formula (1.9) by making sure that Jacobian at every zero of W is non-degenerate.

Weiner's spherical formula is particularly useful when applied to the tangent indicatrix of a closed space curve. Suppose $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ is a smooth closed space curve which is arc-length parametrized. Then $\gamma' : \mathbb{S}^1 \to \mathbb{S}^2$ is the tangent indicatrix of γ. It is understood that γ is generic if γ' is generic as a spherical curve. The formula (1.9) holds for γ'. Hence, a reinterpretation of the variables in terms of γ will give a new formula. In fact, the correspondence is

a double point of γ'	$\longleftrightarrow$	a pair of parallel tangent vectors of γ pointing in the same direction
an antipodal pair of γ'	$\longleftrightarrow$	a pair of parallel tangent vectors of γ pointing in the opposite directions
an inflection point of γ'	$\longleftrightarrow$	a torsion-vanishing point of γ
a bitangent great circle	$\longleftrightarrow$	a pair of parallel osculating planes of γ.

The two types of bitangent great circles depend on how the curve γ passes through the two parallel osculating planes in question. External (internal) corresponds to a pair of parallel osculating planes where γ passes through each of them in the same (opposite) direction and the parallel osculating planes are called concordant (disconcordant).

Weiner's formula (1.9) then translates to the new relation

$$t - s = d - a + \frac{1}{2}i, \tag{1.10}$$

where

$$
\begin{aligned}
i &= \text{ the number of vertices of } \gamma,\\
d &= \text{ the number of pairs of directly parallel tangents of } \gamma,\\
a &= \text{ the number of pairs of oppositely parallel tangents of } \gamma,\\
t &= \text{ the number of concordant parallel osculating planes of } \gamma,\\
s &= \text{ the number of disconcordant parallel osculating planes of } \gamma.
\end{aligned}
$$

Remark. Again it is possible to recover the formula of Weiner for space curves (1.10) using Halpern's method. Apply Lemma 1.5 to the vector field $W : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^2$ given by

$$(s,t) \mapsto \begin{pmatrix} [\gamma''(s), \gamma'(s), \gamma'(t)] \\ [\gamma''(t), \gamma'(t), \gamma'(s)] \end{pmatrix}.$$

Thompson's generalizations to curves in real projective plane

Abigail Thompson generalized in [36] the cusps formula of Fabricius-Bjerre (1.8) to generic closed curves in the real projective plane $\mathbb{R}P^2$ when restricted to the cusps of type 1 only. Her formula contains some additional terms which vanish when specialized to $\mathbb{R}^2$ by considering curves lying in a small disk in $\mathbb{R}P^2$. Through a duality of $\mathbb{R}P^2$ she obtained another formula. For simplicity only the specialized version, i.e., the $\mathbb{R}^2$ version of the dual formula, is presented here.

Let $\alpha : \mathbb{S}^1 \to \mathbb{R}^2$ be a generic curve with cusps of type 1 (see [36] for genericity conditions). Call a pair $s \neq t \in \mathbb{S}^1$ satisfying $T(s) = \pm N(t)$ a tangent-normal pair. It is of type 2 if $\alpha(s)$ lies between $\alpha(t)$ and the center of curvature of α at $\alpha(t)$, and of type 1 otherwise. Distinguish also between crossings of two types according to the following rule. The tangent lines at a crossing divide $\mathbb{R}^2$ into four regions. Precisely one region is free from the curve in a neighborhood of the crossing. Call the crossing of type 1 if the angle of that region is larger than $\pi/2$ and of type 2 otherwise. The types of crossings and tangent-normal pairs are shown in Figure 1.3.

The dual formula of Thompson for generic plane curves without inflection points and with cusps only of type 1 is

$$\mathrm{cr}_1 - \mathrm{cr}_2 = \mathrm{ext} + \mathrm{int} + \frac{1}{2}(c_1 - n_1 + n_2), \tag{1.11}$$

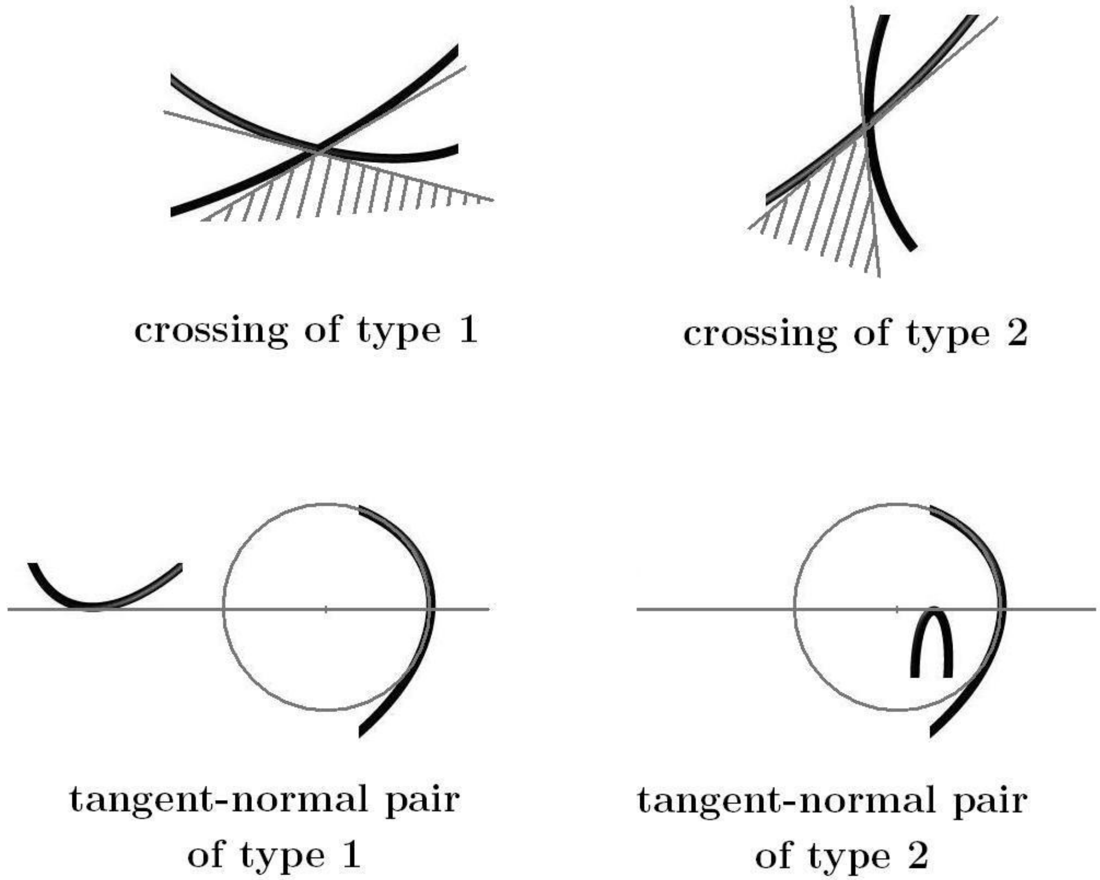


Figure 1.6: The types of crossings and tangent-normal pairs of Thompson's formula.

where

$$\begin{aligned}
\mathrm{cr}_1 &= \text{ the number of crossings of type 1,} \\
\mathrm{cr}_2 &= \text{ the number of crossings of type 2,} \\
c_1 &= \text{ the number of cusps of type 1,} \\
n_1 &= \text{ the number of tangent-normal pairs of type 1,} \\
n_2 &= \text{ the number of tangent-normal pairs of type 2.}
\end{aligned}$$

Chapter 2

Formula for parallel tangents

The idea of generalizing the Fabricius-Bjerre formula from bitangent pairs to parallel tangents arose in collaboration with Jason Cantarella during a research stay of the author at the University of Georgia (Athens) in the winter of 2010. The main goal of the generalization was to solve a problem associated to a space curve - the integration of the Fabricius-Bjerre formula over all orthogonal projections of the space curve (compare Corollary 7.10). Some more discussion of that problem can be found in Chapter 12.

Halpern included in [20] a further formula that involves so called *double normal pairs*, i.e., pairs of points on a plane curve, at which the normal vectors lie on one line. This additional formula was a result of replacing in his proof of Fabricius-Bjerre's Theorem in the key vector field given by (1.6) a determinant by a scalar product. Jason Cantarella's idea was to consider at once all the linear combinations of scalar product with the determinant. This section describes the result of this idea. In particular, the formula presented below includes Halpern's formula for double normal pairs and it also gives a much simpler criterion to distinguish between positive and negative types of double normal pairs.

Let $\alpha : \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth immersion of a circle into the plane $\mathbb{R}^2$. Denote by T and N the unit tangent and the principal normal vector field respectively. Let κ denote as usually the curvature of α and ρ the center of curvature circles of α, i.e., $\rho(s) = \alpha(s) + \frac{1}{\kappa(s)} N(s)$ for all $s \in \mathbb{S}^1$. Recall from Definition 1.1 that an inflection point, i.e., a point $s \in \mathbb{S}^1$ with $\kappa(s) = 0$, is regular if $\kappa'(s) \neq 0$.

Definition 2.1. 1. Let $\theta \in (0, \pi)$. An unordered pair $\{s, t\}$ is called a *parallel tangents pair (at angle θ)* if $\alpha(s) \neq \alpha(t)$ and the tangent lines at $\alpha(s)$ and $\alpha(t)$, when rotated in the counterclockwise direction about these points by the angle θ, coincide with the line passing through $\alpha(s)$ and $\alpha(t)$. See Figure 2.1.

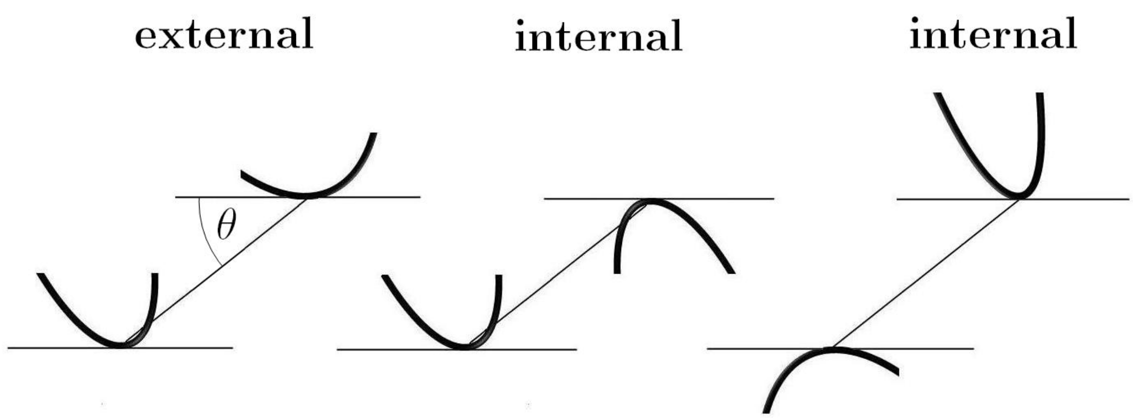


Figure 2.1: External and internal parallel tangents pairs.

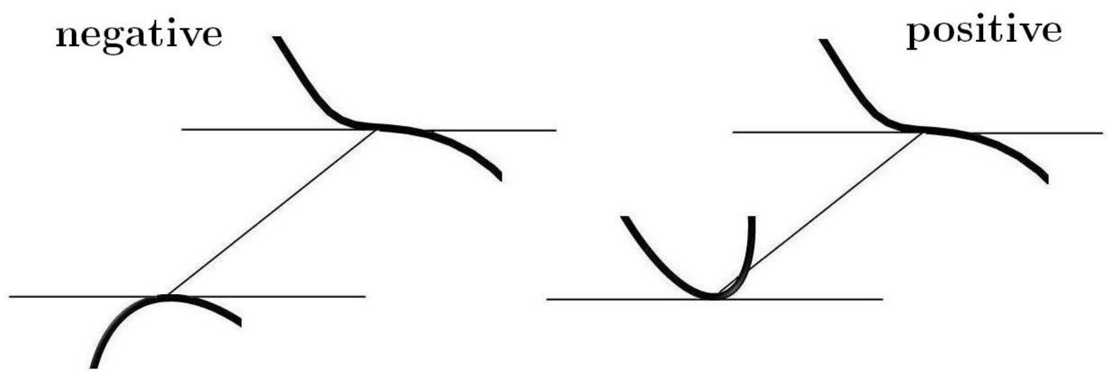


Figure 2.2: A negative and a positive parallel tangents pair involving an inflection point.

2. Given a parallel tangents pair $\{s, t\}$ project the points $\rho(s)$ and $\rho(t)$ on the line through $\alpha(s)$ and $\alpha(t)$ and call the resulting points p and q respectively (see Figure 2.3). A parallel tangents pair is called *regular* if at least one of $\kappa(s)$ and $\kappa(t)$ is non-zero and if whenever $\kappa(s)\kappa(t) \neq 0$ then $p - q \neq 0$.

3. Call a parallel tangents pair $\{s, t\}$ with $\kappa(s)\kappa(t) \neq 0$ *external* (*internal*) if the vectors $\kappa(s)N(s)$ and $\kappa(t)N(t)$ point in the same direction (in opposite directions). See Figure 2.1. Let $\sigma = \sigma(\{s, t\})$ equal $+1$ in the first case and -1 in the second case.

4. A *double point* or a *crossing* is an unordered pair $\{s, t\}$ with $\alpha(s) = \alpha(t)$. It is *regular* if the tangent lines to α at $\alpha(s)$ and $\alpha(t)$ do not coincide.

Observe, that according to the criterion given by 2. above, all parallel tangents pairs at angle $\theta = \pi/2$ of a circle are non-regular. Now, the parallel tangents pairs

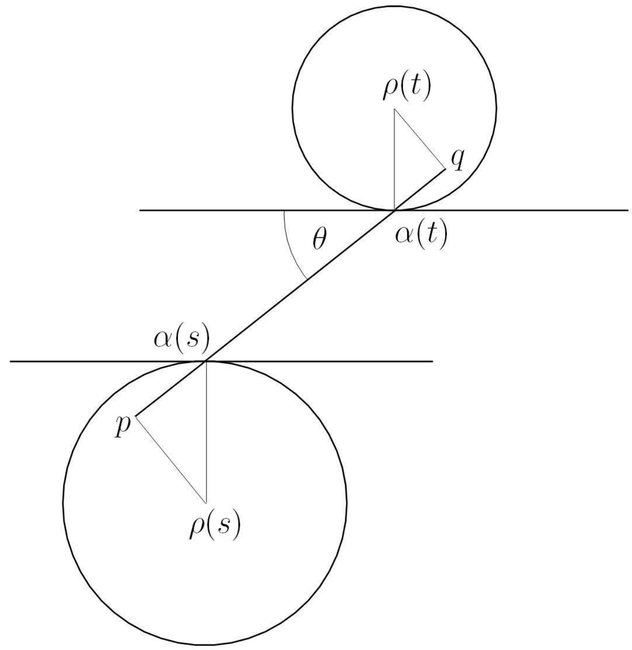


Figure 2.3: Attributes of a parallel tangents pair determining its sign.

at angle θ will be partitioned into two classes that will be referred to as positive and negative.

Definition 2.2. Let $\{s, t\}$ be a parallel tangents pair.

1. If $\kappa(s) = 0$ and $\kappa(t) \neq 0$, then the parallel tangents pair is of *positive* type if $\langle \kappa(t)N(t), \alpha(s) - \alpha(t)\rangle > 0$ and of *negative* type, otherwise (see Figure 2.2).

2. Suppose $\kappa(s)\kappa(t) \neq 0$ and p and q are the projections of $\rho(s)$ and $\rho(t)$ onto the line through $\alpha(s)$ and $\alpha(t)$. Then the parallel tangents pair is said to be of *positive* type if
$$\sigma\langle p - q, \alpha(s) - \alpha(t)\rangle > 0 \tag{2.1}$$
and of *negative* type otherwise.

Theorem 2.3. *Let $\theta \in (0, \pi)$ and let $\alpha : \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth immersion with only regular inflection points, double points and parallel tangents pairs at angle θ. Let* cr *denote the number of double points,* $\mathrm{pt}^+(\theta)$ *the number of positive and* $\mathrm{pt}^-(\theta)$ *the number of negative parallel tangents pairs at angle θ of the curve α. Then these numbers are finite and*
$$\mathrm{cr} = \mathrm{pt}^+(\theta) - \mathrm{pt}^-(\theta) \tag{2.2}$$
holds.

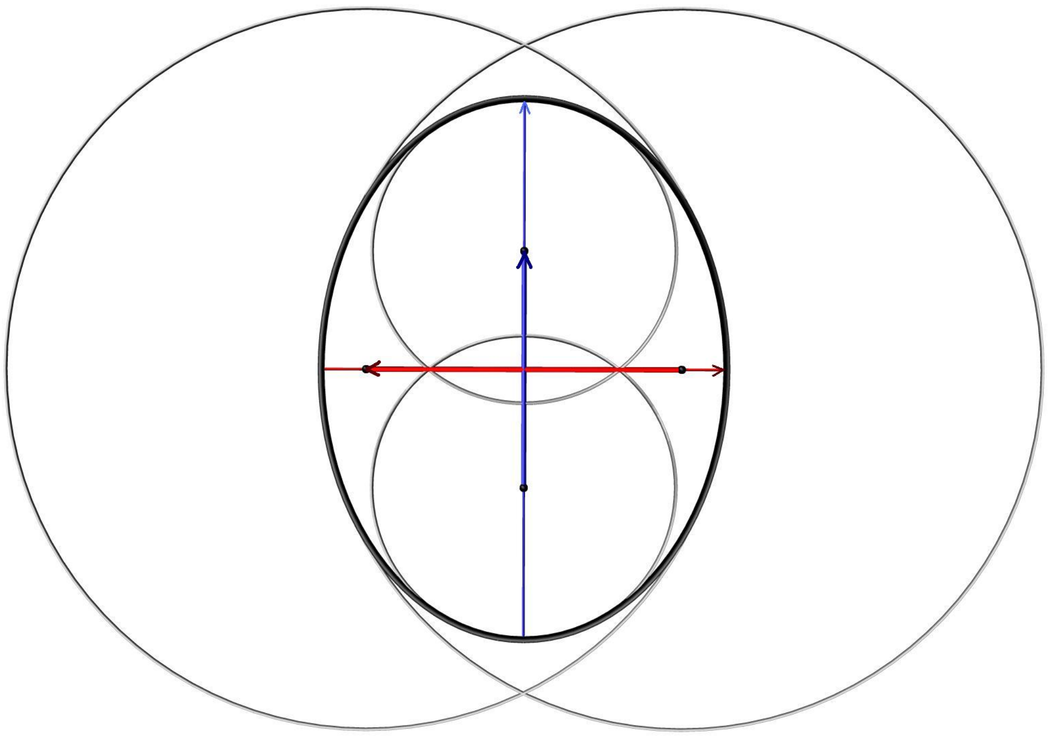


Figure 2.4: An ellipse has two pairs of parallel tangents at angle $\pi/2$. The positive one is indicated in red (the horizontal pair of arrows) and the negative in blue (the vertical pair of arrows).

Example 2.4. Consider an ellipse and the angle $\theta = \pi/2$. There are precisely two parallel tangents pairs as shown in Figure 2.4, both of them are internal. There are no crossings. Hence, the theorem suggests that one of the pair is positive and one negative. Indeed, comparing the orientations of the vector joining the center of curvatures with the vector joining the two points on the curve, according to the criterion given by (2.1), gives that the two parallel tangents pairs have opposite signs.

The proof of Theorem 2.3 will use the method of Halpern [20], summarized by Lemma 1.5, that will be applied to a more general vector field. Before proceeding with the proof some preparation is needed.

Let R_θ be the rotation matrix in $\mathbb{R}^2$ by an angle θ, i.e.,

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Then for two vectors $v,\, w \in \mathbb{R}^2$

$$[R_\theta v, w] = \det(R_\theta v, w) = [v, w]\cos\theta - \langle v, w\rangle \sin\theta$$

holds.

Proof of Theorem 2.3. Consider the vector field $W^\theta : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^2$ given by

$$W^\theta(s,t) = \begin{pmatrix} [R_\theta\alpha'(s), \alpha(s) - \alpha(t)] \\ \\ [R_\theta\alpha'(t), \alpha(s) - \alpha(t)] \end{pmatrix}.$$

Clearly, W^θ vanishes at the diagonal Δ. Further zeros are exactly all the double points and parallel tangents pairs at angle θ (as ordered pairs). Assume w.l.o.g. that α is parametrized by arc-length. In order to apply Lemma 1.5 to W^θ, it must be checked that the zeros of W^θ not in Δ are non-degenerate, i.e., that the Jacobian matrix of W^θ at each of these zeros in non-singular. This technical detail is handled by Lemma 2.5 below. Further, by Lemma 2.6 below, the vector field W^θ has only finitely many zeros not on the diagonal. Hence, Lemma 1.5 applies to W^θ. In particular the variables cr, $\mathrm{pt}^+(\theta)$ and $\mathrm{pt}^-(\theta)$ are finite.

Note that each parallel tangents pair and each crossing appears twice in $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$ as ordered pairs. All crossings have the same index. Also, in case of parallel tangents pair $\{s,t\}$ the index of (s,t) and (t,s) is the same. Hence, using Lemma 2.5 below, the right-hand side of (1.5) is equal to

$$2(-\mathrm{cr} + \mathrm{pt}^+(\theta) - \mathrm{pt}^-(\theta)). \tag{2.3}$$

It remains to check the quantities on the left-hand side of (1.5). Let $\varepsilon > 0$ and

$$\gamma^\pm : \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta, \;\; s \mapsto \pm(s, s+\varepsilon)$$

be as in Lemma 1.5. An easy computation using Taylor series gives

$$\begin{aligned} W^\theta(\gamma^+(s)) &= \begin{pmatrix} [R_\theta\alpha'(s) \quad\;\;, \alpha(s) - \alpha(s+\varepsilon)] \\ \\ [R_\theta\alpha'(s+\varepsilon), \alpha(s) - \alpha(s+\varepsilon)] \end{pmatrix} \\ &= -\varepsilon \sin\theta(1,1) + o(\varepsilon) \end{aligned}$$

and similarly

$$W^\theta(\gamma^-(s)) = \quad \varepsilon \sin\theta(1,1) + o(\varepsilon)$$

for all $s \in \mathbb{S}^1$.

Hence, ε can be chosen small enough so that $W^\theta \circ \gamma^+$ remains in the third and $W^\theta \circ \gamma^-$ in the first quadrant of $\mathbb{R}^2$. Hence, the degrees of both $\frac{W^\theta \circ \gamma^\pm}{\|W^\theta \circ \gamma^\pm\|}$ are zero and the left-hand side of (1.5) is zero. The proof is established by setting (2.3) to be equal to zero. $\square$

Lemma 2.5. *Let $\theta \in (0, \pi)$ and let α be as in Theorem 2.3. Zeros of the vector field W^θ not in Δ are all non-degenerate, i.e.,* $\det J_p W^\theta \neq 0$ *for all $p \in W^{-1}(0) \setminus \Delta$. Moreover, indices of the zeros are*

$$\operatorname{ind}_p W^\theta = \begin{cases} -1 & p \text{ is a crossing,} \\ +1 & p \text{ is a positive parallel tangents pair,} \\ -1 & p \text{ is a negative parallel tangents pair.} \end{cases}$$

Proof. Assume w.l.o.g. that α is parametrized by arc-length. The Jacobian of W^θ at a pair (s,t), denoted $J_{(s,t)}W^\theta$, is

$$\begin{pmatrix} [R_\theta \alpha''(s), \alpha(s) - \alpha(t)] + [R_\theta \alpha'(s), \alpha'(s)] & -[R_\theta \alpha'(s), \alpha'(t)] \\ [R_\theta \alpha'(t), \alpha'(s)] & [R_\phi \alpha''(t), \alpha(s) - \alpha(t)] - [R_\theta \alpha'(t), \alpha'(t)] \end{pmatrix}.$$

Noticing that $[R_\theta v, v] = -\sin\theta$ for a unit vector v, the Jacobian reduces to

$$J_{(s,t)}W^\theta = \begin{pmatrix} [R_\theta \alpha''(s), \alpha(s) - \alpha(t)] - \sin\theta & -[R_\theta \alpha'(s), \alpha'(t)] \\ [R_\theta \alpha'(t), \alpha'(s)] & [R_\theta \alpha''(t), \alpha(s) - \alpha(t)] + \sin\theta \end{pmatrix}.$$

At a crossing, i.e., at a pair (s,t) with $\alpha(s) = \alpha(t)$ with $s \neq t$ the Jacobian is

$$J_{(s,t)}W^\theta = \begin{pmatrix} -\sin\theta & -[R_\theta \alpha'(s), \alpha'(t)] \\ [R_\theta \alpha'(t), \alpha'(s)] & \sin\theta \end{pmatrix}.$$

The index of a crossing is the sign of

$$\begin{aligned}\det(J_{(s,t)}W^\theta) &= -\sin^2\theta + [R_\theta\alpha'(s),\alpha'(t)][R_\theta\alpha'(t),\alpha'(s)]\\ &= -\sin^2\theta + \langle\alpha'(s),\alpha'(t)\rangle^2\sin^2\theta - [\alpha'(s),\alpha'(t)]^2\cos^2\theta.\end{aligned}$$

Let β be the angle spanned between the tangent vector at $\alpha(s)$ and $\alpha(t)$ such that $\langle\alpha'(s),\alpha'(t)\rangle = \cos\beta$ and $[\alpha'(s),\alpha'(t))] = \sin\beta$. Then $\det(J_{(s,t)}W^\theta) = -\sin^2\beta$. Hence, a crossing is a non-degenerate zero of W^θ if and only if it is regular. Moreover, the index of a regular crossing is always negative.

At (s,t) representing a parallel tangents pair $[\alpha'(s),\alpha'(t)] = 0$ holds and the Jacobian matrix of W^θ reduces to

$$\begin{pmatrix} [R_\theta\alpha''(s),\alpha(s)-\alpha(t)] - \sin\theta & \langle\alpha'(s),\alpha'(t)\rangle\sin\theta \\ -\langle\alpha'(t),\alpha'(s)\rangle\sin\theta & [R_\theta\alpha''(t),\alpha(s)-\alpha(t)] + \sin\theta \end{pmatrix}.$$

Since $\langle\alpha'(s),\alpha'(t)\rangle = \pm1$, the determinant of the Jacobian matrix at a parallel tangents pair is

$$\begin{aligned}\det(J_{(s,t)}W^\theta) = {}&[R_\theta\alpha''(s),\alpha(s)-\alpha(t)][R_\theta\alpha''(t),\alpha(s)-\alpha(t)]\\ &+ \sin\theta[R_\theta(\alpha''(s)-\alpha''(t)),\alpha(s)-\alpha(t)].\end{aligned} \tag{2.4}$$

First, assume that $\kappa(s) = 0$ and $\kappa(t) \neq 0$. Then

$$\det(J_{(s,t)}W^\theta) = -\sin\theta[R_\theta\alpha''(t),\alpha(s)-\alpha(t)].$$

And the index of W^θ at (s,t) is

$$\begin{aligned}\operatorname{ind}_{(s,t)} W^\theta &= \operatorname{sign}(\det(J_{(s,t)}W^\theta)) = -\operatorname{sign}[R_\theta\alpha''(t),\alpha(s)-\alpha(t)]\\ &= -\operatorname{sign}\langle R_{(\theta+\frac{\pi}{2})}\alpha''(t),\alpha(s)-\alpha(t)\rangle\end{aligned}$$

and since $R_{(\theta+\frac{\pi}{2})}\alpha''(t)$ and $\alpha(s)-\alpha(t)$ are parallel

$$\begin{aligned}&= -\operatorname{sign}\left(\cos(\theta+\frac{\pi}{2})\langle\alpha''(t),\alpha(s)-\alpha(t)\rangle\right)\\ &= \operatorname{sign}\langle\alpha''(t),\alpha(s)-\alpha(t)\rangle.\end{aligned}$$

In particular, (s,t) is a non-degenerate zero of W^θ and its index is $+1$ when it is positive and -1 if it is negative according to Definition 2.2. In the symmetric case when $\kappa(s) \neq 0$ and $\kappa(t) = 0$, the determinant is

$$\det(J_{(s,t)}W^\theta) = \sin\theta[R_\theta\alpha''(s), \alpha(s) - \alpha(t)].$$

Similar computation gives that (s,t) is non-degenerate and its index equals the sign of the pair as in Definition 2.2.

To understand better the quantity (2.4) in the remaining case, i.e., when $\kappa(s)\kappa(t) \neq 0$, let $\sigma_1 = \operatorname{sign}\langle\alpha''(s), \alpha(s) - \alpha(t)\rangle$ and $\sigma_2 = \operatorname{sign}\langle\alpha''(t), \alpha(s) - \alpha(t)\rangle$. Then $\sigma := \sigma_1\sigma_2$ equals 1 if the parallel tangents pair is *external* and -1 if it is *internal*. Set $l = \alpha(s) - \alpha(t)$. The determinant then becomes

$$\begin{aligned}
\det(J_{(s,t)}W^\theta) &= \sigma\frac{1}{r_1}\frac{1}{r_2}\|l\|^2 + \left(\frac{\sigma_2}{r_2} - \frac{\sigma_1}{r_1}\right)\|l\|\sin\theta \\
&= \sigma\frac{\|l\|}{r_1 r_2}\left(\|l\| + \sigma_1 r_1\sin\theta - \sigma_2 r_2\sin\theta\right) \\
&= \sigma\frac{1}{r_1 r_2}\langle l + (p - \alpha(s)) - (q - \alpha(t)), l\rangle \\
&= \sigma\frac{1}{r_1 r_2}\langle p - q, \alpha(s) - \alpha(t)\rangle.
\end{aligned}$$

Thus, by genericity of α, a pair (s,t) is a non-degenerate zero of W^θ. The index of W^θ at (s,t) also coincides with sign of the pair according to the Definition 2.2. □

Lemma 2.6. *Let $\theta \in (0,\pi)$ and let α be as in Theorem 2.3. Then the vector field W^θ has finitely many zeros inside $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$.*

Proof. By the preceding lemma, all zeros of W^θ inside $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$ are isolated. So, if there were infinitely many zeros in the complement of the diagonal Δ, they would have an accumulation point at Δ. By genericity conditions of α this is impossible by the same argument as in the proof of Lemma 1.6. □

In Theorem 2.3 the angle θ was not allowed to be a multiple of π. This excluded case is exactly the case of the Fabricius-Bjerre formula. Note that when θ approaches 0 or π in (2.1) the vector $p - q$ approaches $\alpha(s) - \alpha(t)$ and so the scalar product of these vectors approaches 1. Hence, the sign of a parallel tangents pair is determined by σ, i.e., parallel tangents pairs will be of different types depending whether they are external or internal, just as in the Fabricius-Bjerre formula. Inflection points can be therefore viewed as degenerate pairs of parallel tangents. When $\theta \to 0$ or π

they disappear from the connected region of $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$ enclosed by $\gamma^{\pm}$ (compare Lemma 1.5 and its proof). Instead their number contributes to the left-hand side of (1.5) as the sum of the degrees of $W^{\theta} \circ \gamma^{\pm}$.

Corollary 2.7. *The Fabricius-Bjerre formula and Theorem 2.3 provide formulas for parallel tangents pairs at all angles.*

It is possible to generalize Theorem 2.3 even further. Instead of taking the vector field W^{θ}, as in the proof above, take the vector field $W^{\theta,\phi} : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^2$ given by

$$W^{\theta,\phi}(s,t) = \begin{pmatrix} [R_{\theta}\alpha'(s), \alpha(s) - \alpha(t)] \\ \\ [R_{\phi}\alpha'(t), \alpha(s) - \alpha(t)] \end{pmatrix}.$$

Then the zeros of $W^{\theta,\phi}$ are double points and pairs of points (s,t) such that the tangent vectors of α at these points make angles θ and ϕ respectively with the line through the points $\alpha(s)$ and $\alpha(t)$, as in Figure 2.5. The left-hand side of (1.5) will then vanish as long as neither angle is a multiple of π. Under suitable regularity conditions the general formula will then relate all the zeros of $W^{\theta,\phi}$ according to their indices.

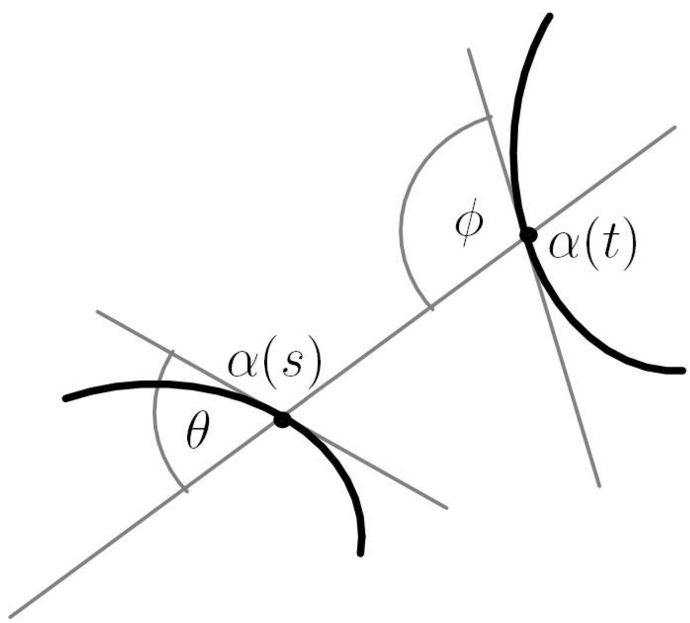


Figure 2.5: The pair (s,t) is a zero of the vector field $W^{\theta,\phi}$.

Chapter 3

Arnold's invariants versus Fabricius-Bjerre formula

This chapter contains a discussion on Arnold's basic invariants of the space of immersions of a circle into the plane, satisfying certain genericity condition. An introduction of well known results on that subject, mainly by Arnold [4], is presented in Section 3.1. It turns out that there is a close relation between Arnold's invariants and invariants occurring in the Fabricius-Bjerre formula. In particular the idea of the proof of Fabricius-Bjerre's Theorem due to Banchoff is much in the spirit of Arnold's invariants. Section 3.2 contains some remarks on possible details of that proof. Section 3.3 confirms the close relation between the two, on the first glimpse, independent subjects. In particular, a formula relating the Arnold's invariants with invariants of FB-generic curves due to Ferrand [18] is presented there.

3.1 Arnold's invariants

The content of this section is a summary of results of Arnold [4].

A *generic* curve *in the sense of Arnold*, shortly *A-generic* is a smooth immersion of a circle $\mathbb{S}^1$ into the plane $\mathbb{R}^2$ with transverse double points as the only singularities. A point of non-transverse intersection is called a *direct* self-tangency if the two tangent vectors (tangent to the curve at the two points of singularity) point in the same direction; and an *inverse* self-tangency if the vectors point in the opposite directions (see Figure 3.1). The set of non-A-generic curves called the *discriminant* (or *A-discriminant*) forms a stratified space in $C^\infty(\mathbb{S}^1, \mathbb{R}^2)$. The strata of the highest dimension are of codimension 1 in $C^\infty(\mathbb{S}^1, \mathbb{R}^2)$ and these are:

Figure 3.1: A direct and inverse self-tangency.

Σ^+: The set of all smooth immersions of a circle into the plane having as singularities transverse double points and exactly one point of direct self-tangency.

Σ^-: The set of all smooth immersions of a circle into the plane having as singularities transverse double points and exactly one point of inverse self-tangency.

Σ^{St}: The set of all smooth immersions of a circle into the plane having as singularities transverse double points and exactly one triple point, at which each strand intersects the other two transversely.

A *generic homotopy* is a regular homotopy of smooth immersions of a circle into the plane such that, when viewed as a path in the space $C^\infty(\mathbb{S}^1, \mathbb{R}^2)$, it intersects the discriminant transversely in the main strata a finite number of times. A change experienced by a curve while generic homotopy takes it through the discriminant is called a *perestroika*. The possible types of perestroikas are shown in Figure 3.2 and Figure 3.3.

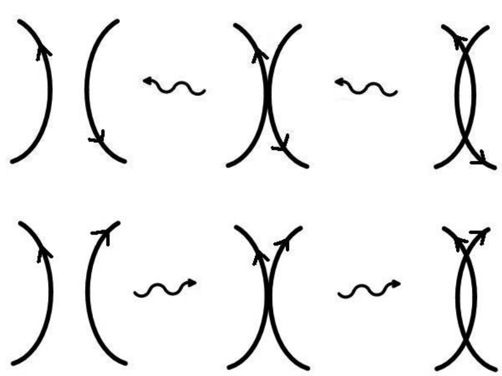


Figure 3.2: Perestroikas of an inverse (top) and a direct (bottom) self-tangency point.

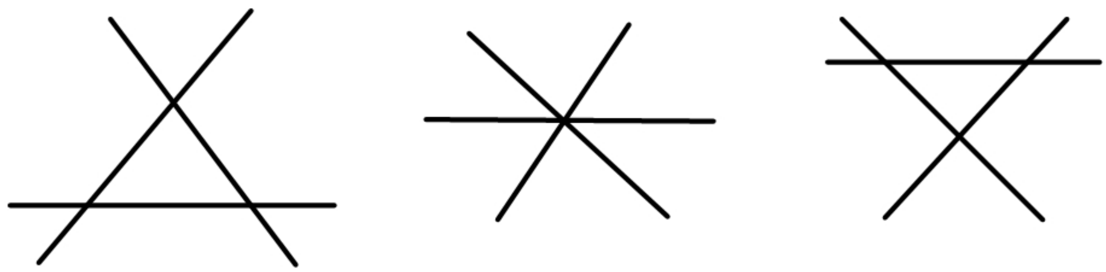


Figure 3.3: A Perestroika of a triple point.

The discriminant divides the set $C^\infty(\mathbb{S}^1, \mathbb{R}^2)$ into path-connected components. Two curves are considered to be equivalent if they belong to the same component. An invariant in the space of generic curves is a function that is constant on each of the components and its changes between adjacent components have a prescribed behavior.

The basic Arnold's Invariants

Arnold defined the basic invariants St and $J^\pm$ by giving a *coorientation* to the main strata of the discriminant and describing the changes of the invariants according to this coorientation. In general, a coorientation is a choice of one of the sides of the strata, locally around a generic point of the strata. The chosen side is considered to be *positive* and the other *negative* respectively. For the strata Σ^+, *positive* is the side on which the number of crossings is higher and for the strata Σ^- it is the side on which the number of crossings is lower. For the stratum Σ^{St} the coorientation is explained in terms of so called *vanishing triangles*.

Call a triangle formed by strands of a curve occurring just before a triple point singularity *vanishing*. The orientation of the curve prescribes a cyclic order on the sides of that triangle. Let q be the number of sides of the vanishing triangle on which the orientation of the cyclic order coincides with the curve's orientation. Then the *sign* of a vanishing triangle is set to $(-1)^q$ and it does not depend on the orientation of the curve. Figure 3.4 demonstrates the only two possibilities of the perestroikas of a triple point. The arrows in both Figure 3.2 and Figure 3.4 suggest the coorientation of the strata showing the direction from the negative to the positive side of the strata.

The space $C^\infty(\mathbb{S}^1, \mathbb{R}^2)$ has infinitely many connected components which are in one-to-one correspondence with integers given by the *Whitney index* of a curve, i.e., a rotation number of the positive tangent vector to the curve. In order to define an invariant for generic curves choose a representative in each of these components

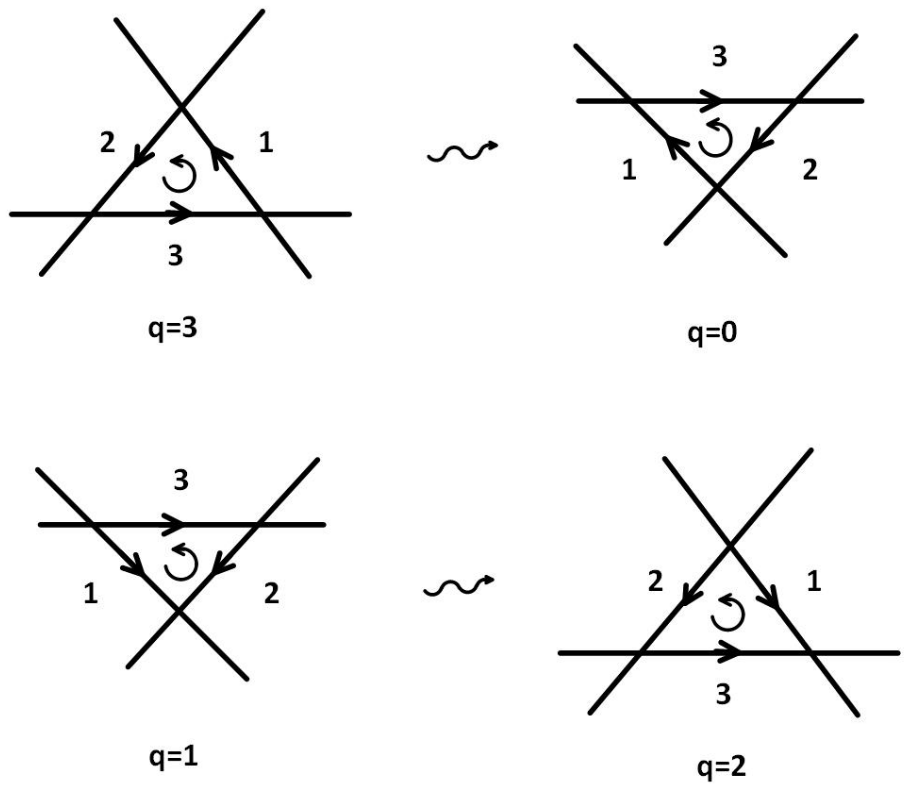


Figure 3.4: Two types of perestroika of a triple point according to the orientation of the vanishing triangle.

and assign some value to the invariant and then describe the changes of the invariant due to perestroikas. A generally accepted choice of representatives is given by the Figure 3.5.

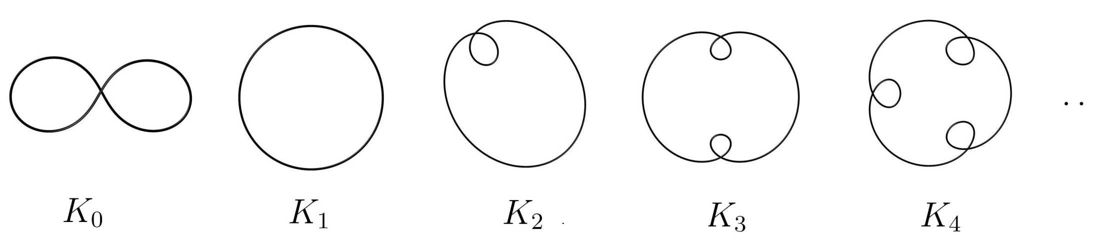


Figure 3.5: Standard curves of the Whitney index $0, \pm 1, \pm 2 \pm 3, \pm 4, ...$

The Arnold basic invariants are defined through the following axioms:

Theorem 3.1 (Arnold [3]). *There exist three unique discrete invariants of an A-generic curve $\alpha : \mathbb{S}^1 \to \mathbb{R}^2$ with respect to homotopy in the space of A-generic curves, denoted by $\mathrm{St}(\alpha)$, $J^+(\alpha)$ and $J^-(\alpha)$, that are given by the following properties:*

(1) The invariant St *does not change during perestroikas of direct and inverse self-tangencies, increases (decreases) by 1 during a positive (negative) perestroika of a triple point.*

(2) The invariant J^+ does not change during perestroikas of triple points and inverse self-tangencies, increases (decreases) by 2 during a positive (negative) perestroika of direct self-tangency.

(3) The invariant J^- does not change during perestroikas of triple points and direct self-tangencies, increases (decreases) by 2 during a positive (negative) perestroika of inverse self-tangency.

(4) For the standard curves $K_0, K_1, K_2, ...,$ shown in Figure 3.5, the invariants $\mathrm{St}(\alpha)$, *$J^+(\alpha)$ and $J^-(\alpha)$ have the following values:*

$$\begin{array}{rlrll} \mathrm{St}(K_0) = & 0, & \mathrm{St}(K_{i+1}) = & i & (i = 0, 1, 2, ...); \\ J^+(K_0) = & 0, & J^+(K_{i+1}) = & -2i & (i = 0, 1, 2, ...); \\ J^-(K_0) = & -1, & J^-(K_{i+1}) = & -3i & (i = 0, 1, 2, ...). \end{array}$$

The above choice of the values for the standard curves assures a property that the invariants are additive with respect to the connected sum. Moreover, an important property

$$\mathrm{cr} = J^+ - J^- \tag{3.1}$$

holds, where cr is the number of double points of the curve α.

In general it is quite tedious to compute the values of the Arnold's invariants for a given curve α using their definition. First, one has to find a generic homotopy that connects the curve α with one of the standard curves K_i's. The values of the invariants for K_i's are prescribed. Then one goes back on the same path and updates the values of the invariants after each perestroika. An example of that procedure can be found in Figure 3.6.

Soon after Arnold's invariants were introduced, a couple of explicit formulas for the invariants appeared, due to Viro [38], Schumakovitch [34] and Polyak [31]. An exposition treating these result due to Chmutov and Duzhin can be found in [13]. Another explicit formula for $J^\pm$ was given by Ferrand [18]. His formula is of a particular interest as it connects $J^\pm$ invariants with the Fabricius-Bjerre formula. Section 3.3 is devoted entirely to Ferrand's result.

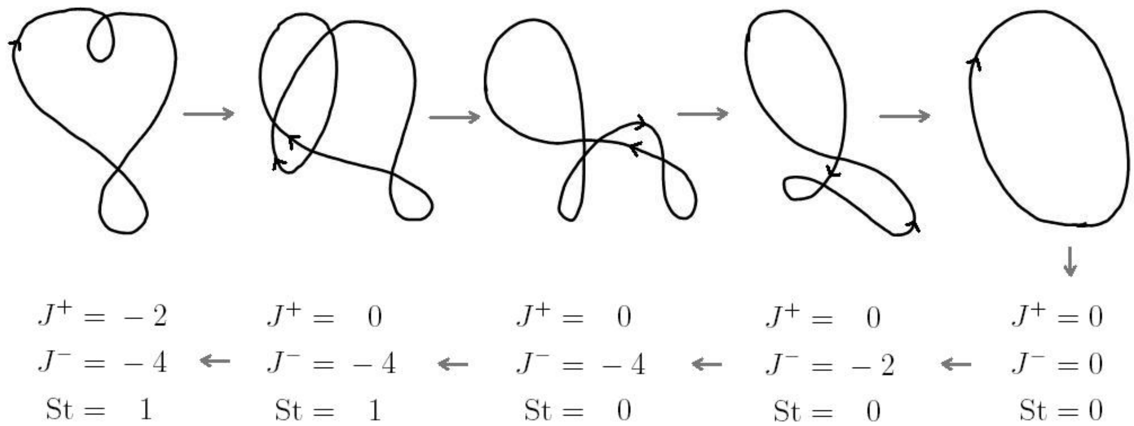


Figure 3.6: An example of computation of Arnold's invariants from the definition.

3.2 A proof of Fabricius-Bjerre formula according to Banchoff's idea

This section discusses the idea of the proof of Fabricius-Bjerre's Theorem by Thomas Banchoff (compare Section 1.2.3). The details are left out here. Filling out the missing details can be considered as an exercise after complete reading of this thesis.

It should be clear now, that the proof of the Fabricius-Bjerre formula according to Banchoff's idea is much in the spirit of Arnold invariants. The following steps are needed for the complete proof:

- First, show that the set of non-FB-generic curves, called the *FB-discriminant*, forms a stratified space in the space $C^\infty(\mathbb{S}^1, \mathbb{R}^2)$ with top strata of codimension 1. It actually suffices to show that generic points of the FB-discriminant form a hypersurface in $C^\infty(\mathbb{S}^1, \mathbb{R}^2)$. The main strata, i.e., the strata of codimension 1, correspond to curves having a single occurrence of irregular singularity as in the middle pictures of Figure 1.3. Up to some simplification these main strata are:

  Σ^{λ} : curves which satisfy FB-genericity conditions at all but one point, which is a cusp (see Figure 1.3 a));

  $\Sigma^{\asymp}$: curves which satisfy FB-genericity conditions at all but one pair, which is an irregular crossing, i.e., a non-transverse self-intersection point (see Figure 1.3 b) and c));

  Σ^{bp}: curves which satisfy FB-genericity conditions at all but one pair, which is an irregular bitangent pair (see Figure 1.3 d));

Σ^{κ} : curves which satisfy FB-genericity conditions at all but one point s, which is an irregular inflection point with $\kappa(s) = \kappa'(s) = 0 \neq \kappa''(s)$ (Figure 1.3 e)).

- Argue, that the variables ext, int, cr and infl are constant on the connected components of the space of FB-generic curves: Clearly, cr can only change its value while passing through $\Sigma^{\curlywedge}$ and $\Sigma^{\asymp}$. Also infl can only change while passing through Σ^{κ}. To show that ext and int are also well-defined invariants on the space of FB-generic curves, the Halpern's vector field given by (1.6) can be used (if there is no better option).

- Show, that through each generic point of the FB-discriminant, i.e., a point belonging to a codimension 1 strata, there is a path passing through that point connecting the two sides of the FB-discriminant. Moreover, on both of the sides of the FB-discriminant the value of the expression

$$\mathrm{ext} - \mathrm{int} - \mathrm{cr} - \frac{1}{2}\,\mathrm{infl} \tag{3.2}$$

  remains unchanged. Such transversal intersections of the discriminant can be locally described by the table in Figure 1.3.

- Fix one FB-generic curve for which the expression (3.2) is easy to check, e.g., a circle.

- The proof then follows from the fact that any FB-generic curve can be connected with the circle via a path that crosses the FB-discriminant transversely only in the codimension 1 strata.

This section is finalized with a couple of remarks.

Remark. In contrast to Arnold's invariants it is not necessary here that the path joining the circle with an arbitrary curve crosses the strata finitely many of times. The invariant of the space of FB-generic curves that is considered here is trivial, i.e., identically equal zero on the whole space, and there are no jumps recorded during perestroikas.

Remark. The path joining the representative curve with the given curve can be made a regular homotopy (i.e., each curve on the path is an immersion). For this, as with Arnold's invariants, the whole family of representative curves is required, e.g., the standard curves K_i's given by Figure 3.5. The regular homotopy will then connect an arbitrary curve with a standard curve of the same Whitney index.

Remark. In order to define Arnold's invariants the global coorientation of the main strata of Arnold discriminant was needed. Here it is the other way around. The four variables ext, int, cr and infl are well-defined and in particular, their jumps through the strata are known. Here, it is not necessary to prove that the strata are globally coorientable.

Remark. In case of Arnold's invariants there was a natural duality between the main strata and the three invariants in the sense that an invariant changed if and only if the path crossed its dual strata. Here there is no such duality between the strata $\Sigma^{\asymp}$, Σ^{bp}, Σ^{κ}, Σ^{infl}, $\Sigma^{\curlywedge}$ and the variables cr, ext, int, infl together with the Whitney index. In particular, it cannot be deduced immediately that the strata are coorientable.

3.3 Formula for $J^{\pm}$-invariants.

There is a connection between the Fabricius-Bjerre formula and Arnold's $J^{\pm}$-invariants. On one hand $\mathrm{cr} = J^{+} - J^{-}$ and on the other hand $\mathrm{cr} = \mathrm{ext} - \mathrm{int} - \frac{1}{2}\,\mathrm{infl}$. The natural question that arises is whether $J^{\pm}$ invariants can be expressed in terms of other invariants of FB-generic curves. The positive answer was given by Emanuel Ferrand in [18]. John Sullivan and the author independently came up with the same result given by Theorem 3.2 below. In this section the connection between the two subjects, Arnold's invariants and Fabricius-Bjerre's formula, is briefly sketched.

Consider a smooth FB-generic immersion of a circle into the plane. For such a curve the variables ext, int, infl and cr as well as $J^{\pm}$ are well-defined. Adopting Arnold's notation, call a bitangent pair $\{s, t\}$ *direct* if the tangent vectors at s and t point in the same direction and *inverse* otherwise (see Figure 3.7). Denote by ext^{+} the number of direct and by ext^{-} the number of inverse external bitangent pairs. Analogously, denote by int^{+} the number of direct and by int^{-} the number of inverse internal bitangent pairs. Clearly, $\mathrm{ext} = \mathrm{ext}^{+} + \mathrm{ext}^{-}$ and $\mathrm{int} = \mathrm{int}^{+} + \mathrm{int}^{-}$. These new variables $\mathrm{ext}^{\pm}$ and $\mathrm{int}^{\pm}$ are also well-defined for a FB-generic curve since the only possibility to change the type of a bitangent pair from positive to negative would be by passing through the strata $\Sigma^{\curlywedge}$.

The connection between Arnold's invariants and the Fabricius-Bjerre formula is given by the following theorem.

Theorem 3.2 (Ferrand [18, Theorem 4.2]). *Let $\alpha : \mathbb{S}^1 \to \mathbb{R}^2$ be a FB-generic curve.*

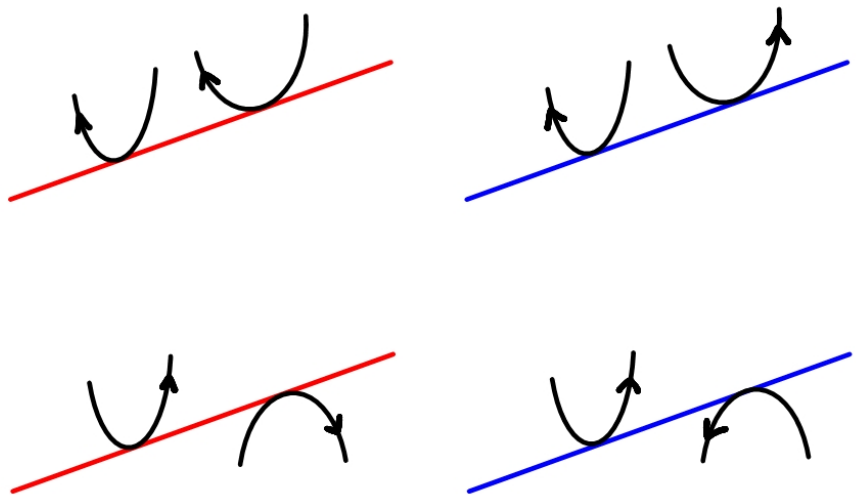


Figure 3.7: Examples of *direct* (red) and *inverse* (blue) bitangent pairs.

The following formulas

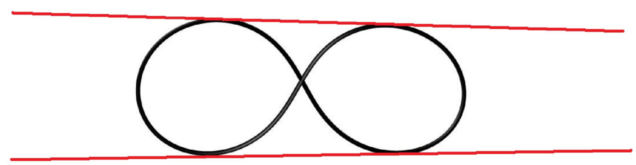
$$
\begin{aligned}
J^- + \omega^2 - 1 &= -\operatorname{ext}^- + \operatorname{int}^- \\
J^+ + \omega^2 - 1 &= \quad \operatorname{ext}^+ - \operatorname{int}^+ - \frac{1}{2}\operatorname{infl},
\end{aligned}
\tag{3.3}
$$

hold, where ω denotes the Whitney index of the curve α.

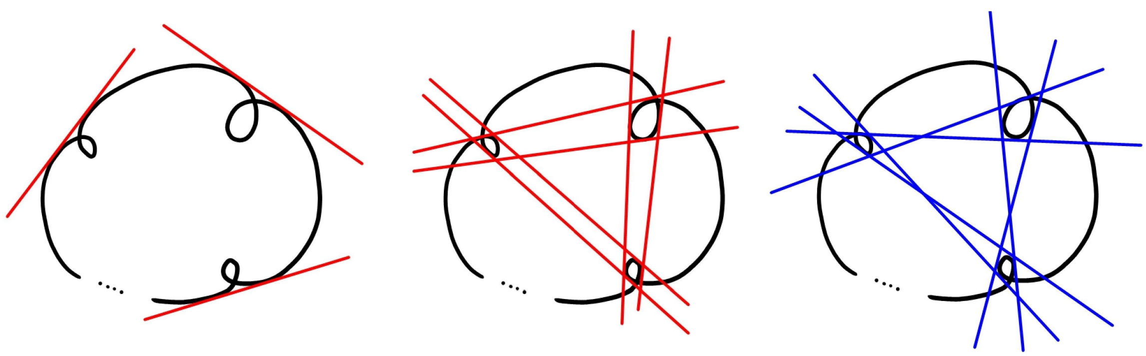
Ferrand's original proof goes back to the generalization of Arnold's invariants to so called spherical *wave fronts*. Roughly speaking a generic spherical wave front can be considered as a spherical curve with an even number of cusps satisfying some additional conditions. The set of smooth plane curves is injected in the set of all spherical wave fronts by mapping it into a small disc on the sphere. The formula (3.1) has its corresponding generalized version in case of the spherical wave fronts and its dual version under certain spherical duality yield the formulas (3.3).

The theorem can be also proved in an analogous way to Banchoff's idea of proving the Fabricius-Bjerre formula.

Sketch of the proof due to Sullivan and the author. Check that the equations (3.3) are satisfied for a set of standard curves and that these equations are also satisfied while experiencing perestroikas. The proof follows from the fact that any FB-generic curve can be connected with one of the standard curves via a path that is a regular homotopy and that meets only the strata of FB-discriminant of the codimension 1. Assume w.l.o.g. that the standard curve K_0 is given by an "∞"-shaped curve with precisely two inflection points and that the standard curves with the non-zero Whitney index



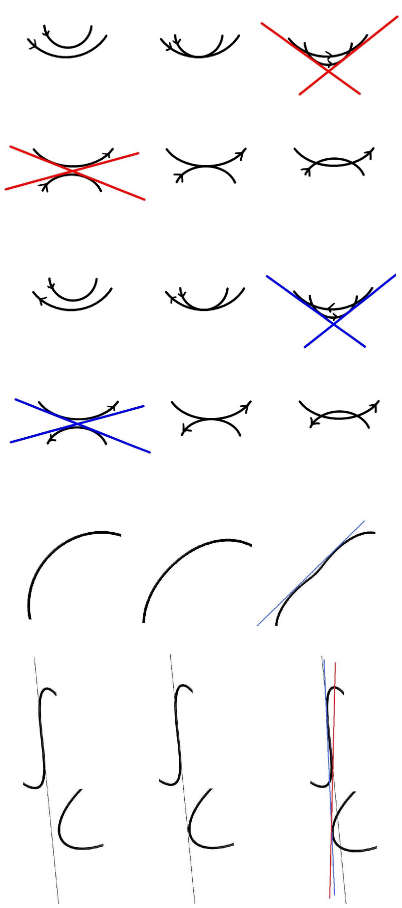
ω	ext^+	int^+	infl	J^+	ext^-	int^-	J^-
0	0	0	2	0	2	0	-1



ω	ext^+	int^+	infl	J^+	ext^-	int^-	J^-
ω	$(\lvert\omega\rvert - 1)^2$	0	0	$-2(\lvert\omega\rvert - 1)$	0	$(\lvert\omega\rvert - 1)^2 - (\lvert\omega\rvert - 1)$	$-3(\lvert\omega\rvert - 1)$

Figure 3.8: Values of the variables in (3.3) obtained from their definitions.

have curvature always positive and are shaped as shown in Figure 3.8. Tables of initial values of standard curves are given in Figure 3.8. Tables with changes during perestroikas can be found in Figure 3.9. $\square$



$\Delta\,\mathrm{ext}^+$	$-\Delta\,\mathrm{int}^+$	$-\frac{1}{2}\Delta\,\mathrm{infl}$	J^+	$-\Delta\,\mathrm{ext}^-$	$\Delta\,\mathrm{int}^-$	J^-
2			2			
	2		2			
				-2		-2
					-2	-2
1		-1				
1	1			1	1	

Figure 3.9: Jumps of the variables involved in (3.3) when passing through the strata of FB-discriminant.

Part II

Closed space curves

Chapter 4

Preliminaries

Let $\mathbb{S}^1$ be the quotient space $\mathbb{R}/\mathbb{Z}$ and let $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ denote the space of all smooth mappings of $\mathbb{S}^1$ to the Euclidean space $\mathbb{R}^3$. It is a topological vector space of infinite dimension. More about the topology of this space can be found in Chapter 9. An element of $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ is referred to as a *(smooth closed) space curve.*

4.1 Notation

Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ denote a smooth closed space curve. The curve γ is an *immersion* if the tangent vector $\gamma'(t)$ at each $s \in \mathbb{S}^1$ is non-zero.

Curvature and Torsion

The letters κ and τ are reserved throughout this work to denote the curvature and torsion of an immersed curve γ, whenever defined. In particular, the curvature of γ at $t \in \mathbb{S}^1$ is equal to

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}$$

and if $\kappa(t)$ is non-zero the torsion of γ at t is well-defined as

$$\tau(t) = \frac{\langle \gamma'(t) \times \gamma''(t), \gamma'''(t) \rangle}{\|\gamma'(t) \times \gamma''(t)\|^2} = \frac{[\gamma'(t), \gamma''(t), \gamma'''(t)]}{\|\gamma'(t) \times \gamma''(t)\|^2}.$$

Frenet-Serret frame

Whenever $\gamma'(t) \times \gamma''(t) \neq 0$ at $t \in \mathbb{S}^1$, i.e., $\kappa(t) \neq 0$, the Frenet-Serret frame is well-defined at t. It consists of the unit tangent vector $T(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}$, the principal normal vector $N(t) := \frac{T'(t)}{\|T'(t)\|}$ and the binormal vector $B(t) := T(t) \times N(t)$. If γ has nowhere vanishing curvature, then the Frenet-Serret frame is globally well-defined.

The plane spanned by $T(t)$ and $N(t)$ is called the *osculating plane* of γ at t. The plane through $\gamma(t)$ parallel to the osculating plane at $\gamma(t)$ will be referred to as the *affine* osculating plane of γ at t.

Orthogonal projections

To each point v of the unit sphere $\mathbb{S}^2 = \{v \in \mathbb{R}^3 \mid \|v\| = 1\}$ associate a curve that is obtained by orthogonal projection of γ onto the plane $v^\perp$ perpendicular to v. Throughout this work such a curve will be denoted by $\gamma_v : \mathbb{S}^1 \to \mathbb{R}^3$, which can be explicitly written as

$$\gamma_v(t) := \pi_v \circ \gamma(t) = \gamma(t) - \langle \gamma(t), v \rangle v. \tag{4.1}$$

The signed curvature of the curve γ_v will be given by

$$\kappa_{\gamma_v} = \frac{\langle \gamma' \times \gamma'', v \rangle}{\|\gamma_v'\|} = \frac{[\gamma', \gamma'', v]}{\|\gamma_v'\|}.$$

This definition of signed curvature coincides with the definition of curvature of plane curves given in Section 1.1, provided that the basis of vectors defining positive orientation of the plane $v^\perp$ extended by v gives the positive orientation of $\mathbb{R}^3$.

Geodesic curvature of a curve on the sphere $\mathbb{S}^2$

Let $c : (-\varepsilon, \varepsilon) \to \mathbb{S}^2$ be an immersed curve on the unit sphere (oriented by choice of the outer normal). Then the *geodesic curvature* of c is given by

$$\kappa_c^{\mathrm{geo}} = \frac{[c, c', c'']}{\|c'\|}. \tag{4.2}$$

Call a point $t \in (-\varepsilon, \varepsilon)$ at which $c'(t) \neq 0$ and $\kappa_c^{\mathrm{geo}}(t) = 0$ *an inflection point* of the curve c. An inflection point t of the curve c is *regular* if additionally

$$\kappa_c^{\mathrm{geo}\prime}(t) \neq 0$$

holds, otherwise it is *irregular*. Distinguish an irregular point t of c with $\kappa_c^{\mathrm{geo}\prime\prime}(t) \neq 0$ and call it a *special irregular inflection point*. It will be useful to have the following straightforward equivalences at hand

$$\begin{array}{c}t \text{ is a regular}\\ \text{inflection point of } c\end{array} \iff [c,c',c''](t)=0\neq[c,c',c'''](t);$$
$$\begin{array}{c}t \text{ is a special irregular}\\ \text{inflection point of } c\end{array} \iff \left\{\begin{array}{c}[c,c',c''](t)=[c,c',c'''](t)=0 \text{ and}\\ [c,c',c^{IV}](t)\neq 0\end{array}\right\}. \tag{4.3}$$

If $\gamma:\mathbb{S}^1\to\mathbb{R}^3$ is parametrized by arc-length then $\gamma':\mathbb{S}^1\to\mathbb{S}^2$ is the *tangent indicatrix* of γ and its geodesic curvature can be expressed in terms of the curvature and the torsion of γ in the following way

$$\kappa^{\text{geo}}_{\gamma'}(t)=\frac{\tau(t)}{\kappa(t)} \quad \text{for all } t\in\mathbb{S}^1.$$

Order of contact

Let $\gamma:\mathbb{S}^1\to\mathbb{R}^3$ be an injective immersion. Given a plane P consider a point $p\in\gamma(\mathbb{S}^1)\cap P$. Then the *(local) order of contact of* γ *with* P *at* p is the integer $n\geq 0$ satisfying the condition

$$\gamma'(p),\gamma''(p),\dots\gamma^{(n)}(p)\in P \text{ and } \gamma^{(n+1)}(p)\notin P.$$

The *total order of contact of* γ *with* P is then the sum of the local orders of contact of p for all $p\in\gamma(\mathbb{S}^1)\cap P$.

4.2 General position

Definition 4.1. A closed space curve $\gamma:\mathbb{S}^1\to\mathbb{R}^3$ is in *general position* if it is an injective immersion with non-vanishing curvature and it additionally satisfies

(i) all points of vanishing torsion are non-degenerate, i.e., for all $s\in\mathbb{S}^1$

$$\tau(s)=0 \implies \tau'(s)\neq 0$$

holds and

(ii) the following situations *cannot* occur for any distinct s,t in $\mathbb{S}^1$:

(a) $\gamma'(s)$ and $\gamma(s)-\gamma(t)$ are parallel, and $\gamma'(t)$ lies in the osculating plane at $\gamma(s)$;

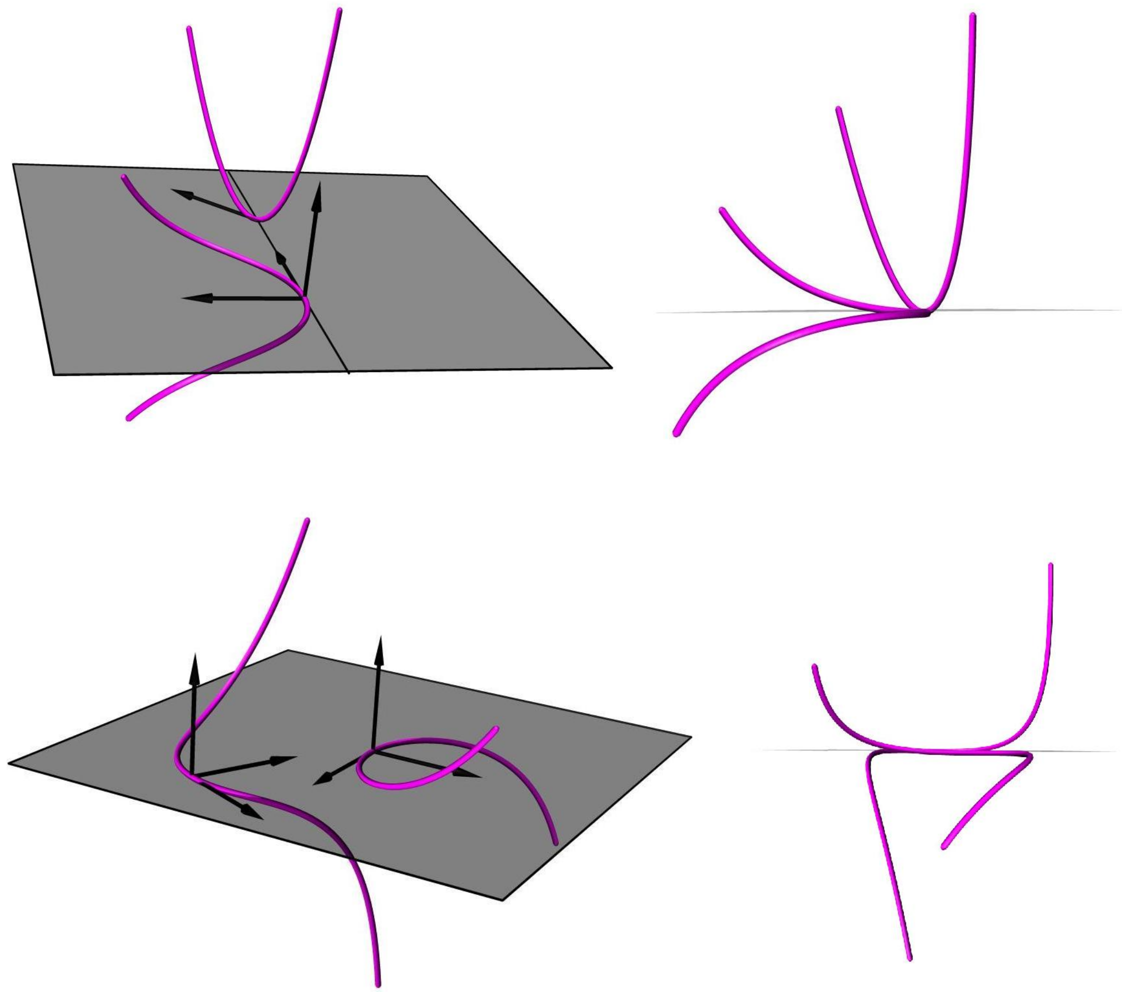


Figure 4.1: Situations not allowed for a curve in general position by Definition 4.1 (ii): top (a) and bottom (b). Both of the situations are depicted from a random direction (left) and from direction $\gamma(s) - \gamma(t)$ (right) at a pair $\{s, t\}$ violating (ii).

(b) the affine osculating planes of γ at $\gamma(t)$ and $\gamma(s)$ coincide.

Denote by $\mathcal{G}$ the set of all closed space curves in general position and by $\mathcal{NG}$ its complement in $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$.

The forbidden situations of the above definitions are illustrated in Figure 4.1. (This figure as well as most of the remaining 3D-figures in this work are not provided with labels, which would substantially lower the quality of the images.)

To justify the usage of the term *generic* in the above definition, it needs to be shown, that the set $\mathcal{G}$ is open and dense in the space $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$. This proof is postponed to Section 10.1. The conditions of the above definition can be equivalently described as in the following lemma.

Lemma 4.2. *A curve $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ is in general position if and only if it satisfies both of the conditions*

$$\begin{pmatrix} [\gamma'(s), \gamma''(s), \gamma'''(s)] \\ [\gamma'(s), \gamma''(s), \gamma^{IV}(s)] \end{pmatrix} \neq 0 \quad \forall s \in \mathbb{S}^1, \tag{G1}$$

$$\begin{pmatrix} [\gamma'(s), \gamma'(t), \gamma(s) - \gamma(t)] \\ [\gamma''(s), \gamma'(t), \gamma(s) - \gamma(t)] \\ [\gamma'(s), \gamma''(t), \gamma(s) - \gamma(t)] \end{pmatrix} \neq 0 \quad \forall s \neq t \in \mathbb{S}^1. \tag{G2}$$

Proof. The condition (G1) implies that

$$\gamma'(s) \neq 0 \ \text{ and } \ \gamma'(s) \times \gamma''(s) \neq 0 \quad \text{for all } s \in \mathbb{S}^1.$$

Thus, a curve γ satisfying (G1) is necessarily an immersion with nowhere vanishing curvature and the torsion of γ is well-defined for all $s \in \mathbb{S}^1$. Furthermore, (G1) is equivalent to

$$[\gamma'(s), \gamma''(s), \gamma'''(s)] = 0 \implies [\gamma'(s), \gamma''(s), \gamma^{IV}(s)] \neq 0 \quad \text{for all } s \in \mathbb{S}^1,$$

which translates to

$$\tau(s) = 0 \implies \tau'(s) \neq 0 \quad \text{for all } s \in \mathbb{S}^1.$$

Hence, (G1) is equivalent to γ being an immersion of nowhere vanishing curvature satisfying the condition (i) of Definition 4.1.

Now, assume that γ satisfies (G1). Suppose that at some distinct pair $\{s, t\} \subset \mathbb{S}^1$ the condition (G2) does *not* hold for γ. This means either that $\gamma(s) = \gamma(t)$ or that $\gamma(s) \neq \gamma(t)$ and one of the following situations appear:

- the two vectors $\gamma'(s)$ and $\gamma(s) - \gamma(t)$ are parallel, i.e.,

  $$\gamma'(s) \times (\gamma(s) - \gamma(t)) = 0,$$

  in which case $[\gamma''(s), \gamma'(t), \gamma(s) - \gamma(t)] = 0$ means that the vector $\gamma'(t)$ lies in the osculating plane to γ at $\gamma(s)$ (or the fully symmetric case where s and t are exchanged), or

- none of the two vectors $\gamma'(s)$ and $\gamma'(t)$ is parallel to $\gamma(s)-\gamma(t)$, i.e.,

$$\gamma'(s)\times(\gamma(s)-\gamma(t))\neq 0 \text{ and } \gamma'(t)\times(\gamma(s)-\gamma(t))\neq 0.$$

In this case, the osculating planes to γ at $\gamma(s)$ and $\gamma(t)$ coincide as affine planes.

Hence, under the assumption that (G1) holds, the condition (G2) is equivalent to γ additionally being injective and satisfying the condition (ii) of Definition 4.1. □

Remark 4.3. The condition (G1) captures all local and (G2) all global properties of a generic closed space curve. In fact, (G2) can be regarded as a regularity property of the *tangent developable surface* to the curve γ, i.e., the surface given by $\mathbb{S}^1\times\mathbb{R}\to\mathbb{R}^3$, $(s,\lambda)\mapsto\gamma(s)+\gamma'(s)\lambda$. A double point is a self-intersection of the tangent developable surface at its singular part. Coincidence of the osculating planes (condition (b) of Definition 4.1) corresponds to a self-tangency of the tangent developable surface. The remaining case (condition (a) of Definition 4.1) occurs whenever the tangent developable surface is tangent to the curve itself (see Figure 4.2).

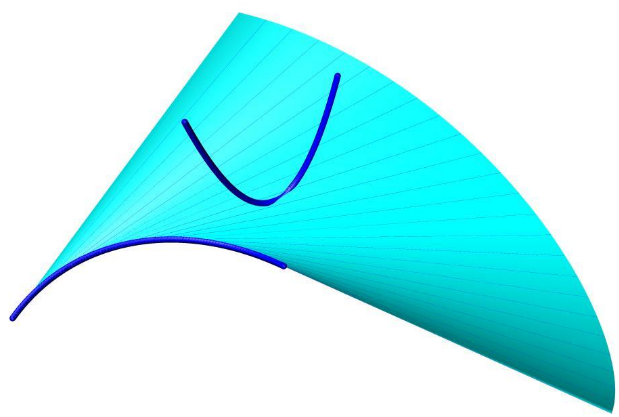


Figure 4.2: A global singularity: curve is tangent to its tangent developable surface. (Here only a part of the tangent developable surface with $\lambda\geq 0$ is pictured.)

Chapter 5

Reidemeister Curves

Let γ be a smooth immersion of a circle into $\mathbb{R}^3$. In this chapter three subsets of the sphere $\mathbb{S}^2$ will be constructed that capture directions $v \in \mathbb{S}^2$, from which when looking at γ, it appears to have certain singularities (see Figure 5.1). In each of the three cases

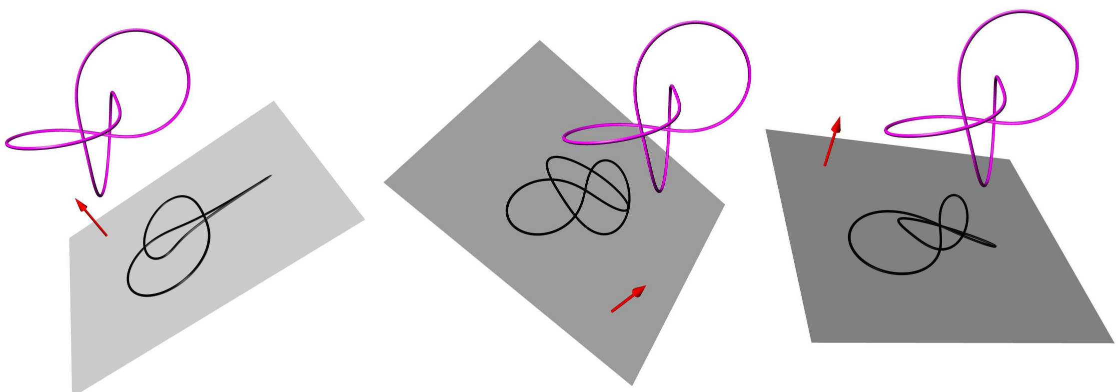


Figure 5.1: Three different orthogonal projections of the same space curve onto planes showing certain singularities.

it turns out that, under certain genericity conditions, the sets can be parametrized and so they will be referred to as curves. The name of these curves comes from the connection to the so called *Reidemeister moves* of a diagram of a knot.

In the knot theory, a *knot* (*link*) is an equivalence class of smooth embeddings of a circle (circles) into $\mathbb{R}^3$ up to continuous deformations (ambient isotopy). A *knot (link) diagram* of a knot (link) representative $\gamma : (\mathbb{S}^1)^{\times k} \to \mathbb{R}^3$ with $k = 1$ ($k > 1$) in direction v, denoted $D_v(\gamma)$, is a projection of γ in direction v with certain *decorations*. By a decoration an additional information is understood that makes it possible to distinguish which of the two strands at a crossing was under and which over. Then the Reidemeister moves are local moves (due to the ambient isotopy) of a knot (link) diagram, depicted in Figure 5.2.

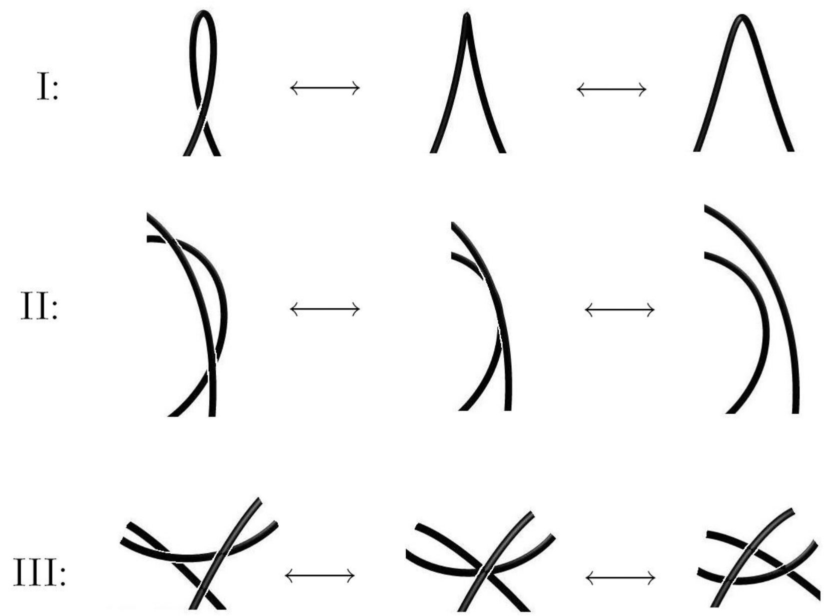


Figure 5.2: The three Reidemeister moves.

In geometric knot theory, a *geometric knot* is simply a smooth embedding of $\mathbb{S}^1$ into $\mathbb{R}^3$, i.e., the geometry is kept fixed. Also here, the Reidemeister moves of a geometric knot are observed when varying the direction of the projection, or in other words, when rotating the knot. See examples in Figure 5.3.

The subsets of $\mathbb{S}^2$, that will be constructed here, will capture precisely the singularities of the Reidemeister moves (pictured in the middle column of Figure 5.2) in the orthogonal projections in this directions. Perturbing the singular direction of projection slightly will resolve the singularity of the projected curve and a Reidemeister move can be observed. This idea was first introduced by Adams et al. in [1] in case of piecewise linear knots.

In this chapter the Reidemeister curves of *smooth* geometric knots are introduced and analyzed. Only the most important properties are listed that will be crucial for Chapter 6 and eventually for the new proof of Fabricius-Bjerre's Theorem. A special emphasis is given to the Reidemeister II curve. The Reidemeister I curve is well-known and the Reidemeister III curve does not appear in the rest of the thesis. Therefore, they will be mentioned just briefly.

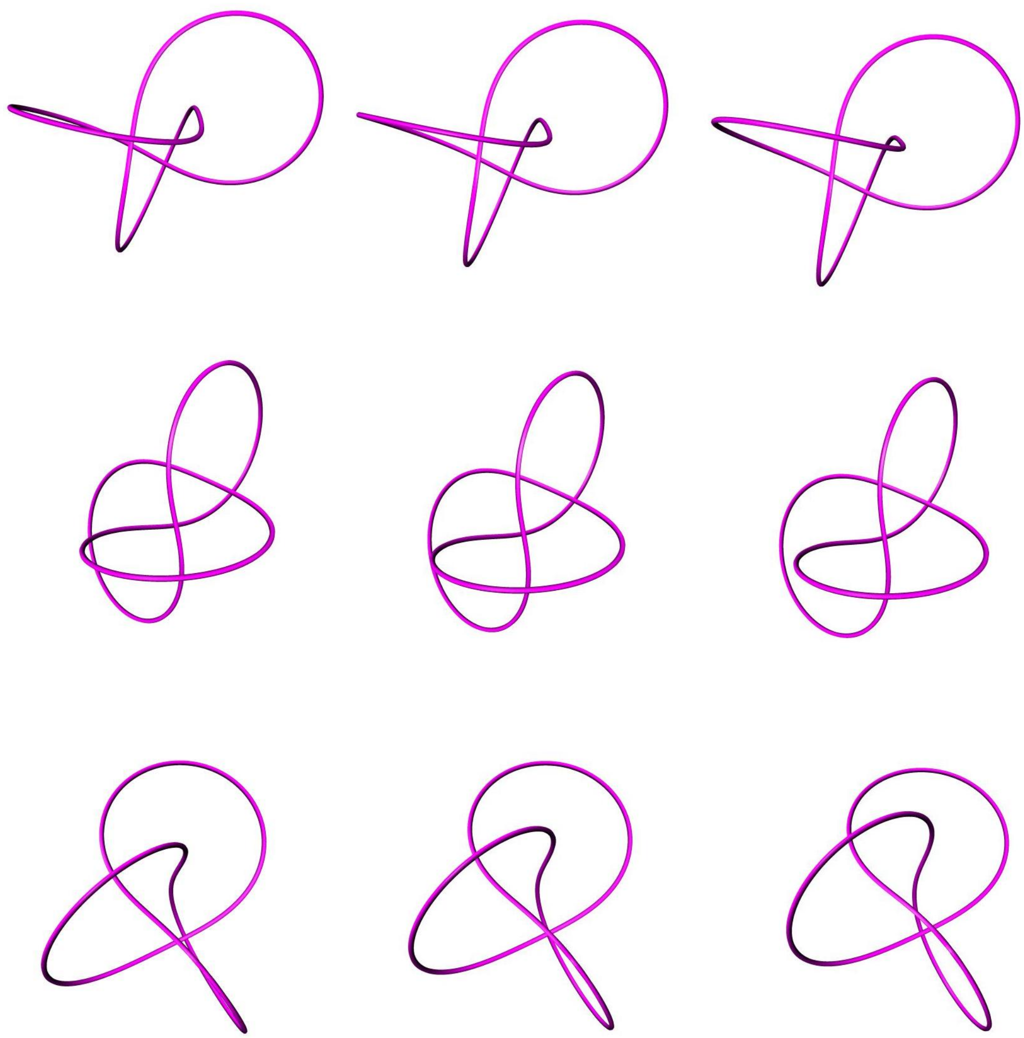


Figure 5.3: The three Reidemeister moves in the projections of a geometric knot.

The knot $\mathcal{K}$

Most of the figures of curves and surfaces in this thesis are based on the same curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$, given by

$$\theta \mapsto \frac{1}{1 - b\sin(p\theta)}(a\cos(q\theta), a\sin(q\theta), b\cos(p\theta))$$

with $\theta \in [0, 2\pi)$, $p = 3$, $q = 2$, $a = 0.7$ and $a^2 + b^2 = 1$. The image of this curve is a trefoil knot depicted in Figure 5.4. It also appeared in Figures 5.1 and 5.3. This special curve and its image will be from now on referred to as the knot $\mathcal{K}$.

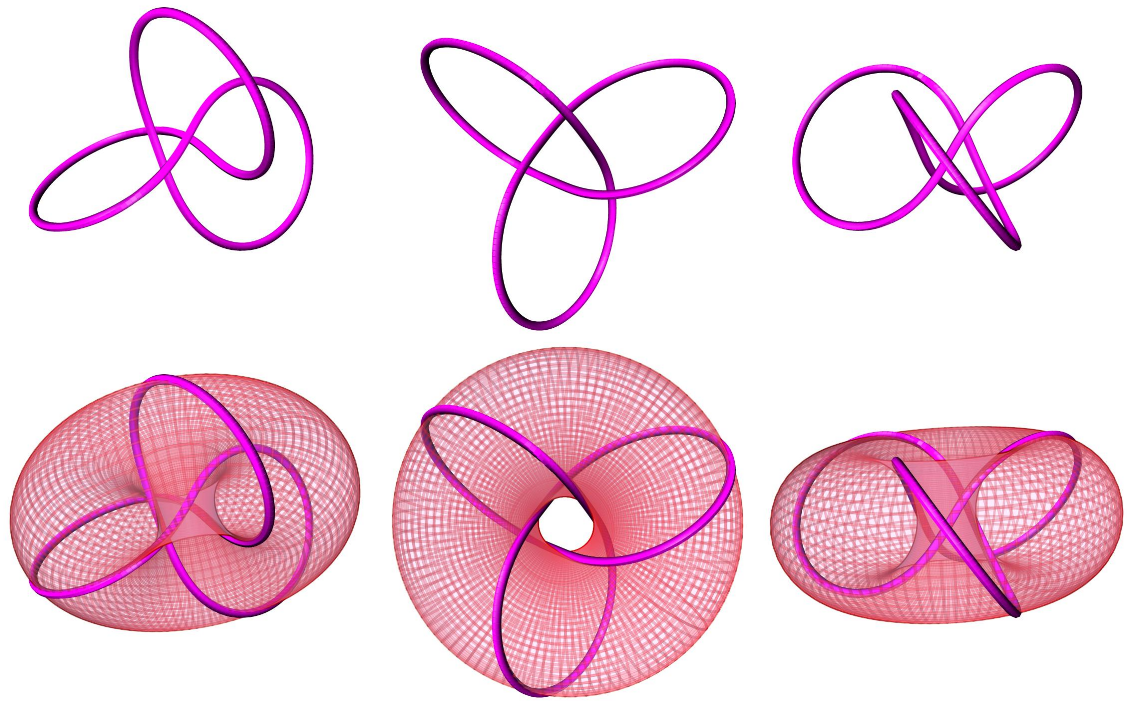


Figure 5.4: The torus knot $\mathcal{K}$ viewed from different angles and together with the underlying torus.

5.1 I curve

Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a smooth immersion with nowhere vanishing curvature. Let $\mathcal{R}_I(\gamma)$ be the set of all those points $v \in \mathbb{S}^2$ (regarded as unit vectors) such that the curve γ_v, obtained through the orthogonal projection onto $v^\perp$, fails to be an immersion. Clearly, $\mathcal{R}_I(\gamma)$ is just the set of all unit tangent directions to γ, i.e.,

$$\mathcal{R}_I(\gamma) = \{v \in \mathbb{S}^2 \mid \gamma_v \text{ is not an immersion}\} = \left\{\pm\frac{\gamma'(t)}{\|\gamma'(t)\|} \mid t \in \mathbb{S}^1\right\}.$$

The above set for the knot $\mathcal{K}$ is shown in Figure 5.5. The notation will be shortened to $\mathcal{R}_I := \mathcal{R}_I(\gamma)$ whenever no ambiguity arises. A natural parametrization of $\mathcal{R}_I$ is given by $\pm\gamma' : \mathbb{S}^1 \to \mathbb{R}^3$. Call the set $\mathcal{R}_I$ *the Reidemeister I curve of* γ or simply *the I curve.*

Note that the parametrization $\pm\gamma'$ of the I curve of γ is an immersion since γ has nowhere vanishing curvature. The inflection points (of this parametrization) of the I curve correspond to the torsion vanishing points. If, moreover, γ satisfies the condition

$$\tau(s) = 0 \implies \tau'(s) \neq 0$$

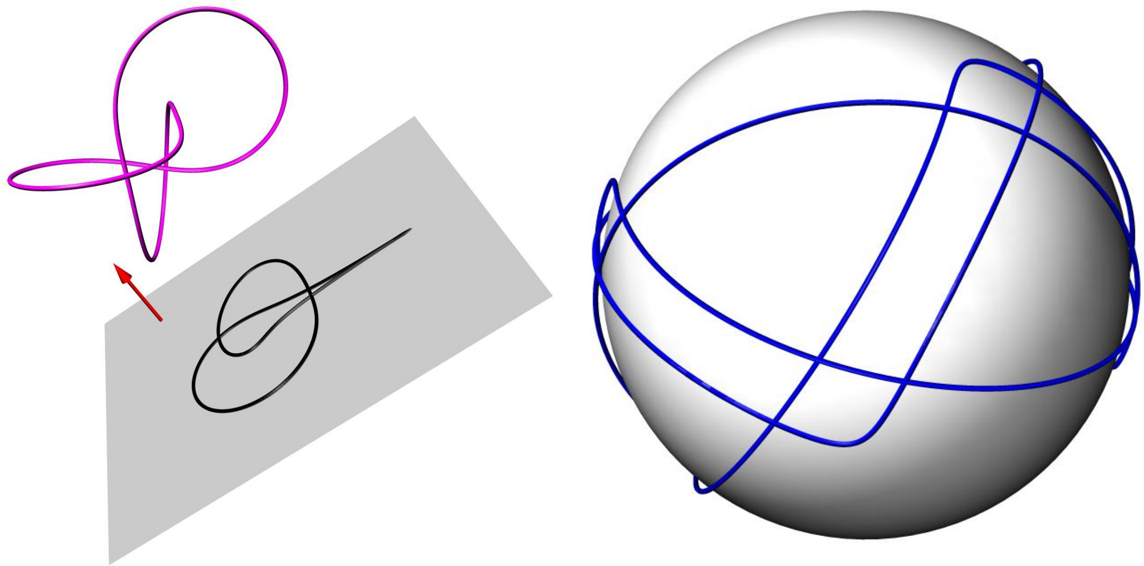


Figure 5.5: The trefoil knot $\mathcal{K}$ with an orthogonal projection (left) characterizing the Reidemeister I curve (right).

for all $s \in \mathbb{S}^1$, then the inflection points of the I curve are all regular and there are two types of singularities that γ_v with $v \in \mathcal{R}_I$ may experience. Namely, recall the two types of cusps pictured in Figure 1.5 and suppose that $\gamma_v'(s) = 0$. Then locally the image of γ_v around $\gamma_v(s)$ looks like the cusp of type 1, provided $\tau(s) \neq 0$ and like the cusp of type 2, provided $\tau(s) = 0$.

5.2 II curve

Consider a curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ in general position as defined in Definition 4.1. What is the set of all unit vectors $v \in \mathbb{S}^2$ such that the curve γ_v obtained from γ by orthogonal projection onto $v^\perp$ has a non-transverse double point (as, e.g., on the left of Figure 5.6)? For this to happen, there must exist a plane tangent to γ at two distinct points, i.e.,

$$[\gamma'(t), \gamma'(s), \gamma(t) - \gamma(s)] = 0 \quad \text{for some } s \neq t \in \mathbb{S}^1.$$

Then the direction v is simply the segment $\gamma(s) - \gamma(t)$ normalized to the unit length.

Definition 5.1. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a curve in generic position.

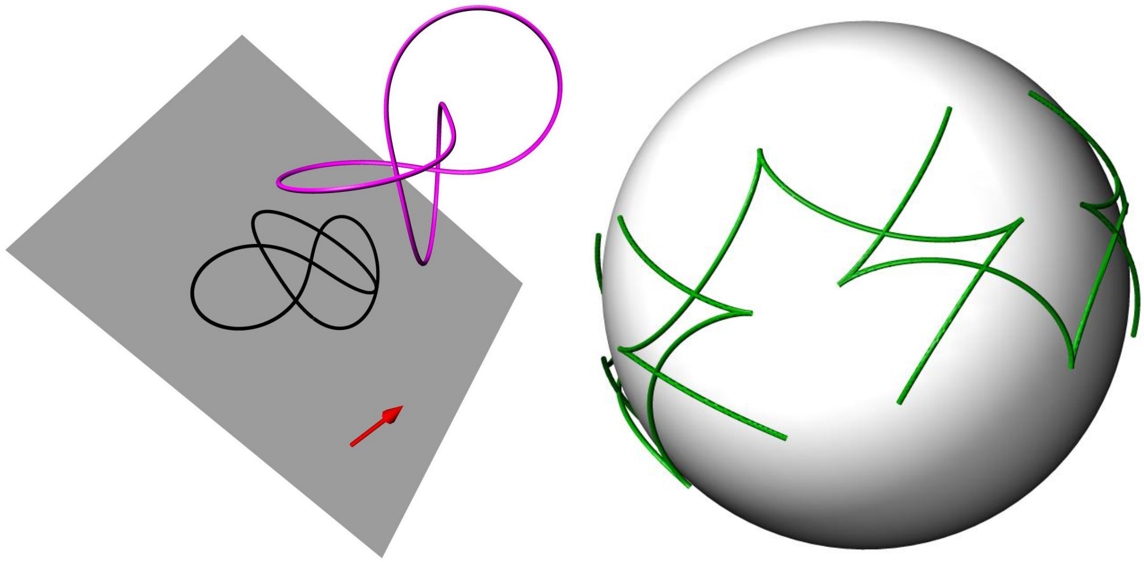


Figure 5.6: The trefoil knot $\mathcal{K}$ with an orthogonal projection (left) characterizing the Reidemeister II curve (right) of $\mathcal{K}$.

(i) Define $b : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ by $b(s,t) = [\gamma'(t), \gamma'(s), \gamma(t) - \gamma(s)]$ and let

$$\mathcal{B}(\gamma) := \left\{(s,t) \in \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta \mid b(s,t) = 0\right\}. \tag{5.1}$$

(ii) Let $f : \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta \to \mathbb{S}^2 \subset \mathbb{R}^3$ be the Gauss map

$$f(s,t) = \frac{\gamma(t) - \gamma(s)}{\|\gamma(t) - \gamma(s)\|}. \tag{5.2}$$

(iii) Call the set given by

$$\mathcal{R}_{II}(\gamma) := f(\mathcal{B}(\gamma)) = \{v \in \mathbb{S}^2 \mid \gamma_v \text{ has non-transverse double points}\}$$

the Reidemeister II curve of γ or shortly the *II curve.*

In the further work $\mathcal{B}(\gamma)$ will often be replaced simply by $\mathcal{B}$ and $\mathcal{R}_{II}(\gamma)$ by $\mathcal{R}_{II}$ whenever no confusion is created. The II curve of the trefoil knot $\mathcal{K}$ is presented on the right of Figure 5.6. In order to justify the usage of the name "curve", it needs to be shown that a parametrization of the set $\mathcal{R}_{II}$, with a 1-manifold as the domain, can be found.

Lemma 5.2. *Let a curve* $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ *be in general position. Then the set* $\mathcal{B}(\gamma)$ *is a* *properly embedded smooth submanifold of* $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$ *of dimension 1. Call* $\mathcal{B}(\gamma)$ *the* bitangency manifold *of* γ.

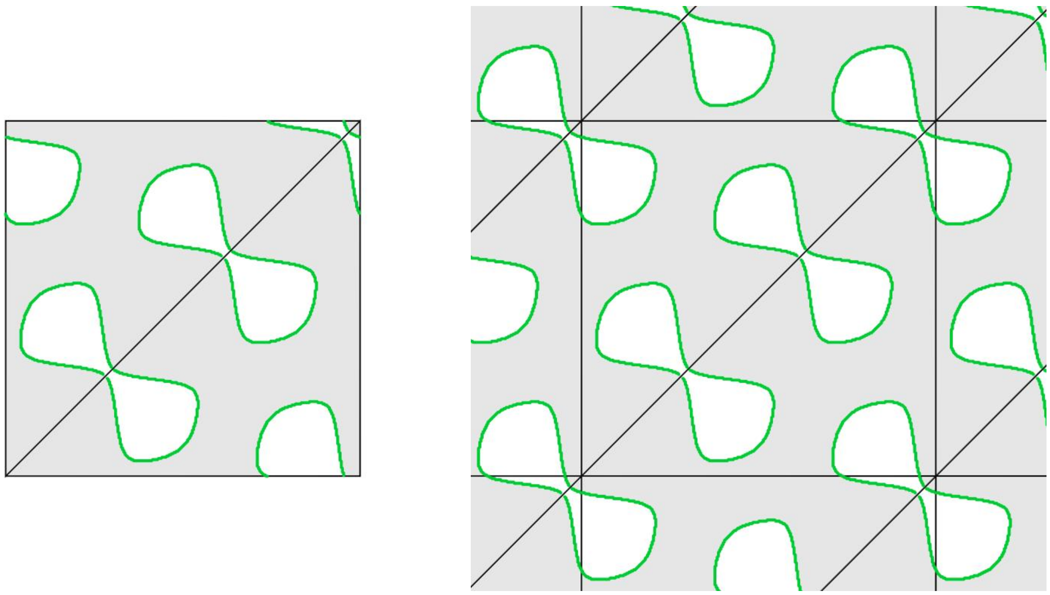


Figure 5.7: The bitangency manifold of the knot $\mathcal{K}$: in $\mathbb{S}^1 \times \mathbb{S}^1$ (left) and pulled back to $\mathbb{R} \times \mathbb{R}$ (right).

The bitangency manifold of the knot $\mathcal{K}$ is shown in Figure 5.7. Before proceeding with the proof of Lemma 5.2, one more map and some identities are stated that will be useful throughout Part II of this thesis.

Definition 5.3. Define $g : \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta \to \mathbb{R}$ by

$$g(s,t) = \frac{b(s,t)}{\|\gamma(t) - \gamma(s)\|^3}. \tag{5.3}$$

Note that the partial derivatives of f are

$$\begin{aligned} f_s(s,t) &= \quad \frac{1}{\|\gamma(s) - \gamma(t)\|}\Big(\gamma'(s) - \langle \gamma'(s), f\rangle\, f\Big)(s,t) \\ f_t(s,t) &= -\frac{1}{\|\gamma(s) - \gamma(t)\|}\Big(\gamma'(t) - \langle \gamma'(t), f\rangle\, f\Big)(s,t). \end{aligned} \tag{5.4}$$

An elementary computation then leads to the following identity

$$g(s,t) = [f_s, f_t, f](s,t). \tag{5.5}$$

The partial derivatives of f of the second order are

$$
\begin{aligned}
f_{ss}(s,t) &= \frac{1}{\|\gamma(s)-\gamma(t)\|}\Big(\gamma''(s) - 2\langle\gamma'(s), f\rangle f_s - \langle\gamma''(s), f\rangle f - \langle\gamma'(s), f_s\rangle f\Big)(s,t)\\
f_{tt}(s,t) &= -\frac{1}{\|\gamma(s)-\gamma(t)\|}\Big(\gamma''(t) - 2\langle\gamma'(t), f\rangle f_t - \langle\gamma''(t), f\rangle f - \langle\gamma'(t), f_t\rangle f\Big)(s,t)\\
f_{st}(s,t) &= \frac{1}{\|\gamma(s)-\gamma(t)\|}\Big(\langle\gamma'(t), f\rangle f_s - \langle\gamma'(s), f_t\rangle f - \langle\gamma'(s), f\rangle f_t\Big)(s,t)\\
&= \frac{1}{\|\gamma(s)-\gamma(t)\|}\Big(\langle\gamma'(t), f\rangle f_s + \langle\gamma'(t), f_s\rangle f - \langle\gamma'(s), f\rangle f_t\Big)(s,t).
\end{aligned}
\tag{5.6}
$$

Under the assumption that $g(s,t)=0$, one has:

$$\big[\gamma''(s), \gamma'(t), \gamma(s)-\gamma(t)\big] = 0 \iff \big[f_{ss}, f_t, f\big](s,t) = 0 \tag{5.7}$$

$$\big[\gamma'(s), \gamma''(t), \gamma(s)-\gamma(t)\big] = 0 \iff \big[f_s, f_{tt}, f\big](s,t) = 0. \tag{5.8}$$

Proof of Lemma 5.2. Since $\mathcal{B} := \mathcal{B}(\gamma) = b^{-1}(0) \setminus \Delta$, it remains to check that for a curve γ in general position one of the partial derivatives b_s or b_t is non-zero at every point of $\mathcal{B}$. The result will then follow from the Regular Value Theorem (see, e.g., Spivak [35, Proposition 12, Chapter 2]). At a point $(s,t) \in \mathcal{B}$

$$
\begin{aligned}
b_s(s,t) &= [\gamma'(t), \gamma''(s), \gamma(t)-\gamma(s)] \quad \text{and}\\
b_t(s,t) &= [\gamma''(t), \gamma'(s), \gamma(t)-\gamma(s)].
\end{aligned}
$$

Simultaneous vanishing of b_s and b_t at a point of $\mathcal{B}$ is excluded by Lemma 4.2. $\square$

Let B be a set homeomorphic to $\mathcal{B}$. Then the name of $\mathcal{R}_{II}$ is justified as it is the image of a curve given by $f \circ \varphi : B \to \mathbb{S}^2$, where $\varphi : B \to \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$ is an immersion satisfying $\varphi(B) = \mathcal{B}$. The following corollary is an immediate consequence of the proof of Lemma 5.2 and the identity (5.3).

Corollary 5.4. *The vector field $v : \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta \to \mathbb{R}^2$ given by*

$$v(p) = (-g_t(p), g_s(p))$$

is tangent to the manifold $\mathcal{B}$. Moreover, $v(p) \neq 0$ at each $p \in \mathcal{B}$.

Remark 5.5. Note that at a point $p = (s,t) \in \mathcal{B}$, by the genericity conditions, $g_s(p)$ and $g_t(p)$ cannot both vanish at the same time. The same is true for $f_s(p)$ and $f_t(p)$.

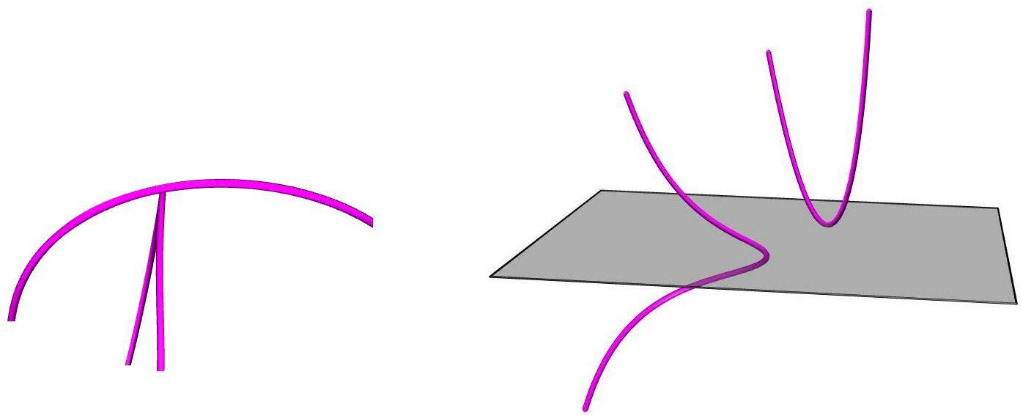


Figure 5.8: The two possibilities of a zero of g_s in $\mathcal{B}$: a cross-tangent (left) and an osculating bitangent plane (right).

If $g_s(p) = 0$ then either the bitangent plane spanned by f, f_s and f_t at p is simultaneously an osculating plane to γ at $\gamma(s)$ or $f_t(p) = 0$. In the latter case $\gamma'(t)$ is parallel to $\gamma(s)-\gamma(t)$ and it is called a *cross-tangent* (see Figure 5.8). In fact, later in Lemma 5.18, it will be shown that the cross-tangents of a generic curve are isolated.

Let $\mu : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$, $\mu(s,t) = (t,s)$ denote the reflection in $\mathbb{S}^1 \times \mathbb{S}^1$ with respect to Δ. Since there always exist some supporting planes to γ, the bitangency manifold $\mathcal{B}$ is non-empty and, in particular, consists of at least one component. By symmetry of the bitangency function b w.r.t. the diagonal Δ, i.e., $b \circ \mu = \mu$, so is $\mathcal{B}$ symmetric w.r.t. Δ, i.e., $\mu(\mathcal{B}) = \mathcal{B}$. Can a component C of $\mathcal{B}$ be symmetric to itself? Lemma 5.7 below gives an answer to this question.

Remark 5.6. Note that μ reverses the orientation of $\mathbb{S}^1 \times \mathbb{S}^1$. Let $\phi : A \to C$ be an immersion of a component $C \subset \mathcal{B}$ (with A homeomorphic to an open interval or a circle). Then b has opposite signs (sufficiently close) on both sides of C. If b is positive (negative) to the right of C, then b is negative (positive) to the right of $\mu(C)$.

Lemma 5.7. *Let γ be a curve in general position. Let μ be the reflection with respect to Δ. The bitangency manifold $\mathcal{B}$ of γ is non-empty. For any component C of $\mathcal{B}$*

$$\mu(C) \neq C.$$

Hence, $\mathcal{B}$ has an even number of components.

Proof. Suppose C is a non-empty component of $\mathcal{B}$ with $\mu(C) = C$. Then, for some $p = (s,t) \in C$ also $\bar{p} := \mu(p) \in C$ and $p \neq \bar{p}$. Hence, there exists an injective path in C joining p and $\bar{p}$. Let $\psi : [0,1] \to C$ be a parametrization of this path with

$\psi'(a) \neq 0$ for all $a \in [0,1]$, $p = \psi(0)$ and $\bar{p} = \psi(1)$. Denote by Ψ the image $\psi([0,1])$. Now, $\mu \circ \psi$ is a path in C from $\bar{p}$ to p. If C is homeomorphic to an open interval, then $\mu(\Psi)$ must be the same as Ψ. In case when C is homeomorphic to a circle, then there are the two possibilities: either (i) $\mu(\Psi) = \Psi$ or (ii) $\mu(\Psi) \cup \Psi = C$. Suppose that (ii) holds. Then

$$\mathbb{S}^1 \to C, \quad t \mapsto \begin{cases} \psi(2t) & \text{if } t \in [0, 1/2] \\ \mu \circ \psi(2t-1) & \text{if } t \in [1/2, 1] \end{cases}$$

is a parametrization of C that contradicts Remark 5.6. Hence, in both cases $\mu(\Psi) = \Psi$. Now, if a point q traverses Ψ starting at p with orientation prescribed by ψ, then the point $\mu(q)$ traverses Ψ starting at $\bar{p}$ with orientation opposite to that of ψ. Since both points q and $\bar{q}$ traverse the entire path Ψ in the opposite directions, there must be a point $q_0 \in \Psi$ with $q_0 = \mu(q_0)$. However, the only fixed points of μ are all points of Δ but this is in contradiction with $C \cap \Delta = \emptyset$. $\square$

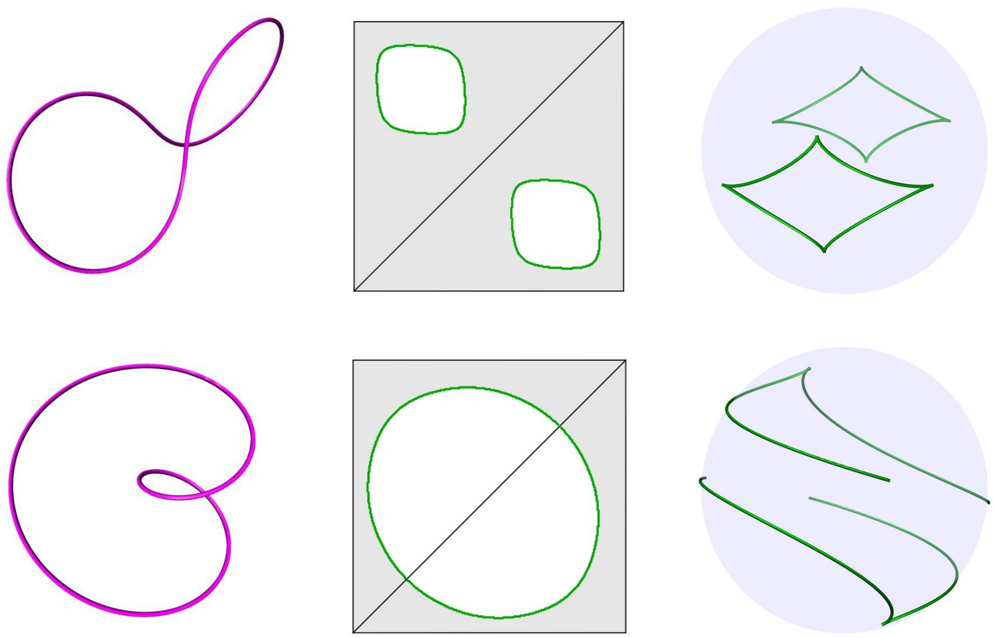


Figure 5.9: A generic space curve (left), its bitangency manifold $\mathcal{B}$ (center) and its II curve (right). The top curve is a (1,2)- and the bottom curve is a (2,1)-curve on a round torus.

The components of the bitangency manifold may be as well homeomorphic to an open interval or to $\mathbb{S}^1$, as shown by the examples in Figure 5.9. The author is convinced that a (1,1)-torus curve as a bitangency component is possible, but has no example of a generic curve with such a component at hand. (There are numerical examples of space curves possessing such components but it is difficult to determine whether these space curves are generic.)

Bitangent planes

Given $p=(s,t)\in\mathcal{B}$ denote the plane through the origin spanned by $\gamma'(s),\gamma'(t)$ and $\gamma(s)-\gamma(t)$ (or equivalently by f_s, f_t and f at p) by $\omega(p)$. Denote the affine plane through $f(p)$ parallel to $\omega(p)$ - the actual bitangent plane - by $\omega_f(p)$. In this work both of these planes will be referred to as *bitangent planes*. To each plane $\omega(p)$ or $\omega_f(p)$ choose a normal vector $\nu(p)$ and associate an index given by

$$\operatorname{ind}(\omega(p)) = \operatorname{ind}(\omega_f(p)) = \operatorname{sign}\langle f_{ss},\nu\rangle\langle f_{tt},\nu\rangle(p). \tag{5.9}$$

This definition does not depend on the choice of ν. In particular, this index distinguishes the three types of bitangent planes of a generic curves, namely

+1: the two arcs tangent to the bitangent plane $\omega_f(p)$ lie on the same side of $\omega_f(p)$;

0: one of the arcs tangent to $\omega_f(p)$ has order of contact 2 with the plane;

-1: the two tangent arcs lie on the opposite sides of $\omega_f(p)$.

Call a bitangent plane with index 1 *external* and with index -1 *internal*. A bitangency plane with index 0 is a plane which osculates at either $\gamma(s)$ or $\gamma(t)$. See Figure 5.10.

In particular, the normal vector ν can be chosen, e.g., to be $\gamma'(s)\times(\gamma(s)-\gamma(t))$, whenever non-zero. Hence, the index of a bitangent plane can be equivalently described as

$$\operatorname{ind}(\omega(p)) = \operatorname{sign}\begin{cases}[f,f_s,f_{ss}][f,f_s,-f_{tt}](p) & \text{if } f_s(p)\neq 0\\ [f,f_t,f_{ss}][f,f_t,-f_{tt}](p) & \text{if } f_t(p)\neq 0\end{cases} \tag{5.10}$$

$$= \operatorname{sign}\begin{cases}[\gamma(s)-\gamma(t),\gamma'(s),\gamma''(s)][\gamma(s)-\gamma(t),\gamma'(s),\gamma''(t)] & \text{if } \gamma'(s)\times(\gamma(s)-\gamma(t))\neq 0\\ [\gamma(s)-\gamma(t),\gamma'(t),\gamma''(s)][\gamma(s)-\gamma(t),\gamma'(t),\gamma''(t)] & \text{if } \gamma'(t)\times(\gamma(s)-\gamma(t))\neq 0.\end{cases}$$

The closure of the bitangency manifold

The closure of $\mathcal{B}$ in $\mathbb{S}^1\times\mathbb{S}^1$ is characterized by the following lemma.

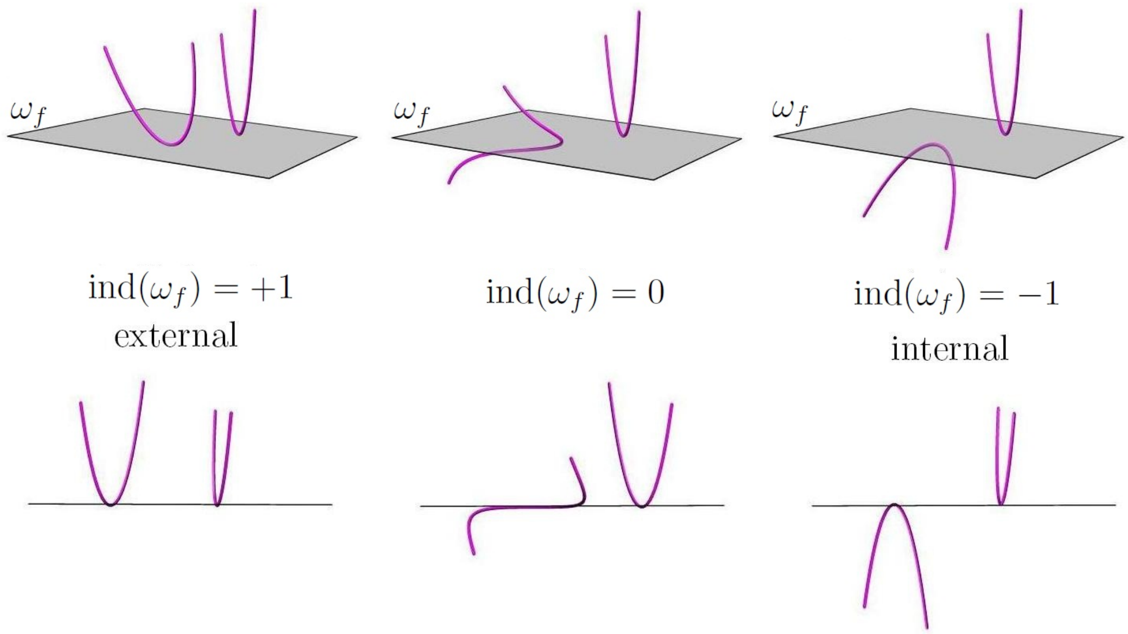


Figure 5.10: Types of bitangent planes and their indices.

Lemma 5.8. *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a curve in general position. The closure $\bar{\mathcal{B}}$ of the bitangency manifold $\mathcal{B}$ in $\mathbb{S}^1 \times \mathbb{S}^1$ is given by*

$$\bar{\mathcal{B}} = \mathcal{B} \cup \{(s,s) \in \Delta \mid \tau(s) = 0\}.$$

Moreover, $\bar{\mathcal{B}} \subset \mathbb{S}^1 \times \mathbb{S}^1$ is a smooth compact submanifold without boundary of dimension 1. For $p \in \mathcal{B}$ in a sufficiently small neighborhood of Δ the bitangent plane $\omega_f(p)$ is external. Since $\bar{\mathcal{B}}$ is compact it has finitely many components.

Proof. By Ozawa's proof of Lemma 5.2 in [25] the above statement is true for a curve γ in general position which additionally satisfies:

$$\textit{Each plane in } \mathbb{R}^3 \textit{ has total order of contact with } \gamma \textit{ less than } 4. \tag{G3}$$

In fact this particular proof of Ozawa's paper uses a much weaker condition than this global one. Instead of requiring (G3) it requires only:

$$\textit{Each plane } P \subset \mathbb{R}^3 \textit{ has order of contact with } \gamma \textit{ less than } 4 \textit{ at each } p \in \gamma(\mathbb{S}^1) \cap P. \tag{G4}$$

This condition (G4) is satisfied by γ in generic position since $\tau(s) = 0$ implies $\tau'(s) \neq 0$ for all $s \in \mathbb{S}^1$. Hence, the proof of Ozawa will work equally well to prove Lemma 5.8, i.e., in the case when γ is generic according to Definition 4.1 and does not necessarily satisfy (G3).

Ozawa uses in his proof the theory of unfoldings of smooth maps. An unfolding of a smooth function f is briefly speaking a family of functions containing f. The family unfolds to reveal all these functions which are in close relation with f. The reader is asked to refer to [10] for details. The proof of Ozawa will be quoted below so that the reader can convince himself or herself that (G3) can be weakened to (G4). An explanation of the terminology will be omitted as it would be too extensive.

Ozawa's proof: Suppose $t_0 \in \mathbb{S}^1$ with $\tau(t_0)$. For each unit vector $z \in \mathbb{S}^2 \subset \mathbb{R}^3$, define a smooth function $h_z : \mathbb{S}^1 \to \mathbb{R}$, by $h_z(s) = \langle \gamma(t_0 + s) - \gamma(t_0), z \rangle$. At $(s, z) = (0, B(t_0))$, where B is the binormal vector

$$h_z' = h_z'' = h_z''' = 0 \text{ and } h_z^{IV} \neq 0.$$

Hence, if h_z is regarded as an unfolding of $h_{B(t_0)}$ with the parameter $z \in \mathbb{S}^2$, $z \mapsto h_z$ is a versal unfolding, and equivalent to the unfolding $(u, v) \mapsto H_{(u,v)}$ defined by

$$H_{(u,v)}(w) = (w^4/4) + (uw^2/2) + vw \text{ for } u, v, w \in \mathbb{R}.$$

For each $a \in \mathbb{R}$, the function $H_{(-a^2,0)}$ has three critical points $-a$, 0, a, and $-a$ and a have the same critical value. This and the equivalence of the unfoldings imply that there exist two smooth functions: $\mathbb{R} \ni a \mapsto s_i(a) \in \mathbb{S}^1$ $(i = 1, 2)$ defined on a small neighborhood of $0 \in \mathbb{R}$ such that

(i) $s_1(0) = s_2(0) = 0$,

(ii) the derivatives $s_1'(0)$ and $s_2'(0)$ are non-zero, and have opposite signs,

(iii) there exists a certain $z = z(a) \in \mathbb{S}^2$ smoothly dependent on $a \in \mathbb{R}$ with $z(0) = B(t_0)$ such that the function $f_{z(a)}$ has three critical points $s_1(a)$, 0, $s_2(a)$ in a certain neighborhood of 0 in $\mathbb{S}^1$ and $s_1(a)$ and $s_2(a)$ have the same critical value.

For $a \neq 0$, the second derivatives of h_z at $s_1(a)$ and $s_2(a)$ are non-zero. Hence, $(t_0 + s_1(a), t_0 + s_2(a)) \in \mathcal{B}$ for each $a \in \mathbb{R}$ with $a \neq 0$. [In particular, for a sufficiently close to a, the bitangent plane $\omega_f(a)$ is external.] Therefore, the pair (t_0, t_0) with $\tau(t_0)$ is on $\bar{\mathcal{B}}$, and near (t_0, t_0), $\bar{\mathcal{B}}$ is a smooth manifold. Other pairs (t, t) with $\tau(t) \neq 0$ are not on $\bar{\mathcal{B}}$ since the functions $G_u(w) = w^3 + uw$ for any fixed u cannot have a pair of critical points with the same critical value. This proves the lemma. $\square$

Remark 5.9. The above lemma implies that the endpoints of the closure of the II curve $\mathrm{cl}(\mathcal{R}_{II})$ lie on the I curve, more precisely at the I curve's inflection points.

The bitangency manifold $\mathcal{B}$ of a curve in generic position can be oriented. For the purpose of the thesis one choice of orientation is particularly useful. By Lemma 5.7 the components of $\mathcal{B}$ come in pairs. Call $\{C, \mu(C)\}$ for some component $C \subset \mathcal{B}$ a *symmetric pair of components* of $\mathcal{B}$. Out of each pair pick one component, e.g., C and call it the *main* one. Denote the component $\mu(C)$ by $\bar{C}$ and call it the *symmetric* component of the pair. Orient the main component C respectively $\bar{C}$ in such a way that the function b (or g) is positive respectively negative to the right (as shown in Figure 5.11). The image of the main components of $\mathcal{B}$ under the map f will also be referred to as the *main components of the II curve.*

Remark 5.10. By Lemma 5.8 the bitangent planes $\omega_f(p)$ for points $p \in \mathcal{B}$ sufficiently close to a point (s_0, s_0) with $\tau(s_0) = 0$ are all external. Moreover, note that the planes $\omega(p)$ converge to the osculating plane of γ at s_0 as p tends to (s_0, s_0). More precisely, for $(s,t) \in \mathcal{B}$ sufficiently close to (s_0, s_0)

$$\omega(s,t) = \operatorname{span}\left\{ \frac{\gamma'(s) - \gamma'(t)}{\|\gamma'(s) - \gamma'(t)\|}, \gamma'(s) \right\}$$

and as $(s,t) \to (s_0, s_0)$

$$\omega(s,t) \quad \to \quad \omega(s_0, s_0) := \operatorname{span}\left\{\gamma''(s_0), \gamma'(s_0)\right\}.$$

Denote the set of all great circles on $\mathbb{S}^2$ representing the osculating planes at torsion vanishing points of γ by $\mathcal{P}^{\tau=0}_{\mathrm{osc}}(\gamma)$, i.e.,

$$\mathcal{P}^{\tau=0}_{\mathrm{osc}}(\gamma) = \bigcup_{s \in \mathbb{S}^1, \tau(s)=0} \mathbb{S}^2 \cap \omega(s,s).$$

Tangent developable surface versus II curve

There is a correspondence between the bitangency manifold $\mathcal{B}$ of a generic curve γ and the set of double points of the tangent developable surface of γ (compare also with Remark 4.3). The tangent developable surface of γ is given by

$$\mathbb{S}^1 \times \mathbb{R} \to \mathbb{R}^3,\ (s, \lambda) \mapsto \gamma(s) + \lambda\gamma'(s).$$

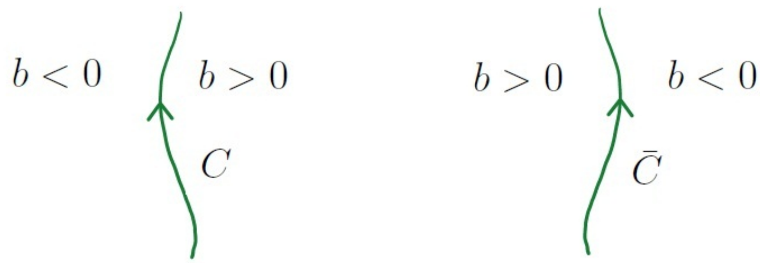


Figure 5.11: Orientation of the bitangency manifold.

For each pair $(s,t) \in \mathcal{B}$ such that the tangent vectors $\gamma'(s)$ and $\gamma'(t)$ are not parallel, the tangent lines through the points $\gamma(s)$ and $\gamma(t)$ meet at some point. This point will be a self-intersection point of the tangent developable surface of γ. On the other hand, a self-intersection point of the tangent developable surface, say

$$\gamma(s) + \lambda_1\gamma'(s) = \gamma(t) + \lambda_2\gamma'(t),$$

for some $s \neq t$ suggests that the three vectors $\gamma'(s)$, $\gamma'(t)$ and $\gamma(s) - \gamma(t)$ span a plane. Hence, $(s,t) \in \mathcal{B}$.

Lemma 5.11. *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a curve in generic position. The set*

$$\bar{\mathcal{B}} \setminus \{(s,t) \in \mathbb{S}^1 \times \mathbb{S}^1 \mid \gamma'(s) \times \gamma'(t) = 0\}$$

modulo μ (the reflection w.r.t. Δ) is in 1-1 correspondence with the set of self-intersection points of the tangent developable surface of γ.

With the above correspondence, Lemma 5.8 has a nice geometric interpretation in the local form of the tangent developable surface close to the curve γ. Namely, Cleave [14] proved that the tangent developable surface of a generic curve γ, in a neighborhood of each point $\gamma(s)$, can be written, in a suitable system of coordinates x, y, z in $\mathbb{R}^3$, in the following normal forms

1. $(x,y) \mapsto (x, y^2, y^3)$ if $\tau(s) \neq 0$,

2. $(x,y) \mapsto (x, y^2, xy^3)$ if $\tau(s) = 0$ and $\tau'(s) \neq 0$.

Both local forms are illustrated in Figure 5.12. More details can be found in [14].

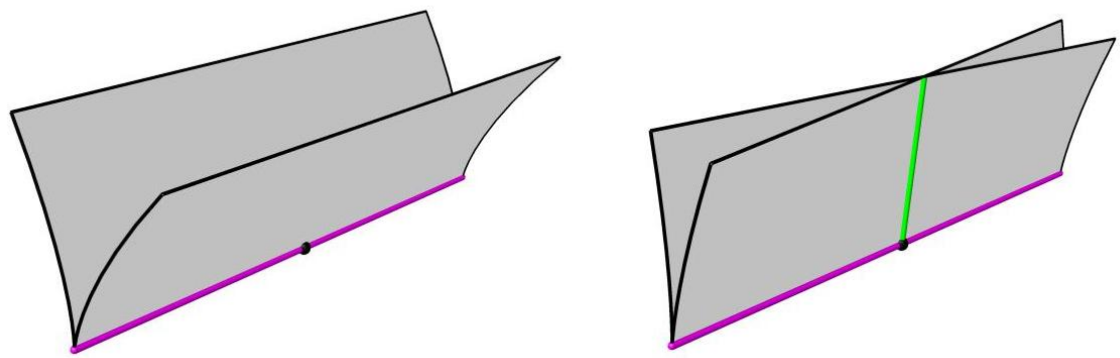


Figure 5.12: The local form of the tangent developable surface around a point of vanishing torsion (left) and around a non-degenerate point of vanishing torsion (right).

Parametrization of the II curve

From now on, suppose that given $p \in C_i \subset \mathcal{B}$, where C_i is a main component of $\mathcal{B}$, the map $\phi = (s,t) : (-\varepsilon, \varepsilon) \to \mathcal{B}$ is a local parametrization of the bitangency manifold $\mathcal{B}$ at $p = \phi(0)$ with a small $\varepsilon > 0$ and such that $\phi'(a) = (-g_t, g_s)(\phi(a))$. Note that this choice of ϕ' agrees with the convention chosen for the orientation of $\mathcal{B}$ (compare Figure 5.11). If p belonged to one of the symmetric components $\bar{C}_i$, the tangent map ϕ' would have to be chosen to be of the opposite sign in order to preserve the orientation. Let $u := f \circ \phi : (-\varepsilon, \varepsilon) \to \mathbb{S}^2$ be a local parametrization of a main component of the II curve at some point $q = u(0)$. Then, for instance, $\omega_f \circ \phi$ is a path in the space of all affine planes in $\mathbb{R}^3$.

Now, the differentiation of u w.r.t. the parameter a gives

$$u'(a) = (-g_t f_s + g_s f_t)(\phi(a)). \tag{5.11}$$

Corollary 5.12. *Let ϕ be as above and $u = f \circ \phi$. If $u'(a) \neq 0$, then*

$$\operatorname{span}\{u(a), u'(a)\} = \operatorname{span}\{\gamma'(s), \gamma'(t), \gamma(s) - \gamma(t)\} = \operatorname{span}\{f, f_s, f_t\}(s,t).$$

In particular, this means that the great circle tangent to the II curve at $f(s,t)$ spans the plane $\omega(a)$, i.e., the plane through the origin parallel to the bitangent plane tangent to γ at $\gamma(s)$ and $\gamma(t)$. This dependency can be observed in Figure 5.13.

The shape of the II curve of a trefoil knot depicted in that same Figure 5.13 with cusps suggests that the parametrization of the II curve might not necessarily be an immersion. What are the points of $\mathcal{B}$ with $u'(a) = 0$? To answer this question consider the orthogonal projection of the curve γ into the plane perpendicular to

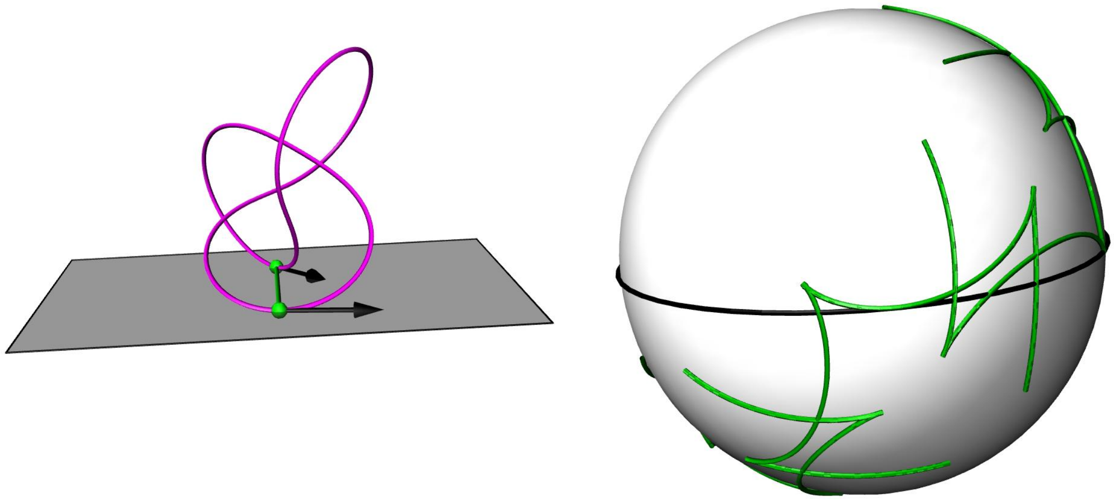


Figure 5.13: A plane spanned by a great circle tangent to the II curve (right) corresponds to a bitangent plane (left).

$u(a) = f(\phi(a))$. The goal is to investigate the two arcs of the curve $\gamma_{u(a)}$ (as defined by (4.1)) locally at the point of self-tangency (see Figure 5.14).

In general $\gamma_{u(a)}$ is not parametrized by arc-length. By definition $\gamma_u(s) = \gamma_u(t)$ and $\gamma_u'(s)||\gamma_u'(t)$ and

$$\begin{aligned} f_s \circ \phi &= \quad \frac{1}{\|\gamma(s) - \gamma(t)\|}{\gamma_u}'(s), \\ f_t \circ \phi &= -\frac{1}{\|\gamma(s) - \gamma(t)\|}{\gamma_u}'(t) \end{aligned} \tag{5.12}$$

at each s, t with $\phi(a) = (s,t) \in \mathcal{B}$.

Observe that at each point $p = (s,t) \in \mathcal{B}$ the partial derivatives $f_s(p)$ and $f_t(p)$ are collinear (compare with (5.4)). Hence, let

$$\begin{aligned} \sigma(p) &:= -\operatorname{sign}\Big\langle f_s(p), f_t(p)\Big\rangle \\ &= \quad \operatorname{sign}\Big\langle \gamma_u'(s), \gamma_u'(t)\Big\rangle. \end{aligned} \tag{5.13}$$

Adopt the notation of Arnold from Chapter 3 and call a bitangent pair $p = (s,t) \in \mathcal{B}$, the bitangent plane $\omega_f(p)$ and the plane $\omega(p)$ with $\sigma(p) = 1$ *direct* and with $\sigma(p) = -1$ *inverse*. In case of a cross-tangent, i.e., when $\gamma'(s)$ or $\gamma'(t)$ is collinear with $\gamma(s) - \gamma(t)$, $\sigma(p)$ degenerates to zero (see Figure 5.15). Before proceeding with further investiga-

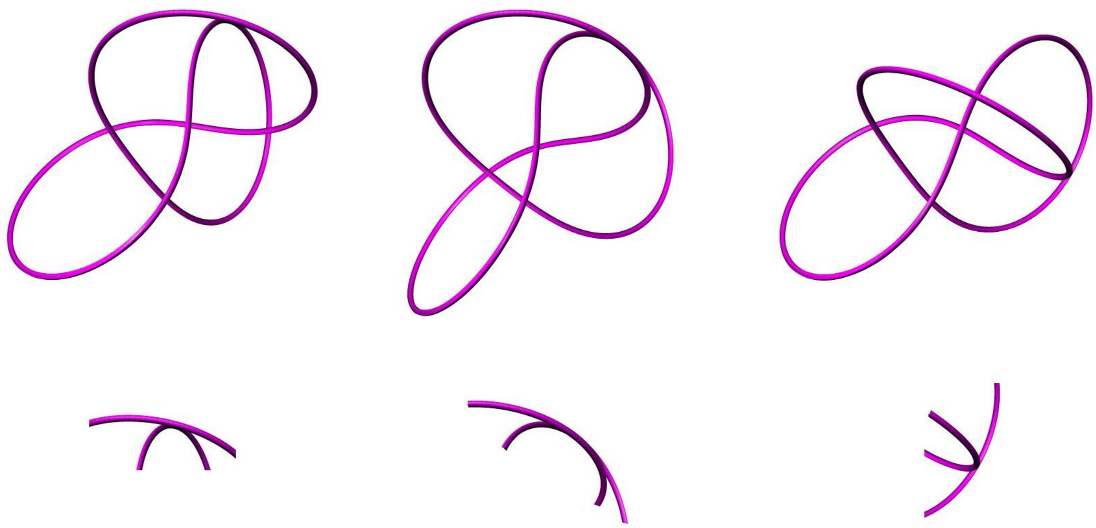


Figure 5.14: Projections onto the plane perpendicular to $u(a)$ (top) give locally two tangent arcs (bottom). In the middle picture the two arcs have 2^{nd} order of contact.

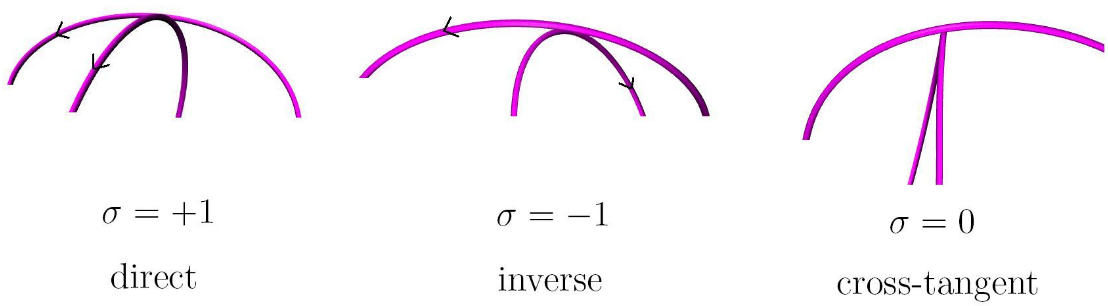


Figure 5.15: The three possibilities of σ: direct and inverse bitangent planes and a cross-tangent.

tion some useful identities are summarized below. At $p = (s, t) \in \mathcal{B}$ with $\sigma(p) \neq 0$

$$\begin{aligned} \frac{f_s(p)}{\|f_s(p)\|} &= \operatorname{sign}\langle f_s(p), f_t(p)\rangle \frac{f_t(p)}{\|f_t(p)\|} \\ &= -\sigma(p)\frac{f_t(p)}{\|f_t(p)\|} \end{aligned} \tag{5.14}$$

and further

$$
\begin{aligned}
[f, f_s, f_{ss}] &= \quad \sigma[f_{ss}, f_t, f]\frac{\|f_s\|}{\|f_t\|} = \quad \sigma g_s\frac{\|f_s\|}{\|f_t\|} \\
[f, f_t, f_{tt}] &= -\sigma[f_s, f_{tt}, f]\frac{\|f_t\|}{\|f_s\|} = -\sigma g_t\frac{\|f_t\|}{\|f_s\|}
\end{aligned} \tag{5.15}
$$

$$
[f, f_s, f_{ss}][f, f_s, f_{tt}] = \sigma g_s g_t\frac{\|f_t\|}{\|f_s\|}. \tag{5.16}
$$

Hence,

$$
\operatorname{ind}(\omega) = -\sigma\,\operatorname{sign}(g_s g_t) \tag{5.17}
$$

and

$$
\operatorname{sign}[f, f_s, f_{ss}] = \sigma\,\operatorname{ind}(\omega)\,\operatorname{sign}[f, f_t, f_{tt}]. \tag{5.18}
$$

Lemma 5.13. *Let $u = f \circ \phi : (-\varepsilon, \varepsilon) \to \mathbb{S}^2$ be a local parametrization of a main component of the II curve as above. Let κ^a denote the unsigned curvature of the curve $\gamma_{u(a)}$. Then,*

$$
\begin{aligned}
u'(a) = 0 \quad \iff \quad & \kappa^a(s(a)) = \kappa^a(t(a)) \neq 0 \textit{ and the plane } \omega(\phi(a)) \textit{ is external,} \\
& \textit{i.e., the curve } \gamma_{u(a)} \textit{ has a contact of } 2^{nd} \textit{ order at } \gamma_{u(a)}(s(a)).
\end{aligned}
$$

Moreover,

$$
u'(a) = 0 \implies g_s(\phi(a)) \neq 0 \;\; \textit{and} \;\; g_t(\phi(a)) \neq 0.
$$

Proof. Using (5.4), (5.6) and (5.12) observe that

$$
\begin{aligned}
[f_s, f_{ss}, f] \circ \phi &= \frac{1}{\|\gamma(s) - \gamma(t)\|}\Big[f_s \circ \phi, \gamma_u''(s), f \circ \phi\Big] \\
&= \frac{1}{\|\gamma(s) - \gamma(t)\|^2}\Big[\gamma_u'(s), \gamma_u''(s), f \circ \phi\Big] \\
&= \|\gamma_u'(s) \times \gamma_u''(s)\|\frac{\operatorname{sign}([f_s, f_{ss}, f] \circ \phi)}{\|\gamma(s) - \gamma(t)\|^2}
\end{aligned}
$$

and so

$$
\begin{aligned}
\frac{[f_s, f_{ss}, f]}{\|f_s\|^3} \circ \phi &= \frac{\|\gamma_u'(s) \times \gamma_u''(s)\|}{\|\gamma_u'(s)\|^3}\big\{\|\gamma(s) - \gamma(t)\|(\operatorname{sign}[f_s, f_{ss}, f] \circ \phi)\big\} \\
&= \kappa^a(s)\|\gamma(s) - \gamma(t)\|\,\operatorname{sign}([f_s, f_{ss}, f] \circ \phi).
\end{aligned} \tag{5.19}
$$

Similarly,

$$\begin{aligned} \frac{[f_t, f_{tt}, f]}{\|f_t\|^3} \circ \phi &= \frac{\|\gamma_u'(t) \times \gamma_u''(t)\|}{\|\gamma_u'(t)\|^3} \big\{ \|\gamma(s) - \gamma(t)\| (\operatorname{sign}[f_t, f_{tt}, f] \circ \phi) \big\} \\ &= \kappa^a(t) \big\{ \|\gamma(s) - \gamma(t)\| \operatorname{sign}([f_t, f_{tt}, f] \circ \phi) \big\}. \end{aligned} \tag{5.20}$$

The tangent to the II curve is

$$\begin{aligned} u' &\overset{(5.11)}{=} (-g_t f_s + g_s f_t)(\phi) \\ &= (-[f_s, f_{tt}, f] f_s + [f_{ss}, f_t, f] f_t)(\phi). \end{aligned}$$

Under the assumption that f_s and f_t are both non-zero at $\phi(a)$ and using (5.15), (5.18), (5.19), (5.20) one gets:

$$\begin{aligned} u' &= \left(\sigma [f_t, f_{tt}, f] \frac{\|f_s\|}{\|f_t\|} f_s + [f_{ss}, f_s, f] \frac{\|f_t\|^2}{\|f_s\|^2} f_s \right) (\phi) \\ &= \left(-\|f_s\| \|f_t\|^2 \left(\frac{[f_s, f_{ss}, f]}{\|f_s\|^3} - \sigma \frac{[f_t, f_{tt}, f]}{\|f_t\|^3} \right) f_s \right) (\phi) \\ &= -\|\gamma(s) - \gamma(t)\| (\kappa^a(s) - \operatorname{ind}(\omega) \kappa^a(t)) \big\{ \|f_s\| \|f_t\|^2 \operatorname{sign}[f_s, f_{ss}, f] f_s \big\} (\phi). \end{aligned}$$

Summing up, the derivative of u w.r.t. a is

$$u' = \begin{cases} -\|\gamma(s) - \gamma(t)\| (\kappa^a(s) - \operatorname{ind}(\omega) \kappa^a(t)) \cdot & \\ \qquad \cdot \big\{ \|f_s\| \|f_t\|^2 \operatorname{sign}[f_s, f_{ss}, f] f_s \big\} (\phi) & \text{if } g_s(\phi) \neq 0 \neq g_t(\phi) \\ -g_t f_s(\phi) & \text{if } g_s(\phi) = 0 \\ g_s f_t(\phi) & \text{if } g_t(\phi) = 0. \end{cases}$$

In the last two cases, i.e., when either one of $\gamma'(s(a))$, $\gamma'(t(a))$ is a cross-tangent or the bitangent plane osculates at one of the $\gamma(s(a))$, $\gamma(t(a))$, the genericity of γ implies $u'(a) \neq 0$ (compare with Remark 5.5). In the remaining case $\operatorname{ind}(\omega) \neq 0$ and $[f_s, f_{ss}, f] \neq 0$ at $\phi(a)$ and hence,

$$\begin{aligned} u'(a) = 0 \qquad &\iff \qquad \kappa^a(s(a)) - \operatorname{ind}(\omega(\phi(a))) \kappa^a(t(a)) = 0 \\ &\iff \qquad \kappa^a(s(a)) = \kappa^a(t(a)) \text{ and } \operatorname{ind}(\omega(\phi(a))) > 0 \\ &\iff \qquad \kappa^a(s(a)) = \kappa^a(t(a)) \\ &\qquad\qquad \text{and the bitangent plane } \omega(\phi(a)) \text{ is external.} \end{aligned}$$

If $\kappa^a(s(a)) = \kappa^a(t(a)) = 0$ this would mean that the osculating planes to γ at $s(a)$ and $t(a)$ coincide, which contradicts the genericity of γ. $\square$

Corollary 5.14. *Under the assumptions of the above lemma,*

$$u'(a) = 0 \implies f_s(\phi(a)) \neq 0 \text{ and } f_t(\phi(a)) \neq 0,$$

i.e., neither $s(a)$ nor $t(a)$ is a cross-tangent of γ.

Inflection points of the II curve

Further important aspects of the II curve are its inflection points.

Lemma 5.15. *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a curve in general position, $\mathcal{B}$ its bitangency manifold, $\phi = (s,t) : (-\varepsilon, \varepsilon) \to \mathcal{B}$ with $\phi'(a) = (-g_t, g_s)(\phi(a))$ a local parametrization of $\mathcal{B}$ and $u = f \circ \phi$ a local parametrization of a main component of the II curve. Then at points $a \in (-\varepsilon, \varepsilon)$ with $u'(a) \neq 0$*

(i) *a is an inflection point of u*

$\iff$ *the plane $\omega(\phi(a))$ is an osculating plane*

either at $s(a)$ or at $t(a)$.

(ii) *a is a regular inflection point of u*

$\iff$ *the plane $\omega(\phi(a))$ is an osculating plane*

either at $s(a)$ with $\tau(s(a)) \neq 0$

or at $t(a)$ with $\tau(t(a)) \neq 0$

(iii) *a is a special irregular inflection point of u*

$\iff$ *the plane $\omega(a)$ is an osculating plane*

either at $s(a)$ with $\tau(s(a)) = 0$ and $\tau'(s(a)) \neq 0$,

or at $t(a)$ with $\tau(t(a)) = 0$ and $\tau'(t(a)) \neq 0$.

There are no further irregular inflection points. Furthermore, at any $a \in (-\varepsilon, \varepsilon)$ with $u'(a) \neq 0$

$$\operatorname{sign}[u, u', u''](a) = \operatorname{ind}(\omega(\phi(a))) \tag{5.21}$$

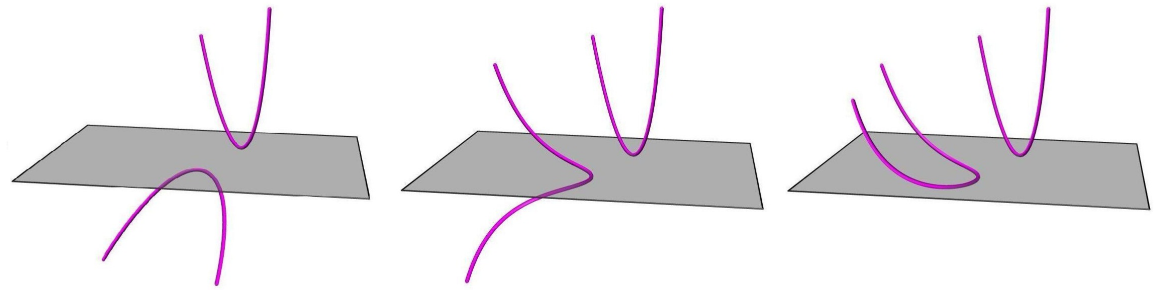


Figure 5.16: The only three possibilities of geodesic curvature of the II curve.

and this expression changes the sign precisely at the osculating bitangent plane described by (ii).

The three cases for geodesic curvature of the II curve are illustrated in Figure 5.16.

Proof. Observe first that

$$\begin{aligned} u'(a) = 0 \quad &\iff \quad g_t f_s = g_s f_t \qquad \text{at } \phi(a) \\ &\iff \quad g_t \|f_s\| = -\sigma g_s \|f_t\| \quad \text{at } \phi(a) \end{aligned} \tag{5.22}$$

Use (4.3) to determine the inflection points of u and for any a with $u'(a) \neq 0$ compute

$$\begin{aligned} [u, u', u''](a) &= \Big[f(\phi(a)), (-g_t f_s + g_s f_t)(\phi(a)), \{(-g_t f_s + g_s f_t) \circ \phi\}'(a)\Big] \\ &= [f, -g_t f_s + g_s f_t, g_t^2 f_{ss} + g_s^2 f_{tt}](\phi(a)) \\ &= \big([f_{ss}, f_s, f] g_t^3 - [f_{ss}, f_t, f] g_s g_t^2 \\ &\qquad - [f_s, f_{tt}, f] g_s^2 g_t + [f_t, f_{tt}, f] g_s^3\big)(\phi(a)). \end{aligned} \tag{5.23}$$

Suppose that $f_s(\phi(a)) = 0$, i.e., s is a cross-tangent of γ. Then the definition of g gives $g_t(\phi(a)) = 0$. Notice that by Corollary 5.14 this implies $u'(a) \neq 0$. By genericity of γ (or simply by (5.22)) both $f_t(\phi(a))$ and $g_s(\phi(a))$ must be non-zero (the same holds when s and t are interchanged). Hence, under the assumption that $f_s(\phi(a)) = 0$:

$$[u, u', u''](a) = [f_t, f_{tt}, f] g_s^3(\phi(a))$$

and this gives (5.21) in this special case. Furthermore,

$$[u, u', u''](a) = 0 \quad \Longleftrightarrow \quad [f_t, f_{tt}, f](\phi(a)) = 0$$

and so the bitangent plane $\omega(\phi(a))$ is at the same time an osculating plane to γ at $t(a)$. Under the assumption that f_s and f_t are both non-zero at $\phi(a)$ with $u'(a) \neq 0$, the computation started in (5.23) continues as follows

$$\begin{aligned}
[u, u', u''](a) &\overset{(5.14)}{=} \Big(-\sigma \frac{\|f_s\|}{\|f_t\|} [f_{ss}, f_t, f] g_t^3 - g_s g_s g_t^2 - g_t g_s^2 g_t \\
&\qquad\qquad - \sigma \frac{\|f_t\|}{\|f_s\|} [f_s, f_{tt}, f] g_s^3 \Big)(\phi(a)) \\
&= \frac{-\sigma g_s g_t}{\|f_s\| \|f_t\|} \big(\|f_s\|^2 g_t^2 + \sigma \|f_s\| \|f_t\| 2 g_s g_t + \|f_t\|^2 g_s^2 \big)(\phi(a)) \\
&= \frac{-\sigma g_s g_t}{\|f_s\| \|f_t\|} \underbrace{(\|f_t\| g_s + \sigma \|f_s\| g_t)^2}_{\neq 0 \text{ by } (5.22)} (\phi(a)).
\end{aligned}$$

This proves (i). In particular, from the above computation follows that

$$\operatorname{sign}[u, u', u''](a) = \operatorname{sign}(-\sigma g_s g_t) \overset{(5.17)}{=} \operatorname{ind}(\omega(\phi(\mathrm{a}))) \,.$$

and (5.21) is proved. To show (ii) again use (4.3) and via an elementary but long and tedious computation get

$$\begin{aligned}
&[u, u', u''] = 0 \text{ and } [u, u', u'''] \neq 0 \text{ at } \phi(a). \\
\Longleftrightarrow \quad & (g_s = 0 \text{ and } [f, f_s, f_{sss}] \neq 0) \text{ or } (g_t = 0 \text{ and } [f, f_t, f_{ttt}] \neq 0) \text{ at } \phi(a) \\
\Longleftrightarrow \quad & (f \times f_s \neq 0 \text{ and } [f, f_s, f_{ss}] = 0 \text{ and } [f_s, f_{ss}, f_{sss}] \neq 0) \\
& \text{or } (f \times f_t \neq 0 \text{ and } [f, f_t, t_{tt}] = 0 \text{ and } [f_t, f_{tt}, f_{ttt}] \neq 0) \text{ at } \phi(a).
\end{aligned}$$

Similarly, to show (iii) again use (4.3) and via an elementary but even longer and more tedious computation get

$$
\begin{aligned}
&[u, u', u''] = [u, u', u'''] = 0 \text{ and } [u, u', u^{IV}] \neq 0 \text{ at } \phi(a) \\
\iff\ & (g_s = [f, f_s, f_{sss}] = 0 \text{ and } [f, f_s, f_{ssss}] \neq 0) \\
& \text{or } (g_t = [f, f_t, f_{ttt}] = 0 \text{ and } [f, f_t, f_{tttt}] \neq 0) \text{ at } \phi(a) \\
\iff\ & (f \times f_s \neq 0 \text{ and } [f, f_s, f_{ss}] = [f_s, f_{ss}, f_{sss}] = 0 \text{ and } [f_s, f_{ss}, f_{ssss}] \neq 0) \\
& \text{or } (f \times f_t \neq 0 \text{ and } [f, f_t, t_{tt}] = [f_t, f_{tt}, f_{ttt}] = 0 \text{ and } [f_t, f_{tt}, f_{tttt}] \neq 0) \text{ at } \phi(a).
\end{aligned}
$$

Clearly, for a generic curve γ all possibilities of an inflection point of its II curve have been exhausted. □

Definition 5.16. Call a plane $\omega(s,t)$ with $(s,t) \in \mathcal{B}$, such that $\omega(s,t)$ is an osculating plane of γ at either s or t, a *bitangent osculating plane.* Denote the union of great circles $\omega(s,t) \cap \mathbb{S}^2$ for all bitangent osculating planes $\omega(s,t)$ by $\mathcal{P}^{\mathrm{bit}}_{\mathrm{osc}}(\gamma)$.

The following lemma shows that for a generic curve γ there are finitely many bitangent osculating planes. In particular, this implies that $\mathcal{P}^{\mathrm{bit}}_{\mathrm{osc}}(\gamma)$ has measure zero.

Corollary 5.17. *A generic curve γ has finitely many bitangent osculating planes. In particular, it has an even number of bitangent planes described by (ii) of Lemma 5.15 or in the middle of Figure 5.16.*

Proof. Combining Lemma 5.13 and Lemma 5.15 the bitangent osculating planes are in one-to-one correspondence with the inflection points of the main components of the II curve. A parametrization of the II curve of a curve γ can only have regular or special irregular inflection points. By (4.3) they are all isolated. Hence, on the components of the bitangency manifold homeomorphic to $\mathbb{S}^1$ there are only finitely many instances. Recall, from Lemma 5.8, that points of a component of $\mathcal{B}$ homeomorphic to an open interval, sufficiently close to Δ, all correspond to the external bitangent planes. Therefore, also those components of $\mathcal{B}$ can have only finitely many inflection points. Moreover, $\mathcal{B}$ has finitely many components.

With the above facts and using the final part of Lemma 5.15 the even parity of the set of the regular inflection points follows. □

Bitangent osculating planes are zeros in $\mathcal{B}$ of b_s and b_t with b defined in Definition 5.1. Further zeros in $\mathcal{B}$ of these functions are pairs (s,t) such that $\gamma'(s)$ or $\gamma'(t)$ are cross-tangents. It turns out that there are also finitely many of those.

Lemma 5.18. *A cross-tangent $\gamma'(t)$ is an isolated zero of b_s in $\mathcal{B}$. In particular, a curve γ in generic position has only finitely many cross-tangents.*

Proof. Let $\phi = (s,t) : (-\varepsilon, \varepsilon) \to C$ be a local parametrization of a component C of $\mathcal{B}$ with $\phi' = (-b_t, b_s) \circ \phi$. Suppose $b_s(\phi(a_0)) = 0$ at some $a_0 \in (-\varepsilon, \varepsilon)$ and $(s_0, t_0) := \phi(a_0)$ and $\gamma'(t_0)$ is a cross-tangent, i.e., $\gamma'(t_0) \times (\gamma(s_0) - \gamma(t_0)) = 0$. Recall that by genericity condition $b_t(\phi(a_0)) \neq 0$. Then

$$\begin{aligned}(b_s \circ \phi)'(a_0) &= \big(- b_{ss} \cdot b_t + b_{st} \cdot \underbrace{b_s}_{=0} \big) \circ \phi(a_0) \\ &= \Big\{ \underbrace{[\gamma'''(s_0), \gamma'(t_0), \gamma(s_0) - \gamma(t_0)]}_{=0} + [\gamma''(s_0), \gamma'(t_0), \gamma'(s_0)] \Big\} \cdot \underbrace{b_t(\phi(a_0))}_{\neq 0} \\ &\neq 0,\end{aligned}$$

since $[\gamma''(s_0), \gamma'(t_0), \gamma'(s_0)] = 0$ would mean that the plane $\omega(a_0)$ is at the same time an osculating plane of γ at s_0 and this is not allowed by Definition 4.1(ii)(a).

By interchanging s with t, the same argument shows that all cross-tangents are isolated in $\mathcal{B}$. If C is a component of $\mathcal{B}$ that is homeomorphic to $\mathbb{S}^1$ then the finiteness follows from compactness. Otherwise, note that since γ has nowhere vanishing curvature, there is a maximal curvature on $\mathbb{S}^1$. If C is homeomorphic to an interval then the accumulation point of infinitely many cross-tangents would necessarily lie on Δ. However, this constraint on the curvature prohibits cross-tangents to appear as pairs in $\mathcal{B}$ arbitrarily close to the diagonal Δ. $\square$

Remark. Having defined the II curve a natural question arises, what are the properties of the II curve that follow from knottedness of the space curve. One of such properties, that was long believed in, could be a self-intersection of the II curve. However, a numerical simulation of a trefoil knot shown in Figure 5.17 proves that this suggestion is not true. The trefoil used for the example is obtained from the parametrization

$$\theta \mapsto \frac{1}{1 - b\sin(p\theta)} (a\cos(q\theta), a\sin(q\theta), b\cos(p\theta))$$

with $\theta \in [0, 2\pi)$, $p = 3$, $q = 2$, $a = 0.985$ and $a^2 + b^2 = 1$.

5.3 III curve

For a smooth immersion $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ let $\mathcal{R}_{III}$ be the set of all those unit vectors $v \in \mathbb{S}^2$ such that γ_v has multiple points. That is $v \in \mathcal{R}_{III}$ if and only if there exist

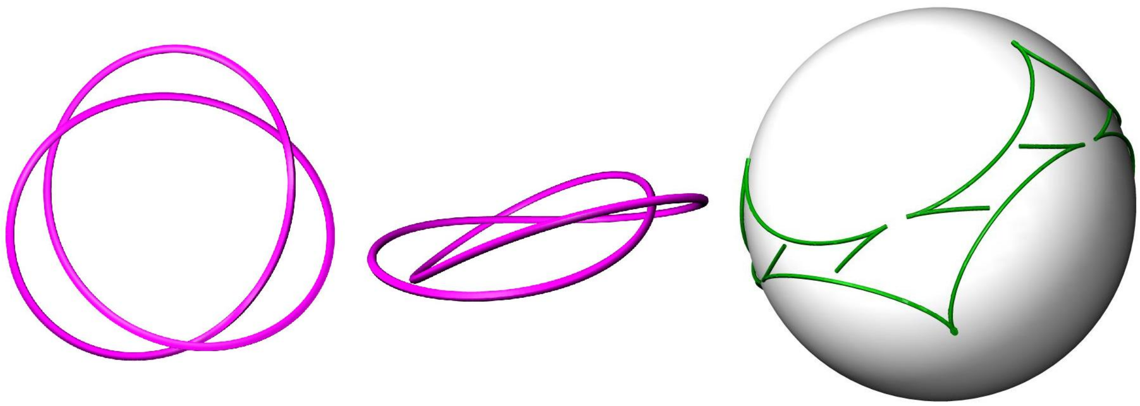


Figure 5.17: A flat trefoil knot and its non-intersecting II curve.

three pairwise distinct points $r, s, t \in \mathbb{S}^1$ with $\gamma_v(r) = \gamma_v(s) = \gamma_v(t)$. An example based on the knot $\mathcal{K}$ is shown in Figure 5.18. It is straightforward that this set can be equivalently (but not necessarily most efficiently) described as

$$\mathcal{R}_{III} = \{f(s,t) \mid (r,s,t) \in (\mathbb{S}^1)^{\times 3} \setminus \Delta_3 \text{ and } \gamma(r) - \gamma(s) \parallel \gamma(t) - \gamma(s)\},$$

where $\Delta_3 := \{(r,s,t) \in (\mathbb{S}^1)^{\times 3} \mid r = s \text{ or } s = t \text{ or } r = t\}$. In order to use the name "curve" a parametrization of $\mathcal{R}_{III}$ has to be found. Let $\varphi : (\mathbb{S}^1)^{\times 3} \setminus \Delta_3 \to \mathbb{R}$ be the map given by $(r,s,t) \mapsto \langle \gamma(s) - \gamma(t), \gamma(s) - \gamma(r) \rangle$. Notice, that if γ additionally satisfies the condition:

For any triple $(r,s,t) \in (\mathbb{S}^1)^{\times 3} \setminus \Delta_3$ *with* $\varphi(r,s,t) = 0$ *not all tangent vectors* $\gamma'(r), \gamma'(s), \gamma'(t)$ *are parallel to the segment* $\gamma(s) - \gamma(t)$. (G5)

then zero is a regular value of φ. By the Regular Value Theorem (see, e.g., Spivak [35, Proposition 12, Chapter 2]) the zero set $\mathcal{T} := \varphi^{-1}(0)$ is a smooth properly embedded submanifold of $(\mathbb{S}^1)^{\times 3} \setminus \Delta_3$ of dimension 1. In particular, (G5) is satisfied by a generic (in the sense of Definition 4.1) curve since a bitangent line is excluded. Find a parametrization of $\mathcal{T}$, say $\phi = (r,s,t) : A \to \mathcal{T}$ and then for example $f(s,t)(a) : A \to \mathbb{S}^2$, where f is the Gauss map from Definition 5.1, will give a possible parametrization of the set $\mathcal{R}_{III}$. Hence, the name "curve" is justified. Call the set $\mathcal{R}_{III}$ *the Reidemeister III curve of* γ or for short *the III curve*. There are certainly more suitable parametrizations of the III curve than the one proposed here. Exploring those and further properties of the set $\mathcal{R}_{III}$ are left to an interested reader.

Remark. Note that for a non-trivially knotted curve (i.e., not ambient isotopic to

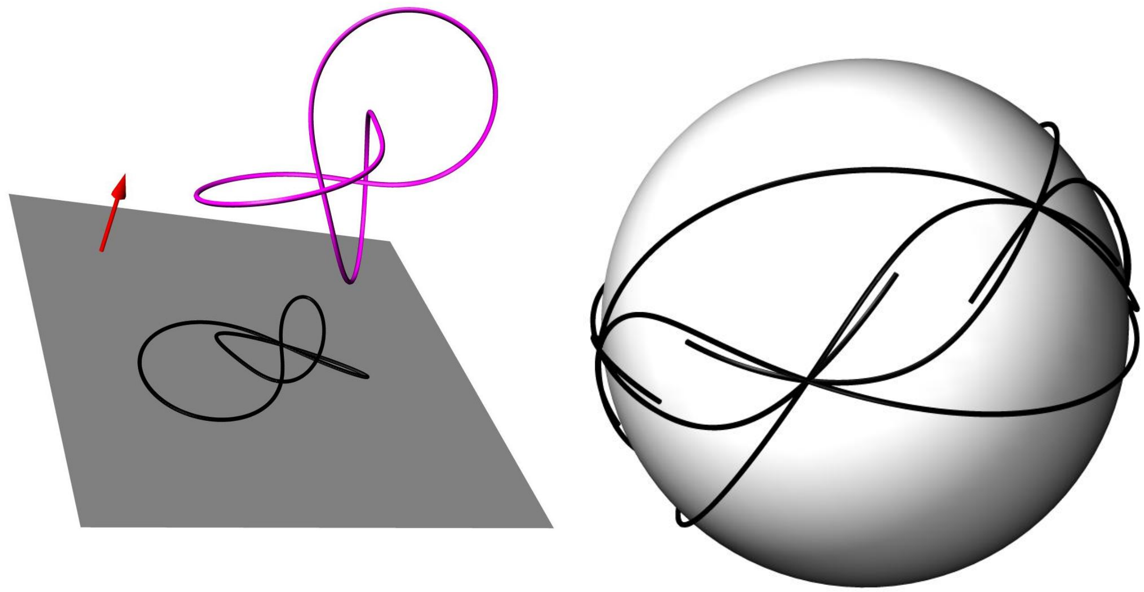


Figure 5.18: The trefoil knot $\mathcal{K}$ with an orthogonal projection (left) characterizing the Reidemeister III curve (right) of $\mathcal{K}$.

a circle) there is necessarily a self-intersection of the III curve. It was first proved by Pannwitz [26] in case of polygonal knots and later by Kuperberg [23] in case of smooth knots that any non-trivially knotted closed space curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ (satisfying some generic properties) has a *quadrisecant*, i.e., a line that passes through $\gamma(r), \gamma(s), \gamma(t), \gamma(u)$ for some pairwise distinct points $r, s, t, u \in \mathbb{S}^1$. In fact, Kuperberg's genericity conditions are stronger than γ being a smooth immersion satisfying (G5). However, the proof works well with these significantly weakened conditions.

Reidemeister Curves versus Arnold discriminant

If γ is in generic position in the sense of Definition 4.1, then it was already shown (see Remark 5.9) that the endpoints of the closure of $\mathcal{R}_{II}$ lie on $\mathcal{R}_I$. Moreover, the endpoints of the closure of $\mathcal{R}_{III}$ lie on the intersection points of $\mathcal{R}_{II}$ with $\mathcal{R}_I$ and these points correspond to the cross-tangents of the curve γ. This can be seen by observing that since $\mathcal{T}$ is properly embedded in $(\mathbb{S}^1)^3 \setminus \Delta_3$, the limit points of $\mathcal{T}$, different from $\mathcal{T}$ itself, lie necessarily in Δ_3. Suppose that $(r_0, s_0, t_0) \in \Delta_3$ is such a limit point. This means that two of the pairwise distinct points $r, s, t \in \mathbb{S}^1$ with $\gamma(r), \gamma(s), \gamma(t)$ collinear, converge to one another, say $r \to r_0 = s_0 \leftarrow s$ when approaching Δ_3. But then clearly $f(r, s) \to \pm\gamma'(r_0)$. So the endpoints of the closure of $\mathcal{R}_{III}$ lie on $\mathcal{R}_I$. Since, $\pm\gamma'(r_0)$ is a cross-tangent, it also lie on $\mathcal{R}_{II}$. The converse might not be true,

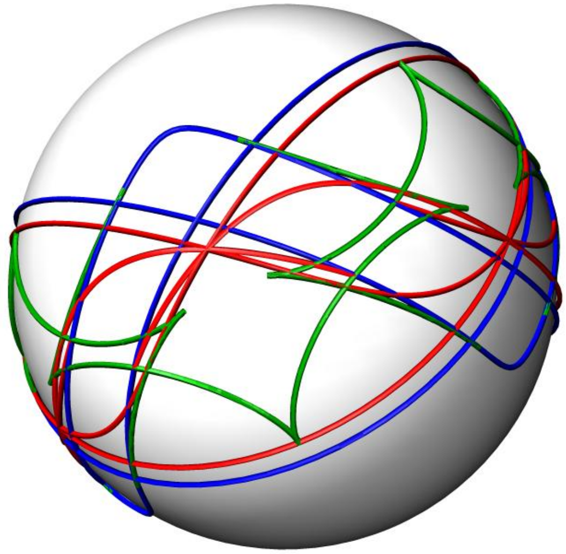


Figure 5.19: All three Reidemeister curves of the trefoil knot $\mathcal{K}$.

i.e., an intersection v of the I curve with the II curve does not necessarily have to correspond to a cross-tangent. The singularities - a cusp and a self-tangency of γ_v - may be independent of each other. To summarize, the union of all three curves - $\mathcal{R}_I \cup \mathcal{R}_{II} \cup \mathcal{R}_{III}$ - form a graph on a sphere, without endpoints (see Figure 5.19).

There is a remarkable connection between the Reidemeister curves and the Arnold discriminant. Namely, the set $\mathbb{S}^2 \setminus \mathcal{R}_I$ can be viewed as a 2-parameter family of A-generic curves γ_v with $v \in \mathbb{S}^2 \setminus \mathcal{R}_I$. The II curve is then an intersection of this family with self-tangency stratum $\Sigma^{\asymp}$and the III curve with the triple point stratum Σ^{St} of the Arnold discriminant (compare Section 3.1). Hence, the graph made of the images of the Reidemeister curves can be a helpful tool in analyzing and understanding the Arnold discriminant.

Chapter 6

Surfaces

Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a curve in generic position. In this chapter three surfaces will be associated to the curve γ that describe certain geometric features of projections of γ. To each of the three surfaces a smooth map will be assigned that maps the surface to a 2-dimensional unit sphere $\mathbb{S}^2$. The image of the first of the surfaces, called here the *crossing surface* is already well known, e.g., in [15] under the name the *writhe mesh*. The second surface called here *the inflection surface* might be partially described in the literature in connection to the self-linking number of a space curve (compare with Chapter 8). In this work some important properties of that surface are presented. The last of the surfaces called here the *bitangency surface* is to the best of the author's knowledge not known so far. Therefore, the most of the attention of this chapter is paid to its definition and properties.

In order to describe the properties of the three surfaces, some preliminary material regarding mapping between Euclidean spaces is introduced first. Also a general construction of a special type of a spherical surface made of great circles is described.

Quasi smooth maps

In this work a very special type of maps between surfaces will be used. In particular, smooth maps whose domains are surfaces without boundaries will be continuously extended to the boundaries. In order not to worry about the smoothness of this extended maps the following definitions have been chosen to fit the thesis.

Let M and N be smooth manifolds of dimension m and n respectively. Let $f : M \to N$ be a continuous map and suppose that there exists a finite graph Γ (i.e., composed of finitely many edges and vertices) smoothly embedded in M and f is smooth on each face of $M \setminus (\partial M \cup \Gamma)$. If Γ is empty, then f is simply a smooth map on the interior of M. Call a point $p \in M \setminus (\partial M \cup \Gamma)$ *regular* if $D_p f$ has maximal rank and *singular* otherwise. Call $p \in M$ a *critical point* of f if $p \in \Gamma \cup \partial M$ or p is singular. Call $v \in N$ a *critical value* of f if $f^{-1}(v)$ contains a critical point and a *regular value* otherwise.

With the above definition, the set of critical points has measure zero since Γ and ∂M are of measure zero, and the rest follows by Sard. Also, if v is a regular value of f, then there exists a neighborhood U of v such that the preimage $f^{-1}(U)$ consists entirely of regular points of f.

Degree of a map

In this thesis several maps between surfaces, i.e., manifolds of dimension 2, and a unit sphere $\mathbb{S}^2$ will be considered. The orientation of the images of those surfaces on the sphere will play a crucial role. Hence, this issue has to be treated with some special care.

Orient the unit sphere $\mathbb{S}^2 := \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ by choosing the outer normal, so that from outside one sees its positively oriented side. Let $\mathcal{F}$ be an oriented surface and $f : \mathcal{F} \to \mathbb{S}^2$ a smooth map. For each regular point $p \in \mathcal{F}$ of f define its index $\mathrm{ind}_{f,\mathcal{F}}(p)$ to be $+1$ if for a sufficiently small neighborhood U of p the restriction $f_{|U}$ preserves the orientation and -1 if it reverses the orientation. Then, given an oriented chart of $(V, v : V \to \mathcal{F})$ with $V \subset \mathbb{R}^2$ with coordinates (x, y) (and the usual orientation of the XY-plane) and with $v(0) = p$, the index of a point p is given by

$$\mathrm{ind}_{f,\mathcal{F}}(p) = \mathrm{ind}_{f \circ v, V}(0) = \mathrm{sign}[f \circ v, (f \circ v)_x, (f \circ v)_y](0).$$

Definition 6.1. Let M be a connected closed oriented surface (a 2-manifold) and $f : M \to \mathbb{S}^2$ a continuous map. The second homology groups of both M and $\mathbb{S}^2$ are both homeomorphic to $\mathbb{Z}$, i.e., $H_2(M) \approx \mathbb{Z} \approx H_2(\mathbb{S}^2)$. Then the induced map $f_* : H_2(M) \to H_2(\mathbb{S}^2)$ is a multiplication by an integer and this integer is defined to be the *degree of* f, denoted $\deg(f)$. Equivalently if $[a]$ is a generator of $H_2(M)$ and $[b]$ a generator of $H_2(\mathbb{S}^2)$, then $f_*([a]) = \deg(f) \cdot [b]$.

If the continuous map f from the above definition additionally satisfies the condition that it is smooth away of some finite smoothly embedded graph in M and for any regular value v of f the number of preimages $f^{-1}(v)$ is finite, then there is an easy way to compute the degree of f. Note that if f is smooth and M a closed surface, then the finiteness of $f^{-1}(v)$ follows automatically.

Lemma 6.2. *Let M be a closed oriented surface, Γ a finite graph smoothly embedded in M and $f : M \to \mathbb{S}^2$ a continuous map such that it is smooth when restricted to $M \setminus \Gamma$. Suppose that for any regular value $v \in \mathbb{S}^2$ of f the set $f^{-1}(v)$ is finite, i.e., it can be written as $f^{-1}(v) = \{x_1, x_2, \ldots, x_k\}$ for some $k \in \mathbb{N}$. Then the degree of f is*

$$\deg(f) = \sum_{i=1}^{k} \operatorname{ind}_{f,M}(x_i). \tag{6.1}$$

Proof. The above fact is well-known in case when f is smooth, i.e., $\Gamma = \emptyset$ (see, e.g., Spivak [35] or Berger and Gostiaux [8]), in which case the finiteness of the preimage set $f^{-1}(v)$ is guaranteed by the smoothness. In general case, the proof of Corollary 7.5 of Bredon [9] can be slightly modified to fit the setup of this lemma and provide a proof. □

Whitney's results on folds

This section presents a summary of results of Whitney [40] on maps between Euclidean spaces of dimension 2. In the case of a map from a 2-manifold to a sphere $\mathbb{S}^2$ these results can be directly applied to the corresponding map between charts.

Consider a smooth map $f : U \to \mathbb{R}^2$ with U a neighborhood of $\mathbb{R}^2$. Then a point $p \in U$ is regular if $\det D_p f \neq 0$ and singular if $\det D_p f = 0$. (Here the notions of a critical point (value) and singular point (value) coincide). A point $p \in U$ is said to be *good* if either $\det D_p f \neq 0$ or at least one of $(\det D_p f)_x$ or $(\det D_p f)_y$ is non-zero. The map f is *good* if every point of U is good.

If $f : U \to \mathbb{R}^2$ is good, then

(i) for each $p \in U$, the image space of $D_p f$ is of dimension 2 or 1, according to whether p is a regular or a singular point and

(ii) the singular points of f form smooth curves in U that will be called the *general folds* of f.

Let $f : U \to \mathbb{R}^2$ be smooth and good. Take any singular point p, and let $\phi(t)$ be a smooth parametrization of the general fold C through p, with $\phi(0) = p$. Then p is a *fold point* of f if

$$\frac{d}{dt} f \circ \phi(t) \neq 0 \quad \text{at } t = 0.$$

The definition is independent of the parametrization chosen for C. An example is shown in Figure 6.1.

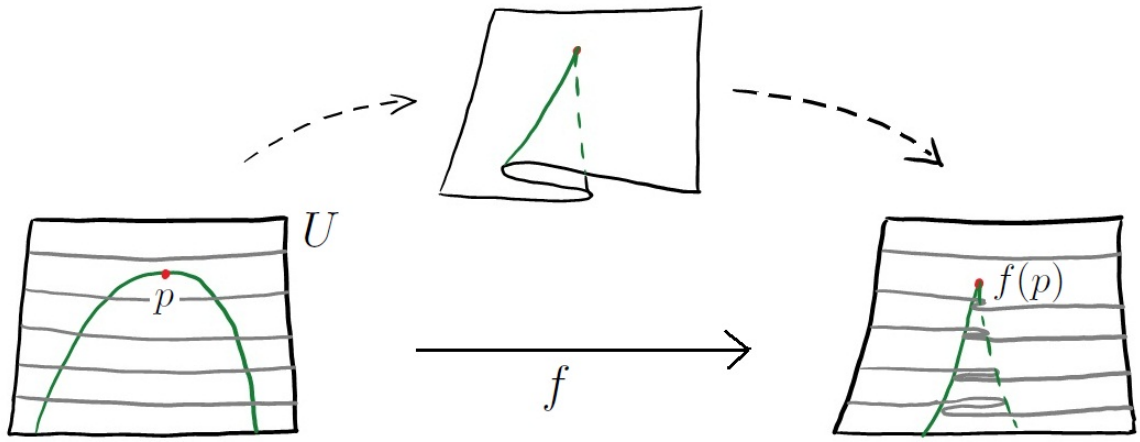


Figure 6.1: An illustration of a good map f, a general fold and its image (in green). The point p belongs to the general folds of f but it is not a fold point.

Theorem 6.3 (The normal form for folds, Whitney [40]). *Let p be a fold point of the smooth mapping $f : U \to \mathbb{R}^2$. Then smooth coordinate systems (x, y), (u, v) may be introduced about p and $f(p)$ respectively, in terms of which f takes the form*

$$u = x^2, \quad v = y.$$

For the proof see Whitney [40]. The above results hold as well when $f : U \to \mathbb{S}^2$ is a smooth map from a neighborhood U of $\mathbb{R}^2$ to the unit sphere. The map f is regular at $p \in U$ if $\big[f_x, f_y, f\big](p) \neq 0$ and singular if $\big[f_x, f_y, f\big](p) = 0$. A point $p \in U$ is said to be *good* if

$$\big[f_x, f_y, f\big](p) \neq 0 \ \text{ or } \ \Big(\big[f_x, f_y, f\big]_x(p) \neq 0 \ \text{ or } \ \big[f_x, f_y, f\big]_y(p) \neq 0\Big). \tag{6.2}$$

The map f is *good* if every point of U is good. The definition of fold points is analogous.

A surface made of great circles

Here a special type of a surface made of great circles tangent to a spherical curve is described, together with some of its properties. Let $c : (-\varepsilon, \varepsilon) \to \mathbb{S}^2$ be a smooth curve on a sphere. If $c'(a) \neq 0$ then $\pm\frac{c'(a)}{\|c'(a)\|}$ are two unit vectors tangent to the curve c at a point a. In more general case, when $c'(a) \neq 0$ is not necessarily guaranteed, let $w : (-\varepsilon, \varepsilon) \to \mathbb{S}^2$ be another spherical curve that will play the role of the unit tangent map to c. Hence, let w satisfy

$$c(a) \perp w(a) \quad \text{and} \quad \big[c, c', w\big](a) = 0 \quad \text{for all } a \in (-\varepsilon, \varepsilon).$$

Then define a surface made of great circles tangent to c by

$$\phi : (-\varepsilon, \varepsilon) \times \mathbb{S}^1, \qquad (a, \theta) \mapsto c(a)\cos(\theta) + w(a)\sin(\theta). \tag{6.3}$$

(Here, a point on $\mathbb{S}^1 \approx \mathbb{R}/2\pi\mathbb{Z}$ is represented by an angle $\theta \in [0, 2\pi)$.)

Lemma 6.4. *Let $\phi : (-\varepsilon, \varepsilon) \times \mathbb{S}^1 \to \mathbb{S}^2$ be as in (6.3). Then a point $(a, \theta) \in (-\varepsilon, \varepsilon) \times \mathbb{S}^1$ is singular if and only if*

$$\theta = 0 \text{ or } \theta = \pi \text{ or } \big[c, w, w'\big](a) = 0. \tag{6.4}$$

Proof. Since the image of ϕ lies on the unit sphere $\mathbb{S}^2$,

$$\begin{aligned}
\phi_a \times \phi_\theta(a, \theta) = 0 \quad &\iff \quad \big[\phi, \phi_a, \phi_\theta\big](a, \theta) = 0 \\
&\iff \quad \big[c(a)\cos\theta + w(a)\sin\theta, c'(a)\cos\theta + w'(a)\sin\theta, \\
&\qquad\qquad - c(a)\sin\theta + w(a)\cos\theta\big] = 0 \\
&\overset{[c,c',w]=0}{\iff} \quad \big[w(a)\sin\theta, w'(a)\sin\theta, -c(a)\sin\theta\big] + \\
&\qquad\qquad + \big[c(a)\cos\theta, w'(a)\sin\theta, w(a)\cos\theta\big] = 0 \\
&\iff \quad \big[c(a), w'(a), w(a)\big]\sin\theta = 0.
\end{aligned}$$

□

Remark. Under the assumption that $c'(a) \neq 0$ the condition (6.4) reads as

$$\theta = 0 \;\text{ or }\; \theta = \pi \;\text{ or }\; \big[c, c', c''\big](a) = 0.$$

This means, that whenever $c'(a) \neq 0$, the set of singular points is

$$(-\varepsilon, \varepsilon) \times \{0, \pi\} \cup \{(a, \theta) \mid a \text{ is an inflection point of } c\}$$

and the set of singular values consists of: the image of c and its antipodal copy, as well as great circles tangent to c at the inflection points of the curve c. It cannot be determined, without further properties of c, whether ϕ is an immersion at points (a, θ) with $c'(a) = 0$ or not.

Let us now investigate under what circumstances the surface ϕ is good according to (6.2). The computation gives

$$\begin{aligned}
\big[\phi_a, \phi_\theta, \phi\big](a,\theta) &= \big[c, w', w\big](a)\sin\theta \\
\big[\phi_a, \phi_\theta, \phi\big]_a(a,\theta) &= \big[c', w', w\big](a)\sin\theta + \big[c, w'', w\big](a)\sin\theta = \big[c, w'', w\big](a)\sin\theta \\
\big[\phi_a, \phi_\theta, \phi\big]_\theta(a,\theta) &= \big[c, w', w\big](a)\cos\theta
\end{aligned}$$

and for all three terms to vanish the following

$$\big[c, w', w\big](a) = 0 \quad \text{and} \quad \big[c, w'', w\big](a)\sin\theta = 0$$

must hold.

It turns out that all the curves in this thesis, which will serve as a basis for this special type of a surface, additionally satisfy the condition

$$[c, w, w'](a) \neq 0 \qquad \text{or} \qquad c'(a) \neq 0 \qquad \text{for all } a \in (-\varepsilon, \varepsilon). \tag{6.5}$$

Lemma 6.5. *The map $\phi : (-\varepsilon, \varepsilon) \times \mathbb{S}^1 \to \mathbb{S}^2$ defined by (6.3) satisfying (6.5) fails to be good at (a, θ) precisely when*

(i) θ is a multiple of π and a is a regular inflection point of c, i.e.,

$$[c, c', c''](a) = 0 \neq [c, c', c'''](a)$$

or

(ii) a is an irregular inflection point at a, i.e.,

$$c'(a) \neq 0 \;\; \textit{and} \;\; [c, c', c''](a) = [c, c', c'''](a) = 0.$$

Moreover, the fold points of ϕ are all points (a, θ) with

(j) $[c, c', c''](a) \neq 0$ and $\theta = 0, \pi$ and

(jj) $[c, c', c''](a) = 0 \neq [c, c', c'''](a)$ and $\theta \neq 0, \pi$.

The singular values of ϕ consists of the image of the curve c and its antipodal copy, as well as great circles tangent to the curve c at its inflection points.

It is now possible to construct locally the image of ϕ on the sphere $\mathbb{S}^2$ of points around the fold points. By Lemma 6.5 the image of the fold points of ϕ is the union of

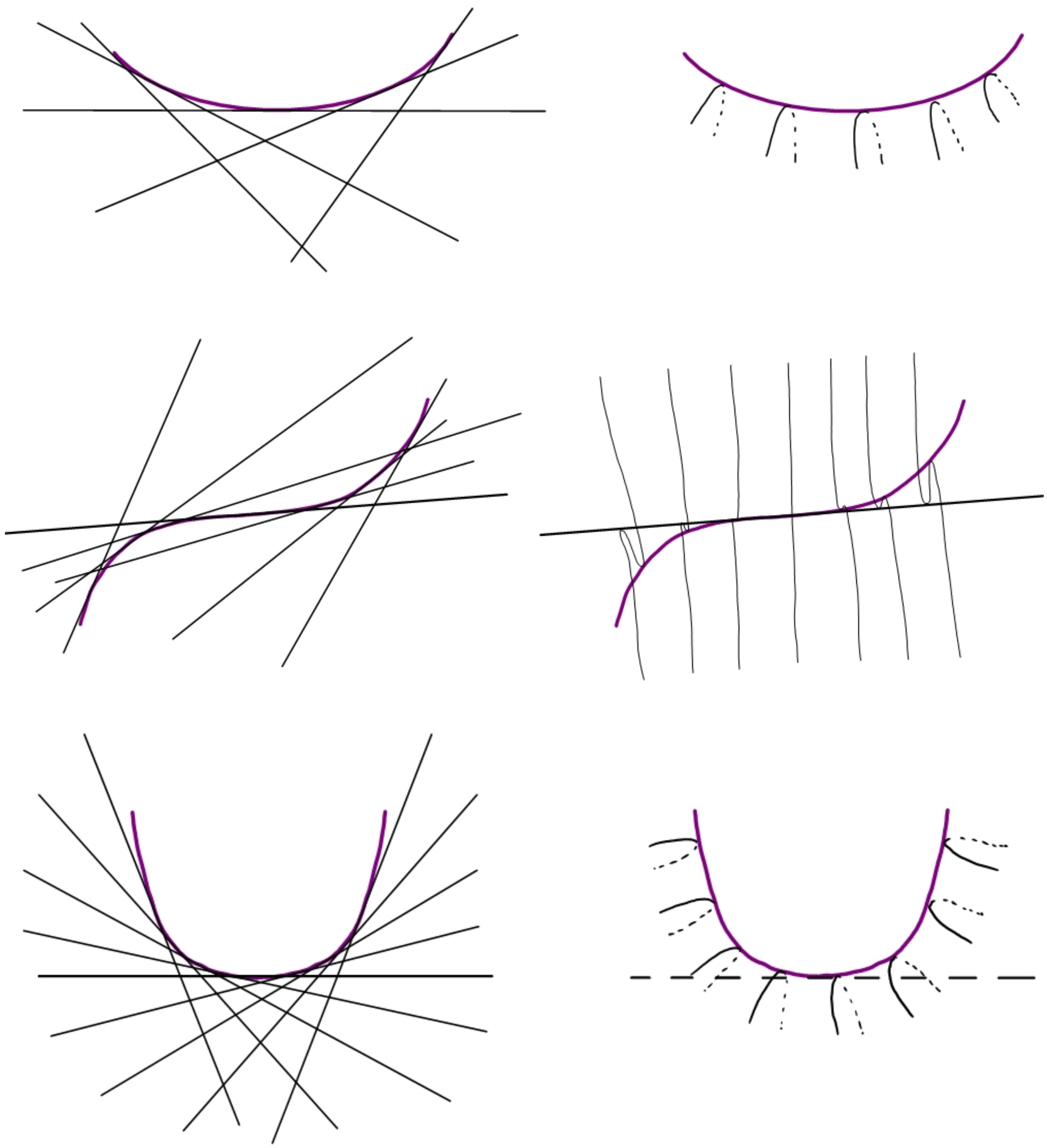


Figure 6.2: Two local pictures of the image of ϕ around the curve c with $c'(a) \neq 0$. Great circles are locally depicted as straight lines tangent to the arc (left) and as arbitrary lines depicting the folding (right). Top: away of inflection points of c, middle: around a regular inflection point and bottom: around a special irregular inflection point of c.

- the image of the curve c and its antipodal copy, whenever $c'(a) \neq 0$ and away from the inflection points of c (see top of Figure 6.2) and

- great circles tangent to the curve c at any regular inflection point a of c, apart of the two points $\pm c(a)$ (see middle of Figure 6.2).

The only type of irregular inflection points that will appear in this thesis will be a special irregular inflection point a that by definition satisfies

$$\big[c, c', c''\big](a) = \big[c, c', c'''\big](a) = 0 \quad \text{and} \quad \big[c, c', c^{IV}\big](a) \neq 0.$$

This means that the curve c has order of contact exactly 3 with its tangent great circle at such a point. Clearly, the inflection points of this type are isolated. The local picture of the image of ϕ around a special inflection point is shown in the bottom of Figure 6.2. In fact, the surface is not an immersion at the great circle tangent to c at this point. However with certainty the surface doesn't fold there, in the following sense. Suppose a is a special irregular inflection point of c and a neighborhood U of $\{a\} \times \mathbb{S}^1$ in $(-\varepsilon, \varepsilon) \times \mathbb{S}^1$ is divided by $\{a\} \times \mathbb{S}^1$ into two parts U_1 and U_2. Then both of these neighborhoods will be mapped onto $\mathbb{S}^2$ with the same orientation.

Recall that the tangent indicatrix of a smooth embedding $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ with nowhere vanishing curvature has only regular inflection points and the II curve of a curve in general position has only regular or special irregular inflection points.

Let us now discuss the orientation of the image on the sphere of the surface $\mathcal{S} := (-\varepsilon, \varepsilon) \times \mathbb{S}^1$ under the map ϕ. More precisely, the points of $\mathcal{S}$ away from the general folds are to be divided into two classes, according to whether ϕ locally around such a point is orientation preserving or reversing. It has been discussed previously in this chapter that this distinction can be made by deciding of the index

$$\mathrm{ind}_{\phi,\mathcal{S}}(a, \theta) = \mathrm{sign}\,\big[\phi_a, \phi_\theta, \phi\big](a, \theta) = \pm 1$$

at a regular point (a, θ) of $\mathcal{S}$. Denote by A the set $\{(a, 0) \mid a \in (-\varepsilon, \varepsilon)\}$ and by $\bar{A}$ the set $\{(a, \pi) \mid a \in (-\varepsilon, \varepsilon)\}$ both with orientation induced by the orientation of $(-\varepsilon, \varepsilon)$. Note that $\phi(a, \pi)$ is the antipodal point of $\phi(a, 0)$. Hence, $\phi(\bar{A})$ is the antipodal copy of $\phi(A)$. Let $(a, \theta) \notin A \cup \bar{A}$ and $\big[c, w, w'\big](a) \neq 0$. Then, by computation as in the proof of Lemma 6.4,

$$\mathrm{sign}\,\big[\phi_a, \phi_\theta, \phi\big](a, \theta) = -\,\mathrm{sign}\Big(\big[c(a), w(a), w'(a)\big] \sin\theta\Big) \tag{6.6}$$

and if moreover $c'(a) \neq 0$, then

$$\operatorname{sign}\left[\phi_a, \phi_\theta, \phi\right](a,\theta) = -\operatorname{sign}\Big(\left[c(a), c'(a), c''(a)\right]\sin\theta\Big)$$
$$= \begin{cases} +1 & \text{if } \left[c, c', c''\right](a) > 0 \text{ and } \theta \in (\pi, 2\pi) \text{ or} \\ & \quad \left[c, c', c''\right](a) < 0 \text{ and } \theta \in (0, \pi), \\ -1 & \text{if } \left[c, c', c''\right](a) < 0 \text{ and } \theta \in (\pi, 2\pi) \text{ or} \\ & \quad \left[c, c', c''\right](a) > 0 \text{ and } \theta \in (0, \pi). \end{cases}$$

To summarize the above discussion assume that ϕ with two spherical curves c and w satisfies (6.5). Suppose that the interval $(-\varepsilon, \varepsilon)$ contains precisely one inflection point of c (either a regular or a special irregular one) and $c'(a) \neq 0$ for all $a \in (-\varepsilon, \varepsilon)$. Then the orientation on the sphere can be described by Figure 6.3.

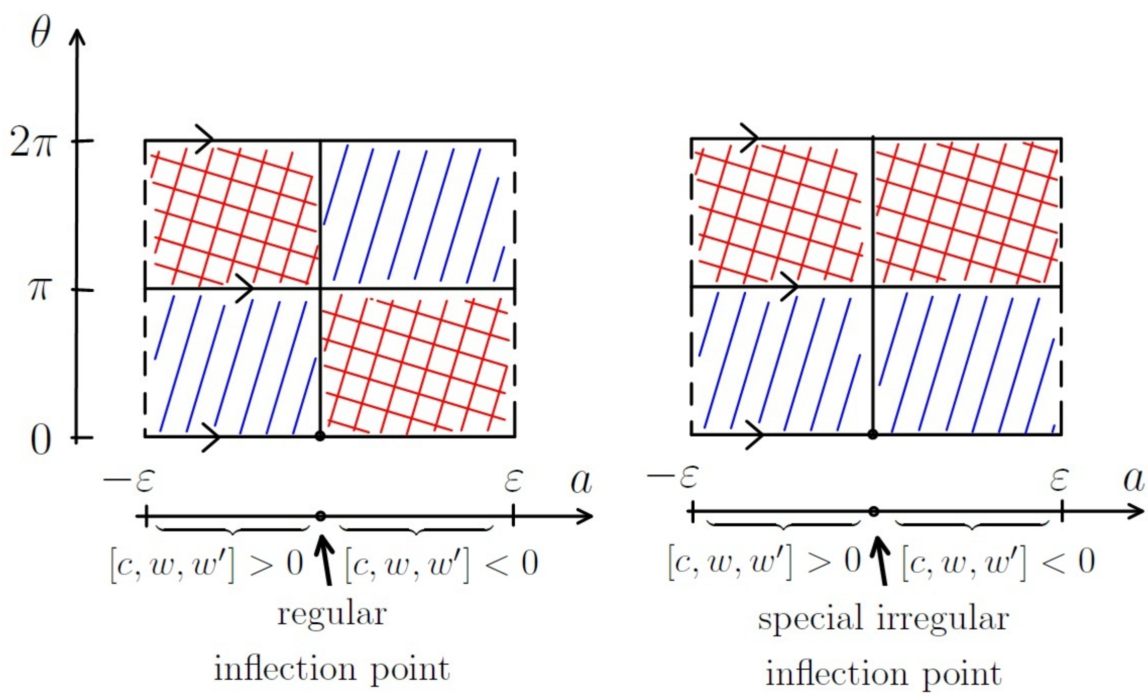


Figure 6.3: Orientation on the sphere of a surface made of tangent great circles to a spherical curve: positive (red/checked) and negative (blue/lined).

6.1 Crossing surface

The first surface is a closed annulus. It can be realized in the following way: remove from a torus $\mathbb{S}^1 \times \mathbb{S}^1$ its diagonal $\Delta := \{(s, s) \mid s \in \mathbb{S}^1\} \approx \mathbb{S}^1$ to obtain an open annulus and compactify it by adding the missing boundaries, denoted by Δ_+ and Δ_-. When $\mathbb{S}^1$ is considered as $\mathbb{R}/\mathbb{Z}$, the closed annulus can be explicitly described as shown in the Figure 6.4. Since both Δ_+ and Δ_- are homeomorphic to $\mathbb{S}^1$, a point on one of the diagonals will often be regarded as a point on a curve.

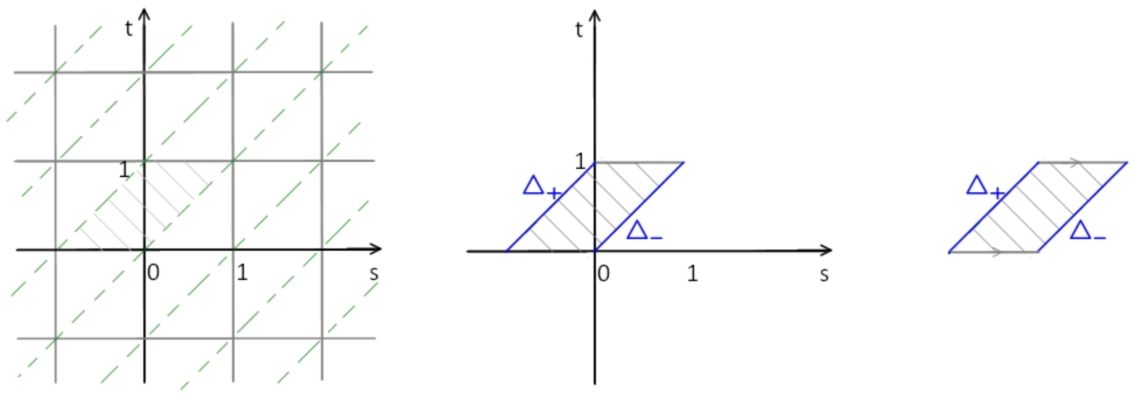


Figure 6.4: Construction of the crossing surface.

Definition 6.6. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a smooth, injective map. Call a *crossing surface* the closed annulus $\mathcal{C} := (\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta) \cup \Delta_+ \cup \Delta_-$ and the *crossing map of* γ the map given by

$$f : \mathcal{C} \to \mathbb{S}^2, \quad (s,t) \mapsto \begin{cases} \frac{\gamma(s)-\gamma(t)}{\|\gamma(s)-\gamma(t)\|} & \text{for } (s,t) \in \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta \\ \gamma'(s) & \text{for } (s,t) \in \Delta_+ \\ -\gamma'(s) & \text{for } (s,t) \in \Delta_- \end{cases}. \tag{6.7}$$

The orientation of $\mathcal{C}$ is given by the order of elements of the tuple (s,t).

The crossing map f is continuous since

$$\lim_{s \searrow s_0} \frac{\gamma(s) - \gamma(s_0)}{\|\gamma(s) - \gamma(s_0)\|} = \gamma'(s_0) \quad \text{and} \quad \lim_{s \nearrow s_0} \frac{\gamma(s) - \gamma(s_0)}{\|\gamma(s) - \gamma(s_0)\|} = -\gamma'(s_0).$$

An important property of the crossing surface concerns its fold points, when it is mapped onto the sphere via the crossing map.

Lemma 6.7. *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a curve in general position. Then the crossing map of γ is good at every point of the interior $\mathcal{C}^\circ$ of $\mathcal{C}$. The set of singular points of f in $\mathcal{C}^\circ$, i.e., the general folds, is precisely the bitangency manifold $\mathcal{B}$ of γ.*

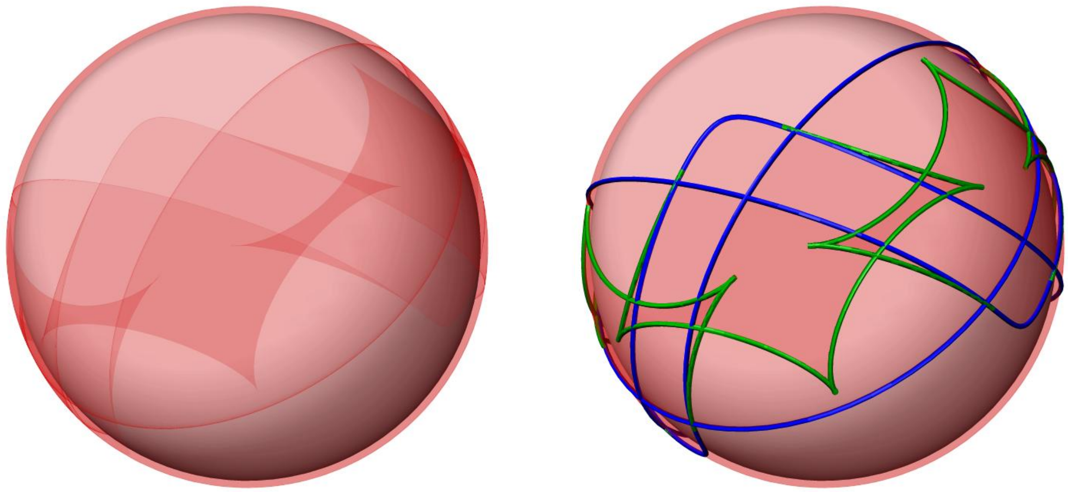


Figure 6.5: The image of the crossing surface $f(\mathcal{C})$ of the knot $\mathcal{K}$ (left) with its boundary and general folds marked (right): blue - the I curve $\mathcal{R}_{\mathcal{I}}$ and green - the II curve $\mathcal{R}_{\mathcal{II}}$.

Proof. By (6.2) the map f fails to be good at $p \in \mathcal{C}^\circ$ if $\big[f_s, f_t, f\big](p) = 0$ and

$$\big[f_s, f_t, f\big]_s(p) = \big[f_{ss}, f_t, f\big](p) = 0 \ \text{ and } \ \big[f_s, f_t, f\big]_t(p) = \big[f_s, f_{tt}, f\big](p) = 0.$$

Vanishing of all three conditions is ruled out by the genericity condition (G2) of γ together with (5.7). Hence, f is good on $\mathcal{C}^\circ$. By (5.1) and (5.3) the set of general folds of f is precisely the bitangency manifold $\mathcal{B}$. $\square$

It is not necessarily the case that all points of the general folds are indeed fold points. The fold points of the crossing surface are characterized by Lemma 5.13. Namely, a point $p = (s,t) \in \mathcal{B}$, belonging to the set of general folds, is not a fold point if and only if the curve $\gamma_{f(p)}$ has a contact of 2^{nd} order at $\gamma_{f(p)}(s)$ (as shown in the middle of Figure 5.14). Such a point $p \in \mathcal{B}$, that is not a fold point, may give rise to a cusp on the II curve, as can be seen in the example in Figure 6.5. It is not clear whether the set of fold points of a curve in generic position must be non-empty.

Remark. By Lemma 6.7, the bitangency manifold $\mathcal{B}$ is the set of critical points of the smooth crossing map restricted to its interior $f|_{\mathcal{C}^\circ}$. By Sard's theorem (see, e.g., 4.3.1. in [8]) the set of critical values of $f|_{\mathcal{C}^\circ}$, which is precisely the II curve $\mathcal{R}_{II}$, has measure zero in $\mathbb{S}^2$.

A critical value of the crossing map f is each point of the II curve and (by the definition chosen) each point of the image of the boundary of $\mathcal{C}$, i.e., the I curve.

Recall that for any $v \in \mathbb{S}^2$ the plane curve obtained by orthogonal projection of γ

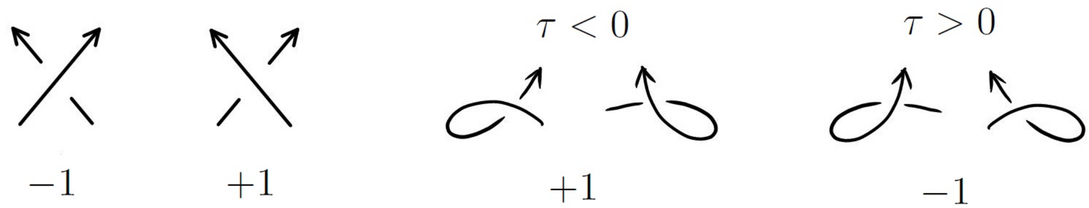


Figure 6.6: The index of a crossing: in general (left) and close to $\Delta_\pm$ (right).

onto the plane perpendicular to v was given by $\gamma_v(s) = \gamma(s) - \langle \gamma(s), v \rangle v$. In case where γ_v is an immersion, a double point of γ_v was defined in Definition 1.1. Then a straightforward result of the construction of f is the following.

Lemma 6.8. *Let γ be in general position and $v \in \mathbb{S}^2$ be a regular value of the crossing map, i.e., $v \notin \mathcal{R}_{\mathcal{I}} \cup \mathcal{R}_{\mathcal{II}}$. Then γ_v is a smooth immersion with only regular double points. Moreover, the number of preimages of v under the crossing map f equals the number of double points of γ_v, i.e.,*

$$\# f^{-1}(v) = \mathrm{cr}(\gamma_v).$$

Let us now determine the orientation on the sphere of the image of the crossing surface under the crossing map. For a regular point (s,t) of f, i.e., $v \in \mathcal{C}^\circ \setminus \mathcal{B}$, its index is precisely

$$\mathrm{ind}_{f,\mathcal{C}}(s,t) = \mathrm{sign}[f, f_s, f_t](s,t) = \mathrm{sign}(g(s,t)),$$

with g defined in (5.5).

A comparison with the orientation chosen on $\mathcal{B}$ (see Figure 5.11) shows that the points of the region to the right of a main component C of $\mathcal{B}$ all have positive index and to the left all have negative index. The situation around a symmetric component $\bar{C}$ of C is precisely the opposite. See the top part of Figure 6.7.

In general, the index of a point of the crossing surface can be determined by a simple rule. Looking at the curve from the direction $f(s,t)$, the point $\gamma(t)$ appears to lie over $\gamma(s)$. Taking into account the orientation of the two strands of the curve γ locally about the points s and t, the index is as depicted on the left of Figure 6.6. Additionally, the index of a regular point (s,t) sufficiently close to the diagonals $\Delta_\pm$ can be recognized by the sign of the torsion of the short arc between $\gamma(s)$ and $\gamma(t)$, as shown on the right of Figure 6.6. More precisely, if $\tau(s) \neq 0$ and $U \subset \mathbb{S}^1$ is a

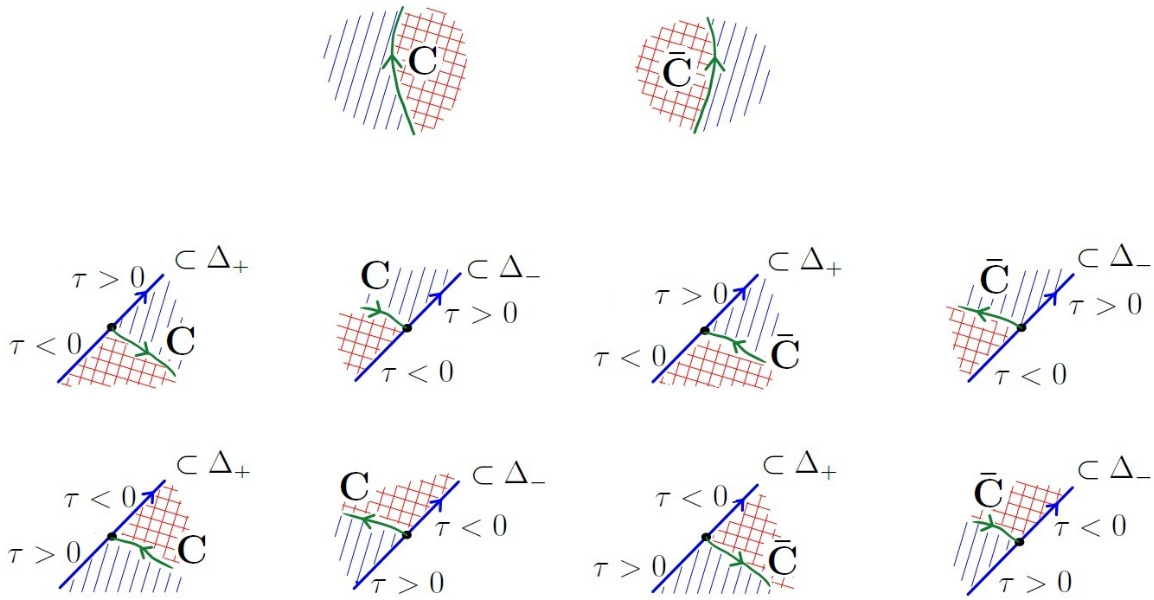


Figure 6.7: Orientation of the crossing surface on the sphere locally about the general folds $\mathcal{B}$ with a main component C and its symmetric one $\bar{C}$: points with positive index - checked/red and negative index -lined/blue.

neighborhood of s such that $\tau(r) \neq 0$ for all $r \in U$, then applying the general rule (left of Figure 6.6) to a suitable projection of the arc $\gamma(U)$ (as for example those depicted in on the right of Figure 6.6) gives that for $t \in U$ the index $\mathrm{ind}_{f,\mathcal{C}}(s,t) = -\operatorname{sign}(\tau(r))$ for any $r \in U$. Hence, the points (s,t), lying sufficiently close to the diagonals Δ_+ and Δ_-, have positive (respectively negative) index if at the closest points on the diagonals, the curve γ has negative (respectively positive) torsion. See the bottom part of Figure 6.7.

It is now possible to determine a local picture of the image of the crossing surface around its boundary and around its fold points. The local image around the tangent indicatrix follows immediately from the above discussion and is depicted in Figure 6.8. In the remaining case, for a curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ in general position, consider an open interval I, belonging to a main component C of the bitangency manifold $\mathcal{B}$ of γ, that consists entirely of fold points. Let $\phi : (-\varepsilon, \varepsilon) \to I$ be a parametrization of I with $\phi'(a) = (-g_t(\phi(a)), g_s(\phi(a)))$ for all $a \in (-\varepsilon, \varepsilon)$. Then $u := f \circ \phi$ is a local parametrization of the II curve preserving the orientation chosen on $\mathcal{B}$. Knowing the orientation on the sphere, it follows that the crossing surface always falls to the right of $u(I)$ (see Figure 6.9). Since the orientation of the symmetric components is always opposite to that of the main components of $\mathcal{B}$, it immediately follows that the

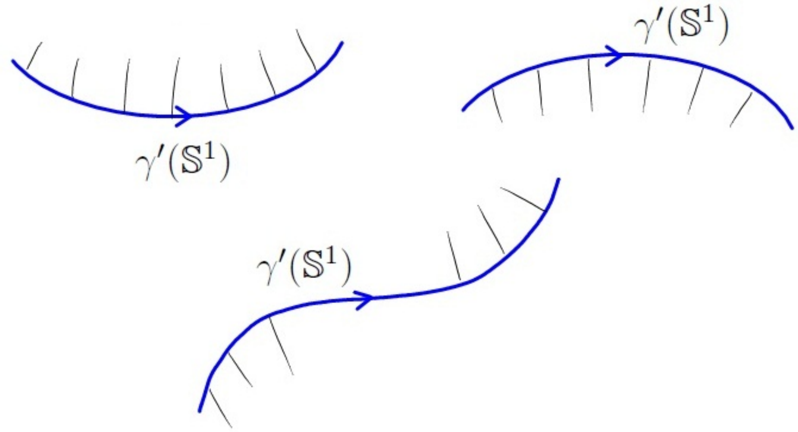


Figure 6.8: A local picture of the image of the crossing surface about the tangent indicatrix.

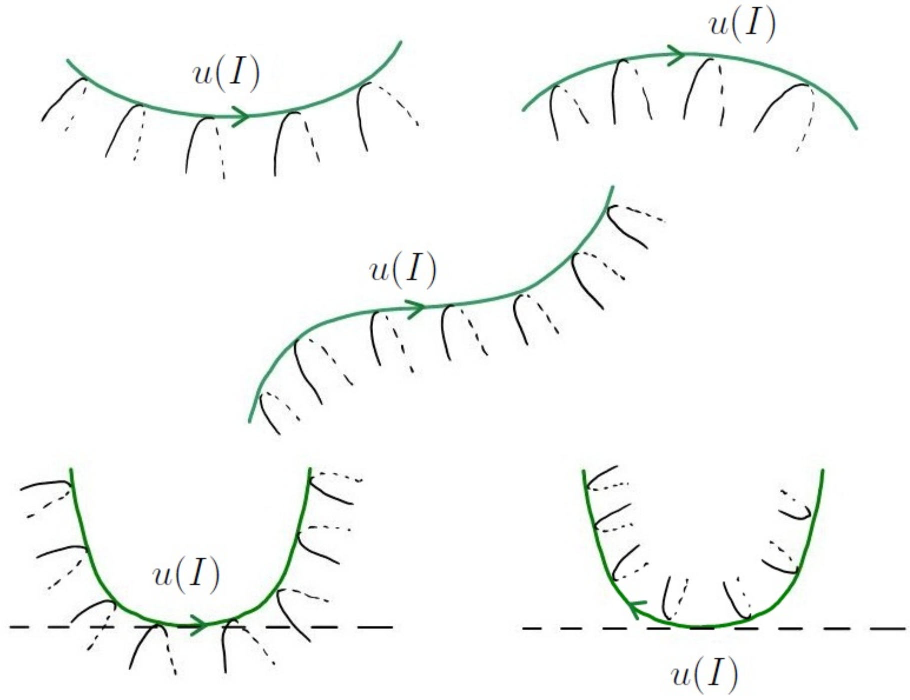


Figure 6.9: A local picture of the image of the crossing surface around the fold points of a *main* component: (top) an arc with non-vanishing geodesic curvature, (middle) a regular inflection point, (bottom) a special irregular inflection point.

crossing surface falls to the left of $-u(I)$.

The above facts together with (5.21) imply that an arc of $u(I)$ with positive geodesic curvature corresponds to external and with negative to internal bitangency planes. There is a simple geometric argument to justify that rule. By Corollary 5.12, the bitangent plane corresponding to a point on the II curve is parallel to the plane spanned by the great circle tangent to the II curve at that point (this all provided that the II curve is immersed at the considered point). Recall that for a given $v \in \mathbb{S}^2$, all preimages under the crossing map correspond to the crossings of the projected plane curve γ_v. Hence, if the direction v, lying initially on the II curve, travels along the tangent great circle, then the self-tangency of the curve γ_v resolves either to two new crossings (if the bitangency plane was external) or to no crossing (if the bitangency plane was internal). This agrees with the fact that traveling along the great circle, the direction v either remains on the surface or falls off of it. See Figure 6.10.

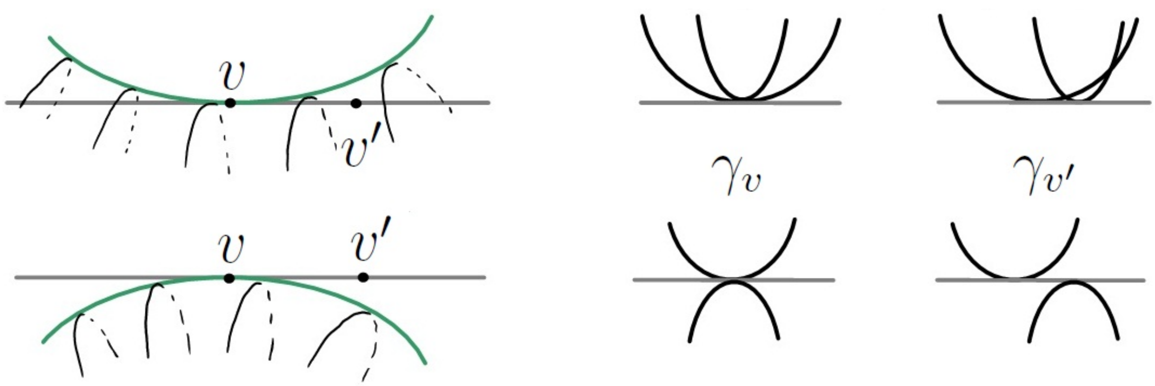


Figure 6.10: Left: local image of the crossing surface about its fold points with v being a direction on the II curve and v' on the great circle tangent to the II curve at v. Right: a self-tangency of γ_v resolves to two crossings of $\gamma_{v'}$ or none.

Remark. If there exists a component of the closure $\mathrm{cl}_{\mathcal{C}}(\mathcal{B})$ of the bitangency manifold $\mathcal{B}$ homeomorphic to an interval, with both endpoints lying on the different diagonals (i.e., one endpoint lies on Δ_+ and the other on Δ_-), then this implies that γ has at least 4 points at which torsion vanishes. Suppose a main component of $\mathrm{cl}_{\mathcal{C}}(\mathcal{B})$ starts off at a point $(s_0, s_0) \in \Delta_+$ with $\tau(s_0) = 0$ and $\tau'(s_0) > 0$, then from the proceeding discussion follows that if this component ends up at a point $(t_0, t_0) \in \Delta_-$ with $\tau(t_0) = 0$, then necessarily also $\tau'(t_0) > 0$. Lemma 5.7 implies that a component of $\mathrm{cl}_{\mathcal{C}}(\mathcal{B})$ cannot be symmetric to itself. Hence, $s_0 \neq t_0$ and there are two different points s_0 and t_0 at which torsion vanishes and torsion's first derivative is positive. Since γ is a generic curve there necessarily must exist two further points $u_0 \neq w_0$ with $\tau(u_0) = \tau(w_0) = 0$, satisfying $\tau'(u_0) < 0$ and $\tau'(w_0) < 0$.

Notation for the crossing surface

This section ends with the introduction of some notation that will be used until the end of Part II of this thesis. Suppose that the curve γ has exactly $2M$ points ($M \in \mathbb{N}$) at which torsion vanishes, i.e., $s_1 < s_2 < \ldots < s_{2M} \in \mathbb{S}^1$ such that $\tau(s_i) = 0$ for all $i = 1, ..., 2M$. Assume, w.l.o.g., that $\tau'(s_1) < 0$. Hence,

$$\begin{array}{l} \tau'(s_{2i}) > 0 \\ \tau'(s_{2i-1}) < 0 \end{array} \quad \text{for } i = 1, ..., M.$$

Denote by ${}_{\mathcal{C}}P^i_+$ the points $(s_i, s_i) \in \Delta_+ \subset \mathcal{C}$ and by ${}_{\mathcal{C}}P^i_-$ the points $(s_i, s_i) \in \Delta_- \subset \mathcal{C}$ for $i = 1, \ldots, 2M$. Moreover, denote by ${}_{\mathcal{C}}\Delta^i_\pm$ the subsets of $\Delta_\pm$ starting at ${}_{\mathcal{C}}P^i_\pm$ and ending at ${}_{\mathcal{C}}P^{i+1}_\pm$ oriented according to the orientation of $\Delta_\pm$. (Indices are taken modulo $2M$.) More precisely,

$$\begin{aligned} {}_{\mathcal{C}}\Delta^i_+ &:= \{(s, s) \in \Delta_+ \mid s_i \leq s \leq s_{i+1}\} \subset \Delta_+ \text{ and} \\ {}_{\mathcal{C}}\Delta^i_- &:= \{(s, s) \in \Delta_- \mid s_i \leq s \leq s_{i+1}\} \subset \Delta_- \end{aligned}$$

for $i = 1, ..., 2M$. Hence, for $i = 1, ..., M$

$$\begin{aligned} \tau(s) \geq 0 \ \text{ if } (s, s) \in {}_{\mathcal{C}}\Delta^{2i}_\pm \\ \tau(s) \leq 0 \ \text{ if } (s, s) \in {}_{\mathcal{C}}\Delta^{2i-1}_\pm. \end{aligned} \tag{6.8}$$

In case when the torsion of γ does not vanish, rename $\Delta_\pm$ by ${}_{\mathcal{C}}\Delta^0_\pm$ if the torsion is positive and by ${}_{\mathcal{C}}\Delta^{-1}_\pm$, otherwise. This non-intuitive choice of notation has the advantage that also (6.8) holds for $i = 0$, in the case when $M = 0$. In general, an even index $2i$ will be chosen to represent a non-negative or positive instance of τ' and τ respectively.

Now, consider the closure $\mathrm{cl}_{\mathcal{C}}(\mathcal{B})$ of the bitangency manifold $\mathcal{B}$ in $\mathcal{C}$. If $2M$ with $M \in \mathbb{N}$ is the number of torsion vanishing points of γ, then $\mathrm{cl}_{\mathcal{C}}(\mathcal{B})$ has $2M$ components (M pairs of symmetric components) homeomorphic to a closed interval. Denote them by ${}_{\mathcal{C}}C_1$, ${}_{\mathcal{C}}C_2, \ldots$, ${}_{\mathcal{C}}C_M$, ${}_{\mathcal{C}}\bar{C}_1$, ${}_{\mathcal{C}}\bar{C}_2, \ldots$, ${}_{\mathcal{C}}\bar{C}_M$ so that $\{{}_{\mathcal{C}}C_j, {}_{\mathcal{C}}\bar{C}_j\}$ represents a symmetric pair of components for $j = 1, \ldots, M$. Suppose that there are $2N$ with $N \in \mathbb{N}$ components (N pairs of symmetric components) of $\mathrm{cl}_{\mathcal{C}}(\mathcal{B})$ that are homeomorphic to $\mathbb{S}^1$. Denote them analogously by ${}_{\mathcal{C}}C_{M+1}, \ldots, {}_{\mathcal{C}}C_{M+N}, {}_{\mathcal{C}}\bar{C}_{M+1}, \ldots, {}_{\mathcal{C}}\bar{C}_{M+N}$ so that $\{{}_{\mathcal{C}}C_j, {}_{\mathcal{C}}\bar{C}_j\}$ represents a pair of symmetric components for $j = M + 1, \ldots, M + N$. The indices j of the components of $\mathrm{cl}_{\mathcal{C}}(\mathcal{B})$ are in no relation with the indices i of the

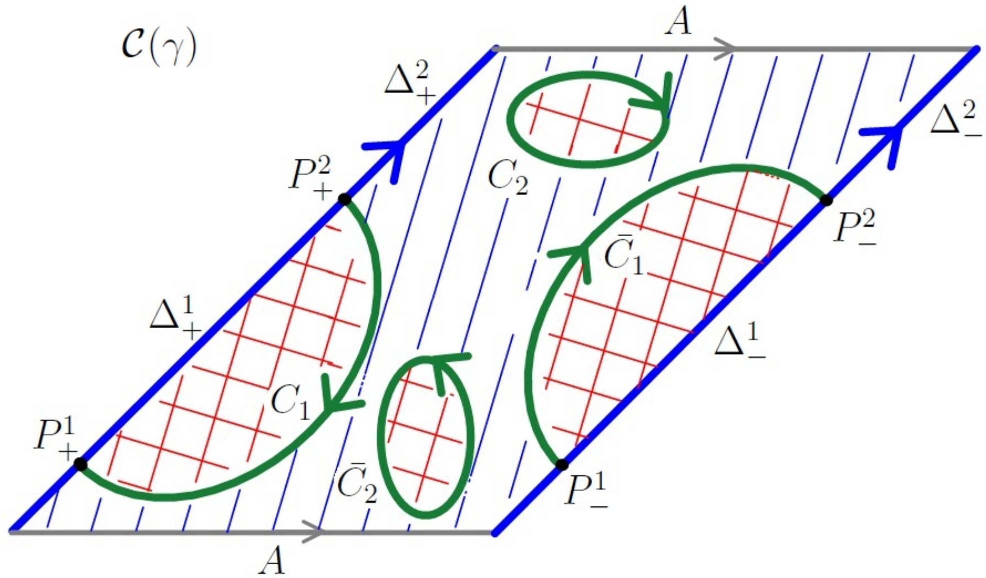


Figure 6.11: An example illustrating the notation for the crossing surface $\mathcal{C}(\gamma)$ of some curve γ.

torsion vanishing points s_i's.

The crossing surface $\mathcal{C}$ was so far introduced simply as a set that is independent of γ. Denote by $\mathcal{C}(\gamma)$ the set $\mathcal{C}$ together with the notation introduced above. The prefix in the form of a subscript $_\mathcal{C}$ was introduced in order for later to distinguish objects lying on different surfaces. Whenever it does not create confusion, this prefix will be dropped. An illustration of the notation (with the prefix dropped) can be found in Figure 6.11.

6.2 Inflection surface

The second surface associated to a space curve γ is a torus $\mathbb{S}^1 \times \mathbb{S}^1$.

Definition 6.9. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a smooth immersion with nowhere vanishing curvature and let $T, N : \mathbb{S}^1 \to \mathbb{R}^3$ denote its unit tangent and unit normal vectors respectively. Call a torus $\mathcal{I} := \mathbb{S}^1 \times \mathbb{S}^1$, an *inflection surface* and the map

$$\iota : \mathcal{I} \to \mathbb{S}^2, \quad (t, \theta) \mapsto \cos(\theta)T(t) + \sin(\theta)N(t), \tag{6.9}$$

the *inflection map of* γ. The orientation of $\mathcal{I}$ is given by the order of the tuple (t, θ).

Note that the sets $\mathbb{S}^1 \times \{0\} \subset \mathcal{I}$ and $\mathbb{S}^1 \times \{\pi\} \subset \mathcal{I}$ are mapped via ι to the tangent indicatrix and its antipodal copy, respectively. Denote these two longitude circles of the torus $\mathcal{I}$ by ${}_{\mathcal{I}}\Delta_+$ and ${}_{\mathcal{I}}\Delta_-$ respectively. Set ${}_{\mathcal{C}}\Delta_\pm := \Delta_\pm \subset \mathcal{C}$. Then for each $s \in \mathbb{S}^1$ the following relations

$$\begin{aligned} f(s,s) &= \iota(s,0) \quad \text{ if } (s,s) \in {}_{\mathcal{C}}\Delta_+ \text{ and} \\ f(s,s) &= \iota(s,\pi) \quad \text{ if } (s,s) \in {}_{\mathcal{C}}\Delta_-. \end{aligned} \tag{6.10}$$

are true. With this notation also

$$\begin{aligned} \gamma'(\mathbb{S}^1) &= f({}_{\mathcal{C}}\Delta_+) = \iota({}_{\mathcal{I}}\Delta_+) \text{ and} \\ -\gamma'(\mathbb{S}^1) &= f({}_{\mathcal{C}}\Delta_-) = \iota({}_{\mathcal{I}}\Delta_-) \end{aligned}$$

holds.

The image of the inflection surface under the inflection map is swept out by great circles tangent to the I curve $\mathcal{R}_I$. The name of this surface is explained by Lemma 6.10 below. Recall that for a vector $v \in \mathbb{S}^2$, the curve $\gamma_v = \gamma - \langle \gamma, v\rangle v$ represents the orthogonal projection of γ onto the plane perpendicular to the vector v and let κ_{γ_v} denote the signed curvature of γ_v.

Lemma 6.10. *Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be a smooth immersion with nowhere vanishing curvature and $v \in \mathbb{S}^2 \setminus \left\{\pm\frac{\gamma'(0)}{\|\gamma'(0)\|}\right\}$. Then*

1. *v lies in the osculating plane of γ at 0 if and only if the curvature of γ_v at 0 vanishes, i.e.,*
$$v \in \operatorname{span}\{\gamma'(0), \gamma''(0)\} \iff \kappa_{\gamma_v}(0) = 0.$$

2. *If $\kappa_{\gamma_v}(0) = 0$, then*
$$\gamma_v \text{ has a regular inflection point at } 0 \text{ if and only if } \tau(0) \neq 0.$$

Proof. 1. Let $\gamma_v : \mathbb{S}^1 \to v^\perp$ be the plane curve obtained as the orthogonal projection of γ in direction v, i.e., $\gamma_v(t) = \gamma(t) - \langle \gamma(t), v\rangle v$. Let the orientation of the plane $v^\perp$ be as described in Chapter 4. Then the signed curvature of γ_v is

$$\kappa_{\gamma_v}(t) = \frac{[\gamma_v'(t), \gamma_v''(t), v]}{\|\gamma_v'(t)\|^3} = \frac{[\gamma'(t), \gamma''(t), v]}{\|\gamma_v'(t)\|^3}.$$

2. For $v \in \operatorname{span}\{\gamma'(0), \gamma''(0)\}$ the derivative of the curvature κ_{γ_v} at 0 is

$$\kappa_{\gamma_v}'(0) = \frac{[\gamma'(0), \gamma'''(0), v]}{\|\gamma_v'(0)\|^3}. \tag{6.11}$$

Hence, under the assumption $\kappa_{\gamma_v}(0) = 0$

$$\kappa'_{\gamma_v}(0) = 0 \quad \Longleftrightarrow \quad [\gamma'(0), \gamma''(0), \gamma'''(0)] = 0 \quad \Longleftrightarrow \quad \tau(0) = 0.$$

□

By the above lemma each point $(t, \theta) \in \mathcal{I}$ (provided that $\iota(t, \theta)$ does not lie on the I curve and $\tau(t)$ is non-zero) represents an inflection point of the curve $\gamma_{\iota(t,\theta)}$. Moreover, no two different points $(t_1, \theta_1) \neq (t_2, \theta_2)$ represent the same inflection point of γ_v with $v := \iota(t_1, \theta_1) = \iota(t_2, \theta_2)$. For if they did, then necessarily $t_1 = t_2$ and the existence of v implies $\theta_1 = \theta_2$. On the other hand, if t is an inflection point of a curve γ_v for some v, then the above lemma implies that v lies in an osculating plane of γ at t and so there exists some $\theta \in \mathbb{S}^1$ such that $v = \iota(t, \theta)$. Hence, the name of the inflection surface is justified as it represents all inflection points of plane curves γ_v with $v \in \mathbb{S}^2 \setminus \mathcal{R}_I$.

Let us now investigate where the inflection surface folds when mapped onto the sphere. The partial derivatives of the inflection map are

$$\begin{aligned}
\iota_t(t, \theta) &= -\kappa(t)\sin(\theta)T(t) + \kappa(t)\cos(\theta)N(t) + \tau(t)\sin(\theta)B(t), \\
\iota_\theta(t, \theta) &= -\sin(\theta)T(t) + \cos(\theta)N(t).
\end{aligned}$$

Hence,

$$\big[\iota, \iota_t, \iota_\theta\big](t, \theta) = -\tau(t)\sin(\theta). \tag{6.12}$$

Lemma 6.11. *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be an immersion with nowhere vanishing curvature.*

(i) Then the inflection map of γ is good at

$$\tilde{\mathcal{I}} := \mathbb{S}^1 \times \mathbb{S}^1 \setminus \{(t, \theta) \mid \tau(t) = \sin(\theta) = 0 \ \text{or} \ \tau(t) = \tau'(t) = 0\}.$$

(ii) The general folds of the inflection map are

$$\{(t, \theta) \in \tilde{\mathcal{I}} \mid \tau(t) = 0 \ \text{or} \ \theta = 0 \ \text{or} \ \theta = \pi\}.$$

(iii) The fold points of the inflection map are precisely all the points of the general folds.

Proof. (i): Differentiation of (6.12) gives

$$\big[\iota, \iota_t, \iota_\theta\big]_t(t, \theta) = -\tau'(t)\sin(\theta),$$

$$\big[\iota, \iota_t, \iota_\theta\big]_\theta(t,\theta) = -\tau(t)\cos(\theta)$$

and

$$\big[\iota, \iota_t, \iota_\theta\big](t,\theta) = \big[\iota, \iota_t, \iota_\theta\big]_t(t,\theta) = \big[\iota, \iota_t, \iota_\theta\big]_\theta(t,\theta) = 0$$
$$\iff \quad \tau(t) = \sin(\theta) = 0 \ \text{ or } \ \tau(t) = \tau'(t) = 0.$$

Part (ii) follows from the above computation. (iii): A great circle has a well-defined tangent vector at each point. Also, since the curve γ has nowhere vanishing curvature, the tangent indicatrix is an immersed curve. $\square$

Remark 6.12. The set of critical values of the inflection map is precisely the I curve $\mathcal{R}_I$ and great circles tangent to the I curve at points corresponding to vanishing torsion, denoted by $\mathcal{P}_{\text{osc}}^{\tau=0}$. From Lemma 6.11 follows that if $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ is in general position, its inflection map is good everywhere except at finitely many points of the inflection surface, i.e., $\tilde{\mathcal{I}} = \mathcal{I} \setminus \{(t,\theta) \in \mathbb{S}^1 \times \mathbb{S}^1 \mid \tau(t) = \sin(\theta) = 0\}$. So the inflection surface "folds" at almost all critical values, i.e., along the I curve and the great circles of $\mathcal{P}_{\text{osc}}^{\tau=0}$ (except the finitely many mentioned). See Figure 6.12.

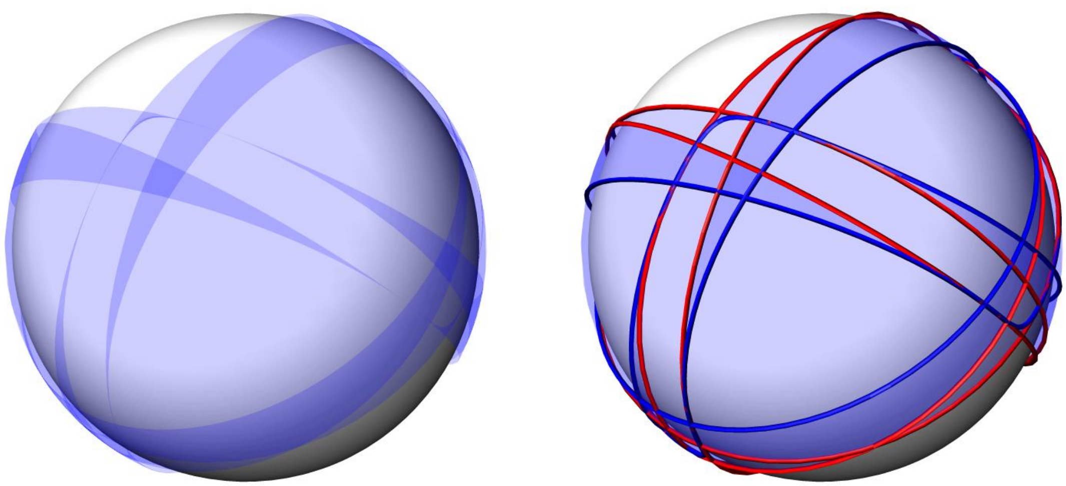


Figure 6.12: The image of the inflection surface $\iota(\mathcal{I})$ of the knot $\mathcal{K}$ (left) with its general folds marked (right): blue - the tangent indicatrix and red - great circles corresponding to osculating planes at torsion vanishing points.

An immediate consequence of the above lemmas is the significant property of the inflection map given by

Theorem 6.13. *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a smooth immersion with nowhere vanishing curvature. For a regular value v of the inflection map ι, i.e., $v \notin \mathcal{R}_I \cup \mathcal{P}_{\text{osc}}^{\tau=0}$, the* *curve γ_v is a smooth immersion with only regular inflection points. Moreover, the number of preimages of v, i.e., the cardinality of $\iota^{-1}(v)$ equals the number of regular inflection points of γ_v, i.e.,*

$$\#\iota^{-1}(v) = \mathrm{infl}(\gamma_v).$$

Next, the orientation of the image of the surface on the sphere will be investigated.

Lemma 6.14. *Let $(t, \theta) \in \mathcal{I}$ be a regular point of the inflection map ι. Then, the index of (t, θ) under the inflection map can be decided by the geometry of the corresponding inflection point t of curve $\gamma_{\iota(t,\theta)}$, as shown in Figure 6.13. More precisely,*

$$\mathrm{ind}_{\iota,\mathcal{I}}(t, \theta) = \mathrm{sign}\, \kappa'_{\gamma_{\iota(t,\theta)}}(t).$$

Figure 6.13: The index of a regular inflection point.

Proof. Using (6.12) the index of an inflection point (t, θ) is given by

$$\mathrm{ind}_{\iota,\mathcal{I}}(t, \theta) = \mathrm{sign}[\iota, \iota_t, \theta](t, \theta) = -\,\mathrm{sign}\,\big(\tau(t)\sin(\theta)\big). \tag{6.13}$$

Now, fix $(t_0, \theta_0) \in \mathcal{I}$. By Lemma 6.10, the signed curvature of γ_v with $v = \iota(t_0, \theta_0)$ vanishes at t_0, i.e., $\kappa_{\gamma_v}(t_0) = 0$. However, using (6.11),

$$\kappa'_{\gamma_v}(t_0) = \frac{[\gamma'(t_0), \gamma'''(t_0), v]}{\|\gamma'_v(t_0)\|^3} = \frac{[\gamma'(t_0), \gamma'''(t_0), \sin(\theta_0)\gamma''(t_0)]}{\|\gamma'_v(t_0)\|^3\|\gamma''(t_0)\|} = \frac{-\tau(t_0)\sin(\theta_0)}{\|\gamma'_v(t_0)\|^3\|\gamma''(t_0)\|}.$$

□

Since the two types of inflection points with positive and negative index always appear alternately, provided the curve γ is in general position, it immediately follows

Corollary 6.15. *The degree of the inflection map ι is zero.*

The orientation of the image of the inflection surface on the sphere is summarized in Figure 6.14, which also contains some additional notation described in the section below.

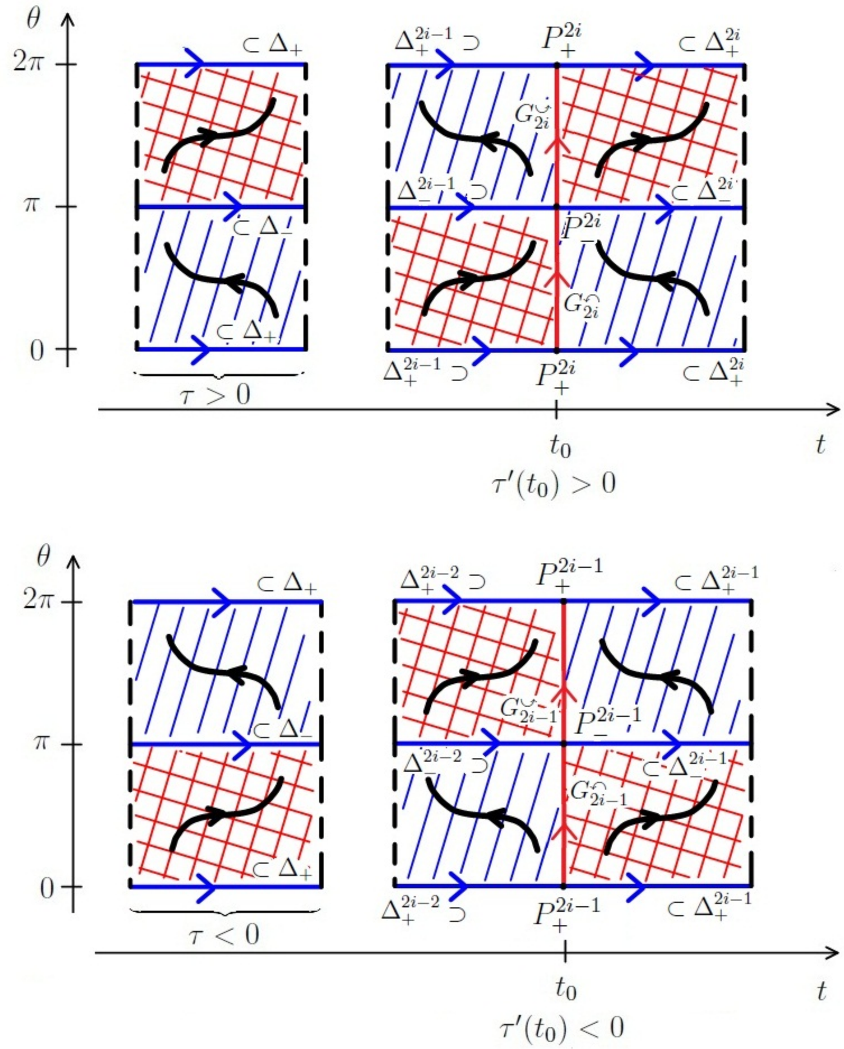


Figure 6.14: Orientation of the inflection surface: points of negative index - blue/lined and points of positive index - red/checked.

Notation for the inflection surface

Again the section ends with the introduction of some notation that will be used for later purposes. The idea is to name the subsets of all three surfaces of this chapter, whose images on $\mathbb{S}^2$ coincide, using the same notation. The notation shall vary only by a prefix that lets to distinguish to which surface the set belongs to. Again, whenever there is no ambiguity (as for example in the figures of this section), the prefix is dropped.

Suppose that the curve γ is in general position (!) and has exactly $2M$ with $M \neq 0$ points at which torsion vanishes. Index them precisely in the same way it has already been done in section about the crossing surface, i.e., with $s_1 < s_2 < \ldots < s_{2M} \in \mathbb{S}^1$. For $i = 1, \ldots, 2M$, denote by ${}_{\mathcal{I}}P^i_+$ the points $(s_i, 0) \in \mathcal{I}$ and by ${}_{\mathcal{I}}P^i_-$ the points $(s_i, \pi) \in \mathcal{I}$. Then

$$f({}_{\mathcal{C}}P^i_+) = \iota({}_{\mathcal{I}}P^i_+) \text{ and } f({}_{\mathcal{C}}P^i_-) = \iota({}_{\mathcal{I}}P^i_-) \text{ for } i = 1, \ldots, 2M.$$

For $i = 1, \ldots, 2M$, denote by

$$\begin{aligned} {}_{\mathcal{I}}\Delta^i_+ &:= \{(s, 0) \in \mathcal{I} \mid s_i \leq s \leq s_{i+1}\} \\ {}_{\mathcal{I}}\Delta^i_- &:= \{(s, \pi) \in \mathcal{I} \mid s_i \leq s \leq s_{i+1}\} \end{aligned}$$

the line segments starting at ${}_{\mathcal{I}}P^i_\pm$ and ending at ${}_{\mathcal{I}}P^{i+1}_\pm$. Then, for each $s \in \mathbb{S}^1$,

$$(s, s) \in {}_{\mathcal{C}}\Delta^i_\pm \quad \Longleftrightarrow \quad (s, 0) \in {}_{\mathcal{I}}\Delta^i_\pm$$

and, with the pointwise correspondence given by (6.10),

$$f({}_{\mathcal{C}}\Delta^i_+) = \iota({}_{\mathcal{I}}\Delta^i_+) \;\text{ and }\; f({}_{\mathcal{C}}\Delta^i_-) = \iota({}_{\mathcal{I}}\Delta^i_-) \tag{6.14}$$

for $i = 1, \ldots, 2M$. Moreover, for each $i = 1, \ldots, 2M$, let

$$\begin{aligned} {}_{\mathcal{I}}G^{\frown}_i &:= \{(s_i, \theta) \in \mathcal{I} \mid \theta \in [0, \pi]\} \text{ and} \\ {}_{\mathcal{I}}G^{\smile}_i &:= \{(s_i, \theta) \in \mathcal{I} \mid \theta \in [\pi, 2\pi]\} \end{aligned}$$

and orient them according to the orientation of $\mathbb{S}^1 \equiv \mathbb{R}/2\pi\mathbb{Z}$. Notice that $\iota({}_{\mathcal{I}}G^{\frown}_i)$ is the first half and $\iota({}_{\mathcal{I}}G^{\smile}_i)$ the second half of the great circle tangent to γ' at $\gamma'(s_i)$.

In case when the torsion of γ does not vanish, rename as in the previous section, $\Delta_\pm$ by ${}_{\mathcal{I}}\Delta^0_\pm$ if the torsion is positive and by ${}_{\mathcal{I}}\Delta^{-1}_\pm$, otherwise. This way also (6.14) holds for $i = 0$ or $i = -1$. Denote analogously by $\mathcal{I}(\gamma)$ the torus $\mathcal{I}$ together with the above notation specific to a particular curve γ.

6.3 Bitangency surface

Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a curve in general position. To each point (s, t) on the bitangency manifold $\mathcal{B}$ of γ there is a bitangent plane associated. Namely, the plane

$$\omega(s, t) = \operatorname{span}\{\gamma'(s), \gamma'(t), \gamma(s) - \gamma(t)\}. \tag{6.15}$$

From the genericity of γ follows that the partial derivatives of f cannot both vanish at the same time, at any point of $\mathcal{B}$ (compare with Remark 5.5). Hence,

$$\omega(s,t) = \operatorname{span}\{f(s,t), f_s(s,t), f_t(s,t)\}. \tag{6.16}$$

By Corollary 5.12, the above plane $\omega(s,t)$ is tangent to the II curve at $f(s,t)$, provided that the II curve can be immersed around this point (in other words (s,t) is a fold point of the crossing map). Moreover, $f(s,t)$ is perpendicular to both $f_s(s,t)$ and $f_t(s,t)$ at each $(s,t) \in \mathcal{B}$.

The aim of this section is to construct a surface, whose image consists of great circles, each representing a bitangent plane. Topologically, one attaches $\mathbb{S}^1$ to each point of $\mathcal{B}$, so in case of a component of $\mathcal{B}$ homeomorphic to an interval, such a surface will be an annulus. In case of a component of $\mathcal{B}$ homeomorphic to $\mathbb{S}^1$, the surface obtained may be a torus or a Klein bottle.

The aim of the following lemma is to find an orthonormal basis for the planes $\omega(s,t)$ with $(s,t) \in \mathcal{B}$, consisting of vectors belonging to the II curve and vectors perpendicular to it.

Lemma 6.16. *Let C be a main component of the closure $\mathrm{cl}_{\mathcal{C}}(\mathcal{B})$ of $\mathcal{B}$ in $\mathcal{C}$ (oriented as in Figure 5.11). Let $\phi : [0,1] \to C$ be a smooth immersion such that if C is homeomorphic to $\mathbb{S}^1$, the function ψ, obtained through the natural factorization of ϕ through $[0,1] \to \mathbb{S}^1 \xrightarrow{\psi} C$, is smooth. Let $c := f \circ \phi$. Then there exists a smooth vector field $w : [0,1] \to \mathbb{S}^2$ such that $c(a) \perp w(a)$ and $[c, c', w](a) = 0$ for all $a \in [0,1]$. Moreover, for each $a \in [0,1]$ with $\phi(a) \in \mathcal{B}$ (i.e., $a \in (0,1)$ for $C \approx [0,1]$ and $a \in [0,1]$ for $C \approx \mathbb{S}^1$) the vector field w satisfies*

$$[c, w, w'](a) = 0 \quad \Longrightarrow \quad c'(a) \neq 0 \tag{6.17}$$

and w is unique up to sign. When C is homeomorphic to an interval, then the endpoints of w are parallel to $\gamma''(s(0))$ and $\gamma''(s(1))$ respectively.

Proof. By Corollary 5.18, γ has finitely many cross-tangents and therefore, there are finitely many zeros of f_s in $\mathcal{B}$. Each of the two open sets

$$\{a \in (0,1) \mid f_s(s(a), t(a)) \neq 0\} \quad \text{and} \quad \{a \in (0,1) \mid f_t(s(a), t(a)) \neq 0\}$$

consists of finitely many open intervals which form a cover $\mathcal{A}$ of $(0,1)$. For some point $a_0 \in (0,1)$ with $f_s(s(a_0), t(a_0)) \neq 0$, choose w to be either $w(a_0) = +\frac{f_s(s,t)}{\|f_s(s,t)\|}(a_0)$ or

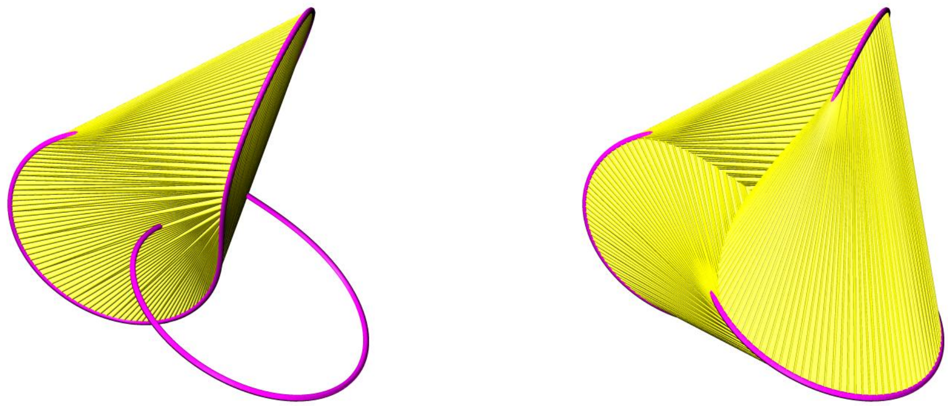


Figure 6.15: The developable surface made of bitangent segments of the knot $\mathcal{K}$ from a single component of $\mathcal{B}$ (left) and from the entire $\mathcal{B}$ (right).

$w(a_0) = -\frac{f_s(s,t)}{\|f_s(s,t)\|}(a_0)$. Extend this choice on the remaining sets of the cover $\mathcal{A}$ so that w agrees on the intersection sets. Set the endpoints to be $w(0) := \lim_{a\to 0} w(a)$ and $w(1) := \lim_{a\to 1} w(a)$. Clearly, w can be extended globally on $[0,1]$ in case when C is homeomorphic to $\mathbb{S}^1$. When C is homeomorphic to $[0,1]$, then by Remark 5.10, for a sufficiently close to $\bar{a} \in \partial[0,1]$, the bitangent plane $\omega(s(a), t(a))$ is external. Moreover, it tends to an osculating plane at a point $\bar{s} := s(\bar{a}) = t(\bar{a})$ with $\tau(\bar{s}) = 0$ and the vectors $\frac{f_s(s,t)}{\|f_s(s,t)\|}(a)$ and $\frac{f_t(s,t)}{\|f_t(s,t)\|}(a)$ both tend to $\pm\frac{\gamma''(\bar{s})}{\|\gamma''(\bar{s})\|}$ as a tends to $\bar{a}$.

The condition that $[c, c', w](a) = 0$ for all $a \in [0,1]$ follows directly from Corollary 5.12. It remains to show (6.17). Suppose $c'(a_0) = 0$ at some $a_0 \in [0,1]$ with $\phi(a_0) \in \mathcal{B}$. Then from Lemma 5.13 follows that $g_s(\phi(a_0)) \neq 0$ and $g_t(\phi(a_0)) \neq 0$, which immediately implies that $f_s(\phi(a_0)) \neq 0$ and $f_t(\phi(a_0)) \neq 0$. The preceding discussion implies that $w(a_0) = \pm\frac{f_s(\phi(a_0))}{\|f_s(\phi(a_0))\|}$ and

$$\begin{aligned}[c, w, w'](a_0) = 0 \quad &\iff \quad g_t \cdot [f, f_s, f_{ss}](\phi(a_0)) = 0 \\ &\iff \quad g_s \cdot g_t(\phi(a_0)) = 0.\end{aligned}$$

Hence, the proof is established. □

Remark 6.17. A natural question to ask is, whether the vector field w of the preceding lemma can be continuously defined on $\mathbb{S}^1$, in case of a component C homeomorphic to $\mathbb{S}^1$. The question is simply whether $w(0) = w(1)$ always holds, or is it possible to

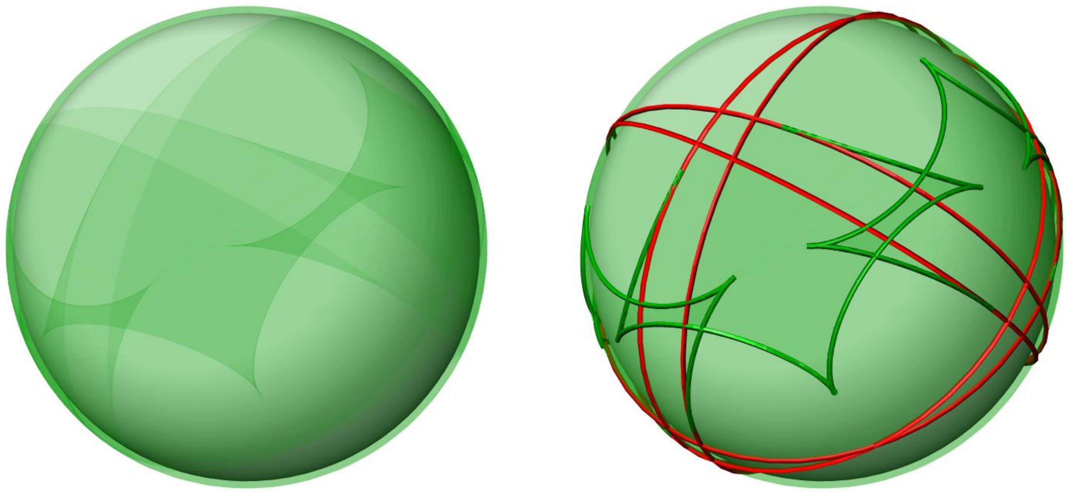


Figure 6.16: The image of the bitangency surface $h(\mathcal{H})$ of the knot $\mathcal{K}$ (left) with its general folds marked green and boundaries marked red (right).

have $w(0) = -w(1)$. This, in turns, is equivalent to deciding whether the image of the developable surface made of bitangent segments $x : C \times [0,1] \to \mathbb{R}^3$,

$$x(a,\lambda) = \gamma(s(a)) + \gamma(t(a)) - \gamma(s(a))\lambda, \tag{6.18}$$

has a continuous normal (see Figure 6.15). All numerical examples considered by the author suggest that $w(0) = w(1)$ holds. However, the author was unable to rule out the case $w(0) = -w(1)$.

It is now possible to define a surface made of great circles with c and w as in the lemma above.

Definition 6.18. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a curve in general position. Let $C_1 := {}_{\mathcal{C}}C_1, \ldots, C_{M+N} := {}_{\mathcal{C}}C_{M+N}$ be the main components of $\mathrm{cl}_{\mathcal{C}}(\mathcal{B})$. Let J be a disjoint union of $M+N$ intervals $I_j := [0,1]$ with $j = 1,\ldots,M+N$. Let $\phi = (s,t) : J \to \mathrm{cl}_{\mathcal{C}}(\mathcal{B})$ and $w : J \to \mathbb{S}^2$ be maps such that for each $j = 1,\ldots,M+N$, the restriction $\phi|_{I_j} : [0,1] \to C_j$ and $w|_{I_j} : [0,1] \to \mathbb{S}^2$ is a parametrization of C_j and the corresponding vector field from Lemma 6.16, respectively. Call the set $\mathcal{H} := J \times \mathbb{S}^1$ the *bitangency surface* and the map

$$h : \mathcal{H} \to \mathbb{S}^2, \quad (a,\theta) \mapsto \cos(\theta)c(a) + \sin(\theta)w(a)$$

the *bitangency map* of γ. The orientation of $\mathcal{H}$ is given by the order of (a,θ).

The bitangency surface is defined as $M+N$ closed annuli. The image of (the interior of) the bitangency surface under the bitangency map is swept out by great circles tangent to the II curve, whenever the II curve is immersed (see the left of Figure 6.16). More generally, it is made of great circles each representing a bitangent plane of γ.

Denote by 0_j and 1_j the endpoints of the interval I_j, i.e., $0_j := 0 \in I_j$ and $1_j := 1 \in I_j$, for any $j = 1, \ldots, M+N$. Consider the case when the component C_j is homeomorphic to an interval, i.e., $j = 1, \ldots, M$. Then the image under h of the boundary of $I_j \times \mathbb{S}^1$ consists of two great circles spanning the osculating planes at torsion vanishing points of γ corresponding to the endpoints of C_j. That is, $h(0_j \times \mathbb{S}^1)$ and $h(1_j \times \mathbb{S}^1)$ are both in $\mathcal{P}_{\text{osc}}^{\tau=0}(\gamma)$. Suppose now that a component C_j is homeomorphic to $\mathbb{S}^1$, i.e., $j = M+1, \ldots, M+N$. The image of the boundaries of the component $I_j \times \mathbb{S}^1 \subset \mathcal{H}$ under the bitangency map is the same great circle $h(0_j, \mathbb{S}^1) = h(1_j, \mathbb{S}^1)$. Hence, these boundaries of an annulus could be glued together to form a compact manifold. However, by Remark 6.17, it cannot be excluded that due to the gluing a Klein bottle instead of a torus arises. Since it is more convenient here to work with orientable surfaces, the author prefers to stick to the closed annuli as the domain of the bitangency map. Also keep in mind that the point $(s(0_j), t(0_j)) = (s(1_j), t(1_j)) \in C_j$ can be chosen arbitrarily and can be moved around if necessary. So the great circle $h(0_j, \mathbb{S}^1) = h(1_j, \mathbb{S}^1)$ won't be treated separately, whenever it will not be necessary. In particular, these great circles are not true critical values of the map h, although the choice of the definition treats them as such. However, this inconvenience in the notation will be resolved in the next chapter.

In the spirit of the above comment, let the subset of $\mathcal{H}$ given by

$$\mathcal{H}^\circ := \dot{\bigcup}_{j=1}^{M} \operatorname{int}(I_j) \times \mathbb{S}^1 \quad \dot\cup \quad \dot{\bigcup}_{j=M+1}^{M+N} I_j \times \mathbb{S}^1,$$

be the quasi-interior of $\mathcal{H}$. Then the information on general folds of $\mathcal{H}^\circ$ follows immediately from the introductory section of this chapter, devoted to surfaces made of great circles tangent to a spherical curve. From Lemma 6.5 follows that the bitangency map is *not* good precisely at $(a, \theta) \in \mathcal{H}^\circ$, when

(i) the II curve has a regular inflection point at a, i.e., the plane $\omega(s(a), t(a))$ is an osculating bitangent plane at either s with $\tau(s) \neq 0$ or at t with $\tau(t) \neq 0$, and moreover, θ is a multiple of π or

(ii) the II curve has a special irregular inflection point at a, i.e., $\omega(s, t)$ is an osculating bitangent plane at either s with $\tau(s) = 0$ or at t with $\tau(t) = 0$.

Moreover, the bitangency map h has general folds at $(a, \theta) \in \mathcal{H}^\circ$ when h is good at (a, θ) and θ is a multiple of π.

Furthermore, from Lemma 6.5 follows that the set of singular values of the bitangency map is the union of the II curve and great circles tangent to the II curve at its inflection points. The regular inflection points correspond to the bitangent osculating planes of γ and irregular inflection points to the osculating planes at torsion vanishing points of γ.

Corollary 6.19. *A regular value $v \in \mathbb{S}^2$ of the bitangency map h of γ is any vector different from*

- *the II curve $\mathcal{R}_{II}(\gamma)$ and*
- *$\mathcal{P}_{\text{osc}}^{\text{bit}}(\gamma)$, i.e., great circles representing bitangent osculating planes of γ and*
- *$\mathcal{P}_{\text{osc}}^{\tau=0}(\gamma)$, i.e., the great circles representing osculating planes at torsion-vanishing points of γ,*
- *(the union of the N great circles $P_N := \bigcup_{j=M+1}^{M+N} h(0_j, \mathbb{S}^1)$).*

Now, given a point $(a, \theta) \in \mathcal{H}^\circ$, its image $h(a, \theta)$ represents a direction in the bitangent plane $\omega(a)$ which is at the angle θ to the segment $\gamma(s(a)) - \gamma(t(a))$. In the projected curve $\gamma_{h(a,\theta)}$ the point $(s(a), t(a))$ represents a bitangent pair, provided that θ is not a multiple of π. On the other hand, assuming that a direction $v \in \mathbb{S}^2$ is different from the II curve, let (s, t) be a bitangent pair of the curve γ_v. Then there exists a bitangent plane to the curve γ at s and t, i.e., $(s, t) \in \mathcal{B}$. Set $a := \phi^{-1}(s, t)$ and let $\theta \neq 0, \pi$ be the positive angle from $\gamma(s) - \gamma(t)$ to v in the plane $\omega(s, t)$, oriented by $(c(a), v(a))$. Then (a, θ) is the unique point on the bitangency surface $\mathcal{H}$ representing the bitangent pair (s, t) of $\gamma_v = \gamma_{h(a,\theta)}$. To summarize, the points on the bitangency surface with $\theta \neq 0, \pi$ are in 1-to-1 correspondence with the bitangent pairs of all plane curves obtained as an orthogonal projection of γ. This explains the name of the surface and the map. Another formulation of the above discussion is summarized in the following

Theorem 6.20. *Let $\gamma : \mathbb{S}^1 \to \mathbb{S}^2$ be a curve in general position. For any unit vector $v \in \mathbb{S}^2 \setminus \mathcal{R}_I \cup \mathcal{R}_{II} \cup \mathcal{P}_{\text{osc}}^{\tau=0} \cup \mathcal{P}_{\text{osc}}^{\text{bit}}$, the curve γ_v is a smooth immersion of a circle into the plane $v^\perp$ with only regular bitangent pairs. The number of preimages of v, i.e., the cardinality of the set $h^{-1}(v)$, equals the number of bitangent pairs of the plane curve γ_v, i.e.,*

$$\#h^{-1}(v) = \text{ext}(\gamma_v) + \text{int}(\gamma_v). \tag{6.19}$$

By Lemma 6.16, there are always two choices of the vector field w that differ in the sign. However, with either of them and with the choice of the main components of the bitangency manifold $\mathcal{B}$ (that depends on the choice of orientation of $\mathcal{B}$), w satisfies the following important property.

Lemma 6.21. *Let ϕ and w be as in Definition 6.18 and set $c := f \circ \phi$. Then, for any a in the interior of J*

$$\begin{aligned} \big[c, w, w'\big](a) > 0 &\iff \text{the bitangency plane } \omega(a) \text{ is external} \\ \big[c, w, w'\big](a) < 0 &\iff \text{the bitangency plane } \omega(a) \text{ is internal} \end{aligned}$$

holds.

Proof. This follows from a straightforward computation. Suppose $a \in \operatorname{int}(J)$ with $f_s(\phi(a)) \neq 0$. Then

$$\begin{aligned} \big[c, w, w'\big](a) > 0 &\iff [f, f_s, -g_t \cdot f_{ss}](\phi(a)) > 0 \\ &\iff [f(s,t), f_s(s,t), \gamma''(t)][f(s,t), f_s(s,t), \gamma''(s)](a) > 0. \end{aligned}$$

□

The orientation of the bitangency surface follows immediately from (6.6) and Lemma 6.21 and is

$$\begin{aligned} \operatorname{ind}_{h,\mathcal{H}}(a,\theta) &= \operatorname{sign}\big[h_a, h_\theta, h\big](a,\theta) \\ &= -\operatorname{sign}\Big(\big[c(a), v(a), v'(a)\big]\sin\theta\Big) \\ &= \begin{cases} +1 & \begin{array}{l} \text{if } \ \omega(s,t) \ \text{ is external and } \ \theta \in (\pi, 2\pi) \ \text{ or} \\ \qquad \omega(s,t) \ \text{ is internal and } \ \theta \in (0,\pi), \end{array} \\ \\ -1 & \begin{array}{l} \text{if } \ \omega(s,t) \ \text{ is internal and } \ \theta \in (\pi, 2\pi) \ \text{ or} \\ \qquad \omega(s,t) \ \text{ is external and } \ \theta \in (0,\pi). \end{array} \end{cases} \end{aligned} \tag{6.20}$$

An illustration of the orientation of $\mathcal{H}$ on the sphere is presented in Figure 6.17.

Notation for the bitangency surface

Suppose that $\operatorname{cl}_{\mathcal{C}}(\mathcal{B})$ has $2M$ and $2N$ (with $M,\ N \in \mathbb{N} \cup \{0\}$ and $M + N \neq 0$) components homeomorphic to an interval and $\mathbb{S}^1$ respectively, denoted by

$${}_{\mathcal{C}}C_1, \ldots, {}_{\mathcal{C}}C_{M+N}, {}_{\mathcal{C}}\bar{C}_1, \ldots, {}_{\mathcal{C}}\bar{C}_{M+N}$$

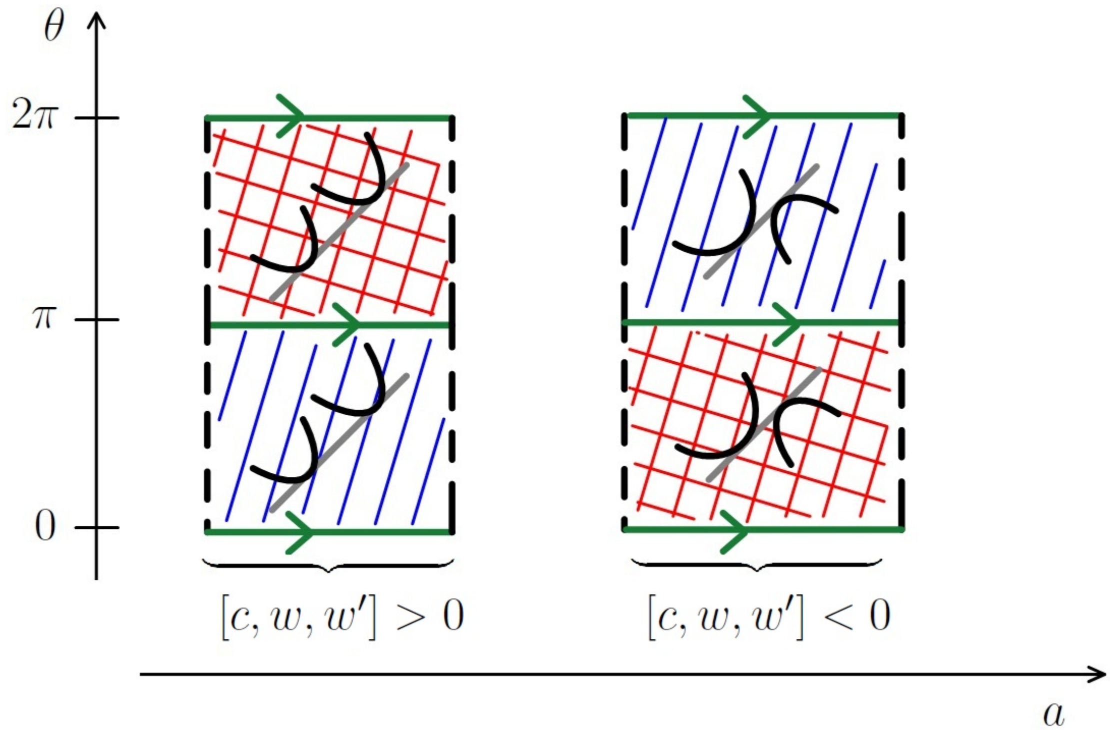


Figure 6.17: Orientation of the bitangency surface: points of negative index - blue/lined and points of positive index - red/checked.

as in section on crossing surface. Suppose that $\phi : J := \dot{\bigcup}_{j=1}^{M+N} I_j \to \mathrm{cl}_{\mathcal{C}}(\mathcal{B})$ is the map from Definition 6.18. Then for any $a \in J$ with $\phi(a) = (s,t)$

$$h(a,0) = f(s,t) \quad \text{and} \quad h(a,\pi) = f(t,s). \tag{6.21}$$

Let

$$\begin{aligned} {}_{\mathcal{H}}C_j &:= I_j \times \{0\} \subset \mathcal{H} \text{ and} \\ {}_{\mathcal{H}}\bar{C}_j &:= I_j \times \{\pi\} \subset \mathcal{H} \end{aligned}$$

for all $j = 1, \ldots, M+N$ so that for any $a \in J$ with $\phi(a) = (s,t)$

$$\begin{aligned} (a,0) \in {}_{\mathcal{H}}C_j \quad &\Longleftrightarrow \quad (s,t) \in {}_{\mathcal{C}}C_j \text{ and} \\ (a,\pi) \in {}_{\mathcal{H}}\bar{C}_j \quad &\Longleftrightarrow \quad (t,s) \in {}_{\mathcal{C}}\bar{C}_j. \end{aligned} \tag{6.22}$$

Therefore,

$$\begin{aligned} h({}_{\mathcal{H}}C_j) &= f({}_{\mathcal{C}}C_j) \\ h({}_{\mathcal{H}}\bar{C}_j) &= f({}_{\mathcal{C}}\bar{C}_j) \end{aligned} \quad \text{for all } j = 1, \ldots, M+N. \tag{6.23}$$

Next, suppose that $M > 0$ and $s_1 < s_2 < \ldots < s_{2M}$ denote, as in the previous two sections, the torsion vanishing points of γ. For $i = 1, \ldots, 2M$ denote by ${}_{\mathcal{H}}P^i_+$ a point $(0_j, 0) \in \mathcal{H}$ or $(1_j, 0) \in \mathcal{H}$ for some $j \in \{1, \ldots, M\}$ if $\phi(0_j) = (s_i, s_i) \in {}_{\mathcal{C}}\Delta_+$ or $\phi(1_j) = (s_i, s_i) \in {}_{\mathcal{C}}\Delta_+$, respectively. Analogously, denote by ${}_{\mathcal{H}}P^i_-$ a point $(0_j, 0) \in \mathcal{H}$ or $(1_j, 0) \in \mathcal{H}$ if $\phi(0_j) = (s_i, s_i) \in {}_{\mathcal{C}}\Delta_-$ or $\phi(1_j) = (s_i, s_i) \in {}_{\mathcal{C}}\Delta_-$, respectively. This way

$$h({}_{\mathcal{H}}P^i_\pm) = f({}_{\mathcal{C}}P^i_\pm) = \iota({}_{\mathcal{I}}P^i_\pm) \text{ for } i = 1, \ldots, 2M. \tag{6.24}$$

Further, for $j = M+1, \ldots, M+N$ let $Q_j := (0_j, 0)$, ${}_{\mathcal{H}}Q_{j*} = (1_j, 0)$, ${}_{\mathcal{H}}\bar{Q}_j = (0_j, \pi)$ and ${}_{\mathcal{H}}\bar{Q}_{j*} = (1_j, \pi)$. Then

$$h({}_{\mathcal{H}}Q_j) = h({}_{\mathcal{H}}Q_{j*}) \quad \text{and} \quad h({}_{\mathcal{H}}\bar{Q}_j) = h({}_{\mathcal{H}}\bar{Q}_{j*})$$

for all $j = M+1, \ldots, M+N$.

The last family of subsets of $\mathcal{H}$ to describe are the boundaries. Let us start with those of $I_j \times \mathbb{S}^1$ for $j = M+1, \ldots, M+N$. Clearly, both of the sets $0_j \times \mathbb{S}^1$ and $1_j \times \mathbb{S}^1$ map to the same great circle. Denote by ${}_{\mathcal{H}}G^{\frown}_j$ and ${}_{\mathcal{H}}G^{\smile}_j$ the sets $0_j \times [0, \pi]$ and $0_j \times [\pi, 2\pi]$ oriented according to the orientation of $\mathbb{S}^1 \ni \theta$. Recall, that for each $j = M+1, \ldots, M+N$ there are the two possibilities: (1) either $w(0_j) = w(1_j)$ or (2) $w(0_j) = -w(1_j)$, where w is the vector field from Definition 6.18. In case (1), i.e., when $h(0_j, \theta) = h(1_j, \theta)$ holds for each $\theta \in \mathbb{S}^1$, denote by ${}_{\mathcal{H}}G^{\frown}_{j*}$ and ${}_{\mathcal{H}}G^{\smile}_{j*}$ the sets $1_j \times [0, \pi]$ and $1_j \times [\pi, 2\pi]$ oriented according to the orientation of $\mathbb{S}^1 \ni \theta$. In the remaining case (2), i.e., when $h(0_j, \theta) = h(1_j, 2\pi - \theta)$ holds for each $\theta \in \mathbb{S}^1$, denote by ${}_{\mathcal{H}}G^{\frown}_{j*}$ and ${}_{\mathcal{H}}G^{\smile}_{j*}$ the sets $1_j \times [\pi, 2\pi]$ and $1_j \times [0, \pi]$ oriented *oppositely* to the orientation of $\mathbb{S}^1 \ni \theta$. See Figure 6.18. With this notation the equalities

$$h({}_{\mathcal{H}}G^{\frown}_j) = h({}_{\mathcal{H}}G^{\frown}_{j*}) \quad \text{and} \quad h({}_{\mathcal{H}}G^{\smile}_j) = h({}_{\mathcal{H}}G^{\smile}_{j*})$$

hold for all $j = M+1, \ldots, M+N$.

It remains to describe the boundaries of $I_j \times \mathbb{S}^1$ for $j = 1, \ldots, M$. From Lemma 6.16 follows that each $h(0_j \times \mathbb{S}^1)$ and $h(1_j \times \mathbb{S}^1)$ is a great circle of $\mathbb{S}^2$ that spans an osculating plane at a point of γ of vanishing torsion. More precisely, if $(e_j, 0) = P^i_\pm$ (with $e_j = 0_j$ or 1_j) for some $i \in \{1, \ldots, 2M\}$, then $h(e_j \times \mathbb{S}^1)$ spans the osculating plane of γ at s_i. (Recall that no direct dependence between the indices i and j can be established,

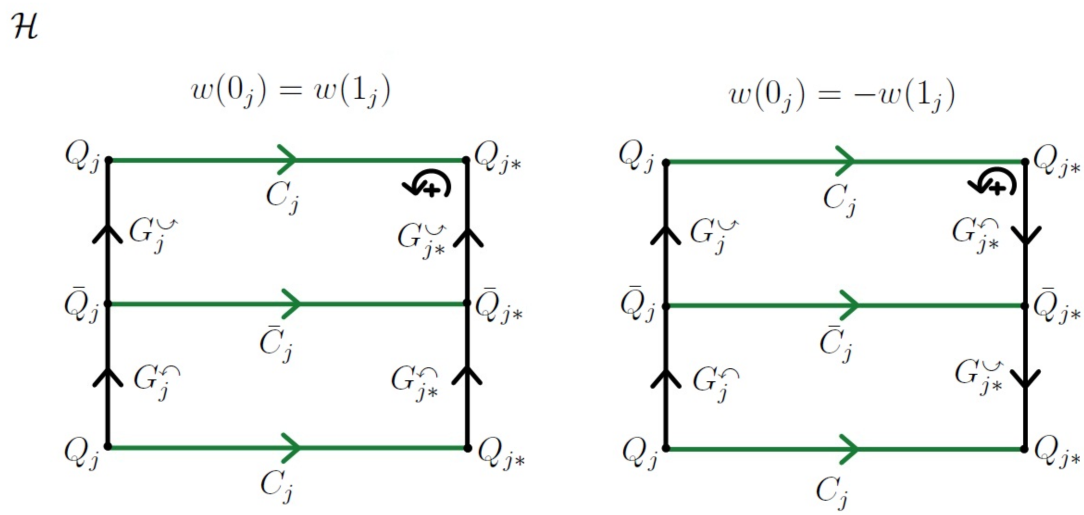


Figure 6.18: Notation of a component $I_j \times \mathbb{S}^1$ of $\mathcal{H}$ for $j = M+1, \ldots, M+N$.

as the components of $\mathcal{B}$ may start and end at the same or opposite diagonals.) The aim is to denote the boundaries of $\mathcal{H}$ with ${}_{\mathcal{H}}G_i^{\frown}$, ${}_{\mathcal{H}}G_i^{\smile}$ in such a way that

$$h({}_{\mathcal{H}}G_i^{\frown}) = \iota({}_{\mathcal{I}}G_i^{\smile}) \quad \text{and} \quad h({}_{\mathcal{H}}G_i^{\frown}) = \iota({}_{\mathcal{I}}G_i^{\smile}) \tag{6.25}$$

hold (pointwise) for each $i = 1, \ldots, 2M$. So, define

$$\left.\begin{cases} {}_{\mathcal{H}}G_i^{\frown} := \{s_i\} \times [0, \ \pi] \\ {}_{\mathcal{H}}G_i^{\smile} := \{s_i\} \times [\pi, 2\pi] \end{cases}\right\} \text{ if } \ h(s_i, \theta) = \iota(s_i, \theta) \ \text{ for all } \ \theta \in \mathbb{S}^1 \ \text{ or}$$
$$\left.\begin{cases} {}_{\mathcal{H}}G_i^{\frown} := \{s_i\} \times [\pi, 2\pi] \\ {}_{\mathcal{H}}G_i^{\smile} := \{s_i\} \times [0, \ \pi] \end{cases}\right\} \text{ if } \ h(s_i, \theta) = \iota(s_i, 2\pi - \theta) \ \text{ for all } \ \theta \in \mathbb{S}^1 \tag{6.26}$$

for $i = 1, \ldots, 2M$ so that (6.25) holds. In order to match the orientation, in the first of the two cases above G_i's are oriented according to θ and in the second case in the opposite manner. Note that this notation depends on the choice of the vector field w. A change in the sign of w would result in the change of notation between the two cases above. However, it would not change the orientation of the surface on the sphere. The index of each point remains unchanged when w is replaced by $-w$.

To illustrate the two possibilities fix a $j \in \{1, \ldots, M\}$ and take a main component ${}_{\mathcal{C}}C := {}_{\mathcal{C}}C_j$ of $\mathrm{cl}_{\mathcal{C}}(\mathcal{B})$ homeomorphic to an interval and the corresponding subset ${}_{\mathcal{H}}C := {}_{\mathcal{H}}C_j$ of $\mathcal{H}$. Now, by the section about the crossing surface, there are precisely two possibilities for a *main* component ${}_{\mathcal{C}}C$ to start off the diagonals $\Delta_{\pm}$ and precisely two possibilities to end up at the diagonals, all four depicted on the left of Figure 6.19

(compare also with Figure 6.7). The middle part of Figure 6.19 represents locally the image of the sets ${}_{\mathcal{C}}\Delta_{\pm}$, ${}_{\mathcal{H}}G_i^{\frown}$ and ${}_{\mathcal{H}}G_i^{\smile}$ under the corresponding maps (with appropriate subscript i). The right two columns of Figure 6.19 represent the two possibilities of the surface $\mathcal{H}$ (together with its orientation on the sphere).

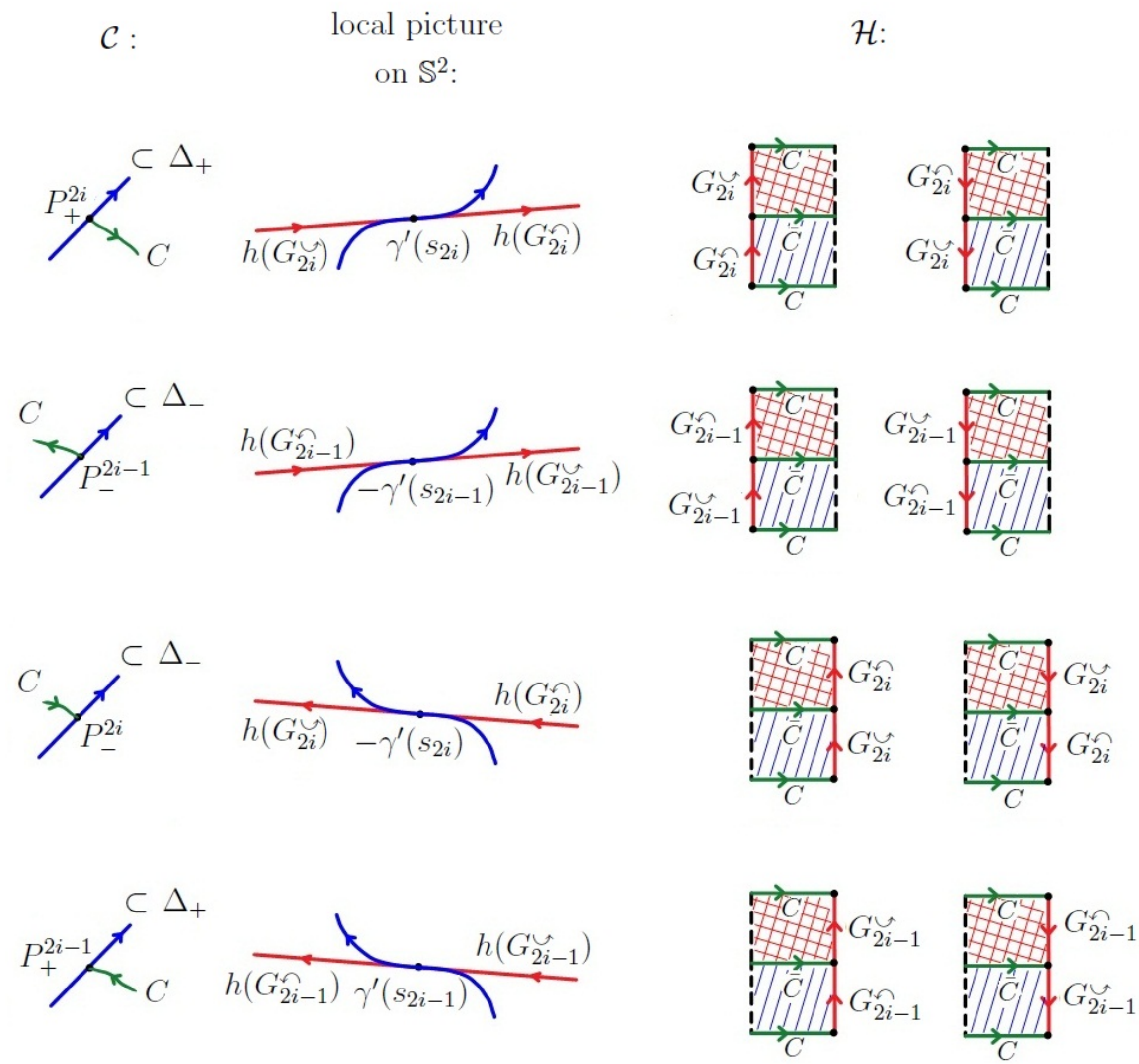


Figure 6.19: Left: 4 cases of a component ${}_{\mathcal{C}}C$ to start and end at ${}_{\mathcal{C}}\Delta_{\pm}$; middle: a local picture of the I curve and $h({}_{\mathcal{H}}G^{\frown} \cup {}_{\mathcal{H}}G^{\smile})$ for each case; right: the two possibilities of the bitangency surface at its boundary together with the orientation on the sphere for each case.

Chapter 7

Surfaces related to Fabricius-Bjerre formula

Throughout this chapter let γ denote a curve in general position in the sense of Definition 4.1. Here, from each of the three surfaces - the crossing, the inflection and the bitangency surface (defined in the previous chapter) a new surface is derived. This is achieved by "cutting" the original surfaces into pieces, reorienting those pieces, and performing some identifications on their boundaries. Then, it turns out, that these surfaces can be glued together to form an oriented closed surface (Theorem 7.4). The three maps - the crossing, the inflection and the bitangency map - give rise to a continuous map (which is smooth except at a finite graph) from the constructed closed 2-manifold to the sphere $\mathbb{S}^2$. The degree of this new map is closely related to the Fabricius-Bjerre formula (1.1) and will be later in Part III of the thesis used to give a new proof of Fabricius-Bjerre's Theorem.

Cutting and Gluing

Here, cutting of a surface $\mathcal{S}$ along an oriented 1-manifold A with endpoints P and Q lying in the boundary of $\mathcal{S}$, is understood as removing A from $\mathcal{S}$. Moreover, to keep the modified surface compact, two copies of the removed 1-manifold, say A^* and A^{**} (with the orientation inherited from A) are attached at the place of the missing A. (The endpoints are accordingly relabeled into P^*, P^{**} and Q^*, Q^{**}). A gluing is understood as the reverse process, i.e., identifying A^* with A^{**} together with the endpoints and will be denoted by $A^* \leftrightarrow A^{**}$. (When gluing together, the pointwise identifications must be made precise.) See Figure 7.1.

The renaming of the copies of A is done arbitrarily and it will make no difference,

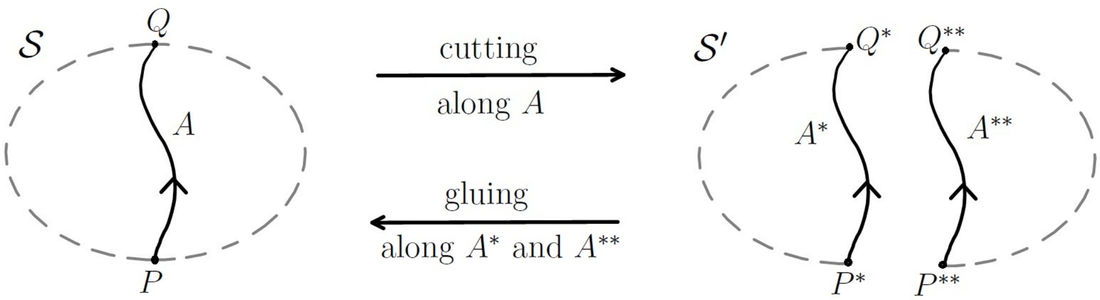


Figure 7.1: Cutting and gluing of a surface along an oriented 1-manifold.

which boundary is named A^*. Later on, different surfaces will be cut and glued together, but along different edges. Precisely this freedom in the labeling of the new boundaries (created in the cutting process) will be the reason for non-uniqueness of the gluing.

The surface $\mathcal{C}'$

Recall the crossing surfaces $\mathcal{C} = \mathcal{C}(\gamma)$ and the crossing map $f = f(\gamma)$ of a generic curve γ. Let $\mathrm{cl}_{\mathcal{C}}(\mathcal{B})$ be the closure of the bitangency manifold $\mathcal{B}$ of γ in $\mathcal{C}$. Denote by $\mathcal{C}' = \mathcal{C}'(\gamma)$ a surface that arises from the crossing surface $\mathcal{C}$ by cutting it along $\mathrm{cl}_{\mathcal{C}}(\mathcal{B})$, oriented as described below. Since $\mathrm{cl}_{\mathcal{C}}(\mathcal{B})$ is a properly embedded manifold, the surface $\mathcal{C}'$ consists of a finite number of connected surfaces, which are homeomorphic to discs or discs with holes, whose boundaries consist of

$$\begin{aligned} {}_{\mathcal{C}'}C_j^*, {}_{\mathcal{C}'}C_j^{**}, {}_{\mathcal{C}'}\bar{C}_j^*, {}_{\mathcal{C}'}\bar{C}_j^{**} \quad & \text{for } j = 1, \dots, M+N && \text{and} \\ {}_{\mathcal{C}'}\Delta_\pm^i \quad & \text{for } i = 1, \dots, 2M && \text{if } M > 0 \text{ or} \\ {}_{\mathcal{C}'}\Delta_\pm^0 \text{ or } {}_{\mathcal{C}'}\Delta_\pm^{-1} \quad & && \text{if } M = 0. \end{aligned} \tag{7.1}$$

Each of the connected components of $\mathcal{C}'$ is embedded in $\mathcal{C}$. Let $i : \mathcal{C}' \to \mathcal{C}$ be the map that is an embedding of each component. Consider a connected component X of $\mathcal{C}'$. Then f is an immersion on the interior $\mathrm{int}(i(X))$ of the corresponding set in $\mathcal{C}$ since the cut was performed precisely along the critical points of f. In particular, $\mathrm{ind}_{f,\mathcal{C}}(x)$ is constant for each $x \in \mathrm{int}(i(X))$. Orient surface $\mathcal{C}'$ using this orientation on the sphere of the image $f(\mathcal{C})$ of the crossing surface. Namely, let the component X inherit the orientation of $\mathcal{C}$ if $\mathrm{ind}_{f,\mathcal{C}}(\mathrm{int}(i(X))) \equiv +1$. Otherwise, reverse the orientation of X, i.e., when $\mathrm{ind}_{f,\mathcal{C}}(\mathrm{int}(i(X))) \equiv -1$. An illustration of construction of the surface $\mathcal{C}'$ based on the example from Figure 6.11 is shown in Figure 7.2.

Now, let f also denote the pullback of $f : \mathcal{C} \to \mathbb{S}^2$ to $\mathcal{C}'$ via i. Then the regular

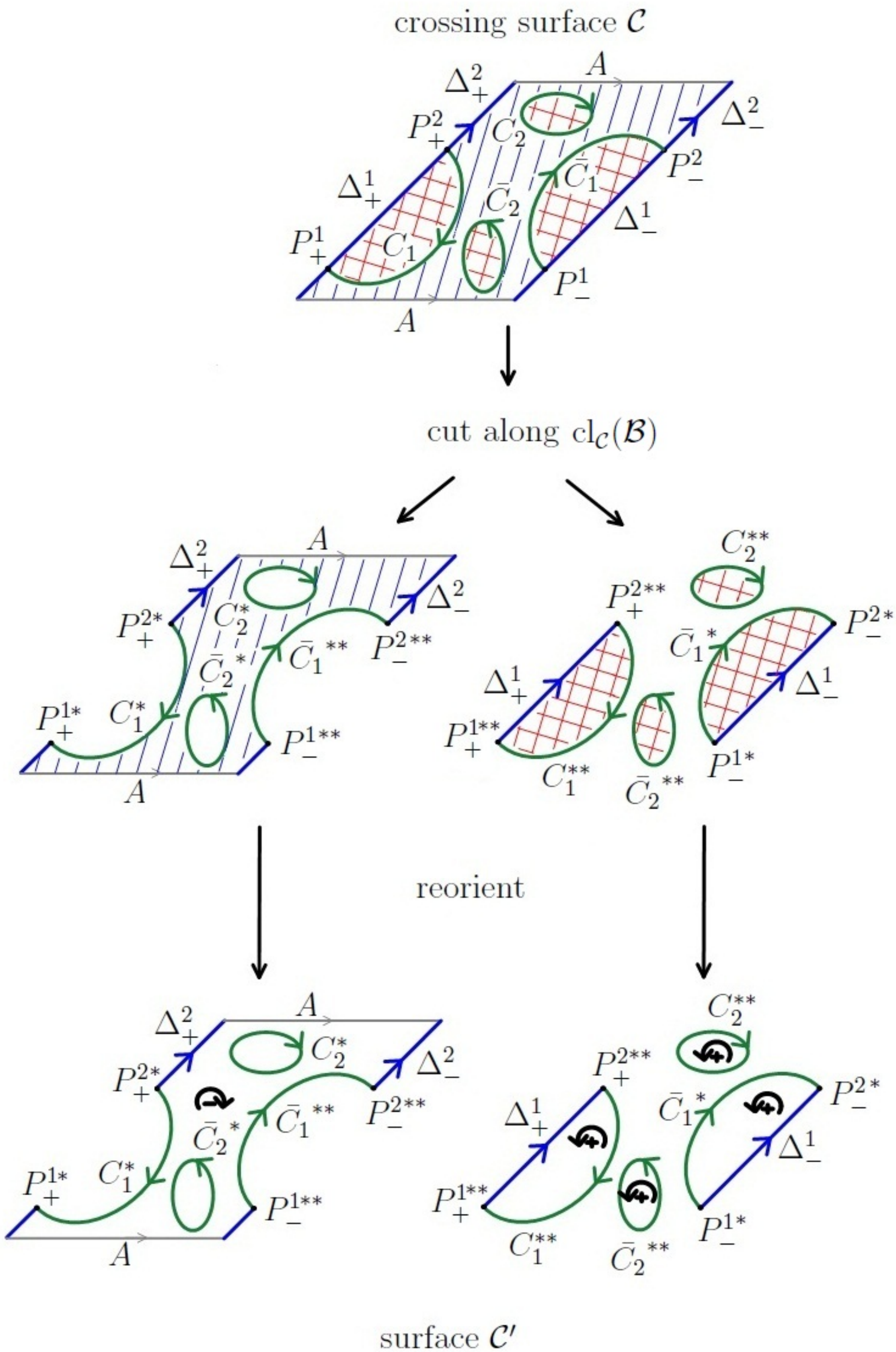


Figure 7.2: An example of creation of the surface $\mathcal{C}'$.

values of $f : \mathcal{C}' \to \mathbb{S}^2$ are precisely those of $f : \mathcal{C} \to \mathbb{S}^2$. By construction, the new function f has the following useful property.

Lemma 7.1. *Let $v \in \mathbb{S}^2$ be a regular value of $f : \mathcal{C}' \to \mathbb{S}^2$, i.e., $v \notin \mathcal{R}_I \cup \mathcal{R}_{II}$. Then*

(i)
$$\#f^{-1}(v) = \mathrm{cr}(\gamma_v);$$

(ii)
$$\mathrm{ind}_{f,\mathcal{C}'}(x) = +1 \text{ for all } x \in \mathrm{int}(\mathcal{C}');$$

(iii) the orientation of the surface $\mathcal{C}'$ close to its boundaries corresponds to that of Figure 7.6.

Proof. Part (i) of the lemma follows immediately from Lemma 6.8. Parts (ii) and (iii) follow immediately from the construction of $\mathcal{C}'$, in particular, from the orientation chosen on $\mathcal{C}'$. □

The surface $\mathcal{I}'$

Recall the inflection surface $\mathcal{I} = \mathcal{I}(\gamma)$ and the inflection map $\iota = \iota(\gamma)$ of a generic curve γ. Define a surface $\mathcal{I}' = \mathcal{I}'(\gamma)$ by

$$\begin{aligned} \mathcal{I}' &:= \mathcal{I} \setminus \{x \in \mathcal{I} \mid \mathrm{ind}_{\iota,\mathcal{I}}(x) = -1\} \\ &\overset{(6.13)}{=} \{(t,\theta) \in \mathcal{I} \mid \tau(t)\sin(\theta) \le 0\} \end{aligned} \tag{7.2}$$

with orientation inherited from $\mathcal{I}$. If the number M of torsion vanishing points is zero, then the surface $\mathcal{I}'$ is given by $\mathbb{S}^1 \times [0,\pi]$ or $\mathbb{S}^1 \times [\pi, 2\pi]$ if the torsion of γ is negative or positive respectively. Otherwise it consists of M rectangles joined together at the vertices:

$$\begin{aligned} \mathcal{I}' &= \{(t,\theta) \in \mathcal{I} \mid \tau(t) \le 0 \text{ and } 0 \le \theta \le \pi\} \cup \{(t,\theta) \in \mathcal{I} \mid \tau(t) \ge 0 \text{ and } \pi \le \theta \le 2\pi\} \\ &= \bigcup_{i=1}^{M} \{t \in \mathbb{S}^1 \mid (t,0) \in {}_{\mathcal{I}}\Delta_+^{2i-1}\} \times [0,\pi] \cup \{t \in \mathbb{S}^1 \mid (t,0) \in {}_{\mathcal{I}}\Delta_+^{2i}\} \times [\pi, 2\pi]. \end{aligned}$$

The boundary of $\mathcal{I}'$ consists of

$$\begin{aligned} &{}_{\mathcal{I}'}\Delta_\pm^i, {}_{\mathcal{I}'}G_i^\frown, {}_{\mathcal{I}'}G_i^\smile \quad \text{for } i = 1, \dots, 2M \quad \text{if } M > 0 \text{ or} \\ &{}_{\mathcal{I}'}\Delta_\pm^0 \text{ or } {}_{\mathcal{I}'}\Delta_\pm^{-1} \qquad\qquad\qquad\qquad \text{if } M = 0. \end{aligned} \tag{7.3}$$

An example of a surface $\mathcal{I}'$ is shown in Figure 7.3. The interior $\mathrm{int}(\mathcal{I}')$ of $\mathcal{I}'$ consists

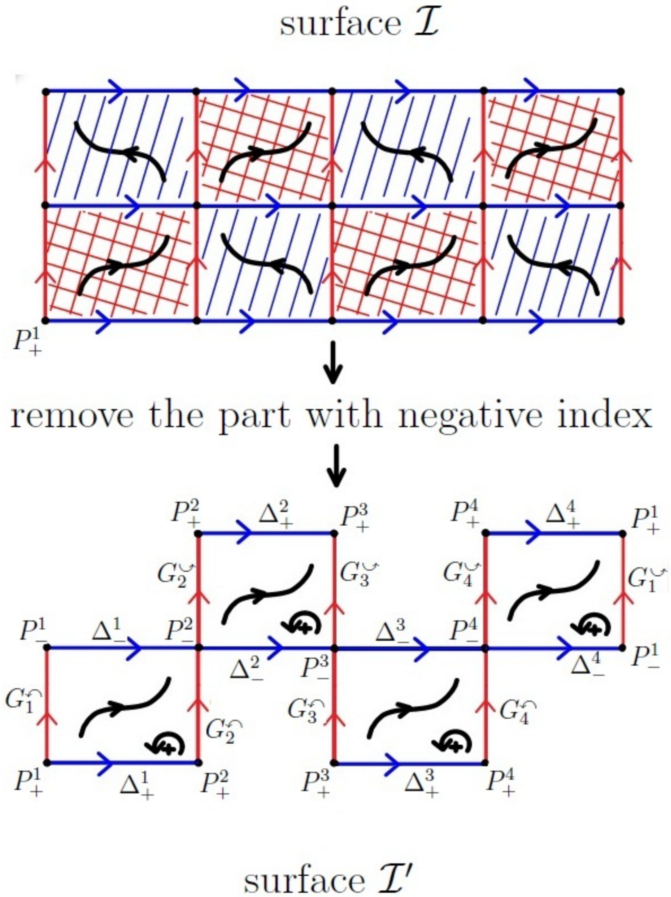


Figure 7.3: An example of creation of the surface $\mathcal{I}'$.

of (a subset of the set of) regular points. The boundary of $\mathcal{I}'$ consists of all critical points of ι. Moreover, by Lemma 6.14, each point $(t,\theta) \in \operatorname{int}(\mathcal{I}')$ represents a regular inflection point of $\gamma_{\iota(t,\theta)}$ of the same type, shown on the left of Figure 6.13. If $v \in \mathbb{S}^2$ is a regular value of ι, then γ_v has only regular inflection points and the two types of regular inflection points appear in γ_v in an alternating manner. Therefore, γ_v has the same number of inflection points of each type. In particular, $\iota(\mathcal{I}) = \iota(\mathcal{I}')$. Hence, the regular values of ι are precisely the regular values of the restriction $\iota|_{\mathcal{I}'}$ of ι to $\mathcal{I}'$ (compare with Remark 6.12). Denote this restriction also by ι throughout the rest of this chapter. By construction the surface $\mathcal{I}'$ has the following useful properties.

Lemma 7.2. *Let $v \in \mathbb{S}^2$ be a regular value of $\iota : \mathcal{I}' \to \mathbb{S}^2$, i.e., $v \notin \mathcal{R}_I \cup \mathcal{P}_{\text{osc}}^{\tau=0}$. Then*

(i)
$$\#\iota^{-1}(v) = \frac{1}{2}\operatorname{infl}(\gamma_v);$$

(ii) $$\mathrm{ind}_{\iota,\mathcal{I}'}(x) = +1 \textit{ for all } x \in \mathrm{int}(\mathcal{I}');$$

(iii) the orientation of the surface $\mathcal{I}'$ close to its boundaries corresponds to that of Figure 7.7.

Proof. Part (i) follows immediately from Theorem 6.13 and the construction of $\mathcal{I}'$. Also (ii) follows from the construction, in particular, from the way the orientation was chosen on $\mathcal{I}'$. For the orientation of $\mathcal{I}'$ close to its boundaries, compare Figure 6.14 with the construction of $\mathcal{I}'$. □

With this new notation the equivalence (6.14) translates to

$$\begin{array}{lll} f({}_{\mathcal{C}'}\Delta^0_\pm) = \iota({}_{\mathcal{I}'}\Delta^0_\pm) & \text{or } f({}_{\mathcal{C}'}\Delta^{-1}_\pm) = \iota({}_{\mathcal{I}'}\Delta^{-1}_\pm) & \text{if } \ M = 0 \ \text{ or} \\ f({}_{\mathcal{C}'}\Delta^i_\pm) = \iota({}_{\mathcal{I}'}\Delta^i_\pm) & & \text{for } i = 1, \ldots, 2M \ \text{ if } \ M \neq 0, \end{array} \tag{7.4}$$

where the equalities hold pointwise for any $s \in \mathbb{S}^1$, according to the rule

$$\begin{array}{ll} f(s,s) = \iota(s,0) & \text{for } (s,s) \in {}_{\mathcal{C}'}\Delta^i_+ \text{ and} \\ f(s,s) = \iota(s,\pi) & \text{for } (s,s) \in {}_{\mathcal{C}'}\Delta^i_- \end{array} \tag{7.5}$$

for $i = 1, \ldots, 2M$ if $M \neq 0$ or, in the remaining case, for either $i = 0$ or $i = -1$.

The surface $\mathcal{H}'$

Finally, recall the bitangency surface $\mathcal{H} = \mathcal{H}(\gamma)$ and the bitangency map $h = h(\gamma)$ of a generic curve γ. It consists of $M + N$ sets each homeomorphic to a closed annulus, where M and N are the numbers of pairs of symmetric components of the bitangency manifold homeomorphic to an interval and $\mathbb{S}^1$ respectively. Denote by $\mathcal{H}' = \mathcal{H}'(\gamma)$ the surface obtained by cutting $\mathcal{H}$ along ${}_{\mathcal{H}}C_j$ and ${}_{\mathcal{H}}\bar{C}_j$ for $j = 1, \ldots, M+N$, with orientation and some additional identifications as described below. The cut surface $\mathcal{H}$ is the disjoint union of $2(M+N)$ subsets of $\mathcal{H}$ given by $I_j \times [0,\pi]$ and $I_j \times [\pi, 2\pi]$ for $j = 1, \ldots, M+N$, where the edges ${}_{\mathcal{H}}C_j$, ${}_{\mathcal{H}}\bar{C}_j \subset \mathcal{H}$ are relabeled into ${}_{\mathcal{H}'}C^*_j$, ${}_{\mathcal{H}'}C^{**}_j$, ${}_{\mathcal{H}'}\bar{C}^*_j$, ${}_{\mathcal{H}'}\bar{C}^{**}_j$ for $j = 1, \ldots, M+N$, since due to the cutting now each of them appears twice. (Also the end points of the "old" edges have to be provided accordingly with a superscript $*$ and $**$.) Orient this cut surface $\mathcal{H}$ so that the components $J \times [0,\pi]$ inherit the orientation of $\mathcal{H}$ and the components $J \times [\pi, 2\pi]$ have the *opposite* orientation to that of $\mathcal{H}$. For the components that arise from components of $\mathcal{B}$ homeomorphic to an interval, i.e., for $j = 1, \ldots, M$, the construction of $\mathcal{H}'$ is finished and an example is illustrated in Figure 7.4. To complete

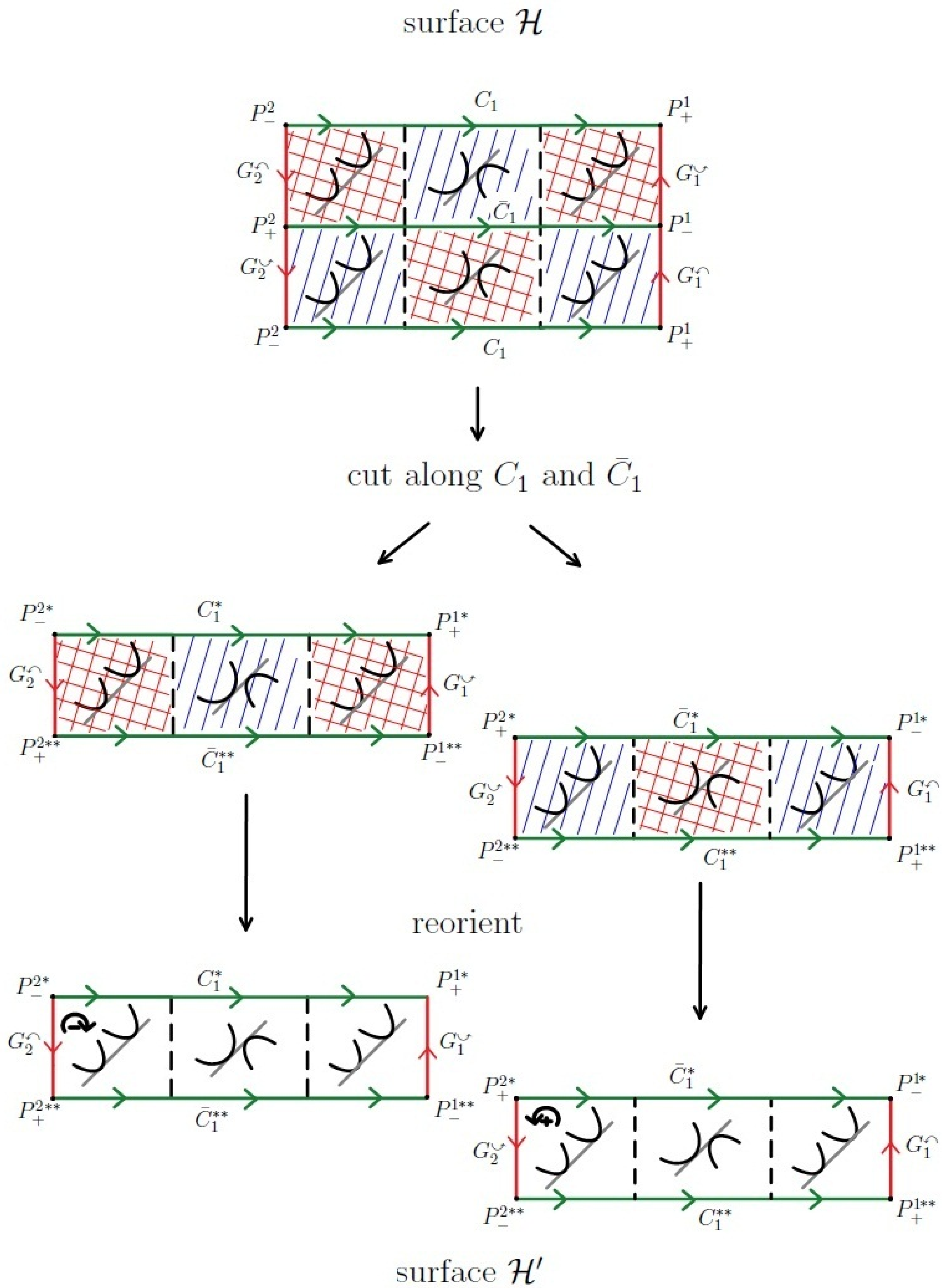


Figure 7.4: An example of the construction of a component of the surface $\mathcal{H}'$ corresponding to a component of $\mathcal{B}$ homeomorphic to an interval.

the construction of $\mathcal{H}'$, some additional identifications for components of the cut surface $\mathcal{H}$, that arise from component of $\mathcal{B}$ homeomorphic to $\mathbb{S}^1$, are made. Namely, the set ${}_{\mathcal{H}}G_j^{\frown}$ is identified with ${}_{\mathcal{H}}G_{j*}^{\frown}$ and ${}_{\mathcal{H}}G_j^{\smile}$ with ${}_{\mathcal{H}}G_{j*}^{\smile}$ for $j = M+1, \ldots, M+N$ (according to their orientation) and then these sets are no longer distinguished in the new surface $\mathcal{H}'$. Recall the two cases for a component of $\mathcal{H}$ with $j = M+1, \ldots, M+N$, depicted in Figure 6.18, that is, when $I_j \times \mathbb{S}^1$ is homeomorphic to a torus or to a Klein bottle (if the identification of the boundaries were performed in $\mathcal{H}$ without previous reorientation). Figure 7.5 shows that in both cases the identification leaves the new components of $\mathcal{H}'$ oriented. In the first case (on the left of Figure 7.5), two closed annuli arise in $\mathcal{H}'$ from $I_j \times \mathbb{S}^1$, due to the identification. In the second case (on the right of Figure 7.5), a single component of $\mathcal{H}'$ arises, that is homeomorphic to an annulus. In both of the two cases, the boundaries are given by the four edges ${}_{\mathcal{H}'}C_j^*$, ${}_{\mathcal{H}'}C_j^{**}$, ${}_{\mathcal{H}'}\bar{C}_j^*$, ${}_{\mathcal{H}'}\bar{C}_j^{**}$ for $j = M+1, \ldots, M+N$.

In general, the boundary of $\mathcal{H}'$ consists of

$$\begin{gathered} {}_{\mathcal{H}'}C_j^*, {}_{\mathcal{H}'}C_j^{**}, {}_{\mathcal{H}'}\bar{C}_j^*, {}_{\mathcal{H}'}\bar{C}_j^{**} \text{ for } j = 1, \ldots, M+N \text{ and} \\ {}_{\mathcal{H}'}G_i^{\frown}, {}_{\mathcal{H}'}G_i^{\smile} \text{ for } i = 1, \ldots, 2M \text{ if } M > 0. \end{gathered} \tag{7.6}$$

Now, each connected component of $\mathcal{H}'$ is embedded in $\mathcal{H}$. Let $i : \mathcal{H}' \to \mathcal{H}$ be the componentwise embedding and denote also by h the pullback of the bitangency map to $\mathcal{H}'$. The regular points of h in $\mathcal{H}'$ contain the regular points of h in $\mathcal{H}$ since the cuts where made along (a subset of) the critical points of h in $\mathcal{H}$. The regular points are all points $(a, \theta) \in \mathrm{int}(\mathcal{H}')$, provided that $h(a, \theta)$ does not lie in a bitangent osculating plane of γ (or in other words a is not an inflection point of the parametrization of the II curve). Also since $h(\mathcal{H}) = h(\mathcal{H}')$ and due to the gluing, the sets ${}_{\mathcal{H}'}G_j^{\frown}$, ${}_{\mathcal{H}'}G^{\smile}{}_j$ for $j = M+1, \ldots M+N$ no longer lie in the boundary of $\mathcal{H}'$, the set of critical values of the new map $h : \mathcal{H}' \to \mathbb{S}^2$ is $\mathcal{R}_{II} \cup \mathcal{P}_{\mathrm{osc}}^{\tau=0} \cup \mathcal{P}_{\mathrm{osc}}^{\mathrm{bit}}$ (compare with Corollary 6.19).

Due to the above construction the new surface $\mathcal{H}'$ has the following useful properties.

Lemma 7.3. *Let $v \in \mathbb{S}^2$ be a regular value of $h : \mathcal{H}' \to \mathbb{S}^2$, i.e., $v \notin \mathcal{R}_{II} \cup \mathcal{P}_{\mathrm{osc}}^{\mathrm{bit}} \cup \mathcal{P}_{\mathrm{osc}}^{\tau=0}$, and additionally let $v \notin \mathcal{R}_I$.*

(i) Then

$$\# h^{-1}(v) = \mathrm{ext}(\gamma_v) + \mathrm{int}(\gamma_v).$$

(ii) Moreover, let $\varphi = (s, t)$ be the map from Definition 6.18. For all regular

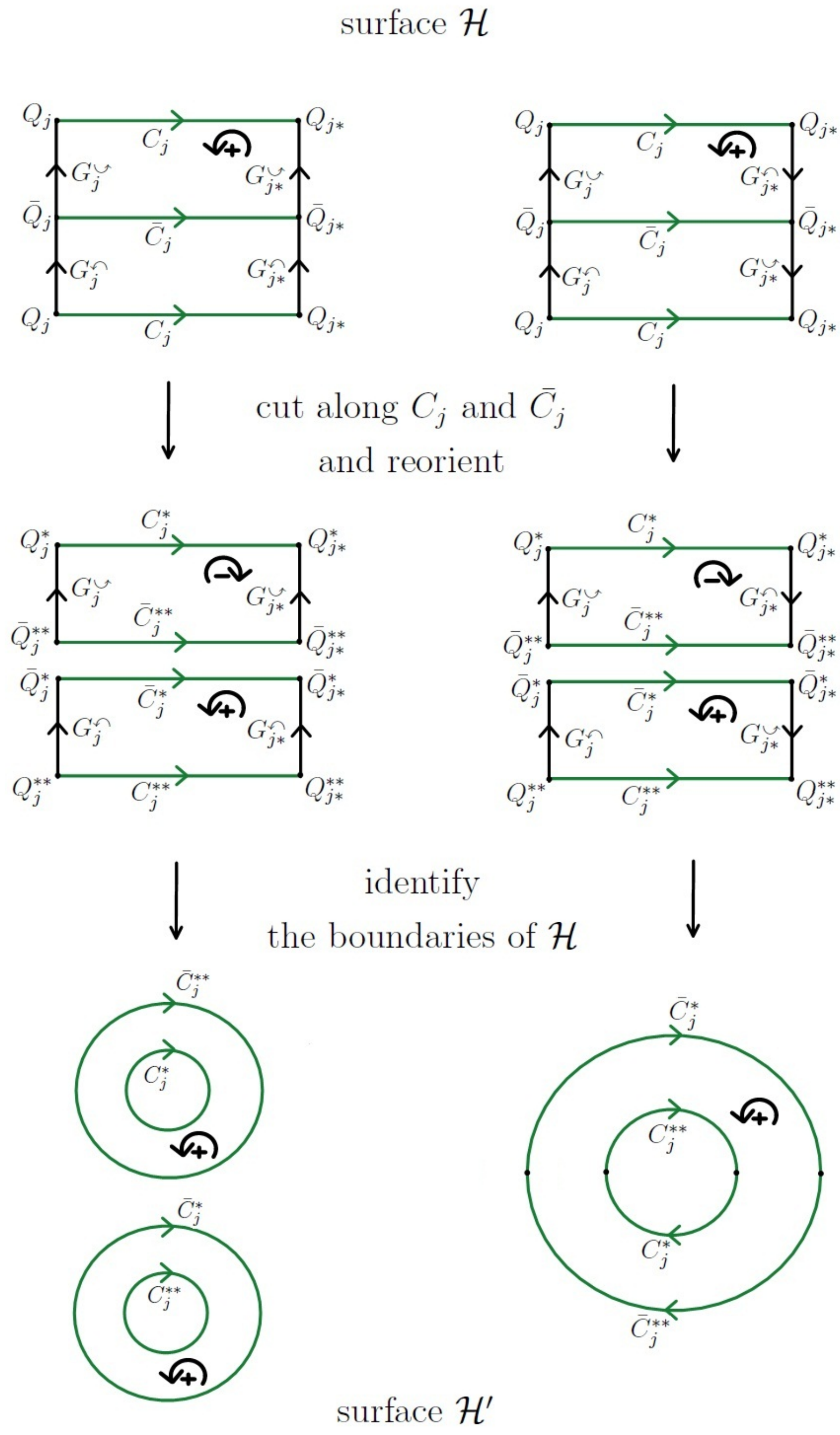


Figure 7.5: An illustration of the construction of the two types of components of the surface $\mathcal{H}'$ corresponding to a component of $\mathcal{B}$ homeomorphic to $\mathbb{S}^1$.

points (a,θ) of h

$$\begin{aligned}
\mathrm{ind}_{h,\mathcal{H}'}(a,\theta)=+1 &\iff \textit{the plane } \omega(\varphi(a)) \textit{ is internal}\\
&\iff \textit{the bitangent pair } \{s(a),t(a)\} \textit{ of } \gamma_v \textit{ is internal}\\
\mathrm{ind}_{h,\mathcal{H}'}(a,\theta)=-1 &\iff \textit{the plane } \omega(\varphi(a)) \textit{ is external}\\
&\iff \textit{the bitangent pair } \{s(a),t(a)\} \textit{ of } \gamma_v \textit{ is external.}
\end{aligned}$$

(iii) The orientation of the surface $\mathcal{H}'$ close to its boundaries corresponds to Figure 7.8.

Proof. The part (i) follows immediately from Theorem 6.20. For (ii) and (iii) compare (6.20) with the way the orientation was chosen on $\mathcal{H}'$. □

Finally, the equalities (6.23) and (6.25) need to be adjusted to the new notation. For $i=1,\dots,2M$, each of ${}_{\mathcal{I}'}G_i^{\frown}$, ${}_{\mathcal{I}'}G_i^{\smile}$, ${}_{\mathcal{H}'}G_i^{\frown}$, ${}_{\mathcal{H}'}G_i^{\smile}$ is just a direct copy of ${}_{\mathcal{I}}G_i^{\frown}$, ${}_{\mathcal{I}}G_i^{\smile}$, ${}_{\mathcal{H}}G_i^{\frown}$, ${}_{\mathcal{H}}G_i^{\smile}$, respectively. By the choice of the definition in (6.26) clearly

$$\iota({}_{\mathcal{I}'}G_i^{\frown})=h({}_{\mathcal{H}'}G_i^{\frown}),\quad \iota({}_{\mathcal{I}'}G_i^{\smile})=h({}_{\mathcal{H}'}G_i^{\smile})\quad \text{for } i=1,\dots,2M \quad \text{if } M\neq 0 \tag{7.7}$$

holds with the obvious pointwise correspondence

$$\begin{aligned}
&\left\{\begin{array}{l} {}_{\mathcal{H}'}G_i^{\frown}=\{s_i\}\times[0,\ \pi]\\ {}_{\mathcal{H}'}G_i^{\smile}=\{s_i\}\times[\pi,2\pi]\end{array}\right\} \ \text{ iff }\ h(s_i,\theta)=\iota(s_i,\theta)\ \text{ for all }\ \theta\in\mathbb{S}^1\ \text{ and}\\
&\left\{\begin{array}{l} {}_{\mathcal{H}'}G_i^{\frown}=\{s_i\}\times[\pi,2\pi]\\ {}_{\mathcal{H}'}G_i^{\smile}=\{s_i\}\times[0,\ \pi]\end{array}\right\} \ \text{ iff }\ h(s_i,\theta)=\iota(s_i,2\pi-\theta)\ \text{ for all }\ \theta\in\mathbb{S}^1
\end{aligned} \tag{7.8}$$

for $i=1,\dots,2M$. Furthermore, since each of ${}_{\mathcal{C}'}C_j^{*}$ and ${}_{\mathcal{C}'}C_j^{**}$ is just a copy of ${}_{\mathcal{C}}C_j$ and each of ${}_{\mathcal{H}'}C_j^{*}$ and ${}_{\mathcal{H}'}C_j^{**}$ is a copy of ${}_{\mathcal{H}}C_j$ for $j=1,\dots,M+N$ the following is true

$$\begin{array}{l} f({}_{\mathcal{C}'}C_j^{*})=f({}_{\mathcal{C}'}C_j^{**})=h({}_{\mathcal{H}'}C_j^{*})=h({}_{\mathcal{H}'}C_j^{**})\\ f({}_{\mathcal{C}'}\bar{C}_j^{*})=f({}_{\mathcal{C}'}\bar{C}_j^{**})=h({}_{\mathcal{H}'}\bar{C}_j^{*})=h({}_{\mathcal{H}'}\bar{C}_j^{**})\end{array} \quad \text{for } j=1,\dots,M+N. \tag{7.9}$$

In particular, the above equalities hold pointwise analogously to (6.21) and (6.22). Namely, if J and ϕ are as in Definition 6.18, then for any $a\in J$ with $\phi(a)=(s,t)$

$$\begin{aligned}
h(a,0)=f(s,t) &\quad \text{for } (a,0)\in {}_{\mathcal{H}'}C_j^{*}\,\dot\cup\,{}_{\mathcal{H}'}C_j^{**} \text{ and } (s,t)\in {}_{\mathcal{C}'}C_j^{*}\,\dot\cup\,{}_{\mathcal{C}'}C_j^{**} \text{ and}\\
h(a,\pi)=f(t,s) &\quad \text{for } (a,\pi)\in {}_{\mathcal{H}'}\bar{C}_j^{*}\,\dot\cup\,{}_{\mathcal{H}'}\bar{C}_j^{**} \text{ and } (t,s)\in {}_{\mathcal{C}'}\bar{C}_j^{*}\,\dot\cup\,{}_{\mathcal{C}'}\bar{C}_j^{**}
\end{aligned} \tag{7.10}$$

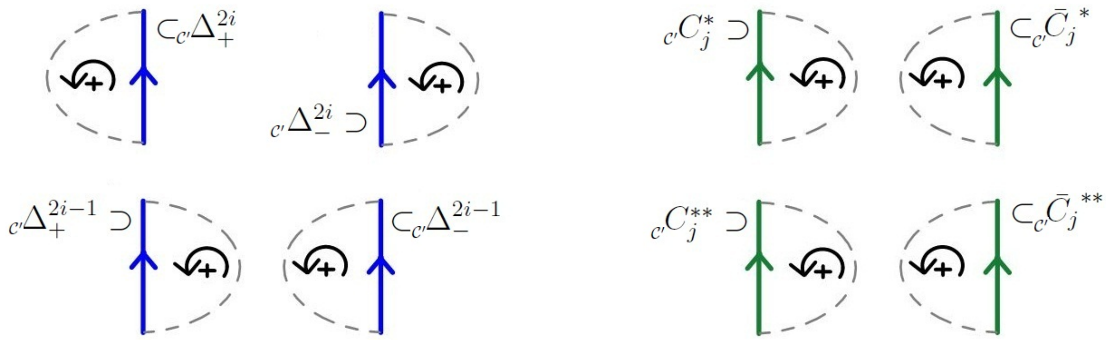


Figure 7.6: Orientation of the surface $\mathcal{C}'$ locally around its boundary.

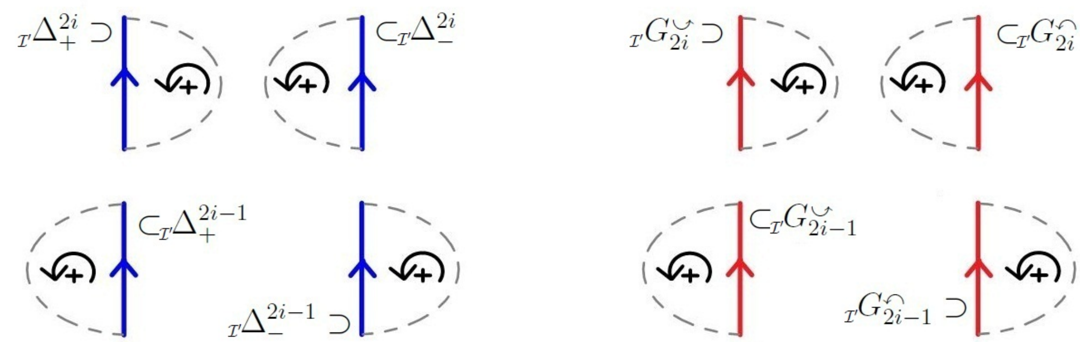


Figure 7.7: Orientation of the surface $\mathcal{I}'$ locally around its boundary.

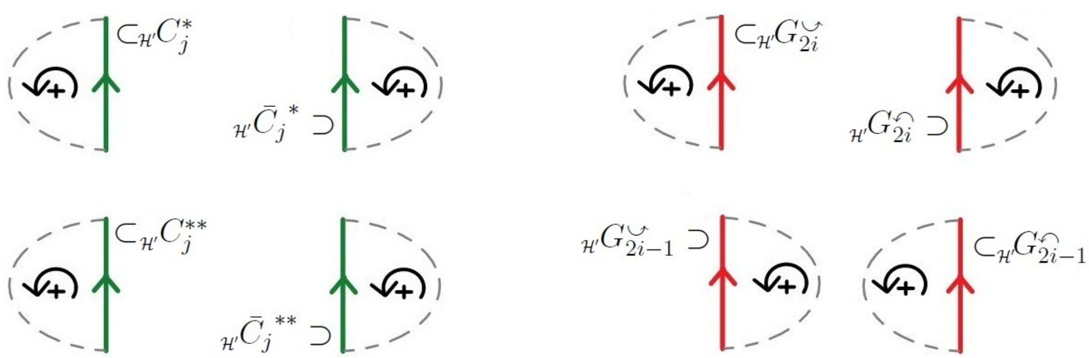


Figure 7.8: Orientation of the surface $\mathcal{H}'$ locally around its boundary.

for $j = 1, \ldots, M + N$.

Gluing

Let us summarize some properties of the surfaces just constructed. The surface $\mathcal{C}'$ consists of finitely many components homeomorphic to discs or discs with holes. The surface $\mathcal{I}'$ consists of a number of rectangles joined at vertices if the number M of torsion-vanishing points of γ is non-zero, or of two annuli, otherwise. The surface $\mathcal{H}'$ consists of a finite number of components homeomorphic to a disc or an annulus. The boundaries of all three surfaces $\mathcal{C}'$, $\mathcal{I}'$ and $\mathcal{H}'$ are

$$\begin{array}{rl} {}_{\mathcal{C}'}C_j^*, {}_{\mathcal{C}'}C_j^{**}, {}_{\mathcal{C}'}\bar{C}_j^*, {}_{\mathcal{C}'}\bar{C}_j^{**} & \\ {}_{\mathcal{H}'}C_j^*, {}_{\mathcal{H}'}C_j^{**}, {}_{\mathcal{H}'}\bar{C}_j^*, {}_{\mathcal{H}'}\bar{C}_j^{**} & \text{for } j = 1, \ldots, M+N \text{ and} \end{array}$$

$$\begin{array}{rl} {}_{\mathcal{C}'}\Delta_\pm^i & \\ {}_{\mathcal{I}'}\Delta_\pm^i, {}_{\mathcal{I}'}G_i^{\frown}, {}_{\mathcal{I}'}G_i^{\smile} & \text{for } i = 1, \ldots, 2M \text{ if } M > 0 \text{ or} \\ {}_{\mathcal{H}'}G_i^{\frown}, {}_{\mathcal{H}'}G_i^{\smile} & \end{array}$$

$$\begin{array}{rl} {}_{\mathcal{C}'}\Delta_\pm^0 \text{ or } {}_{\mathcal{C}'}\Delta_\pm^{-1} & \\ {}_{\mathcal{I}'}\Delta_\pm^0 \text{ or } {}_{\mathcal{I}'}\Delta_\pm^{-1} & \text{if } M = 0 \end{array}$$

(compare with (7.1), (7.3) and (7.6)). All labels of the above list appear in pairs, differing only in the prefix that states to which surface the edge belongs. It turns out that the surfaces can be glued together along these edges, i.e., by identifying the labels that appear in pairs.

Theorem 7.4. *Let $\gamma \in \mathcal{G}$ be a curve in general position. Let $2M$ be the number of torsion vanishing points of γ and $2N$ the number of components of $\mathcal{B}(\gamma)$ homeomorphic to $\mathbb{S}^1$. Identify the boundaries of surfaces $\mathcal{C}'(\gamma)$, $\mathcal{I}'(\gamma)$ and $\mathcal{H}'(\gamma)$ in the following way:*

- *glue together the surfaces $\mathcal{C}'$ and $\mathcal{H}'$ along the edges*

$$\begin{array}{ll} {}_{\mathcal{C}'}C_j^* \leftrightarrow {}_{\mathcal{H}'}C_j^*, \quad {}_{\mathcal{C}'}C_j^{**} \leftrightarrow {}_{\mathcal{H}'}C_j^{**} & \\ {}_{\mathcal{C}'}\bar{C}_j^* \leftrightarrow {}_{\mathcal{H}'}\bar{C}_j^*, \quad {}_{\mathcal{C}'}\bar{C}_j^{**} \leftrightarrow {}_{\mathcal{H}'}\bar{C}_j^{**} & \textit{for } j = 1, \ldots, M+N; \end{array}$$

  via the pointwise identifications

$$\begin{aligned} {}_{\mathcal{H}'}C_j^\circ \ni (a, 0) &= (s, t) \in {}_{\mathcal{C}'}C_j^\circ \textit{ and} \\ {}_{\mathcal{H}'}\bar{C}_j^\circ \ni (a, \pi) &= (t, s) \in {}_{\mathcal{C}'}\bar{C}_j^\circ \end{aligned}$$

where $\circ$ *is either* $*$ *or* $**$, $j = 1, \ldots, M+N$, $a \in J$, *and* $\phi(a) = (s,t)$ *with* J *and* ϕ *from Definition 6.18;*

- *glue together the surfaces* $\mathcal{C}'$ *and* $\mathcal{I}'$ *along the edges*

$$\begin{array}{lll} {}_{\mathcal{C}'}\Delta^0_\pm \leftrightarrow {}_{\mathcal{I}'}\Delta^0_\pm, \quad {}_{\mathcal{C}'}\Delta^{-1}_\pm \leftrightarrow {}_{\mathcal{I}'}\Delta^{-1}_\pm & & \textit{if } M = 0 \textit{ or} \\ {}_{\mathcal{C}'}\Delta^i_\pm \leftrightarrow {}_{\mathcal{I}'}\Delta^i_\pm & & \textit{for } i = 1, \ldots, 2M \textit{ if } M \neq 0; \end{array}$$

via the pointwise identifications

$$\begin{aligned} {}_{\mathcal{C}'}\Delta^i_+ \ni (s,s) &= (s,0) \in {}_{\mathcal{I}'}\Delta^i_+ \textit{ and} \\ {}_{\mathcal{C}'}\Delta^i_- \ni (s,s) &= (s,\pi) \in {}_{\mathcal{I}'}\Delta^i_- \end{aligned}$$

for all $s \in \mathbb{S}^1$ *and* $i \in \{-1, 0, \ldots, 2M\}$.

- *glue together the surfaces* $\mathcal{H}'$ *and* $\mathcal{I}'$ *along the edges*

$${}_{\mathcal{I}'}G_i^{\frown} \leftrightarrow {}_{\mathcal{H}'}G_i^{\frown}, \quad {}_{\mathcal{I}'}G_i^{\smile} \leftrightarrow {}_{\mathcal{H}'}G_i^{\smile} \quad \textit{for } i = 1, \ldots, 2M \textit{ if } M > 0.$$

via the pointwise identifications

$$\begin{aligned} {}_{\mathcal{H}'}G_i^{\frown} \,\dot\cup\, {}_{\mathcal{H}'}G_i^{\smile} \ni (s_i,\theta) &= (s_i,\theta) \in {}_{\mathcal{I}'}G_i^{\frown} \,\dot\cup\, {}_{\mathcal{I}'}G_i^{\smile} \textit{ if } \left(s_i, \frac{\pi}{2}\right) \in {}_{\mathcal{H}'}G_i^{\frown} \\ {}_{\mathcal{H}'}G_i^{\frown} \,\dot\cup\, {}_{\mathcal{H}'}G_i^{\smile} \ni (s_i,\theta) &= (s_i, 2\pi - \theta) \in {}_{\mathcal{I}'}G_i^{\frown} \,\dot\cup\, {}_{\mathcal{I}'}G_i^{\smile} \textit{ otherwise} \end{aligned}$$

for $\theta \in \mathbb{S}^1$ *and* $i = 1, \ldots, 2M$.

The surface obtained in this way is an orientable (in general disconnected) closed 2-manifold.

Proof. Clearly, due to the gluing a surface without boundary arises since all the boundaries of the three surfaces were used in the identification. The orientability follows from the fact that also the orientation of the surfaces agree along the identified edges. See Figures 7.6, 7.7 and 7.8 and compare Lemma 7.1(iii), 7.2(iii) and 7.3(iii). □

In general, it is possible to obtain different manifolds through the gluing. Recall that the renaming of the edges ${}_{\mathcal{C}}C_j$ and ${}_{\mathcal{C}}\bar{C}_j$ as well as ${}_{\mathcal{H}}C_j$ and ${}_{\mathcal{H}}\bar{C}_j$, due to the cutting process, by adding a superscript $*$ or $**$, was done arbitrarily. All other identifications of the edges of the three surfaces are unique. There are two different combinations of gluing along these four edges for each $j = 1, \ldots, M+N$. Hence,

2^{M+N} different manifolds are possible. However, each of the manifolds has the same Euler characteristic number. Furthermore, the manifold obtained by the gluing is not necessarily connected. It can be easily checked that for the curve shown in the bottom of Figure 5.9 (i.e., whose bitangency manifold consists of one pair of symmetric components homeomorphic to an interval), a manifold obtained from Lemma 7.4 consists of two spheres.

Connection with the Fabricius-Bjerre formula

The pointwise identifications of Theorem 7.4 are chosen in such a way that each pair of points (to be identified) have the same image on the sphere under one of the three maps: the crossing map f, the inflection map ι or the bitangency map h (compare with the properties (7.5), (7.10) and (7.7)). So there is a well-defined continuous map from any glued surface of Theorem 7.4 to the sphere, which is smooth almost everywhere.

Definition 7.5. Let $\mathcal{M} = \mathcal{M}(\gamma)$ be a manifold from Theorem 7.4. Define a map $\mathcal{F} = \mathcal{F}(\gamma) : \mathcal{M}(\gamma) \to \mathbb{S}^2$ by

$$\mathcal{F}(x) = \begin{cases} f(x) & \text{if } x \in \mathcal{C}' \\ \iota(x) & \text{if } x \in \mathcal{I}' \\ h(x) & \text{if } x \in \mathcal{H}' \end{cases} .$$

An illustration to the identification of the boundaries of the surfaces' images is shown in Figure 7.9. In general,

- the images $f(\mathcal{C}')$ and $h(\mathcal{H}')$ are glued together along the Reidemeister II curve $\mathcal{R}_{II}$ (the green curves in Figure 7.9),

- the images $f(\mathcal{C}')$ and $\iota(\mathcal{I}')$ are glued together along the Reidemeister I curve $\mathcal{R}_I$ (the blue curves in Figure 7.9) and

- the images $h(\mathcal{H}')$ and $\iota(\mathcal{I}')$ are glued together along the great circles corresponding to the osculating planes at torsion vanishing points (the red curves in Figure 7.9).

Given this continuous (almost everywhere smooth) map from the closed oriented surface to $\mathbb{S}^2$, it is natural to ask, what the degree of this map is. To answer this question, first, observe that the set of critical values of the map $\mathcal{F}$ is the sum of the

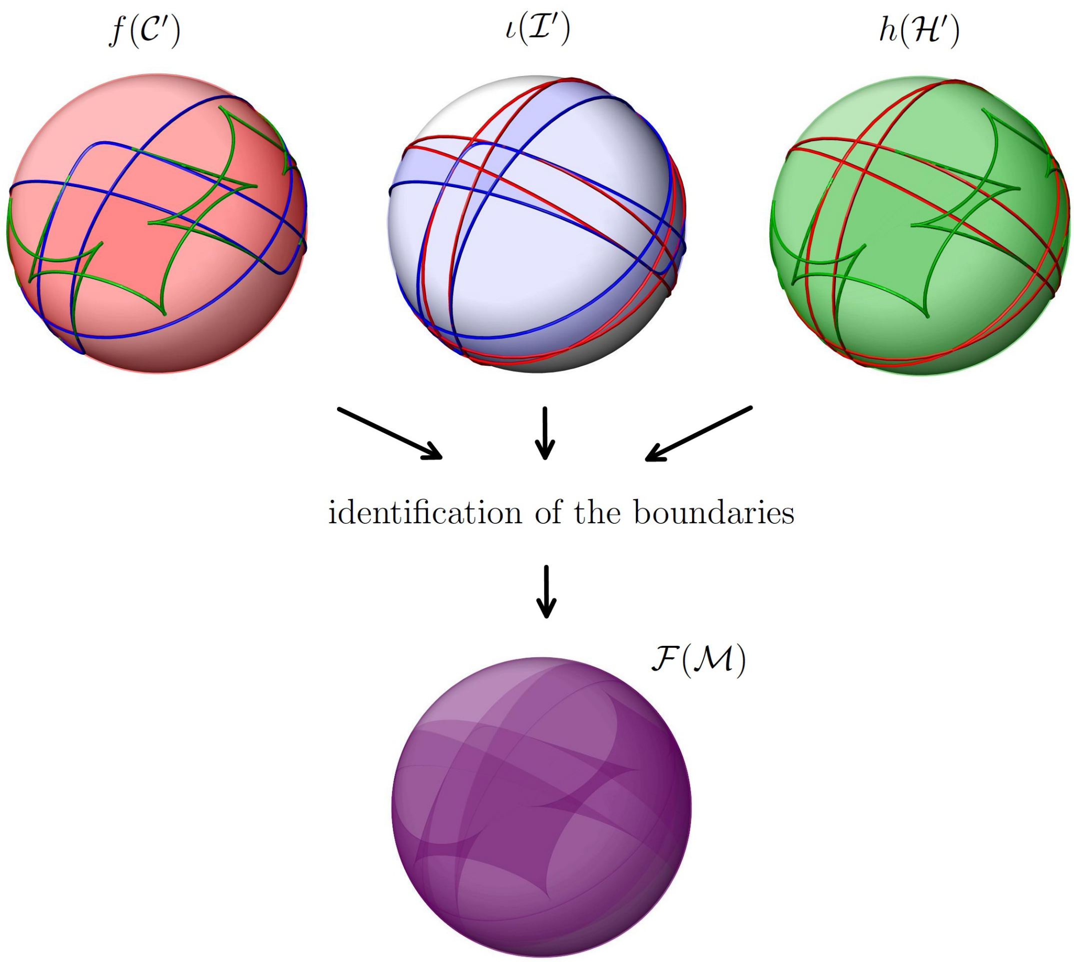


Figure 7.9: Image of a closed surface obtained via gluing of the images of the surfaces $\mathcal{C}'$, $\mathcal{I}'$ and $\mathcal{H}'$ along their boundaries.

critical values of the corresponding maps $f : \mathcal{C}' \to \mathbb{S}^2$, $\iota : \mathcal{I}' \to \mathbb{S}^2$ and $h : \mathcal{H}' \to \mathbb{S}^2$ (compare Lemma 7.1, 7.2 and 7.3).

Lemma 7.6. *For each generic space curve γ the set of critical values of the map $\mathcal{F} : \mathcal{M} \to \mathbb{S}^2$ consists of the union of*

(i) the Reidemeister I curve $\mathcal{R}_I$,

(ii) the Reidemeister II curve $\mathcal{R}_{II}$,

(iii) great circles spanning osculating planes at torsion vanishing points $\mathcal{P}_{\text{osc}}^{\tau=0}$ and

(iv) great circles spanning bitangent osculating planes $\mathcal{P}_{\text{osc}}^{\text{bit}}$.

This set of critical values of $\mathcal{F}$ has measure zero. In particular the set of regular values of $\mathcal{F}$ is non-empty.

Since the curve γ to start with was in generic position, then naturally some properties of the curve γ_v arise for a regular value v of $\mathcal{F}$. Surprisingly or not, this properties are already encountered in Part I of this thesis.

Lemma 7.7. *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a generic space curve. Then $v \in \mathbb{S}^2$ is a regular value of the map $\mathcal{F} : \mathcal{M} \to \mathbb{S}^2$ if and only if the plane curve γ_v obtained from γ through the orthogonal projection of γ onto $v^\perp$ is FB-generic (i.e., generic in the sense of Definition 1.2).*

Proof. For each element of the list of Lemma 7.6, there are the following equivalences:

(i) $v \notin \mathcal{R}_I$ if and only if γ_v is an immersion;

(ii) $v \notin \mathcal{R}_{II}$ if and only if γ_v has no non-transverse double points, i.e., all double points of γ_v are regular;

(iii) by Lemma 6.10

$$\begin{aligned} v \notin \mathcal{P}_{\text{osc}}^{\tau=0} \quad &\iff \quad \kappa_{\gamma_v}(s) = 0 \implies \kappa'_{\gamma_v}(s) \neq 0 \;\text{ for all } s \in \mathbb{S}^1 \\ &\iff \quad \text{all inflection points of } \gamma_v \text{ are regular;} \end{aligned}$$

(iv) $v \notin \mathcal{P}_{\text{osc}}^{\text{bit}}$ if and only if all bitangent pairs of γ_v are regular.

□

It will be convenient to have a name for v of the lemma above, also in the case when the space curve is not generic.

Definition 7.8. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a smooth immersion. Call $v \in \mathbb{S}^2$ a *regular projection direction of* γ if the projected plane curve γ_v is FB-generic. If such a regular projection direction exists, then γ is said to *regularly project* to a plane curve.

The set of regular values of the map $\mathcal{F}(\gamma)$ can be viewed as a (non-empty) 2-parameter family of FB-generic curves. On the other hand, later Theorem 11.1 shows that also for each FB-generic curve α there exists a generic curve γ and a regular value v of $\mathcal{F}(\gamma)$ such that γ_v is precisely equal to α.

Finally it is possible to compute the degree.

Theorem 7.9. *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a generic space curve, $\mathcal{M}(\gamma)$ a manifold obtained via gluing of three surfaces from Theorem 7.4 and $\mathcal{F}(\gamma) : \mathcal{M}(\gamma) \to \mathbb{S}^2$ the map from*

Definition 7.5. If v is a regular value of the map $\mathcal{F}(\gamma)$, then the set $\mathcal{F}^{-1}(\gamma)(v)$ is finite and the degree of $\mathcal{F}(\gamma)$ is

$$\deg(\mathcal{F}(\gamma)) = -\operatorname{ext}(\gamma_v) + \operatorname{int}(\gamma_v) + \operatorname{cr}(\gamma_v) + \frac{1}{2}\operatorname{infl}(\gamma_v), \tag{7.11}$$

where γ_v is the plane curve obtained from γ through projection onto $v^\perp$. In particular, this degree does not depend on the manifold $\mathcal{M}$ chosen.

Proof. By part (i) of Lemma 7.1, 7.2 and 7.3, the cardinality of the set $\mathcal{F}(\gamma)^{-1}(v)$ is precisely $\operatorname{ext}(\gamma_v) + \operatorname{int}(\gamma_v) + \operatorname{cr}(\gamma_v) + \frac{1}{2}\operatorname{infl}(\gamma_v)$. To see that this is a finite number, let first Γ be the graph in $\mathcal{M}(\gamma)$ that consists of the edges that were identified through the gluing (as in Theorem 7.4). Since v is a regular value of $\mathcal{F}(\gamma)$ and $\mathcal{F}(\gamma)$ is smooth away of Γ, there is an open neighborhood U around Γ such that $\mathcal{F}(\gamma)(U)$ doesn't meet v. Hence, $\mathcal{F}(\gamma)$ restricted to $\mathcal{M}(\gamma) \setminus U$ is a smooth map with a compact domain, and so the cardinality of the set of preimages of v under that map is finite.

Now, by Lemma 6.2

$$\deg(\mathcal{F}(\gamma)) = \sum_{x\in f^{-1}(v)} \operatorname{ind}_{f,\mathcal{C}'}(x) + \sum_{x\in\iota^{-1}(v)} \operatorname{ind}_{\iota,\mathcal{I}'}(x) + \sum_{x\in h^{-1}(v)} \operatorname{ind}_{h,\mathcal{H}'}(x)$$

and the result follows from combining the parts (i) and (ii) of Lemma 7.1, 7.2 and 7.3. □

The degree of the constructed map turned out to be in a strong relation with the Fabricius-Bjerre formula. In particular, the Fabricius-Bjerre formula implies that the degree of the map $\mathcal{F}(\gamma)$ is zero for a generic curve γ. The aim of the thesis however is to provide a new proof of Fabricius-Bjerre's Theorem. This will be achieved in Chapter 11 of Part III by combining mainly the two facts

- Any FB-curve α arise from a projection of some generic curve γ_α in a regular direction of $\mathcal{F}(\gamma_\alpha)$ (Theorem 11.1).

- The degree of $\mathcal{F}$ is identically zero on the space of generic curves $\mathcal{G}$ (Theorem 11.4).

Another consequence of Theorem 7.4 is a formula that arises by averaging of the Fabricius-Bjerre formula over the sphere of projection directions. Namely, given a generic space curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ and the set $\mathcal{V}$ of all regular directions of the map $\mathcal{F}(\gamma)$ the Fabricius-Bjerre formula can be integrated over $\mathcal{V}$ and one obtains

$$0 = \int_{\mathcal{V}} \left(\operatorname{ext}(\gamma_v) - \operatorname{int}(\gamma_v) - \operatorname{cr}(\gamma_v) - \frac{1}{2} \operatorname{infl}(\gamma_v) \right) d\mathbb{S}^2,$$

where $d\mathbb{S}^2$ is the standard volume form on $\mathbb{S}^2$. Call $\operatorname{bit}(\alpha) := \operatorname{ext}(\alpha) - \operatorname{int}(\alpha)$ the *bitangency number* of a FB-generic plane curve α. Then the integration gives the following result.

Corollary 7.10. *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a generic space curve. Let $\mathcal{V}$ be the open and dense subset of $\mathbb{S}^2$ of regular projection directions of γ. Then, the difference of the average bitangency number of γ and the average crossing number, both averaged over $\mathcal{V}$, equals the average of the absolute torsion of γ averaged over $\mathbb{S}^1$. More precisely,*

$$\frac{1}{4\pi} \int_{\mathcal{V}} \operatorname{bit}(\gamma_v) d\mathbb{S}^2 - \frac{1}{4\pi} \int_{\mathcal{V}} \operatorname{cr}(\gamma_v) d\mathbb{S}^2 = \frac{1}{2\pi} \int_{\mathbb{S}^1} |\tau(s)| ds, \tag{7.12}$$

where τ is the torsion of γ and $d\mathbb{S}^2$ is the standard volume form on $\mathbb{S}^2$.

Proof. It only needs to be proved that

$$\frac{1}{2} \int_{\mathcal{V}} \operatorname{infl}(\gamma_v) d\mathbb{S}^2 = 2 \int_{\mathbb{S}^1} |\tau(s)| ds.$$

But the expression on the left-hand side can be rewritten with a suitable change of variables as an integral over the surface $\mathcal{I}'$ (compare with the definition of $\mathcal{I}'$ given by (7.2)) in the following way

$$\begin{aligned} \frac{1}{2} \int_{\mathcal{V}} \operatorname{infl}(\gamma_v) d\mathbb{S}^2 &= \int_{\mathcal{V}} \sum_{i=1}^{\#\iota^{-1}(v) \cap \mathcal{I}'} 1 d\mathbb{S}^2 \\ &= \int_{\mathcal{I}'} \left\langle \frac{\partial}{\partial t} \iota \times \frac{\partial}{\partial \theta} \iota, \iota \right\rangle (t, \theta) dt d\theta \end{aligned}$$

Let $C_{\tau>0} := \{t \in \mathbb{S}^1 \mid \tau(t) > 0\}$ and $C_{\tau<0} := \{t \in \mathbb{S}^1 \mid \tau(t) < 0\}$. Then the above computation continues with

$$
\begin{aligned}
&\overset{(6.12)}{=} \int_{C_{\tau<0}} \int_0^{\pi} -\tau(t)\sin(\theta)dtd\theta + \int_{C_{\tau>0}} \int_{\pi}^{2\pi} -\tau(t)\sin(\theta)dtd\theta \\
&=2\int_{C_{\tau<0}} -\tau(t)dt + 2\int_{C_{\tau>0}} \tau(t)dt \\
&=2\int_{\mathbb{S}^1} |\tau(t)|dt.
\end{aligned}
$$

$\square$

Chapter 8

A Self-Linking Number Formula

This chapter is dedicated to the self-linking number - an invariant of smooth immersions of a circle into the Euclidean 3-space with nowhere vanishing curvature. This number is defined as a degree of a certain map between a torus and the sphere $\mathbb{S}^2$. With methods developed in this thesis, it is possible to prove a theorem due to Banchoff [5] (in case of polygonal curves) and Aicardi [2] (in the smooth case) that expresses the self-linking number in terms of some geometric features of a projected curve. Namely, the self-linking number can be written as a sum of the writhe (of a diagram) and half of the signed sum of inflection points of a projected space curve, where each inflection point contributes ± 1 according to the sign of the torsion of the space curve at that point.

The contribution of this new proof is to provide a direct connection between the formula of the invariant, expressed as a degree of a certain function, and the "planar formula" of a projected curve. In particular, a surface and a map will be constructed such that the result follows as a direct application of Lemma 6.2, i.e., as an expression "local vs global degree" - an approach that seems to be missing in the literature.

Linking number

The *linking number* of two disjoint smooth curves $\gamma_1, \gamma_2 : \mathbb{S}^1 \to \mathbb{R}^3$ is the degree of the map

$$l = l_{\gamma_1,\gamma_2} : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^2, \qquad (s,t) \mapsto \frac{\gamma_2(t) - \gamma_1(s)}{\|\gamma_2(t) - \gamma_1(s)\|}, \tag{8.1}$$

where the orientation of the domain is given by the order of the elements of the tuple (s,t), and the sphere $\mathbb{S}^2$ is oriented by the choice of the outer normal. Equivalently, the linking number is the value of the Gauss integral

$$\begin{aligned}\operatorname{LK}(\gamma_1,\gamma_2) :&= \frac{1}{4\pi}\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{\big[\gamma_1'(s),\gamma_2'(t),\gamma_1(s)-\gamma_2(t)\big]}{\|\gamma_1(s)-\gamma_2(t)\|^3} ds dt \\ &= \frac{1}{4\pi}\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \big[l, l_s, l_t\big](s,t) ds dt\end{aligned}$$

which is just the total area (counted according to the orientation) of the image of the surface defined by (8.1). In other words it is the integral of the local degree of l as of Lemma 6.2, i.e.,

$$\begin{aligned}\deg(l) &= \frac{1}{4\pi}\int_{\mathbb{S}^2\setminus\operatorname{crit}(l)} \deg(l) d\mathbb{S}^2 \\ &= \frac{1}{4\pi}\int_{v\in\mathbb{S}^2\setminus\operatorname{crit}(l)} \sum_{x\in l^{-1}(v)} \operatorname{ind}_{l,\mathbb{S}^1\times\mathbb{S}^1}(x) d\mathbb{S}^2 \\ &= \operatorname{LK}(\gamma_1,\gamma_2),\end{aligned}$$

where $\operatorname{crit}(l)$ denotes the set of critical values of l (which by Sard has measure zero) and $d\mathbb{S}^2$ is the standard volume form on the sphere.

It is a well-known fact (see, e.g., Kauffman [22]) that the linking number can be computed from an oriented link diagram $D_v(\gamma_1,\gamma_2)$ of the two curves γ_1 and γ_2 in some regular direction v of l (i.e., a projection of γ_1 and γ_2 in direction v with decorations allowing to distinguish over- and undercrossings). Each element $(s,t)\in l^{-1}(v)$ represents a crossing of the two curves in the link diagram in direction v such that the curve γ_2 lies over γ_1. Its index is given by

$$\operatorname{ind}_{l,\mathbb{S}^1\times\mathbb{S}^1}(s,t) = \operatorname{sign}[l,l_s,l_t](s,t) = \operatorname{sign}\big[\gamma_1'(s),\gamma_2'(t),\gamma_1(s)-\gamma_2(t)\big].$$

Hence, the linking number can be determined by adding the indices of the crossings whenever γ_2 crosses over γ_1, with the index given as on the left of Figure 8.1. (Note that here, the index of a crossing is opposite to that from Figure 6.6). The same is true for a link diagram in a regular direction v and $-v$. Hence, combining the two calculations results in

$$2\operatorname{LK}(\gamma_1,\gamma_2) = \sum_{\substack{x \text{ a crossing} \\ \text{of } \gamma_{1v} \text{ and } \gamma_{2v}}} \operatorname{index}(x), \tag{8.2}$$

where v is a regular value of l. The quantity on the right-hand side of the above formula is known as the *writhe of the link diagram* $D_v(\gamma_1, \gamma_2)$.

The common approach for this planar formula uses two antipodal directions. In order to express the quantity on the right-hand side of (8.2) as a local degree of a single function, observe that

$$\mathrm{LK}(\gamma_2, \gamma_1) = \mathrm{LK}(\gamma_1, \gamma_2)$$

and for any $x \in l^{-1}_{\gamma_1,\gamma_2}(v) \cup l^{-1}_{\gamma_2,\gamma_1}(v)$

$$\mathrm{ind}_{l_{\gamma_1,\gamma_2},\mathbb{S}^1\times\mathbb{S}^1}(x) = \mathrm{ind}_{l_{\gamma_2,\gamma_1},\mathbb{S}^1\times\mathbb{S}^1}(x).$$

Let $T_i := \mathbb{S}^1 \times \mathbb{S}^1$ for $i = 1, 2$ and $K := T_1 \dot{\cup} T_2$. Define $\omega : K \to \mathbb{S}^2$ by

$$\omega(x) = \begin{cases} l_{\gamma_1,\gamma_2}(x) & \text{if } x \in T_1 \\ l_{\gamma_2,\gamma_1}(x) & \text{if } x \in T_2 \end{cases}.$$

Then, by construction, the degree of ω is precisely $2\,\mathrm{LK}(\gamma_1, \gamma_2)$ and an application of Lemma 6.2 gives precisely (8.2) as an expression for a local degree of the map ω.

Self-linking number

The self-linking number is an invariant characterizing a single smooth immersion $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ with nowhere vanishing curvature. It is defined as a linking number of a curve and a copy slightly pushed in the direction of the principal normal, i.e.,

$$\mathrm{SLK}(\gamma) = \mathrm{LK}(\gamma, \gamma + \varepsilon N), \tag{8.3}$$

for any sufficiently small $\varepsilon > 0$.

It was Călugăreanu [11, 12] who first observed that in this special case the Gauss integral can be decomposed into two quantities: the *writhe* $\mathrm{WR}(\gamma)$ and the total torsion:

$$\mathrm{SLK}(\gamma) = \mathrm{WR}(\gamma) + \frac{1}{2\pi}\int_{\mathbb{S}^1} \tau(s)ds, \tag{8.4}$$

where the writhe is defined as

$$\mathrm{WR}(\gamma) = \frac{1}{4\pi}\int_{\mathbb{S}^1\times\mathbb{S}^1\setminus\Delta} \frac{\big[\gamma'(s), \gamma'(t), \gamma(s) - \gamma(t)\big]}{\|\gamma(s) - \gamma(t)\|^3} dsdt.$$

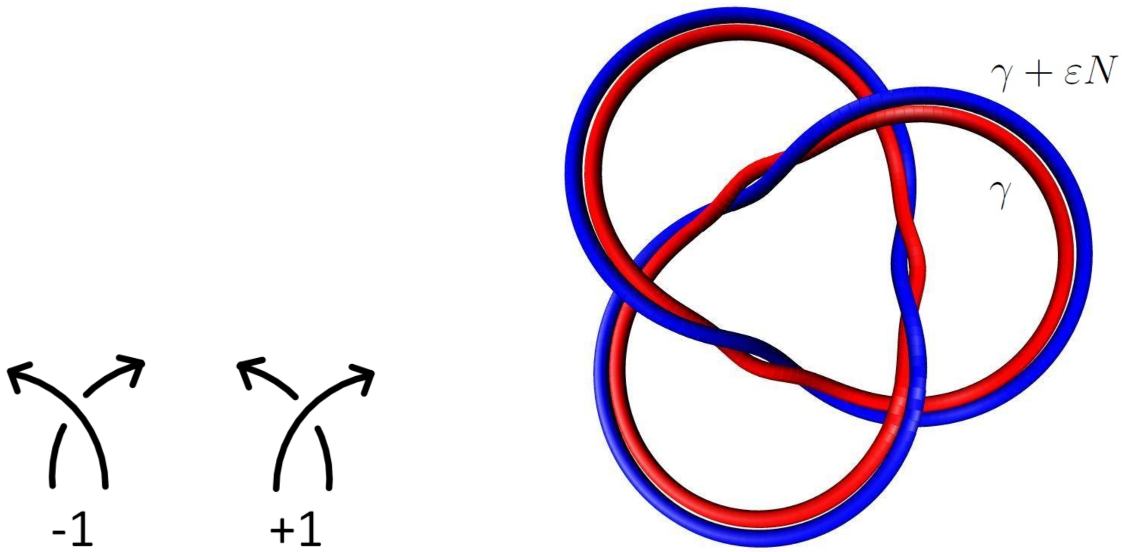


Figure 8.1: Left: index of a crossing; right: a trefoil knot and its copy pushed slightly in the principal normal direction.

Later Pohl [30] gave various new characterizations of the self-linking invariant, e.g., as an algebraic intersection number of a curve $\gamma + \varepsilon N$ (for sufficiently small $\varepsilon > 0$) with any disc bounded by γ and transversal to $\gamma + \varepsilon N$. (The disc should be oriented so that the induced orientation of its boundary agrees with the orientation of γ.)

The first person to observe connections between the total torsion of a closed space curve and inflection points of its projections was Banchoff [5], who provided a proof in the polygonal case. The dependence was later rediscovered by Aicardi [2], who worked in the smooth case. Below the smooth version of the theorem is presented.

Before proceeding with the theorem, note that since γ is not necessarily in general position in the sense of Definition 4.1, the set

$$\mathcal{B} := \{(s,t) \in \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta \mid [\gamma'(s), \gamma'(t), \gamma(s) - \gamma(t)] = 0\}$$

of singular points of f is not necessarily a manifold. Hence, the set of the singular values of f given by $\mathcal{R}_{II}(\gamma) := f(\mathcal{B})$ is in general not an image of a curve. Therefore, it will be referred to as the Reidermeister II *set* of γ instead of a Reidemeister II curve. Denote the set of great circles which span osculating planes of γ at torsion vanishing points, as before, by $\mathcal{P}^{\tau=0}_{\text{osc}}$.

Theorem 8.1 (Banchoff [5], Aicardi [2]). *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a smooth immersion with nowhere vanishing curvature. Let $v \in \mathbb{S}^2 \setminus \mathcal{R}_I \cup \mathcal{R}_{II} \cup \mathcal{P}^{\tau=0}_{\text{osc}}$. Then the self-linking* *of γ is*

$$2\,\mathrm{SLK}(\gamma) = 2 \sum_{\substack{x \text{ crossing} \\ \text{of } \gamma_v}} \mathrm{index}(x) + \sum_{\substack{u \text{ inflection} \\ \text{point of } \gamma_v}} \mathrm{sign}(\tau(u)), \tag{8.5}$$

where the index $\mathrm{index}(x)$ *of a crossing x of γ_v is given by the rule of Figure 8.1.*

Half of the first summand of the right-hand side of (8.5) is analogously known as the *writhe* of a knot diagram $D_v(\gamma)$ of γ in direction v.

Banchoff proves the polygonal version of the theorem using deformation arguments, showing that the expression on the right-hand side of (8.5) remains preserved under certain moves. Aicardi uses Pohl's result and computes the algebraic intersection number of a curve with a disc, that is in this case, represented as a half cylinder over this curve in the viewing direction v or $-v$. The proof presented here aims to show the equality (8.5) as a local degree of certain function between a closed manifold and the sphere $\mathbb{S}^2$. Below the list of ingredients and a recipe required for this proof is presented.

The ingredients

Take two copies of the crossing surfaces $\mathcal{C} := \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$. Orient one of them according to the order of the elements $(s,t) \in \mathcal{C}$ and denote it by $\mathcal{C}^{\circlearrowleft}$. Orient the other copy in the opposite way and denote by $\mathcal{C}^{\circlearrowright}$. Take the inflection surface $\mathcal{I} := \mathbb{S}^1 \times \mathbb{S}^1$ and cut along $\mathbb{S}^1 \times \{0\}$ and $\mathbb{S}^1 \times \{\pi\}$. This way one obtains two pieces $[0,\pi] \times \mathbb{S}^1$ and $[\pi, 2\pi] \times \mathbb{S}^1$. Orient the first of them oppositely to the order of the elements (t,θ) and denote it by $\mathcal{I}^{\circlearrowright}_{[0,\pi]}$ or simply by $\mathcal{I}^{\circlearrowright}$. Orient the second of the two pieces according to the original orientation and denote it by $\mathcal{I}^{\circlearrowleft}_{[\pi,2\pi]}$ or simply by $\mathcal{I}^{\circlearrowleft}$. The four surfaces, each of which is a closed annulus, are presented together with their orientations in Figure 8.2.

In order to glue the edges of the four cylinders together, some notation needs to be introduced. Recall the notation for the two diagonals of the crossing surface from Section 6.1, namely $\Delta_\pm$. There the choice of the sign as a subscript was motivated by the crossing map property, namely that $f(\Delta_+)$ was the tangent indicatrix and $f(\Delta_-)$ its antipodal copy. Here, the same notation will be rather confusing, so denote the diagonals by $\Delta_1 := \Delta_+$ and $\Delta_2 := \Delta_-$ as in Figure 8.2. As before, use a prefix (whenever necessary) to denote which surface the edge belongs to, e.g., ${}_{\mathcal{C}^{\circlearrowleft}}\Delta_1$ and ${}_{\mathcal{C}^{\circlearrowright}}\Delta_2$. The notation for the boundaries of the pieces of the inflection surface is

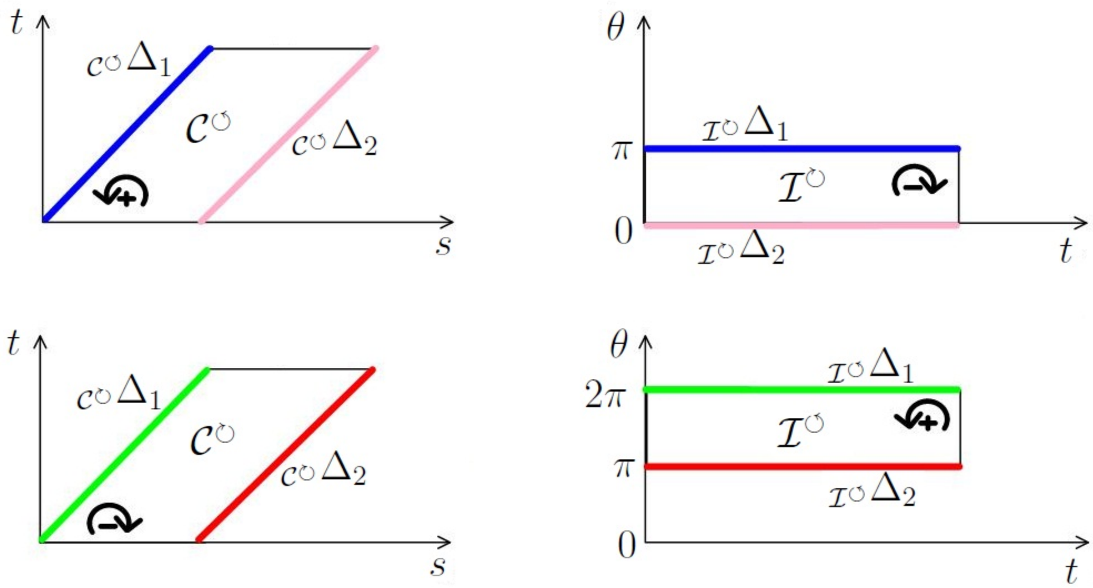


Figure 8.2: The four ingredients for the SLK-surface.

as follows:

$$
\begin{aligned}
{}_{\mathcal{I}^{\circlearrowright}}\Delta_1 &:= \mathbb{S}^1 \times \{\pi\} \subset \mathcal{I}^{\circlearrowright} \\
{}_{\mathcal{I}^{\circlearrowright}}\Delta_2 &:= \mathbb{S}^1 \times \{0\} \subset \mathcal{I}^{\circlearrowright} \\
{}_{\mathcal{I}^{\circlearrowleft}}\Delta_1 &:= \mathbb{S}^1 \times \{2\pi\} \subset \mathcal{I}^{\circlearrowleft} \\
{}_{\mathcal{I}^{\circlearrowleft}}\Delta_2 &:= \mathbb{S}^1 \times \{\pi\} \subset \mathcal{I}^{\circlearrowleft}.
\end{aligned}
$$

The recipe

Glue the surface $\mathcal{C}^{\circlearrowleft}$ with $\mathcal{I}^{\circlearrowright}$ and $\mathcal{C}^{\circlearrowright}$ with $\mathcal{I}^{\circlearrowleft}$ along their edges via the identifications

$$
\begin{aligned}
{}_{\mathcal{C}^{\circlearrowleft}}\Delta_i &\longleftrightarrow {}_{\mathcal{I}^{\circlearrowright}}\Delta_i \\
{}_{\mathcal{C}^{\circlearrowright}}\Delta_i &\longleftrightarrow {}_{\mathcal{I}^{\circlearrowleft}}\Delta_i
\end{aligned}
$$

for $i = 1, 2$. As a result two tori arise, say T_1 is the first one and T_2 the other one and set $M := T_1 \dot{\cup} T_2$. Now define the following map $\Omega : M \to \mathbb{S}^2$ by

$$
\Omega : M \to \mathbb{S}^2, \quad x \mapsto \begin{cases} f(x) & \text{if } x \in \mathcal{C}^{\circlearrowright} \\ -f(x) & \text{if } x \in \mathcal{C}^{\circlearrowleft} \\ \iota(x) & \text{if } x \in \mathcal{I}^{\circlearrowleft} \cup \mathcal{I}^{\circlearrowright}. \end{cases} \tag{8.6}
$$

It can be easily checked that this map is continuous since $f(s,s) = T(s) = \iota(s,0)$ for $s \in \mathbb{S}^1$ with $(s,s) \in \Delta_1$ etc.

Lemma 8.2. *The degree of the map $\Omega(\gamma)$ is equal to twice the self-linking number of γ, i.e.,*

$$\deg(\Omega(\gamma)) = 2\,\mathrm{SLK}(\gamma).$$

The proof of the lemma is postponed for a moment. With this lemma it is now easy to prove the theorem of Banchoff and Aicardi.

Proof of Theorem 8.1. Assuming Lemma 8.2, it suffices to show

$$\deg(\Omega(\gamma)) = 2 \sum_{\substack{x \text{ crossing} \\ \text{of } \gamma_v}} \mathrm{index}(x) + \sum_{\substack{u \text{ inflection} \\ \text{point of } \gamma_v}} \mathrm{sign}(\tau(u)).$$

This will be achieved by an application of Lemma 6.2 to the map Ω (with Γ being the tangent indicatrix and its antipodal). The set of critical points of Ω is precisely the union of $\mathcal{R}_I$, $\mathcal{R}_{II}$ and $\mathcal{P}_{\mathrm{osc}}^{\tau=0}$. The fact that for any regular value v of Ω the set $\Omega^{-1}(v)$ has finitely many elements, is proved by Lemma 8.3 below. Then, for a regular value $v \in \mathbb{S}^2$ of Ω, by Lemma 6.2,

$$\deg(\Omega(\gamma)) = \sum_{x \in \Omega^{-1}(v)} \mathrm{ind}_{\Omega,M}(x)$$

and with $f(s,t) = -f(t,s)$ further

$$= \sum_{(s,t)\in f^{-1}(v)} (\mathrm{ind}_{f,\mathcal{C}^{\circlearrowleft}}(s,t) + \mathrm{ind}_{-f,\mathcal{C}^{\circlearrowleft}}(t,s)) + \sum_{x \in \iota^{-1}(v)} \mathrm{ind}_{\iota,\mathcal{I}^{\circlearrowleft}\dot{\cup}\mathcal{I}^{\circlearrowleft}}(x).$$

Let us first consider the first summand of the above sum. By symmetry of the crossing map f, the index of (s,t) is equal to the index of (t,s) (independently on the orientation of the domain of f). Moreover, since the orientation of $\mathcal{C}^{\circlearrowright}$ is opposite to that of $\mathcal{C}^{\circlearrowleft}$ for each $x \in f^{-1}(v)$,

$$\mathrm{ind}_{f,\mathcal{C}^{\circlearrowleft}}(x) = \mathrm{ind}_{-f,\mathcal{C}^{\circlearrowleft}}(x)$$

and the index of the crossing x is precisely the opposite to the convention from

Section 6.1, and corresponds to that on the left of Figure 8.1. Hence,

$$\begin{aligned}\sum_{(s,t)\in f^{-1}(v)} \left(\operatorname{ind}_{f,\mathcal{C}^{\circlearrowleft}}(s,t) + \operatorname{ind}_{-f,\mathcal{C}^{\circlearrowleft}}(t,s)\right) &= 2 \sum_{x\in f^{-1}(v)} \operatorname{ind}_{f,\mathcal{C}^{\circlearrowleft}}(x) \\ &= 2 \sum_{\substack{x \text{ crossing} \\ \text{of } \gamma_v}} \operatorname{index}(x).\end{aligned}$$

As for the second summand, by the properties of the inflection surface, each $x \in \iota^{-1}(v)$ represents an inflection point in the orthogonal projection γ_v. The index of $x := (t, \theta)$ lying in $\mathcal{I}^{\circlearrowleft}$ is precisely the same as on the inflection surface $\mathcal{I}$, i.e.,

$$\operatorname{ind}_{\iota,\mathcal{I}^{\circlearrowleft}}(t,\theta) = \operatorname{ind}_{\iota,\mathcal{I}}(x) = -\operatorname{sign}\left(\tau(t)\sin\theta\right) \overset{\sin\theta<0}{=} \operatorname{sign}(\tau(t))$$

since the orientation of the two surfaces is the same (compare with (6.13)). The index of $x := (t, \theta)$ in $\mathcal{I}^{\circlearrowright}$ is opposite to that in $\mathcal{I}$, i.e.,

$$\operatorname{ind}_{\iota,\mathcal{I}^{\circlearrowright}}(t,\theta) = -\operatorname{ind}_{\iota,\mathcal{I}}(x) = \operatorname{sign}\left(\tau(t)\sin\theta\right) \overset{\sin\theta>0}{=} \operatorname{sign}(\tau(t))$$

since the orientations of the two surfaces are opposite. Hence,

$$\sum_{x\in\iota^{-1}(v)} \operatorname{ind}_{\iota,\mathcal{I}^{\circlearrowleft}\dot\cup\mathcal{I}^{\circlearrowright}}(x) = \sum_{\substack{t \text{ inflection} \\ \text{point of } \gamma_v}} \operatorname{sign}(\tau(t))$$

and (8.5) follows, what proves the theorem. $\square$

Proof of Lemma 8.2. From now on, equip the maps depending on the curve γ with a subscript, e.g., Ω_γ instead of Ω. Consider the restriction $\chi_\gamma := \Omega_\gamma|_{T_1}$ of the map Ω_γ to the torus T_1. The degree of this map is known to be the self-linking number of γ (see, e.g., [15]). An illustration of its construction is shown in Figure 8.3. Let τ_γ be the torsion of γ. Then, $\tau_{-\gamma} = -\tau_\gamma$ and $\operatorname{SLK}(\gamma) = -\operatorname{SLK}(-\gamma)$ immediately follows from (8.4). Hence,

$$2\operatorname{SLK}(\gamma) = \operatorname{SLK}(\gamma) - \operatorname{SLK}(-\gamma)$$

and in order to prove the lemma it suffices to show that

$$-\deg(\Omega_\gamma|_{T_2} : T_2 \to \mathbb{S}^2) = \operatorname{SLK}(-\gamma) = \deg(\chi_{-\gamma} : T_1 \to \mathbb{S}^2). \tag{8.7}$$

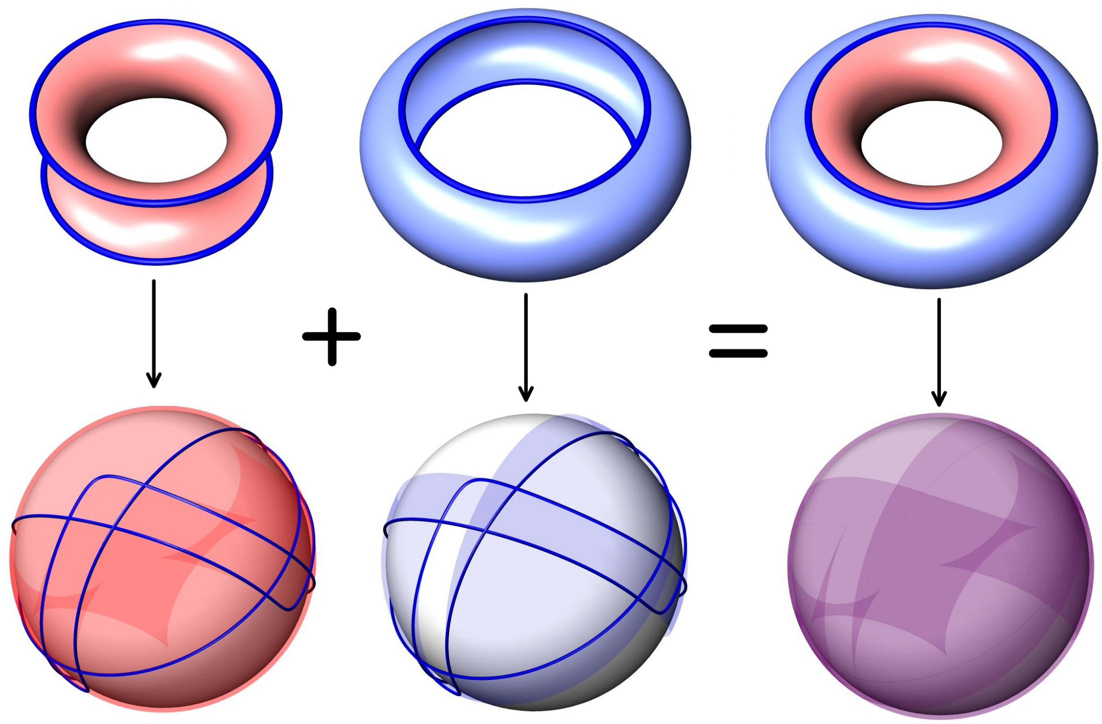


Figure 8.3: A crossing surface and map (left) + half of inflection surface and map (middle) = a torus with a map whose degree is the self-linking number (right).

The two functions to be compared are

$$\Omega_\gamma|_{T_2}(x) = \begin{cases} f_\gamma(x) & \text{if } x \in \mathcal{C}^{\circlearrowleft} \\ \iota_\gamma(x) & \text{if } x \in \mathcal{I}^{\circlearrowleft}_{[\pi,2\pi]} \end{cases} \quad \text{and} \quad \chi_{-\gamma}(x) = \begin{cases} -f_{-\gamma}(x) & \text{if } x \in \mathcal{C}^{\circlearrowleft} \\ \iota_{-\gamma}(x) & \text{if } x \in \mathcal{I}^{\circlearrowleft}_{[0,\pi]}. \end{cases}$$

Note that if $f_\gamma := f$ denote the crossing map of the curve γ, then $-f_{-\gamma} = f_\gamma$. Furthermore observe that

$$\iota_{-\gamma}(t,\theta) = \iota_\gamma(t,\theta+\pi) \quad \text{for all } (t,\theta) \in \mathbb{S}^1 \times \mathbb{S}^1$$

and so the images of the two maps $\Omega_\gamma|_{T_2}$ and $\chi_{-\gamma}$ are the same. Also the indices

$$\operatorname{ind}_{\iota_{-\gamma},\mathcal{I}}(t,\theta) = -\operatorname{sign}(\tau_{-\gamma}(t)\sin\theta) = -\operatorname{sign}(\tau_\gamma(t)\sin(\theta+\pi)) = \operatorname{ind}_{\iota_\gamma,\mathcal{I}}(t,\theta+\pi)$$

coincide. The negative of the degree is obtained by the reorientation of the domain in the opposite way and so (8.7) and the lemma is proved. □

Lemma 8.3. *For any regular value v of the map Ω defined in (8.6), i.e.,*

$v \notin \mathcal{R}_I \cup \mathcal{R}_{II} \cup \mathcal{P}_{\text{osc}}^{\tau=0}$, the number of preimages $\Omega^{-1}(v)$ is finite.

Proof. By Theorem 6.13, the set $\iota^{-1}(v)$ is finite. It remains to show that $f^{-1}(v)$ is finite as well. The set $f^{-1}(v)$ consists of double points (s,t) of the projected curve γ_v, each of which is regular (compare Definition 1.1). Hence, the double points are isolated in $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$. (Similar as in Lemma 1.6, double points of γ_v are zeros of the vector field $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^2$, $(s,t) \mapsto \gamma_v(s) - \gamma_v(t)$ whose Jacobian doesn't vanish at the zeros.) If γ_v had infinitely many double points, their accumulation points would have to lie on the diagonal Δ. This is ruled out by the fact that γ_v is an immersion. $\square$

A similar approach to the proof of Theorem 8.1 via comparison of a local with the global degree of a certain map, can be used to prove a more general version of the theorem for framed curves (compare with [15]). In fact the idea is nothing else but taking a closed surface with a map to the sphere with a certain degree and taking another copy of the surface and applying the negative of the map. Then, one only needs to be careful about the proper orientation of the two copies of the surfaces. (If the orientation of both copies is the same, then the sum of degrees will trivially be zero).

Part III

A new proof of Fabricius-Bjerre's Theorem

Chapter 9

The space $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$

This chapter contains descriptions of some structures on the space $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ of smooth closed curves and introduces tools helpful for further analysis of this space. Section 9.1 describes the Whitney $C^\infty-$topology via jet bundles. Moreover, it introduces the powerful machinery of Thom's transversality which will later be used in Section 10.1 and Section 11.1. Section 9.2 deals with Fréchet spaces and concludes with a version of the Regular Value Theorem that holds in the category of so-called tame Fréchet spaces. This Theorem 9.21 will play an essential role in Section 10.2 in proving that the space $\mathcal{NG}$ of non-generic curves forms a singular hypersurface in the Fréchet space $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$.

9.1 Transversality Theorems

This section summarizes results of Golubitsky and Guillemin [19]. The author also consulted Petters et al. [27] and Vassiliev [37].

9.1.1 Jet Bundles

Let X and Y be smooth manifolds, and let $C^\infty(X, Y)$ denote the space of all smooth maps between X and Y. Consider two maps $f, g \in C^\infty(X, Y)$ with equal values $f(x) = g(x) =: y$ at some $x \in X$. Two functions f and g are said to have *k-th order contact at x* (denoted $f \sim_k g$ at x) if for some (and therefore any) choice of local coordinates around x and y the Taylor polynomials up to (and including) order k of these maps coincide. The property to have k-th order contact at x defines an equivalence relation among maps of $C^\infty(X, Y)$ whose value at x is y and the set of equivalence classes is denoted by $J^k(X,Y)_{x,y}$. More precisely

$$J^k(X,Y)_{x,y} = \{f \in C^\infty(X,Y) \mid f(x) = y\} \,/\, \text{“} \sim_k \text{ at } x\text{”}.$$

Definition 9.1. 1. Let

$$J^k(X,Y) = \dot{\bigcup}_{(x,y)\in X\times Y} J^k(X,Y)_{x,y}$$

(disjoint union). An element of $\sigma \in J^k(X,Y)$ is called a *k-jet* of maps from X to Y or simply a *k-jet* and $J^k(X,Y)$ is the *space of k-jets from X to Y*.

2. For a k-jet $\sigma \in J^k(X,Y)$ there exist $p \in X$ and $q \in Y$ such that $\sigma \in J^k(X,Y)_{p,q}$. Call p the *source* and q the *target* of σ, respectively.

3. The *k-jet of a map* $f : X \to Y$ is the canonical map $j^k f : X \to J^k(X,Y)$ defined for every $x \in X$ by the equivalence class of f w.r.t. “$\sim_k$ at x” of mappings $g \in \mathcal{C}^\infty(X,Y)$ with $g(x) = f(x)$, i.e.,

$$j^k f(x) = \{g : X \to Y \mid g \sim_k f \text{ at } x\} \in J^k(X,Y)_{x,f(x)}.$$

The k-jet space $J^k(X,Y)$ can be given a structure of a smooth manifold and then the map $j^k f : X \to J^k(X,Y)$ is smooth. For full details see Golubitsky and Guillemin [19]. The space of k-jets has then a natural structure of a smooth bundle over $X \times Y$ (vector bundle if $Y = \mathbb{R}^m$) with fiber $J^k(X,Y)_{x,y}$. Below the charts in the case when $Y = \mathbb{R}^m$ (for some $m \in \mathbb{N}$) are described as in [19] adopting the original notation.

Let A_n^k be the vector space of polynomials in n variables of degree at most k with zero as the constant term. Let the coefficients of the polynomials be the coordinates for A_n^k. Then A_n^k is a smooth manifold since it is isomorphic to the Euclidean space of dimension $(n+k)!/(n!k!) - 1$. Then, the direct sum $B_{n,m}^k = \oplus_{i=1}^m A_n^k$ is again a smooth manifold.

Let $f : U \to \mathbb{R}$ be a smooth map, where U is an open subset of $\mathbb{R}^n$. Let $T_k f : U \to A_n^k$ at $x_0 \in U$ be defined as the polynomial in x of degree k given by the first k terms of the Taylor series of f at x_0 after the constant term.

Let V be any open subset of $\mathbb{R}^m$. Then there is a canonical bijective map $T_{U,V} : J^k(U,V) \to U \times V \times B_{n,m}^k$ given by

$$T_{U,V}(\sigma) = (x_0, y_0, T_k f_1(x_0), \ldots, T_k f_m(x_0)), \tag{9.1}$$

where x_0 is the source and y_0 the target of σ represented by the smooth map $f : U \to V$ and $f_i : U \to \mathbb{R}$ for $i = 1, \ldots, m$ are the coordinate functions associated to f. In particular the k-jet of a map $f : U \to V$, i.e., $j^k f : U \to J^k(U, V)$ can be written as

$$j^k f(x) = (x, f(x), T_k f_1(x), \ldots, T_k f_m(x)) \tag{9.2}$$

and this representation will be used in Chapter 10 and Chapter 11.

Let X be a smooth manifold of dimension n and $Y = \mathbb{R}^m$. Let U be the domain for a chart ϕ of X and V an open subset of $\mathbb{R}^m$. Then $J^k(X, Y)$ is a smooth manifold with the charts $\tau_{U,V}$ described below.

Set $U' = \phi(U)$. Define $(\phi^{-1})^* : J^k(U, V) \to J^k(U', V)$ in the following way. Let σ be in $J^k(U, V)_{p,q}$ and let $g : U \to V$ represent σ. Then

$$(\phi^{-1})^*(\sigma) = \text{ the equivalence class of } g \circ \phi^{-1} \text{ in } J^k(U', V)_{\phi(p),q}.$$

Then the chart $\tau_{U,V}$ is defined by

$$\tau_{U,V} := T_{U',V} \circ (\phi^{-1})^* : J^k(U, V) \to U' \times V \times B^k_{n,m}.$$

That this chart indeed makes $J^k(X, Y = \mathbb{R}^m)$ a smooth manifold is proved in [19, Theorem 2.7, p.40] (where also charts in case of any smooth manifold Y can be found).

The jet bundles can be naturally generalized to so-called multi-jet bundles. For smooth manifolds X and Y define

$$X^s = X \times \cdots \times X \quad (s \text{ times}),$$

$$X^{(s)} = \{(x_1, \ldots, x_s) \in X^s \mid x_i \neq x_j \text{ for } 1 \leq i < j \leq s\}.$$

If $\alpha : J^k(X, Y) \to X$ is the projection sending $p \in J^k(X, Y)_{(x,y)}$ to x, then let $\alpha^s : J^k(X, Y)^s \to X^s$ be the componentwise projection. Define the *s-fold k-jet bundle* to be $J^k_s(X, Y) = (\alpha^s)^{-1}(X^{(s)})$. A *multijet bundle* is an s-fold k-jet bundle for some s and k. As an open subset of X^s, the set $X^{(s)}$ is a smooth manifold and as an open subset of $J^k(X, Y)^s$, the set $J^k_s(X, Y)$ is also a smooth manifold. For a smooth map $f : X \to Y$ define $j^k_s f : X^{(s)} \to J^k_s(X, Y)$ to be

$$j^k_s f(x_1, \ldots, x_s) = (j^k f(x_1), \ldots, j^k f(x_s)). \tag{9.3}$$

9.1.2 The Whitney C^∞-topology

From the topology of the jet spaces $J^k(X,Y)$ one can define a topology on the space $C^\infty(X,Y)$.

Definition 9.2. For a fixed non-negative integer k, let U be an open subset of $J^k(X,Y)$. Consider the set

$$M(U) := \{f \in C^\infty(X,Y) \mid j^k f(X) \subset U\}.$$

The family of sets $\{M(U)\}$ with U an open subset of $J^k(X,Y)$ form a basis for a topology on $C^\infty(X,Y)$ called the *Whitney C^k-topology.* Open subsets of $C^\infty(X,Y)$ in the Whitney C^k-topology for all k form a basis for the *Whitney C^∞-topology.*

A neighborhood basis in the Whitney C^k-topology for a function f is given by

$$B^k_\delta(f) = \{g \in C^\infty(X,Y) \mid \forall x \in X,\ d(j^k f(x), j^k g(x)) < \delta(x)\}, \tag{9.4}$$

where d is a metric on $J^k(X,Y)$ compatible with its topology and $\delta : X \to (0,\infty)$ is a continuous mapping. Then $B^k_\delta(f)$ consists of those smooth mappings of $X \to Y$ all of whose first k partial derivatives are δ-close to those of f. A subset U of $C^\infty(X,Y)$ is open in the Whitney C^∞-topology if and only if for every $f \in U$ there is a continuous function $\delta : X \to (0,\infty)$ and an integer $k \geq 0$ such that $B^k_\delta(f) \subset U$.

If X is a compact manifold, then $B^k_n(f) = B^k_{\delta_n}(f)$ with $\delta_n(x) = 1/n$ for all x in X form a countable neighborhood basis of f and so $C^\infty(X,Y)$ satisfies the first axiom of countability (with C^k- and C^∞-topology). Moreover, if X is compact then a sequence of functions f_n in $C^\infty(X,Y)$ converges to f in the Whitney C^k-topology if and only if f_n and all of its partial derivatives of f_n of order less or equal k converge uniformly to f. For the details again see [19].

The Whitney C^∞-topology guaranties that the space $C^\infty(X,Y)$ satisfies an important property described below that is a prerequisite for the transversality theorems.

Lemma 9.3. *Let X and Y be smooth manifolds and consider $C^\infty(X,Y)$ with the Whitney C^∞-topology. Then any countable intersection of open dense subsets of $C^\infty(X,Y)$ is dense.*

This lemma is a version of the Baire Category Theorem and its proof can be found in [19, Proposition 3.3.].

9.1.3 Transversality

Definition 9.4. Let X and Y be smooth manifolds and $f : X \to Y$ a smooth mapping. Let W be a submanifold of Y and x a point in X. Then f is *transverse* to W at x (denoted by $f \pitchfork W$ at x) if either

- $f(x) \notin W$, or
- $f(x) \in W$ and $T_{f(x)}Y = T_{f(x)}W + D_x f(T_x X)$.

If $f \pitchfork W$ at every $x \in X$ then f is *transverse* to W (denoted by $f \pitchfork W$).

Note that if f misses W then automatically $f \pitchfork W$. Moreover, if

$$\dim(D_x f(T_x X)) + \dim W < \dim Y,$$

then transversality means precisely that W misses the image of f at x.

The initial transversality theorem due to Rene Thom deals with transversality of a map to a submanifold of a jet bundle. For the purpose of this work a further generalization is needed where a submanifold is replaced by a so-called Whitney stratified space.

A *stratification* of a closed subspace $\mathcal{S}$ of a smooth manifold Y is a partition of $\mathcal{S}$ into disjoint subsets, each of which is a smooth manifold. These manifolds are called *strata* and they have to fit together in a certain way. The space $\mathcal{S}$ with a stratification is called a *(Whitney) stratified* space. More precisely, there exists a finite filtration by closed subsets

$$S = S^m \supset S^{m-1} \supset S^{m-2} \supset \ldots \supset S^0 \supset S^{-1} = \emptyset$$

such that the strata $S_i = S^i \setminus S^{i-1}$ are manifolds of dimension i. There are some additional local conditions that $\mathcal{S}$ has to satisfy. In particular, if the so-called Whitney (B)-condition holds then the stratification is called *Whitney.* For further details see Pflaum [28]. These technical definitions will be omitted here, since the stratified spaces occurring in this thesis will be algebraic varieties and a result of Whitney [41] states that every real or complex algebraic variety possesses a Whitney regular stratification.

A smooth map $f : X \to Y$ is *transverse* to a (Whitney) stratified space $\mathcal{S}$ if it is transverse to each open stratum of $\mathcal{S}$.

The following version of the transversality theorem is obtained from Vassiliev [37] (by combining Theorem 4.6, 4.7.2 and the theorem of Section 4.8 of his book).

Theorem 9.5 (Multijet transversality theorem to stratified spaces [37])**.** *Let X and Y be smooth manifolds and $\mathcal{S}$ an arbitrary closed Whitney stratified subspace of $J^k_s(X,Y)$. Then the set*

$$T_{\mathcal{S}} = \{f \in C^\infty(X,Y) \mid j^k_s f \pitchfork \mathcal{S}\}$$

forms a dense subset in the space of all smooth maps $X \to Y$ equipped with the C^∞-topology. If $s = 1$ and X is compact then $T_{\mathcal{S}}$ is also open in $C^\infty(X,Y)$.

Let us end this section with a version of the transversality theorem for the case when a countable family of submanifolds instead of a Whitney stratification is given.

Lemma 9.6 (Corollary 7.18 p. 226 [27])**.** *Let $\{W_i\}_{i\in I}$ be a family of countably many submanifolds of $J^k_s(X,Y)$. Let*

$$T_{W_i} = \{f \in C^\infty(X,Y) \mid j^k_s f \pitchfork W_i\}.$$

Then $\bigcap_{i\in I} T_{W_i}$ is a dense subset of $C^\infty(X,Y)$.

For the proof see [27].

9.2 Fréchet spaces

The aim of this section is to provide a tool that will be later used in Section 10.2 to analyze the set $\mathcal{N}G$ of non-generic space curves. That key tool is contained in Proposition 9.15 which is an analog to the Regular Value Theorem in case of a special type of infinite-dimensional manifolds useful for this thesis. In the general case, the Regular Value Theorem requires the Inverse Mapping Theorem to hold. In the case of smooth mappings between Banach spaces, there is a generalization of the Inverse Mapping Theorem (see [21]). In case of so-called Fréchet spaces - a generalization of Banach spaces - a straightforward generalization of the Inverse Mapping Theorem does not exist and there are in fact counterexamples, e.g., due to Łojasiewicz and Zehnder [24]. Fortunately, John Nash and Jürgen Moser found a very useful class of so-called *tame* Fréchet spaces so that smooth maps between these spaces can be inverted. The space $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ endowed with the Whitney C^∞-topology is tame.

Section 9.2.1 describes the basics of the Fréchet spaces needed to state the key tool, i.e., Proposition 9.15. Section 9.2.2 introduces some definitions of the category of tame Fréchet spaces and the tame version of the Regular Value Theorem and finally ends with the proof of Proposition 9.15.

This section summarizes the results of Hamilton [21].

9.2.1 Fréchet spaces and manifolds

Definition 9.7. Let F be a vector space. A *seminorm* on F is a real-valued function $\|\cdot\| : F \to \mathbb{R}$ satisfying the following properties:

(i) $\|f\| \geq 0$ for all vectors $f \in F$;

(ii) $\|f + g\| \leq \|f\| + \|g\|$ for all vectors f, $g \in F$;

(iii) $\|cf\| = |c|\|f\|$ for all scalars $c \in \mathbb{R}$ and vectors $f \in F$.

A countable collection of seminorms $\{\|\cdot\|_n : n \in \mathbb{N}\}$ defines a unique topology such that a sequence f_j converges to f (denoted $f_j \to f$) if and only if $\|f_j - f\|_n \to 0$ for all $n \in \mathbb{N}$. A *metrizable locally convex topological vector space* is a vector space with a topology which arises from some countable collection of seminorms. The topology is *Hausdorff* if and only if $f = 0$ when all $\|f\|_n = 0$. A sequence f_j is *Cauchy* if for each n, $\|f_j - f_k\|_n \to 0$ as j and k approach infinity. The space is (sequentially) *complete* if every Cauchy sequence converges.

Definition 9.8.

1. A *Fréchet space* is a metrizable locally convex topological vector space which is complete and Hausdorff.

2. A *grading* of a Fréchet space F is a choice of a collection of seminorms satisfying

$$\|f\|_0 \leq \|f\|_1 \leq \|f\|_2 \leq \cdots$$

   and defining the topology. The space F together with a grading is referred to as a *graded* (Fréchet) space.

Given any collection of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}_0}$ the collection of seminorms given by $\{\sum_{i=0}^n \|\cdot\|_i\}_{n\in\mathbb{N}_0}$ is a grading.

Every Banach space with a norm $\|\cdot\|$ is trivially a Fréchet space with the collection of seminorms given by $\|\cdot\|_n := \|\cdot\|$ for all $n = 0, 1, \ldots$. The cartesian product of two Fréchet spaces is also a Fréchet space with seminorms

$$\|(f, g)\|_n = \|f\|_n + \|g\|_n.$$

Example 9.9 ([21, Example I.1.1.4.])**.** Let $C^\infty[a,b]$ be the vector space of smooth real-valued functions on the interval $[a,b]$. Put

$$\|f\|_n = \sum\nolimits_{j=0}^{n} \sup_{x\in[a,b]} |f^{(j)}(x)|.$$

Then $C^\infty[a,b]$ is a graded Fréchet space.

Example 9.10 ([21, Example II.1.1.2.])**.** Let $\Sigma(B)$ denote the space of all sequences $\{f_k\}$ of elements in a Banach space B such that

$$\|\{f_k\}\|_n = \sum_{k=0}^{\infty} e^{nk}\|f_k\|_B < \infty$$

for all $n \geq 0$. Then $\Sigma(B)$ is a graded space with the norms above.

Example 9.11. Let X be a compact manifold of finite-dimension m. Then $C^\infty(X,\mathbb{R})$ is a graded space with

$$\|f\|_n = \sup_{|\alpha|\leq n} \sup_{x\in X} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(u) \right|, \tag{9.5}$$

where the outer supremum runs over all partial derivatives of degree $|\alpha|$ at most n. Here $\frac{\partial^{|\alpha|}}{\partial x^\alpha}$ denotes a higher order mixed partial derivative: let $\alpha = (\alpha_1, ..., \alpha_m)$ be an m-tuple of non-negative integers, then

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_m^{\alpha_m}} f \quad \text{where } |\alpha| = \alpha_1 + ... + \alpha_m.$$

Golubitsky and Guillemin [19] comment that the "standard" topology of $C^\infty(X,Y)$ (presumably induced by the seminorms of the above example) coincides with the Whitney C^∞-topology when X is compact. Below this equivalence of topologies is shown for the space $C^\infty(\mathbb{S}^1,\mathbb{R}^3)$.

Proposition 9.12. *The topology of the space $C^\infty(\mathbb{S}^1,\mathbb{R}^3)$ induced by the family of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ given by (9.5) coincides with the Whitney C^∞-topology.*

Proof. For any curve $c \in C^\infty(\mathbb{S}^1,\mathbb{R}^3)$ the seminorms are given by

$$\|c\|_n = \sup_{s\in\mathbb{S}^1} \sup_{0\leq k\leq n} |c^{(k)}(s)|_\infty,$$

where on $\mathbb{R}^3$ the maximum norm $|(x_1,x_2,x_3)|_\infty = \max_{i=1,2,3} |x_i|$ is chosen.

There is a canonical bijection $J^k(\mathbb{S}^1, \mathbb{R}^3) \to \mathbb{S}^1 \times \mathbb{R}^3 \times \mathbb{R}^{3k}$ given by

$$j^k c(s) = (s, c(s), c'(s), c''(s), \ldots, c^{(k)}(s)).$$

Let the metric d_k on $J^k(\mathbb{S}^1, \mathbb{R}^3)$ be given by

$$d_k(j^k c(s), j^k d(t)) = \max\big\{|s-t|, |c(s)-d(t)|_\infty, |c'(s)-d'(t)|_\infty, \ldots, |c^{(k)}(s)-d^{(k)}(t)|_\infty\big\},$$

where on $\mathbb{S}^1$ $|s-t|$ is, e.g., the smaller distance between s and t. Then an ε-stripe around c w.r.t. the seminorm $\|\cdot\|_k$ is

$$D^k_\varepsilon(c) = \{d \in C^\infty(\mathbb{S}^1, \mathbb{R}^3) \mid \|c-d\|_k < \varepsilon\}$$

and it is precisely the same as the (δ, k)-ball $B^k_{\delta_\varepsilon} := B^k_\delta$ in the C^∞-topology given by (9.4) for $\delta : \mathbb{S}^1 \to \mathbb{R}$ being a constant function $\delta(s) = \varepsilon$ for all $s \in \mathbb{S}^1$, i.e.,

$$B^k_{\delta_\varepsilon}(c) = \{d \in C^\infty(\mathbb{S}^1, \mathbb{R}^3) \mid \forall s \in \mathbb{S}^1,\ d_k(j^k c(s), j^k d(s)) < \varepsilon\}.$$

Since both families of open balls for all k and $\varepsilon > 0$ form a neighborhood basis of c in the respective topologies, the topologies are the same. $\square$

Definition 9.13. Let F and G be Fréchet spaces, U an open subset of F, and $P : U \to G$ a continuous map.

(i) The *derivative* of P at the point $u \in U$ in the direction $v \in F$ is defined by

$$DP(u, v) = D_u P(v) = \lim_{\tau \to 0} \frac{P(u + \tau v) - P(u)}{\tau}.$$

P is *differentiable* at u in the direction v if the limit exists. The *k-th derivative* of P at the point $u \in U$ in directions $v_1, \ldots, v_k \in F$ denoted by $D^k_u P(v_1, ..., v_k)$ is defined as the derivative of $u \mapsto D^{k-1}_u P(v_1, \ldots, v_{k-1})$ with respect to u in direction v_k. P is *smooth* if P is k-times differentiable for every k.

(ii) The *tangent* $TP : U \times F \to G \times G$ of the map P is defined by

$$TP(u, v) := T_u P(v) = (P(u), D_u P(v)).$$

There is a natural generalization of the notion of a manifold to Fréchet spaces.

Definition 9.14. (i) A *Fréchet manifold* $\mathcal{M}$ is a Hausdorff topological space with an atlas of coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ such that each ϕ_α is a homeomorphism from a neighborhood U_α of $\mathcal{M}$ to a Fréchet space and all coordinate transition functions (defined in the usual way) are smooth maps between Fréchet spaces.

(ii) Let $\mathcal{M}$ be a Fréchet manifold and $\{(U_\alpha, \phi_\alpha)\}$ be an atlas of coordinate charts. A subset $\mathcal{N}$ of $\mathcal{M}$ is a *submanifold* of $\mathcal{M}$ if for every U_α with $U_\alpha \cap \mathcal{N} \neq \emptyset$ there exist Fréchet spaces F and G such that $\phi_\alpha(U_\alpha) \subset F \times G$ and

$$x \in U_\alpha \cap \mathcal{N} \quad \Longleftrightarrow \quad \phi_\alpha(x) \in F \times 0.$$

If G is finite dimensional with dimension k then k is called the *codimension* of $\mathcal{N}$ in $\mathcal{M}$.

Let $\mathcal{M}$ and $\mathcal{N}$ be Fréchet manifolds with atlases of coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ respectively.

(iii) A map $P : \mathcal{M} \to \mathcal{N}$ is a *smooth* map of Fréchet manifolds if for any $x \in \mathcal{M}$ there exist charts (U_α, ϕ_α) with $x \in U_\alpha$ and (V_β, ψ_β) with $P(x) \in V_\beta$ such that the local representative of P in these charts, i.e., $\psi_\beta \circ P \circ \phi_\alpha^{-1} : U_\alpha \to V_\beta$ is a smooth map of Fréchet spaces.

(iv) Let $\alpha : (0,1) \to \mathcal{M}$ be a differentiable curve. Then $\alpha'(t) := D_t\alpha(1)$ is a tangent vector to $\mathcal{M}$ at $\alpha(t)$ for all $t \in (0,1)$.

(v) The *tangent space* $T_f\mathcal{M}$ to $\mathcal{M}$ at $f \in \mathcal{M}$ is the space of all tangent vectors to $\mathcal{M}$ at f. The *tangent bundle* $T\mathcal{M}$ over $\mathcal{M}$ is the disjoint union of tangent spaces to $\mathcal{M}$ at all $f \in \mathcal{M}$, i.e., $T\mathcal{M} = \dot{\bigcup}_{f\in\mathcal{M}} T_f\mathcal{M}$ with a Fréchet manifold structure whose coordinate transition functions are the tangents TP of the coordinate transition functions P for $\mathcal{M}$.

(vi) A smooth map $P : \mathcal{M} \to \mathcal{N}$ of Fréchet manifolds induces a *tangent map* $TP : T\mathcal{M} \to T\mathcal{N}$ of their tangent bundles which maps linearly $T_f\mathcal{M}$ into $T_{P(f)}\mathcal{N}$. The local representatives for the tangent map TP are the tangents of the local representatives for P. The *derivative* of P at f is the linear map

$$D_fP : T_f\mathcal{M} \to T_{P(f)}\mathcal{N}$$

induced by TP on the tangent space. (When the manifolds are Fréchet spaces this agrees with the previous definition.)

Also the notion of *transversality* can be naturally extended to the case of a submanifolds of a Fréchet space by simply replacing the manifold Y with a Fréchet space F in Definition 9.4.

All the important notions have been defined and the key tool of this section can now be stated.

Proposition 9.15. *Let $F := C^\infty(\mathbb{S}^1, \mathbb{R}^3) \times \mathbb{R}^k$ for some positive integer k and let $P : F \to \mathbb{R}^{k+1}$ be a smooth map. Suppose that for some $g_0 \in F$ the derivative $D_{g_0}P$ is surjective. Then the level set $\mathcal{N} := P^{-1}(P(g_0))$ is a smooth submanifold of codimension $k+1$ in a neighborhood of g_0 with tangent space*

$$T_g\mathcal{N} = \ker D_g P.$$

As already mentioned the prerequisite for the proof of the above statement is the Regular Value Theorem which in turns requires the Inverse Mapping Theorem. They both hold in the category of tame Fréchet spaces described below. The proof of Proposition 9.15 is located at the end of this section.

9.2.2 Tame Fréchet spaces.

The definition of a tame Fréchet space is introduced for the sake of completeness in the argumentation. It however won't be used at any further point of this work. In fact Corollary 9.20 will guarantee that the spaces occurring in this thesis all are tame.

Definition 9.16. (i) A linear map $L : F \to G$ of one graded Fréchet space into another is *tame* if it satisfies the estimate

$$\|Lf\|_n \leq C\|f\|_{n+r}$$

for each $n \geq b$ (with a constant C which may depend on n) for some b and r. A tame linear map is automatically continuous in the Fréchet space topologies.

(ii) For any Banach space B, let $\Sigma(B)$ denote the graded space of exponentially decreasing sequences defined in Example 9.10. A graded space F is *tame* if one can find B and tame linear maps $L : F \to \Sigma(B)$ and $M : \Sigma(B) \to F$ such that the composition $ML : F \to F$ is the identity

$$F \xrightarrow{L} \Sigma(B) \xrightarrow{M} F.$$

Remark 9.17. Any Banach space B is trivially tame using the maps $L : B \to \Sigma(B)$ and $M : \Sigma(B) \to B$ given by $L(x) = \{x, 0, 0, 0, ...\}$ and $M(\{f_k\}) = f_0$.

Lemma 9.18 (Hamilton [21]). *The Cartesian product $F_1 \times F_2$ of two tame spaces F_1 and F_2 is tame.*

Proof. If $F_1 \xrightarrow{L_1} \Sigma(B_1) \xrightarrow{M_1} F_1$ and $F_2 \xrightarrow{L_2} \Sigma(B_2) \xrightarrow{M_2} F_2$ are identity maps with L_i and M_i linear and B_i Banach for $i = 1, 2$, then

$$F_1 \times F_2 \xrightarrow{L_1 \times L_2} \Sigma(B_1) \times \Sigma(B_2) = \Sigma(B_1 \times B_2) \xrightarrow{M_1 \times M_2} F_1 \times F_2$$

makes $F_1 \times F_2$ a tame space. $\square$

Also the spaces of functions from a compact manifold turn out to be tame. For the proof of the following statement see Hamilton [21, Theorem 1.3.6].

Theorem 9.19 (Hamilton [21]). *If X is a compact manifold then $C^\infty(X, \mathbb{R})$ is a tame Fréchet space.*

From Remark 9.17, Lemma 9.18 and Theorem 9.19 one immediately obtains the following Corollary.

Corollary 9.20. *The space $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ as well as its cartesian product with any Banach space is a tame Fréchet space.*

Below is a slightly weaker statement of Hamilton's Regular Value Theorem. For the proof see [21, §2.3.1, p. 196].

Theorem 9.21 (Hamilton [21]). *Let F be a tame space and $P : U \to V$ is a smooth map from an open subset $U \subset F$ to a finite dimensional vector space V. Suppose that for some $f_0 \in U$ the derivative $D_{f_0}P$ is surjective. Then the level set*

$$\mathcal{N} := \{f \in U \mid P(f) = P(f_0)\}$$

is a smooth submanifold in a neighborhood of f_0 with tangent space

$$T_f N = \ker D_f P.$$

In fact Hamilton's theorem in its full strength states that the submanifold of the theorem is also a so-called *tame* submanifold, which is just a submanifold satisfying certain conditions. These additional properties won't be used in this thesis and therefore, they are omitted here.

Finally, Proposition 9.15 is just a special case of Hamilton's theorem above.

Proof of Proposition 9.15. By Corollary 9.20 the space F is tame for any $k \in \mathbb{N}$ and the result follows from Theorem 9.21. $\square$

Chapter 10

The spaces of generic and non-generic curves

This chapter analyzes the structure of the two spaces: the set of generic closed space curves $\mathcal{G}$ and its complement $\mathcal{NG} = C^\infty(\mathbb{S}^1, \mathbb{R}^3) \setminus \mathcal{G}$. In order to justify the use of term "generic", Section 10.1 provides the proof of the fact that the set $\mathcal{G}$ is an open and dense subset of the space of all smooth closed space curves. Section 10.2 shows that non-generic space curves $\mathcal{NG}$ form a singular hypersurface in the space of all smooth closed curves. Moreover, the codimension 1-strata of this by V.I. Arnold traditionally called *discriminant hypersurface* are extracted and explicitly described.

10.1 The set of generic space curves is open and dense

With standard techniques from transversality theory summarized in Section 9.1 it is now possible to prove the following result (compare [32, 19, 33]).

Lemma 10.1. *The set $\mathcal{G}$ of closed space curves in generic position is open and dense in the space $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ equipped with the Whitney C^∞-topology.*

The proof will be split into several steps, first showing the density and later the openness. Recall the conditions (G1) and (G2) of Lemma 4.2 and define two sets:

$$\mathcal{G}_1 := \{\ c \in C^\infty(\mathbb{S}^1, \mathbb{R}^3) \mid c \ \text{ satisfies (G1)} \ \ \forall s \in \mathbb{S}^1\} \quad \text{and}$$

$$\mathcal{G}_2 := \{\ c \in C^\infty(\mathbb{S}^1, \mathbb{R}^3) \mid c \ \text{ satisfies (G2)} \ \ \forall s, t \in \mathbb{S}^1\}.$$

Clearly, the set of generic curves $\mathcal{G}$ is the intersection of the two sets above, i.e., $\mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2$, and it remains to show that each of the sets $\mathcal{G}_i$ for $i = 1, 2$ is dense, to show that $\mathcal{G}$ is dense. It should now be clear that the key to this proof is the Transversality Theorem 9.5. To show openness different methods are used.

Lemma 10.2. *The set $\mathcal{G}_1$ is both open and dense in $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$.*

Proof. Consider the 3-jet of a map of $c \in C^\infty(\mathbb{S}^1, \mathbb{R}^3)$, i.e., $j^3c : \mathbb{S}^1 \to J^3(\mathbb{S}^1, \mathbb{R}^3)$, given locally by

$$j^3c(s) = (s, c(s), c'(s), c''(s), c'''(s)).$$

Consider the subspace of $J^3(\mathbb{S}^1, \mathbb{R}^3)$ given by

$$\mathcal{S} := \{j^3c(s) \in J^3(\mathbb{S}^1, \mathbb{R}^3) \mid [c', c'', c'''](s) = [c', c'', c^{IV}](s) = 0\}.$$

It is a closed variety, hence a space possessing a Whitney regular stratification (by Whitney [41]), all of whose strata are smooth manifolds. The main stratum is of codimension 2. By the Transversality Theorem 9.5 the set

$$W_1 := \{c \in C^\infty(\mathbb{S}^1, \mathbb{R}^3) \mid j^3c \pitchfork \mathcal{S}\}$$

is an open dense subset of the space of smooth space curves. The condition $j^3c \pitchfork \mathcal{S}$ is equivalent to $j^3c(\mathbb{S}^1)$ missing $\mathcal{S}$. Thus

$$W_1 = \{c \in C^\infty(\mathbb{S}^1, \mathbb{R}^3) \mid [c', c'', c'''](s) \neq 0 \text{ or } [c', c'', c^{IV}](s) \neq 0 \; \forall s \in \mathbb{S}^1\}$$

and so $\mathcal{G}_1 = W_1$ is open and dense. $\square$

Lemma 10.3. *The set $\mathcal{G}_2$ is a dense subset of $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$.*

Proof. Consider the 2-fold 2-jet map of $c \in C^\infty(\mathbb{S}^1, \mathbb{R}^3)$, i.e., $j_2^2 : \mathbb{S}^{1(2)} \to J_2^2(\mathbb{S}^1, \mathbb{R}^3)$ given locally by

$$j_2^2c(s,t) = (s, c(s), c'(s), c''(s), t, c(t), c'(t), c''(t)).$$

Consider the subspace of $J_2^2(\mathbb{S}^1, \mathbb{R}^3)$ given by

$$\begin{aligned}\mathcal{S} := \big\{j_2^2c(s,t) \in J_2^2(\mathbb{S}^1, \mathbb{R}^3) \mid [c'(s), c'(t), c(s) - c(t)] = \\ = [c''(s), c'(t), c(s) - c(t)] = [c'(s), c''(t), c(s) - c(t)] = 0\big\}.\end{aligned}$$

It is a closed algebraic variety and hence a stratified space $\mathcal{S}$. By the Transversality Theorem 9.5, the set

$$W_2 := \{c \in C^\infty(\mathbb{S}^1, \mathbb{R}^3) \mid j_2^3 c \pitchfork \mathcal{S}\}$$

is a dense set. The codimension of the main stratum of $\mathcal{S}$ is equal to 3 (see [33]). The transversality condition $j_2^2 c \pitchfork \mathcal{S}$ is equivalent to $j_2^2 c(\mathbb{S}^{1(2)})$ missing $\mathcal{S}$. Hence,

$$\begin{aligned} W_2 := \{c \in C^\infty(\mathbb{S}^1, \mathbb{R}^3) \mid\ & [c'(s), c'(t), c(s) - c(t)] \neq 0 \ \text{ or} \\ & [c''(s), c'(t), c(s) - c(t)] \neq 0 \ \text{ or} \\ & [c'(s), c''(t), c(s) - c(t)] \neq 0 \ \ \forall s, t \in \mathbb{S}^1 \text{ with } s \neq t\big\} \end{aligned}$$

and so $\mathcal{G}_2 = W_2$ is dense.

□

The proof that $\mathcal{G}$ is open will require a separate method. A rather technical fact that can be found (together with its proof) in Sedykh [33, Lemma 4.3] will be needed first. Denote by $\mathcal{E}$ the subspace of $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ consisting of injective immersions. This set is already known to be open and dense (see [19, Theorem 5.7 and Proposition 5.8]) and $\mathcal{G} \subset \mathcal{E}$. Below is Sedykh's lemma adjusted to the notation used here.

Lemma 10.4 (Sedykh [33])**.** *Let $c_0 \in C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ be injective and satisfy the condition (G1), i.e., $c_0 \in \mathcal{G}_1 \cap \mathcal{E}$. Then for any point $s \in \mathbb{S}^1$ there exist open neighborhoods $U_s \subset \mathbb{S}^1$ and $W_{c_0}^s \subset \mathcal{G}_1 \cap \mathcal{E}$ such that:*

a) if $\tau(s) \neq 0$, then for each curve $c \in W_{c_0}^s$ and for any two points s_1, $s_2 \in U_s$, $s_1 \neq s_2$, there is no plane tangent simultaneously to the curve c at the points $c(s_1)$ and $c(s_2)$, i.e.,

$$[c'(s_1), c'(s_2), c(s_1) - c(s_2)] \neq 0 \quad \forall (s_1, s_2) \in {U_s}^{(2)};$$

b) if $\tau(s) = 0$, then for every curve $c \in W_{c_0}^s$ and for any point $s_1 \in U_s$ there exists at most one point $s_2 \in U_s$ such that points $c(s_1)$ and $c(s_2)$ have a common tangent plane ω to the curve c and, moreover, $\tau(s_1) \neq 0 \neq \tau(s_2)$ and the vectors $c'(s_1)$, $c'(s_2)$ are not collinear with the vector $c(s_1) - c(s_2)$ and the vectors $c''(s_1)$ and $c''(s_2)$ do not lie in ω.

The openness of the set of generic curves can now be shown.

Lemma 10.5. *The set of generic space curves $\mathcal{G}$ is an open subset of $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ in the Whitney $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ topology.*

Proof. The set $\mathcal{G}_1$ is open and dense and so is $\mathcal{E} \cap \mathcal{G}_1$. Suppose $\mathcal{G}$ is not open. Then there exists a map $c \in \mathcal{G}$ such that each open neighborhood of c in $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ contains elements not in $\mathcal{G}$. Since $\mathbb{S}^1$ is compact and $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ satisfies the first axiom of countability, there is a sequence of functions c_n converging to c each of which is not in $\mathcal{G}$. W.l.o.g. each c_n is in $\mathcal{G}_1 \cap \mathcal{E}$. Let $\{(s_n, t_n)\}_{n\in\mathbb{N}}$ be a sequence of tuples in $\mathbb{S}^{1(2)}$ satisfying

$$\begin{aligned}[][c_n'(s_n), c_n'(t_n), c_n(s_n) - c_n(t_n)] = [c_n''(s_n), c_n'(t_n), c_n(s_n) - c_n(t_n)] = \\ = [c_n'(s_n), c_n''(t_n), c_n(s_n) - c_n(t_n)] = 0.\end{aligned}$$

Take a convergent subsequence $\{(s_{n_k}, t_{n_k})\}_{k\in\mathbb{N}}$ and set $(s,t) := \lim_{k\to\infty}(s_{n_k}, t_{n_k})$. Since c_{n_k} converges uniformly to c

$$0 = \lim_{n\to\infty} \begin{pmatrix} [c_{n_k}'(s_{n_k}), c_{n_k}'(t_{n_k}), c_{n_k}(s_{n_k}) - c_{n_k}(t_{n_k})] \\ [c_{n_k}''(s_{n_k}), c_{n_k}'(t_{n_k}), c_{n_k}(s_{n_k}) - c_{n_k}(t_{n_k})] \\ [c_{n_k}'(s_{n_k}), c_{n_k}''(t_{n_k}), c_{n_k}(s_{n_k}) - c_{n_k}(t_{n_k})] \end{pmatrix} = \begin{pmatrix} [c'(s), c'(t), c(s) - c(t)] \\ [c''(s), c'(t), c(s) - c(t)] \\ [c'(s), c''(t), c(s) - c(t)] \end{pmatrix}.$$

The above contradicts the fact that $c \in \mathcal{G}_2$ unless $s = t$. That the latter is not the case will be shown using Sedykh's lemma below.

By Lemma 10.4 applied to the curve c, the diagonal Δ of the set $\mathbb{S}^1 \times \mathbb{S}^1$ can be covered by the open subsets $U_r \times U_r$ for each $r \in \mathbb{S}^1$. Since Δ is compact there is a finite subcover say $Z := \bigcup_{i=1}^m (U_{r_i} \times U_{r_i})$ for some positive integer m and the set $W := W_c^{r_1} \cap \ldots \cap W_c^{r_m}$ is an open neighborhood of c. Then $V := \mathbb{S}^1 \times \mathbb{S}^1 \setminus Z$ is a compact subset of $\mathbb{S}^{1(2)}$ and for n big enough all $c_n \in W$. For n even bigger also $(s_n, t_n) \in V$. Otherwise, $(s_n, t_n) \in U_r \times U_r$ for all $n > N$ with some constant $N \in \mathbb{N}$ and with $r \in \{r_1, \ldots, r_m\}$ (and w.l.o.g. $c_n \in W$ for all $n > N$). Clearly, $\tau(r) \neq 0$ cannot hold because of part a) of Lemma 10.4 which implies $[c_n'(s_n), c_n'(t_n), c(s_n) - c(t_n)] \neq 0$ for all $n > N$. If $\tau(r) = 0$, then by Sedykh's lemma part b) for $n > N$

$$\begin{aligned}[][c_n'(s_n), c_n'(t_n), c_n(s_n) - c_n(t_n)] &= 0 \\ [c_n''(s_n), c_n'(t_n), c_n(s_n) - c_n(t_n)] &\neq 0 \\ [c_n'(s_n), c_n''(t_n), c_n(s_n) - c_n(t_n)] &\neq 0.\end{aligned}$$

This is again a contradiction. Hence, $(s_n, t_n) \in V$ for n big enough and also the limit of the convergent subsequence $\{(s_{n_k}, t_{n_k})\}$ must lie in V, i.e., $(s,t) \in V$. Since $V \cap \Delta = \emptyset$, the points s and t are distinct. Hence, $\mathcal{G}$ is open. $\square$

Proof of Lemma 10.1. Combine Lemma 10.2, 10.3 and Lemma 10.5. □

10.2 Hypersurface of non-generic curves

With the preparation tool from Section 9.2, the fact that elements of the space $\mathcal{NG}$ of non-generic curves lie on a singular hypersurface in $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$, can now be proved. This hypersurface of non-generic curves is called traditionally by V.I. Arnold the *discriminant hypersurface* or simply the *discriminant*.

For each curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$, let γ also denote the associated periodic curve with $\mathbb{R}$ as the domain, i.e., $\gamma : \mathbb{R} \to \mathbb{R}^3$. For the two genericity conditions (G1) and (G2) of Lemma 4.2, there are smooth maps $f_k : C^\infty(\mathbb{S}^1, \mathbb{R}^3) \times \mathbb{R}^k \to \mathbb{R}^{k+1}$ $(k = 1, 2)$ constructed below that capture the corresponding singularities. Namely, for $k = 1$ let

$$f_1(\gamma, s) = \begin{pmatrix} [\gamma'(s), \gamma''(s), \gamma'''(s)] \\ [\gamma'(s), \gamma''(s), \gamma^{IV}(s)] \end{pmatrix} \tag{10.1}$$

and for $k = 2$ let

$$f_2(\gamma, s, t) \mapsto \begin{pmatrix} [\gamma'(s), \gamma'(t), \gamma(s) - \gamma(t)] \\ [\gamma''(s), \gamma'(t), \gamma(s) - \gamma(t)] \\ [\gamma'(s), \gamma''(t), \gamma(s) - \gamma(t)] \end{pmatrix}, \tag{10.2}$$

where in both cases γ maps both $\mathbb{S}^1$ and $\mathbb{R}$ to $\mathbb{R}^3$. The maps are actually periodic in the variable in $\mathbb{R}^k$ so that they factor through a projection ρ_k as follows

$$C^\infty(\mathbb{S}^1, \mathbb{R}^3) \times \mathbb{R}^k \xrightarrow{\rho_k} C^\infty(\mathbb{S}^1, \mathbb{R}^3) \times (\mathbb{S}^1)^k \longrightarrow \mathbb{R}^{k+1}.$$

The key tool for this proof is Proposition 9.15 which will give that the level set

$$\mathcal{M}_k := f_k^{-1}(0) \cap \{m \in C^\infty(\mathbb{S}^1, \mathbb{R}^3) \times \mathbb{R}^k \mid D_m f_k \text{ is surjective}\}$$

is a manifold of codimension $k + 1$ in $C^\infty(\mathbb{S}^1, \mathbb{R}^3) \times \mathbb{R}^k$. Project the manifold onto the space of smooth closed space curves via the projection $\pi_k : C^\infty(\mathbb{S}^1, \mathbb{R}^3) \times \mathbb{R}^k \to C^\infty(\mathbb{S}^1, \mathbb{R}^3)$, given by $(\gamma, s_1, ..., s_k) \mapsto \gamma$. The set $\pi_k(\mathcal{M}_k)$ is no longer a manifold, it is a stratified space and the curves γ belonging to the codimension 1 strata of this singular hypersurface are precisely those satisfying

1. $\rho_k\big(\big(\{\gamma\} \times \mathbb{R}^{(k)}\big) \cap \mathcal{M}_k\big) = \big(\{\gamma\} \times \mathbb{S}^{1(k)}\big) \cap \rho_k(\mathcal{M}_k) = \{m_0\}$ contains precisely one element and

2. $\ker D_m\pi_k \cap T_m\mathcal{M}_k = \{0\}$ at any $m \in \rho_k^{-1}(\{m_0\})$.

The differential of the projection π_k at any point (γ, s) in direction $(\xi, \dot{s})$ is simply $D_{(\gamma,s)}\pi_k(\xi, \dot{s}) = \xi$ and so the kernel $\ker D_m\pi_k = \{(\xi, \dot{s}) \in T_m\pi_k \mid \xi = 0\}$. The tangent space $T_m\mathcal{M}_k$ will by Proposition 9.15 be equal to $\ker D_m f_k$.

Below this general procedure is applied to the two different conditions (G1) and (G2) of general position of Lemma 4.2.

Lemma 10.6. *The set of curves for which the condition (G1) of Lemma 4.2 is not satisfied form a subset of $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ with codimension 1. A curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ belongs to the codimension 1 stratum if and only if there is only one point $s \in \mathbb{S}^1$ at which the condition (G1) fails and moreover,*

$$[\gamma', \gamma''', \gamma^{IV}](s) + [\gamma', \gamma'', \gamma^V](s) \neq 0 \tag{10.3}$$

holds.

Proof. Let f_1 be the map from (10.1). The differential of f_1 at $(\gamma, s) \in C^\infty(\mathbb{S}^1, \mathbb{R}^3) \times \mathbb{S}^1$ in direction $(\xi, \dot{s}) \in C^\infty(\mathbb{S}^1, \mathbb{R}^3) \times \mathbb{R}$ is

$$(D_{(\gamma,s)}f_1)(\xi, \dot{s}) = \begin{pmatrix} \dot{s}[\gamma', \gamma'', \gamma^{IV}] + [\xi', \gamma'', \gamma'''] + [\gamma', \xi'', \gamma'''] + [\gamma', \gamma'', \xi'''] \\ \dot{s}[\gamma', \gamma''', \gamma^{IV}] + \dot{s}[\gamma', \gamma'', \gamma^V] + [\xi', \gamma'', \gamma^V] + [\gamma', \xi'', \gamma^V] + [\gamma', \gamma'', \xi^V] \end{pmatrix}(s).$$

Consider the points $(\gamma, s) \in f_1^{-1}(0)$ at which the differential map is surjective. By Proposition 9.15 the set

$$\mathcal{M}_1 = f_1^{-1}(0) \cap \{(\gamma, s) \in C^\infty(\mathbb{S}^1, \mathbb{R}^3) \times \mathbb{R} \mid D_{(\gamma,s)}f_1 \text{ is surjective}\}$$

is a smooth manifold of codimension 2. Curves γ which belong to the codimension 1 strata of $\pi_1(\mathcal{M}_1)$ are those which have precisely one point $s_\gamma \in \mathbb{S}^1$ at which the condition (G1) is violated and $\{(\gamma, s_\gamma)\} = \rho_1(\mathcal{M}_1) \cap (\{\gamma\} \times \mathbb{S}^1)$ is the only element. Then at $m = (\gamma, s) \in \rho_1^{-1}(\{(\gamma, s_\gamma)\})$ the condition $\ker D_m\pi_1 \cap T_m\mathcal{M}_1 = \{0\}$ is equivalent to

$$D_m f_1(0, \dot{s}) = \dot{s}\begin{pmatrix} 0 \\ [\gamma', \gamma''', \gamma^{IV}](s) + [\gamma', \gamma'', \gamma^V](s) \end{pmatrix} = 0 \quad \Longleftrightarrow \quad \dot{s} = 0$$

and so equivalent to (10.3). Conversely, if $(\gamma, s) \in f_1^{-1}(0)$ satisfies (10.3), then automatically the differential $D_{(\gamma,s)}f_1$ is surjective. The lemma follows. $\square$

Note that curves that are not immersions do not lie on the codimension 1 strata of $\mathcal{NG}$. Moreover, if the curvature of γ (denoted κ_γ) vanishes at s (i.e., $\gamma'(s) \times \gamma''(s) = 0$) then (10.3) implies that $[\gamma', \gamma''', \gamma^{IV}](s) \neq 0$. On the other hand, if at some point s the expression $[\gamma', \gamma''', \gamma^{IV}](s)$ vanishes, then (10.3) implies that $[\gamma', \gamma'', \gamma^{V}](s) \neq 0$. In this case, the curvature doesn't vanish at s so it is justified to speak about torsion of γ (denoted τ_γ) in a neighborhood of s. The condition that γ lies on a codimension 1 stratum then translates to

$$\tau_\gamma(s) = \tau_\gamma'(s) = 0 \quad \Longrightarrow \quad \tau_\gamma''(s) \neq 0.$$

Definition 10.7. (i) Consider curves γ in $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ that satisfy the condition $(G2)$ of Lemma 4.2 for all pairs $s \neq t \in \mathbb{S}^1$ and satisfy the condition $(G1)$ at all but one $s \in \mathbb{S}^1$ and at that s the equation (10.3) holds for γ (the point s may be different for different curves). Denote this set by $\Sigma^{(G1)}$.

(ii) Partition further $\Sigma^{(G1)}$ into two subsets Σ^κ and Σ^τ, where

$$\Sigma^\kappa := \left\{\gamma \in \Sigma^{(G1)} \mid \kappa_\gamma(s) = 0 \ \text{ and } \ [\gamma', \gamma''', \gamma^{IV}](s) \neq 0 \ \text{ for some } \ s \in \mathbb{S}^1\right\}$$

and

$$\Sigma^\tau := \Big\{\gamma \in \Sigma^{(G1)} \mid \kappa_\gamma(s) \neq 0 \ \text{ and } \\ \tau_\gamma(s) = \tau_\gamma'(s) = 0 \ \text{ and } \ \tau_\gamma''(s) \neq 0 \ \text{ for some } \ s \in \mathbb{S}^1\Big\}$$

with κ_γ and τ_γ denoting the curvature and torsion of γ.

A similar procedure can be applied to the condition (G2). In the following lemma use the abbreviation γ_s for $\gamma(s)$ to represent the matrix.

Lemma 10.8. *The set of curves for which the condition (G2) of Lemma 4.2 is not satisfied forms a subset of $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ with codimension 1. A curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ belongs to the codimension 1 strata if and only if there is a single pair of points $s \neq t \in \mathbb{S}^1$ at which the condition (G2) fails for γ, and at this pair the three vectors $\gamma'(s)$, $\gamma'(t)$ and $\gamma(s) - \gamma(t)$ are not all parallel and moreover,*

$$\det \begin{pmatrix} [\gamma_s'', \gamma_t', \gamma_s'] + [\gamma_s''', \gamma_t', \gamma_s - \gamma_t] & [\gamma_s'', \gamma_t'', \gamma_s - \gamma_t] \\ [\gamma_s'', \gamma_t'', \gamma_s - \gamma_t] & -[\gamma_s', \gamma_t'', \gamma_t'] + [\gamma_s', \gamma_t''', \gamma_s - \gamma_t] \end{pmatrix} \neq 0 \qquad (10.4)$$

holds.

Proof. Let f_2 be the map defined in (10.2). The differential of f_2 in direction $(\xi, \dot{s}, \dot{t})$ at a point $u = (\gamma, s, t) \in f_2^{-1}(0)$ is

$$(D_u f_2)(\xi, \dot{s}, \dot{t}) = \begin{pmatrix} 0 \\ \dot{s}[\gamma_s'', \gamma_t', \gamma_s'] + \dot{t}[\gamma_s'', \gamma_t'', \gamma_s - \gamma_t] + \dot{s}[\gamma_s''', \gamma_t', \gamma_s - \gamma_t] \\ -\dot{t}[\gamma_s', \gamma_t'', \gamma_t'] + \dot{t}[\gamma_s', \gamma_t''', \gamma_s - \gamma_t] + \dot{s}[\gamma_s'', \gamma_t'', \gamma_s - \gamma_t] \end{pmatrix}$$
$$+ \begin{pmatrix} [\gamma_s', \gamma_t', \xi_s - \xi_t] + [\gamma_s', \xi_t', \gamma_s - \gamma_t] + [\xi_s', \gamma_t', \gamma_s - \gamma_t] \\ [\gamma_s'', \gamma_t', \xi_s - \xi_t] + [\gamma_s'', \xi_t', \gamma_s - \gamma_t] + [\xi_s'', \gamma_t', \gamma_s - \gamma_t] \\ [\gamma_s', \gamma_t'', \xi_s - \xi_t] + [\gamma_s', \xi_t'', \gamma_s - \gamma_t] + [\xi_s', \gamma_t'', \gamma_s - \gamma_t] \end{pmatrix}.$$

Consider the set of elements $u \in f_2^{-1}(0)$ for which the differential $D_u f_2$ is surjective. Note, that at (γ, s, t) with

- $s = t$ and
- $s \neq t$ and all three vectors $\gamma'(s)$, $\gamma'(t)$ and $\gamma(s) - \gamma(t)$ parallel

the differential is not surjective. Proceed analogously to the proof of Lemma 10.6 and obtain from Proposition 9.15 that the set

$$\mathcal{M}_2 = f_2^{-1}(0) \cap \{u \in C^\infty(\mathbb{S}^1, \mathbb{R}^3) \times \mathbb{R}^2 \mid D_u f_2 \text{ is surjective}\}$$

is a manifold of codimension 3. Project $\mathcal{M}_2$ onto $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ via π_2 and find that the elements γ of $\pi_2(\mathcal{M}_3)$ belong to codimension 1 strata if there is precisely one pair of distinct points $\{s_\gamma, t_\gamma\} \subset \mathbb{S}^1$ such that $f_2(\gamma, s, t) = 0$ for all $(\gamma, s, t) \in \rho_2^{-1}(\{(\gamma, s_\gamma, t_\gamma)\})$. Moreover, $\{(\gamma, s_\gamma, t_\gamma)\} = \rho_2(\mathcal{M}_2) \cap \big(\{\gamma\} \times \mathbb{S}^{1(2)}\big)$ is the only element. At any $m = (\gamma, s, t) \in \rho_2^{-1}(\{(\gamma, s_\gamma, t_\gamma\})$

$$D_m f_2(0, \dot{s}, \dot{t}) = \dot{s} \begin{pmatrix} 0 \\ [\gamma_s'', \gamma_t', \gamma_s'] + [\gamma_s''', \gamma_t', \gamma_s - \gamma_t] \\ [\gamma_s'', \gamma_t'', \gamma_s - \gamma_t] \end{pmatrix} + \dot{t} \begin{pmatrix} 0 \\ [\gamma_s'', \gamma_t'', \gamma_s - \gamma_t] \\ -[\gamma_s', \gamma_t'', \gamma_t'] + [\gamma_s', \gamma_t''', \gamma_s - \gamma_t] \end{pmatrix}$$

and the condition

$$\ker D_m \pi_2 \cap T_m \mathcal{M}_2 = \{0\}$$

is equivalent to

$$D_m f_2(0, \dot{s}, \dot{t}) = 0 \quad \Longleftrightarrow \quad (\dot{s}, \dot{t}) = 0$$

and so to (10.4). Conversely, if $(\gamma, s, t) \in f_2^{-1}(0)$ with $s \neq t$ satisfies (10.4) and the three vectors $\gamma'(s)$, $\gamma'(t)$ and $\gamma(s) - \gamma(t)$ are not all parallel, then automatically the differential $D_{(\gamma,s,t)}f_2$ is surjective. The lemma follows. $\square$

The list of different types of codimension 1 strata of the discriminant $\mathcal{NG}$ can now be completed. There are three types of singularities corresponding to the global genericity condition (G2). Apart from a double point, the two remaining singularities for a given γ and $s \neq t \in \mathbb{S}^1$ are:

(a) $\gamma'(s)$ and $\gamma(s) - \gamma(t)$ are collinear, and $\gamma'(t)$ lies in the osculating plane at $\gamma(s)$;

(b) the affine osculating planes of γ at $\gamma(t)$ and $\gamma(s)$ coincide, i.e., the five vectors $\gamma'(s)$, $\gamma''(s)$, $\gamma'(t)$, $\gamma''(t)$ and $\gamma(s) - \gamma(t)$ lie in a plane

(compare with Definition 4.1). Notice, that if γ has a double point at $s \neq t \in \mathbb{S}^1$ then γ belongs to the codimension 1 strata if

$$[\gamma''(s), \gamma'(t), \gamma'(s)] \neq 0 \ \text{ and } \ [\gamma'(s), \gamma''(t), \gamma'(t)] \neq 0.$$

This means that the tangent vectors $\gamma'(s)$ and $\gamma'(t)$ span a plane different from the osculating planes of γ at s and t.

Definition 10.9. (i) Consider curves γ in $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ which satisfy the condition (G1) of Lemma 4.2 for all points $s \in \mathbb{S}^1$ and satisfy the condition (G2) at all but some one pair $\{s,t\} \in \mathbb{S}^{1(2)}$. Moreover, at that pair $\{s,t\}$ (which may be different for different curves) the three vectors $\gamma'(s)$, $\gamma'(t)$ and $\gamma(s)-\gamma(t)$ are not all parallel and the equation (10.4) holds for γ. Denote this set by $\Sigma^{(\mathrm{G2})}$.

(ii) Partition further the set $\Sigma^{(\mathrm{G2})}$ into three subsets $\Sigma^\times$, Σ^a and Σ^b where

$$\Sigma^\times = \big\{\gamma \in \Sigma^{(\mathrm{G2})} \mid \gamma(s) = \gamma(t) \ \text{ and } \\ [\gamma''(s), \gamma'(t), \gamma'(s)][\gamma'(s), \gamma''(t), \gamma'(t)] \neq 0 \ \text{ for some } \ s \neq t \in \mathbb{S}^1\big\};$$

$$\Sigma^a = \big\{\gamma \in \Sigma^{(\mathrm{G2})} \mid \gamma'(s) \times (\gamma(s) - \gamma(t)) = 0 \ \text{ and } \\ [\gamma'(t), \gamma'(s), \gamma''(s)] = 0 \ \text{ for some } \ s \neq t \in \mathbb{S}^1\big\};$$

$$\Sigma^b = \big\{\gamma \in \Sigma^{(\mathrm{G2})} \mid \gamma'(s), \gamma''(s), \gamma'(t), \gamma''(t), \gamma(s) - \gamma(t) \ \text{ are coplanar, and } \\ \gamma'(s) \times (\gamma(s) - \gamma(t)) \neq 0 \ \text{ and } \ \gamma'(t) \times (\gamma(s) - \gamma(t)) \neq 0 \ \text{ for some } \ s \neq t \in \mathbb{S}^1\big\}.$$

(iii) Set

$$\Sigma := \Sigma^{(\mathrm{G1})} \cup \Sigma^{(\mathrm{G2})}.$$

The space of non-generic space curves $\mathcal{NG}$ is a singular hypersurface in $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ and the set Σ is the codimension 1 stratum of $\mathcal{NG}$. The elements of Σ satisfy the following important property.

Lemma 10.10. *For each element $\gamma \in \Sigma$ there are infinitely many directions $v \in \mathbb{S}^2$ such that the projected curve γ_v is a FB-generic plane curve. In particular*

(i) if $\gamma \in \Sigma^{(\mathrm{G1})}$ and $s \in \mathbb{S}^1$ is the point at which the condition $(G1)$ fails, then there exists $v \in \mathbb{S}^2$ such that γ_v is FB-generic and

1. *$v \notin \operatorname{span}\{\gamma'(s), \gamma'''(s)\}$ if $\gamma \in \Sigma^\kappa$ (i.e., if $\gamma'(s) \times \gamma''(s) = 0$) and*
2. *$v \notin \operatorname{span}\{\gamma'(s), \gamma''(s)\}$ if $\gamma \in \Sigma^\tau$;*

(ii) if $\gamma \in \Sigma^{(\mathrm{G2})}$ and $\{s,t\}$ is the pair of distinct points of $\mathbb{S}^1$ at which the condition (G2) fails, then there exists $v \in \mathbb{S}^2$ such that γ_v is FB-generic and

$$v \notin \operatorname{span}\{\gamma'(s), \gamma'(t), \gamma(s) - \gamma(t)\}.$$

Proof. Suppose that $\gamma \in \Sigma^{(\mathrm{G1})}$ and $s \in \mathbb{S}^1$ is the point at which the condition $(G1)$ fails. The curve γ_v is a smooth immersion if and only if $v \notin \pm\gamma(\mathbb{S}^1)$. Moreover, γ_v has only regular inflection points if and only if

- $v \notin \operatorname{span}\{\gamma'(t), \gamma''(t)\}$ whenever $t \neq s$ and $\tau(t) = 0$ and
- $v \notin \operatorname{span}\{\gamma'(s), \gamma'''(s)\}$ if $\gamma \in \Sigma^\kappa$ and
- $v \notin \operatorname{span}\{\gamma'(s), \gamma''(s)\}$ if $\gamma \in \Sigma^\tau$.

So far v is different than the tangent indicatrix of γ and a finite number of great circles. Furthermore, to be on the safe side, s should not be allowed to be part of a bitangent pair or a double point of the curve γ_v. Since (G2) is satisfied for all distinct pairs of points of $\mathbb{S}^1$, there are only finitely many bitangent planes of γ containing $\gamma'(s)$. Hence, v should be different from $\operatorname{span}\{\gamma'(s), \gamma'(t), \gamma(s) - \gamma(t)\}$ whenever $[\gamma'(s), \gamma'(t), \gamma(s) - \gamma(t)] = 0$ for $t \in \mathbb{S}^1$. The set of directions v allowed so far, say $\mathcal{V}$, is open and dense in $\mathbb{S}^1$. Now, for any $v \in \mathcal{V}$, if γ_v is not FB-generic then, (by an argument similar to Lemma 7.7) v can be perturbed to another $v' \in \mathcal{V}$ such that the irregularity of double points and bitangent pairs of γ_v are resolved and $\gamma_{v'}$ is FB-generic.

If γ is in $\Sigma^{(G2)}$ then the situation is even simpler. Let $\{s,t\}$ be the pair of distinct points of $\mathbb{S}^1$ at which the condition (G2) fails. First, choose as above $v \notin \pm\gamma(\mathbb{S}^1)$ so that γ_v is a smooth immersion and $v \notin \mathrm{span}\{\gamma'(r) \times \gamma''(r)\}$ at the finitely many points $r \in \mathbb{S}^1$ with $\tau(r) = 0$. Choose v different from

$$\mathrm{span}\{\gamma'(s), \gamma'(t), \gamma(s) - \gamma(t)\}$$

to assure that the pair $\{s,t\}$ is neither a double point nor a bitangent pair of γ_v. If $\mathcal{V}$ is the set of directions allowed so far, then $\mathcal{V}$ is again open and dense in $\mathbb{S}^1$. If γ_v with $v \in \mathcal{V}$ is not FB-generic then, (again by an argument similar to Lemma 7.7) v can be perturbed to another $v' \in \mathcal{V}$ such that the irregularity of double points and bitangent pairs of γ_v are resolved and $\gamma_{v'}$ is FB-generic. $\square$

Chapter 11

New proof of Fabricius-Bjerre's Theorem

This chapter describes how to obtain a new proof of Fabricius-Bjerre's Theorem in two main steps. The first step is to show that any FB-generic plane curve can be obtained through a projection of some generic space curve, which is done in Section 11.1. On the other hand, it was already shown in Chapter 7, that every generic closed space curve projects to FB-generic curve. This gives an equivalence between Fabricius-Bjerre's Theorem and the fact that the map constructed in Chapter 7 (whose degree is closely related to the Fabricius-Bjerre formula) has degree zero for any generic closed space curve. The latter will be proved in Section 11.2 using a similar strategy to that of V.I. Arnold, which is the following.

Since the degree map is constant in each connected component of the space $\mathcal{G}$ of generic closed space curves, it needs to be shown that the degree is equal to zero for some chosen representative of $\mathcal{G}$. Moreover, it has to be checked that the degree is the same on any pair of neighboring components of $\mathcal{G}$ separated by the discriminant hypersurface Σ. This will be achieved by constructing for each point of Σ a path (i.e., a one-parameter family of space curves) in $\mathcal{G} \cup \Sigma$ that intersects the discriminant hypersurface transversely precisely in that point and such that the degree of each member of the family is the same.

11.1 Generic plane curves lift to generic space curves

Recall the notion of a FB-generic plane curve from Definition 1.2. The main result of this section is demonstrated in the following theorem.

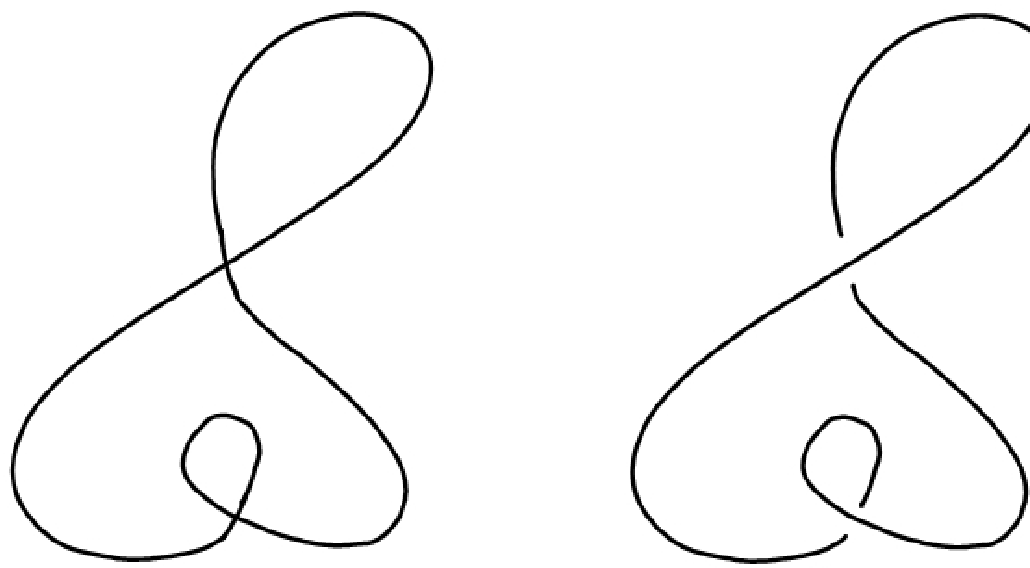


Figure 11.1: A generic plane curve (left) and a generic space curve (right) obtained by lifting the plane curve off its plane.

Theorem 11.1. *For any FB-generic plane curve α there exists a generic space curve γ_α that projects orthogonally to α.*

The strategy will be to place the plane curve in the xy-plane at height 0 and then to modify its height in such a way that the space curve obtained is generic. Intuitively, there are enough degrees of freedom for this approach to work. Then projecting the obtained generic space curve in the direction of the z-axis will clearly have the same image as the initial plane curve. See Figure 11.1.

The key to the proof will again be transversality theorems. Below two functions with domains in the jet spaces will be considered. The coordinates of the local representation of elements of the jet spaces are organized as in (9.1) and (9.3), i.e., for $h \in C^\infty(\mathbb{S}^1, \mathbb{R})$ and $s, t \in \mathbb{S}^1$ an element of a 4-jet space $J^4(\mathbb{S}^1, \mathbb{R})$ shall be locally represented by

$$j^k h(s) = (s, h(s), h'(s), \dots, h^{(4)}(s)).$$

and an element of the 2-fold 2-jet space $J_2^2(\mathbb{S}^1, \mathbb{R})$ by

$$j_2^2 h(s) = (s, h(s), h'(s), h''(s), t, h(t), h'(t), h''(t)).$$

A space curve is generic if it satisfies the genericity conditions (G1) and (G2) of Lemma 4.2. Given a FB-generic plane curve $\alpha : \mathbb{S}^1 \to \mathbb{R}^2$ parametrized by arc-length, define accordingly two maps $g_1 : J^4(\mathbb{S}^1, \mathbb{R}) \to \mathbb{R}^2$, by

$$(s, x_0, x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} \left[\begin{pmatrix} \alpha'(s) \\ x_1 \end{pmatrix}, \begin{pmatrix} \alpha''(s) \\ x_2 \end{pmatrix}, \begin{pmatrix} \alpha'''(s) \\ x_3 \end{pmatrix} \right] \\ \left[\begin{pmatrix} \alpha'(s) \\ x_1 \end{pmatrix}, \begin{pmatrix} \alpha''(s) \\ x_2 \end{pmatrix}, \begin{pmatrix} \alpha^{(IV)}(s) \\ x_4 \end{pmatrix} \right] \end{pmatrix}$$

with $s \in \mathbb{S}^1$ and $x_0, \ldots, x_4 \in \mathbb{R}$ and $g_2 : J_2^2(\mathbb{S}^1, \mathbb{R}) \to \mathbb{R}^3$, by

$$\begin{matrix} (s, x_0, x_1, x_2, \\ \quad t, y_0, y_1, y_2) \end{matrix} \mapsto \begin{pmatrix} \left[\begin{pmatrix} \alpha'(s) \\ x_1 \end{pmatrix}, \begin{pmatrix} \alpha'(t) \\ y_1 \end{pmatrix}, \begin{pmatrix} \alpha(s) - \alpha(t) \\ x_0 - y_0 \end{pmatrix} \right] \\ \left[\begin{pmatrix} \alpha''(s) \\ x_2 \end{pmatrix}, \begin{pmatrix} \alpha'(t) \\ y_1 \end{pmatrix}, \begin{pmatrix} \alpha(s) - \alpha(t) \\ x_0 - y_0 \end{pmatrix} \right] \\ \left[\begin{pmatrix} \alpha'(s) \\ x_1 \end{pmatrix}, \begin{pmatrix} \alpha''(t) \\ y_2 \end{pmatrix}, \begin{pmatrix} \alpha(s) - \alpha(t) \\ x_0 - y_0 \end{pmatrix} \right] \end{pmatrix}$$

with $s \neq t \in \mathbb{S}^1$ and $x_0, x_1, x_2, y_0, y_1, y_2 \in \mathbb{R}$.

The proof of Theorem 11.1 will require that the zero sets of g_1 and g_2 can be written as a finite union of manifolds of high enough codimension.

Lemma 11.2. *The set $g_1^{-1}(0)$ is a finite union of submanifolds W_a (for a in some finite index set A) of codimension at least 2, i.e.,*

$$g_1^{-1}(0) = \bigcup_{a \in A} W_a.$$

Proof. Indeed by genericity of plane curves, α has finitely many inflection points, double points and bitangent pairs. Let these numbers be l, m and n respectively. Denote by

$\tilde{s}_i \in \mathbb{S}^1$ for $i = 1, \ldots, l$ the inflection points,

$\{\bar{s}_j, \bar{t}_j\}$ for $j = 1, \ldots, m$ the bitangent pairs and

$\{s_k^\times, t_k^\times\}$ for $k = 1, \ldots, n$ the double points

of α.

Consider first the set $g_1^{-1}(0)$. The differential of the map g_1 is surjective at $x = (s, x_0, \ldots, x_4)$ whenever s is not an inflection point of α, i.e., $s \neq \tilde{s}_i$ for all $i = 1, \ldots, l$ (i.e., $\alpha''(s) \neq 0$ since α is parametrized by arc-length). This can be seen by verifying that $\frac{\partial}{\partial x_3} g_1$ and $\frac{\partial}{\partial x_4} g_1$ are linearly independent at any such x. By the Regular Value Theorem the set

$$W_1 := g_1^{-1}(0) \cap \{(s, \dots, x_4) \mid \alpha''(s) \neq 0\}$$

is a smooth manifold of codimension 2.

Consider $x = (s, \dots, x_4)$ with $\alpha''(s) = 0$ (and with $[\alpha', \alpha'''](s) \neq 0$ by genericity of α), i.e., $s = \tilde{s}_i$ for some $i = 1, \dots, l$. From

$$0 = g_1(x) = x_2 \begin{pmatrix} [\alpha', \alpha'''](s) \\ [\alpha', \alpha^{(IV)}](s) \end{pmatrix}$$

follows that x_2 must vanish. Hence,

$$W_2 \;:=\; g_1^{-1}(0) \cap \{(s, \dots, x_4) \mid \alpha''(s) = 0\} \;=\; \{(\tilde{s}_i, x_0, x_1, 0, x_3, x_4) \mid i = 1, \dots, l\}$$

is a manifold of codimension 2. Thus $g_1^{-1}(0) = W_1 \cup W_2$ is a union of two manifolds of codimension 2. $\square$

Lemma 11.3. *The set $g_2^{-1}(0)$ is a finite union of submanifolds W_b (for b in some finite index set B) of codim at least 3, i.e.,*

$$g_2^{-1}(0) = \bigcup_{b \in B} W'_b.$$

Proof. Consider now the set $g_2^{-1}(0)$. Let $(s, x_0, x_1, t, y_0, y_1)$ be an element of this set. Now, $g_2^{-1}(0)$ will be partitioned into 5 subsets

$$g_2^{-1}(0) = \bigcup_{b=1,\dots,5} W'_b \tag{11.1}$$

according to the following conditions on s and t:

1. the pairs of vectors $\alpha'(s)$, $\alpha(s) - \alpha(t)$ and $\alpha'(t)$, $\alpha(s) - \alpha(t)$ are linearly independent, i.e.,

$$\begin{aligned} W'_1 = \{(s, x_0, x_1, x_2, t, y_0, y_1, y_2) \in g_2^{-1}(0) \mid \\ [\alpha'(s), \alpha(s) - \alpha(t)] \neq 0 \;\text{ and }\; [\alpha'(t), \alpha(s) - \alpha(t)] \neq 0\}; \end{aligned}$$

2. the vectors $\alpha'(s)$ and $\alpha'(t)$ are linearly independent but $\alpha'(s)$ and $\alpha(s) - \alpha(t)$ are parallel and $\alpha(s) \neq \alpha(t)$, i.e.,

$$W_2' = \{(s, x_0, x_1, x_2, t, y_0, y_1, y_2) \in g_2^{-1}(0) \mid [\alpha'(s), \alpha'(t)] \neq 0 \text{ and } [\alpha'(s), \alpha(s) - \alpha(t)] = 0 \text{ and } \alpha(s) \neq \alpha(t)\};$$

3. the vectors $\alpha'(s)$ and $\alpha'(t)$ are linearly independent but $\alpha'(t)$ and $\alpha(s) - \alpha(t)$ are parallel and $\alpha(s) \neq \alpha(t)$, i.e.,

$$W_3' = \{(s, x_0, x_1, x_2, t, y_0, y_1, y_2) \in g_2^{-1}(0) \mid [\alpha'(s), \alpha'(t)] \neq 0 \text{ and } [\alpha'(t), \alpha(s) - \alpha(t)] = 0 \text{ and } \alpha(s) \neq \alpha(t)\};$$

4. $\{s, t\}$ is a bitangent pair of α, i.e.,

$$W_4' = \{(s, x_0, x_1, x_2, t, y_0, y_1, y_2) \in g_2^{-1}(0) \mid [\alpha'(s), \alpha'(t)] = [\alpha(s) - \alpha(t), \alpha'(s)] = 0\};$$

5. $\{s, t\}$ is a double point of α, i.e.,

$$W_5' = \{(s, x_0, x_1, x_2, t, y_0, y_1, y_2) \in g_2^{-1}(0) \mid \alpha(s) = \alpha(t)\}.$$

To see that (11.1) holds, compare the table below, where the conditions defining each subset of $g_2^{-1}(0)$ are listed.

	$\alpha(s) = \alpha(t)$	$[\alpha'(s), \alpha(s) - \alpha(t)] = 0$	$[\alpha'(t), \alpha(s) - \alpha(t)] = 0$	$[\alpha'(s), \alpha'(t)] = 0$
W_1'	false	false	false	true or false
W_2'	false	true	false	false
W_3'	false	false	true	false
W_4'	false	true	true	true
W_5'	true	true	true	false

Indeed all possible cases are exhausted, since genericity of α excludes the case

$$[\alpha'(s), \alpha'(t)] = 0 \text{ and } \alpha(s) = \alpha(t) \text{ at } s \neq t \in \mathbb{S}^1$$

and since under the assumption $\alpha(s) \neq \alpha(t)$ and $[\alpha'(s), \alpha(s) - \alpha(t)] = 0$

$$[\alpha'(s), \alpha'(t)] = 0] \quad \Longleftrightarrow \quad [\alpha'(t), \alpha(s) - \alpha(t)] = 0$$

for all $s \neq t \in \mathbb{S}^1$. Moreover, the sets are disjoint.

Each of these five sets can be shown to be a manifold of a codimension equal or higher than 3.

W_1': The differential of the map g_2 is surjective on the set W_1'. To check this, verify that the partial derivatives of g_2 w.r.t. the variables x_1, x_2 and y_2 are linearly independent at each points of W_1'. Hence, W_1' is a manifold of codimension 3.

W_2': The set W_2' can be equivalently described as a zero set of the map $f : U \to \mathbb{R}^3$ given by

$$\begin{aligned} f(s, x_0, x_1, x_2, \\ t, y_0, y_1, y_2) \end{aligned} = \begin{pmatrix} [\alpha'(s), \alpha(s) - \alpha(t)] \\ \left[\begin{pmatrix} \alpha'(s) \\ x_1 \end{pmatrix}, \begin{pmatrix} \alpha'(t) \\ y_1 \end{pmatrix}, \begin{pmatrix} \alpha(s) - \alpha(t) \\ x_0 - y_0 \end{pmatrix}\right] \\ \left[\begin{pmatrix} \alpha''(s) \\ x_2 \end{pmatrix}, \begin{pmatrix} \alpha'(t) \\ y_1 \end{pmatrix}, \begin{pmatrix} \alpha(s) - \alpha(t) \\ x_0 - y_0 \end{pmatrix}\right] \end{pmatrix}.$$

where $U \subset J_2^2(\mathbb{S}^1, \mathbb{R})$ is the open set

$$U := \{(s, x_0, x_1, x_2, t, y_0, y_1, y_2) \in J_2^2(\mathbb{S}^1, \mathbb{R}) \mid [\alpha'(s), \alpha'(t)] \neq 0 \text{ and } \alpha(s) \neq \alpha(t)\}.$$

Namely, if $p := (s, x_0, x_1, x_2, t, y_0, y_1, y_2) \in f^{-1}(0)$ then the vectors $(\alpha(s) - \alpha(t), x_0 - y_0)$ and $(\alpha'(s), x_1)$ are linearly dependent what implies $g_2(p) = 0$ and hence $p \in W_2'$. That the converse holds is clear and so $f^{-1}(0) = W_2'$. This map f is a submersion along the zero set (the partial derivatives of f w.r.t. the variables x_2, t and y_0 are linearly independent at each point of W_2') and hence the set W_2' is a manifold of codimension 3.

W_3': Observe the symmetry between the sets W_2' and W_3'. Analogously to the above case, the set W_3' is a manifold of codimension 3.

W'_4: In case of a bitangent pair, an elementary computation shows that

$$W'_4 = \Bigg\{ (\bar{s}_j, x_0, x_1, x_2, \bar{t}_j, y_0, y_1, y_2) \in J^2_2(\mathbb{S}^1, \mathbb{R}) \mid j = 1, \dots, m,$$

$$x_1 = -\frac{(x_0 - y_0)[\alpha'(\bar{s}_j), \alpha''(\bar{t}_j)]}{[\alpha''(\bar{t}_j), \alpha(\bar{s}_j) - \alpha(\bar{t}_j)]},\ y_1 = \frac{(x_0 - y_0)[\alpha''(\bar{s}_j), \alpha'(\bar{t}_j)]}{[\alpha''(\bar{s}_j), \alpha(\bar{s}_j) - \alpha(\bar{t}_j)]} \Bigg\}.$$

It is a submanifold of codimension 4 in $J^4_2(\mathbb{S}^1, \mathbb{R})$.

W'_5: In case of a double point, $x_0 = y_0$ must hold and

$$W'_5 = \{(s^\times_k, x_0, x_1, x_2, t^\times_k, x_0, y_1, y_2) \in J^2_2(\mathbb{S}^1, \mathbb{R}) \mid k = 1, \dots, n\}.$$

It is clearly a manifold of codimension 3.

□

With the above preparation the proof of Theorem 11.1 can now be completed.

Proof of Theorem 11.1. Let $\alpha : \mathbb{S}^1 \to \mathbb{R}^2$ be any FB-generic plane curve parametrized by arc-length. For any smooth height function $h : \mathbb{S}^1 \to \mathbb{R}$, define a space curve

$$\gamma_{\alpha,h} : \mathbb{S}^1 \to \mathbb{R}^3, \ \ t \mapsto (\alpha(t), h(t)).$$

Consider the set

$$U_\alpha := \{h \in C^\infty(\mathbb{S}^1, \mathbb{R}) \mid \gamma_{\alpha,h} \text{ is a generic space curve}\}$$

of all possible height functions yielding a generic space curve $\gamma_{\alpha,h}$. It needs to be shown that the set U_α is non-empty. The following equivalence

$$h \in U_\alpha \iff j^4 h\big(\mathbb{S}^1\big) \cap g_1^{-1}(0) = \emptyset \text{ and } j^2_2 h\big(\mathbb{S}^{1(2)}\big) \cap g_2^{-1}(0) = \emptyset$$

holds. With Lemma 11.2 and 11.3, and by the definition of "$\pitchfork$"

$$\begin{aligned} j^4 h \pitchfork W_a \text{ for all } a \in A &\iff j^4 h\left(\mathbb{S}^1\right) \cap g_1^{-1}(0) = \emptyset \quad \text{and} \\ j^2_2 h \pitchfork W'_b \text{ for all } b \in B &\iff j^2_2 h\left(\mathbb{S}^{1(2)}\right) \cap g_2^{-1}(0) = \emptyset. \end{aligned}$$

Then

$$U_\alpha = \bigcap_{a \in A} T_{W_a} \cap \bigcap_{b \in B} T_{W_b'},$$

where

$$T_{W_a} = \{h \in C^\infty(\mathbb{S}^1, \mathbb{R}) \mid j^4 h \pitchfork W_a\} \text{ for } a \in A$$

and

$$T_{W_b'} = \{h \in C^\infty(\mathbb{S}^1, \mathbb{R}) \mid j_2^2 h \pitchfork W_b'\} \text{ for } b \in B.$$

By Lemma 9.6, both $\bigcap_{a \in A} T_{W_a}$ and $\bigcap_{b \in B} T_{W_b}$ are dense and hence, so is their intersection. Therefore, for a given generic arc-length parametrized plane curve α, the set U_α is non-empty and any $\gamma_\alpha = \gamma_{\alpha,h}$ with $h \in U_\alpha$ projects orthogonally to α.

For a generic curve α which is not arc-length parametrized proceed as follows. Reparametrize it first via a smooth diffeomorphism $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ (with $\phi'(s) \neq 0$ for all $s \in \mathbb{S}^1$) to obtain an arc-length parametrized curve $\tilde{\alpha} = \alpha \circ \phi$. Find a space curve $\gamma_{\tilde{\alpha},h}$ for some height function $h \in U_{\tilde{\alpha}}$. Then $\gamma_{\alpha,h} := \gamma_{\tilde{\alpha},h} \circ \phi^{-1}$ is a generic space curve clearly satisfying the assumptions of the theorem for α. □

11.2 The degree map vanishes on $\mathcal{G}$

Given $\gamma \in \mathcal{G}$, recall from Chapter 7 the closed surface $\mathcal{M}(\gamma)$ (constructed from pieces of the three surfaces associated to γ) given by Theorem 7.4 and the map $\mathcal{F}(\gamma) : \mathcal{M}(\gamma) \to \mathbb{S}^2$ from Definition 7.5. Define the *degree map* $\mathcal{D} : \mathcal{G} \to \mathbb{Z}$ by

$$\mathcal{D}(\gamma) = \deg(\mathcal{F}(\gamma)).$$

This map is well-defined since by Theorem 7.9 the degree of $\mathcal{F}(\gamma)$ does not depend on the manifold $\mathcal{M}(\gamma)$ chosen. By Lemma 7.7, v being a regular projection direction of γ (i.e., γ_v is FB-generic) is equivalent to v being a regular value of the map $\mathcal{F}(\gamma)$. This means that any generic space curve regularly projects to a FB-generic plane curve. Conversely, by Theorem 11.1 any FB-generic plane curve can be obtained as the orthogonal projection of some generic space curve. More precisely, a FB-generic plane curve can be lifted off its plane to a generic space curve. The direction perpendicular to the plane, say v, is then a regular direction of the generic space curve and projection along v yields back the plane curve. Recall from Theorem 7.9

that the degree map at γ is

$$\mathcal{D}(\gamma) = -\mathrm{ext}(\gamma_v) + \mathrm{int}(\gamma_v) + \frac{1}{2}\mathrm{infl}(\gamma_v) + \mathrm{cr}(\gamma_v), \tag{11.2}$$

where v is *any* regular projection direction $v \in \mathbb{S}^2$ of γ (i.e., v is a regular value of the map $\mathcal{F}(\gamma)$). The above discussion shows that Fabricius-Bjerre's Theorem is equivalent to vanishing of the degree map $\mathcal{D}$ on the entire $\mathcal{G}$.

Theorem 11.4. *The degree map $\mathcal{D} : \mathcal{G} \to \mathbb{Z}$ is identically zero on the whole $\mathcal{G}$, i.e.,*

$$\mathcal{D}(\gamma) = 0 \textit{ for all } \gamma \in \mathcal{G}. \tag{11.3}$$

In particular, this condition is equivalent to Fabricius-Bjerre's Theorem.

Hence, in order to give a new proof of Fabricius-Bjerre's Theorem it suffices to show (11.3). The proof will be carried out with a strategy analogous to that of Arnold (see [4]), used to introduce the basic invariants of closed plane curves (compare Chapter 3).

Recall from Section 10.2 of the previous chapter that the set $\mathcal{NG} = C^\infty(\mathbb{S}^1, \mathbb{R}^3) \setminus \mathcal{G}$ called the discriminant is a singular hypersurface, whose top stratum is

$$\Sigma = \Sigma^\kappa \,\dot\cup\, \Sigma^\tau \,\dot\cup\, \Sigma^\times \,\dot\cup\, \Sigma^a \,\dot\cup\, \Sigma^b.$$

A continuous deformation of a curve γ within the set of generic curves $\mathcal{G}$ results in a continuous deformation of the surfaces associated to γ and hence, of the manifold $\mathcal{M}(\gamma)$. The map $\mathcal{D}$ is continuous on $\mathcal{G}$ and in particular $\mathcal{D}$ is constant in each connected component of the space $\mathcal{G}$. Following Arnold, one verifies the value of the degree map for one chosen generic curve, e.g., a curve $\gamma_\circ \in \mathcal{G}$ whose projection into some regular direction is a circle. That such a curve exists, follows again from Theorem 11.1. Since $\gamma_\circ$ projects to a circle for some $v \in \mathbb{S}^2$, the variables on the right-hand side of (11.2) clearly vanish for that v and $\mathcal{D}(\gamma_\circ) = 0$. An arbitrary curve $\gamma \in \mathcal{G}$ is then connected with $\gamma_\circ$ via a path in the space $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$, that intersects the set of non-generic curves $\mathcal{NG}$ transversely, only in the strata of codimension 1, i.e., in Σ. While passing from one component of $\mathcal{G}$ to a neighboring one through the discrimant Σ, it will be shown that the degree map $\mathcal{D}$ does not change its value. Hence, $\mathcal{D}$ is constant on $\mathcal{G}$ and since $\mathcal{D}(\gamma_\circ) = 0$, the degree map $\mathcal{D}$ is identically zero on $\mathcal{G}$.

Given a curve $\gamma \in \Sigma$ lying on a codimension 1 stratum of $\mathcal{NG}$, an explicit continuous one-parameter family of curves can be constructed, that crosses Σ transversely, only at that one point γ. In particular, this family can be chosen in such a way that all the curves differ from γ only along some fixed direction $v \in \mathbb{R}^3$. If additionally, v is a regular projection direction of γ (i.e., γ_v is a FB-generic plane curve) then automatically, v is a regular projection direction for each curve from the family. Namely, all the curves from the family, when projected onto the plane perpendicular to v, give the same generic plane curve γ_v. Consider for example γ that has a double point. Imagine looking at the curve from some direction such that a nice transversal double point is observed as in Figure 11.2. Now, move one of the strands either towards or

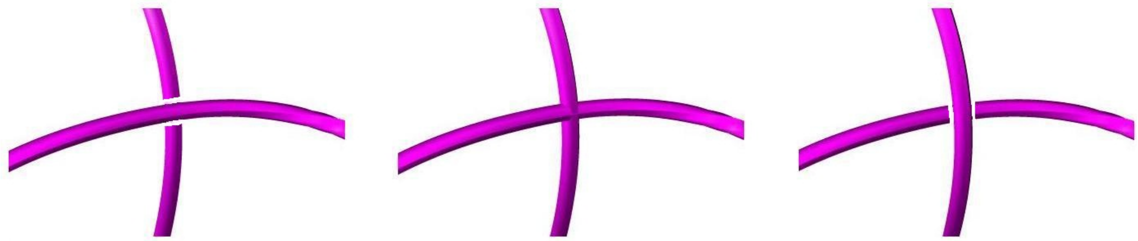


Figure 11.2: Resolving a double point singularity.

away from yourself. The double point then resolves into an over- or undercrossing. However, the images of the curves, projected onto the plane perpendicular to the viewing direction, are the same (see also Figure 11.3). Another example is shown in

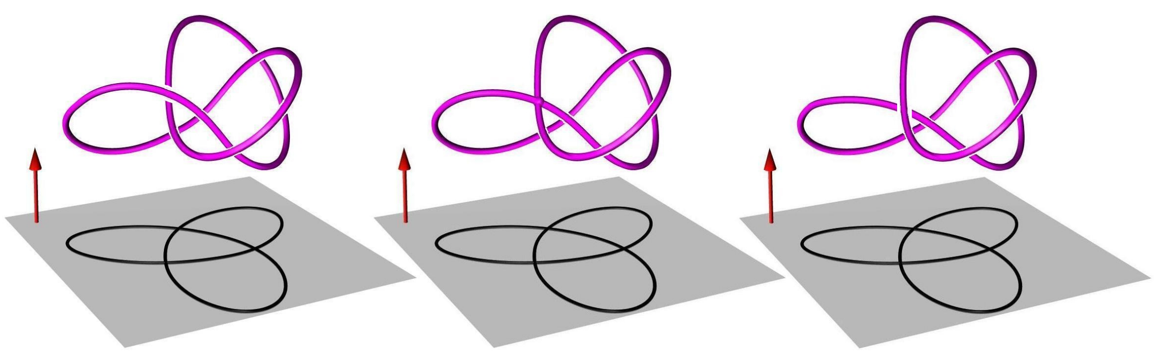


Figure 11.3: Resolving a double point singularity along a fixed direction (red arrow). Projections onto the plane perpendicular to that direction are the same.

Figure 11.4, where γ belongs to a stratum of global singularity $\Sigma^{(G2)}$. The singularity is represented by a tangency of the tangent developable surface of γ to the curve itself. Look at the the singularity from some fixed direction and move one of the arcs that are involved in the singularity again towards or away from yourself (as shown in Figure 11.4). The singularity will then resolve. However, the images obtained as orthogonal projections of the modified curves do not differ. In this case, for each curve

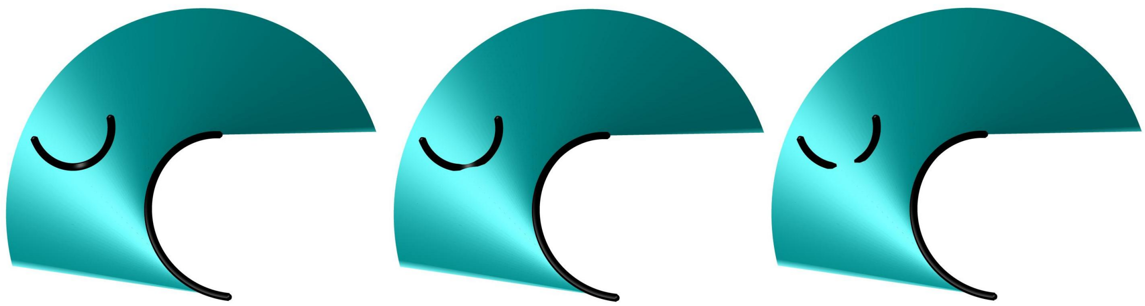


Figure 11.4: Resolving a singularity corresponding to a tangency of $\tilde{\gamma}$ to its tangent developable surface.

of the family, the variables on the right-hand side of (11.2) do not change. Hence, the degree map $\mathcal{D}$ has the same value in the (possibly disconnected) components of $\mathcal{G}$ on the different sides of Σ around γ.

For the proof of Theorem 11.4 the following technical lemma is needed, whose proof will be postponed for a moment.

Lemma 11.5. *For $\gamma \in \Sigma$, let U be a neighborhood of γ with $U \cap \Sigma = U \cap \mathcal{NG}$. Then, for some sufficiently small $\varepsilon > 0$, there is a regular projection direction v of γ (i.e., γ_v is FB-generic) and a family of curves $\Gamma : (-\varepsilon, \varepsilon) \to C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ given by $a \mapsto \Gamma(a)$ with*

$$\Gamma(a)(s) = \gamma(s) + f(a, s)v, \tag{11.4}$$

where $f : (-\varepsilon, \varepsilon) \times \mathbb{S}^1 \to \mathbb{R}$ is a map satisfying $f(0, s) = 0$ for all $s \in \mathbb{S}^1$ and moreover, $\Gamma(a) \in \mathcal{G}$ for all $a \neq 0$, and Γ intersects $\Sigma \cap U$ transversely at the single point γ.

With the above fact the proof of Theorem 11.4 is now possible.

Proof of Theorem 11.4. Let γ be any curve that lies on a codimension 1 stratum of $\mathcal{NG}$, that is $\gamma \in \Sigma$. Then, there is an open neighborhood $U \subset C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ of γ such that $U \cap \Sigma = U \cap \mathcal{NG}$ is a smooth connected manifold and the set $U \setminus \Sigma$ has two connected components, say U_1 and U_2. By Lemma 11.5 there exist a path $\Gamma : (-\varepsilon, \varepsilon) \to C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ with $\Gamma(0) = \gamma$ and $\Gamma(a) \in \mathcal{G}$ for any $a \neq 0$ connecting elements of the set U_1 with elements of the set U_2 (this is guaranteed by the transversality of the intersection). Moreover, there is a direction $v \in \mathbb{S}^2$ that is a regular projection direction for all $\Gamma(a)$ with $a \in (-\varepsilon, \varepsilon)$ and $\Gamma(a)_v = \gamma_v$ for all $a \in (-\varepsilon, \varepsilon)$. Hence, the degree map at each element of the family Γ except γ is

$$\mathcal{D}(\Gamma(a)) = -\mathrm{ext}(\gamma_v) + \mathrm{int}(\gamma_v) + \frac{1}{2}\mathrm{infl}(\gamma_v) + \mathrm{cr}(\gamma_v)$$

for each $a \neq 0$. So the degree map $\mathcal{D}$ has the same value at the connected components of U separated by Σ. As in the preceding discussion, there exists a space curve $\gamma_\circ \in \mathcal{G}$ whose projection is some direction is a round circle and clearly $\mathcal{D}(\gamma_\circ) = 0$. Now, $C^\infty(\mathbb{S}^1, \mathbb{R}^3)$ is a vector space, $\mathcal{G}$ is its open and dense subset, and Σ is the top-stratum of $\mathcal{NG}$. Hence, $\mathcal{NG} \setminus \Sigma$ has codimension strictly greater than 1 and $\mathcal{G} \cup \Sigma$ is path-connected. Passing from one component of $\mathcal{G}$ to another, $\mathcal{D}$ does not change its value, so must $\mathcal{D}$ be identically equal to zero on the entire $\mathcal{G}$. □

It remains to prove the following lemma.

Proof of Lemma 11.5. Suppose γ is an element of $\Sigma^{(G1)}$ and w.l.o.g. the local singularity occurs at $s = 0 \in \mathbb{S}^1$. Let $h : \mathbb{S}^1 \to \mathbb{R}$ be a function which is identically equal to 1 on a small neighborhood around $s = 0$ and identically equal to 0 on the complement of some bigger neighborhood around $s = 0$. Choose a constant $R \in \mathbb{R}$ such that $\gamma'(0) \times (\gamma'''(0) - 3R\gamma''(0)) \neq 0$. That such a choice of R is possible is clear when $\gamma'(0) \times \gamma''(0) \neq 0$. Otherwise, the choice is guaranteed by (10.3). Set $f : (-\varepsilon, \varepsilon) \times \mathbb{S}^1 \to \mathbb{R}$ to be

$$f(a, s) = a(s^2 + Rs^3)h(s).$$

Then h can be adjusted to guarantee that Γ given by (11.4) lies entirely in U for small enough a. Recall the map f_1 defined in (10.1) whose zero set was used to define $\Sigma^{(G1)}$. For Γ to intersect $\Sigma \cap U$ transversely at γ, it must be checked that the Jacobian of the map $(a, s) \mapsto f_1(\Gamma(a), s)$ is invertible at $(a, s) = 0$. Indeed, the Jacobian at $(a, s) = 0$ is

$$\begin{pmatrix} [\gamma', 2v, \gamma'''] + [\gamma', \gamma'', 6Rv] & [\gamma', \gamma'', \gamma^{IV}] \\ * & [\gamma', \gamma''', \gamma^{IV}] + [\gamma', \gamma'', \gamma^{V}] \end{pmatrix} (0).$$

Note, that $[\gamma', \gamma'', \gamma^{IV}](0) = 0$ since the condition $(G1)$ fails and $[\gamma', \gamma''', \gamma^{IV}](0) + [\gamma', \gamma'', \gamma^{V}](0) \neq 0$ by (10.3). Hence, the invertibility is equivalent to

$$[\gamma'(0), \gamma'''(0) - 3R\gamma''(0), v] \neq 0.$$

By the choice of R, for any $v \notin \operatorname{span}\{\gamma'(0), \gamma'''(0) - 3R\gamma''(0)\}$ the path Γ crosses the strata $\Sigma^{(G1)}$ transversely. Moreover, by Lemma 10.10, v can be chosen so that $\Gamma(0)_v$ is FB-generic and hence, v is a regular projection direction for each space curve $\Gamma(a)$ with $a \in (-\varepsilon, \varepsilon)$.

Now, suppose γ is an element of $\Sigma^{(G2)}$ and the global singularity occurs at some distinct pair of points $\{\bar{s}, \bar{t}\} \subset \mathbb{S}^1$. First choose $R \in \mathbb{R}$ such that

$$\gamma'(\bar{s}) \times [\gamma(\bar{s}) - \gamma(\bar{t}) - (1 + R\bar{t})\gamma'(\bar{t})] \neq 0.$$

Such a number exists, since not all three vectors $\gamma'(\bar{s})$, $\gamma'(\bar{t})$ and $\gamma(\bar{s}) - \gamma(\bar{t})$ are parallel. This time let $h : \mathbb{S}^1 \to \mathbb{R}$ be non-zero only in some neighborhood of $\bar{t}$ (not containing $\bar{s}$) and moreover, h is equal to 1 in some even smaller neighborhood of $\bar{t}$. These neighborhoods can be chosen sufficiently small so that Γ defined on $(-\varepsilon, \varepsilon)$ as in (11.4) with

$$f(a, s) = -a(1 + Rs)h(s)$$

lies entirely in U for some small $\varepsilon > 0$. Then, locally around $\bar{s}$ and $\bar{t}$ the map $\Gamma(a)$ is given by

$$\begin{aligned} \Gamma(a)(s) &= \gamma(s) && \text{for } s \text{ sufficiently close to } \bar{s}; \\ \Gamma(a)(t) &= \gamma(t) - a(1 + Rt)v && \text{for } t \text{ sufficiently close to } \bar{t}. \end{aligned}$$

Recall the map f_2 defined in (10.2) which was used to describe $\Sigma^{(G2)}$. Again for Γ to intersect $\Sigma \cap U$ transversely at γ, it must be checked that the Jacobian of the map $(a, s, t) \mapsto f_2(\Gamma(a), s, t)$ is invertible at $(a, s, t) = (0, \bar{s}, \bar{t})$. In fact, the Jacobian is

$$\begin{pmatrix} \begin{array}{l} [\gamma'_{\bar{s}}, -v, \gamma_{\bar{s}} - \gamma_{\bar{t}}] \\ + \quad [\gamma'_{\bar{s}}, \gamma'_{\bar{t}}, -(1 + Rt)v] \end{array} & 0 & 0 \\ [\gamma''_{\bar{s}}, \gamma'_{\bar{t}}, v] & [\gamma''_{\bar{s}}, \gamma'_{\bar{t}}, \gamma'_{\bar{s}}] + [\gamma'''_{\bar{s}}, \gamma'_{\bar{t}}, \gamma_{\bar{s}} - \gamma_{\bar{t}}] & [\gamma''_{\bar{s}}, \gamma''_{\bar{t}}, \gamma_{\bar{s}} - \gamma_{\bar{t}}] \\ [\gamma'_{\bar{s}}, \gamma''_{\bar{t}}, v] & [\gamma''_{\bar{s}}, \gamma''_{\bar{t}}, \gamma_{\bar{s}} - \gamma_{\bar{t}}] & [\gamma'_{\bar{s}}, \gamma'''_{\bar{t}}, \gamma_{\bar{s}} - \gamma_{\bar{t}}] - [\gamma'_{\bar{s}}, \gamma''_{\bar{t}}, \gamma'_{\bar{t}}] \end{pmatrix}.$$

(The convention of the notation is the same as in Section 10.2, e.g., $\gamma'_{\bar{s}}$ means $\gamma'(\bar{s})$ in the above matrix.) Taking into account that $\gamma \in \Sigma^{(G2)}$ satisfies (10.4) as of Lemma 10.8, the condition for transversal intersection reduces to

$$[\gamma'(\bar{s}), v, \gamma(\bar{s}) - \gamma(\bar{t}) - (1 + R\bar{t})\gamma'(\bar{t})] \neq 0.$$

Choose $v \notin \operatorname{span}\{\gamma'(\bar{s}), \gamma'(\bar{t}), \gamma'(\bar{s}) - \gamma(\bar{t})\}$. Then, Γ intersects Σ transversely. Moreover, by Lemma 10.10, v can be chosen to be a regular projection direction of γ. The proof is completed. $\square$

The new proof of Fabicius-Bjerre's Theorem is established.

Chapter 12

Final remarks

This work is finalized with a few remarks.

Differentiability class of the new proof

Halpern [20] assumes for his proof of the Fabricius-Bjerre formula that the plane curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ is four times differentiable. Any of the proofs using the deformation argument within the differentiability class of the curve (i.e., the proof of Banchoff and the proof of the author) indeed requires smoothness. Since at some stage Thom's Transversality Theorem comes into play, the plane curve has to belong to $C^\infty(\mathbb{S}^1, \mathbb{R}^2)$.

The integral version of Fabricius-Bjerre's formula

The methods used by the author in her proof of Fabricius-Bjerre's Theorem led to the integral version of the formula given by Corollary 7.10. The author invested a large amount of time investigating the term

$$\int_{\mathcal{V}} \mathrm{bit}(\gamma_v) d\mathbb{S}^2 \tag{12.1}$$

(with the notation of the corollary) appearing on the left-hand side of the formula (7.12). This is eventually just the integration of the bitangency map over the bitangency surface, additionally taking into account the orientation. The major problem here, however, is that the bitangency surface is not given explicitly. Namely, the bitangency manifold $\mathcal{B}(\gamma)$ of a generic space curve γ, which is a base for the bitangency surface, is defined through the set of singularities of the crossing map.

One of the ideas to tackle this problem was the one from Jason Cantarella that led to the generalization of the Fabricius-Bjerre formula to parallel tangents pairs instead (Theorem 2.3). He hoped that application of the Halpern's method to the more general vector field would give a formula where only one side depends on the angle and its integral over all possible angles could be related to the above term (12.1).

It is still an interesting question for the author whether the term (12.1) can be expressed in terms of some other geometric features. Perhaps a relation can be established in connection with the developable surface made of the bitangent segments.

Can a component of a bitangency surface be a Klein bottle?

In the construction of the bitangency surface it was assumed that a component derived from a component of the bitangency manifold $\mathcal{B}$ homeomorphic to $\mathbb{S}^1$ can be a torus or a Klein bottle. The examples considered by the author all give tori. The question whether it is possible to obtain the Klein bottle is still unanswered.

Projections of a plane curve to all possible lines

Clint McCrory suggested to the author to consider projections of a plane curve to all possible lines through the origin. Indeed there is a connection between the variables occurring in the Fabricius-Bjerre formula and the singularities of the surface constructed as follows. Suppose that a plane curve $\alpha : \mathbb{S}^1 \to \mathbb{R}^2$ is placed in the (y,z)-plane. Define a surface $f : \mathbb{S}^1 \times [0, 2\pi] \to \mathbb{R}^3$ by rotating the plane curve α in the (y,z)-plane and simultaneously moving it in the direction of the x-axis, i.e.,

$$f(s,\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \alpha(s) + (\theta, 0, 0).$$

Project further this surface to the (x,y)-plane (i.e. perpendicular to $v = (0,0,1)$.) Then the obtained surface, say f_v, captures the idea of considering projections of the plane curve to all lines (through the origin). An example of such a surface is shown in Figure 12. There is the following duality between geometric features of the plane curve α and the singularities of "fold lines" of the surface f_v:

- "cusps" of the fold lines correspond to inflection points of the plane curve and
- intersections of the fold lines correspond to the bitangencies of the plane curve.

Double points of the plane curves are not captured by the image under f_v. They are, however, captured as intersection lines in the image of the surface under f.

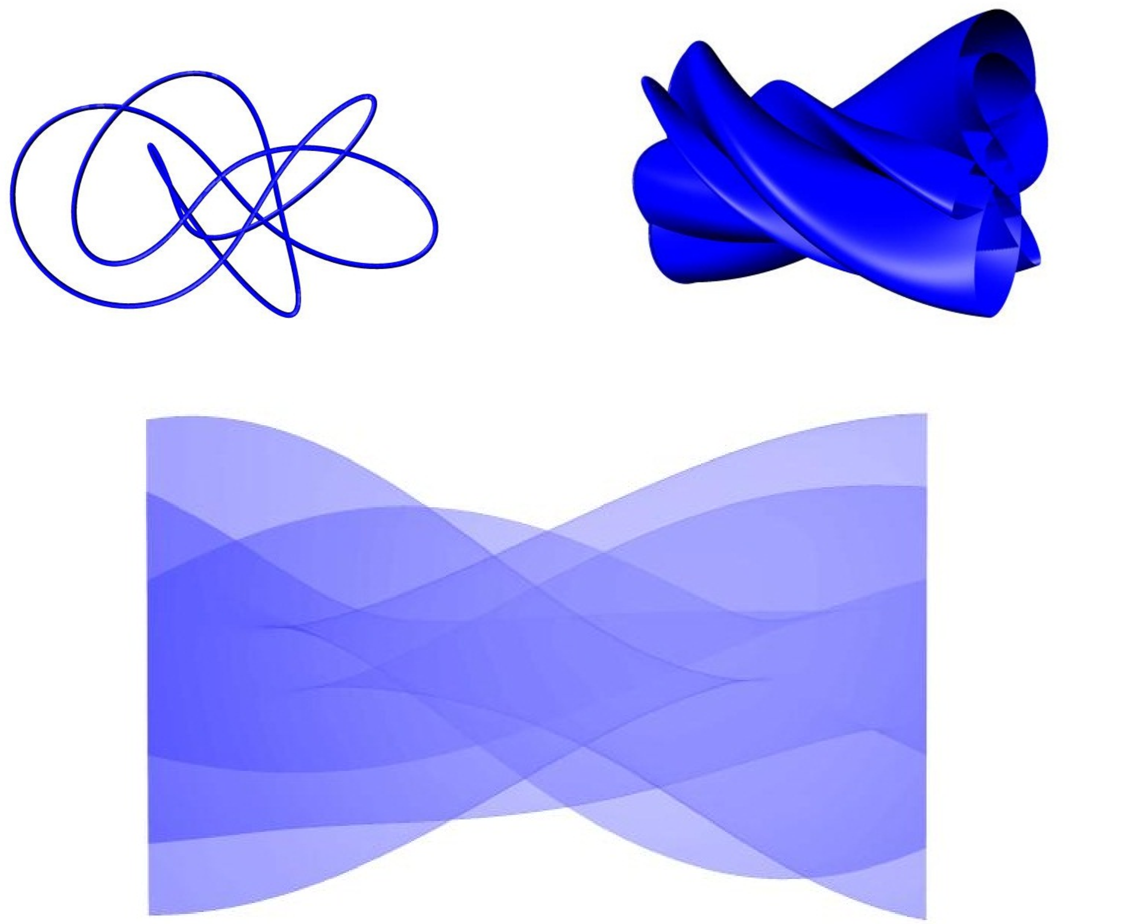


Figure 12.1: An example of a plane curve (top left) and the images of the surface capturing projection of this plane curve to all lines through the origin: the image under f (top right) and the image under f_v (bottom).

The reader is invited to establish a connection between the Fabricius-Bjerre formula and the above duality.

More generalizations of the Fabricius-Bjerre formula

Having the surface (obtained from gluing), with a map to the sphere whose degree is related to the Fabricius-Bjerre formula, now the cusps formula of Fabricius-Bjerre (1.8) can be interpreted. It is a generalization, in which one prescribes how to count preimages of the map at a non-regular projection direction, which is allowed to be a point of the tangent indicatrix or its antipodal or of the set $\mathcal{P}_{\mathrm{osc}}^{\tau=0}$ (i.e., great circles representing osculating planes at torsion vanishing points). More generalizations of the Fabricius-Bjerre formula can be obtained by prescribing how to count singularities of the projected curves for the remaining non-regular projection directions.

More surfaces

Thompson's formula (1.11) or the formula for parallel tangents pairs given by Theorem 2.3 suggest that other surfaces can be constructed that give the formulas in an analogous way. Perhaps, as pointed out by Thomas Banchoff, there exists an entire family of closed surfaces such that the "Fabricius-Bjerre surface" (given by Theorem 7.4) is just a special case of the family and the construction of each element of that family is more intuitive. This way the tedious work of cutting and gluing of the surfaces could be avoided.

Reidemeister curves and the three surfaces generalize to links

Clearly, the approach of the Reidemeister curves and surfaces derived from a single space curve can be extended to links, i.e., to embeddings of several disjoint circles to $\mathbb{R}^3$. This way also Halpern's formula involving a pair of curves (compare Exercise 1.8) can be proved using analogous methods.

Let $\gamma_1, \gamma_2 : \mathbb{S}^1 \to \mathbb{R}^3$ be disjoint curves. The Reidemeister I curve of the pair is just the union of the Reidemeister I curves for each of the two curves separately. Similarly, the inflection surface and the inflection map are just the unions of the corresponding objects for each curve separately. The crossing surface of the pair will consist of the crossing surface for each of the curves, i.e., $\mathcal{C}(\gamma_1)$ and $\mathcal{C}(\gamma_2)$ together with the crossing surface of the pair given by $\mathcal{C}(\gamma_1, \gamma_2) := \mathbb{S}^1 \times \mathbb{S}^1$. The crossing map has to be extended to the new component by $\mathcal{C}(\gamma_1, \gamma_2) \to \mathbb{S}^2$, $(s, t) \mapsto \frac{\gamma_1(s)-\gamma_2(t)}{\|\gamma_1(s)-\gamma_2(t)\|}$. The bitangency manifold (under suitable genericity conditions involving both curves) is again defined as the set of singularities of the crossing map and it consists of the bitangency manifold of each of the curve separately, i.e. $\mathcal{B}(\gamma_1) \cup \mathcal{B}(\gamma_2)$ as well as the set of singularities lying in $\mathcal{C}(\gamma_1, \gamma_2)$ denoted by $\mathcal{B}(\gamma_1, \gamma_2)$. Analogously, the Reidemeister II curve of the pair of curves is the image of the bitangency manifold under the crossing map. The bitangency surface is defined in the same way as in the case of a single curve. It is clear now, that the only new objects are the crossing surface representing non-local double points and components of the bitangency surface representing non-local bitangent pairs (i.e., the points of the pair belong to both of the different curves). To obtain a closed surface from the pieces of surfaces one has to apply Theorem 7.4 for each curve separately and proceed analogously in case of the non-local objects. The Reidemeister III curve of a pair of curves is left as an exercise.

For a link of more than two components the non-local objects have to be defined for each pair of components of the link.

Index of Symbols

J^+, 31
J^-, 31
$[\cdot,\cdot], [\cdot,\cdot,\cdot]$, 6
$\mathcal{P}_{\text{osc}}^{\text{bit}}$, 69
$\mathcal{P}_{\text{osc}}^{\tau=0}$, 59
$\Sigma^{\times}$, 159
Σ, 160
$\Sigma^{\asymp}$, 33
$\Sigma^{\curlywedge}$, 33
Σ^+, 28
Σ^-, 29
Σ^{κ}, 157
Σ^{τ}, 157
Σ^a, 159
Σ^b, 159
$\Sigma^{(G1)}$, 157
$\Sigma^{(\mathrm{G2})}$, 159
Σ^{St}, 29
Σ^{bp}, 33
γ_v, 41
ι, the inflection map, 90
κ_c^{geo}, 41
$\mathcal{K}$, 48
$\mathcal{B}$, $\mathcal{B}(\gamma)$, 51
$\mathcal{C}$, $\mathcal{C}(\gamma)$, 83
$\mathcal{D}$, $\mathcal{D}(\gamma)$, 169
$\mathcal{F}$, $\mathcal{F}(\gamma)$, 120
$\mathcal{G}$, 43
$\mathcal{H}$, $\mathcal{H}(\gamma)$, 99
$\mathcal{I}$, $\mathcal{I}(\gamma)$, 90
$\mathcal{M}$, $\mathcal{M}(\gamma)$, 120
$\mathcal{R}_{III}$, $\mathcal{R}_{III}(\gamma)$, 70
$\mathcal{R}_{II}$, $\mathcal{R}_{II}(\gamma)$, 51
$\mathcal{R}_I$, $\mathcal{R}_I(\gamma)$, 49
$\mathcal{NG}$, 43
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ω_f, 56
St, 31
bit, 124
cr, 6
ext, 7
$\text{ind}_{f,\mathcal{F}}(p)$, 75
infl, 7
int, 7
f, the crossing map, 83
h, the bitangency map, 99

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