Persistence of sums of independent random variables, iterated processes and fractional Brownian motion

vorgelegt von
Dipl.-Math. Christoph Baumgarten

aus Henstedt-Ulzburg

Von der Fakultät II – Mathematik und Naturwissenschaften
der Technischen Universität Berlin
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften
– Dr. rer. nat. –

genehmigte Dissertation

Promotionsausschuss

Vorsitzender:	Prof. Dr. Reinhard Nabben
Berichter:	Prof. Dr. Frank Aurzada
	Prof. Dr. Michael Scheutzow

Tag der wissenschaftlichen Aussprache: 24. April 2013

Berlin 2013
D 83

Summary

The persistence probability up to time T of a stochastic process is the probability that the process does not cross a certain constant barrier until time T. For many processes of interest, this probability converges to zero at polynomial or exponential speed, and it is typically nontrivial to determine the rate of the decay. Sometimes this problem is also referred to as one-sided exit problem. Although it is a classial problem, the behaviour of the persistence probability is unknown for most processes except for random walks, Lévy processes, certain integrated processes such as integrated Brownian motion, fractional Brownian motion and a few other Gaussian processes.

In this thesis, we study persistence of different stochastic processes. We start by considering weighted sums of independent and identically distributed (i.i.d.) random variables. Such processes have independent increments, but in contrast to random walks, the increments are not stationary, and classical results from fluctuation theory are therefore not applicable. Here we focus on weighted sums with weights that grow polynomially or exponentially. In the former case, we determine the polynomial rate of decay of the persistence probability, and the rate is shown to be universal over a class of centred distributions.

Autoregressive processes are another example of weighted sums of i.i.d. random variables. Here the weights are the solution to a certain linear difference equation. The behaviour of the persistence probability can range from exponential or polynomial decay to convergence to a positive constant. We derive various results that allow for a characterisation of the behaviour according to the weights. Particular emphasis is put on autoregressive processes of order 2.

Given two independent real-valued stochastic processes $(X(t), t \in \mathbb{R})$ and $(Y(t), t \geq 0)$, we can define a new process Z by composition: $Z = X \circ Y$. Such processes are referred to as iterated processes. The persistence problem for Z reduces to studying the supremum of the process X over the image of $[0, T]$ under the independent process Y. This is a challenging problem if Y is discontinuous so that its image is not an interval in general. We determine the polynomial rate of the persistence probability for Lévy processes and fractional Brownian motion composed with (the absolute value of) a random walk or a Lévy process.

In the last part of the thesis, we discuss persistence probabilities related to fractional Brownian motion (FBM). We study persistence of FBM involving a moving boundary that is allowed to increase or decrease like some power of a logarithm. Our results show that the presence of such a boundary does not change the persistence probability of FBM up to terms of lower order. As an application, we determine the asymptotic behaviour of an integral functional related to a physical model involving FBM.

Acknowledgement

First of all, I would like to thank my supervisor Frank Aurzada for introducing me to the subject of the thesis and for his invaluable support during the last three years. His door was always open for me, and I enjoyed our discussions that have helped me a lot. His encouragement and the assistance that Frank provided in many ways made the last years a very positive experience. Finally, I would like to thank Frank for the funding that made my stay in Paris in 2012 possible, and of course, Zhan Shi for his invitation. I am indebted to Zhan Shi for his hospitality during these three months I spent at Université Pierre et Marie Curie (Paris VI).
I also thank the professors in the committee for undertaking this task. I gratefully acknowledge the fincancial support of the DFG Emmy Noether-Programm, Berlin Mathematical School and the former international research training group 'Stochastic Models of Complex Processes' that allowed me to visit various conferences on three continents. I am also indebted to the probability group at TU Berlin for a very pleasant atmosphere. In particular, I thank Tanja Kramm for reading parts of this thesis and many helpful suggestions. Moreover, I appreciate the help of Karl-Heinz Förster (TU Berlin) concerning some questions from functional analysis.
Last but not least, I thank my parents and Leonie for their continuing support.

Contents

Chapter 1

Introduction

1.1 Statement of the problem

Let $X = (X_t)_{t\in\mathbb{T}}$ denote a real-valued stochastic process defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{T} = \mathbb{R}$ for continuous time processes and $\mathbb{T} = \mathbb{N}$ for discrete time processes. We define its persistence probability up to time T with barrier x by

$$p_T(x) := \mathbb{P}\left(X_t \leq x, \forall t \in [0,T] \cap \mathbb{T}\right), \quad x, T \geq 0. \tag{1.1}$$

The probability above is also called survival probability up to time T. The main focus of this thesis is to determine the asymptotic behaviour of $p_T(x)$ as $T \to \infty$ for fixed x and for certain classes of stochastic processes. This question is sometimes called one-sided exit problem in the literature since $p_T(x) = \mathbb{P}(\tau_x > T)$, where $\tau_x := \inf\{t \geq 0 : X_t > x\}$ is the first exit time of X from the set $(-\infty, x]$.
Without any additional assumptions on the process X, we cannot say much about the behaviour of the survival probability (it could be equal to any number between zero and one for all T, or it could go to zero arbitrarily fast). Generally, we assume that X is not a nonnegative process in order to distinguish the one-sided exit problem from small deviation and small ball problems described below. For instance, a typical situation where finding the asymptotics of the survival probability is an interesting problem is as follows: if X is a process that oscillates, i.e. almost surely,

$$-\infty = \liminf_{t\to\infty} X_t < \limsup_{t\to\infty} X_t = \infty,$$

then clearly $p_T(x) \to 0$ as $T \to \infty$ for every x. The difficulty then consists of determining the speed of convergence to zero. For most processes considered here, the persistence probability decays either polynomially or exponentially with time. To be more precise, we say that $p_T(x)$ decreases polynomially if $p_T(x) = T^{-\theta+o(1)}$ as $T \to \infty$, i.e.

$$\lim_{T\to\infty} -\frac{\log p_T(x)}{\log T} = \theta.$$

In that case, θ is called the persistence or survival exponent which typically does not depend on the value of x.
If the persistence probability decays exponentially, that is,

$$\log p_T(x) = -(\lambda(x) + o(1)) \cdot T, \quad T \to \infty,$$

then the rate $\lambda = \lambda(x)$ is usually sensitive to the value of the barrier x.
Although (1.1) is a classical problem, it has not been studied very intensively so far except for a few Gaussian processes such as fractional Brownian motion, processes having independent and stationary increments such as random walks and Lévy processes, and integrated processes such as integrated Brownian motion. We refer to [AS12] for a recent survey on the subject. We will also introduce most of the known results in Section 1.2 below that will be relevant later on.
Note that the probability in (1.1) is generally the probability of a *rare event*. In contrast to large deviation probabilities ([DS89, DZ10]), survival probabilites describe the event that a process stays *below* a certain threshold, and it is not surprising that the techniques used differ significantly. A more related question are small deviation probabilities, that is,

$$\mathbb{P}\left(|X_t| \le x, \forall t \in [0,T]\right), \quad T \to \infty, \tag{1.2}$$

or small ball probabilites, i.e.

$$\mathbb{P}\left(|X_t| \le \epsilon, \forall t \in [0,1]\right), \quad \epsilon \downarrow 0. \tag{1.3}$$

In fact, (1.2) is obviously a special case of (1.1), but as explained above, persistence is generally related to processes that can also become negative and therefore, the persistence probability (one-sided barrier) is usually much larger than the small deviation probability (two-sided barrier).
Note that the problem of finding the asymptotics in (1.2) and (1.3) is equivalent if X is self-similar. Small deviation and small ball probabilites are usually studied on a logarithmic scale, and in contrast to persistence probabilites, extensive research has been carried out in this field. We refer to [LS01] for a survey and to [Lif13] for an up-to-date bibliography.

Let us now mention some mathematical questions that are related to persistence probabilities. For instance, research on persistence probabilities of integrated processes was motivated by the investigation of the inviscid Burgers equation, see e.g. [Sin92b, SAF92, Ber98, MK04]. If the initial condition of the Burgers equation is given by a stochastic process such as (fractional) Brownian motion, persistence of the integrated (fractional) Brownian motion has been found to be connected with the Hausdorff dimension of the so-called Lagrangian regular points of the solution.
Moreover, survival probabilities also arise in the study of zeros of random polynomials $f_n(z) := \sum_{k=0}^n a_k \xi_k z^k$, where $z \in \mathbb{C}$, $(\xi_n)_{n\ge0}$ is a sequence of i.i.d. random variables, and $(a_n)_{n\ge0}$ is a deterministic sequence of complex numbers. Various questions such

as the distribution of the zeros in the complex plane and the expected number of real zeros have been considered in the literature (see [BRS86, EK95, DPSZ02] for further information and references). In the context of persistence, Dembo et al. ([DPSZ02]) consider so-called Kaç polynomials ($a_n \equiv 1$ for all n) for centred random variables such that $\mathbb{E}\left[|\xi_0|\right]^p$ is finite for all $p \geq 1$. It is shown that the probability that f_{2n} has no real zeros decays polynomially in n, i.e.

$$\mathbb{P}\left(f_{2n}(x) \neq 0, \forall x \in \mathbb{R}\right) = n^{-4\theta+o(1)}, \quad n \to \infty,$$

and θ is the survival rate of a certain related stationary Gaussian process. This interesting connection between the probability of no real zeros and persistence has been further investigated for different classes of polynomials in [SM08, DM12].
A more detailed outline of the two examples can be found in [AS12, Section 4]. We also refer to [LS04] where pursuit problems are mentioned as another application.

The problem of determining the persistence exponent is relevant in various physical models such as diffusion equations with random initial condition, reaction diffusion systems, granular media, and Lotka-Volterra models for population dynamics, see the survey of Majumdar ([Maj99]) for references for these and other examples. Typically, physicists are interested in the following question: given a random field $\Phi(x,t) = \Phi(x,t,\omega)$ $(t \geq 0, x \in \mathbb{R}^d)$ that evolves in space and time according to some dynamics, what is the probability that $\Phi(x,\cdot)$ did not change its sign up to time T for a given point $x \in \mathbb{R}^d$? For instance, the evolution of Φ could be described by the heat equation with random inital condition, that is,

$$\partial_t \Phi = \Delta_d \Phi, \qquad \Phi(x,0) = \Phi_0(x),$$

where $\Phi_0(\cdot)$ is a Gaussian random field, and Δ_d is the Laplace operator in $\mathbb{R}^d$. If $\Phi_0(\cdot)$ is white noise, it is worth noting that the persistence exponent of Φ is related to the exponent corresponding to the non-zero probability of random polynomials described above, see [SM08, DM12].

1.2 Related work

Let us briefly summarise some important known results on persistence probabilities. The processes of interest comprise random walks, Lévy processes, integrated processes and fractional Brownian motion, which will reappear throughout this thesis. For a more comprehensive account, we refer to [AS12].

1.2.1 Brownian motion, random walks and Lévy processes

For a Brownian motion $(B_t)_{t\geq 0}$, the survival exponent is easily seen to be $\theta = 1/2$, since it is a well-known consequence of the reflection principle that $\sup\{B_t, t \in [0,T]\}$ and

$|B_T|$ have the same law for every $T > 0$. By scaling, as $xT^{-1/2} \to 0$, we therefore have that

$$\mathbb{P}(B_t \leq x, \forall t \in [0,T]) = \mathbb{P}(|B_T| \leq x) = \mathbb{P}\left(|B_1| \leq x/\sqrt{T}\right) \sim \sqrt{2/\pi}\, x\, T^{-1/2}. \quad (1.4)$$

Survival probabilites of random walks have been studied in the context of fluctuation theory, and very precise results have been obtained. Let $(S_n)_{n\geq 1}$ denote a random walk, i.e. $S_n = Y_1 + \cdots + Y_n$ where $Y_1, Y_2, \ldots$ is a sequence of i.i.d. random variables. Moreover, let $\tau_0 := \inf\{n \geq 1 : S_n > 0\}$ denote the first time that the random walk enters $(0, \infty)$. τ_0 is often called the first (ascending) ladder epoch and S_{τ_0} the first (ascending) ladder height, see [Fel71, Section XII.1]. The famous Sparre-Andersen formula expresses the generating function of the probabilities $(\mathbb{P}(\tau_0 > n))_{n\geq 0}$ in terms of the probabilities $(\mathbb{P}(S_n \leq 0))_{n\geq 1}$:

$$\sum_{n=0}^{\infty} s^n \mathbb{P}(\tau_0 > n) = \exp\left(\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbb{P}(S_n \leq 0)\right), \quad s \in (-1,1). \quad (1.5)$$

(1.5) is stated in [AS12, Equation 2.2] and can be derived directly from [Fel71, Theorem XII.7.1]. In particular, if $\mathbb{P}(S_n \leq 0) = \rho \in (0,1)$ for all n, (1.5) implies that

$$\sum_{n=0}^{\infty} s^n \mathbb{P}(\tau_0 > n) = \exp\left(\rho \sum_{n=1}^{\infty} \frac{s^n}{n}\right) = \frac{1}{(1-s)^{\rho}},$$

and if one writes down the Taylor series of the function $s \mapsto (1-s)^{-\rho}$, one finds that

$$\mathbb{P}(\tau_0 > n) = \frac{\Gamma(n+\rho)}{n!\Gamma(\rho)} \sim \frac{n^{\rho-1}}{\Gamma(\rho)}, \quad n \to \infty.$$

It is quite remarkable that for $\rho \in (0,1)$, the survival probability is exactly the same for any random walk such that $\mathbb{P}(S_n \leq 0) = \rho$. For instance, if S_1 has a continuous and symmetric distribution, one has $\mathbb{P}(S_n \leq 0) = 1/2$ for all n and $\mathbb{P}(\tau_0 > N) \sim N^{-1/2}/\sqrt{\pi}$. More generally, we say that the random walk S fulfills Spitzer's condition if there is $\rho \in [0,1]$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{P}(S_k \leq 0) = \rho. \quad (1.6)$$

In [Don95, BD97], this is shown to be equivalent to $\lim_{n\to\infty} \mathbb{P}(S_n \leq 0) = \rho \in [0,1]$. For instance, if $\mathbb{E}[S_1] = 0$ and $\mathbb{E}[S_1^2] < \infty$, we have that $\rho = 1/2$ by the CLT.
In view of (1.5), we have that

$$\sum_{n=0}^{\infty} s^n \mathbb{P}(\tau_0 > n) = (1-s)^{-\rho} \exp\left(\sum_{n=1}^{\infty} \frac{s^n}{n} \left(\mathbb{P}(S_n \leq 0) - \rho\right)\right) =: (1-s)^{-\rho} \ell(1/(1-s)).$$

If Spitzer's condition holds, it can be shown that the function ℓ is slowly varying at infinity (see e.g. [Rog71, Lemma 1]), and in combination with a Tauberian theorem for power series ([Fel71, Theorem XIII.5.5]), this yields that $\mathbb{P}(\tau_0 > N) \sim N^{\rho-1}\,\ell(N)/\Gamma(\rho)$, see also [Rog71, Bin73]. In fact, Spitzer's condition and regular variation of the function $\mathbb{P}(\tau_0 > \cdot)$ are even equivalent. In order to state this result in full generality, let

$$V(x) := 1 + \sum_{n=1}^{\infty} H^{n*}(x), \tag{1.7}$$

where $H(x) := \mathbb{P}(S_{\tau_0} \le x)$ is the distribution function of the first ladder height and H^{n*} denotes n-fold convolution. V is called the renewal function of the first ladder height. The following theorem is from [BGT87, Theorem 8.9.12].

Theorem 1.2.1. *Let $\rho \in (0,1)$. The random walk S fulfills Spitzer's condition with ρ if and only if for some $x \ge 0$, there is a constant $C_x > 0$ and a function ℓ slowly varying at infinity such that*

$$\mathbb{P}(S_n \le x, \forall n = 1, \dots, N) \sim C_x N^{\rho-1} \ell(N), \quad N \to \infty. \tag{1.8}$$

Moreover, if (1.8) holds for some $x \ge 0$, it holds for all continuity points x of the renewal function $V(\cdot)$ of the first ladder height given in (1.7) and $C_x = V(x)/\Gamma(\rho)$. Finally, if $\mathbb{E}[S_1] = 0$ and $\mathbb{E}[S_1^2] < \infty$, for all continuity points x of V, there is a constant $\tilde{C}_x > 0$ such that

$$\mathbb{P}(S_n \le x, \forall n = 1, \dots, N) \sim \tilde{C}_x N^{-1/2}, \quad N \to \infty.$$

The last part of the theorem follows from the absolute convergence of the series $\sum_{n=1}^{\infty} n^{-1}(\mathbb{P}(S_n \le 0) - 1/2)$ if $\mathbb{E}[S_1] = 0$ and $\mathbb{E}[S_1^2] < \infty$ ([Ros62]) implying that the slowly varying function ℓ is asymptotically constant.
Let us also mention that Dembo et al. ([DDG12]) give an elegant new proof of the fact that $p_N(x) \asymp N^{-1/2}$ if $\mathbb{E}[S_1] = 0$ and $\mathbb{E}[S_1^2] < \infty$. Here and in the sequel, $f_N \asymp g_N$ means that the ratio f_N/g_N is bounded away from zero and infinity for large values of N. Moreover, we remark that the persistence probability may decay polynomially even in case $\mathbb{E}[S_1] > 0$, see [Don89] for details. On the other hand, if $\mathbb{E}[S_1] < 0$, it is clear that $\rho = 1$, and it is known that $\mathbb{P}(S_n \le 0, \forall n \in \mathbb{N}) > 0$. More generally, without any moment assumption, it holds that

$$S_n \to -\infty \quad \text{a.s.} \iff A := \sum_{n=1}^{\infty} \frac{\mathbb{P}(S_n > 0)}{n} < \infty,$$

see [Fel71, Theorem XII.7.2]. By (1.5), we conclude that

$$\sum_{n=0}^{\infty} s^n \mathbb{P}(\tau_0 > n) = \exp\left(\sum_{n=1}^{\infty} \frac{s^n}{n} - \sum_{n=1}^{\infty} \frac{s^n}{n} \mathbb{P}(S_n > 0) \right) \sim \frac{1}{1-s}\, e^{-A}, \quad s \uparrow 1,$$

and using again Tauberian arguments, it follows that

$$\mathbb{P}(S_n \le 0, \forall n \in \mathbb{N}) = e^{-A}.$$

If $(X_t)_{t\ge 0}$ is a Lévy process, classical results from fluctutation theory (in particular, Wiener-Hopf factorisation, see [Don07, Section 4]) imply similar statements for the continuous-time case. As before, we say that X satisfies Spitzer's condition if

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}(X_s \le 0)\, ds = \rho \in [0,1].$$

By [Don07, Theorem 23], this is equivalent to $\lim_{t\to\infty} \mathbb{P}(X_t \le 0) = \rho \in [0,1]$. Spitzer's condition implies again that the persistence exponent is equal to $1-\rho$, see e.g. [Bin73, Theorem 3]. In fact, we have the following equivalence, see [Don07, Proposition 6]:

Theorem 1.2.2. *Let $(X_t)_{t\ge 0}$ be a Lévy process and $\rho \in (0,1)$. Then X satisfies Spitzer's condition with ρ if and only if there is a function ℓ which is slowly varying at infinity, and for some, and then all, $x > 0$, there is a constant $C_x > 0$ such that*

$$\mathbb{P}(X_t \le x, \forall t \in [0,T]) \sim C_x \ell(T)\, T^{\rho-1}, \quad T \to \infty.$$

Finally, let us mention that instead of considering a constant barrier, one could also study survival probabilities involving a moving barrier, that is, $\mathbb{P}(X_t \le f(t), \forall t \in [0,T])$ where f is some measurable function. For instance, the probability that a Brownian motion does not hit a moving boundary has been studied in [Uch80], and the same problem is investigated for general Lévy processes in [AKS12].

1.2.2 Integrated processes

If $(S_n)_{n\ge 1}$ is a random walk, the process $(X_n)_{n\ge 1}$ given by $X_n = S_1 + \cdots + S_n$ is called integrated random walk. The persistence exponent of an integrated simple random walk ($\mathbb{P}(S_1 = 1) = \mathbb{P}(S_1 = -1) = 1/2$) has been shown to be $\theta = 1/4$ by Sinaĭ ([Sin92a, Theorem 3]). This result has been extended in several subsequent articles to more general distributions, see [AD13], [Vys10] and [Vys12b]. Recently, Dembo et al. ([DDG12]) proved the following result:

Theorem 1.2.3 ([DDG12])**.** *If $\mathbb{E}[S_1] = 0$ and $\mathbb{E}[S_1^2] < \infty$, it holds that*

$$\mathbb{P}(X_n \le 1, \forall n = 1, \ldots, N) \asymp N^{-1/4}, \quad N \to \infty.$$

Finally, assuming slightly more than finite variance, the following precise asymptotics have been derived:

Theorem 1.2.4 ([DW12])**.** *If $\mathbb{E}[S_1] = 0$ and $\mathbb{E}\left[S_1^{2+\delta}\right] < \infty$ for some $\delta > 0$, it holds that*

$$\mathbb{P}(X_n \le 0, \forall n = 1, \ldots, N) \sim C N^{-1/4}, \quad N \to \infty.$$

For continuous-time processes, it has been shown that the survival exponent of integrated Brownian motion is $\theta = 1/4$ ([McK63, Gol71, Sin92a, IW94]). To be more precise, it is shown in [IW94] that

$$\mathbb{P}\left(\int_0^t B_s\, ds \le 1, \forall t \in [0,T]\right) \sim CT^{-1/4}, \quad T \to \infty,$$

where $C > 0$ is some explicit constant. Since integrated Brownian motion is self-similar with index $H = 3/2$, this entails that

$$\mathbb{P}\left(\sup_{t\in[0,T]} \int_0^t B_s\, ds \le x\right) = \mathbb{P}\left(\sup_{t\in[0,x^{-2/3}T]} \int_0^t B_s\, ds \le 1\right) \sim Cx^{1/6}T^{-1/4},$$

as $x^{-2/3}T \to \infty$.
Aurzada and Dereich ([AD13]) use strong approximation techniques to show that the persistence exponent is also equal to $\theta = 1/4$ for integrated Lévy processes under the assumption of exponential moments. Moreover, they consider fractionally integrated processes and derive some results on their survival exponents. Let us finally mention that results on integrated stable Lévy processes can be found in [Sim07].

1.2.3 Fractional Brownian motion

Recall that fractional Brownian motion (FBM) with Hurst index $H \in (0,1)$ is a centred Gaussian process $(X_t)_{t\in\mathbb{R}}$ with covariance

$$\mathbb{E}\left[X_s X_t\right] = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right), \quad s,t \in \mathbb{R}. \tag{1.9}$$

We remark that X has stationary increments and is self-similar of index H, i.e. $(X_{ct})_{t\in\mathbb{R}}$ and $(c^H X_t)_{t\in\mathbb{R}}$ have the same distribution for any $c > 0$. Let us remark that X is non-Markovian unless $H = 1/2$ (see e.g. [MVN68]). The study of persistence for this process has been motivated by the analysis of Burgers equation with random inital conditions ([Sin92b]) and the linear Langevin equation ([KKM+97]). Sinaĭ also derived estimates on the persistence probability in a subsequent article ([Sin97]), and the exponent was shown to equal $\theta = 1 - H$ by Molchan ([Mol99]), where H is the Hurst parameter of the FBM. The estimates on the persistence probability have recently been improved by Aurzada ([Aur11]): there is a constant $c = c(H) > 0$ such that

$$T^{-(1-H)}(\log T)^{-c} \precsim \mathbb{P}\left(X_t \le 1, \forall t \in [0,T]\right) \precsim T^{-(1-H)}(\log T)^{c}, \quad T \to \infty. \tag{1.10}$$

The notation $f(T) \precsim g(T)$ means that $\limsup_{T\to\infty} f(T)/g(T) < \infty$. However, it is still an open problem to show that $p_T(1) \asymp T^{-(1-H)}$ as $T \to \infty$. Note that in view of the self-similarity, (1.10) translates into

$$|\log \epsilon|^{-c}\, \epsilon^{(1-H)/H} \precsim \mathbb{P}\left(X_t \le \epsilon, \forall t \in [0,1]\right) \precsim |\log \epsilon|^{c}\, \epsilon^{(1-H)/H}, \quad \epsilon \downarrow 0. \tag{1.11}$$

Finally, by [Mol12, Proposition 5], it holds for $\alpha \in [0,1]$ that

$$\mathbb{P}\left(X_t \leq 1, \forall t \in [-T^\alpha, T]\right) = T^{-(\alpha H + 1 - H) + o(1)}, \quad T \to \infty.$$

In particular, if $\alpha = 1$, the survival exponent is equal to $\theta = 1$, independent of H, which had been shown in [Mol99, Theorem 3].

1.2.4 Other results

As a matter of fact, the explicit value of the persistence exponent is still unknown for most processes that were not mentioned in the previous subsections except for a few special cases. For instance, persistence of the process $(X_n)_{n\geq 1}$ given by $X_n = Y_n + \epsilon Y_{n-1}$, where $-1 \leq \epsilon \leq 1$ and $Y_0, Y_1, \ldots$ are i.i.d., is studied in [MD01]. If $\epsilon = 1$ and if Y_0 has an absolutely continuous and symmetric law, it is shown that

$$\lim_{N\to\infty} -N^{-1} \log \mathbb{P}\left(X_n \leq 0, \forall n = 1, \ldots, N\right) = \log(\pi/2).$$

An explicit computation of a survival exponent was also achieved by Castell et al. ([CGPPS13]) who consider processes called *random stable Lévy processes in random scenery*. To define such a process, recall that if $(Y_t)_{t\geq 0}$ is a strictly stable Lévy process with index $\alpha \in (1,2]$, there exists a continuous version $(L_t(x))_{x\in\mathbb{R}, t\geq 0}$ of its local time process. If $(W(x))_{x\in\mathbb{R}}$ is a two-sided Brownian motion defined on the same probability space, define the process in random scenery by

$$X_t = \int_{\mathbb{R}} L_t(x)\, dW(x), \quad t \geq 0.$$

The process X is non-Markovian, self-similar with index $H = 1 - 1/(2\alpha)$, and has stationary increments. Castell et al. show that the persistence exponent is given by $\theta = 1 - H = 1/(2\alpha)$. This confirms the findings in [Red97, Maj03].

Let us finally mention some articles where upper or lower bounds on persistence exponents have been established. For Gaussian processes, one of the main tools is Slepian's inequality. Slepian ([Sle62]) studied survival probabilities for stationary Gaussian processes and among other things, he derived this important inequality that will also be a very relevant tool throughout this work. Roughly speaking, Slepian's inequality allows for a comparison of persistence probabilities of different Gaussian processes based on estimates of their covariances. For future reference, it will be useful to state Slepian's inequality at this point.

Lemma 1.2.5. *1. Let $(\mathbb{T}, d)$ be a separable metric space, and let $(X_t)_{t\in\mathbb{T}}$ and $(Y_t)_{t\in\mathbb{T}}$ denote two real-valued, separable, centred Gaussian processes such that*

$$\mathbb{E}\left[X_t^2\right] = \mathbb{E}\left[Y_t^2\right], \quad \forall t \in \mathbb{T}, \qquad \mathbb{E}\left[X_s X_t\right] \leq \mathbb{E}\left[Y_s Y_t\right], \quad s, t \in \mathbb{T}.$$

Let $f\colon \mathbb{T} \to \mathbb{R}$ be a measurable function that is continuous on $\mathbb{T} \setminus D$, where D is at most countable. Then the following inequality holds:

$$\mathbb{P}\left(X_t \leq f(t), \forall t \in \mathbb{T}\right) \leq \mathbb{P}\left(Y_t \leq f(t), \forall t \in \mathbb{T}\right).$$

2. *With $f\colon [0,\infty) \to \mathbb{R}$ as in part 1, if $(Y_t)_{t\geq 0}$ is a separable, centred Gaussian process with $\mathbb{E}[Y_tY_s] \geq 0$ for all $s,t \geq 0$, it holds for all $S,T > 0$ that*

$$\mathbb{P}(Y_t \leq f(t), \forall t \in [0,S+T]) \geq \mathbb{P}(Y_t \leq f(t), \forall t \in [0,S]) \cdot \mathbb{P}(Y_t \leq f(t), \forall t \in [S,S+T]). \tag{1.12}$$

In particular, if Y is also stationary, it holds for every $x \in \mathbb{R}$ that

$$-\lim_{T\to\infty} \frac{\log \mathbb{P}(X_t \leq x, \forall t \in [0,T])}{T} = -\sup_{T>0} \frac{\mathbb{P}(X_t \leq x, \forall t \in [0,T])}{T} \in (0,\infty].$$

The proof can be found in appendix A. Both versions of Slepian's inequality stated in Lemma 1.2.5 are frequently applied in [NR62], [LS04] and [Mol12] to derive general upper and lower bounds on the survival probability of Gaussian processes. Let us also mention the article [LS02] containing some more comparison inequalities that are useful in the context of persistence probabilities.

1.3 Main results of the thesis

In this thesis, we study persistence for different stochastic processes such as weighted sums of independent random variables, iterated processes, autoregressive processes and fractional Brownian motion. The results are based on the articles [AB11], [Bau12], [Bau13] and [AB13].

1.3.1 Weighted sums of i.i.d. random variables

In Chapter 2, we investigate the behaviour of persistence probabilites of processes $Z = (Z_n)_{n\geq 1}$ defined by

$$Z_n := \sum_{k=1}^{n} \sigma(k) Y_k, \quad n \geq 1,$$

where $Y_1, Y_2, \dots$ are i.i.d. random variables such that $\mathbb{E}[Y_1] = 0$ and $\sigma\colon [0,\infty) \to (0,\infty)$ is a measurable function. We call Z a weighted random walk with weight function σ. The results of Chapter 2 have been published in [AB11].

Note that Z has independent increments, but the increments are not stationary unless σ is constant. For such processes, there is virtually no theory available so far, and despite the obvious resemblance, the methods for computing the persistence probability of (unweighted) random walks ($\sigma(n) \equiv 1$) do not carry over since they strongly rely upon the stationarity of increments.

We mainly consider weight functions that grow polynomially or exponentially. In the former case, that is, $\sigma(N) \asymp N^p$ as $N \to \infty$ for some $p > 0$, one can compute the persistence exponent $\theta = \theta(p)$ explicitly if the law of Y_1 is standard Gaussian. Moreover, under the assumption of Gaussian increments, it holds that

$$\mathbb{P}(Z_n \leq 1, \forall n = 1,\dots,N) = \mathbb{P}\left(B_{\kappa(n)} \leq 1, \forall n = 1,\dots,N\right), \tag{1.13}$$

where $\kappa(n) = \sum_{k=1}^{n} \sigma(k)^2$ and $(B_t)_{t\geq 0}$ is a Brownian motion. Hence, one needs to study the persistence problem for Brownian motion evaluated at discrete time points. The following question is natural in this context: for what kind of functions κ is the asymptotic behaviour of the probability in (1.13) and the probability $\mathbb{P}(B_t \leq 1, \forall t \in [0, \kappa(N)])$ the same (up to terms of lower order)? Intuitively, we expect that the distance between the points $\kappa(n)$ and $\kappa(n+1)$ must not be too large. Among other things, we show that this is the case for functions $\kappa(x) = x^p$ $(p > 0)$.
Using strong approximation techniques, one can then infer that the exponent $\theta(p)$ is universal for a larger class of distributions under suitable moment conditions for polynomial weight functions $\sigma(N) \asymp N^p$. For exponential weight functions, such a universal behaviour of the survival probability does not hold in general, and the rate of decay has to be determined separately for different distributions of Y_1. We present various results for the exponential rate in the Gaussian framework. For instance, the rate can be characterised as the largest spectral value of a certain integral operator which leads to a useful variational characterisation of the rate.

1.3.2 Iterated processes

In Chapter 3, we consider the one-sided exit problem for processes $Z = (X \circ |Y_t|)_{t\geq 0}$, where $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ are independent stochastic processes and $Z = (X \circ Y_t)_{t\geq 0}$ if $X = (X_t)_{t\in\mathbb{R}}$ ($\circ$ denotes function composition). Such processes will be referred to as *iterated processes*. For instance, if $(X_t)_{t\in\mathbb{R}}$ is a two-sided process, the persistence probability of the iterated process is then

$$\mathbb{P}(Z_t \leq 1, \forall t \in [0,T]) = \mathbb{P}(X(Y_t) \leq 1, \forall t \in [0,T]) = \mathbb{P}(X_t \leq 1, \forall t \in Y([0,T])),$$

where $Y([0,T]) = \{Y_t : t \in [0,T]\}$ is the image of $[0,T]$ under Y. If Y is a continuous process, $Y([0,T])$ is just the interval with the minimum of Y and the maximum of Y over $[0,T]$ as endpoints. Hence, if the persistence exponent of X is known and if one can control the lenght of $Y([0,T])$, by independence of X and Y, it is not difficult to obtain estimates on the persistence probability of Z. For instance, for continuous self-similar processes $(Y_t)_{t\geq 0}$, we show that the rate of decay of the survival probability of Z can be inferred directly from the persistence probability of X and the index of self-similarity of Y. As a corollary, we obtain that the survival probability of iterated Brownian motion decays asymptotically like $T^{-1/2}$.
If Y is discontinuous, the range of Y possibly contains gaps, and the persistence problem becomes much harder in that case. We determine the polynomial rate of decay for X being a Lévy process (possibly two-sided if $I = \mathbb{R}$) or a fractional Brownian motion and Y being a Lévy process or random walk under suitable moments conditions.
Let us remark that if $(B_t)_{t\geq 0}$ is a Brownian motion and $(S_n)_{n\geq 1}$ is a centred random walk, the persistence probability of the iterated proces $(B(|S_n|))_{n\geq 1}$ given by

$$\mathbb{P}(B(|S_n|) \leq 1, \forall n = 1, \dots, N)$$

is reminiscent of the second probability in (1.13). Instead of the deterministic function κ in (1.13), the time points are now random, and again, it will be important to control the gaps of the random set $\{|S_1|, \dots, |S_N|\}$.

1.3.3 Autoregressive processes

For fixed $p \geq 1$, define $X_n = \sum_{k=1}^{p} a_k X_{n-k} + Y_n$, $n \geq 0$, where $(Y_n)_{n\geq 1}$ is a sequence of i.i.d. random variables, $a_1, \dots, a_p \in \mathbb{R}$, and by convention, $X_n = 0$ for $n \leq 0$. $(X_n)_{n\geq 1}$ is called an autoregressive process of order p (AR(p)-process in short). In this context, the random variables $(Y_n)_{n\geq 1}$ are often referred to as *innovations* in the literature. Autoregressive processes are frequently used to model time series in many applications, see [BD87].

Note that random walks and integrated random walks are special cases of AR-processes with $p = 1, a_1 = 1$ resp. $p = 2, a_1 = 2, a_2 = -1$, and the corresponding results on persistence have been described in Section 1.2.

In Chapter 4, we investigate the behaviour of the persistence probability p_N for such processes under various conditions on the distribution of the innovations. An AR(p)-process X can be written as $X_n = \sum_{k=1}^{n} c_{n-k} Y_k$ where $(c_n)_{n\geq 0}$ solves the difference equation $c_n = a_1 c_{n-1} + \dots + a_p c_{n-p}$ with a certain initial condition. In particular, X_n is again a weighted sum of i.i.d. random variables, but in contrast to the processes considered in Chapter 2, the weights now depend on n as well.

We search criteria for the sequence $(c_n)_{n\geq 0}$ that allow us to characterise the asymptotics of the survival probability. Specifically, we are interested in the following question for AR(p)-processes: when is p_N of polynomial order, when does p_N converge to a positive limit, and when is the decay faster than any polynomial? This classification seems natural if one recalls the results for AR(1)-processes $X_n = \rho X_{n-1} + Y_n$ where $c_n = \rho^n$ for all n. In this case, the behaviour of the persistence probability ranges from exponential decay for $\rho < 1$ ([NK08]), polynomial decay if $\rho = 1$ and $\mathbb{E}[Y_1] = 0$ to convergence to a positive constant if $\rho > 1$.

As we will see, the sequence $(c_n)_{n\geq 0}$ often has a much more complex form if $p \geq 2$, so the results for AR(1)-processes generally cannot be extended directly to higher order processes. We will derive criteria that allow for the classification of the asymptotic behaviour of the p_N as above. Particular emphasis is put on AR(2)-processes. The results of Chapter 4 have been accepted for publication ([Bau13]).

1.3.4 Fractional Brownian motion with moving boundaries

In Chapter 5, we consider various problems related to the persistence probability of fractional Brownian motion (FBM). The results presented have been accepted for publication ([AB13]).

Recently, Oshanin et al. ([ORS12]) study a physical model where persistence properties of FBM are shown to be related to scaling properties of a quantity J_N, called steady-

state current. It turns out that for this analysis it is important to determine persistence probabilities of FBM with a moving boundary f, i.e. $\mathbb{P}\left(X_t \leq f(t), \forall t \in [0,T]\right)$.
We show that one can add a boundary of logarithmic order to a FBM without changing the polynomial rate of decay of the corresponding persistence probability which proves a result needed in [ORS12]. Moreover, we complement their findings by considering the continuous-time version of J_N. Finally, we use the results for moving boundaries in order to improve estimates by Molchan ([Mol99]) concerning the persistence properties of other quantities of interest such as the time when a FBM reaches its maximum on the time interval $(0,1)$ or the last zero in the interval $(0,1)$.

1.4 Notation

Let us introduce some notation and conventions: If $f, g \colon \mathbb{R} \to \mathbb{R}$ are two functions, we write $f \precsim g$ as $x \to x_0$ if $\limsup_{x \to x_0} f(x)/g(x) < \infty$, and $f \asymp g$ as $x \to x_0$ if $f \precsim g$ and $g \precsim f$ as $x \to x_0$. Moreover, $f \sim g$ as $x \to x_0$ if $f(x)/g(x) \to 1$ as $x \to x_0$. We use the usual Landau notation $a_n = o(b_n)$ if $a_n/b_n \to 0$ as $n \to \infty$. $1_{\{A\}}$ denotes the indicator function of a set A.
If X is a random variable, we always assume that X is not concentrated at a single point. If $(X_t)_{t \geq 0}$ is a stochastic process, it will often be convenient to write $X(t)$ instead of X_t. If X and Y are random variables, we write $X \stackrel{d}{=} Y$ to denote equality in distribution. If X and Y are processes, $X \stackrel{d}{=} Y$ means that they have the same finite dimensional distribution. Moreover, we say that $(X_t)_{t \in I}$ is self-similar of index H if $(Y_{ct})_{t \in I} \stackrel{d}{=} (c^H Y_t)_{t \in I}$ for all $c > 0$.
Finally, $x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$, $x^+ := x \vee 0$, $x^- := (-x) \vee 0$, $\lfloor t \rfloor := \sup\{k \in \mathbb{Z} : k \leq t\}$, and $\lceil t \rceil := \inf\{k \in \mathbb{Z} : t \leq k\}$.

Chapter 2

Sums of weighted i.i.d. random variables

In this chapter, we consider persistence probabilities of weighted sums of i.i.d. random variables. Recall that the process $(Z_n)_{n\geq 1}$ given by

$$Z_n := \sum_{k=1}^{n} \sigma(k) Y_k, \tag{2.1}$$

is called a weighted random walk with weight function σ, where $\sigma\colon [0,\infty) \to (0,\infty)$ is a measurable function and $(Y_n)_{n\geq 1}$ denotes a sequence of i.i.d. centred random variables. We are interested in the persistence probability of Z mainly for the two classes of weight functions σ that either grow polynomially or exponentially.

The remainder of this chapter is organised as follows. We begin by reviewing the main results in Section 2.1. In Section 2.2, we consider the persistence probability of Gaussian weighted random walks: polynomial weight functions are studied in Section 2.2.1, and after discussing some extensions of these results, we turn to exponential weight functions in Section 2.2.4. In Section 2.3, the results of the Gaussian case for a polynomial weight function are extended to a broader class of weighted random walks whose increments obey certain moment conditions.

2.1 Main results

Note that if the Y_k have a standard normal distribution, then the processes $(Z_n)_{n\geq 1}$ and $(B_{\kappa(n)})_{n\geq 1}$ have the same law where $\kappa(n) := \sigma(1)^2 + \cdots + \sigma(n)^2$ and B is a standard Brownian motion. Therefore, the computation for the weighted Gaussian random walk reduces to the case of Brownian motion evaluated at discrete time points. In this setup, we prove the following theorem.

Theorem 2.1.1. *Let* $\kappa\colon [0,\infty) \to (0,\infty)$ *be a measurable function such that* $\kappa(N) \asymp N^q$ *for some* $q > 0$. *If there is some* $\delta < q$ *such that* $\kappa(N+1) - \kappa(N) \precsim N^\delta$, *then*

$$\mathbb{P}\left(B_{\kappa(n)} \leq 0, \forall n = 1, \dots, N\right) = N^{-q/2+o(1)}, \quad N \to \infty.$$

In particular, we have under the assumptions of Theorem 2.1.1 that

$$\mathbb{P}\left(B_{\kappa(n)} \leq 0, \forall n = 1, \dots, N\right) = \mathbb{P}\left(B_t \leq 0, \forall t \in [1, \kappa(N)]\right) N^{o(1)} = N^{-q/2+o(1)}. \quad (2.2)$$

In the Gaussian framework, the weight function $\sigma(n) = n^p$ corresponds to $\kappa(n) = \sum_{k=1}^n \sigma(k)^2 \asymp n^{2p+1}$ as remarked above. This implies that the survival exponent for the weighted Gaussian random walk Z is equal to $\theta = p + 1/2$ in that case.
In fact, we show that this survival exponent is universal over a much larger class of weighted random walks in case the Y_k are not necessarily Gaussian:

Theorem 2.1.2. *Let* $(Y_k)_{k\geq 1}$ *be a sequence of i.i.d. random variables with* $\mathbb{E}\left[Y_1\right] = 0$ *and* $\mathbb{E}\left[e^{a|Y_1|}\right] < \infty$ *for some* $a > 0$. *If* σ *is increasing and* $\sigma(N) \asymp N^p$, *then for the weighted random walk* Z *defined in* (2.1), *we have that*

$$\mathbb{P}\left(Z_n \leq 0, \forall n = 1, \dots, N\right) = N^{-(p+1/2)+o(1)}, \quad N \to \infty.$$

The proof of the lower bound for the survival probability in Theorem 2.1.2 under weaker assumptions (Theorem 2.3.2) is based on the Skorokhod embedding. The upper bound (Theorem 2.3.3) is established using strong approximation results in [KMT76]. In either case, the problem is reduced to finding the survival exponent for Gaussian increments, i.e. to the case treated in Theorem 2.1.1.

As noted in (2.2), Theorem 2.1.1 shows that the survival exponent does not change if one samples the Brownian motion at the discrete points $(\kappa(n))_{n\geq 1}$ or over the corresponding interval if κ increases polynomially. This result can be generalised to functions of the type $\kappa(n) = \exp(n^\alpha)$, $n \geq 0$, at least for $\alpha < 1/4$ (Theorem 2.2.11). This fact turns out to be wrong however for the case $\alpha = 1$ in general. Namely, if we consider an exponential function $\kappa(n) = \exp(\beta n)$ for $n \geq 0$ and some $\beta > 0$, in the Gaussian case, it follows from Slepian's inequality and a subadditivity argument that

$$\lim_{N\to\infty} -\frac{1}{N} \log \mathbb{P}\left(B\left(e^{\beta n}\right) \leq 0, \forall n = 0, \dots, N\right) =: \lambda_\beta$$

exists for every $\beta > 0$, and that $\beta \mapsto \lambda_\beta$ is non-decreasing. However, we prove that

$$\lambda_\beta < \beta/2 = \lim_{N\to\infty} -\frac{1}{N} \log \mathbb{P}\left(B\left(e^{\beta t}\right) \leq 0, \forall t \in [0, N]\right)$$

at least for $\beta > 2\log 2$ showing that the rates of decay in the discrete and continuous time framework do not coincide in contrast to (2.2). Additionally, the rate of decay of the survival probability for an exponentially weighted random walk now depends on the distribution of Y_1 even under an exponential moment condition, that is, a universality

result similar to the polynomial case found in Theorem 2.1.2 does not hold.
In the Gaussian case, we state upper and lower bounds on the rate of decay λ_β in Theorem 2.2.14 and characterise $\exp(-\lambda_\beta)$ as an eigenvalue of a certain integral operator in Proposition 2.2.20. Upon applying a transformation, it can be shown that $\exp(-\lambda_\beta)$ is the largest spectral value of a related compact and self-adjoint operator which leads to a useful variational characterisation of λ_β in Theorem 2.2.28. Unfortunately, an explicit computation of λ_β does not seem to be possible easily.

2.2 The Gaussian case

Let $B = (B_t)_{t\geq 0}$ denote a standard Brownian motion. Given a sequence $(Y_n)_{n\geq 1}$ of independent standard Gaussian random variables and a weight function $\sigma(\cdot)$, let Z be the corresponding weighted random walk defined in (2.1). Note that

$$(Z_n)_{n\geq 1} \stackrel{d}{=} (B_{\kappa(n)})_{n\geq 1}, \qquad \kappa(n) := \sum_{k=1}^{n} \sigma(k)^2. \tag{2.3}$$

The problem therefore amounts to determining the asymptotics of

$$\mathbb{P}\left(B_{\kappa(n)} \leq 0, \forall n = 1, \ldots, N\right). \tag{2.4}$$

Intuitively speaking, if $B_{\kappa(1)} \leq 0, \ldots, B_{\kappa(N)} \leq 0$, then typically $B_{\kappa(N-1)}$ and $B_{\kappa(N)}$ are quite far away from the point 0 if N is large. One therefore expects that also $B_t \leq 0$ for $t \in [\kappa(N-1), \kappa(N)]$ unless the difference $\kappa(N) - \kappa(N-1)$ is so large that the Brownian motion has enough time to cross the x-axis with sufficiently high probability in the meantime. So if $\kappa(N) - \kappa(N-1)$ does not grow too fast, one would expect that the probability in (2.4) behaves asymptotically just as in the case where the supremum is taken continuously over the corresponding interval (modulo terms of lower order). In the proof of Theorem 2.2.2 and 2.2.11, this idea will be made explicit in a slightly different way: we will require that the Brownian motion stays below a moving boundary on the intervals $[\kappa(N-1), \kappa(N)]$ where the moving boundary increases sufficiently slowly compared to $\kappa(N)$ in order to leave the survival exponent unchanged. We therefore split our results as follows: In Section 2.2.1, we consider polynomial functions $\kappa(N) = N^q$ for $q > 0$ (so $\kappa(N) - \kappa(N-1) \asymp N^{q-1}$). In Section 2.2.3, we discuss the subexponential case $\kappa(N) = \exp(N^\alpha)$ for $0 < \alpha < 1$ (here $\kappa(N) - \kappa(N-1) \asymp \kappa(N)N^{\alpha-1}$) before finally turning to the exponential case $\kappa(N) = \exp(\beta N)$ for $\beta > 0$ (now $\kappa(N) - \kappa(N-1) \asymp \kappa(N)$) in Section 2.2.4.

Remark 2.2.1. In the statement of Theorem 2.2.2 and 2.2.11, the value 0 of the barrier can be replaced by any $c \in \mathbb{R}$ without changing the result. Indeed, let $\kappa\colon [0, \infty) \to (0, \infty)$ be such that $\kappa(N) \to \infty$ as $N \to \infty$, and let $a := \inf\{\kappa(n) : n \in \mathbb{N}\} > 0$. Note that for $c, d \in \mathbb{R}$, it holds that

$$\begin{aligned}\mathbb{P}\left(B_{\kappa(n)} \leq c, \forall n = 1, \ldots, N\right) &\geq \mathbb{P}\left(B_{a/2} \leq c - d, \sup_{n=1,\ldots,N} B_{\kappa(n)} - B_{a/2} \leq d\right) \\ &= \mathbb{P}\left(B_{a/2} \leq c - d\right) \mathbb{P}\left(B_{\kappa(n)-a/2} \leq d, \forall n = 1, \ldots, N\right) > 0.\end{aligned}$$

Now $\tilde{\kappa}(n) := \kappa(n) - a/2$ is positive for all n and satisfies the same growth conditions as κ stated in all theorems. Hence, it suffices to prove Theorem 2.2.2 and 2.2.11 for the barrier 1.

2.2.1 Polynomial weight functions

The first result is a slightly more precise version of Theorem 2.1.1.

Theorem 2.2.2. *Let $\kappa\colon [0,\infty) \to (0,\infty)$ be a measurable function such that for some $q > 0$ and $\delta < q$*

$$\kappa(N) \asymp N^q \qquad \text{and} \qquad \kappa(N) - \kappa(N-1) \lesssim N^{\delta}, \quad N \to \infty. \tag{2.5}$$

Then for any $\gamma \in (\delta/2, q/2)$, it holds that

$$N^{-q/2} \lesssim \mathbb{P}\left(B_{\kappa(n)} \leq 0, \forall n = 1, \ldots, N\right) \lesssim N^{-q/2} (\log N)^{q/(4\gamma - 2\delta)}, \quad N \to \infty.$$

Proof. By assumption, there are constants $c_1, c_2 > 0$ such that $c_1 n^q \leq \kappa(n) \leq c_2 n^q$ for n large enough. The constant c_2 may be chosen so large that the second inequality holds for all $n \geq 1$. The lower bound is then easily established by comparison to the continuous time case if the barrier 0 is replaced by 1: Using (1.4), one finds that

$$\mathbb{P}\left(B_{\kappa(n)} \leq 1, \forall n = 1, \ldots, N\right) \geq \mathbb{P}\left(B_t \leq 1, \forall t \in [0, c_2 N^q]\right) \asymp N^{-q/2}, \quad N \to \infty.$$

This also implies the same asymptotic order of the lower bound for any other barrier, see Remark 2.2.1.
For the proof of the upper bound, we will assume without loss of generality that κ is non-decreasing. Otherwise, consider the continuous non-decreasing function $\tilde{\kappa}$ with $\tilde{\kappa}(n) = \max\{\kappa(l) : l = 0, \ldots, n\}$ for $n \in \mathbb{N}$ and $\tilde{\kappa}$ linear on $[n, n+1]$ for all $n \in \mathbb{N}$. Then $\tilde{\kappa}(N) \asymp N^q$ as $N \to \infty$. Moreover, $\tilde{\kappa}(N) - \tilde{\kappa}(N-1) = 0$ if $\kappa(N) \leq \tilde{\kappa}(N-1)$, and for $\kappa(N) > \tilde{\kappa}(N-1)$, we have

$$\tilde{\kappa}(N) - \tilde{\kappa}(N-1) = \kappa(N) - \tilde{\kappa}(N-1) \leq \kappa(N) - \kappa(N-1).$$

Thus, $\tilde{\kappa}$ satisfies the same growth conditions as κ. Clearly, for all N,

$$\mathbb{P}\left(B_{\kappa(n)} \leq 1, \forall n = 1, \ldots, N\right) \leq \mathbb{P}\left(B_{\tilde{\kappa}(n)} \leq 1, \forall n = 1, \ldots, N\right),$$

since $\{\kappa(n) : n = 1, \ldots, N\} \subseteq \{\tilde{\kappa}(n) : n = 1, \ldots, N\}$, so it suffices to prove the assertion of the theorem for a non-decreasing function κ.

Choose any γ such that $\delta/2 < \gamma < q/2$ and set $g(N) = \lceil (K \cdot \log N)^{\frac{1}{2\gamma-\delta}} \rceil$ for some $K > 0$ to be specified later. Next, note that

$$\bigcap_{n=g(N)}^{N} \{B_{\kappa(n)} \leq 1\} \subseteq \bigcap_{n=g(N)}^{N-1} \left\{ \sup_{t\in[\kappa(n),\kappa(n+1)]} B_t \leq n^\gamma + 1 \right\}$$
$$\cup \bigcup_{n=g(N)}^{N-1} \left\{ \sup_{t\in[\kappa(n),\kappa(n+1)]} B_t - B_{\kappa(n)} > n^\gamma \right\} =: G_N \cup H_N.$$

Clearly, it holds that

$$\mathbb{P}\left(\sup_{n=1,\dots,N} B_{\kappa(n)} \leq 1 \right) \leq \mathbb{P}\left(\sup_{n=g(N),\dots,N} B_{\kappa(n)} \leq 1 \right) \leq \mathbb{P}\left(G_N\right) + \mathbb{P}\left(H_N\right).$$

Let us first show that the term $\mathbb{P}(H_N)$ decays faster than $N^{-q/2}$ if we choose the constant K in the definition of g large enough: Using the stationarity of increments and the scaling property of Brownian motion, we obtain the following estimates:

$$\begin{aligned}\mathbb{P}\left(H_N\right) &\leq \sum_{n=g(N)}^{N-1} \mathbb{P}\left(\sup_{t\in[0,\kappa(n+1)-\kappa(n)]} B_t > n^\gamma \right) \\ &= \sum_{n=g(N)}^{N-1} \mathbb{P}\left(\sup_{t\in[0,1]} B_t > \frac{n^\gamma}{\sqrt{\kappa(n+1)-\kappa(n)}} \right).\end{aligned}$$

Let c denote a constant such that $\kappa(n+1) - \kappa(n) \leq c\, n^\delta$ for all n sufficiently large. In particular, for N large enough,

$$\begin{aligned}\mathbb{P}\left(H_N\right) &\leq N \max_{n=g(N),\dots,N} \mathbb{P}\left(\sup_{t\in[0,1]} B_t > c^{-1/2} n^{\gamma-\delta/2} \right) \\ &= N\, \mathbb{P}\left(\sup_{t\in[0,1]} B_t > c^{-1/2} g(N)^{\gamma-\delta/2} \right),\end{aligned}$$

since γ was chosen such that $\gamma - \delta/2 > 0$. Next, recall that

$$\mathbb{P}\left(\sup_{t\in[0,1]} B_t > u \right) = \mathbb{P}\left(|B_1| > u\right) = \sqrt{\frac{2}{\pi}} \int_u^\infty e^{-x^2/2}\, dx \leq e^{-u^2/2}, \quad u \geq 0, \tag{2.6}$$

so we finally conclude that

$$\mathbb{P}\left(H_N\right) \leq N \exp\left(-\frac{g(N)^{2\gamma-\delta}}{2c} \right) \leq N^{1-\frac{K}{2c}}.$$

By choosing K large enough, the assertion that $\mathbb{P}(H_N)$ decreases faster than $N^{-q/2}$ is verified.
It remains to show that $\mathbb{P}(G_N) \precsim N^{-q/2}(\log N)^{q/(4\gamma-2\delta)}$. To this end, note that $\kappa(n) \le t$ implies that $n \le (t/c_1)^{1/q}$ for n sufficiently large. Using also that $\kappa(\cdot)$ is non-decreasing, we obtain that

$$\begin{aligned}\mathbb{P}(G_N) &\le \mathbb{P}\left(\bigcap_{n=g(N)}^{N-1}\left\{\sup_{t\in[\kappa(n),\kappa(n+1)]} B_t - (t/c_1)^{\gamma/q} \le 1\right\}\right)\\ &= \mathbb{P}\left(B_t - (t/c_1)^{\gamma/q} \le 1, \forall t \in [\kappa(g(N)), \kappa(N)]\right) =: q_1(N). \end{aligned} \tag{2.7}$$

Let us define the continuous, non-decreasing function F by

$$F(t) := \begin{cases} c_1^{-\gamma/q} t^{\gamma/q}, & t \ge c_1, \\ 1, & t < c_1. \end{cases}$$

Clearly, we have for N large enough that

$$\begin{aligned} q_1(N) &= \mathbb{P}\left(B_t - c_1^{-\gamma/q} t^{\gamma/q} \le 1, \forall t \in [\kappa(g(N)), \kappa(N)]\right)\\ &= \mathbb{P}\left(B_t - F(t) \le 1, \forall t \in [\kappa(g(N)), \kappa(N)]\right). \end{aligned}$$

Since $\mathbb{E}[B_s B_t] \ge 0$ for all $s, t \ge 0$, Slepian's inequality (cf. (1.12) of Lemma 1.2.5 here) implies that

$$\mathbb{P}(B_t - F(t) \le 1, \forall t \in [\kappa(g(N)), \kappa(N)]) \le \frac{\mathbb{P}\left(\sup_{t\in[0,\kappa(N)]} B_t - F(t) \le 1\right)}{\mathbb{P}\left(\sup_{t\in[0,\kappa(g(N))]} B_t - F(t) \le 1\right)}.$$

One has to determine the probability that a Brownian motion does not hit the moving boundary $1 + F(\cdot)$. Now

$$\mathbb{P}(B_t \le c\,t^\alpha + 1,\ \forall t \in [0,T]) \asymp \mathbb{P}(B_t \le 1, \forall t \in [0,T]) \asymp T^{-1/2}, \quad T \to \infty$$

if $\alpha < 1/2$ and $c > 0$ by [Uch80, Theorem 5.1], i.e. adding a drift of order t^α ($\alpha < 1/2$) to a Brownian motion does not change the rate $T^{-1/2}$. Since $\gamma/q < 1/2$, this implies for the boundary $1 + F(\cdot)$ that

$$\begin{aligned} &\frac{\mathbb{P}\left(\sup_{t\in[0,\kappa(N)]} B_t - F(t) \le 1\right)}{\mathbb{P}\left(\sup_{t\in[0,\kappa(g(N))]} B_t - F(t) \le 1\right)} \asymp \frac{\mathbb{P}\left(\sup_{t\in[0,\kappa(N)]} B_t \le 1\right)}{\mathbb{P}\left(\sup_{t\in[0,\kappa(g(N))]} B_t \le 1\right)}\\ &\quad \asymp \kappa(g(N))^{1/2}\,\kappa(N)^{-1/2} \asymp (\log N)^{q/(4\gamma-2\delta)}\,N^{-q/2}. \end{aligned}$$

□

Remark 2.2.3. The assertion of Theorem 2.2.2 above becomes false if we remove the condition $\kappa(N+1)-\kappa(N) \precsim N^\delta$ for some $\delta < q$. Indeed, let $q > 0$ and for $n \in \mathbb{N}$, set

$$\kappa(n) = \exp(qk), \quad \text{if } e^k \le n < e^{k+1} \text{ for some } k \in \mathbb{N}.$$

Then $\kappa(N) \asymp N^q$ as $N \to \infty$. Moreover, $\kappa(N+1)-\kappa(N) = 0$ if there is $k \in \mathbb{N}$ such that $N, N+1 \in [e^k, e^{k+1})$, and

$$\kappa(N+1)-\kappa(N) = \exp(q(k+1)) - \exp(qk) = \kappa(N)(e^q - 1)$$

for $k \in \mathbb{N}$ such that $e^k \le N < e^{k+1} \le N+1$. In particular, $\kappa(N+1)-\kappa(N) \precsim N^q$, whereas for every $\delta < q$,

$$\limsup_{N\to\infty} N^{-\delta}(\kappa(N+1)-\kappa(N)) = \infty,$$

so κ does not fulfill the assumptions of Theorem 2.2.2. Next, note that

$$\begin{aligned}
&\mathbb{P}\left(B_{\kappa(n)} \le 0, \forall n = 1,\dots,N\right) = \mathbb{P}\left(B(e^{qk}) \le 0, k \in \mathbb{N}, e^k \le N\right) \\
&\quad = \mathbb{P}\left(B(e^{qk}) \le 0, \forall k = 1,\dots,\lfloor \log N \rfloor\right) \\
&\quad \ge \prod_{k=1}^{\lfloor \log N \rfloor} \mathbb{P}\left(B(e^{qk}) \le 0\right) \ge (1/2)^{\log N} = N^{-\log 2}.
\end{aligned}$$

The first inequality holds by Slepian's inequality (see also (2.20)). Hence, $N^{-q/2}$ cannot be an upper bound for the survival probability if $q > 2\log 2$.

2.2.2 Some extensions

The proof of the upper bound does not require properties that are specific to Brownian motion. As a matter of fact, the conclusion of Theorem 2.1.1 remains true if we replace the Brownian motion by other Gaussian processes such as fractional Brownian motion or integrated Brownian motion. Moreover, we may also consider Lévy processes instead of the Brownian motion. Let us discuss the necessary modifications in the proof of Theorem 2.1.1 in more detail.

First of all, we state a technical lemma that reduces the problem of finding an upper bound on the survival probability to controlling the probability that the process stays below a certain moving barrier and the probability that the increments of the process are very large.

Lemma 2.2.4. *Let $(X_t)_{t\ge 0}$ denote a stochastic process and $\kappa\colon [0,\infty) \to (0,\infty)$ some measurable function with $\kappa(N) \asymp N^q$ for some $q > 0$. Let $\gamma > 0$ and $(g_n)_{n\ge 1}$ some sequence with $g_n \in \mathbb{N}$, $g_n < n$ and $g_n \to \infty$ as $n \to \infty$. Then there are constants $c_1, N_0 >$*

0 such that for all $N \geq N_0$*, it holds that* $\mathbb{P}\left(\sup_{n=1,\dots,N} X_{\kappa(n)} \leq 1\right) \leq q_1(N) + q_2(N)$*, where*

$$q_1(N) := \mathbb{P}\left(\sup_{t\in[\kappa(g_N),\kappa(N)]} X_t - (t/c_1)^{\gamma/q} \leq 1\right),$$

$$q_2(N) := \sum_{n=g_N}^{N-1} \mathbb{P}\left(\sup_{t\in[\kappa(n),\kappa(n+1)]} X_t - X_{\kappa(n)} > n^\gamma\right).$$

Proof. We have seen in the proof of Theorem 2.2.2 that we can assume without loss of generality that κ is non-decreasing. As before, one verifies that

$$\bigcap_{n=g_N}^{N} \left\{X_{\kappa(n)} \leq 1\right\} \subseteq \bigcap_{n=g_N}^{N-1} \left\{\sup_{t\in[\kappa(n),\kappa(n+1)]} X_t \leq n^\gamma + 1\right\} \cup \bigcup_{n=g_N}^{N-1} \left\{\sup_{t\in[\kappa(n),\kappa(n+1)]} X_t - X_{\kappa(n)} > n^\gamma\right\} =: G_N \cup H_N,$$

Hence,

$$\mathbb{P}\left(X_{\kappa(n)} \leq 1, \forall n = 1,\dots,N\right) \leq \mathbb{P}\left(X_{\kappa(n)} \leq 1, \forall n = g_N,\dots,N\right) \leq \mathbb{P}(G_N) + \mathbb{P}(H_N).$$

By assumption, there is a constant $c_1 > 0$ such that $c_1 n^q \leq \kappa(n)$ for n large enough. Therefore, $\kappa(n) \leq t$ implies that $n \leq (t/c_1)^{1/q}$ for all n sufficiently large. Using also that $\kappa(\cdot)$ is non-decreasing, we obtain that

$$\mathbb{P}(G_N) \leq \mathbb{P}\left(\bigcap_{n=g_N}^{N-1} \left\{\sup_{t\in[\kappa(n),\kappa(n+1)]} X_t - (t/c_1)^{\gamma/q} \leq 1\right\}\right) = q_1(N),$$

whenever N is large enough (so that g_N is large). Moreover, the inequality $\mathbb{P}(H_N) \leq q_2(N)$ is trivial by subadditivity. □

As in the proof of Theorem 2.2.2, the goal is to show that the moving boundary $f(t) = (t/c_1)^{\gamma/q}$ does not change the survival exponent of the process X for a suitable choice of γ, whereas the term q_2 is of lower order. Let us carry this out for fractional Brownian motion, integrated Brownian motion and Lévy processes.

Fractional Brownian motion

In this section, we consider fractional Brownian motion (FBM), see Section 1.2.3 for a definition and properties. In particular, recall that the persistence exponent is equal to $1 - H$ for a FBM with Hurst index $H \in (0,1)$. The following theorem is the analogue of Theorem 2.2.2.

Theorem 2.2.5. *Let X be a FBM with Hurst index $H \in (0,1)$. Assume that κ satisfies the assumptions of Theorem 2.2.2. Then*

$$\mathbb{P}\left(X_{\kappa(n)} \leq 1, \forall n = 1, \ldots, N\right) = N^{-q(1-H)+o(1)}, \quad N \to \infty.$$

Of course, the proof is again based upon estimates on the quantities q_1 and q_2 of Lemma 2.2.4. In partiular, we have to determine the asymptotic behaviour of the survival probability of FBM involving a moving boundary f in order to control the term q_1 stated in the lemma. To do so, we need the concept of the reproducing kernel Hilbert space (RKHS).
Let us just give a very concise description of the RKHS. For a thorough introduction, we refer to [Wei82]. Let $(X_t)_{t\in\mathbb{T}}$ denote a mean-zero Gaussian process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance function $R(t,s) = \mathbb{E}\left[X_s X_t\right]$, $s, t \in \mathbb{T}$, where $\mathbb{T}$ denotes a closed subset of $\mathbb{R}$. The covariance function R determines a Hilbert space $\mathcal{H}$ with scalar product $(\cdot,\cdot)_{\mathcal{H}}$, called the RKHS $\mathcal{H}$ of X, which consists of real functions $f\colon \mathbb{T} \to \mathbb{R}$ such that

$$R(\cdot,t) \in \mathcal{H}, \quad \forall t \in \mathbb{T}, \tag{2.8}$$
$$(f, R(\cdot,t))_{\mathcal{H}} = f(t), \quad \forall t \in \mathbb{T}. \tag{2.9}$$

Property (2.9) is often called the reproducing property of $\mathcal{H}$. To give a more constructive description, let

$$\mathbb{H}_0 := \operatorname{span}\left\{X_t : t \in \mathbb{T}\right\}, \quad \mathbb{H} := \overline{\mathbb{H}_0},$$

where the closure is taken w.r.t. the norm $L^2(\Omega)$. If we equip the closed subspace $\mathbb{H}$ of $L^2(\Omega)$ with the scalar product $(h_1,h_2)_{\mathbb{H}} := \mathbb{E}\left[h_1 h_2\right]$, it is a Hilbert space.
Define now the map

$$\mathcal{J} : \mathbb{H} \to \mathbb{R}^{\mathbb{T}}, \qquad (\mathcal{J}h)(t) := \mathbb{E}\left[h X_t\right], \quad t \in \mathbb{T}.$$

Let $\mathcal{H} := \mathcal{J}(\mathbb{H})$, and $(\mathcal{J}h_1, \mathcal{J}h_2)_{\mathcal{H}} := (h_1,h_2)_{\mathbb{H}} = \mathbb{E}\left[h_1 h_2\right]$.
Indeed, it is not hard to show that $\mathcal{J}(\mathbb{H})$ fulfills the properties (2.8) and (2.9). If $h = X_t \in \mathbb{H}$, then $(\mathcal{J}h) = R(t,\cdot) \in \mathcal{H}$, and if $h = \sum_{j=1}^n \alpha_j X_{t_j} \in \mathbb{H}_0$ for some $t_1, \ldots, t_n \in \mathbb{T}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, then $\mathcal{J}h = \sum_{j=1}^n \alpha_j R(t_j,\cdot) \in \mathcal{H}$ and

$$(\mathcal{J}h, R(\cdot,t))_{\mathcal{H}} = (\mathcal{J}h, \mathcal{J}(X_t))_{\mathcal{H}} = (h, X_t)_{\mathbb{H}} = \mathbb{E}\left[h X_t\right] = \sum_{j=1}^n \alpha_j R(t_j,t) = (\mathcal{J}h)(t).$$

Moreover, if $(h_n)_{n\geq 1}$ is a sequence in $\mathbb{H}_0$ such that $h_n \to h$ in $\mathbb{H}$, then $\|\mathcal{J}h_n - \mathcal{J}h\|_{\mathcal{H}} = \|h_n - h\|_{\mathbb{H}} \to 0$, and it follows that $(\mathcal{J}h, R(t,\cdot))_{\mathcal{H}} = (\mathcal{J}h)(t)$, i.e. (2.9) holds for all $f \in \mathcal{J}(\mathbb{H})$.

Let us now state the result that we need below to deal with moving boundaries: If $(Z_t)_{t\geq 0}$ is a centred Gaussian process and f is an element of the RKHS $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$

of Z, the following estimate is proved in [AD13, Proposition 3.1]: With $p_T(1) := \mathbb{P}(Z_t \le 1, \forall t \in [0,T])$, it holds that

$$e^{-\sqrt{2\|f\|_{\mathcal{H}}^2 \log(1/p_T(1))} - \|f\|_{\mathcal{H}}/2} \le \frac{\mathbb{P}(Z_t - f(t) \le 1, \forall t \in [0,T])}{\mathbb{P}(Z_t \le 1, \forall t \in [0,T])} \le e^{\sqrt{2\|f\|_{\mathcal{H}}^2 \log(1/p_T(1))} - \|f\|_{\mathcal{H}}/2}. \tag{2.10}$$

In particular, if $p_T(1) = T^{-\theta+o(1)}$ for some $\theta > 0$, (2.10) implies that

$$\mathbb{P}(Z_t - f(t) \le 1) = T^{-\theta+o(1)}, \quad T \to \infty.$$

Let $\mathcal{H}_H$ denote the RKHS corresponding to a FBM $(X_t)_{t\in\mathbb{R}}$ with Hurst index $H \in (0,1)$. Various characterisations of this space are known, see [BP88, DÜ99, PT00]. In general, it is not easy to say whether a given function f belongs to $\mathcal{H}_H$. Since we are interested in an upper bound on q_1 given in Lemma 2.2.4, it is enough to find a function $g \in \mathcal{H}_H$ such that $f \le g$. Then

$$\mathbb{P}(X_t - f(t) \le 1, \forall t \in [0,T]) \le \mathbb{P}(X_t - g(t) \le 1, \forall t \in [0,T]),$$

and the probability on the right-hand side can be estimated in view of (2.10). The following lemma will be helpful:

Lemma 2.2.6. *Let $1/2 \ne H \in (0,1)$ and $0 < \rho < H$. Let f be some measurable, locally bounded function such that $f(T) \precsim T^\rho$ as $T \to \infty$. Let $T_0 > 0$. Then there is a function $g \in \mathcal{H}_H$ such that $g \ge 0$ and $f \le g$ on $[T_0, \infty)$.*

Proof. Let $\eta > 1/2$ (to be specified later), $T_0 > 0$, and set $h(t) := t^{-\eta}$ for $t \ge T_0/2$ and $h(t) = 0$ for $t < T_0/2$. Note that $h \in L^2(\mathbb{R})$. Moreover, the assumptions on f imply that there is a constant $C_1 > 0$ such that $f(t) \le C_1 t^\rho$ for all $t \ge T_0$.
Case $H > 1/2$:
Since $h \in L^2(\mathbb{R})$, according to [BP88, Corollary 4.2], the function

$$H(t) := \int_0^t \int_{-\infty}^s (s-u)^{H-3/2} h(u)\, du\, ds, \qquad t \in \mathbb{R},$$

is an element of $\mathcal{H}_H$. Note that $H \equiv 0$ on $(-\infty, T_0/2]$, and for $t > T_0/2$, we have that

$$\begin{aligned} H(t) &= \int_0^t \int_{T_0/2}^s (s-u)^{H-3/2} u^{-\eta}\, du\, ds \ge \int_{T_0/2}^t s^{-\eta} \int_{T_0/2}^s (s-u)^{H-3/2}\, du\, ds \\ &= \int_{T_0/2}^t s^{-\eta} \int_0^{s-T_0/2} u^{H-3/2}\, du\, ds = \frac{1}{H-1/2} \int_{T_0/2}^t s^{-\eta}(s-T_0/2)^{H-1/2}\, ds \\ &\ge \frac{t^{-\eta}}{H-1/2} \int_0^{t-T_0/2} s^{H-1/2}\, ds = \frac{t^{-\eta}(t-T_0/2)^{H+1/2}}{(H+1/2)(H-1/2)} > 0. \end{aligned}$$

In particular, if $t \ge T_0$, there is a constant $C_2 = C_2(H, T_0)$ such that $H(t) \ge C_2\, t^{H+1/2-\eta}$ for all $t \ge T_0$. Since $\rho < H$, we may take $\eta := H - \rho + 1/2 > 1/2$, so $H(t) \ge C_2\, t^\rho$ for all $t \geq T_0$. Since $f(t) \leq C_1 t^\rho$ for all $t \geq T_0$, we see that $f(t) \leq (C_1/C_2) \cdot H(t)$ for all $t \geq T_0$. Hence, the function $g(t) := (C_1/C_2) \cdot H(t) \in \mathcal{H}_H$ satisfies $g \geq 0$ and $f \leq g$ on $[T_0, \infty)$.

Case $H < 1/2$:

According to [PT00, Proposition 6.1],

$$\mathcal{H}_H = \left\{ K_H v : v \in L^2(\mathbb{R}) \right\},$$

where

$$(K_H v)(t) := \int_{-\infty}^{\infty} v(u) \, (D_-^{1/2-H} 1_{\{[0,t)\}})(u) \, du,$$

where D_-^β is the Marchaud fractional derivative of order $\beta \in (0,1)$, see [SKM93]. For our purposes, it is enough to know that for $H < 1/2$, by [PT00, Lemma 3.1],

$$\Gamma(H+1/2) \, (D_-^{1/2-H} 1_{\{[0,t)\}})(u) = (\max\{t-u, 0\})^{H-1/2}, \quad t, u > 0.$$

With h as above, this implies that

$$\begin{aligned} H(t) &:= \Gamma(H+1/2) \cdot (K_H h)(t) = \int_{T_0/2}^{\infty} u^{-\eta} \Gamma(H+1/2)(D_-^{1/2-H} 1_{\{[0,t)\}})(u) \, du \\ &= 1_{\{t > T_0/2\}} \cdot \int_{T_0/2}^{t} u^{-\eta} (t-u)^{H-1/2} \, du. \end{aligned}$$

In particular, $H(t) = 0$ for $t \leq T_0/2$, and for $t > T_0/2$,

$$\begin{aligned} H(t) &\geq t^{-\eta} \cdot \int_{T_0/2}^{t} (t-u)^{H-1/2} \, du = t^{-\eta} \cdot \int_{0}^{t-T_0/2} u^{H-1/2} \, du \\ &= \frac{1}{(H+1/2)} \, t^{-\eta} (t - T_0/2)^{H+1/2}. \end{aligned}$$

In particular, there is a constant $C_3 > 0$ such that $H(t) \geq C_3 t^{H-\eta+1/2}$ for all $t \geq T_0$. As above, with $\eta := H - \rho + 1/2 > 1/2$, the function $g(t) := (C_1/C_3) \cdot H(t)$ has the desired properties. □

Proof of Theorem 2.2.5. In view of Lemma 2.2.4, it suffices to show that for a suitable choice of g_N and γ, q_1 yields the correct order whereas q_2 is of lower order. We start with the term q_2. Let c denote a constant such that $\kappa(n+1) - \kappa(n) \leq c\, n^\delta$ for all n sufficiently large and write $X_1^* = \sup\{X_t : t \in [0,1]\}$. Using first that X has stationary increments and then that X is self-similar, we find that

$$\begin{aligned} q_2(N) &= \sum_{n=g_N}^{N-1} \mathbb{P}\left(\sup_{t \in [0, \kappa(n+1)-\kappa(n)]} X_t > n^\gamma \right) \leq \sum_{n=g_N}^{N-1} \mathbb{P}\left(\sup_{t \in [0, cn^\delta]} X_t > n^\gamma \right) \\ &\leq \sum_{n=g_N}^{N-1} \mathbb{P}\left(X_1^* > c^{-H} n^{\gamma - \delta H} \right). \end{aligned}$$

If we take $\gamma > \delta H$, it follows that $q_2(N) \le N\,\mathbb{P}\left(X_1^* > c^{-H} g_N^{\gamma-\delta H}\right)$. Now set $g_N := \lceil (K\log N)^{1/((\gamma-\delta H)2)} \rceil$ and apply the Gaussian concentration inequality (see e.g. [LT91, Section 3.1]) to obtain for suitable constants $d_1, d_2 > 0$ that

$$q_2(N) \le d_1 N \exp\left(-d_2 g(N)^{2(\gamma-\delta H)}\right) \le d_1 N \exp\left(-d_2 K \log N\right) = d_1 N^{1-d_2 K}.$$

By choosing K large enough, we see that $q_2(N) = o(N^{-q(1-H)})$.
Let us now show that $q_1(N) \precsim N^{-q(1-H)+o(1)}$. Let

$$F(t) := c_1^{-\gamma/q} t^{\gamma/q}, \quad t \ge 0.$$

Since $\delta < q$, we can fix $\gamma \in (\delta H, qH)$ such that $\rho := \gamma/q < H$. According to Lemma 2.2.6, there is a function $G\colon [0,\infty) \to [0,\infty)$ such that $F(t) \le G(t)$ for all $t \ge 1$, and G is an element of the RKHS of X. Combining this with Slepian's inequality ($\mathbb{E}\left[X_s X_t\right] \ge 0$ for all s, t) and the fact that $G \ge 0$, we find that

$$\begin{aligned} q_1(N) = \mathbb{P}\left(\sup_{t\in[\kappa(g_N),\kappa(N)]} X_t - F(t) \le 1\right) &\le \mathbb{P}\left(\sup_{t\in[\kappa(g_N),\kappa(N)]} X_t - G(t) \le 1\right) \\ \le \frac{\mathbb{P}\left(\sup_{t\in[0,\kappa(N)]} X_t - G(t) \le 1\right)}{\mathbb{P}\left(\sup_{t\in[0,\kappa(g_N)]} X_t - G(t) \le 1\right)} &\le \frac{\mathbb{P}\left(\sup_{t\in[0,\kappa(N)]} X_t - G(t) \le 1\right)}{\mathbb{P}\left(\sup_{t\in[0,\kappa(g_N)]} X_t \le 1\right)}. \end{aligned}$$

Since G is an element of the RKHS and $\mathbb{P}\left(X_t \le 1, \forall t \in [0,T]\right) = T^{-(1-H)+o(1)}$, we conclude from (2.10)) that

$$\mathbb{P}\left(X_t - G(t) \le 1,\ \forall t \in [0,T]\right) = T^{-(1-H)+o(1)}, \quad T \to \infty,$$

i.e. the moving boundary G does not not change the survival exponent $\theta = 1 - H$ of FBM. Hence, since $g_N = N^{o(1)}$ and $\kappa(N) \asymp N^q$, it follows that

$$q_1(N) \le \frac{\mathbb{P}\left(\sup_{t\in[0,\kappa(N)]} X_t - G(t) \le 1\right)}{\mathbb{P}\left(\sup_{t\in[0,\kappa(g_N)]} X_t \le 1\right)} = \frac{\kappa(N)^{-(1-H)+o(1)}}{\kappa(g_N)^{-(1-H)+o(1)}} = N^{-q(1-H)+o(1)}.$$

□

Remark 2.2.7. If X is a FBM, set $Y_k := X_k - X_{k-1}$ for all $k \ge 1$, i.e. $X_n = \sum_{k=1}^n Y_k$. Note that the Y_k are correlated standard normal random variables unless $H = 1/2$. In particular, since X has stationary increments and $X_0 = 0$, we see from (1.9) that

$$\begin{aligned} \mathbb{E}\left[Y_k Y_{k+n}\right] &= \mathbb{E}\left[(X_k - X_{k-1})(X_{k+n} - X_{k+n-1})\right] = \mathbb{E}\left[X_1(X_{n+1} - X_n)\right] \\ &= \frac{1 + (n+1)^{2H} - n^{2H}}{2} - \frac{1 + n^{2H} - (n-1)^{2H}}{2}. \end{aligned}$$

With $f(x) = (1+x)^{2H}$, this can be written as

$$\begin{aligned}\mathbb{E}\left[Y_k Y_{k+n}\right] &= \frac{n^{2H-2}}{2} \cdot \frac{f(1/n) + f(-1/n) - 2f(0)}{1/n^2} \\ &\sim n^{2H-2} f''(0)/2 = H(2H-1)\, n^{2H-2}, \quad n \to \infty.\end{aligned}$$

In particular, since $H \in (0,1)$, it holds that $\mathbb{E}\left[Y_k Y_{k+n}\right] \to 0$, and if we apply Theorem 2.2.5 with $\kappa(n) = n$, we obtain that

$$\mathbb{P}\left(\sum_{k=1}^{n} Y_k \leq 1, \forall n = 1, \ldots, N\right) = N^{-(1-H)+o(1)}, \quad N \to \infty.$$

This shows that the persistence exponent for sums of *correlated* random variables $(Y_n)_{n\geq 1}$ depends on the speed of the decay of $\mathbb{E}\left[Y_k Y_{k+n}\right]$ as $n \to \infty$ in this special case. In contrast to the general results for random walks (see Section 1.2.1), persistence of sums of correlated random variables has not been studied in the literature so far, and although we do not pursue this here, it would be interesting to make progress in this direction.

Integrated Brownian motion

Let B denote a Brownian motion, and define the integrated Brownian motion $(I_t)_{t\geq 0}$ by $I_t := \int_0^t B_s \, ds$. Recall that the persistence exponent of I is $\theta = 1/4$, see Section 1.2.2. If κ is as in Theorem 2.2.2, we have again that

$$\mathbb{P}\left(I_{\kappa(n)} \leq 1, \forall n = 1, \ldots, N\right) = \mathbb{P}\left(I_t \leq 1, \forall t \in [0, \kappa(N)]\right) \cdot N^{o(1)}, \quad N \to \infty,$$

i.e. the persistence exponent in the continuous and discrete time framework coincides:

Theorem 2.2.8. *Let $\kappa\colon [0,\infty) \to (0,\infty)$ be as in Theorem 2.2.2. It holds that*

$$\mathbb{P}\left(I_{\kappa(n)} \leq 1, \forall n = 1, \ldots, N\right) = N^{-q/4+o(1)}, \quad N \to \infty.$$

Proof. With the notation of Lemma 2.2.4, we consider the quantities q_1 and q_2 defined there. First, note that, for $t > s$, we have that

$$I_t - I_s = \int_s^t B_u \, du \leq (t-s) \sup_{u \in [s,t]} B_u \leq (t-s) \sup_{u \in [0,t]} B_u.$$

Therefore, for all N sufficiently large, we obtain that

$$\begin{aligned} q_2(N) &= \sum_{n=g_N}^{N-1} \mathbb{P}\left(\sup_{u\in[\kappa(n),\kappa(n+1)]} I_u - I_{\kappa(n)} > n^\gamma \right) \\ &\le \sum_{n=g_N}^{N-1} \mathbb{P}\left((\kappa(n+1)-\kappa(n)) \sup_{u\in[0,\kappa(n+1)]} B_u > n^\gamma \right) \\ &\le \sum_{n=g_N}^{N-1} \mathbb{P}\left(\sup_{u\in[0,1]} B_u > c^{-1} n^{\gamma-\delta} \kappa(n+1)^{-1/2} \right) \\ &\le \sum_{n=g_N}^{N-1} \mathbb{P}\left(\sup_{u\in[0,1]} B_u > d_1 n^{\gamma-\delta-q/2} \right). \end{aligned}$$

We have used that $\kappa(n+1)-\kappa(n) \le c\,n^\delta$ in the third inequality, and $c_1 n^q \le \kappa(n)$ in the last inequality. Take $\gamma > \delta + q/2$ so that

$$q_2(N) \le N\mathbb{P}\left(\sup_{u\in[0,1]} B_u > d_1 g_N^{\gamma-\delta-q/2} \right) = N\mathbb{P}\left(|B_1| > d_1 g_N^{\gamma-\delta-q/2} \right).$$

In view of (2.6), it is easy to conclude that q_2 decreases faster than $N^{-q/4}$ if we set $g_N := \lfloor K(\log N)^{1/(2(\gamma-\delta)-q)} \rfloor$, where K is some suitably large constant.
It remains to show that $q_1(N) \precsim N^{-q/4+o(1)}$. This follows again along similar lines as in the proof of Theorem 2.2.2: Since $\mathbb{E}\left[I_s I_t\right] \ge 0$ for all $s,t \ge 0$, Slepian's inequality implies that

$$q_1(N) \le \mathbb{P}\left(\sup_{t\in[\kappa(g(N)),\kappa(N)]} I_t - c_1^{-\gamma/q} t^{\gamma/q} \le 1 \right) \le \frac{\mathbb{P}\left(\sup_{t\in[0,\kappa(N)]} I_t - c_1^{-\gamma/q} t^{\gamma/q} \le 1 \right)}{\mathbb{P}\left(\sup_{t\in[0,\kappa(g(N))]} I_t \le 1 \right)}. \tag{2.11}$$

The moving barrier does not change the survival exponent of I for a suitable choice of γ, since

$$\mathbb{P}\left(I_t \le c\,t^\alpha + 1, \forall t \in [0,T] \right) = T^{-1/4+o(1)}, \quad T \to \infty,$$

if $0 < \alpha < 3/2$ and $c > 0$, see [AD13, Example 3.2]. For our moving barrier, this amounts to taking $\gamma/q < 3/2$. Taking into account the condition on γ required for treating the term q_2, we must choose γ such that $\delta + q/2 < \gamma < 3q/2$. Note that this is always possible since $\delta < q$. Hence, for such γ, the assertion $q_1(N) \precsim N^{-q/4+o(1)}$ follows easily from (2.11). □

Lévy processes

Finally, let us consider Lévy processes. Recall from Section 1.2.1 that the persistence exponent is equal to $\theta = 1/2$ if X is a Lévy process with $\mathbb{E}\left[X_1\right] = 0$ and $\mathbb{E}\left[X_1^2\right] < \infty$, see also Theorem 3.3.1 below for a slightly more precise result. Under a subexponential moment condition, we can prove the analogue of Theorem 2.2.2 for Lévy processes.

Theorem 2.2.9. *Let $X = (X_t)_{t\geq 0}$ denote a centred Lévy process with $\mathbb{E}[\exp(|X_1|^\alpha)] < \infty$ for some $\alpha > 0$, and let $\kappa\colon [0,\infty) \to (0,\infty)$ be as in Theorem 2.2.2. It holds that*

$$\mathbb{P}\left(\sup_{n=1,\dots,N} X_{\kappa(n)} \leq 1\right) = N^{-q/2+o(1)}, \quad N \to \infty.$$

Proof. Let us again show that the term q_2 of Lemma 2.2.4 is of lower order than $N^{-q/2}$ for a suitable choice of γ and g_N. To this end, note that the stationarity of increments and the fact that $\kappa(n+1) - \kappa(n) \leq cn^\delta$ for all n large enough imply that

$$q_2(N) = \sum_{n=g_N}^{N-1} \mathbb{P}\left(\sup_{t\in[0,\kappa(n+1)-\kappa(n)]} X_t > n^\gamma\right) \leq \sum_{n=g_N}^{N-1} \mathbb{P}\left(\sup_{t\in[0,cn^\delta]} X_t > n^\gamma\right).$$

Recall the following maximal inequality for Lévy processes:

$$\mathbb{P}\left(\sup_{t\in[0,T]} |X_t| \geq x\right) \leq 9\,\mathbb{P}\left(|X_T| \geq x/30\right), \quad T, x > 0. \tag{2.12}$$

This follows from Montgomery-Smith's inequality for sums of centred i.i.d. random variables ([MS93, Corollary 4]) since

$$\mathbb{P}\left(\sup_{t\in[0,T]} |X_t| \geq x\right) = \lim_{n\to\infty} \mathbb{P}\left(\sup_{k=1,\dots,n} \left|X_{kT/n}\right| \geq x\right) \leq 9\mathbb{P}\left(|X_T| \geq x/30\right).$$

The application of Montgomery-Smith's inequality is possible since $X_{kT/n} = Y_{1,T/n} + \cdots + Y_{k,T/n}$ $(k = 1,\dots,n)$ where $Y_{1,T/n},\dots,Y_{n,T/n}$ are i.i.d. random variables with $Y_{1,T/n} \overset{d}{=} X_{T/n}$.
Write $m_n = \lceil cn^\delta \rceil$. The inequality (2.12) implies that

$$q_2(N) \leq \sum_{n=g_N}^{N-1} \mathbb{P}\left(\sup_{t\in[0,m_n]} X_t > \left(\frac{m_n}{2c}\right)^{\gamma/\delta}\right) \leq 9 \sum_{n=g_N}^{N-1} \mathbb{P}\left(|X(m_n)| > \frac{m_n^{\gamma/\delta}}{(2c)^{\gamma/\delta}30}\right).$$

We can then conclude in view of the following large deviation result (see Lemma 2.2.10 below): Let $\rho > 1/2$ and $d > 0$. Since $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[\exp(|X_1|^\alpha)] < \infty$, there are constants $C, \eta > 0$ such that for all n large enough,

$$\mathbb{P}\left(|X_n| > dn^\rho\right) \leq C\, \exp(-n^\eta).$$

In particular, if $\gamma/\delta > 1/2$, it follows with $\rho = \gamma/\delta$ and $d = 1/((2c)^{\gamma/\delta}30)$ that

$$q_2(N) \leq 9C \sum_{n=g_N}^{N-1} \exp(-m_n^{\eta\gamma/\delta}) \leq 9C\, N \exp(-g_N^{\eta\gamma/\delta}),$$

and if we set $g_N := \lceil K(\log N)^{\delta/(\eta\gamma)}\rceil$, we conclude that $q_2(N) \precsim N^{1-K}$ which is of lower order than $N^{-q/2}$ if we take $K = q/2 + 2$.
Let us now show that $q_1(N) \precsim N^{-q/2+o(1)}$. Let $F(t) := c_1^{-\gamma/q} t^{\gamma/q}$ for $t \geq c_1$ and $F(t) = 1$ for $t < c_1$. Clearly, we have that

$$\begin{aligned} q_1(N) &= \mathbb{P}\left(\sup_{t\in[\kappa(g_N),\kappa(N)]} X_t - F(t) \leq 1\right) \\ &\leq \mathbb{P}\left(X_n - F(n) \leq 1, \forall n = \lceil\kappa(g_N)\rceil, \ldots, \lfloor\kappa(N)\rfloor\right) \\ &\leq \frac{\mathbb{P}\left(X_n - F(n) \leq 1, \forall n = 1, \ldots, \lfloor\kappa(N)\rfloor\right)}{\mathbb{P}\left(X_n - F(n) \leq 1, \forall n = 1, \ldots, \lceil\kappa(g_N)\rceil - 1\right)}. \end{aligned} \tag{2.13}$$

The second inequality follows from [EPW67]. Indeed, we have that $X_n = Y_1 + \cdots + Y_n$ where $Y_k = X_k - X_{k-1}$ are i.i.d. Moreover, if $u_1, \ldots, u_N \in \mathbb{R}$, $1 \leq K < L \leq N$, note that the function

$$g_{K,L}(x_1, \ldots, x_N) \mapsto \begin{cases} -1, & \sum_{k=1}^n x_k \leq u_n \quad \text{for all } n = K, \ldots, L, \\ 0, & \text{else,} \end{cases}$$

is non-decreasing in every component. Since independent random variables are associated ([EPW67]), the very definition of association of random variables implies that

$$\mathbb{E}\left[g_{1,N_0}(Y_1, \ldots, Y_N) g_{N_0+1,N}(Y_1, \ldots, Y_N)\right] \geq \mathbb{E}\left[g_{1,N_0}(Y_1, \ldots, Y_N)\right] \cdot \mathbb{E}\left[g_{N_0+1,N}(Y_1, \ldots, Y_N)\right],$$

i.e. for $1 \leq N_0 < N$, it holds that

$$\mathbb{P}\left(\bigcap_{k=1}^{N} \{X_n \leq u_k\}\right) \geq \mathbb{P}\left(\bigcap_{k=1}^{N_0} \{X_k \leq u_k\}\right) \cdot \mathbb{P}\left(\bigcap_{k=N_0+1}^{N} \{X_k \leq u_k\}\right), \tag{2.14}$$

so (2.13) follows.
By [Nov82, Theorem 1], it holds that

$$\mathbb{P}\left(X_n - n^s \leq 1, \forall n = 1, \ldots, N\right) \asymp N^{-1/2}, \quad N \to \infty,$$

whenever $s \in (0, 1/2)$. For our function F, this amounts to $\gamma/q < 1/2$. Combining this with the condition $\gamma/\delta > 1/2$, let us fix $\gamma \in (\delta/2, q/2)$. Since $\kappa(g_N) = N^{o(1)}$, the result then follows easily from (2.13). □

Let us remark that one could also use the results of [AKS12] on moving boundaries for Lévy processes to show that the boundary F does not change the persistence exponent of X in the previous proof under a slighlty more restrictive assumption on the distribution of X_1, see [AKS12, Theorem 2].

The large deviation estimate for a Lévy process needed in the proof of Theorem 2.2.9 follows from the very general results of [Nag79]. One could derive a more precise estimate than the one we give in the Lemma 2.2.10, but in order not to make things more complicated than necessary, we state the following result that is sufficient for our purposes.

Lemma 2.2.10. *Let $Y_1, Y_2, \dots$ denote a sequence of centred i.i.d. random variables such that $\mathbb{E}[\exp(|Y_1|^\alpha)] < \infty$ for some $\alpha > 0$. Let $S_n = \sum_{k=1}^n Y_k$. Let $\rho > 1/2$ and $c > 0$. Then there are constant $C, \eta, N_0 > 0$ such that*

$$\mathbb{P}(|S_n| \geq cn^\rho) \leq C\exp(-n^\eta), \quad n \geq N_0.$$

Proof. We can assume w.lo.g. that $\rho < 1$. Let $t \geq 2$ and set $b = t/(t+2)$ and $a = 1-b$. By [Nag79, Corollary 1.7], if $x, y > 0$, it holds that

$$\mathbb{P}(S_n \geq x) \leq n\mathbb{P}(Y_1 \geq y) + \exp\left(-\frac{a^2x^2}{2e^t nB(y)}\right) + \left(\frac{nA(t,y)}{bxy^{t-1}}\right)^{bx/y},$$

where $B(y) = \mathbb{E}[Y_1^2; Y_1 < y]$ and $A(t,y) = \mathbb{E}[Y_1^t; 0 < X_1 < y]$.
If we apply this inequality with $t = 3$, $x = cn^\rho$, $y = n^{1-\rho}$. Since $B(y) \leq \mathbb{E}[Y_1^2] =: \sigma^2$ and $A(3,y) \leq \mathbb{E}\left[|Y_1|^3\right] =: A$ for any $y > 0$, we see that

$$\mathbb{P}(S_n \geq cn^\rho) \leq n\mathbb{P}\left(Y_1 \geq n^{1-\rho}\right) + \exp\left(-C_1\, c^2\, n^{2\rho-1}\right) + \left(C_2 n^{\rho-1}/c\right)^{bcn^{2\rho-1}},$$

where $C_1 := a^2/(2e^3\sigma^2)$ and $C_2 := A/b$ are constants that neither depend on n nor c. Taking into account that $\mathbb{P}(Y_1 \geq x) \leq \mathbb{E}[\exp(|Y_1|^\alpha)]\exp(-x^\alpha)$ for any $x > 0$ and also that $1/2 < \rho < 1$, we see that, whenever $n \geq n_0 = n_0(c)$,

$$\mathbb{P}(S_n \geq cn^\rho) \leq n\mathbb{E}[\exp(|Y_1|^\alpha)]\exp(-n^{\alpha(1-\rho)}) + 2\exp\left(-C_1\, c^2\, n^{2\rho-1}\right).$$

In particular, if we take η such that $0 < \eta < \min\{\alpha(1-\rho), 2\rho - 1\}$, it follows that

$$\lim_{n\to\infty} \mathbb{P}(S_n \geq cn^\rho) \cdot \exp(n^\eta) = 0.$$

Replacing S by $-S$ in the above estimates, we find in the same way that also

$$\lim_{n\to\infty} \mathbb{P}(S_n \leq -cn^\rho) \cdot \exp(n^\eta) = 0.$$

Hence, the result follows since $\mathbb{P}(|S_n| \geq cn^\rho) = \mathbb{P}(S_n \leq -cn^\rho) + \mathbb{P}(S_n \geq cn^\rho)$. □

2.2.3 Subexponential weight functions

Let us turn again to the persistence problem for Brownian motion at discrete time points $\kappa(1), \kappa(2), \dots$. Here we consider functions $\kappa(\cdot)$ that grow faster than any polynomial but slower than any exponential function, i.e.

$$\lim_{N\to\infty} \frac{N^q}{\kappa(N)} = 0, \quad \forall q > 0, \qquad \lim_{N\to\infty} \frac{\kappa(N)}{e^{\beta N}} = 0, \quad \forall \beta > 0.$$

For simplicity, we restrict our attention to the natural choice $\kappa(n) \asymp \exp(\nu\, n^\alpha)$ for $\nu > 0, \alpha \in (0,1)$. Under certain additional assumptions, the proof of Theorem 2.2.2 can be adapted to yield the following result:

Theorem 2.2.11. *Let $\kappa\colon [0,\infty) \to (0,\infty)$ be a measurable function such that*

$$\kappa(N) \asymp \exp(\nu\, N^\alpha), \qquad \kappa(N+1) - \kappa(N) \precsim \kappa(N)\, N^{-\gamma}, \quad N \to \infty,$$

where $\alpha, \nu > 0$ and $\gamma > 3\alpha$. Then

$$\lim_{N\to\infty} N^{-\alpha} \log \mathbb{P}\left(B_{\kappa(n)} \le 0, \forall n = 1, \dots, N\right) = -\nu/2.$$

More precisely, for $\Lambda := \alpha/(\gamma - 2\alpha) < 1$, one has

$$\exp\left(-\frac{\nu}{2}\, N^\alpha\right) \precsim \mathbb{P}\left(B_{\kappa(n)} \le 0, \forall n = 1, \dots, N\right) \precsim \exp\left(-\frac{\nu}{2}\, N^\alpha\right) \cdot \exp\left(N^{\Lambda\alpha + o(1)}\right).$$

Proof. For simplicity of notation, we again use the barrier 1 instead of 0. The result then follows in view of Remark 2.2.1. By assumption, there are constants $N_0, c_1, c_2 > 0$ such that $c_1 \exp(\nu\, n^\alpha) \le \kappa(n) \le c_2 \exp(\nu\, n^\alpha)$ for all $n \ge N_0$. Since the lower bound follows easily by comparison to the continuous time case, we only prove the upper bound. We can assume w.l.o.g. that κ is non-decreasing (see the proof of Theorem 2.2.2). The assumption $\gamma > 3\alpha$ allows us to find constants ρ, δ with $\alpha < \rho < \gamma/2$ and $\alpha/(\gamma - 2\rho) < \delta < 1$. Set

$$f(t) := \exp\left(\frac{\nu}{2}\, t^\alpha\right)\, t^{-\rho}, \qquad g(t) := \lceil t^\delta \rceil, \quad t > 0.$$

As in the proof of Theorem 2.2.2, it holds that

$$\mathbb{P}\left(\sup_{n=1,\dots,N} B_{\kappa(n)} \le 1\right) \le \mathbb{P}\left(\sup_{n=g(N),\dots,N} B_{\kappa(n)} \le 1\right) \le \mathbb{P}\left(G_N\right) + \mathbb{P}\left(H_N\right),$$

where

$$G_N := \bigcap_{n=g(N)}^{N-1} \left\{ \sup_{t\in[\kappa(n),\kappa(n+1)]} B_t \le f(n) + 1 \right\},$$
$$H_N := \bigcup_{n=g(N)}^{N-1} \left\{ \sup_{t\in[\kappa(n),\kappa(n+1)]} B_t - B_{\kappa(n)} > f(n) \right\}.$$

Next, using the stationarity and the scaling property of Brownian motion, we have that

$$\mathbb{P}\left(H_N\right) \le \sum_{n=g(N)}^{N-1} \mathbb{P}\left(\sup_{t\in[0,1]} B_t > \frac{f(n)}{\sqrt{\kappa(n+1) - \kappa(n)}} \right).$$

We first show that the term $\mathbb{P}\left(H_N\right)$ is of lower order than $\exp(-N^\alpha)$. To this end, since $\kappa(N+1) - \kappa(N) \le c_4 \kappa(N) N^{-\gamma} \le c_5 \exp(\nu N^\alpha)\, N^{-\gamma}$ for all N sufficiently large and some constants $c_4, c_5 > 0$, we get

$$\mathbb{P}(H_N) \leq N \max_{n=g(N),\dots,N} \mathbb{P}\left(\sup_{t\in[0,1]} B_t > c_5^{-1/2}\, n^{\gamma/2-\rho}\right)$$
$$= N\,\mathbb{P}\left(\sup_{t\in[0,1]} B_t > c_5^{-1/2}\, g(N)^{\gamma/2-\rho}\right),$$

since $\gamma/2 - \rho > 0$ by the choice of ρ. By (2.6), we obtain

$$\mathbb{P}(H_N) \leq N \exp\left(-\frac{1}{2c_5}\, g(N)^{\gamma-2\rho}\right) \precsim N \exp\left(-\frac{1}{2c_5}\, N^{\delta(\gamma-2\rho)}\right), \quad N \to \infty.$$

Now $\delta(\gamma - 2\rho) > \alpha$ by the choice of δ, so this term is $o(\exp(-N^\alpha))$.
It remains to show that

$$\mathbb{P}(G_N) \precsim \exp\left(-\frac{\nu}{2}N^\alpha\right) \cdot \exp\left(\frac{\nu}{2}N^{\delta\alpha}\right), \quad N \to \infty.$$

To this end, note that $t \geq \kappa(n)$ implies that $n \leq (\log(t/c_1)/\nu)^{1/\alpha} =: h(t)$ if $n \geq N_0$. Keeping in mind that $f(\cdot)$ is ultimately increasing and using that

$$F(t) := f(h(t)) = \frac{\nu^{\rho/\alpha}}{\sqrt{c_1}} \cdot \frac{\sqrt{t}}{(\log(t/c_1))^{\rho/\alpha}} =: c_3 \cdot \frac{\sqrt{t}}{(\log(t/c_1))^{\rho/\alpha}}, \quad t \geq c_1,$$

we obtain the following estimates for large N:

$$\mathbb{P}(G_N) \leq \mathbb{P}\left(\bigcap_{n=g(N)}^{N-1} \left\{\sup_{t\in[\kappa(n),\kappa(n+1)]} B_t - f(h(t)) \leq 1\right\}\right)$$
$$= \mathbb{P}\left(B_t - F(t) \leq 1, \forall t \in [\kappa(g(N)), \kappa(N)]\right) =: q_1(N).$$

If we set $F(t) = 0$ for $t \leq c_1$, by Slepian's inequality, one has for N sufficiently large

$$q_1(N) = \mathbb{P}\left(\sup_{t\in[\kappa(g(N)),\kappa(N)]} B_t - F(t) \leq 1\right) \leq \frac{\mathbb{P}\left(\sup_{t\in[0,\kappa(N)]} B_t - F(t) \leq 1\right)}{\mathbb{P}\left(\sup_{t\in[0,\kappa(g(N))]} B_t \leq 1\right)}.$$

Since $\rho > \alpha$ entails that

$$\int_1^\infty F(t) t^{-3/2}\, dt = c_3 \cdot \int_{\min\{c_1,1\}}^\infty \frac{1}{t(\log(t/c_1))^{\rho/\alpha}} < \infty,$$

the drift $F(\cdot)$ does not change the rate of the survival probability by [Uch80, Theorem 5.1]. Therefore,

$$\frac{\mathbb{P}\left(\sup_{t\in[0,\kappa(N)]} B_t - F(t) \leq 1\right)}{\mathbb{P}\left(\sup_{t\in[0,\kappa(g(N))]} B_t \leq 1\right)} \asymp \frac{\mathbb{P}\left(\sup_{t\in[0,\kappa(N)]} B_t \leq 1\right)}{\mathbb{P}\left(\sup_{t\in[0,\kappa(g(N))]} B_t \leq 1\right)}$$
$$\asymp \kappa(g(N))^{1/2}\, \kappa(N)^{-1/2} \asymp \exp\left(-\frac{\nu}{2}\, N^\alpha\right) \cdot \exp\left(\frac{\nu}{2}\, N^{\delta\alpha}\right), \quad N \to \infty.$$

Finally, in order to make $\delta \in (\alpha/(\gamma - 2\rho), 1)$ as small as possible, we can let $\rho \downarrow \alpha$, so that $\delta = \alpha/(\gamma - 2\alpha) + o(1) = \Lambda + o(1)$ as $\rho \downarrow \alpha$. ☐

Corollary 2.2.12. *If $\kappa(n) = \exp(\nu\, n^\alpha)$ for some $\nu > 0$ and $\alpha \in (0, 1/4)$, then*

$$\lim_{N\to\infty} N^{-\alpha} \log \mathbb{P}\left(B_{\kappa(n)} \le 0, \forall n = 1, \dots, N\right) = -\nu/2.$$

Proof. Note that

$$\begin{aligned}\kappa(N+1) - \kappa(N) &= \kappa(N)(e^{\nu((N+1)^\alpha - N^\alpha)} - 1) \sim \nu\kappa(N)((N+1)^\alpha - N^\alpha) \\ &= \nu\kappa(N)N^{\alpha-1}\frac{(1+1/N)^\alpha - 1}{1/N} \sim \alpha\nu\,\kappa(N)N^{\alpha-1}, \quad N \to \infty.\end{aligned}$$

Hence, we can apply Theorem 2.2.11 with $\gamma = 1 - \alpha$ if $\gamma > 3\alpha$, i.e. for $\alpha \in (0, 1/4)$. ☐

Remark 2.2.13. The case $\alpha \ge 1/4$ remains unsolved. In view of the heuristics presented below (2.4), it would be interesting to know whether

$$\lim_{N\to\infty} N^{-\alpha}\frac{\log \mathbb{P}\left(\sup_{n=1,\dots,N} B_{\kappa(n)} \le 1\right)}{N^\alpha} = -\nu/2 = \lim_{N\to\infty} \frac{\log \mathbb{P}\left(\sup_{t\in[0,\kappa(N)]} B_t \le 1\right)}{N^\alpha}$$

for all $\alpha \in (0, 1)$. At least for $\alpha = 1$, the rate of decay of the continuous time and discrete time survival probability is different in general as we prove in the next subsection, cf. (2.19).

2.2.4 Exponential weight functions

In this section, we consider the asymptotic behaviour of

$$p_N := \mathbb{P}\left(B(e^{\beta n}) \le 0, \forall n = 0, \dots, N\right),$$

as $N \to \infty$ for $\beta > 0$. It will be helpful to rewrite the process as a discrete Ornstein-Uhlenbeck process. Indeed, observe that

$$p_N = \mathbb{P}\left(e^{-\beta n/2} B(e^{\beta n}) \le 0, \forall n = 0, \dots, N\right) = \mathbb{P}\left(U_{\beta n} \le 0, \forall n = 0, \dots, N\right), \qquad (2.15)$$

where $(U_t)_{t\ge 0}$ is an Ornstein-Uhlenbeck process, i.e. a centred stationary Gaussian process with covariance function

$$\rho(t, s) = \mathbb{E}\left[U_t U_s\right] = e^{-|t-s|/2}, \quad s, t \in \mathbb{R}.$$

In particular, $(e^{-t/2}B(e^t))_{t\in\mathbb{R}}$ is an Ornstein-Uhlenbeck process.
To our knowlegde, the survival probability of the discrete Ornstein-Uhlenbeck process

has not been computed in the literature. For the continuous time case, it is has been shown in [Sle62] that

$$\mathbb{P}\left(U_t \leq 0, \forall t \in [0,T]\right) = \frac{1}{\pi} \arcsin(e^{-T/2}), \quad T \geq 0. \tag{2.16}$$

In fact, this relation can be established by direct computation using an integral formula (see [GR00, Eq. 6.285.1]). It is important to remark that the survival exponent of the Ornstein-Uhlenbeck process depends on the value of the barrier, i.e. for $c \geq 0$,

$$\mathbb{P}\left(U_t \leq c, \forall t \in [0,T]\right) \asymp \exp(-\theta(c)\,T), \quad T \to \infty,$$

for some decreasing function $\theta\colon [0,\infty) \to (0,1/2]$. Moreover, persistence of the Ornstein-Uhlenbeck process is directly related to that of Brownian motion with a square root boundary since

$$\mathbb{P}\left(U_t \leq c, \forall t \in [0,T]\right) = \mathbb{P}\left(B(e^t) \leq c e^{t/2}, \forall t \in [0,T]\right) = \mathbb{P}\left(B_t \leq c\sqrt{t}, \forall t \in [1, e^T]\right).$$

We refer to [Bee75, Sat77] for more details and related results. In the sequel, we work with the barrier $c = 0$, although the techniques presented are applicable for $c \neq 0$ as well.

Since U is stationary, recall from Slepian's inequality (part 2 of Lemma 1.2.5) that there is $\lambda_\beta \in (0,\infty]$ such that

$$\lim_{N\to\infty} -\frac{1}{N} \log \mathbb{P}\left(U_{\beta n} \leq 0, \forall n = 0,\dots,N\right) = \lambda_\beta. \tag{2.17}$$

Slepian's inequality further implies that $\beta \mapsto \lambda_\beta$ is non-decreasing. Unfortunately, we are not able to obtain an explicit expression for λ_β. However, we provide several estimates which are summarised in the next theorem.

Theorem 2.2.14. *For all $\beta > 0$, we have that*

$$\lambda_\beta \geq \begin{cases} \log(2) - c(\beta), & \beta > \beta_0, \\ (\log(2) - c(\beta m))/m, & \beta \in (0,\beta_0], \quad m = \lceil \beta_0/\beta \rceil, \end{cases} \tag{2.18}$$

where $\beta_0 := 2\log(1 + 1/\log 2) \approx 1.786$, $c(\beta_0) = \log 2$, and

$$c(x) := \frac{e^{-x/2}}{1 - e^{-x/2}}, \quad x > 0.$$

Moreover,

$$\lambda_\beta \leq \begin{cases} \beta/2, & \beta \in (0,\beta_1], \\ \log(2) - \log\left(1 + \frac{2}{\pi}\arcsin\left(e^{-\beta/2}\right)\right), & \beta \in [\beta_1,\infty), \end{cases} \tag{2.19}$$

where $\beta_1 \approx 0.472$ is the unique solution on $(0,\infty)$ to the equation

$$\frac{\beta}{2} = \log(2) - \log\left(1 + \frac{2}{\pi}\arcsin\left(e^{-\beta/2}\right)\right).$$

A plot of a numerical approximation of λ_β and of the upper / lower bounds can be found in Figure 2.1 on page 52.

Remark 2.2.15. Let us briefly comment on the lower bound for $\beta \in (0, \beta_0]$ in (2.18). Let $k \in \mathbb{N}$ such that $\beta \in [\beta_0/(k+1), \beta_0/k)$. Then $k < \beta_0/\beta \le k+1$, so $m = \lceil \beta_0/\beta \rceil = k+1$, and by (2.18),

$$\lambda_\beta \ge \frac{\log(2) - c(\beta(k+1))}{k+1} =: \ell(\beta), \quad \beta \in [\beta_0/(k+1), \beta_0/k).$$

Note that ℓ is increasing on $[\beta_0/(k+1), \beta_0/k)$ with $\ell(\beta_0/(k+1)) = 0$ since $c(\beta_0) = \log 2$. On the other hand, upon considering ℓ on $[\beta_0/(k+2), \beta_0/(k+1))$, we have that

$$\lim_{\beta \uparrow \beta_0/(k+1)} \ell(\beta) = \frac{\log 2 - c\left(\frac{\beta_0}{k+1}(k+2)\right)}{k+2} = \frac{\log 2 - c\left(\beta_0(1 + \frac{1}{k+1})\right)}{k+2} > 0.$$

In particular, the lower bound ℓ is discontinuous at the points β_0/n for every $n \in \mathbb{N}$. Since we know that λ_β in non-decreasing in β, one can improve the lower bound in (2.18) upon setting $\tilde{\ell}(\beta) = \sup\{\ell(u) : u \le \beta\}$. In particular, $\tilde{\ell}$ is positive, continuous and non-decreasing.

Remark 2.2.16. For $\beta > \beta_0$, the above theorem implies that

$$\frac{2}{\pi} e^{-\beta/2} \sim \log\left(1 + \frac{2}{\pi} \arcsin\left(e^{-\beta/2}\right)\right) \le \log(2) - \lambda_\beta \le c(\beta) \sim e^{-\beta/2}, \quad \beta \to \infty,$$

i.e. $\lambda_\beta \uparrow \log 2$ exponentially fast as $\beta \uparrow \infty$.
However, it remains an open question whether λ_β is stricly less than $\beta/2$ also for $\beta < \beta_1$ (this would imply that the rate in the discrete time and continuous time framework does not coincide for all β) and whether $\lambda_\beta \sim \beta/2$ as $\beta \downarrow 0$.

Upper bounds for the survival probability

Here we prove the first part of the inequality (2.18).

Lemma 2.2.17. *Let $\beta > \beta_0 = 2\log(1 + 1/\log 2)$. Then for all N*

$$\mathbb{P}\left(B(e^{\beta n}) \le 0, \forall n = 0, \dots, N\right) \le \frac{1}{2} \exp\left(-\left(\log 2 - c(\beta)\right) N\right), \quad N \ge 0,$$

where $c(\beta) \in (0, \log 2)$ is defined in Theorem 2.2.14.

Proof. First, note that $c(\cdot)$ is decreasing with $c(\beta_0) = \log 2$. Since $\mathbb{P}\left(B(e^{\beta n}) \le 0\right) = \mathbb{P}\left(U_{\beta n} \le 0\right) = 1/2$, we have by [LS02, Corollary 2.3] that

$$\mathbb{P}\left(\sup_{n=0,\dots,N} B(e^{\beta n}) \le 0\right) \le \prod_{n=1}^{N+1} \mathbb{P}\left(U_{\beta(n-1)} \le 0\right) \exp\left(\sum_{1 \le i < j \le N+1} e^{-\beta|i-j|/2}\right)$$
$$= 2^{-(N+1)} \exp\left(\sum_{1 \le i < j \le N+1} e^{-\beta|i-j|/2}\right).$$

One computes

$$\sum_{1\le i<j\le N+1} e^{-\beta|i-j|/2} = \sum_{i=1}^{N}\sum_{j=i+1}^{N+1} e^{-\beta(j-i)/2} = \sum_{i=1}^{N}\sum_{j=1}^{N+1-i} e^{-\beta j/2}$$
$$= c(\beta)\sum_{i=1}^{N}(1-e^{-\beta(N+1-i)/2}) \le c(\beta)\,N.$$

□

Next, we prove the second part of (2.18). For small β, we rescale the exponent of the weight function in order to apply Lemma 2.2.17.

Lemma 2.2.18. *Let* $0<\beta<\beta_0$ *and set* $m=m_\beta=\lceil\beta_0/\beta\rceil$. *Then*

$$\mathbb{P}\left(\sup_{n=0,\dots,N} B(e^{\beta n})\le 0\right) \le \exp\left(-\frac{\log 2 - c(\beta m)}{m}N - c(\beta m)\right), \quad N>m.$$

Proof. Clearly, for $N>m$,

$$\mathbb{P}\left(\sup_{n=0,\dots,N} B(e^{\beta n})\le 0\right) \le \mathbb{P}\left(\sup_{n\in\{0,1,2,\dots,\lfloor N/m\rfloor\}} B(e^{\beta mn})\le 0\right) \le \frac{1}{2}e^{-(\log 2 - c(\beta m))\lfloor N/m\rfloor}$$

by Lemma 2.2.17 whenever $\beta m>\beta_0$. Using that $\lfloor N/m\rfloor \ge N/m-1$, the assertion follows.
If $\beta m=\beta_0$, then $\log 2-\log(\beta m)=0$, and the lower bound holds trivially, cf. Remark 2.2.15. □

Lower bounds for the survival probability

We now prove (2.19). In view of (2.16), a comparison to the continuous time framework yields

$$\mathbb{P}\left(B(e^{\beta n})\le 0, \forall n=0,\dots,N\right) \ge \mathbb{P}\left(U_{\beta t}\le 0, \forall t\in[0,N]\right) \sim \pi^{-1}\,e^{-\beta N/2}, \quad N\to\infty.$$

Obviously, for any sequence $0=t_0<t_1<\dots<t_N$, we have

$$\mathbb{P}\left(\sup_{n=1,\dots,N} B(t_n)\le 0\right) \ge \mathbb{P}\left(B(t_1)\le 0, \sup_{n=2,\dots,N} B(t_n)-B(t_{n-1})\le 0\right) = 2^{-N}, \quad (2.20)$$

by independence and symmetry of the increments (or simply Slepian's inequality again). For the exponential case, simple lower bounds are therefore

$$\mathbb{P}\left(B(e^{\beta n})\le 0, \forall n=0,\dots,N\right) \gtrsim \exp(-(\tfrac{\beta}{2}\wedge\log 2)\cdot N), \qquad N\to\infty.$$

In particular, this shows that $\lambda_\beta\le\beta/2$ for all $\beta>0$. The fact that the probability $\mathbb{P}\left(B_t\le 0, B_s\le 0\right)$ admits an explicit formula in terms of s and t can be used to establish a new lower bound that improves the trivial bound $\log 2$ and completes the proof of (2.19).

Lemma 2.2.19. *It holds that*

$$\mathbb{P}\left(\sup_{n=0,\dots,N} B(e^{\beta n}) \le 0\right) \ge \frac{1}{2}\left(\frac{1}{2}+\frac{1}{\pi}\arcsin\left(e^{-\beta/2}\right)\right)^N, \quad N \ge 0.$$

Proof. Let $A_n := \left\{B(e^{\beta k}) \le 0, \forall k = 0,\dots,n\right\}$. Then

$$\begin{aligned}\mathbb{P}(A_N) &= \mathbb{P}\left(B(e^{\beta N}) \le 0 | A_{N-1}\right)\mathbb{P}(A_{N-1}) = \mathbb{P}(X_0 \le 0)\prod_{n=1}^N \mathbb{P}\left(B(e^{\beta n}) \le 0 | A_{n-1}\right) \\ &\ge \frac{1}{2}\prod_{n=1}^N \mathbb{P}\left(B(e^{\beta n}) \le 0 | B(e^{\beta(n-1)}) \le 0\right),\end{aligned}$$

where the inequality follows from [Bra78, Lemma 5]. Next, recall that

$$\mathbb{P}(B_s \le 0, B_t \le 0) = \frac{1}{4}+\frac{1}{2\pi}\arctan\left(\sqrt{\frac{s}{t-s}}\right), \quad s < t,$$

see e.g. [GS01, Exercise 8.5.1]. In particular,

$$\mathbb{P}\left(B(e^{\beta n}) \le 0 | B(e^{\beta(n-1)}) \le 0\right) = \frac{1}{2}+\frac{1}{\pi}\arctan\left(\frac{1}{\sqrt{\exp(\beta)-1}}\right), \quad n \ge 1,$$

independent of n. Now use that $\arctan(x) = \arcsin(x/\sqrt{x^2+1})$. □

A related Fredholm integral equation

If $(Y_n)_{n\ge 0}$ is a sequence of independent standard normal random variables, set

$$X_0 = Y_0, \quad X_n = e^{-\beta/2}X_{n-1} + (1-e^{-\beta})^{1/2}Y_n, \quad n \ge 1. \tag{2.21}$$

It follows immediately that $(X_n)_{n\ge 0}$ is a stationary Markov chain with transition density

$$p(x,y) := \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-\rho x)^2}{2\sigma^2}\right), \quad x, y \in \mathbb{R},$$

where $\rho = e^{-\beta/2}$ and $\sigma = \sqrt{1-e^{-\beta}}$. Moreover, the recursion equation (2.21) is a special case of an autogregressive process of order 1 (AR(1)-model). Persistence of such processes is studied in [Lar04]. Larralde explicitly computes the generating function of the first hitting time of the set $(0,\infty)$ if the Y_n have a two-sided exponential distribution. Conditions ensuring that exponential moments of the first hitting time of the set $[x,\infty)$ $(x \ge 0)$ exist for an AR(1) process can be found in [NK08]. These results will be presented in Chapter 4 where persistence of more general autoregressive processes will

be discussed.
If we iterate the recursion equation for X_n, we see that

$$X_n = e^{-\beta n/2} Y_0 + (1-e^{-\beta})^{1/2} \sum_{k=1}^{n} e^{-\beta(n-k)/2} Y_k, \quad n \geq 0,$$

and a simple computation shows that

$$\mathbb{E}\left[X_n X_{n+m}\right] = e^{-\beta m/2}, \quad n, m \geq 0.$$

Since $(X_n)_{n\geq 0}$ is a Gaussian process, this implies that $(X_n)_{n\geq 0}$ and $(U(\beta n))_{n\geq 0}$ are equal in distribution, where U is again the Ornstein-Uhlenbeck process. Hence, taking (2.15) into account, we have that

$$\mathbb{P}\left(\sup_{n=0,\dots,N} X_n \leq 0\right) = \mathbb{P}\left(\sup_{n=0,\dots,N} U_{\beta n} \leq 0\right) = \mathbb{P}\left(\sup_{n=0,\dots,N} B(e^{\beta n}) \leq 0\right).$$

Studying persistence of the Gaussian AR(1)-process X is therefore equivalent to the persistence problem for $(B(e^{\beta n}))_{n\geq 0}$. In this subsection, we mainly use the fact that $(X_n)_{n\geq 1}$ is Markovian in order to tackle this problem. The approach here provides an alternative method to gain some new information about the rate λ_β defined in (2.17). In particular, we show that $\exp(-\lambda_\beta)$ is the eigenvalue of a certain integral operator T related to the transition density $p(\cdot,\cdot)$ of X. If we apply a suitable transfomation to T, we can show that $\exp(-\lambda_\beta)$ is the largest spectral value of a related compact and self-adjoint operator S which leads to a useful variational characterisation of the rate λ_β.
Let us now carry out the necessary steps to prove the results just outlined. To begin, set $A_n := \{X_0 \leq 0, \dots, X_n \leq 0\}$, and let π_n be the cumulative distribution function of X_n given A_n, i.e.

$$\pi_n(u) := \mathbb{P}\left(X_n \leq u | A_n\right), \quad u \leq 0.$$

Proposition 2.2.20. *With λ_β defined in* (2.17)*, it holds that*

$$\mathbb{P}\left(X_n \leq 0 | A_{n-1}\right) \nearrow \exp(-\lambda_\beta), \quad n \to \infty.$$

Moreover, the sequence $(\pi_n)_{n\geq 0}$ converges weakly to a distribution function π on $(-\infty, 0]$ which is absolutely continuous w.r.t. the Lebesgue measure on $(-\infty, 0]$. Denote its density by φ. Then φ satisfies the following Fredholm integral equation of second kind:

$$\exp(-\lambda_\beta)\,\varphi(u) = \int_{-\infty}^{0} p(y,u)\,\varphi(y)\,dy, \quad u \leq 0.$$

Proof. Let $F_n(u) := \mathbb{P}\left(X_n \leq u | A_{n-1}\right)$, $u \leq 0$. Note that

$$\pi_n(u) = \frac{\mathbb{P}\left(X_n \leq u, A_{n-1}\right)}{\mathbb{P}\left(A_n\right)} = \frac{\mathbb{P}\left(X_n \leq u | A_{n-1}\right)}{\mathbb{P}\left(X_n \leq 0 | A_{n-1}\right)} = \frac{F_n(u)}{F_n(0)}, \quad u \leq 0. \tag{2.22}$$

Moreover, for $u \le 0$, we have

$$F_n(u) = \mathbb{P}\left(X_n \le u | A_{n-1}\right) = \int_{-\infty}^{0} \mathbb{P}\left(X_n \le u | X_{n-1} = y\right) \mathbb{P}\left(X_{n-1} \in dy | A_{n-1}\right)$$
$$= \int_{-\infty}^{0} \int_{-\infty}^{u} p(y,z)\, dz\, d\pi_{n-1}(y).$$

Assume for a moment that $F_n(u)$ converges to $F(u)$ for all $u \le 0$ and that $(\pi_n)_{n\ge 1}$ converges weakly to some distribution function π. Then the last equation and (2.22) imply that

$$\pi(u) = \frac{F(u)}{F(0)} = \frac{1}{F(0)} \int_{-\infty}^{0} \int_{-\infty}^{u} p(y,z)\, dz\, d\pi(y), \quad u \le 0.$$

Applying Fubini's theorem, the previous equation reads

$$F(0)\, \pi(u) = \int_{-\infty}^{u} \int_{-\infty}^{0} p(y,z)\, d\pi(y)\, dz, \quad u \le 0. \tag{2.23}$$

The right-hand side of (2.23) is clearly differentiable in u, so π is absolutely continuous w.r.t. the Lebesgue measure. Denote its density by φ. Differentiating (2.23) w.r.t. u, we conclude that

$$F(0)\, \varphi(u) = \int_{-\infty}^{0} p(y,u)\, \varphi(y)\, dy \quad u < 0. \tag{2.24}$$

In order to prove convergence of $F_n(u)$ for $u \le 0$, it suffices to show that $F_n(u)$ is non-decreasing in n. Indeed,

$$F_{n+1}(u) = \mathbb{P}\left(X_{n+1} \le u | X_0 \le 0, \dots, X_n \le 0\right)$$
$$\ge \mathbb{P}\left(X_{n+1} \le u | X_1 \le 0, \dots, X_n \le 0\right) = F_n(u).$$

The inequality follows from [Bra78, Lemma 5], the last equality is due to the stationarity of X. In view of (2.22), we can set $\pi(u) := \lim_{n\to\infty} \pi_n(u) = F(u)/F(0)$. Obviously, $\pi(0) = 1$, and since $F_n(\cdot)$ is non-decreasing, so are F and π. In order to show that $\pi(-\infty) = 0$, recall that $(X_n)_{n\ge 0}$ and $(U_{\beta n})_{n\ge 0}$ have the same law (U is the Ornstein-Uhlenbeck process). Therefore, we can apply [LS02, Corollary 2.3] to infer that

$$\mathbb{P}\left(X_n \le u, A_{n-1}\right) \le \mathbb{P}\left(X_n \le u\right) \mathbb{P}\left(A_{n-1}\right) C_\beta, \quad u \le 0,$$

where $C_\beta < \infty$ is independent of n and u. Moreover, by Slepian's inequality, it holds that

$$\mathbb{P}\left(A_n\right) \ge \mathbb{P}\left(A_{n-1}\right) \mathbb{P}\left(X_n \le 0\right) = \mathbb{P}\left(A_{n-1}\right) \cdot \frac{1}{2}.$$

Since $X_n \overset{d}{=} X_0 \sim \mathcal{N}(0,1)$, if Φ denotes the cumulative distribution function of a standard Gaussian random variable, this shows that

$$\pi(u) \le 2C_\beta \mathbb{P}\left(X_n \le u\right) = 2C_\beta \Phi(u), \quad u \le 0. \tag{2.25}$$

In particular, $\pi(-\infty) = 0$ proving that π is a distribution function, and by (2.22), the sequence $(\pi_n)_{n\geq 0}$ converges weakly to π.
Next, since

$$F_n(0) = F(0)\,(1 + g(n)), \quad n \geq 0,$$

where $g(n) \to 0$ as $n \to \infty$, we get

$$\begin{aligned} \mathbb{P}\left(X_n \leq 0, \forall n = 0, \dots, N\right) &= \mathbb{P}\left(X_N \leq 0 | A_{N-1}\right) \mathbb{P}\left(A_{N-1}\right) \\ &= \mathbb{P}\left(X_0 \leq 0\right) \prod_{n=1}^{N} \mathbb{P}\left(X_n \leq 0 | A_{n-1}\right) \\ &= \frac{1}{2} F(0)^N \exp\left(\sum_{n=1}^{N} \log(1 + g(n))\right) = F(0)^N\, e^{o(N)}. \end{aligned}$$

One concludes (recall (2.17)) that

$$\lim_{N\to\infty} -\frac{1}{N} \log \mathbb{P}\left(X_n \leq 0, \forall n = 0, \dots, N\right) = -\log F(0) = \lambda_\beta.$$

□

Proposition 2.2.20 shows that $\exp(-\lambda_\beta)$ is an eigenvalue corresponding to a positive eigenfunction φ of the positive (i.e. $Tf \geq 0$ if $f \geq 0$ a.e.) bounded linear operator

$$T\colon L^1((-\infty, 0]) \to L^1((-\infty, 0]), \quad (Tf)(z) := \int_{-\infty}^{0} p(y, z) f(y)\, dy, \quad z \leq 0. \tag{2.26}$$

Let us briefly discuss some properties of T. If Z is standard Gaussian and $\Phi(x) := \mathbb{P}\left(Z \leq x\right)$, it holds that

$$\int_{-\infty}^{0} p(y, z)\, dz = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{0} e^{-(z-\rho y)^2/(2\sigma^2)}\, dz = \Phi\left(-\frac{\rho y}{\sigma}\right), \quad y \in \mathbb{R}. \tag{2.27}$$

Using this identity and Fubini's theorem, one obtains

$$\|Tf\|_1 \leq \int_{-\infty}^{0} |f(y)|\, \Phi(-\rho y/\sigma)\, dy \leq \|f\|_1, \quad f \in L^1.$$

In particular, $\|T\| \leq 1$. In fact, $\|T\| = 1$ since $\|Tf_n\|_1 \to 1$ if $f_n = 1_{[-n,-n+1]}$ since $\Phi(x) \to 1$ as $x \to \infty$. Moreover, $Tf_n \to 0$ as $n \to \infty$ pointwise showing that T is not compact ($Tf_n(u) \to 0$ as $n \to \infty$ for every u and $\|Tf_n\|_1 \to 1$ as $n \to \infty$ implies that there cannot be a subsequence that converges in L^1).
One might suspect that $\exp(-\lambda_\beta)$ is the largest spectral value of T, i.e. $\exp(-\lambda_\beta) = r(T)$ where $r(T) := \lim_{n\to\infty} \|T^n\|^{1/n}$ denotes the spectral radius of T. For instance, such a result holds for positive matrices (by Perron-Frobenius type results, [Sch74, Corollary I.2.3]). However, in our case, it can be shown that $r(T) = 1 > \exp(-\lambda_\beta)$ (see

Lemma B.1.1 in the appendix). Moreover, 1 cannot be an eigenvalue of T: if $Tf = f$ for some $f \in L^1 \setminus \{0\}$, it would follow that

$$\|f\|_1 = \|Tf\|_1 \leq \int_{-\infty}^{0} |f(y)| \, \Phi(-\rho y/\sigma) \, dy < \|f\|_1,$$

a contradiction. If T were compact, this could not occur in view of [Sch74, Theorem V.6.6].
It remains unclear if $\exp(-\lambda_\beta) \geq |\mu|$ for every other eigenvalue μ of T. Results of this type are known ([KLS89, Theorem 11.4]), but not applicable in our case.
The preceding discussion shows that the operator T does not have certain nice properties such as compactness, and therefore, it is not easily amenable to methods from functional analysis to conclude that $\exp(-\lambda_\beta)$ is the largest eigenvalue.

Remark 2.2.21. Majumdar et al. ([MBE01]) heuristically derived the same integral equation when studying persistence of the discrete Ornstein-Uhlenbeck process. The authors were motivated by the following question: How accurately can one estimate the persistence exponent θ of a continuous time process if one merely observes a discrete-time sample? They also solved the integral equation numerically to obtain an approximate value $\hat{\lambda}_\beta$ for different choices of β.

We cannot show that $\exp(-\lambda_\beta)$ is the largest eigenvalue of T. However, we prove that $\exp(-\lambda_\beta)$ is the largest spectral value of a related integral operator S which is compact and self-adjoint. This leads to a nice variational characterisation of $\exp(-\lambda_\beta)$ that can be used for numerical computations and even analytic bounds on λ_β. Let us describe the necessary steps in the sequel.
For simplicity of notation, let $L^p := L^p((-\infty, 0])$ for $p \geq 1$ with the usual norm $\|\cdot\|_p$, and let $\mathcal{L}(L^p, L^q)$ denote the space of continuous linear functionals mapping from L^p to L^q.
We know that there is a positive function $\varphi \in L^1$ such that $T\varphi = \exp(-\lambda_\beta)\varphi$, where $T \in \mathcal{L}(L^1, L^1)$ is defined in (2.26). Set

$$\begin{aligned} K(x,y) &:= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1+\rho^2}{4\sigma^2}(x^2+y^2) + \frac{\rho}{\sigma^2} xy \right) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1+\rho^2}{4\sigma^2}(x-y)^2 - \frac{(1-\rho)^2}{2\sigma^2} xy \right), \quad x, y \leq 0. \end{aligned}$$

Note that K is symmetric and bounded on $(-\infty, 0]^2$, and define the corresponding integral operator S by

$$(Sf)(x) := \int_{-\infty}^{0} K(x,y) f(y) \, dy, \quad x \leq 0$$

for suitable functions f. The next lemma shows that $S \colon L^p \to L^q$ is well-defined:

Lemma 2.2.22. *Let $p, q \in [1,\infty]$. Then $S \in \mathcal{L}(L^p, L^q)$. Moreover, $S\colon L^p \to L^q$ is compact if $p, q \in (1,\infty)$.*

Proof. Consider first the case $p, q \in (1,\infty)$. Let $p' \in (1,\infty)$ such that $1/p + 1/p' = 1$. For $f \in L^p$, an application of Hölder's inequality yields that

$$\begin{aligned}\int_{-\infty}^0 |Sf(x)|^q \, dx &\le \int_{-\infty}^0 \left(\int_{-\infty}^0 K(x,y)\,|f(y)|\,dy\right)^q dx \\ &\le \int_{-\infty}^0 \left(\int_{-\infty}^0 K(x,y)^{p'}\,dy\right)^{q/p'} \left(\int_{-\infty}^0 |f(y)|^p\,dy\right)^{q/p} dx \\ &= \|f\|_p^q \cdot \int_{-\infty}^0 \left(\int_{-\infty}^0 K(x,y)^{p'}\,dy\right)^{q/p'} dx.\end{aligned}$$

Let us show that

$$\int_{-\infty}^0 \left(\int_{-\infty}^0 K(x,y)^{p'}\,dy\right)^{q/p'} dx < \infty. \tag{2.28}$$

To prove (2.28), recall that

$$\int_{-\infty}^0 e^{-\alpha y^2+\beta y}\,dy = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} e^{\beta^2/(4\alpha)} \mathrm{Erfc}(\beta/(2\sqrt{\alpha})), \quad \alpha > 0, \beta \in \mathbb{R}, \tag{2.29}$$

where $\mathrm{Erfc}(\cdot)$ is the complementary error function given by

$$\mathrm{Erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,dt, \quad x \in \mathbb{R}.$$

Let $r > 0$, $A := (1+\rho^2)/(4\sigma^2) > 0$ and $B := \rho/\sigma^2 > 0$. Using (2.29) and the fact that $\mathrm{Erfc}(u) \in (0,2)$ for all $u \in \mathbb{R}$, one obtains that

$$\begin{aligned}\int_{-\infty}^0 K(x,y)^r\,dy &= \frac{1}{(\sqrt{2\pi}\sigma)^r} e^{-rAx^2} \int_{-\infty}^0 e^{-Ary^2+Brxy}\,dy \\ &= \frac{1}{(\sqrt{2\pi}\sigma)^r} e^{-rAx^2} \cdot \frac{\sqrt{\pi}}{2\sqrt{Ar}} e^{(Brx)^2/(4Ar)} \mathrm{Erfc}\left(\frac{Brx}{2\sqrt{Ar}}\right) &(2.30)\\ &\le \frac{\sqrt{\pi}}{(\sqrt{2\pi}\sigma)^r\sqrt{Ar}} \cdot \exp\left(-r\left(A - \frac{B^2}{4A}\right)x^2\right). &(2.31)\end{aligned}$$

Since $\rho > 0$, it follows that $A - B^2/(4A) > 0$ since

$$A - \frac{B^2}{4A} > 0 \iff 2A > B \iff 2\frac{1+\rho^2}{4\sigma^2} > \frac{\rho}{\sigma^2} \iff (\rho-1)^2 > 0. \tag{2.32}$$

Now (2.31) clearly implies that (2.28) holds, so $S \in \mathcal{L}(L^p, L^q)$. Compactness follows from [Alt02, Satz 8.15] or [DS58, Excercise VI.9.52].
If $p = 1$ and $q \in [1,\infty)$, let

$$K^*(x) := \sup\{K(x,y) : y \le 0\} = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\inf_{y\le 0}(Ay^2 - Bxy) - Ax^2\right), \quad x \le 0.$$

The infimum is attained at $y = Bx/(2A)$ which yields that

$$K^*(x) := \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-(A - B^2/(4A))x^2\right).$$

It follows that

$$\int_{-\infty}^0 |Sf(x)|^q \, dx \le \int_{-\infty}^0 \left(\int_{-\infty}^0 K(x,y)\,|f(y)| \, dy\right)^q dx \le \|f\|_1^q \int_{-\infty}^0 K^*(x)^q \, dx,$$

and since $A - B^2/(4A) > 0$ by (2.32), it is clear that $\|K^*\|_q < \infty$.
Similarly, if $p = 1, q = \infty$,

$$|(Sf)(x)| \le \int_{-\infty}^0 K(x,y)\,|f(y)| \, dy \le K^*(x)\|f\|_1,$$

so $S \in \mathcal{L}(L^1, L^\infty)$ is well-defined.
The remaining cases $p = \infty$ and $q \in [1,\infty]$ are similar, and the proof is omitted. □

Remark 2.2.23. If $S\colon L^p \to L^q$ for $p, q \in (1,\infty)$, the preceding proof shows that $\|Sf\|_q \le \|K\|_{p,q}\|f\|_p$ where

$$\|K\|_{p,q} := \left(\int_{-\infty}^0 \left(\int_{-\infty}^0 K(x,y)^{p'} \, dy\right)^{q/p'} dx\right)^{1/q}, \quad 1/p + 1/p' = 1.$$

If $p = q = 2$, we can compute the integral explicitly in view of the following formula ([GR00, Eq. 6.285]):

$$\int_{-\infty}^0 e^{-\alpha x^2} \mathrm{Erfc}(\beta x) \, dx = \frac{\pi - \arctan(\sqrt{\alpha}/\beta)}{\sqrt{\pi\alpha}}, \quad \alpha, \beta > 0, \tag{2.33}$$

After some calculation (see Appendix B), we conclude from (2.30) that

$$\|K\|_{2,2} = \sqrt{\frac{\pi - \arctan\left(\left|1-\rho^2\right|/(2\rho)\right)}{2\pi\left|1-\rho^2\right|}}. \tag{2.34}$$

This identity will be useful in order to derive an estimate on λ_β later.

Set

$$h(x) := \exp\left(\frac{1-\rho^2}{4\sigma^2}x^2\right), \quad x \le 0.$$

For our choice $\rho = e^{-\beta/2}$ and $\sigma^2 = 1 - e^{-\beta}$, $h(x) = \exp(x^2/4)$. The following lemma shows that that the operators S and T are related via a transformation involving the function h. This observation is due to [MBE01].

Lemma 2.2.24. *Let $p \in [1, \infty)$. If $f \in L^1$ and $fh \in L^p$, it holds that $Tf(x) = (1/h(x))S(hf)(x)$ for all $x \leq 0$.*

Proof. We have that

$$\begin{aligned} p(y,x) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\rho^2 y^2 - 2\rho xy + x^2}{2\sigma^2}\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1+\rho^2}{4\sigma^2}(x^2+y^2) + \frac{\rho}{\sigma^2}xy\right) \exp\left(\frac{1-\rho^2}{4\sigma^2}y^2\right) \exp\left(-\frac{1-\rho^2}{4\sigma^2}x^2\right) \\ &= K(x,y)h(y)/h(x), \end{aligned}$$

and therefore,

$$Tf(x) = \int_{-\infty}^0 p(y,x)f(y)\,dy = \frac{1}{h(x)} \int_{-\infty}^0 K(x,y)h(y)f(y)\,dy = \frac{1}{h(x)} S(hf)(x).$$

□

In particular, Lemma 2.2.24 shows that if $\mu \in \mathbb{R}$ is an eigenvalue of T with eigenfunction $f \in L^1$, i.e. $Tf = \mu f$, then $\mu(hf) = S(hf)$ if $hf \in L^1$, and vice versa. Of course, since $h(x) = \exp(x^2/4)$ in our case, $hf \notin L^1$ for $f \in L^1$ in general. Fortunately, we can show that the eigenfunction φ of T satisfies $h\varphi \in L^p$ for every $p \in [1, \infty]$.

Lemma 2.2.25. *If φ is the eigenfunction defined in Proposition 2.2.20, then $x \leq 0 \mapsto e^{\alpha x^2}\varphi(x) \in L^1$ for every $\alpha < 1/2$. Moreover, $h\varphi \in L^p$ for every $p \in [1, \infty]$.*

Proof. If $Z \leq 0$ is a random variable distributed according to the density φ, we have to show that $\mathbb{E}\left[e^{\alpha Z^2}\right] < \infty$ for $\alpha < 1/2$. Assume w.l.o.g. that $\alpha > 0$, so we have that

$$\mathbb{E}\left[e^{\alpha Z^2}\right] = \int_0^\infty \mathbb{P}\left(e^{\alpha Z^2} > t\right) dt \leq 2 + \int_2^\infty \mathbb{P}\left(Z < -\alpha^{-1/2}\sqrt{\log t}\right) dt. \tag{2.35}$$

Recall from (2.25) that $\mathbb{P}(Z < -t) \leq C\Phi(-t)$, where C is a finite constant depending on β, and Φ is the cumulative distribution function of a standard Gaussian random variable. Hence,

$$\mathbb{E}\left[e^{\alpha Z^2}\right] \leq 2 + C \int_2^\infty \Phi(-\alpha^{-1/2}\sqrt{\log t})\,dt \leq 2 + C_1 \int_2^\infty \exp\left(-\frac{\log t}{2\alpha}\right) dt,$$

and the last quantity is finite whenever $\alpha < 1/2$.
In particular, for $\alpha = 1/4$, we see that $h\varphi \in L^1$, and since $e^{-\lambda_\beta}\varphi = T\varphi$, Lemma 2.2.24 implies that $e^{-\lambda_\beta}(h\varphi) = S(h\varphi)$. Since $S\colon L^1 \to \bigcap_{p\geq 1} L^p$ by Lemma 2.2.22, we conclude that $h\varphi \in L^p$ for every $p \in [1, \infty]$. □

Just as in the case of positive square matrices (Perron-Frobenius theorem, see [Sch74, Chapter I]), one can show that a positive eigenfunction of S must correspond to the largest eigenvalue of S.

Lemma 2.2.26. *Let $p \in [1,\infty]$. If $Sf = \lambda f$ for some $f \in L^p \setminus \{0\}$ with $f > 0$ a.e., then*
$$\lambda = \max\{|\mu| : \mu \in \mathbb{R}, \exists g \in L^p \setminus \{0\} \quad \text{s.t.} \quad Sg = \mu g\}.$$

Proof. Assume that $Sg = \mu\, g$, where $g \in L^p \setminus \{0\}$ and $\mu \in \mathbb{R}$. Since S maps from L^p to $L^p \cap L^2$ for every $p \in [1,\infty]$ (Lemma 2.2.22), $Sf, Sg \in L^p \cap L^2$, and since f and g are eigenfunctions, it follows that $f, g \in L^p \cap L^2$. Let $(\cdot,\cdot)$ denote the usual scalar product in L^2. Then $(f, |g|) < \infty$, and

$$\lambda(f, |g|) = (Sf, |g|) = (f, S(|g|)) \geq (f, |Sg|) = |\mu|\,(f, |g|).$$

The inequality comes from the fact that S is a positive linear operator, i.e. $S\xi \geq 0$ if $\xi \in L^p$ with $\xi \geq 0$ a.e. (apply this with $\xi = |g| - g$). In the second equality, we have used that the kernel K is symmetric, so Fubini's Theorem implies that

$$(Sf, |g|) = \int_{-\infty}^{0} \int_{-\infty}^{0} K(x,y) f(y)\, dy\, |g(x)|\; dx = (f, S\,|g|).$$

Since $f > 0$ and g is not identically zero, it holds that $(f, |g|) > 0$, and therefore, we necessarily have that $\lambda \geq |\mu|$. □

It is now easy to deduce the following proposition from the above lemmata.

Proposition 2.2.27. *For $p \in [1,\infty]$, it holds that*

$$e^{-\lambda_\beta} = \max\{|\mu| : \mu \in \mathbb{R}, \exists f \in L^p \setminus \{0\} \;\; s.t. \;\; Sf = \mu f\}.$$

Proof. Since $h\varphi \in L^p$ (Lemma 2.2.25) and $T\varphi = e^{-\lambda_\beta}\varphi$, we see from Lemma 2.2.24 that $e^{-\lambda_\beta}(h\varphi) = S(h\varphi)$. Now $h\varphi > 0$ on $(-\infty, 0]$, so the assertion follows from Lemma 2.2.26. □

The above proposition has been derived by elementary arguments. If we use some known results from functional analysis, we obtain a better result if we focus on the case of the Hilbert space L^2.

Theorem 2.2.28. *Let $S\colon L^2 \to L^2$, and let $r(S)$ denote its spectral radius. It holds that*
$$e^{-\lambda_\beta} = r(S) = \|S\| = \sup\left\{(Sf, f) : f \in L^2, \|f\|_2 = 1\right\}.$$
Moreover, if μ is another eigenvalue of S, it holds that $|\mu| < e^{-\lambda_\beta}$.

Proof. Since $S\colon L^p \to L^p$ is compact for every $p \in (1,\infty)$ (Lemma 2.2.22), we have by [Sch74, Theorem V.6.6] that $r(S)$ is an eigenvalue of S with a unique eigenfunction f satisfying $f > 0$ a.e. and $\|f\|_p = 1$. Moreover, if μ is another eigenvalue, it holds that $|\mu| < r(S)$ by [Sch74, Theorem V.6.6] as well.
The equality $r(S) = e^{-\lambda_\beta}$ follows from Proposition 2.2.27.

We are only interested in the case $p = 2$. Symmetry of K implies that S is self-adjoint, and it is well-known that $r(S) = \|S\|$ ([RS72, Theorem VI.6]) and

$$\|S\| = \sup \left\{ (Sf, f) : f \in L^2, \|f\|_2 = 1 \right\}$$

in that case. □

One can derive upper and lower bounds on λ_β from Theorem 2.2.28. For instance, recall from (2.34) that $\exp(-\lambda_\beta) \leq \|K\|_{2,2}$, and with $\rho = e^{-\beta/2}$, this amounts to

$$\lambda_\beta \geq \frac{1}{2} \cdot \log \left(\frac{2\pi(1 - e^{-\beta})}{\pi - \arctan(\sinh(\beta/2))} \right) =: L(\beta). \tag{2.36}$$

One sees that $L(\beta) \to \log 2$ as $\beta \to \infty$, whereas for small values of β, $L(\beta)$ does not provide a useful lower bound since $L(\beta) < 0$ in that case.
Moreover, an upper bound on λ_β is given by $\lambda_\beta \leq -\log(Sf, f)$ for every $f \in L^2$ with $\|f\|_2 = 1$. For instance, if one uses the test functions

$$f_\alpha(x) := (8\alpha/\pi)^{1/4}\, e^{-\alpha x^2}, \quad x \leq 0, \alpha > 0,$$

then $\|f_\alpha\|_2 = 1$, and a direct calculation (see Appendix B.2) using (2.29) and (2.33) shows that

$$I(\alpha) := (f_\alpha, Sf_\alpha) = \frac{\sqrt{\alpha}}{\pi\sigma} \cdot \frac{\pi - \arctan\left(\frac{2\sqrt{(A+\alpha)^2 - B^2/4}}{B} \right)}{\sqrt{(A+\alpha)^2 - B^2/4}}, \tag{2.37}$$

where

$$A = \frac{1+\rho^2}{4\sigma^2} = \frac{1+e^{-\beta}}{4(1-e^{-\beta})} = \frac{1}{4\tanh(\beta/2)}, \quad B = \frac{\rho}{\sigma^2} = \frac{e^{-\beta/2}}{1-e^{-\beta}} = \frac{1}{2\sinh(\beta/2)}.$$

Then $\lambda_\beta \leq -\sup \{\log I(\alpha) : \alpha > 0\}$, and the last expression can be computed numerically. Moreover, one can also use test functions

$$g_{\alpha,\beta}(x) := C(x+\beta)\, e^{-\alpha x^2}, \quad x \leq 0, \quad \alpha, \beta > 0.$$

Even though the integral $(Sg_{\alpha,\beta}, g_{\alpha,\beta})$ can be computed explicitly, this leads to very unwieldy expressions, and we shall refrain from stating them here.
In order to obtain an approximate solution $\hat{\lambda}_\beta$, the integral equation can be solved numerically, see [Hac95, Chapter 4] for a description of different methods.

In Figure 2.1, we have plotted the upper and lower bounds on λ_β from Theorem 2.2.14, (2.36) and (2.37) for $\beta \in (0, 2.5]$. Moreover, the lower dotted line has been computed by a numerical approximation of the largest eigenvalue of S. To do so, on a grid

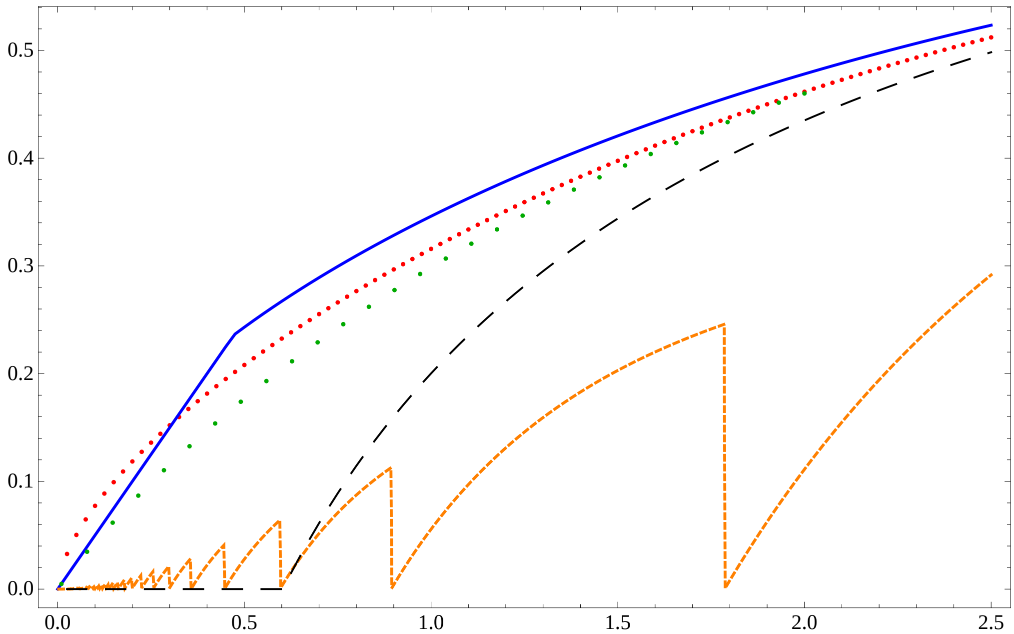

Figure 2.1: Numerical approximation of λ_β (lower dotted line). The upper dotted line is computed by numerically maximising (2.37), the solid line is the upper bound from Theorem 2.2.14. The dashed lines are the lower bounds from Theorem 2.2.14 (short dashes) and from (2.36) (long dashes).

$x_n = -n \cdot T/N$ $(n = 1, \dots, N)$ for large values of T and N, one can use the simple approximation

$$\int_{-\infty}^{0} K(x,y)f(y)\,dy \approx \int_{-T}^{0} K(x,y)f(y)\,dy \approx (T/N)\sum_{j=1}^{N} K(x,x_j)f(x_j).$$

If μ is an eigenvalue of S with eigenfunction f, then

$$\mu f(x_i) = \int_{-\infty}^{0} K(x_i,y)f(y)\,dy \approx (T/N)\sum_{j=1}^{N} K(x_i,x_j)f(x_j).$$

We rewrite this as $(\mu N/T) f_N \approx K_N f_N$, where $f_N = (f(x_1), \dots, f(x_N))$ and the matrix K_N is given by $(K(x_i,x_j))_{i,j=1,\dots,N}$. One can then compute the largest eigenvalue $\hat{\mu}_N$ of K_N numerically in order to obtain an approximation for $\exp(-\lambda_\beta)$.
We see from Figure 2.1 that the upper bound obtained by maximising (2.37) numerically and the numerical approximation of λ_β are very close if β is not too small, whereas for values of β close to zero, the simple estimate $\lambda_\beta \leq \beta/2$ stated in Theorem 2.2.14 is better.
Concerning the lower bounds, we see that the lower bound from Theorem 2.2.14 (short dashes in Figure 2.1) is quite irregular due to rounding in (2.18), cf. Remark 2.2.15.

Note also that the lower bound stated in (2.36) (long dashes in Figure 2.1) improves that of Theorem 2.2.14 for β not too close to zero.

Finally, Figure 2.2 shows the numerical solution for small values of β. In view of these results, it seems reasonable to conjecture that $\lambda_\beta < \beta/2$ for all $\beta > 0$ and that $\lambda_\beta \sim \beta/2$ as $\beta \to 0$, cf. Remark 2.2.16.

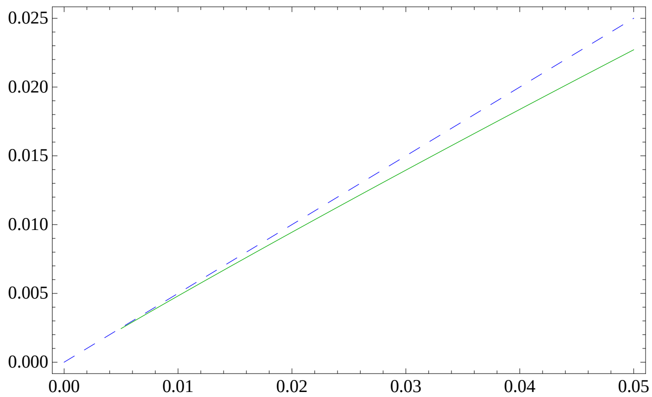

Figure 2.2: Numerical solution of λ_β for $\beta \in [0.005, 0.05]$ in comparison to the upper bound $\beta/2$ (dashed line).

2.3 Universality results

Up to now, we have studied the persistence problem of weighted random walks only for Gaussian increments $Y_1, Y_2, \ldots$. In the sequel, we consider different distributions. As before, we distinguish polynomial and exponential weight functions.

2.3.1 Polynomial weight functions

Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables such that $\mathbb{E}[Y_1] = 0$ and $\mathbb{E}[Y_1^2] = 1$ and $\sigma\colon [0,\infty) \to (0,\infty)$ some measurable function. Let Z denote the corresponding weighted random walk defined in (2.1). For a sequence $(Y_n)_{n\geq 1}$ of standard normal random variables, the persistence problem has already been solved for $\sigma(n) = n^p$. Indeed, the survival exponent is equal to $p + 1/2$ in view of (2.3) and Theorem 2.2.2 applied to the function $\kappa(\cdot)$ defined by $\kappa(n) = \sigma(1)^2 + \cdots + \sigma(n)^2$ such that

$$\kappa(N) \asymp N^{2p+1}, \qquad \kappa(N+1) - \kappa(N) = \sigma(N+1)^2 \asymp N^{2p}, \quad N \to \infty.$$

It is a natural question to ask whether the same results holds for any sequence of random variables that obey a suitable moment condition. This is the subject of Theorem 2.3.2 and Theorem 2.3.3.

Remark 2.3.1. Theorem 2.3.2 and Theorem 2.3.3 also hold if the barrier 0 is replaced by any $c \in \mathbb{R}$. The proof of Theorem 2.3.3 can be easily modified to cover this case. We briefly indicate below how to adapt the proof of the lower bound. The proofs will be then carried out again for the barrier 1 instead of 0.
Let $c \in \mathbb{R}$. Take any $x > 0$ such that $\mathbb{P}(Y_1 \leq -x) > 0$. Choose N_0 such that $-x(\sigma(1) + \cdots + \sigma(N_0)) \leq c - 1$. On $A_0 := \{Y_1 \leq -x, \dots, Y_{N_0} \leq -x\}$, it holds that $Z_{N_0} \leq c - 1$ by construction. Then, for $N > N_0$,

$$\mathbb{P}\left(\sup_{n=1,\dots,N} Z_n \leq c\right) \geq \mathbb{P}\left(A_0, \sup_{n=N_0+1,\dots,N} Z_n - Z_{N_0} \leq 1\right)$$
$$= \mathbb{P}(A_0)\,\mathbb{P}\left(\sup_{n=1,\dots,N-N_0} \sum_{k=1}^{n} \sigma(k+N_0) Y_k \leq 1\right).$$

Hence, it suffices to prove a lower bound for the survival probability of the weighted random walk $\tilde{Z}$ with $\tilde{\sigma}(k) := \sigma(k + N_0)$ $(k \geq 1)$ and the barrier 1 since $\tilde{\sigma}(N) \asymp \sigma(N) \asymp N^p$.

Lower bound via Skorokhod embedding

Here we prove the lower bound of Theorem 2.1.2 under weaker assumptions.

Theorem 2.3.2. *Let $(Y_n)_{n\geq 1}$ be a sequence of i.i.d. centred random variables such that $\mathbb{E}[Y_1^2] = 1$. Denote by $Z = (Z_n)_{n\geq 1}$ the corresponding weighted random walk defined in (2.1). Let $\sigma(N) \asymp N^p$ for some $p > 0$. Assume that $\mathbb{E}[|Y_1|^\alpha] < \infty$ for some $\alpha > 4p + 2$. Then*

$$\mathbb{P}(Z_n \leq 0, \forall n = 1, \dots, N) \gtrsim N^{-(p+1/2)}, \quad N \to \infty.$$

Proof. *Step 1:* Since the Y_i are independent centred random variables, Z is a martingale, and one can use a Skorokhod embedding: there exists a Brownian motion B and an increasing sequence of stopping times $(\tau(n))_{n\in\mathbb{N}}$ such that $(Z_n)_{n\in\mathbb{N}}$ and $(B_{\tau(n)})_{n\in\mathbb{N}}$ have the same finite dimensional distributions. Moreover,

$$\mathbb{E}[\tau(N)] = \mathbb{E}\left[B_{\tau(N)}^2\right] = \mathbb{E}\left[Z_N^2\right] = \sum_{k=1}^{N} \sigma(k)^2 =: \kappa(N),$$

see e.g. Proposition 11.1 in the survey on the Skorokhod problem of Obłój ([Obł04]). In particular, this implies that $(B_{t\wedge\tau(n)})_{t\geq 0}$ is uniformly integrable.
From the contruction of the stopping times described in the cited article (Section 11.1), one deduces that the increments of $(\tau(n))_{n\geq 1}$ are independent since those of Z are.
Note that there exist constants $c_1, c_2 > 0$ such that $c_1 N^{2p+1} \leq \kappa(N) \leq c_2 N^{2p+1}$ for all N sufficiently large. W.l.o.g. assume that c_2 is so large that the upper bound holds for

all N. Then one has for $\epsilon > 0$ and N large enough

$$\begin{aligned}\mathbb{P}\left(Z_n \le 1, \forall n = 1, \ldots, N\right) &= \mathbb{P}\left(B_{\tau(n)} \le 1, \forall n = 1, \ldots, N\right) \\ &\ge \mathbb{P}\left(\sup_{t\in[0,(1+\epsilon)\kappa(N)]} B_t \le 1,\ \tau(N) \le (1+\epsilon)\kappa(N)\right) \\ &\ge \mathbb{P}\left(\sup_{t\in[0,(1+\epsilon)c_2N^{2p+1}]} B_t \le 1\right) - \mathbb{P}\left(\tau(N) - \kappa(N) > \epsilon c_1 N^{2p+1}\right).\end{aligned} \tag{2.38}$$

Clearly, by (1.4), we have that

$$\mathbb{P}\left(\sup_{t\in[0,(1+\epsilon)c_2N^{2p+1}]} B_t \le 1\right) \sim \sqrt{\frac{2}{\pi(1+\epsilon)c_2}}\, N^{-(p+1/2)}, \quad N \to \infty. \tag{2.39}$$

The second term in (2.38) may be estimated with Chebychev's inequality if one can control the centred moments of the stopping times $\tau(N)$. Concretely, we claim that for all N and $\gamma \ge 2$ such that $\mathbb{E}\left[|Y_1|^{2\gamma}\right] < \infty$, it holds that

$$\mathbb{E}\left[|\tau(N) - \kappa(N)|^\gamma\right] = \mathbb{E}\left[|\tau(N) - \mathbb{E}\left[\tau(N)\right]|^\gamma\right] \le CN^{(2p+1/2)\gamma}, \tag{2.40}$$

where $C > 0$ is some constant depending only on γ. If (2.40) is true, Chebychev's inequality yields

$$\begin{aligned}\mathbb{P}\left(\tau(N) - \kappa(N) > \epsilon c_1 N^{2p+1}\right) &\le \mathbb{E}\left[|\tau(N) - \kappa(N)|^\gamma\right] (\epsilon c_1)^{-\gamma} N^{-\gamma(2p+1)} \\ &\le C\,(c_1\epsilon)^{-\gamma} N^{-\gamma/2}.\end{aligned}$$

By choosing $\gamma > 2p+1$, this term is of lower order than $N^{-(p+1/2)}$. The assertion of the theorem follows from (2.38), (2.39), and Remark 2.3.1.
Step 2: It remains to verify the validity of (2.40). Choose $\gamma > 2p+1$ such that $\mathbb{E}\left[|Y_1|^{2\gamma}\right] < \infty$. Since $(B_{\tau(n)\wedge t})_{t\ge 0}$ is a uniformly integrable martingale, we deduce from the Burkholder-Davis-Gundy (BDG) inequality (see e.g. [Obł04, Proposition 2.1]) that

$$\mathbb{E}\left[\tau(n)^\gamma\right] \le C(\gamma)\mathbb{E}\left[\left|B_{\tau(n)}\right|^{2\gamma}\right] = C(\gamma)\mathbb{E}\left[|Z_n|^{2\gamma}\right] < \infty.$$

The finiteness of the last expectation follows from our choice of γ and the assumption $\mathbb{E}\left[|Y_1|^{2\gamma}\right] < \infty$. This shows that $\tau(n)^\gamma$ is integrable.
Recall that

$$\tilde{B} = (B_{t+\tau(n-1)} - B_{\tau(n-1)})_{t\ge 0}$$

is a Brownian motion w.r.t. the filtration $\mathbb{G}^{(n)} = (\mathcal{G}_t^{(n)})_{t\ge 0} := (\mathcal{F}_{t+\tau(n-1)})_{t\ge 0}$ if B is a Brownian motion w.r.t. $(\mathcal{F}_t)_{t\ge 0}$. Note that $\tau(n) - \tau(n-1)$ is a $\mathbb{G}^{(n)}$-stopping time for all n. Using again the BDG inequality, we get

$$\begin{aligned}\mathbb{E}\left[(\tau(n)-\tau(n-1))^\gamma\right] &\le c_\gamma\,\mathbb{E}\left[\left|\tilde{B}_{\tau(n)-\tau(n-1)}\right|^{2\gamma}\right]\\ &= c_\gamma\,\mathbb{E}\left[\left|B_{\tau(n)}-B_{\tau(n-1)}\right|^{2\gamma}\right]\\ &= c_\gamma\,\mathbb{E}\left[|Z_n-Z_{n-1}|^{2\gamma}\right] = c_\gamma\,\mathbb{E}\left[|Y_1|^{2\gamma}\right]\sigma(n)^{2\gamma}, \qquad (2.41)\end{aligned}$$

where c_γ is a constant depending on γ only and $\mathbb{E}\left[|Y_1|^{2\gamma}\right] < \infty$ by assumption. For $n = 1, 2, \ldots,$ let

$$A_n := \tau(n) - \tau(n-1) - \mathbb{E}\left[\tau(n)-\tau(n-1)\right] = \tau(n)-\tau(n-1)-\sigma(n)^2.$$

As remarked at the beginning of the proof, the A_i are independent centred random variables. Using the Marcinkiewicz-Zygmund inequality (or the BDG-inequality), we get

$$\begin{aligned}\mathbb{E}\left[|\tau(N)-\kappa(N)|^\gamma\right] = \mathbb{E}\left[\left|\sum_{n=1}^N A_n\right|^\gamma\right] &\le C(\gamma)\mathbb{E}\left[\left(\sum_{n=1}^N A_n^2\right)^{\gamma/2}\right]\\ &= C(\gamma)\|\sum_{n=1}^N A_n^2\|_{\gamma/2}^{\gamma/2},\end{aligned}$$

where $C(\gamma)$ is again some constant that depends only on γ and $\|\cdot\|_p$ denotes the L^p-norm (here we need that $\gamma \ge 2$). An application of the triangle inequality yields

$$\left(\mathbb{E}\left[|\tau(N)-\kappa(N)|^\gamma\right]\right)^{2/\gamma} \le C(\gamma)^{2/\gamma}\sum_{n=1}^N \|A_n^2\|_{\gamma/2} = C(\gamma)^{2/\gamma}\sum_{n=1}^N \left(\mathbb{E}\left[|A_n|^\gamma\right]\right)^{2/\gamma}.$$

Clearly $|A_n|^\gamma \le 2^\gamma(|\tau(n)-\tau(n-1)|^\gamma + \sigma(n)^{2\gamma})$ implying that

$$\begin{aligned}\mathbb{E}\left[|\tau(N)-\kappa(N)|^\gamma\right]^{2/\gamma} &\le 4C(\gamma)^{2/\gamma}\sum_{n=1}^N \left(\mathbb{E}\left[|\tau(n)-\tau(n-1)|^\gamma\right]+\sigma(n)^{2\gamma}\right)^{2/\gamma}\\ &\le 4C(\gamma)^{2/\gamma}\sum_{n=1}^N \left((c_\gamma\mathbb{E}\left[|Y_1|^{2\gamma}\right]+1)\sigma(n)^{2\gamma}\right)^{2/\gamma}\\ &\le 4\left\{C(\gamma)(c_\gamma\mathbb{E}\left[|Y_1|^{2\gamma}\right]+1)\right\}^{2/\gamma}\sum_{n=1}^N \sigma(n)^4.\end{aligned}$$

In the above estimates, the second inequality follows from (2.41). We finally arrive at

$$\begin{aligned}\mathbb{E}\left[|\tau(N)-\kappa(N)|^\gamma\right] &\le 2^\gamma C(\gamma)(c_\gamma\mathbb{E}\left[|Y_1|^{2\gamma}\right]+1)\left(\sum_{n=1}^N \sigma(n)^4\right)^{\gamma/2}\\ &\le 2^\gamma C(\gamma)(c_\gamma\mathbb{E}\left[|Y_1|^{2\gamma}\right]+1)\,c_2^{2\gamma}\,N^{(4p+1)\gamma/2},\end{aligned}$$

proving (2.40) with $C = 2^\gamma C(\gamma)(c_\gamma\mathbb{E}\left[|Y_1|^{2\gamma}\right]+1)c_2^{2\gamma} < \infty$. □

Upper bound via coupling

The upper bound in Theorem 2.1.2 is a consequence of the following more precise statement.

Theorem 2.3.3. *Let $(Y_n)_{n\geq 1}$ be a sequence of i.i.d. centred random variables such that $\mathbb{E}\left[e^{a|Y_1|}\right] < \infty$ for some $a > 0$. Let σ be increasing such that $\sigma(N) \asymp N^p$ for some $p > 0$. If $Z = (Z_n)_{n\geq 1}$ denotes the corresponding random walk defined in (2.1), it holds for any $\rho > 4p + 2$ that*

$$\mathbb{P}\left(Z_n \leq 0, \forall n = 1, \dots, N\right) \precsim N^{-(p+1/2)}(\log N)^{\rho/2}, \quad N \to \infty.$$

Proof. We may assume w.l.o.g. that $\mathbb{E}\left[Y_1^2\right] = 1$. Let $\tilde{Z}_n := \sum_{k=1}^n \sigma(k)\tilde{Y}_k$, where the $\tilde{Y}_k$ are independent standard normal random variables constructed on the same probability space as the Y_k. As usual, denote by $S_n = Y_1 + \cdots + Y_n$ the corresponding random walk, and define $\tilde{S}$ analogously. Let

$$E_N := \left\{ \sup_{n=1,\dots,N} \left| S_n - \tilde{S}_n \right| \leq C \log N \right\}$$

for some constant $C > 0$ to be specified later. We now use a coupling of the random walks S and $\tilde{S}$ that allows us to work with the Gaussian process $\tilde{Z}$ instead of the original process Z. Since $\mathbb{E}\left[e^{a|Y_1|}\right] < \infty$ for some $a > 0$, we may assume by [KMT76, Theorem 1] that the sequences $(Y_n)_{n\geq 1}$ and $(\tilde{Y}_n)_{n\geq 1}$ are constructed on a common probability space such that for all N and some $C > 0$ sufficiently large

$$\mathbb{P}\left(E_N^c\right) = \mathbb{P}\left(\sup_{n=1,\dots,N} \left| S_n - \tilde{S}_n \right| > C \log N \right) \leq K\, N^{-(p+1/2)}, \tag{2.42}$$

where K is a constant that depends only on the distribution of Y_1 and on C.
In order to relate the quantities $\left|S_n - \tilde{S}_n\right|$ and $\left|Z_n - \tilde{Z}_n\right|$, recall Abel's inequality: If $(a_k)_{k\geq 1}, (b_k)_{k\geq 1}$ are two sequences with $a_k \leq a_{k+1}$, it holds that

$$\left| \sum_{k=1}^n a_k b_k \right| \leq (2a_n - a_1) \sup_{k=1,\dots,n} \left| \sum_{j=1}^k b_j \right|, \quad n \geq 1. \tag{2.43}$$

The inequality follows easily from Abel's transformation, which is the discrete analogue of integration by parts: If $B_k = b_1 + \cdots + b_k$, one has

$$\sum_{k=1}^n a_k b_k = a_n B_n - \sum_{k=1}^{n-1} B_k (a_{k+1} - a_k), \quad n \geq 1.$$

Since $\sigma(\cdot)$ is increasing and nonnegative, we can apply (2.43) to conclude that on E_N, it holds for all $n \leq N$ that

$$\sup_{k=1,\dots,n} \left| Z_k - \tilde{Z}_k \right| = \sup_{k=1,\dots,n} \left| \sum_{j=1}^{n} \sigma(j)(Y_j - \tilde{Y}_j) \right| \tag{2.44}$$
$$\leq 2\,\sigma(n) \sup_{k=1,\dots,n} \left| S_k - \tilde{S}_k \right| \leq 2\,C\,\sigma(n) \log N.$$

Therefore, on $E_N \cap \left\{ \sup_{n=1,\dots,N} Z_n \leq 1 \right\}$, one has

$$\tilde{Z}_n = \tilde{Z}_n - Z_n + Z_n \leq 2C\sigma(n)\log N + 1, \quad n \leq N.$$

We may now estimate

$$\mathbb{P}\left(\sup_{n=1,\dots,N} Z_n \leq 1 \right) \leq \mathbb{P}\left(\sup_{n=1,\dots,N} Z_n \leq 1, E_N \right) + \mathbb{P}\left(E_N^c\right)$$
$$\leq \mathbb{P}\left(\sup_{n=1,\dots,N} \tilde{Z}_n - 2C\sigma(n)\log N \leq 1 \right) + \mathbb{P}\left(E_N^c\right).$$

In view of (2.42), the term $\mathbb{P}\left(E_N^c\right)$ is at most of order $N^{-(p+1/2)}$. It remains to show that the order of the first term is $N^{-(p+1/2)}(\log N)^{\rho/2}$ for $\rho > 4p+2$. Let $\kappa(n) := \sigma(1)^2 + \cdots + \sigma(n)^2$. If B is a Brownian motion, one has in view of (2.3) that

$$\mathbb{P}\left(\sup_{n=1,\dots,N} \tilde{Z}_n - 2C\sigma(n)\log N \leq 1 \right) = \mathbb{P}\left(\sup_{n=1,\dots,N} B_{\kappa(n)} - 2C\sigma(n)\log N \leq 1 \right).$$

One can now proceed similarly to the proof of Theorem 2.2.2. Note that

$$\bigcap_{n=1}^{N} \left\{ B_{\kappa(n)} - 2C\sigma(n)\log N \leq 1 \right\} \subseteq \bigcap_{n=1}^{N-1} \left\{ \sup_{t\in[\kappa(n),\kappa(n+1)]} B_t - 3C\sigma(n)\log N \leq 1 \right\}$$
$$\cup \bigcup_{n=1}^{N-1} \left\{ \sup_{t\in[\kappa(n),\kappa(n+1)]} B_t - B_{\kappa(n)} > C\sigma(n)\log N \right\} =: G_N \cup H_N.$$

Clearly, since $\kappa(n+1) - \kappa(n) = \sigma(n+1)^2$, we have that

$$\mathbb{P}\left(H_N\right) \leq \sum_{n=1}^{N-1} \mathbb{P}\left(\sup_{t\in[0,1]} B_t > \frac{C\sigma(n)\log N}{\sqrt{\kappa(n+1) - \kappa(n)}} \right) \leq N\,\mathbb{P}\left(\sup_{t\in[0,1]} B_t > \tilde{C}\log N \right),$$

where $\tilde{C} = C \inf\left\{\sigma(n)/\sigma(n+1) : n \geq 1\right\} \in (0, C)$ since $\sigma(\cdot)$ is increasing and $\sigma(n) \asymp n^p$. By (2.6), the last term is $o(N^{-\alpha})$ for any $\alpha > 0$.
It remains to estimate $\mathbb{P}\left(G_N\right)$. Set $c_1 = \inf\left\{\kappa(n)/n^{2p+1} : n \geq 1\right\} \in (0, \infty)$ since $\sigma(n) \asymp$

n^p and $\sigma(n) > 0$ for all $n \geq 1$ by monotonicity. Hence, $\kappa(n) \geq c_1 n^{2p+1}$, and $t \geq \kappa(n)$ implies that $(t/c_1)^{1/(2p+1)} \geq n$ and therefore,

$$\mathbb{P}(G_N) \leq \mathbb{P}\left(\bigcap_{n=1}^{N-1}\left\{\sup_{t\in[\kappa(n),\kappa(n+1)]} B_t - 3C\sigma\left((t/c_1)^{1/(2p+1)}\right)\log N \leq 1\right\}\right)$$
$$\leq \mathbb{P}\left(\sup_{t\in[\kappa(1),\kappa(N)]} B_t - c_2 t^{p/(2p+1)}\log N \leq 1\right).$$

Let $\rho > 2(2p+1)$, i.e. $1/\rho + p/(2p+1) < 1/2$. Then $t^{p/(2p+1)}\log N \leq t^{p/(2p+1)+1/\rho}$ for $t \geq (\log N)^\rho$ and

$$\mathbb{P}(G_N) \leq \mathbb{P}\left(\sup_{t\in[(\log N)^\rho,\kappa(N)]} B_t - c_2 t^{p/(2p+1)+1/\rho} \leq 1\right).$$

The last expression is already familiar from (2.7): using Slepian's inequality, and recalling that the moving barrier t^α does not change the survival exponent of Brownian motion for $\alpha < 1/2$, one deduces analogously that

$$\mathbb{P}(G_N) \lesssim \frac{\mathbb{P}\left(\sup_{t\in[0,\kappa(N)]} B_t \leq 1\right)}{\mathbb{P}\left(\sup_{t\in[0,(\log N)^\rho]} B_t \leq 1\right)} \asymp \kappa(N)^{-1/2}(\log N)^{\rho/2} \asymp N^{-(p+1/2)}(\log N)^{\rho/2}.$$

□

Remark 2.3.4. In the proof of Theorem 2.3.3, we applied the Komlós-Major-Tusnády (KMT) coupling to the random walk S where $S_n = Y_1 + \cdots + Y_n$. If the Y_i are independent, but not necessarily identically distributed, one could use the coupling for non-i.i.d. random variables introduced by Sakhanenko:

Theorem 2.3.5. *([Sak84]) Assume that $(Y_n)_{n\geq 1}$ is a sequence of independent centred random variables and that there is $\lambda > 0$ such that*

$$\lambda\,\mathbb{E}\left[e^{\lambda Y_n}|Y_n|^3\right] \leq \mathbb{E}\left[Y_n^2\right], \quad \forall n \geq 1,. \tag{2.45}$$

Without changing the distribution of $(Y_n)_{n\geq 1}$, one can construct $(Y_n)_{n\geq 1}$ and centred Gaussian random variables $(\tilde{Y}_n)_{n\geq 1}$ on a common probability space such that $\mathbb{E}[Y_n^2] = \mathbb{E}\left[\tilde{Y}_n^2\right]$ for all n, and for some absolute constant $A > 0$, it holds that

$$\mathbb{P}\left(\sup_{n=1,\ldots,N}\left|\sum_{k=1}^n Y_k - \sum_{k=1}^n \tilde{Y}_k\right| > C\log N\right) \leq \left(1 + \lambda\sum_{n=1}^N \mathbb{E}\left[Y_n^2\right]\right) N^{-\lambda A C}, \quad N \geq 1.$$

Note that one can find $\lambda > 0$ such that (2.45) is satisfied if $Y_1, Y_2 \ldots$ is a sequence of i.i.d. random variables with $\mathbb{E}\left[e^{\lambda_0|Y_1|}\right] < \infty$ for some $\lambda_0 > 0$, so the KMT coupling

follows as a special case. Another simple condition that ensures that (2.45) holds for some $\lambda > 0$ is that the Y_n are uniformly bounded, i.e. $\mathbb{P}(|Y_n| \leq K) = 1$ for all n and some constant K.
Moreover, (2.45) implies by Jensen's inequality that

$$\lambda \left(\mathbb{E}\left[Y_n^2\right]\right)^{3/2} \leq \lambda \mathbb{E}\left[|Y_n|^3\right] \leq \lambda \mathbb{E}\left[e^{\lambda|Y_n|}\,|Y_n|^3\right] \leq \mathbb{E}\left[Y_n^2\right],$$

i.e. $0 < \lambda \leq \left(\mathbb{E}\left[Y_n^2\right]\right)^{-1/2}$ for all n implying that $(\mathbb{E}\left[Y_n^2\right])_{n\geq 1}$ is necessarily bounded.

Under the assumptions of Theorem 2.3.5, if $Z_n = \sum_{k=1}^n \sigma(k) Y_k$, we can control the term $\mathbb{P}(E_N^c)$ in the proof above as before, and thus, one could generalise Theorem 2.3.3 to independent, not necessarily i.i.d. sequence $(Y_n)_{n\geq 1}$ in that case.

2.3.2 Exponential weight functions

In this section, we briefly comment on universality in the case of an exponential weight function, i.e. $\sigma(n) = e^{\beta n}$ for some $\beta > 0$. The situation here is completely different compared to the polynomial case.
First of all, the rate of decay for the discretised process and for the continuous time process is not the same in general. This was observed already in the Brownian case where

$$\mathbb{P}\left(B(e^{\beta t}) \leq 0, \forall t \in [0, N]\right) \sim \frac{1}{\pi}\, e^{-\beta N/2}, \quad N \to \infty,$$

in view of (2.16) and the fact that $(e^{-\beta t/2} B(e^{\beta t}))_{t\geq 0}$ is an Ornstein-Uhlenbeck process. In particular, for $\beta > 2\log 2$, the decay is faster than 2^{-N} which is a universal lower bound in the discrete framework (cf. (2.20)).
Secondly, the universality of the survival exponent that one observes in the polynomial case no longer persists even under the assumption of exponential moments as the following example shows.

Example 2.3.6. Let $\sigma(n) = \exp(\beta n)$ for some $\beta \geq \log 2$ and assume that $\mathbb{P}(Y_n = 1) = \mathbb{P}(Y_n = -1) = 1/2$ for all n. Then for all $N \geq 1$

$$Z_n \leq 0 \ \forall n = 1, \ldots, N \quad \Longleftrightarrow \quad Y_1 = \cdots = Y_N = -1. \tag{2.46}$$

The implication "$\Leftarrow$" is trivial. On the other hand, if $Y_1 = \cdots = Y_{k-1} = -1$ and $Y_k = 1$, for some $k \leq N$, then

$$Z_k = -\sum_{j=1}^{k-1} e^{\beta j} + e^{\beta k} = e^{\beta(k-1)} \frac{e^{\beta} - 2 + e^{\beta(2-k)}}{e^{\beta} - 1} > 0$$

since $\beta \geq \log 2$. This proves the implication "$\Rightarrow$".
Note that (2.46) implies that $\mathbb{P}(Z_n \leq 0, \forall n = 1, \ldots, N) = 2^{-N} = \exp(-\log(2)\, N)$. If we consider $(B(e^{\beta n}))_{n\geq 0}$, the corresponding survival probability is strictly greater than

2^{-N} by Lemma 2.2.19. To be very precise, we actually have to consider $(B_{\kappa(n)})_{n\geq 1}$ where

$$\kappa(n) = \sum_{k=1}^{n} \sigma(k)^2 = e^{2\beta} \frac{e^{2\beta n} - 1}{e^{2\beta} - 1}.$$

In particular, by scaling,

$$\mathbb{P}\left(B_{\kappa(n)} \leq 0, \forall n = 1, \ldots, N\right) = \mathbb{P}\left(B(e^{2\beta n} - 1) \leq 0, \forall n = 1, \ldots, N\right),$$

and the same arguments used in Lemma 2.2.19 show that

$$\lim_{N\to\infty} -\frac{1}{N} \log \mathbb{P}\left(B_{\kappa(n)} \leq 0, \forall n = 1, \ldots, N\right) < \log 2.$$

Chapter 3

Persistence of iterated processes

In this chapter, we consider persistence of processes $Z = (X \circ |Y_t|)_{t \geq 0}$, where $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ are independent stochastic processes and $Z = (X \circ Y_t)_{t \geq 0}$ if $X = (X_t)_{t \in \mathbb{R}}$ ($\circ$ denotes function composition). Such processes will be referred to as *iterated processes*. Starting with the work of Burdzy ([Bur93]), the study of iterated Brownian motion has attracted a lot of interest ([KL99]). Moreover, there are interesting connections between the exit times of iterated processes and the solution of certain fourth-order PDEs ([AZ01, Nan08]). The asymptotic behaviour of the survival probabilities of subordinated Brownian motion is also relevant for the study of Green functions ([GR08]). However, the one-sided exit problems for iterated processes has not been studied systematically so far. Here we investigate how the survival exponent of $X \circ |Y|$ and $X \circ Y$ is related to that of the outer process X and properties of the inner process Y. The relevant scenario affecting the survival probability can be identified so that the results are quite intuitive. For small ball probabilities (i.e. two-sided exit problems mentioned in (1.3)), this problem has been investigated in [AL09].

The remainder of the chapter is organised as follows. We start by reviewing the main results of the chapter in Section 3.1. In Section 3.2, we assume that the inner process Y is a continuous self-similar process. We compute the survival exponent of $X \circ |Y|$ (Theorem 3.1.1). Next, we turn to discontinuous processes Y. The survival exponent of $X \circ |Y|$ is determined for X being a Lévy process or fractional Brownian motion and Y being a random walk or Lévy process (Theorem 3.1.2 and 3.1.3) in Section 3.3. Finally, we extend the previous results to two-sided processes (Theorem 3.1.4 and Theorem 3.1.5) in Section 3.4.

3.1 Main results

First, we consider processes $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ where Y is self-similar and continuous. In this setup, the following result can be established without much difficulty:

Theorem 3.1.1. *Let $(X_t)_{t\geq 0}$ be a stochastic process with*

$$\mathbb{P}\left(X_t \leq 1, \forall t \in [0,T]\right) \asymp T^{-\theta}, \quad T \to \infty,$$

for some $\theta > 0$. Let $(Y_t)_{t\geq 0}$ be an independent stochastic processes which is self-similar of index H, has continuous paths, satisfies $Y_0 = 0$, and for some $\rho > \theta$, it holds that

$$\mathbb{P}\left(|Y_t| \leq \epsilon, \forall t \in [0,1]\right) \precsim \epsilon^{\rho}, \quad \epsilon \downarrow 0. \tag{3.1}$$

Then

$$\mathbb{P}\left(X(|Y_t|) \leq 1, \forall t \in [0,T]\right) \asymp T^{-\theta H}, \quad T \to \infty.$$

Moreover, $A := \mathbb{E}\left[(\sup\{|Y_t| : t \in [0,1]\})^{-\theta}\right] < \infty$, and if $\mathbb{P}\left(X_t \leq 1, \forall t \in [0,T]\right) \sim cT^{-\theta}$ for some $c > 0$, it holds that

$$\mathbb{P}\left(X(|Y_t|) \leq 1, \forall t \in [0,T]\right) \sim cA\,T^{-\theta H}, \quad T \to \infty.$$

We remark that the assumption (3.1) is very weak since this so-called small deviations probability usually decays faster than any polynomial as $\epsilon \downarrow 0$. Moreover, the result can be explained intuitively: by self-similarity of Y, typical fluctuations of $|Y|$ up to time T are of order T^H. The rare event that X stays below 1 until time T^H is then of order $T^{-\theta H}$. The assumption (3.1) prevents a contribution of the event that Y stays close to the origin to the survival exponent of $Z = X \circ |Y|$. In short, the survival probability of Z is determined by a rare event for X and a typical scenario for Y.
The assumption of continuity of the inner process Y allows us to write

$$\mathbb{P}\left(X(|Y_t|) \leq 1, \forall t \in [0,T]\right) = \mathbb{P}\left(X_t \leq 1, \forall t \in [0, (-I_T) \vee M_T]\right),$$

where I and M denote the running infimum resp. supremum process of Y. This will simplify the proof of the upper bound of Theorem 3.1.1 very much. If Y is discontinuous, the equality sign has to be replaced by $\geq$ in the preceding equation. It is then by far more challenging to find the survival exponent of $X \circ |Y|$ since the gaps in the range of $|Y|$ have to be taken into account. We prove the following theorem for X being a Lévy process and Y being a random walk or a Lévy process.

Theorem 3.1.2. *Let $(X_t)_{t\geq 0}$ be a centred Lévy process such that $|X_1|^{\alpha}$ has a finite exponential moment for some $\alpha > 0$. Let $(Y_t)_{t\geq 0}$ denote an independent random walk or Lévy process such that $|Y_1|^{\beta}$ has a finite exponential moment for some $\beta > 0$. It holds that*

$$\mathbb{P}\left(X(|Y_t|) \leq 1, \forall t \in [0,T]\right) = T^{-\theta + o(1)}, \quad T \to \infty,$$

where $\theta = 1/4$ if $\mathbb{E}[Y_1] = 0$, and $\theta = 1/2$ if $\mathbb{E}[Y_1] \neq 0$.

Again, the results are intuitive: If $\mathbb{E}[Y_1] = 0$, the random walk oscillates, and typical fluctuations up to time N are of magnitude $\sqrt{N}$. Since the survival exponent θ of a centred Lévy process with second finite moments is $1/2$, it is very plausible that the

survival exponent of $X \circ |Y|$ is $1/4$ at least if the gaps in the range of the random walk are not too large. If $\mathbb{E}[Y_1] > 0$, then $\mathbb{E}[Y_N]/N \to \mathbb{E}[Y_1]$ by the law of large numbers, and one expects the survival exponent of $X \circ |Y|$ to be $1/2$ by the same reasoning.
The methods to prove Theorem 3.1.2 can be extended to the case that the outer process is a fractional Brownian motion.

Theorem 3.1.3. *Let $(X_t)_{t\geq 0}$ denote a fractional Brownian motion with Hurst parameter $H \in (0,1)$. Let $(Y_t)_{t\geq 0}$ denote a Lévy process or a random walk such that $|Y_1|^\beta$ possesses a finite exponential moment for some $\beta > 0$. It holds that*

$$\mathbb{P}\left(X(|Y_t|) \leq 1, \forall t \in [0,T]\right) = T^{-\theta+o(1)}, \quad T \to \infty,$$

where $\theta = (1-H)/2$ if $\mathbb{E}[Y_1] = 0$, and $\theta = 1 - H$ if $\mathbb{E}[Y_1] \neq 0$.

Note that the outer processes in Theorem 3.1.2 and 3.1.3 share the property of stationary increments. We provide an example showing that an analogous result can fail without this property.
Up to now, the outer process $X = (X_t)_{t\geq 0}$ had the index set $[0,\infty)$, so it was only possible to evaluate X over the range of the non-negative process $|Y|$. In order to consider the one-sided exit problem for $X \circ Y$, we consider two-sided processes $X = (X_t)_{t\in\mathbb{R}}$ where

$$X_t := \begin{cases} X_t^+, & t \geq 0, \\ X_{-t}^-, & t < 0, \end{cases} \tag{3.2}$$

and $(X^+)_{t\geq 0}$ and $(X_t^-)_{t\geq 0}$ are stochastic processes. We refer to X^+ and X^- as the branches of X. We prove that the previous results can be extended in a natural way for two-sided processes.

Theorem 3.1.4. *Let $(X_t)_{t\in\mathbb{R}}$ be a two-sided process with*

$$\mathbb{P}\left(X_t \leq 1, \forall t \in [-T,T]\right) \asymp T^{-\theta}$$

for some $\theta > 0$. Let $(Y_t)_{t\in\mathbb{R}}$ denote an independent self-similar process of index H with $Y_0 = 0$ and continuous paths such that $\mathbb{E}\left[|I|^{-\theta}\right] + \mathbb{E}\left[M^{-\theta}\right] < \infty$ where $I = \inf\{Y_t : t \in [-1,1]\}$ and $M = \sup\{Y_t : t \in [-1,1]\}$. Then

$$\mathbb{P}\left(X(Y_t) \leq 1, \forall t \in [-T,T]\right) \asymp T^{-H\theta}, \quad T \to \infty.$$

As a corollary to Theorem 3.1.4, we compute the survival exponent of n-times iterated Brownian motions.
The result corresponding to Theorem 3.1.2 in the two-sided setup is

Theorem 3.1.5. *Let $(X_t)_{t\in\mathbb{R}}$ denote a two-sided Lévy process with branches X^+, X^- such that $\mathbb{E}\left[X_1^\pm\right] = 0$, $\mathbb{E}\left[\exp\left(\left|X_1^\pm\right|^\alpha\right)\right] < \infty$ for some $\alpha > 0$. Let $(Y_t)_{t\geq 0}$ denote another Lévy process or random walk independent of X with $\mathbb{E}\left[\exp\left(|Y_1|^\beta\right)\right] < \infty$ for some $\beta > 0$. Then*

$$\mathbb{P}\left(X(Y_t) \leq 1, \forall t \in [0,T]\right) = T^{-1/2+o(1)}, \quad T \to \infty.$$

Theorem 3.1.5 shows that the survival exponent is equal to 1/2 no matter if $\mathbb{E}[Y_1] = 0$ or not (in contrast to Theorem 3.1.2, see Remark 3.4.5 for an explanation).

3.2 Taking the supremum over the range of a continuous self-similar process

If $Y = (Y_t)_{t\geq 0}$ is a stochastic process, denote by $\mathcal{F}_t^Y := \sigma(Y_s : 0 \leq s \leq t)$ the filtration generated by Y up to time t. Let us now prove Theorem 3.1.1 announced in Section 3.1.

Proof of Theorem 3.1.1. Let us write $Y_t^* := \sup_{s\in[0,t]} |Y_s|$. Note that our assumption (3.1) implies that $(Y_1^*)^{-\theta}$ is integrable, see Lemma 3.2.2 below.

Upper bound: By assumption, there are constants $C, T_0 > 0$ such that for any $T > T_0$, we have that $\mathbb{P}\left(\sup_{t\in[0,T]} X_t \leq 1\right) \leq CT^{-\theta}$. Clearly, we can choose C so large that the inequality holds for all $T > 0$. By continuity of Y, the fact that $Y_0 = 0$ and independence of X and Y, and self-similarity of Y, we have that

$$\begin{aligned}\mathbb{P}\left(X(|Y_t|) \leq 1, \forall t \in [0,T]\right) &= \mathbb{E}\left[\mathbb{P}\left(X_t \leq 1, \forall t \in [0, Y_T^*] | \mathcal{F}_T^Y\right)\right] \\ &\leq C\mathbb{E}\left[(Y_T^*)^{-\theta}\right] = C\mathbb{E}\left[(Y_1^*)^{-\theta}\right] T^{-\theta H}.\end{aligned}$$

Lower bound: Note that for any $C > 0$, it holds that

$$\begin{aligned}&\mathbb{P}\left(X(|Y_t|) \leq 1, \forall t \in [0,T]\right) \geq \mathbb{P}\left(Y_T^* \leq CT^H, \{X_t \leq 1, \forall t \in [0, Y_T^*]\}\right) \\ &\quad\geq \mathbb{P}\left(Y_T^* \leq CT^H\right) \mathbb{P}\left(X_t \leq 1, \forall t \in [0, CT^H]\right) = \mathbb{P}\left(Y_1^* \leq C\right) \mathbb{P}\left(X_t \leq 1, \forall t \in [0, CT^H]\right).\end{aligned}$$

If C is large enough, $\mathbb{P}(Y_1^* \leq C) = \mathbb{P}\left(Y^*(c^{-1/H}) \leq 1\right) > 0$ by continuity and the fact that $Y_0 = 0$. This proves the lower bound.

If $\mathbb{P}\left(\sup_{t\in[0,T]} X_t \leq 1\right) \sim cT^{-\theta}$, we can find for all $\epsilon > 0$ small enough a constant $T_0(\epsilon)$ such that for all $T \geq T_0(\epsilon)$, it holds that $(c-\epsilon)T^{-\theta} \leq \mathbb{P}\left(\sup_{t\in[0,T]} X_t \leq 1\right) \leq (c+\epsilon)T^{-\theta}$. Hence,

$$\begin{aligned}T^{\theta H}\, \mathbb{P}\left(\sup_{t\in[0,T]} X(|Y_t|) \leq 1\right) &\geq T^{\theta H}\, \mathbb{E}\left[1_{\{Y_T^* \geq T_0(\epsilon)\}}\, \mathbb{P}\left(X_t \leq 1, \forall t \in [0, Y_T^*] | \mathcal{F}_T^Y\right)\right] \\ \geq (c-\epsilon)T^{\theta H}\, \mathbb{E}\left[1_{\{Y_T^* \geq T_0(\epsilon)\}} (Y_T^*)^{-\theta}\right] &= (c-\epsilon)\, \mathbb{E}\left[1_{\{Y_1^* \geq T_0(\epsilon)T^{-H}\}} (Y_1^*)^{-\theta}\right].\end{aligned}$$

Letting $T \to \infty$, monotone convergence implies for all $\epsilon > 0$ small enough that

$$\liminf_{T\to\infty} T^{\theta H} \mathbb{P}\left(X(|Y_t|) \leq 1, \forall t \in [0,T]\right) \geq (c-\epsilon)\mathbb{E}\left[(Y_1^*)^{-\theta}\right],$$

i.e. $\liminf_{T\to\infty} T^{\theta H} \mathbb{P}\left(X(|Y_t|) \leq 1, \forall t \in [0,T]\right) \geq c \cdot \mathbb{E}\left[(Y_1^*)^{-\theta}\right]$.

For the proof of the upper bound, note that

$$\begin{aligned}\mathbb{P}\left(X_t \leq 1, \forall t \in [0, Y_T^*]\right) &\leq \mathbb{P}\left(Y_T^* \leq T_0\right) + \mathbb{E}\left[1_{\{Y_T^* \geq T_0\}}\, \mathbb{P}\left(X_t \leq 1, \forall t \in [0, Y_T^*] | \mathcal{F}_T^Y\right)\right] \\ &\leq \mathbb{P}\left(Y_1^* \leq T_0 T^{-H}\right) + (c+\epsilon)\mathbb{E}\left[1_{\{Y_T^* \geq T_0\}} (Y_T^*)^{-\theta}\right].\end{aligned}$$

The assumption on the small deviation probability of Y implies that

$$T^{\theta H}\mathbb{P}\left(Y_1^* \le T_0(\epsilon)T^{-H}\right) \precsim T^{H(\theta-\rho)}T_0(\epsilon)^\rho \to 0, \quad T \to \infty.$$

Hence, as in the proof of the lower bound, we obtain that

$$\limsup_{T\to\infty} T^{\theta H}\mathbb{P}\left(X(|Y_t|) \le 1, \forall t \in [0,T]\right) \le (c+\epsilon)\mathbb{E}\left[(Y_1^*)^{-\theta}\right],$$

which finishes the proof upon letting $\epsilon \downarrow 0$. □

Remark 3.2.1. The proof reveals that the lower bounds of Theorem 3.1.1 are also valid without continuity of paths of Y and the assumption (3.1) on the small deviations of Y. Moreover, we remark that the proof can be easily adapted to cover the case that $\mathbb{P}\left(\sup_{t\in[0,T]} X_t \le 1\right) = T^{-\theta+o(1)}$.

As already mentioned in the proof, the small deviations probability in (3.1) is linked to integrability of $\sup\{|Y_t| : t \in [0,1]\}$. For convenience and later reference, let us state this fact without proof in the following lemma.

Lemma 3.2.2. *Let Z be a random variable such that $Z > 0$ a.s. and $\mathbb{P}(Z \le \epsilon) \precsim \epsilon^\rho$ as $\epsilon \downarrow 0$ for some $\rho > 0$. Then for $\eta \in (0,\rho)$, it holds that $\mathbb{E}[Z^{-\eta}] < \infty$. Conversely, if $\mathbb{E}[Z^{-\eta}] < \infty$ for some $\eta > 0$, then $\mathbb{P}(Z \le \epsilon) \precsim \epsilon^\eta$ as $\epsilon \downarrow 0$.*

To conclude this section, let us give a simple application of Theorem 3.1.1.

Example 3.2.3. If X and Y are independent Brownian motions, recall from (1.4) that $\mathbb{P}(X_t \le 1, \forall t \in [0,T]) \sim \sqrt{2/\pi}\, T^{-1/2}$ as $T \to \infty$. Moreover, since

$$\lim_{\epsilon\downarrow 0} \epsilon^2 \log \mathbb{P}(|Y_t| \le \epsilon, \forall t \in [0,1]) = -\pi^2/8,$$

see e.g. [LS01, Theorem 6.3], Lemma 3.2.2 implies that $(\sup_{t\in[0,1]}|Y_t|)^{-\eta}$ is integrable for every $\eta > 0$. Hence, Theorem 3.1.1 implies that the survival exponent $X \circ |Y|$ of is 1/4.
More generally, if W and $B^{(1)}, \dots, B^{(n)}$ are independent Brownian motions, it follows for any $n \ge 1$ that

$$\mathbb{P}\left(W\left(\left|B^{(1)}\right| \circ \dots \circ \left|B_t^{(n)}\right|\right) \le 1, \forall t \in [0,T]\right) \sim c_n T^{-2^{-(n+1)}}, \quad T \to \infty,$$

with

$$c_n = \sqrt{\frac{2}{\pi}} \prod_{k=1}^n \mathbb{E}\left[(W_1^*)^{-2^{-k}}\right] < \infty, \quad n \ge 1, \qquad W_1^* = \sup_{t\in[0,1]} |W_t|.$$

3.3 Taking the supremum over the range of discontinuous processes

The goal of this section is to find the asymptotics of

$$\mathbb{P}\left(X(|S_n|) \le 1, \forall n = 1, \dots, N\right), \quad \mathbb{P}\left(X(|Y_t|) \le 1, \forall t \in [0,T]\right) \qquad N, T \to \infty.$$

Here $X = (X_t)_{t \ge 0}$ is a centred Lévy process or a fractional Brownian motion, S is a random walk, and Y is a Lévy process. First, we recall known results on survival probabilities of Lévy processes and prove a slight generalisation. If X is a centred Lévy process with $\mathbb{E}\left[X_1^2\right] < \infty$, recall that

$$\mathbb{P}\left(X_t \le 1, \forall t \in [0,T]\right) = T^{-1/2}\, l(T), \quad T \to \infty,$$

where l is slowly varying at infinity, see Section 1.2.1. Our first goal is to show that the function l may be chosen asymptotically constant which is suggested by the analogous result for random walks: If $(S_n)_{n \ge 1}$ is a centred random walk with finite variance, then $\mathbb{P}\left(\sup_{n=1,\dots,N} S_n \le 0\right) \sim cN^{-1/2}$. However, to the author's knowledge, an analogous result for Lévy processes has not been proved in the literature so far.
Clearly, $\mathbb{P}\left(\sup_{t \in [0,T]} X_t \le 1\right) \le \mathbb{P}\left(\sup_{n=1,\dots,\lfloor T \rfloor} X_n \le 1\right) \asymp T^{-1/2}$, since $(X_n)_{n \ge 1}$ is a centred random walk with finite variance. Moreover, if $\mathbb{E}\left[X_1^{2+\epsilon}\right] < \infty$ for some $\epsilon > 0$, then also $\mathbb{P}\left(\sup_{n=1,\dots,\lfloor T \rfloor} X_n \le 1\right) \gtrsim T^{-1/2}$, see [AD13, Proposition 2.1]. The next theorem states the precise asymptotic decay of $\mathbb{P}\left(\sup_{t \in [0,T]} X_t \le 1\right)$ as $T \to \infty$ under the assumption of finite variance. The idea to approximate the integral over $\mathbb{P}\left(X_t > 0\right)$ by the sum over $\mathbb{P}\left(X_n > 0\right)$ in the proof below is due to Ron Doney.

Theorem 3.3.1. *Let $(X_t)_{t \ge 0}$ be a centred Lévy process with $\mathbb{E}\left[X_1^2\right] < \infty$. For any $x > 0$, there is a constant $c(x) > 0$ such that*

$$\mathbb{P}\left(X_t \le x, \forall t \in [0,T]\right) \sim c(x)\, T^{-1/2}, \quad T \to \infty.$$

Proof. Let τ_x be the first hitting time of the set (x, ∞), $x > 0$. According to [Don07, Eq. 4.4.7], it holds that

$$1 - \mathbb{E}\left[e^{-\lambda \tau_x}\right] \sim U(x)\kappa(\lambda), \quad \lambda \downarrow 0, \tag{3.3}$$

where U is a renewal function (see [Don07, Eq. 4.4.6]), and

$$\kappa(u) = \exp\left(\int_0^\infty \frac{e^{-t} - e^{-ut}}{t}\, \mathbb{P}\left(X_t > 0\right)\, dt\right), \quad u \ge 0.$$

Using that $\int_0^\infty t^{-1}(e^{-t} - e^{-ut})\, dt = \log u$ for $u > 0$ (a Frullani integral), it follows that

$$\kappa(u) = \sqrt{u}\, \exp\left(\int_0^\infty \frac{e^{-t} - e^{-ut}}{t}\left(\mathbb{P}\left(X_t > 0\right) - 1/2\right) dt\right). \tag{3.4}$$

We will show that

$$\lim_{\lambda\downarrow 0}\int_0^\infty \frac{e^{-\lambda t}-e^{-t}}{t}\left(\mathbb{P}\left(X_t>0\right)-1/2\right)dt=\int_0^\infty \frac{1-e^{-t}}{t}\left(\mathbb{P}\left(X_t>0\right)-1/2\right)dt=:A<\infty, \tag{3.5}$$

implying that $\kappa(\lambda)\sim\sqrt{\lambda}e^{-A}$ as $\lambda\downarrow 0$. By a Tauberian theorem (see e.g. [Fel71, Theorem XIII.5.4]), we conclude from (3.3) that

$$\mathbb{P}\left(\tau_x>T\right)\sim\frac{U(x)e^{-A}}{\Gamma(1/2)}\cdot T^{-1/2}=\frac{U(x)e^{-A}}{\sqrt{\pi}}\cdot T^{-1/2},\quad T\to\infty,$$

so the theorem follows.
In order to prove (3.5), we approximate the term $\mathbb{P}\left(X_t>0\right)$ by $\mathbb{P}\left(X_n>0\right)$ for $t\in(n,n+1]$, which allows us to use classical results from fluctuation theory of random walks to show that the integral in (3.4) converges as $u\to 0$. To this end, note that for $u\in(0,1)$, we have the following estimates:

$$\begin{aligned}0&\le\int_0^\infty\frac{e^{-ut}-e^{-t}}{t}\left|\mathbb{P}\left(X_t>0\right)-1/2\right|dt\le\int_0^1\frac{1-e^{-t}}{t}\left|\mathbb{P}\left(X_t>0\right)-1/2\right|dt\\&\quad+\sum_{n=1}^\infty\int_n^{n+1}\frac{e^{-ut}-e^{-t}}{t}\left(\left|\mathbb{P}\left(X_t>0\right)-\mathbb{P}\left(X_n>0\right)\right|+\left|\mathbb{P}\left(X_n>0\right)-1/2\right|\right)dt\\&\le c+\sum_{n=1}^\infty n^{-1}\sup_{t\in[n,n+1]}\left|\mathbb{P}\left(X_t>0\right)-\mathbb{P}\left(X_n>0\right)\right|+\sum_{n=1}^\infty n^{-1}\left|\mathbb{P}\left(X_n>0\right)-1/2\right|.\end{aligned}\tag{3.6}$$

By [Ros62, Theorem 3], it is known that the series $\sum_{n=1}^\infty n^{-1}(\mathbb{P}\left(X_n>0\right)-1/2)$ converges absolutely if $\mathbb{E}\left[X_1\right]=0$ and $\mathbb{E}\left[X_1^2\right]\in(0,\infty)$, so the second series in (3.6) converges. Next, we show that the first series also converges using results on the speed of convergence in the CLT. To this end, let $t\in(n,n+1]$. By independence and stationarity of increments of X, we have that

$$\begin{aligned}\mathbb{P}\left(X_t\le 0\right)-\mathbb{P}\left(X_n\le 0\right)&=\int_{-\infty}^\infty\left(\mathbb{P}\left(X_n\le -y\right)-\mathbb{P}\left(X_n\le 0\right)\right)\mathbb{P}\left(X_{t-n}\in dy\right)\\&=\int_{-\infty}^\infty\big(F_n(-y/\sqrt{n})-F_n(0)\big)\,\mathbb{P}\left(X_{t-n}\in dy\right),\end{aligned}\tag{3.7}$$

where $F_n(x):=\mathbb{P}\left(X_n/\sqrt{n}\le x\right)$. Let Φ denote the cumulative distribution function of a standard Gaussian variable. With $\Delta_n:=\sup\left\{|F_n(x)-\Phi(x)|:x\in\mathbb{R}\right\}$, we get for $y\in\mathbb{R}$ that

$$\begin{aligned}\left|F_n(y/\sqrt{n})-F_n(0)\right|&\le\left|F_n(y/\sqrt{n})-\Phi(y/\sqrt{n})\right|+\left|\Phi(y/\sqrt{n})-\Phi(0)\right|+\left|\Phi(0)-F_n(0)\right|\\&\le 2\Delta_n+\left|\Phi(y/\sqrt{n})-\Phi(0)\right|\le 2\Delta_n+(2\pi)^{-1/2}\left|y\right|/\sqrt{n}.\end{aligned}$$

In view of (3.7), we obtain that

$$\begin{aligned}|\mathbb{P}(X_t \le 0) - \mathbb{P}(X_n \le 0)| &\le 2\Delta_n + (2\pi)^{-1/2}\mathbb{E}[|X_{t-n}|]/\sqrt{n} \\ &\le 2\Delta_n + (2\pi)^{-1/2}\mathbb{E}[|X_1|]/\sqrt{n},\end{aligned}$$

where we have used that $0 \le t-n \le 1$ and that $(|X_t|)_{t\ge 0}$ is a submartingale in the last inequality. Since $\mathbb{E}[X_1^2] < \infty$, $\sum_{n=1}^{\infty} \Delta_n/n$ is finite by [Ego73, Theorem 1]. Hence, the first series in (3.6) is also finite, so by dominated convergence,

$$\int_0^\infty \frac{1-e^{-t}}{t}\, |\mathbb{P}(X_t > 0) - 1/2|\, dt = \lim_{u\downarrow 0} \int_0^\infty \frac{e^{-ut}-e^{-t}}{t}\, |\mathbb{P}(X_t > 0) - 1/2|\, dt < \infty.$$

□

Having determined the asymptotic behaviour of the survival probability for X, let us continue to give some heuristics concerning the survival exponent of $X \circ |S|$. If $\mathbb{E}[S_1] = 0$ and $\mathbb{E}[S_1^2] = 1$, it follows from the invariance principle that

$$\lim_{N\to\infty} \mathbb{P}\left(|S_n| \le \sqrt{N}\, x, \forall n = 1,\dots,N\right) = \mathbb{P}(|B_t| \le x, \forall t \in [0,1]), \quad x > 0.$$

Here, B denotes a standard Brownian motion. Intuitively, one would therefore expect that

$$\mathbb{P}(X(|S_n|) \le 1, \forall n = 1,\dots,N) \asymp \mathbb{P}\left(X_t \le 1, \forall t \in [0,\sqrt{N}]\right) \asymp N^{-1/4},$$

at least if the points $|S_1|,\dots,|S_N|$ are sufficiently "dense" in $[0,\sqrt{N}]$. Under a subexponential moment condition on the random walk, we show that the survival exponent is indeed 1/4. For simplicity of notation, we denote by $\mathcal{X}(\gamma)$ the class of non-degenerate random variables X with $\mathbb{E}\left[e^{|X|^\gamma}\right] < \infty$ where $\gamma > 0$.

Before proving the upper bound of Theorem 3.1.2, we need the following auxiliary result:

Lemma 3.3.2. *Let $(f_n)_{n\ge 1}$ denote a sequence of positive numbers with $f_N \to \infty$ and $f_N/\sqrt{N} \to 0$ as $N \to \infty$. Let $(S_n)_{n\ge 1}$ denote a centred random walk with $\mathbb{E}[S_1^2] < \infty$ and let $M_n := \max\{S_1,\dots,S_n\}$. There are a constants C, N_0 independent of the sequence (f_n) such that*

$$\mathbb{P}(M_N \le f_N) \le C f_N\, N^{-1/2}, \qquad f_N, N \ge N_0.$$

Proof. Recall from (2.14) that for $1 \le N_0 < N$, it holds that

$$\mathbb{P}(M_N \le 0) \ge \mathbb{P}(M_{N_0} \le 0) \cdot \mathbb{P}(S_n \le 0, \forall n = N_0+1,\dots,N).$$

Now

$$\begin{aligned}\mathbb{P}\left(\sup_{n=N_0+1,\dots,N} S_n \le 0\right) &\ge \mathbb{P}\left(S_{N_0} \le -f_N, \sup_{n=N_0+1,\dots,N} S_n - S_{N_0} \le f_N\right) \\ &= \mathbb{P}(S_{N_0} \le -f_N)\, \mathbb{P}\left(\sup_{n=1,\dots,N-N_0} S_n \le f_N\right) \ge \mathbb{P}(S_{N_0} \le -f_N)\, \mathbb{P}(M_N \le f_N).\end{aligned}$$

Hence, we get that

$$\mathbb{P}\left(M_N \leq f_N\right) \leq \frac{\mathbb{P}\left(M_N \leq 0\right)}{\mathbb{P}\left(M_{N_0} \leq 0\right)\mathbb{P}\left(S_{N_0} \leq -f_N\right)}.$$

With $N_0 = \lfloor f(N) \rfloor^2$, it follows from the CLT that $\mathbb{P}\left(S_{N_0} \leq -f_N\right) \to \mathbb{P}\left(Z \leq -1\right)$, where Z is centred Gaussian with variance $\mathbb{E}\left[Y_1^2\right]$. Moreover, since $\mathbb{P}\left(M_N \leq 0\right) \sim cN^{-1/2}$, we conclude that

$$\frac{\mathbb{P}\left(M_N \leq 0\right)}{\mathbb{P}\left(M_{N_0} \leq 0\right)\mathbb{P}\left(S_{N_0} \leq -f_N\right)} \sim \frac{N^{-1/2}}{N_0^{-1/2}\mathbb{P}\left(Z \leq -1\right)} \sim \frac{f_N N^{-1/2}}{\mathbb{P}\left(Z \leq -1\right)}.$$

□

Remark 3.3.3. We frequently need to apply Lemma 3.3.2 to Lévy processes in the following situation: Let $(X_t)_{t\geq 0}$ denote a centred Lévy process such that $\mathbb{E}\left[X_1^2\right] < \infty$. Let $g\colon [0,\infty) \to (0,\infty)$ be a function such that $g(T) \to \infty$ as $T \to \infty$. Since $(X_n)_{n\geq 1}$ is a random walk, for any $c, \rho > 0$ and T large enough, we see from Lemma 3.3.2 that

$$\mathbb{P}\left(\sup_{t\in[0,g(T)]} X_t \leq c(\log T)^\rho\right) \leq \mathbb{P}\left(\sup_{n=1,\ldots,\lfloor g(T)\rfloor} X_n \leq c(\log T)^\rho\right) \leq C\,\frac{c(\log T)^\rho}{\sqrt{g(T)}}.$$

We are now ready to establish the upper bounds of Theorem 3.1.2.

Proof of the upper bound of Theorem 3.1.2.

Let us first observe that it suffices to prove the upper bound for the case that the inner process Y is a random walk. Indeed, if Y is a Lévy process, $(Y_n)_{n\geq 1}$ is a random walk and we have for all $T > 0$ that

$$\mathbb{P}\left(X(|Y_t|) \leq 1, \forall t \in [0,T]\right) \leq \mathbb{P}\left(X(|Y_n|) \leq 1, \forall n = 1,\ldots,\lfloor T\rfloor\right). \tag{3.8}$$

In the sequel, we denote the random walk by $(S_n)_{n\geq 0}$ and write $S_n = Y_1 + \cdots + Y_n$ where $(Y_n)_{n\geq 1}$ is a sequence of i.i.d. random variables. Let us begin to develop a method to deal with the gaps in the range of the random walk. The idea is to fill the gaps in the range, which will only result in a term of lower order if the gaps are not too large. Let $t(1) \leq t(2) \leq \ldots$, $N \geq 2$, $k \geq 0$, $x, y > 0$. Using that $(X_{T-t} - X_T)_{t\in[0,T]} \overset{d}{=} (-X_t)_{t\in[0,T]}$, observe that

$$\begin{aligned}
&\mathbb{P}\left(\bigcap_{n=1}^{N} \left\{X_t \leq x + ky, \forall t \in [t(n)-k, t(n)+k]\right\}\right)\\
&\leq \mathbb{P}\left(\bigcap_{n=1}^{N} \left\{X_t \leq x + (k+1)y, \forall t \in [t(n)-(k+1), t(n)+(k+1)]\right\}\right)\\
&+ \sum_{n=1}^{N-1} \mathbb{P}\left(\sup_{t\in[0,1]} X_{t(n)-k-t} - X_{t(n)-k} \geq y\right) + \sum_{n=1}^{N-1} \mathbb{P}\left(\sup_{t\in[0,1]} X_{t(n)+k+t} - X_{t(n)+k} \geq y\right)\\
&\leq \mathbb{P}\left(\bigcap_{n=1}^{N} \left\{\sup_{t\in[t(n)-(k+1),t(n)+k+1]} X_t \leq x + (k+1)y\right\}\right) + 2N\mathbb{P}\left(\sup_{t\in[0,1]} |X_t| \geq y\right).
\end{aligned}$$

(Here and below, the interval $[t(n)-k, t(n)+k]$ stands for $[0, t(n)+k]$ whenever $t(n)-k < 0$.) Let $p_N := \mathbb{P}\left(\sup_{n=1,\dots,N} X(|S_n|) \le 1\right)$. Conditioning on $S_1, \dots, S_N$ and using the previous inequality with $x = 1$ and $y = (2\log N)^{1/\alpha}$ iteratively for $k = 0, \dots, L$, we obtain that

$$\begin{aligned} p_N &\le \mathbb{P}\left(\bigcap_{n=0}^{N} \left\{X_t \le 1 + (2\log N)^{1/\alpha}, \forall t \in [|S_n| - 1, |S_n| + 1]\right\}\right) \\ &\qquad + 2(N+1)\mathbb{P}\left(\sup_{t\in[0,1]} |X_t| \ge (2\log N)^{1/\alpha}\right) \\ &\le \cdots \le \mathbb{P}\left(\bigcap_{n=0}^{N} \left\{X_t \le 1 + L(2\log N)^{1/\alpha}, \forall t \in [|S_n| - L, |S_n| + L]\right\}\right) \\ &\qquad + L2(N+1)\mathbb{P}\left(\sup_{t\in[0,1]} |X_t| \ge (2\log N)^{1/\alpha}\right). \end{aligned} \tag{3.9}$$

Since $X_1 \in \mathcal{X}(\alpha)$ and X is a martingale, it is not hard to show that

$$C_1 := \mathbb{E}\left[\sup\left\{\exp\left(|X_t|^\alpha\right) : t \in [0,1]\right\}\right] < \infty.$$

Indeed, note that the function $x \mapsto \exp(x^\alpha)$ is increasing on $[0,\infty)$ and convex $[x_0,\infty)$, where $x_0 > 0$ is some suitable constant. In particular, if we set $h(x) = \exp(x_0^\alpha)$ on $[0, x_0]$ and $h(x) = \exp(x^\alpha)$ for $x \ge x_0$, then h is a non-decreasing convex function, so $(h(|X_t|))_{t\ge0}$ is a submartingale. By Doob's inequality, it follows that C_1 is finite.
By Chebychev's inequality, we find that

$$\mathbb{P}\left(\sup_{t\in[0,1]} |X_t| \ge (2\log N)^{1/\alpha}\right) \le e^{-2\log N}\, C_1 = C_1\, N^{-2}. \tag{3.10}$$

Let $S_N^* := \max\left\{|S_1|, \dots, |S_N|\right\}$. With $L = \lfloor C(\log N)^\gamma \rfloor$, we obtain from (3.9) and (3.10) that

$$\begin{aligned} p_N &\le \mathbb{P}\left(\bigcap_{n=0}^{N} \left\{\sup_{t\in[|S_n|-L,|S_n|+L]} X_t \le 1 + 2^{1/\alpha} C(\log N)^{\gamma+1/\alpha}\right\}\right) + 4CC_1(\log N)^\gamma N^{-1} \\ &\le \mathbb{P}\left(X_t \le 2^{1+1/\alpha} C(\log N)^{\gamma+1/\alpha}, \forall t \in [0, S_N^*]\right) + \mathbb{P}\left(A_N\right) + C_2(\log N)^\gamma N^{-1}, \end{aligned} \tag{3.11}$$

where A_N is the event that the set $\{0, |S_1|, \dots, |S_N|\}$ contains a gap larger than $L = \lfloor C(\log N)^\gamma \rfloor$. In particular, the event A_N implies that the random walk must have a jump larger than L up to time N. If $Y_1 \in \mathcal{X}(\beta)$, take $\gamma = 1/\beta$, and note that

$$\begin{aligned} \mathbb{P}\left(A_N\right) &\le \mathbb{P}\left(\max_{n=1,\dots,N} |Y_n| \ge \lfloor C(\log N)^{1/\beta} \rfloor\right) \le N\mathbb{P}\left(|Y_1| \ge (C/2)(\log N)^{1/\beta}\right) \\ &\le N e^{-(C/2)^\beta \log N}\, \mathbb{E}\left[e^{|Y_1|^\beta}\right] = o(N^{-1/2}), \end{aligned} \tag{3.12}$$

where the last equality holds for C large enough. Now combining (3.11) and (3.12), we arrive at

$$p_N \leq \mathbb{P}\left(X_t \leq 2^{1+1/\alpha} C (\log N)^{1/\beta+1/\alpha}, \forall t \in [0, S_N^*]\right) + o(N^{-1/2}). \tag{3.13}$$

We need to distinguish the cases $\mathbb{E}[S_1] = 0$ and $\mathbb{E}[S_1] \neq 0$.
Case $\mathbb{E}[Y_1] = 0$*:* First, note that

$$\begin{aligned}
&\mathbb{P}\left(X_t \leq C_3 (\log N)^{1/\beta+1/\alpha}, \forall t \in [0, S_N^*]\right) \\
&\quad \leq \mathbb{P}\left(X_t \leq C_3 (\log N)^{1/\beta+1/\alpha}, \forall\, t \in [0, \sqrt{N/\log N}]\right) + \mathbb{P}\left(S_N^* \leq \sqrt{N/\log N}\right).
\end{aligned} \tag{3.14}$$

By [dA83, Corollary 4.6] (or [Mog74, Theorem 4]), one has

$$\lim_{N\to\infty} a_N^{-2} \log \mathbb{P}\left(S_N^* \leq \sqrt{N}/a_N\right) = -\pi^2/8, \tag{3.15}$$

whenever $0 < a_N \to \infty$ and $a_N^2/N \to 0$. This shows that

$$\mathbb{P}\left(S_N^* \leq \sqrt{N/\log N}\right) = N^{-\pi^2/8+o(1)} = o(N^{-1}). \tag{3.16}$$

Finally, (3.13), (3.14), (3.16) and Remark 3.3.3 imply that

$$\begin{aligned}
\mathbb{P}\left(\sup_{n=1,\dots,N} X(|S_n|) \leq 1\right) &\leq \mathbb{P}\left(\sup_{t\in[0,\sqrt{N/\log N}]} X_t \leq C_3 (\log N)^{1/\beta+1/\alpha}\right) + o(N^{-1/4}) \\
&\lesssim (\log N)^{1/\alpha+1/\beta+1/4}\, N^{-1/4}.
\end{aligned}$$

Case $\mathbb{E}[Y_1] \neq 0$*:* Similarly, note that

$$\begin{aligned}
&\mathbb{P}\left(X_t \leq C_2 (\log N)^{1/\beta+1/\alpha}, \forall t \in [0, S_N^*]\right) \leq \mathbb{P}\left(X_t \leq C_2 (\log N)^{1/\beta+1/\alpha}, \forall t \in [0, |S_N|]\right) \\
&\quad \leq \mathbb{P}\left(X_t \leq C_2 (\log N)^{1/\beta+1/\alpha}, \forall t \in [0, N\,|\mathbb{E}[Y_1]|/2]\right) + \mathbb{P}\left(|S_N| \leq N\,|\mathbb{E}[Y_1]|/2\right).
\end{aligned}$$

Write $\tilde{S}_n := S_n - n\mathbb{E}[Y_1]$, so $\tilde{S}$ is a centred random walk, and note that

$$\begin{aligned}
\mathbb{P}\left(|S_N| \leq N\,|\mathbb{E}[Y_1]|/2\right) &\leq \mathbb{P}\left(N\,|\mathbb{E}[Y_1]| - \left|\tilde{S}_N\right| \leq N\,|\mathbb{E}[Y_1]|/2\right) \\
&\leq \mathbb{P}\left(\left|\tilde{S}_N\right| \geq |\mathbb{E}[Y_1]|\, N/2\right) \leq 4\frac{\mathbb{E}\left[\tilde{S}_N^2\right]}{\mathbb{E}[Y_1]^2 N^2} = C_3 N^{-1}.
\end{aligned}$$

As above, in combination with (3.13) and Remark 3.3.3, we conclude that

$$\begin{aligned}
p_N &\leq \mathbb{P}\left(X_t \leq C_2 (\log N)^{1/\beta+1/\alpha}, \forall t \in [0, N\,|\mathbb{E}[Y_1]|/2]\right) + o(N^{-1/2}) \\
&\lesssim (\log N)^{1/\alpha+1/\beta} N^{-1/2}, \qquad N \to \infty.
\end{aligned}$$

□

Let us now prove the lower bound of Theorem 3.1.2. We only prove the lower bound if the inner process Y is a Lévy process. If Y is a random walk, the proof is almost identical.

Proof of the lower bound of Theorem 3.1.2.

Case $\mathbb{E}[Y_1] = 0$*:*

By independence of X and Y, we have that

$$\mathbb{P}\left(\sup_{t\in[0,T]} X(|Y_t|) \leq 1\right) \geq \mathbb{P}\left(\sup_{t\in[0,c\sqrt{T}]} X_t \leq 1\right) \mathbb{P}\left(\sup_{t\in[0,T]} |Y_t| \leq c\sqrt{T}\right)$$

Note that by Doob's inequality applied to the submartingale $(Y_t^2)_{t\geq 0}$, we obtain that

$$\mathbb{P}\left(\sup_{t\in[0,T]} |Y_t| \leq c\sqrt{T}\right) = 1 - \mathbb{P}\left(\sup_{t\in[0,T]} Y_t^2 > c^2 T\right) \geq 1 - \frac{EY_T^2}{c^2 T} = 1 - EY_1^2/c^2 = 1/2$$

for $c := \sqrt{2\,\mathbb{E}[Y_1^2]}$. We have used that $\mathbb{E}[Y_t^2] = t \cdot EY_1^2$ for a square integrable Lévy martingale. This proves the lower bound if $\mathbb{E}[Y_1] = 0$.

Case $\mathbb{E}[Y_1] \neq 0$*:*

As before, for any $c > |\mathbb{E}[Y_1]|$, we have

$$\mathbb{P}\left(\sup_{t\in[0,T]} X(|Y_t|) \leq 1\right) \geq \mathbb{P}\left(\sup_{t\in[0,cT]} X_t \leq 1\right) \mathbb{P}\left(\sup_{t\in[0,T]} |Y_t| \leq cT\right).$$

Next, since $|Y_t| \leq |Y_t - \mathbb{E}[Y_t]| + |\mathbb{E}[Y_t]|$ and $\mathbb{E}[Y_t] = \mathbb{E}[Y_1] \cdot t$ for a Lévy process, it follows that

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,T]} |Y_t| \leq cT\right) &\geq \mathbb{P}\left(\sup_{t\in[0,T]} |Y_t - \mathbb{E}[Y_t]| \leq (c - |\mathbb{E}[Y_1]|)T\right) \\ &\geq 1 - \frac{\mathbb{E}\left[|Y_T - \mathbb{E}[Y_T]|^2\right]}{(c - |\mathbb{E}[Y_1]|)^2\, T^2} = 1 - \frac{\mathbb{E}\left[|Y_1 - \mathbb{E}[Y_1]|^2\right]}{(c - |\mathbb{E}[Y_1]|)^2\, T} \to 1\end{aligned}$$

as $T \to \infty$. We have again used Doob's inequality and the fact that $\mathbb{E}\left[|Y_T - \mathbb{E}[Y_T]|^2\right] = \mathbb{E}\left[|Y_1 - \mathbb{E}[Y_1]|^2\right] \cdot T$. This completes the proof of the lower bound. □

Remark 3.3.4. The proof reveals that under the assumptions of Theorem 3.1.2, if $\mathbb{E}[Y_1] = 0$, it holds that

$$N^{-1/4} \precsim \mathbb{P}\left(\sup_{n=1,\ldots,N} X(|S_n|) \leq 1\right) \precsim N^{-1/4} (\log N)^{1/\alpha + 1/\beta + 1/4}, \quad N \to \infty.$$

Note that the lower bounds of Theorem 3.1.2 hold whenever $\mathbb{E}[X_1^2] + \mathbb{E}[Y_1^2] < \infty$. The upper bound of Theorem 3.1.2 can be improved if X is a symmetric Lévy process and Y is a subordinator. Assume w.l.o.g. that $Y_1 \geq 0$ a.s. Then $Z := X \circ Y$ is a symmetric Lévy process. In particular,

$$\mathbb{P}\left(Z_t \leq 1, \forall t \in [0,T]\right) \leq \mathbb{P}\left(Z_n \leq 1, \forall n = 1, \ldots, \lfloor T \rfloor\right) \asymp T^{-1/2},$$

without any additional assumption of moments, see e.g. [DDG12, Proposition 1.4]. This observation suggests that Theorem 3.1.2 remains true under much weaker integrability conditions. Indeed, Vysotsky ([Vys12a]) shows that if $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are independent centred Lévy processes such that $\mathbb{E}\left[X_1^2\right] + \mathbb{E}\left[Y_1^2\right] < \infty$, it holds that

$$\mathbb{P}\left(X(|Y_t|) \leq 1, \forall t \in [0,T]\right) \asymp T^{-1/4}, \qquad T \to \infty.$$

The proof of the crucial upper bound relies on an identity from [Sin92a]. To explain the idea in some more detail, let $\mathcal{T}_n$ denote the sequence of (ascending) ladder moments of the random walk $(Y_n)_{n\geq 0}$, i.e.

$$\mathcal{T}_0 := 0, \quad \mathcal{T}_n = \inf\left\{k > \mathcal{T}_{n-1} : Y_k > Y_{\mathcal{T}_{n-1}}\right\},$$

and $\mathcal{H}_n := Y(\mathcal{T}_n)$ the corresponding ladder heights (see [Fel71, Section XII.1]). Set $Z_n := X(\mathcal{H}_n)$. Then $(\mathcal{T}_n, Z_n)_{n\geq 1}$ defines a bivariate random walk. Now

$$\begin{aligned}\mathbb{P}\left(X(Y_t) \leq 1, \forall t \in [0,T]\right) &\leq \mathbb{P}\left(X(\mathcal{H}_n) \leq 1, \forall n \text{ s.t. } \mathcal{T}_n \leq T\right) \\ &= \mathbb{P}\left(Z_n \leq 1, \forall n \text{ s.t. } \mathcal{T}_n \leq T\right).\end{aligned}$$

Let $\xi := \inf\{n \geq 0 : Z_n > 0\}$. It can be shown that

$$\mathbb{P}\left(Z_n \leq 0, \forall n \text{ s.t. } \mathcal{T}_n \leq T\right) = \mathbb{P}\left(\mathcal{T}_\xi > T\right). \tag{3.17}$$

Indeed, let $\theta(T) := \inf\{n > 0 : \mathcal{T}_n > T\}$. Then

$$\{Z_n \leq 0, \forall n \text{ s.t. } \mathcal{T}_n \leq T\} = \left\{Z_1 \leq 0, \ldots, Z_{\theta(T)-1} \leq 0\right\} = \{\xi \geq \theta(T)\},$$

and the latter event amounts to $\mathcal{T}_\xi > T$.
One can conclude from (3.17) in view of the results in [Sin92a] where the Laplace transform of $(\xi, \mathcal{T}_\xi)$ is given in terms of the probabilities $\mathbb{P}\left(\mathcal{T}_n = k, Z_n < 0\right)$ for $k, n \in \mathbb{N}$. This is a generalisation of Sparre-Andersen's formula (1.5) to bivariate random walks. The result $\mathbb{P}\left(\mathcal{T}_\xi > T\right) \asymp T^{-1/4}$ then follows by Tauberian arguments, see [Vys12a] for the details. Let us also remark that the preceding arguments imply that the rather sparse set of maxima $\{M_1, \ldots, M_N\} \subseteq \{Y_1, \ldots, Y_N\}$ of the random walk $(Y_n)_{n\geq 1}$ suffices to give the right order for the upper bound.
In the proof of Theorem 3.1.2, we needed stretched exponential moments in order to ensure that the probability of a gap of size $(C \log N)^\gamma$ in the set $\{0, |S_1|, \ldots, |S_N|\}$ is asymptotically irrelevant, i.e. of lower order than $N^{-1/2}$. This allowed us (at the cost of a lower order term) to consider the supremum of the process X over the whole interval from 0 to the maximum of the absolute value of the random walk up to time

N instead of the set $\{0, |S_1|, \dots, |S_N|\}$. In contrast to the method in [Vys12a] that is very specific to the random walk case, the technique presented here is also applicable to other processes such as fractional Brownian motion (Theorem 3.1.3).
In (3.12), we have seen that the probability of a gap of size $C(\log N)^{1/\beta}$ up to time N can be made of arbitrarily small polynomial order by increasing the constant C under the assumption that $S_1 \in \mathcal{X}(\beta)$. However, if we only assume that $\mathbb{E}[|S_1|^p]$ is finite for some $p \geq 2$, it does not seem easy to get a polynomial upper bound on this probability. Moreover, it is easy to see that a gap of size $(\log N)^\gamma$ is much more likely in that case. For simplicity, assume that $\mathbb{E}[S_1^2] < \infty$ and that $\mathbb{P}(S_1 > x) \asymp x^{-p}$ as $x \to \infty$ with $p > 2$. The event that the random walk jumps above L at once and stays above the level S_1 after that up to time N clearly implies that the set $\{0, |S_1|, \dots, |S_N|\}$ has a gap of size L. Hence, the probability of a gap of size L is bounded below by

$$\mathbb{P}\left(S_1 \geq L, \sup_{n=2,\dots,N} S_n - S_1 \geq 0\right) = \mathbb{P}(S_1 \geq L)\, \mathbb{P}\left(\sup_{n=1,\dots,N-1} S_n \geq 0\right),$$

and if $L = C(\log N)^\gamma$, the product is of order $(\log N)^{-p\gamma} N^{-1/2} = N^{-1/2+o(1)}$. However, it does not seem easy to find an upper bound of this order.
Moreover, as we have seen in Chapter 2, even for a deterministic increasing sequence $(s_n)_{n\geq 1}$ such that $s_N \to \infty$ as $N \to \infty$ and a Brownian motion $(B_t)_{t\geq 0}$, it is not obvious to find conditions on $(s_n)_{n\geq 1}$ such that

$$\mathbb{P}(B(s_n) \leq 1, \forall n = 1, \dots, N) \asymp \mathbb{P}(B_t \leq 1, t \in [0, s_N]) \asymp s_N^{-1/2}.$$

Let us now prove Theorem 3.1.3 for fractional Brownian motion.
Proof of Theorem 3.1.3. Let X be a FBM with Hurst index H, and recall that $\mathbb{P}(X_t \leq 1, \forall t \in [0, T]) = T^{-(1-H)+o(1)}$, see [Mol99] or Section 1.2.3 here.
Upper bound: We can almost repeat the proof of Theorem 3.1.2. It suffices again to prove the upper bound for the case that the inner process is a random walk $(S_n)_{n\geq 1}$. With $c = \mathbb{E}[\sup\{X_t : t \in [0,1]\}]$, recall that

$$\mathbb{P}\left(\sup_{t\in[0,1]} |X_t| > (4\log N)^{1/2}\right) = 2\mathbb{P}\left(\sup_{t\in[0,1]} X_t > (4\log N)^{1/2}\right)$$
$$\leq C_1 \exp\left(-((4\log N)^{1/2} - c)^2/2\right) = N^{-2+o(1)},$$

by the Gaussian concentration inequality (see e.g. [LT91, Section 3.1]), which is the equivalent of (3.10).
Write $S_N^* := \max\{|S_1|, \dots, |S_N|\}$. Since $(X_{t+T} - X_T)_{t\in[0,T]}$ and $(X_{T-t} - X_T)_{t\in[0,T]}$ are equal in law to $(X_t)_{t\in[0,T]}$, we can proceed as in the proof of Theorem 3.1.2 to obtain that

$$\mathbb{P}\left(\sup_{n=1,\dots,N} X(|S_n|) \leq 1\right) \leq \mathbb{P}\left(X_t \leq C(\log N)^{1/\beta+1/2}, \forall t \in [0, S_N^*]\right) + o(N^{-1}).$$

Set $g_N := C(\log N)^{1/\beta+1/2}$. If $\mathbb{E}[S_1] = 0$, in view of (3.15) and the self-similarity of X, we obtain that

$$\begin{aligned}\mathbb{P}(X_t \le g_N, \forall t \in [0, S_N^*]) &\le \mathbb{P}\left(X_t \le g_N, \forall t \in [0, \sqrt{N/\log N}]\right) + o(N^{-1}) \\ &= \mathbb{P}\left(X_t \le 1, \forall t \in [0, g_N^{-1/H}\sqrt{N/\log N}]\right) + o(N^{-1}) \\ &= (g_N^{-1/H}\sqrt{N/\log N})^{-(1-H)+o(1)} + o(N^{-1}) = N^{-(1-H)/2+o(1)}.\end{aligned}$$

If $\mathbb{E}[S_1] \neq 0$, a similar argument yields the upper bound. The proof of the lower bound poses no difficulty and is omitted. □

Remark 3.3.5. In Theorem 3.1.2 and 3.1.3, the outer process X had stationary increments in both cases. One might wonder if this assumption can be relaxed. In view of Theorem 3.1.1, one might guess that if X has a survival exponent $\theta > 0$ and $\mathbb{E}[S_1] = 0$, it would follow that

$$\mathbb{P}\left(\sup_{n=1,\dots,N} X(|S_n|) \le 1\right) = N^{-\theta/2+o(1)}, \quad N \to \infty,$$

under suitable moment conditions. However, this turns out to be false in general. As an example, consider a sequence $\tilde{X}_1, \tilde{X}_2, \dots$ of independent random variables with $\mathbb{P}\left(\tilde{X}_n = 2\right) = 1 - \mathbb{P}\left(\tilde{X}_n = 0\right) = 1/(n+1)$ for $n \ge 1$ and define $X = (X_t)_{t\ge0}$ by

$$X_t = \tilde{X}_n \quad \text{if } t = (2n-1)/2 \text{ for some } n \in \mathbb{N}, \quad X_t = 0 \quad \text{else.}$$

Obviously, X does not have stationary increments. Moreover, it is not hard to check that

$$\mathbb{P}\left(\sup_{t\in[0,T]} X_t \le 1\right) \asymp \mathbb{P}\left(\tilde{X}_1 = 0, \dots, \tilde{X}_{\lfloor T\rfloor} = 0\right) = \prod_{n=1}^{\lfloor T\rfloor}(1 - 1/(n+1)) \asymp T^{-1}.$$

If $(S_n)_{n\ge1}$ is a symmetric simple random walk, one has by construction that $X(|S_n|) = 0$ for all n, i.e. $\mathbb{P}(X(|S_n|)) \le 1, \forall n \ \ge 1) = 1$.

3.4 Two-sided processes

In Sections 3.2 and 3.3, the outer process $X = (X_t)_{t\ge0}$ had the index set $[0, \infty)$, so it was only possible to evaluate X over the range of the absolute value of the inner process Y. In this section, we work with two-sided processes $X = (X_t)_{t\in\mathbb{R}}$ allowing us to consider the one-sided exit problem for the process $X \circ Y$.

In Section 3.4.1, we assume that X is a two-sided process defined in (3.2) and that the inner process Y is a self-similar continuous process before turning to the case of random walks and Lévy processes in Section 3.4.2.

3.4.1 Continuous self-similar processes

Let us first prove Theorem 3.1.4. As a corollary, we obtain the survival exponent of iterated Brownian motions and iterated fractional Brownian motions.

Proof of Theorem 3.1.4.

The lower bound can be proved as in Theorem 3.1.1, so we only give the proof of the upper bound. Denote by I and M the infimum and maximum process of Y, i.e. $I_t = \inf_{u\in[-t,t]} Y_u$ and $M_t = \sup_{u\in[-t,t]} Y_u$. By assumption, we can choose a constant C such that for all $T > 0$

$$\mathbb{P}\left(X_t \leq 1, \forall t \in [-T,T]\right) \leq C\, T^{-\theta}.$$

Since Y is independent of X and has continuous paths, we have

$$\begin{aligned}\mathbb{P}\left(X(Y_t) \leq 1, \forall t \in [0,T]\right) &= \mathbb{P}\left(X_t \leq 1, \forall t \in [I_T, M_T]\right) \\ &\leq \mathbb{P}\left(X_t \leq 1, \forall t \in [-(|I_T| \wedge M_T), |I_T| \wedge M_T]\right) \\ &\leq C\,\mathbb{E}\left[(|I_T| \wedge M_T)^{-\theta}\right] = C\,T^{-\theta H}\mathbb{E}\left[(|I_1| \wedge M_1)^{-\theta}\right].\end{aligned}$$

Now $\mathbb{E}\left[(|I_1| \wedge M_1)^{-\theta}\right] \leq \mathbb{E}\left[(-I_1)^{-\theta}\right] + \mathbb{E}\left[M_1^{-\theta}\right]$, and the last expectation is finite by assumption. This completes the proof. □

Let us apply Theorem 3.1.4 to iterated fractional Brownian motions.

Corollary 3.4.1. *Let $(Y_n(t))_{t\in\mathbb{R}}$ be a FBM with Hurst index H_n for every $n \geq 1$, all independent. For $t \in \mathbb{R}$, set $X_1(t) := Y_1(t)$ and $X_n(t) := X_{n-1} \circ Y_n(t)$. Let $\theta_1 = 1$ and $\theta_n = H_2 \cdot \ldots \cdot H_n$. It holds that*

$$\mathbb{P}\left(X_n(t) \leq 1, \forall t \in [-T,T]\right) = T^{-\theta_n + o(1)}, \qquad T \to \infty, \quad n \geq 1.$$

Proof. By [Mol99, Theorem 3], if B^H is a FBM with Hurst index H, it holds that

$$\mathbb{P}\left(B^H(t) \leq 1, \forall t \in [-T,T]\right) = T^{-1+o(1)}, \quad T \to \infty.$$

In view of the self-similarity, this amounts to $\mathbb{P}\left(\sup_{t\in[-1,1]} B^H(t) \leq \epsilon\right) = \epsilon^{1/H+o(1)}$ as $\epsilon \downarrow 0$. Hence, by symmetry, $\mathbb{E}\left[\left(-\inf_{t\in[-1,1]} B^H(t)\right)^{-\eta}\right] + \mathbb{E}\left[\left(\sup_{t\in[-1,1]} B^H(t)\right)^{-\eta}\right] < \infty$ for any $\eta < 1/H$ by Lemma 3.2.2. Since $\theta_n \leq 1$ for all n, the assertion follows now easily by induction in view of Theorem 3.1.4. □

If we know the precise behaviour of $\mathbb{P}\left(X_t \leq 1, \forall t \in [-T_1, T_2]\right)$ for $T_1, T_2 \to \infty$, we can get a stronger result than Theorem 3.1.4. In particular, if X has independent branches such as in the case of two-sided Brownian motion, the next theorem allows us to determine the exact asymptotics of the survival probability (see Corollary 3.4.4).

Theorem 3.4.2. *Let* $(X_t)_{t\geq 0}$ *be a stochastic process such that*

$$\mathbb{P}\left(X_t^+ \leq 1, \forall t \in [-T_1, T_2]\right) \sim c\, T_1^{-\theta^-} T_2^{-\theta^+}, \qquad T_1, T_2 \to \infty.$$

Let $(Y_t)_{t\in\mathbb{R}}$ *denote an independent self-similar process of index* H *with* $Y_0 = 0$ *and continuous paths such that for some* $\rho > \theta^+ + \theta^-$*, it holds that*

$$\mathbb{P}\left(Y_t \geq -\epsilon, \forall t \in [-1,1]\right) + \mathbb{P}\left(Y_t \leq \epsilon, \forall t \in [-1,1]\right) \leq C\epsilon^\rho, \quad \epsilon \downarrow 0. \tag{3.18}$$

Then $A := \mathbb{E}\left[|\inf\{Y_t, t \in [-1,1]\}|^{-\theta^-} \left(\sup\{Y_t : t \in [-1,1]\}\right)^{-\theta^+}\right] < \infty$ *and*

$$\mathbb{P}\left(X(Y_t) \leq 1, \forall t \in [-T,T]\right) \sim A\, c\, T^{-H(\theta^+ + \theta^-)}, \quad T \to \infty.$$

Proof. Let $I_T := \inf\{Y_t : t \in [-T,T]\}$ and $M_T := \sup_{t\in[-T,T]} Y_t$. By Lemma 3.2.2, we know that $\mathbb{E}\left[|I_1|^{-\eta}\right] + \mathbb{E}\left[M_1^{-\eta}\right] < \infty$ for $\eta \in (0,\rho)$. The finiteness of A then follows from Lemma 3.4.3 below. The rest of the proof is analogous to the one of Theorem 3.1.1. We only sketch the proof of the upper bound. For $\epsilon > 0$, we can find T_0 such that for all $T_1, T_2 \geq T_0$, we have that

$$\mathbb{P}\left(X_t \leq 1, \forall t \in [-T_1, T_2]\right) \leq (c+\epsilon)\, T_1^{-\theta^-} T_2^{-\theta^+}.$$

Using the independence of X and Y, we see that

$$\begin{aligned}
&\mathbb{P}\left(X(Y_t) \leq 1, \forall t \in [-T,T]\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left(X_t^+ \leq 1, t \in [I_T, M_T] | \mathcal{F}_T^Y\right)\right] \\
&\quad \leq \mathbb{P}\left(I_T \geq -T_0\right) + \mathbb{P}\left(M_T \leq T_0\right) + (c+\epsilon)\mathbb{E}\left[|I_T|^{-\theta^-} M_T^{-\theta^+}\right].
\end{aligned}$$

Next, note that $\limsup_{T\to\infty} T^{H(\theta^+ + \theta^-)} \left(\mathbb{P}\left(I_T \geq -T_0\right) + \mathbb{P}\left(M_T \leq T_0\right)\right) = 0$. Indeed, since $\rho > \theta^+ + \theta^-$, this follows in view of (3.18):

$$\mathbb{P}\left(I_T \geq -T_0\right) + \mathbb{P}\left(M_T \leq T_0\right) = \mathbb{P}\left(I_1 \geq -T_0 T^{-H}\right) + \mathbb{P}\left(M_1 \leq T_0 T^{-H}\right) \leq C T_0^\rho T^{-\rho H}.$$

Hence, writing $\theta := \theta^+ + \theta^-$ and noting that$\mathbb{E}\left[|I_T|^{-\theta^-} M_T^{-\theta^+}\right] = T^{-H\theta}\mathbb{E}\left[|I_1|^{-\theta^-} M_1^{-\theta^+}\right]$, we conclude that

$$\limsup_{T\to\infty} T^{H\theta}\, \mathbb{P}\left(X(Y_t) \leq 1, \forall t \in [-T,T]\right) \leq (c+\epsilon)\mathbb{E}\left[|I_1|^{-\theta^-} M_1^{-\theta^+}\right].$$

Letting $\epsilon \downarrow 0$ establishes the desired upper bound. □

The following lemma is stated separately for better readability and is needed in the preceding proof.

Lemma 3.4.3. *Let* X_1, X_2 *denote nonnegative random variables with* $\mathbb{E}\left[X_i^{\alpha_i}\right] < \infty$ *(*$i = 1,2$*) for some* $\alpha_1, \alpha_2 > 0$*. Then for* $\beta_i \in (0, \alpha_i)$*, it holds that* $\mathbb{E}\left[X_1^{\beta_1} X_2^{\beta_2}\right] < \infty$*.*

Proof. We have that $\mathbb{E}\left[X_1^{\beta_1}X_2^{\beta_2}\right] \leq \mathbb{E}\left[X_1^{\beta_1}\right] + \mathbb{E}\left[X_2^{\beta_2}\right] + \mathbb{E}\left[X_1^{\beta_1}X_2^{\beta_2}1_{\{X_1>1,X_2>1\}}\right]$. It suffices to show that the last expectation is finite. If $1/p + 1/q + 1/r = 1$, we deduce from a generalised version of Hölder's theorem that

$$\mathbb{E}\left[X_1^{\beta_1}X_2^{\beta_2}1_{\{X_1>1,X_2>1\}}\right] \leq \mathbb{E}\left[X_1^{\beta_1 p}\right]^{1/p}\mathbb{E}\left[X_2^{\beta_2 q}\right]^{1/q}\mathbb{E}\left[1_{\{X_1>1,X_2>1\}}\right]^{1/r}.$$

With $p = \alpha_1/\beta_1 > 1$, $q = \alpha_2/\beta_2 > 1$ and appropriate r, the claim follows. □

Theorem 3.4.2 allows us to state the precise behaviour of the survival probability for n-times iterated Brownian motion.

Corollary 3.4.4. *Let $(B_n)_{n\geq 1}$ denote a sequence of independent two-sided Brownian motions. Set $W_t^{(1)} := B_1(t)$ and $W_t^{(n)} := B_n(W^{(n-1)}(t))$. For every $n \geq 1$, let $\theta_n := 2^{-(n-1)}$. It holds that*

$$\mathbb{P}\left(W_t^{(n)} \leq 1, \forall t \in [-T,T]\right) \sim \frac{2}{\pi}c_n T^{-\theta_n}, \qquad T\to\infty, \quad n\geq 1,$$

where $c_1 = 1$ and for $n \geq 2$,

$$c_n = \mathbb{E}\left[\left|\inf\left\{W_t^{(n-1)} : t\in[-1,1]\right\}\right|^{-1/2}\left(\sup\left\{W_t^{(n-1)} : t\in[-1,1]\right\}\right)^{-1/2}\right] < \infty.$$

Proof. If B is a two-sided Brownian motion, note that the branches are independent Brownian motions by (1.9). Hence, we have that

$$\mathbb{P}\left(\sup_{t\in[-T_1,T_2]} B_t \leq 1\right) = \mathbb{P}\left(\sup_{t\in[0,T_1]} B_t \leq 1\right)\mathbb{P}\left(\sup_{t\in[0,T_2]} B_t \leq 1\right) \sim \frac{2}{\pi}T_1^{-1/2}T_2^{-1/2},$$

whenever $T_1, T_2 \to \infty$. The assertion is therefore clear for $n = 1$. By induction, if the assertion holds for some $n \geq 1$, we can apply Theorem 3.4.2 with $X = B_{n+1}$ and $Y = W^{(n)}$. Indeed, $W^{(n)}$ is 2^{-n}-selfsimilar. Moreover, since $W^{(n)}$ is symmetric, and by the induction hypothesis, we have that

$$\mathbb{P}\left(W_t^{(n)} \geq -\epsilon, \forall t\in[-1,1]\right) + \mathbb{P}\left(W_t^{(n)} \leq \epsilon, \forall t\in[-1,1]\right) = 2\mathbb{P}\left(W_t^{(n)} \leq \epsilon, \forall t\in[-1,1]\right)$$
$$= 2\mathbb{P}\left(W_t^{(n)} \leq 1, \forall t \in [-\epsilon^{-2^n}, \epsilon^{-2^n}]\right) \sim (4/\pi)c_n\epsilon^{2^n\theta_n} = (4/\pi)c_n\epsilon^2, \qquad \epsilon\downarrow 0.$$

Hence, we infer from Theorem 3.4.2 ($c = 2/\pi, \theta^+ = \theta^- = 1/2, \rho = 2, H = 2^{-n}$) that

$$c_{n+1} = \mathbb{E}\left[\left|\inf\left\{W_t^{(n)} : t\in[-1,1]\right\}\right|^{-1/2}\left(\sup\left\{W_t^{(n)} : t\in[-1,1]\right\}\right)^{-1/2}\right] < \infty$$

and

$$\mathbb{P}\left(W_t^{(n+1)} \leq 1, \forall t\in[-T,T]\right) \sim (2/\pi)c_{n+1}T^{-2^{-n}}, \quad T\to\infty.$$

□

3.4.2 Two-sided Lévy processes at random walk or Lévy times

Let us now consider the one-sided exit problem for the process $(X(S_n))_{n\geq 0}$, where S is again a random walk and X is a two-sided Lévy process, i.e. the branches of X are independent Lévy processes. Theorem 3.1.5 shows that the survival exponent is 1/2 under suitable integrability conditions regardless of the sign of $\mathbb{E}[S_1]$ in contrast to Theorem 3.1.2, see Remark 3.4.5 below.

We now give a proof of Theorem 3.1.5 for the case that the inner process Y is a random walk. As before, the upper bound for the case that Y is a Lévy process follows immediately, whereas the proof of the lower bound is similar and is omitted.

Proof of Theorem 3.1.5.

The lower bound can be established as in the proof of Theorem 3.1.2 if $\mathbb{E}[Y_1] = 0$. If $\mathbb{E}[Y_1] > 0$ (say), using that $\inf_{n\geq 1} S_n$ is a finite random variable a.s., the result follows along similar lines.

The proof of the upper bound is also similar to that of Theorem 3.1.2, though it is a bit more technical. Let $p_N := \mathbb{P}\left(\sup_{n=1,\dots,N} X(S_n) \leq 1\right)$. Repeating the steps before (3.9), we obtain that

$$p_N \leq \mathbb{P}\left(\bigcap_{n=0}^{N} \left\{X_t \leq 1 + L(2\log N)^{1/\alpha}, \forall t \in [S_n - L, S_n + L]\right\}\right) \tag{3.19}$$
$$+ 2L(N+1)\left(\mathbb{P}\left(\sup_{t\in[0,1]} \left|X_t^+\right| \geq (2\log N)^{1/\alpha}\right) + \mathbb{P}\left(\sup_{t\in[0,1]} \left|X_t^-\right| \geq (2\log N)^{1/\alpha}\right)\right).$$

Take $L = C(\log N)^{1/\beta}$, and let $\tilde{A}_N$ denote that the event that the set $\{0, S_1, \dots, S_N\}$ contains a gap larger than L. Let $g_N := 2^{1+1/\alpha}C(\log N)^{1/\alpha+1/\beta}$. Let

$$M_n := \max\{0, S_1, \dots, S_n\}, \quad I_n := \min\{0, S_1, \dots, S_n\}.$$

Since $X_1^+, X_1^- \in \mathcal{X}(\alpha)$, we get in view of (3.10) and (3.11) that

$$\begin{aligned} p_N &\leq \mathbb{P}\left(X_t \leq g_N, \forall\, t \in [I_N, M_N]\right) + \mathbb{P}\left(\tilde{A}_N\right) + C_2(\log N)^{1/\beta}N^{-1} \\ &\leq \mathbb{P}\left(X_t \leq g_N, \forall\, t \in [I_N, M_N]\right) + o(N^{-1/2}). \end{aligned}$$

The last inequality follows from an estimate on $\mathbb{P}\left(\tilde{A}_N\right)$ as in (3.12).

Let us again consider two cases:

Case $\mathbb{E}[Y_1] \neq 0$*:* Assume first that $\mathbb{E}[Y_1] > 0$. If $\mathbb{E}[Y_1] < 0$, the proof is almost identical. Note that

$$\begin{aligned} \mathbb{P}\left(X_t \leq g_N, \forall\, t \in [0, M_N]\right) &\leq \mathbb{P}\left(M_N \leq \delta N\right) + \mathbb{P}\left(X_t \leq g_N, \forall\, t \in [(2\log N^{1/\alpha}), \delta N]\right) \\ &\leq C_2 N^{-1} + C_3 g_N N^{-1/2} \precsim N^{-1/2}(\log N)^{1/\alpha+1/\beta}, \end{aligned}$$

where we have used that $\mathbb{P}\left(M_N \leq \delta N\right) \leq \mathbb{P}\left(S_N \leq \delta N\right) = o(N^{-1})$ for δ small enough, and Remark 3.3.3 in the second inequality.

Case $\mathbb{E}[Y_1] = 0$: Let us finally consider the case $\mathbb{E}[Y_1] = 0$. With g_N as above, it suffices to show that

$$h_N := \mathbb{P}\left(X_t \le g_N, \forall\, t \in [I_N, M_N]\right) \precsim N^{-1/2+o(1)}, \quad N \to \infty.$$

Let $f_N := \sqrt{N/\log N}$, $N \ge 2$. Note that

$$\begin{aligned} h_N &\le \mathbb{P}\left(\sup_{n=1,\dots,N} |S_n| \le f_N\right) + \mathbb{P}\left(M_N \le f_N, -I_N > f_N, X_t \le g_N, \forall t \in [I_N, M_N]\right) \\ &\quad + \mathbb{P}\left(M_N > f_N, -I_N \le f_N, X_t \le g_N, \forall t \in [I_N, M_N]\right) \\ &\quad + \mathbb{P}\left(M_N > f_N, -I_N > f_N, X_t \le g_N, \forall t \in [I_N, M_N]\right) \\ &=: J_1(N) + J_2(N) + J_3(N) + J_4(N). \end{aligned}$$

First, recall that $J_1(N) = o(N^{-1/2})$ (cf. (3.15)). It remains to estimate the terms J_2 and J_4. The term J_3 can be dealt with analogously to J_2.
Step 1: Note that

$$\begin{aligned} J_2(N) \le \mathbb{P}&\left(M_N \le N^{1/4}, -I_N > f_N, \sup_{t\in[I_N,M_N]} X_t \le g_N\right) \\ &+ \mathbb{P}\left(N^{1/4} \le M_N \le f_N, -I_N > f_N, \sup_{t\in[I_N,M_N]} X_t \le g_N\right) =: K_{2,1}(N) + K_{2,2}(N). \end{aligned}$$

Let us now find upper bounds for $K_{2,j}$ for $j = 1, 2$. First, note that

$$K_{2,1}(N) \le \mathbb{P}\left(M_N \le N^{1/4}\right) \mathbb{P}\left(\sup_{t\in[-f_N,0]} X_t \le g_N\right).$$

Applying Lemma 3.3.2 with $\tilde{f}_N := N^{1/4}$ to the first factor and Remark 3.3.3 to the second, we conclude that

$$K_{2,1}(N) \precsim N^{-1/4}\, g_N f_N^{-1/2} \asymp N^{-1/2}\, (\log N)^{1/\alpha+1/\beta+1/4}. \tag{3.20}$$

Let us now find an upper bound on $K_{2,2}$. Set $a(k) := \sum_{l=1}^{k} 2^{-(l+1)} = (1-2^{-k})/2$, $k \ge 1$. Since $a(N) \to 1/2$, we can find $\gamma(N)$ such that $N^{a(\gamma(N))} \ge f_N = \sqrt{N/\log N}$. Indeed, this just amounts to

$$a(\gamma(N)) = \frac{(1-2^{-\gamma(N)})}{2} \ge \frac{\log f_N}{\log N} = \frac{1}{2} - \frac{\log\log N}{2\log N}, \tag{3.21}$$

i.e.

$$\gamma(N) \ge \frac{1}{\log 2}\, \log\left(\frac{\log N}{\log\log N}\right).$$

Hence, it suffices to set $\gamma(N) := \lceil (\log\log N)/\log 2 \rceil$.
Next, note that $\left\{N^{1/4} \le M_N \le N^{1/2}/f_N\right\} \subseteq \left\{N^{a(1)} \le M_N \le N^{a(\gamma(N))}\right\}$, so

$$\begin{aligned} K_{2,2}(N) &\le \sum_{k=1}^{\gamma(N)-1} \mathbb{P}\left(N^{a(k)} \le M_N \le N^{a(k+1)}, -I_N > f_N, \sup_{t\in[I_N,M_N]} X_t \le g_N\right) \\ &\le \sum_{k=1}^{\gamma(N)-1} \mathbb{P}\left(N^{a(k)} \le M_N \le N^{a(k+1)}\right) \mathbb{P}\left(\sup_{t\in[-f_N,N^{a(k)}]} X_t \le g_N\right) \\ &\le \mathbb{P}\left(\sup_{t\in[0,f_N]} X_t^- \le g_N\right) \sum_{k=1}^{\gamma(N)-1} \mathbb{P}\left(M_N \le N^{a(k+1)}\right) \mathbb{P}\left(\sup_{t\in[0,N^{a(k)}]} X_t^+ \le g_N\right). \end{aligned}$$

In view of Lemma 3.3.2, we can find constants C_1 and N_0 such that for $N \ge N_0$

$$\mathbb{P}\left(M_N \le N^{a(k+1)}\right) \le C_1\, N^{a(k+1)-1/2}, \qquad k = 1, 2, \ldots$$

Similarly, for all N large enough,

$$\mathbb{P}\left(\sup_{t\in[0,N^{a(k)}]} X_t^+ \le g_N\right) \le C_2\, g_N N^{-a(k)/2}, \qquad k = 1, 2, \ldots$$

Hence, for N large enough, we obtain that

$$\begin{aligned} K_{2,2}(N) &\le C_3 g_N/\sqrt{f_N} \sum_{k=1}^{\gamma(N)-1} N^{a(k+1)-1/2}\, g_N N^{-a(k)/2} \\ &= C_3 g_N^2 (\log N)^{1/4} N^{-1/4} \sum_{k=1}^{\gamma(N)-1} N^{a(k+1)-a(k)/2-1/2} \\ &= C_4 (\log N)^{2/\alpha+2/\beta+1/4} (\gamma(N)-1) N^{-1/2}, \end{aligned}$$

since $a(k+1) - a(k)/2 = 1/4$. By definition of $\gamma(N)$, we arrive at $K_{2,2}(N) \precsim (\log\log N)\,(\log N)^{2/\alpha+2/\beta+1/4}\, N^{-1/2}$. Combining this with (3.20), it follows that

$$J_2(N) \precsim (\log\log N)\,(\log N)^{2/\alpha+2/\beta+1/4}\, N^{-1/2}, \quad N \to \infty. \tag{3.22}$$

Step 3:
Finally, with g_N as above, note that

$$\begin{aligned} J_4(N) &\le \mathbb{P}\left(\sup_{t\in[-f_N,f_N]} X_t \le g_N\right) = \mathbb{P}\left(\sup_{t\in[0,f_N]} X_t^- \le g_N\right) \mathbb{P}\left(\sup_{t\in[0,f_N]} X_t^+ \le g_N\right) \\ &\precsim (g_N/\sqrt{f_N})^2 \asymp (\log N)^{2/\alpha+2/\beta+1/2}\, N^{-1/4}. \end{aligned}$$

□

Remark 3.4.5. The proof reveals that the survival exponent is equal to 1/2 no matter if $\mathbb{E}[Y_1] = 0$ or not for quite different reasons. If $\mathbb{E}[Y_1] > 0$, $S_N/N \to \mathbb{E}[Y_1]$ by the law of large numbers, so the random walk diverges to $+\infty$ with speed N and the survival probability is determined by the right branch X^+ of X.
If $\mathbb{E}[Y_1] = 0$, the random walks oscillates and typical fluctuations are of order $\pm\sqrt{N}$. The survival probability up to time N is therefore approximately equal to the probability that both X^+ and X^- stay below 1 until time $\sqrt{N}$. By independence of X^+ and X^-, this probability is equal to the product of these two probabilities which are each of order $N^{-1/4}$.

Chapter 4

Persistence of autoregressive processes

In the following chapter, we study persistence of autoregressive processes. Recall that an autoregressive process X of order $p \in \mathbb{N}$ (AR(p) in short) is defined as

$$X_n = \sum_{k=1}^{p} a_k X_{n-k} + Y_n, \quad n \geq 1, \tag{4.1}$$

with the convention that $X_n = 0$ for $n \leq 0$. Here $Y_1, Y_2, \dots$ denote a sequence of i.i.d. random variables, often referred to as *innovations* in this context, and $a_1, \dots, a_p \in \mathbb{R}$. One verifies that $X_n = \sum_{k=1}^{n} c_{n-k} Y_k$, where

$$c_n = 0, \quad n < 0, \qquad c_0 = 1, \qquad c_n = \sum_{k=1}^{p} a_k c_{n-k}, \quad n \geq 1.$$

In particular, X_n is again a weighted sum of i.i.d. random variables, but in contrast to the processes considered in Chapter 2, the weights now depend on n as well.
Let again

$$p_N(x) := \mathbb{P}\left(X_n \leq x, \forall n = 1, \dots, N\right), \quad N \geq 1, x \in \mathbb{R},$$

denote the persistence probability of X. We write p_N instead of $p_N(0)$ in the sequel. Persistence of AR(1)-processes with $a_1 \in (0,1)$ has been studied in [NK08], and it is shown that p_N decays at least exponentially under a mild moment condition. Note that the AR(1)-process X with $X_n = \rho X_{n-1} + Y_n$ is given by $X_n = \sum_{k=1}^{n} \rho^{n-k} Y_k$, and therefore, if $\rho > 0$, we clearly have that

$$\mathbb{P}\left(X_n \leq 0, n = 1, \dots, N\right) = \mathbb{P}\left(\sum_{k=1}^{n} \rho^{-k} Y_k \leq 0, n = 1, \dots, N\right).$$

In other words, the persistence probability $p_N(0)$ of an AR(1)-process with $a_1 = \rho > 0$ is equal to that of a weighted random walk with weight function $\sigma(x) := \rho^{-x}$ defined in Section 2.1. In particular, if $\rho \in (0,1)$, σ increases exponentially, and bounds on the

exponential rate of decay for Gaussian innovations were stated in Section 2.2.4.
To the author's knowledge, persistence of other AR-processes has not been studied in the literature, so taken as a whole, very little is known about persistence of AR-processes. As noted in [DDG12], this would be of much interest in view of the frequent appearance of AR-processes and persistence probabilities in physical and ecomomic models.

The remainder of this chapter is organised as follows. We begin by presenting some preliminaries on AR-processes in Section 4.1 before presenting the main results for AR(2) processes in Section 4.2. In Section 4.3, we state general conditions ensuring that p_N decays exponentially or at least faster than any polynomial. Special emphasis is put on the case that $(c_n)_{n\geq 0}$ is absolutely summable and AR(2)-processes. We also prove exponential lower bounds for certain classes of AR-processes. We then determine the pairs (a_1, a_2) where the persistence probability decays polynomially for AR(2)-processes in Section 4.4, before briefly treating the case that p_N converges to a positive constant in Section 4.5.

4.1 Preliminaries

We begin by recalling a few facts about autoregressive processes that we need in the sequel. For more details, the reader may consult [BD87, Chapter 3].
In order to determine the coefficients $(c_n)_{n\geq 0}$ corresponding to an AR(p)-process, one needs to solve the linear difference equation

$$c_n = a_1 c_{n-1} + \dots a_p c_{n-p}, \quad n \geq p,$$

with initial conditions

$$c_0 = 1, \quad c_1 = a_1 c_0, \quad c_2 = a_1 c_1 + a_2 c_0, \quad \dots, \quad c_{p-1} = a_1 c_{p-2} + \dots + a_{p-1} c_0.$$

Solving this equation amounts to finding the roots $s_1, \dots, s_p \in \mathbb{C}$ of the characteristic polynomial $f_p(\cdot)$, given by $f_p(x) := x^p - \sum_{k=1}^p a_k x^{p-k}, x \in \mathbb{R}$.
In the sequel, special emphasis is put on AR(2)-processes. In that case, the roots s_1, s_2 of $f_2(\lambda) = \lambda^2 - a_1\lambda - a_2$ are given by

$$s_1 := (a_1 + h)/2, \quad s_2 := (a_1 - h)/2, \quad h := \sqrt{a_1^2 + 4a_2} \in \mathbb{C}. \tag{4.2}$$

Taking into account the inital conditions $c_0 = 1, c_1 = a_1$, one can show that

$$c_n = \begin{cases} h^{-1}\left(s_1^{n+1} - s_2^{n+1}\right), & n \geq 0, \quad a_1^2 + 4a_2 \neq 0, \\ (a_1/2)^n (n+1), & n \geq 0, \quad a_1^2 + 4a_2 = 0. \end{cases} \tag{4.3}$$

If $a_1^2 + 4a_2 < 0$, writing $s_1 = re^{i\varphi}$ and $s_2 = \overline{s}_1 = re^{-i\varphi}$ in polar form, elementary manipulations show that the solution is given by

$$c_n = |a_2|^{(n+1)/2} \cdot 2\sin((n+1)\varphi)/\tilde{h}, \tag{4.4}$$

where

$$\tilde{h} = \sqrt{-(a_1^2 + 4a_2)} > 0, \quad \varphi = \begin{cases} \arctan(\tilde{h}/a_1) \in (0, \pi/2), & a_1 > 0, \\ \pi/2, & a_1 = 0, \\ \pi + \arctan(\tilde{h}/a_1) \in (\pi/2, \pi), & a_1 < 0. \end{cases}$$

Note that the behaviour of the sequence $(c_n)_{n \geq 0}$ may change significantly for different values of (a_1, a_2): it can grow or decay exponentially, oscillate, converge to a constant, grow polynomially,... For later reference, let us remark that $c_n \to 0$ if and only if $\max\{|s_1|, |s_2|\} < 1$, which is easily seen to be equivalent to the conditions

$$a_1 + a_2 < 1, \quad a_2 < 1 + a_1, \quad a_2 > -1, \tag{4.5}$$

see [Ela99, Theorem 2.37].

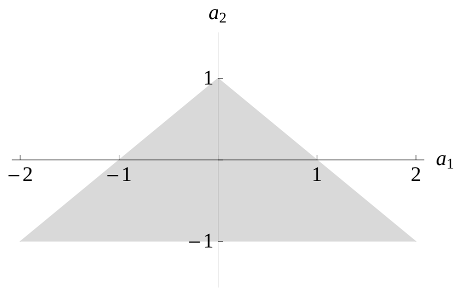

Figure 4.1: The region of parameters (a_1, a_2) where $c_n \to 0$

Remark 4.1.1. The convention that $X_n = 0$ for $n < 0$ is not standard to define autoregressive processes. It is often customary to define AR(p)-processes as follows, see e.g. [BD87, Chapter 3]: If $(Y_n)_{n \in \mathbb{Z}}$ is a sequence of i.i.d. random variables, $X = (X_n)_{n \in \mathbb{Z}}$ is AR(p) if

$$X_n = a_1 X_{n-1} + \cdots + a_p X_{n-p} + Y_n, \quad n \in \mathbb{Z}.$$

Moreover, X is called causal if there exists a deterministic sequence $(c_n)_{n \geq 0}$ with $\sum |c_n| < \infty$ such that $X_n = \sum_{k=0}^{\infty} c_k Y_{n-k}$. From a practical perspective, it is natural to consider only causal processes such that X_n only depends on the past values $Y_n, Y_{n-1}, \ldots$. By [BD87, Theorem 3.1.1], X is causal if and only if the polynomial $p(z) = 1 - a_1 z - \cdots - a_p z^p$ has no zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$. In that case, the coefficients c_n are determined by the relation $\sum_{k=0}^{\infty} c_k z^k = 1/p(z)$ for $|z| \leq 1$. Equating the coefficients of z^k, one easily verifies (or see [BD87, Section 3.3]) that the sequence $(c_n)_{n \geq 0}$ satisfies the same recursion equation with the same initial conditions as above. Hence, if X is a causal AR(p)-process, we can decompose it for $n \geq 1$ in the following way:

$$X_n = \sum_{k=0}^{n-1} c_k Y_{n-k} + \sum_{k=n}^{\infty} c_k Y_{n-k} = \sum_{k=1}^{n} c_{n-k} Y_k + \sum_{k=0}^{\infty} c_{n+k} Y_{-k} = X_n^{(1)} + X_n^{(2)}.$$

Note that $X^{(1)}$ and $X^{(2)}$ are independent and that $X^{(1)}$ is an AR(p)-process in the sense of this chapter. The term $X^{(2)}$ can be seen as a small perturbation for large values of n in general. For instance, if $\mathbb{E}\left[|Y_0|\right] < \infty$, it follows that $X_n^{(2)} \to 0$ in probability, since

$$\mathbb{P}\left(\left|X_n^{(2)}\right| \geq \epsilon\right) \leq \epsilon^{-1}\,\mathbb{E}\left[\left|\sum_{k=0}^{\infty} c_{n+k} Y_{-k}\right|\right] \leq \epsilon^{-1}\,\mathbb{E}\left[|Y_0|\right] \sum_{k=0}^{\infty} |c_{n+k}|$$
$$= \epsilon^{-1}\,\mathbb{E}\left[|Y_0|\right] \sum_{k=n}^{\infty} |c_k| \to 0, \quad n \to \infty.$$

By using the alternative definition in (4.1), we do not have to assume that the c_k are summable in order to define AR-processes, and therefore get a much larger class of processes including, for example, random walks. Moreover, Theorem 4.3.5 below provides a general upper bound on the persistence probability for a class of processes that contains AR-processes in the sense of Brockwell and Davis as a special case.

We will use different methods to prove certain statements about the persistence probability depending on the parameters (a_1, a_2). To this end, set

$$E_1 := \{(a_1, a_2) : a_1 < 0, a_2 > 0, a_2 > 1 + a_1\}, \quad E_2 := (-\infty, 0]^2,$$
$$E_3 := \left\{(a_1, a_2) : a_1 > 0, a_1^2 + 4a_2 < 0\right\}.$$

Figure 4.2 will be helpful to visualise the regions that will be considered separately below.

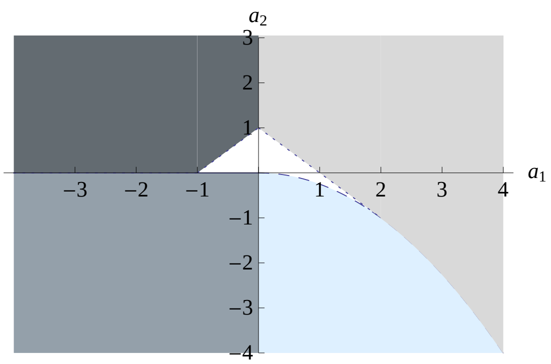

Figure 4.2: The regions E_1, E_2, E_3 and C.

Let us also comment briefly on the dependence of the persistence probability on the barrier x for AR(p)-processes. In principle, the behaviour of the persistence probability can vary significantly for different barriers. An extreme example is an AR(1)-process $Z_n = \rho Z_{n-1} + Y_n$ where $\rho \in (0,1)$ with $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = 1/2$. It is known that $p_N \precsim \exp(-\lambda N)$ for some $\lambda > 0$ (see Theorem 4.3.1 below), whereas $p_N(x) = 1$ for all $x \geq 1/(1-\rho)$ since $|X_n| = \left|\sum_{k=1}^{n} \rho^{n-k} Y_k\right| \leq \sum_{k=0}^{\infty} \rho^k = 1/(1-\rho)$.
On the other hand, if $c_n \geq \delta > 0$ for all $n \geq 0$ and $\mathbb{P}(Y_1 \leq -\epsilon) > 0$ for some $\epsilon > 0$, one

can show that $p_N(x) \asymp p_N$ as $N \to \infty$ for all $x \geq 0$. Indeed, note that if $Y_1 \leq -\epsilon$, it follows that $X_n = c_{n-1}Y_1 + \sum_{k=2}^n c_{n-k}Y_k \leq -\epsilon\delta + \sum_{k=2}^n c_{n-k}Y_k$, so that

$$p_N = \mathbb{P}\left(\sup_{n=1,\dots,N} X_n \leq 0\right) \geq \mathbb{P}(Y_1 \leq -\epsilon)\, \mathbb{P}\left(\sup_{n=2,\dots,N} \sum_{k=2}^n c_{n-k}Y_k \leq \delta\epsilon\right)$$
$$\geq \mathbb{P}(Y_1 \leq -\epsilon)\, p_N(\delta\epsilon).$$

Iteration shows that $p_N \geq \mathbb{P}(Y_1 \leq -\epsilon)^L p_N(L\delta\epsilon)$ for $L = 1, \dots, N$. Hence, if $x \geq 0$, take L with $L\delta\epsilon \geq x$ to get that $\mathbb{P}(Y_1 \leq -\epsilon)^L p_N(x) \leq p_N \leq p_N(x)$ for all N large enough.

4.2 Main results for AR(2) processes

Let us illustrate our main result when X is AR(2), i.e. $X_n = a_1 X_{n-1} + a_2 X_{n-2} + Y_n$ with $(Y_n)_{n\geq 1}$ i.i.d. Recall that $X_n = \sum_{k=1}^n c_{n-k}Y_k$ for $n \geq 1$. We decompose $\mathbb{R}^2$ into three disjoint regions C, E and P (see Figure 4.3) defined as follows:

$$\begin{aligned}
C &:= \left\{(a_1, a_2) : a_1 \geq 2, a_1^2 + 4a_2 > 0\right\} \cup \{(a_1, a_2) : a_1 \in (0, 2), a_1 + a_2 > 1\} \\
&\quad \cup \left\{(a_1, a_2) : a_1^2 + 4a_2 = 0, a_1 > 2\right\} \cup \{(a_1, a_2) : a_1 = 0, a_2 > 1\}, \\
P &:= \{(a_1, a_2) : a_1 + a_2 = 1, a_2 \in [-1, 1]\}, \\
E &:= \mathbb{R}^2 \setminus (C \cup P).
\end{aligned}$$

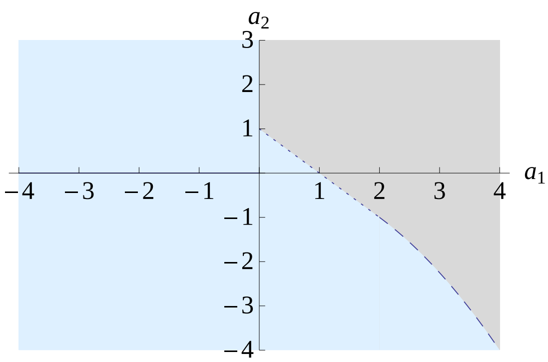

Figure 4.3: The regions C and E. P corresponds to the dotted line. The dashed line is the boundary of C whereas E is open.

Depending on the membership of (a_1, a_2) to one of these sets, we can characterise the behaviour of the persistence probability under certain conditions on the law of Y_1. The main purpose is to determine whether the persistence probability decays polynomially or exponentially or whether it converges to a positive constant.

If $(a_1, a_2) \in P$, the persistence probability decays polynomially if $\mathbb{E}[Y_1] = 0$ under suitable moment conditions. The choice $a_1 = 2, a_2 = -1$ corresponds to an integrated random walk where $p_N \asymp N^{-1/4}$ if $\mathbb{E}[Y_1] = 0$ and $\mathbb{E}[Y_1^2] < \infty$ ([DDG12]). If $a_1 + a_2 = 1$ with $|a_2| < 1$, we will see that X can be seen as a perturbed random walk since $c_n = c + C\epsilon^n$ where $|\epsilon| < 1$. Moreover, X can also be written as an integrated AR(1)-process. The process corresponding to $a_1 = 0, a_2 = 1$ describes two independent random walks such that its persistence probability is the square of that of a random walk.

Theorem 4.2.1. *Let $(a_1, a_2) \in P \setminus \{(2, -1)\}$. Assume that $\mathbb{E}[Y_1] = 0$ and that $\mathbb{E}\left[e^{|Y_1|^\alpha}\right] < \infty$ for some $\alpha > 0$. Then*

$$p_N = N^{-1/2+o(1)} \quad (|a_2| < 1), \qquad p_N \sim CN^{-1} \quad (a_2 = 1).$$

Next, we also prove that the persistence probability decays faster than any polynomial if $(a_1, a_2) \in E$ under certain conditions on the law of Y_1.

Theorem 4.2.2. *Let $(a_1, a_2) \in E$. Assume that $\mathbb{P}(Y_1 > 0) \in (0, 1)$, $\mathbb{E}\left[e^{|Y_1|^\alpha}\right] < \infty$ for some $\alpha > 0$ and that the characteristic function φ of Y_1 satisfies $\varphi(t) \to 0$ as $|t| \to \infty$. Then $p_N \precsim \exp(-\lambda N/ \log N)$ for some $\lambda = \lambda(a_1, a_2) > 0$.*

Actually, we can show that $p_N \precsim \exp(-\lambda N)$ on most parts of E under much weaker conditions on the distribution of Y_1. For instance, on E_2, we will see that the persistence probability decays exponentially for a trivial reason, whereas the same holds on E_3 by Proposition 4.3.16.
The reason for the rapid decay of the persistence probability on E can be explained as follows: either $c_n \to 0$ exponentially fast or (c_n) oscillates and diverges to $\pm\infty$.
If $(a_1, a_2) \in C$, we will see that $c_n = \exp(\lambda n(1 + o(1))$ for some $\lambda > 0$. One therefore expects that the process stays below a constant barrier at all times with positive probability. This is confirmed by the following theorem:

Theorem 4.2.3. *Let $(a_1, a_2) \in C$. Assume that $\mathbb{P}(Y_1 < 0) > 0$ and $\mathbb{P}(Y_1 \geq x) \precsim (\log x)^{-\alpha}$ as $x \to \infty$ for some $\alpha > 1$. Then it holds that*

$$\mathbb{P}\left(\sup_{n\geq 1} X_n \leq x\right) = \lim_{N\to\infty} p_N(x) > 0, \quad x \geq 0.$$

Note that the assumption $\mathbb{E}[Y_1] = 0$ is essential for the polynomial behaviour of p_N if $(a_1, a_2) \in P$. For instance, if $(S_n)_{n\geq 1}$ is a random walk, it is known that the persistence probability can decay polynomially or exponentially if $\mathbb{E}[S_1] > 0$ (see [Don89]) whereas it converges to a positive constant if $\mathbb{E}[S_1] < 0$. In contrast, if $(a_1, a_2) \in E \cup C$, the behaviour of p_N is more stable in the sense that Theorem 4.2.3 and Theorem 4.2.2 do not rely on the condition $\mathbb{E}[Y_1] = 0$.
The best results can be obtained if the innovations are Gaussian, where we can actually prove that p_N admits an exponential upper bound for all $(a_1, a_2) \in E$. Summing up, this leads to the following theorem:

Theorem 4.2.4. *If Y_1 is Gaussian with zero mean, the following statements hold:*

1. $\lim_{N\to\infty} p_N = p_\infty > 0$ *if and only if* $(a_1, a_2) \in C$,
2. $p_N \sim cN^{-1}$ *iff* $(a_1, a_2) = (0, 1)$, *and* $p_N \asymp N^{-1/4}$ *iff* $(a_1, a_2) = (2, -1)$,
3. $p_N = N^{-1/2+o(1)}$ *if and only if* $(a_1, a_2) \in P$ *and* $|a_2| < 1$, *and*
4. $p_N \lesssim e^{-\lambda N}$ *for some* $\lambda > 0$ *if and only if* $(a_1, a_2) \in E$.

The theorems above are mostly corollaries to more general theorems that are also applicable to AR(p)-processes if $p \geq 3$ (see e.g. Theorem 4.3.2 and 4.3.10 and Proposition 4.3.17 and 4.5.1 below). We will indicate possible extensions throughout this chapter. The main advantage of focussing on AR(2)-processes consists of the fact that we have an explicit solution of the difference equation for the sequence $(c_n)_{n\geq 0}$. For instance, this allows us to explicitly describe the parameters (a_1, a_2) such that $c_n \to 0$. However, even for AR(2)-processes, one is forced to distinguish a variety of cases that require different treatment. It is clear that this becomes much more complicated for processes of higher order.

4.3 Exponential bounds

4.3.1 Exponential upper bounds

Let us begin with a trivial observation: If $a_1 \leq 0, \dots, a_p \leq 0$, we have that

$$\mathbb{P}\left(\sup_{n=1,\dots,N} X_n \leq 0\right) \leq \mathbb{P}\left(Y_1 \leq 0\right)^N,$$

since $X_1 \leq 0, \dots, X_n \leq 0$ implies that $Y_k \leq -a_1 X_{k-1} - \cdots - a_p X_{k-p} \leq 0$ for all $k = 1, \dots, n$. If $p = 2$, this shows that p_N decays at least exponentially on E_2, see Figure 4.2.

As we will see in the sequel, exponential decay of p_N occurs for two differnt reasons: first, if $c_n \to 0$ (exponentially fast) and second, if $(c_n)_{n\geq 0}$ oscillates and diverges exponentially fast.

Let us first consider the case that c_n goes to zero. Recall that for AR(1)-processes $(Z_n)_{n\geq 1}$ with $Z_n = \rho Z_{n-1} + Y_n$ for $\rho \in (0,1)$, $c_n = \rho^n \to 0$, and p_N decays exponentially under mild assumptions on the distribution of Y_1:

Theorem 4.3.1 ([NK08]). *Let* $0 < \rho < 1$, $x > 0$ *and assume that* $\mathbb{E}\left[(Y_1^-)^\delta\right] < \infty$ *for some* $\delta \in (0, 1)$ *and* $\mathbb{P}\left(Y_1 > x(1-\rho)\right) > 0$. *Then* $\mathbb{E}\left[\exp(\alpha\tau_x)\right] < \infty$ *for some* $\alpha > 0$.

Let us remark that Theorem 4.3.1 implies also that $\mathbb{E}\left[\exp(\alpha\tau_0)\right] < \infty$ for some $\alpha > 0$ if $\mathbb{E}\left[(Y_1^-)^\delta\right] < \infty$ and $\mathbb{P}\left(Y_1 > 0\right) > 0$.

We now state a similar weaker result that provides a simple criterion for AR(p)-processes to ensure that p_N decays faster to zero than any polynomial.

Theorem 4.3.2. *Let $(c_k)_{k\geq 0}$ denote a sequence with $c_0 = 1$, $A := \sum_{k=0}^{\infty} |c_k| < \infty$ and $\sum_{k=q}^{\infty} |c_k| \leq Ce^{-\lambda q}$ for every $q \geq 1$ where $C, \lambda > 0$ are constants. Assume that there are constants $\gamma, \delta > 0$ with $\min\{\mathbb{P}(Y_1 < -\gamma), \mathbb{P}(Y_1 > \gamma)\} > 0$ and that $\mathbb{E}\left[|Y_1|^{\delta}\right] < \infty$. Let $X_n = \sum_{k=1}^{n} c_{n-k} Y_k$. Then for $x \in [0, \gamma A)$, there is $c(x) > 0$ such that*

$$p_N(x) \precsim \exp\left(-c(x)\sqrt{N}\right), \quad N \to \infty.$$

Moreover, if $\mathbb{E}[\exp(|Y_1|^{\alpha})] < \infty$ for some $\alpha > 0$ and $x \in [0, \gamma A)$, there is $c(x) > 0$ such that

$$p_N(x) \precsim \exp\left(-c(x)\, N/\log N\right), \quad N \to \infty.$$

Proof. For $q \geq 1$, define $Z_{q,n} = \sum_{k=n-q}^{n} c_{n-k} Y_k$ for $n \geq q+1$. Note that $Z_{q,n}$ is measurable w.r.t. $\sigma(Y_{n-q}, \ldots, Y_n)$ which implies that $(Z_{q,n(q+1)+1})_{n\geq 1}$ defines a sequence of i.i.d. random variables with $Z_{q,q+2} \stackrel{d}{=} X_{q+1}$. We will show that $Z_{q,n}$ is a good approximation of X_n if q is large. We then obtain an estimate on $p_N(x)$ by computing the persistence probability of the independent random variables $(Z_{q,(q+1)n+1})_{n\geq 1}$.
First, observe that

$$\begin{aligned}\mathbb{P}\left(\sup_{n=q+2,\ldots,N} |X_n - Z_{q,n}| > u\right) &\leq \sum_{n=q+2}^{N} \mathbb{P}\left(\left|\sum_{k=1}^{n-q-1} c_{n-k} Y_k\right| > u\right) \\ &= \sum_{n=q+2}^{N} \mathbb{P}\left(\left|\sum_{k=q+1}^{n-1} c_k Y_k\right| > u\right) =: h_N(u). \qquad (4.6)\end{aligned}$$

In the first equality, we have used that the Y_k are i.i.d., and therefore exchangeable. Hence,

$$\begin{aligned}\mathbb{P}\left(\sup_{n=1,\ldots,N} X_n \leq x\right) &\leq \mathbb{P}\left(\sup_{n=q+2,\ldots,N} Z_{q,n} \leq x + \epsilon\right) + h_N(\epsilon) \\ &\leq \mathbb{P}\left(\sup_{n=1,\ldots,\lfloor (N-1)/(q+1)\rfloor} Z_{q,n(q+1)+1} \leq x + \epsilon\right) + h_N(\epsilon) \\ &= \mathbb{P}\left(Z_{q,q+2} \leq x + \epsilon\right)^{\lfloor (N-1)/(q+1)\rfloor} + h_N(\epsilon), \qquad (4.7)\end{aligned}$$

where we have used the fact that $(Z_{q,n(q+1)+1})_{n\geq 1}$ is an i.i.d. sequence. Using again the exchangeability, we get for $y \in \mathbb{R}$ that

$$\mathbb{P}\left(Z_{q,q+2} \leq y\right) = \mathbb{P}\left(\sum_{k=0}^{q} c_k Y_{k+1} \leq y\right) \to \mathbb{P}\left(\sum_{k=0}^{\infty} c_k Y_{k+1} \leq y\right), \quad q \to \infty, \qquad (4.8)$$

since the series $\sum_{k=0}^{\infty} c_k Y_{k+1} =: Z$ converges a.s. by Kolmogorov's Three Series Theorem. Next, $\mathbb{P}(Z \leq y) < 1$ for every $0 \leq y < \gamma A$. To see this, let us construct an event Ω_0 of positive probability that implies $Z > y$: Fix $m \in \mathbb{N}$, and let

$$\Omega_0 := \bigcap_{k=0}^{m} \{c_k Y_{k+1} > \gamma |c_k|\} \cap \bigcap_{k=m+1}^{\infty} \left\{|Y_{k+1}| \leq (1-e^{-\lambda/2})e^{\lambda k/2}\right\}.$$

Using that $|c_k| \leq Ce^{-\lambda k}$ for all k, it holds on Ω_0 that

$$\begin{aligned} Z &\geq \sum_{k=0}^{m} c_k Y_{k+1} - \sum_{k=m+1}^{\infty} |c_k| |Y_{k+1}| \geq \gamma \sum_{k=0}^{m} |c_k| - C(1-e^{-\lambda/2}) \sum_{k=m+1}^{\infty} e^{-\lambda k/2} \\ &= \gamma \sum_{k=0}^{m} |c_k| - Ce^{-\lambda(m+1)/2}. \end{aligned}$$

Since $\sum_{k=0}^{m} |c_k| \to A$ and $y < \gamma A$, we can choose m so large that indeed $Z > y$ on Ω_0. Finally, using that $\min\{\mathbb{P}(Y_1 > \gamma), \mathbb{P}(Y_1 < -\gamma)\} > 0$ and $\mathbb{E}\left[|Y_1|^\delta\right] < \infty$, one verifies readily that $\mathbb{P}(\Omega_0) > 0$.
Then for $0 \leq y < \gamma A$, by (4.8), there is $\rho = \rho(y) < 1$ such that $\mathbb{P}(Z_{2,q+2} \leq y) \leq \rho$ for all q sufficiently large.
Note that in view of our assumption on the sequence (c_n),

$$\sup_{n=q+1,\dots,N} \left| \sum_{k=q+1}^{n} c_k Y_k \right| \leq \sup_{l=q+1,\dots,N} |Y_k| \sum_{k=q+1}^{\infty} |c_k| \leq Ce^{-\lambda q} \sup_{l=q+1,\dots,N} |Y_k|,$$

so we deduce that

$$h_N(u) \leq \sum_{n=q+1}^{N} \mathbb{P}\left(\sup_{k=q+1,\dots,N} |Y_k| > e^{\lambda q} u/C \right) \leq N^2 \mathbb{P}\left(|Y_1| > e^{\lambda q} u/C\right).$$

Let $q = q_N := \lfloor \sqrt{N} \rfloor$ - 1. Since $\mathbb{E}\left[|Y_1|^\delta\right] < \infty$, we can apply Chebychev's inequality:

$$h_N(u) \leq N^2 \mathbb{E}\left[|Y_1|^\delta\right] e^{-\delta\lambda\sqrt{N}} (u/C)^{-\delta}. \tag{4.9}$$

For $x \in [0, \gamma A)$, let $\epsilon > 0$ such that $x + \epsilon < \gamma A$, and recall that we can fix $\rho \in (0,1)$ such that $\mathbb{P}(Z_{q,q+2} \leq x + \epsilon) \leq \rho$ for q large enough. Combining (4.7) and (4.9), we obtain that

$$p_N(x) \precsim \rho^{\sqrt{N}} + N^2 e^{-\delta\lambda\sqrt{N}},$$

so the theorem follows under the assumption $\mathbb{E}\left[|Y_1|^\delta\right] < \infty$. If $\mathbb{E}[\exp(|Y_1|^\alpha)] < \infty$, the estimate on h_N can be improved as follows:

$$h_N(u) \leq N^2 \mathbb{P}\left(|Y_1| > e^{\lambda q} u/C\right) \leq N^2 \exp\left(-e^{\alpha\lambda q} (u/C)^\alpha\right) \mathbb{E}[\exp(|Y_1|^\alpha)].$$

In particular, with $q = q_N = \lfloor \kappa \log N \rfloor$, if κ is large enough, this implies together with (4.7) that, for some $c(x) > 0$,

$$p_N(x) \lesssim N^2 e^{-N^2} + \rho^{\lfloor (N-1)/(q_N+1) \rfloor} \lesssim \exp(-c(x)N/\log N), \quad N \to \infty.$$

□

The proof of Theorem 4.3.2 reveals that fast decay of p_N can be explained intuitively as follows: if we write $X_n = \sum_{k=1}^{n-q-1} c_{n-k}Y_k + \sum_{k=n-q}^{n} c_{n-k}Y_k$, the first summand is typically small if q is large and $c_n \to 0$. Hence, heuristically,

$$\mathbb{P}\left(\sup_{n=1,\dots,N} X_n \le 0 \right) \approx \mathbb{P}\left(\sup_{n=q+1,\dots,N} \sum_{k=n-q}^{n} c_{n-k}Y_k \le 0 \right) \approx \mathbb{P}\left(\sum_{k=1}^{q+1} c_{n-k}Y_k \le 0 \right)^{N/q}.$$

Remark 4.3.3. If $(c_k)_{k\ge 0}$ denote a sequence with $c_0 = 1$ and $\sum_{k=0}^{\infty} |c_k| < \infty$ and $|Y_1| \le M$ a.s. for some $M < \infty$, one can prove in an analogous way that even $p_N \lesssim \exp(-cN)$ for some $c > 0$ since $h_N(u)$ in the proof of Theorem 4.3.2 vanishes for q large enough.

Remark 4.3.4. As it was already remarked by [NK08], if $(c_k)_{k\ge 0}$ denotes a sequence of positive numbers, one has that

$$X_n = \sum_{k=1}^{n} c_{n-k}Y_k \ge \sum_{k=1}^{n} c_{n-k}Y_k 1_{\{Y_k \le M\}} = \sum_{k=1}^{n} c_{n-k}\tilde{Y}_k =: \tilde{X}_n,$$

such that $\mathbb{P}(X_n \le x, \forall n \le N) \le \mathbb{P}\left(\tilde{X}_n \le x, \forall n \le N\right)$. Hence, if the c_n are positive, one can assume without loss of generality that the innovations are bounded from above in order to establish an upper bound on the persistence probability. Hence, the moment conditions of Theorem 4.3.2 only apply to Y_1^- in that case.

For AR(2)-processes, Theorem 4.3.2 is applicable if $a_1 + a_2 < 1$, $a_2 < a_1 + 1$ and $a_2 > -1$, cf. (4.5) and Figure 4.1. Moreover, the preceding theorem can be generalised easily to cover more general processes $(X_n)_{n\in\mathbb{Z}}$ that can be written as

$$X_n = \sum_{k=-\infty}^{\infty} c_{n-k}Y_k, \quad n \in \mathbb{Z},$$

where $(c_n)_{n\in\mathbb{Z}}$ is a deterministic sequence. This class contains autoregressive moving average models (ARMA(p,q)) and moving average processes of infinte order (MA(∞)), see [BD87, Section 3].

Theorem 4.3.5. *Let $(c_k)_{k\in\mathbb{Z}}$ denote a sequence with $c_0 = 1$, $A := \sum_{k=-\infty}^{\infty} |c_k| < \infty$ and $\sum_{|k|\ge q} |c_k| \le Ce^{-\lambda q}$ for all $q \ge 1$ and some $\lambda > 0$. Let $(Y_k)_{k\in\mathbb{Z}}$ be a sequence of i.i.d. random variables such that* $\min\{\mathbb{P}(Y_1 > \gamma), \mathbb{P}(Y_1 < -\gamma)\} > 0$ *for some $\gamma > 0$,* *and* $\mathbb{E}\left[|Y_1|^\delta\right] < \infty$ *for some* $\delta > 0$. *Let* $X_n := \sum_{k=-\infty}^{\infty} c_{n-k} Y_k$ *for* $n \in \mathbb{Z}$. *If* $x \in [0, \gamma A)$, *it holds for some* $c(x) > 0$ *that*

$$\mathbb{P}\left(\sup_{|n|\le N} X_n \le x\right) \precsim \exp(-c(x)\sqrt{N}), \quad N \to \infty.$$

Moreover, if $\mathbb{E}\left[\exp(|Y_1|^\alpha)\right] < \infty$ *for some* $\alpha > 0$ *and* $x \in [0, \gamma A)$, *there is* $c(x) > 0$ *such that*

$$\mathbb{P}\left(\sup_{|n|\le N} X_n \le x\right) \precsim \exp(-c(x)N/\log N), \quad N \to \infty.$$

Proof. Note that X_n is well defined for every $n \in \mathbb{Z}$ by Kolmogorov's Three Series Theorem. The proof is then very similar to that of Theorem 4.3.2. We define $Z_{q,n} := \sum_{k=n-q}^{n+q} c_{n-k} Y_k$. Note that $(Z_{q,n(2q+1)})_{n\in\mathbb{Z}}$ forms a sequence of i.i.d. random variables with $Z_{q,0} = \sum_{k=-q}^{q} c_k Y_k$. The remainder of the proof is along the same lines of the proof of Theorem 4.3.2. □

In certain special cases, we can improve Theorem 4.3.2. Namely, if (c_n) is a sequence of positive numbers and $c_n = \rho^n(1+o(1))$ where $\rho \in (0,1)$, it follows from Theorem 4.3.1 that p_N goes to zero exponentially fast under mild assumptions on Y_1:

Proposition 4.3.6. *Let* $(c_n)_{n\ge 0}$ *be a sequence such that* $\alpha C\rho^n \le c_n \le C\rho^n$ *for all* $n \ge 0$ *where* $\rho \in (0,1)$, $0 < \alpha < 1$, $C > 0$. *Assume that* $\mathbb{E}\left[(Y_1^-)^\delta\right] < \infty$ *for some* $\delta \in (0,1)$. *Let* $x \ge 0$ *be such that* $\mathbb{P}\left(Y_1 > x(1-\rho)/(\alpha C)\right) > 0$, *and* $X_n := \sum_{k=1}^{n} c_{n-k} Y_k$. *Then there is some* $\lambda = \lambda(x) > 0$ *such that* $p_N(x) \precsim \exp(-\lambda N)$.

Proof. Define the i.i.d. random variables $\tilde{Y}_k := Y_k 1_{\{Y_k<0\}} + \alpha Y_k 1_{\{Y_k>0\}}$, $k \ge 0$. Since $c_k \ge 0$ for all k, we obtain that

$$X_n = \sum_{k=1}^{n} c_{n-k} Y_k \ge \sum_{k=1}^{n} C\rho^{n-k} Y_k 1_{\{Y_k<0\}} + \sum_{k=1}^{n} \alpha C\rho^{n-k} Y_k 1_{\{Y_k>0\}} = C\sum_{k=1}^{n} \rho^{n-k}\tilde{Y}_k =: CZ_n,$$

where $Z_n := \rho Z_{n-1} + \tilde{Y}_n$. In particular, we conclude that

$$\mathbb{P}\left(\sup_{n=1,\dots,N} X_n \le x\right) \le \mathbb{P}\left(\sup_{n=1,\dots,N} Z_n \le x/C\right).$$

Now $\mathbb{P}\left(\tilde{Y}_1 > x(1-\rho)/C\right) = \mathbb{P}\left(Y_1 > x(1-\rho)/(\alpha C)\right) > 0$ by the choice of x. Hence, the result follows from [NK08, Theorem 1] (Theorem 4.3.1 above). □

The preceding proposition yields the following corollary for AR(2)-processes:

Corollary 4.3.7. *Let* $a_1 \in (0,2), a_2 < 0$ *with* $a_1 + a_2 < 1$ *and* $a_1^2 + 4a_2 > 0$. *Assume that* $\mathbb{E}\left[(Y_1^-)^\delta\right] < \infty$ *for some* $\delta \in (0,1)$ *and* $\mathbb{P}\left(Y_1 \ge y\right) > 0$ *for every* y. *For every* $x \ge 0$, *there is* $\lambda = \lambda(x) > 0$ *such that* $p_N(x) \precsim \exp(-\lambda N)$.

Proof. It is not hard to check that $0 < s_2 < s_1 < 1$. Hence, $c_n = s_1^n(s_1 - s_2(s_2/s_1)^n)/h$ and $h^{-1}(s_1 - s_2)s_1^n \leq c_n \leq h^{-1}s_1^{n+1}$ for all n. The result follows from Proposition 4.3.6. □

If $|Y_1| \leq M$ a.s., the preceding corollary is not applicable. However, we already know that $p_N \precsim e^{-cN}$ for some $c > 0$ in that case, see Remark 4.3.3.

Let us now establish exponential upper bounds for p_N for certain distributions if the sequence (c_n) oscillates and diverges exponentially. The proof relies on the following proposition.

Proposition 4.3.8. *Let $\rho \in (-1, 1)$ ($\rho \neq 0$) and set $Z := \sum_{n=1}^{\infty} \rho^n Y_n$. Moreover, suppose that $\mathbb{E}\left[|Y_1|^\delta\right] < \infty$ for some $\delta > 0$. Let φ denote the characteristic function of Y_1 and assume that there are $\Delta \in (0, |\rho|)$ and $t_0 > 0$ such that $|\varphi(t)| \leq \Delta$ for all $|t| \geq t_0$. It follows that $\mathbb{P}(|Z| \leq \epsilon) \precsim \epsilon$ as $\epsilon \downarrow 0$.*

Proof. Z is well-defined, and its characteristic function $\tilde{\varphi}$ is given by $\tilde{\varphi}(t) = \prod_{n=1}^{\infty} \varphi(\rho^n t)$, see e.g. [Luk70, Section 3.7]. Let us show that $\tilde{\varphi}$ is absolutely integrable. If this holds, by [Luk70, Theorem 3.2.2], Z admits a continuous density g given by

$$g(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} \tilde{\varphi}(t)\, dt, \quad x \in \mathbb{R}.$$

In particular, g is bounded by $C := \|\tilde{\varphi}\|_1/(2\pi)$ implying that $\mathbb{P}(|Z| \leq \epsilon) \leq C\,\epsilon$ for any $\epsilon \geq 0$.
To prove the integrability of $\tilde{\varphi}$, let Δ and t_0 be as in the statement of the proposition and note that

$$|\tilde{\varphi}(t)| = \prod_{n=1}^{\infty} |\varphi(\rho^n t)| \leq \Delta^{N(t)},$$

where $N(t) = \#\{n \geq 1 : |\rho^n t| \geq t_0\} = \lfloor (\log|t| - \log(t_0))/\log(1/|\rho|) \rfloor$. In particular,

$$|\tilde{\varphi}(t)| \leq \exp\left(\log\Delta\left(\frac{\log|t| - \log(t_0)}{\log(1/|\rho|)} - 1\right)\right) = C\,|t|^{-\alpha},$$

where C depends on t_0, ρ and Δ only and $\alpha := \log(1/\Delta)/\log(1/|\rho|) > 1$. This shows that $|\tilde{\varphi}(t)|$ is integrable over $\mathbb{R}$. □

Remark 4.3.9. Recall that $\lim_{|t|\to\infty} \mathbb{E}\left[e^{itX}\right] = 0$ if X has an absolutely continuous distribution, see e.g. [Luk70, Section 2.2]. In general, even if the characteristic function does not tend to zero as $|t| \to \infty$, the preceding proof might still be applicable if one can control the quantity $N(t)$ above.
However, if the distribution of X is purely discrete, $\limsup_{|t|\to\infty} \left|\mathbb{E}\left[e^{itX}\right]\right| = 1$, and in general, it is a very challenging problem to find conditions such that the random series $\sum_{n=1}^{\infty} \rho^n Y_n$ has a density. This question has attracted a lot of attention for so-called infinite Bernoulli convolutions. We refer to the survey of Peres and Solomyak([PSS00]).

We can now prove the following theorem.

Theorem 4.3.10. *Let* $X_n := \sum_{k=1}^n c_{n-k} Y_k$ *where* $c_n = d\rho^n + \beta_n r^n$ *where* $d \neq 0$, $\rho < -1$ *and* $|\rho| > |r|$ *and* $|\beta_n| e^{-\lambda n} \to 0$ *as* $n \to \infty$ *for every* $\lambda > 0$. *Assume* $\mathbb{E}\left[|Y_1|^\delta\right] < \infty$ *for some* $\delta > 0$. *Moreover, suppose that the characteristic function* φ *of* Y_1 *satisfies* $|\varphi(t)| \leq \Delta < 1/|\rho|$ *for all* $|t| \geq t_0$. *Then there is a constant* $C > 0$ *such that for every* $x \geq 0$, *it holds that*

$$\liminf_{N\to\infty} -N^{-1} \log \mathbb{P}\left(\sup_{n=1,\dots,N} X_n \leq x\right) \geq C.$$

If $\mathbb{E}\left[\exp(|Y_1|^\alpha)\right] < \infty$ *for some* $\alpha > 0$, *then*

$$C \geq \begin{cases} \log|\rho/r|, & |r| > 1, \\ \log|\rho|, & \textit{else}. \end{cases}$$

Proof. Assume w.l.o.g. that $d = 1$ (write $X_n = \sum_{k=1}^n (c_{n-k}/d)(dY_k)$). Let $\hat{\beta}_n := \sup\{|\beta_0|, \dots, |\beta_n|\}$ and $E_N := \{|Y_1| \leq f_N, \dots, |Y_N| \leq f_N\}$ where $1 \leq f_N \to \infty$ is to be specified later. On E_N, it holds for $n = 1, \dots, N$ that

$$\begin{aligned} X_n &= \sum_{k=1}^n c_{n-k} Y_k = \sum_{k=1}^n \rho^{n-k} Y_k + \sum_{k=1}^n \beta_{n-k} r^{n-k} Y_k \\ &\geq \sum_{k=1}^n \rho^{n-k} Y_k - \hat{\beta}_n f_N \sum_{k=1}^n |r|^{n-k} \geq \sum_{k=1}^n \rho^{n-k} Y_k - \hat{\beta}_N f_N \sum_{k=0}^N |r|^k . \end{aligned}$$

Case 1: Consider first the case that $\beta_n \neq 0$ for some n. Let $R_N := \sum_{k=0}^N |r|^k$. Then

$$p_N(x) \leq \mathbb{P}(E_N^c) + \mathbb{P}\left(\sup_{n=1,\dots,N} \sum_{k=1}^n \rho^{n-k} Y_k \leq x + \hat{\beta}_N f_N R_N, E_N\right). \tag{4.10}$$

Note that $Z_n := \sum_{k=1}^n \rho^{n-k} Y_k$ is an AR(1)-process satisfying $Z_n = \rho Z_{n-1} + Y_n$. Let us begin with the following useful observation: if $Z_{N-1} \leq z$ and $Z_N \leq z$ for some large $z > 0$, we have with high probability that $|Z_{N-1}| \leq z$. This will allow us to reduce the estimation of $p_N(x)$ to controlling $\mathbb{P}(|Z_N| \leq z_N)$ where $z_N \to \infty$ as $N \to \infty$. To be precise, recall that $\rho < -1$, and note that

$$\begin{aligned} \{Z_{N-1} \leq z, Z_N \leq z\} &\subseteq \{|Z_{N-1}| \leq z\} \cup \{Z_{N-1} < -z, Z_N \leq z\} \\ &\subseteq \{|Z_{N-1}| \leq z\} \cup \{Y_N \leq -(|\rho| - 1)z\}. \end{aligned} \tag{4.11}$$

For the last inclusion, we have used that the event $\{Z_{N-1} < -z, Z_N \leq z\}$ implies that $z \geq Z_N = \rho Z_{N-1} + Y_N \geq -\rho z + Y_N$. Hence, combining this with (4.10), we obtain that

$$\begin{aligned} p_N(x) &\leq \mathbb{P}(E_N^c) + \mathbb{P}\left(Z_{N-1} \leq x + \hat{\beta}_N f_N R_N, Z_N \leq x + \hat{\beta}_N f_N R_N\right) \\ &\leq \mathbb{P}(E_N^c) + \mathbb{P}\left(|Z_{N-1}| \leq x + \hat{\beta}_N f_N R_N\right) + \mathbb{P}\left(Y_N \leq -(|\rho| - 1)(x + \hat{\beta}_N f_N R_N)\right). \end{aligned} \tag{4.12}$$

It remains to estimate the three probabilities above. Clearly,

$$\mathbb{P}\left(E_N^c\right) = \mathbb{P}\left(\bigcup_{n=1}^{N} \{|Y_N| > f_N\}\right) \leq N\mathbb{P}\left(|Y_1| > f_N\right).$$

Next, since $|\rho| > 1$ and $\hat{\beta}_N \geq \beta > 0$ for some $\beta > 0$ and for all $N \geq N_0$ large enough and $R_N \geq 1$, it follows that

$$\mathbb{P}\left(Y_N \leq -(|\rho|-1)(x + \hat{\beta}_N f_N R_N)\right) \leq \mathbb{P}\left(|Y_1| \geq (|\rho|-1)\beta f_N\right), \quad N \geq N_0.$$

For large N, using the last two inequalities in (4.12), we arrive at

$$p_N(x) \leq (N+1)\mathbb{P}\left(|Y_1| \geq C_1 f_N\right) + \mathbb{P}\left(|Z_{N-1}| \leq 2\hat{\beta}_N f_N R_N\right), \tag{4.13}$$

where $C_1 := \min\{1, (|\rho|-1)\beta\}$. Set $\tilde{Z}_n := \rho^{-n} Z_n = \sum_{k=1}^{n} \rho^{-k} Y_k$. Then

$$\mathbb{P}\left(|Z_{N-1}| \leq 2\hat{\beta}_N f_N R_N\right) = \mathbb{P}\left(\left|\tilde{Z}_{N-1}\right| \leq 2\,|\rho|^{-(N-1)}\,\hat{\beta}_N f_N R_N\right).$$

Note that $\tilde{Z}_n$ converges a.s. to a random variable $\tilde{Z}_\infty$ by Kolmogorov's Three Series Theorem. Moreover, for $u, v > 0$,

$$\begin{aligned}
\mathbb{P}\left(\left|\tilde{Z}_\infty\right| \leq u+v\right) &\geq \mathbb{P}\left(\left|\tilde{Z}_\infty - \tilde{Z}_{N-1}\right| \leq u+v-\left|\tilde{Z}_{N-1}\right|, \left|\tilde{Z}_{N-1}\right| \leq u\right) \\
&\geq \mathbb{P}\left(\left|\tilde{Z}_\infty - \tilde{Z}_{N-1}\right| \leq v, \left|\tilde{Z}_{N-1}\right| \leq u\right) = \mathbb{P}\left(\left|\tilde{Z}_\infty - \tilde{Z}_{N-1}\right| \leq v\right)\mathbb{P}\left(\left|\tilde{Z}_{N-1}\right| \leq u\right).
\end{aligned}$$

The last equality follows from the independence of increments of $\tilde{Z}$. Hence,

$$\mathbb{P}\left(\left|\tilde{Z}_{N-1}\right| \leq u\right) \leq \frac{\mathbb{P}\left(\left|\tilde{Z}_\infty\right| \leq u+v\right)}{1-\mathbb{P}\left(\left|\tilde{Z}_\infty - \tilde{Z}_{N-1}\right| > v\right)}, \quad u, v > 0, N \geq 1.$$

Using this inequality with $u = v = 2\,|\rho|^{-(N-1)}\,\hat{\beta}_N f_N R_N$, we obtain that

$$\begin{aligned}
\mathbb{P}\left(\left|\tilde{Z}_{N-1}\right| \leq 2\,|\rho|^{-(N-1)}\,\hat{\beta}_N f_N R_N\right) &\leq \frac{\mathbb{P}\left(\left|\tilde{Z}_\infty\right| \leq 4\,|\rho|^{-(N-1)}\,\hat{\beta}_N f_N R_N\right)}{1-\mathbb{P}\left(\left|\tilde{Z}_\infty - \tilde{Z}_{N-1}\right| > 2\,|\rho|^{-(N-1)}\,\hat{\beta}_N f_N R_N\right)} \\
&\leq 2\,\mathbb{P}\left(\left|\tilde{Z}_\infty\right| \leq 4\,|\rho|^{-(N-1)}\,\hat{\beta}_N f_N R_N\right),
\end{aligned}$$

where the last inequality holds for all N sufficiently large in view of the following estimates: Since $\mathbb{E}\left[|Y_1|^\delta\right] < \infty$ for some $\delta \in (0,1)$ and $f_N \to \infty$, we have that

$$\begin{aligned}
&\mathbb{P}\left(\left|\tilde{Z}_\infty - \tilde{Z}_{N-1}\right| > 2\,|\rho|^{-(N-1)}\,\hat{\beta}_N f_N R_N\right) \\
&\quad = \mathbb{P}\left(\left|\sum_{n=N}^{\infty} \rho^{-n} Y_n\right|^\delta > 2^\delta\,|\rho|^{-\delta(N-1)}\,(\hat{\beta}_N f_N R_N)^\delta\right) \\
&\quad \le \mathbb{P}\left(\sum_{n=N}^{\infty} |\rho|^{-\delta n}\,|Y_n|^\delta > 2^\delta\,|\rho|^{-\delta(N-1)}\,(\hat{\beta}_N f_N R_N)^\delta\right) \\
&\quad \le \frac{\mathbb{E}\left[|Y_1|^\delta\right]\sum_{n=N}^{\infty}|\rho|^{-\delta n}}{2^\delta\,|\rho|^{-\delta(N-1)}\,(\hat{\beta}_N f_N R_N)^\delta} \le C_2 \frac{|\rho|^{-\delta N}}{|\rho|^{-\delta N}\,\hat{\beta}_N^\delta f_N^\delta R_N^\delta} = C_2 \frac{1}{\hat{\beta}_N^\delta f_N^\delta R_N^\delta} \to 0.
\end{aligned}$$

In the first inequality, we have used that $(x+y)^\delta \le x^\delta + y^\delta$ for $x, y \ge 0$ and $\delta \in (0,1)$, and for the convergence to 0, recall that $\hat{\beta}_N \ge \beta$ and $R_N \ge 1$ for all N. We have shown that (4.13) implies for all N large enough that

$$p_N(x) \le (N+1)\mathbb{P}\left(|Y_1| \ge C_1 f_N\right) + 2\,\mathbb{P}\left(\left|\tilde{Z}_\infty\right| \le 4\,|\rho|^{-(N-1)}\,\hat{\beta}_N f_N R_N\right). \tag{4.14}$$

If $f_N \to \infty$ is chosen such that $|\rho|^{-N}\,\hat{\beta}_N f_N R_N \to 0$, we conclude from (4.14) and Proposition 4.3.8 (recall that $\Delta < 1/|\rho|$) that

$$p_N(x) \le (N+1)\mathbb{P}\left(|Y_1| \ge C_1 f_N\right) + C_3\,|\rho|^{-N}\,\hat{\beta}_N f_N R_N, \quad N \to \infty. \tag{4.15}$$

Let us now state the suitable choice for f_N. First, recall that by assumption, we have that $\hat{\beta}_N = e^{o(N)}$.

Assume first that $|r| \le 1$. Then $R_N \le N$. One can set $f_N := A^N$ where $1 < A < |\rho|$, use Chebychev's inequality (recall that $\mathbb{E}\left[|Y_1|^\delta\right] < \infty$) and (4.15) to show that

$$p_N(x) \precsim N A^{-\delta N} + |\rho/A|^{-N}\,e^{o(N)} N = e^{o(N)}\left(A^\delta \wedge (|\rho|/A)\right)^{-N}, \quad N \to \infty.$$

If $|r| > 1$, $R_N \asymp |r|^N$, take $f_N := A^N$ where $1 < A < |\rho/r|$, and as above, one sees that

$$p_N(x) \precsim N A^{-\delta N} + |\rho/(Ar)|^{-N}\,e^{o(N)} = e^{o(N)}\left(A^\delta \wedge (|\rho/(rA)|)\right)^{-N}, \quad N \to \infty.$$

If $\mathbb{E}\left[\exp(|Y_1|^\alpha)\right] < \infty$ for some $\alpha > 0$, it suffices to take $f_N := N^{2/\alpha}$ to obtain

$$p_N(x) \le (N+1)\mathbb{E}\left[\exp(|Y_1|^\alpha)\right]\exp(-C_1^\alpha N^2) + C_3\,|\rho|^{-N}\,e^{o(N)} N^{2/\alpha} R_N,$$

and it is then easy to conclude that $\liminf -N^{-1}\log p_N(x) \ge -\log(1/|\rho|) = \log(|\rho|)$ if $|r| \le 1$ and $\liminf -N^{-1}\log p_N(x) \ge \log(|\rho/r|)$ if $|r| > 1$.

Case 2: Finally, assume that $\beta_n = 0$ for all n. Then $X_n = Z_n = \sum_{k=1}^n \rho^{n-k} Y_k$. Let $0 \le f_N \to \infty$ to be specified later. Clearly, for large N,

$$\mathbb{P}\left(\sup_{n=1,\dots,N} Z_n \le x\right) \le \mathbb{P}\left(Z_{N-1} \le x, Z_N \le x\right) \le \mathbb{P}\left(Z_{N-1} \le f_N, Z_N \le f_N\right)$$
$$\le \mathbb{P}\left(|Z_{N-1}| \le f_N\right) + \mathbb{P}\left(Y_1 \le -(|\rho|-1) f_N\right),$$

where we have used (4.11) in the last inequality. But the last line is just a special case of (4.12) with $x = 0, \hat{\beta}_N = R_N = 1$, so we can proceed as above. □

We can apply Theorem 4.3.10 to prove that p_N decays exponentially for $(a_1, a_2) \in E_1$, cf. Figure 4.2.

Corollary 4.3.11. *Let $(a_1, a_2) \in E_1$. Assume that Y_1 satisfies the conditions of Theorem 4.3.10. Then there is a constant $C > 0$ such that for every $x \ge 0$, it holds that*

$$\liminf_{N\to\infty} -N^{-1} \log \mathbb{P}\left(\sup_{n=1,\dots,N} X_n \le x\right) \ge C.$$

If $\mathbb{E}\left[\exp(|Y_1|^\alpha)\right] < \infty$ for some $\alpha > 0$, then

$$C \ge \begin{cases} \log(|s_2|/s_1), & a_1 + a_2 > 1, \\ \log|s_2|, & a_1 + a_2 \le 1. \end{cases}$$

Proof. For $(a_1, a_2) \in E_1$, we have that $s_2 < -1$ and $|s_2| > s_1 > 0$. Hence, we can apply Theorem 4.3.10 with $\rho = s_2$ and $r = s_1$. To get the lower bound on C, note that $|r| = s_1 \le 1$ amounts to $a_1 + a_2 \le 1$. □

Remark 4.3.12. One can show by direct computation that the correlation coefficient ρ_n of X_{n-1} and X_n, given by

$$\rho_n = \mathbb{E}\left[X_{n-1}X_n\right] / \sqrt{\mathbb{E}\left[X_{n-1}^2\right]\mathbb{E}\left[X_n^2\right]},$$

satisfies $\rho_n = -1 + O(|s_1/s_2|^n)$. Clearly, $p_N \le \mathbb{P}\left(X_{N-1} \le 0, X_N \le 0\right)$, and if Y_1 is a centred Gaussian random variable, we get in view of a well-known formula for Gaussian random variables (see e.g. [GS01, Exercise 8.5.1]) that

$$\mathbb{P}\left(X_{N-1} \le 0, X_N \le 0\right) = \frac{1}{2\pi}\left(\frac{\pi}{2} + \arcsin \rho_N\right).$$

Since $\pi/2 + \arcsin x \sim \sqrt{2(1+x)}$ as $x \downarrow -1$ (by l'Hôpital's rule), it follows that $p_N \precsim |s_1/s_2|^{N/2}$.

Note that the previous results do not cover the case $a_1 + 1 = a_2$ if $a_2 \in (0,1)$. Let us now turn to this particular case. One verifies that $c_n = (a_1^{n+1} + (-1)^n)/(a_1+1)$, i.e. c_n oscillates but does not diverge as in Theorem 4.3.10. We show that p_N still decreases at least exponentially in this case.

Proposition 4.3.13. *Let $a_1 + 1 = a_2$ and set $Z_n = a_2 Z_{n-1} + Y_n$ for $n \geq 1$. Then, for all $x \geq 0$ and $N \geq 1$,*

$$\mathbb{P}\left(\sup_{n=1,\dots,N} X_n \leq x\right) \leq \mathbb{P}\left(\sup_{n=1,\dots,N} Z_n \leq 2x\right).$$

In particular, if $a_2 \in (0,1)$, $\mathbb{E}\left[(Y_1^-)^\alpha\right] < \infty$ for some $\alpha > 0$ and $\mathbb{P}\left(Y_1 > 2x(1-a_2)\right) > 0$, it holds that $p_N(x) \precsim \exp(-\lambda N)$ for some $\lambda = \lambda(x) > 0$.

Proof. Note that $X_{n+1} + X_n = (a_1+1)X_n + a_2 X_{n-1} + Y_{n+1} = a_2(X_n + X_{n-1}) + Y_{n+1}$. Hence, $(Z_n)_{n\geq 1}$ can be written in the form $Z_n := X_n + X_{n-1}$. In particular, $X_n \leq x$ for $n = 1, \dots, N$ implies that $Z_n \leq 2x$ for $n = 1, \dots, N$.
If $a_2 \in (0,1)$, we deduce from Theorem 4.3.1 that $p_N(x)$ decays exponentially under the conditions stated above. □

In fact, the proof of Proposition 4.3.13 can be generalised as follows: if X is AR(p), one can try to determine $b > 0$ such that $(Z_n)_{n\geq 1}$ is AR($p-1$) where $Z_n := X_n + bX_{n-1}$. Then we always have that $X_n \leq 0$ for $n = 1, \dots, N$ implies $Z_n \leq 0$ for $n = 1, \dots, N$. We carry this out for $p = 2$.

Proposition 4.3.14. *Let $a_1^2 + 4a_2 > 0$. Moreover, assume that either $a_1, a_2 < 0$ or that $a_1 + a_2 < 1$ if $a_2 > 0$. Then $s_2 < 0$, $-a_2/s_2 < 1$ and $Z_n := X_n - s_2 X_{n-1}$ satisfies $Z_n = -a_2/s_2 Z_{n-1} + Y_n$. In particular,*

$$\mathbb{P}\left(\sup_{n=1,\dots,N} X_n \leq x\right) \leq \mathbb{P}\left(\sup_{n=1,\dots,N} Z_n \leq (1-s_2)x\right), \quad x \geq 0.$$

Proof. Let $\rho := -a_2/s_2$. We claim that $Z_n := X_n - s_2 X_{n-1}$ is an AR(1)-process with

$$Z_n = \rho Z_{n-1} + Y_n.$$

Moreover, the assumptions on the coefficients a_1, a_2 imply that $s_2 < 0$, so by definition of Z, we have that

$$\bigcap_{n=1}^{N} \{X_n \leq x\} \subseteq \bigcap_{n=1}^{N} \{Z_n \leq (1-s_2)x\}, \quad x \geq 0.$$

In order to verify our claim, let us determine $b > 0$ such that $(Z_n)_{n\geq 1}$ defined by $Z_n := X_n + bX_{n-1}$ is an AR(1)-process: we have that

$$Z_n = (a_1 + b)X_{n-1} + a_2 X_{n-2} + Y_n = (a_1 + b)X_{n-1} + \frac{a_2}{b} b X_{n-2} + Y_n.$$

Hence, if $a_1 + b = a_2/b$, it follows indeed that $Z_n = a_2/b Z_{n-1} + Y_n$. Now $a_1 + b = a_2/b$ amounts to $b^2 + a_1 b - a_2 = 0$, and the solutions to this equation are $-s_1$ and $-s_2$. Since $a_1^2 + 4a_2 > 0$, we have that $s_2 < s_1$. Hence, we can only find $b > 0$ such that Z defines

an AR(1)-process if $s_2 < 0$, and $b := -s_2$ in that case. Now $s_2 < 0$ amounts to $a_1 \leq 0$ or $a_1, a_2 > 0$ since $h = \sqrt{a_1^2 + 4a_2} > 0$.
Finally, $\rho = a_2/(-s_2) < 1$ if and only if $a_1 + 2a_2 < h$. If $a_1, a_2 > 0$, this amounts to $a_1 + a_2 < 1$.
In the remaining cases, we necessarily have that $a_1 \leq 0$. If also $a_1 + 2a_2 \leq 0$ (in particular, if $a_1, a_2 \leq 0$), the inequality $a_1 + 2a_2 \leq 0$ is obviously satisfied. Finally, if $a_1 + 2a_2 > 0$, $a_1 + 2a_2 < h$ is equivalent to $a_1^2 + 4a_1a_2 + 4a_2^2 < a_1^2 + 4a_2$, i.e. $a_1 + a_2 < 1$ since $a_2 > 0$. □

The preceding proposition allows us to find exponential upper bounds for the persistence probability p_N for a wide class of distributions. Specifically, we obtain exponential upper bounds for certain parameters a_1 and a_2 and distributions that do not fulfill the requirements of Theorem 4.3.10. Let us record this result as a corollary:

Corollary 4.3.15. *Let a_1, a_2 be such that $a_2 > 0$ and $a_1 + a_2 < 1$. Assume that $\mathbb{E}\left[(Y_1^-)^\alpha\right] < \infty$ for some $\alpha > 0$. Let $x \geq 0$ such that $\mathbb{P}\left(Y_1 > x(1 - s_2)(1 + a_2/s_2)\right) > 0$. Then $p_N(x) \precsim \exp(-\lambda N)$ for some $\lambda = \lambda(x) > 0$.*

Proof. Set $\rho := -a_2/s_2$ and let $(Z_n)_{n\geq 1}$ satisfy $Z_n = \rho Z_{n-1} + Y_n$. By Proposition 4.3.14, we have that $\rho \in (0, 1)$ and that $p_N(x) \leq \mathbb{P}\left(\sup_{n=1,\ldots,N} Z_n \leq x(1 - s_2)\right)$. The claim now follows from Theorem 4.3.1. □

Let us finally turn to the region $a_1 > 0$ and $a_1^2 + 4a_2 < 0$ (E_3 in Figure 4.2) so that the expression of the sequence c_n involves the function sine, cf. (4.4).

Proposition 4.3.16. *Let $(a_1, a_2) \in E_3$. Assume that $\mathbb{P}(Y_1 > 0) > 0$. Then there exists $\lambda > 0$ such that $p_N \precsim \exp(-\lambda N)$ as $N \to \infty$.*

Proof. The recursion $X_n = a_1X_{n-1} + a_2X_{n-2} + Y_n$ allows us to express X_n as follows ($n \geq k + 2$):

$$X_n = \alpha_k X_{n-k} + \beta_k X_{n-k-1} + L_k(Y_{n-k+1}, \ldots, Y_n)$$

where $L_k(x_1, \ldots, x_k)$ is some linear combination of $x_1, \ldots, x_k$. Clearly, $\alpha_1 = a_1$, $\beta_1 = a_2$ and $L_1(x_1) = x_1$ and iteratively, we get that $\alpha_{k+1} = a_1\alpha_k + \beta_k$, $\beta_{k+1} = a_2\alpha_k$ and $L_{k+1}(x_1, \ldots, x_{k+1}) = \alpha_k x_1 + L_k(x_2, \ldots, x_{k+1})$ for $k \geq 1$. In particular, $\alpha_k = a_1\alpha_{k-1} + a_2\alpha_{k-2}$ for $k \geq 2$ with $\alpha_0 = 1$ and $\alpha_1 = a_1$, hence,

$$\alpha_k = c_k, \quad \beta_k = a_2 c_{k-1}, \quad L_k(x_1, \ldots, x_k) = \sum_{j=1}^{k} c_{k-j} x_j.$$

Let $q := \inf\{k \geq 1 : c_k \leq 0\} \geq 2$ ($c_0 = 1, c_1 = a_1 > 0$). Assume for a moment that $q < \infty$. Then, if $X_n \leq 0$ for all $n \leq N$, it follows that

$$\begin{aligned} 0 \geq X_n &= c_q X_{n-q} + a_2 c_{q-1} X_{n-q-1} + L_q(Y_{n-q+1}, \ldots, Y_n) \\ &\geq 0 + 0 + L_q(Y_{n-q+1}, \ldots, Y_n), \quad n = q + 2, \ldots, N, \end{aligned}$$

where we have used that $a_2 c_{q-1} < 0$ since $a_2 < 0$ on E_3 and $c_{q-1} > 0$ by the definition of q.
In particular, we have that

$$\begin{aligned}\mathbb{P}\left(\sup_{n=1,\dots,N} X_n \leq 0\right) &\leq \mathbb{P}\left(\sup_{n=q+2,\dots,N} L_q(Y_{n-q+1},\dots,Y_n) \leq 0\right)\\ &\leq \mathbb{P}\left(\sup_{k=1,\dots,\lfloor (N-2)/q\rfloor} L_q(Y_{(k-1)q+3},\dots,Y_{kq+2})) \leq 0\right)\\ &\leq \mathbb{P}\left(L_q(Y_3,\dots,Y_{q+2}) \leq 0\right)^{\lfloor (N-2)/q\rfloor},\end{aligned}$$

since $(L_q(Y_{(k-1)q+3},\dots,Y_{kq+2})_{k=0,1,\dots}$ are i.i.d. Next, note that X_q and $L_q(Y_3,\dots,Y_{q+3})$ have the same law. Hence, using that $c_0,\dots,c_{q-1} > 0$ and $\mathbb{P}(Y_1 > 0) > 0$, we have that

$$\mathbb{P}(X_q > 0) = \mathbb{P}\left(\sum_{k=1}^{q} c_{q-k} Y_k > 0\right) \geq \mathbb{P}(Y_1 > 0)^q > 0.$$

It remains to show that $q < \infty$. Let $\varphi \in (0, \pi/2)$ (since $a_1 > 0$) be the angle associated with (a_1, a_2) in (4.4). We see from (4.4) that $c_n \leq 0$ for some n if $\sin((n+1)\varphi) \leq 0$ for some n. Take $n = \lceil \pi/\varphi \rceil$. Clearly, $\pi < (n+1)\varphi \leq (\pi/\varphi + 2)\varphi < 2\pi$ since $\varphi < \pi/2$. In particular, we have shown that $q \leq \lceil \pi/\varphi \rceil$. □

We are now ready to give a proof of Theorem 4.2.2 which is a corollary of the previous results. A look at Figure 4.2 will be helpful to distinguish the different cases.
Proof of Theorem 4.2.2. On E_1, the assertion follows from Corollary 4.3.11. On $E_2 = (-\infty, 0]^2$, the assertion is trivial. If $(a_1, a_2) \in E_3$, we can apply Proposition 4.3.16. The remaining cases covered by Theorem 4.3.2 and Proposition 4.3.13 (the latter is needed for the strip $a_2 = 1 + a_1$ with $a_1 \in (-1, 0)$ only). □

Note that we have established exponential upper bounds on p_N under various conditions on the distribution of Y_1 in the region Δ_2 defined as the set of (a_1, a_2) such that c_n goes to 0 for AR(2)-processes (cf. (4.5)). Indeed, on $\Delta_2 \cap E_2$, the assertion is trivial, and on $\Delta_2 \cap E_3$, we can use Proposition 4.3.16. Taking into account the corollaries 4.3.7 and 4.3.15, we see that we have obtained exponential upper bounds on p_N under different conditions on Y_1 except for the the curve $a_1^2 + 4a_2 = 0$ with $a_1 \in (0, 2)$. In that case, $c_n = (a_1/2)^n (n+1)$, and by Theorem 4.3.2, we know that $p_N \precsim \exp(-\lambda N/\log N)$ if $\mathbb{E}[\exp(|Y_1|^\alpha)]$ is finite. If Y_1 has a Gaussian law with zero mean, the next proposition establishes an exponential upper bound on p_N for these values of (a_1, a_2). In particular, in combination with the Theorems 4.2.1, 4.2.2 and 4.2.3, we directly obtain Theorem 4.2.4.

Proposition 4.3.17. *Let Y_1 have a Gaussian law. Let $\rho \in (0, 1)$ and $(\alpha_n)_{n\geq 0}$ denote a sequence of positive numbers with the following properties:*

$$\exists C > 0 \text{ such that } \alpha_{n+m} \leq C\alpha_n\alpha_m \quad \forall n, m \geq 0, \qquad \lim_{n\to\infty} e^{-\lambda n}\alpha_n = 0 \quad \forall \lambda > 0.$$

Set $X_n := \sum_{k=1}^{n} \alpha_{n-k}\rho^{n-k} Y_k$. *It holds that*

$$\liminf_{N\to\infty} -N^{-1} \log \mathbb{P}\left(\sup_{n=1,\dots,N} X_n \le x\right) > 0, \quad x \in \mathbb{R}.$$

Proof. Clearly, we may suppose that $\mathbb{E}\left[(Y_1 - \mathbb{E}\left[Y_1\right])^2\right] = 1$. Moreover, it suffices to consider the case $\mathbb{E}\left[Y_1\right] = 0$. To see this, set $\sum_{k=1}^{n} \alpha_{n-k}\rho^{n-k}(Y_k - \mu)$. If $\mu := \mathbb{E}\left[Y_1\right] < 0$, we have that

$$X_n = \sum_{k=1}^{n} \alpha_{n-k}\rho^{n-k}(Y_k - \mu) + \mu \sum_{k=0}^{n-1} \alpha_k \rho^k \ge \tilde{X}_n + \mu \sum_{k=0}^{\infty} \alpha_k \rho^k,$$

where $A := \sum_{k=0}^{\infty} \alpha_k \rho^k < \infty$ since $\rho < 1$ and $\alpha_n = e^{o(n)}$. Hence,

$$\mathbb{P}\left(\sup_{n=1,\dots,N} X_n \le x\right) \le \mathbb{P}\left(\sup_{n=1,\dots,N} \tilde{X}_n \le x - \mu A\right).$$

Similarly, if $\mu > 0$, $X_n \ge \tilde{X}_n$ for all n, and therefore

$$\mathbb{P}\left(\sup_{n=1,\dots,N} X_n \le x\right) \le \mathbb{P}\left(\sup_{n=1,\dots,N} \tilde{X}_n \le x\right).$$

Hence, we can assume from now on that $\mathbb{E}\left[Y_1\right] = 0$ and $\mathbb{E}\left[Y_1^2\right] = 1$. Let $\rho < \delta < 1$ and set

$$\gamma_n := \sqrt{\frac{\sum_{k=0}^{n-1} \rho^{2k}\alpha_k^2}{\sum_{k=0}^{n-1} \delta^{2k}}}, \qquad Z_n := \gamma_n \sum_{k=1}^{n} \delta^{n-k} Y_k.$$

We would like to apply Slepian's inequality (part 1 of Lemma 1.2.5) to compare the probabilities that X and Z stay below 0 until time N. By construction, we have that $\mathbb{E}\left[X_n^2\right] = \mathbb{E}\left[Z_n^2\right]$ for all $n \ge 1$. Next, note that $\gamma_n \ge \alpha_0\sqrt{1-\delta^2}$ for all $n \ge 1$. Hence, if $n > m \ge 1$, we have that

$$\mathbb{E}\left[Z_n Z_m\right] = \gamma_n\gamma_m \sum_{k=1}^{m} \delta^{n-k}\delta^{m-k} \ge \alpha_0^2(1-\delta^2)\delta^{n-m} \sum_{k=1}^{m} \delta^{2(m-k)} \ge C_1\delta^{n-m},$$

where $C_1 := \alpha_0^2(1-\delta^2)$. Moreover,

$$\begin{aligned}\mathbb{E}\left[X_n X_m\right] &= \sum_{k=1}^{m} \alpha_{n-k}\alpha_{m-k}\rho^{m-k}\rho^{n-k} = \rho^{n-m} \sum_{k=1}^{m} \alpha_{(n-m)+m-k}\alpha_{m-k}\rho^{2(m-k)} \\ &\le C\rho^{n-m}\alpha_{n-m} \sum_{k=1}^{m} \alpha_{m-k}^2\rho^{2(m-k)} \le C\rho^{n-m}\alpha_{n-m} \sum_{k=0}^{\infty} \alpha_k^2\rho^{2k} =: C_2\rho^{n-m}\alpha_{n-m}.\end{aligned}$$

In the last equality, we have used that $\sum_{k=0}^{\infty} \alpha_k^2\rho^{2k}$ converges since $\alpha_n = e^{o(n)}$. Now $C_1\delta^{n-m} \ge C_2\alpha_{n-m}\rho^{n-m}$ holds whenever $n - m \ge q$ for some $q \ge 1$ since $\delta > \rho$ and

α_n grows slower than any exponential. In particular, $\mathbb{E}[X_n X_m] \le \mathbb{E}[Z_n Z_m]$ whenever $|n-m| \ge q$.
Hence, using Slepian's inequality, we obtain that

$$\mathbb{P}\left(\sup_{n=1,\dots,N} X_n \le x\right) \le \mathbb{P}\left(\sup_{n=1,\dots,\lfloor N/q\rfloor} X_{nq} \le x\right) \le \mathbb{P}\left(\sup_{n=1,\dots,\lfloor N/q\rfloor} Z_{nq} \le x\right).$$

Let $\tilde{Z}_n := \delta^{-nq} Z_{nq}/\gamma_{nq} = \sum_{k=1}^{nq} \delta^{-k} Y_k$. One verifies easily that $(\tilde{Z}_n)_{n\ge1}$ is equal in distribution to $(B(t_n))_{n\ge1}$ where $(B_t)_{t\ge0}$ is a one-dimensional Brownian motion and $t_n := \sum_{k=1}^{nq} \delta^{-2k} = C_\delta(\delta^{-2nq}-1)$, so

$$\begin{aligned}\mathbb{P}\left(\sup_{n=1,\dots,N} X_n \le x\right) &\le \mathbb{P}\left(\sup_{n=1,\dots,\lfloor N/q\rfloor} Z_{nq} \le x\right) = \mathbb{P}\left(\bigcap_{n=1}^{\lfloor N/q\rfloor} \left\{\tilde{Z}_n \le x\delta^{-nq}/\gamma_{nq}\right\}\right) \\ &= \mathbb{P}\left(\bigcap_{n=1}^{\lfloor N/q\rfloor} \left\{B(C_\delta(\delta^{-2qn}-1)) \le x\delta^{-nq}/\gamma_{nq}\right\}\right) \le \mathbb{P}\left(\sup_{n=1,\dots,\lfloor N/q\rfloor} B(\delta^{-2qn}-1) \le \tilde{x}\right),\end{aligned}$$

where we have used the scaling property of Brownian motion and the fact that $\gamma_n \ge C_1\alpha_0/\delta^n$ for all n (i.e. $\tilde{x} := x/(C_1\alpha_0 C_\delta^{1/2})$). Next, note that

$$\begin{aligned}\mathbb{P}\left(\sup_{n=1,\dots,N} B(\delta^{-2qn}) \le 0\right) &\ge \mathbb{P}\left(B_1 \le -\tilde{x}, \sup_{n=1,\dots,N} B(\delta^{-2qn}) - B_1 \le \tilde{x}\right) \\ &= \mathbb{P}\left(B_1 \le -\tilde{x}\right) \mathbb{P}\left(\sup_{n=1,\dots,N} B(\delta^{-2qn}-1) \le \tilde{x}\right).\end{aligned}$$

Hence,

$$\mathbb{P}\left(\sup_{n=1,\dots,N} X_n \le x\right) \le \frac{\mathbb{P}\left(B(\delta^{-2qn}) \le 0, \forall n = 1,\dots,\lfloor N/q\rfloor\right)}{\mathbb{P}\left(B_1 \le -\tilde{x}\right)}.$$

If $(U_t)_{t\ge0}$ is the Ornstein-Uhlenbeck process, an application of Slepian's inequality together with a subadditivity argument (see (2.17) above) yields for $a := \delta^{-2q} > 1$ that

$$\begin{aligned}&\lim_{N\to\infty} -N^{-1}\log\mathbb{P}\left(B(a^n) \le 0, \forall n = 1,\dots,N\right) \\ &\quad = \lim_{N\to\infty} -N^{-1}\log\mathbb{P}\left(U(\log(a)n) \le 0, \forall n = 1,\dots,N\right) > 0,\end{aligned}$$

so the claim follows. □

4.3.2 Exponential lower bounds

Let us now comment on exponential lower bounds for AR-processes. In general, we cannot expect to find exponential lower bounds in the whole region where we have established exponential upper bounds. The following example illustrates this point for AR(2)-processes.

Example 4.3.18. If X is AR(p) and the innovation Y_1 takes only the values $\pm y$ for some $y > 0$ and $a_1 < -1$, then $p_2 = \mathbb{P}(X_1 \leq 0, X_2 \leq 0) = 0$. Indeed, on $\{X_1 \leq 0\} = \{Y_1 = -y\}$, we have that $X_2 = a_1 Y_1 + Y_2 \geq -ya_1 - y = -y(a_1 + 1) > 0$.
Similarly, if $a_1 \in [-1, 0]$ and $a_1(a_1 + 1) + a_2 < -1$, one has that $p_3 = 0$.

Note that even if one chooses a very large boundary $x > 0$, it can happen that $p_N(x) = 0$ for all N large enough. Indeed, if $Z_n = \rho Z_{n-1} + Y_n$ with $\rho < -2$ and $\mathbb{P}(Y_1 = 1) = 1 - \mathbb{P}(Y_1 = -1)$, note that

$$E_N := \bigcap_{n=1}^{N} \{Z_n \leq x\} \subseteq \bigcap_{n=1}^{N-1} \{Z_n \in [(x+1)/\rho, x]\},$$

since $x \geq Z_n \geq \rho Z_{n-1} - 1$. Hence, the event E_{N+1} implies that the absolute value of Z remains bounded by a constant independent of N until time N. However, note that

$$\begin{aligned} |Z_N| &= \left| \sum_{k=1}^{N} \rho^{N-k} Y_k \right| \geq |\rho|^{N-1} |Y_1| - \left| \sum_{k=2}^{N} \rho^{N-k} Y_k \right| \geq |\rho|^{N-1} - \sum_{k=0}^{N-2} |\rho|^k \\ &= |\rho|^{N-1} - \frac{|\rho|^{N-1} - 1}{|\rho| - 1} = \frac{|\rho|^{N-1}(|\rho| - 2) + 1}{|\rho| - 1}. \end{aligned}$$

Since $|\rho| > 2$, it is clear that $\mathbb{P}(Z_N \in [(x+1)/\rho, x]) = 0$ for all N large enough, and hence, $p_N(x) = 0$.
Let us also remark that if X is AR(p) with $a_1 \geq 0, \ldots, a_p \geq 0$, it is trivial to obtain the exponential lower bound $p_N(x) \geq p_N \geq \mathbb{P}(Y_1 \leq 0)^N$.
The following theorem states a simple condition on the coefficients $a_1, \ldots, a_p$ such that the persistence probability cannot decay faster than exponentially.

Theorem 4.3.19. *If X is AR(p) with $\sum_{k=1}^{p} |a_k| < 1$, it holds for some $c \in (0,1)$ that $c^N \precsim p_N$ as $N \to \infty$. Moreover, if $a_k > 0$ for some $k \in \{1, \ldots, p\}$, one may take*

$$c := \sup \{\mathbb{P}(Y_1 \in [\alpha(1 - a_+), \alpha |a_-|]) : \alpha < 0\}$$

where (with the convention that $\sum_{\emptyset} = 0$)

$$a_+ := \sum_{k \in I_+} a_k, \quad a_- := \sum_{k \in I_-} a_k, \quad I_+ = \{k : a_k > 0\}, \quad I_- = \{k : a_k < 0\}.$$

Proof. The goal is to find intervals $([\alpha_n, \beta_n])_{n \geq 1}$ such that

$$\bigcap_{k=1}^{n} \{Y_k \in [\alpha_k, \beta_k]\} \subseteq \bigcap_{k=1}^{n} \{X_k \in [\gamma_k, 0]\}, \quad n \geq 1. \tag{4.16}$$

If (4.16) holds and $\mathbb{P}(Y_n \in [\alpha_n, \beta_n]) \geq c > 0$ for all $n \geq 1$, we immediately obtain that $c^N \precsim \mathbb{P}(X_n \leq 0, \forall n = 1, \ldots, N)$.

Using the recursive definition of X, we can iteratively define the sequences $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 1}$,$(\gamma_n)_{n\geq 1}$ as follows: Start with $\gamma_1 = \alpha_1 < \beta_1 \leq 0$. Define successively (with the convention $\gamma_n = 0$ for $n \leq 0$)

$$\beta_k := -\sum_{j\in I_-} a_j \gamma_{k-j}, \quad \alpha_k < \beta_k, \quad \gamma_k := \sum_{j\in I_+} a_j\gamma_{k-j} + \alpha_k.$$

It is clear that $\gamma_k \leq 0$ and $\beta_k \leq 0$ for all k. We claim that (4.16) holds for such sequences $(\alpha_n), (\beta_n)$ and (γ_n). For $n = 1$, this is obvious, and inductively, if the statement holds for some $n - 1 \geq 1$, we have that

$$X_n = \sum_{j=1}^{p} a_k X_{n-j} + Y_n \leq \sum_{j\in I_-}^{p} a_j X_{n-j} + \beta_n \leq \sum_{j\in I_-} a_j\gamma_{n-j} + \beta_n = 0,$$

and

$$X_n = \sum_{j=1}^{p} a_k X_{n-j} + Y_n \geq \sum_{j\in I_+}^{p} a_j X_{n-j} + \alpha_n \geq \sum_{j\in I_+} a_j\gamma_{n-j} + \alpha_n = \gamma_n.$$

Note that the above inequalities hold even if $I_+ = \emptyset$ or if $I_- = \emptyset$.
Let us now state a suitable choice for the sequences $(\alpha_n), (\beta_n)$ and (γ_n). Fix $\alpha_1 = \gamma_1 < \beta_1 = 0$ and let $\alpha_k = -\alpha_1(a_+ - 1)$ for all $k \geq 2$. We claim that $\gamma_k \geq \alpha_1$. Inductively, if the claim holds for all $k \leq n - 1$, we have that

$$\gamma_n = \sum_{j\in I_+} a_j\gamma_{n-j} - \alpha_1(a_+ - 1) \geq \alpha_1 a_+ - \alpha_1(a_+ - 1) = \alpha_1.$$

It follows that $\beta_n \geq -\alpha_1 a_-$ and in particular, $\alpha_k < \beta_k$ since

$$\alpha_k - \beta_k \leq -\alpha_1(a_+ - 1) + \alpha_1 a_- = -\alpha_1 \left(\sum_{k=1}^{p} |a_k| - 1\right) < 0.$$

In view of (4.16), we obtain that

$$\begin{aligned}\mathbb{P}\left(\sup_{k=1,\ldots,n} X_k \leq 0\right) &\geq \prod_{k=1}^{n} \mathbb{P}\left(Y_k \in [\alpha_k, \beta_k]\right) \\ &\geq \mathbb{P}\left(Y_1 \in [\alpha_1, 0]\right) \mathbb{P}\left(Y_1 \in [-\alpha_1(a_+ - 1), -\alpha_1 a_-]\right)^{n-1}\end{aligned}$$

□

Remark 4.3.20. In general, there is no reason to believe that the lower bound of Theorem 4.3.19 is sharp.

Corollary 4.3.21. *Let $(Y_n)_{n\geq 0}$ be a sequence of i.i.d. standard Gaussian random variables. Using the notation of Theorem 4.3.19, if I_- and I_+ are nonempty, we have that $p_N \geq c^N$ where*

$$c = \mathbb{P}\left(\alpha^*(1-a_+) \leq Y_1 \leq \alpha^* |a_-|\right) = \mathbb{P}\left(-\sqrt{\frac{-\log A^2}{1-A^2}} \leq Y_1 \leq -A\sqrt{\frac{-\log A^2}{1-A^2}}\right)$$

and

$$\alpha^* := -\sqrt{\frac{\log(1-a_+)^2 - \log|a_-|^2}{(1-a_+)^2 - |a_-|^2}} < 0, \quad A := \frac{|a_-|}{1-a_+} \in (0,1).$$

Proof. By Theorem 4.3.19, we have to determine

$$\sup_{\alpha\leq 0} \mathbb{P}\left(\alpha(1-a_+) \leq Y_1 \leq \alpha |a_-|\right) = \sup_{\alpha\leq 0} \left\{\Phi(\alpha |a_-|) - \Phi(\alpha(1-a_+))\right\},$$

where Φ is the cumulative distribution function of a standard normal random variable. It is not hard to verify that the unique maximum is attained at

$$\alpha^* := -\sqrt{\frac{\log(1-a_+)^2 - \log|a_-|^2}{(1-a_+)^2 - |a_-|^2}} < 0.$$

□

4.4 Polynomial order

If X is an AR(2)-process and $\mathbb{E}[Y_1] = 0$ and $\mathbb{E}[Y_1^2] < \infty$, it is known that p_N decays polynomially if X is a random walk ($a_1 = 1, a_2 = 0$) or an integrated random walk ($a_1 = 2, a_2 = -1$). These results have been outlined in Section 1.2.1. For instance, if $S_n = \sum_{k=1}^n Y_k$, recall that $\mathbb{P}(S_n \leq 0, n = 1, \ldots, N) \sim C\, N^{-1/2}$. Moreover, note that the process $X_n = 2X_{n-1} - X_{n-1} + Y_n$ is given by $X_n = \sum_{k=1}^n (n-k+1)Y_k = \sum_{k=1}^n S_k$, so X is indeed an integrated random walk and $p_N \asymp N^{-1/4}$.

4.4.1 Integrated AR-processes

In this subsection, we will prove that $p_N = N^{-1/2+o(1)}$ under suitable moment conditions if $a_1 + a_2 = 1$ and $|a_2| < 1$. As we will see shortly, these AR(2)-processes can be written as integrated AR(1)-processes.

Let us begin by characterising the behaviour of the sequence $(c_n)_{n\geq 0}$ for such a_1, a_2. Instead of manipulating the explicit expression for c_n in (4.3), we give a short proof of the following lemma.

Lemma 4.4.1. *The sequence (c_n) converges to a constant $c \neq 0$ if and only if $a_1 + a_2 = 1$ and $|a_2| < 1$. In that case, $\lim_{n\to\infty} c_n = 1/(1 + a_2)$. Moreover, if $a_1 + a_2 = 1$, $c_n = (1 - (-a_2)^{n+1})/(1 + a_2)$ if $a_2 \neq -1$, and $c_n = n + 1$ if $a_2 = -1$ for $n \geq 0$.*

Proof. Assume that $a_1 + a_2 = 1$. Then $c_{n+1} = (a_1 + a_2 - a_2)c_n + a_2 c_{n-1} = c_n - a_2(c_n - c_{n-1})$, i.e. $c_{n+1} - c_n = -a_2(c_n - c_{n-1})$. Iteration yields $c_{n+1} - c_n = (-a_2)^n(c_1 - c_0) = (-a_2)^{n+1}$ since $c_1 - c_0 = a_1 - 1 = -a_2$. Hence, since $c_0 = 1$, we find that

$$c_n = 1 + \sum_{k=1}^{n}(c_k - c_{k-1}) = \begin{cases} 1 + \sum_{k=1}^{n}(-a_2)^k = \frac{1-(-a_2)^{n+1}}{1-(-a_2)}, & a_2 \neq -1, \\ n + 1, & a_2 = -1, \end{cases}$$

and therefore, $c_n \to c = 1/(1 + a_2) \neq 0$ if and only if $|a_2| < 1$.
On the other hand, if $\lim c_n = c \neq 0$, then the recursion equation implies that $c = a_1 c + a_2 c$, i.e. $a_1 + a_2 = 1$. By the preceding lines, convergence implies that $|a_2| < 1$. □

In particular, the preceding lemma shows for $a_1 + a_2 = 1$ and $|a_2| < 1$ that

$$X_n = \frac{1}{1 + a_2}\left(\sum_{k=1}^{n} Y_k - \sum_{k=1}^{n}(-a_2)^{n-k+1}Y_k\right), \quad n \geq 1,$$

and since $|a_2| < 1$, one expects that the behaviour of X is similar to that of a random walk.
Moreover, AR(2)-processes with $a_1 + a_2 = 1$ and $|a_2| < 1$ can also be regarded as integrated AR(1)-processes. Let us explain this in more detail.
If $\tilde{X}$ is AR(p) with coefficients $a_1, \ldots, a_p$, set $X_n := \sum_{k=1}^{n} \tilde{X}_k$. Then

$$\begin{aligned} X_n &= X_{n-1} + \sum_{k=1}^{p} a_k \tilde{X}_{n-k} + Y_n = X_{n-1} + \sum_{k=1}^{p} a_k(X_{n-k} - X_{n-k-1}) + Y_n \\ &= (1 + a_1)X_{n-1} + \sum_{k=2}^{p}(a_k - a_{k-1})X_{n-k} - a_p X_{n-p-1} + Y_n, \end{aligned}$$

i.e. X is AR($p+1$) and the transfomation of the coefficients $T_p\colon \mathbb{R}^p \to \mathbb{R}^{p+1}$ is given by

$$T_p(a_1, \ldots, a_p) = (a_1 + 1, a_2 - a_1, \ldots, a_p - a_{p-1}, -a_p). \tag{4.17}$$

Note that T_p is one-to-one and that $T_p(\mathbb{R}^p)$ is an affine subspace of $\mathbb{R}^{p+1}$.
Now, if $\tilde{X}$ is AR(1) with $\tilde{X}_n = \rho\tilde{X}_{n-1} + Y_n$, we have that X with $X_n = \sum_{k=1}^{n} \tilde{X}_k$ is AR(2) with coefficients $T_1(\rho) = (\rho - 1, -\rho) =: (a_1, a_2)$. In other words, AR(2)-processes with $a_1 + a_2 = 1$ and $|a_2| < 1$ are integrated AR(1)-processes with $\rho = -a_2$ and $|\rho| < 1$.
Finally, let us mention that the class of AR(p)-processes contains p-times integrated random walks $S^{(p)}$ as a special case (i.e. $S^{(1)}$ is a centred random walk, and $S_n^{(p)} = \sum_{k=1}^{n} S_k^{(p-1)}$). Here, the behaviour of the persistence probability is not known for $p \geq 3$.
The next theorem states conditions under which the persistence probability of an integrated process behaves like $N^{-1/2+o(1)}$.

Theorem 4.4.2. *Let $(\tilde{c}_k)_{k\geq 0}$ denote a sequence of real numbers such that $\sum_{k=1}^{\infty} k\,|\tilde{c}_k| < \infty$ and $\sum_{k=0}^{\infty} \tilde{c}_k \neq 0$. Let $\tilde{X}_n = \sum_{k=1}^{n} \tilde{c}_{n-k} Y_k$ where $Y_1, Y_2, \ldots$ is a sequence of centred i.i.d. random variables. Set $X_n := \sum_{k=1}^{n} \tilde{X}_k$.*

1. *If $|Y_1| \leq M < \infty$ a.s., there is $x_0 \geq 0$ such that for all $x \geq x_0$, it holds that*
$$\mathbb{P}\left(\sup_{n=1,\ldots,N} X_n \leq x\right) \asymp N^{-1/2}, \quad N \to \infty.$$
2. *If $\mathbb{E}\left[\exp(|Y_1|^{\alpha})\right] < \infty$, it holds for all $x \geq 0$ that*
$$\mathbb{P}\left(\sup_{n=1,\ldots,N} X_n \leq x\right) \precsim N^{-1/2}(\log N)^{1/\alpha}, \quad N \to \infty.$$
3. *If $\mathbb{E}\left[\exp(|Y_1|^{\alpha})\right] < \infty$ and $\sum_{k=0}^{n} \tilde{c}_k \geq 0$ for all $n \geq 0$, it holds for all $x \geq 0$ that*
$$N^{-1/2}(\log N)^{-1/\alpha+o(1)} \precsim \mathbb{P}\left(\sup_{n=1,\ldots,N} X_n \leq x\right), \quad N \to \infty.$$

Proof. First, note that
$$X_n = \sum_{k=1}^{n}\sum_{j=1}^{k} \tilde{c}_{k-j} Y_j = \sum_{j=1}^{n} Y_j \sum_{k=j}^{n} \tilde{c}_{k-j} = \sum_{j=1}^{n} Y_j \sum_{k=0}^{n-j} \tilde{c}_k = \sum_{k=1}^{n} c_{n-k} Y_k$$
where $c_n := \sum_{k=0}^{n} \tilde{c}_k \to c = \sum_{k=0}^{\infty} \tilde{c}_k \neq 0$. Set $S_n := \sum_{k=1}^{n} cY_k$, so that for all $n \geq 1$,
$$|S_n - X_n| = \left|\sum_{k=1}^{n} (c - c_{n-k}) Y_k\right|,$$
In particular, if $|Y_1| \leq M < \infty$ a.s., it follows that
$$|S_n - X_n| \leq M \sum_{k=0}^{n-1} |c - c_k| \leq M \sum_{k=0}^{n-1}\sum_{j=k+1}^{\infty} |\tilde{c}_j| = M \sum_{j=1}^{\infty} j\,|\tilde{c}_j| =: \tilde{M} < \infty.$$
Hence, $S_n - \tilde{M} \leq X_n \leq S_n + \tilde{M}$ for all n, and we get for $x \geq \tilde{M}$ that
$$\mathbb{P}\left(\sup_{n=1,\ldots,N} S_n \leq 0\right) \leq \mathbb{P}\left(\sup_{n=1,\ldots,N} X_n \leq x\right) \leq \mathbb{P}\left(\sup_{n=1,\ldots,N} S_n \leq x + \tilde{M}\right),$$
and the proof of part 1. is complete since S is a centred random walk with finite variance. The proof of part 2. is similar. Let $E_N := \left\{|Y_k| \leq (2\log N)^{1/\alpha}, k = 1, \ldots, N\right\}$. On E_N, we get as above that
$$|S_n - X_n| \leq (2\log N)^{1/\alpha} \sum_{k=0}^{n-1} |c - c_k| \leq (2\log N)^{1/\alpha} \sum_{j=1}^{\infty} j\,|\tilde{c}_j| =: C(\log N)^{1/\alpha}. \quad (4.18)$$

Hence,

$$\mathbb{P}\left(\sup_{n=1,\dots,N} X_n \leq x\right) \leq \mathbb{P}\left(E_N^c\right) + \mathbb{P}\left(\sup_{n=1,\dots,N} S_n \leq x + C(\log N)^{1/\alpha}\right).$$

By Chebyshev's inequality,

$$\mathbb{P}\left(E_N^c\right) \leq N\mathbb{P}\left(|Y_1| \geq (2\log N)^{1/\alpha}\right) \leq N\mathbb{E}\left[\exp(|Y_1|^\alpha)\right] N^{-2} \asymp N^{-1}.$$

Finally, by Lemma 3.3.2, it holds that

$$\mathbb{P}\left(\sup_{n=1,\dots,N} S_n \leq x + C\left(\log N\right)^{1/\alpha}\right) \precsim \left(\log N\right)^{1/\alpha} N^{-1/2},$$

which proves part 2.

It suffices to prove the lower bound of part 3 for $x = 0$. Moreover, we use that independent random variables $Y_1, \dots, Y_N$ are associated for every N, cf. [EPW67]. Since $c_n = \sum_{k=0}^n \tilde{c}_k \geq 0$ for every n by assumption, the function

$$f_{K,L}(x_1, \dots, x_N) \mapsto \begin{cases} -1, & \sum_{k=1}^n c_{n-k}x_k \leq 0 \quad \text{for all } n = K, \dots, L, \\ 0, & \text{else}, \end{cases}$$

is non-decreasing in every component. Hence, the very definition of associated random variables implies for $1 \leq N_0 < N$ that

$$\operatorname{cov}\left(f_{1,N_0}(Y_1, \dots, Y_N), f_{N_0+1,N}(Y_1, \dots, Y_N)\right) \geq 0,$$

or equivalently,

$$\mathbb{P}\left(\sup_{n=1,\dots,N} X_n \leq 0\right) \geq \mathbb{P}\left(\sup_{n=1,\dots,N_0} X_n \leq 0\right) \mathbb{P}\left(\sup_{n=N_0+1,\dots,N} X_n \leq 0\right). \tag{4.19}$$

Hence, we can bound the persistence probability p_N of X from below as follows:

$$\begin{aligned} p_N &\geq p_{N_0} \cdot \mathbb{P}\left(\sup_{n=N_0+1,\dots,N} X_n \leq 0, E_N\right) \\ &\geq p_{N_0} \cdot \mathbb{P}\left(\sup_{n=N_0+1,\dots,N} S_n \leq -C\left(\log N\right)^{1/\alpha}, E_N\right). \end{aligned} \tag{4.20}$$

Note that we have used (4.18) in the second inequality. Next,

$$\begin{aligned} &\mathbb{P}\left(\sup_{n=N_0+1,\dots,N} S_n \leq -C\left(\log N\right)^{1/\alpha}, E_N\right) \\ &\geq \mathbb{P}\left(\sup_{n=N_0+1,\dots,N} S_n \leq -C\left(\log N\right)^{1/\alpha}\right) - \mathbb{P}\left(E_N^c\right) \\ &\geq \mathbb{P}\left(\sup_{n=N_0+1,\dots,N} S_n - S_{N_0} \leq 0, S_{N_0} \leq -C\left(\log N\right)^{1/\alpha}\right) - \mathbb{P}\left(E_N^c\right) \\ &\geq \mathbb{P}\left(\sup_{n=1,\dots,N} S_n \leq 0\right) \mathbb{P}\left(S_{N_0} \leq -C\left(\log N\right)^{1/\alpha}\right) - \mathbb{P}\left(E_N^c\right). \end{aligned}$$

Let $N_0 := \lfloor \log N \rfloor^{2/\alpha}$. Then $\mathbb{P}\left(S_{N_0} \le -C\,(\log N)^{1/\alpha}\right) \ge \mathbb{P}\left(S_{N_0}/\sqrt{N_0} \le -C\right)$ and the r.h.s. converges to a positive constant by the central limit theorem. Using the estimate on $\mathbb{P}\left(E_N^c\right)$ from above and (4.20), we have for N large enough that

$$p_N \ge C_1\, p_{N_0} \cdot N^{-1/2} = C_1\, \mathbb{P}\left(\sup_{n=1,\dots,\lfloor \log N \rfloor^{2/\alpha}} X_n \le 0 \right) N^{-1/2}. \tag{4.21}$$

Since $c_n \ge 0$ for all n, we can now use the trivial estimate $p_{N_0} \ge \mathbb{P}\left(Y_1 \le 0\right)^{N_0} = e^{-\kappa N_0}$ implying for N large enough that

$$p_N \ge C_1 \,\exp(-\kappa \lfloor \log N \rfloor^{2/\alpha})\, N^{-1/2}.$$

Using this as an a priori estimate for p_{N_0}, we get for large N in view of (4.21) that

$$\begin{aligned} p_N &\ge C_1^2 \,\exp(-\kappa \lfloor \log N_0 \rfloor^{2/\alpha})\, N_0^{-1/2}\, N^{-1/2} \\ &= C_1^2 \exp\left(-\kappa \lfloor \log\left(\lfloor \log N \rfloor^{1/\alpha}\right) \rfloor^{2/\alpha}\right)\, \lfloor \log N \rfloor^{-1/\alpha}\, N^{-1/2} \\ &\ge C_2 \exp(-C_3 (\log\log N)^{2/\alpha})\, (\log N)^{-1/\alpha}\, N^{-1/2}. \end{aligned}$$

Using this improved estimate again to obtain a lower bound on p_{N_0}, we deduce from (4.21) that $(\log N)^{-1/\alpha + o(1)} N^{-1/2} \precsim p_N$. □

Remark 4.4.3. One cannot expect to get a useful lower bound without any restriction on the weights c_n. For instance, if Y_1 takes only values ± 1 and $X_n = \sum_{k=1}^n c_{n-k} Y_k$ with $c_0 = 1, c_1 = -3$, then $\mathbb{P}\left(X_1 \le 0, X_2 \le 0\right) = \mathbb{P}\left(X_1 \le 0, X_1 + X_2 \le 0\right) = 0$.

Corollary 4.4.4. *Assume that* $\mathbb{E}\left[Y_1\right] = 0$. *Let* $a_1 + a_2 = 1$ *with* $|a_2| < 1$ *and* $x \ge 0$.

1. *If* $|Y_1| \le M$ *a.s., it holds that* $p_N(x) \asymp N^{-1/2}$ *as* $N \to \infty$.
2. *If* $\mathbb{E}\left[\exp(|Y_1|^\alpha)\right] < \infty$ *for some* $\alpha > 0$, *it holds that* $p_N(x) = N^{-1/2 + o(1)}$ *as* $N \to \infty$.

Proof. If X is AR(2) with coefficients a_1, a_2 as in the statement of the corollary, we have seen that $X_n = \sum_{k=1}^n Z_k$ where Z is AR(1) with $Z_n = -a_2 Z_{n-1} + Y_n$, i.e. $Z_n = \sum_{k=1}^n (-a_2)^{n-k} Y_k$. Since $\sum_{k=0}^n (-a_2)^k > 0$ for all n, part 2 and part 3 of Theorem 4.4.2 imply part 2 of the corollary. Similarly, by part 1 of Theorem 4.4.2 and the fact that $p_N(x) \asymp p_N$ (see the comment on p. 88), we obtain part 1 of the corollary. □

In analogy to the results for random walks, it is very likely that the assertion of Corollary 4.4.4 remains true under the much weaker integrability assumption $\mathbb{E}\left[Y_1^2\right] < \infty$. Depending on the sign of a_2, we can improve the preceding corollary by proving an upper or lower bound of order $N^{-1/2}$:

Proposition 4.4.5. *Let* $a_1 + a_2 = 1$ *with* $|a_2| < 1$. *Assume that* $\mathbb{E}\left[Y_1\right] = 0$, $\mathbb{E}\left[Y_1^2\right] < \infty$.

1. *If* $a_2 > 0$, *we have that* $p_N(x) \precsim N^{-1/2}$ *for all* $x \ge 0$.

2. If $a_2 < 0$, we have that $p_N(x) \gtrsim N^{-1/2}$ for all $x \geq 0$.

Proof. For $n \geq 1$, set $S_n := X_n + a_2 X_{n-1}$, and since $a_1 + a_2 = 1$, note that

$$S_n = a_1 X_{n-1} + a_2 X_{n-2} + Y_n + a_2 X_{n-1} = X_{n-1} + a_2 X_{n-2} + Y_n = S_{n-1} + Y_n,$$

i.e. $(S_n)_{n\geq 1}$ defines a centred random walk. In particular, if $a_2 > 0$, it holds that $X_n \leq x$ for $n = 1, \ldots, N$ implies that $S_n \leq (1 + a_2)x$ for $n = 1, \ldots, N$ and therefore,

$$p_N(x) \leq \mathbb{P}\left(\sup_{n=1,\ldots,N} S_n \leq a_2 x \right) \lesssim N^{-1/2}.$$

Similarly, we may write $X_n = -a_2 X_{n-1} + S_n$. If $a_2 < 0$, this yields by induction that $S_n \leq 0$ for $n = 1, \ldots, N$ implies that $X_n \leq 0$ for $n = 1, \ldots, N$. Hence, the lower bound follows. □

Let us finally remark that Theorem 4.4.2 is also applicable to integrated AR(p)-processes such that the roots $s_1, \ldots, s_p$ of the corresponding characteristic polynomial lie inside the unit disc. Let us just state the simplest case of bounded innovations Y_n. Set

$$\Delta_p := \left\{ (a_1, \ldots, a_p) : \max_{k=1,\ldots,p} |s_k| < 1 \right\},$$

where $s_1, \ldots, s_p$ are the roots of the characteristic polynomial, see p. 86.

Corollary 4.4.6. *Let X be the AR(p)-process corresponding to $(a_1, \ldots, a_p) \in \Delta_p$. Assume that $|Y_1| \leq M < \infty$ a.s. Then there is $x_0 \geq 0$ such that for all $x \geq x_0$, we have that*

$$\mathbb{P}\left(\sup_{n=1,\ldots,N} \sum_{k=1}^{n} X_k \leq x \right) \asymp N^{-1/2}.$$

Since we know the region Δ_2 explicitly (cf. Figure 4.1), we obtain the following result for AR(3)-processes:

Corollary 4.4.7. *Let X be AR(3) with a_1, a_2, a_3 satisfying*

$$a_1 + a_2 + a_3 = 1, \quad a_2 < \min\{1, 3 - 2a_1\}, \quad a_2 > -a_1.$$

Assume that $|Y_1| \leq M$ a.s. for some $M < \infty$. Then there is $x_0 \geq 0$ such that $p_N(x) \asymp N^{-1/2}$ for all $x \geq x_0$.

Proof. Let us show that X is an integrated AR(2)-process $\tilde{X}$ with parameters in Δ_2. Since $a_1 + a_2 + a_3 = 1$, we have that $T_2(a_1 - 1, a_1 + a_2 - 1) = (a_1, a_2, a_3)$ where T_2 was defined in (4.17). Hence, by Corollary 4.4.6, we only need to show that

$$(a_1 - 1, a_1 + a_2 - 1) \in \Delta_2 = \{(\tilde{a}_1, \tilde{a}_2) : \tilde{a}_1 + \tilde{a}_2 < 1, \tilde{a}_2 < 1 + \tilde{a}_2, \tilde{a}_2 > -1\},$$

(see (4.5)) whenever (a_1, a_2, a_3) satisfy the constraints stated in the corollary. Let $\tilde{a}_1 = a_1 - 1$ and $\tilde{a}_2 = a_1 + a_2 - 1$. Now $a_2 < 3 - 2a_1$ amounts to $\tilde{a}_1 + \tilde{a}_2 = 2a_1 + a_2 - 2 < 1$. Next, $\tilde{a}_2 < 1 + \tilde{a}_1$ is equivalent to $a_2 < 1$, whereas $\tilde{a}_2 > -1$ translates into $a_1 > -a_2$. □

4.4.2 The case $a_1 = 0$

We still have to consider the case $X_n = X_{n-2} + Y_n$ which is a special case of the equation $X_n = \rho X_{n-2} + Y_n$. The solution of the latter equation is given by

$$X_n = \begin{cases} \sum_{j=1}^k \rho^{k-j} Y_{2j-1}, & n = 2k-1, k \in \mathbb{N}, \\ \sum_{j=1}^k \rho^{k-j} Y_{2j}, & n = 2k, k \in \mathbb{N}. \end{cases}$$

In particular, (X_{2n}) and (X_{2n-1}) define two independent sequences with the same law as $(Z_n)_{n\geq 1}$ given by $Z_n = \rho Z_{n-1} + Y_n$. Hence,

$$\begin{aligned} \mathbb{P}\left(\sup_{n=1,\dots,2N} X_n \leq x\right) &= \left(\mathbb{P}\left(\sup_{n=1,\dots,N} Z_n \leq x\right)\right)^2, \\ \mathbb{P}\left(\sup_{n=1,\dots,2N-1} X_n \leq x\right) &= \mathbb{P}\left(\sup_{n=1,\dots,N} Z_n \leq x\right)\mathbb{P}\left(\sup_{n=1,\dots,N-1} Z_n \leq x\right). \end{aligned} \tag{4.22}$$

In particular, the behaviour of the persistence probability can be determined by the persistence probabilities of AR(1)-processes. If $\rho = 1$, X defines two indpendent random walk, so we immediately obtain the following lemma:

Lemma 4.4.8. *Assume that $\mathbb{E}[Y_1] = 0$, $\mathbb{E}[Y_1^2] < \infty$, and consider the AR(2)-process $X_n = X_{n-2} + Y_n$. For any $x \geq 0$, there is a constant $c(x)$ such that*

$$\mathbb{P}\left(\sup_{n=1,\dots,N} X_n \leq x\right) \sim c(x) N^{-1}, \quad N \to \infty.$$

Proof. By the preceding discussion, (X_{2n}) and (X_{2n-1}) define two independent centred random walks with finite variance that have the same law. Recall from Section 1.2.1 that $\mathbb{P}\left(\sup_{n=1,\dots,N} \sum_{k=1}^n Y_k \leq x\right) \sim d(x) N^{-1/2}$, so the assertion follows in view of (4.22). □

Remark 4.4.9. By the same reasoning, if $X_n = X_{n-p} + Y_n$ ($p \geq 1$), we have that $p_N(x) \sim \tilde{c}(x) N^{-p/2}$ for any $x \geq 0$ if $\mathbb{E}[Y_1] = 0$ and $\mathbb{E}[Y_1^2] < \infty$.

4.5 A positive limit

We now turn to the case that the persistence probability converges to a positive limit, i.e. $p_N(x) \to p_\infty(x) > 0$ as $N \to \infty$, implying that the process $(X_n)_{n\geq 1}$ stays below x at all times with positive probability. If $X_n = \sum_{k=1}^n c_{n-k} Y_k$, one would expect that this happens if $0 < c_n \to \infty$ and $c_n - c_{n-1} \to \infty$ sufficiently fast. Indeed, if c_n is very large compared to c_k for $k \leq n-1$, then $Y_1 \leq -\delta$ for some $\delta > 0$ implies that $X_n \leq -\delta c_n + \sum_{k=2}^n c_{n-k} Y_k$, and one expects that the expression on the r.h.s. stays below a fixed barrier with high probability. In fact, we can transform this idea directly into a proof.

Proposition 4.5.1. *Let $(\alpha_n)_{n\geq 0}$ denote a sequence of positive numbers. Let $\rho > 1$ and assume that $\mathbb{P}(Y_1 < 0) > 0$ and $\mathbb{P}(Y_1 \geq x) \precsim (\log x)^{-\alpha}$ as $x \to \infty$ for some $\alpha > 1$. Let $X_n := \sum_{k=1}^{n} \rho^{n-k}\alpha_{n-k}Y_k$.*

1. *If $(\alpha_n)_{n\geq 0}$ is non-decreasing, there is a constant $c > 0$ such that*

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty}\left\{X_n \leq -c\alpha_{n-1}\rho^{n-1}\right\}\right) > 0.$$

2. *If $0 < l \leq \alpha_n \leq u < \infty$ for all $n \geq 0$, there is a constant $c > 0$ such that*

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty}\left\{X_n \leq -c\rho^{n-1}\right\}\right) > 0.$$

Proof. We first prove part 1. Let $\delta > 0$ such that $\mathbb{P}(Y_1 \leq -\delta) > 0$, and fix $\beta > 0$ such that $\beta\sum_{k=1}^{\infty} k^{-2} \leq \delta/2$. Then

$$A_N := \{Y_1 \leq -\delta\} \cap \bigcap_{n=2}^{N}\left\{Y_n \leq \rho^{n-1}\beta n^{-2}\right\} \subseteq \bigcap_{n=1}^{N}\left\{X_n \leq -\delta\alpha_{n-1}\rho^{n-1}/2\right\}$$

Indeed, since (α_n) is non-decreasing, the event A_N implies that $X_1 = \alpha_0 Y_1 \leq -\alpha_0\delta$ and for all $n = 2, \dots, N$ that

$$\begin{aligned}X_n &= \rho^{n-1}\alpha_{n-1}Y_1 + \sum_{k=2}^{n}\rho^{n-k}\alpha_{n-k}Y_k \leq -\delta\alpha_{n-1}\rho^{n-1} + \rho^{n-1}\sum_{k=2}^{n}\alpha_{n-k}\beta k^{-2} \\ &\leq -\delta\alpha_{n-1}\rho^{n-1} + \rho^{n-1}\alpha_{n-1}\beta\sum_{k=1}^{\infty}k^{-2} = \alpha_{n-1}\rho^{n-1}\left(\beta\sum_{k=1}^{\infty}k^{-2} - \delta\right) \leq -\delta\alpha_{n-1}\rho^{n-1}/2.\end{aligned}$$

Finally, in view of the assumption on the tail behaviour of Y_1, it is not hard to show that

$$\lim_{N\to\infty}\mathbb{P}(A_N) = \mathbb{P}(Y_1 \leq -\delta)\lim_{N\to\infty}\prod_{n=2}^{N}\left(1 - \mathbb{P}\left(Y_1 > \beta\rho^{n-1}n^{-2}\right)\right) > 0.$$

The proof of part 2 is very similar. Let A_N be defined as above. Then, using the bounds on (α_n), we get for $n = 2, \dots, N$ that

$$\begin{aligned}X_n &\leq -\delta\alpha_{n-1}\rho^{n-1} + \rho^{n-1}\sum_{k=2}^{n}\alpha_{n-k}\beta k^{-2} \leq -\delta l\rho^{n-1} + \rho^{n-1}u\beta\sum_{k=1}^{\infty}k^{-2} \\ &= \rho^{n-1}\left(\beta u\sum_{k=1}^{\infty}k^{-2} - \delta l\right).\end{aligned}$$

For $\beta > 0$ sufficiently small, this implies that $X_n \leq -(\delta\, l/2)\, \rho^{n-1}$ for all $n = 2, \ldots, N$. □

We can now prove Theorem 4.2.3 showing that the persistence probability converges to a positive constant if X is AR(2) with $(a_1, a_2) \in C$ (cf. Figure 4.3) under mild conditions.

Proof of Theorem 4.2.3.

Let $(a_1, a_2) \in C$. Assume first that $a_1 > 0$ and $a_2 \in \mathbb{R}$ such that $a_1^2 + 4a_2 > 0$. Moreover, assume that either $a_1 \geq 2$ or $a_1 + a_2 > 1$ if $a_1 < 2$. Recall from (4.3) that $c_n = s_1^{n+1}/h - s_2^{n+1}/h$ where $h > 0$ since $a_1^2 + 4a_2 > 0$. Note that $s_1 = (a_1 + h)/2 > 1$ if and only if either $a_1 \geq 2$ or if $a_1 + a_2 > 1$ in case $a_1 < 2$. Moreover $|s_2| < s_1$ if and only if $a_1 > 0$ and $h > 0$. Hence, in view of our assumptions, we have that $c_n = s_1^n s_1/h\,(1 - (s_2/s_1)^{n+1}) =: s_1^n \alpha_n \geq 0$ for all n. Note that $\alpha_n \to s_1/h > 0$. Hence, the assertion follows by part 2 of Proposition 4.5.1.

If $a_1^2 + 4a_2 = 0$ and $a_1 > 2$, $c_n = (a_1/2)^n(n+1)$ by (4.3). Hence, the result follows from part 1 of Proposition 4.5.1 with $\rho = a_1/2 > 1$ and $\alpha_n = n + 1$.

Finally, if $a_1 = 0$ and $a_2 > 1$, the claim follows in view of (4.22) and Proposition 4.5.1. □

Chapter 5

Persistence of fractional Brownian motion with moving boundaries

In this chapter, we discuss persistence probabilities related to fractional Brownian motion (FBM). Recall that FBM with Hurst index $H \in (0,1)$ is a centered Gaussian process $(X_t)_{t\in\mathbb{R}}$ with covariance

$$\mathbb{E}\left[X_s X_t\right] = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right), \quad s,t \in \mathbb{R}.$$

We remark that X has stationary increments and is self-similar of index H, i.e. $(X_{ct})_{t\in\mathbb{R}}$ and $(c^H X_t)_{t\in\mathbb{R}}$ have the same distribution for any $c > 0$. Results concerning persistence of FBM have been described in Section 1.2.3.

The main motivation of this chapter comes from a physical model involving FBM that has been studied recently in [ORS12]. In general, FBM has received a lot of interest in physical applications. Since the behavior of many dynamical systems exhibits long-range correlations, one observes so-called *anomalous dynamics* which are typically characterised by a nonlinear growth in time (i.e. $\mathbb{E}\left[X_t^2\right] \propto t^{2H}$ where $H \neq 1/2$) where X models the evolution of the corresponding quantity ([BG90]). In order to take such features into account, FBM has been proposed in [MVN68] in 1968. For instance, FBM has been used in polymers models ([ZRM09, WFCV12]) and in finance to describe longe-range dependence of stock prices and volatility ([CR98, Øks07]). We also refer to [EK08] and [ES12] where the emergence of FBM in certain complex systems is investigated.

Oshanin et al. ([ORS12]) study an extension to the Sinaĭ model involving FBM. If $(X_t)_{t\geq 0}$ denotes a FBM with Hurst index H, the authors are interested in the asymptotics of the k-th moment $\mathbb{E}\left[J_N^k\right]$ of the quantity J_N, called the steady-state current J_N through a finite segment of length N, given by

$$J_N := \frac{1}{2}\left(1 + \sum_{n=1}^{N-1} \exp(X_n)\right)^{-1}.$$

Oshanin et al. find that $\mathbb{E}\left[J_N^k\right] = N^{-(1-H)+o(1)}$ as $N \to \infty$ for any $k > 0$. In particular, the exponent is independent of k. In order to prove the lower bound, the authors need the following estimate: If $Y_0, Y_1 > 0$ are some constants, then

$$N^{-(1-H)}(\log N)^{-c} \precsim \mathbb{P}\left(X_n \le Y_0 - Y_1 \log(1+n), \forall n = 1, \dots, N\right), \qquad N \to \infty. \quad (5.1)$$

In general, the following question arises: What kind of functions f are admissible such that $\mathbb{P}\left(X_t \le f(t), \forall t \in [0,T]\right) = T^{-(1-H)+o(1)}$, i.e. what kind of moving boundaries f do not change the persistence exponent of a FBM? Given the increasing relevance of FBM for various applications, it is important to understand such questions since they convey information about the path behavior of FBM. In the remainder of this chapter, we take a further step in this direction. Let us now briefly summarise our main results.
We begin to study the persistence probability of FBM involving a moving boundary that is allowed to increase or decrease like some power of a logarithm. Our results show that the presence of such a boundary does not change the persistence exponent of FBM, and (5.1) will follow as a special case.

Proposition 5.0.2. *Let $Y_0, Y_1 > 0$ and X denote a FBM with Hurst index $H \in (0,1)$.*

1. *Let $\gamma \ge 1$ and $f_-(t) := Y_0 - Y_1(\log(1+t))^\gamma$. There is a constant $c = c(H,\gamma) > 0$ such that*
$$T^{-(1-H)}(\log T)^{-c} \precsim \mathbb{P}\left(X_t \le f_-(t), \forall t \in [0,T]\right) \precsim T^{-(1-H)}, \qquad T \to \infty.$$
2. *Let $\gamma > 0$ and $f_+(t) := Y_0 + Y_1(\log(1+t))^\gamma$. There is a constant $c = c(H,\gamma) > 0$ such that*
$$T^{-(1-H)}(\log T)^{-c} \precsim \mathbb{P}\left(X_t \le f_+(t), \forall t \in [0,T]\right) \precsim T^{-(1-H)}(\log T)^{c}, \qquad T \to \infty.$$

Considering the continuous-time version of J, we prove the following result:

Proposition 5.0.3. *Set*

$$J_T := \left(\int_0^T e^{X_t}\, dt\right)^{-1}, \quad T > 0.$$

For any $k > 0$, there is $c = c(H,k) > 0$ such that

$$T^{-(1-H)}(\log T)^{-c} \precsim \mathbb{E}\left[J_T^k\right] \precsim T^{-(1-H)}(\log T)^{c}, \qquad T \to \infty. \quad (5.2)$$

Solving the case $k = 1$ was actually the key to the computation of the persistence exponent in [Mol99] where it is shown that $\mathbb{E}\left[J_T\right] \sim CT^{-(1-H)}$ for some constant $C > 0$. Our proof is based on estimates of the persistence probability of FBM in [Aur11], an estimate on the modulus of continuity of FBM in [Sch09] and Proposition 5.0.2.
Finally, we discuss various related quantities such as the time when a FBM reaches its maximum on the time interval $(0,1)$, the last zero in the interval $(0,1)$ and the Lebesgue measure of the set of points in time when X is positive on $(0,1)$. If ξ denotes any of these quantities, we are interested in the probability of small values, i.e. $\mathbb{P}\left(\xi < \epsilon\right)$ as ϵ goes to zero. In Proposition 5.3.1 below, we improve the estimates given in [Mol99].
These issues are addressed in Sections 5.1, 5.2 and 5.3 respectively.

5.1 FBM and moving boundaries

In this section, we prove Proposition 5.0.2. We need to distinguish between increasing and decreasing boundaries. Let us begin with a simple general upper bound on the probabilitiy that a FBM stays below a function f until time T when $f(x) \to -\infty$ as $x \to \infty$.

Lemma 5.1.1. *Let f be some measurable function such that there is a constant $b > 0$ such that $\int_0^\infty e^{bf(s)}\, ds < \infty$. Then*

$$\mathbb{P}\left(X_t \le f(b^{1/H}t), \forall t \in [0,T]\right) \precsim T^{-(1-H)}.$$

Proof. Recall from [Mol99, Statement 1] that

$$\lim_{T\to\infty} T^{1-H}\mathbb{E}\left[J_T\right] \in (0,\infty).$$

Therefore, there is a constant $c > 0$ such that, for T large enough,

$$\begin{aligned} cT^{-(1-H)} &\ge \mathbb{E}\left[\frac{1}{\int_0^T e^{X_t}\,dt}\right] \ge \mathbb{E}\left[\frac{1}{\int_0^T e^{X_t}\,dt} 1_{\{X_t \le bf(t), \forall t\in[0,T]\}}\right] \\ &\ge \frac{1}{\int_0^T e^{bf(t)}\,dt}\,\mathbb{P}\left(X_t \le bf(t), \forall t \in [0,T]\right) \\ &\ge \frac{1}{\int_0^\infty e^{bf(t)}\,dt}\,\mathbb{P}\left(X_{b^{-1/H}t} \le f(t), \forall t \in [0,T]\right) \\ &= C(b)\,\mathbb{P}\left(X_t \le f(b^{1/H}t), \forall t \in [0, b^{-1/H}T]\right), \end{aligned}$$

and the lemma follows. □

The next lemma provides a lower bound on the survial probability if the function f does not decay faster than some power of the logarithm.

Lemma 5.1.2. *Let f be a continuous function with $f(0) > 0$. Assume that there are constants $T_0, K, \alpha > 0$ such that $f(T) \ge -K(\log T)^\alpha$ for all $T \ge T_0$. Then there is a constant $c > 0$ such that*

$$\mathbb{P}\left(X_t \le f(t), \forall t \in [0,T]\right) \succsim T^{-(1-H)}(\log T)^{-c}.$$

Proof. Set $g(T) := \mathbb{P}\left(X_t \le f(t), \forall t \in [0,T]\right)$ and fix $s_0 > 0$ (to be chosen later). Since $\mathbb{E}\left[X_s X_t\right] \ge 0$ for all $t, s \ge 0$, Slepian's inequality yields

$$g(T) \ge \mathbb{P}\left(X_t \le f(t), \forall t \in [0, s_0(\log T)^{\alpha/H}]\right) \cdot \mathbb{P}\left(X_t \le f(t), \forall t \in [s_0(\log T)^{\alpha/H}, T]\right).$$

Note that

$$\begin{aligned}
&\mathbb{P}\left(X_t \le f(t), \forall t \in [s_0(\log T)^{\alpha/H}, T]\right)\\
&\quad = \mathbb{P}\left(X_{(\log T)^{\alpha/H} t} \le f\left((\log T)^{\alpha/H} t\right), \forall t \in [s_0, T/(\log T)^{\alpha/H}]\right)\\
&\quad = \mathbb{P}\left((\log T)^{\alpha} X_t \le f\left((\log T)^{\alpha/H} t\right), \forall t \in [s_0, T/(\log T)^{\alpha/H}]\right)\\
&\quad = \mathbb{P}\left(X_t \le \frac{f((\log T)^{\alpha/H} t)}{(\log T)^{\alpha}}, \forall t \in [s_0, T/(\log T)^{\alpha/H}]\right). \qquad (5.3)
\end{aligned}$$

Certainly, for all T large enough,

$$\begin{aligned}
&\inf_{t\in[s_0,T/(\log T)^{\alpha/H}]} \frac{f((\log T)^{\alpha/H} t)}{(\log T)^{\alpha}}\\
&\quad = \inf_{t\in[s_0,T/(\log T)^{\alpha/H}]} \frac{f((\log T)^{\alpha/H} t)}{(\log[(\log T)^{\alpha/H} t])^{\alpha}} \cdot \frac{(\log[(\log T)^{\alpha/H} t])^{\alpha}}{(\log T)^{\alpha}} \ge -K.
\end{aligned}$$

Thus, the term in (5.3) can be estimated from below by

$$\mathbb{P}\left(X_t \le -K, \forall t \in [s_0, T/(\log T)^{1/H}]\right). \qquad (5.4)$$

Let us first consider the case $H \ge 1/2$. Recall that the increments of FBM are positively correlated if and only if $H \ge 1/2$, so using Slepian's lemma in the second inequality, we obtain the following lower bound for the term in (5.4):

$$\begin{aligned}
\mathbb{P}\left(X_t \le -K, \forall t \in [s_0, T]\right) &\ge \mathbb{P}\left(X_{s_0} \le -(K+1), \sup_{t\in[s_0,T]} X_t - X_{s_0} \le 1\right)\\
&\ge \mathbb{P}\left(X_{s_0} \le -(K+1)\right) \cdot \mathbb{P}\left(X_t - X_{s_0} \le 1, \forall t \in [s_0, T]\right)\\
&\ge c(s_0, K) \cdot \mathbb{P}\left(X_t \le 1, \forall t \in [0, T]\right).
\end{aligned}$$

Hence,

$$g(T) \ge c(s_0, K) g(s_0(\log T)^{\alpha/H}) \cdot \mathbb{P}\left(X_t \le 1, \forall t \in [0, T]\right),$$

and (1.10) implies that there is $c > 0$ such that, for all large T,

$$g(T) \ge g(s_0(\log T)^{\alpha/H}) T^{-(1-H)} (\log T)^{-c}.$$

Let us now prove that a similar inequality also holds if $H < 1/2$. In this case, we cannot use Slepian's inequality since the increments of FBM are negatively correlated. Applying [Aur11, Lemma 5] (and the specific choice of s_0 there), the term in (5.4) is lower bounded by

$$\mathbb{P}\left(X_t \le 1, \forall t \in \left[0, k\,T/(\log T)^{1/H} (\log\log T)^{1/(4H)}\right]\right) \cdot (\log T)^{-o(1)},$$

where k is some constant. Finally, by (1.10), this term admits $T^{-(1-H)}(\log T)^{-c}$ as a lower bound with some appropriate constant $c > 0$ and all T large enough. Thus, we have seen that

$$g(T) \ge g(s_0(\log T)^{1/H}) T^{-(1-H)} (\log T)^{-c} \qquad (5.5)$$

for some constants $s_0, c > 0$.
If we combine this result with the case $H \geq 1/2$, this shows that for any $H \in (0,1)$, there are constants $c = c(H), \beta = \beta(H), s_0 = s_0(H) > 0$ such that

$$g(T) \geq g(s_0(\log T)^\beta)T^{-(1-H)}(\log T)^{-c}. \tag{5.6}$$

Using this inequality iteratively, we will prove the preliminary estimate $g(T) \geq T^{-\theta_1}$ for some $\theta_1 > 1-H$ and all T large enough. Once we have this estimate, (5.6) shows that

$$g(T) \gtrsim (\log T)^{-(\theta_1\beta+c)}\, T^{-(1-H)}, \qquad T \to \infty,$$

and the proof is complete for all $H \in (0,1)$.
Let us now establish the preliminary lower bound. (5.6) implies that if $\beta_1 > \beta$ and $\theta > 1-H$, there is a constant $T_0 \geq 1$ such that

$$g(T) \geq g((\log T)^{\beta_1}) \cdot T^{-\theta}, \qquad T \geq T_0. \tag{5.7}$$

The idea is to iterate this inequality until $\log(\log(\dots)^{\beta_1})^{\beta_1}$ is smaller than some constant. As we will see, the number of iterations that are needed is very small and merely leads to a term of logarithmic order. Since each iteration is valid only for large values of T depending on the number of iterations, and the number of iterations is itself a function of T, some care is needed to perform this step. To this end, fix $\beta_2 > \beta_1$ and set $T_0' := \max\left\{\log(T_0)/\beta_2, \beta_2^{\beta_1/(\beta_2-\beta_1)}\right\}$. Define $\log^{(1)} x = \log x$ for $x > x_1 = 1$ and $\log^{(i)} x = \log^{(i-1)}(\log x)$ for $x > x_i := \exp(x_{i-1})$. For any $j \geq 1$ and $T > 0$, the following implication holds:

$$\log^{(j+1)} T \geq T_0' \Longrightarrow g((\log^{(j)} T)^{\beta_2}) \geq g((\log^{(j+1)} T)^{\beta_2}) \cdot (\log^{(j)} T)^{-\theta\beta_2}. \tag{5.8}$$

Indeed, note that $\log^{(j+1)} T \geq T_0'$ translates into

$$(\log^{(j)} T)^{\beta_2} \geq T_0 \quad \text{ and } \quad \beta_2^{\beta_1}(\log^{(j+1)} T)^{\beta_1} \leq (\log^{(j+1)} T)^{\beta_2}.$$

Hence, in view of (5.7), we find that

$$\begin{aligned} g((\log^{(j)} T)^{\beta_2}) &\geq g((\log((\log^{(j)} T)^{\beta_2}))^{\beta_1}) \cdot (\log^{(j)} T)^{-\beta_2\theta} \\ &= g(\beta_2^{\beta_1}(\log^{(j+1)} T)^{\beta_1}) \cdot (\log^{(j)} T)^{-\beta_2\theta} \\ &\geq g((\log^{(j+1)} T)^{\beta_2}) \cdot (\log^{(j)} T)^{-\beta_2\theta}, \end{aligned}$$

so (5.8) follows. Denote by $a(T) := \min\left\{n \in \mathbb{N} : \log^{(n)} T \leq T_0'\right\}$. By definition, $\log^{(a(T))} T \leq T_0' < \log^{(a(T)-1)} T$, so we can apply (5.8) iteratively for all $j \leq a(T)-2$ to obtain that

$$\begin{aligned} g((\log T)^{\beta_2}) &\geq g((\log^{(2)} T)^{\beta_2})(\log T)^{-\beta_2\theta} \geq \dots \\ &\geq g((\log^{(a(T)-1)} T)^{\beta_2}) \prod_{j=1}^{a(T)-2} (\log^{(j)} T)^{-\beta_2\theta} \\ &\geq g(e^{T_0'\beta_2}) \prod_{j=1}^{a(T)-2} (\log^{(j)} T)^{-\beta_2\theta}, \end{aligned} \tag{5.9}$$

which holds for all $T \geq \exp(\exp(\exp(T_0')))$, i.e. such that $a(T) \geq 3$. Moreover, $g(e^{T_0'\beta_2}) > 0$ since f is continuous with $f(0) > 0$. Finally,

$$\prod_{j=1}^{a(T)-2} (\log^{(j)} T)^{-\beta_2\theta} \geq (\log T)^{-\beta_2\theta} \cdot (\log^{(2)} T)^{-\beta_2\theta a(T)}.$$

In view of

$$a(T) = 1 + a(\log T) = j + a(\log^{(j)} T),$$

which holds for any $j \in \mathbb{N}$ and T large enough and the simple observation that $a(T) \leq T$, we obtain that $a(T) = o(\log^{(j)} T)$ for any $j \in \mathbb{N}$. Hence, for all T large enough,

$$\begin{aligned}(\log^{(2)} T)^{-\beta_2\theta\, a(T)} &\geq (\log^{(2)} T)^{-\beta_2\theta \log^{(3)} T} = \exp\left(-\beta_2\theta(\log^{(3)} T)^2\right)\\ &\geq \exp\left(-\log^{(2)} T\right) = (\log T)^{-1}. \qquad (5.10)\end{aligned}$$

Combining (5.7), (5.9) and (5.10), we conclude that $T^{-\theta_1} \precsim g(T)$ for any $\theta_1 > \theta$. □

Combining Lemma 5.1.1 and Lemma 5.1.2, we obtain part 1 of Proposition 5.0.2.

Proof of part 1 of Proposition 5.0.2.

Lower bound: With $f(t) := Y_0 - Y_1(\log(1+t))^\gamma \geq -2Y_1(\log(1+t))^\gamma$ for all large t, the lower bound follows directly from Lemma 5.1.2.

Upper bound: If $\gamma > 1$, we can directly apply Lemma 5.1.1 with $f(t) := Y_0 - Y_1(\log(1+t))^\gamma$ and $b = 1$ to obtain the upper bound.

If $\gamma = 1$, take $b > 0$ such that $bY_1 > 1$ and set $f(t) := Y_0 - Y_1(\log(1 + b^{-1/H}t))$, so that $\int_0^\infty e^{bf(t)}\,dt < \infty$ and by Lemma 5.1.1,

$$T^{-(1-H)} \succsim \mathbb{P}\left(X_t \leq f(b^{1/H}t), \forall t \in [0,T]\right) = \mathbb{P}\left(X_t \leq Y_0 - Y_1 \log(1+t), \forall t \in [0,T]\right).$$

□

Remark 5.1.3.

1. We remark that the removal of the boundary by a change of measure argument (Cameron-Martin-formula) results in less precise estimates of the form

$$T^{-(1-H)}e^{-c\sqrt{\log T}} \precsim \mathbb{P}\left(X_t \leq Y_0 - Y_1(\log(1+t))^\gamma, \forall t \in [0,T]\right) \precsim T^{-(1-H)}e^{c\sqrt{\log T}},$$

see [AD13] ((2.10) here), [Mol99] or [Mol12]. Moreover, as we have seen in Lemma 2.2.6, it requires some tedious computations to show that a function belongs to the RKHS of FBM.

2. In view of the results for Brownian motion (i.e. $H = 1/2$, see [Uch80]), it is reasonable to expect that the upper bound in part 1 of Proposition 5.0.2 has the correct order.

3. The restriction $\gamma \geq 1$ is necessary in order to apply Lemma 5.1.1. However, for any $\gamma > 0$, (1.10) immediately implies the following weaker bound:

$$\begin{aligned}&\mathbb{P}\left(X_t \leq Y_0 - Y_1(\log(1+t))^{\gamma}, \forall t \in [0,T]\right)\\&\quad\leq \mathbb{P}\left(X_t \leq Y_0, \forall t \in [0,T]\right) \precsim T^{-(1-H)}(\log T)^c.\end{aligned}$$

4. Let $f(x) = Y_0 - Y_1 \log(1+x)$. Trivially, if we consider discrete time,

$$\mathbb{P}\left(X_k \leq f(k), k = 1, \dots, N\right) \geq \mathbb{P}\left(X_t \leq f(t), \forall t \in [0,N]\right) \succsim N^{-(1-H)} \log(N)^{-c}.$$

This estimate is needed in [ORS12] (see Eq. (15) there) when proving a lower bound for $\mathbb{E}\left[J_N^k\right]$.

Clearly, Lemma 5.1.2 is only applicable if the boundary f satisfies $f(x) \to -\infty$ as $x \to \infty$. It is natural to suspect that the persistence exponent does not change if we introduce a barrier that increase like some power of a logarithm. This is part 2 of Proposition 5.0.2 which follows from the next lemma:

Lemma 5.1.4. *Let $f\colon [0,\infty) \to \mathbb{R}$ denote a continuous function with $f(0) > 0$. Moreover, we assume that there are constants $A, \alpha, T_0 > 0$ such that $f(x) \geq -A$ for all $x \geq 0$ and $f(x) \leq (\log x)^{\alpha}$ for all $x \geq T_0$. Then there is a constant $c > 0$ such that*

$$T^{-(1-H)}(\log T)^{-c} \precsim \mathbb{P}\left(X_t \leq f(t), \forall t \in [0,T]\right) \precsim T^{-(1-H)}(\log T)^c.$$

Proof. *Lower bound:* We can directly apply Lemma 5.1.2 directly since $f(x) \geq -A$ on $[0,\infty)$.
Upper bound: Note that

$$\begin{aligned}\mathbb{P}\left(X_t \leq f(t), \forall t \in [0,T]\right) &\leq \mathbb{P}\left(X_t \leq f(t), \forall t \in [T_0,T]\right)\\&\leq \mathbb{P}\left(X_t \leq (\log t)^{\alpha}, \forall t \in [T_0,T]\right)\\&\leq \mathbb{P}\left(X_t \leq (\log(2+T))^{\alpha}, \forall t \in [T_0,T]\right)\\&\leq \frac{\mathbb{P}\left(X_t \leq (\log(2+T))^{\alpha}, \forall t \in [0,T]\right)}{\mathbb{P}\left(X_t \leq (\log(2+T))^{\alpha}, \forall t \in [0,T_0]\right)}\\&\sim \mathbb{P}\left(X_t \leq (\log(2+T))^{\alpha}, \forall t \in [0,T]\right), \quad T \to \infty.\end{aligned}$$

We have used Slepian's inequality in the last inequality. Using once more the self-similarity of X and (1.10), the upper bound follows. ☐

5.2 Higher moments of J_N

We are now ready to prove Proposition 5.0.3. The lower bound follows easily from our result on moving boundaries in Proposition 5.0.2, whereas the proof of the upper bound is more involved.

Proof of Proposition 5.0.3.

Lower bound: Let $\gamma > 1$.

$$\mathbb{E}\left[J_T^k\right] \geq \mathbb{E}\left[\left(\int_0^T e^{X_t}\,dt\right)^{-k} 1_{\{X_t \leq 1-(\log(1+t))^\gamma,\,\forall t\in[0,T]\}}\right]$$
$$\geq \left(\int_0^T \frac{e}{(1+t)^\gamma}\,dt\right)^{-k} \mathbb{P}\left(X_t \leq 1-(\log(1+t))^\gamma,\,\forall t\in[0,T]\right).$$

The lower bound now follows by part 1 of Proposition 5.0.2.

Upper bound: Let $H/2 < \gamma < H$ and fix a such that $a > 2/H > 1/\gamma$ and $\gamma < H - 1/a$. Self-similarity and stationarity of increments imply for all $s, t \in [0,1]$ that

$$\mathbb{E}\left[|X_t - X_s|^a\right] = \mathbb{E}\left[\left|X_{|t-s|}\right|^a\right] = |t-s|^{aH}\,\mathbb{E}\left[|X_1|^a\right] = |t-s|^{(aH-1)+1}\,\mathbb{E}\left[|X_1|^a\right].$$

Since $aH - 1 > 0$, it follows from [Sch09, Lemma 2.1] that there is a positive random variable S such that

$$\mathbb{E}\left[S^a\right] \leq \left(\frac{2}{1-2^{-\gamma}}\right)^a \cdot \frac{\mathbb{E}\left[|X_1|^a\right]}{2^{aH-1-a\gamma}-1}, \tag{5.11}$$

and for all $\epsilon \in (0,1)$,

$$|X_t - X_s| \leq S\epsilon^\gamma, \qquad \forall\, s,t \in [0,1], |t-s| \leq \epsilon. \tag{5.12}$$

Write $X_1^* := \sup\{X_t : t \in [0,1]\}$, and let u^* denote a point where the supremum is attained. Using the self-similarity of X and (5.12) in the second inequality, we obtain the following estimates:

$$\mathbb{E}\left[J_T^k\right] = \mathbb{E}\left[\left(\int_0^1 e^{T^H X_t} T\,dt\right)^{-k}\right]$$
$$= T^{-k}\,\mathbb{E}\left[e^{-kT^H X_1^*}\left(\int_0^1 e^{-T^H(X_1^* - X_t)}\,dt\right)^{-k}\right]$$
$$\leq T^{-k}\,\mathbb{E}\left[e^{-kT^H X_1^*}\left(\int_{\min\{u^*-\epsilon,0\}}^{\max\{u^*+\epsilon,1\}} e^{-T^H(X_{u^*} - X_t)}\,dt\right)^{-k}\right]$$
$$\leq T^{-k}\,\mathbb{E}\left[e^{-kT^H X_1^*}\left(\int_{\min\{u^*-\epsilon,0\}}^{\max\{u^*+\epsilon,1\}} e^{-T^H S\epsilon^\gamma}\,dt\right)^{-k}\right]$$
$$\leq T^{-k}\,\mathbb{E}\left[e^{-kT^H X_1^*}\epsilon^{-k} e^{kT^H S\epsilon^\gamma}\right].$$

Set $\epsilon := \min\left\{(T^H S)^{1/\gamma}, 1\right\}$. Then $T^H S\epsilon^\gamma \leq 1$ and $\epsilon^{-1} \leq (T^H S)^{1/\gamma} + 1$, and we find that

$$\begin{aligned}\mathbb{E}\left[J_T^k\right] &\leq T^{-k} e^k \mathbb{E}\left[e^{-kT^H X_1^*}((T^H S)^{1/\gamma} + 1)^k\right] \\ &\leq T^{-k}(2e)^k \left(\mathbb{E}\left[e^{-kT^H X_1^*} S^{k/\gamma}\right] T^{kH/\gamma} + \mathbb{E}\left[e^{-kT^H X_1^*}\right]\right). \end{aligned} \tag{5.13}$$

Applying Hölder's inequality ($1/p + 1/q = 1$), we have that

$$\mathbb{E}\left[e^{-kT^H X_1^*} S^{k/\gamma}\right] \leq \mathbb{E}\left[e^{-qkT^H X_1^*}\right]^{1/q} \mathbb{E}\left[S^a\right]^{1/p} \leq \mathbb{E}\left[e^{-kT^H X_1^*}\right]^{1/q} \mathbb{E}\left[S^a\right]^{1/p}, \tag{5.14}$$

where $a = kp/\gamma$, and the last inequality holds for all $T > 0, a > 2/H$ and $H/2 < \gamma < H - 1/a$. Fix $\delta \in (0, 1)$ and set

$$a := (\log\log T)^{-\delta} \log T, \quad \gamma = H - 2/a.$$

(Since $a = kp/\gamma$, this amounts to $p = (H(\log\log T)^{-\delta} \log T + 2)/k$, $a = (kp - 2)/H$ and $\gamma = H - 2/a$). Assume for a moment that there are constants $M, \nu \in (0, \infty)$ such that for all a large enough, it holds that

$$(\mathbb{E}\left[|X_1|^a\right])^{1/a} \leq Ma^\nu. \tag{5.15}$$

Then in view of the relations $1/p = k/(a\gamma)$ and $aH - a\gamma = 2$, we obtain that

$$\mathbb{E}\left[S^a\right]^{1/p} \leq \frac{\left((\mathbb{E}\left[|X_1|^a\right])^{1/a}\right)^{k/\gamma}}{(2^{aH-1-a\gamma} - 1)^{1/p}} = \frac{M^{k/\gamma} a^{\nu k/\gamma}}{(2^{2-1} - 1)^{1/p}} = M^{k/\gamma} a^{k\nu/H + o(1)}, \quad a \to \infty.$$

In particular, $(\mathbb{E}\left[S^a\right])^{1/p} = o((\log T)^\eta)$ as $a \to \infty$, or equivalently, $T \to \infty$, for every $\eta > k\nu/H$. For such η, combining (5.13) and (5.14), we find for T large enough that

$$\mathbb{E}\left[J_T^k\right] \leq 2(2e)^k T^{kH/\gamma - k} (\log T)^\eta \mathbb{E}\left[e^{-kT^H X_1^*}\right]^{1/q}. \tag{5.16}$$

By Karamata's Tauberian theorem (see [BGT87, Theorem 1.7.1]), (1.11) implies that (with the same $c > 0$ as in (1.11))

$$\lambda^{-(1-H)/H} (\log\lambda)^{-c} \precsim \mathbb{E}\left[e^{-\lambda X_1^*}\right] \precsim \lambda^{-(1-H)/H} (\log\lambda)^c, \qquad \lambda \to \infty. \tag{5.17}$$

(In fact, the lower bound is easy since $\mathbb{E}\left[e^{-\lambda X_1^*}\right] \geq e^{-1}\mathbb{P}\left(X_1^* \leq 1/\lambda\right)$. For our purposes, it is enough to note that $\mathbb{E}\left[e^{-\lambda X_1^*}\right] \leq \mathbb{P}\left(X_1^* \leq \log(\lambda)/\lambda\right) + e^{-\log\lambda}$, so the upper bound follows from (1.11) with some $\tilde{c} > c$.) By (5.17), we conclude that

$$\begin{aligned}T^{Hk/\gamma - k}\mathbb{E}\left[e^{-kT^H X_1^*}\right]^{1/q} &= T^{Hk/\gamma - k}\mathbb{E}\left[e^{-kT^H X_1^*}\right]^{1 - k/(a\gamma)} \\ &\leq C' T^{Hk/\gamma - k} T^{-(1-H)(1-k/(a\gamma))} (\log T)^{c(1-k/(a\gamma))} \\ &= C' T^{k/(a\gamma)(aH - a\gamma + (H-1))} T^{-(1-H)} (\log T)^{c + o(1)}.\end{aligned}$$

Using again that $aH - a\gamma = 2$, note that by definition of a,

$$T^{k/(a\gamma)(Ha-a\gamma+(H-1))} = T^{(H+1)/(a\gamma)} = \exp(\gamma^{-1}(H+1)(\log\log T)^{\delta}) = (\log T)^{o(1)}.$$

Hence, we have shown that

$$\mathbb{E}\left[J_T^k\right] \precsim (\log T)^{\eta+o(1)} T^{-(1-H)}, \quad T \to \infty,$$

as soon as we prove that (5.15) holds. Since X_1 is standard Gaussian, it is well-known that $\mathbb{E}\left[|X_1|^a\right] = 2^{a/2}\Gamma((a+1)/2)/\sqrt{\pi}$ for every $a > 0$, and therefore, it is not hard to show that $\mathbb{E}\left[|X_1|^a\right]^{1/a} \leq M\sqrt{a}$ for some M and all a large enough. This completes the proof. □

Remark 5.2.1. Note that if X is a self-similar process with stationary increments (SSSI) satisfying (5.15), the proof above shows that (5.16) holds in that case as well. By (5.16), if we already know a lower bound on $\mathbb{E}\left[J_T^k\right]$, we get a lower bound on the Laplace transform of X_1^*, whereas a upper bound on the Laplace transform yields an upper bound on $\mathbb{E}\left[J_T^k\right]$. Since the behaviour of the Laplace transform $\mathbb{E}\left[\exp(-\lambda X_1^*)\right]$ as $\lambda \to \infty$ is related to that of the probability $\mathbb{P}\left(X_1^* \leq \lambda\right)$ as $\lambda \downarrow 0$ via Tauberian theorems, this approach could be useful to study persistence of other SSSI-processes, see [CGPPS13].

5.3 Some related quantities

Given a FBM X, the following quantities are studied in [Mol99]:

$$\tau_{\max} := \text{argmax}_{t\in[0,1]} X_t, \tag{5.18}$$
$$z_- := \sup\{t \in (0,1) : X_t = 0\}, \tag{5.19}$$
$$s_+ := \lambda(\{t \in (0,1) : X_t > 0\}), \tag{5.20}$$

where λ denotes the Lebesgue measure. We remark that the definition of $\tau_{\max}$ is unambiguous since a FBM attains its maximum at a unique point on $[0,1]$ a.s. ([KP90, Lemma 2.6]). If ξ denotes any of the three random variables above, by [Mol99, Theorem 2], there is $c > 0$ such that

$$\epsilon^{1-H}\exp(-(1/c)\sqrt{|\log\epsilon|}) \precsim \mathbb{P}\left(\xi < \epsilon\right) \precsim \epsilon^{1-H}\exp(c\sqrt{|\log\epsilon|}), \qquad \epsilon \downarrow 0.$$

Upon combining our results (Proposition 5.0.2), the arguments used in [Mol99] and the more precise estimate for the persistence probability of FBM in [Aur11], we obtain the following improvement:

Proposition 5.3.1. *If* ξ *denotes any of the random variables in* (5.18), (5.19) *or* (5.20), *there is* $c > 0$ *such that*

$$\epsilon^{1-H}\left|\log\epsilon\right|^{-c} \precsim \mathbb{P}\left(\xi < \epsilon\right) \precsim \epsilon^{1-H}\left|\log\epsilon\right|^{c}, \qquad \epsilon \downarrow 0. \tag{5.21}$$

Proof. Let us recall the relations of the probabilities involving the quantities $\tau_{\max}, s_+$ and z_- that are used in the proof of [Mol99, Theorem 2]:
The symmetry and continuity of X imply that

$$\mathbb{P}\left(X_t < 0, \forall t \in (\epsilon, 1)\right) = \tfrac{1}{2}\,\mathbb{P}\left(X_t \neq 0, \forall t \in (\epsilon, 1)\right) = \tfrac{1}{2}\,\mathbb{P}\left(z_- < \epsilon\right), \quad 0 < \epsilon < 1. \tag{5.22}$$

Moreover, we clearly have the following inequalities:

$$\mathbb{P}\left(X_t < 0, \forall t \in (\epsilon, 1)\right) \le \mathbb{P}\left(s_+ < \epsilon\right), \quad \mathbb{P}\left(X_t < 0, \forall t \in (\epsilon, 1)\right) \le \mathbb{P}\left(\tau_{\max} < \epsilon\right). \tag{5.23}$$

We will show that

$$\epsilon^{1-H} \left|\log \epsilon\right|^{-c} \precsim \mathbb{P}\left(X_t < 0, \forall t \in (\epsilon, 1)\right) \precsim \epsilon^{1-H} \left|\log \epsilon\right|^{c}. \tag{5.24}$$

If (5.24) holds, (5.22) proves the statement for z_- whereas the lower bounds in (5.21) for $\xi = s_+$ and $\xi = \tau_{\max}$ follow from (5.23).
Before establishing the remaining lower bound, let us prove (5.24). Note that the self-similarity of X implies that $\mathbb{P}\left(X_t < 0, \forall t \in (\epsilon, 1)\right) = \mathbb{P}\left(X_t < 0, \forall t \in (1, 1/\epsilon)\right)$. By Slepian's inequality, it holds that

$$\begin{aligned} \mathbb{P}\left(X_t < 0, \forall t \in (1, 1/\epsilon)\right) &\le \mathbb{P}\left(X_t < 1, \forall t \in (1, 1/\epsilon)\right) \\ &\le \mathbb{P}\left(X_t < 1, \forall t \in (0, 1/\epsilon)\right) / \mathbb{P}\left(X_t < 1, \forall t \in (0, 1)\right). \end{aligned}$$

In view of (1.10), this proves the upper bound of (5.24). The lower bound follows from part 2 of Proposition 5.0.2 since

$$\mathbb{P}\left(X_t < 0, \forall t \in (1, 1/\epsilon)\right) \ge \mathbb{P}\left(X_t \le 1 - \log(1 + 3t), \forall t \in [0, 1/\epsilon]\right).$$

Let us now turn to the upper bound for $\mathbb{P}\left(\tau_{\max} < \epsilon\right)$. Note that

$$\mathbb{P}\left(\tau_{\max} < \epsilon\right) \le \mathbb{P}\left(X_1^* < h\right) + \mathbb{P}\left(X_\epsilon^* > h\right), \qquad h > 0.$$

Take $h = \epsilon^H \left|\log \epsilon\right|^\alpha$ where $\alpha > 1/2$. Using (1.11), we obtain that

$$\mathbb{P}\left(X_1^* < h\right) = \mathbb{P}\left(X_1^* < \epsilon^H \left|\log \epsilon\right|^\alpha\right) \precsim \epsilon^{1-H} \left|\log \epsilon\right|^{\alpha(1-H)/H + c + o(1)},$$

whereas, for some constants $A, B > 0$, an application of the Gaussian concentration inequality (or Fernique's estimate stated in [Mol99]) yields that

$$\mathbb{P}\left(X_\epsilon^* > h\right) = \mathbb{P}\left(X_1^* > \epsilon^{-H} h\right) = \mathbb{P}\left(X_1^* > \left|\log \epsilon\right|^\alpha\right) \le A e^{-B\left|\log \epsilon\right|^{2\alpha}},$$

i.e. this term decays faster than any polynomial since $2\alpha > 1$.
Finally, to establish the upper bound on $\mathbb{P}\left(s_+ < \epsilon\right)$, it suffices to note that the arguments in the proof of Theorem 2 of [Mol99] show that there is a constant c such that $\mathbb{P}\left(s_+ < \epsilon\right) \le 2\mathbb{P}\left(X_{1/\epsilon}^* < c\left|\log \epsilon\right|^{1/2}\right)$ for all $\epsilon > 0$ small enough. It is now straightforward to conclude in view of the self-similarity and (1.11). □

Remark 5.3.2. As already remarked in [Mol99], $1/z_- \stackrel{d}{=} z_+ := \inf\left\{s > 1 : X_s = 0\right\}$ since $(X_t)_{t>0}$ and $(t^{2H} X_{1/t})_{t>0}$ have the same law. Hence, Proposition 5.3.1 shows that $\mathbb{P}\left(z_+ > T\right)$ decays like $T^{-(1-H)}$ modulo logarithmic terms as $T \to \infty$.

Chapter 6

Open problems

Let us summarise some open questions at the end of this thesis. First of all, one could try to strengthen the assertions of many theorems or weaken their assumptions. For instance, concerning the result on weighted sums of independent random variables of Theorem 2.1.2, it seems reasonable to make the following conjecture:

Conjecture 1. *Let $(Y_n)_{n\geq 1}$ be a sequence of centered i.i.d. random variables with $\mathbb{E}[Y_1^2] < \infty$ and $(\sigma(n))_{n\geq 1}$ a sequence of positive numbers such that $\sigma(N) \asymp N^p$ for some $p > 0$. Set $Z_n := \sum_{k=1}^{n} \sigma(k) Y_k$. Then*

$$\mathbb{P}(Z_n \leq 0, \forall n = 1, \ldots, N) \asymp N^{-(p+1/2)}, \qquad N \to \infty.$$

Let us continue to mention some open problems in the context of Chapter 2. In the case of exponential weight functions, it does not seem easy to compute the exponential rate of decay λ_β (cf. (2.17)) even in the Gaussian case. To obtain new insights, the variational characterisation of $\exp(-\lambda_\beta)$ involving an integral operator in Theorem 2.2.28 appears to be useful (for instance, one could use suitable test functions to obtain better estimates on λ_β, cf. (2.37)). Moreover, it is still open if $\lambda_\beta \sim \beta/2$ as $\beta \to 0$ or if $\lambda_\beta < \beta/2$ for all $\beta > 0$, see Remark 2.2.16. The latter question is directly related to the following one: for which functions κ with $\kappa(x) \to \infty$ is $\mathbb{P}\left(B_{\kappa(n)} \leq 1, \forall n = 1, \ldots, N\right) = \kappa(N)^{-1/2+o(1)}$? This problem was mentioned in Remark 2.2.3 and 2.2.13.
Finally, almost nothing seems to be known about persistence of sums of correlated random variables, cf. Remark 2.2.7.

Concerning persistence of iterated processes, Theorem 3.1.2 has been strengthened in [Vys12a]. The assertions of the Theorems 3.1.2, 3.1.5 and 3.1.3 could be improved as well if better estimates on the probability of gaps in the range of random walks were known, cf. Remark 3.3.4.
Moreover, it would be interesting to study persistence of iterated Lévy processes if the inner process does not have a finite second moment. For instance, let $(X_t)_{t\geq 0}$ be a

symmetric Lévy process and $(Y_t)_{t\geq 0}$ an independent α-stable subordinator with index $\alpha \in (0,1)$. Then $(X(Y_t))_{t\geq 0}$ is a symmetric Lévy process and

$$T^{-1/2} \asymp \mathbb{P}\left(X(Y_t) \leq 1, \forall t \in [0,T]\right), \qquad T \to \infty.$$

In particular, note that the lower bound

$$\mathbb{P}\left(X_t \leq 1, \forall t \in [0, Y_T]\right) \geq \mathbb{P}\left(Y_T \leq cT^{1/\alpha}\right) \mathbb{P}\left(X_t \leq 1, \forall t \in [0, cT^{1/\alpha}]\right) \asymp T^{-1/(2\alpha)}$$

does not provide the correct order. Hence, the gaps in the range of the inner process cannot be neglected as in the case of finite variance. If $(Y_t)_{t\geq 0}$ is α-stable, but not a subordinator, it seems challenging to compute the persistence exponent of $X \circ |Y|$.

For autoregressive processes, it appears very difficult to compute the persistence exponent of higher-order integrated random walks $S^{(n)}$ ($S^{(1)}$ is a usual centred random walk and $S_k^{(n)} := \sum_{j=1}^{k} S_j^{(n-1)}$).
For AR(p)-processes $X_n = a_1 X_{n-1} + \cdots + a_p X_{n-p} + Y_n$, it would be interesting to find general conditions on the coefficients $a_1, \ldots, a_p$ such that the survival probability decays exponentially. In particular, a generalisation of Theorem 4.3.1 to higher order processes would be desirable.
If $(c_n)_{n\geq 1}$ is a sequence with $\sum |c_n| < \infty$, let $Z := \sum_{n=1}^{\infty} c_n Y_n$, where (Y_n) is a sequence of i.i.d. random variables such that Z is a.s. a finite (for instance, if $\mathbb{E}\left[Y_1^2\right] < \infty$). As we have seen in Theorem 4.3.10, it can be important to understand the behaviour of $\mathbb{P}\left(|Z| \leq \epsilon\right)$ as $\epsilon \downarrow 0$. In Proposition 4.3.8, we have considered the case $c_n = \rho^n$ with $|\rho| < 1$. However, this results does not cover purely discrete distributions. As already mentioned in Remark 4.3.9, it would be interesting to make progess in this direction.

Finally, if $(B_t^H)_{t\geq 0}$ is a fractional Brownian motion with Hurst index $H \in (0,1)$, it is still an open problem to prove that $\mathbb{P}\left(B_t^H \leq 1, \forall t \in [0,T]\right) \asymp T^{H-1}$. Moreover, if $\gamma < H$ and $f(t) = 1 + ct^{\gamma}$ with $c \in \mathbb{R}$, it can be shown by a change of measure that

$$\exp(-c\sqrt{\log T})T^{H-1} \precsim \mathbb{P}\left(B_t^H \leq f(t), \forall t \in [0,T]\right) \precsim \exp(c\sqrt{\log T})T^{H-1},$$

see [AD13, Proposition 3.1] (equation (2.10) here). I suspect that the lower order terms could be removed or at least reduced to a logarithmic error as in Chapter 5.

Appendix A

Slepian's inequality

Let us now give the proof of Lemma 1.2.5.

Proof. *Part 1:* Let $t_1, t_2, \ldots$ denote a sequence in $\mathbb{T}$. Using Slepian's inequality as stated in [LT91, Corollary 3.12], it holds that

$$\mathbb{P}\left(X(t_k) \leq f(t_k), \forall k = 1, \ldots, n\right) \leq \mathbb{P}\left(Y(t_k) \leq f(t_k), \forall k = 1, \ldots, n\right), \quad n \geq 1. \tag{A.1}$$

Clearly, the last inequality also holds for $n = \infty$. Hence, part 1 follows if we can find a sequence $(t_n)_{n\geq 1}$ such that $\mathbb{P}\left(X(t_k) \leq f(t_k), \forall k \geq 1\right) = \mathbb{P}\left(X(t) \leq f(t), \forall t \in \mathbb{T}\right)$ and $\mathbb{P}\left(Y(t_k) \leq f(t_k), \forall k \geq 1\right) = \mathbb{P}\left(Y(t) \leq f(t), \forall t \in \mathbb{T}\right)$.

Recall that separability of X means that there is a negligible set N, and a countable set $I_X \subset \mathbb{T}$ such that for all $t \in \mathbb{T}$, all $\epsilon > 0$ and all $\omega \notin N$,

$$X_t(\omega) \in \overline{\{X_s(\omega) : s \in I_X, d(s,t) < \epsilon\}},$$

see [LT91, p.45]. In particular, this implies that there are countable sets $I_X, I_Y \subseteq \mathbb{T}$ such that a.s.

$$\sup\left\{X_t : t \in I_X\right\} = \sup\left\{X_t : t \in \mathbb{T}\right\}, \quad \sup\left\{Y_t : t \in I_Y\right\} = \sup\left\{Y_t : t \in \mathbb{T}\right\}.$$

By enlarging the sets I_X and I_Y if necessary, we may assume that they are countable and dense in $\mathbb{T}$ since $\mathbb{T}$ is separable. Let $I_X \cup I_Y \cup D = \{t_1, t_2, \ldots\}$. By continuity of f on $\mathbb{T} \setminus D$ and separability of X and Y, (A.1) implies that

$$\mathbb{P}\left(X(t) \leq f(t), \forall t \in \mathbb{T}\right) \leq \mathbb{P}\left(Y(t) \leq f(t), \forall t \in \mathbb{T}\right). \tag{A.2}$$

Indeed, on the event $\{X(t_n) \leq f(t_n), \forall n \geq 1\}$, then a.s., for every $t \in \mathbb{T} \setminus D$, there is a subsequence $t_{n_k} \to t$, and using the separability of X in the first inequality, and the continuity of f on $\mathbb{T} \setminus D$ in the last equality, we obtain that

$$X_t \leq \limsup_{k\to\infty} X(t_{n_k}) \leq \limsup_{k\to\infty} f(t_{n_k}) = f(t),$$

and the same holds for Y.
Part 2: If $(Y_t)_{t\geq 0}$ is a centred Gaussian process with $\mathbb{E}[Y_tY_s] \geq 0$, let $(Y_t')_{t\geq 0}$ denote an independent copy of Y on the same probability space. Let $S, T > 0$. Define $X_t = Y_t$ on $[0, S]$ and $X_t = Y_t'$ on (S, ∞). Note that X is also a centred, separable Gaussian process. Clearly $\mathbb{E}[X_t^2] = \mathbb{E}[Y_t^2]$. Moreover, $0 = \mathbb{E}[X_tX_s] \leq \mathbb{E}[Y_tY_s]$ whenever $s \in [0, S]$ and $t > S$, whereas $\mathbb{E}[X_tX_s] = \mathbb{E}[Y_tY_s]$ for $s, t \leq S$ and $s, t > S$. Hence, by part 1,

$$\mathbb{P}\left(X_t \leq f(t), \forall t \in [0, T+S]\right) \leq \mathbb{P}\left(Y_t \leq f(t), \forall t \in [0, T+S]\right),$$

and by independence of Y and Y',

$$\begin{aligned}\mathbb{P}\left(X_t \leq f(t), \forall t \in [0, T+S]\right) &= \mathbb{P}\left(Y_t \leq f(t), \forall t \in [0, S]\right) \cdot \mathbb{P}\left(Y_t' \leq f(t), \forall t \in (S, T+S]\right) \\ &= \mathbb{P}\left(Y_t \leq f(t), \forall t \in [0, S]\right) \cdot \mathbb{P}\left(Y_t \leq f(t), \forall t \in (S, T+S]\right).\end{aligned}$$

This proves (1.12).
If Y is additionally stationary, we can apply (1.12) with the constant function $f \equiv x$ to find that $h(T) := -\log \mathbb{P}\left(X_t \leq x, \forall t \in [0, T]\right)$ is subadditive. Hence, the existence of the limit and its representation as a supremum follows from the usual subaddivity argument (Fekete's Lemma). □

Appendix B

An integral operator related to persistence

B.1 The spectral radius of T

Lemma B.1.1. *The spectral radius* $r(T)$ *of* T *defined in* (2.26) *is equal to* 1*.*

Proof. Let $T'\colon L^\infty \to L^\infty$ denote the adjoint of T. Since the spectral radius of T and T' coincides (see e.g.[RS72, Theorem VI.7]), it suffices to show that $r(T') = 1$. In fact, we show that $\|(T')^n\| = 1$ for all $n \geq 1$.
It is easy to verify that $T'g(u) := \int_{-\infty}^0 p(u,y)g(y)\,dy$, $g \in L^\infty$, is the adjoint of T: if $f \in L^1, g \in L^\infty$, denote by $\langle f, g\rangle = \int_{-\infty}^0 f(y)g(y)\,dy$ the duality pairing. By Fubini's theorem, it holds that

$$\langle Tf, g\rangle = \int_{-\infty}^0 \int_{-\infty}^0 p(x,y)f(x)\,dx\,g(y)\,dy = \int_{-\infty}^0 \int_{-\infty}^0 p(x,y)g(y)\,dy\,f(x)\,dx = \langle f, T'g\rangle,$$

which proves that T' is the adjoint of T.
Let $g_0(u) = 1_{\{(-\infty,0]\}}(u)$. We claim for all $n \geq 1$ that

$$\lim_{u\to-\infty} (T')^n g_0(u) = 1, \qquad \text{and} \qquad \|(T')^n\| = 1.$$

If this is true, it is evident that $r(T') = 1$, and the lemma follows.
For $n = 1$, we get in view of (2.27) that

$$T'g_0(u) = \int_{-\infty}^0 p(u,y)\,dy \to 1, \quad u \to -\infty. \tag{B.1}$$

Moreover, by (2.27), we have that

$$|T'g(u)| \leq \|g\|_\infty \cdot \int_{-\infty}^0 p(u,y)\,dy \leq \|g\|_\infty,$$

so clearly $\|T'\| = 1$.
If the claim holds for some $n \geq 1$, then for any $u < 0$, we have

$$(T')^{n+1} g_0(u) \geq \int_{-\infty}^{\rho u/2} p(u,y)(T')^n g_0(y)\, dy \geq \left\{ \inf_{z \in (-\infty, \rho u/2]} (T')^n g_0(z) \right\} \int_{-\infty}^{\rho u/2} p(u,y)\, dy.$$

By the induction hypothesis, the term $\{\dots\} \to 1$ as $u \to -\infty$. Moreover, one computes

$$\begin{aligned}\int_{-\infty}^{\rho u/2} p(u,y)\, dy &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\rho u/2} \exp\left(-\tfrac{(y-\rho u)^2}{2\sigma^2}\right) dy \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{-\rho u/2} \exp\left(-\tfrac{y^2}{2\sigma^2}\right) dy \to 1, \quad u \to -\infty.\end{aligned}$$

Hence, $\liminf_{u \to -\infty} (T')^{n+1} g_0(u) \geq 1$. Since $\|(T')^{n+1}\| \leq \|T'\| \|(T')^n\| = \|(T')^n\| = 1$ by the induction hypothesis, it follows easily that $\|(T')^{n+1} g_0\|_\infty \leq 1$. Consequently, it holds that $\lim_{u \to -\infty} (T')^{n+1} g_0(u) = 1$ and $\|(T')^{n+1}\| = 1$. □

B.2 Bounds on the eigenvalues

Here we give a proof of (2.34) and (2.37). Let $A := (1+\rho^2)/(4\sigma^2) > 0$ and $B := \rho/\sigma^2$. We first compute $\|K\|_{2,2}$ stated in (2.34). In view of (2.30) and (2.33), we have that

$$\begin{aligned}\|K\|_{2,2}^2 &= \int_{-\infty}^0 \int_{-\infty}^0 K(x,y)^2\, dy\, dx \\ &= \frac{1}{2\pi\sigma^2} \cdot \int_{-\infty}^0 e^{-2Ax^2} \frac{\sqrt{\pi}}{2\sqrt{2A}} e^{B^2x^2/(2A)} \mathrm{Erfc}\left(\frac{Bx}{\sqrt{2A}}\right) dx \\ &= \frac{1}{4\sqrt{2\pi}\sqrt{A}\sigma^2} \int_{-\infty}^0 e^{-2(A - B^2/(4A))x^2} \mathrm{Erfc}\left(\frac{Bx}{\sqrt{2A}}\right) dx \\ &= \frac{1}{4\sqrt{2\pi}\sqrt{A}\sigma^2} \cdot \frac{\pi - \arctan\left(\frac{\sqrt{2(A - B^2/(4A))}}{B/\sqrt{2A}}\right)}{\sqrt{\pi}\sqrt{2(A - B^2/(4A))}} \\ &= \frac{1}{8\pi\sigma^2} \cdot \frac{\pi - \arctan\left(\frac{2\sqrt{A^2 - B^2/4}}{B}\right)}{\sqrt{A^2 - B^2/4}}.\end{aligned}$$

Note that

$$A^2 - \frac{B^2}{4} = \frac{1 + 2\rho^2 + \rho^4}{16\sigma^4} - \frac{\rho^2}{4\sigma^4} = \frac{(1-\rho^2)^2}{16\sigma^4},$$

so it follows that

$$\|K\|_{2,2}^2 = \frac{1}{2\pi} \cdot \frac{\pi - \arctan\left(\left|1-\rho^2\right| / (2\rho)\right)}{\left|1-\rho^2\right|}.$$

This proves (2.34).
In order to prove (2.37), let $f_\alpha(x) := (8\alpha/\pi)^{1/4}\, e^{-\alpha x^2}$ and $I(\alpha) := (Sf_\alpha, f_\alpha)$. Then

$$\begin{aligned}
I(\alpha) &= \int_{-\infty}^{0}\int_{-\infty}^{0} f(y)K(x,y)f(x)\,dy\,dx = \int_{-\infty}^{0}\int_{-\infty}^{0} f(y)\frac{e^{-A(x^2+y^2)+Bxy}}{\sqrt{2\pi}\sigma} f(x)\,dy\,dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\cdot\sqrt{\frac{8\alpha}{\pi}}\cdot\int_{-\infty}^{0} e^{-(A+\alpha)x^2}\int_{-\infty}^{0} e^{-(A+\alpha)y^2+Bxy}\,dy\,dx \\
&= \frac{2\sqrt{\alpha}}{\pi\sigma}\cdot\int_{-\infty}^{0} e^{-(A+\alpha)x^2}\cdot\frac{\sqrt{\pi}}{2\sqrt{A+\alpha}}\, e^{B^2x^2/(4(A+\alpha))}\mathrm{Erfc}\left(\frac{Bx}{2\sqrt{A+\alpha}}\right)dx \\
&= \frac{\sqrt{\alpha}}{\sigma\sqrt{\pi}\sqrt{A+\alpha}}\cdot\int_{-\infty}^{0} e^{-(A+\alpha-B^2/(4(A+\alpha)))x^2}\mathrm{Erfc}\left(\frac{Bx}{2\sqrt{A+\alpha}}\right)dx.
\end{aligned}$$

We have used (2.29) in the third equality. Let $\Delta := B/(2\sqrt{A+\alpha})$. Using (2.33), we obtain that

$$\begin{aligned}
I(\alpha) &= \frac{\sqrt{\alpha}}{\sigma\sqrt{\pi}\sqrt{A+\alpha}}\cdot\int_{-\infty}^{0} e^{-(A+\alpha-\Delta^2)x^2}\mathrm{Erfc}(\Delta x)\,dx \\
&= \frac{\sqrt{\alpha}}{\sigma\sqrt{\pi}\sqrt{A+\alpha}}\cdot\frac{\pi-\arctan(\sqrt{A+\alpha-\Delta^2}/\Delta)}{\sqrt{\pi}\sqrt{A+\alpha-\Delta^2}} \\
&= \frac{\sqrt{\alpha}}{\sigma\pi}\cdot\frac{\pi-\arctan\left(\frac{2\sqrt{(A+\alpha)^2-B^2/4}}{B}\right)}{\sqrt{(A+\alpha)^2-B^2/4}}.
\end{aligned}$$

Bibliography

[AB11] F. Aurzada and C. Baumgarten, *Survival probabilities of weighted random walks*, ALEA Lat. Am. J. Probab. Math. Stat. **8** (2011), 235–258.

[AB13] ———, *Persistence of fractional Brownian motion with moving boundaries*, J. Phys. A **46** (2013), no. 12.

[AD13] F. Aurzada and S. Dereich, *Universality of the asymptotics of the one-sided exit problem for integrated processes*, Ann. Inst. Henri Poincaré Probab. Stat. **49** (2013), 236–251.

[AKS12] F. Aurzada, T. Kramm, and M. Savov, *First passage times of Lévy processes over a one-sided moving boundary*, arXiv:1201.1118 (2012).

[AL09] F. Aurzada and M. Lifshits, *On the small deviation problem for some iterated processes*, Electron. J. Probab. **14** (2009), no. 68, 1992–2010.

[Alt02] H. W. Alt, *Lineare Funtkionalanalysis*, Springer-Verlag, Berlin, 2002.

[AS12] F. Aurzada and T. Simon, *Persistence probabilities and exponents*, arXiv:1203.6554 (2012).

[Aur11] F. Aurzada, *On the one-sided exit problem for fractional Brownian motion*, Electron. Commun. Probab. **16** (2011), 392–404.

[AZ01] H. Allouba and W. Zheng, *Brownian-time processes: the PDE connection and the half-derivative generator*, Ann. Probab. **29** (2001), no. 4, 1780–1795.

[Bau12] C. Baumgarten, *Survival probabilities of some iterated processes*, arXiv:1106.2999 (2012).

[Bau13] ———, *Survival probabilities of autoregressive processes*, to appear in ESAIM P&S (2013).

[BD87] P. J. Brockwell and R. A. Davis, *Time series: theory and methods*, Springer Series in Statistics, Springer-Verlag, New York, 1987.

[BD97] J. Bertoin and R. A. Doney, *Spitzer's condition for random walks and Lévy processes*, Ann. Inst. Henri Poincaré Probab. Stat. **33** (1997), no. 2, 167–178.

[Bee75] J. A. Beekman, *Asymptotic distributions for the Ornstein-Uhlenbeck process*, J. Appl. Probability **12** (1975), 107–114.

[Ber98] J. Bertoin, *The inviscid Burgers equation with Brownian initial velocity*, Comm. Math. Phys. **193** (1998), no. 2, 397–406.

[BG90] J.-P. Bouchaud and A. Georges, *Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications*, Physics Reports **195** (1990), no. 4 & 5, 127 – 293.

[BGT87] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular variation*, Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge University Press, Cambridge, 1987.

[Bin73] N. H. Bingham, *Limit theorems in fluctuation theory*, Advances in Appl. Probability **5** (1973), 554–569.

[BP88] R. J. Barton and H. V. Poor, *Signal detection in fractional Gaussian noise*, IEEE Trans. Inform. Theory **34** (1988), no. 5, part 1, 943–959.

[Bra78] M. D. Bramson, *Maximal displacement of branching Brownian motion*, Comm. Pure Appl. Math. **31** (1978), no. 5, 531–581.

[BRS86] A. T. Bharucha-Reid and M. Sambandham, *Random polynomials*, Probability and Mathematical Statistics, Academic Press Inc., Orlando, FL, 1986.

[Bur93] K. Burdzy, *Some path properties of iterated Brownian motion*, Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992), Progr. Probab., vol. 33, Birkhäuser Boston, Boston, MA, 1993, pp. 67–87.

[CGPPS13] F. Castell, N. Guillotin-Plantard, F. Pène, and B. Schapira, *On the one-sided exit problem for stable processes in random scenery*, to appear in Electron. Commun. Probab. (2013).

[CR98] F. Comte and E. Renault, *Long memory in continuous-time stochastic volatility models*, Math. Finance **8** (1998), no. 4, 291–323.

[dA83] A. de Acosta, *Small deviations in the functional central limit theorem with applications to functional laws of the iterated logarithm*, Ann. Probab. **11** (1983), no. 1, 78–101.

[DDG12] A. Dembo, J. Ding, and F. Gao, *Persistence of iterated partial sums*, arXiv:1205.5596, to appear in Ann. Inst. Henri Poincaré Probab. Stat. (2012).

[DM12] A. Dembo and S. Mukherjee, *No zero-crossings for random polynomials and the heat equation*, arXiv:1208.2382 (2012).

[Don89] R. A. Doney, *On the asymptotic behaviour of first passage times for transient random walk*, Probab. Theory Related Fields **81** (1989), no. 2, 239–246.

[Don95] ______, *Spitzer's condition and ladder variables in random walks*, Probab. Theory Related Fields **101** (1995), no. 4, 577–580.

[Don07] ______, *Fluctuation theory for Lévy processes*, Lecture Notes in Mathematics, vol. 1897, Springer, Berlin, 2007.

[DPSZ02] A. Dembo, B. Poonen, Q.-M. Shao, and O. Zeitouni, *Random polynomials having few or no real zeros*, J. Amer. Math. Soc. **15** (2002), no. 4, 857–892 (electronic).

[DS58] N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory*, With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7, Interscience Publishers, Inc., New York, 1958.

[DS89] J.-D. Deuschel and D. W. Stroock, *Large deviations*, Pure and Applied Mathematics, vol. 137, Academic Press Inc., Boston, MA, 1989.

[DÜ99] L. Decreusefond and A. S. Üstünel, *Stochastic analysis of the fractional Brownian motion*, Potential Anal. **10** (1999), no. 2, 177–214.

[DW12] D. Denisov and V. Wachtel, *Exit times for integrated random walks*, arXiv:1207.2270 (2012).

[DZ10] A. Dembo and O. Zeitouni, *Large deviations techniques and applications*, Stochastic Modelling and Applied Probability, vol. 38, Springer-Verlag, Berlin, 2010, Corrected reprint of the second (1998) edition.

[Ego73] V. A. Egorov, *The rate of convergence to the normal law that is equivalent to the existence of the second moment*, Theory Prob. Appl. **18** (1973), 175–180.

[EK95] A. Edelman and E. Kostlan, *How many zeros of a random polynomial are real?*, Bull. Amer. Math. Soc. (N.S.) **32** (1995), no. 1, 1–37.

[EK08] I. Eliazar and J. Klafter, *Markov-breaking and the emergence of long memory in Ornstein-Uhlenbeck systems*, J. Phys. A **41** (2008), no. 12, 122001.

[Ela99] S. N. Elaydi, *An introduction to difference equations*, second ed., Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1999.

[EPW67] J. D. Esary, F. Proschan, and D. W. Walkup, *Association of random variables, with applications*, Ann. Math. Statist. **38** (1967), 1466–1474.

[ES12] I. Eliazar and M. F. Shlesinger, *Langevin unification of fractional motions*, J. Phys. A **45** (2012), no. 16, 162002.

[Fel71] W. Feller, *An introduction to probability theory and its applications. Vol. II.*, Second edition, John Wiley & Sons Inc., New York, 1971.

[Gol71] M. Goldman, *On the first passage of the integrated Wiener process*, Ann. Mat. Statist. **42** (1971), 2150–2155.

[GR00] I.S. Gradshteyn and M. Ryzhik, *Table of integrals, series, and products. sixth edition.*, Academic Press, San Diego, 2000.

[GR08] T. Grzywny and M. Ryznar, *Two-sided optimal bounds for Green functions of half-spaces for relativistic α-stable process*, Potential Analysis **28** (2008), no. 3, 201–239.

[GS01] G. R. Grimmett and D. R. Stirzaker, *Probability and random processes*, third ed., Oxford University Press, New York, 2001.

[Hac95] W. Hackbusch, *Integral equations*, International Series of Numerical Mathematics, vol. 120, Birkhäuser Verlag, Basel, 1995, Theory and numerical treatment, Translated and revised by the author from the 1989 German original.

[IW94] Y. Isozaki and S. Watanabe, *An asymptotic formula for the Kolmogorov diffusion and a refinement of Sinai's estimates for the integral of Brownian motion*, Proc. Japan Acad. Ser. A Math. Sci. **70** (1994), no. 9, 271–276.

[KKM$^+$97] J. Krug, H. Kallabis, S. N. Majumdar, S. J. Cornell, A. J. Bray, and C. Sire, *Persistence exponents for fluctuating interfaces*, Phys. Rev. E **56** (1997), 2702–2712.

[KL99] D. Khoshnevisan and T. M. Lewis, *Iterated Brownian motion and its intrinsic skeletal structure*, Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1996), Progr. Probab., vol. 45, Birkhäuser, Basel, 1999, pp. 201–210.

[KLS89] M. A. Krasnosel′skij, Je. A. Lifshits, and A. V. Sobolev, *Positive linear systems*, Sigma Series in Applied Mathematics, vol. 5, Heldermann Verlag, Berlin, 1989, The method of positive operators, Translated from the Russian by Jürgen Appell.

[KMT76] J. Komlós, P. Major, and G. Tusnády, *An approximation of partial sums of independent RV's, and the sample DF. II*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **34** (1976), no. 1, 33–58.

[KP90] J. K. Kim and D. Pollard, *Cube root asymptotics*, Ann. Statist. **18** (1990), no. 1, 191–219.

[Lar04] H. Larralde, *A first passage time distribution for a discrete version of the Ornstein-Uhlenbeck process*, J. Phys. A **37** (2004), no. 12, 3759–3767.

[Lif13] M. Lifshits, www.proba.jussieu.fr/pageperso/smalldev/biblio.pdf, 2013.

[LS01] W. V. Li and Q.-M. Shao, *Gaussian processes: inequalities, small ball probabilities and applications*, Stochastic processes: theory and methods, Handbook of Statist., vol. 19, North-Holland, Amsterdam, 2001, pp. 533–597.

[LS02] ______, *A normal comparison inequality and its applications*, Probab. Theory Related Fields **122** (2002), no. 4, 494–508.

[LS04] ______, *Lower tail probabilities for Gaussian processes*, Ann. Probab. **32** (2004), no. 1A, 216–242.

[LT91] M. Ledoux and M. Talagrand, *Probability in Banach spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 23, Springer-Verlag, Berlin, 1991, Isoperimetry and processes.

[Luk70] E. Lukacs, *Characteristic functions*, Hafner Publishing Co., New York, 1970, Second edition, revised and enlarged.

[Maj99] S. N. Majumdar, *Persistence in nonequlilibrium systems*, Current Sci. **77** (1999), no. 3, 370–375.

[Maj03] S. N. Majumdar, *Persistence of a particle in the Matheron-de Marsily velocity field*, Phys. Rev. E **68** (2003), 050101.

[MBE01] S. N. Majumdar, A. J. Bray, and G. C. M. A. Ehrhardt, *Persistence of a continuous stochastic process with discrete-time sampling*, Phys. Rev. E **64** (2001), 015101–(1–4).

[McK63] H. P. McKean, Jr., *A winding problem for a resonator driven by a white noise*, J. Math. Kyoto Univ. **2** (1963), 227–235.

[MD01] S. N. Majumdar and D. Dhar, *Persistence in a stationary time series*, Phys. Rev. E **64** (2001), 046123.

[MK04] G. M. Molchan and A. Khokhlov, *Small values of the maximum for the integral of fractional Brownian motion*, J. Statist. Phys. **114** (2004), no. 3-4, 923–946.

[Mog74] A. A. Mogul′skiĭ, *Small deviations in the space of trajectories*, Theor. Prob. Appl. **19** (1974), no. 4, 726–736.

[Mol99] G. M. Molchan, *Maximum of a fractional Brownian motion: probabilities of small values*, Comm. Math. Phys. **205** (1999), no. 1, 97–111.

[Mol12] ———, *Survival exponents for some Gaussian processes*, Int. J. Stoch. Anal. (2012), 20 pages, Article ID 137271.

[MS93] S. J. Montgomery-Smith, *Comparison of sums of independent identically distributed random vectors*, Probab. Math. Statist. **14** (1993), no. 2, 281–285 (1994).

[MVN68] B. B. Mandelbrot and J. W. Van Ness, *Fractional Brownian motions, fractional noises and applications*, SIAM Rev. **10** (1968), 422–437.

[Nag79] S. V. Nagaev, *Large deviations of sums of independent random variables*, Ann. Probab. **7** (1979), no. 5, 745–789.

[Nan08] E. Nane, *Higher order PDE's and iterated processes*, Trans. Amer. Math. Soc. **360** (2008), no. 5, 2681–2692.

[NK08] A. Novikov and N. Kordzakhia, *Martingales and first passage times of* AR(1) *sequences*, Stochastics **80** (2008), no. 2-3, 197–210.

[Nov82] A. A. Novikov, *The crossing time of a one-sided nonlinear boundary by sums of independent random variables*, Theory Probab. Appl. **27** (1982), no. 4, 688–702.

[NR62] G. F. Newell and M. Rosenblatt, *Zero crossing probabilities for Gaussian stationary processes*, Ann. Math. Statist. **33** (1962), 1306–1313.

[Obł04] J. Obłój, *The Skorokhod embedding problem and its offspring*, Probab. Surv. **1** (2004), 321–390.

[Øks07] B. Øksendal, *Fractional Brownian motion in finance*, Stochastic economic dynamics, Cph. Bus. Sch. Press, Frederiksberg, 2007, pp. 11–56.

[ORS12] G. Oshanin, A. Rosso, and G. Schehr, *Anomalous fluctuations of currents in Sinai-type random chains with strongly correlated disorder*, arXiv:1209.3313v2 (2012).

[PSS00] Y. Peres, W. Schlag, and B. Solomyak, *Sixty years of Bernoulli convolutions*, Fractal geometry and stochastics, II (Greifswald/Koserow, 1998), Progr. Probab., vol. 46, Birkhäuser, Basel, 2000, pp. 39–65.

[PT00] V. Pipiras and M. S. Taqqu, *Integration questions related to fractional Brownian motion*, Probab. Theory Related Fields **118** (2000), no. 2, 251–291.

[Red97] S. Redner, *Survival probability in a random velocity field*, Phys. Rev. E **56** (1997), 4967–4972.

[Rog71] B. A. Rogozin, *Distribution of the first ladder moment and height, and fluctuations of a random walk*, Theory Prob. Appl. **16** (1971), no. 4, 575–595.

[Ros62] B. Rosén, *On the asymptotic distribution of sums of independent identically distributed random variables*, Ark. Mat. **4** (1962), 323–332.

[RS72] M. Reed and B. Simon, *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, New York, 1972.

[SAF92] Z.-S. She, E. Aurell, and U. Frisch, *The inviscid Burgers equation with initial data of Brownian type*, Comm. Math. Phys. **148** (1992), no. 3, 623–641.

[Sak84] A. I. Sakhanenko, *Rate of convergence in the invariance principle for variables with exponential moments that are not identically distributed*, Limit theorems for sums of random variables, Trudy Inst. Mat., vol. 3, "Nauka" Sibirsk. Otdel., Novosibirsk, 1984, pp. 4–49.

[Sat77] S. Sato, *Evaluation of the first-passage time probability to a square root boundary for the Wiener process*, J. Appl. Probability **14** (1977), no. 4, 850–856.

[Sch74] H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, New York, 1974, Die Grundlehren der mathematischen Wissenschaften, Band 215.

[Sch09] M. Scheutzow, *Chaining techniques and their application to stochastic flows*, Trends in stochastic analysis, London Math. Soc. Lecture Note Ser., vol. 353, Cambridge Univ. Press, Cambridge, 2009, pp. 35–63.

[Sim07] T. Simon, *The lower tail problem for homogeneous functionals of stable processes with no negative jumps*, ALEA Lat. Am. J. Probab. Math. Stat. **3** (2007), 165–179.

[Sin92a] Ya. G. Sinaĭ, *Distribution of some functionals of the integral of a random walk*, Teoret. Mat. Fiz. **90** (1992), no. 3, 323–353.

[Sin92b] ______, *Statistics of shocks in solutions of inviscid Burgers equation*, Comm. Math. Phys. **148** (1992), no. 3, 601–621.

[Sin97] ______, *On the distribution of the maximum of fractional Brownian motion*, Uspekhi Mat. Nauk **52** (1997), no. 2(314), 119–138.

[SKM93] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives*, Gordon and Breach Science Publishers, Yverdon, 1993, Theory and applications, Edited and with a foreword by S. M. Nikol′skiĭ, Translated from the 1987 Russian original, Revised by the authors.

[Sle62] D. Slepian, *The one-sided barrier problem for Gaussian noise*, Bell System Tech. J. **41** (1962), 463–501.

[SM08] G. Schehr and S. N. Majumdar, *Real roots of random polynomials and zero crossing properties of diffusion equation*, J. Stat. Phys. **132** (2008), no. 2, 235–273.

[Uch80] K. Uchiyama, *Brownian first exit from and sojourn over one-sided moving boundary and application*, Z. Wahrsch. Verw. Gebiete **54** (1980), no. 1, 75–116.

[Vys10] V. Vysotsky, *On the probability that integrated random walks stay positive*, Stochastic Process. Appl. **120** (2010), no. 7, 1178–1193.

[Vys12a] ______, *Persistence of iterated random walks*, Preprint available under `http://math.la.asu.edu/~vysotsky/index_files/iterated.pdf` (2012).

[Vys12b] ______, *Positivity of integrated random walks*, to appear in Ann. Inst. Henri Poincaré Probab. Stat. (2012+).

[Wei82] H.L. Weinert, *Reproducing kernel Hilbert spaces: applications in statistical signal processing*, Benchmark papers in electrical engineering and computer science, Hutchinson Ross Pub. Co., 1982.

[WFCV12] J.-C. Walter, A. Ferrantini, E. Carlon, and C. Vanderzande, *Fractional Brownian motion and the critical dynamics of zipping polymers*, Phys. Rev. E **85** (2012), 031120.

[ZRM09] A. Zoia, A. Rosso, and S. N. Majumdar, *Asymptotic behavior of self-affine processes in semi-infinite domains*, Phys. Rev. Lett. **102** (2009), 120602.