Properties of linear Brownian motion with variable drift

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Summary

Path properties of Brownian motion is an important and well studied area of probability theory. In this thesis we want to investigate what happens to path properties of linear Brownian motion if we add a drift function. The path properties we are interested in are the following in this thesis.

It is a well known fact that standard one-dimensional Brownian motion $B(t)$ has no isolated zeros almost surely. In chapter II we will address the question whether one can get isolated zeros by adding a function to Brownian motion. In particular, we will show that for any $\alpha < 1/2$ there are α-Hölder continuous functions f for which the process $B - f$ has isolated zeros with positive probability. En route we will see that the Cantor function added to one-dimensional Brownian motion has zeros in the middle α-Cantor set, $\alpha \in (0, 1)$, with positive probability if and only if $\alpha \neq 1/2$. Besides Cantor functions we also take a look at another class of functions that can provide isolated zeros when added to Brownian motion.

Motivated by the results of chapter II we take an even closer look at middle 1/2-Cantor functions in chapter III. We will define a generalized class of Cantor functions by allowing the middle 1/2 intervals to vary in size around the value 1/2 at each iteration step. That way we can give a refined picture of the result above. We will see that there is a class of generalized Cantor functions such that if these are added to one-dimensional Brownian motion, there are no zeros that lie in the corresponding Cantor set almost surely.

In chapter IV we prove that for any continuous function f, the zero set of $B - f$ has Hausdorff dimension at least 1/2 with positive probability, and 1/2 is an upper bound on the Hausdorff dimension if f is 1/2-Hölder continuous or of bounded variation.

A famous result of Orey and Taylor gives the Hausdorff dimension of the set of fast times, that is the set of points where linear Brownian motion moves faster than according to the law of iterated logarithm. In chapter V we examine what happens to the set of fast times if a variable drift is added to linear Brownian motion. In particular, we will show that the Hausdorff dimension of the set of fast times cannot be decreased by adding a function to Brownian motion. Furthermore, we will look at the intersection of the set of fast times and the zero set.

Chapters II and IV are joint work with Tonci Antunović, Kris Burdzy and Yuval Peres, and were published in the paper [ABPR].

Zusammenfassung

Pfadeigenschaften der Brownschen Bewegung ist ein wichtiges und gut erforschtes Gebiet der Wahrscheinlichkeitstheorie. In dieser Arbeit untersuchen wir wie sich Pfadeigenschaften der linearen Brownschen Bewegung verhalten, wenn wir eine Driftfunktion dazu addieren. Folgende Pfadeigenschaften werden wir in dieser Arbeit betrachten:

Es ist bekannt, dass die eindimensionale Standard-Brownsche Bewegung $B(t)$ fast sicher keine isolierten Nullstellen hat. Im Kapitel II widmen wir uns der Frage, ob man isolierte Nullstellen erhalten kann, wenn man eine Funktion zur Brownschen Bewegung addiert. Insbesondere zeigen wir, dass für jedes $\alpha < 1/2$ es eine α-Hölder stetige Funktion f gibt, sodass der Prozess $B - f$ isolierte Nullstellen mit positiver Wahrscheinlichkeit hat. Dabei werden wir sehen, dass mit positiver Wahrscheinlichkeit die Cantor-Funktion, addiert zur Brownschen Bewegung, Nullstellen hat in der Mittel-α-Cantor-Menge, $\alpha \in (0,1)$, genau dann wenn $\alpha \neq 1/2$. Neben Cantor-Funktionen schauen wir uns auch noch eine weitere Klasse von Funktionen an, die isolierte Nullstellen liefern kann, falls man sie zur Brownschen Bewegung addiert.

Motiviert durch die Ergebnisse vom Kapitel II werden wir uns im Kapitel III die Mittel-1/2-Cantor-Funktion nochmals anschauen. Wir definieren eine verallgemeinerte Klasse von Cantor-Funktionen indem wir zulassen, dass die Längen der Mittel-1/2-Intervalle um den Wert 1/2 schwanken dürfen bei jedem Iterationsschritt. Dadurch erhalten wir ein verfeinertes Bild der obigen Resultate. Wir werden zeigen, dass es eine Klasse von verallgemeinerten Cantor-Funktionen gibt, für die gilt, wenn man sie zur Brownschen Bewegung addiert, liegen fast sicher keine Nullstellen in der Cantor-Menge.

Im Kapitel IV werden wir beweisen, dass für jede stetige Funktion f, die Hausdorff Dimension der Nullstellenmenge von $B - f$ mit positiver Wahrscheinlichkeit mindestens 1/2 ist, und 1/2 ist eine obere Schranke der Hausdorff dimension, falls f 1/2-Hölder-stetig oder von beschränkter Variation ist.

Ein berühmtes Resultat von Orey und Taylor bestimmt die Hausdorff Dimension der Schnellzeitenmenge - das ist die Menge von Punkten, wo die lineare Brownsche Bewegung sich schneller bewegt als bezüglich des Gesetzes des iterierten Logarithmus. Im Kapitel V untersuchen wir die Schnellzeitenmenge von der linearer Brownscher Bewegung mit variablem Drift. Insbesondere zeigen wir, dass die Hausdorff Dimension der Schnellzeitenmenge nicht kleiner werden kann, wenn man eine Funktion zur Brownschen Bewegung addiert. Des Weiteren werden wir uns die Schnittmenge von Schnellzeiten und Nullstellen anschauen.

Kapitel II und IV sind in Zusammenarbeit mit Tonci Antunović, Kris Burdzy and Yuval Peres entstanden, und wurden veröffentlicht in [ABPR].

Preliminaries and Notation

Throughout this thesis we will call B a linear Brownian motion with $B(0) = 0$. We will use common properties of Brownian motion in various places. [MP], [KS91] and [RY] are good sources to learn about them.
One important property of one-dimensional Brownian motion that we will use several times is Lévy's modulus of continuity, that is

$$\lim_{h \downarrow 0} \sup_{0 \le t \le 1-h} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log(1/h)}} = 1,$$

almost surely.
In chapters IV and V an important tool will be Hausdorff dimension. Therefore, we recall the definition at the beginning of chapter IV. For a nice introduction to Hausdorff dimension and its properties we recommend [Fal].
We will repeatedly use the following simple observations.
(i) Suppose that F_k, $k \ge 1$, are events and for some $p > 0$ and all k we have $\mathbb{P}(F_k) \ge p$. Recall that $\limsup_k F_k = \bigcap_{n \ge 1} \bigcup_{k \ge n} F_k$ is the event that infinitely many F_k's occur. Then $\mathbb{P}(\limsup_k F_k) \ge p$.
(ii) As an easy consequence of the Cauchy-Schwarz inequality we have $\mathbb{P}(Z > 0)\mathbb{E}(Z^2) \ge (\mathbb{E}Z)^2$, for any non-negative variable Z. See Lemma 3.23 in [MP]. Often this inequality is referred to as Paley-Zygmund inequality or second moment method.
We list often used notation in the following.

$B(t)$	linear Brownian motion at time t
$B^{(H)}$	fractional Brownian motion with Hurst index $0 < H < 1$
$C(I)$	space of all continuous functions on the interval I
C_γ, C_b	middle $(1 - 2\gamma)$-Cantor set defined in II.3 and generalized Cantor set defined in III.2, respectively
$\mathfrak{C}_{\gamma,n}$, $\mathfrak{C}_{b_n}$	union of all the intervals from $\mathfrak{C}_{\gamma,n}$, and union of all the intervals from $\mathfrak{C}_{b_n}$, respectively
$C_{\gamma,n}$, C_{b_n}	approximating Cantor sets at level n, see definition in II.3 and in III.2, respectively

$\mathbf{D}(I)$	Dirichlet space, also called Cameron-Martin space, defined at the beginning of Chapter I
$\dim(A)$	Hausdorff dimension of the set A
$\mathbb{E}$	expectation
f_γ, f_b	middle $(1-2\gamma)$-Cantor function defined in II.3 and generalized Cantor function defined in III.2, respectively
$f_{\gamma,n}$, f_{b_n}	n-th iteration of the approximating middle $(1-2\gamma)$-Cantor function defined in II.3 and generalized Cantor function defined in III.2, respectively
$\mathrm{Int}(A)$	interior of the set A
$\mathbf{1}(A)$	indicator function on the set A or of the event A, respectively
$\mathbf{L}^2(I)$	is the Banach space of equivalence classes of functions on the interval I with finite $\mathbf{L}^2$-norm
$M_g(t)$	$= \max_{0\le s\le t} g(s)$ for a function g
$\mathrm{Rec}(g)$	record times of g, i.e. $\{t > 0 : M_g(t) = g(t)\}$
$\mathbb{R}^+$	set of non-negative real numbers
$\mathbb{V}$	variance
$\mathcal{Z}(g)$	the set of zeros of g in $(0,\infty)$

Chapter I

Introduction

A lot is known a about path properties of Brownian motion. See e.g. [MP], [KS91], [RY] to name a few sources among a wide range of literature on Brownian motion. In the past few years people considered path properties of Brownian motion with non-Camerion-Martin-drift. According to the Cameron Martin theorem (Theorem 1.38 in [MP], Theorem 2.2 in Chapter 8 in [RY]) a linear Brownian motion B and $B - f$ have the same almost sure path properties if f is a function in the Cameron-Martin space $\mathbf{D}(I)$ (often also called Dirichlet space), where I is a closed interval. That is,

$$\mathbf{D}(I) := \left\{ f \in C(I) : \ \exists \ F \in \mathbf{L}^2(I) \ \text{ s.t. } \ f(t) = \int_0^t F(s)ds, \ \forall \ t \in I \right\}.$$

In general, it is not clear what kind of path properties $B - f$ has if f does not lie in the Cameron-Martin space. This is interesting not only for applications, for example in finance and physics, but also to learn more about Brownian motion itself - to see what happens if we add a variable drift gives us also an idea how strongly path properties are "incorporated" in Brownian motion. On the one hand, we can ask for what kind of drifts does Brownian motion maintain or loose a certain property, and on the other hand which types of drifts makes Brownian motion gain a property that it does not possess. Therefore, investigating for what kind of drifts path properties of Brownian motion change (how "wild" has to be the drift such that path properties change?) is also a way of studying Brownian motion itself. Additionally, for applications it might be desirable to have a simple process as of the form $B - f$ fulfilling a certain property. Hence, as well for a property that Brownian motion does not possess it is worthwhile to investigate for which functions f the process $B - f$ is able to gain this particular property.

A nice list of open problems concerning path properties of Brownian motion with variable drift can be found at the end of the book [MP].

In this thesis properties that we are interested in are, roughly speaking, isolated zeros, the Hausdorff dimension of zeros sets, zeros that are lying in Cantor sets and fast times. In other words, we will try to particularly answer the following questions.

- Can Brownian motion with variable drift have isolated zeros?
- If we add a Cantor function to Brownian motion, can zeros lie in the Cantor set?
- How big is the zero set of Brownian motion with variable drift?

- How big is the set where Brownian motion with variable drift grows exceptionally fast?

We will go into details for each of these in the next chapters. Let us briefly review a few results that have been proven recently about path properties of Brownian motion with variable drift.

I.1 Related work

In this thesis we will later observe that an important rôle is played by functions fulfilling some Hölder continuity property. A connection between Hölder continuity of a drift f and path properties of $B - f$ has already been observed. For $d \geq 2$, a function $f\colon \mathbb{R}^+ \to \mathbb{R}^d$ is called polar if, for d-dimensional Brownian motion B started at the origin and any point $x \in \mathbb{R}^d$, the probability that there is a $t > 0$ such that $B(t) - f(t) = x$ is positive. In [Grav] Graversen constructed α-Hölder continuous functions which are polar for two dimensional Brownian motion, for $\alpha < 1/2$. Le Gall ([LeG88]) showed that 1/2-Hölder continuous functions are not polar for two dimensional Brownian motion and that the same conclusion holds in higher dimensions when f satisfies a slightly stronger condition than $1/d$-Hölder continuity. In a recent paper [APV] it was shown that for any $\alpha < 1/d$ there are α-Hölder continuous functions f such that the image of $B - f$ covers an open set almost surely.

Peres und Sousi [PS10] considered double points (points that are hit by Brownian motion at least twice) of d-dimensional Brownian motion with 1/2-Hölder continuous drift. They proved that for $d \leq 3$ there are double points almost surely and for $d \geq 4$ there almost surely no double points. Furthermore, they showed that for $d \geq 1$, the Hausdorff dimension of the image (graph) of d-dimensional Brownian motion with continuous drift function is at least maximum of the dimension of the image (graph) of Brownian motion and the dimension of the image (graph) of the continuous drift, respectively.

In a later paper [PS11] they obtained an analogous result for the expected volume of the Wiener sausage. Namely, for $d \geq 1$, a collection of open sets $(O_s)_s$ in $\mathbb{R}^d$, and for all $s \geq 0$, let $\mathbf{B}_s$ be a ball centered at 0 such that the volume $\mathbf{B}_s$ equals the volume of O_s, then it holds that the expected volume of the unions over all $s \leq t$ of $B_s + O_s$ is at least as big as the expected volume of the unions over all $s \leq t$ of $B_s + \mathbf{B}_s$, for all t.

Chapter II

Isolated Zeros for Brownian motion with variable drift

This chapter (together with Chapter IV) is joint work with Tonci Antunović, Kris Burdzy and Yuval Peres. Most parts were published in the paper [ABPR] in the Electronic Journal of Probability.

II.1 Introduction

Let B be standard one-dimensional Brownian motion and $f\colon I \to \mathbb{R}$ a continuous function defined on some interval $I \subset \mathbb{R}^+$.

A standard result is that the zero set of B has no isolated points almost surely, see Theorem 2.28 in [MP]. Recall from the introduction, Chapter I, that by the Cameron-Martin theorem the zero set of the process $B - f$ has no isolated points almost surely if f is in the Cameron-Martin space. We will prove in this chapter that the same is true for any function f which is 1/2-Hölder continuous. Since all functions in the Cameron-Martin space are 1/2-Hölder continuous, this is a stronger statement than the one implied by the Cameron-Martin theorem.

For any function g defined on some subset (or the whole) of $\mathbb{R}^+$ denote by $\mathcal{Z}(g)$ the set of zeros of g in $(0, \infty)$. We remove the origin from consideration since the origin is an isolated zero of the process $B - f$ for any f growing fast enough in the neighborhood of the origin, say $f(t) > t^{1/3}$.

Proposition II.1.1. *For $f\colon \mathbb{R}^+ \to \mathbb{R}$ which is 1/2-Hölder continuous on compact intervals, the set $\mathcal{Z}(B - f)$ has no isolated points almost surely.*

We will give a proof and a general condition for zeros to be isolated in the next section. In Proposition II.1.1 the condition that f is 1/2-Hölder continuous is sharp in the following sense.

Theorem II.1.2. *For every $\alpha < 1/2$ there is an α-Hölder continuous function $f\colon \mathbb{R}^+ \to \mathbb{R}$ such that the set $\mathcal{Z}(B - f)$ has isolated points with positive probability.*

Theorem II.1.2 will follow directly from Proposition II.3.12. An example of function f satisfying Theorem II.1.2 is given in Section II.3. For $\gamma < 1/2$, let C_γ denote the middle $(1 - 2\gamma)$-Cantor set and let f_γ be the corresponding Cantor function, shifted to an interval

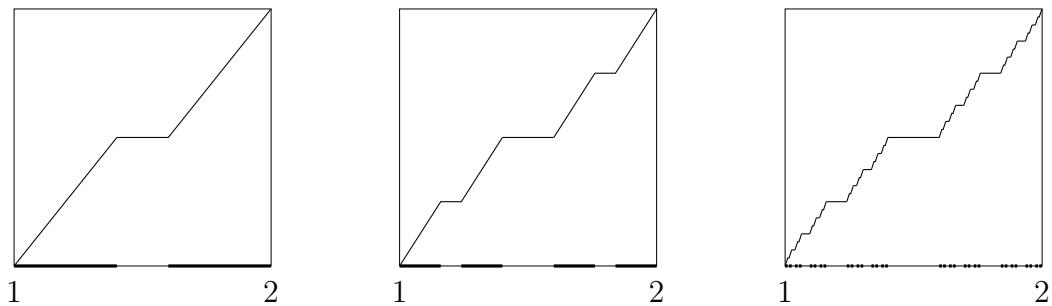

Figure II.1: Approximations of the Cantor function on the interval $[1,2]$ (functions $f_{\gamma,n}$ from the construction in Section II.3) for $\gamma = 0.4$ and $n = 1,2,5$. Approximations of the Cantor set (sets $C_{\gamma,n}$ from the construction in Section II.3) are drawn in bold.

away from the origin; see Figure II.1 and Section II.3 for a precise definition. For $\gamma < 1/4$, the set $\mathcal{Z}(B - f_\gamma)$ has isolated zeros with positive probability, and all such zeros are contained in the Cantor set C_γ. The proof of this claim consists of constructing a subset of C_γ which contains zeros of $B - f_\gamma$ with positive probability, and in which any zero is isolated. En route we obtain the following result of independent interest.

Theorem II.1.3. *Let f_γ be a Cantor function. Then $\mathbb{P}(\mathcal{Z}(B - f_\gamma) \cap C_\gamma \neq \emptyset) > 0$ if and only if $\gamma \neq 1/4$.*

The case $\gamma = 1/4$ of the above theorem has already been resolved by Taylor and Watson (see Example 3 in [TW]). Their interest in the graph of the restriction $f_\gamma|_{C_\gamma}$ stemmed from the fact that, although the projection of this set on the vertical axis is an interval, the graph of Brownian motion does not intersect this set almost surely.

In section II.4 we will discuss some modifications such as the following remark.

Remark II.1.4. *For a function $g\colon \mathbb{R}^+ \to \mathbb{R}$, define $M_g(t) = \max_{0\le s\le t} g(s)$ and denote the set of its record times by $\mathrm{Rec}(g) = \{t > 0 : M_g(t) = g(t)\}$. For standard Brownian motion B, a result of Lévy says that the processes $(M_B(t) - B(t))_t$ and $(|B(t)|)_t$ have the same distribution (see e.g. Theorem 2.34 in [MP]), which implies that sets $\mathcal{Z}(B)$ and $\mathrm{Rec}(B)$ have the same distribution (on the Borel sigma algebra of families of closed subsets of $\mathbb{R}^+$, generated by the Hausdorff metric). In general, for Brownian motion with drift there is no such correspondence. Actually, one can see that there are no isolated points in the set of record times of the process $B - f$ almost surely. This is proven in part (ii) of Proposition II.4.3.*

We will give in Section II.5 a second example of a class of functions such that Brownian motion added to these functions can have isolated zeros. The example will be a distribution function of a measure supported on a set of Hausdorff dimension strictly less than 1.

II.2 General results and proof of Proposition II.1.1

For an interval I we denote its length by $|I|$ and say it is *dyadic* if it is of the form $I = [k2^m, (k+1)2^m]$ for integers $k > 0$ and m.

For intervals I and J (that intersect in at most one point), we will write $I < J$ if J is located to the right of I.

Proposition II.2.1. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function.*

(i) Let $A \subset \mathbb{R}^+$ be a set such that for any $t \in A$ there is an $\alpha < 1/2$ such that $\liminf_{s\to t} \frac{|f(s)-f(t)|}{|t-s|^\alpha} > 0$. Then, almost surely any point in $\mathcal{Z}(B-f) \cap A$ is isolated in $\mathcal{Z}(B-f)$.

(ii) Almost surely all isolated points of $\mathcal{Z}(B-f)$ are located inside the set $A_f^+ \cup A_f^-$, where $A_f^+ = \{t \in \mathbb{R}^+ : \lim_{h\downarrow 0} \frac{f(t+h)-f(t)}{\sqrt{h}} = \infty\}$ and $A_f^- = \{t \in \mathbb{R}^+ : \lim_{h\downarrow 0} \frac{f(t+h)-f(t)}{\sqrt{h}} = -\infty\}$.

Proof. (i) First assume that the set A is contained in $[0, N]$ for some large N. If a zero $s \in A \cap \mathcal{Z}(B-f)$ is not isolated, then we can find a sequence $(s_n) \subset \mathcal{Z}(B-f)$, converging to s, which, for some $\alpha < 1/2$, necessarily satisfies

$$\liminf_{n\to\infty} \frac{|f(s_n) - f(s)|}{|s_n - s|^\alpha} > 0 \quad \text{and} \quad \liminf_{n\to\infty} \frac{|B(s_n) - B(s)|}{|s_n - s|^\alpha} > 0.$$

However, this is impossible since, by Levy's modulus of continuity, almost surely, there exists a (random) $h_1 > 0$ such that for all $h \in (0, h_1)$ and all $0 \le t \le N$ we have $|B(t+h) - B(t)| \le 3\sqrt{h \log(1/h)}$, see e. g. Theorem 1.14 in [MP]. If A is unbounded, apply the above reasoning to $A_N = A \cap [0, N]$ and let N go to infinity.

(ii) First define $\tau_q = \min\{t \ge q : B(t) = f(t)\}$ and notice that any isolated zero of the process $B - f$ must equal τ_q, for some $q \in \mathbb{Q}$. This is because, for any zero $s \in \mathcal{Z}(B-f)$, not of the form τ_q, and a sequence of rational numbers (q_n) converging to s from below, we have $\lim_n \tau_{q_n} = s$. Therefore, it is enough to prove that for each $q \in \mathbb{Q}^+$, the event that $\tau_q \notin A_f^+ \cup A_f^-$ and that τ_q is isolated in the set $\mathcal{Z}(B-f)$, has probability zero.

Fix a positive integer M and define sequences of functions

$$s_n^-(t) = \max\{0 \le h \le 1/n : f(t+h) - f(t) \le M\sqrt{h}\}$$

and

$$s_n^+(t) = \max\{0 \le h \le 1/n : f(t+h) - f(t) \ge -M\sqrt{h}\}.$$

Since f is continuous, it is easy to see that for each n, the functions s_n^+ and s_n^- are measurable. Also define

$$\overline{A}_f(M) = \Big\{t : \liminf_{h\downarrow 0} \frac{f(t+h) - f(t)}{\sqrt{h}} < M, \limsup_{h\downarrow 0} \frac{f(t+h) - f(t)}{\sqrt{h}} > -M\Big\}.$$

For all $t \in \overline{A}_f(M)$ it holds that $s_n^+(t) > 0$ and $s_n^-(t) > 0$, for all n. Since τ_q is a stopping time, the process $B_q(t) = B(\tau_q + t) - B(\tau_q)$, is, by the strong Markov property, a Brownian

motion independent of the sigma algebra $\mathcal{F}_{\tau_q}$. Let F_- denote the event that $B_q(s_n^-(\tau_q)) \geq M\sqrt{s_n^-(\tau_q)}$ happens for infinitely many n's. Since the random variables $s_n^-(\tau_q)$ are measurable with respect to $\mathcal{F}_{\tau_q}$, Blumenthal's 0-1 law implies that $\mathbb{P}(F_- \mid \mathcal{F}_{\tau_q})$ is equal to 0 or 1 on the event $\{\tau_q \in \overline{A}_f(M)\}$. On the event $\{\tau_q \in \overline{A}_f(M)\}$, for every n, we have

$$\mathbb{P}\Big(B_q(s_n^-(\tau_q)) \geq M\sqrt{s_n^-(\tau_q)} \mid \mathcal{F}_{\tau_q}\Big) = \mathbb{P}(B_q(1) \geq M \mid \mathcal{F}_{\tau_q}) > 0.$$

Since the right hand side does not depend on n, by observation (i) in our Preliminaries, $\mathbb{P}(F_- \mid \mathcal{F}_{\tau_q}) = 1$ on the event $\{\tau_q \in \overline{A}_f(M)\}$. Similarly, if F_+ denotes the event that $B_q(s_n^+(\tau_q)) \leq -M\sqrt{s_n^+(\tau_q)}$ happens for infinitely many n's then $\mathbb{P}(F_+ \mid \mathcal{F}_{\tau_q}) = 1$ on the event $\{\tau_q \in \overline{A}_f(M)\}$. By the definition of the sequences $(s_n^-(t))$ and $(s_n^+(t))$, if $F_- \cup F_+$ holds then τ_q is not an isolated zero from the right. Therefore, the probability that $\tau_q \in \overline{A}_f(M)$ and that τ_q is an isolated point of $\mathcal{Z}(B-f)$ is equal to zero. Taking the union over all rational q's and observing that $(A_f^- \cup A_f^+)^c = \bigcup_{M=1}^{\infty} \overline{A}_f(M)$ proves the claim. □

Remark II.2.2. *By Proposition II.2.1 (ii) and the time reversal property of Brownian motion (i.e. for an $s \in \mathbb{R}^+$ the process $\hat{B}(t) := B(s-t) - B(s)$ is again a Brownian motion with $t \in [0,s]$) it follows that, almost surely all isolated points of $\mathcal{Z}(B-f)$ are contained in the set*

$$\Big\{t \in \mathbb{R}^+ : \lim_{h \uparrow 0} \frac{f(t+h)-f(t)}{\sqrt{|h|}} = \infty\Big\} \cup \Big\{t \in \mathbb{R}^+ : \lim_{h \uparrow 0} \frac{f(t+h)-f(t)}{\sqrt{|h|}} = -\infty\Big\}.$$

Every point t for which the limits

$$\lim_{h \downarrow 0} \frac{f(t+h)-f(t)}{\sqrt{h}} = \lim_{h \uparrow 0} \frac{f(t+h)-f(t)}{\sqrt{|h|}}$$

are equal to ∞ or $-\infty$ is a strict local minimum or maximum. Since a function can have only countably many strict local extrema, almost surely all isolated points of $\mathcal{Z}(B-f)$ are contained in the set

$$\Big\{t \in \mathbb{R}^+ : \lim_{h \to 0} \frac{f(t+h)-f(t)}{\sqrt{|h|}\,\mathrm{sign}(h)} = \infty\Big\} \cup \Big\{t \in \mathbb{R}^+ : \lim_{h \to 0} \frac{f(t+h)-f(t)}{\sqrt{|h|}\,\mathrm{sign}(h)} = -\infty\Big\}.$$

In particular, all isolated points of $\mathcal{Z}(B-f)$ are contained in the set of points of increase or decrease of f (recall that t is a point of increase of f if for some $\epsilon > 0$ we have $f(s) < f(t)$ for $t-\epsilon < s < t$ and $f(t) < f(s)$ for $t < s < t+\epsilon$ and points of decrease are defined analogously). Therefore, if f is a function with at most countably many points of increase or decrease, then $\mathcal{Z}(B-f)$ has no isolated points almost surely. Examples of such functions include functions constructed by Loud in [Loud] which satisfy a certain local reverse Hölder property at each point (see also the construction in [MP53]). These functions are defined as $g(t) = \sum_{k=1}^{\infty} g_k(t)$ where $g_k(t) = 2^{-2A\alpha k} g_0(2^{2Ak}t)$, for $0 < \alpha < 1$, a positive integer A such that $2A(1-\alpha) > 1$, and a continuous function g_0 which has value 0 at even integers, value 1 at odd integers and is linear at all other points. To prove that these functions have at most countably many points of increase or decrease, we proceed analogously as in the proof of the lower bound in [Loud]. We will take $t \notin \mathbb{Q}$ and show that t is not a point of increase. First

observe that $\lfloor 2^{2Am}t \rfloor$ is odd for infinitely many integers m. For such an integer m assume that $g_m(t) \geq 2^{-2A\alpha m-1}$ and denote $t_m = t + 2^{-2A(m+1)}$. Then by construction $g_k(t) = g_k(t_m)$ for all $k > m$, $g_m(t_m) = g_m(t) - 2^{-2A(\alpha m+1)}$ and $|g_k(t_m) - g_k(t)| \leq 2^{2Ak(1-\alpha)-2A(m+1)}$, for $k < m$. Therefore

$$\begin{aligned} g(t_m) - g(t) &\leq -2^{-2A(\alpha m+1)} + 2^{-2A(m+1)} \sum_{k=1}^{m-1} 2^{2Ak(1-\alpha)} \\ &= \frac{2^{-2A(\alpha m+1)+1} - 2^{-2A\alpha(m+1)} - 2^{-2A(m+\alpha)}}{2^{2A(1-\alpha)} - 1}. \end{aligned}$$

Using the fact that $2A(1-\alpha) > 1$, it is easy to check that the right hand side above is negative, and since $t_m > t$ can be arbitrarily close to t the claim follows. If $g_m(t) < 2^{-2A\alpha m-1}$ then define $t_m = t - 2^{-2A(m+1)}$, which now satisfies $g_m(t_m) = g_m(t) + 2^{-2A(\alpha m+1)}$ and proceed analogously to prove that $g(t_m) > g(t)$.

Proof of Proposition II.1.1. This is straightforward from part (ii) of Proposition II.2.1. □

II.3 First example and proofs of Theorems II.1.2 and II.1.3

For $0 < \gamma < 1/2$, we will define the middle $(1-2\gamma)$-Cantor set and denote it by C_γ. Take a compact interval I of length $|I|$. Define $\mathfrak{C}_{\gamma,1}$ as the set consisting of two disjoint closed subintervals of I of length $\gamma|I|$, the left one (for which the left endpoint coincides with the left endpoint of I) and the right one (for which the right endpoint coincides with the right endpoint of I). Continue recursively, if $J \in \mathfrak{C}_{\gamma,n}$, then include in the set $\mathfrak{C}_{\gamma,n+1}$ its left and right closed subintervals of length $\gamma^{n+1}|I|$. Define the set $C_{\gamma,n}$ as the union of all the intervals from $\mathfrak{C}_{\gamma,n}$. For any n, the family $\mathfrak{C}_{\gamma,n}$ is the set of all connected components of the set $C_{\gamma,n}$. The Cantor set is a compact set defined as $C_\gamma = \bigcap_{n=1}^{\infty} C_{\gamma,n}$. It is easy to show that $\dim(C_\gamma) = \log 2/\log(1/\gamma)$.

Now we recall the construction of the standard Cantor function. Define the function $f_{\gamma,1}$ so that it has values 0 and 1 at the left and the right endpoint of the interval I, respectively, value $1/2$ on $I \backslash C_{\gamma,1}$ and interpolate linearly on the intervals in $\mathfrak{C}_{\gamma,1}$. Recursively, construct the function $f_{\gamma,n+1}$ so that for every interval $J = [s,t] \in \mathfrak{C}_{\gamma,n}$, the function $f_{\gamma,n+1}$ agrees with $f_{\gamma,n}$ at s and t, it has value $(f_{\gamma,n}(s) + f_{\gamma,n}(t))/2$ on $J \backslash C_{\gamma,n+1}$ and interpolate linearly on the intervals in $\mathfrak{C}_{\gamma,n+1}$. See Figure II.1. It is easy to see that the sequence of functions $(f_{\gamma,n})$ converges uniformly on I. We define the Cantor function f_γ as the limit $f_\gamma = \lim_n f_{\gamma,n}$. Note that for any n and all $m \leq n$ the functions f_γ and $f_{\gamma,n}$ agree at the endpoints of intervals $J \in \mathfrak{C}_{\gamma,m}$.

Another way to characterize the Cantor set C_γ and the Cantor function f_γ is by representing it as fixed points of certain transformations. Define linear bijections $g_\gamma \colon [0,\gamma] \to [0,1]$ and $h_\gamma \colon [1-\gamma, 1] \to [0,1]$ by $g_\gamma(t) = t/\gamma$ and $h_\gamma(t) = (t-1+\gamma)/\gamma$. The Cantor set C_γ defined on $[0,1]$ is the unique nonempty compact set that satisfies $C_\gamma = g_\gamma^{-1}(C_\gamma) \cup h_\gamma^{-1}(C_\gamma)$, and the corresponding Cantor function is the unique continuous function that satisfies

$$f_\gamma(t) = \begin{cases} f_\gamma(g_\gamma(t))/2, & 0 \leq t \leq \gamma, \\ 1/2, & \gamma \leq t \leq 1-\gamma, \\ 1/2 + f_\gamma(h_\gamma(t))/2, & 1-\gamma \leq t \leq 1. \end{cases} \tag{II.3.1}$$

It is not hard to see that f_γ is $\log 2/\log(1/\gamma)$-Hölder continuous. The value $\gamma = 1/4$ is the threshold at which the functions f_γ become 1/2-Hölder continuous. For $\gamma < 1/4$ the function $f = f_\gamma$ will give an example for Theorem II.1.2. This threshold is sharp, since by Proposition II.1.1, for $\gamma \geq 1/4$ there are no isolated points in the zero set $\mathcal{Z}(B - f_\gamma)$ almost surely. For simplicity, we will assume the initial interval I to be $[1, 2]$, but we note that the analysis works for all compact intervals. To satisfy the assumptions of Theorem II.1.2, the function f_γ should of course be extended to $\mathbb{R}^+\backslash[1, 2]$, by, say, value 0 on $[0, 1)$ and value 1 on $(2, \infty)$.

Proof of Theorem II.1.3. For an interval $I = [r, s] \in \mathfrak{C}_{\gamma,n}$, define $Z_n(I)$ as the event $B(s) \in [f_\gamma(r), f_\gamma(s)]$, and the random variable $Z_{\gamma,n} = \sum_{I\in\mathfrak{C}_{\gamma,n}} \mathbf{1}(Z_n(I))$, where $\mathbf{1}(Z_n(I))$ is the indicator function of the event $Z_n(I)$. Note that there is a constant $c_1 > 0$, such that for any $0 < \gamma < 1/2$ we have

$$c_1 2^{-n} \leq \mathbb{P}(Z_n(I)) \leq 2^{-n} \quad \text{and} \quad c_1 \leq \mathbb{E}(Z_{\gamma,n}) \leq 1. \tag{II.3.2}$$

If $Z_{\gamma,n} > 0$ happens for infinitely many n's, then we can find a sequence of intervals $I_k = [r_k, s_k] \in \mathfrak{C}_{\gamma,n_k}$, such that $f_\gamma(r_k) \leq B(s_k) \leq f_\gamma(s_k)$, thus $|B(s_k) - f_\gamma(s_k)| \leq 2^{-n_k}$. Since $s_k \in C_{\gamma,n_k}$, the sequence (s_k) will have a subsequence converging to some $s \in C_\gamma$, which obviously satisfies $B(s) = f_\gamma(s)$. Therefore

$$\mathbb{P}(\mathcal{Z}(B - f_\gamma) \cap C_\gamma \neq \emptyset) \geq \mathbb{P}(\limsup_{n\to\infty}\{Z_{\gamma,n} > 0\}). \tag{II.3.3}$$

To estimate the probabilities $\mathbb{P}(Z_{\gamma,n} > 0)$ from below we will use the Paley-Zygmund inequality, for which we need to bound the second moment $\mathbb{E}(Z^2_{\gamma,n})$ from above. First we express it as

$$\mathbb{E}(Z^2_{\gamma,n}) = 2 \sum_{\substack{I,J\in\mathfrak{C}_{\gamma,n}\\ I<J}} \mathbb{P}(Z_n(I))\mathbb{P}(Z_n(J) \mid Z_n(I)) + \mathbb{E}(Z_{\gamma,n}). \tag{II.3.4}$$

Now fix n and intervals $I = [s_1, t_1]$ and $J = [s_2, t_2]$ from $\mathfrak{C}_{\gamma,n}$, so that $I < J$, and denote $a_i = f_\gamma(s_i)$ and $b_i = f_\gamma(t_i)$, for $i = 1, 2$. Let $\widetilde{B}$ be the process

$$\widetilde{B}(t) = (t_2 - t_1)^{-1/2}\Big(B(t_1 + (t_2 - t_1)t) - B(t_1)\Big),$$

which is, by the Markov property and Brownian scaling, again a Brownian motion, independent of $\mathcal{F}_{t_1}$, and thus independent of the event $Z_n(I) \in \mathcal{F}_{t_1}$. The event $Z_n(J)$ happens when $\widetilde{B}(1) \in \overline{J}$ for the interval $\overline{J} = [(t_2 - t_1)^{-1/2}(a_2 - B(t_1)), (t_2 - t_1)^{-1/2}(b_2 - B(t_1))]$ of length $(t_2 - t_1)^{-1/2}2^{-n}$.

Case $\gamma < 1/4$: Fix intervals I^0 and J^0 in $\mathfrak{C}_{\gamma,\ell+1}$, which are contained in a single interval in $\mathfrak{C}_{\gamma,\ell}$. Label the intervals from $\mathfrak{C}_{\gamma,n}$ contained in I^0 by $I_1, \ldots, I_{2^{n-\ell-1}}$, and those contained in J^0 by $J_1, \ldots, J_{2^{n-\ell-1}}$, so that $I_{i+1} < I_i$ and $J_j < J_{j+1}$. Set $I = I_i$ and $J = J_j$ for some $1 \leq i, j \leq 2^{n-\ell-1}$, and define a_i, b_i, $\widetilde{B}$ and $\overline{J}$ as before. Conditional on $Z_n(I)$, the left endpoint of the interval $\overline{J}$ is at least $(a_2 - b_1)(t_2 - t_1)^{-1/2}$, and since $a_2 - b_1 = (i + j - 2)2^{-n}$

we have $\overline{J} \subset [(i+j-2)2^{-n}\gamma^{-\ell/2}, \infty)$. Because $\overline{J}$ has length at most $(1-2\gamma)^{-1/2}2^{-n}\gamma^{-\ell/2}$ we obtain

$$\begin{aligned}\mathbb{P}(Z_n(J_j) \mid Z_n(I_i)) &= \mathbb{P}(\widetilde{B}(1) \in \overline{J} \mid Z_n(I)) \\ &\le \frac{2^{-n}\gamma^{-\ell/2}}{\sqrt{2\pi(1-2\gamma)}} \exp\Big(-\frac{((j+i-2)2^{-n}\gamma^{-\ell/2})^2}{2}\Big),\end{aligned}$$

which, by summing over $1 \le i, j \le 2^{n-\ell-1}$ gives

$$\begin{aligned}&\sum_{1\le i,j\le 2^{n-\ell-1}} \mathbb{P}(Z_n(J_j) \mid Z_n(I_i)) \\ &\qquad \le 1 + \frac{1}{\sqrt{2\pi(1-2\gamma)}} \sum_{k=1}^{\infty}(k+1)2^{-n}\gamma^{-\ell/2} \exp\Big(-\frac{(k2^{-n}\gamma^{-\ell/2})^2}{2}\Big). \qquad \text{(II.3.5)}\end{aligned}$$

Here we used the trivial bound for $i = j = 1$. The sum on the right hand side can be written as

$$S(a) = a\sum_{k=1}^{\infty}(k+1)\exp(-(ka)^2/2),$$

for $a = 2^{-n}\gamma^{-\ell/2}$. Since $\exp(-t^2/2) \le \exp(-t+1/2)$, we see that

$$S(a) \le e^{1/2}a\sum_{k\ge 1}(k+1)e^{-ka} = e^{1/2}a^{-1}\Big(\frac{a}{1-e^{-a}}\Big)^2(2e^{-a}-e^{-2a}).$$

Since $a \mapsto (\frac{a}{1-e^{-a}})^2(2e^{-a}-e^{-2a})$ is a bounded function on $\mathbb{R}^+$, (II.3.5) implies that for any fixed I^0 and J^0 as above

$$\sum_{\substack{I,J\in\mathfrak{C}_{\gamma,n}\\ I\subset I^0, J\subset J^0}} \mathbb{P}(Z_n(J) \mid Z_n(I)) \le 1 + c_2 2^n\gamma^{\ell/2}, \qquad \text{(II.3.6)}$$

for some $c_2 > 0$. Therefore, summing the inequality in (II.3.6) over all I^0 and J^0 and $\ell = 0, \dots, n-1$, and using it together with (II.3.4) and (II.3.2), we have

$$\mathbb{E}(Z_{\gamma,n}^2) \le 2^{-n+1}\sum_{\ell=0}^{n-1}\Big(2^\ell + 2^\ell c_2 2^n\gamma^{\ell/2}\Big) + 1 \le 2\sum_{\ell=0}^{n}\Big(2^{-(n-\ell)} + c_2(2\sqrt{\gamma})^\ell\Big),$$

which is bounded since $2\sqrt{\gamma} < 1$.

Thus we have shown that, for a fixed $\gamma < 1/4$, the second moments $\mathbb{E}(Z_{\gamma,n}^2)$ are bounded from above. Now the lower bound in the second inequality in (II.3.2) and the Paley-Zygmund inequality imply that $\mathbb{P}(Z_{\gamma,n} > 0) \ge \mathbb{E}(Z_{\gamma,n})^2/\mathbb{E}(Z_{\gamma,n}^2)$ is bounded from below and the claim follows from (II.3.3) and our observation (i) of the Preliminaries.

Case $\gamma > 1/4$: Again pick $I, J \in \mathfrak{C}_{\gamma,n}$ such that $[s_1, t_1] = I < J = [s_2, t_2]$ and define a_i, b_i, $\widetilde{B}$ and $\overline{J}$ as before. By ℓ denote the largest integer such that both I and J are contained in a single interval from $\mathfrak{C}_{\gamma,\ell}$. Assume that $Z_n(I)$ happens. Clearly the endpoints of the interval $\overline{J}$ satisfy

$$\frac{a_2 - B(t_1)}{(t_2-t_1)^{1/2}} \ge 0 \quad \text{and} \quad \frac{b_2 - B(t_1)}{(t_2-t_1)^{1/2}} \le \frac{1}{(1-2\gamma)^{1/2}(2\sqrt{\gamma})^\ell}.$$

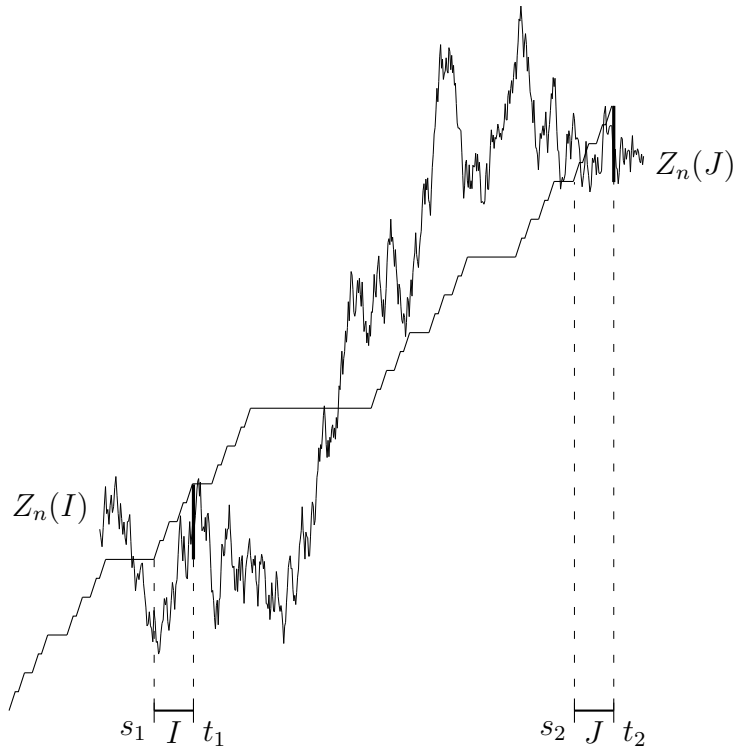

Figure II.2: Events $Z_n(I)$ and $Z_n(J)$ (graph of Brownian motion intersects two bold vertical intervals).

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The sequence $((2\sqrt{\gamma})^{-\ell})$ is bounded, and therefore the interval $\overline{J}$ is contained in a compact interval, which does not depend on the choice of n, ℓ, I or J. Using this and the fact that the length of $\overline{J}$ is bounded from above by $(1-2\gamma)^{-1/2}2^{-n}\gamma^{-\ell/2}$ and from below by $c'2^{-n}\gamma^{-\ell/2}$, we get that for some positive constants c_3 and c_4 we have

$$c_3 2^{-n}\gamma^{-\ell/2} \leq \mathbb{P}(Z_n(J) \mid Z_n(I)) = \mathbb{P}(\widetilde{B}(1) \in \overline{J} \mid Z_n(I)) \leq c_4 2^{-n}\gamma^{-\ell/2}. \tag{II.3.7}$$

Note that, since the sequence $((2\sqrt{\gamma})^{-\ell})$ is bounded for $\gamma = 1/4$, estimates (II.3.7) also hold for $\gamma = 1/4$. Substituting (II.3.7) and the upper bounds from (II.3.2) into (II.3.4), and summing over all intervals I and J, we obtain

$$\mathbb{E}(Z_{\gamma,n}^2) \leq 1 + 2^{-n+1}\sum_{\ell=0}^{n-1} 2^{\ell}2^{2(n-\ell-1)}c_4 2^{-n}\gamma^{-\ell/2} = 1 + \frac{c_4}{2}\sum_{\ell=0}^{n-1}(2\sqrt{\gamma})^{-\ell}.$$

Since $2\sqrt{\gamma} > 1$, we have bounded $\mathbb{E}(Z_{\gamma,n}^2)$ from above by a constant not depending on n, and the claim follows as in the case $\gamma < 1/4$.

Case $\gamma = 1/4$: Assume that $\mathcal{Z}(B - f_\gamma) \cap C_\gamma \neq \emptyset$ and define τ as the first zero of $B - f_\gamma$ in the Cantor set C_γ (τ exists since $\mathcal{Z}(B - f_\gamma) \cap C_\gamma$ is a closed set). For an interval $I = [s, t] \in \mathfrak{C}_{\gamma,n}$ assume that $\tau \in I$. Since τ is a stopping time, and by Brownian scaling, the conditional probability $\mathbb{P}\Big(Z_n(I) \mid \mathcal{F}_\tau, \tau \in I\Big)$ is equal to the probability that Brownian motion at time 1 is between $y_1 = (f_\gamma(s) - f_\gamma(\tau))(t - \tau)^{-1/2}$ and $y_2 = (f_\gamma(t) - f_\gamma(\tau))(t - \tau)^{-1/2}$. Since $f_\gamma(s) \leq f_\gamma(\tau) \leq f_\gamma(t)$ we see that $y_1 \leq 0$ and $y_2 \geq 0$. Moreover, the assumption $\gamma = 1/4$ implies that $t - \tau \leq 4^{-n} = (f_\gamma(t) - f_\gamma(s))^2$ which leads to $y_2 - y_1 \geq 1$. Thus we can bound the probability

$$\mathbb{P}\Big(Z_n(I) \mid \mathcal{F}_\tau, \tau \in I\Big) \geq \mathbb{P}(0 \leq B(1) \leq 1) = K^{-1}, \tag{II.3.8}$$

for some $K > 0$, see also Figure II.3.

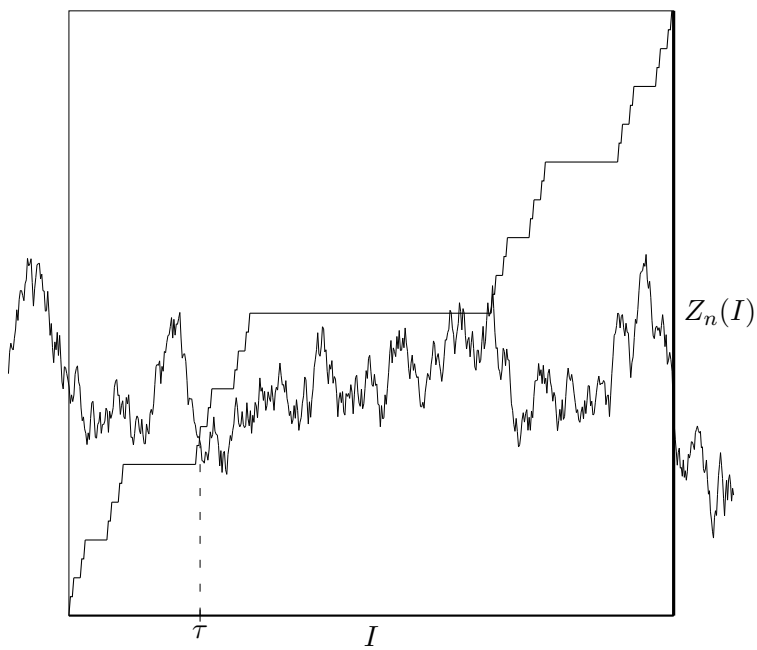

Figure II.3: If $\gamma \leq 1/4$, conditional on the event that there is a zero of $B - f_\gamma$ in an interval $I \in \mathfrak{C}_{\gamma,n}$ (τ is the first such zero) the probability of the event $Z_n(I)$ (Brownian motion intersecting the right hand side of the rectangle) is bounded from below.

Therefore, $\mathbb{P}(Z_{\gamma,n} > 0 \mid \tau \in I) \geq K^{-1}$ and, since the events $\{\tau \in I\}$ are disjoint for different $I \in \mathfrak{C}_{\gamma,n}$, we have $\mathbb{P}(Z_{\gamma,n} > 0 \mid \mathcal{Z}(B - f) \cap C_\gamma \neq \emptyset) \geq K^{-1}$ for every n. Thus we obtain

$$\mathbb{P}(\mathcal{Z}(B - f_\gamma) \cap C_\gamma \neq \emptyset) \leq K \inf_n \mathbb{P}(Z_{\gamma,n} > 0). \tag{II.3.9}$$

The arguments leading to (II.3.8) are true for all $\gamma \leq 1/4$ and therefore (II.3.9) holds for $\gamma \leq 1/4$.

For $I = [s,t] \in \mathfrak{C}_{\gamma,n}$ and $0 \le \ell < n$, let I^ℓ denote the interval from $\mathfrak{C}_{\gamma,\ell}$ that contains I, and let $I_1^\ell, I_2^\ell \in \mathfrak{C}_{\gamma,\ell+1}$ be the left and right subintervals of I^ℓ, respectively. Let $s = 1.a_1a_2\dots a_n$ denote the 4-ary expansion of the left endpoint of I (note that the 4-ary expansion of s contains only the digits 0 and 3 and has length at most n, here we add zeros at the end, if necessary, to make it of length n). It is easy to see that $I \subset I_1^\ell$ if $a_{\ell+1} = 0$ and $I \subset I_2^\ell$ if $a_{\ell+1} = 3$. Call an interval I *balanced* if the sequence $a_1,\dots,a_n$ contains at least $n/3$ zeros and otherwise unbalanced. For a balanced interval $I = [s,t] \in \mathfrak{C}_{\gamma,n}$ let A_I denote the event that I is the leftmost balanced interval for which $Z_n(I)$ happens, and, as before, let $1.a_1\dots a_n$ denote the 4-ary expansion of s. Assume that for some $0 \le \ell < n$ we have $a_{\ell+1} = 0$ and pick an interval $J \in \mathfrak{C}_{\gamma,n}$ such that $J \subset I_2^\ell$. Since $A_I \in \mathcal{F}_{t_1}$ we can use the same arguments that lead to the lower bound in (II.3.7) to conclude that the probability $\mathbb{P}(Z_n(J) \mid A_I) \ge c_5 2^{-(n-\ell)}$, for some constant $c_5 > 0$. Summing over all $J \subset I_2^\ell$ and over all ℓ such that $a_{\ell+1} = 0$ gives

$$\mathbb{E}(Z_{\gamma,n} \mid A_I) \ge c_5|\{1 \le \ell \le n : a_\ell = 0\}| \ge c_5 n/3.$$

By (II.3.2) and since events A_I are disjoint, we have

$$\mathbb{P}(Z_n(I) \text{ for some balanced interval } I) \le \frac{\mathbb{E}(Z_{\gamma,n})}{\mathbb{E}(Z_{\gamma,n} \mid Z_n(I) \text{ for some balanced interval } I)} \le \frac{3}{c_5 n}. \quad \text{(II.3.10)}$$

To estimate the probability that $Z_n(I)$ happens for some unbalanced interval I notice that the number of such intervals is bounded from above by $e^{-c_6 n}2^n$ for some $c_6 > 0$. By (II.3.2) this gives

$$\mathbb{P}(Z_n(I) \text{ for some unbalanced interval } I) \le e^{-c_6 n}. \quad \text{(II.3.11)}$$

Now (II.3.10) and (II.3.11) yield $\lim_{n\to\infty} \mathbb{P}(Z_{\gamma,n} > 0) = 0$ and the claim follows from (II.3.9). □

Since the Cantor function f_γ is $\log 2/\log(1/\gamma)$-Hölder continuous, the following proposition proves Theorem II.1.2.

Proposition II.3.12. *For $\gamma < 1/4$ consider the Cantor function f_γ. Then the set $\mathcal{Z}(B - f_\gamma)$ has isolated points with positive probability.*

Proof. For $A \subset \mathbb{R}^+$ define $Z(A)$ as the event $\{\mathcal{Z}(B - f_\gamma) \cap A \ne \emptyset\}$. We claim that there exists a constant c_1, such that for any interval $J \subset [0,1]$ of length $|J|$, we have

$$\mathbb{P}(Z(C_\gamma \cap f_\gamma^{-1}(J))) \le c_1|J|. \quad \text{(II.3.13)}$$

To prove this, fix an interval J and take the largest integer n such that $|J| \le 2^{-n}$. Notice that J can be covered by no more than two consecutive binary intervals J_1 and J_2 of length 2^{-n}. Moreover, there are consecutive $I_1, I_2 \in \mathfrak{C}_{\gamma,n}$ such that $f_\gamma(I_i) = J_i$ for $i = 1,2$, and $C_\gamma \cap f_\gamma^{-1}(J) \subset I_1 \cup I_2$, see Figure II.4. Now using the notation from the proof of Theorem II.1.3 and the arguments that lead to (II.3.8) we obtain $\mathbb{P}(Z_n(I_i) \mid Z(C_\gamma \cap I_i)) \ge K^{-1}$ which yields

$$\mathbb{P}(Z(C_\gamma \cap I_i)) \le K\mathbb{P}(Z_n(I_i)). \quad \text{(II.3.14)}$$

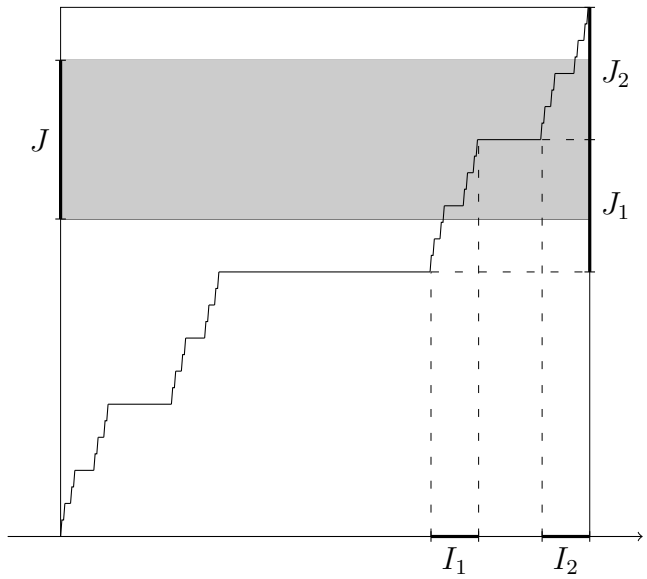

Figure II.4: An interval $J \subset [0,1]$ can be covered by two dyadic intervals J_1 and J_2 of comparable size. Intervals $I_1, I_2 \in \mathfrak{C}_{\gamma,n}$ are such that $f_\gamma(I_i) = J_i$, for $i = 1,2$.

But by the first inequality in (II.3.2), the probability on the right hand side is bounded from above by 2^{-n}. Using this fact in (II.3.14) and summing the expression for $i = 1,2$, we obtain (II.3.13).

By Theorem II.1.3 the set $\mathcal{Z}(B - f_\gamma) \cap C_\gamma$ is non-empty with some probability $p > 0$. Take an arbitrary $\gamma < \gamma_1 < 1/4$ and n_0 such that $\sum_{n \geq n_0} (2\sqrt{\gamma_1})^n \leq p/(2c_1)$.

For $n \geq n_0$ consider the interval $J_{k,n} = [k2^{-n} - \gamma_1^{n/2}/2, k2^{-n} + \gamma_1^{n/2}/2]$ and define the set $M_{n_0} = \bigcup_{n \geq n_0} \bigcup_{0 \leq k \leq 2^n} J_{k,n}$. By (II.3.13) and the choice of n_0, we have that $\mathbb{P}(Z(C_\gamma \cap f_\gamma^{-1}(M_{n_0}))) \leq p/2$. Therefore, the probability that there is a zero of $B(t) - f_\gamma(t)$ in the set $C_\gamma \cap \mathrm{Int}(C_{\gamma,n_0}) \backslash f_\gamma^{-1}(M_{n_0})$ is at least $p/2$ (here $\mathrm{Int}(C_{\gamma,n_0})$ is the interior of the set C_{γ,n_0}). Now the claim will be proven if we show that any such zero is isolated. Take $t \in C_\gamma \cap \mathrm{Int}(C_{\gamma,n_0}) \backslash f_\gamma^{-1}(M_{n_0})$ and any $s \neq t$ in the same connected component of $\mathrm{Int}(C_{\gamma,n_0})$. The largest integer ℓ such that both s and t are contained in the same interval of $\mathfrak{C}_{\gamma,\ell}$ satisfies $\ell \geq n_0$. Moreover, $|f_\gamma(s) - f_\gamma(t)| \geq \gamma_1^{(\ell+1)/2}/2$ and $|s - t| \leq \gamma^\ell$. Now it is clear that t satisfies the condition in part (i) of Proposition II.2.1 with $\alpha = \log \gamma_1/(2 \log \gamma) < 1/2$. Therefore, the statement follows from part (i) of Proposition II.2.1. □

II.4 Modifications

Let us discuss a few modifications. For instance, whether we can strengthen Theorem II.1.2 to an almost sure result.

Remark II.4.1. *It is not difficult to construct a continuous function f such that the set $\mathcal{Z}(B-f)$ has isolated points almost surely. Let f_γ be the Cantor function with $\gamma < 1/4$ defined on the interval $[0,1]$. Construct the function $f\colon \mathbb{R}^+ \to \mathbb{R}$ such that for every $n \geq 1$ and $0 \leq t \leq 1$ we have $f(4^{-n}(1+3t)) = 2^{-n}(1+f_\gamma(t))$ and define f on $(1,\infty)$ arbitrarily, see Figure II.5. By Proposition II.3.12 and the Cameron-Martin theorem, the probability that $\mathcal{Z}(B-f)$ has an isolated point in the interval $[1/4,1]$ is positive, denote it by p. By Brownian scaling the probability that there is an isolated point of $\mathcal{Z}(B-f)$ in the interval $[4^{-n-1},4^{-n}]$ is also equal to p, for any $n \geq 1$. Therefore, in view of observation (i) of our Preliminaries, the probability that there is an isolated point of $\mathcal{Z}(B-f)$ in the interval $[4^{-n-1},4^{-n}]$, for infinitely many n's is bounded from below by p. By Blumenthal's zero-one law this event has probability one, which proves the claim.*

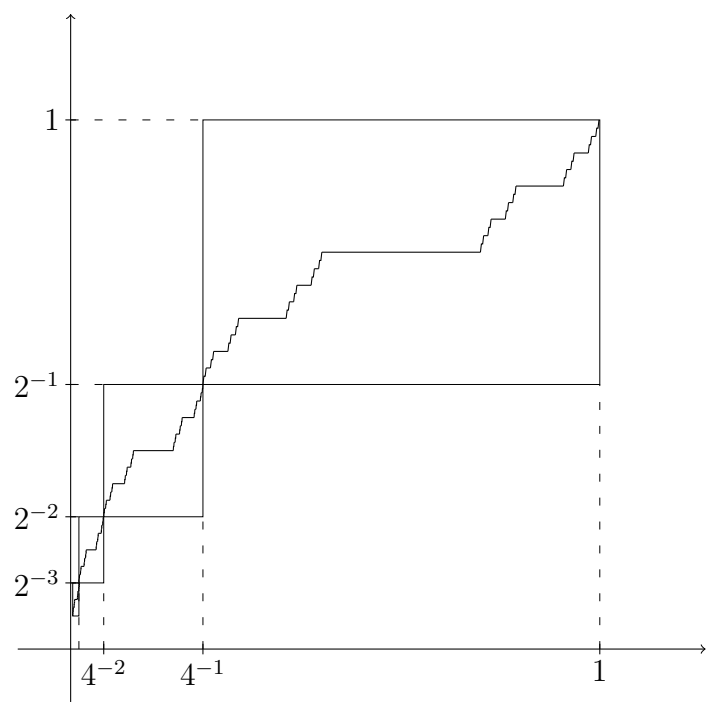

Figure II.5: Construction of the function f from Remark II.4.1. On each interval $[4^{-n-1},4^{-n}]$ function f is a scaled and shifted copy of the Cantor function (on the interval $(1,\infty)$ the function is defined arbitrarily).

The following proposition shows that, with positive probability, the set $\mathcal{Z}(B-f)$ can have only isolated points, or even only one point. However, later we will see that $\mathcal{Z}(B-f)$ cannot contain only isolated points almost surely (follows from Theorem IV.1.2).

Proposition II.4.2. *There exists a continuous function f such that, with positive probability, $\mathcal{Z}(B-f)$ is a singleton.*

Proof. Take f_γ to be the Cantor function with $\gamma < 1/4$ defined on the interval $[1, 2]$. Since $\mathcal{Z}(B - f_\gamma)$ has isolated points with positive probability, there are two rational numbers $q_1 < q_2$ such that, with positive probability, there will be only one zero of $B - f_\gamma$ in the interval (q_1, q_2). Denote this event by D and on this event the unique zero by τ. Note that on the event $D \cap \{(B(q_1) - f_\gamma(q_1))(B(q_2) - f_\gamma(q_2)) > 0\}$ the unique zero τ is necessarily a local extremum of the process $B - f_\gamma$ and by part (i) of the upcoming Proposition II.4.3 this event has probability zero. Furthermore, on the event $D \cap \{B(q_1) < f_\gamma(q_1)\} \cap \{B(q_2) > f_\gamma(q_2)\}$ the unique zero τ is necessarily a local point of increase of the Brownian motion. By a result of Dvoretzky, Erdös and Kakutani in [DEK], almost surely, Brownian motion has no points of increase and thus

$$\mathbb{P}(D, B(q_1) < f_\gamma(q_1), B(q_2) > f_\gamma(q_2)) = 0,$$

see also Theorem 5.14 in [MP]. Next define

$$S = \{y > f_\gamma(q_1) : \mathbb{P}(D \mid B(q_1) = y) > 0\},$$

and notice that by the Markov property and the discussion above $\mathbb{P}(B(q_1) \in S) > 0$. This implies that the set S is of positive Lebesgue measure and so is $S_1 = S \cap (f_\gamma(q_1) + \epsilon, \infty)$, for ϵ small enough. Now the claim will follow if we prove that there is a modification f of the function f_γ on the intervals $(0, q_1)$ and (q_2, ∞), such that

1) with positive probability there are no zeros of $B - f$ in $(0, q_1)$ and $B(q_1) \in S_1$,

2) for any $y < f_\gamma(q_2) - \epsilon$, conditional on $B(q_2) = y$, with positive probability there are no zeros of $B - f$ in (q_2, ∞).

For 1) define f to be linear on $[0, q_1]$ with $f(0) = -\epsilon$ and $f(q_1) = f_\gamma(q_1)$, if we do not require $f(0) = 0$. To prove that the probability that both $\{\mathcal{Z}(B - f) \cap (0, q_1) = \emptyset\}$ and $\{B(q_1) \in S_1\}$ happen is positive, by the Cameron-Martin theorem, it is enough to prove that

$$\mathbb{P}\Big(\min_{t \in [0, q_1]} B(t) > -\epsilon, B(q_1) \in S_2 = S_1 - f_\gamma(q_1) - \epsilon\Big) > 0.$$

While this is intuitively obvious it can be proven by using the reflection principle at the first time the Brownian motion hits the level $-\epsilon$ to conclude that the probability that both $\min_{0<t<q_1} B(t) \leq -\epsilon$ and $B(q_1) \in S_2$ happen is equal to the probability that $B(q_1) \in S_3$, where S_3 is obtained by reflecting the set S_2 around $-\epsilon$. Since $S_2 \subset \mathbb{R}^+$ we have $\mathbb{P}(B(q_1) \in S_3) < \mathbb{P}(B(q_1) \in S_2)$ and therefore

$$\mathbb{P}\Big(\min_{t \in [0, q_1]} B(t) > -\epsilon, B(q_1) \in S_2\Big) = \mathbb{P}(B(q_1) \in S_2) - \mathbb{P}(B(q_1) \in S_3) > 0.$$

Now the probability that there is a unique zero in the interval $(0, q_2)$ is equal to some $p > 0$. If we do require $f(0) = 0$, redefine f on the interval $(0, \delta)$ as $f(t) = -ct^{1/3}$. Here $c > 0$ is chosen large enough so that for $\delta > 0$, for which f is continuous, we have $\mathbb{P}(\mathcal{Z}(B - f) \cap (0, \delta)) < p$.

To satisfy the condition 2), replace f_γ on (q_2, ∞) by a linear function of slope 1, that is $f(q_2 + t) - f_\gamma(q_2) = t$. To prove that this f satisfies the required condition on (q_2, ∞) it

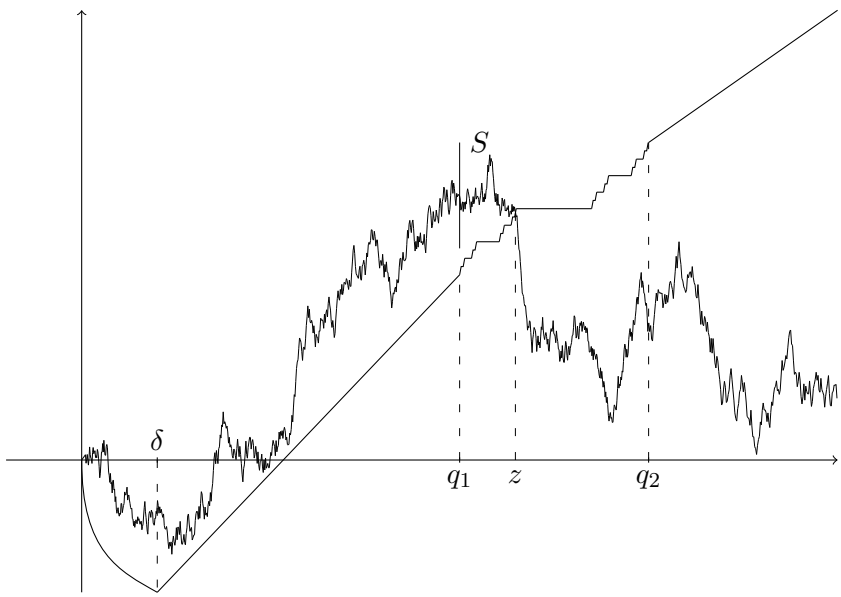

Figure II.6: Function f from Proposition II.4.2. It is a continuous function defined as $f(t) = -ct^{1/3}$ on $[0, \delta]$, linear on both $[\delta, q_1]$ and $[q_2, \infty)$ and is a part of the Cantor function on $[q_1, q_2]$. For this trajectory of B, there is only one zero of the process $B - f$, labeled as z.

is enough to prove that, for standard Brownian motion B, with positive probability there are no zeros of the process $B(t) - t - \epsilon$. This probability is equal to $1 - e^{-2\epsilon}$ by (5.13), Sect. 3.5 in [KS91]. See Figure II.6.

□

The next proposition justifies Remark II.1.4.

Proposition II.4.3. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function and define the process $X(t) = B(t) - f(t)$.*

(i) Almost surely there are no points in $\mathcal{Z}(X)$ which are local extrema.

(ii) Define $M_X(t) = \max_{0 \le s \le t} X(s)$ and the set of record times of the process X as $\mathrm{Rec}(X) = \{t > 0 : X(t) = M_X(t)\}$. Almost surely there are no isolated points in the set $\mathrm{Rec}(X)$.

Proof. (i) Take an interval $[q_1, q_2]$ with $q_1 > 0$ and let M be the maximum of the process X on this interval. Then, since the process X has independent increments, $X(q_1)$ and $M - X(q_1)$ are independent. Since $X(q_1)$ has a continuous distribution, so has $M = (M - X(q_1)) + X(q_1)$, and therefore $\mathbb{P}(M = x) = 0$ for any $x \in \mathbb{R}$. Taking $x = 0$ and a union over all rational $q_1 < q_2$ proves the claim for local maxima. Similarly the statement holds for local minima. See Figure II.7.

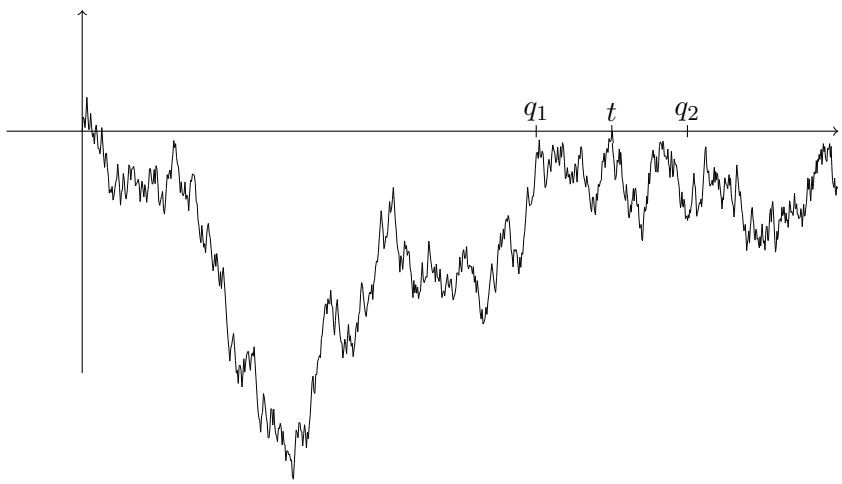

Figure II.7: Process $X(t) = B(t) - f(t)$ having a zero and a local extremum at some $t > 0$; this is an event of probability zero.

(ii) For any continuous function g, any record time $s \in \mathrm{Rec}(g)$ is a maximum of g on the interval $[s-\epsilon, s]$, for every $\epsilon > 0$. Let $s > 0$ be an isolated point in $\mathrm{Rec}(g)$. Then s is a local maximum, because otherwise we would have record times to the right of s, arbitrarily close to s. Since g is continuous and there are no record times in the interval $(s-\epsilon, s)$ for some $\epsilon > 0$, there has to be an $r \in \mathrm{Rec}(g) \cup \{0\}$, which is also a local maximum and such that $r < s$ and $g(r) = g(s)$. Applying these observations to $g(t) = X(t) = B(t) - f(t)$, we see that, in order to prove the claim, it is enough to show that the process $X = B - f$ does not have two equal local maxima almost surely. See Figure II.8. This is well known for standard Brownian motion and can be proven in the same way for the process X. Namely, for two intervals $[q_1, r_1]$ and $[q_2, r_2]$, with $r_1 < q_2$, define the random variables $Y_1 = X(r_1) - \max_{q_1 \le t \le r_1} X(t)$ and $Y_3 = \max_{q_2 \le t \le r_2} X(t) - X(q_2)$, and let $Y_2 = X(q_2) - X(r_1)$. Clearly these three random variables are independent. Since Y_2 is a continuous random variable, so is $Y_1 + Y_2 + Y_3$ and $\mathbb{P}(Y_1 + Y_2 + Y_3 = 0) = 0$. Therefore, almost surely the maxima on $[q_1, r_1]$ and $[q_2, r_2]$ are different. Taking the union over all possible rational q_1, r_1, q_2 and r_2 as above proves the claim.

□

II.5 Second example

In this section we will discuss another example of a drift which will cause Brownian motion to have isolated zeros with positive probability. The example will be a distribution function of a measure supported on a set of Hausdorff dimension strictly less than 1.

Let $\mathcal{F}_n = \{I = [k2^{-n}, (k+1)2^{-n}] : I \subset [1,2]\}$ be the set of dyadic subintervals of $[1,2]$ of generation n. Notice that for any $I \in \mathcal{F}_n$ we can find a unique sequence $a_1, \ldots, a_n$,

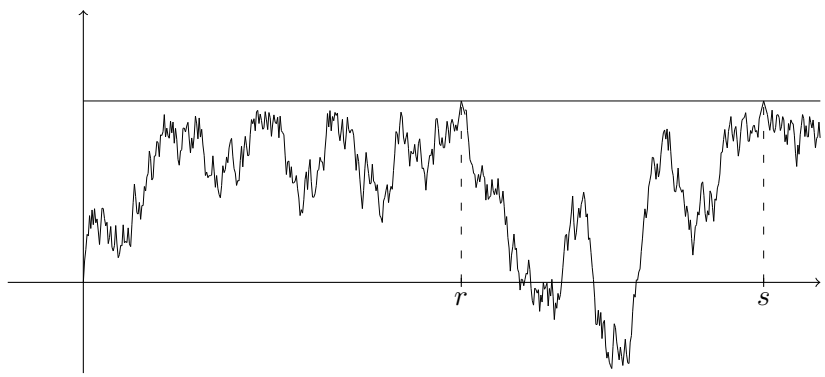

Figure II.8: If $X(t) = B(t) - f(t)$ has an isolated record time at time s then there exists a record time $r < s$ for which $X(r) = X(s)$. In this case both record times r and s have to be local maxima. This is an event of probability zero.

$a_i \in \{0, 1\}$, such that

$$I = \{t : t \text{ has a binary expansion starting with } 1.a_1 \dots a_n\}.$$

We will say that the sequence $a_1, \dots, a_n$ is the labeling of I, or that I corresponds to the sequence $a_1, \dots, a_n$. Also define $N_I(k)$ as the number of 1's in the sequence $a_1, \dots a_k$.

For a fixed $0 < \beta < 1$ let μ_β be the measure supported on $[1, 2]$, defined recursively as follows. First define μ_β on intervals in $\mathcal{F}_1$ as $\mu_\beta([1, 3/2]) = 1-\beta$ and $\mu_\beta([3/2, 2]) = \beta$. Now, for an interval $I \in \mathcal{F}_n$, $I = [k2^{-n}, (k+1)2^{-n}]$, its subintervals $I_0 = [k2^{-n}, k2^{-n} + 2^{-n-1}]$ and $I_1 = [k2^{-n} + 2^{-n-1}, (k+1)2^{-n}]$ will be assigned $\mu_\beta(I_0) = (1-\beta)\mu_\beta(I)$ and $\mu_\beta(I_1) = \beta\mu_\beta(I)$. Equivalently, we can define $\mu_\beta(I) = \beta^{N_I(n)}(1-\beta)^{n-N_I(n)}$. Define g_β as the distribution function of μ_β, that is $g_\beta \colon \mathbb{R}^+ \to \mathbb{R}$ is such that $g_\beta(t) = \mu_\beta([0, t])$. We will prove that, for suitably chosen values of the parameter β, the process $B(t) - g_\beta(t)$ will have isolated zeros with positive probability.

By $\mu_{\beta,n}$ denote the n-th approximation of μ_β constructed above. More precisely, $\mu_{\beta,n}$ is a measure which is uniform on any $I \in \mathcal{F}_n$, and for all $I \in \mathcal{F}_n$ satisfies $\mu_{\beta,n}(I) = \mu_\beta(I)$, and by $g_{\beta,n} \colon \mathbb{R}^+ \to \mathbb{R}$ denote its distribution function. Obviously, the sequence of functions $g_{\beta,n}$ converges uniformly to g_β. See figure II.9.

For $I \in \mathcal{F}_n$ the value of $\mu_{\beta,n}(I) = \mu_\beta(I)$ can also be interpreted in the following way. Let Y_i be independent identically distributed random variables, with $\mathbf{P}_\beta(Y_i = 1) = \beta$ and $\mathbf{P}_\beta(Y_i = 0) = 1 - \beta$ (we denote the probability measure by $\mathbf{P}_\beta$ to distinguish it from the probability measure on the Wiener space). If an interval $I \in \mathcal{F}_n$ corresponds to the sequence $a_1, \dots, a_n$, we have $\mu_{\beta,n}(I) = \mathbf{P}_\beta(Y_1 = a_1, \dots, Y_n = a_n)$. The right hand side will be denoted by $\mathbf{P}_\beta(a_1, \dots, a_n)$ in the text. By using the notation for the partial sums $S_k = Y_1 + \dots + Y_k$, we have that if an interval I is labeled by $Y_1, \dots, Y_n$, then $N_I(k) = S_k$.

Define the function $\phi \colon (0, 1) \to \mathbb{R}$ as $\phi(\beta) = \beta^\beta(1-\beta)^{1-\beta}$. It is easy to check that ϕ is convex and symmetric ($\phi(\beta) = \phi(1-\beta)$) around $1/2$, where the minimum $\phi(1/2) = 1/2$ is achieved. Moreover, it is strictly decreasing on $(0, 1/2)$ and strictly increasing on $(1/2, 1)$

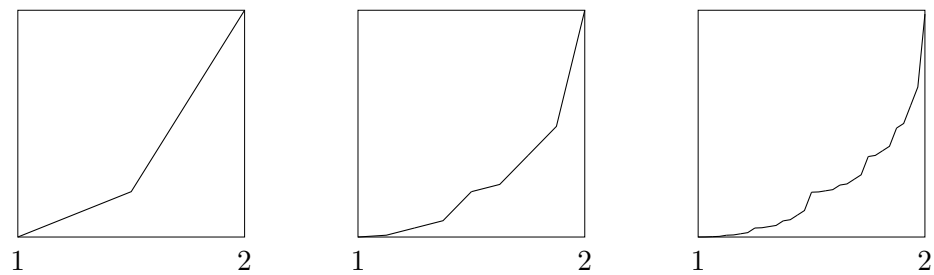

Figure II.9: Functions $g_{\beta,n}$ for $\beta = 0.8$ and $n = 1, 3, 5$

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and $\lim_{\beta\uparrow 1} \phi(\beta) = \lim_{\beta\downarrow 0} \phi(\beta) = 1$. Therefore, there is a unique $\beta_0 > 1/2$ such that $\phi(\beta_0) = 2^{-1/2}$.

The value of β_0 is the threshold for the parameter $\beta > 1/2$, which makes the function g_β be 1/2-Hölder continuous μ_β-almost everywhere. That is, for $\beta < \beta_0$ the function g_β is 1/2-Hölder continuous μ_β-almost everywhere, while for $\beta > \beta_0$ it is not.

It turns out that for $\beta > \beta_0$ the process $B(t) - g_\beta(t)$ has isolated zeros with positive probability.

As for the Cantor function from a previous section, the proof is an application of the second moment method. The sequence of variables to which the method is applied will be the number of linear segments in the graph of $g_{\beta,n}$, which are intersected by Brownian motion, but we will count only the intervals for which the density of 1's in the labeling is concentrated around β on all scales. More precisely, we will look at the following family of intervals

$$\mathcal{G}_n(\beta, \epsilon, k) = \{I \text{ labeled by } a_1, \dots, a_n : a_n = 1 \\ \text{and } |N_I(\ell) - \ell\beta| \le \ell\epsilon, \text{ for all } k \le \ell < n\}. \quad \text{(II.5.1)}$$

Moreover, for intervals I and J in $\mathcal{F}_n$ such that $I < J$ define $h_{\beta,r}(I, J) = \sum_{I<\tilde{I}\le J} \mu_\beta(\tilde{I})$ and $h_{\beta,l}(I, J) = \sum_{I\le\tilde{I}<J} \mu_\beta(\tilde{I})$. For an interval $I \in \mathcal{F}_n$ define $I + 2^{-\ell}$ as the interval obtained by translating I by $2^{-\ell}$ to the right. If $I + 2^{-\ell} \in \mathcal{F}_n$, we denote it by I_ℓ. For two positive sequences (a_n) and (b_n) we write $a_n \asymp b_n$ if the quotient a_n/b_n is bounded from above and bounded away from zero. First we prove a technical lemma.

Lemma II.5.2. *For any $\beta \ne 1/2$ and $\delta > 0$ there is a positive integer k_0 and positive numbers ϵ, c_+ and c_- such that for any positive integer $n \ge k_0$, any $I \in \mathcal{F}_n$ and any $\ell \le n$ either of $I \in \mathcal{G}_n(\beta, \epsilon, k_0)$ or $I_\ell \in \mathcal{G}_n(\beta, \epsilon, k_0)$ implies*

$$c_-\phi(\beta)^{\ell(1+\delta)} \le \min(h_{\beta,l}(I, I_\ell), h_{\beta,r}(I, I_\ell)) \\ \le \max(h_{\beta,l}(I, I_\ell), h_{\beta,r}(I, I_\ell)) \le c_+\phi(\beta)^{\ell(1-\delta)} \quad \text{(II.5.3)}$$

Proof. Without loss of generality we can assume that $\beta > 1/2$. Let k_0 be a positive integer and $\epsilon > 0$, both to be chosen later. Take positive integers n and $\ell \leq n$. Let interval $I \in \mathcal{F}_n$ be labeled by $a_1, \dots, a_n$ and distinguish two cases: $a_\ell = 0$ and $a_\ell = 1$.

In the case that $a_\ell = 0$, every $J \in \mathcal{F}_n$ such that $I < J \leq I_\ell$ will correspond to a sequence of the form $a_1, \dots, a_{\ell-1}, b_\ell, c_{\ell+1}, \dots c_n$, where b_ℓ can be either 0 or 1. Moreover, for any combination $c_{\ell+1}, \dots, c_n$ there is a $b_\ell \in \{0, 1\}$ and an interval J as above which corresponds to a sequence $a_1, \dots, a_{\ell-1}, b_\ell, c_{\ell+1}, \dots, c_n$. The same is true if the condition $I < J \leq I_\ell$ is replaced by the condition $I \leq J < I_\ell$. If $\ell > k_0$, either of conditions $I \in \mathcal{G}_n(\beta, \epsilon, k_0)$ or $I_\ell \in \mathcal{G}_n(\beta, \epsilon, k_0)$ implies that the number of 1's in the sequence $a_1, \dots, a_{\ell-1}$ is equal to $p = N_I(\ell - 1) = N_{I_\ell}(\ell - 1) \geq (\beta - \epsilon)(\ell - 1)$. Also, $p \leq (\beta + \epsilon)(\ell - 1)$.

Therefore

$$\begin{aligned} h_{\beta,r}(I, I_\ell) &= \sum_{I<J\leq I_\ell} \beta^{N_J(n)}(1-\beta)^{n-N_J(n)} \\ &\geq \beta^p(1-\beta)^{\ell-p} \sum_{c_{\ell+1},\dots,c_n} \prod_{i=\ell+1}^{n} \beta^{c_i}(1-\beta)^{1-c_i} \\ &= \beta^p(1-\beta)^{\ell-p} \\ &\geq \beta^{(\beta+\epsilon)(\ell-1)}(1-\beta)^{\ell-(\beta-\epsilon)(\ell-1)} \\ &\geq (\beta^{-1}-1)^{\beta+\epsilon}(\beta^{\beta+\epsilon}(1-\beta)^{1-\beta-\epsilon})^\ell, \end{aligned} \tag{II.5.4}$$

where in the second and in the last inequality we used the assumption that $\beta > 1/2$.

Therefore for all ϵ with $\beta\delta \geq \epsilon > 0$ holds that $\beta^{\beta+\epsilon}(1-\beta)^{1-\beta-\epsilon} \geq \phi(\beta)^{1+\delta}$ and with k_0 such that $\mathcal{G}_n(\beta, \epsilon, k_0)$ is nonempty, we get $h_{\beta,r}(I, I_\ell) \geq c_-\phi(\beta)^{1+\delta}$ for all $k_0 < \ell \leq n$. Since everything is true if the sum on the right hand side of the first line in (II.5.4) is replaced by $\sum_{I\leq J<I_\ell} \beta^{N_J(n)}(1-\beta)^{n-N_J(n)}$, the same holds for $h_{\beta,l}(I, I_\ell)$.

The case $a_\ell = 1$ is just slightly more complicated. Assume $a_\ell = 1$ and let $t \geq 1$ be such that $a_{\ell-t} = 0$ and $a_{\ell-t+1} = a_{\ell-t+2} = \cdots = a_\ell = 1$. Then, any $I < J \leq I_\ell$ will correspond to the sequence $a_1, \dots, a_{\ell-t-1}$, $b_{\ell-t}, \dots, b_\ell$, $c_{\ell+1}, \dots, c_n$, where block $b_{\ell-t}, \dots, b_\ell$ is either $0, 1, \dots, 1$ or $1, 0, \dots, 0$. As before, for any choice of $c_{\ell+1}, \dots, c_n$ we can find such a sequence $b_{\ell-t}, \dots, b_\ell$ and an interval J satisfying $I < J \leq I_\ell$ which is labeled by $a_1, \dots, a_{\ell-t-1}$, $b_{\ell-t}, \dots, b_\ell$, $c_{\ell+1}, \dots, c_n$. The same is true if the condition $I < J \leq I_\ell$ is replaced by the condition $I \leq J < I_\ell$. Therefore

$$\begin{aligned} h_{\beta,r}(I, I_\ell) &= \sum_{I<J\leq I_\ell} \beta^{N_J(n)}(1-\beta)^{n-N_J(n)} \\ &\geq \beta^{N_I(\ell-t-1)}(1-\beta)^{\ell-t-1-N_I(\ell-t-1)}\beta(1-\beta)^t \qquad \text{(II.5.5)} \\ &\qquad \cdot \sum_{c_{\ell+1},\dots,c_n} \prod_{i=\ell+1}^{n} \beta^{c_i}(1-\beta)^{1-c_i} \\ &= \beta^{N_I(\ell-t-1)}(1-\beta)^{\ell-t-1-N_I(\ell-t-1)}\beta(1-\beta)^t \\ &= \beta^{N_{I_\ell}(\ell)}(1-\beta)^{\ell-N_{I_\ell}(\ell)}. \end{aligned} \tag{II.5.6}$$

If $I_\ell \in \mathcal{G}_n(\beta, \epsilon, k_0)$ and $\ell \geq k_0$, we proceed as in the case $a_\ell = 0$. If $I \in \mathcal{G}_n(\beta, \epsilon, k_0)$, we argue as follows. Assume $\ell > \frac{k_0}{1-\beta-\epsilon}$. Then $\ell - k_0 > (\beta + \epsilon)\ell \geq N_I(\ell) \geq t$. Thus, we have

$\ell - t - 1 \geq k_0$. Now we can calculate

$$(\beta+\epsilon)\ell \geq N_I(\ell) = N_I(\ell - t - 1) + t \geq (\beta - \epsilon)(\ell - t - 1) + t.$$

This implies $t \leq \frac{2\epsilon\ell+\beta-\epsilon}{1-\beta+\epsilon}$. Since $\ell - t - 1 \geq k_0$, the third inequality of (II.5.5) gives

$$\begin{aligned} h_{\beta,r}(I, I_\ell) &\geq \beta^{(\beta+\epsilon)\ell-t+1}(1-\beta)^{\ell-t-1-(\beta-\epsilon)(\ell-t-1)} \\ &\geq \beta^{(\beta+\epsilon)\ell+1}(1-\beta)^{\ell(1-\beta+\epsilon)+\beta(1+t)} \\ &\geq \beta^{(\beta+\epsilon)\ell+1}(1-\beta)^{\ell(1-\beta+\epsilon)+\beta(1+\frac{2\epsilon\ell+\beta}{1-\beta+\epsilon})}. \end{aligned}$$

Therefore, for all ϵ with $\delta\frac{(1-\beta)^2}{(1+\beta)} \geq \epsilon > 0$ holds that $\beta + \epsilon \leq \beta(1+\delta)$ and $1 - \beta + \epsilon + \frac{2\epsilon\beta}{1-\beta+\epsilon} \leq (1-\beta)(1+\delta)$. Thus, for k_0 such that $\mathcal{G}_n(\beta, \epsilon, k_0)$ is nonempty we obtain $h_{\beta,r}(I, I_\ell) \geq c_-\phi(\beta)^{\ell(1+\delta)}$, for all $\frac{k_0}{1-\beta-\epsilon} < \ell \leq n$ and for some constant $c_- > 0$ not depending on ϵ.

The modification for $h_{\beta,\ell}$ is the same as in the case $a_\ell = 0$. Finally, by adjusting the constants the claim is true for any $\ell \leq n$.

The upper bound in (II.5.3) can be proven by replacing the inequalities in (II.5.4) and (II.5.5) by

$$h_{\beta,r}(I, I_\ell) \leq \beta^{p+1}(1-\beta)^{\ell-p-1} \text{ and } h_{\beta,r}(I, I_\ell) \leq \beta^{N_I(\ell)}(1-\beta)^{\ell-N_I(\ell)},$$

respectively and arguing as before.

□

Proposition II.5.7. *For $\beta \in (0, 1-\beta_0) \cup (\beta_0, 1)$ the set $\mathcal{Z}(B - g_\beta)$ has isolated points with positive probability.*

Proof. We prove the claim only for $\beta \in (\beta_0, 1)$, since the proof $\beta \in (0, 1-\beta_0)$ is analogous.

As before set (Y_i) to be a sequence of independent identically distributed random variables such that $\mathbf{P}_\beta(Y_i = 1) = \beta$ and $\mathbf{P}_\beta(Y_i = 0) = 1-\beta$ and $S_k = Y_1 + \cdots + Y_k$. Since $\phi(\beta) > 2^{-1/2}$, by Lemma II.5.2 we can find an $\epsilon > 0$ and a positive integer k_0 so that (II.5.3) is fulfilled with $\delta > 0$ such that $\phi(\beta)^{1+\delta} = 2^{-\alpha} > 2^{-1/2}$. Moreover, we can decrease $\epsilon > 0$ and increase k_0 so that we also have $\beta^{\beta-\epsilon}(1-\beta)^{1-\beta+\epsilon} > 2^{-1/2}$, and for all $n \geq k_0$

$$\mathbf{P}_\beta\Big(\bigcup_{k=k_0}^{n} |S_k - k\beta| > k\epsilon\Big) \leq \frac{1}{2}. \tag{II.5.8}$$

The latter is possible since the probabilities $\mathbf{P}_\beta(|S_k - k\beta| > k\epsilon)$ decay exponentially in k.

For an interval $I = [r, s] \in \mathcal{F}_n$, denote by $Z_n(I)$ as the event $B(s) \in [g_{\beta,n}(r), g_{\beta,n}(s)]$, and the random variable $Z_{\beta,n} = \sum_{I \in \mathcal{G}_n(\beta,\epsilon,k_0)} \mathbf{1}(Z_n(I))$, where $\mathbf{1}(Z_n(I))$ is the indicator function of the event $Z_n(I)$. Note that

$$\mathbb{P}(Z_n(I)) \asymp \mu_\beta(I) = \mathbf{P}_\beta(a_1, \ldots, a_n), \tag{II.5.9}$$

where $a_1, \ldots, a_n$ is the labeling of I. Thus, the first moment of the random variable $Z_{\beta,n} = \sum_{I \in \mathcal{G}_n(\beta,\epsilon,k_0)} \mathbf{1}(Z_n(I))$ can be estimated as

$$\mathbb{E}(Z_{\beta,n}) = \sum_{I\in\mathcal{G}_n(\beta,\epsilon,k_0)} \mathbb{P}(Z_n(I)) \asymp \mathbf{P}_\beta(\mathcal{G}_n(\beta,\epsilon,k_0))$$
$$= \beta\mathbf{P}_\beta(\bigcap_{k=k_0}^{n-1} |S_k - k\beta| \le k\epsilon) \ge \frac{\beta}{2}. \quad \text{(II.5.10)}$$

Next we prove that the second moment $\mathbb{E}(Z^2_{\beta,n})$ is bounded from above. To end this notice that

$$\mathbb{E}(Z^2_{\beta,n}) = 2\sum_{\substack{I,J\in\mathcal{G}_n(\beta,\epsilon,k_0)\\ I<J}} \mathbb{P}(Z_n(I))\mathbb{P}(Z_n(J)|Z_n(I)) + \mathbb{E}(Z_{\beta,n}). \quad \text{(II.5.11)}$$

From (II.5.10) we see that $\mathbb{E}(Z_{\beta,n})$ is bounded from above, so we only need to prove the boundedness of the first summand in (II.5.11). Let $I, J \in \mathcal{G}_n(\beta,\epsilon,k_0)$ be such that $I < J$.

Use the notation $I = [a2^{-n}, (a+1)2^{-n}]$, $J = [b2^{-n}, (b+1)2^{-n}]$ and $h(I,J) = g_{\beta,n}(b2^{-n}) - g_{\beta,n}((a+1)2^{-n})$. We see that for some constant $C_1 > 0$ holds

$$\mathbb{P}(Z_n(J)|Z_n(I)) \quad \text{(II.5.12)}$$
$$\le C_1\mathbb{P}\Big(B((b+1)2^{-n}) \in [g_{\beta,n}(b2^{-n}), g_{\beta,n}((b+1)2^{-n})]\Big|Z_n(I)\Big)$$
$$\le C_1\beta^{N_J(n)}(1-\beta)^{n-N_J(n)}\sqrt{\frac{2^n}{2\pi(b-a-1)}}\exp\Big(-\frac{2^n h(I,J)^2}{2(b-a+1)}\Big). \quad \text{(II.5.13)}$$

Assume now that $1 \le \ell \le n$ is such that $2^{-\ell} < (b-a)2^{-n} \le 2^{-\ell+1}$. Continuing (II.5.12) we have

$$\mathbb{P}(Z_n(J)|Z_n(I)) \le C_1\beta^{N_J(n)}(1-\beta)^{n-N_J(n)}2^{\ell/2}\exp(-2^\ell h(I,J)^2/8). \quad \text{(II.5.14)}$$

Using the usual notation $I_\ell = I + 2^{-\ell}$ we have $I < I_\ell < J$ and thus $h_r(I, I_\ell) \le h(I,J)$. Lemma II.5.2 now yields $h(I,J) \ge c2^{-\alpha\ell}$ for some $\alpha < 1/2$ and $c > 0$. This implies that

$$2^{\ell/2}\exp(-2^\ell h(I,J)^2/8) \le 2^{\ell/2}\exp(-c^2 2^{(1-2\alpha)\ell}/8)$$

is bounded from above, uniformly in ℓ. Therefore, (II.5.14) implies

$$\mathbb{P}(Z(J)|Z(I)) \le C_2\beta^{N_J(n)}(1-\beta)^{n-N_J(n)},$$

for some $C_2 > 0$. Using (II.5.9), the first summand on the right hand side in (II.5.11) can now be bounded from above as

$$\sum_{\substack{I,J\in\mathcal{G}_n(\beta,\epsilon,k_0)\\ I<J}} \mathbb{P}(Z_n(I))\mathbb{P}(Z_n(J)|Z_n(I))$$
$$\le C_3 \sum_{\substack{I,J\in\mathcal{G}_n(\beta,\epsilon,k_0)\\ I<J}} \beta^{N_I(n)}(1-\beta)^{n-N_I(n)}\beta^{N_J(n)}(1-\beta)^{n-N_J(n)}, \quad \text{(II.5.15)}$$

for some $C_3 > 0$. The right hand side is bounded from above by $C_3\mathbb{E}(Z_{\beta,n})^2$. By (II.5.10) the first moments $\mathbb{E}(Z_{\beta,n})$ are also bounded from above, which gives an upper bound for $\mathbb{E}(Z^2_{\beta,n})$ in (II.5.11).

By Payley-Zygmund inequality the probabilities $\mathbb{P}(Z_{\beta,n} > 0) \geq \frac{\mathbb{E}(Z_{\beta,n})^2}{\mathbb{E}(Z^2_{\beta,n})}$ are bounded away from zero. Thus $\mathbb{P}(Z_{\beta,n} > 0 \text{ infinitely often}) > 0$.

This event implies that we can find a sequence of intervals $I_k = [r_k, s_k] \in \mathcal{G}_n(\beta, \epsilon, k_0)$, such that $g_{\beta,n}(r_k) \leq B(s_k) \leq g_{\beta,n}(s_k)$, thus $|B(s_k) - g_{\beta,n}(s_k)| \leq \mu(I_k)$. And, the sequence s_k will have a subsequence which converges to some t satisfying $B(t) = g_\beta(t)$. For $\ell > k_0$ take an s such that $2^{-\ell} < (s-t)/3 \leq 2^{-\ell+1}$. For n large enough we also have $2^{-\ell} < (s - t_n)/3$, and denoting $I^n_\ell = I^n + 2^{-\ell}$, we get

$$g_\beta(s) - g_\beta(t_n) \geq h_r(I^n, I^n_\ell) \geq c_- 2^{-\alpha\ell} \geq c_2(s-t)^\alpha.$$

Similarly, for s such that $2^{-\ell} < (t-s)/3 \leq 2^{-\ell+1}$ and n large enough

$$g_\beta(t_n) - g_\beta(s) \geq h_l(I^n_{-\ell}, I^n) \geq c_- 2^{-\alpha\ell} \geq c_2(t-s)^\alpha,$$

where $I^n_{-\ell} = I^n - 2^{-\ell}$. Taking the limit as $n \to \infty$ in the above expressions we obtain $|g_\beta(s) - g_\beta(t)| \geq c_2|s-t|^\alpha$, for s close enough to t. Now by part (i) of Proposition II.2.1 we see that t is an isolated zero of $B - g_\beta$. Since this happens on the event $\{Z_{\beta,n} > 0 \text{ infinitely often}\}$ which has positive probability, the claim follows.

□

Remarks and open problems

An open question of the list of selected open problems at the end of the book [MP] is to characterize those continuous functions f such that if added to Brownian motion they have almost surely no isolated zeros (problem 1(c) of [MP]). This chapter did not completely answer this open question. In particular, Proposition II.2.1 provides a necessary and a sufficient criterion for isolated points but they do not coincide. We will revisit this question later, in the next chapter, see III.4, but we will not completely resolve it.

Other questions that might be of interest are the following. For instance, is there another way/ property to characterize the zeros of $B - f$? Theorem II.1.3 showed a surprising property of the Cantor function when added to Brownian motion. Are there other functions with a similar phenomenon?

Chapter III

Cantor set zeros

III.1 Introduction

Recall that by C_γ we denoted the middle $(1-2\gamma)$-Cantor set for $\gamma < 1/2$ on the interval $[1,2]$ and was $f_\gamma\colon [1,2] \to [0,1]$ be the corresponding Cantor function. Furthermore, for any function g defined on some subset (or the whole) of $\mathbb{R}^+$ we denoted by $\mathcal{Z}(g)$ the set of zeros of g in $(0,\infty)$.

Taylor and Watson (Example 3 in [TW]) showed that Brownian motion does not intersect the graph of the middle 1/2-Cantor function restricted to the Cantor set, even though the projection of this set on the vertical axis is an interval. We have seen in Theorem II.1.3 that surprisingly, the middle 1/2-Cantor function is an exception in the following sense. $\mathbb{P}(\mathcal{Z}(B-f_\gamma)\cap C_\gamma \neq \emptyset) > 0$ holds if and only if $\gamma \neq 1/4$. In the previous chapter we used this result to prove that for every $\beta < 1/2$ there is a β-Hölder continuous function $f\colon \mathbb{R}^+ \to \mathbb{R}$ such that the set $\mathcal{Z}(B-f)$ has isolated points with positive probability.

Theorem II.1.3 is the main motivation for this chapter. We investigate a more general class of Cantor functions defined in section III.2. Recall from chapter II.3 that the n-th approximation of the middle 1/2-Cantor function increases on some intervals of length 4^{-n} and is constant elsewhere. We now allow these intervals to vary in length for every iteration step meaning that for a positive, real sequence (a_n) the length of intervals where the function increases is $4^{-n}a_n^2$ at iteration level n. See Figure III.1 for an example. Theorems III.3.1 and III.3.9 give conditions for which of these generalized Cantor functions f_b the process $B-f_b$ has zeros in the corresponding generalized Cantor set with positive probability or with probability 0, respectively.

III.2 Generalized Cantor sets and generalized Cantor function

We now define a generalized Cantor set analogously to the Cantor set we have defined in chapter II.3.

For a given positive, real sequence (a_n) we define a sequence (b_n) by $b_n := (2^{-n}\cdot a_n)^2$. For this sequence we define a corresponding Cantor-type set and denote it by C_b (to distinguish it from the classical Cantor set we defined in the last chapter we will use Latin instead of Greek indices). Take a closed interval I of length $b_0 = a_0^2$. Let $\mathfrak{C}_{b_1}$ be the set consisting of two disjoint closed subintervals of I of length b_1, the left one (for which the left endpoint

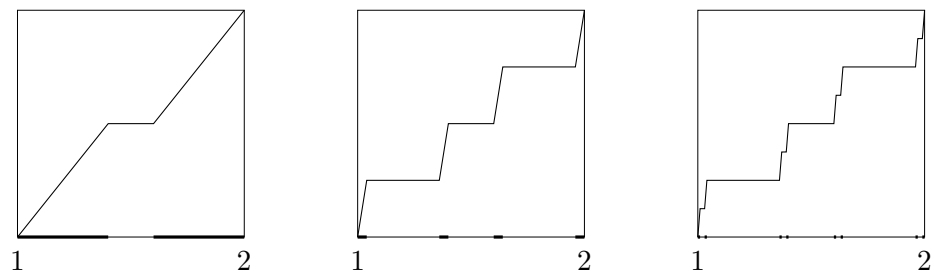

Figure III.1: First three approximations of the generalized Cantor function on the interval $[1,2]$ (functions f_{b_n} from the construction for $b_1 = 0.4$, $b_2 = 0.04$ and $b_3 = 0.01$, or $a_1 = \sqrt{1.6}$, $a_2 = \sqrt{0.64}$ and $a_3 = 0.8$, respectively. Approximations of the generalized Cantor set C_b are drawn in bold).

coincides with the left endpoint of I) and the right one (for which the right endpoint coincides with the right endpoint of I). Now continue recursively, if $J \in \mathfrak{C}_{b_n}$, then include in the set $\mathfrak{C}_{b_{n+1}}$ its left and right closed subintervals of length b_{n+1}. We define the set C_{b_n} as the union of all the intervals from $\mathfrak{C}_{b_n}$. For any n, the family $\mathfrak{C}_{b_n}$ is the set of all connected components of the set C_{b_n}. The generalized Cantor set is a compact set defined as $C_b = \bigcap_{n=1}^{\infty} C_{b_n}$.

Now we construct a Cantor-type function corresponding to the generalized Cantor set above. Define the function f_{b_1} so that it has values 0 and 1 at the left and the right endpoint of the interval I, respectively, value 1/2 on $I \backslash C_{b_1}$ and interpolate linearly on the intervals in $\mathfrak{C}_{b_1}$. Recursively, construct the function $f_{b_{n+1}}$ so that for every interval $J = [s,t] \in \mathfrak{C}_{b_n}$, the function $f_{b_{n+1}}$ agrees with f_{b_n} at s and t, it has value $(f_{b_n}(s) + f_{b_n}(t))/2$ on $J \backslash C_{b_{n+1}}$ and interpolate linearly on the intervals in $\mathfrak{C}_{b_{n+1}}$.

The sequence of functions (f_{b_n}) converges uniformly on I. We define the generalized Cantor function f_b as the limit $f_b = \lim_n f_{b_n}$. For any n and all $m \leq n$ the functions f_b and f_{b_n} agree at the endpoints of intervals $J \in \mathfrak{C}_{b_m}$. See Figure III.1 for an example.

Further, we fix an arbitrary small $\epsilon > 0$ and require the sequence (a_n) to fulfill the condition $a_n^2 - \frac{1}{2}a_{n+1}^2 \geq \epsilon a_n^2$ (or equivalently $b_n - 2b_{n+1} \geq \epsilon b_n$) for all n. Corollary III.3.26 addresses the more general case of having a weaker condition $b_n - 2b_{n+1} > 0$ for all n. So we will only consider non-degenerate Cantor sets/ functions. Note that if the sequence (a_n) is non-increasing, then $a_n^2 - \frac{1}{2}a_{n+1}^2 \geq \frac{1}{2}a_n^2$ holds for all n.

For simplicity, we will assume the initial interval I to be $[1,2]$, and we can extend the function f_b to $\mathbb{R}^+$, for instance by value 0 on $[0,1)$ and value 1 on $(2,\infty)$.

III.3 Zeros in the generalized Cantor set

III.3.1 Condition for having Cantor set zeros

The following theorem gives a condition for which sequences (b_n) (or (a_n) respectively) $B - f_b$ has zeros in the generalized Cantor set with positive probability.

Theorem III.3.1. $\mathbb{P}(\mathcal{Z}(B - f_b) \cap C_b \neq \emptyset) > 0$ *holds if either*
$\sum_{n=1}^{\infty} a_n < \infty$, *or*
$\sum_{n=1}^{\infty} \frac{1}{a_n} < \infty$.

Remarks III.3.2. *(i) Note that for geometric series* (a_n) *the result was already shown in Theorem II.1.3. For* $a_n = \frac{1}{x^n}$ *with some* $x > 1$ *gives* $b_n := (2^{-n} \cdot \frac{1}{x^n})^2 = (2x)^{-2n}$ *which corresponds to* $\gamma = \frac{1}{(2x)^2}$ *in Theorem II.1.3. So for* $\sum_{n=1}^{\infty} a_n < \infty$ *Theorem III.3.1 extends Theorem II.1.3 for convergent series* $(\sum a_n)$ *that increase slower than geometric series, for instance take* $a_n = \frac{1}{n^d}$ *with* $d > 1$.

(ii)The generalized Cantor function f_b *is not necessarily* α*-Hölder continuous for some* $\alpha \in (0, 1)$. *If the sequence* a_n *fulfills that there is a* $\sigma \in (0, 1)$ *that is the smallest value fulfilling* $1/\sigma \geq 2 - \liminf_{n\to\infty} \frac{2 \log a_n}{n \log 2}$, *then the corresponding generalized Cantor function is* σ*-Hölder continuous. For example, the generalized Cantor function corresponding to the sequence* $a_n = \exp -2n$ *does not satisfy a Hölder condition of any positive order.*

To prove Theorem III.3.1 we will use essentially the same method as for the proof of Theorem II.1.3.

Proof of Theorem III.3.1. For an interval $I = [r, s] \in \mathfrak{C}_{b_n}$, define $Z_n(I)$ as the event $B(s) \in [f_b(r), f_b(s)]$, and the random variable $Z_{b_n} = \sum_{I \in \mathfrak{C}_{b_n}} \mathbf{1}(Z_n(I))$, where $\mathbf{1}(Z_n(I))$ is the indicator function of the event $Z_n(I)$.

Note that, by simple bounds on the transition density of Brownian motion, there is a constant $c_1 > 0$, such that for any sequence $b_n := (2^{-n} \cdot a_n)^2$,

$$c_1 2^{-n} \leq \mathbb{P}(Z_n(I)) \leq 2^{-n} \quad \text{and} \quad c_1 \leq \mathbb{E}(Z_{b_n}) \leq 1. \tag{III.3.3}$$

If the event $Z_{b_n} > 0$ happens for infinitely many n's, then we can find a sequence of intervals $I_k = [r_k, s_k] \in \mathfrak{C}_{b_{n_k}}$, such that $f_b(r_k) \leq B(s_k) \leq f_b(s_k)$, thus $|B(s_k) - f_b(s_k)| \leq 2^{-n_k}$. Since $s_k \in C_{b_{n_k}}$, the sequence (s_k) will have a subsequence converging to some $s \in C_b$, which satisfies $B(s) = f_b(s)$. Therefore

$$\mathbb{P}(\mathcal{Z}(B - f_b) \cap C_b \neq \emptyset) \geq \mathbb{P}(\limsup_{n\to\infty} \{Z_{b_n} > 0\}). \tag{III.3.4}$$

To bound the probabilities $\mathbb{P}(Z_{b_n} > 0)$ from below we apply the Paley-Zygmund inequality,

$$\mathbb{P}(Z_{b_n} > 0) \geq (\mathbb{E} Z_{b_n})^2 / \mathbb{E}(Z_{b_n}^2).$$

Therefore, we have to bound the second moment $\mathbb{E}(Z_{b_n}^2)$ from above. We will use the following expression for the second moment

$$\mathbb{E}(Z_{b_n}^2) = 2 \sum_{\substack{I, J \in \mathfrak{C}_{b_n} \\ I < J}} \mathbb{P}(Z_n(I)) \mathbb{P}(Z_n(J) \mid Z_n(I)) + \mathbb{E}(Z_{b_n}), \tag{III.3.5}$$

(where as previously by $I < J$ we mean that the interval I is located to the left of the interval J and their intersection is at most one point).

Now we fix n and intervals $I = [s_1, t_1]$ and $J = [s_2, t_2]$ from $\mathfrak{C}_{b_n}$, so that $I < J$. Let $x_i = f_b(s_i)$ and $y_i = f_b(t_i)$, for $i = 1, 2$. By the Markov property and the scaling property of Brownian motion, the process

$$\widetilde{B}(t) = (t_2 - t_1)^{-1/2}\Big(B(t_1 + (t_2 - t_1)t) - B(t_1)\Big),$$

is again a Brownian motion, independent of $\mathcal{F}_{t_1}$, and thus independent of the event $Z_n(I) \in \mathcal{F}_{t_1}$.

The event $Z_n(J)$ happens when $\widetilde{B}(1) \in \overline{J}$ for the interval $\overline{J} = [(t_2 - t_1)^{-1/2}(x_2 - B(t_1)), (t_2 - t_1)^{-1/2}(y_2 - B(t_1))]$ of length $(t_2 - t_1)^{-1/2}2^{-n}$.

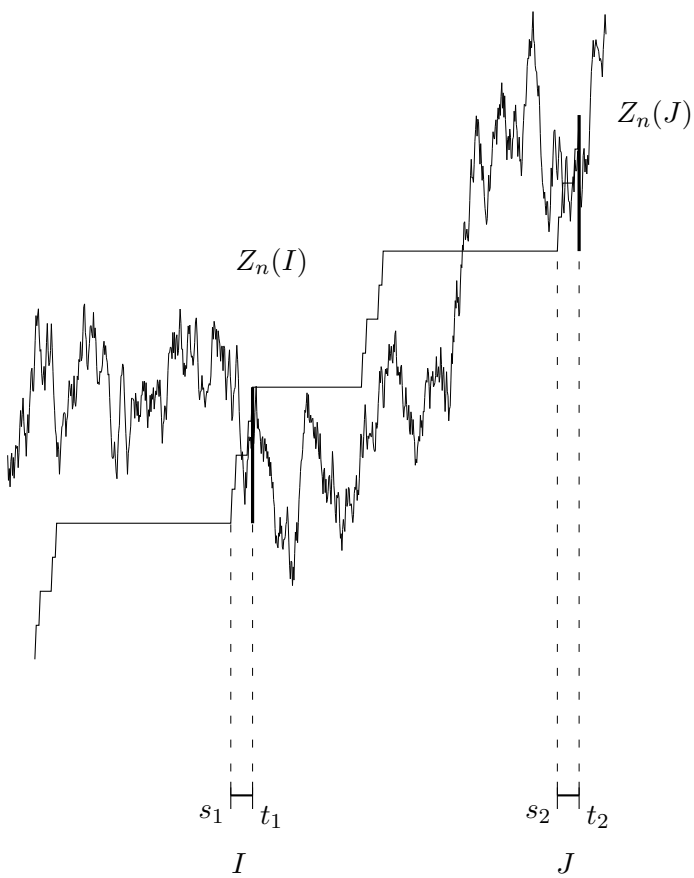

Figure III.2: Events $Z_n(I)$ and $Z_n(J)$ (graph of Brownian motion intersects two bold vertical intervals).

Fix intervals I^0 and J^0 in $\mathfrak{C}_{b_{\ell+1}}$, which are contained in a single interval in $\mathfrak{C}_{b_\ell}$. Assume $I^0 < J^0$ and label the intervals from $\mathfrak{C}_{b_n}$ contained in I^0 by $I_1, \ldots, I_{2^{n-\ell-1}}$, and those contained in J^0 by $J_1, \ldots, J_{2^{n-\ell-1}}$, so that $I_{i+1} < I_i$ and $J_j < J_{j+1}$. Set $I = I_i$ and $J = J_j$ for some

$1 \le i,j \le 2^{n-\ell-1}$, and define x_i, y_i, $\widetilde{B}$ and $\overline{J}$ as before. For $t_2 - t_1$ we use the estimates $\epsilon b_\ell = \epsilon 2^{-2\ell} a_\ell^2 \le 2^{-2\ell}(a_\ell^2 - \frac{1}{2}a_{\ell+1}^2) = b_\ell - 2b_{\ell+1} \le t_2 - t_1 \le b_\ell$. Conditional on $Z_n(I)$, the left endpoint of the interval $\overline{J}$ is at least $(x_2 - y_1)(t_2 - t_1)^{-1/2}$ and the right endpoint is at most $(y_2 - x_1)(t_2 - t_1)^{-1/2}$. Since $x_2 - y_1 = (i+j-2)2^{-n}$ and $y_2 - x_1 = (i+j)2^{-n}$ we have $\overline{J} \subset [(i+j-2)2^{-n} b_\ell^{-1/2}, (i+j)2^{-n}(\epsilon b_\ell)^{-1/2}]$. Because $\overline{J}$ has length at least $(b_\ell)^{-1/2}2^{-n} = \frac{2^{\ell-n}}{a_\ell}$ and at most $(\epsilon b_\ell)^{-1/2}2^{-n} = \frac{2^{\ell-n}}{\sqrt{\epsilon}a_\ell}$ we obtain

$$\begin{aligned}\mathbb{P}(Z_n(J_j) \mid Z_n(I_i)) &= \mathbb{P}(\widetilde{B}(1) \in \overline{J} \mid Z_n(I)) \\ &\le \frac{2^{\ell-n}}{\sqrt{2\pi\epsilon}\, a_\ell} \exp\Big(-\frac{1}{2}((j+i-2)\frac{2^{\ell-n}}{a_\ell})^2\Big).\end{aligned}$$

By summing over $1 \le i,j \le 2^{n-\ell-1}$ it follows that

$$\sum_{1\le i\, j\le 2^{n-\ell-1}} \mathbb{P}(Z_n(J_j) \mid Z_n(I_i)) \le 1 + \frac{1}{\sqrt{2\pi\epsilon}} \sum_{k=1}^{\infty}(k+1)\frac{2^{\ell-n}}{a_\ell} \exp\Big(-\frac{1}{2}(\frac{2^{\ell-n}k}{a_\ell})^2\Big), \quad \text{(III.3.6)}$$

where we used the trivial bound for $i = j = 1$. The sum on the right hand side can be written as

$$S(z) = z\sum_{k=1}^{\infty}(k+1)\exp(-(kz)^2/2),$$

for $z = \frac{2^{\ell-n}}{a_\ell}$. Since $\exp(-t^2/2) \le \exp(-t + 1/2)$, we see that

$$S(z) \le e^{1/2} z \sum_{k\ge 1}(k+1)e^{-kz} = e^{1/2}z^{-1}\Big(\frac{z}{1-e^{-z}}\Big)^2(2e^{-z} - e^{-2z}).$$

Since $z \mapsto (\frac{z}{1-e^{-z}})^2(2e^{-z} - e^{-2z})$ is a bounded function on $\mathbb{R}^+$, (III.3.6) implies that for any fixed I^0 and J^0 as above

$$\sum_{\substack{I,J\in\mathfrak{C}_{b_n}\\ I\subset I^0, J\subset J^0}} \mathbb{P}(Z_n(J) \mid Z_n(I)) \le 1 + c_2\frac{a_\ell}{2^{\ell-n}}, \quad \text{(III.3.7)}$$

for some constant $c_2 > 0$.

Therefore, summing the inequality in (III.3.7) over all I^0 and J^0 and $\ell = 0, \dots, n-1$, and using it together with (III.3.5) and (III.3.3), we have

$$\mathbb{E}(Z_{b_n}^2) \le 2^{-n+1}\sum_{\ell=0}^{n-1}\Big(2^\ell + 2^\ell c_2\frac{a_\ell}{2^{\ell-n}}\Big) + 1 \le 2\sum_{\ell=0}^{n}\Big(2^{-(n-\ell)} + c_2 a_\ell\Big).$$

Now we see that $\mathbb{E}(Z_{b_n}^2)$ is bounded from above if $\sum_{n=1}^{\infty} a_n < \infty$. The lower bound in the second inequality in (III.3.3) and the Paley-Zygmund inequality imply that $\mathbb{P}(Z_{b_n} > 0) \ge \mathbb{E}(Z_{b_n})^2/\mathbb{E}(Z_{b_n}^2)$ is bounded from below and the claim follows for the first case of the theorem from (III.3.4).

Now for the second case pick $I, J \in \mathfrak{C}_{b_n}$ such that $[s_1, t_1] = I < J = [s_2, t_2]$ and define x_i, y_i, $\widetilde{B}$ and $\overline{J}$ as before. By ℓ denote the largest integer such that both I and J are contained in a single interval from $\mathfrak{C}_{b_\ell}$. Assume that $Z_n(I)$ happens. We see that the endpoints of the interval $\overline{J}$ are satisfying that

$$\frac{x_2 - B(t_1)}{(t_2 - t_1)^{1/2}} \geq 0 \quad \text{and} \quad \frac{y_2 - B(t_1)}{(t_2 - t_1)^{1/2}} \leq 2^{-\ell} \cdot \sqrt{\frac{1}{\epsilon}} b_\ell^{-1/2} = \frac{1}{\sqrt{\epsilon} a_\ell}.$$

If $\frac{1}{a_\ell}$ is bounded, then the interval $\overline{J}$ is contained in a compact interval, which does not depend on the choice of n, ℓ, I or J. Using this and the fact that the length of $\overline{J}$ is bounded with $(b_\ell)^{-1/2} 2^{-n} \leq |\overline{J}| \leq (\epsilon b_\ell)^{-1/2} 2^{-n}$, we get that for some positive constants c_3 and c_4 we have

$$c_3 (b_\ell)^{-1/2} 2^{-n} \leq \mathbb{P}(Z_n(J) \mid Z_n(I)) = \mathbb{P}(\widetilde{B}(1) \in \overline{J} \mid Z_n(I)) \leq c_4 (\epsilon b_\ell)^{-1/2} 2^{-n}. \tag{III.3.8}$$

Substituting (III.3.8) and the upper bounds from (III.3.3) into (III.3.5), and summing over all intervals I and J, we obtain

$$\mathbb{E}(Z_{b_n}^2) \leq 1 + 2^{-n+1} \sum_{\ell=0}^{n-1} 2^\ell 2^{2(n-\ell-1)} c_4 (\epsilon b_\ell)^{-1/2} 2^{-n} = 1 + c_5 \sum_{\ell=0}^{n-1} \frac{1}{a_\ell}$$

for some constant c_5. If $\sum_{n=1}^\infty \frac{1}{a_n} < \infty$, then $\mathbb{E}(Z_{b_n}^2)$ is bounded from above by a constant not depending on n, and the claim follows.

□

III.3.2 Condition for having no Cantor set zeros

The following theorem gives a condition for which sequences (b_n) (or (a_n) respectively) $B - f_b$ has no zeros in the generalized Cantor set almost surely.

Theorem III.3.9. *(i) If $\liminf_n a_n < \infty$ and if there is a sequence (c_n) with $\sum_{n=1}^\infty c_n = \infty$ and a fixed but arbitrary small $\delta > 0$ and an $n_0 > 0$ such that for all $n \geq n_0$*

$$\frac{1}{c_n} \geq a_n \geq \sqrt{\frac{1}{(8-\delta)\epsilon \ln \frac{1}{c_n}}}, \tag{III.3.10}$$

then $\mathbb{P}(\mathcal{Z}(B - f_b) \cap C_b \neq \emptyset) = 0$.

(ii) If $\liminf_n a_n = \infty$ and $\frac{1}{a_n} \sum_{\ell=1}^n \frac{1}{a_\ell} \to \infty$ for $n \to \infty$, then $\mathbb{P}(\mathcal{Z}(B - f_b) \cap C_b \neq \emptyset) = 0$.

Examples III.3.11. *The sequence (a_n) defined by $a_n = \sqrt{\frac{1}{\ln n}}$ fulfills the conditions of the Theorem III.3.9(i) (to see that choose $c_n = \frac{1}{n}$) and sequence (a_n) defined by $a_n = n^d$ with some $d < 1/2$ fulfills the conditions of the Theorem III.3.9(ii) since $n^{-d} \int_1^n x^{-d} dx$ goes to infinity for $n \to \infty$.*

By Theorem III.3.9(i) also applies to sequences (a_n) where every element of the sequence is chosen from a fixed finite set of a numbers. Therefore, we see that $\mathbb{P}(\mathcal{Z}(B - f_b) \cap C_b \neq \emptyset) = 0$ holds for all these sequences.

Note that, if two sequences (a_n) and (a'_n) only differ by finitely many numbers, and if one of the sequences fulfills the conditions of one of the Theorems III.3.1 or III.3.9, then the other sequence fulfills the conditions of the same theorem.

Proof of Theorem III.3.9. For an interval $I \in \mathfrak{C}_{b_n}$, define $Y_n(I)$ as the event that Brownian motion hits the graph of f_{b_n} on the interval I, that is a diagonal of the rectangle $I \times f_{b_n}(I)$, and the random variable $Y_{b_n} = \sum_{I \in \mathfrak{C}_{b_n}} \mathbf{1}(Y_n(I))$.

For an interval $I = [x, y] \in \mathfrak{C}_{b_n}$ define R to be the corresponding rectangle $I \times f_b(I)$ and let R_1 be the triangle with the vertices $(x, f_b(x))$, $(y, f_b(y))$ and $(x, f_b(x) + 2^{-n})$ (so it is the upper left triangle of R with respect to the diagonal of R) and $R_2 = R \backslash R_1$ is the lower right triangle part of R.

Fix an $n > 0$ and let C be the event that $B - f_{b_n}$ has a zero that is contained in $\mathfrak{C}_{b_n}$. Define $\overline{R}$ to be the event that $B - f_b$ has a zero that is contained in $\mathfrak{C}_b$ and the corresponding intersection point (by definition it is contained in a rectangle R of described form) of the graph of Brownian motion and the graph of f_b is contained in R_1. Analogously, by $\underline{R}$ denote the the event that $B - f_b$ has a zero that is contained in $\mathfrak{C}_b$ and the corresponding intersection point of the graph of Brownian motion and the graph of f_b is contained in R_2.

Let $\overline{\tau}$ be the first time that $\overline{R}$ happens. Then $\mathbb{P}(B(y) \leq f_b(\overline{\tau})) = 1/2$. The event $\{B(y) \leq f_b(\overline{\tau})\}$ implies that there is an $s \in I$ such that $B(s) = f_{b_n}(s)$, that is $\mathbb{P}(C|\overline{R}) \geq 1/2$.

Now we go backwards in time. By the time reversal property of Brownian motion the process $\bar{B}(t) = B(2) - B(2-t)$ for $t \in [0, 2]$ is again a Brownian motion. Let $\underline{\tau}$ be the first time that $\underline{R}$ happens for the time reversed Brownian motion $B(2-t)$, and let $\bar{x} = 2 - x$. We want to show that $\mathbb{P}(\bar{B}(\underline{\tau}) - \bar{B}(\bar{x}) > 0 | \bar{B}(\underline{\tau}) = B(2) - f_b(2 - \underline{\tau})) \geq \alpha$ for some $\alpha > 0$. In general, for a Brownian motion B the random vector $(B(t) - B(x), B(t))$ has the density

$$\begin{aligned}\psi(p, q) &= \frac{1}{(2\pi)\sqrt{t(t-x)\frac{x}{t}}} \exp\Big(- \frac{t}{2x}\big(\frac{p^2}{(t-x)} - \frac{2pq\sqrt{t-x}}{t\sqrt{t-x}} + \frac{q^2}{t}\big)\Big) \\ &= \frac{1}{(2\pi)\sqrt{x(t-x)}} \exp\Big(- \frac{1}{2}\big(\frac{tp^2}{x(t-x)} - \frac{2pq}{x} + \frac{q^2}{x}\big)\Big).\end{aligned}$$

Therefore, with the substitutions $g = B(2) - f_b(2 - \underline{\tau})$, $p_1 = p - \frac{(\underline{\tau} - \bar{x})g}{\underline{\tau}}$ and $p_2 = p_1\sqrt{\frac{\underline{\tau}}{\bar{x}(\underline{\tau} - \bar{x})}}$ we get

$$\begin{aligned}\mathbb{P}(\bar{B}(\underline{\tau}) - \bar{B}(\bar{x}) > 0 | \bar{B}(\underline{\tau}) = g) &= \int_0^\infty \psi(p, g) dp \\ &\geq \frac{\exp(-\frac{g^2}{2\underline{\tau}})}{(2\pi)\sqrt{\bar{x}(\underline{\tau} - \bar{x})}} \int_{-\frac{\underline{\tau} - \bar{x}}{\underline{\tau}} g}^\infty \exp\Big(- \frac{\underline{\tau} p_1^2}{2\bar{x}(\underline{\tau} - \bar{x})}\Big) dp_1 \\ &= \frac{\exp(-\frac{g^2}{2\underline{\tau}})}{(2\pi)\sqrt{\underline{\tau}}} \int_{-\sqrt{\frac{\underline{\tau} - \bar{x}}{\underline{\tau}\bar{x}}} g}^\infty \exp\Big(- \frac{p_2^2}{2}\Big) dp_2.\end{aligned}$$

Since the right hand side is bounded from below, and the event $\big\{\bar{B}(\underline{\tau}) - \bar{B}(\bar{x}) > 0\big\}$ implies the event C, we get $\mathbb{P}(C|\underline{R}) \geq \alpha$.

Thus, it follows

$$\begin{aligned}\mathbb{P}(\mathcal{Z}(B-f_b)\cap C_b\neq\emptyset) &\leq \mathbb{P}(\overline{R}\cup\underline{R})\\ &\leq \mathbb{P}(\overline{R})+\mathbb{P}(\underline{R})\\ &\leq 2\mathbb{P}(C|\overline{R})\mathbb{P}(\overline{R})+\frac{1}{\alpha}\mathbb{P}(C|\underline{R})\mathbb{P}(\underline{R})\\ &\leq 2\mathbb{P}(C)+\frac{1}{\alpha}\mathbb{P}(C).\end{aligned}$$

Therefore,

$$\mathbb{P}(\mathcal{Z}(B-f_b)\cap C_b\neq\emptyset)\leq(2+\frac{1}{\alpha})\inf_n\mathbb{P}(Y_{b_n}>0). \tag{III.3.12}$$

For a given rectangle that is contained in the square $[1,2]\times[0,1]$ and has the four sides, right side $\mathbf{r}$, left side $\mathbf{l}$, bottom side $\mathbf{b}$ and top side $\mathbf{t}$. Call the events that Brownian motion hits these sides $\overline{r},\overline{l},\overline{b}$ and $\overline{t}$, respectively. Let D be the event that Brownian motion hits the diagonal of the rectangle.

Then, analogously to the above argument, by assuming that the events that the graph of Brownian motion hits each side happen instead of the events $\overline{R}$ or $\underline{R}$, there is a constant $\beta>0$ such that

$$\beta\max\left\{\mathbb{P}(\overline{r}),\mathbb{P}(\overline{l}),\mathbb{P}(\overline{b}),\mathbb{P}(\overline{t})\right\}\leq\mathbb{P}(D). \tag{III.3.13}$$

If Brownian motion hits the diagonal of the rectangle, then it has to intersect at least one of the sides of the rectangle. That gives the following inequality

$$\mathbb{P}(D)\leq\mathbb{P}(\overline{r})+\mathbb{P}(\overline{l})+\mathbb{P}(\overline{b})+\mathbb{P}(\overline{t})\leq 4\max\left\{\mathbb{P}(\overline{r}),\mathbb{P}(\overline{l}),\mathbb{P}(\overline{b}),\mathbb{P}(\overline{t})\right\}. \tag{III.3.14}$$

Assume that the graph hits the bottom side of the rectangle and let τ' be the first such time. Since τ is a stopping time, by strong Markov property the process $B'(t)=B(t+\tau')-B(\tau')$ is a Brownian motion. If there is a constant $c_1>0$ such that $\sqrt{|\mathbf{b}|}\leq c_1|\mathbf{r}|$, we can find a constant $\hat{C}>0$, only depending on c_1, such that the maximum of B' on the interval $[0,|\mathbf{b}|]$ is less than $|\mathbf{r}|$ with probability at least $1/\hat{C}$. Since this event implies the event $\overline{r}$, we have $\mathbb{P}(\overline{r}|\overline{b})\geq 1/\hat{C}$, which implies $\mathbb{P}(\overline{b})\leq\hat{C}\mathbb{P}(\overline{r})$. The inequality $\mathbb{P}(\overline{t})\leq\hat{C}\mathbb{P}(\overline{r})$ can be proven analogously.

If there is a constant $c_2>0$ such that $\sqrt{|\mathbf{b}|}\geq c_2|\mathbf{r}|$, then we can show analogously that there is a constant $C'>0$, only depending on c_2, such that $\mathbb{P}(\overline{l})\leq C'\mathbb{P}(\overline{t})$, and $\mathbb{P}(\overline{l})\leq C'\mathbb{P}(\overline{b})$.

Note that there are constants $c_3>0$ and $c_4>0$ such that $c_3|\mathbf{r}|\leq\mathbb{P}(\overline{r})\leq c_4|\mathbf{r}|$ (follows by simply bounding the transition densities). Assume $\sqrt{|\mathbf{b}|}=|\mathbf{r}|$. By the above inequalities, we get that $\frac{c_3}{C'}\sqrt{|\mathbf{b}|}\leq\mathbb{P}(\overline{b})\leq\hat{C}c_4\sqrt{|\mathbf{b}|}$.

Therefore, and since the graph of the restriction of f_{b_n} to an interval $I\in\mathcal{C}_{b_n}$ is a diagonal of a rectangle of width $2^{-2n}a_n^2$ and height 2^{-n}, there are constants C_1 and C_2 such that

$$\left.\begin{array}{ll}C_1 2^{-n}, & \text{if } a_n\leq 1,\\ C_1 2^{-n}a_n, & \text{if } a_n>1,\end{array}\right\}\leq\mathbb{P}(Y_n(I))\leq\left\{\begin{array}{ll}C_2 2^{-n}, & \text{if } a_n\leq 1,\\ C_2 2^{-n}a_n, & \text{if } a_n>1.\end{array}\right. \tag{III.3.15}$$

Therefore,

$$\left.\begin{array}{ll} C_1, & \text{if } a_n \le 1, \\ C_1 a_n, & \text{if } a_n > 1, \end{array}\right\} \le \mathbb{E}(Y_{b_n}) \le \left\{\begin{array}{ll} C_2, & \text{if } a_n \le 1, \\ C_2 a_n, & \text{if } a_n > 1. \end{array}\right. \tag{III.3.16}$$

For $I \in \mathfrak{C}_{b_n}$ and $0 \le \ell < n$, let I^ℓ denote the interval from $\mathfrak{C}_{b_\ell}$ that contains I, and let $I_1^\ell, I_2^\ell \in \mathfrak{C}_{b_{\ell+1}}$ be the left and right subintervals of I^ℓ, respectively.

Now we define an binary address $v_1 v_2 \dots v_n$ for I. Namely, if $I \subset I_1^\ell$ then $v_{\ell+1} = 0$ and if $I \subset I_2^\ell$ then $v_{\ell+1} = 1$.

Call an interval I *balanced* if the sequence $v_1, \dots, v_n$ contains at least $n/3$ zeros and otherwise unbalanced. For a balanced interval $I \in \mathfrak{C}_{b_n}$ let A_I denote the event that I is the leftmost balanced interval for which $Y_n(I)$ happens, and, as before, let $v_1 \dots v_n$ denote the binary address of I. Let τ be the first time that $f_{b_n}(t) = B(t)$ with $t \in I$ happens.

We look again at the event $Z_n(I)$ (see proof of Theorem III.3.1). Fix $n > 0$, and an interval $J = [x, y] \in \mathfrak{C}_{b_n}$, so that $I < J$. Let $\widetilde{B}$ be the process

$$\widetilde{B}(t) = (y-\tau)^{-1/2}\Big(B(\tau + (y-\tau)t) - B(\tau)\Big),$$

which is, by the Markov property and Brownian scaling, again a Brownian motion, independent of $\mathcal{F}_\tau$.

Let $\overline{J}$ be the interval $[(y-\tau)^{-1/2}(f_{b_n}(x) - B(\tau)), (y-\tau)^{-1/2}(f_{b_n}(y) - B(\tau))]$ of length $(y-\tau)^{-1/2}2^{-n}$.

Assume that for some $0 \le \ell < n$ we have $v_{\ell+1} = 0$. I_2^ℓ contains $2^{n-\ell-1}$ intervals, we label them by $J_1, \dots, J_{2^{n-\ell-1}}$ with $J_j < J_{j+1}$. $\overline{J_j}$ has length at least $(b_\ell)^{-1/2}2^{-n} = \frac{2^{\ell-n}}{a_\ell}$ and at most $(\epsilon b_\ell)^{-1/2}2^{-n} = \frac{2^{\ell-n}}{\sqrt{\epsilon}a_\ell}$ and $\overline{J_j} \subset [(j-1)2^{-n}b_\ell^{-1/2}, (j+2^{n-\ell-1})2^{-n}(\epsilon b_\ell)^{-1/2}]$. Then

$$\mathbb{P}(Z_n(J_j) \mid A_I) = \mathbb{P}(\widetilde{B}(1) \in \overline{J_j} \mid A_I) \tag{III.3.17}$$

$$\ge \int_{(j+2^{n-\ell-1})\frac{2^{\ell-n}}{\sqrt{\epsilon}a_\ell} - \frac{2^{\ell-n}}{a_\ell}}^{(j+2^{n-\ell-1})\frac{2^{\ell-n}}{\sqrt{\epsilon}a_\ell}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}t^2)dt. \tag{III.3.18}$$

We can use the following lower bound

$$\mathbb{P}(Z_n(J_j) \mid A_I) \ge \frac{2^{\ell-n}}{\sqrt{2\pi}a_\ell} \exp\Big(-\frac{1}{2}((j+2^{n-\ell-1})\frac{2^{\ell-n}}{\sqrt{\epsilon}a_\ell})^2\Big). \tag{III.3.19}$$

Summing over all $J_1, \dots, J_{2^{n-\ell-1}}$ and over all ℓ such that $v_{\ell+1} = 0$ gives

$$\mathbb{E}(Z_{b_n} \mid A_I) \ge \sum_{1 \le \ell \le n : v_\ell = 0} \sum_{1 \le j \le 2^{n-\ell-1}} \frac{2^{\ell-n}}{\sqrt{2\pi}a_\ell} \exp\Big(-\frac{1}{2}((j+2^{n-\ell-1})\frac{2^{\ell-n}}{\sqrt{\epsilon}a_\ell})^2\Big).$$

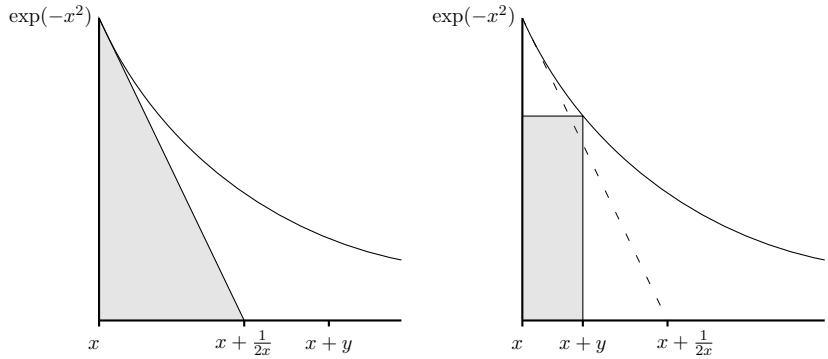

Figure III.3: If $x + \frac{1}{2x} \leq x + y$, then we bound $\int_x^{x+y} \exp(-t^2)dt$ from below by the area of a triangle. If $x + \frac{1}{2x} > x + y$, then we use the area of a rectangle to bound this integral from below.

Estimating the inner sum by integration gives

$$\sum_{1\leq j\leq 2^{n-\ell-1}} \frac{2^{\ell-n}}{\sqrt{2\pi}a_\ell} \exp\Big(-\frac{1}{2}((j+2^{n-\ell-1})\frac{2^{\ell-n}}{\sqrt{\epsilon}a_\ell})^2\Big) \tag{III.3.20}$$

$$\geq \int_1^{2^{n-\ell-1}+1} \frac{2^{\ell-n}}{\sqrt{2\pi}a_\ell} \exp\Big(-\frac{1}{2}((j+2^{n-\ell-1})\frac{2^{\ell-n}}{\sqrt{\epsilon}a_\ell})^2\Big)dj \tag{III.3.21}$$

$$= \sqrt{\frac{\epsilon}{\pi}} \int_{\frac{2^{\ell-n}+1/2}{\sqrt{2\epsilon}a_\ell}}^{\frac{2^{\ell-n}+1}{\sqrt{2\epsilon}a_\ell}} \exp(-t^2)dt. \tag{III.3.22}$$

If $2\epsilon a_\ell^2 \leq 2^{\ell-n} + 1/2$ holds we can use the estimate $\int_x^{x+y} \exp(-t^2)dt \geq \frac{\exp(-x^2)}{4x}$ (see Figure III.3). In case $\epsilon a_\ell^2 \leq 1/4$ we will use this estimate and $\int_x^{x+y} \exp(-t^2)dt \geq y\exp(-(x+y)^2)$ otherwise. Using the first estimate we get that (III.3.20) is at least $\frac{\sqrt{2}\epsilon a_\ell}{4\sqrt{\pi}(2^{\ell-n}+1/2)} \exp(-(\frac{2^{\ell-n}+1/2}{\sqrt{2\epsilon}a_\ell})^2)$ and in the second $\frac{1}{2\sqrt{2}a_\ell} \exp(-(\frac{2^{\ell-n}+1}{\sqrt{2\epsilon}a_\ell})^2)$.

Therefore, using the trivial bound 0 for $\ell = n$ and $\ell = n-1$, we get

$$\mathbb{E}(Z_{b_n} \mid A_I) \geq \sum_{1\leq\ell\leq n-2:\, v_\ell=0,\, \epsilon a_\ell^2\leq 1/4} \frac{\sqrt{2}\epsilon a_\ell}{4\sqrt{\pi}(2^{\ell-n}+1/2)} \exp(-(\frac{2^{\ell-n}+1/2}{\sqrt{2\epsilon}a_\ell})^2) \tag{III.3.23}$$

$$+ \sum_{1\leq\ell\leq n-2:\, v_\ell=0,\, \epsilon a_\ell^2>1/4} \frac{1}{2\sqrt{2\pi}a_\ell} \exp(-(\frac{2^{\ell-n}+1}{\sqrt{2\epsilon}a_\ell})^2)$$

$$\geq \sum_{1\leq\ell\leq \frac{5}{6}n:\, v_\ell=0,\, \epsilon a_\ell^2\leq 1/4} \frac{\sqrt{2}\epsilon a_\ell}{3\sqrt{\pi}} \exp(-(\frac{2^{\ell-n}+1/2}{\sqrt{2\epsilon}a_\ell})^2)$$

$$+ \sum_{1\leq\ell\leq n-2:\, v_\ell=0,\, \epsilon a_\ell^2>1/4} \frac{1}{2\sqrt{2\epsilon}a_\ell} \exp(-(\frac{5}{4\sqrt{2\pi}a_\ell})^2).$$

Since the events A_I are disjoint, we have

$$\mathbb{P}(Y_n(I) \text{ for some balanced interval } I) \leq \frac{\mathbb{E}(Y_{b_n})}{\mathbb{E}(Y_{b_n} \mid Y_n(I) \text{ for some balanced interval } I)}. \quad \text{(III.3.24)}$$

Recall (III.3.13). To estimate the probability that $Y_n(I)$ happens for some unbalanced interval I notice that the number of such intervals is bounded from above by $e^{-c_5 n}2^n$ for some constant $c_5 > 0$. By (III.3.15) this gives

$$\mathbb{P}(Y_n(I) \text{ for some unbalanced interval } I) \leq \begin{cases} e^{-c_5 n}, & \text{if } a_n \leq 1, \\ e^{-c_5 n} a_n, & \text{if } a_n > 1. \end{cases} \quad \text{(III.3.25)}$$

But note for the case of $a_n > 1$ that we note that $a_n \leq 2^{n/2}$. Thus, (III.3.25) goes to 0 for $n \to \infty$.

We consider the following two cases.

(i) Assume now that $\liminf_n a_n < \infty$. That means that there is an $k > 0$ such that $a_n \leq k$ holds for infinitely many n's. Also, assume there is a sequence (c_n) with $\sum_{n=1}^{\infty} c_n = \infty$ and fixed but arbitrary small $\delta > 0$ and an $n_0 > 0$ such that for all $n \geq n_0$

$$\frac{1}{c_n} \geq a_n \geq \sqrt{\frac{1}{(8-\delta)\epsilon \ln \frac{1}{c_n}}}.$$

Now take the subsequence (a_{n_k}) such that for all $n_k \geq n_0$

$$\frac{1}{2\sqrt{\epsilon}} \geq a_{n_k} \geq \sqrt{\frac{1}{(8-\delta)\epsilon \ln \frac{1}{c_{n_k}}}}.$$

Note that this subsequence does not have to be an infinite sequence. The right inequality is equivalent to

$$-\frac{1}{(8-\delta)\epsilon a_{n_k}^2} \geq \ln c_{n_k}.$$

From this it follows that

$$-\frac{1}{8\epsilon a_{n_k}^2} + \ln a_{n_k} \geq \ln c_{n_k},$$

and

$$\exp\left(-\frac{1}{8\epsilon a_{n_k}^2}\right) \cdot a_{n_k} \geq c_{n_k}.$$

Now take the complementary subsequence $(a_{n_{k'}})$ such that for all $n_{k'} \geq n_0$

$$\frac{1}{c_{n_{k'}}} \geq a_{n_{k'}} \geq \frac{1}{2\sqrt{\epsilon}}.$$

The left inequality is equivalent to $\frac{1}{a_{n_{k'}}} \geq c_{n_{k'}}$, and if $\sum_{n_{k'}} (1-\delta)\frac{1}{a_{n_{k'}}} = \infty$ for a fixed but arbitrary small $\delta > 0$, then it also holds that

$$\sum_{n_{k'}} \frac{1}{a_{n_{k'}}} \exp\left(-\left(\frac{5}{4\sqrt{2\epsilon} a_{n_{k'}}}\right)^2\right) = \infty.$$

It follows that those sequences (a_n) fulfilling the above assumptions yield that the liminf of (III.3.24) is 0 by (III.3.23), (**??**) and (III.3.16). Thus together with (III.3.25) it follows $\inf_n \mathbb{P}(Y_{b_n} > 0) = 0$.

(ii) Assume now that $\liminf_n a_n = \infty$. That means that $a_n > k$ holds for all n's with only finitely many exceptions for a arbitrarily chosen but fixed $k > 0$. Also, assume $\frac{1}{a_n}\sum_{\ell=1}^{n}\frac{1}{a_\ell} \to \infty$ for $n \to \infty$. It follows that those sequences (a_n) fulfilling these assumptions yield that the liminf of (III.3.24) is 0 by (III.3.23), (**??**) and (III.3.16). Thus together with (III.3.25) it follows $\inf_n \mathbb{P}(Y_{b_n} > 0) = 0$.

The claim follows from (III.3.12).

□

III.3.3 Modifications

If we require the sequence (a_n) to fulfill $a_n^2 - \frac{1}{2}a_{n+1}^2 \geq x_n$ for some positive sequence (x_n) with $x_n < a_n^2$ for all n instead of the condition $a_n^2 - \frac{1}{2}a_{n+1}^2 \geq \epsilon a_n^2$ for all n that we used so far, then the analogue to the Theorems III.3.1 and III.3.9 is the following result.

Corollary III.3.26. *If the sequence (a_n) fulfills $a_n^2 - \frac{1}{2}a_{n+1}^2 \geq x_n$ for some positive sequence (x_n) with $x_n < a_n^2$ for all n, then $\mathbb{P}(\mathcal{Z}(B - f_b) \cap C_b \neq \emptyset) > 0$ holds if either $\sum_{n=1}^{\infty}\frac{a_n^2}{\sqrt{x_n}} < \infty$, or $\sum_{n=1}^{\infty}\frac{1}{\sqrt{x_n}} < \infty$, and $\mathbb{P}(\mathcal{Z}(B - f_b) \cap C_b \neq \emptyset) = 0$ holds if*

(i) $\liminf_n a_n < \infty$ and there is a sequence (c_n) with $\sum_{n=1}^{\infty} c_n = \infty$ and a fixed but arbitrary small $\delta > 0$ and an $n_0 > 0$ such that for all $n \geq n_0$

$$\frac{1}{c_n^2} \geq x_n \geq \frac{1}{(8-\delta)\ln\frac{1}{c_n}},$$

or if

(ii) $\liminf_n a_n = \infty$ and $\frac{1}{a_n}\sum_{\ell=1}^{n}\frac{1}{\sqrt{x_\ell}} \to \infty$ for $n \to \infty$.

Proof. Analogously to the proofs of the Theorems III.3.1 and III.3.9. □

Theorems III.3.1 and III.3.9 do not give an answer for certain sequences (a_n), for instance $a_n = \frac{1}{n}$, whether or not the zero set of $B - f_b$ contains points of the corresponding generalized Cantor set with positive probability. It is natural to ask if we can strengthen the methods we used to get a stronger result.

Note that by (III.3.13) and (III.3.14) the probability of the event that Brownian motion hits a diagonal of a rectangle is up to a constant the maximum of the probabilities of the events that that Brownian motion hits the lower horizontal side or the right vertical side of the rectangle. Thus, it worth looking at the event that Brownian motion hits the lower horizontal side of the rectangle instead of the event $Z_n(J_j)$ for the estimate (III.3.17). Call the former event $X_n(J_j)$, then we get instead of (III.3.17)

$$\mathbb{P}(X_n(J_j) \mid A_l) \geq \int_x^{x+2^{-2n}a_n^2} \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(2^{-\ell-1} + 2^{-n}j)^2}{2t})dt$$

for some point $x \in [\sqrt{\epsilon}2^{-2\ell}a_\ell^2, 2^{-2\ell}a_\ell^2)$. But here we see that summing over all j and ℓ would not give an expression that goes to infinity for $n \to \infty$ for any possible sequence (a_n). Thus, looking at the event that that Brownian motion hits the lower horizontal side cannot provide an improvement of the result III.3.9.

Comparing

$$\int_{(j+2^{n-\ell-1})\frac{2^{\ell-n}}{\sqrt{\epsilon}a_\ell}-\frac{2^{\ell-n}}{a_\ell}}^{(j+2^{n-\ell-1})\frac{2^{\ell-n}}{\sqrt{\epsilon}a_\ell}} \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}t^2)dt \leq \frac{2^{\ell-n}}{\sqrt{2\pi}a_\ell}\exp\Big(-\frac{1}{2}((j+2^{n-\ell-1})\frac{2^{\ell-n}}{\sqrt{\epsilon}a_\ell}-\frac{2^{\ell-n}}{a_\ell})^2\Big),$$

to the estimate (III.3.19), and

$$\begin{aligned}\sum_{1\leq j\leq 2^{n-\ell-1}} \frac{2^{\ell-n}}{\sqrt{2\pi}a_\ell}\exp\Big(-\frac{1}{2}((j+2^{n-\ell-1})\frac{2^{\ell-n}}{\sqrt{\epsilon}a_\ell}-\frac{2^{\ell-n}}{a_\ell})^2\Big)&\\ \leq \int_0^{2^{n-\ell-1}} \frac{2^{\ell-n}}{\sqrt{2\pi}a_\ell}\exp\Big(-\frac{1}{2}((j+2^{n-\ell-1})\frac{2^{\ell-n}}{\sqrt{\epsilon}a_\ell}-\frac{2^{\ell-n}}{a_\ell})^2\Big)dj&\\ = \sqrt{\frac{\epsilon}{\pi}}\int_{\frac{\frac{1}{2}-\sqrt{\epsilon}2^{\ell-n}}{\sqrt{2\epsilon}a_\ell}}^{\frac{1-\sqrt{\epsilon}2^{\ell-n}}{\sqrt{2\epsilon}a_\ell}} \exp(-t^2)dt&\\ \leq \frac{1}{2\sqrt{2\epsilon}a_\ell}\exp(-(\frac{\frac{1}{2}-\sqrt{\epsilon}2^{\ell-n}}{\sqrt{2\epsilon}a_\ell})^2),&\end{aligned}$$

to (III.3.20) and (III.3.23) we see that for the inequality (III.3.10) we cannot improve the result III.3.9 by using better bounds.

For the proof of Theorem III.3.1 it is easy to check that better estimates can not provide a stronger result. In particular, by bounding $S(z)$ from above we see that the second moment can at most be improved by a constant factor.

Remark III.3.27. *Theorems III.3.1 and III.3.9 can be extended to more classes of Cantor-like functions. We consider for a given positive, real sequence (a_n) a sequence $(b_{k,n})$ defined by $b_{k,n} := (k^{-n}\cdot a_n)^2$ with $k > 0$. For this sequence we define a corresponding k-Cantor-type set and denote it by C_{b_k}. Take a closed interval I of length $b_{k,0}$. Define $\mathfrak{C}_{b_{k,1}}$ as the set consisting of k disjoint closed subintervals of I of length $b_{k,1}$, the left one (for which the left endpoint coincides with the left endpoint of I), the right one (for which the right endpoint coincides with the right endpoint of I) and $k-2$ intervals such that two neighboring intervals have the distance $\frac{b_{k,0}-k\,b_{k,1}}{k-1}$. Continue recursively, if $J \in \mathfrak{C}_{b_n}$, then include in the set $\mathfrak{C}_{b_{n+1}}$ its k closed subintervals of length $b_{k,n+1}$. Define the set C_{b_n} as the union of all the intervals from $\mathfrak{C}_{b_n}$. For any n, the family $\mathfrak{C}_{b_{k,n}}$ is the set of all connected components of the set $C_{b_{k,n}}$. The k-Cantor set is a compact set defined as $C_{b_k} = \bigcap_{n=1}^{\infty} C_{b_{k,n}}$. Now we construct a k-Cantor-type function corresponding to the k-Cantor type set above. Define the function $f_{b_{k,1}}$ so that it has values 0 and 1 at the left and the right endpoint of the interval I, respectively, values $1/k$ on the most left, $2/k$ on the second most left, ..., and $(k-1)/k$ on the least most left of the $k-1$ disjoint intervals of the set $I\backslash C_{b_{k,1}}$, and interpolate*

linearly on the intervals in $\mathfrak{C}_{b_{k,1}}$. *Recursively, construct the function* $f_{b_{k,n+1}}$ *so that for every interval* $J = [s,t] \in \mathfrak{C}_{b_{k,n}}$, *the function* $f_{b_{k,n+1}}$ *agrees with* $f_{b_{k,n}}$ *at* s *and* t, *it has values* $(f_{b_{k,n}}(t) - f_{b_{k,n}}(s)) \cdot \frac{1}{k} + f_{b_{k,n}}(s)$ *on the most left,* $(f_{b_{k,n}}(t) - f_{b_{k,n}}(s)) \cdot \frac{2}{k} + f_{b_{k,n}}(s)$ *on the second most left, ..., and* $(f_{b_{k,n}}(t) - f_{b_{k,n}}(s)) \cdot \frac{k-1}{k} + f_{b_{k,n}}(s)$ *on the least most left of the* $k-1$ *disjoint intervals of the set* $J \backslash C_{b_{k,n+1}}$ *and interpolate linearly on the intervals in* $\mathfrak{C}_{b_{k,n+1}}$. *See Figure III.4 for an example of* $k = 3$.

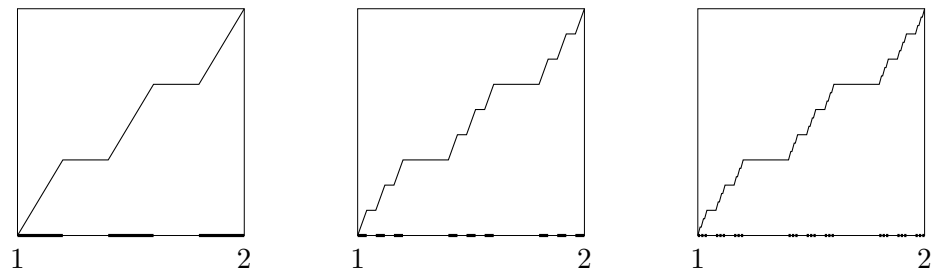

Figure III.4: First three approximations of the 3-Cantor type function on the interval $[1,2]$ (functions $f_{b_{3,n}}$ from the construction for $b_1 = 0.2$, $b_2 = 0.04$ and $b_3 = 0.01$, or $a_1 = \sqrt{1.8}$, $a_2 = 1.8$ and $a_3 = 2.7$, respectively. Approximations of the Cantor set C_{b_3} are drawn in bold).

As in section III.2, for a fixed $k > 1$ *we fix an arbitrary small* $\epsilon > 0$ *and require the sequence* (a_n) *to fulfill the condition* $a_n^2 - \frac{1}{k} a_{n+1}^2 \geq \epsilon a_n^2$ *(or equivalently* $b_{k,n} - k b_{k,n+1} \geq \epsilon b_{k,n}$*) for all* n. *Then, Theorems III.3.1 and III.3.9 hold also for the* k*-Cantor-type function.*

Further, note that if we have the weaker condition $b_{k,n} - k b_{k,n+1} > 0$ *for all* n, *then we can apply Corollary III.3.26.*

III.4 Isolated zeros - general criteria revisited

We will now state two more criteria determining whether a zero of $B - f$ for a continuous function f is almost surely isolated or not isolated. Recall that for any function g defined on some subset (or the whole) of $\mathbb{R}^+$ we denote by $\mathcal{Z}(g)$ the set of zeros of g in $(0, \infty)$.

Proposition III.4.1. *Let* $f : \mathbb{R}^+ \to \mathbb{R}$ *be a continuous function.*

(i) Let A *be a closed subset of* $\mathbb{R}^+$ *such that for any* $t \in A$

$$\liminf_{s \to t} \frac{|f(s) - f(t)|}{\sqrt{2|t-s| \log\log \frac{1}{|t-s|}}} > 1.$$

Then, almost surely any point in $\mathcal{Z}(B - f) \cap A$ *is isolated in* $\mathcal{Z}(B - f)$.

(ii) Let $A \subset \mathbb{R}^+$ *be a set such that for any* $t \in A$

$$\limsup_{s \to t} \frac{|f(s) - f(t)|}{\sqrt{2|t-s| \log\log \frac{1}{|t-s|}}} < 1.$$

Then, almost surely any point in $\mathcal{Z}(B-f)\cap A$ is not isolated in $\mathcal{Z}(B-f)$.

Proof. (i) We define a sequence of stopping times (τ_n). Let $\tau_0 = \min\{t \in A : B(t) = f(t)\}$ and $\tau_n = \min\{t \in A, t > \tau_{n-1} : B(t) = f(t)\}$. Since τ_n is a stopping time for every n, $t \mapsto B(\tau_n + t) - B(\tau_n)$ is a Brownian motion if $\tau_n < \infty$. We can apply the law of iterated logarithm (see Theorem 5.1 in [MP]) to get that almost surely for all n we have

$$\liminf_{t\downarrow 0} \frac{B(\tau_n+t)-B(\tau_n)}{\sqrt{2t\log\log\frac{1}{t}}} = -1 \quad \text{and} \quad \limsup_{t\downarrow 0} \frac{B(\tau_n+t)-B(\tau_n)}{\sqrt{2t\log\log\frac{1}{t}}} = 1.$$

Thus all τ_n's are isolated from the right. By the reverse property of Brownian motion all τ_n's are also isolated from the left. This implies that τ_n converges to ∞ since A is a closed set. Therefore, every zero in A is contained in the sequence (τ_n).

(ii) Assume that there exists a an isolated zero in the set A with positive probability. Then, there is a $q \in \mathbb{Q}$ such that $\tau_q = \min\{t \geq q : B(t) = f(t)\}$ is an isolated zero in A.

Since τ_q is a stopping time, the process $B_q(t) = B(\tau_q + t) - B(\tau_q)$, is, by the strong Markov property, a Brownian motion independent of the sigma algebra $\mathcal{F}_{\tau_q}$. By the law of the iterated logarithm it follows that τ_q is not isolated.

□

Note that (i) is a stronger statement than Proposition II.2.1(i) for closed subsets of $\mathbb{R}^+$. Recall that according to Proposition II.2.1(ii) almost surely all isolated points of $\mathcal{Z}(B-f)$ are located inside the set $A_f^+ \cup A_f^-$, where

$A_f^+ = \{t \in \mathbb{R}^+ : \lim_{h\downarrow 0} \frac{f(t+h)-f(t)}{\sqrt{h}} = \infty\}$ and

$A_f^- = \{t \in \mathbb{R}^+ : \lim_{h\downarrow 0} \frac{f(t+h)-f(t)}{\sqrt{h}} = -\infty\}$. In particular, (ii) shows that not all points in $\{A_f^+ \cup A_f^-\} \cap \mathcal{Z}(B-f)$ have to be isolated in $\mathcal{Z}(B-f)$.

III.5 Isolated zeros of Brownian Motion minus Cantor function

We will now go back to the question of having isolated zeros, and extend the results of chapter II for the generalized Cantor functions we defined in this chapter. We showed for the middle $1-2\gamma$-Cantor function f_γ that $B - f_\gamma$ has isolated zeros with positive probability if $\gamma < 1/4$ (cf. Proposition II.3.12). Recall that for $a_n = \frac{1}{x^n}$ with some $x > 1$ gives $b_n := (2^{-n} \cdot \frac{1}{x^n})^2 = (2x)^{-2n}$ which corresponds to $\gamma = \frac{1}{(2x)^2}$. Therefore, the following proposition extends this result of Proposition II.3.12.

Proposition III.5.1. *The set $\mathcal{Z}(B-f_b)$ has isolated points with positive probability if $\sum_{n=1}^\infty a_n < \infty$, and no isolated points almost surely if $\sum_{n=1}^\infty \frac{1}{a_n} < \infty$.*

Proof. First we will prove that if $\sum_{n=1}^\infty a_n < \infty$, then $\mathcal{Z}(B-f_b)$ has isolated points with positive probability. For $A \subset \mathbb{R}^+$ define $Z(A)$ as the event $\{\mathcal{Z}(B-f_b)\cap A \neq \emptyset\}$. We claim that there exists a constant c_1, such that for any interval $J \subset [0,1]$ of length $|J|$, we have

$$\mathbb{P}(Z(C_b \cap f_b^{-1}(J))) \leq c_1|J|. \tag{III.5.2}$$

In order to show this statement, fix an interval J and take the biggest integer n satisfying $|J| \leq 2^{-n}$. Notice that J can be covered by two consecutive binary intervals J_1 and J_2 of length 2^{-n}. Moreover, there are consecutive $I_1, I_2 \in \mathfrak{C}_{b_n}$ such that $f_b(I_i) = J_i$ for $i = 1, 2$, and $C_b \cap f_b^{-1}(J) \subset I_1 \cup I_2$.

Now we will again use the notation that we have used already in the proof of Theorem III.3.1. Assume that $\mathcal{Z}(B - f_b) \cap C_b \neq \emptyset$ and denote the first zero of $B(t) - f_b(t)$ in the generalized Cantor set C_b by τ (τ exists since $\mathcal{Z}(B - f_b) \cap C_b$ is a closed set). For an interval $I = [s, t] \in \mathfrak{C}_{b_n}$ assume that $\tau \in I$. Since τ is a stopping time, and by Brownian scaling, the conditional probability $\mathbb{P}\Big(Z_n(I) \mid \mathcal{F}_\tau, \tau \in I\Big)$ is equal to the probability that Brownian motion at time 1 is between $y_1 = (f_b(s) - f_b(\tau))(t - \tau)^{-1/2}$ and $y_2 = (f_b(t) - f_b(\tau))(t - \tau)^{-1/2}$. Since $f_b(s) \leq f_b(\tau) \leq f_b(t)$ we see that $y_1 \leq 0$ and $y_2 \geq 0$. Moreover, for big enough n, $t - \tau \leq 4^{-n} = (f_b(t) - f_b(s))^2$ leads to $y_2 - y_1 \geq 1$.

Thus we can bound the probability

$$\mathbb{P}\Big(Z_n(I) \mid \mathcal{F}_\tau, \tau \in I\Big) \geq \mathbb{P}(0 \leq B(1) \leq 1) = \alpha, \tag{III.5.3}$$

for some $\alpha > 0$.Therefore,

$$\alpha \mathbb{P}(Z(C_b \cap I_i)) \leq \mathbb{P}(Z_n(I_i)). \tag{III.5.4}$$

But by the first inequality in (III.3.3), the probability on the right hand side of (III.5.4) is bounded from above by 2^{-n}. Using this fact in (III.5.4) and summing the expression for $i = 1, 2$, we obtain (III.5.2).

By Theorem III.3.1 the set $\mathcal{Z}(B - f_b) \cap C_b$ is non-empty with some probability $p > 0$. Take $n_0 > 0$ such that for all $n \geq n_0$ we can find an arbitrary $b'_n = (2^{-n} a'_n)^2$ with $b_n < b'_n < 2^{-2n}$ and such that $\sum_{n \geq n_0} a'_n \leq p/(2c_1)$.

For $n \geq n_0$ consider the interval $J_{k,n} = [k2^{-n} - \frac{1}{2}\sqrt{b'_n}, k2^{-n} + \frac{1}{2}\sqrt{b'_n}]$ and define the set $M_{n_0} = \bigcup_{n \geq n_0} \bigcup_{0 \leq k \leq 2^n} J_{k,n}$. By (III.5.2) and the choice of n_0, we see that $\mathbb{P}(Z(C_b \cap f_b^{-1}(M_{n_0}))) \leq p/2$. Hence, the event that there is a zero of $B(t) - f_b(t)$ in the set $C_b \cap \mathrm{Int}(C_{b_{n_0}}) \backslash f_b^{-1}(M_{n_0})$ has probability of at least $p/2$ (here $\mathrm{Int}(C_{b_{n_0}})$ is the interior of the set $C_{b_{n_0}}$). Now the claim follows if we prove that any such zero is isolated. Take $t \in C_b \cap \mathrm{Int}(C_{b_{n_0}}) \backslash f_b^{-1}(M_{n_0})$ and any $s \neq t$ in the same connected component of $\mathrm{Int}(C_{b_{n_0}})$. The biggest integer ℓ such that both s and t are contained in the same interval of $\mathfrak{C}_{b_\ell}$ satisfies $\ell \geq n_0$. Further, $|f_b(s) - f_b(t)| \geq \frac{1}{2}\sqrt{b'_\ell}$ and $b_{\ell+1} \leq |s - t| \leq b_\ell$. With Proposition III.4.1(i) the claim follows by taking for instance $a'_n = \sqrt{a_n}$ for n big enough.

For the second part of the claim we just need to apply the Proposition II.2.1(i). □

Remarks and open problems

Instead of varying the size of the intervals where the approximating Cantor function increases at each iteration step as we did in this chapter we could vary the length of the image of these intervals, that is the vertical size of the increasing parts of the graph, at each iteration step. So far, at the iteration step n the vertical size was 2^{-n}. It is natural to ask whether there are Cantor set zeros in this case if we add Brownian motion? For instance, let the vertical size of the two steep parts of the first iteration be β and $1 - \beta$, $\beta \in (0, 1)$ instead of $1/2$

and 1/2 as for the classical Cantor function (see Figure III.5), and continue iterating like this (with a fixed β or even with different sizes at each iteration level). Then, we conjecture that the limiting function might behave similarly to the second example of the previous chapter in the sense that the bound for which values β there are isolated zeros when the function is added to Brownian motion (cf. Proposition II.5.7) might not be explicit.

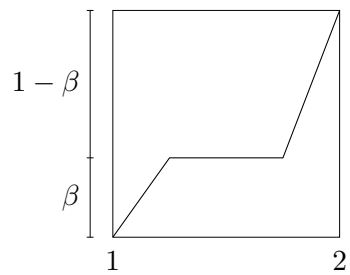

Figure III.5: First iteration of another Cantor-like function with $\beta \in (0, 1)$.

Chapter IV

On Hausdorff dimension of zero sets

IV.1 Introduction

This chapter (together with Chapter II) is joint work with Tonci Antunović, Kris Burdzy and Yuval Peres. Most parts were published in the paper [ABPR] in the Electronic Journal of Probability.

In this chapter we investigate "how big" the zero set of the process $B - f$ is. In order to quantify the size of the zero set Hausdorff dimension is the appropriate tool to use. Let us briefly recall the definition.

Hausdorff dimension

Let $A \in \mathbb{R}^d$, then for every $\alpha \geq 0$ the α-*Hausdorff measure* of the set A is defined by

$$\mathcal{H}^\alpha(A) = \lim_{\delta \downarrow 0} \inf \Big\{ \sum_{i=0}^{\infty} |U_i|^\alpha \;\Big|\; A \subset \bigcup_{i=0}^{\infty} U_i, |U_i| \leq \delta, \ \forall\ i \Big\}.$$

Now the *Hausdorff dimension* of the set A is defined by $\dim(A) = \inf\{\alpha : \mathcal{H}^\alpha(A) = 0\}$. To learn more about Hausdorff dimension and its properties we recommend the book [Fal].

Part (ii) of Proposition II.2.1 shows that isolated zeros of the process $B - f$ can occur only where the function f increases or decreases very quickly. In the following theorem we bound the Hausdorff dimension of such sets.

Theorem IV.1.1. *For any continuous function $f : \mathbb{R}^+ \to \mathbb{R}$ there exists a set A_f, such that $\dim(A_f) \leq 1/2$ and such that, almost surely, all isolated points of $\mathcal{Z}(B - f)$ are contained in A_f.*

It is a classical result that the zero set of Brownian motion has Hausdorff dimension 1/2 almost surely, see Theorem 4.24 in [MP]. Of course, for any compact interval I not containing 0 and any continuous function $f : I \to \mathbb{R}$ the event $\{\mathcal{Z}(B - f) = \emptyset\}$ will have a non-zero probability, and it is easy to construct a function $f : \mathbb{R}^+ \to \mathbb{R}$ with the same property. However, we prove that adding a continuous drift can not decrease the Hausdorff dimension of the zero set almost surely. This is the content of the following theorem that we want to prove in this chapter.

Theorem IV.1.2. *For continuous function $f\colon \mathbb{R}^+ \to \mathbb{R}$, the set $\mathcal{Z}(B-f)$ has Hausdorff dimension greater than or equal to $1/2$ with positive probability.*

As the following example shows, upper bounds on the Hausdorff dimension of the zero set can not be obtained without additional assumptions on the drift f. Recall that fractional Brownian motion $B^{(H)}\colon \mathbb{R}^+ \to \mathbb{R}$ with Hurst index $0 < H < 1$ is a continuous, centered Gaussian process, such that $\mathbb{E}(|B^{(H)}(t) - B^{(H)}(s)|^2) = |t-s|^{2H}$. Taking the drift f to be an independent sample of fractional Brownian motion with Hurst index H, one gets that the Hausdorff dimension $\dim(\mathcal{Z}(B-f))$ is bounded from below by $1-H$, almost surely. This follows from the proof of the same fact for unperturbed fractional Brownian motion in Theorem 4 in Chapter 18 of [Kah] (see also Proposition 5.1 and Section 7.2 in [BDG]).

Theorem IV.1.3. *Let $f\colon \mathbb{R}^+ \to \mathbb{R}$ be either $1/2$-Hölder continuous on compact intervals or of bounded variation on compact intervals. Then the Hausdorff dimension of $\mathcal{Z}(B-f)$ is at most $1/2$, almost surely.*

Arguments from the proof of Theorem IV.1.2 and Theorem IV.1.3 imply the following corollary.

Corollary IV.1.4. *If $f\colon [0,1] \to \mathbb{R}$ is a $1/2$-Hölder continuous function, such that $f(0) = 0$, then the Hausdorff dimension of $\mathcal{Z}(B-f)$ is equal to $1/2$, almost surely.*

IV.2 On Hausdorff dimension of zero sets

In this Section we prove Theorems IV.1.1, IV.1.2, IV.1.3 and Cororllary IV.1.4 introduced in the previous section.

Recall that an interval I is called *dyadic* if it is of the form $I = [k2^m, (k+1)2^m]$ for some integers $k > 0$ and m. For a dyadic interval I, the set of its subintervals which are dyadic and of length $2^{-n}|I|$, will be denoted by $\mathcal{G}_n(I)$. Also, for intervals I and J that overlap in at most one point we will write $I < J$ if J is located to the right of I.

IV.2.1 Set containing all isolated zeros

Proof of Theorem IV.1.1. Recall the definition of sets A_f^+ and A_f^- from part (ii) of Proposition II.2.1. It is enough to prove that both of these sets have Hausdorff dimension at most $1/2$. This follows from the case $\alpha = 1/2$ of part (i) of the following lemma, which estimates the Hausdorff dimension of larger sets. □

Lemma IV.2.1. *(i) For a locally bounded function $f\colon \mathbb{R}^+ \to \mathbb{R}$ and $0 < \alpha < 1$ the sets $B_f^+ = \{t \in \mathbb{R}^+ : \liminf_{h\downarrow 0} \frac{f(t+h)-f(t)}{h^\alpha} > 0\}$ and $B_f^- = \{t \in \mathbb{R}^+ : \limsup_{h\downarrow 0} \frac{f(t+h)-f(t)}{h^\alpha} < 0\}$ have Hausdorff dimension at most α.*

(ii) Assume $f\colon \mathbb{R}^+ \to \mathbb{R}$ has bounded variation on compact sets. For any $0 < \alpha < 1$ the set

$$B_f = \left\{ t \in \mathbb{R}^+ : \limsup_{h\downarrow 0} \frac{|f(t+h) - f(t)|}{h^\alpha} > 0 \right\}$$

has Hausdorff dimension at most α.

Proof. (i) Since $B_f^- = B_{-f}^+$ it is enough to prove the claim for B_f^+. First cover the set B_f^+ by sets $B_{f,n}^+ = \{t \in \mathbb{R}^+ : f(t+h) - f(t) \geq h^\alpha/n, \text{ for all } 0 < h \leq 2^{-n}\}$, that is $B_f^+ = \bigcup_{n=1}^\infty B_{f,n}^+$. Note that for any positive integer k, and any k points $t_1 < t_2 < \cdots < t_k$ from $B_{f,n}^+$ such that $t_k \leq t_1 + 2^{-n}$, we have

$$f(t_k) \geq f(t_{k-1}) + \frac{(t_k - t_{k-1})^\alpha}{n} \geq \cdots \geq f(t_1) + \sum_{i=1}^{k-1} \frac{(t_{i+1} - t_i)^\alpha}{n}.$$

Therefore, for any dyadic interval $I \subset \mathbb{R}^+$ of length 2^{-n} there is a constant c_1 such that, for any k points $t_1 < \cdots < t_k$ from $I \cap B_{f,n}^+$, we have

$$\sum_{i=1}^{k-1} (t_{i+1} - t_i)^\alpha \leq c_1. \tag{IV.2.2}$$

Now for any positive integer m and any dyadic interval $J \in \mathcal{G}_m(I)$ such that $B_{f,n}^+ \cap J \neq \emptyset$ define $r_J^- = \inf\{B_{f,n}^+ \cap J\}$ and $r_J^+ = \sup\{B_{f,n}^+ \cap J\}$. The family $\mathcal{A}_m = \{[r_J^-, r_J^+] : J \in \mathcal{G}_m(I), B_{f,n}^+ \cap J \neq \emptyset\}$ is a cover of the set $B_{f,n}^+ \cap I$ with intervals of diameter at most 2^{-n-m}. From (IV.2.2) it is easy to see that

$$\sum_{J \in \mathcal{A}_m} |J|^\alpha = \sum_{\substack{J \in \mathcal{G}_m(I) \\ B_{f,n}^+ \cap J \neq \emptyset}} (r_J^+ - r_J^-)^\alpha \leq c_1.$$

Since m was arbitrary, by the definition of Hausdorff dimension we obtain $\dim(B_{f,n}^+ \cap I) \leq \alpha$, for any dyadic interval I of length 2^{-n}. Therefore $\dim(B_{f,n}^+) \leq \alpha$ and taking the union over all n gives $\dim(B_f^+) \leq \alpha$.

(ii) It is enough to prove that for any $c > 0$ the set

$$B_f(c) = \left\{ t \in \mathbb{R}^+ : \limsup_{h \downarrow 0} \frac{|f(t+h) - f(t)|}{h^\alpha} > c \right\}$$

has Hausdorff dimension bounded by α from above. Representing B_f as the union $B_f = \bigcup_{n=1}^\infty B_f(1/n)$ will prove the claim. Fix a dyadic interval I and an $\epsilon > 0$. For every $t \in B_f(c) \cap I$ we can find $0 < h_t < \epsilon$ such that $|f(t+h_t) - f(t)| \geq c h_t^\alpha$. Denote the interval $[t - h_t, t + h_t]$ by I_t. Clearly the family of intervals $\{I_t : t \in B_f(c) \cap I\}$ is a cover of the set $B_f(c) \cap I$. By Besicovitch's covering theorem (see Theorem 2.6 in [Matt]) we can find an integer k, not depending on ϵ, and a subcover which can be represented as a union of k at most countable disjoint subfamilies $\{I_t : t \in S_i\}$. More precisely, there are sets $S_i \subset B_f(c) \cap I$, $1 \leq i \leq k$ such that

$$B_f(c) \cap I \subset \bigcup_{j=1}^k \bigcup_{t \in S_j} I_t \quad \text{and} \quad I_s \cap I_t = \emptyset, \text{ for all } s, t \in S_i, s \neq t.$$

Clearly for any i we have

$$\sum_{t \in S_i} |I_t|^\alpha = 2^\alpha \sum_{t \in S_i} h_t^\alpha \leq \frac{2^\alpha}{c} \sum_{t \in S_i} |f(t+h_t) - f(t)| \leq \frac{2^\alpha M}{c},$$

where M is the total variation of f on the interval I. The last inequality follows form the fact that the intervals I_t, $t \in S_i$ are disjoint. Now summing the above inequality over $1 \le i \le k$ we obtain

$$\sum_{i=1}^{k} \sum_{t \in S_i} |I_t|^\alpha \le \frac{2^\alpha M k}{c}.$$

Since $\{I_t : t \in S_i, 1 \le i \le k\}$ is a cover of $B_f(c) \cap I$ of diameter at most 2ϵ and neither k nor M depend on ϵ, the α-dimensional Hausdorff measure of $B_f(c) \cap I$ is finite. Therefore $\dim(B_f(c) \cap I) \le \alpha$ and, since the dyadic interval I is arbitrary, the claim follows. □

IV.2.2 General lower bound using the fractal percolation method

The proof of Theorem IV.1.2 will be an application of the percolation method due to Hawkes [Haw], which we now describe. See Chapter 9 in [MP] for more on this method. Fix a dyadic interval I, a real number $0 < \beta < 1$ and set $p = 2^{-\beta}$ (the construction and the results can be stated for any interval I). Construct the set $\mathrm{S}_\beta(1)$ by including in it (as subsets) each of the two dyadic intervals from $\mathcal{G}_1(I)$ with probability p, independently of each other. To construct $\mathrm{S}_\beta(m+1)$, for each dyadic interval $J \in \mathcal{G}_m(I)$ such that $J \subset \mathrm{S}_\beta(m)$, include each of its dyadic subintervals from $\mathcal{G}_{m+1}(I)$ in $\mathrm{S}_\beta(m+1)$ with probability p, independently of each other. We obtain a decreasing sequence of compact sets $(\mathrm{S}_\beta(m))$ whose intersection we denote by $\Gamma[\beta]$. Comparing this to the Galton-Watson process with binomial offspring distribution $B(2,p)$, we see that $\mathbb{P}(\Gamma[\beta] \neq \emptyset) > 0$. The following theorem is due to Hawkes (Theorem 6 in [Haw]).

Theorem IV.2.3 (Hawkes)**.** *For any set $A \subset I$, if $\mathbb{P}(A \cap \Gamma[\beta] \neq \emptyset) > 0$ then* $\dim(A) \ge \beta$.

In the above construction of the percolation set we can change the retention probabilities of intervals at each level. If $p_n = 2^{-\beta_n}$ is a retention probability at level n, we denote the union of intervals kept at level m by $S_{(\beta_n)}(m)$, and the limiting percolation set by $\Gamma[(\beta_n)]$. We will use the following result which is an easy corollary of Hawkes' theorem.

Corollary IV.2.4. *Let $0 < \beta < 1$ and let (β_n) be a sequence converging to β. For any set $A \subset I$, if $\mathbb{P}(A \cap \Gamma[(\beta_n)] \neq \emptyset) > 0$ then* $\dim(A) \ge \beta$.

Proof. Let $0 < \alpha < \beta$ and m_0 be such that $\alpha < \beta_m$ for all $m \ge m_0$. There is a realization B of $S_{(\beta_n)}(m_0)$ for which $\mathbb{P}(A \cap \Gamma[(\beta_n)] \neq \emptyset \mid S_{(\beta_n)}(m_0) = B) > 0$. We have

$$\mathbb{P}(A \cap \Gamma[(\beta_n)] \neq \emptyset \mid S_{(\beta_n)}(m_0) = B) \le \mathbb{P}(A \cap \Gamma[\alpha] \neq \emptyset \mid S_\alpha(m_0) = B),$$

since we can couple two percolation processes so that for $m \ge m_0$, if an interval $J \in \mathcal{G}_m(I)$ is retained in the percolation process with retention probabilities $(2^{-\beta_n})$, it is also retained in the process with retention probability $2^{-\alpha}$. Since $\mathbb{P}(S_\alpha(m_0) = B) > 0$ we get $\mathbb{P}(A \cap \Gamma[\alpha] \neq \emptyset) > 0$ and, by Hawkes' theorem, $\dim(A) \ge \alpha$. Since $\alpha < \beta$ was arbitrary the claim follows. □

Proof of Theorem IV.1.2. It is enough to prove that the Hausdorff dimension of $\mathcal{Z}(B - f) \cap [1,2]$ is greater than or equal to $1/2$ with positive probability. Let (β_n) be a sequence converging to $1/2$ from below, to be chosen later. Consider the percolation process on the

interval $[1,2]$ with retention probabilities $(2^{-\beta_n})$, independent of Brownian motion. Fix a positive integer m. For a dyadic interval $I \in \mathcal{G}_m([1,2])$ denote by t_I its center and, for a fixed $\epsilon > 0$, consider the event

$$F_{m,\epsilon}(I) = \{I \subset \mathrm{S}_{(\beta_n)}(m), |B(t_I) - f(t_I)| \le \epsilon\}.$$

Define $Y_{m,\epsilon} = \sum_{I \in \mathcal{G}_m([1,2])} \mathbf{1}(F_{m,\epsilon}(I))$, where $\mathbf{1}(F_{m,\epsilon}(I))$ is the indicator function of the event $F_{m,\epsilon}(I)$. Using trivial bounds on the transition density of Brownian motion, the first moment of $Y_{m,\epsilon}$ can be estimated simply by

$$c_1 2^{m-\gamma_m}\epsilon \le \mathbb{E}(Y_{m,\epsilon}) \le 2\epsilon 2^{m-\gamma_m}, \tag{IV.2.5}$$

for some constant c_1 depending only on $\max_{t\in[1,2]} |f(t)|$, and where $\gamma_m = \beta_1 + \cdots + \beta_m$. In the same way, for $I < J$ we can estimate the conditional probability

$$\mathbb{P}\Big(|B(t_J) - f(t_J)| \le \epsilon \;\Big|\; |B(t_I) - f(t_I)| \le \epsilon\Big) \le 2\epsilon(t_J - t_I)^{-1/2}. \tag{IV.2.6}$$

For $I, J \in \mathcal{G}_m([1,2])$ such that $I < J$ let ℓ be the largest integer so that both I and J are contained in a single interval from $\mathcal{G}_\ell([1,2])$. In other words there are consecutive intervals $I^0, J^0 \in \mathcal{G}_{\ell+1}([1,2])$, contained in a single interval from $\mathcal{G}_\ell([1,2])$, such that $I \subset I^0$ and $J \subset J^0$. Then

$$\mathbb{P}(I \cup J \subset \mathrm{S}_{(\beta_n)}(m)) = 2^{-2\gamma_m+\gamma_\ell}. \tag{IV.2.7}$$

Using independence, (IV.2.6) and (IV.2.7)

$$\mathbb{P}(F_{m,\epsilon}(I) \cap F_{m,\epsilon}(J)) \le (2\epsilon)^2 2^{-2\gamma_m+\gamma_\ell}(t_J - t_I)^{-1/2}. \tag{IV.2.8}$$

Summing (IV.2.8) over all $I \subset I^0$ and $J \subset J^0$ for a fixed I^0 and J^0 as above

$$\begin{aligned}\sum_{I\subset I^0, J\subset J^0} \mathbb{P}(F_{m,\epsilon}(I) \cap F_{m,\epsilon}(J)) &\le (2\epsilon)^2 2^{-2\gamma_m+\gamma_\ell} \sum_{k=1}^{2^{m-\ell}} k(k2^{-m})^{-1/2} \\ &\le c_2(2\epsilon)^2 2^{m/2-2\gamma_m+\gamma_\ell} 2^{3/2(m-\ell)} \le 4c_2\epsilon^2 2^{2(m-\gamma_m)+\gamma_\ell-3\ell/2},\end{aligned}$$

for some universal constant $c_2 > 0$. Summing this over all I^0 and J^0 and all $0 \le \ell \le m-1$, and using (IV.2.5), we can estimate the second moment

$$\begin{aligned}\mathbb{E}(Y_{m,\epsilon}^2) &= 2 \sum_{I,J\in\mathcal{G}_m([1,2]), I<J} \mathbb{P}(F_{m,\epsilon}(I) \cap F_{m,\epsilon}(J)) + \mathbb{E}(Y_{m,\epsilon}) \\ &\le 8c_2\epsilon^2 \sum_{\ell=1}^{m-1} \Big(2^\ell 2^{2(m-\gamma_m)+\gamma_\ell-3\ell/2}\Big) + 2\epsilon 2^{m-\gamma_m} \\ &\le 8c_2\epsilon^2 2^{2(m-\gamma_m)} \sum_{\ell=0}^{m-1} 2^{\gamma_\ell-\ell/2} + 2\epsilon 2^{m-\gamma_m}.\end{aligned} \tag{IV.2.9}$$

Now choose a sequence (β_n) which converges to $1/2$ from below, and so that the series $\sum_{\ell=0}^\infty 2^{\gamma_\ell-\ell/2}$ converges (for example take $\beta_n = 1/2 - 2/(n\log 2)$, then γ_ℓ is up to an additive

constant equal to $\ell/2 - 2\log_2 \ell$). With such (β_n) and for the sequence $\epsilon_m = 2^{-(m-\gamma_m)}$ that converges to zero, we define $V_m = Y_{m,\epsilon_m}$. By (IV.2.5) we have $\mathbb{E}(V_m) \geq c_1 > 0$ and by (IV.2.9) we have $\mathbb{E}(V_m^2) \leq C_3 < \infty$, where C_3 is a universal constant. The Paley-Zygmund inequality yields $\mathbb{P}(V_m > 0) \geq c_1^2/C_3$. Thus the probability of the event $\limsup_m \{V_m > 0\}$ is also bounded from below by c_1^2/C_3. On this event there is a sequence (s_m), such that each s_m is the center of a dyadic interval from $\mathcal{G}_{k_m}([1,2])$ contained in $S_{(\beta_n)}(k_m)$, and such that $|B(s_m) - f(s_m)| \leq \epsilon_{k_m}$. The sequence (s_m) contains a subsequence that converges to some $s \in \Gamma[(\beta_n)]$ such that $B(s) = f(s)$. Therefore we have $\mathbb{P}(\mathcal{Z}(B-f) \cap \Gamma[(\beta_n)] \neq \emptyset) \geq c_1^2/C_3 > 0$. By the independence of Brownian motion and the percolation process and Corollary IV.2.4 it follows that

$$\mathbb{P}(\dim(\mathcal{Z}(B-f)) \geq 1/2) \geq c_1^2/C_3 > 0. \tag{IV.2.10}$$

□

Remark IV.2.11. *Note that, for a continuous function f defined on* $[1,2]$*, the lower bound from* (IV.2.10) *depends only on* $\max_{t\in[1,2]} |f(t)|$.

IV.2.3 Upper bound for functions which are $1/2$-Hölder or of bounded variation

Proof of Theorem IV.1.3. It is enough to prove that $\dim(\mathcal{Z}(B-f) \cap I) \leq 1/2$, for any dyadic interval I. First we will prove the claim for functions which are 1/2-Hölder on compact intervals.

Assume f is a 1/2-Hölder continuous function on I, that is $|f(t) - f(s)| \leq c_0|t-s|^{1/2}$ for some $c_0 > 0$ and all $s, t \in I$. For an interval $J = [s_1, s_2] \subset I$ set $Z(J) = 1$, if there exists $t \in J$ such that $B(t) = f(t)$, and $Z(J) = 0$ otherwise, and define the interval $\overline{J} = [f(s_1) - c_0\sqrt{s_2 - s_1}, f(s_1) + c_0\sqrt{s_2 - s_1}]$. On the event $Z(J) = 1$ define the stopping time $\tau = \min\{\mathcal{Z}(B-f) \cap J\}$. Since $(\tau, B(\tau)) \in J \times \overline{J}$, by the strong Markov property, conditional on the sigma algebra $\mathcal{F}_\tau$ and on the event $\{Z(J) = 1\}$, the probability p_1 that $B(s_2) \in \overline{J}$ is equal to the probability that $B(1) \in \overline{J}_\tau$, where $\overline{J}_\tau$ is the interval $\overline{J}$ shifted by $-B(\tau)$ and scaled by $(s_2 - \tau)^{-1/2}$. Since the interval $\overline{J}_\tau$ has length at least $2c_0$ and contains the origin, p_1 is bounded from below by a constant not depending on the choice of the interval J; see also arguments in the proof of Theorem II.1.3 leading to (II.3.8). Therefore, for some $c_1 < \infty$ we obtain $\mathbb{P}(B(t) \in \overline{J} \mid Z(J) = 1) \geq c_1^{-1}$. This implies

$$\mathbb{P}(Z(J) = 1) \leq c_1 \mathbb{P}(B(t) \in \overline{J}) \leq c_2 |J|^{1/2}, \tag{IV.2.12}$$

for some $c_2 > 0$.

Now consider the covering $\mathcal{A}_k$ of the set $\mathcal{Z}(B-f) \cap I$, consisting of the dyadic intervals from $\mathcal{G}_k(I)$ which intersect the set $\mathcal{Z}(B-f) \cap I$. Since every interval in $\mathcal{A}_k$ has length $2^{-k}|I|$, by (IV.2.12) we have

$$\mathbb{E}\left(\sum_{J\in\mathcal{A}_k} |J|^{\frac{1}{2}}\right) = \mathbb{E}\left(\sum_{J\in\mathcal{G}_k(I)} Z(J) 2^{-\frac{k}{2}} |I|^{1/2}\right) \leq 2^k c_2 2^{-\frac{k}{2}} |I|^{1/2} 2^{-\frac{k}{2}} |I|^{1/2} = c_2 |I|.$$

Fatou's lemma implies

$$\mathbb{E}\left(\liminf_{k\to\infty}\sum_{J\in\mathcal{A}_k}|J|^{1/2}\right)\le\liminf_{k\to\infty}\mathbb{E}\left(\sum_{J\in A_k}|J|^{1/2}\right)\le c_2|I|.$$

Therefore, almost surely, we can find a sequence of coverings $\{J_n^k, n\ge 1\}$ of $\mathcal{Z}(B-f)\cap I$, with $\lim_{k\to\infty}\sup_{n\ge1}|J_n^k|=0$ and $\limsup_{k\to\infty}\sum_{n\ge1}|J_n^k|^{1/2}<\infty$. This implies that $\dim(\mathcal{Z}(B-f)\cap I)\le 1/2$, a.s.

Now let f be of bounded variation on compact intervals and define the set

$$B_f=\left\{t\in\mathbb{R}^+:\limsup_{h\downarrow 0}|f(t+h)-f(t)|\,h^{-1/2}\ge 1\right\}.$$

By part (ii) of Lemma IV.2.1 we have $\dim(B_f)\le 1/2$. Therefore, it is enough to bound the dimension of the zero set in the complement, $\mathcal{Z}(B-f)\cap B_f^c$, where $B_f^c=I\backslash B_f$. We can cover B_f^c by a countable union of the sets

$$D_{f,n}=\left\{t\in I:|f(t+h)-f(t)|\le\sqrt{h},\ \text{for all } 0\le h<2^{-n}\right\}.$$

For a fixed n, all $t_1,t_2\in D_{f,n}$ with $|t_1-t_2|<2^{-n}$ satisfy $|f(t_2)-f(t_1)|\le|t_1-t_2|^{1/2}$. Therefore, the restriction $f|_{D_{f,n}}$ is 1/2-Hölder continuous, that is, for some positive constant c_3 and all $t_1,t_2\in D_{f,n}$ we have $|f(t_2)-f(t_1)|\le c_3|t_1-t_2|^{1/2}$. Define the function $f_n\colon I\to\mathbb{R}$ to be $f_n(t)=f(t)$ for all $t\in D_{f,n}$. We will define f_n for $t\in I\setminus D_{f,n}$ using linear interpolation, in a sense. If $D_{f,n}=\emptyset$ then set f_n to be any constant function. Assume that $D_{f,n}\ne\emptyset$. Note that the set $D_{f,n}$ is closed from the left, that is if (s_k) is an increasing sequence of points in $D_{f,n}$ converging to some s, then $s\in D_{f,n}$. Since f has bounded variation, by assumption, the right limit $\lim_{t\downarrow s}f(t)$ exists for all $s\in I$. Thus if $t\in I\backslash D_{f,n}$ define $t_l=\max\{s\in D_{f,n}:s<t\}$, $t_r=\inf\{s\in D_{f,n}:t<s\}$ as well as $a_l=f(t_l)$ and $a_r=\lim_{s\downarrow t_r}f(s)$ and notice that $|a_r-a_l|\le c_3\sqrt{t_r-t_l}$. Now define f_n on $[t_l,t_r]$ to be linear with $f_n(t_l)=a_l$ and $f_n(t_r)=a_r$. If t_l does not exist, define t_l as the left endpoint of I and $a_l=a_r$, and do similarly if t_r does not exist. Clearly f_n is 1/2-Hölder continuous with the same constant c_3. We can apply the first part of the proof to f_n, to conclude that the set $\mathcal{Z}(B-f_n)$ has Hausdorff dimension at most 1/2 almost surely. Since $\mathcal{Z}(B-f)\cap D_{f,n}\subset\mathcal{Z}(B-f_n)$, we obtain $\dim(\mathcal{Z}(B-f)\cap D_{f,n})\le 1/2$ and $\dim\left(\mathcal{Z}(B-f)\cap\bigcup_{n\in\mathbb{N}}D_{f,n}\right)\le 1/2$. Since $B_f^c\subset\bigcup_{n\in\mathbb{N}}D_{f,n}$ the claim follows. □

We finish the chapter with the proof of Corollary IV.1.4.

Proof of Corollary IV.1.4. Assume that $|f(t)-f(s)|\le C\sqrt{|t-s|}$ for all $0<s,t<1$. For every positive integer n define a continuous function $f_n\colon[1,2]\to\mathbb{R}$ as $f_n(t)=2^{n/2}f(t2^{-n})$, for all $t\in[1,2]$. Clearly $\max_{t\in[1,2]}|f_n(t)|\le C\sqrt{2}$ and, by Remark IV.2.11 and Brownian scaling, there is a constant $c>0$ such that

$$\mathbb{P}(\dim(\mathcal{Z}(B-f)\cap[2^{-n},2^{-n+1}])\ge 1/2)=\mathbb{P}(\dim(\mathcal{Z}(B-f_n))\ge 1/2)\ge c.$$

Thus by Blumenthal's zero-one law, almost surely, there are infinitely many integers n such that the Hausdorff dimension of the set $\mathcal{Z}(B-f)\cap[2^{-n},2^{-n+1}]$ is greater or equal than 1/2. The upper bound follows from Theorem IV.1.3. □

Chapter V

Fast times

V.1 Introduction

In 1974 Orey and Taylor [OT74] studied the so-called fast points of Brownian motion. That is, for a given $a > 0$ a time $t \in [0, 1]$ with

$$\limsup_{h \downarrow 0} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log(1/h)}} \geq a$$

is called an *a-fast time* of linear Brownian motion. By the law of iterated logarithm follows that almost surely the set of a-fast times has Lebesgue measure zero. Therefore, to quantify how often these a-fast times occur we use Hausdorff dimension. Orey and Taylor [OT74] showed for every $a \in [0, 1]$,

$$\dim \left\{ t \in [0,1] \middle| \limsup_{h \downarrow 0} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\} = 1 - a^2, \tag{V.1.1}$$

almost surely. By Lévy's modulus of continuity it follows that the set above is empty for $a > 1$.

Khoshnevisan and Shi ([KS]) extended Orey's and Taylor's results [OT74] in several different ways. One of which is the intersection of the set of fast points with the zero set of Brownian motion.

As in previous chapters we will denote the zero set of a process Y by $\mathcal{Z}(Y)$, but in this chapter we only consider the intersection with the interval $(0, 1]$.

Theorem V.1.2 ([KS]). *Let $\mathcal{Z}(B) := \{t \in (0,1] | B(t) = 0\}$. For every $a \in (0, 1]$,*

$$\dim \left\{ t \in \mathcal{Z}(B) \middle| \limsup_{h \downarrow 0} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\} = \max\left\{ \frac{1}{2} - a^2, 0 \right\},$$

almost surely.

In this chapter we will first give some general remarks on fast times of Brownian motion with variable drift, see section V.2. We can extend the result of Orey and Taylor by adding a continuous function f to Brownian motion and giving a general lower bound on the Hausdorff dimension of the set of a-fast times. In particular, this theorem implies that by adding a function to Brownian motion the Hausdorff dimension of the set of a-fast times cannot be decreased.

Theorem V.1.3. *Suppose $f : \mathbb{R}^+ \to \mathbb{R}$ is an arbitrary function and $X(t) := B(t) - f(t)$. For every $a \in (0, 1]$*

$$\dim \left\{ t \in [0,1] \middle| \limsup_{h \downarrow 0} \frac{|X(t+h) - X(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\} \geq 1 - a^2,$$

almost surely.

To prove this theorem we will bound the Hausdorff dimension of a so-called limsup fractal that is contained in a set of points fulfilling that at least "half of the points" are a-fast times. An example of a function f where the dimension of a-fast times is strictly greater than $1 - a^2$ is given in the next section (see Proposition V.2.2 and the subsequent example). The following result is an upper bound analogue of Theorem V.1.2 for 1/2-Hölder continuous functions added to one-dimensional Brownian motion. Recall from Theorem IV.1.4 of the previous chapter that the Hausdorff dimension of the zero set of the latter process is 1/2.

Theorem V.1.4. *Suppose $f : \mathbb{R}^+ \to \mathbb{R}$ is a 1/2-Hölder continuous function with $f(0) = 0$, $X(t) := B(t) - f(t)$ and let $\mathcal{Z}(X) := \{t \in (0,1] | X(t) = 0\}$. For every $a \in (0, 1]$*

$$\dim \left\{ t \in \mathcal{Z}(X) \middle| \limsup_{h \downarrow 0} \frac{|X(t+h) - X(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\} \leq \max \left\{ \frac{1}{2} - a^2, 0 \right\}$$

almost surely.

We will prove the upper using the method of [KS] in section V.3 . A general lower bound for continuous functions can be given as well.

Theorem V.1.5. *Suppose $f : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function, $X(t) := B(t) - f(t)$ and let $\mathcal{Z}(X) := \{t \in (0,1] | X(t) = 0\}$. For every $a \in (0, 1]$*

$$\dim \left\{ t \in \mathcal{Z}(X) \middle| \limsup_{h \downarrow 0} \frac{|X(t+h) - X(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\} \geq \max \left\{ \frac{1}{2} - a^2, 0 \right\}$$

with positive probability.

V.2 Some first remarks on fast times of Brownian motion with variable drift

By the Cameron-Martin theorem (see Chapter I) we see that the Theorem of Orey and Taylor, see (V.1.1), holds as well if we replace Brownian motion by a function f added to Brownian motion where f is in the Cameron-Martin space. We will show that the same is true for any function f which is locally 1/2-Hölder continuous. Since all functions in the Cameron-Martin space are 1/2-Hölder continuous, this is a stronger statement than the one implied by the Cameron-Martin theorem.

Corollary V.2.1. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a locally $1/2$-Hölder continuous function and let $X(t) := B(t) - f(t)$. Then, for every $a \in [0,1]$*

$$\dim\left\{t \in [0,1] \middle| \limsup_{h\downarrow 0} \frac{|X(t+h) - X(t)|}{\sqrt{2h\log(1/h)}} \geq a\right\} = 1 - a^2$$

almost surely.

Proof. By the definition of 1/2-Hölder continuity and the triangle inequality we get that for every $t \geq 0$,

$$\begin{aligned}
\limsup_{h\downarrow 0} \frac{|B(t+h) - B(t)|}{\sqrt{2h\log(1/h)}} &= \limsup_{h\downarrow 0} \frac{|B(t+h) - B(t)| - |f(t+h) - f(t)|}{\sqrt{2h\log(1/h)}} \\
&\leq \limsup_{h\downarrow 0} \frac{|X(t+h) - X(t)|}{\sqrt{2h\log(1/h)}} \\
&\leq \limsup_{h\downarrow 0} \frac{|B(t+h) - B(t)| + |f(t+h) - f(t)|}{\sqrt{2h\log(1/h)}} \\
&= \limsup_{h\downarrow 0} \frac{|B(t+h) - B(t)|}{\sqrt{2h\log(1/h)}}.
\end{aligned}$$

□

The function f can be random in the statement of Corollary V.2.1 (but independent of the Brownian motion B). Also, note that the statement of corallary V.2.1 also holds if f is not locally 1/2-Hölder continuous on a countable subset of $\mathbb{R}^+$; by that we mean it is enough if f fulfills that there is $c > 0$ such that $|f(t) - f(s)| \leq c\sqrt{|s-t|}$ with $t \in [0,1]\backslash M$ and $s \in [0,1]$ where M is a countable set.

A natural question to ask is if we can perturb linear Brownian motion by a function such that the Hausdorff dimension of the set of a-fast times differs from the result (V.1.1). The following proposition gives a positive answer. Also, this is an example for a function such that a strict inequality holds in Theorem V.1.3 (for some a). More examples are given below.

Proposition V.2.2. *Let $f_\gamma : [0,1] \to [0,1]$ be a middle $(1-2\gamma)$-Cantor function with $\gamma < 1/4$ and let $X_\gamma(t) := B(t) - f_\gamma(t)$. Then, for every $a \in [0,1]$*

$$\dim\left\{t \in [0,1] \middle| \limsup_{h\downarrow 0} \frac{|X_\gamma(t+h) - X_\gamma(t)|}{\sqrt{2h\log(1/h)}} \geq a\right\} = \max\left\{1 - a^2, -\frac{\log 2}{\log \gamma}\right\},$$

almost surely.

Note here that $-\frac{\log 2}{\log \gamma}$ is also both the Hausdorff dimension of the Cantor set and the Hölder exponent of the Cantor function.

Proof. Recall from section II.3 that for $n > 0$ we call the n-th approximation of the Cantor set $C_{\gamma,n}$, $\mathfrak{C}_{\gamma,n}$ the set of all connected components of $C_{\gamma,n}$, and $f_{\gamma,n}$ the corresponding n-th approximation of the Cantor function (see II.3 for precise definition). Take an arbitrary $\gamma < \gamma_1 < 1/4$. There is an $n_0 > 0$ such that $\sum_{n\geq n_0}(2\sqrt{\gamma_1})^n \leq 1/2$. For $n \geq n_0$ consider the

interval $J_{k,n} = [k2^{-n} - \gamma_1^{n/2}/2, k2^{-n} + \gamma_1^{n/2}/2]$ and define the set $M_{n_0} = \bigcup_{n \geq n_0} \bigcup_{0 \leq k \leq 2^n} J_{k,n}$. Take $t \in C_\gamma \backslash f_\gamma^{-1}(M_{n_0})$ and any $s \neq t$ in the same connected component of the interior of C_{γ,n_0}. The largest integer ℓ such that both s and t are contained in the same interval of $\mathfrak{C}_{\gamma,\ell}$ satisfies $\ell \geq n_0$. Moreover, $|f_\gamma(s) - f_\gamma(t)| \geq \gamma_1^{(\ell+1)/2}$ and $|s-t| \leq \gamma^\ell$. We see that t satisfies

$$\limsup_{h \downarrow 0} \frac{|f_\gamma(t+h) - f_\gamma(t)|}{h^\beta} > 0.$$

with $\beta = \frac{\log \gamma_1}{2 \log \gamma} < 1/2$. Hence, t is an a-fast time of the process X_γ.

Because $\sum_{n \geq n_0} (2\sqrt{\gamma_1})^n \leq 1/2$ note that for every n holds $|C_{\gamma,n} \backslash f_\gamma^{-1}(M_{n_0})| \geq 1/2 |C_{\gamma,n}|$. Therefore the Hausdorff dimension of the fast times of the process X_γ on the set $C_\gamma \backslash f_\gamma^{-1}(M_{n_0})$ equals the Hausdorff dimension of the Cantor set (that is $-\frac{\log 2}{\log \gamma}$).

The Hausdorff dimension of fast times on the set $[0,1] \backslash C_\gamma$, that is the union of open intervals where the function f_γ is constant, is $1 - a^2$. Note, that the set $f_\gamma^{-1}(M_{n_0}) \cap C_\gamma$ has at most the Hausdorff dimension $-\frac{\log 2}{\log \gamma}$. Then, by the definition of Hausdorff dimension we see that for two sets A and B it holds $\dim(A \cup B) = \sup\{\dim A, \dim B\}$. The claim follows. □

Note that there are functions such that for all $a > 0$ the Hausdorff dimension of the set of a-fast times of these functions added to Brownian motion is 1 almost surely. For instance, Loud in [Loud] constructed functions which satisfy a certain local reverse Hölder property at each point (see also the construction in [MP53]). We have already mentioned these functions in a different context in Remark II.2.2. Recall that these functions are defined as $g(t) = \sum_{k=1}^\infty g_k(t)$ where $g_k(t) = 2^{-2A\alpha k} g_0(2^{2Ak} t)$, for $0 < \alpha < 1$, a positive integer A such that $2A(1-\alpha) > 1$, and a continuous function g_0 which has value 0 at even integers, value 1 at odd integers and is linear at all other points. It holds that there is a positive constant c such that $|g(t+h) - g(t)| > ch^\alpha$ for infinitely many arbitrarily small $h > 0$ (see Theorem of [Loud]). Therefore, if we choose $\alpha < 1/2$, then for every $a \geq 0$,

$$\dim \left\{ t \in [0,1] \middle| \limsup_{h \downarrow 0} \frac{|(B-g)(t+h) - (B-g)(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\} = 1,$$

almost surely.

For fractional Brownian motion Khoshnevisan and Shi ([KS]) proved the following analogue result to (V.1.1).

Theorem V.2.3 ([KS]). *Let $B^{(H)} \colon \mathbb{R}^+ \to \mathbb{R}$ be a fractional Brownian motion with Hurst index $H \in]0,1[$ and $B^{(H)}(0) = 0$. For every $a \in (0,1]$*

$$\dim \left\{ t \in [0,1] \middle| \limsup_{h \downarrow 0} \frac{|B^{(H)}(t+h) - B^{(H)}(t)|}{\sqrt{2} \cdot h^H \sqrt{\log(1/h)}} \geq a \right\} = 1 - a^2$$

almost surely.

As before we can extend this result.

Corollary V.2.4. *Let $B^{(H)} \colon \mathbb{R}^+ \to \mathbb{R}$ be a fractional Brownian motion with Hurst index $H \in]0,1[$ and $B^{(H)}(0) = 0$.*

(i)Let $f^{(H)}\colon \mathbb{R}^+ \to \mathbb{R}$ be a locally H-Hölder continuous function and let $X^{(H)}(t) := B^{(H)}(t) - f^{(H)}(t)$. Then, for every $a \in [0,1]$

$$\dim\left\{t \in [0,1] \middle| \limsup_{h\downarrow 0} \frac{|X^{(H)}(t+h) - X^{(H)}(t)|}{\sqrt{2}\cdot h^H\sqrt{\log(1/h)}} \geq a\right\} = 1 - a^2,$$

almost surely.

(ii)Let $f_\gamma\colon [0,1] \to [0,1]$ be a middle $(1-2\gamma)$-Cantor function and let $X_\gamma^{(H)}(t) := B^{(H)}(t) - f_\gamma(t)$. Then, for every $a \in [0,1]$, and every $\gamma < 2^{-\frac{1}{H}}$

$$\dim\left\{t \in [0,1] \middle| \limsup_{h\downarrow 0} \frac{|X_\gamma^{(H)}(t+h) - X_\gamma^{(H)}(t)|}{\sqrt{2}\cdot h^H\sqrt{\log(1/h)}} \geq a\right\} = \max\left\{1 - a^2, \frac{\log 2}{-\log\gamma}\right\},$$

almost surely.

Proof. Analogously to the proofs of Corollary V.2.1 and Proposition V.2.2. □

As we have already mentioned, Khoshnevisan and Shi ([KS]) also looked at the intersection set of a-fast times and the zero set of Brownian motion (see V.1.2). Unfortunately, it is not known whether an analogue statement holds for fractional Brownian motion. (The proof cannot be adapted for fractional Brownian motion with $H \neq 1/2$ since the increments of the process are not independent.)

V.3 Theorem V.1.4: Upper bound

First we denote the set of a-fast times for every $a \in (0,1]$ by $F(a)$, that is

$$F(a) := \left\{t \in [0,1] | \limsup_{h\downarrow 0} \frac{|X(t+h) - X(t)|}{\sqrt{2h\log(1/h)}} \geq a\right\}.$$

By the proof of corollary V.2.1 we see that,

$$F(a) = \left\{t \in [0,1] | \limsup_{h\downarrow 0} \frac{|B(t+h) - B(t)|}{\sqrt{2h\log(1/h)}} \geq a\right\}.$$

Further we define for every $a \in (0,1]$ and $h > 0$,

$$\mathfrak{F}(a,h) := \left\{t \in [0,1] | \sup_{t\leq s\leq t+h} |B(s) - B(t)| \geq a\sqrt{2h\log(1/h)}\right\}.$$

Then, for all $0 < b < a$, we have that $F(a) \subset \bigcap_{h>0}\bigcup_{0<\delta<h}\mathfrak{F}(b,\delta)$. Now let $I_{k,j}^\eta := [k\beta^{-\eta j}, (k+1)\beta^{-\eta j}]$ for any $\beta, \eta > 1$, all $j \geq 1$, and every integer $0 \leq k < \beta^{\eta j}$. For all $t \in \mathfrak{F}(b,\delta)$ it holds for $\delta < h < 1$ with $\beta^{-j} \leq \delta \leq \beta^{1-j}$ that

$$\sup_{t\leq s\leq t+\beta^{1-j}} |B(s) - B(t)| \geq b\beta^{-j/2}\sqrt{2\log(\beta^{j-1})}$$
$$= b\beta^{-1/2}\beta^{-(j-1)/2}\sqrt{2\log(\beta^{j-1})}. \quad \text{(V.3.1)}$$

It follows $t \in \mathfrak{F}(b\beta^{-1/2}, \beta^{1-j})$. We fix $\beta, \eta > 1$, $\theta \in]0,1[$, then we get for any integer $i \geq 1$,

$$F(a) \subset \bigcup_{j \geq i} \bigcup_{k \geq 1} I^{\eta}_{k,j} \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}).$$

f is a 1/2-Hölder continuous function, that is $|f(t) - f(s)| \leq c_0 |t-s|^{1/2}$ for some $c_0 > 0$ and all $s, t \in [0,1]$. Now we will bound the probability of the event $|B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}$ from above with $c_1 := \max\{2c_0, 2\sqrt{2}\}$. By the scaling property of Brownian motion we get,

$$\begin{aligned}
&\mathbb{P}\big\{|B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}\big\} \\
&\qquad = \mathbb{P}\big\{B(k\beta^{-\eta j}) \in [-c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)} + f(k\beta^{-\eta j}), \\
&\qquad\qquad\qquad\qquad c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)} + f(k\beta^{-\eta j})]\big\} \\
&\qquad = \mathbb{P}\big\{B(1) \in [-c_1 k^{-1/2} \sqrt{\eta j \log(\beta)} + f(k\beta^{-\eta j}) k^{-1/2} \beta^{\eta j/2}, \\
&\qquad\qquad\qquad\qquad c_1 k^{-1/2} \sqrt{\eta j \log(\beta)} + f(k\beta^{-\eta j}) k^{-1/2} \beta^{\eta j/2}]\big\}.
\end{aligned}$$

Because $f(0) = 0$, we obtain $0 \leq |f(k\beta^{-\eta j})| k^{-1/2} \beta^{\eta j/2} \leq c_0$. By symmetry and the unimodality property of the normal distribution, we get that

$$\begin{aligned}
&\mathbb{P}\big\{|B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}\big\} \\
&\qquad \leq \mathbb{P}\big\{B(1) \in [-c_1 k^{-1/2} \sqrt{\eta j \log(\beta)}, c_1 k^{-1/2} \sqrt{\eta j \log(\beta)}]\big\} \\
&\qquad \leq 2 c_1 k^{-1/2} \sqrt{\eta j \log(\beta)}. \qquad \text{(V.3.2)}
\end{aligned}$$

Also, note that by Levy's modulus of continuity there exists a finite random variable K, depending on η and β, such that for all $j \geq K$ almost surely $\mathbf{1}_{\{|B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}\}} \geq \mathbf{1}_{\{I^{\eta}_{k,j} \cap \mathcal{Z}(X) \neq \emptyset\}}$. Therefore,

$$F(a) \cap \mathcal{Z}(X) \subset \bigcup_{j \geq i} \bigcup_{\substack{k: |B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \\ \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}}} I^{\eta}_{k,j} \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}).$$

The next step is to show that this is a good covering. With (V.3.2) we get, for any $\gamma > 0$,

$$\begin{aligned}
&\sum_{j \geq i} \sum_{0 \leq k < \beta^{\eta j}} |I^{\eta}_{k,j}|^{\gamma} \mathbb{P}(I^{\eta}_{k,j} \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}) \neq \emptyset, I^{\eta}_{k,j} \cap \mathcal{Z}(X) \neq \emptyset) \\
&\qquad \leq \sum_{j \geq i} \sum_{0 \leq k < \beta^{\eta j}} |I^{\eta}_{k,j}|^{\gamma} \mathbb{P}\big(I^{\eta}_{k,j} \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}) \neq \emptyset, \\
&\qquad\qquad\qquad\qquad |B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}\big) \\
&\qquad \leq \sum_{j \geq i} \sum_{0 \leq k < \beta^{\eta j}} |I^{\eta}_{k,j}|^{\gamma} \mathbb{P}(I^{\eta}_{k,j} \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}) \neq \emptyset) \\
&\qquad\qquad\qquad\qquad \cdot \mathbb{P}\big(|B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}\big),
\end{aligned}$$

where we used the independence of increments of Brownian motion in the last step.

In order to bound $\mathbb{P}(I^{\eta}_{k,j} \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}) \neq \emptyset)$ from above we will need the following Lemma.

Lemma V.3.3 (see Lemma 3.1. of [KS]). *For all $b > 0$, $0 < \varepsilon < 1$, $\eta > 1$, and all $\beta > 1$, there is a $2 \leq J < \infty$ depending on ε, η, b and β such that for all $j \geq J$ and all $k \geq 0$,*

$$\mathbb{P}(I^{\eta}_{k,j} \cap \mathfrak{F}(b, \beta^{-j}) \neq \emptyset) \leq \beta^{-b^2(1-\varepsilon)j}.$$

Hence, we obtain that for all $\mu \in]0,1[$ there is a $\infty > J \geq 2$ depending on μ, η, a, β and θ such that for $j \geq J$ and all $k > 0$, $\mathbb{P}(I^{\eta}_{k,j} \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}) \neq \emptyset)$ is bounded from above by $\beta^{-\theta^2 a^2 \beta^{-1}(1-\mu)(j-1)}$. Note that $\beta^{-\theta^2 a^2 \beta^{-1}(1-\mu)(j-1)} \leq \beta^{-\theta^2 a^2 \beta^{-2}(1-\mu)j}$ for large enough j. Thus, for large enough i,

$$\begin{aligned}
\sum_{j \geq i} \sum_{0 \leq k < \beta^{\eta j}} & |I^{\eta}_{k,j}|^{\gamma} \mathbb{P}(I^{\eta}_{k,j} \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}) \neq \emptyset) \\
& \cdot \mathbb{P}\big(|B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}\big) \\
& \leq \sum_{j \geq i} \beta^{-\eta\gamma j} \beta^{-\theta^2 a^2 \beta^{-2}(1-\mu)j} \Big(1 + \sum_{k=1}^{\beta^{\eta j}-1} 2c_1 k^{-1/2} \sqrt{\eta j \log(\beta)}\Big) \\
& \leq \sum_{j \geq i} \beta^{-\eta\gamma j} \beta^{-\theta^2 a^2 \beta^{-2}(1-\mu)j} \big(\beta^{\eta j/2} \cdot 2c_1 \sqrt{\eta j \log(\beta)} + 1\big).
\end{aligned}$$

That means, if $\eta\gamma - \eta/2 + \theta^2 a^2 \beta^{-2}(1-\mu) > 0$, then almost surely

$$\begin{aligned}
\lim_{i \to \infty} \sum_{j \geq i} \sum_{0 \leq k < \beta^{\eta j}} & |I^{\eta}_{k,j}|^{\gamma} \mathbb{P}(I^{\eta}_{k,j} \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}) \neq \emptyset) \\
& \cdot \mathbb{P}\big(|B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}\big) = 0.
\end{aligned}$$

By letting $\mu \downarrow 0$, $\theta \uparrow 1$, $\beta \downarrow 1$ and $\eta \downarrow 1$ the claim follows.

■

V.4 Proof of Theorem V.1.5

In order to prove Theorem V.1.5 we will give a proof of the following theorem which is an analogue of Theorem 8.1. of [KS]. The statement of Theorem V.4.1 might be of independent interest.

Theorem V.4.1. *Let $E \subset [0,1]$ be a compact set. If $\dim(E) > a^2 + 1/2$, then the set*

$$\Big\{ t \in \mathcal{Z}(X) \cap E \Big| \limsup_{h \downarrow 0} \frac{|X(t+h) - X(t)|}{\sqrt{2h \log(1/h)}} \geq a \Big\}$$

is non-empty with positive probability.

Now the lower bound of Theorem V.1.5 follows using the following stochastic codimension argument. For a random set $M \subset \mathbb{R}_+$ the *upper stochastic codimension* $\overline{\mathrm{codim}}(M)$ is defined by the smallest value γ such that for all Borel measurable sets G with $\dim(G) > \gamma$ holds that

$\mathbb{P}(G \cap M \neq \emptyset) > 0$. Then $\overline{\text{codim}}(M) + \dim(M) \geq 1$ with positive probability, see [Kho02], p. 436 and also [Kho03], p. 238. This is essentially again the method of Hawkes' [Haw] as we have already seen in the previous chapter when we used the fractal percolation method. The idea is if the intersection of a random set with test sets of a certain Hausdorff dimension (we used limiting percolation sets in the last chapter) is non-empty with positive probability, then we obtain a positive probability lower bound on the Hausdorff dimension of the random set. More precisely, recall from Theorem IV.2.3 that for an set $A \subset [0,1]$, if $\mathbb{P}(A \cap \Gamma[\beta] \neq \emptyset) > 0$ then $\dim(A) \geq \beta$, where $\Gamma[\beta]$ was a limiting percolation set with parameter $\beta \in (0,1)$. It is easy to check that $\mathbb{P}(\dim \Gamma[\beta] = 1 - \beta \mid \Gamma[\beta] \neq \emptyset) = 1$. So we see that instead of using percolation limit sets of Hausdorff dimension $a^2 + 1/2$ we are going to use now compact sets of Hausdorff dimension $a^2 + 1/2$ (or larger but arbitrarily close to $a^2 + 1/2$, respectively) to achieve a lower bound on the Hausdorff dimension of the intersection of fast times and zeros.

In order to prove Theorem V.4.1 we need some technical lemmas. First we give some definitions. For $\eta > 0$ and an atomless probability measure μ, call

$$A_\eta(\mu) := \sup_{0<h\leq 1/2} \sup_{t\in[h,1-h]} \frac{\mu[t-h, t+h]}{h^\eta}.$$

Further, define for $h > 0$

$$S_h(\mu) := \sup_{0\leq s\leq h} \int_s^h \frac{1}{\sqrt{t-s}} d\mu(t), \textit{ and}$$

$$\tilde{S}_h(\mu) := \sup_{0\leq s\leq 1} \int_s^{(s+h\wedge 1)} \frac{1}{\sqrt{t-s}} d\mu(t).$$

The first lemma is a version of the famous Frostman's lemma.

Lemma V.4.2 (Frostman, cf. [Kah], p. 130)**.** *Let $\eta > 0$, and let $E \subset [0,1]$ be Borel measurable set satisfying $\eta < \dim(E)$, then there is an atomless probability measure μ on E for which $A_\eta(\mu) < \infty$.*

Lemma V.4.3 ([KS], Lemma 8.2)**.** *Let μ be an atomless probability measure on a compact set $E \subset [0,1]$, and for $h > 0$ and $\eta > 1/2$,*

$$S_h(\mu) \leq \frac{2\exp(\eta)}{2\eta - 1} A_\eta(\mu) h^{\eta - 1/2},$$

$$\tilde{S}_h(\mu) \leq \frac{2\exp(\eta)}{2\eta - 1} A_\eta(\mu) h^{\eta - 1/2}.$$

Lemma V.4.4 ([KS], Theorem 2.5)**.** *Let (E_n) be a countable collection of open random sets. If $\sup_{n\geq 1} \overline{\text{codim}}(E_n) < 1$, then*

$$\overline{\text{codim}}(\bigcap_{n=1}^\infty E_n) = \sup_{n\geq 1} \overline{\text{codim}}(E_n).$$

Proof of Theorem V.4.1. First, for $h > 0$ we define the two sets

$$\mathcal{S}^+(h) := \{t \in [0,1] : f(t+h) - f(t) \geq 0\},$$

and

$$\mathcal{S}^-(h) := \{t \in [0,1] : f(t+h) - f(t) \leq 0\},$$

Now for an atomless probability measure μ on E let

$$\mathcal{S}^\circ(h) := \begin{cases} \mathcal{S}^-(h), & \text{if } \int_0^1 \mathbf{1}_{\mathcal{S}^-(h)}(s)d\mu(s) \geq \int_0^1 \mathbf{1}_{\mathcal{S}^+(h)}(s)d\mu(s), \\ \mathcal{S}^+(h), & \text{if } \int_0^1 \mathbf{1}_{\mathcal{S}^-(h)}(s)d\mu(s) < \int_0^1 \mathbf{1}_{\mathcal{S}^+(h)}(s)d\mu(s). \end{cases}$$

Since μ is a probability measure on the set $E \subset [0,1]$ it follows $1 \geq \int_0^1 \mathbf{1}_{\mathcal{S}^\circ(h)} d\mu(s) \geq 1/2$.

Define

$$J_\mu(h,a) := \int_0^1 \mathbf{1}_{\{B(s)\in(f(s)-h,f(s)+h)\}} \cdot \mathbf{1}_{\{B(s+h)-B(s)>a\sqrt{2h\log(1/h)} \text{ if } \mathcal{S}^\circ(h)=\mathcal{S}^-(h), B(s+h)-B(s)<-a\sqrt{2h\log(1/h)} \text{ if } \mathcal{S}^\circ(h)=\mathcal{S}^+(h)\}} d\mu(s).$$

In the following we will denote the event $B(s+h)-B(s) > a\sqrt{2h\log(1/h)}$ if $\mathcal{S}^\circ(h) = \mathcal{S}^-(h)$ and $B(s+h) - B(s) < -a\sqrt{2h\log(1/h)}$ if $\mathcal{S}^\circ(h) = \mathcal{S}^+(h)$ by $\mathcal{K}_a(s,h)$.

For $h > 0$ and $s \in [0,1]$, there are constants $C_1 > 0$ and $C_2 > 0$ (only depending on $\max_{x\in[0,1]} |f(x)|$) with

$$C_1 s^{-1/2} h \leq \mathbb{P}(B(s) \in (f(s)-h, f(s)+h)) \leq C_2 s^{-1/2} h. \tag{V.4.5}$$

Note that, by the independence of increments of Brownian motion,

$$\mathbb{E}(J_\mu(h,a)) = \frac{1}{\sqrt{2\pi}} \int_{a\sqrt{2\log(1/h)}}^\infty \exp(-\frac{u^2}{2})du \int_0^1 \mathbf{1}_{\mathcal{S}^\circ(h)}(t)d\mu(t) \cdot \int_0^1 \mathbb{P}(B(s) \in (f(s)-h, f(s)+h))d\mu(s)$$

Applying V.4.5 we get

$$\mathbb{E}(J_\mu(h,a)) \geq \frac{C_1}{2\sqrt{2\pi}} h \int_{a\sqrt{2\log(1/h)}}^\infty \exp(-\frac{u^2}{2})du \int_{h^2}^1 s^{-1/2}\mu(ds).$$

We fix an $h' > 0$, then there is a constant $c_1 > 0$ (depending on $\max_{x\in[0,1]} |f(x)|$) for all $0 < h \leq h'$ such that

$$\mathbb{E}(J_\mu(h,a)) \geq c_1 h \int_{a\sqrt{2\log(1/h)}}^\infty \exp(-\frac{u^2}{2})du. \tag{V.4.6}$$

Later we will apply the second moment method to $J_\mu(h,a)$. Therefore, we need to bound the second moment of $J_\mu(h,a)$ from above.

$$\begin{aligned}
\mathbb{E}(J_\mu^2(h,a)) &= 2\mathbb{E}\Big[\int_0^1 \mathbf{1}_{\{B(t)\in(f(t)-h,f(t)+h)\}}\mathbf{1}_{\mathcal{K}_a(t,h)} \\
&\qquad\qquad \cdot \int_0^t \mathbf{1}_{\{B(s)\in(f(s)-h,f(s)+h)\}}\mathbf{1}_{\mathcal{K}_a(s,h)}\,d\mu(s)d\mu(t)\Big] \\
&= \int_0^1 \mathbf{1}_{\mathcal{S}^\circ(h)}(r)d\mu(r)\cdot\sqrt{\frac{2}{\pi}}\int_{a\sqrt{2\log(1/h)}}^\infty \exp(-\frac{u^2}{2})du \\
&\quad \cdot \mathbb{E}\Big[\int_0^1 \mathbf{1}_{\{B(t)\in(f(t)-h,f(t)+h)\}}\int_0^t \mathbf{1}_{\{B(s)\in(f(s)-h,f(s)+h)\}}\mathbf{1}_{\mathcal{K}_a(s,h)}d\mu(s)d\mu(t)\Big] \\
&\le \sqrt{\frac{2}{\pi}}\int_{a\sqrt{2\log(1/h)}}^\infty \exp(-\frac{u^2}{2})du\cdot(T_1+T_2), \qquad \text{(V.4.7)}
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= \mathbb{E}\Big[\int_h^1 \mathbf{1}_{\{B(t)\in(f(t)-h,f(t)+h)\}}\cdot\int_0^{(t-h)^+}\mathbf{1}_{\{B(s)\in(f(s)-h,f(s)+h)\}}\mathbf{1}_{\mathcal{K}_a(s,h)}d\mu(s)d\mu(t)\Big], \\
T_2 &= \mathbb{E}\Big[\int_0^1 \mathbf{1}_{\{B(t)\in(f(t)-h,f(t)+h)\}}\int_{(t-h)^+}^t \mathbf{1}_{\{B(s)\in(f(s)-h,f(s)+h)\}}d\mu(s)d\mu(t)\Big].
\end{aligned}$$

First, we will estimate T_1, note that

$$T_1 = \int_h^1\int_0^{(t-h)^+}\mathbb{P}\Big(B(t)\in(f(t)-h,f(t)+h),B(s)\in(f(s)-h,f(s)+h),\mathcal{K}_a(s,h)\Big) \\ d\mu(s)d\mu(t).$$

Take a $t\in[h,1]$ and an $s\in[0,t-h]$. Then we have $s\le s+h\le t$ and,

$$\begin{aligned}
&\mathbb{P}(B(t)\in(f(t)-h,f(t)+h)|B(r) \text{ with } r\le s+h) \\
&\qquad = \mathbb{P}(B(t)-B(s+h)+B(s+h)\in(f(t)-h,f(t)+h)|B(r) \\
&\qquad\qquad \text{with } r\le s+h) \\
&\qquad \le \sup_{\zeta\in\mathbb{R}}\mathbb{P}(B(t-s-h)+\zeta\in(f(t)-h,f(t)+h)).
\end{aligned}$$

Since $B(t-s-h)$ is normally distributed we know by the unimodality property of the normal distribution that,

$$\sup_{\zeta\in\mathbb{R}}\mathbb{P}(B(t-s-h)+\zeta\in(f(t)-h,f(t)+h))\le\mathbb{P}(B(t-s-h)\in(-h,h)).$$

Hence, we get for T_1 that,

$$
\begin{aligned}
T_1 &\leq \int_h^1 \int_0^{(t-h)^+} \mathbb{P}(B(t-s-h) \in (-h,h))\mathbb{P}\big(B(s) \in (f(s)-h, f(s)+h), \mathcal{K}_a(s,h)\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad d\mu(s)d\mu(t) \\
&\leq \int_0^1 \mathbf{1}_{\mathcal{S}^\circ(h)}(r)d\mu(r) \cdot \frac{1}{\sqrt{2\pi}} \int_{a\sqrt{2\log(1/h)}}^{\infty} \exp(-\frac{u^2}{2})du \\
&\quad \cdot \int_h^1 \int_0^{(t-h)^+} \mathbb{P}(B(t-s-h) \in (-h,h))\mathbb{P}(B(s) \in (f(s)-h, f(s)+h))\, d\mu(s)d\mu(t).
\end{aligned}
$$

Now, by applying (V.4.5),

$$
T_1 \leq c_2 h^2 \int_{a\sqrt{2\log(1/h)}}^{\infty} \exp(-\frac{u^2}{2})du \int_h^1 \int_0^{(t-h)^+} \frac{1}{\sqrt{s(t-s-h)}} d\mu(s)d\mu(t),
$$

with some positive constant c_2 (depending on $\max_{x\in[0,1]} |f(x)|$). Further, we get

$$
\begin{aligned}
T_1 &\leq c_2 h^2 \int_{a\sqrt{2\log(1/h)}}^{\infty} \exp(-\frac{u^2}{2})du \int_0^{1-h} \frac{1}{\sqrt{s}} \int_{s+h}^1 \frac{1}{\sqrt{(t-s-h)}} d\mu(t)d\mu(s) \\
&\leq c_2 h^2 \int_{a\sqrt{2\log(1/h)}}^{\infty} \exp(-\frac{u^2}{2})du \cdot S_1^2(\mu) \\
&\leq \frac{4c_2 \exp(2\eta)}{(2\eta-1)^2} A_\eta^2(\mu) h^2 \int_{a\sqrt{2\log(1/h)}}^{\infty} \exp(-\frac{u^2}{2})du, \qquad \text{(V.4.8)}
\end{aligned}
$$

where the last step follows from Lemma V.4.3, with $\eta > 1/2$.

The next step is to estimate T_2. Again, we use the unimodality argument as before. For all $t \geq s$ and $h > 0$,

$$
\begin{aligned}
&\mathbb{P}(B(t) \in (f(t)-h, f(t)+h), B(s) \in (f(s)-h, f(s)+h)) \\
&\qquad \leq \mathbb{P}(B(t-s) \in (-h,h))\mathbb{P}(B(s) \in (f(s)-h, f(s)+h)) \\
&\qquad \leq \mathbb{P}(B(t-s) \in (-h,h))\mathbb{P}(B(s) \in (-h,h)).
\end{aligned}
$$

Now we can use the same calculations as in [KS], p.413 to bound T_2 from above. For

the sake of completeness we perform them in the following. With (V.4.5) we get that,

$$\begin{aligned}
T_2 &\le C_2^2 h^2 \int_0^1 \int_{(t-h)^+}^t \frac{1}{\sqrt{s(t-s)}} d\mu(s) d\mu(t) \\
&= C_2^2 h^2 \Big[\int_0^h \int_0^t \frac{1}{\sqrt{s(t-s)}} d\mu(s) d\mu(t) + \int_h^1 \int_{t-h}^t \frac{1}{\sqrt{s(t-s)}} d\mu(s) d\mu(t) \Big] \\
&\le C_2^2 h^2 \Big[\int_0^h \frac{1}{\sqrt{s}} \int_s^h \frac{1}{\sqrt{t-s}} d\mu(t) d\mu(s) \\
&\qquad\qquad + \int_0^1 \frac{1}{\sqrt{s}} \int_s^{(s+h)\wedge 1} \frac{1}{\sqrt{t-s}} d\mu(t) d\mu(s) \Big] \\
&\le C_2^2 h^2 \Big[S_h^2(\mu) + S_1(\mu) \tilde{S}_h(\mu) \Big] \\
&\le C_2^2 h^2 \Big[\frac{4 \exp(2\eta)}{(2\eta-1)^2} A_\eta^2(\mu) h^{2\eta-1} + \frac{4 \exp(2\eta)}{(2\eta-1)^2} A_\eta^2(\mu) h^{\eta-1/2} \Big] \\
&\le \frac{8 C_2^2 \exp(2\eta)}{(2\eta-1)^2} A_\eta^2(\mu) h^{\eta+3/2} \qquad \text{(V.4.9)}
\end{aligned}$$

Therefore, with (V.4.7), (V.4.8) and (V.4.9) we can now bound $\mathbb{E}(J_\mu^2(h,a))$ from above. There is a constant $c_3 > 0$ such that

$$\mathbb{E}(J_\mu^2(h,a)) \le \frac{c_3 \exp(2\eta)}{(2\eta-1)^2} A_\eta^2(\mu) (h^{\eta+3/2}\Phi + h^2\Phi^2), \qquad \text{(V.4.10)}$$

where $\Phi = \frac{1}{\sqrt{2\pi}} \int_a^\infty \sqrt{2\log(1/h)} \exp(-\frac{u^2}{2}) du$.

The next step is applying the second moment method. First, we define the four sets,

$$\mathfrak{G}(a,h) := \Big\{ t \in [0,1] \Big| \sup_{0 \le s \le h} \frac{|X(t+s) - X(t)|}{\sqrt{2s\log(1/s)}} > a \Big\},$$

and

$$\mathfrak{G}^+(a,h) := \Big\{ t \in [0,1] \Big| \sup_{\substack{0 \le s \le h \\ s: t \in \mathcal{S}^-(s)}} \frac{B(t+s) - B(t)}{\sqrt{2s\log(1/s)}} > a \Big\},$$

and

$$\mathfrak{G}^-(a,h) := \Big\{ t \in [0,1] \Big| \sup_{\substack{0 \le s \le h \\ s: t \in \mathcal{S}^+(s)}} \frac{B(t+s) - B(t)}{\sqrt{2s\log(1/s)}} < -a \Big\},$$

and $\mathcal{Z}_h(X) := \{t \in [0,1] : |X(t)| < h\}$. Note that $\mathfrak{G}^+(a,h) \cup \mathfrak{G}^-(a,h) \subset \mathfrak{G}(a,h)$. By Lemma V.4.2, if $\eta < \dim(E)$, then there is an atomless probability measure μ on E with $A_\eta(\mu) < \infty$. Fix such a measure μ and an η such that $a^2 + 1/2 < \eta < \dim(E)$.

By the Paley-Zygmund inequality, $\mathbb{P}(J_\mu(h,a) > 0) \ge \frac{(\mathbb{E}[J_\mu(h,a)])^2}{\mathbb{E}(J_\mu^2(h,a))}$. Using the fact that $\int_x^\infty \exp(-\frac{u^2}{2}) du \ge \frac{x}{x^2+1} \exp(-\frac{x^2}{2})$ (see for instance [MP], Lemma 12.9), we see that $\Phi \ge$

$\frac{h^{a^2} a\sqrt{\log(1/h)}}{\sqrt{\pi}(2a^2\log(1/h)+1)}$. Now, for small enough h and some positive constant c_4 we get

$$\begin{aligned}\frac{(\mathbb{E}[J_\mu(h,a)])^2}{\mathbb{E}(J_\mu^2(h,a))} &\geq \frac{c_4(2\eta-1)^2}{\exp(2\eta)A_\eta^2(\mu)}\Big[\frac{h^{\eta-1/2}}{\Phi}+1\Big]^{-1}\\ &\geq \frac{c_4(2\eta-1)^2}{\exp(2\eta)A_\eta^2(\mu)}\Big[h^{\eta-a^2-1/2}\frac{\sqrt{\pi}(2a^2\log(1/h)+1)}{a\sqrt{\log(1/h)}}+1\Big]^{-1}.\end{aligned}$$

Since $h^{\eta-a^2-1/2}\frac{\sqrt{\pi}(2a^2\log(1/h)+1)}{a\sqrt{\log(1/h)}}$ goes to 0 as h goes to 0, it follows that there is a number ρ depending on η for small enough h with $\liminf_{h\to 0^+}\mathbb{P}(J_\mu(h,a)>0)>\rho>0$. The event $J_\mu(h,a)>0$ implies $\mathfrak{G}(a,h)\cap\mathcal{Z}_h(X)\cap E\neq\emptyset$. Note that if $h\leq h'$, then $\{\mathfrak{G}(a,h)\cap\mathcal{Z}_h(X)\}\subset\{\mathfrak{G}(a,h')\cap\mathcal{Z}_{h'}(X)\}$. $\bigcap_{h>0}\{\mathfrak{G}(a,h)\cap\mathcal{Z}_h(X)\}\cap E$ equals to

$$\Big\{t\in\mathcal{Z}(X)\cap E\Big|\limsup_{h\downarrow 0}\frac{|X(t+h)-X(t)|}{\sqrt{2h\log(1/h)}}\geq a\Big\}.$$

Observe that for every $h>0$, $\mathfrak{G}(a,h)\cap\mathcal{Z}_h(X)$ is an open subset of $[0,1]$. We can apply Lemma V.4.4 since $\overline{\text{codim}}(\mathfrak{G}(a,h)\cap\mathcal{Z}_h(X))\leq\dim(E)$ for all small enough $h>0$). It follows that

$$\mathbb{P}\Big(\Big\{t\in\mathcal{Z}(X)\cap E\Big|\limsup_{h\downarrow 0}\frac{|X(t+h)-X(t)|}{\sqrt{2h\log(1/h)}}\geq a\Big\}\neq\emptyset\Big)>0.$$

□

V.5 Proof of Theorem V.1.3

We will need some notation of the previous chapter. Recall that an interval I is called *dyadic* if it is of the form $I=[k2^m,(k+1)2^m]$ for some integers $k\geq 0$ and m. For each positive integer m let F_m be the collection of dyadic intervals $[k2^{-m},(k+1)2^{-m}]$ for $k=0,\ldots,2^m-1$ and F be the union over all such collections. For each interval $I\in F$ let $L(I)$ be a random variable that takes only the values 0 and 1. Define the sets

$$\mathcal{J}_m:=\bigcup_{\substack{I\in F_m:\\ L(I)=1}} I,$$

and

$$\mathcal{J}:=\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}\mathcal{J}_m.$$

$\mathcal{J}$ is called *limsup fractal* since $\mathbf{1}_{\mathcal{J}}=\limsup_{m\to\infty}\mathbf{1}_{\mathcal{J}_m}$. In order to prove Theorem V.1.3 we will show a lower bound on the Hausdorff dimension of a certain limsup fractal. For more on this method see for instance [MP].

Fix an $\epsilon>0$ and an integer $m>0$. For an interval $I=[t_I,s_I]$ of the form $[k2^{-m},(k+1)2^{-m}]$ we set $L(I)=1$ if $|B(t_I+m2^{-m})-B(t_I)|>a(1+\epsilon)\sqrt{m2^{-m+1}\log(m^{-1}2^m)}$ holds.

Now we want to show that the set $\mathcal{J}$ associated with this family of random variables $\{L(I),I\in F\}$ is contained in a set of points fulfilling that at least "half of the points" are

a-fast times. Then, a lower bound of the Hausdorff dimension of the set of $\mathcal{J}$ is also a lower bound of the set of a-fast times of the process X.

Note that there is a constant $c_1 > 0$ such that for all $s, t \in [0,2]$ with $|s-t| \le h'$ with random $h' > 0$,

$$|B(s) - B(t)| \le c_1 \sqrt{|s-t| \log \frac{1}{|s-t|}},$$

almost surely (see Theorem 1.12 of [MP]). Let $t \in \mathcal{J}$ and also $t \in I = [t_I, s_I] \in F_m$ with $L(I) = 1$. Then, by the triangle inequality it follows

$$\begin{aligned} &|B(t+m2^{-m}) - B(t)| \\ &\qquad \ge |B(t_I + m2^{-m}) - B(t_I)| - |B(t+m2^{-m}) - B(t_I + m2^{-m})| - |B(t_I) - B(t)|. \end{aligned}$$

Now we see that for m (larger than some random $m' > 0$ and) large enough such that $a\epsilon\sqrt{2m\log(m^{-1}2^m)} \ge 2c_1\sqrt{\log 2^m}$ the following inequalities hold,

$$\begin{aligned} |B(t_I + m2^{-m}) - B(t_I)| - |B(t+m2^{-m}) - B(t_I + m2^{-m})| - |B(t_I) - B(t)| \\ \ge a(1+\epsilon)\sqrt{m2^{-m+1}\log(m^{-1}2^m)} - 2c_1\sqrt{2^{-m}\log 2^m} \\ \ge a\sqrt{m2^{-m+1}\log(m^{-1}2^m)}. \end{aligned}$$

This event happens for infinitely many m's. Therefore, t is an a-fast time of B.

Further, we define a process $\hat{B}$ depending on the Brownian motion B by tossing a coin,

$$\hat{B} = \begin{cases} B, & \text{with probability } 1/2, \\ -B, & \text{with probability } 1/2. \end{cases}$$

If the time t is an a-fast time of B, then it is also an a-fast time of $\hat{B}$. Note that if

$$|\hat{B}(t+m2^{-m}) - \hat{B}(t)| \ge a\sqrt{m2^{-m+1}\log(m^{-1}2^m)}$$

holds, then conditional on this the event $\hat{B}(t+m2^{-m}) - \hat{B}(t) \ge a\sqrt{m2^{-m+1}\log(m^{-1}2^m)}$ happens with probability of at least $1/2$ and $\hat{B}(t+m2^{-m}) - \hat{B}(t) \le -a\sqrt{m2^{-m+1}\log(m^{-1}2^m)}$ happens with probability of at least $1/2$ as well. Therefore, we see that the event that $\hat{B}(t+m2^{-m}) - \hat{B}(t) \ge a\sqrt{m2^{-m+1}\log(m^{-1}2^m)}$ if $f(t+m2^{-m}) - f(t) \le 0$ or $\hat{B}(t+m2^{-m}) - \hat{B}(t) \le -a\sqrt{m2^{-m+1}\log(m^{-1}2^m)}$ if $f(t+m2^{-m}) - f(t) > 0$ happens with probability of at least $1/2$. Since $\hat{B}$ is also a Brownian motion, it follows that t is an a-fast time of the process X with probability of at least $1/2$.

Now set $\dim \mathcal{J} = \alpha$ (we actually know by Orey, Taylor (V.1.1) and Theorem V.5.1 that $\dim \mathcal{J} = 1 - a^2$ almost surely). Let $\epsilon > 0$, then there exists a probability measure μ on $\mathcal{J}$ such that the energy

$$\mathbb{E}\Big(\int_{[0,1]}\int_{[0,1]} \frac{1}{|x-y|^{\alpha-\epsilon}} d\mu(x) d\mu(y)\Big) < \infty,$$

see for instance [MP], p.113 or [Matt]. Define $\hat{\mathcal{J}} := \{t \in \mathcal{J} | t \text{ is an } a\text{-fast time of } X\}$, and a probability measure on $\hat{\mathcal{J}}$ by $\mu'(A) = \frac{\mu(A)}{\mu(\hat{\mathcal{J}})}$, where A are measurable sets with respect to μ. Note that

$$\mu(\hat{\mathcal{J}}) = \int_{[0,1]} \mathbb{P}(t \text{ is an } a\text{-fast time of } X \mid \mathfrak{F}) d\mu(t),$$

where $\mathfrak{F}$ is the sigma algebra of B. Then $\mu(\hat{\mathcal{J}}) \geq \frac{1}{2}\mu(\mathcal{J}) = \frac{1}{2}$. Therefore,

$$\mathbb{E}\Big(\int_{[0,1]}\int_{[0,1]} \frac{1}{|x-y|^{\alpha-\epsilon}} d\mu'(x)d\mu'(y)\Big) \leq \mathbb{E}\Big(4 \cdot \int_{[0,1]}\int_{[0,1]} \frac{1}{|x-y|^{\alpha-\epsilon}} d\mu(x)d\mu(y)\Big) < \infty.$$

This implies $\dim \hat{\mathcal{J}} > \alpha - \epsilon$ almost surely (see Theorem 4.27 of [MP]), and by letting $\epsilon \downarrow 0$, it follows that $\dim \hat{\mathcal{J}} \geq \dim \mathcal{J}$ almost surely.

The rest of the proof is the same as in [MP] and we give the details for the sake of completeness. The next step is to bound the first moment of $L(I)$ for $I \in F_m$ from below. Note that

$$\begin{aligned}\mathbb{P}(L(I)=1) &\geq \mathbb{P}\big(B(t_I + m2^{-m}) - B(t_I) > a(1+\epsilon)\sqrt{m2^{-m+1}\log(m^{-1}2^m)}\big)\\ &\geq \mathbb{P}(B(1) > a(1+\epsilon)\sqrt{2\log(m^{-1}2^m)})\\ &\geq 2^{-ma^2(1+\epsilon)^3},\end{aligned}$$

for large enough m and where the last step follows from the fact that $\int_x^\infty \exp(-\frac{u^2}{2})du \geq \frac{x}{x^2+1}\exp(-\frac{x^2}{2})$ (see for instance [MP], Lemma 12.9) and $\frac{a(1+\epsilon)\sqrt{2\log(m^{-1}2^m)}}{\sqrt{2\pi}(1+2a^2(1+\epsilon)^2\log(m^{-1}2^m))}\exp(-a^2(1+\epsilon)^2\log(m^{-1}2^m)) \geq 2^{-ma^2(1+\epsilon)^3}$ for sufficiently large enough m.

In order to prove Theorem V.1.3 we will apply the following theorem.

Theorem V.5.1 (Theorem 10.6 of [MP]). *Let $\mathcal{J}$ be a limsup fractal associated to the family of random variables $\{L(I), I \in F\}$. Suppose $p_k := \mathbb{P}(L(I)=1)$ is the same for all $I \in F_k$, and for an interval $I \in F_m$ let $M_n(I) := \sum_{I' \subset I, I' \in F_n} L(I')$ with $m \leq n$. If there are $\eta_n \geq 1$ and $\gamma \in (0,1)$ such that*

$$\mathbb{V}(M_n(I)) \leq \eta_n \mathbb{E}(M_n(I)) = \eta_n p_n 2^{n-m},$$

and also

$$\lim_{n\to\infty} 2^{(\gamma-1)n} \cdot \frac{\eta_n}{p_n} = 0,$$

then $\dim \mathcal{J} \geq \gamma$, almost surely.

In order to be able to apply Theorem V.5.1 it is left to bound the variance $\mathbb{V}(M_n(I))$ from above. To achieve this, we see that

$$\begin{aligned}\mathbb{E}(M_n^2(I)) &= \sum_{\substack{I_1, I_2 \subset I,\\ I_1, I_2 \in F_n}} \mathbb{E}\big[L(I_1)L(I_2)\big]\\ &\leq \sum_{\substack{I_1 \subset I,\\ I_1 \in F_n}} \Big[(2n+1)\mathbb{E}(L(I_1)) + \mathbb{E}(L(I_1)) \sum_{\substack{I_2 \subset I,\\ I_2 \in F_n}} \mathbb{E}(L(I_2))\Big],\end{aligned}$$

where we used that the random variables $L(I_1)$ and $L(I_2)$ are independent if distance of the two intervals I_1 and I_2 is at least $n2^{-n}$, and further the trivial estimate $\mathbb{E}[L(I_1)L(I_2)] \leq$

$\mathbb{E}[L(I_1)]$. It follows that

$$\mathbb{V}(M_n(I)) = \mathbb{E}(M_n^2(I)) - \mathbb{E}(M_n(I))^2 \leq \sum_{\substack{I_1 \subset I, \\ I_1 \in F_n}} (2n+1)p_n = 2^{n-m}(2n+1)p_n.$$

Applying Theorem V.5.1 for $\gamma < 1 - a^2(1+\epsilon)^3$, the claim follows by letting $\epsilon \downarrow 0$.

■

Remarks and open problems

We looked at fast times, conversely it is natural to ask whether there are also times with unusual slow growth. That is, for a given $a > 0$ a time $t \in [0,1]$ with

$$\limsup_{h \downarrow 0} \frac{|B(t+h) - B(t)|}{\sqrt{h}} \leq a$$

is called an *a-slow time* of linear Brownian motion. The Hausdorff dimension of slow times was considered by Perkins ([Pe83]). For α-Hölder continuous functions f with $\alpha > 1/2$ an a-slow time of Brownian motion B is also an a-slow time of $B - f$. What can one say about the slow times of $B - f$ for other functions f?

Also, can we strenghten Theorem V.1.5 to an almost sure result for functions f that are 1/2-Hölder continuous?

We have seen that we can state some results even for certain random functions f, for instance Theorem V.1.3. Can we say even more about fast times of Brownian motion with random drift?

Bibliography

[ABPR] T. Antunović, K. Burdzy, Y. Peres, and J. Ruscher. *Isolated zeros for Brownian motion with variable drift*. Electronic Journal of Probability. Vol. 16, No. 65: 1793–1814, 2011.

[APV] T. Antunović, Y. Peres, and B. Vermesi. *Brownian motion with variable drift can be space-filling*. Proceedings of the American Mathematical Society, 139(9), 3359-3373, 2011.

[BDG] B. Boufoussi, M. Dozzi, and R. Guerbaz. *Path properties of a class of locally asymptotically self similar processes*. Electron. J. Probab., 13:no. 29, 898–921, 2008.

[BB01] R. Bass and K. Burdzy. *The supremum of Brownian local times on Hölder curves* Ann. Inst. H. Poincaré 37, 627–642, 2001.

[BB02] R. Bass and K. Burdzy. *Erratum to "The supremum of Brownian local times on Hölder curves"* Ann. Inst. H. Poincaré 38, 799–800, 2002.

[BCS] K. Burdzy, Z.-Q. Chen and J. Sylvester. *The heat equation and reflected Brownian motion in time dependent domains. II Singularities of solutions* J. Func. Anal. 204, 1–34, 2003.

[DEK] A. Dvoretzky, P. Erdős, and S. Kakutani. *Nonincrease everywhere of the Brownian motion process*. In Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Vol. II, pages 103–116. Univ. California Press, Berkeley, Calif., 1961.

[Fal] K. J. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. John Wiley & Sons, New York, Second Edition, 2003.

[Grav] S. E. Graversen. *"Polar"-functions for Brownian motion*. Z. Wahrsch. Verw. Gebiete, 61(2):261–270, 1982.

[Haw] J. Hawkes. *Trees generated by a simple branching process*. J. London Math. Soc. (2), 24(2):373–384, 1981.

[HT] G. G. Hamedani, M. N. Tata, *On the determination of the bivariate normal distribution from distributions of linear combinations of the variables*. The American Mathematical Monthly. Vol. 82, No. 9: 913–915, 1975.

[K74] R. Kaufman. *Large increments of Brownian Motion*. Nagoya Math. J. 56, 139-145, 1974.

[Kah] J.-P. Kahane. *Some random series of functions*, volume 5 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1985.

[Kho02] D. Khoshnevisan. *Multiparameter processes: an introduction to random fields*. Springer monographs in mathematics, Springer Verlag, 2002.

[Kho03] D. Khoshnevisan. *The Codimension of the Zeros of a Stable Process in Random Scenery*. Séminaire de Probabilités XXXVII, Lecture Notes in Mathematics - Springer Verlag Issue 1832, 236-245, 2003.

[KPX] D. Khoshnevisan, Y. Peres, Y. Xiao. *Limsup Random Fractals*. Electronic Journal of Probability. Vol. 5, No. 4: 1–24, 2000.

[KS] D. Khoshnevisan, Z. Shi. *Fast Sets and Points for Fractional Brownian Motion*. Séminaire de Probabilités XXXIV, Lecture Notes in Mathematics - Springer Verlag Issue 1729, 393-416, 2000.

[KS91] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus, Second Edition*. Graduate Texts in Mathematics **113**. Springer, 1991.

[LeG88] J.-F. Le Gall. *Sur les fonctions polaires pour le mouvement brownien*. In Séminaire de Probabilités, XXII, volume 1321 of *Lecture Notes in Math.*, pages 186–189. Springer, Berlin, 1988.

[Loud] W. S. Loud. *Functions with prescribed Lipschitz condition*. Proc. Amer. Math. Soc., 2:358–360, 1951.

[Matt] P. Mattila. *Geometry of sets and measures in Euclidean spaces: Fractals and rectifiability*. Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995.

[MP] P. Mörters and Y. Peres. *Brownian Motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2010.

[MP53] I. Marx and G. Piranian. *Lipschitz functions of continuous functions*. Pacific J. Math., 3:447–459, 1953.

[OT74] S. Orey and S.J. Taylor. *How often on a Brownian path does the law of the iterated logarithm fail?* Proc. London Math. Soc. 28, 174192 (1974).

[Pe83] E. Perkins. *On the Hausdorff dimension of Brownian slow points*. Z. Wahrscheinlichkeitstheorie verw. Gebiete 64, 369399 (1983).

[PS10] Y. Peres and P. Sousi. *Brownian motion with variable drift: 0-1 laws, hitting probabilities and Hausdorff dimension*. Available at http://arxiv.org/abs/1010.2987v1.

[PS11] Y. Peres and P. Sousi. *An isoperimetric inequality for the Wiener sausage*. Available at http://arxiv.org/abs/1103.6059v1.

[R12a] J. Ruscher. *Cantor set zeros of one-dimensional Brownian motion minus Cantor function* Available at http://arxiv.org/abs/1203.0753v1.

[R12b] J. Ruscher. *A note on fast times of Brownian motion with variable drift* Available at http://arxiv.org/abs/1203.0752v1.

[RY] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[Tay86] S.J. Taylor. *The measure theory of random fractals*. Math. Proc. Camb. Phil. Soc. 100, 383–406 (1986).

[TW] S. J. Taylor and N. A. Watson. *A Hausdorff measure classification of polar sets for the heat equation*. Math. Proc. Cambridge Philos. Soc., 97(2):325–344, 1985.

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