Stability and Controllability of Double Integrator Consensus Systems in Heterogeneous Networks

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STABILITY AND CONTROLLABILITY
OF DOUBLE INTEGRATOR CONSENSUS SYSTEMS IN
HETEROGENEOUS NETWORKS

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A consensus system is a set of communicating agents that agree on a variable of interest using only local communication. Some possible uses for consensus systems include smart grids, wireless sensor networks, social networks and various applications of unmanned aerial vehicles such as surveillance, mapping and environmental monitoring. If the agents have double integrator dynamics, then the consensus system is called a double integrator consensus system.

In this thesis a double integrator consensus system in continuous time is studied. The agents are assumed to be communicating their position and velocity information along some possibly different, weighted directed communication networks that are modeled by weighted directed graphs. The mathematical model of the system has the form

\[ \ddot{x}(t) = -L_x \dot{x}(t) - \beta L_x \ddot{x}(t), \]  

where \( x : \mathbb{R}^+ \rightarrow \mathbb{R}^n \) and \( \dot{x} : \mathbb{R}^+ \rightarrow \mathbb{R}^n \) are the collected positions and velocities of the agents, \( L_x \) and \( \dot{L}_x \) are the Laplacians of the communication graphs and \( \beta \in \mathbb{R}^+ \setminus \{0\} \) is a gain. A stability concept for such systems is derived in the thesis. The system \((S_1)\) is said to be consensus stable if all of its velocity differences approach zero in time, and asymptotically consensus stable, if all of its position differences tend to zero.

The standard assumption made in the literature, that the communication networks are the same, i.e. homogeneous, for both information, is too strong in many cases. Thus, the algorithm is investigated under the assumption that the communication graphs may be different and possibly even disconnected. Necessary and sufficient conditions for consensus stability and asymptotic consensus stability are presented in the special case that the communication graphs are weighted and undirected. These conditions are subsequently relaxed to some cases of weighted directed graphs. Moreover, we consider the convergence rate of the system and the final convergence value.

Following the stability analysis, it is shown how the autonomous consensus system \((S_1)\) can be transformed into a control system by introducing an external control input to a subset of the agents. The resulting system is given by

\[ \ddot{x}(t) = -A_x \dot{x}(t) - A_x \ddot{x}(t) + B u(t), \]  

where \( x : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_x} \) and \( \dot{x} : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_x} \) are the states of the controlled, “follower” agents, \( u : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_u} \) is the control input.
provided by the “leader” agents, \( n_f + n_1 = n \), and the matrices \( A_f^x, A_\xi \in \mathbb{R}^{n_f \times n_f}, B \in \mathbb{R}^{n_f \times n_1} \) are obtained from the Laplacian matrices \( L_x, L_\xi \). This network is called a leader-follower network and its controllability is investigated. Necessary conditions for controllability of the system (S2) are given on both algebraic and graph-theoretic levels. It is made apparent that the graph-theoretic conditions are highly dependent on the underlying graph symmetries.
Ein Konsensussystem besteht aus einer Menge von kommunizierenden Agenten, die sich mittels ausschließlich lokaler Kommunikation auf eine Variable einigen. Einige denkbare Anwendungen für Konsensussysteme sind beispielsweise elektrische Netzerwerke (so genannte “smart grids”), drahtlose Sensornetzwerke, soziale Netzwerke, und viele Anwendungen aus dem Bereich unbemannte Flugmaschinen, wie etwa Überwachung, Kartographie und Beobachtung von Ökosystemen. Wenn die Agenten, aus denen das Netzwerk besteht, eine Doppelintegratordynamik besitzen, so spricht man vom Doppelintegratorkonsensussystem.

Diese Arbeit beschäftigt sich mit einem Doppelintegratorkonsensussystem in kontinuierlicher Zeit. Es wird angenommen, dass die Agenten ihre Position und Geschwindigkeit entlang von gewichteten Kommunikationsnetzwerken übermitteln können, die verschieden sein dürfen und mittels gewichteter gerichteter Graphen modelliert werden. Das mathematische Modell des Systems hat die Form

\[ \ddot{x}(t) = -L_x x(t) - \beta L_x \dot{x}(t), \]  

wobei \( x : \mathbb{R}^+ \rightarrow \mathbb{R}^n \) und \( \dot{x} : \mathbb{R}^+ \rightarrow \mathbb{R}^n \) die gesammelten Positionen und Geschwindigkeiten der Agenten repräsentieren, \( L_x \) und \( L_{\dot{x}} \) die Laplacematrizen der Kommunikationsgraphen sind und \( \beta \in \mathbb{R}^+ \setminus \{0\} \) ein Verstärkungsfaktor ist. Für solche Systeme wird in dieser Arbeit ein Stabilitätskonzept hergeleitet. Das System (S1) wird konsensusstabil genannt, wenn alle Geschwindigkeitsunterschiede im System mit laufender Zeit null werden. Es wird asymptotisch konsensusstabil genannt, wenn alle Positionsunterschiede mit laufender Zeit null werden.

Im Anschluß an die Stabilitätsanalyse wird aufgezeigt, wie das autonome Konsensusystem \((S_1)\) in ein Regelsystem überführt werden kann, indem ein Teil der Agenten mit einem externen Regler versehen wird. Es ergibt sich das System

\[
\dot{x}(t) = -A_f^t x(t) - A_{\dot{x}}^t \dot{x}(t) + Bu(t), \quad (S_2)
\]

wobei \(x: \mathbb{R}^+ \rightarrow \mathbb{R}^{n_f}\) und \(\dot{x}: \mathbb{R}^+ \rightarrow \mathbb{R}^{n_f}\) die Zustände der zu regelnden, "folgenden" Agenten sind, \(u: \mathbb{R}^+ \rightarrow \mathbb{R}^{n_l}\) die Stellgröße ist, die von den "führenden" Agenten übermittelt wird, \(n_f + n_l = n\), und die Matrizen \(A_f^t, A_{\dot{x}}^t \in \mathbb{R}^{n_f \times n_f}\), \(B \in \mathbb{R}^{n_f \times n_l}\) aus den Laplacematrizen \(L_x, L_{\dot{x}}\) hergeleitet werden. Dieses Netzwerk wird als ein so genanntes "leader-follower" Netzwerk bezeichnet. Es wird auf Steuerbarkeit hin untersucht. Notwendige Bedingungen für die Steuerbarkeit von \((S_2)\) werden sowohl auf graphentheoretischer als auch auf algebraischer Ebene hergeleitet. Es wird aufgezeigt, dass die graphentheoretischen Bedingungen stark von den in den Graphen vorhandenen Symmetrien abhängen.
Some ideas and figures have appeared previously in the following publications:


Für Niko.
When you can’t run anymore, you crawl.
And when you can’t crawl,
when you can’t do that...
You find someone to carry you.
— Firefly

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iSCC  independent strongly connected component 17

LF  leader-follower 108

MASS  multi-agent supporting system 43

ODE  ordinary differential equation 33

QEP  quadratic eigenvalue problem 134

QMP  quadratic matrix polynomial 48

SOLF  second order leader-follower 114

LIST OF SYMBOLS

\((A)_{i,1}\)  
i-th column of a matrix A 9

\((A)_{i,:}\)  
i-th row of a matrix A 9

\((A)_{i,j}\)  
i, j-th entry of a matrix A 9

\(A^{-T}\)  
transpose of the inverse of the matrix A 9

\(I_k\)  
k x k identity matrix 9

\(\mathbb{R}\)  
field of real numbers 9

\(\mathbb{R}^+\)  
set of nonnegative real numbers 9

\(\text{asym}(A)\)  
asymmetric part of a matrix A 89, 131

\(\text{diag}(k_1, \ldots, k_n)\)  
diagonal matrix with entries \(k_1, \ldots, k_n\) on the diagonal 10

\(i\)  
imaginary unit 10

\(\text{ker}(A)\)  
kernel of a matrix A 10

\(\|v\|\)  
euclidian vector norm, 10

\(\mathbb{C}\)  
field of complex numbers 9

\(\mathbb{N}\)  
set of natural numbers 9

\(\text{Im}(a)\)  
imaginary part of a complex number a, 10
Re(a) real part of a complex number $a$ 10
alg($\lambda$) algebraic multiplicity of an eigenvalue $\lambda$ 10
blockdiag($K_1, \ldots, K_n$) block diagonal matrix with matrix blocks $K_1, \ldots, K_n$ on the diagonal 10
geo($\lambda$) geometric multiplicity of an eigenvalue $\lambda$ 10
$1_n$ $n \times 1$ vector of ones 9
$1_{k \times m}$ $k \times m$ matrix of ones 9
span($v_1, \ldots, v_k$) linear span of the vectors $v_1, \ldots, v_k$ 10
spec($A$) ordered list of the eigenvalues of a matrix $A$ 10
sym($A$) symmetric part of a matrix $A$ 89, 131
$\mathbb{0}_n$ $n \times 1$ vector of zeroes 9
$\mathbb{0}_{k \times m}$ $k \times m$ matrix of zeroes 9
$f(t) \rightarrow g(t)$ $f$ behaves like $g$ for large $t$ 10, 63
$x^*$ conjugate transpose of a vector $x$ 9
$x^T$ transpose of a vector $x$ 9
$|N|$ cardinality of a set $N$ 10
$|a|$ absolute value of a number $a$, 10

$A_G$ adjacency matrix of a graph 13
$A_{\dot{x}}$ system matrix of the double integrator leader-follower consensus system 113
$A_{\dot{\dot{x}}}$ system matrix of the double integrator leader-follower consensus system 113
$A_f$ system matrix of the single integrator leader-follower consensus system 107

$B_{\dot{x}}$ control matrix of the double integrator leader-follower consensus system 113
$B_{\dot{x}, \text{int}}$ control matrix of the double integrator leader-follower consensus system 114
$B_{\dot{x}}$ control matrix of the double integrator leader-follower consensus system 113
$B_f$ control matrix of the single integrator leader-follower consensus system 107

$D_G$ degree matrix of a graph 13
d($v_i$) degree of a node $v_i$ in an undirected graph 12
d_{in}($v_i$) in-degree of node $v_i$ 12
d_{out}($v_i$) out-degree of node $v_i$ 12
List of Symbols

E  edge set of a graph 11

\( G^f_x = (V^f, E^f_x, w^f_x) \)  double integrator velocity follower graph 114
\( G^p_x = (V^p, E^p_x, w^p_x) \)  double integrator position follower graph 114
\( G^vf_x = (V^f \cup V^l, E^vf_x, w^vf_x) \)  double integrator velocity leader-follower graph 114
\( G^lf^f_{2\text{int}} = (V^f_{2\text{int}}, E^f_{2\text{int}}, w^f_{2\text{int}}) \)  double integrator leader-follower graph 114
\( G^lf^l_x = (V^f \cup V^l, E^lf^l_x, w^lf^l_x) \)  double integrator position leader-follower graph 114
\( G^l = (V^l, \emptyset) \)  leader graph 108
\( G = (V, E, w) \)  graph G with the node set V, the edge set E and the weight function w 11
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INTRODUCTION

The musicians of the Perviy Simfonicheskiy Ansambl’ (First Symphony Ensemble, Persimfans), which existed in Moscow between 1922 and 1932, performed extremely complex musical compositions without a conductor. The string section formed a full circle (partly with their backs to the audience), while the wind section was situated inside of that circle. Every musician not only heard, but was also able to see the others. This way the magic chemistry among the performers, harmony and dynamic coordination between all the participants brought such a synchronisation, that it became the substitute of the role of the conductor. Amazingly, one of the Persimfans’ distinguishing features was their ability, according to the testimony of the extremely demanding and tough critics, to maintain a particularly subtle and profoundly individual approach to the interpretation of musical pieces, normally unthinkable without the help of the conductor.

— P. Chebotarev and R. Agaev [24]

The word “consensus” stems from Latin and its literal translation means “feel together”, though it is usually used in the sense “agreement”. Consensus problems are problems that require different entities to agree on a variable of interest. Just as the musicians of the Persimfans use visual communication to agree on their performance, a flock of birds agree on a maneuver in the air, or a group of robots agree on a trajectory chosen to fulfill a task.

Consensus problems are usually formulated for a communicating network of agents, like sensors, robots or power generators. The systems are said to have achieved consensus when some of their state variables have synchronised, i.e. become the same. It is desirable that the networks do so in a decentralised manner, only using information provided by their neighbours. Some examples of multi-agent systems with a goal of synchrony include smart grids, wireless sensor networks, and of course various applications of unmanned aerial vehicles, such as surveillance, mapping and environmental monitoring. Another example of a system that can be modelled by consensus algorithms is a social network. In fact, Aschinger, [6] studied consensus algorithms in connection with social interaction as early as 1974,
decades before modern technological advances made wireless communication possible.

All of these systems share a common property, namely that they consist of individual dynamic units that are equipped with the means to transmit and receive information. Moreover, these units are interconnected in a network along which this information is transmitted. This network can be straightforwardly modelled as a graph. Thus a large number of different applications can be uniformly studied with a set of system-theoretic tools. Adapting a top-bottom approach to networked systems, one is able to make statements about the network, often without even knowing the precise dynamic equations of the individual agents. Reaching consensus is usually only a small part of a decentralised control problem. However, in most cases it is an important element of the control strategy and often it determines the stability and controllability of the system.

The study of networked systems has thus established itself as a highly active interdisciplinary field in control theory research. Some fundamental works on this subject are those by Ren and Beard, [84], Bullo et al., [18], Ren and Cao, [85] and Mesbahi and Egerstedt, [64]. In particular, the work of Ren and Beard on consensus algorithms with agents that have double integrator dynamics was the inspiration for this thesis. In studying networked systems, a wide range of tools from different areas of mathematics prove useful. In this thesis for example, graph theory, Godsil and Royle, [37] and matrix polynomial theory, Gohberg et al., [38] will be used heavily.

**DOUBLE INTEGRATOR CONSENSUS ALGORITHM**

This work is devoted to studying a consensus algorithm, namely the double integrator consensus algorithm in continuous time. It is formally introduced in Chapter 4, along with the concepts of consensus stability and asymptotic consensus stability that are used to describe its behaviour.

The double integrator consensus algorithm shares many properties with the single integrator consensus algorithm, defined for agents with single integrator dynamics. Therefore, in Chapter 4 we first introduce the single integrator consensus algorithm and accumulate a number of existing results on its asymptotic consensus stability.

Hereafter, the main object of this thesis, the double integrator consensus algorithm, is introduced. In this algorithm, the agents are assumed to be communicating some of their states (often referred to as position and velocity) among each other. The standard assumption made in the literature, e.g. Ren and Atkins [83], Yu et al., [110], Yu et al., [109], Zhu et al., [114], Wen et al., [99], Zhu, [113], Li et al., [58], Gao et al., [35], is that the communication networks are the same, i.e. homogeneous, for both position and velocity.
In many cases this assumption turns out to be too strong, for example when communication links are weighted differently, the agents are equipped with different sensors and do not have the ability to measure or estimate both position and velocity, or in the event of partial network failure. Thus, we investigate the algorithm under the assumption that the communication networks may be different and possibly even disconnected. This we define as heterogeneous communication topologies.

The study of the double integrator consensus algorithm with heterogeneous communication topologies is an original problem setting that has not been studied prior to the present work. In fact, the term “heterogeneous communication topologies” has not appeared in literature prior to Goldin et al., [44].

**Consenus Stability of the Double Integrator Consensus Algorithm in Heterogeneous Networks**

After formally introducing the double integrator consensus algorithm in Chapter 4, we study its consensus stability in Chapter 5-7. The results presented here have been partially published in Goldin et al., [44], Goldin and Raisch, [42], and Schiffer et al., [90]. As heterogeneous networks had not previously been considered in the literature, new theory and models have to be developed. A related problem setting was considered by Ren, [82] utilising a Lyapunov approach. However, the results indicated shed no light on the relationship between behaviour of the considered system and the underlying graph structures.

The contribution of this work is an improved understanding of some intricate mechanisms behind the algorithm, the graph topologies and the system dynamics. It will turn out that some concepts from single integrator consensus translate intuitively into double integrator consensus algorithms. Most surprisingly however, the intuition fails for a large number of properties, particularly when the communication networks stop being bidirectional. Necessary and sufficient conditions for consensus stability and asymptotic consensus stability of the algorithm are obtained in the case that the communication networks are heterogeneous, weighted and undirected. For directed communication topologies, sufficient conditions for consensus stability and asymptotic consensus stability are obtained for a subset of admissible systems and an explanation is offered as to why some systems will never be stable.

Chapter 5 establishes necessary and sufficient conditions for double integrator consensus algorithms to achieve consensus. This is done by evaluating the eigenstructure of the quadratic matrix polynomial which corresponds to the system model. It is shown that the kernel of the polynomial is closely related to the connectivity of the corre-
sponding communication graphs. A necessary and sufficient graph-theoretic condition for the kernel of the matrix polynomial to have the desired form is derived. Building on these results, necessary and sufficient conditions for consensus stability and asymptotic consensus stability of the double integrator consensus algorithm are presented.

The results presented so far make no assumptions on the information structure of the communication graphs. This is different in Chapter 6, where only undirected graphs are considered. The specific property that the Laplacian of the undirected graph is symmetric allows a straightforward eigenvalue analysis of the system matrix of the double integrator consensus algorithm with undirected communication topologies. Complex and purely imaginary eigenvalues of the matrix polynomial are considered and related to the graph structure of the communication graphs and controllability of an induced subsystem. With these results, the necessary and sufficient conditions obtained in Chapter 5 are simplified for the special case that the communication graphs are undirected. Thereafter, the final convergence value of the algorithm and the corresponding convergence rate are explicitly derived.

Chapter 7 considers the more general case where the communication topologies are directed. In this chapter, the matrices comprising the algorithm are no longer symmetric and the analysis grows more complicated. We give a number of results that can be derived for general weighted directed graphs and indicate how the structure of the communication topologies is related to the spectrum of the corresponding system matrix. In particular, we outline a connection between cycles in graphs and properties of the double integrator consensus system. We then focus on the special case of acyclic graphs in order to make more specific statements.

**Controllability of the Double Integrator Leader-Follower Consensus Algorithm**

The ability of a system to synchronise is only one of the many properties of networked systems that are interesting from a control-theoretic standpoint. In the case of consensus systems, synchronisation can be used as a base layer to provide system cohesion, while a suitably chosen control input drives the system to a desired common state.

For example, the purpose of a swarm system will often be to achieve and maintain a specific formation that is described by the relative positions between the agents. In this case, we would like to have a large number of the agents implement the relatively simple consensus protocol and to have a leader provide an external control input to the system. This was defined by Tanner, [93] as the leader-follower consensus algorithm. The question considered here is how the leader
may be chosen if the networked system is to become controllable. For agents with single integrator dynamics, the single integrator leader-follower consensus algorithm was studied by Lou and Hong, [59], Egerstedt et al., [31], and references therein. For double integrator dynamics, the corresponding leader-follower consensus algorithm was first defined Jiang et al., [53] and further studied in Partovi et al., [76].

The contribution of the present work is the extension of the results formulated for a single integrator leader-follower consensus system in Egerstedt et al., [31] and Lou and Hong, [59] to the double integrator system. It turns out that for both single and double integrator consensus algorithms, controllability is largely dependent on symmetries in the communication topologies. The results given in this part have been partially published in Goldin and Raisch, [41], Goldin and Raisch, [43] and Goldin, [39].

Controllability of multi-agent networks is defined in Chapter 8. Here we show how the agent network that follows a consensus algorithm can be transformed into a leader-follower network, where the leader agents serve as the control input to the followers. As before, this is done for the single integrator consensus algorithm first and the results obtained by other authors are listed as a reference. The double integrator leader-follower consensus system is then derived and available single integrator results are extended to hold for it. We further consider the special case that the communication topologies are homogeneous or that only one of the states is available for control.

ADDITIONAL CONTENTS

An overview of the notation used globally in the present work is given in Chapter 2.

The study of networked systems and consensus algorithms is deeply intertwined with algebraic graph theory. Graphs are used to model the communication networks and to design the control algorithms. Therefore, in order to make this work self-contained, it is convenient to present a brief introduction to graph theory in Chapter 3 before formulating the consensus algorithm to be studied. In particular the Laplacian matrix is introduced here which plays a major role in the modelling of consensus algorithms. Some special graph structures, namely the path and circle graph, are further discussed. For better understanding we collect various connectivity notions for digraphs and relate the kernels of Laplacian matrices to the connectivity of the corresponding graphs. The original contribution here is the accumulation and classification of the different existing definitions of connectivity of directed graphs. Graph iso- and automorphisms and some partitions of graphs are also presented.
All the other external results that are used in this thesis are collected in Appendix A. Linear-algebraic and calculus basics are listed in Section A.1 and control theory basics in Section A.2.

Chapter 9 is the last chapter of this thesis and contains a conclusion, as well as suggestions for further work.

For better readability, each chapter opens with a detailed overview of its contents.
A clear and consequent notation is indispensable for any kind of scientific work. In this thesis we use the established notation from the fields of consensus theory, graph theory and control theory as much as possible. In this chapter we define basic notation that will be consistently used throughout the thesis. Additional notation will be introduced throughout the chapters whenever it is required for the analysis they contain.

The symbols $\mathbb{N}$, $\mathbb{R}$, $i\mathbb{R}$ and $\mathbb{C}$ denote the natural, real, purely imaginary and complex numbers. The notation $\mathbb{R}^{k \times m}$, $\mathbb{C}^{k \times m}$, $\{0,1\}^{k \times m}$ is used to denote the field of matrices with $k$ rows, $m$ columns and with entries that are real, complex, or from the set $\{0,1\}$. $\mathbb{R}^+$ denotes the set of nonnegative real numbers. For two numbers $a, b \in \mathbb{R}$, unless mentioned otherwise, $(a, b)$ and $[a, b]$ denote the open respectivey closed interval with the endpoints $a, b$ and $\{a, b\}$ the set containing $a, b$.

As is the habit in control theory, we use a capital latin letter to denote a matrix and lowercase latin letters to denote vectors and scalars. Greek letters are also sometimes used to denote scalars. Additionally, $I_k$ denotes the $k \times k$ identity matrix, and $1_{k \times m}$ and $0_{k \times m}$ the one and zero matrix of size $k \times m$, respectively. We do not generally underline vectors, however two exceptions are made: $1_n$ is the $n \times 1$ vectors of ones, while $0_n$ is the $n \times 1$ vector of zeroes. We generally assume all vectors to be column vectors. Consequently, if $x \in \mathbb{C}^n$ is an $n \times 1$ vector, then $x^T$ is its $1 \times n$ transpose, while $x^*$ denotes its conjugate transpose.

We indicate dimensions of matrices and vectors as often as necessary, but drop them when they are clear from the context. The $(i,j)$-th element of a matrix $A$ is denoted $(A)_{i,j}$, the $i$-th row is $(A)_{i,:}$ and the $i$-th column $(A)_{:,i}$. The $i$-th element of a vector $v$ is denoted $(v)_i$. The notation $A^T$ and $A^{-1}$ denotes the transpose and, if it exists, the inverse of $A$, $A^{-T}$ denotes the transpose of the inverse of $A$. Matrix powers are not employed in this thesis. Therefore with the previously mentioned exceptions, the notation $A^K$, $A^k$, etc. always denotes the matrix $A$ with the corresponding superscript $K$, $k$. Given an $n \times m$
matrix $A$ and an $n \times k$ matrix $B$, the notation $(A \mid B)$ denotes the $n \times (m + k)$ composite matrix.

For the numbers $k_1, \ldots, k_n \in \mathbb{C}$, with $n \in \mathbb{N}$, the notation $\text{diag}(k_1, \ldots, k_n)$ denotes a $n \times n$ matrix with $k_1, \ldots, k_n$ on the diagonal and zero entries otherwise. For the matrices $K_1 \in \mathbb{C}^{p_1 \times q_1}, \ldots, K_n \in \mathbb{C}^{p_n \times q_n}$, with $q_1, p_1 \in \mathbb{N}$, the notation $\text{blockdiag}(K_1, \ldots, K_n)$ denotes the block diagonal matrix with the blocks $K_1, \ldots, K_n$ on the diagonal and zeros otherwise.

Throughout this thesis we generally encounter matrices with entries in $\mathbb{R}$. However sometimes we will consider them over the field of complex numbers $\mathbb{C}$. Since the conjugate transpose of a real matrix over $\mathbb{C}$ is equivalent to its transpose, we denote both the transpose and the conjugate transpose of a real matrix $A$ as $A^T$. When dealing with complex numbers, $i$ denotes the imaginary unit, while $\text{Re}(a)$ and $\text{Im}(a)$ denote the real and imaginary parts of a complex number $a$, respectively. The absolute value of a number $a$ is denoted $|a|$, the euclidian vector norm of a vector $v$ is denoted $\|v\|$. The linear span of a set of vectors $(v_1, \ldots, v_k)$, $k \in \mathbb{N}$, is denoted $\text{span}(v_1, \ldots, v_k)$.

Given a set $N$, its cardinality is denoted by $|N|$. The union of two sets $N_1, N_2$ is denoted $N_1 \cup N_2$ and their intersection $N_1 \cap N_2$. The notation $N_1 \subseteq N_2$ denotes that $N_1$ is a subset of or equivalent to $N_2$, while the notation $N_1 \subset N_2$ means that $N_1$ is a proper subset of $N_2$.

Given a complex matrix $A$ with the eigenspace of size $n$, its spectrum is the ordered list of its eigenvalues, given by $\text{spec}(A) = \{\lambda_1, \ldots, \lambda_n\}$. The ordering is chosen according to $|\text{Re}(\lambda_1)| \leq |\text{Re}(\lambda_2)| \leq \ldots \leq |\text{Re}(\lambda_n)|$. The algebraic multiplicity, denoted $\text{alg}(\lambda)$ is the multiplicity of $\lambda$ as the root of the characteristic polynomial of $A$. The geometric multiplicity, denoted $\text{geo}(\lambda)$ is the dimension of the corresponding eigenspace. The eigenvalue is called simple eigenvalue if $\text{alg}(\lambda) = \text{geo}(\lambda) = 1$, it is called semi-simple eigenvalue if $\text{alg}(\lambda) = \text{geo}(\lambda) > 1$. An eigenvalue that is neither simple nor semi-simple is called deficient eigenvalue. The kernel of a matrix is denoted by $\ker(A)$. For further details on linear algebra please see the Section A.1.

A matrix or a vector is called positive (non-negative, non-positive, negative) if all of its entries are positive (non-negative, non-positive, negative). A vector $v \in \mathbb{C}^n$ is called a unit vector, if it satisfies $v^*v = 1$.

The variable $t$ is reserved to denote time in dynamical systems. Unless stated otherwise, $t \geq 0$ is assumed.

For two functions $f$, $g$, the notation $f(t) \to g(t)$ denotes that $f$ behaves like $g$ for large values of $t$. 

Graphs and algebraic graph theory have proven to be powerful tools when working with agent networks. We will show in later chapters how the network of agents can be straightforwardly modelled as a graph. Graphs and graph properties are indispensable for this thesis, as the consensus system \( (S_2) \) on page 35, introduced in Section 4.2, and its consensus stability will turn out to be directly connected to the corresponding graph structure. For this reason, we give an in depth introduction to the standard graph theoretic notions in this chapter, instead of including it in the appendix. The materials in this chapter will be the foundation for most of the subsequent chapters and the notation introduced here is established globally for this thesis.

In the following we briefly recap the standard graph theoretic notions, focusing on algebraic properties of directed and undirected weighted graphs in Section 3.1 and on matrix representations of graphs, especially the Laplacian matrix, in Section 3.2. In Section 3.3 we consider some particular graphs, namely path, circle and fully connected graphs. Then in Section 3.4 the connectivity concepts adopted in this work are listed. Based on these properties the kernels of Laplacian matrices are derived in Section 3.5. In Section 3.5.1 the kernel of the weighted undirected graph Laplacian is given, while in Section 3.5.2 the left and right kernels of the weighted digraph Laplacian are presented. Section 3.6 introduces graph iso- and automorphisms. Subsequently, graph partitions, in particular equitable partitions are defined in Section 3.7.

Some standard works on graph theory are, e.g., Godsil and Royle, [37], or Diestel, [29]. Unless noted otherwise, the presented results and the terminology are taken from the cited monographs.

3.1 Basic Definitions

A weighted directed graph (digraph) of order \( n \) is a triple \( G = (V, E, w) \), where
- \( V = \{v_1, v_2, \ldots, v_n\} \) is the set of nodes,
- \( E \subseteq V \times V \) is the set of edges, i.e. ordered pairs of nodes and
\( \cdot \) is an associated weight function, to be defined later.

For \( v_i, v_j \in V \) and \( i \neq j \), the symbol \( v_i \rightarrow v_j \) denotes an edge from \( v_i \) to \( v_j \). The node \( v_i \) is then referred to as the initial node and is said to have an outgoing edge, while \( v_j \) is the final node and has an incoming edge. The weight function \( \omega : V \times V \rightarrow \mathbb{R}^+ \) has the following property. For all \( v_i, v_j \in V \) it holds that \( \omega(v_j, v_i) = 0 \) if and only if \( v_i \rightarrow v_j \notin E \) and \( \omega(v_i, v_j) > 0 \) otherwise. For better readability we write \( \omega_{ij} \) instead of \( \omega(v_j, v_i) \) from here on.

A weighted undirected graph is given by \( G = (V, E, \omega) \) with the node set \( V \) and the edge set \( E \). For \( v_i, v_j \in V \), \( i \neq j \), \( v_i \leftrightarrow v_j \) denotes an undirected edge. The nodes \( v_i \) and \( v_j \) are then said to be neighbours. For an undirected edge \( v_i \leftrightarrow v_j \) the weight function has the additional property that \( \omega_{ij} = \omega_{ji} \) holds.

A directed version of the undirected graph is a graph \( G' = (V, E', \omega) \) with the property that if \( v_i \leftrightarrow v_j \in E \) then \( v_i \rightarrow v_j \in E' \) and \( v_j \rightarrow v_i \in E' \). Every undirected graph can be represented by a digraph, therefore, undirected graphs can be considered a special case of digraphs.

As we will show in the next sections, matrices associated with undirected graphs often have nice properties that in many cases allow an easier treatment. Therefore we will sometimes state results for only undirected graphs, or prove results for digraphs and undirected graphs separately. Chapter 6 even exclusively treats systems modelled by undirected graphs. In order to avoid confusion, we will hereafter tag all the presented results and definitions with either \( \omega \rightarrow \) or \( \omega \leftarrow \), that indicate whether the result is applicable to only weighted undirected graphs or weighted undirected and directed graphs.

Unless stated otherwise, we consider simple graphs, i.e. graphs that do not contain an edge \( v \rightarrow v \), for \( v \in V \) (a so-called self-loop) and that do not contain multiple edges between the same pair of nodes.

The neighbourhood of a node \( v_i \in V \) is given by the set \( N(v_i) \) and contains all the nodes that are its neighbours. The degree of a node \( v_i \) in an undirected graph is given by \( d(v_i) := \sum_{j=1}^{n} \omega_{ij} \). In a digraph, we differentiate between the in-neighbourhood of \( v_i \), given by \( N_{\text{in}}(v_i) := \{ k : k \in \{1, \ldots, n\} \setminus \{i\}, \ v_k \rightarrow v_i \in E \} \) and the out-neighbourhood, given by \( N_{\text{out}}(v_i) := \{ k : k \in \{1, \ldots, n\} \setminus \{i\}, \ v_i \rightarrow v_k \in E \} \). The in-degree of \( v_i \) is then \( d_{\text{in}}(v_i) := \sum_{j=1}^{n} \omega_{ij} \) and the out-degree \( d_{\text{out}}(v_i) := \sum_{j=1}^{n} \omega_{ji} \). A node with no incoming edges is called a root node, while a node with no outgoing edges is called a leaf node.

A digraph \( G' = (V', E', \omega') \) is a subgraph of \( G = (V, E, \omega) \), if \( V' \subseteq V \), \( E' \subseteq E \) and for any edge \( v_i \rightarrow v_j \in E' \), \( \omega'(v_j, v_i) = \omega(v_j, v_i) \). It is called a spanning subgraph if it is a subgraph and \( V' = V \). A subgraph of \( G \) induced by \( V' \subset V \) is the digraph \( (V', E', \omega') \), where \( E' \) contains all the edges in \( E \) between two nodes in \( V' \).
The *union* of \( k \) graphs \( G_i = (V_i, E_i, w_i) \), where \( i = 1, \ldots, k \), is defined as \( G := (V, U_i E_i, w) \). Here \( w_u : V \times V \rightarrow \mathbb{R}^+ \) takes the values
\[
  w_u(v_i, v_j) = \sum_{i=1}^k w_i(v_i, v_j).
\]

A weighted balanced graph, sometimes simply called weight-balanced, is a digraph with the additional property that each node has the same in- and out-degree (though the degrees may vary between the nodes).

If the weight function of some weighted digraph \( G = (V, E, w) \) is given by \( w : V \times V \rightarrow \{0, c\} \) for some \( c \in \mathbb{R}^+ \setminus \{0\} \), the graph is called uniformly weighted. A special case of the uniformly weighted digraph is the unweighted digraph. This graph has the property that \( w : V \times V \rightarrow \{0, 1\} \), i.e. that all edges of the graph have weight one. In some cases, uniformly weighted graphs have properties and symmetries that do not translate to the set of all weighted graphs with the same edge set. Therefore, we will sometimes state results only for unweighted graphs, for example in Section 6.1. In order to avoid cluttered notation, in writing unweighted graphs the explicit mention of the one weight function is omitted, i.e. we write \( G = (V, E) \) instead of writing \( G = (V, E, w) \), \( w : V \times V \rightarrow \{0, 1\} \). In order to avoid confusion, results and definitions given for unweighted digraphs and graphs are tagged with \( \rightarrow \) respectively \( \leftarrow \). However, unless explicitly stated otherwise, all results presented in this thesis are given for weighted digraphs.

### 3.2 Matrix Representations of a Graph

Associated with weighted digraphs are several matrix representations, of which we at this point introduce the adjacency matrix \( A_G \), the degree matrix \( D_G \) and the Laplacian matrix \( L \). All these matrices depend on the corresponding graph, however, for better readability we from here on drop the reference to the graph whenever it does not create confusion and simply write \( A, D \) and \( L \).

An algebraic representation of a weighted digraph \( G = (V, E, w) \) is given by the \( n \times n \) adjacency matrix \( A \), with the entries
\[
  (A)_{ij} = w(v_i, v_j), \quad i, j = 1, \ldots, n.
\]

As we have defined all non-zero weights in the graph to be positive, the adjacency matrix is a non-negative matrix. It is symmetric if the graph is undirected.

The degree matrix of \( G \), \( D \), is a diagonal matrix given by
\[
  (D)_{ii} = \sum_{j=1}^n (A)_{ij}, \quad i = 1, \ldots, n.
\]
For digraphs, the degree matrix contains the in-degrees of the nodes. The graph Laplacian matrix \( L \) is given by

\[
(L)_{ij} = \begin{cases} 
-(A_G)_{ij} & i \neq j, \\
\sum_{j=1}^{n}(A_G)_{ij} & i = j, \quad i, j = 1, \ldots, n.
\end{cases}
\] (3.3)

For undirected graphs the Laplacian will be a symmetric positive semi-definite matrix.

### 3.3 Special Graphs

We now list some particular graph terms and structures that will be considered in the following chapters. As these structures depend only on the graph edges but not on weights, they are without loss of generality listed for unweighted graphs.

A **path** in a graph on \( n \) nodes is an ordered sequence of nodes such that any pair of consecutive nodes in the sequence are connected by an edge. A path in a digraph is an ordered sequence of nodes such that, apart from the first and the last node, every node is the initial node of an edge and the next node in the sequence is the final node of the edge. The first and last node are only initial respectively only final nodes of an edge.

The length of a path is the number of edges it contains. A **shortest path** from \( v_i \) to \( v_j \), \( i, j = 1, \ldots, n \), if it exists, is the path that contains the minimal number of edges. A **path graph** is a graph that is itself a path. With appropriate node indexing the Laplacian of an unweighted undirected path graph on \( n \) nodes is given by the \( n \times n \) matrix

\[
L = \begin{pmatrix}
1 & -1 & 0 & 0 & \ldots & 0 \\
-1 & 2 & -1 & 0 & \ldots & 0 \\
0 & -1 & 2 & -1 & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \\
0 & \ldots & 0 & -1 & 2 & -1 \\
0 & \ldots & \ldots & 0 & -1 & 1
\end{pmatrix},
\] (3.4)

while the Laplacian of an unweighted path digraph on \( n \) nodes is given by the \( n \times n \) matrix

\[
L = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
-1 & 1 & 0 & \ddots & \\
0 & -1 & 1 & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & -1 & 1
\end{pmatrix}.
\] (3.5)
A circle in a graph is a path with an additional edge that connects the last to the first node. A circle graph is a graph that is itself a circle. The Laplacian of an unweighted circle graph on $n$ nodes is given by the $n \times n$ matrix

$$L = \begin{pmatrix}
2 & -1 & 0 & \ldots & 0 & -1 \\
-1 & 2 & -1 & 0 & \ldots & 0 \\
0 & -1 & 2 & -1 & \ddots & \\
\vdots & \ddots & \ddots & \ddots \ddots & \ddots & 0 \\
0 & \ldots & 0 & -1 & 2 & -1 \\
-1 & 0 & \ldots & 0 & -1 & 2
\end{pmatrix}, \quad (3.6)$$

while the Laplacian of an unweighted circle digraph on $n$ nodes is given by the $n \times n$ matrix

$$L = \begin{pmatrix}
1 & 0 & \ldots & 0 & -1 \\
-1 & 1 & 0 & \ldots & 0 \\
0 & -1 & 1 & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & -1 & 1
\end{pmatrix}. \quad (3.7)$$

A graph is acyclic if it contains no circles. A tree in a graph is an acyclic subgraph with the property that there is a path between every two nodes and that adding any edge to it would create a cycle. In a digraph a tree is an acyclic subgraph with the property of there being one node that has no incoming edges, called the root node, every other node has exactly one incoming edge, and there is a path from the root to any other node in the graph. A tree is called spanning if it touches every node. A tree graph is a (di-)graph that is itself a tree. With appropriate node indexing the Laplacian of a tree digraph is a lower or upper triangular matrix.

A fully connected graph on $n$ nodes is a graph with an edge from every node to every node. The Laplacian of an unweighted fully connected graph is given by the $n \times n$ matrix

$$L = \begin{pmatrix}
n-1 & -1 & \ldots & -1 \\
-1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & -1 \\
-1 & \ldots & -1 & n-1
\end{pmatrix}, \quad (3.8)$$

3.4 Graph Connectivity

Graph connectivity is a term that describes the information flow in the graph. Graph connectivity can often be related to whether or not
the networked system modelled by the graph can achieve consensus. Different connectivity notions exist for directed and undirected graphs. These properties can be directly related to the eigenstructure of the Laplacian matrices. All definitions in this section are given for weighted graphs. However, they do not depend on the particular choice of the weight function. That is, if a graph $G = (V, E, w)$ has certain connectivity properties, then so do all graphs with the same node set $V$ and edge set $E$.

A weighted undirected graph $G = (V, E, w)$ can be connected or disconnected. It is connected if there exists a path between any two nodes and is otherwise disconnected. If the graph is disconnected, then it has several connected components. The extreme case is the graph with no edges, $E = \emptyset$, that has $n$ connected components. The converse case, $E = \{V \times V \setminus \bigcup_{i=1}^{n} (v_i \leftrightarrow v_i)\}$, is the fully connected graph introduced in Section 3.3.

The connectivity concept is more sophisticated when it comes to digraphs. Several, sometimes conflicting, definitions exist. In this thesis, we will use the concept of a reach, taken from the work of Caughman and Veerman, [22]. Some other common and similar (though in general not equivalent) concepts are: Strong component, in- and out-forest [2], tree, rooted out-branching [64], etc. Connectivity of digraphs can be defined using any of these terms. Here, we choose to define reaches, because this construction is very intuitive and corresponds well with the concept of a connected component in an undirected graph. Alongside it we will also introduce the notion of an independent strongly connected component defined in Wieland, [100].

For any node $v_i \in V$, $i = 1, \ldots, n$, we define $R(v_i)$ to be the set containing $v_i$ and all nodes $v_j$, $j \in \{1, \ldots, n\}$, such that there is a directed path from $v_i$ to $v_j$. We call $R(v_i)$ the reachable set of $v_i$, i.e. the set of all nodes that can be reached from $v_i$ by a directed path. By convention, $v_i \in R(v_i)$.

A set of nodes $R \subseteq V$ in a digraph is called a reach if it is a maximal reachable set. That is, $R$ is a reach if $R = R(v_i)$ for some $i \in \{1, \ldots, n\}$ and there is no $j \in \{1, \ldots, n\}$ such that $R(v_i) \subset R(v_j)$. We only consider graphs on a finite number of nodes, therefore maximal reachable sets exist and are uniquely determined by the graph. Suppose that a digraph consists of $k \leq n$ reaches. For each reach $R_i$, $i = 1 \ldots k$, its exclusive part is given by $R h_i := R_i \setminus \cup_{j=1, j \neq i}^{k} R_j$. Its common part is given by the set $R c_i := R_i \setminus R h_i$. The common part of the digraph is then defined as $R c := \cup_{i=1}^{k} R c_i$.

A digraph is weakly connected if it is connected ignoring the orientation of the edges. Thus, a directed graph is weakly connected if and only if, for any two reaches $R_i, R_j$, where $i, j = 1, \ldots, k$, there is a node $v$ such that $v \in R_i$ and $v \in R_j$. A digraph that is not weakly connected is disconnected. Thus a disconnected digraph consists of several components that are at least weakly connected.
A digraph is called strongly connected if $\mathcal{R}(v_i) = \mathcal{R}(v_j)$, for all $v_i, v_j \in V$, i.e. if the reachable set of each node contains all the other nodes. A strongly connected component of a digraph is a subgraph $(V', G', w')$ that is a strongly connected graph.

An independent strongly connected component (iSCC), introduced by Wieland, \[100\], is a subgraph $G' = (V', E', w')$ of $G = (V, E, w)$ with the property that $G'$ is strongly connected and there are no incoming edges from the nodes in $V \setminus V'$ to the nodes in $V'$.

Clearly, given a node $v \in V$ such that $v \in V'$, it follows that the nodes in $V'$ belong to the reachable set of $v$. Furthermore, the reach that contains $v$ by definition also contains $V'$. We denote the reach that contains $v$ by $\mathcal{R}_v$. As there are no incoming edges to $V'$ from outside $V'$, it holds that $V'$ belongs to the exclusive part of $\mathcal{R}_v$. As the exclusive parts of reaches are mutually node disjoint, it follows that for any graph, its number of reaches and its number of iSCC coincide. Therefore, when speaking of digraph connectivity, we may equivalently say that the graph consists of $k$ reaches or $k$ iSCC. It will turn out that reaches determine the right nullspace of the Laplacian matrix, while iSCC determine its left nullspace. A schematic overview of the realationship between reaches and independent strongly connected components is given in Figure 3.1 for a graph consisting of two reaches. Furthermore, the following example illustrates the concepts.

**Example 3.1** The digraph illustrated on the left has two reaches $\{v_3, v_4\}$ and $\{v_1, v_2, v_4\}$, with the common part $\{v_4\}$ and the exclusive parts $\{v_3\}$ and $\{v_1, v_2\}$. It is weakly connected but does not have a spanning tree. There are two iSCC, $(\{v_1\}, \emptyset)$ and $(\{v_3\}, \emptyset)$.

**Remark 3.2** In consensus-related work, the connectivity of a digraph is often described by whether or not the graph contains a spanning tree. This is equivalent to saying that the digraph consists of one reach, for there is a spanning tree rooted at $v_i \in V$ if and only if $\mathcal{R}(v_i) = V$.

## 3.5 Properties of the Laplacian Matrix

As we will show in Chapter 5, the Laplacian matrix can be directly linked to the properties of consensus algorithms. Therefore it plays a pivotal role in this thesis. Early studies of properties of Laplacian matrices are due to Kelmans, e.g. [55]. In the 1990s the results were reviewed by Merris, [63] and published in the books Godsil and Royle,
Figure 3.1: Sketch of the relationship between reaches and independent strongly connected components.

[37] and Chung, [25]. However, most of these publications, as well as recent works, focus on unweighted undirected graphs and the Laplacian matrix was initially defined only for undirected graphs. Only a small number of results are available for both unweighted and weighted digraphs. This, as we will see, is mostly due to the fact that the Laplacian of a digraph in general has a complex spectrum, while the Laplacian of an undirected graph has a real spectrum. Thus determining properties of digraph Laplacians is much more difficult than the corresponding problem for an undirected graph. For digraphs the definition of the Laplacian is not unique, cf. Chung, [25], or Chebotarev and Agaev, [23] for a discussion. Our formulation is often referred to as a row Laplacian, which stems from the fact that the row-sums of $L$ are zero. We use this formulation because, as we will see in Chapter 4, it is best suited to describe some consensus algorithms.

Unless stated otherwise, all results in this section hold for weighted graphs.

3.5.1 Undirected Graphs

The Laplacian matrix $L$ of a weighted undirected graph $G = (V, E, w)$, or weighted undirected graph Laplacian for short, is a symmetric quadratic matrix. Thus $L$ has a set of $n$ linearly independent eigenvectors (due to its symmetry, the left and the right eigenvectors coincide) and all its eigenvalues are semi-simple. By construction, $L$ has non-negative diagonal and non-positive offdiagonal entries. As all the rows of $L$ sum up to zero, $L$ has at least one zero eigenvalue. By the Gershgorin disk theorem, $L$ is positive semi-definite and all of its eigenvalues are non-negative.

---

1 Properties of symmetric matrices are listed in Section A.1.3 on page 131 of the appendix.

2 Cf. Section A.1.4 on page 132 of the appendix.
non-zero eigenvalues are positive and real. The numerical range\(^3\) of \(L\) is limited by
\[
0 \leq \frac{v^*Lv}{v^*v} \leq \lambda_n, \tag{3.9}
\]
where \(v \in \mathbb{C}^n \setminus \{0_n\}\) and \(\lambda_n\) is the largest eigenvalue of \(L\). As \(L\) is symmetric, its numerical range is precisely the section of the real axis given by \([0, \lambda_n]\).

The smallest non-zero eigenvalue of the unweighted undirected graph Laplacian is called algebraic connectivity and is a measure of how well a graph is connected. It was introduced by Fiedler, \([34]\) and further studied by Mohar, \([66]\). Several bounds exist for the eigenvalues of the Laplacian matrix. For unweighted undirected graphs, an overview is given in \([66]\). We will need the following result for unweighted graphs:

**Lemma 3.3 (Mohar, \([66]\))** Let an unweighted undirected graph \(G\) have \(k\) connected components of size \(k_1, \ldots, k_k\). Let \(G_{k_i}\) be the graph corresponding to the connected component of size \(k_i\). Then, the largest eigenvalue \(\lambda_n\) of the corresponding Laplacian is bounded by
\[
\min_{i \in \{1, \ldots, k\}} \frac{k_i}{k_i - 1} \max_{j \in \{1, \ldots, n\}} d(v_j \in G_{k_i}) \leq \lambda_n \leq \max_{i \in \{1, \ldots, k\}} k_i. \tag{3.10}
\]
Here \(d\) denotes the degree of a node. Clearly the degree of a node in an unweighted graph is bounded by the size of the connected component, i.e.
\[
\max_{j \in \{1, \ldots, n\}} d(v_j \in G_{k_i}) \leq k_i - 1. \tag{3.11}
\]

The famous Matrix-Tree-Theorem by Kelmans, \([55]\) relates the number of connected components in a graph to the spectrum of its Laplacian:

**Theorem 3.4 (Matrix-Tree-Theorem, Kelmans, \([55]\))** Let \(G\) be a weighted undirected graph on \(n\) nodes. \(G\) has \(k\) connected components if and only if \(L(G)\) has rank \(n - k\).

Note that if \(G\) is connected, i.e. \(k = 1\), then \(L(G)\) has rank \(n - 1\). Therefore, as already mentioned above, \(L\) always has at least one zero eigenvalue. By construction \(L\) has zero column- and row-sums, i.e. \(L1_n = 0_n\) and the vector \(1_n\) always lies in the kernel of \(L\).

We give a sketch of the proof of the Matrix-Tree-Theorem, which will lead us to a formulation of the right nullspace of \(L\). If some graph \(\tilde{G}\) consists of \(k\) connected components of size \(k_1, \ldots, k_k\), then there is an isomorphic graph \(G\) and a permutation matrix \(P\) such

\(^3\) Cf. Section A.1.3 on page 131 of the appendix.
that \( L(G) = P \tilde{L}(G) P^T \) is block diagonal. Each block of the permuted matrix has dimension \( k_i \times k_i \), \( i = 1, \ldots, k \) and is, per definition, a full Laplacian matrix. Denote the block matrices as \( L_1, \ldots, L_k \). Then \( \text{spec}(L) = \bigcup_{i=1}^{k} \text{spec}(L_i) \). Furthermore, each \( L_i \) has a simple eigenvalue zero and an eigenvector \( 1_{k_i} \) affording it. Thus, \( L \) has \( k \) linearly independent eigenvectors given by the set

\[
\left\{ \begin{pmatrix} 1_{k_1} \\ 0_{n-k_1} \end{pmatrix}, \begin{pmatrix} 0_{k_1} \\ 1_{k_2} \end{pmatrix}, \ldots, \begin{pmatrix} 0_{n-k_k} \\ 1_{k_k} \end{pmatrix} \right\}. \tag{3.12}
\]

We will need a basis of the right nullspace of \( L \) quite frequently, either in the above version or in the following formulation:

\[
\left\{ 1_n, \begin{pmatrix} 1_{k_1} \\ 0_{n-k_1} \end{pmatrix}, \begin{pmatrix} 0_{k_1} \\ 1_{k_2} \end{pmatrix}, \ldots, \begin{pmatrix} 0_{n-k_k-k_{k-1}} \\ 1_{k_{k-1}} \end{pmatrix} \right\} \tag{3.13}
\]

which replaces the last eigenvector with \( 1_n \), the common eigenvector of all Laplacian matrices.

As \( L \) is symmetric, its left and right nullspaces coincide. Note that the nullspace of the undirected graph Laplacian does not depend on the particular choice of weight function. The fact that the left and right nullspaces of the Laplacian matrix of a weighted undirected graph can be given in such a simple analytical manner will allow us to clearly formulate some convergence results.

### 3.5.2 Directed Graphs

Unlike the undirected graph Laplacian, the Laplacian matrix of a weighted directed graph (from here on: the weighted digraph Laplacian) is not symmetric and in general does not have a real spectrum or zero column sums. It still has the property that its diagonal entries are non-negative and its off-diagonal entries are non-positive and the corresponding row sums are zero. Thus, the Gershgorin disk theorem implies that all of its eigenvalues have non-negative real parts. In general the left and right eigenvectors of a directed graph Laplacian do not coincide.

The numerical range of a digraph Laplacian is a convex section of the complex plane given by, cf. (A.9) on page 131 of the appendix,

\[
\lambda_{\min} \left( 0.5(L + L^T) \right) \leq \Re \left( \frac{v^* L v}{v^* v} \right) \leq \lambda_{\max} \left( 0.5(L + L^T) \right), \\
\lambda_{\min} \left( -i0.5(L - L^T) \right) \leq \Im \left( \frac{v^* L v}{v^* v} \right) \leq \lambda_{\max} \left( -i0.5(L - L^T) \right), \tag{3.14}
\]

where \( \lambda_{\max}, \lambda_{\min} \) denote the largest and smallest real eigenvalue of a matrix and \( v \in \mathbb{C}^n \setminus \{0_n\} \).
The following result corresponds to the Matrix-Tree-Theorem for digraphs and provides a basis for the eigenspace of the Laplacian. It was formulated by Caughman and Veerman, [22], though some older related results are available from Chebotarev and Agaev, particularly in [2], [23]:

**Lemma 3.5 (Right nullspace of digraph Laplacians, Caughman and Veerman, [22], Theorem 3.3)** Let $G$ denote a weakly connected weighted directed graph and let $L$ denote its Laplacian matrix. Suppose $G$ has $n$ nodes and $k$ reaches. Then the algebraic and geometric multiplicity of the eigenvalue zero equals $k$. Furthermore, the associated eigenspace has a basis \{v_1, v_2, \ldots, v_k\} in $\mathbb{R}^n$. For every $i = 1, \ldots, k$, the basis elements satisfy:

1. $(v_i)_v = 0$ for $v \in V \setminus R_i$,
2. $(v_i)_v = 1$ for $v \in R\ell_i$,
3. $(v_i)_v \in (0, 1)$ for $v \in Rc_i$,
4. $\sum_{i=1}^{k} v_i = 1_{n}$.

The above is the directed version of the basis (3.12). It can be easily obtained by noting that the following decomposition holds. Let $L$ be the Laplacian of a weakly connected digraph on $n$ nodes with $k$ reaches of exclusive size $k_1, \ldots, k_k$ and common size $c = n - \sum_{i=1}^{k} k_i$. Then, with an appropriate node numbering, $L$ has the form

$$L = \begin{pmatrix}
L_1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & \ddots & L_k & 0 \\
L_{k+1,1} & \ldots & L_{k+1,k} & L_{k+1,k+1}
\end{pmatrix}.$$  \hfill (3.15)

Here, $L_i \in \mathbb{R}^{k_i \times k_i}$, $L_{k+1,i} \in \mathbb{R}^{c \times k_i}$ and $L_{k+1,k+1} \in \mathbb{R}^{c \times c}$, $i = 1, \ldots, k$. Each of the matrices $L_i$, $i = 1, \ldots, k$ is itself a digraph Laplacian of the subgraph induced by the nodes that belong to the exclusive part of the $i$-th reach.

Remember that the common part of the graph is given by $\bigcup_{i=1}^{k} Rc_i$, where $Rc_i$ is the common part of the $i$-th reach. Assuming appropriate node numbering, we denote the part of every vector $v_i$, $i = 1, \ldots, k$, that corresponds to the nodes in $Rc_i$ by $r_i$. It suits our work better to choose the following set as a basis of the nullspace of $L$ (note that $L$ is weakly connected):

$$\begin{pmatrix}
1_{n} \\
1_{k_1} \\
\frac{1}{c} \begin{pmatrix}
1_{k_1} \\
0_{n-k_1-c} \\
r_1
\end{pmatrix} \\
\frac{1}{c} \begin{pmatrix}
0_{n-k_1-c} \\
1_{k_2} \\
r_2
\end{pmatrix} \\
\vdots \\
\frac{1}{c} \begin{pmatrix}
0_{n-k_k-c} \\
1_{k_k-1} \\
r_{k-1}
\end{pmatrix} \\
\frac{1}{c} \begin{pmatrix}
0_{n} \\
1_{k_k-1} \\
r_k
\end{pmatrix}
\end{pmatrix}.$$  \hfill (3.16)

4 Remember that the entries of the eigenvectors of $L$ are indexed with the nodes of $G$. 

\[\rightarrow\]
where it is assumed that \( L \) has a common part of size \( c \) and \( k \) reaches with exclusive parts of size \( k_1, \ldots, k_k \). The vectors \( r_i \) contain entries from the interval \((0, 1)\). Note that unlike in Lemma 3.5, the vectors no longer sum up to \( 1 \). This corresponds with the basis (3.13).

As Laplacians of digraphs are not symmetric, their left and right nullspaces are generally different. Ren and Beard, [84] show that the Laplacian of a digraph that consists of one reach has a left eigenvector \( w \) affording the zero eigenvalue with the properties: \( \sum_{i=1}^{n} (w)_i = 1 \), \( (w)_i \geq 0 \), for all \( i = 1, \ldots , n \). Wieland, [100] extended their result to graphs that consist of more than one reach.

**Lemma 3.6 (Left nullspace of a digraph Laplacian, Wieland, [100], Theorem 2.13)** Let \( G \) denote a weakly connected weighted digraph and let \( L \) denote its Laplacian matrix. Suppose \( G \) has \( n \) nodes and \( k \) reaches with the associated iSCC, denoted by \( \text{iSCC}_1, \ldots, \text{iSCC}_k \). Then the algebraic and geometric multiplicity of the eigenvalue zero equals \( k \) and the associated left eigenvectors can be given by \( w_1, w_2, ..., w_k \in \mathbb{R}^n \), whose elements satisfy:

(i) \( (w_i)_v = 0 \) for \( v \in V \setminus \text{iSCC}_i \),

(ii) \( (w_i)_v > 0 \), for \( v \in \text{iSCC}_i \),

(iii) \( \sum_{j=1}^{k} (w_i)_j = 1 \).

Note that Wieland formulated the above result for unweighted digraphs. However, its extension to weighted digraphs is trivial. In either case, the result follows from the following fact. Let \( \tilde{L} \) be the Laplacian of the digraph \( G \) on \( n \) nodes and suppose that \( G \) consists of one reach. Let the corresponding iSCC consist of \( n_1 < n \) nodes. Then, \( \tilde{L} \) is reducible. With appropriate node numbering, it has the form

\[
\tilde{L} = \begin{pmatrix}
L_{11} & 0 \\
L_{21} & L_{22}
\end{pmatrix},
\]

where \( L_{11} \in \mathbb{R}^{n_1 \times n_1} \) is itself the Laplacian of the subgraph induced by the nodes belonging to the iSCC. The matrix \( L_{21} \in \mathbb{R}^{(n-n_1) \times n_1} \) describes the edges from the nodes in the iSCC to the other nodes in the reach, and the matrix \( L_{22} \in \mathbb{R}^{(n-n_1) \times (n-n_1)} \) describes the edges between the nodes that are not part of the iSCC. Clearly, if a vector \( w_1 \in \mathbb{C}^{n_1} \) satisfies \( w_1^T L_{11} = \mathbb{0}_{n_1} \), then the vector \( w = (w_1^T, 0_{n-n_1}^T)^T \) satisfies \( w^T \tilde{L} = \mathbb{0}_n \).

Consider again the decomposition (3.15). It describes the Laplacian of a digraph on \( n \) nodes with \( k \) reaches, that have exclusive parts of size \( k_1, \ldots, k_k \) and a common part of size \( c = \sum_{i=1}^{k} k_i \). It follows from the above discussion, that every block \( \tilde{L}_i, i = 1, \ldots, k \) in (3.15) can be decomposed according to (3.17). Suppose that the corresponding iSCC are of size \( m_1, \ldots, m_k \), where \( m_i \leq k_i \) holds for all \( i = 1, \ldots, k \).
The resulting matrix, assuming appropriate node numbering, can be written as
\[
L = \begin{pmatrix}
L_{111} & 0 & \ldots & 0 & \ldots & 0 \\
L_{121} & L_{122} & \ldots & \ldots & \ldots & \\
0 & 0 & \ddots & \ddots & \ddots & \\
\vdots & \vdots & \ddots & L_{k_{11}} & 0 & 0 \\
0 & 0 & \ldots & L_{k_{21}} & L_{k_{22}} & 0 \\
L_{k_{1+1,1}} & L_{k_{1+1,2}} & \ldots & L_{k_{1+1,k_1}} & L_{k_{1+1,k_2}} & L_{k_{1+1,k_1+1}}
\end{pmatrix}. \tag{3.18}
\]

Here, for \( i = 1, \ldots, k \), the blocks \( L_{i,11} \in \mathbb{R}^{m_i \times m_i} \) describe the edges between the nodes in the iSCC of the \( i \)-th reach, the blocks \( (L_{i,21} \mid L_{i,22}) \in \mathbb{R}^{(k_i-m_i) \times k_i} \) describe the edges incoming to the nodes that belong to the exclusive part of the \( i \)-th reach, but not to the iSCC and the blocks \( (L_{k+1,i,1} \mid L_{k+1,i,2}) \in \mathbb{R}^{c \times k_i} \) describe the edges incoming from the exclusive part of the \( i \)-th reach to the common part of the \( i \)-th reach. Finally, \( L_{k+1,k+1} \in \mathbb{R}^{c \times c} \) describes the edges within the common part of the reaches. By construction, the blocks \( L_{i,22}, i = 1, \ldots, k \) and \( L_{k+1,k+1} \) have non-negative diagonal and non-positive off-diagonal elements. Furthermore, the corresponding row sums are either zero or positive, with at least one row sum being strictly positive by construction. Therefore, the blocks \( L_{i,22}, i = 1, \ldots, k \) and \( L_{k+1,k+1} \) are irreducibly diagonally dominant and, thus, of full rank.

Let \( q_1 \in \mathbb{C}^{m_1} \) be a vector satisfying \( q_1^T L_{i,11} = 0_{m_1}^T \) in (3.18). Then, clearly, it follows that a basis of the left nullspace of a digraph Laplacian is given by
\[
\left\{ \begin{pmatrix} q_1 \\ 0_{n-m_1} \\ \vdots \\ 0_{n-k_1-m_2} \\ \vdots \\ 0_{n-k_{k-1}-m_k} \\ q_k \end{pmatrix} \right\}. \tag{3.19}
\]

If \( G \) has several weakly connected components, then each of them has the left nullspace given by Lemma 3.6. The complete nullspace is achieved by stacking the the vectors accordingly.

For two weighted digraphs \( G_1 = (V,E,w_1) \), \( G_2 = (V,E,w_2) \) the left and right nullspaces of the corresponding Laplacians \( L_1, L_2 \) in general do not coincide. However, it follows from Lemmas 3.6 and 3.5 as well as the decompositions (3.18) and (3.15), respectively, that the eigenvectors in the left and right nullspace of \( L_1 \) and \( L_2 \) will have the same zero/non-zero pattern. The vectors in the right nullspace will further have one entries in the same locations and differ only in the entries corresponding to the common parts of the graphs.

The extension of the bases (3.16) and (3.19) to disconnected digraphs is straightforward. If the graph \( G \) consists of \( p \) weakly con-
connected components of size $p_1, \ldots, p_p$, then, with appropriate node numbering, its Laplacian is given by

$$L = \begin{pmatrix}
L_1 & 0 & \ldots & 0 \\
0 & L_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & L_p
\end{pmatrix} \quad (3.20)$$

where $L_i \in \mathbb{R}^{p_i \times p_i}$, $i = 1, \ldots, p$. Then the left and right nullspaces of each $L_i$ are given by the basis (3.16) respectively (3.19) and the complete nullspaces are obtained by stacking the vectors appropriately.

The following result is obvious.

**Lemma 3.7** The Laplacian matrix of a digraph is irreducible if and only if the digraph is strongly connected.

**Proof:** Remember that a digraph being strongly connected is equivalent to it consisting of one iSCC. Thus, the lemma follows directly from the decomposition (3.18).

We will further require the following technical result in order to prove Theorem 5.4 on page 53.

**Lemma 3.8** (Goldin et al., [44]) Let $L$ be the Laplacian of a weighted digraph. There is no vector $v \in \mathbb{C}^n$ satisfying

$$Lv = 1_n. \quad (3.21)$$

**Proof:** First remember that $1_n$ is an eigenvector of $L$ affording zero. Then, if $v$ exists, it is a generalised eigenvector of $L$ affording zero. By Lemma 3.5 $L$ has a full set of eigenvectors affording zero. Therefore, the zero eigenvalue of $L$ is semi-simple. Thus, the above equation has no solution.

**Example 3.9** The Laplacian of the graph studied in Example 3.1 is

$$\begin{pmatrix}
0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 3
\end{pmatrix}, \quad (3.22)$$

its left and right kernels can be given by

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2/3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1/3 \end{pmatrix}; \quad w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (3.23)$$
3.6 Isomorphisms and Automorphisms

Given two graphs $G_1 = (V_1, E_1, \omega_1)$, $G_2 = (V_2, E_2, \omega_2)$, a graph isomorphism is a bijective mapping $\tau_1 : V_1 \rightarrow V_2$ such that for all $v_i, v_j \in V_1$, for $i, j = 1, \ldots, n$, it holds that $v_i \rightarrow v_j \in E_1$ if and only if $\tau_1(v_i) \rightarrow \tau_1(v_j) \in E_2$ and $\omega_2(\tau_1(v_i), \tau_1(v_j)) = \omega_1(v_i, v_j)$.

Algebraically $\tau_1$ represents a simultaneous permutation of the rows and columns of $L(G_1)$. Let $P = P(\tau_1)$ be a permutation matrix, i.e. for all $i, j = 1, \ldots, n$, it holds that $(P)_{ij} = 1$ if $\tau_1(v_i) = v_j$ and $(P)_{ij} = 0$ otherwise. Two graphs $G_1, G_2$ are isomorphic if and only if there exists a $n \times n$ permutation matrix $P = P(\tau_1)$ such that $L(G_2) = PL(G_1)P^T$. Thus, isomorphic graphs have similar Laplacian matrices. Therefore, they have the same Laplacian spectrum and if $v$ is an eigenvector of $L_1$, then $Pv$ is an eigenvector of $L_2$. The converse, however, is not true. It is possible to find graphs that have the same Laplacian spectrum that are not isomorphic, cf. Biyikoglu et al., [13].

A bijection $\tau_A : V \rightarrow V$ is called an automorphism of $G = (V, E, \omega)$, if it preserves the edge set and weights of $G$. That is, if $\tau_A$ is an automorphism, then for two nodes $v_i, v_j \in V$, for $i = 1, \ldots, n$, it holds that $v_i \rightarrow v_i \in E(G)$ if and only if $\tau_A(v_i) \rightarrow \tau_A(v_i) \in E(\tau_A(V))$ and $\omega(v_i, v_j) = \omega(\tau_A(v_i), \tau_A(v_j))$. In other words, the automorphism is an isomorphism from $G$ to itself. The orbit of a node $v \in V(G)$ under the automorphism is said to have size $\phi$ if $P^\phi(v) = v$. The node $v$ is called a fixed point of the automorphism if $\phi = 1$.

Graph automorphisms will be used in Section 6.1 and Chapter 8. Cvetković et al., [26] show that a permutation matrix $P = P(\tau_A)$ is an automorphism if and only if $PA_G(G) = A_GP(G)$ holds. We extend this result to the Laplacian matrix.

**Lemma 3.10** Let $P$ be a permutation matrix and $G$ a graph on $n$ nodes with the Laplacian $L = D_G - A_G$. $P$ is an automorphism of $G$ if and only if $PL = LP$.

**Proof:** The permutation matrix $P$ is an automorphism if and only if $PA = AP$ holds. The matrix $D_G$ is diagonal, furthermore, if two nodes $v_i, v_j$, $i \in \{1, \ldots, n\}$ belong to the same orbit, they have the same degree and $(D_G)_{ii} = (D_G)_{jj}$ holds. Therefore, $PD_G = D_GP$. Thus we obtain


(3.24)

□

Biyikoglu et al., [13] study the behaviour of the eigenvectors of the undirected graph Laplacian $L$ under automorphisms. We will use the following result.
Lemma 3.11 (Biyikoglu et al., [13], Lemma 2.17) If \( v \in \mathbb{C}^n \) is an eigenvector of \( L \) affording the eigenvalue \( \lambda \in \mathbb{R} \) and \( P \) is an automorphism, then \( P^Tv \) is also an eigenvector of \( L \).

Example 3.12 The graph \( G_1 \) in Figure 3.2a has seven non-trivial automorphisms of orbit size two. The corresponding permutation matrices are given by

\[
P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}
\]

and their products. The node \( v_3 \) is a fixed point of all of them. The graph \( G_2 \) in Figure 3.2b has three non-trivial automorphisms of orbit size two given by \( P_1, P_2 \) and their products.

Weighting all edges of the above graphs with different weights would result in two graphs that have no non-trivial automorphisms.

3.7 Almost Equitable Partitions of Graphs

Almost equitable partitions of graphs are considered in Section 6.1 and Chapter 8. Let \( G = (V, E, w) \). For \( k \in \{1, \ldots, n\} \), the \( k \)-partition \( \pi_k \) of a graph \( G \) is a partition of its node set \( V \). That is, \( \pi_k := (V_1, \ldots, V_k) \), such that \( \bigcup_{i=1}^{k} V_i = V \) and \( V_i \cap V_j = \emptyset \) for \( i \neq j \). The sets \( V_i \) are called cells of the partition. The characteristic vector corresponding with a cell \( V_i \), for \( i = 1, \ldots, k \), is a vector \( p_i \in \{0, 1\}^n \), indexed with the node set \( V \), satisfying

\[
(p_i)_j = \begin{cases} 
1 & \text{if } v_j \in V_i \\
0 & \text{otherwise.}
\end{cases}
\]

The characteristic matrix of the partition is then the matrix \( P_{\text{char}} := (p_1 | p_2 | \ldots | p_k) \in \{0, 1\}^{n \times k} \). It holds that

\[
P_{\text{char}}^T P_{\text{char}} = \text{diag}(|V_i|),
\]
\[ P_{\text{char}} P_{\text{char}}^T = \text{diag} \left( \frac{1}{|V_i|} \times |V_i| \right) . \]  
\[ (3.28) \]

Remember that we denote the neighbourhood of a node \( v_i \in V \) in an undirected graph by \( N(v_i) \). Let us denote the restriction of this neighbourhood to the cell \( V_j \) by \( N^j(v_i) := N(v_i) \cap V_j \).

A \( k \)-partition of a weighted undirected graph \( G = (V, E, w) \) is called \textit{almost equitable} if and only if for any pair \( i, j \in \{1, \ldots, k\} \), where \( i \neq j \) and for all \( v \in V_i \) the numbers

\[ q_{ij} = q_{ij}(v) := \sum_{v_l \in N(v)} w(v_l, v), \]  
\[ (3.29) \]

depend only on \( i \) and \( j \).

Almost equitable partitions were introduced by Cardoso et al., [21] for unweighted and by Gerbaud, [36] for weighted graphs. The almost equitable partition of a digraph can be defined analogously by using the in- and out-neighbourhoods of the nodes. However, we do not formally do this here, as in this thesis we only consider almost equitable partitions of weighted undirected graphs.

Each graph has at least two almost equitable partitions: the 1-partition, where the whole graph is taken as one cell, and the \( n \)-partition, where each node forms its own cell. The orbits of any group of automorphisms of a graph always form an almost equitable partition. However, it is possible to construct graphs that have equitable partitions but no non-trivial automorphisms.

The following lemmas relate graphs that have an almost equitable partition to their Laplacian eigenvectors.

**Lemma 3.13 (Gerbaud, [36], Proposition 4)** Let \( G \) be a weighted graph and \( L(G) \) its Laplacian. For \( k \in \{1, \ldots, n\} \), let \( \pi_k = (V_1, \ldots, V_k) \) be a \( k \)-partition of \( V(G) \) with the characteristic matrix \( P_{\text{char}} \). The partition is almost equitable, if and only if there is a \( k \times k \) matrix \( L_{\pi_k} \) such that \( LP_{\text{char}} = P_{\text{char}} L_{\pi_k} \) holds.

This is equivalent to saying that if \( v_i, v_j \in V_h \), for a \( h \in \{1, \ldots, k\} \), then \( (LP_{\text{char}})_{ij} = (LP_{\text{char}})_{ji} \). The matrix \( L_{\pi_k} \) is given by

\[ L_{\pi_k} = \text{diag} \left( \frac{1}{|V_i|} \right) p_{\text{char}}^T L_{\text{char}} . \]  
\[ (3.30) \]

**Lemma 3.14 (Gerbaud, [36], Corollary 5)** Let \( \pi_k = (V_1, \ldots, V_k) \) be an almost equitable partition of \( G \), let \( b \in \mathbb{C}^k \) and \( \lambda \neq 0 \). Then

\[ (L_{\pi_k} - \lambda I)b = 0_k \]  
\[ (3.31) \]

The name “almost equitable” stems from the fact that this is a relaxation of the more widely known equitable partition, as defined for example in Godsil and Royle, [37]. A partition of the graph is equitable if it is almost equitable, and, additionally, (3.29) also holds for \( i = j \). Equitable partitions will not be used in this thesis.
if and only if
\[(L - \lambda I)P_{\text{char}}b = \mathbf{0}_n,\] (3.32)

where \(L_{\pi_k}\) is the matrix defined in Lemma 3.13.

This implies that if \(G\) has an almost equitable \(k\)-partition and \(v = P_{\text{char}}b\) is an eigenvector of \(L\) corresponding with a non-zero eigenvalue, then \(v\) has at most \(k\) different entries. If two nodes \(i\) and \(j\) belong to the same node set, then the corresponding eigenvector entries are equivalent.

**Example 3.15** The graphs \(G_1\) and \(G_2\) in Figure 3.2a on page 26 both have almost equitable 3-partitions given by \(V_1 = \{v_1, v_2\}, V_2 = \{v_3\}, V_3 = \{v_4, v_5\}\). Therefore, for both graphs,

\[
P_{\text{char}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad L_{\pi_3} = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 1 \end{pmatrix} \]  (3.33)
A general local synchronising algorithm\textsuperscript{1} is defined for a set of agents. The agents broadcast some of their states along some communication network. That is, at each time, an agent receives information from its neighbours in the communication topology. This information is then used by each agent to generate a local control input according to some algorithm that utilises this information.

The control is distributed in the sense that no agent is assumed to obtain the information from every other agent in the system. The advantages of local synchronising algorithms are obvious: The resources are utilised efficiently, as each agent has to store less information in its memory. They are furthermore robust, as only very weak conditions must be imposed on the communication graph, which means that the agents may synchronise even despite link failure in the communication network. Algorithms that achieve synchronisation among the agents are usually called consensus algorithms.

In this chapter we lay the theoretical foundation for the thesis. First, the single integrator consensus algorithm is presented in Section 4.1, followed by the double integrator consensus algorithm in Section 4.2. We define consensus stability and asymptotic consensus stability of double integrator consensus systems in Section 4.2.1. Thereafter, we give formal definition of homogeneous and heterogeneous communication topologies in Section 4.2.2. We then briefly review the possible setups of the double integrator consensus algorithm and the available results. We consider the double integrator consensus in homogeneous networks and in the absence of certain measurements. Finally we arrive at the double integrator consensus algorithm in heterogeneous networks, which is the main object of the thesis.

After the theoretical overview, Section 4.3 provides some examples of double integrator consensus systems. The formation flight of quadcopters, a power system and a multi-agent supporting system for buildings are considered. With these examples, it becomes apparent

\textsuperscript{1}Note that we avoid using the popular term “averaging algorithm” here, because this would imply that the only equilibrium of the system obeying the algorithm is the average of its initial states. This can in fact be achieved for some communication networks, however it will not be the general case.
that the assumptions made in the Section 4.2 hold for a wide range of real-life applications.

Graphs were introduced in Chapter 3. The communication topology between the $n$ agents can be described by a graph $G = (V, E)$ or a weighted graph $G = (V, E, w)$ on $n$ nodes. Each agent $i$ is identified with a node $v_i$. If agent $i$ obtains information from some agent $j$, there is an edge $v_j \rightarrow v_i$ in the corresponding graph, see Chapter 3 for details on graph theory. We use communication topology, network and graph interchangeably hereafter, and, for better readability, sometimes use graph-theoretic terms when discussing the agent network. For example we will speak of the neighbourhood of the agent, and not of the node that represents the agent. The terms consensus system, protocol and algorithm are also used interchangeably.

4.1 Single Integrator Consensus Algorithm

The considered networked multi-agent system is comprised of $n$ individual linear systems with single integrator dynamics that evolve in a one-dimensional space and are coupled via a communication network. The agents transmit their states to their neighbours in the network and use their neighbours’ states to generate their own control input. It is a standard assumption in the literature, cf. Ren and Beard, [84], Bullo et al., [18], Ren and Cao, [85] and Mesbahi and Egerstedt, [64], that the individual agent’s dynamics can be linearised, resulting in $n$ one-dimensional system equations, given by

$$
\dot{x}_i(t) = u_i(t), \\
x_i(0) = x_{i,0},
$$

(4.1)

where

- $i = 1, \ldots, n$ denotes the agent,
- $x_{i,0} \in \mathbb{R}$ are the initial conditions,
- $x_i : \mathbb{R}^+ \to \mathbb{R}$ is the $i$-th agent’s state, and
- $u_i : \mathbb{R}^+ \to \mathbb{R}$ is the corresponding control input.

Agents that can be modeled in this fashion are, for example, wireless sensors and oscillators, cf. Olfati-Saber et al., [74] and references therein. The individual agents are coupled via the control input, i.e.

$$
u_i(t) = f(x_1(t), \ldots, x_n(t))
$$

(4.2)

for some function $f : \mathbb{R}^n \to \mathbb{R}$.

Following Ren and Beard, [84], the control input in equation (4.2) can be chosen as the single integrator consensus algorithm

$$
u_i(x_1(t), \ldots, x_n(t)) = -\sum_{j \in N_i} a_{ij}(x_i(t) - x_j(t)),
$$

(4.3)

where
· $N_i$ denotes the in-neighbourhood of the agent $i$, i.e. the agents that agent $i$ can receive state information from and
· $a_{ij} \in \mathbb{R}^+ \setminus \{0\}$ are weights assigned to different communication channels.

Starting in the 1960s, consensus-type algorithms have been studied systematically in different areas of science. In the context of management science and statistics, Eisenberg and Gale, [32], Norvig, [70], Winkler, [103] and DeGroot, [28] are notable publications. In social sciences, Aschinger, [6], [7] used the algorithm (4.3) in order to model opinion dynamics in social networks. A further related area is the study of the behaviour of Markov chains (Hartfiel, [46]). Consensus was also studied in the setting of clock synchronisation (Schenato and Gamba, [89]) and sensor data fusion (Luo and Kay, [60], Olfati-Saber and Shamma, [73]). In the context of distributed computing and simulation of flocking behaviour consensus algorithms were considered for example by Reynolds, [87], Bertsekas and Tsitsiklis, [9] and Vicsek et al., [98].

The term “consensus”, however, was not widely established until the last decade, when the initial work by Borkar and Varaiya, [16], Papadimitriou and Tsitsiklis, [75], Reynolds, [87] and Vicsek et al., [98] was picked up by Jadbabaie et al., [50]. The book by Ren and Beard, [84] provides a good manual on consensus algorithms and can be considered a standard reference along with the papers by Olfati-Saber et al., [74] and Ren et al., [86]. More recent manuals are the books by Ren and Cao, [85] and Mesbahi and Egerstedt, [64].

The communication network along which information is transmitted among the agents, is described by the graph $G = (V, E, w)$ introduced in Chapter 3. The node set $V$ is identified with the agents, i.e. $v_i \in V$ means that agent $i$ is part of the communication network. The edge set describes the information flow between the agents, i.e. $v_i \rightarrow v_j \in E$ means that agent $i$ can send information to agent $j$. The weight function $w$ assigns a positive weight to each communication channel.

Writing $x = (x_1, \ldots, x_n)^T$ and using the algorithm (4.3), we can restate the system (4.2) in matrix form as

$$
\dot{x}(t) = -L(G)x(t),
\quad x(0) = x_0,
$$

(S1)

where $L$ is the $n \times n$ Laplacian matrix associated with the communication topology $G$ between the agents. The weight function $w$ of $G$ is given by

$$
w(v_i, v_j) = \begin{cases} 
a_{ij} & \text{if } v_j \in N_i \\
0 & \text{otherwise.}
\end{cases}
$$

(4.4)
Thus, in general, \( L \) is the Laplacian of a weighted graph.

In order to discuss properties of (S1) we first need to introduce an appropriate stability definition in the next section.

### 4.1.1 Stability Definitions

We say that the single integrator consensus system (S1) achieves consensus if, as \( t \to \infty \), the states of the agents equalise. Formally, we introduce the following definition.

**Definition 4.1 (asymptotic consensus stability of single integrator consensus algorithms; consensus equilibrium)** System (S1) is asymptotically consensus stable, if for all \( x_0 \in \mathbb{R}^n \),

\[
\lim_{t \to \infty} \| x_i(t) - x_j(t) \| = 0 \quad \text{for all} \ i, j = 1, \ldots, n. \quad (4.5)
\]

If system (S1) is asymptotically consensus stable, then its consensus equilibrium is given by

\[
x_c = \lim_{t \to \infty} x_1(t) = \ldots = \lim_{t \to \infty} x_n(t).
\]

As we will see in the following, the obtained consensus equilibrium depends both on the initial conditions of the system (S1) and the interconnection structure between the agents. The equilibrium is called average consensus if it is the average over the initial states of all agents. We use the following definition.

**Definition 4.2 (average consensus)** System (S1) is said to achieve average consensus, if it holds that

\[
\lim_{t \to \infty} x_1(t) = \ldots = \lim_{t \to \infty} x_n(t) = \frac{1}{n} \sum_{i=1}^{n} x_i(0). \quad (4.6)
\]

That is, an asymptotically consensus stable single integrator consensus system (S1) may, but does not have to, achieve average consensus. This will also become apparent in the results presented in the next section.

### 4.1.2 Results on Stability of Single Integrator Consensus Systems

The following key results on single integrator consensus in continuous time have been collected by Ren and Beard, [84] and Mesbahi and Egerstedt, [64], which we restate here using our terminology. We list them here in order to be able to compare the behaviour of single and double integrator consensus algorithms in Section 4.2. Remember that \( \rightarrow \) and \( \leftarrow \) in the margins indicate whether the result holds for directed or undirected weighted graphs. Not all of the following results are stated for weighted graphs in the cited references. However, the corresponding extension is trivial.
Lemma 4.3 (Ren and Beard, [84] Corollary 2.9) Let $G$ be a weighted undirected graph. System (S1) is asymptotically consensus stable if and only if $G$ is connected.

Lemma 4.4 (Ren and Beard, [84] Theorem 2.8) Let $G$ be a weighted directed graph. System (S1) is asymptotically consensus stable if and only if $G$ consists of one reach. The trajectory generated by system (S1), initialised from $x_0$, satisfies

$$\lim_{t \to \infty} x(t) = (v_0w_0^T)x_0,$$

where $v_0$ and $w_0$ are, respectively, the right and left eigenvectors associated with the zero eigenvalue of $L(G)$, normalised such that $v_0^Tw_0 = 1$.

Lemma 4.5 (Ren and Beard, [84] Corollary 2.9) Let $G$ be a weighted digraph. System (S1) reaches average consensus for every initial condition if and only if $G$ is strongly connected and weight-balanced.

Lemma 4.6 (Mesbahi and Egerstedt, [64] Theorem 3.4) Let $G$ be a weighted connected undirected graph. Then system (S1) is asymptotically consensus stable with a convergence rate that is equivalent to the smallest non-zero eigenvalue of $L(G)$.

These results show that consensus stability of agents with single integrator dynamics, as well as the achieved consensus value, depend only on the connectivity of the underlying communication network and the initial conditions of the system. In fact, one of the most important contributions of Jadbabaie et al., [50], was to connect algebraic graph theory with the study of interconnected networks, while the contribution of Olfati-Saber et al., [74] was to present a systematic framework to analyse the first-order consensus algorithm. The study of consensus algorithms has since been an active research area within the control community. So far, it has been extended to digraphs (Beard and Stepanyan, [8], Olfati-Saber and Murray, [72], Moreau, [67], Fang et al., [33], Ren et al., [86], Arcak, [4]), dynamic communication topologies (Tanner et al., [94], Olfati-Saber and Murray, [72], Olfati-Saber, [71], Tanner et al., [95]), asynchronous protocols (Hatano and Mesbahi, [47], Blondel et al., [15], Cao et al., [20], Hespanha et al., [48]), and quantisation effects and delays (Nedic et al., [69], Nedic and Ozdaglar, [68], You and Xie, [107]) among other topics.

4.2 Double Integrator Consensus Algorithm

Taking into account that many applications cannot be modeled by first-order ordinary differential equations (ODE), it is natural to extend the consensus algorithm (S1) to systems with higher order dynamics. One particular class of systems are those that can be described by second order ODEs, some examples of which will be given.
in Section 4.3. Thus, in this thesis we consider a networked multi-agent system that is comprised of \( n \) individual linear systems that evolve in a two-dimensional space. Their dynamics satisfy (cf. Ren and Beard, [84], Ren and Cao, [85], and references therein)

\[
\dot{x}_i(t) = u_i(t), \\
x_i(0) = x_{i,0}, \quad \dot{x}_i(0) = \dot{x}_{i,0},
\]

(4.8)

where

- \( i = 1, \ldots, n \) denotes the agent,
- \( x_{i,0} \in \mathbb{R} \) and \( \dot{x}_{i,0} \in \mathbb{R} \) are the initial conditions,
- \( x_i : \mathbb{R}^+ \to \mathbb{R} \) is the \( i \)-th agent’s position,
- \( \dot{x}_i : \mathbb{R}^+ \to \mathbb{R} \) is the \( i \)-th agent’s velocity,
- \( u_i : \mathbb{R}^+ \to \mathbb{R} \) is the corresponding control input.

Note that, in general, \( \dot{x} \) and \( x \) do not necessarily represent velocity and position. Their physical meaning depends on the considered system. As \( \dot{x} \) is the derivative of \( x \), they can be called the value and the rate of change of the system. It is, however, customary to name them position and velocity, a nomenclature which we here adopt. The systems that can be modeled in this fashion are, of course, in no way limited to systems describing spatial motion, for example, in Section 4.3, we consider power systems. Therein, \( x \) denotes the phase angle of an inverter and \( \dot{x} \) its frequency. Note that the models of a large class of systems can be feedback linearised as (4.8), such as mobile robots, quadcopters, power grids, cf. Section 4.3, as well as Bullo et al., [18] and references therein.

The individual agents (4.8) are coupled via a communication network. The agents transmit their positions and velocities to their neighbours in the network and use their neighbours’ information to generate their own control input, given by

\[
u_i(t) = f(x_1(t), \ldots, x_n(t), \dot{x}_1(t), \ldots, \dot{x}_n(t))
\]

(4.9)

for some function \( f : \mathbb{R}^{2n} \to \mathbb{R} \).

Under the assumption that some absolute position, velocity, or position and velocity measurements are available to the agents, a consensus algorithm for agents with double integrator dynamics was suggested by Ren and Atkins, [83] as

\[
u_i(x_1(t), \ldots, x_n(t), \dot{x}_1(t), \ldots, \dot{x}_n(t)) =
- \sum_{j \in \mathcal{N}_i} a_{ij}(x_i(t) - x_j(t)) - \beta \sum_{j \in \mathcal{N}_i} b_{ij}(\dot{x}_i(t) - \dot{x}_j(t))
\]

(4.10)

where
\[ N^x_i \text{ denotes the set of agents that agent } i \text{ can obtain position information from,} \]
\[ N^\dot{x}_i \text{ is the set of agents that agent } i \text{ can obtain velocity information from,} \]
\[ a_{ij} \text{ and } b_{ij} \text{ are positive weights that the position respectively velocity communication channels are weighted with, and} \]
\[ \beta \text{ is a positive constant.} \]

Let \( G_x = (V, E_x, w_x) \) denote the \textit{position graph}, i.e. the graph that models the communication network along which the agents exchange their position information. Analogously, let \( G_\dot{x} = (V, E_\dot{x}, w_\dot{x}) \) be the \textit{velocity graph}. The weight functions \( w_x : V \times V \to \mathbb{R}^+ \) and \( w_\dot{x} : V \times V \to \mathbb{R}^+ \) are given by

\[
\begin{align*}
    w_x(v_i, v_j) &= \begin{cases} 
    a_{ij}, & \text{if } v_j \in N^x_i, \\
    0, & \text{otherwise},
    \end{cases} \quad (4.11a) \\

    w_\dot{x}(v_i, v_j) &= \begin{cases} 
    b_{ij}, & \text{if } v_j \in N^\dot{x}_i, \\
    0, & \text{otherwise}.
    \end{cases} \quad (4.11b)
\end{align*}
\]

Let \( L_x \) be the Laplacian matrix of the position graph and \( L_\dot{x} \) the Laplacian matrix of the velocity graph. Then, writing \( x := (x_1, \ldots, x_n)^T \), we can restate equations (4.8) and (4.10) in matrix form as

\[
\begin{align*}
    \ddot{x}(t) &= -L_x x(t) - \beta L_\dot{x} \dot{x}(t), \\
    x(0) &= x_0, \quad \dot{x}(0) = \dot{x}_0. \quad \text{(S2)}
\end{align*}
\]

System (S2) is a natural extension of the single integrator consensus system (S1) on page 31 for agents with double integrator dynamics. It is therefore an intuitive assumption that the properties of system (S1) are inherited by system (S2). While this is partially true, the behaviour of system (S2) is much less straightforward to determine than that of system (S1). One of the main goals of the present work is to uncover some of the intricate dependencies between the communication topologies and the convergence properties of the consensus algorithm. In order to compare the two systems, we first need to define stability of double integrator consensus systems.

\subsection{Stability Definitions}

We define stability of system (S2) in the following manner.

\textbf{Definition 4.7 (consensus stability of double integrator consensus systems)} System (S2) is consensus stable, if for all \( x_0, \dot{x}_0 \in \mathbb{R}^n \)

\[
\lim_{t \to \infty} \| \dot{x}_i(t) - \dot{x}_j(t) \| = 0 \text{ holds for all } i, j = 1, \ldots, n. \quad (4.12)
\]
We sometimes say that the system achieves velocity consensus. If $\dot{x}_i(t)$ is bounded for all $i = 1, \ldots, n$, then we speak of bounded velocity consensus. Note that system (S2) being consensus stable implies that for all $i, j = 1, \ldots, n$ and all initial conditions $\dot{x}_0, x_0$, it holds that
\[
\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = c_{ij}
\]
for some constants $c_{ij} \in \mathbb{R}$.

**Definition 4.8 (asymptotic consensus stability of double integrator consensus systems)** System (S2) is asymptotically consensus stable, if for all $x_0, \dot{x}_0 \in \mathbb{R}^n$
\[
\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0 \quad \text{for all } i, j = 1, \ldots, n. \quad (4.13)
\]

If a system is asymptotically consensus stable, we sometimes say that it achieves position consensus. Note that if the double integrator consensus system is asymptotically consensus stable, then, due to the fact that it is linear, it is also consensus stable.\(^2\) The converse does not necessarily hold.

Due to the interdisciplinary nature and the relative youth of the networked systems field, no widely-acknowledged, uniform definition of stability of networked systems has emerged so far. Ren and Beard, \([84]\) say that a system achieves consensus if it satisfies Definition 4.8. Mesbahi and Egerstedt, \([64]\) as well as Bullo et al., \([18]\) define an agreement subspace, which is identical to the condition in Definition 4.8 and say that an algorithm may or may not drive the system to the agreement subspace. The terms consensusability, see, e.g. You and Xie, \([107]\) and consensualisability, see, e.g. Xi et al., \([104]\) have further appeared in recent publications, and are usually equivalent or comparable to Definition 4.8. All these definitions have in common that they require the final positions of the agents to be synchronised. In Cai et al., \([19]\), a stability concept similar to consensus stability is developed for agents with non-negligible single integrator dynamics and non-decouplable states.

Consensus of the velocities is not generally considered in literature. Therefore, many authors use a definition analogous to asymptotic consensus stability and refer to the described behaviour simply as “achieving second-order consensus”, see, for example, Ren and Atkins, \([83]\), Yu et al., \([109]\). Definition 4.7 is not usually encountered in the literature. It is introduced in this thesis in order to better describe the behaviour of agents with double integrator dynamics, where, as we will show in Section 5.3, the agents might never fully synchronise but still evolve on a common manifold.

One particular property of a consensus equilibrium is its dependence on the initial states of the system. It is useful to define average consensus in analogy with the single integrator average consensus.\(^2\) Note that for non-linear systems it is possible to find examples where the position differences converge to zero but the velocity differences do not.
Definition 4.9 (average consensus) System (S2) is said to achieve average position consensus, if it holds that

$$\lim_{t \to \infty} x_1(t) = \ldots = \lim_{t \to \infty} x_n(t) = \frac{1}{n} \sum_{i=1}^{n} x_i(0). \quad (4.14)$$

Additionally we say that the system achieves average velocity consensus if

$$\lim_{t \to \infty} \dot{x}_1(t) = \ldots = \lim_{t \to \infty} \dot{x}_n(t) = \frac{1}{n} \sum_{i=1}^{n} \dot{x}_i(0). \quad (4.15)$$

4.2.2 Homo- and Heterogeneous Communication Networks

System (S2) has received less attention in the literature than the single integrator consensus algorithm. The existing research on double integrator consensus algorithms can be subdivided in two thematic areas. One operates on the assumption that $G_x = G_\dot{x}$ ([111], [58], [114], [99], [105], [108], [35]), while the other one assumes that there is no communication of velocity information, i.e. $\beta = 0$ ([82], [91], [27], [111]). As the relationship between the position and the velocity graph will play an important role throughout this thesis, we introduce the following definition.

Definition 4.10 (Homo- and heterogeneous communication topologies) The communication topology between the agents is called homogeneous if $L_x = L_\dot{x}$ holds in system (S2). It is called heterogeneous otherwise.

If a network is homogeneous, it means that if an agent transmits its position information it also transmits its velocity information and vice-versa. Furthermore, both information are weighted with the same weight, with the exception of the scaling factor $\beta$ that acts on all velocity information simultaneously. This is the assumption usually adopted in the literature, and it models the reality sufficiently well for a large number of systems. It fails however in cases when the networked system consists of agents equipped with different sensors and not all agents can measure both their velocity and position, or in cases where some of the sensors break down. Furthermore, even if all agents can measure both their position and velocity, it is not guaranteed that both pieces of information arrive at their neighbours due to unreliability of communication. Additionally, homogeneity implies that both position and velocity information are weighted with the same weights, i.e. $a_{ij} = b_{ij}$ must hold for all $i, j = 1, \ldots, n$ in (4.10), which unnecessarily limits the system.

4.2.2.1 Second Order Consensus in Homogeneous Networks

The behaviour of agents comprising system (S2) on page 35 in homogeneous networks was first studied by Ren and Atkins [83].
The assumption that the communication networks are homogeneous severely simplifies the analysis. In particular, they obtain the following result, restated here in our notation and terminology.

**Lemma 4.11** *(Ren and Atkins, [83], Theorem IV.1)* Let $G_x = G_x$, $\beta = 1$ in system $(S2)$. The system is asymptotically consensus stable only if the communication graph consists of one reach.

Remember, that communication graph consisting of one reach is a necessary and sufficient condition for the unweighted single integrator consensus system to be stable. Here we can already see that the extension from single to double integrator dynamics is non-trivial, and the problem is more complicated and challenging than the first-order case. Yu et al., [110] mention that in fact a consensus stable system with homogeneous communication topologies might no longer achieve consensus if one further edge is added between the agents. This is inconsistent with the intuition that more communication should lead to faster agreement, as is indeed the case with single integrator agents.

Yu et al., [109] give a necessary and sufficient condition for system $(S2)$ to be asymptotically consensus stable if the communication topology is homogeneous and weighted.

**Lemma 4.12** *(Yu et al., [109], Theorem 1)* Let $G_x = G_x$ be a weighted digraph and let $L$ be the Laplacian matrix describing the homogeneous communication network. System $(S2)$ on page 35 is asymptotically consensus stable if and only if the communication graph consists of one reach and

$$\beta^2 > \max_{i=2,\ldots,n} \frac{(\text{im}(\lambda_i))^2}{\text{re}(\lambda_i) |\lambda_i|^2},$$

(4.16)

where $\lambda_i$, for all $i = 2,\ldots,n$ are the non-zero eigenvalues of $L$.

Zhu et al., [114] introduce an additional feedback and gains to the algorithm (4.10), obtaining the protocol

$$u_i(t) = -\gamma x_i(t) - \delta \dot{x}_i(t) - \alpha \sum_{j \in N_i} (x_i(t) - x_j(t)) - \beta \sum_{j \in N_i} (\dot{x}_i(t) - \dot{x}_j(t)),$$

(4.17)

for $i = 1,\ldots,n$, where $N_i$ denotes the set of agents that agent $i$ can obtain information from. The additional gains $\alpha, \gamma, \delta \in \mathbb{R}^+ \setminus \{0\}$ can be adjusted, such that the system becomes consensus stable. Zhu, [113] studies the consensus rate of the same generalised consensus algorithm, and shows that it is commonly determined by the largest and the smallest non-zero eigenvalues of the Laplacian matrix. Li et al., [58] further investigate the protocol of Zhu et al., and study its convergence properties and the final states of the agents.
Wen et al., [99] study the algorithm under communication constraints. Here, the network is assumed to be homogeneous and non-switching. However, it is available only at disconnected time intervals. The authors give a sufficient condition for consensus using a Lyapunov approach. An extension of the available results to networks operating in discrete time can be found in Xie and Wang, [105]. The influence of communication delays on the convergence of the algorithm is considered in Yu et al., [108] and the maximal delay that does not destabilise a stable network is given by the authors. Switching systems are considered in Gao et al., [35].

4.2.2.2 Second Order Consensus Without Velocity Measurements

Apart from homogeneous communicatoin topologies, another popular assumption is to assume that there is no communication of velocities between the agents. Depending on the approach, the agents may or may not have access to their own velocity measurements. If the graph $G_x$ has no edges, then system (S2) on page 35 is not consensus stable, as we will formally follow from Chapter 7. Therefore other methods are needed to ensure consensus stability. In fact, the resulting problem is no longer to prove convergence of the existing algorithm, but to design an algorithm that overcomes the shortcomings of the present one. Some solutions so far have involved data sampling or introducing delays in information exchange, cf. Ren, [82], Seuret et al., [91], de Campos and Seuret, [27], Yu et al., [111].

4.2.2.3 Second Order Consensus In Heterogeneous Networks

While the assumption that velocity information is not available to any agent may be applicable to a number of physical systems, it is not generally true. On the other hand, the condition that the communication topology is the same for both velocity and position information, i.e. that $G_x = G_z$ is restrictive in that it only considers agents that have both measurements available to them. In fact, agents that measure and communicate only velocity or only position are likely to be cheaper both in terms of hardware and communication costs. Furthermore, even if the initial communication networks were identical for both velocity and position, information loss and sensor breakdown may create a heterogeneity. Will the networked system (S2) on page 35 be (asymptotically) consensus stable if the communication networks are heterogeneous, and possibly partially disconnected? The assumption that the communication topologies are heterogeneous and weighted is a straightforward extension of the unweighted homogeneous case. Therefore, the results for homogeneous networks will appear as special cases in our results in the following chapters.

To the best of our knowledge heterogeneous networks have only been mentioned once in the literature prior to our work, namely in
Ren, [82], where the case $L_x \neq L_\dot{x}$ is studied for the case that $L_x$ and $L_\dot{x}$ are connected, unweighted and undirected and the control input is bounded and given by

$$u(t) = -\sum_{j=1}^{n} \left\{ (L_x)_{ij} \tanh(K_x(x_i(t) - x_j(t))) \right\} \tag{4.18}$$

$$+ (L_\dot{x})_{ij} \tanh(K_\dot{x}(\dot{x}_i(t) - \dot{x}_j(t)))) \right\}.$$  

Here, $K_x$, $K_\dot{x}$ are positive weights and tanh denotes the tangens hyperbolicus. Using a Lyapunov function, the author obtains that if the graphs are connected, the system is asymptotically consensus stable. However, owing to the form of the Lyapunov function this result bears little insight into the structure of the consensus algorithm. In contrast, Chapter 5-7 relate consensus stability of the algorithm directly to the underlying graph structure.

The considered problem has, to the best of our knowledge, not been formulated and studied prior to Goldin et al., [44] and Goldin and Raisch, [42]. It has, however, since received further attention, cf. Mallada and Tang, [62], Zhang et al., [112], Xue and Yao, [106].

4.3 EXAMPLES

The following three examples illustrate the versatility of the double integrator consensus algorithm in real-life applications and thus further motivate their study in the present work.

4.3.1 Flocking and Formation Flight of Quadcopters

Quadcopters are multicopters that are lifted and propelled by four rotors. They are popular unmanned aerial vehicles owing to their numerous advantages over comparably scaled helicopters. In this example we show how, under certain standard assumptions, a quadcopter can be modeled as the double integrator system (4.8).

The position of the quadcopter is determined by the translational position of its centre of mass in the inertial frame, described by $p := (p_x, p_y, p_z)^T$, and its rotations around the axes of the body frame, described by the Euler angles $\omega(t) := (\phi(t), \theta(t), \psi(t))^T$ which are the rotations around the copter’s x (roll), y (pitch) and z (yaw) axis, respectively. The rotations are performed according to the aerospace conventions (yaw-pitch-roll, DIN9300).
The rotation matrix $R : \mathbb{R}^3 \to \mathbb{R}^{3 \times 3}$ from body frame to inertial frame is therefore given by

$$
R(\phi(t), \theta(t), \psi(t)) := 
\begin{pmatrix}
C_\psi C_\theta & C_\psi S_\theta S_\phi - S_\psi C_\phi & C_\psi S_\theta C_\phi + S_\psi S_\phi \\
S_\psi C_\theta & S_\psi S_\theta S_\phi + C_\psi C_\phi & S_\psi S_\theta C_\phi - C_\psi S_\phi \\
-S_\theta & C_\theta S_\phi & C_\theta C_\phi
\end{pmatrix}
$$

(4.19)

where $C_a = \cos(a)$, $S_a = \sin(a)$ for $a \in \mathbb{R}$.

The total force $F : \mathbb{R}^+ \to \mathbb{R}^3$ on the quadcopter is given by

$$
F(t) := -D_b e_v(t) - m g e_z + \sum_{j=1}^{4} (-T_j(t) R(t) e_T),
$$

(4.20)

where

- $e_v(t)$ is the body velocity direction in the inertial frame,
- $e_z$ is the vertical axis in the inertial frame,
- $e_T(t)$ is the thrust direction in the body frame (usually $(0,0,1)^T$ as we ignore any deflection of the blade plane for simplicity, see Huang et al., [49]),
- $D_b$ is the body drag coefficient, which is usually neglected,
- $m$ is the vehicle mass,
- $g$ is the gravitational constant, and
- $T_j : \mathbb{R}^+ \to \mathbb{R}$ the thrust generated by rotor $j$, for $j = 1, \ldots, 4$.

The total moment, again ignoring blade deflection, and, thus, also the flapping momentum as well as the momentum from aerodynamic drag, is given by

$$
M(t) = \sum_{j=1}^{4} (M_j(t) + r_j \times (-T_j(t) R(t) e_T)).
$$

(4.21)

Here,

- $M_j : \mathbb{R}^+ \to \mathbb{R}$, for $j = 1, \ldots, 4$ is the reaction torque from rotor $j$ and
- $r_j$ denotes the vector from the center of mass to rotor $j$.

Furthermore $\times$ denotes the vector (or cross) product.

The translational and rotational dynamics of a rigid body can generally be described by the well-known Newton-Euler equations. Moreover, the acceleration and angular velocity of the body can be described dependent on the force $F$ and the momentum $M$ that we have just calculated in (4.20) and (4.21):

$$
F(t) = m \ddot{x}(t)
$$

$$
M(t) = I_B \ddot{\omega}(t) + \omega(t) \times (I_B \omega(t)).
$$

(4.22)
Here, $I_B$ denotes the inertia matrix and $x : \mathbb{R}^+ \to \mathbb{R}^3$ is the position of the quadcopter in the inertial frame. In other words, combining (4.20) and (4.21) with (4.22), the system is now modeled as

$$
\dot{x}(t) = -\frac{1}{m} D_B \mathbf{e}_v(t) + -g \mathbf{e}_z + \frac{1}{m} \sum_{j=1}^{4} \left( -T_j(t) \mathbf{R}(t)e_T \right),
$$

$$
I_B \dot{\omega}(t) + \omega(t) \times (I_B \omega(t)) = \sum_{j=1}^{4} \left( M_j(t) + r_j \times (-T_j(t) \mathbf{R}(t)e_T) \right).
$$

As in classical helicopter designs, the forces that control the translational movement of the vehicle are generated by the rotation of its main thrust, cf. Leishman, [57]. Therefore, many control concepts make use of a cascaded structure. In this setup, the translational movement is modeled as a double integrator. The forces necessary for the desired control of this translational movement are the desired reference for the inner control loop, controlling the rotation and thrust of the vehicle. This is possible because the rotational dynamics are very fast compared to the translational dynamics (e.g. Sa and Corke, [88] give a rise time for the rotation of 0.4 seconds). If designing a supervising control scheme for multiple vehicles, concentrating on the position of each vehicle, it is therefore feasible to neglect the inner loop and model the individual vehicles as possessing decoupled double integrator dynamics given by

$$
\ddot{x}_i(t) = u_i(t), \quad i = 1, 2, 3
$$

Then, if several quadcopters are interconnected as a networked system, algorithm (4.10) can be used in order for the copters to achieve consensus. The resulting networked system is then given by (S2) on page 35.

### 4.3.2 Frequency Control of Power Systems

Electric power systems are a highly interesting class of large-scale systems. They consist of a number of power generators, inverters, power lines and loads that together form a complex dynamic system that usually spans a large geographic area. Electrical energy is generated by power plants and transmitted over long distances to load centres, that fulfill the demand of individual consumers. Traditionally, the power is generated by a relatively small number of large nuclear, thermal or hydro plants.

One of the main control objectives in power systems is frequency stability, i.e. the maintaining of a synchronised frequency within the network independently of events like a sudden loss in generation or a sudden change in load. This problem can be treated as a network control problem. The electrical network is then understood as a networked system modeled by a graph, where each node represents a generator or an inverter and the edges model the power...
As long as there are no power line failures in the system it is a reasonable assumption that for small deviations around the nominal operation point the network graph is static, weighted, directed and strongly connected.

Following Schiffer et al., [90], we consider a microgrid formed by \( n \) inverters that are assumed to obey the linearised swing equation

\[
m_i \ddot{x}_i(t) + d_i \dot{x}_i(t) = - \sum_{j \in N_i} k_{ij} (x_i(t) - x_j(t)) + p_i, \quad (4.25)
\]

for \( i = 1, \ldots, n \), where

\[
\begin{align*}
&x_i : \mathbb{R}^+ \to \mathbb{R} \text{ is the phase angle of inverter } i, \\
&m_i \text{ is the virtual inertia coefficient of inverter } i, \\
&d_i \text{ is the virtual damping coefficient of inverter } i, \text{ and} \\
&p_i \text{ is the power setpoint at inverter } i.
\end{align*}
\]

Furthermore, the coefficients \( k_{ij} \) are given by

\[
k_{ij} = \frac{|V_i||V_j|}{b_{ij}}, \quad i, j = 1, \ldots, n, \quad (4.26)
\]

where

\[
\begin{align*}
&V_i \text{ is the voltage of inverter } i, \text{ and} \\
&b_{ij} \text{ is the susceptance of the line between } j \text{ and } i.
\end{align*}
\]

By defining

\[
\begin{align*}
x & := (x_1, \ldots, x_n)^T, \\
P_g & := (p_1, \ldots, p_n)^T, \\
M_g & := \text{diag}(m_1, \ldots, m_n), \\
D_g & := \text{diag}(d_1, \ldots, d_n),
\end{align*} \quad (4.27)
\]

we can restate equation (4.25) in state-space form as

\[
\begin{pmatrix}
\dot{x}(t) \\
\ddot{x}(t)
\end{pmatrix} =
\begin{pmatrix}
\mathcal{O}_{n \times n} & I_n \\
-M_g^{-1}L_x & -M_g^{-1}D_g
\end{pmatrix}
\begin{pmatrix}
x(t) \\
\dot{x}(t)
\end{pmatrix}
+ \begin{pmatrix}
\mathcal{O}_n \\
M_g^{-1}P_g
\end{pmatrix}. \quad (4.28)
\]

Here \( L_x \) is the Laplacian of the electric grid with entries \( (L_x)_{ij} = -k_{ij} \) for \( i \neq j \).

With this model, the inverters will synchronise in frequency if system (4.28) achieves velocity consensus. This system is different from system \((S_2)\) on page 35 in that the Laplacian \( L \) is weighted by the inverse of the inertia matrix \( M_g \), and \( M_g^{-1}D_g \) is not a Laplacian matrix. However, it still fits the more general assumption that the matrices corresponding with the entries \( L_x \) and \( L_x \) in \((S_2)\) are different and can be treated by the same methods.
Networked agents that can support large weights can be used in stabilisation of buildings in earthquake areas. The agents are designed to support the building and balance out terrain movements. Each agent only obtains information from local measurements. The agents then have to agree on a common equilibrium height that the building is kept at.

Ma et al., [61] proposed a multi-agent supporting system (MASS) that is essentially modeled by $n$ agents, each of which is a one-dimensional mechanical mass-spring-damper-system together with an actuator that is assumed to have enough power and enough speed to respond to the earthquake disturbance. The model of agent $i$, for $i = 1, \ldots, n$, is given by

$$m_{\text{MASS}} \ddot{x}_{i}(t) + D_{\text{MASS}} \dot{x}_{i}(t) + k_{\text{MASS}} x_{i}(t) = u_{i}(t) + d_{\text{MASS},i}(t),$$

$$x_{i}(0) = x_{i,0}, \quad \dot{x}_{i}(0) = \dot{x}_{i,0},$$

(4.29)

where

- $m_{\text{MASS}}, D_{\text{MASS}}, k_{\text{MASS}}$ are the mass, damping and stiffness at each agent, assumed to be the same for all agents,
- $x_{i} : \mathbb{R}^+ \rightarrow \mathbb{R}$ the agent’s absolute position,
- $u_{i} : \mathbb{R}^+ \rightarrow \mathbb{R}$ the output force of the actuator mounted at each agent, and
- $d_{\text{MASS},i} : \mathbb{R}^+ \rightarrow \mathbb{R}$ the unknown disturbance at each agent.

Here, $x_{i}$ and $\dot{x}_{i}$ cannot be measured directly, but the difference of position and/or velocity between neighbouring agents is measurable. Ma et al. give a centralised solution for this problem. With the meth-
ods of this thesis a decentralised solution can be offered. Introducing distributed feedback leads to system

\[ \ddot{x}(t) = -M_{\text{mass}} \dot{x}(t) - K_{\text{mass}} x(t) - L_x \dot{x}(t) - L_x x(t) + \tilde{D}_{\text{mass}}(t), \quad (4.30) \]

where

\[ M_{\text{mass}} := \text{diag}(D_{\text{mass}}/m_{\text{mass}}), \quad K_{\text{mass}} := \text{diag}(k_{\text{mass}}/m_{\text{mass}}) \]

are \( n \times n \) constant matrices,

\[ L_x, L_\dot{x} \] are (possibly directed) Laplacian matrices, and

\[ \tilde{D}_{\text{mass}} := \frac{1}{m_{\text{mass}}} (d_{\text{mass},1}, \ldots, d_{\text{mass},n})^T : \mathbb{R}^+ \rightarrow \mathbb{R}^n \] is the disturbance.

Ma et al. suggest a disturbance estimator for \( \tilde{D}_{\text{mass}} \) of system (4.30), and claim that the actuator at each agent acts fast enough to respond to the external disturbance and that the internal dynamics of the system are stable. Therefore, when studying consensus stability of the system, we can neglect the individual agent’s fast dynamics and consider the double integrator consensus system

\[ \dot{x}(t) = -L_x \dot{x}(t) - L_x x(t), \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0. \quad (4.31) \]

This is precisely the double integrator consensus system (S2) studied in this thesis.
In this chapter, we establish the necessary and sufficient conditions for consensus stability and asymptotic consensus stability of system $(S_2)$ on page 35, given by

\[
\begin{align*}
\ddot{x}(t) &= -L_x x(t) - \beta L_x \dot{x}(t), \\
x(0) &= x_0, \\
\dot{x}(0) &= \dot{x}_0,
\end{align*}
\]

where $L_x$ and $\dot{L}_x$ denote the Laplacian matrices of the position graph $G_x$ and the velocity graph $\dot{G}_x$, respectively, and $\beta > 0$. System $(S_2)$ can be written as a first-order system in a straightforward fashion:

\[
\begin{pmatrix}
\dot{x}(t) \\
\ddot{x}(t)
\end{pmatrix} =
\begin{pmatrix}
0_{n \times n} & I_n \\
-L_x & -\beta L_x
\end{pmatrix}
\begin{pmatrix}
x(t) \\
\dot{x}(t)
\end{pmatrix},
\]

\[
\begin{pmatrix}
x(0) \\
\dot{x}(0)
\end{pmatrix} =
\begin{pmatrix}
x_0 \\
\dot{x}_0
\end{pmatrix}.
\]

This system has no external inputs. Its behaviour is determined only by the initial condition and the properties of the matrix $L$, which in turn depend on the properties of the Laplacian matrices $L_x$ and $\dot{L}_x$ and on the value of the parameter $\beta$.

As system $(S_3)$ is linear, we know from Lemma A.4 on page 135 of the appendix that it is asymptotically stable if and only if all eigenvalues of $L$ have negative real parts. An analogous condition can be derived for consensus stability. After obtaining some supporting results, it will be presented at the end of this chapter. First we observe in Section 5.1 that the matrix $L$ is the companion matrix of a quadratic matrix polynomial. Using this connection, we derive the kernel of $L$ for both directed and undirected graphs in Section 5.2. We obtain one of the main results of this thesis in Theorem 5.4, namely that the nullspace of the matrix $L$ is completely determined by the connectivity of the graphs $G_x$ and $\dot{G}_x$. With this result, necessary and sufficient conditions for consensus stability and asymptotic consensus stability of the system $(S_2)$ are given in Theorem 5.6 in Section 5.3.
The graph theoretic definitions and results used throughout this chapter are found in Chapter 3, the notation in Chapter 2 and linear algebraic terms and lemmas in Appendix A.1. Classic control theory notions and results are listed in Appendix A.2. Remember that to improve readability, the signs \( \vec{\leftrightarrow} \) and \( \vec{\leftrightarrow} \) as well as \( \leftrightarrow \) and \( \leftrightarrow \) in the margins indicate whether a result holds for (weighted or unweighted) digraphs or only (weighted or unweighted) undirected graphs. Furthermore, remember that when writing unweighted graphs, the explicit mention of the weight function is omitted, and \( G = (V, E) \) is written instead of \( G = (V, E, w) \).

5.1 **Eigenvectors and Eigenvalues of \( \mathcal{L} \)**

In this section, we use methods from the previous chapters, as well as from matrix polynomial theory in order to establish some general properties of the system matrix \( \mathcal{L} \). These results will then be used to establish one of the main results of this thesis in Theorem 5.6 in Section 5.3. In this section, we would like to obtain results that hold for the most general case, i.e. the case of arbitrary digraphs. Thus, the matrices \( L_x \) and \( L_\dot{x} \) are the Laplacians of the corresponding communication graphs \( G_x \) and \( G_\dot{x} \), that can be directed or undirected, weighted or unweighted, and may consist of several connected components or reaches.

First, we derive an analytic formulation of the eigenvalues and eigenvectors of \( \mathcal{L} \). The matrix \( \mathcal{L} \) is the companion matrix of the quadratic matrix polynomial (QMP)

\[
P(\lambda) = \lambda^2 \mathbf{I} + \lambda \beta L_\dot{x} + L_x, \quad \lambda \in \mathbb{C},
\]

cf. Section A.1.6 on page 133 of the appendix. Thus, we can apply methods for matrix polynomial theory. A good overview of the theory is given in Tisseur and Meerbergen, [96], while a more in-depth introduction is available in the book by Gohberg et al., [38].

In the special case that the graphs are undirected, the corresponding Laplacian matrices are symmetric positive semi-definite, and \( P(\lambda) \), given in (5.1), is a self-adjoint quadratic matrix polynomial. Self-adjoint QMP with positive definite matrices as coefficients have been thoroughly studied in literature due to their regular occurrence in structural mechanics. Fewer results are available on QMP with more general coefficients.

Bilir and Chicone, [12] extend the existing results to QMP with symmetric but semi-definite M-matrix\(^1\) coefficients. However, their result

\(^1\) A matrix \( A \in \mathbb{R}^{n \times n} \) is an M-matrix, if it is weakly diagonally dominant and all of its off-diagonal elements are non-positive. An undirected graph Laplacian is an M-matrix. An overview of properties of M-matrices as well as possible other definitions can be found in Poole, [77].
does not cover the problem under consideration. Guo, [45] studies the solution of the matrix equation
\[ X^2 - EX - F = 0_{n \times n}, \] (5.2)
where \( X \) is an \( n \times n \) unknown matrix, \( E \) is diagonal \( n \times n \) and \( F \) is an \( n \times n \) M-matrix. However, the system (52) does not satisfy the formulated conditions on the matrices \( E, F \).

To the best of our knowledge, there are no other related results that focus on matrix polynomials with M-matrix or symmetric semi-definite coefficients. In fact, no research is available on quadratic matrix polynomials with (possibly not symmetric) Laplacian matrices as coefficients. Thus, the results in this section not only serve the purpose of laying the foundation for Theorem 5.6, but are also an extension of the existing theory to graph Laplacians. Apart from the symmetric matrices arising from undirected graphs, we also study the properties of a matrix polynomial with directed graph Laplacians as coefficients. We show that in both cases, the kernel of \( L \) depends only on the connectivity of the corresponding graphs.

As explained in Section A.1.6 on page 133 of the appendix, \( \lambda_0 \) is an eigenvalue of \( P(\lambda) \) if and only if \( \det(P(\lambda_0)) = 0 \). The corresponding right eigenvectors \( v \) are vectors in \( \mathbb{C}^n \) satisfying the equation
\[ P(\lambda_0)v = 0_n. \] (5.3)
Analogously, the left eigenvectors \( w \in \mathbb{C}^n \) satisfy the equation
\[ w^TP(\lambda_0) = 0_n. \] (5.4)

Since its companion matrix \( L \) is real, the eigenvalues of \( P(\lambda) \) are either real or arise in complex conjugate pairs. Moreover, the corresponding eigenvectors are real or complex conjugate pairs as well. Let \( v \in \mathbb{C}^n \) be a right eigenvector of \( P(\lambda) \). Multiplying equation (5.3) by \( v^* \) from the left we obtain
\[ \lambda_0^2 v^*v + \lambda_0 \beta v^*Lxv + v^*Lxv = 0. \] (5.5)
This equation is quadratic and has two solutions, given by
\[ \lambda_{1/2} = -\beta v^*Lxv \pm \sqrt{\beta^2(v^*Lxv)^2 - 4(v^*v)(v^*Lxv)} \]
\[ 2v^*v \] (5.6)
for any right eigenvector \( v \). Clearly, the same applies if \( v^*P(\lambda) = 0_n \), i.e., (5.6) can be obtained using the left eigenvectors of \( P(\lambda) \) as well. Note that the converse does not hold. A pair \( (\lambda_0, v_0) \) that solves (5.6) is not necessarily an eigenpair of \( P(\lambda) \).

The eigenvectors of the matrix \( L \) are related to those of the QMP \( P(\lambda) \) in the following way.

---

2 Although technically we are dealing with a function \( \lambda \rightarrow P(\lambda) \), we use \( P(\lambda) \) to denote the QMP. This is the common notation, see, for example, Gohberg et al., [38].
Lemma 5.1 (Gohberg et al., [38]) Let \( P(\lambda) \) be a quadratic matrix polynomial and \( L = \begin{pmatrix} 0 & 1 \\ -L_x & -\beta L_x \end{pmatrix} \) be the corresponding companion matrix with an eigenvalue \( \lambda_0 \in \mathbb{C} \). The following two statements are equivalent:

- \( P(\lambda_0) \) has a right eigenvector \( v \) and left eigenvector \( w \) affording \( \lambda_0 \).
- \( L \) has a right eigenvector \( (v^T, \lambda_0 v^T)^T \) and left eigenvector \( (w^T, (\lambda_0 I + \beta L_x) w^T)^T \) affording \( \lambda_0 \).

**Proof:** Suppose that \( (v_1^T, v_2^T)^T \in \mathbb{C}^{2n} \) is a right eigenvector of \( L \). Then,

\[
\begin{pmatrix} 0 & 1 \\ -L_x & -\beta L_x \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda_0 v_1 \\ \lambda_0 v_2 \end{pmatrix} \tag{5.7}
\]

leads to the following set of equations

\[
v_2 = \lambda_0 v_1, \tag{5.8a}
\]

\[
-L_x v_1 - \beta L_x v_2 = \lambda_0 v_2. \tag{5.8b}
\]

Therefore, it holds that

\[
(\lambda_0^2 I + \lambda_0 \beta L_x + L_x) v_1 = 0_n. \tag{5.9}
\]

Thus, \( (v_1^T, \lambda_0 v_1^T)^T \) is a right eigenvector of \( L \). Analogously, taking \( (w_1^T, w_2^T)^T \in \mathbb{C}^n \) as a left eigenvector we obtain

\[
\begin{pmatrix} w_1^T & w_2^T \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -L_x & -\beta L_x \end{pmatrix} = \begin{pmatrix} \lambda_0 w_1^T \\ \lambda_0 w_2^T \end{pmatrix}. \tag{5.10}
\]

Analogously,

\[
-w_2^T L_x = \lambda_0 w_1^T, \tag{5.11a}
\]

\[
w_1^T - \beta w_2^T L_x = \lambda_0 w_2^T, \tag{5.11b}
\]

and, therefore,

\[
w_2^T (\lambda_0^2 I + \lambda_0 \beta L_x + L_x) = 0_n^T. \tag{5.12}
\]

This implies that \( w_2 = w \) and \( w_1^T = w_2^T (\lambda_0 I + \beta L_x) \). This concludes the proof. \( \square \)

In other words, if a vector \( (v_1^T, v_2^T)^T \) is a right eigenvector of \( L \), the vectors \( v_1, v_2 \) differ in length but not in direction.

### 5.2 Zero Eigenvalues and Kernel of \( L \)

An analytic formulation of the kernel of \( L \) will be necessary to state the convergence result in Theorem 5.6. First, we state the following result, which is a special case of Lemma 5.1.
Lemma 5.2 (Kernel of $\mathcal{L}$) Let $\mathcal{L} = \left( \begin{array}{cc} 0 & 1 \\ -L_x & -\beta L_x \end{array} \right)$ be given by (S3) on page 47, where $L_x$ is the Laplacian of the weighted digraph $G_x$, $L_x$ is the Laplacian of the weighted digraph $G_{\dot{x}}$ and $\beta > 0$. Then, $\lambda_0 = 0$ is an eigenvalue of $\mathcal{L}$. Furthermore, its geometric multiplicity is equal to the geometric multiplicity of zero as an eigenvalue of $L_x$. All the corresponding right eigenvectors are then given by $(v^T, \mathbb{Q}_n^T)^T$, and all the corresponding left eigenvectors are given by $(\beta w^T L_{\dot{x}}, w^T)^T$, where $v$ is a right and $w$ a left eigenvector of $L_x$ affording the eigenvalue zero.

Proof: Inserting $\lambda_0 = 0$ in equation (5.7) leads to the eigenproblem

$$
\begin{pmatrix}
0 & 1 \\
-L_x & -\beta L_x
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
=
\begin{pmatrix}
\mathbb{Q}_n \\
\mathbb{Q}_n
\end{pmatrix}.
$$

(5.13)

It follows directly that $v_2 = \mathbb{Q}_n$ and

$$
L_x v_1 = \mathbb{Q}_n
$$

(5.14)

must hold. If $L_x$ has a zero eigenvalue with the algebraic multiplicity $k$, then by Lemma 3.5 there are $k$ linearly independent eigenvectors $v_i$, $i = 1, \ldots, k$, of $L_x$ satisfying equation (5.14). Thus, there are $k$ linearly independent eigenvectors $(v_i^T, \mathbb{Q}_n^T)^T$ in the kernel of $\mathcal{L}$ and the geometric multiplicities of zero as the eigenvalue of $L_x$ and $\mathcal{L}$ coincide.

The corresponding properties for the left eigenvectors can be derived directly from Lemma 5.1 by inserting $\lambda_0 = 0$ in equation (5.11).

Some of the eigenvectors obtained in Lemma 5.2 depend on the shape of the corresponding graph $G_x$. There is, however, one right eigenvector in the kernel of $\mathcal{L}$ that does not depend on the form of the communication graphs. The following result holds for all double integrator consensus systems of the form (S3).

Lemma 5.3 Let $\mathcal{L} = \left( \begin{array}{cc} 0 & 1 \\ -L_x & -\beta L_x \end{array} \right)$ be given by equation (S3) on page 47, where $L_x$ is the Laplacian of the weighted digraph $G_x$, $L_x$ is the Laplacian of the weighted digraph $G_{\dot{x}}$ and $\beta > 0$. Then $\lambda_0 = 0$ is an eigenvalue of $\mathcal{L}$. Moreover, there exists a corresponding right Jordan chain of length two given by

$$
\begin{pmatrix}
1_n \\
\mathbb{Q}_n
\end{pmatrix}, \begin{pmatrix}
\mathbb{Q}_n \\
1_n
\end{pmatrix}
$$

(5.15)

and a corresponding left Jordan chain given by

$$
\begin{pmatrix}
\beta L_x^T p \\
p
\end{pmatrix}, \begin{pmatrix}
\beta L_x^T q + p \\
q
\end{pmatrix}.
$$

(5.16)

Here, $p \in \mathbb{R}^n$ satisfies $p^T L_x = \mathbb{Q}_n^T$ and $q \in \mathbb{R}^n$ satisfies $-q^T L_x = \beta p^T L_x$. 

Proof: Every Laplacian matrix has a right eigenvector $1_n$ affording the zero eigenvalue. Thus, the first part of this lemma is a direct consequence of Lemma 5.2.

Remember that a vector $v_1 \in \mathbb{C}^n$ is a generalised eigenvector of a matrix $A \in \mathbb{R}^{n \times n}$ corresponding with the eigenvector $v_0 \in \mathbb{C}^n$ and the eigenvalue $\lambda_0$ if and only if it satisfies

$$ (A - \lambda_0 I)v_1 = v_0. \tag{5.17} $$

The Jordan chain has length two if additionally there is no nonzero vector $v_2 \in \mathbb{C}^n$ that satisfies

$$ (A - \lambda_0 I)v_2 = v_1. \tag{5.18} $$

Analogously, $w_1 \in \mathbb{C}^n$ is a left generalised eigenvector corresponding with the left eigenvector $w_0 \in \mathbb{C}^n$, if it satisfies

$$ w_1^T(A - \lambda_0 I) = w_0^T. \tag{5.19} $$

The vector $1_n$ is a right eigenvector of $L_x$, therefore, by Lemma 5.2, the vector $(1_n^T, 0_n^T)^T$ is a right eigenvector of $L$. As $L_x$ is a Laplacian matrix, $L_x1_n = 0_n$ holds. Thus, we see directly that the vector $(0^T_n, 1_n^T)$ satisfies

$$ \begin{pmatrix} 0 & I \\ -L_x & -\beta L_x \end{pmatrix} \begin{pmatrix} 0_n \\ 1_n \end{pmatrix} = \begin{pmatrix} 1_n \\ 0_n \end{pmatrix}. \tag{5.20} $$

It follows that $(0_n^T, 1_n^T)^T$ is a generalised right eigenvector of $L$. Furthermore, if this Jordan chain has a length greater than two, then

$$ \begin{pmatrix} 0 & I \\ -L_x & -\beta L_x \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} 0_n \\ 1_n \end{pmatrix}. \tag{5.21} $$

must have a solution $(\tilde{w}^T, \tilde{u}^T)^T \in \mathbb{C}^{2n}$. Then, it must hold that $\tilde{u} = 0_n$ and $L_x\tilde{w} = 1_n$. We have shown in Lemma 3.8 on page 24 that such a $\tilde{w}$ does not exist. Therefore, the computed Jordan chain has exactly length two.

For every right Jordan chain a corresponding left Jordan chain of the same length must exist. Therefore, there is also a generalised left eigenvector of $L$ corresponding with one of its left eigenvectors$^3$ that is not orthogonal on $(1_n^T, 0_n^T)$. From Lemma 5.2 we know that every left eigenvector of $L$ has the form $(\beta p^T L_x, p^T)^T$. Here, $p \in \mathbb{R}^n$ satisfies $p^T L_x = \beta 0_n^T$. It follows by direct computation that the vector $(\beta q^T L_x + p^T, q^T)^T$ satisfies

$$ \begin{pmatrix} q^T \beta L_x + p^T \\ q^T \end{pmatrix} \begin{pmatrix} 0 & I \\ -L_x & -\beta L_x \end{pmatrix} = \begin{pmatrix} \beta p^T L_x \\ p^T \end{pmatrix}. \tag{5.22} $$

$^3$Following Lemma 5.2, the left eigenvector in the kernel of $L$ is unique if $L_x$ consists of one reach. Otherwise, $L_x$ has a left eigenvector corresponding to the right eigenvector $1_n^T$. The left eigenvector of $L$ that corresponds to this left eigenvector of $L_x$ is chosen.
where \( q \) satisfies
\[
- q^T L_x = \beta p^T L_x.
\] (5.23)

This concludes the proof. \( \square \)

We can now state the main result of this section.

**Theorem 5.4** Let \( \mathcal{L} = (L_x - \beta L_\dot{x}) \) be given by equation (5.3) on page 47, where \( L_x \) is the Laplacian of the weighted digraph \( G_x \), \( L_\dot{x} \) is the Laplacian of the weighted digraph \( G_\dot{x} \) and \( \beta > 0 \).

Zero is an eigenvalue of \( \mathcal{L} \) with geometric multiplicity \( k \), where \( k \) is the number of reaches of \( G_x \).

It has algebraic multiplicity \((k + m)\), \( m \geq 1 \), if and only if \( G := G_x \cup G_\dot{x} \) has \( m \) reaches.

The corresponding Jordan chains have lengths two and one.

**Proof:** The proof of Theorem 5.4 has the following structure.

1. First, we show that zero is an eigenvalue of \( \mathcal{L} \) with geometric multiplicity \( k \) if and only if \( G_x \) has \( k \) reaches.

2. Then, we show that if \( G_x \cup G_\dot{x} \) has \( m > 1 \) reaches, then the algebraic multiplicity of zero is exactly \( k + m \). We perform the following steps.

   a) First, we show that if \( G_x \cup G_\dot{x} \) has \( m \) reaches, then the algebraic multiplicity of zero is at least \( k + m \). We do this by directly computing the corresponding left generalised eigenvectors.

   b) Then, we show that the computed Jordan chains have exactly length two.

   c) Then, we show that if \( G_x \) has an additional reach, that is not a reach of \( G_x \cup G_\dot{x} \), then there are no generalised eigenvectors corresponding to the eigenvectors. Together with step 2b we can conclude that the algebraic multiplicity of zero is exactly \( k + m \).

3. Finally, we show that if the algebraic multiplicity of zero is \( k + m \) only if \( G_x \cup G_\dot{x} \) has \( m \) reaches.

**Step 1.** The Laplacian matrix \( L_x \) has a zero eigenvalue with \( k \) corresponding Jordan blocks if and only if the corresponding graph has \( k \) reaches. Thus, the first part of this lemma is a direct consequence of Lemma 5.2.

**Step 2a.** Suppose now that \( G = G_x \cup G_\dot{x} \) has \( m > 1 \) reaches. Then, clearly, \( G_x \) and \( G_\dot{x} \) have at least \( m > 1 \) reaches. Without loss of generality, let \( m = 2 \) and let the reaches of \( G \) have exclusive parts of size \( m_1 \) and \( m_2 \) and a common part of size \( c := n - m_1 - m_2 \). Then,
assuming appropriate node numbering, the matrices $L_x$ and $L_x$ have
the form

$$
L_x = \begin{pmatrix}
1 & 0 & 0 \\
0 & L_{x22} & 0 \\
L_{x31} & L_{x32} & L_{x33}
\end{pmatrix}, \quad L_x = \begin{pmatrix}
1 & 0 & 0 \\
0 & L_{x22} & 0 \\
L_{x31} & L_{x32} & L_{x33}
\end{pmatrix}, \quad (5.24)
$$

where $L_{x11}, L_{x11} \in \mathbb{R}^{m_1 \times m_1}$, $L_{x22}, L_{x22} \in \mathbb{R}^{m_2 \times m_2}$, $L_{x33}, L_{x33} \in \mathbb{R}^{c \times c}$
and the remaining blocks are dimensioned accordingly. Furthermore,
the matrix $L$ has the form

$$
L = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-1 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & -L_{x11} & 0 & -1 & 0 \\
0 & -L_{x22} & 0 & 0 & -1 & 0 \\
-L_{x31} & -L_{x32} & -L_{x33} & -1 & 0 & 0
\end{pmatrix} \quad (5.25)
$$

The matrices $L_{x11}$ and $L_{x22}$ are Laplacian matrices of the subgraphs
induced by the exclusive parts of the reaches of $G_x \cup G_x$. Therefore,
by Lemma 3.6, they each have a zero eigenvalue of geometric multiplicity
at least one, with the corresponding non-negative left eigenvectors
$p_1 \in \mathbb{R}^{m_1}$ and $p_2 \in \mathbb{R}^{m_2}$. For $i = 1, 2$, consider the subsystems
given by $L_i := \begin{pmatrix}
-L_{x11} & 0 \\
0 & -L_{x22}
\end{pmatrix}$. By Lemma 5.2, matrices $L_i$ have the
left eigenvectors $(\beta p_i^T L_{x11}, p_i^T)$ satisfying the eigenvalue zero. By
Lemma 5.2, the corresponding generalised eigenvectors are given by
$(q_i^T \beta L_{x11} + p_i^T, q_i^T)$, where $q_i$ satisfies (5.23), namely

$$
-L_{x11} q_i = \beta L_{x11} q_i.
$$

The matrices $L_i$ are submatrices of $L$. It can be checked by computation
that the vectors

$$
\begin{pmatrix}
p_1^T \beta L_{x11}, & q_1^T & p_2^T & q_2^T & p_1^T & q_1^T
\end{pmatrix}^T \quad \text{and}
\begin{pmatrix}
0, & p_1^T \beta L_{x22}, & q_1^T & p_2^T & q_2^T
\end{pmatrix}^T
$$

lie in the left nullspace of $L$. They are, furthermore, linearly independent
by construction. The corresponding generalised eigenvectors
have to satisfy equation (5.19). They can also be constructed from
the generalised eigenvectors of the matrices $L_1$ and $L_2$. The generalised
left eigenvectors of $L$ are given by

$$
\begin{pmatrix}
q_1^T \beta L_{x11} + p_1^T, & q_1^T & q_2^T & q_2^T
\end{pmatrix}^T
\begin{pmatrix}
0, & q_1^T \beta L_{x22} + p_2^T, & q_1^T & q_2^T
\end{pmatrix}^T
$$

These two generalised eigenvectors are linearly independent. Thus,
we denote two linearly independent generalised eigenvectors exist, i.e. the algebraic
multiplicity of zero as an eigenvalue of $L$ is at least $k + 2$, where $k$
is the number of reaches of $G_x \cup G_x$. 


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Clearly, the above argumentation can be straightforwardly extended for \( m > 2 \).

**Step 2b.** We now show that the Jordan chains computed in the previous step have exactly length two. In this step of the proof we use the right eigenvectors and generalised eigenvectors of \( L \), that we now compute. Without loss of generality, we again assume that \( G_x \cup G_\hat{x} \) consists of two reaches. Following Lemma 5.3, a choice of right eigenvectors of \( L \) corresponding to the zero eigenvalue is given by \( (v^T, \varnothing^T)^T \) where \( v \) is an eigenvector of \( L_x \) affording the eigenvalue zero. Therefore, \( v \) is a linear combination of the eigenvectors in the kernel of \( L_x \), that are given by (3.16). As \( G_x \) has at least two reaches, \( v \) can be chosen as

\[
v := \begin{pmatrix} a_1 1_{m_1} \\ a_2 1_{m_2} \\ a_1 r_1 + a_2 r_2 \end{pmatrix},
\]

(5.28)

where

\[
\cdot r_1, r_2 \in [0, 1]^c \quad \text{(Note that as \( G_x \) may have further reaches, some entries of \( r_1 \) or \( r_2 \) can be zero or one. This, however, has no further impact on this part of the proof as \( r_1 \) and \( r_2 \) are never used explicitly)},
\]

\[
\cdot c \text{ is the size of the common part of the graph } G = G_x \cup G_\hat{x},
\]

\[
\cdot m_1, m_2 \text{ are the sizes of the exclusive parts, and}
\]

\[
\cdot a_1, a_2 \text{ are constants in } \mathbb{R}.
\]

The corresponding additional generalised right eigenvector of \( L \) must satisfy equation (5.17). It can be verified by direct computation that

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -L_{x11} & 0 & -\beta L_{x11} & 0 & 0 \\
0 & -L_{x22} & 0 & 0 & -\beta L_{x22} & 0 \\
-L_{x31} & -L_{x32} & -L_{x33} & -\beta L_{x31} & -\beta L_{x32} & -\beta L_{x33}
\end{pmatrix}
\begin{pmatrix}
\alpha_3 1_{m_1} \\
\alpha_4 1_{m_2} \\
\zeta_3 \\
\alpha_1 1_{m_1} \\
\alpha_2 1_{m_2} \\
\xi_1 + \xi_2 r_2
\end{pmatrix}
= \begin{pmatrix}
a_1 1_{m_1} \\
a_2 1_{m_2} \\
\zeta_3 \\
\xi_1 r_1 + a_2 r_2 \\
\xi_1 \\
\xi_2 r_2
\end{pmatrix}
\]

(5.29)

holds. Here, \( a_3, a_4 \) are constants in \( \mathbb{R} \) and \( \zeta_3 \in \mathbb{R}^c \) is some vector that we do not need to compute explicitly.

We will use this generalised eigenvector in order to show by contradiction that the Jordan chains of \( L \) have exactly length two. Without loss of generality, let \( m = 2 \) again. Suppose the corresponding Jordan chains are of at least length three. Then the equation

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -L_{x11} & 0 & -\beta L_{x11} & 0 & 0 \\
0 & -L_{x22} & 0 & 0 & -\beta L_{x22} & 0 \\
-L_{x31} & -L_{x32} & -L_{x33} & -\beta L_{x31} & -\beta L_{x32} & -\beta L_{x33}
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4 \\
w_5 \\
w_6
\end{pmatrix}
= \begin{pmatrix}
a_3 1_{m_1} \\
a_4 1_{m_2} \\
\zeta_3 \\
\alpha_1 1_{m_1} \\
\alpha_2 1_{m_2} \\
\xi_1 r_1 + a_2 r_2
\end{pmatrix}
\]

is
must have a solution \((\hat{w}^T_1, \hat{w}^T_2, \hat{w}^T_3, \hat{u}^T_1, \hat{u}^T_2, \hat{u}^T_3)^T \in \mathbb{R}^{2n}\). Here, \(w_1, u_1, \hat{w}_1, \hat{u}_1 \in \mathbb{R}^{m_1}, w_2, u_2, \hat{w}_2, \hat{u}_2 \in \mathbb{R}^{m_2}\) and \(w_3, u_3, \hat{w}_3, \hat{u}_3 \in \mathbb{R}^c\).

We obtain a set of six equations. From the first block row of \((5.30)\) we see that

\[
\hat{u}_1 = a_3 1_{m_1} \tag{5.31a}
\]

holds. Inserting \(\hat{u}_1\), the fourth block row then reads

\[
-L_{x_{11}} w_1 - \beta a_3 L_{x_{11}} 1_{m_1} = a_1 1_{m_1}, \tag{5.31b}
\]

As \(L_{x_{11}}\) is a directed graph Laplacian, \(L_{x_{11}} 1_{m_1} = 0_{m_1}\) holds, and \((5.31b)\) reduces to

\[
-L_{x_{11}} w_1 = a_1 1_{m_1}, \tag{5.31c}
\]

which by Lemma 3.8 on page 24 has no solution. As before, this argumentation can be straightforwardly extended to \(m > 2\).

We have now shown that if \(G = G_x \cup G_x\) consists of \(m\) reaches, there are \(m\) Jordan chains of length exactly two corresponding to the zero eigenvalue of \(L\).

**Step 2c.** If \(G_x\) has a reach that is not a reach of \(G_x \cup G_x\), then \(k > m\) holds. We now show that if \(k > m\), the remaining \((k - m)\) Jordan chains have length one. Let \(G_x\) have a reach that is not a reach in \(G_x \cup G_x\). Without loss of generality, suppose again that \(G_x\) has two reaches, but \(G_x \cup G_x\) has one reach. Assuming appropriate node numbering, \(L_x\) has the form

\[
L_x = \begin{pmatrix}
L_{x_{11}} & 0 & 0 \\
0 & L_{x_{22}} & 0 \\
L_{x_{31}} & L_{x_{32}} & L_{x_{33}}
\end{pmatrix}, \tag{5.32}
\]

where \(L_{x_{11}} \in \mathbb{R}^{m_1 \times m_1}, L_{x_{22}} \in \mathbb{R}^{m_2 \times m_2}, L_{x_{33}} \in \mathbb{R}^c \times c\), and the other blocks are dimensioned accordingly. Let \(V\) be the node set of \(G\) and denote the sets of nodes corresponding to the exclusive parts of the two reaches and their common part as \(V_1, V_2, V_3\), respectively. As there is no corresponding reach in \(G = G_x \cup G_x\), there must be an edge in \(G_x\) that goes either from a node from \(V \setminus V_2\) to \(V_2\) or from \(V \setminus V_1\) to \(V_1\). Therefore, the corresponding partitioning of \(L_x\) is given by

\[
L_x = \begin{pmatrix}
L_{x_{11}} & L_{x_{12}} & L_{x_{13}} \\
L_{x_{21}} & L_{x_{22}} & L_{x_{23}} \\
L_{x_{31}} & L_{x_{32}} & L_{x_{33}}
\end{pmatrix}, \tag{5.33}
\]
5.2 Zero Eigenvalues and Kernel of $\mathcal{L}$

There is an edge in $G_\hat{x}$ from $V_3$ to $V_1'$. The matrix $L_{\hat{x}13}$ has a non-zero entry.

There is an edge in $G_\hat{x}$ from $V_2$ to $V_1'$. The matrix $L_{\hat{x}12}$ has a non-zero entry.

Figure 5.1: Illustration of Step 2c in the proof of Theorem 5.4. In both cases $G_\hat{x}$ consists of two reaches but $G_\hat{x} \cup G_\hat{x}$ is connected. $V_1'$ indicates the iSCC of $V_1$.

where at least one of the blocks $L_{\hat{x}12}$, $L_{\hat{x}13}$, $L_{\hat{x}21}$, $L_{\hat{x}23}$ is not identical to zero. Therefore, the corresponding block $L_{\hat{x}11}$ or $L_{\hat{x}22}$ is not identical to zero either. Remember that each reach of $G_\hat{x}$ has a corresponding independent strongly connected component. If $G_\hat{x} \cup G_\hat{x}$ consists of one reach, then there is an incoming edge in $G_\hat{x}$ either to a node in the iSCC of the first reach or of the second reach of $L_\hat{x}$. Let $V_1'$ denote the iSCC of the first reach of $L_\hat{x}$. Suppose that it has size $m_1'$.

Without loss of generality, we assume\(^4\) that there is an edge in $G_\hat{x}$ from a node not in $V_1$ to a node in $V_1'$. This assumption implies that $L_{\hat{x}12}$ or $L_{\hat{x}13}$ has a non-zero entry in the corresponding row. It furthermore implies that $L_{\hat{x}11}$ also has a non-zero entry in the corresponding row and that the corresponding entry of the vector $L_{\hat{x}11} \mathbf{1}_{m_1}$ is positive. We denote this assumption by $(\ast)$. This is illustrated in Figure 5.1.

As before, we know that $L_\hat{x}$ has a right eigenvector $\mathbf{v}^T := (a_1 \mathbf{1}_{m_1}^T, a_2 \mathbf{1}_{m_2}^T, a_1 \mathbf{r}_1^T + a_2 \mathbf{r}_2^T)$ affording the eigenvalue zero, where

- $\mathbf{r}_1, \mathbf{r}_2 \in (0, 1)^c$ are positive vectors that satisfy $\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{1}_c$,
- $a_1 \neq a_2$ are real constants.

From Lemma 5.2 we know that the corresponding eigenvector of $\mathcal{L}$ affording the zero eigenvalue is given by $(\mathbf{v}^T, 0^T)\mathbf{1}_n$. There are no Jor-

\(^4\) If this assumption is not true, then the same assumption must hold true for the iSCC of the second reach. Then, the nodes in the graph can be renamed accordingly and Assumption 1 holds anew.
We now use (•) to show that if $G_x \cup G_x$ consists of one reach, then (5.35b) has no solution, which implies that (5.34) has no solution. This, in turn, implies there is no generalised eigenvector of $L$ corresponding with the vector $(v^T, \mathcal{D}_n^T)$ if $G_x \cup G_x$ consists of one reach.

First, we partition the matrix $L_{x_{11}}$ according to its iSCC. Remember that we denote the exclusive part of the corresponding reach by $V_1$ and its iSCC by $V'_1$ and say that the reach consists of $m_1$ nodes and the iSCC of $m'_1$ nodes. Assuming appropriate node numbering, the corresponding partitioning is given by

$$L_{x_{11}} = \begin{pmatrix} L_{x_{11}^1} \\ L_{x_{11}^2} \end{pmatrix}, \quad L_{x_{11}^1} = \begin{pmatrix} L_{x_{11}^1_{11}} & 0_{m'_1 \times (m_1 - m'_1)} \end{pmatrix},$$

where

- $L_{x_{11}^1_{11}} \in \mathbb{R}^{m'_1 \times m'_1}$ corresponds to $V'_1$,
- $L_{x_{11}^1} \in \mathbb{R}^{m'_1 \times m'_1}$ is the Laplacian of $V'_1$,
- $L_{x_{11}^2} \in \mathbb{R}^{(m_1 - m'_1) \times m_1}$ describes the edges from $V'_1$ to the other nodes in $V_1$ and the remaining edges in $V_1 \setminus V'_1$.

The corresponding partitioning of $L_{x_{11}}$ and $L_{x_{13}}$ is given by

$$L_{x_{11}} = \begin{pmatrix} L_{x_{11}^1} \\ L_{x_{11}^2} \end{pmatrix}, \quad L_{x_{12}} = \begin{pmatrix} L_{x_{12}^1} \\ L_{x_{12}^2} \end{pmatrix}, \quad L_{x_{13}} = \begin{pmatrix} L_{x_{13}^1} \\ L_{x_{13}^2} \end{pmatrix}.$$ (5.37)

We partition the unknown vector $w_1$ accordingly, that is $w_1^T := (w_{11}^T, w_{12}^T)$. Then, equation (5.35b) has a solution only if its first block row, given by

$$L_{x_{11}} w_{11} = -\beta \left( a_1 L_{x_{11}^1_{11}} 1_{m_1} + a_2 L_{x_{12}^1_{12}} 1_{m_2} + L_{x_{13}^1} (a_1 r_1 + a_2 r_2) \right)$$ (5.38)
As $L_x$ is a Laplacian matrix, it has zero row-sums. Thus, it holds that

$$L_{x_1}^1 I_{m_1} = -L_{x_1}^1 I_{m_2} - L_{x_1}^1 I_c.$$  \hspace{1cm} (5.39)

Using (5.39) we can replace $L_{x_1}^1 I_{m_1}$ in (5.38) to obtain

$$L_{x_1}^1 w_1 = -\beta \left( (a_2 - a_1) L_{x_1}^2 I_{m_2} - a_1 L_{x_1}^3 (I_c - r_1) + a_2 L_{x_1}^3 r_2 \right).$$  \hspace{1cm} (5.40a)

Inserting $r_1 + r_2 = I_c$, (5.40a) becomes

$$L_{x_1}^1 w_1 = -\beta \left( (a_2 - a_1) L_{x_1}^2 I_{m_2} + (a_2 - a_1) L_{x_1}^3 r_2 \right).$$  \hspace{1cm} (5.40b)

Finally, we can simplify the above to

$$L_{x_1}^1 w_1 = -\beta (a_2 - a_1) \left( L_{x_1}^2 I_{m_2} + L_{x_1}^3 r_2 \right).$$  \hspace{1cm} (5.40c)

By (*)& at least one row of either $L_{x_1}^2$ or $L_{x_1}^3$ or both is not identically zero. Furthermore, all entries in the two blocks are non-positive. As all entries of $r_2$ are positive, it follows that the vector $L_{x_1}^2 I_{m_2} + L_{x_1}^3 r_2$ has only negative or zero entries, and at least one entry is strictly negative.

The matrix $L_{x_1}^3$ on the left-hand side of (5.40c) is the Laplacian of an isCC. By Lemma 3.6 it has a positive left eigenvector $u \in \mathbb{R}^{m_1}$ affording the eigenvalue zero. Pre-multiplying (5.40c) by $u^T$, we obtain

$$0 = -\beta u^T \left( L_{x_1}^2 I_{m_2} + L_{x_1}^3 r_2 \right) (a_2 - a_1).$$  \hspace{1cm} (5.40d)

As $a_1 \neq a_2$ by assumption, the right-hand side of this equation is zero if and only if

$$0 = u^T \left( L_{x_1}^2 I_{m_2} + L_{x_1}^3 r_2 \right)$$  \hspace{1cm} (5.40e)

holds. We have already established that the vector $L_{x_1}^2 I_{m_2} + L_{x_1}^3 r_2$ has only negative or zero entries, and at least one entry is strictly negative. Therefore, premultiplying it with a positive vector will not lead to a zero solution. Thus, the right-hand side of equation (5.40d) is non-zero. As the multiplication with a vector only enlarges the solution space, it follows that equation (5.38) has no solution. This, in turn, proves that if $G_x$ consists of two reaches but $G_x \cup G_x$ consists of one reach, then the corresponding Jordan chain of $\mathcal{L}$ has length one.

As before, the above argument can be straightforwardly extended to $m > 2$. Therefore, if $G = G_x \cup G_x$ has $m$ connected components, there are exactly $m$ Jordan chains of length two corresponding to the eigenvalue zero of $\mathcal{L}$ and all the other Jordan chains have length one.
Step 3. If $\mathcal{L}$ has $m$ Jordan chains of length two corresponding to the zero eigenvalue, there are $m$ pairs of vectors $(v_1, w_1), \ldots, (v_m, w_m)$ satisfying, for all $i = 1, \ldots, m$,

$$L_x w_i = -\beta L_\dot{x} v_i,$$  \hspace{1cm} (5.41)

with $v_i$ given by the set (3.16) on page 21. We have just shown that if $G_x \cup G_\dot{x}$ has less than $m$ reaches, then we cannot find $m$ linearly independent solutions of the above equation. Therefore, it is necessary that $G_x \cup G_\dot{x}$ consists of $m$ reaches. This concludes the proof. \hfill $\square$

Example 5.5 Consider the system given by the graphs in Figure 5.2. The reaches and iSCC of $G_x$ are discussed in Example 3.1 on page 17. As $G_x$ consists of two reaches, it has two right eigenvectors corresponding to the zero eigenvalue, given e.g. by $(1, 1, 1, 1)^T$ and $(1, 1, 0, \frac{1}{2})^T$. The two corresponding left eigenvectors are given by $(1, 0, 0, 0)^T$ and $(0, 0, 1, 0)^T$. Thus, $\mathcal{L}$ has two right eigenvectors corresponding to the zero eigenvalue, given by $u_1 = (1, 1, 1, 1, 0, 0, 0, 0)^T$, $u_3 = (1, 1, 0, \frac{1}{2}, 0, 0, 0, 0)^T$. As $G_x \cup G_\dot{x}$ consists of one reach, there is one generalised eigenvector, $u_2 = (0, 0, 0, 0, 1, 1, 1, 1)^T$. The corresponding left eigenvectors and generalised left eigenvector of $\mathcal{L}$ are $w_1 = (0, 0, 0, 0, 1, 0, 0, 0)^T$, $w_2 = (1, 0, 0, 0, 0, 0, 0, 0)^T$, $w_3 = (-\beta, 0, \beta, 0, 0, 0, 1, 0)^T$.

5.3 NECESSARY AND SUFFICIENT CONDITIONS FOR CONSENSUS STABILITY

Building on the previous results, we can give the main result of this chapter. It contains necessary and sufficient conditions for consensus stability and asymptotic consensus stability of algorithm (S2) on page 47 in directed and undirected networks.

Theorem 5.6 Consider the double integrator consensus problem (S2) on page 47 for $n$ mobile agents, where $\mathcal{L} = \begin{pmatrix} 0 & \beta \\ -L_x & -L_\dot{x} \end{pmatrix}$, the matrices $L_x, L_\dot{x}$ are $n \times n$ Laplacians of the weighted digraphs $G_x, G_\dot{x}$ and $\beta > 0$. The system is

• consensus stable, if and only if
5.3 Necessary and Sufficient Conditions for Consensus Stability

\(- G_x \cup G_x \) consists of one reach, and
\(- all non-zero eigenvalues of \( \mathcal{L} \) have a negative real part.\)

- asymptotically consensus stable, if and only if it is consensus stable and additionally \( G_x \) consists of one reach.

**Proof:** The proof of Theorem 5.6 has the following structure.
1. First, we give the general equation that describes the behaviour of the system (S2) for \( t \to \infty \).
2. Then, we show that if \( G_x \cup G_x \) consists of one reach, but \( G_x \) consists of k reaches, and all non-zero eigenvalues of \( \mathcal{L} \) have a negative real part, then (S2) is consensus stable.
3. Then, we show that if the system is consensus stable and additionally \( G_x \) consists of one reach, then (S2) is asymptotically consensus stable.
4. Then, we show that (S2) is asymptotically consensus stable only if it is consensus stable and \( G_x \) consists of one reach.
5. Finally, we show that (S2) is consensus stable only if \( G_x \cup G_x \) consists of one reach and all non-zero eigenvalues of \( \mathcal{L} \) have a negative real part. In order to do this, we show the following.
   a) We first show that if \( G_x \cup G_x \) does not consist of one reach, then the system (S2) is not consensus stable.
   b) We then show that if \( \mathcal{L} \) has eigenvalues with a positive real part or a purely imaginary eigenvalues pair, then the system (S2) is not consensus stable.

**Step 1.** Let \( \mathcal{J} \) denote the Jordan canonical form of \( \mathcal{L} \), cf. Section A.1.1 on page 129 of the appendix. The matrix \( \mathcal{J} \) is obtained from

\[
\mathcal{J} := V^{-1} \mathcal{L} V = \begin{pmatrix} w_1^T \\ \vdots \\ w_{2n}^T \end{pmatrix} \mathcal{L} \begin{pmatrix} u_1 & \ldots & u_{2n} \end{pmatrix},
\]

where for \( i = 1, \ldots, 2n \),
- the vectors \( u_i \) can be chosen among the right eigenvectors and generalised eigenvectors of \( \mathcal{L} \),
- \( w_i \) are left eigenvectors and generalised eigenvectors of \( \mathcal{L} \) scaled and allocated such that \( w_i^T u_i = 1 \) holds.

From classic control theory, cf. Section A.2.1 on page 134 of the appendix, we know that the solution of (S2) is given by

\[
\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = e^{\mathcal{L} t} \begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix},
\]

where \( e^{\mathcal{L} t} \), cf. Section A.1.2 of the appendix, satisfies

\[
e^{\mathcal{L} t} = e^{\mathcal{J} t} V^{-1} = Ve^{\mathcal{J}_t} V^{-1}.
\]
We will now prove the theorem step by step by explicitly computing (5.43).

**Step 2.** Suppose that all non-zero eigenvalues of $\mathcal{L}$ have a negative real part. Suppose that $G_x\cup G_x^\dot{}$ consists of one reach, but $G_x$ consists of $k$ reaches with exclusive parts of size $k_1, \ldots, k_k$ and common part of size $c = n - \sum_{i=1}^k k_i$. By Theorem 5.4 on page 53 this implies that $\mathcal{L}$ has a zero eigenvalue with algebraic multiplicity $k + 1$ and geometric multiplicity $k$. We denote a basis for the right eigenspace of $L_x$ corresponding to the zero eigenvalue by $v_1, \ldots, v_k$, and let them be given by the basis (3.16) on page 21. By Lemma 5.2, the corresponding eigenvectors and generalised eigenvector of $\mathcal{L}$ can be chosen as

$$u_1 := \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}, u_2 := \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}, u_3 := \begin{pmatrix} v_2 & \cdots & v_k \end{pmatrix}. \quad (5.45)$$

If all the non-zero eigenvalues of $\mathcal{L}$ have negative real part, then the Jordan matrix has the form

$$J = \begin{pmatrix} J_{02} & 0 & 0 \\ 0 & J_0 & 0 \\ 0 & 0 & J_{01} \end{pmatrix}, \quad (5.46)$$

where

$$J_{02} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (5.47)$$

is the block corresponding to the Jordan chain of length two and $J_0 := 0_{(k - 1) \times (k - 1)}$ are the collected Jordan blocks corresponding to the Jordan chains of length one affording the zero eigenvalue. $J_{01}$ are the remaining Jordan blocks, containing eigenvalues with negative real parts. Thus,

$$e^{\mathcal{L}t} = V \begin{pmatrix} e^{J_{02}t} & 0 & 0 \\ 0 & e^{J_{01}t} & 0 \\ 0 & 0 & e^{J_{02}t} \end{pmatrix} V^{-1}. \quad (5.48)$$

holds. We know that

$$e^{J_{01}t} = \text{blockdiag} \left( e^{\lambda_i t} E_i \right), \quad (5.49a)$$

where $E_i$, for $i = k + 1, \ldots, 2n$, are given by (A.6) on page 131 of the appendix. For $t \to \infty$, the value of $e^{J_{01}t}$ tends to zero, as $E_i$ only contains polynomial expressions of $t$. On the other hand, it holds that

$$e^{J_{02}t} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad (5.49b)$$
and

\[ e^{\beta t} = I_{(k-1)}. \]  \hspace{1cm} (5.49c)

Therefore, combining (5.49a)-(5.49c) with (5.42), we obtain for \( t \to \infty \),

\[ e^{\mathcal{L} t} \to \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_1^T \\ w_2^T \end{pmatrix} + (u_3 \ldots u_{k+1}) \begin{pmatrix} w_3^T \\ \vdots \\ w_{k+1}^T \end{pmatrix}. \]  \hspace{1cm} (5.50)

Here and later in the text, the notation \( f(t) \to g(t) \), for two functions \( f, g \), denotes that \( f \) behaves like \( g \) for large \( t \). Inserting \( u_i = (v_i^T, \varnothing_n^T)^T \) (cf. (5.45), remember that \( (w_i)_j \) denotes the \( j \)-th entry of the vector \( w_i \)) in (5.50), we obtain

\[ e^{\mathcal{L} t} \to \begin{pmatrix} 1_n \left( (w_1)_1 + t(w_2)_1 \right) & \ldots & 1_n \left( (w_1)_{2n} + t(w_2)_{2n} \right) \\ 1_n(w_2)_1 & \ldots & 1_n(w_2)_{2n} \end{pmatrix} + \begin{pmatrix} A_{vw} \\ \varnothing_{n \times 2n} \end{pmatrix}, \]  \hspace{1cm} (5.51)

where \( A_{vw} \) is the \( n \times 2n \) matrix given by (remember that \( v_3, \ldots, v_{k+1} \in \mathbb{C}^n \) are right eigenvectors of \( L_x \) affording the zero eigenvalue)

\[ A_{vw} = \begin{pmatrix} v_3 & \ldots & v_{k+1} \end{pmatrix} \begin{pmatrix} w_3^T \\ \vdots \\ w_{k+1}^T \end{pmatrix}. \]  \hspace{1cm} (5.52)

Thus, inserting (5.51) in (5.43) we obtain

\[ \lim_{t \to \infty} \dot{x}(t) = 1_n \sum_{i=1}^{n} \left( (w_2)_i (x_0)_i + (w_2)_{n+i}(x_0)_i \right). \]  \hspace{1cm} (5.53)

Therefore, \( \lim_{t \to \infty} (x_i(t) - x_j(t)) = 0 \) holds for all \( i, j = 1, \ldots, n \). This implies that \( \lim_{t \to \infty} (x_i(t) - \dot{x}_i(t)) = \text{const} \) for all \( i, j = 1, \ldots, n \). Thus, the system is consensus stable.

**Step 3.** We now show that if the system is consensus stable and additionally \( G_x \) consists of one reach, the system (52) is asymptotically consensus stable.

First, note that the condition that \( G_x \) consists of one reach implies that \( G_x \cup G_x \) consists of one reach. This in turn implies by Theorem 5.4 on page 53 that \( \mathcal{L} \) has a zero eigenvalue with algebraic multiplicity two and geometric multiplicity one. By Lemma 5.3 on page 51 the corresponding right Jordan chain is given by

\[ u_1 := \begin{pmatrix} 1_n \\ \varnothing_n \end{pmatrix}, \quad u_2 := \begin{pmatrix} \varnothing_n \\ 1_n \end{pmatrix}. \]  \hspace{1cm} (5.54)
The Jordan matrix now has the form
\[ J := \begin{pmatrix} d_{02} & 0 \\ 0 & J_\square \end{pmatrix}, \] (5.55)

where
\[ d_{02} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \] (5.56)
is the Jordan block corresponding to the zero eigenvalue. Additionally, \( J_\square \) are the remaining Jordan blocks containing eigenvalues with negative real part. Thus,
\[ e^{Lt} \rightarrow (u_1 \ u_2) \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_1^T \\ w_2^T \end{pmatrix} = \begin{pmatrix} 1_n & 0_n \\ 0_n & 1_n \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_1^T \\ w_2^T \end{pmatrix} \]
\[ = \begin{pmatrix} 1_n \((w_1)_1 + t(w_2)_1) & \ldots & 1_n \((w_1)_2n + t(w_2)_2n) \\ 1_n(w_2)_1 & \ldots & 1_n(w_2)_2n \end{pmatrix}. \] (5.57)

Hence, the limit for \( \dot{x} \) is again given by (5.53). Furthermore, as \( t \rightarrow \infty \),
\[ x(t) \rightarrow 1_n \sum_{i=1}^{n} \left( (w_1)_i x_0 \right) + (w_2)_i \dot{x}_0. \] (5.58)

That is, \( \lim_{t \rightarrow \infty} (x_i(t) - x_j(t)) = 0 \) for all \( i, j = 1, \ldots, n \). Thus, the system is asymptotically consensus stable.

Step 4. The fact that system (S2) is asymptotically consensus stable, only if it is consensus stable and additionally \( G_\dot{x} \) consists of one reach. We will show that the system is not consensus stable if \( G_\dot{x} \) consists of more than one reach, then Jordan matrix has the form (5.46) and as \( t \rightarrow \infty \),
\[ x(t) \rightarrow 1_n \sum_{i=1}^{n} \left( (w_1)_i x_0 \right) + (w_2)_i \dot{x}_0 + A_vw x_0, \] (5.59)
where in general \( A_vw x_0 \neq a1_n \) for some \( a \in \mathbb{R} \). That is, no position consensus is achieved and the system is not asymptotically consensus stable.

Step 5. It remains to show that the system (S2) is consensus stable only if all non-zero eigenvalues of \( \mathcal{L} \) have a negative real part and \( G_\dot{x} \cup G_\dot{x} \) consists of one reach. We will show that the system is not consensus stable if \( G_\dot{x} \cup G_\dot{x} \) consists of more than one reach, \( \mathcal{L} \) has
eigenvalues with positive real parts, or if $L$ has purely imaginary eigenvalues.

**Step 5a** Assume first that $G_x \cup G_x$ consists of more than one reach but all the non-zero eigenvalues of $L$ have a negative real part. Without loss of generality, suppose that $G_x \cup G_x$ consists of two reaches with exclusive parts of size $k_1, k_2$ and common part of size $c = n - k_1 - k_2$. Then, there are two Jordan blocks of the form $J_{02} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, with the corresponding right eigenvectors $u_1$ and $u_3$ and generalised eigenvectors $u_2$ and $u_4$, given by

$$
\begin{align*}
\mathbf{u}_1 &:= \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 := \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix}, \\
\mathbf{u}_3 &:= \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_4 := \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix}, \\
& \quad \text{where} \\
& \cdot \ r_1 \text{ corresponds to the common part of the reach of } G_x \cup G_x \text{ as given in (3.16) on page 21,} \\
& \cdot \ 0 \in \mathbb{R}^c \text{ is a vector, cf. equation (5.29) in the proof of Theorem 5.4.}
\end{align*}
$$

We denote the corresponding left eigenvectors and generalised eigenvectors by $\mathbf{w}_1, \ldots, \mathbf{w}_4$ and assume that they are ordered and scaled accordingly. Without loss of generality, we assume that $G_x$ consists of two reaches as well, so there are no further eigenvectors corresponding to the zero eigenvalue. Then the Jordan matrix of $L$ is given by

$$
J := \begin{pmatrix} J_{02} & 0 & 0 \\ 0 & J_{02} & 0 \\ 0 & 0 & J_{\text{tr}} \end{pmatrix},
$$

and, therefore, for $t \to \infty$,

$$
\begin{pmatrix} 1_{k_1} & t1_{k_1} & 1_{k_1} & (t+1)1_{k_1} \\ 1_{k_2} & t1_{k_2} & 0_{k_2} & 0_{k_2} \\ 1_{c} & t1_{c} & r_1 & tr_1 + \hat{\mathbf{z}} \\ 0_{k_1} & 1_{k_1} & 0_{k_1} & 1_{k_1} \\ 0_{k_2} & 1_{k_2} & 0_{k_2} & 0_{k_2} \\ 0_{c} & 1_{c} & 0_{c} & r_1 \end{pmatrix}
\begin{pmatrix} \mathbf{w}_1^T \\ \mathbf{w}_2^T \\ \mathbf{w}_3^T \\ \mathbf{w}_4^T \end{pmatrix}.
$$

Here, the first three block rows correspond to $\mathbf{x}$ and the last three block rows to $\dot{\mathbf{x}}$. Clearly, the final velocity value depends on whether the agent belongs to the exclusive part of the first or the second reach,
or to their common part. That is, the algorithm is not consensus stable.

**Step 5b.** Assume now that \( G_x \cup G_x \) consists of only one reach but \( \mathcal{L} \) has \( a \leq 2n - 2 \) purely imaginary eigenvalues and \( b \leq 2n - 2 - a \) eigenvalues with a positive real part.\(^5\) For \( i = 1, \ldots, a \), we denote the purely imaginary eigenvalues by \( \imath \alpha_i \), with \( \alpha_i \in \mathbb{R} \) and the corresponding left and right eigenvectors by \( g_i \) and \( h_i \in \mathbb{C}^{2n} \), respectively, where \( h_i^T g_i = 1 \) holds. For \( i = 1, \ldots, b \), we denote the eigenvalues of \( \mathcal{L} \) that have a positive real part by \( \gamma_i + \imath \delta_i \), with \( \gamma_i, \delta_i \in \mathbb{R} \) and \( \gamma_i > 0 \). The corresponding left and right eigenvectors are denoted \( p_i, q_i \in \mathbb{C}^{2n} \), respectively, where \( q_i^T p_i = 1 \) holds.

For simplicity, suppose that all the eigenvalues of \( \mathcal{L} \) that are purely imaginary or have a positive real part are at least semi-simple. If one pair of these eigenvalues is deficient, the argument does not change, but the notation becomes more involved.

Eigenvectors and generalised eigenvectors of a matrix that afford different eigenvalues are mutually orthogonal. Therefore, an eigenvector not affording zero must be orthogonal on at least two rows of the matrices given by different eigenvalues are mutually orthogonal.

We thus obtain for \( t \to \infty \)

\[
e^{\mathcal{L} t} \to \begin{pmatrix} 1_n & 0_n \\ 0_n & 1_n \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_1^T \\ w_2^T \end{pmatrix} + \sum_{i=1}^{a} e^{\imath \alpha_i t} h_i g_i^T + \sum_{i=1}^{b} e^{(\gamma_i + \imath \delta_i) t} q_i p_i^T.
\]

(5.63)

Following the discussion from step 2, the system is consensus stable only if the last \( n \) rows of the matrix \( \sum_{i=1}^{a} e^{\imath \alpha_i t} h_i g_i^T + \sum_{i=1}^{b} e^{(\gamma_i + \imath \delta_i) t} q_i p_i^T \) are identical for \( t \to \infty \).

The blocks \( e^{(\gamma_i + \imath \delta_i) t} \) grow exponentially with different speeds for \( i = 1, \ldots, b \). For the purely imaginary eigenvalues the equivalence \( e^{\imath \alpha_i t} = \cos(\alpha_i t) + \imath \sin(\alpha_i t) \) holds for \( i = 1, \ldots, a \). Thus, they continue to oscillate in time with different phases and amplitudes. It follows that for \( t \to \infty \) at least two of the last \( n \) rows of \( e^{\mathcal{L} t} \) are different. Therefore, there are at least two agents whose velocity grows differently in time for most initial conditions. Thus, for most initial conditions, the position differences between all agents do not approach a constant value. Hence, the system is not consensus stable. \( \square \)

**Remark 5.7** Looking at the shape of the matrix \( \Lambda_{vw} \) in equation (5.51), we further see that if (S2) is consensus stable, then the agents within the exclusive parts of the position graphs achieve position consensus.

---

\(^5\) Remember that if \( G_x \cup G_x \) consists of one reach, then zero is an eigenvalue of \( \mathcal{L} \) with algebraic multiplicity two.
Figure 5.3: Random network on 30 nodes, $G_x$ connected, position and velocity consensus.

Example 5.8 Figure 5.3 shows simulation results for a network of 30 nodes, half of which can communicate both their position and velocity and the other half only their position. Graph $G_x$ is a random connected weighted digraph and $G_x$ is a subgraph of $G_x$ that is obtained by deleting all the edges between nodes that cannot communicate velocities. The initial velocities are chosen from the interval $(-50, 50)$. The network achieves position and velocity consensus.

Example 5.9 For the system introduced in Example 5.5 on page 60,

$$e^{\xi t} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ (5.64)

Thus, $\lim_{t \to \infty} \hat{x}_i(t) = \hat{x}_1(0)$, for all $i = 1, \ldots, 5$, but the final position values are different for the agents $\{1, 2\}$, $\{3\}$ and $\{4\}$. Looking back at Figure 5.2 on page 60 this result is plausible, as $\hat{x}_1$ is the only velocity value available
to all the agents in the system, while different position values are available to the nodes in the two reaches of $G_X$. 
In **Chapter 5** we have presented the necessary and sufficient conditions for the double integrator consensus system \((S_2)\) on page 35, given by

\[
\begin{align*}
\dot{x}(t) &= -L_x x(t) - \beta L_\dot{x} \dot{x}(t), \\
x(0) &= x_0, \quad \dot{x}(0) = \dot{x}_0,
\end{align*}
\]

\((S_2)\)

to be (asymptotically) consensus stable. Here, \(L_x\) and \(L_\dot{x}\) are Laplacian matrices of the position and velocity communication graphs \(G_x\) and \(G_\dot{x}\), respectively, and \(\beta > 0\). These conditions are given in **Theorem 5.6** on page 60 and depend on the eigenvalues of the matrix 

\[
L = \begin{pmatrix} 0 & -I \\ -L_x - \beta L_\dot{x} \end{pmatrix}.
\]

Almost all results that have been presented so far have been given for weighted digraphs. Given that weighted undirected graphs are a special case of digraphs, all results that hold for digraphs also hold for undirected graphs.

As shown in **Section 3.5.1**, however, undirected graph Laplacians are symmetric and, therefore, have a number of useful properties that do not translate to general digraphs. These properties allow a straightforward mathematical treatment of system \((S_2)\) which leads to a concise result formulation. For this reason, in this chapter we study the system \((S_2)\) in the special case that the communication topologies \(G_x\) and \(G_\dot{x}\) are weighted and undirected.

In **Section 6.1** we show that if the networks are undirected, the system matrix \(L\) has no eigenvalues with a positive real part. Furthermore, we connect the existence of purely imaginary eigenvalues to the controllability of a related system. Based on this result, the necessary and sufficient conditions from **Theorem 5.6** are simplified for undirected networks in **Section 6.2**. It turns out that if the graphs are undirected, the eigenvalues of the matrix \(L\), and therefore the consensus stability of the system \((S_2)\) is completely determined by the structure of the graphs \(G_x\) and \(G_\dot{x}\). In **Section 6.3** we further show that if the communication topologies are undirected, every consensus stable system achieves average velocity consensus. In **Section 6.4** the convergence rate of the system is examined based on the properties of the graphs and on the gain \(\beta\).
The graph theoretic definitions and results used throughout this chapter are found in Chapter 3, the notation in Chapter 2 and linear algebraic terms and lemmas in Appendix A.1. Classic control theory notions and results are listed in Appendix A.2. Remember that to improve readability, the signs \( w^{-\rightarrow} \) and \( -\rightarrow \), as well as \( w^{-\leftarrow\rightarrow} \) and \( \leftarrow\rightarrow \) in the margins indicate whether a result holds for (weighted or unweighted) digraphs or only (weighted or unweighted) undirected graphs. Furthermore, remember that when writing unweighted graphs, the explicit mention of the weight function is omitted, and \( G = (V, E) \) is written instead of \( G = (V, E, w) \).

6.1 Non-zero Eigenvalues of \( L \)

As stated in Section 3.5.1 the Laplacian matrix of an undirected graph is symmetric positive semi-definite matrix. This allows us to directly formulate the following result in the case that \( G_x \) and \( G_\dot{x} \) are undirected.

\[ L = \begin{pmatrix} 0 & -I \\ -L_x & -\beta L_x \end{pmatrix}, \beta > 0 \]

\[ \text{Lemma 6.1 (Goldin and Raisch, [42])} \]

Let \( L \) and \( \dot{L} \) be weighted undirected graph Laplacians and \( \beta > 0 \). No eigenvalues of \( L \) are purely imaginary if and only if for the system

\[ \dot{\tilde{x}}(t) = L_x \tilde{x}(t) + \dot{L} \tilde{u}(t), \]

where \( \tilde{x}, \tilde{u} : \mathbb{R}^+ \to \mathbb{R}^n \), zero is the only uncontrollable eigenvalue of \( L_x \).
**Proof:** We prove the lemma by proving its negation. That is, we show that $\mathcal{L}$ has a purely imaginary eigenvalue pair if and only if the system (6.3) or, equivalently, the matrix pair $(L_x, L_\lambda)$ has an uncontrollable non-zero eigenvalue. We first show that if $\mathcal{L}$ has a purely imaginary eigenvalue $\pm \gamma i$, with $\gamma \in \mathbb{R}$, then the matrix pair $(L_x, L_\lambda)$ has an uncontrollable non-zero eigenvalue of $L_x$.

The eigenvalues of $\mathcal{L}$ coincide with the eigenvalues of the QMP $P(\lambda) = \lambda^2 I + \lambda \beta L_x + L_x$. Therefore, we equivalently assume that $P(\lambda)$ has an imaginary eigenvalue $\gamma i$ with the corresponding left eigenvector $v_0^* \in \mathbb{C}^n$. Then

$$v_0^* P(\gamma i) = v_0^* (-\gamma^2 I + \gamma i \beta L_x + L_x) = 0$$

(6.4)

holds. Additionally, $\lambda_{1/2} = \pm \gamma i$ must be a solution of (6.1). As $v_0^* L_x v_0$ and $v_0^* L_\lambda v_0$ are non-negative and real, it follows that (6.1) has the imaginary solution $\lambda_{1/2} = \pm \gamma i$, if and only if

$$v_0^* L_x v_0 = 0$$

(6.5)

holds. As $L_\lambda$ is a symmetric positive semi-definite matrix, $v_0^* L_\lambda v_0 = 0$ holds if and only if $v_0^* L_x = \frac{0}{n}^T$. Inserting this information in $v_0^* P(\gamma i)$ we obtain

$$v_0^* P(\gamma i) = v_0^* (-\gamma^2 I + L_x) = 0$$

(6.6)

It follows that if $P(\lambda)$ has the imaginary eigenvalue $\gamma i$, then $L_x$ has the eigenvalue $\gamma^2$ with the same left eigenvector. We can write this condition together with (6.5) in matrix form as

$$\text{rank } \left( \gamma^2 I - L_x \mid L_x \right) < n.$$  

(6.7)

This is the well-known Popov-Belevitch-Hautus test for controllability, cf. Lemma A.6 on page 136 of the appendix. It is equivalent to $\gamma^2 > 0$ being an eigenvalue of $L_x$, which is uncontrollable for the pair $(L_x, L_\lambda)$.

Next, we show that if the matrix pair $(L_x, L_\lambda)$ has an uncontrollable non-zero eigenvalue, then $\mathcal{L}$ has a purely imaginary eigenvalue pair.

First, note that as $L_x$ is symmetric positive semi-definite, all its eigenvalues are non-negative and real. If $(L_x, L_\lambda)$ has an uncontrollable non-zero eigenvalue $\lambda_0 \in \mathbb{R}$, then there is a corresponding vector $v_0 \in \mathbb{R}^n$ such that

$$v_0^T L_x = \frac{0}{n}^T,$$  

(6.8a)

$$v_0^T L_\lambda = \lambda_0 v_0^T$$  

(6.8b)

holds simultaneously for some $\lambda_0 \in \mathbb{R}^+ \setminus \{0\}$. Premultiplying $P(\lambda)$ by $v_0^T$ we obtain

$$v_0^T P(\lambda) = v_0^T (\lambda^2 I + \lambda \beta L_x + L_x)$$

$$= v_0^T (\lambda^2 I + L_x).$$  

(6.9)
Hence, $\sqrt{\lambda_0} = \pm i\sqrt{\lambda_0}$ is a purely imaginary eigenvalue of $P(\lambda)$ with the corresponding left eigenvector $v_0$. This concludes the proof. □

In particular, Lemma 6.2 implies that $\mathcal{L}$ has purely imaginary eigenvalues only if $G_x$ is disconnected. Formally, we have the following result.

**Corollary 6.3 (Goldin and Raisch, [42])** Let $\mathcal{L} = \begin{pmatrix} 0 & \frac{1}{\beta L_x} \\ -L_x & -\beta L_x \end{pmatrix}$ be given by equation (S3) on page 47, where $L_x$ is the Laplacian of the weighted undirected graph $G_x$, $L_x$ is the Laplacian of the weighted undirected graph $G_x$ and $\beta > 0$. If $G_x$ is connected, then $\mathcal{L}$ has no purely imaginary eigenvalues.

**Proof:** If $G_x$ is connected, then the only vector in the (right and left) kernel of $L_x$ is $\mathbb{1}_n$, which is also in the (right and left) kernel of $L_x$. Therefore,

$$v^T (\lambda I - L_x) = 0 \quad \text{(6.10)}$$

is only satisfied for $v = \mathbb{1}_n$ and $\lambda = 0$. Hence, zero is the only uncontrollable eigenvalue of $L_x$. The result follows from Lemma 6.2. □

If a vector $v \in \mathbb{R}^n$ lies in the kernel of $L_x$, then it can be constructed from the basis $(3.13)$ on page 20. Unless there are no edges in $G_x$, i.e. all connected components of the graph have size one, $v$ is bound to have at least two equal non-zero entries. By Lemma 6.2 if the matrix $\mathcal{L}$ has imaginary eigenvalues, then $v$ is also an eigenvector of $L_x$ corresponding with a non-zero eigenvalue. Therefore, the necessary conditions of Lemma 6.2 are only satisfiable if $L_x$ has an eigenvector with repeated non-zero entries. There are two particular classes of graphs that are guaranteed to have eigenvectors with repeated non-zero entries, namely graphs that contain automorphisms and graphs that contain almost equitable partitions. This observation lets us arrive at the following result.

**Corollary 6.4 (Goldin and Raisch, [42])** Let $\mathcal{L} = \begin{pmatrix} 0 & \frac{1}{\beta L_x} \\ -L_x & -\beta L_x \end{pmatrix}$ be given by (S3) on page 47, where $L_x$ and $L_x$ are Laplacians of the weighted undirected graphs $G_x = (V, E_x, w_x)$ and $G_x = (V, E_x, w_x)$, respectively, and $\beta > 0$. Let $G_x$ consist of $k$ connected components given by the node sets $V_1, \ldots, V_k$, where $k < n$. The matrix $\mathcal{L}$ has purely imaginary eigenvalues if $G_x$ has a non-trivial almost equitable $m$-partition $W_1, \ldots, W_m$, where $m \leq k$, such that $V_1, \ldots, V_k$ is at least as fine as $W_1, \ldots, W_m$.

For $k = 2$, $\mathcal{L}$ has purely imaginary eigenvalues if and only if $(V_1, V_2)$ is a non-trivial almost equitable 2-partition of $G_x$.

---

1 We call an $m$-partition non-trivial if $m > 1$ and there is at least one edge between at least one pair of cells.

2 A partition $V_1, \ldots, V_k$ of a set $V$ is called at least as fine as a partition $W_1, \ldots, W_m$ of the same set, if $m \leq k$ and for each $W_i$ there is an index set $l_i \subseteq \{1, \ldots, k\}$ such that $W_i = \cup_{j \in l_i} V_j$. Because $W_1, \ldots, W_m$ is a partition, $\cup_{i=1}^m W_i = V$ and for all $i \neq j$ it holds that $W_i \cap W_j = \emptyset$. 

---
**Proof:** We start by showing the first statement of the lemma. Let $G_{x}$ have an almost equitable partition $W_1, \ldots, W_m$ of size $m_1, \ldots, m_m$, where $\sum_{i=1}^{m} m_i = n$. Assuming appropriate node numbering, by Lemma 3.14 on page 27, $L_x$ has an eigenvector

$$v = (\alpha_1 \ldots \alpha_1, \alpha_2 \ldots \alpha_2, \ldots, \alpha_m \ldots \alpha_m)^T, \quad \alpha_i \in \mathbb{R}. \quad (6.11)$$

As the partition is non-trivial, $v$ is not in the kernel of $L_x$. Therefore, as $L_x$ is an undirected graph Laplacian, it corresponds to a positive eigenvalue.

Suppose now that the connected components of $G_{x}$ are given by $V_1, \ldots, V_k$ and have size $k_1, \ldots, k_k$ with $\sum_{i=1}^{k} k_i = n$. A basis of the kernel of $L_x$ is given by set $(3.12)$ on page 20. Thus, an eigenvector in the kernel of $L_x$ can be chosen as

$$u = (\delta_1 \ldots \delta_1, \delta_2 \ldots \delta_2, \ldots, \delta_k \ldots \delta_k)^T, \quad \delta_i \in \mathbb{R}. \quad (6.12)$$

As $V_1, \ldots, V_k$ is at least as fine as $W_1, \ldots, W_m$, it follows that we can choose the values $\delta_i$ for $i = 1, \ldots, k$ such that $u = v$ holds. Hence, $v$ is in the kernel of $L_x$. Thus, the corresponding eigenvalue of $L_x$ is uncontrollable for the pair $(L_x, L_x)$. Therefore, it follows from Lemma 6.2 that $\mathcal{L}$ has purely imaginary eigenvalues.

Now consider the case that $k = 2$. It remains to show that $\mathcal{L}$ has purely imaginary eigenvalues only if $(V_1, V_2)$ is a non-trivial almost equitable 2-partition of $G_{x}$. Using $(3.12)$, any eigenvector in the kernel of $L_x$ for $k = 2$ is given by

$$u = \begin{pmatrix} \delta_1 \mathbf{1}_{k_1} \\ \delta_2 \mathbf{1}_{k_2} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{k_1} & \mathbf{0}_{k_1} \\ \mathbf{0}_{k_2} & \mathbf{1}_{k_2} \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}, \quad (6.13)$$

where $k_1$ and $k_2$ are the sizes of the components $V_1$ and $V_2$, respectively, and $\delta_1, \delta_2 \in \mathbb{R}$. Note that $Q$ is the characteristic matrix of the partition $(V_1, V_2)$.

By Lemma 6.2, $\mathcal{L}$ has imaginary eigenvalues only if the matrix pair $(L_x, L_x)$ has an uncontrollable non-zero eigenvalue. We know that $L_x u = \mathbf{0}_n$ holds. Thus, if $k = 2$, $\mathcal{L}$ has an imaginary eigenvalue pair only if there is a value $\gamma \in \mathbb{R} \setminus \{0\}$ that satisfies

$$L_x u = L_x Q \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \gamma Q \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}. \quad (6.14)$$

If the 2-partition is almost equitable, the sum of the weights of incoming edges from the nodes in the set $V_2$ (or $V_1$) to each node in $V_1$ (or $V_2$) is given by $d_{12}$ (or $d_{21}$), i.e.

$$L_x Q = \begin{pmatrix} d_{12} \mathbf{1}_{k_1} & -d_{12} \mathbf{1}_{k_1} \\ -d_{21} \mathbf{1}_{k_2} & d_{21} \mathbf{1}_{k_2} \end{pmatrix}. \quad (6.15)$$
It follows that $d_{12}$ and $d_{21}$ must not be simultaneously zero for $\gamma \neq 0$ to hold. This is the case only if the partition is non-trivial.

We now show by contradiction that if the partition is not almost equitable, $\gamma$ is not an uncontrollable eigenvalue of the pair $(L_x, L_\dot{x})$. Let the partition be not almost equitable. Without loss of generality, let the sum of weights of the incoming edges from $V_2$ to node $v_{k_1} \in V_1$ be $e_{12} \neq d_{12}$. Then,

$$L_x Q \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} d_{12} I_{(k_1-1)} - d_{12} I_{(k_1-1)} \\ e_{12} - e_{12} \\ -d_{21} I_{k_2} \\ d_{21} I_{k_2} \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \gamma Q \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}. \ (6.16)$$

But then $d_{12}(\delta_1 - \delta_2) = \gamma \delta_1$ and $e_{12}(\delta_1 - \delta_2) = \gamma \delta_1$ must hold simultaneously. Thus, either $\gamma = 0$ (this solution exists as $L_x$ always has a zero eigenvalue with the corresponding eigenvector $1_n$), or $\gamma \neq 0$ but $d_{12} = e_{12}$ and the partition is almost equitable. This concludes the proof. \(\square\)

Particularly, if $G_\dot{x}$ has one isolated node $v$ and the remaining graph is connected, then $L$ has imaginary eigenvalues if and only if the isolated node is connected to all nodes of $G_x$ and all the corresponding edges have the same weights.

**Remark 6.5** If a graph has an automorphism, then it also has a corresponding almost equitable partition, though not necessarily vice versa. Thus, it follows from Corollary 6.4 that if $G_\dot{x}$ contains a non-trivial automorphism, there exists a graph $G_\dot{x}$ such that the matrix pair $(L_x, L_\dot{x})$ has a non-zero uncontrollable eigenvalue.

Any partition of the unweighted fully connected graph, shown in Figure 6.1, is an almost equitable (in fact, an equitable) partition. Formally, we have the following result.

**Corollary 6.6 (Goldin and Raisch, [42])** Let $\mathcal{L} = \left(\begin{array}{cc} 0 & 1 \\ -L_x & -\beta L_x \end{array}\right)$ be given by equation (53) on page 47, where $L_x$ is the Laplacian of the weighted undirected graph $G_\times$, $L_x$ is the Laplacian of the weighted undirected graph $G_\times$ and $\beta > 0$. If $G_\times$ is disconnected and $G_\times$ is fully connected with uniform weights, then the matrix $\mathcal{L}$ has purely imaginary eigenvalues.

**Proof:** The result follows from Corollary 6.4 by noting that every partition of the uniformly weighted fully connected graph is an equitable partition. \(\square\)

In summary, we have seen that $L$ having imaginary eigenvalues indicates a symmetry in the position graph. We will elaborate further on the relationship between symmetries of graphs and controllability of the corresponding system in Chapter 8 The following examples illustrate the results of this section.
Example 6.7 Figure 6.1 shows an example of Corollary 6.6. Here, $G_x$ is disconnected while $G_x$ is fully connected. The corresponding consensus system (here with $\beta = 1$) has purely imaginary eigenvalues and, following Theorem 5.6 on page 60, it is not consensus stable.

Example 6.8 Consider the graphs in Figure 6.2. As discussed in Lemma 3.15 on page 28, $G_x$ has an almost equitable partition given by $\{v_1, v_2\}, \{v_3\}, \{v_4, v_5\}$. This partition is at least as coarse as the connected components of $G_x$. Thus, the resulting system matrix $\mathcal{L}$ has purely imaginary eigenvalues.

If the pairs of edges of $G_x$ given by $v_1 \leftrightarrow v_3$ and $v_2 \leftrightarrow v_3$ as well as $v_4 \leftrightarrow v_3$ and $v_5 \leftrightarrow v_3$ are weighted with different weights, it would “destroy” the almost equitable partition and the automorphisms of $G_x$. This, in turn, would result in a different matrix $\mathcal{L}$ that no longer has imaginary eigenvalues.
6.2 Necessary and Sufficient Conditions for Consensus Stability in Undirected Networks

In the previous section, we have determined that the matrix $\mathcal{L}$ has no eigenvalues with a positive real part and that existence of purely imaginary eigenvalues depends on the controllability of the matrix pair $(L_x, L_\dot{x})$. Using these results, we can give a version of Theorem 5.6 on page 60 for undirected graphs that no longer explicitly demands the computation of the eigenvalues of $\mathcal{L}$.

Theorem 6.9 ([Goldin and Raisch, [42]]) Consider the double integrator consensus system (S2) on page 35 for $n$ mobile agents, where $\mathcal{L} = ( -L_x, -\beta L_\dot{x} )$, $\beta > 0$, and $L_x$ and $L_\dot{x}$ are $n \times n$ symmetric Laplacian matrices of the weighted undirected graphs $G_x$ and $G_\dot{x}$, respectively. The system is

- **consensus stable, if and only if**
  - $\text{rank}(\lambda I - L_x \mid L_\dot{x}) = n$ for all $\lambda \in \mathbb{C} \setminus \{0\}$ and
  - $G_x \cup G_\dot{x}$ is connected.

- **asymptotically consensus stable, if and only if** it is consensus stable and additionally $G_x$ is connected.

**Proof:** From Theorem 5.6 on page 60 we know that the necessary and sufficient condition for system (S2) to be consensus stable is that all non-zero eigenvalues of the matrix $\mathcal{L}$ have a negative real part and $G_x \cup G_\dot{x}$ is connected. By Lemma 6.2, $\mathcal{L}$ has no purely imaginary eigenvalues if and only if the matrix pair $(L_x, L_\dot{x})$ has no non-zero uncontrollable eigenvalues. This is equivalent to saying that $\text{rank}(\lambda I - L_x \mid L_\dot{x}) = n$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. By Lemma 6.1 on page 70, all the other non-zero eigenvalues of $\mathcal{L}$ have strictly negative real parts. The rest of the theorem follows from Theorem 5.6.

Theorem 6.9 shows that the system (S2) is consensus stable for any $\beta > 0$. If the graphs are undirected, then the systems' ability to achieve consensus depends entirely on the structures of the corresponding graphs. Furthermore, $G_\dot{x}$ being connected is sufficient for consensus stability of (S2), as we formally state in the following corollary.

Corollary 6.10 Consider the double integrator consensus system (S2) on page 35 for $n$ mobile agents, where $\mathcal{L} = ( -L_x, -\beta L_\dot{x} )$, $\beta > 0$, and $L_x$ and $L_\dot{x}$ are $n \times n$ symmetric Laplacian matrices of the weighted undirected graphs $G_x$ and $G_\dot{x}$, respectively. If $G_\dot{x}$ is connected, then the system is consensus stable.

**Proof:** By Corollary 6.3 on page 72, if $G_\dot{x}$ is connected, then $\mathcal{L}$ has no purely imaginary eigenvalues, i.e. $\text{rank}(\lambda I - L_x \mid L_\dot{x}) = n$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. The result follows directly from Theorem 6.9.
Remember the single integrator consensus system (S1) on page 31, that was given by \( \dot{x}(t) = Lx(t) \), where \( L \) is the Laplacian of the weighted undirected graph \( G \). This system was introduced in Section 4.1. According to Lemma 4.3, for this system, \( G \) being connected is necessary and sufficient for the states of the system to become equal for \( t \to \infty \). Comparing Corollary 6.10 and Theorem 6.9 with Lemma 4.3, we see that the corresponding conditions are more complex for system (S2). In particular, \( G_x \) being connected is sufficient for consensus stability of the system, while \( G_\dot{x} \) being connected is necessary for its asymptotic consensus stability. Therefore, \( G_x \) and \( G_\dot{x} \) being connected simultaneously is sufficient, but not necessary for (S2) to be asymptotically consensus stable.

6.3 Convergence Value

Lemma 4.5 states that a single integrator consensus system achieves average consensus if the communication network is undirected. For double integrator consensus systems with undirected communication networks a similar result can be derived.

**Lemma 6.11** Consider the double integrator consensus system (S2) on page 35 for \( n \) mobile agents, where \( L = (0_{n \times n})_L - \beta 1_L \), \( \beta > 0 \), and \( L_x \) and \( L_\dot{x} \) are \( n \times n \) symmetric Laplacian matrices of the weighted undirected graphs \( G_x \) and \( G_\dot{x} \), respectively. The system achieves average velocity consensus for any initial condition if and only if it is consensus stable. The system achieves average position and velocity consensus for any initial condition if and only if it is asymptotically consensus stable.

**Proof:** Proving the “only if” parts of the statements is trivial. If the system (S2) achieves average velocity consensus for any initial condition, then it is necessary that the system is consensus stable. Likewise, if it achieves average position consensus for any initial condition, then it is necessary that it is asymptotically consensus stable.

We now show that if the system (S2) is consensus stable, then it achieves average velocity consensus. Remember from Section A.2.1 on page 134 of the appendix, that the solution of (S2) is given by

\[
\begin{pmatrix}
    x(t) \\
    \dot{x}(t)
\end{pmatrix} = e^{Lt} \begin{pmatrix}
    x_0 \\
    \dot{x}_0
\end{pmatrix}.
\]  

(6.17)

If the algorithm (S2) achieves velocity consensus, then Theorem 5.6 implies that \( G_x \cup G_\dot{x} \) is connected and all non-zero eigenvalues of \( L \) have a negative real part. Equation (5.51) on page 63 in the proof of Theorem 5.6 states that in this case, as \( t \to \infty \),

\[
\begin{pmatrix}
    1_n (w_1)_1 + t(w_2)_1 \\ \\
    1_n (w_2)_1 \\ \\
    \vdots \\
    1_n (w_1)_n + t(w_2)_n
\end{pmatrix} +
\begin{pmatrix}
    A_{vw} \quad w_1 \\
    \vdots \\
    w_2 \quad 0_{n \times 2n}
\end{pmatrix},
\]
where \( w_1, w_2 \) are the left eigenvector and generalised eigenvector of \( L \) affording the deficient eigenvalue \( 0 \) and \( A_{vw} \) is a \( n \times 2n \) matrix determined by the other left and right vectors in the kernel of \( L \), if they exist. As the graphs are undirected, the vectors \( w_1, w_2 \) are then given by, cf. (3.13),

\[
\begin{align*}
  w_1 &= \frac{1}{n} \begin{pmatrix} 1_n \\ 0_n \end{pmatrix}, \\
  w_2 &= \frac{1}{n} \begin{pmatrix} 0_n \\ 1_n \end{pmatrix}.
\end{align*}
\]

(6.19)

Inserting these in equation (6.18), we obtain for \( t \to \infty \),

\[
e^{Lt} \to \frac{1}{n} \begin{pmatrix} 1_{n \times n} & 1_{n \times n}t \\ 0_{n \times n} & 1_{n \times n} \end{pmatrix} + \begin{pmatrix} A_{vw} \\ 0_{n \times 2n} \end{pmatrix}.
\]

(6.20)

Thus, with (6.17), we obtain

\[
\lim_{t \to \infty} x(t) = \left( \frac{1}{n} \sum_{j=1}^{n} (x_0)_j \right) 1_n
\]

(6.21)

which is bounded average velocity consensus.

Finally, we show that if (S2) is asymptotically consensus stable, then it achieves average position consensus. If (S2) is asymptotically consensus stable, then it follows from the proof of Theorem 5.6 that the matrix \( A_{vw} \) in equation (6.18) is zero. Then, as \( t \to \infty \),

\[
e^{Lt} \to \frac{1}{n} \begin{pmatrix} 1_{n \times n} & 1_{n \times n}t \\ 0_{n \times n} & 1_{n \times n} \end{pmatrix}.
\]

(6.22)

Therefore,

\[
\lim_{t \to \infty} \dot{x}(t) = \left( \frac{1}{n} \sum_{j=1}^{n} (x_0)_j \right) 1_n
\]

(6.23)

which is bounded average velocity consensus. Furthermore, as \( t \to \infty \),

\[
x(t) \to \left( \frac{1}{n} \sum_{j=1}^{n} (x_0)_j \right) 1_n + \left( \frac{1}{n} \sum_{j=1}^{n} (\dot{x}_0)_j \right) 1_n t,
\]

(6.24)

which is average position consensus.

This result is surprising, as we have assumed that the communication topologies are weighted. Thus, even though the information obtained from certain neighbours may be given a higher priority through a greater weight, it does not influence the final consensus.
value. Additionally, unlike in the general case considered in Chapter 5, the velocity consensus value depends only on the initial velocities of the agents, but not on their positions. We will later show that in general the same algorithm does not achieve average consensus if the graphs are directed.

If the graphs are undirected, the eigenvectors in the kernel of $L$, and, therefore, the convergence result, do not depend on the values $\beta$, as long as $\beta > 0$. We can, however, expect that this factor, along with the density of the communication graphs, has an impact on the convergence rate of the consensus algorithm. We study this in the following section.

6.4 CONVERGENCE RATE

It is a known result, see Lemma 4.6 on page 33, that the convergence rate of the single integrator consensus is bounded by the algebraic connectivity of the corresponding Laplacian matrix. We will now derive a similar result for the double integrator consensus algorithm.

For $i = 1, \ldots, 2n$, let $\lambda_i$ be the eigenvalues of $L$. We define

$$\lambda_{\text{crit}} := \min_{i = 1, \ldots, 2n} (|\text{Re}(\lambda_i)| : \lambda_i \neq 0).$$

(6.25)

Lemma 6.12 (Goldin and Raisch, [42]) If the double integrator consensus system (S2) on page 35 is consensus stable, then it reaches velocity consensus exponentially with a rate that is equal to or faster than $\lambda_{\text{crit}}$.

Proof: Suppose that (S2) is consensus stable. Let $\partial_\xi \in \mathbb{R}^n$ be a vector that contains the final velocities of the agents. As the system is consensus stable, it follows that $\partial_\xi = kI_n$ for some constant $k \in \mathbb{R}$. Furthermore, by Lemma 6.11, system (S2) achieves average consensus. Therefore,

$$\partial_\xi = \frac{1}{n} \sum_{i=1}^{n} \dot{x}_i(0) \rightleftharpoons \mathbb{I}_n$$

holds. Note that $L\partial_\xi = \partial_\xi$ for any $n \times n$ Laplacian matrix $L$. Define the group velocity error vector as

$$x_e(t) := \dot{x}(t) - \partial_\xi, \quad x_e(0) = \dot{x}_0 - \partial_\xi,$$

(6.26a)

Then, the disagreement dynamics of the second order consensus algorithm can be derived by differentiating twice:

$$\dot{x}_e(t) = \ddot{x}(t) = -L_x x(t) - \beta L_x \dot{x}(t)$$

(6.27a)

$$= -L_x x(t) - \beta L_x(x_e(t) + \partial_\xi)$$
and

\[ \ddot{x}_e(t) = -L_x \dot{x}(t) - \beta L_x \dot{x}_e(t) \]

\[ = -L_x(x_e(t) + \delta_k) - \beta L_x \dot{x}_e(t). \]

The above equations combined with (6.26) result in

\[ \ddot{x}_e(t) = -L_x x(t) - \beta L_x \dot{x}_e(t). \]

We rewrite the above as a system of first order ODEs, obtaining

\[
\begin{pmatrix}
\dot{x}_e(t) \\
\ddot{x}_e(t)
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-L_x & -\beta L_x
\end{pmatrix}
\begin{pmatrix}
x_e(t) \\
\dot{x}_e(t)
\end{pmatrix},
\]

which is equivalent to

\[
\begin{pmatrix}
\dot{x}_e(t) \\
\ddot{x}_e(t)
\end{pmatrix} = \mathcal{L}
\begin{pmatrix}
x_e(t) \\
\dot{x}_e(t)
\end{pmatrix}.
\]

Using (A.16) on page 134 of the appendix, the solution of system (6.28e) is given by

\[
\begin{pmatrix}
x_e(t) \\
\dot{x}_e(t)
\end{pmatrix} = e^{\mathcal{L}t}
\begin{pmatrix}
x_e(0) \\
\dot{x}_e(0)
\end{pmatrix},
\]

with initial conditions dictated by the initial conditions of the system (S2). Hence, we see that the dynamics of the velocity error are the dynamics of the system (S2). As the system is consensus stable, we know that for \( t \to \infty \) the group velocity error converges to zero.

We have shown in the proof of Theorem 5.6 on page 60 that for \( t \to \infty \),

\[
e^{\mathcal{L}t} \to \frac{1}{n}
\begin{pmatrix}
I_{n \times n} & I_{n \times n} \\
0_{n \times n} & I_{n \times n}
\end{pmatrix} +
\begin{pmatrix}
\Lambda_{vW} \\
0_{n \times 2n}
\end{pmatrix}.
\]

With \( \lambda_{\text{crit}} = \min \{|\text{Re}(\lambda_i)|: \lambda_i \neq 0, i = 1, \ldots, 2n\} \), it follows from (5.49a) on page 62, that \( e^{\mathcal{L}t} \) approaches its solution exponentially with a rate that is equal to or faster than \( \lambda_{\text{crit}} \). Therefore, (6.28e) approaches zero exponentially with a rate that is equal to or faster than \( \lambda_{\text{crit}} \). Thus, the double integrator consensus system (S2) on page 35 reaches consensus exponentially with a rate that is equal to or faster than \( \lambda_{\text{crit}} \).

In the remainder of this chapter we present an informal discussion of how the value of \( \beta \) as well as the connectivity of the graphs influence convergence rate of the system. The discussed concepts are then illustrated with some examples. In order to estimate the value
of $\lambda_{\text{crit}}$ we use the fact that, for a symmetric $n \times n$ Laplacian matrix $L$, cf Section 3.5 on page 17,
\begin{equation}
0 \leq \frac{v^*Lv}{v^*v} \leq \mu_n,
\end{equation}
holds, where $\mu_n$ is the largest eigenvalue of $L$ and $v \in \mathbb{C}^n \setminus \{0_n\}$.

We rewrite equation (5.6) on page 49 as
\begin{equation}
\lambda = -0.5\beta \frac{v^*L_xv}{v^*v} \pm 0.5 \sqrt{\beta^2 \left( \frac{v^*L_xv}{v^*v} \right)^2 - 4 \frac{v^*L_xv}{v^*v}} \quad (6.32)
\end{equation}
and assume that the system is consensus stable. We see that if
\begin{equation}
\beta^2 \left( \frac{v^*L_xv}{v^*v} \right)^2 \geq 4 \frac{v^*L_xv}{v^*v},
\end{equation}
for all $v$ that are eigenvectors of $P(\lambda) = \lambda^2 I + \lambda \beta L_x + L_x$, then $\lambda$ is real. Otherwise it is complex with a real part given by $-0.5\beta \frac{v^*L_xv}{v^*v}$. Consider the eigenvalues of $L$ in dependence of $\beta$. If the non-zero eigenvalue of $L$ that is closest to the imaginary axis is complex, then increasing $\beta$ will lead to a greater $\lambda_{\text{crit}}$. On the other hand, if the non-zero eigenvalue closest to the imaginary axis is real, then decreasing $\beta$ might increase $\lambda_{\text{crit}}$.

In the special case that $\beta = 1$, we can relate $\lambda_{\text{crit}}$ directly to the properties of $L_x$, $L$. For unweighted graphs, a bound for $\mu_n$ can be derived from Lemma 3.3 on page 19. Let $G_x$ have $k^x$ connected components of size $k^x_1, \ldots, k^x_{k^x}$, with $k^x_{\text{max}} = \max_i k^x_i$. Let $G_x$ have $k^x$ connected components, with, analogously, $k^x_{\text{max}} = \max_i k^x_i$. Then, by Lemma 3.3
\begin{equation}
0 \leq \frac{v^*L_xv}{v^*v} \leq k^x_{\text{max}}, \quad 0 \leq \frac{v^*L_xv}{v^*v} \leq k^x_{\text{max}}.
\end{equation}

Setting $\beta = 1$, equation (6.32) then becomes
\begin{equation}
\lambda = -0.5 \frac{v^*L_xv}{v^*v} \pm 0.5 \sqrt{\left( \frac{v^*L_xv}{v^*v} \right)^2 - 4 \frac{v^*L_xv}{v^*v}}. \quad (6.35)
\end{equation}
We see that if $L$ has eigenvalues close to the imaginary axis with a large imaginary part, then $\left( \frac{v^*L_xv}{v^*v} \right)^2 \ll 4 \frac{v^*L_xv}{v^*v}$ must hold. Using the bounds (6.34), we see that if $k^x_{\text{max}} \ll k^x_{\text{max}}$, this situation is more likely to occur. On the other hand, if $k^x_{\text{max}}$ is large and $k^x_{\text{max}}$ is small, it is probable that the eigenvalues of $L$ will be real or have small imaginary parts. This leads us to the observation, that if only a small number of agents can exchange their velocity, i.e. $k^x_{\text{max}}$ is small, the number of agents exchanging their position should be as small as possible, too, in order to reduce the probability of large large oscillations.
This, however, may lead to a small $\lambda_{\text{crit}}$ and, furthermore, increases the probability that $\mathcal{L}$ will have purely imaginary eigenvalues.

We further see that for unweighted undirected graphs $\lambda_{\text{crit}}$ is bounded by

$$0 < \lambda_{\text{crit}} \leq n, \quad (6.36)$$

where the upper bound is attainable (choose $G_x$ fully connected and $G_x$ empty). Analogously, if $\beta \neq 1$,

$$0 < \lambda_{\text{crit}} \leq \beta n. \quad (6.37)$$

**Example 6.13** Figure 6.3 illustrates these behaviours for two particular system matrices $\mathcal{L}_1, \mathcal{L}_2$. For both systems $n = 4$, and the velocity and the position graphs are connected, unweighted and undirected. For first system,
shown in Figure 6.3a, the non-zero eigenvalue of $L_1$ closest to the imaginary axis is real for $\beta = 1$. As we can see from the corresponding root locus, increasing $\beta$ will move it closer to the imaginary axis. On the other hand, the non-zero eigenvalue closest to the imaginary axis is complex for system given by $L_2$ shown in Figure 6.3b. Increasing $\beta$ moves it further away from the imaginary axis, thereby improving $\lambda_{crit}$.

Example 6.14 Figure 6.4 shows the simulation of a double integrator consensus system for $\beta = 1$. Both the position and the velocity graphs are disconnected, unweighted and undirected. However, $G_x \cup G_\dot{x}$ is connected and the non-zero eigenvalues of the corresponding Laplacian matrix pair $(L_x, L_\dot{x})$ are controllable. Therefore, the system has no purely imaginary eigenvalues. As the connected components of $G_x$ consist of the nodes $\{v_1\}$, $\{v_2\}$ and $\{v_3, v_4, v_5\}$, we know from Theorem 6.9 and Remark 5.7 that the agents achieve velocity consensus and the agents $3-5$ achieve position consensus. This is shown in the simulation. We see that the agents continue to move at a fixed distance. The spectrum of $L$ is $\text{spec}(L) = \{0, 0, 0, 0, -2.96, -0.84 \pm 1.26i, -0.30 \pm 0.94i, -0.77\}$, which explains the high amount of oscillation in the velocity plot. Here, $k_x^{\text{max}} = k_{\dot{x}}^{\text{max}} = 3$.

From the previous discussion, we can estimate that adding edges to the velocity graphs would lead to an improvement in consensus rate. This is validated in Figure 6.5. The simulation shows that velocity consensus is achieved faster than in Figure 6.4 and with less oscillations. Here, $k_{\dot{x}}^{\text{max}} = 5$ and the numerical range of $L_\dot{x}$ is now given by $0 \leq \frac{\lambda_\text{max}^{-1}}{\lambda_\text{min}^{\text{negative}}} \leq 3.6$.

Example 6.15 Consider Figure 6.6. Here, $\beta = 1$, and $G_x$ and $G_\dot{x}$ are both connected, unweighted and undirected. Therefore, the system achieves position and velocity consensus. Furthermore, the choice of $G_\dot{x}$ as the complete graph leads to $L$ having only real eigenvalues. Note that the oscillations seen in the velocity plot stem from the initial conditions.
Figure 6.6: Illustration of Example 6.15: Position and velocity consensus.
In Chapter 5 we derived necessary and sufficient conditions for the double integrator consensus system in heterogeneous networks to be consensus stable or asymptotically consensus stable. Remember that the system (S2) on page 35 is given by

\[
\begin{align*}
\ddot{x}(t) &= -L_x x(t) - \beta L_x \dot{x}(t), \\
x(0) &= x_0, \quad \dot{x}(0) = \dot{x}_0,
\end{align*}
\]

(S2)

where \( x : \mathbb{R}^+ \to \mathbb{R}^n \), \( \beta > 0 \) and the matrices \( L_x \) and \( L_\dot{x} \) are Laplacians of the weighted communication digraphs \( G_x \) and \( G_\dot{x} \), respectively. We showed in Theorem 5.6 that consensus stability of (S2) can be related to properties of the matrix \( L = \begin{pmatrix} 0 & -L_x \\ I & -\beta L_\dot{x} \end{pmatrix} \) and to connectivity properties of the communication graphs. System (S2) is consensus stable if and only if all non-zero eigenvalues of the matrix \( L \) have negative real parts and \( G_x \cup G_\dot{x} \) consists of one reach. Moreover, it is asymptotically consensus stable if and only if it is consensus stable and additionally \( G_x \) consists of one reach.

In the previous chapter we considered the special case that the communication networks are undirected. In this case, the corresponding Laplacian matrices are symmetric and positive semi-definite. One of the main results was that \( L \) has no eigenvalues with a positive real part for any choice of undirected communication network. The assumption that the networks are undirected holds in many cases, however, as outlined in Chapter 4, sometimes the communication structure between networked agents has to be directed. Therefore, the convergence conditions in Chapter 5 were already aimed at the general case and hold for digraph communication networks.

In this chapter, our main goal is to derive necessary and sufficient conditions for \( L \) to have no eigenvalues with a positive real part or purely imaginary eigenvalues if the communication networks are directed. If the graphs \( G_x \) and \( G_\dot{x} \) are directed, then the corresponding Laplacian matrices are no longer symmetric. Thus, we show in the following that the methods used in Chapter 6 are not directly applicable.
As we showed in Chapter 5, the spectrum of the matrix $\mathcal{L}$ coincides with the spectrum of the quadratic matrix polynomial (QMP) $P(\lambda) = \lambda^2 I + \lambda \beta L_x + L_x$, $\lambda \in \mathbb{C}$. Thus, in the following, we are essentially investigating the spectrum of a QMP with real unsymmetric matrix coefficients. The typical application area for QMPS is in structural mechanics, where the coefficient matrices are usually symmetric and/or antisymmetric. Thus, to our best knowledge, only few authors consider spectra of QMPS with unsymmetric coefficients, such as Shieh et al., [92], Wimmer, [102] and Diwekar, [30]. However, the problem settings in all these papers are not applicable to Laplacian matrices. Thus, the sufficient conditions for stability of the QMP derived therein are not applicable to the consensus system studied in this thesis. To the best of our knowledge, stability of QMPS with explicitly weighted digraph Laplacian matrix coefficients has not been addressed in the literature to date, outside of, indirectly, the related consensus works listed in Section 4.2.

In this chapter we first illustrate the complications that arise when directed communication topologies are considered in Section 7.1. Then we list properties of the system matrix $\mathcal{L}$ in the case that the matrices $L_x$ and $L_\dot{x}$ are directed graph Laplacians. The numerical range of $L_x$ and $L_\dot{x}$ is the topic of Section 7.2, while the fact that $\mathcal{L}$ has no positive real eigenvalues is proven in Section 7.3. In Section 7.4 we study the cases when $\mathcal{L}$ has purely imaginary eigenvalues. We obtain that, unlike in the previous chapter, the existence of imaginary eigenvalues cannot be characterised only based on the controllability of the system $(L_x, L_\dot{x})$. In fact, it will turn out that some systems have purely imaginary eigenvalues only for distinct values of $\beta$, while other systems have purely imaginary eigenvalues independently of $\beta$.

Thereafter, it remains to consider if and when $\mathcal{L}$ has complex eigenvalues with a positive real part. This is done in Section 7.5. Here, we identify a class of systems for which $\mathcal{L}$ is bound to have eigenvalues in the right half-plane for some values of $\beta$. Furthermore, we show that the existence of such eigenvalues is connected to the existence of circles in the graph $G_x$. Additionally, we obtain that both the structure of $G_x$ and of the pair $G_\dot{x}, G_x$ play a role in the occurrence of eigenvalues with positive real parts. Moreover, we derive that $\mathcal{L}$ has no eigenvalues with a positive real part if $G_\dot{x}$ has no edges. It turns out that if the communication topologies are directed, then $\mathcal{L}$ having eigenvalues with a positive real part depends on $\beta$ as well as on the structure of the communication topologies and weights in the graphs.

In Section 7.6 we consider the special case that the union graph $G_\dot{x} \cup G_x$ is acyclic. Unlike in the previous discussion, in this case we are able to completely characterise consensus stability of the corresponding system based on the structure of $G_\dot{x}$ and $G_x$. Finally, Section 7.7 contains a summary of the findings in this chapter and a brief discussion.
### 7.1 Introductory Example

Before we consider general second integrator consensus systems, we present two examples in order to outline the differences between systems given by directed and undirected graphs and the difficulties that arise in the treatment of the former.

#### Example 7.1

Consider the communication topologies given in Figure 7.1. Here $G_{x,1}$ is an unweighted directed circle on four nodes and $G_{\dot{x},1}$ is the same circle graph but with the orientation of the edges reversed. The corresponding Laplacian matrices are given by

\[
L_{x,1} = \begin{pmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & -1 \\
-1 & 0 & 0 & 1
\end{pmatrix}, \quad L_{\dot{x},1} = \begin{pmatrix}
1 & 0 & 0 & -1 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1
\end{pmatrix}. \quad (7.1)
\]

Note that $L_{x,1} = L_{\dot{x},1}^T$. Both $G_{x,1}$ and $G_{\dot{x},1}$ consist of one reach and are, in fact, balanced strongly connected digraphs. Intuitively we would assume the corresponding double integrator consensus algorithm to achieve position and velocity consensus. However, the eigenvalues of the matrix $L_1 = \begin{pmatrix} 0 & -1 \\ -L_{x,1} & -\beta L_{x,1} \end{pmatrix}$ are, with $\beta = 1$:

\[
\{0, 0.1336 \pm 0.6838i, -1 \pm i, -1.1336 \pm 1.6838i\}, \quad (7.2)
\]

and by Theorem 5.6 on page 60 the system is not consensus stable. Furthermore, changing the value of $\beta$ does not stabilise the system. In Figure 7.2 we show the eigenvalues in dependence of $\beta$. Here, $\beta$ is varied between zero and 15. We can clearly see that for each value of $\beta$, the system has a pair of eigenvalues with positive real parts. These eigenvalues approach the imaginary axis from the right as $\beta$ increases, but only reach it for $\beta \to \infty$. 

![Figure 7.1: Two circle graphs on four nodes.](image-url)
Figure 7.2: Dependence of the eigenvalues of $L_1$ on the values of $\beta$, $0 \leq \beta \leq 15$. Eigenvalues at $\beta = 0$ are marked with dots. Eigenvalues at $\beta = 1$ are marked with crosses.

We know from the previous chapters that if $G_{x,1}$ and $G_{\dot{x},1}$ were undirected circle graphs, the double integrator consensus system would be asymptotically consensus stable. This example illustrates that the necessary and sufficient conditions given in Theorem 6.9 on page 76 do not directly translate to cases when the graphs are directed. Furthermore, as we will now show, the system might have eigenvalues with positive real parts even if $G_x = G_{\dot{x}}$.

**Example 7.2** Let $G_{x,2} = G_{\dot{x},2}$ be given by the graph in Figure 7.3. This is an unweighted directed circle graph, therefore, it is balanced and strongly connected. Yet, if $\beta = 1$, system (S2) on page 35 is not consensus stable, as the eigenvalues of $L_2 = \begin{pmatrix} 0 & -1 \\ -L_{x,2} & -\beta L_{\dot{x},2} \end{pmatrix}$ are given by

$$
\{0, 0.2571 \pm 1.1014i, -0.2571 \pm 1.5291i, -0.8043 \pm 1.4052i, \\
-1 \pm i, -0.9028 \pm 0.6981i, -0.7429 \pm 0.5291i, -0.5500 \pm 0.3943i\}.
$$

Figure 7.4 shows the eigenvalue locations of $L_2$ for different values of $\beta$.

Unlike in the previous example, we see that there is a threshold $\beta_{\text{crit}} \approx 1.72$, such that for $\beta \geq \beta_{\text{crit}}$ all eigenvalues are in the open left half-plane. Thus, according to Theorem 5.6 on page 60, the system becomes consensus stable.

Note that in both examples, if $G_{\dot{x}}$ was chosen in the same way, but $G_x$ was empty, then the systems would achieve velocity consensus, as we formally show in Lemma 7.8. Intuitively, we would expect that adding position communication would improve the convergence rate of the algorithm and possibly ensure position consensus. Instead, the system becomes unstable.

As we have just seen, the graph-theoretic conditions formulated for undirected graphs in Theorem 6.9 on page 76 do not straightforwardly translate to digraphs. In the next sections, we give an in-depth explanation of why this happens. We show why the methods...
Let $G = (V, E, w)$ be a weighted digraph with the corresponding Laplacian $L$. We denote the symmetric and antisymmetric parts of $L$ by

$$
sym(L) := \frac{L + L^T}{2}, \quad \text{asym}(L) := \frac{L - L^T}{2}.
$$

(7.4)

Following Section A.1.3 on page 131 of the appendix the numerical range of the digraph Laplacian $L$ is given by the values of

$$\frac{v^*Lv}{v^*v} = \frac{v^*\text{sym}(L)v}{v^*v} + \frac{v^*\text{asym}(L)v}{v^*v}, \quad \text{for all } v \in \mathbb{C}^n \setminus \{0_n\}.
$$

(7.5)

Here, $\frac{v^*\text{sym}(L)v}{v^*v}$ is real and $\frac{v^*\text{asym}(L)v}{v^*v}$ is purely imaginary. Furthermore, in Section 3.5.1 on page 18 we saw that

$$\lambda_{\min}(\text{sym}(L)) \leq \text{re} \left( \frac{v^*Lv}{v^*v} \right) \leq \lambda_{\max}(\text{sym}(L)),
$$

(7.6)

$$\lambda_{\min}(-i\cdot\text{asym}(L)) \leq \text{im} \left( \frac{v^*Lv}{v^*v} \right) \leq \lambda_{\max}(-i\cdot\text{asym}(L)),
$$

(7.7)

for all $v \in \mathbb{C}^n \setminus \{0_n\}$, where $\lambda_{\min}, \lambda_{\max}$ denote the smallest and the largest eigenvalue of a matrix. As $L$ is not symmetric, its numerical range is complex.

Furthermore, in general, $\text{sym}(L)$ is not weakly diagonally dominant. In most cases it is also not positive semi-definite (see, e.g., Brualdi, [17]). Therefore, in general, for some $v \in \mathbb{C}^n$, $\text{re} \left( \frac{v^*Lv}{v^*v} \right)$ may
Figure 7.5: Illustration of the numerical range of the matrix \( L_1 \) in Example 7.3. The numerical range of its symmetric part is the indicated part of the real axis. The numerical range of its antisymmetric part is the indicated part of the imaginary axis. Locations of the eigenvalues of \( L_1 \) are indicated by crosses.

take on both positive and negative values. Thus, the eigenvalue equation (6.1), that was guaranteed to have only solutions with a non-positive real part in Chapter 6, may have solutions with a positive real part if the graphs are directed.

**Example 7.3** Figure 7.5 shows the numerical range of the digraph Laplacian matrix

\[
L_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 2 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & 0 & 0 \\
-1 & -1 & 0 & 0 & 3 & -1 \\
-1 & -1 & 0 & -1 & 0 & 3
\end{pmatrix}
\] (7.8)

We see that although all eigenvalues of \( L_1 \) lie in the closed right half-plane, its numerical range crosses the imaginary axis. The eigenvalues of \( L_1 \) are given by\(^1\) \{0, 0.4, 3.4 \pm 0.6i, 1.9 \pm 0.5i\}. The eigenvalues of \( \text{sym}(L_1) \) are given by \{−0.6, 0.5, 1.8, 2.3, 3.3, 3.6\} and the eigenvalues of \( \text{asym}(L_1) \) by \{±1.2i, ±0.4i, ±0.8i\}.

### 7.3 Real Eigenvalues of \( L \)

By Theorem 5.6, system (S2) on page 35 is consensus stable only if \( L \) has no eigenvalues with a positive real part. A proof that \( L \) does not have positive real eigenvalues if the graphs \( G_x \) and \( G_\dot{x} \) are directed can be given using the Gershgorin disc theorem.

\[ \rightarrow \]

**Lemma 7.4** (Goldin, [40]) Consider the double integrator consensus system (S2). Let \( L = \left( -1_{nx} - \frac{1}{\beta} L_x \right) \), where \( \beta > 0 \) and \( L_x \) and \( L_\dot{x} \) are \( n \times n \)

\(^1\) rounded to the first decimal
Laplacian matrices of the weighted digraphs $G_x$ and $G_\xi$, respectively. If $\mathcal{L}$ has real nonzero eigenvalues then they are negative.

**Proof:** Remember that if $\lambda_0 \in \mathbb{R}$ is an eigenvalue of $\mathcal{L}$, then it is also an eigenvalue of the QMP $P(\lambda) = \lambda^2 I + \lambda \beta L_x + L_x$. Thus, $\lambda_0$ satisfies $\det(P(\lambda_0)) = 0$. That is, $\lambda_0$ is a real eigenvalue of $P(\lambda)$ if and only if $P(\lambda_0)$, which is a real $n \times n$ matrix, has a zero eigenvalue. We proceed to show that if $\lambda_0 > 0$ then $P(\lambda_0)$ has full rank, which in turn implies that $\lambda_0$ is not an eigenvalue of $P(\lambda)$. We use the Gershgorin disc theorem, stated in Lemma A.1 on page 132 of the appendix.

Consider the $i$-th row of $P(\lambda_0) = \lambda_0^2 I + \lambda_0 \beta L_x + L_x$. Its diagonal entry, i.e. the center of the corresponding Gershgorin disc, is given by

$$ (P(\lambda_0))_{ii} = \lambda_0^2 + \lambda_0 \beta (L_x)_{ii} + (L_x)_{ii}, \quad i = 1, \ldots, n. \quad (7.9) $$

For a Laplacian matrix $L$ it holds that $(L)_{ii} = \sum_{j=1, j \neq i}^{n} |(L)_{ij}|$ by construction. Thus, we can write (7.9) as

$$ (P(\lambda_0))_{ii} = \lambda_0^2 + \lambda_0 \beta \sum_{j=1, j \neq i}^{n} |(L_x)_{ij}| + \sum_{j=1, j \neq i}^{n} |(L_x)_{ij}| \quad (7.10) $$

instead. The radius of the $i$-th Gershgorin disc is given by

$$ r_i = \sum_{j=1, j \neq i}^{n} |\lambda_0 \beta (L_x)_{ij}| + \sum_{j=1, j \neq i}^{n} |(L_x)_{ij}|, \quad i = 1, \ldots, n. \quad (7.11) $$

Suppose that $\lambda_0 > 0$. Then, clearly, $(P(\lambda_0))_{ii} > 0$. Consider the point of the disc that is closest to the imaginary axis. As the center of the disk is a real number, it is located on the real axis. It is given by

$$ (P(\lambda_0))_{ii} - r_i = \lambda_0^2 + \lambda_0 \beta \sum_{j=1, j \neq i}^{n} |(L_x)_{ij}| + \sum_{j=1, j \neq i}^{n} |(L_x)_{ij}| - \sum_{j=1, j \neq i}^{n} |(L_x)_{ij}| - \sum_{j=1, j \neq i}^{n} |(L_x)_{ij}| \quad (7.12) $$

This means that the $i$-th Gershgorin disc of $P(\lambda_0)$ does not touch the imaginary axis if $\lambda_0 > 0$. This holds for every row of $P(\lambda_0)$. Thus, $\det(P(\lambda_0)) \neq 0$. It follows that $\lambda_0 > 0$ is not an eigenvalue of $P(\lambda)$. □

It is straightforward to verify that the same approach cannot be used to exclude that $\mathcal{L}$ has complex eigenvalues with a positive real part or purely imaginary eigenvalues.

### 7.4 purely imaginary eigenvalues of $\mathcal{L}$

In Section 6.1 we have obtained Lemma 6.2 for undirected graphs $G_x$, $G_\xi$. It states that if the communication graphs are undirected, the
corresponding system matrix $\mathcal{L}$ has purely imaginary eigenvalues if and only if $\text{rank}(\lambda I - L_x | L_x) < n$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. If the graphs are directed, this condition is only sufficient.

\textbf{Lemma 7.5} Let $\mathcal{L} = (-\beta L_x - I)$ be given by (S3) on page 47, where $L_x$ and $L_x$ are weighted digraph Laplacians and $\beta > 0$. If the system

$$\dot{x}(t) = L_x \dot{x}(t) + L_x \ddot{u}(t), \quad (7.13)$$

where $\dot{x}, \ddot{u} : \mathbb{R}^+ \to \mathbb{R}^n$, has a non-zero real\(^2\) uncontrollable eigenvalue, then $\mathcal{L}$ has a purely imaginary eigenvalue pair for all values of $\beta$.

\textbf{Proof:} Suppose that the system (7.13), or, equivalently, the matrix pair $(L_x, L_x)$ has a real uncontrollable non-zero eigenvalue $\lambda_0 \in \mathbb{R}^+$. Then, by Lemma A.6 on page 136 of the appendix, it holds that $\text{rank}(\lambda_0 I - L_x | L_x) < n$. Thus, there is a corresponding vector $v_0 \in \mathbb{R}^n$ such that

$$v_0^T L_x = 0_n^T, \quad (7.14a)$$

$$v_0^T (L_x - \lambda_0 I) = 0_n^T \quad (7.14b)$$

holds simultaneously. Remember that the eigenvalues of $\mathcal{L}$ coincide with the eigenvalues of the QMP $P(\lambda) = \lambda^2 I + \lambda \beta L_x + L_x$. Premultiplying $P(\lambda)$ by $v_0^T$ we obtain

$$v_0^T P(\lambda) = v_0^T (\lambda^2 I + \lambda \beta L_x + L_x), \quad (7.15a)$$

which, inserting $v_0^T L_x = 0_n^T$, is equivalent to

$$v_0^T P(\lambda) = 0_n^T (\lambda^2 I + L_x). \quad (7.15b)$$

Comparing (7.15b) with (7.14b), it follows that $v_0^T P(\lambda) = 0_n$ if $\lambda^2 = -\lambda_0$. Therefore, $\sqrt{-\lambda_0} = \pm i \sqrt{-\lambda_0}$ is a purely imaginary eigenvalue of $P(\lambda)$ with the corresponding left eigenvector $v_0$. This concludes the proof.

If the communication networks are undirected, the condition that the real non-zero eigenvalues of $(L_x, L_x)$ be controllable is both necessary and sufficient in order for $\mathcal{L}$ to have no purely imaginary eigenvalues for any $\beta > 0$. This is no longer the case if the communication networks are directed. The matrix $\mathcal{L}$ may have additional purely imaginary eigenvalue pairs for some specific values of $\beta$. Figure 7.4 on page 89 shows the eigenvalues of an example system matrix for varying values of $\beta$. We see that $\mathcal{L}$ has a complex eigenvalue pair that has a positive real part for small values of $\beta$. However, as $\beta$ increases, it this eigenvalue pair crosses the imaginary axis. Therefore there exists a value of $\beta$ such that the corresponding matrix $\mathcal{L}$ has a purely imaginary real part. Hence, some systems may have purely imaginary eigenvalues for some, but not all, values of $\beta$. This was not possible for undirected graphs.

\(^2\)Note that the eigenvalue being real is not explicitly demanded in Lemma 6.2 due to the fact that all eigenvalues of $L_x$ are real if the corresponding graph is undirected.
7.5 Complex Eigenvalues of $\mathcal{L}$

Thus far, we have studied the conditions for the matrix $\mathcal{L}$ to have zero eigenvalues in Section 5.2, positive real eigenvalues in Section 7.3 and imaginary eigenvalues in Section 7.4. It remains to establish the cases when $\mathcal{L}$ has complex eigenvalues with a positive real part. From the examples in Section 7.1 we know that such cases exist, and that, furthermore, they are not related to digraph connectedness and do not reflect our intuition.

In Chapter 6, we derived Lemma 6.1 that states that $\mathcal{L}$ has no complex eigenvalues with a positive real part if the corresponding graphs $G_x$ and $\tilde{G}_x$ are weighted and undirected. In order to obtain this result, we used the fact that if the graphs are undirected, then the numerical ranges of the corresponding Laplacians $L_x$ and $\tilde{L}_x$ are sections of the real line in the closed right half-plane. As we showed in Section 7.2, this is no longer the case with digraph Laplacians. The numerical range of a general weighted digraph Laplacian is a convex section of the complex plane and may take both values with a positive and a negative real part.

The first result of this section states that there are systems which have a complex eigenvalue pair with a positive real part for some values of $\beta$. Here, we do not assume $\beta$ to be fixed, but instead consider $\mathcal{L}$ as a matrix-valued function $\mathcal{L}(\beta) : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^{2n \times 2n}$. The corresponding QMP is given by $P(\lambda, \beta) : \mathbb{C} \times (\mathbb{R}^+ \cup \{0\}) \to \mathbb{R}^{n \times n}$.

**Lemma 7.6** Let $\mathcal{L} = \mathcal{L}(\beta) = \begin{pmatrix} 0 & 1 \\ -L_x & -\beta L_x \end{pmatrix}$ be given by (S3) on page 47, where $\beta > 0$, and $L_x$ are $L_x$ weighted digraph Laplacians. If $L_x$ has at least one complex eigenvalue pair, then there is a value $\beta = \beta_1$ such that $\mathcal{L}(\beta_1)$ has a pair of complex eigenvalues with positive real parts.

**Proof:** First, note that if $\mathcal{L}(\beta)$ is a parameter-dependent matrix, then its eigenvalues $\lambda_i(\beta), \lambda_i : \mathbb{R}^+ \cup \{0\} \to \mathbb{C}$, for $i = 1, \ldots, n$, are continuous functions in $\beta$ (see, e.g., Wilkinson, [101] for details).

Let $\beta \to 0$. Remember that the eigenvalues of $\mathcal{L}(\beta)$ and $P(\lambda, \beta) = \lambda^2I + \lambda \beta L_x + L_x$ coincide. Thus, the eigenvalues of $\lim_{\beta \to 0} \mathcal{L}(\beta)$ are the eigenvalues of $\lim_{\beta \to 0} P(\lambda, \beta) = \lambda^2I + L_x$. Therefore, the eigenvalues of $\lim_{\beta \to 0} \mathcal{L}(\beta)$ are the square roots of the eigenvalues of $-L_x$. Hence, if $L_x$ and, therefore, $-L_x$ has a pair of complex eigenvalues, then $\lim_{\beta \to 0} \mathcal{L}(\beta)$ has a pair of complex eigenvalues with positive real parts. Thus, for small enough $\beta_1 \neq 0$, $\mathcal{L}(\beta_1)$ has a pair of complex eigenvalues with positive real parts. This is illustrated in Figure 7.6. This concludes the proof.

Lemma 7.6 identifies a class of systems for which the corresponding system matrix $\mathcal{L}$ has a complex eigenvalue pair with a positive real part for small values of $\beta$. This class of systems is determined
by the shape of the graph $G_x$. It depends on the shape of the graph $G_x$ whether $L$ has a complex eigenvalue pair with a positive real part for large values of $\beta$ as well. Figure 7.4 and Figure 7.8 both show root loci of double integrator consensus systems with directed communication topologies. The system whose eigenvalues are illustrated in Figure 7.8 has a complex eigenvalue pair with a positive real part for all $\beta$, that becomes purely imaginary for $\beta \to \infty$. On the other hand, the system whose eigenvalues are illustrated in Figure 7.4 has a complex eigenvalue pair with a positive real part only for some $\beta$.

**Remark 7.7** In Section 4.2.2.1, we mentioned Lemma 4.12 that was obtained by Yu et al., [109] for systems with homogeneous directed communication topologies. It states that if the communication topologies are homogeneous and the system consists of one reach, then there exists a $\beta$ such that the system becomes asymptotically consensus stable. The system in Figure 7.8 shows that their result does not hold for general double integrator consensus systems with heterogeneous communication topologies.

If the Laplacian matrix of a digraph has a complex spectrum, the corresponding graph is called **essentially cyclic**. These graphs are studied by Agaev and Chebotarev, [3]. The authors show that if a graph is essentially cyclic, then it contains a directed cycle. The converse does not hold, i.e. the existence of a directed cycle is necessary, but not sufficient for the corresponding Laplacian to have complex eigenvalues. Therefore, if $G_x$ does not contain a directed circle, then Lemma 7.6 is not applicable to the corresponding double integrator consensus system. This, however, does not imply that the corresponding matrix $L$ has no eigenvalues with a positive real part for some values of $\beta$. 

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Figure 7.6: Illustration of the proof of Lemma 7.6. If the matrix $-L_x$ has a complex eigenvalue pair, marked by crosses in the figure, its square roots, marked by circles, are located in both the left and the right-half plane.
7.5 Complex Eigenvalues of $L$

Figure 7.7: Example system.

Figure 7.8: Eigenvalues of the system matrix obtained from the graphs in Figure 7.7 for $0 \leq \beta \leq 100$, eigenvalues for $\beta = 0$ are marked with dots, for $\beta = 1$ with crosses. Arrows indicate increasing values of $\beta$.

In fact, it is straightforward to verify that the eigenvalues of the Laplacian of the graph illustrated in Figure 7.9a are real. They are given by $\{0, 1, 1, 2, 0.4384, 4.5616\}$. However, if the graphs $G_x$ and $\dot{G}_x$ are chosen according to Figure 7.9, the corresponding matrix $L$ has eigenvalues with positive real parts for some $\beta$. The corresponding root loci for $0 \leq \beta \leq 20$ are illustrated in Figure 7.10. Note that as $\beta$ grows larger, the positive eigenvalue pair crosses the imaginary axis and the system becomes consensus stable.

On the other hand, if the communication graphs are directed, it does not follow that the system matrix $L$ has eigenvalues with a positive real part for some $\beta > 0$. This is obvious from the fact that undirected graphs form a subset of digraphs, and all eigenvalues of $L$ lie in the closed left half-plane if the communication graphs are undirected. But even excluding undirected communication topologies, we can find many examples of systems where $G_x$ is not essentially cyclic and the eigenvalues of $L$ are in the closed left half-plane for all $\beta > 0$. One such example is given by the system illustrated in Figure 7.11. The corresponding root locus is shown in Figure 7.12. It is straightfor-

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3 rounded to the fourth decimal
ward to verify that both the graphs $G_x$ and $G_\dot{x}$ consist of one reach. Thus, with the use of Theorem 5.6, the corresponding double integrator consensus system is asymptotically consensus stable for all $\beta > 0$.

Thus, whether or not $L$ has complex eigenvalues with a positive real part for certain values of $\beta$ depends on the shape of the graph $G_x$ and on the combination of the graphs $G_x$ and $G_\dot{x}$. It is interesting to note that if $G_x$ contains no edges and $G_\dot{x}$ consists of one reach, the corresponding double integrator consensus system is consensus stable. Formally, we have the following result.

**Lemma 7.8** Let $L = L(\beta) = \left( \begin{array}{cc} 0 & 1 \\ -L_x & -\beta L_x \end{array} \right)$ be given by (S3) on page 47, where $\beta > 0$, and $L_x$ is a weighted digraph Laplacian. If $L_x = 0_{n \times n}$, then the matrix $L$ has no eigenvalues with a positive real part or purely imaginary eigenvalues for all values of $\beta$.

**Proof:** Remember that the eigenvalues of $L$ coincide with the eigenvalues of $P(\lambda) = \lambda^2 I + \lambda \beta L_x + L_x$. If $L_x = 0_{n \times n}$, then $P(\lambda)$ simplifies to $P(\lambda) = \lambda(\lambda + \beta L_x)$. The eigenvalues of $P(\lambda)$ are all values $\lambda_0 \in \mathbb{C}$ that satisfy $\det(P(\lambda_0)) = 0$. It follows directly that the eigenvalues of
7.5 Complex Eigenvalues of $L$

Figure 7.11: An example system that leads to the corresponding matrix $L$ having no eigenvalues with positive real parts for any $\beta$. The corresponding eigenvalue plot is given in Figure 7.12. Note that $G_x$ is not essentially cyclic, despite containing a directed circle consisting of the nodes $\nu_4, \nu_5, \nu_6$.

Figure 7.12: Eigenvalues of the system matrix obtained from the graphs in Figure 7.11 for $0 \leq \beta \leq 15$.

$P(\lambda)$ in this case are the eigenvalues of $-\beta L_x$ united with $n$ times the eigenvalue zero. As $L_x$ is a digraph Laplacian, all its non-zero eigenvalues have a positive real part. Therefore, all non-zero eigenvalues of $-\beta L_x$ have a negative real part. Thus, $L$ does not have eigenvalues with a positive real part or purely imaginary eigenvalues. This concludes the proof. \hfill \square

With the preceding discussion, the following result is intuitive.

**Lemma 7.9** Let $L = L(\beta) = \begin{pmatrix} 0 & I \\ -L_x & -\beta L_x \end{pmatrix}$ be given by (S3) on page 47, where $\beta > 0$, and $L_x$ is the Laplacian of the weighted digraph $G_x$. If $L_x = 0_{n \times n}$ and $G_x$ contains at least one edge, then the matrix $L$ has either an imaginary eigenvalue pair or a complex eigenvalue pair with positive real parts for all values of $\beta$.

**Proof:** Remember that the eigenvalues of $L$ coincide with the eigenvalues of $P(\lambda) = \lambda^2 I + \lambda \beta L_x + L_x$. If $L_x = 0_{n \times n}$, then $P(\lambda)$ simplifies to $P(\lambda) = \lambda^2 I + L_x$. The eigenvalues of $P(\lambda)$ are all values $\lambda_0 \in \mathbb{C}$.
that satisfy $\det(P(\lambda_0)) = 0$. It follows directly that the eigenvalues of $P(\lambda)$ in this case are square roots of the eigenvalues of $-L_x$ and do not depend on $\beta$. As $L_x$ is a digraph Laplacian, all its non-zero eigenvalues have a positive real part. Therefore, if $L_x$ has a positive real eigenvalue, $P(\lambda)$ has a pair of purely imaginary eigenvalues. If $L_x$ has a complex eigenvalue pair, then $P(\lambda)$ has a complex eigenvalue pair with positive real parts. The number of zero eigenvalues of $L_x$ coincides with the number of reaches in the graph $G_x$. Thus, unless the graph consists of $n$ reaches, i.e. $L_x = 0_{n \times n}$, the matrix $L_x$ has at least one non-zero eigenvalue. This concludes the proof. \qed

### 7.6 Acyclic Communication Topologies

So far in this chapter we have tried to achieve the most general results possible. We have arrived at the observation that the spectrum of $L$, and therefore the consensus stability of the corresponding double integrator consensus system, is highly dependent on the structure of the corresponding communication graphs. We have determined that the existence of circles in the position graph can be connected to the spectrum of $L$.

In this section we consider the special case that the union of the graphs $G_x \cup G_\dot{x}$ is an acyclic graph. Acyclic graphs were introduced in Section 3.3 on page 14. They model hierarchic formations and are, therefore, often encountered in applications, see e.g. Bessler et al., [10] or Mesbahi and Hadaegh, [65]. The defining property of an acyclic graph is that it contains no circles. Therefore, an acyclic graph is not essentially cyclic. A necessary but not sufficient condition for the union to be acyclic is that both $G_x$ and $G_\dot{x}$ are acyclic. This is illustrated in Figure 7.13. If a digraph is acyclic, then its Laplacian can be written in upper triangular form, which greatly facilitates the analysis of the system.
Lemma 7.10 Let $L = \begin{pmatrix} 0 & -L_x \\ -L_x & -\beta L_x \end{pmatrix}$ be the system matrix of (S2) on page 35, where $L_x$ is the Laplacian of the weighted digraph $G_x$, $L_\dot{x}$ is the Laplacian of the weighted digraph $G_\dot{x}$ and $\beta > 0$. Suppose that $G_x \cup G_\dot{x}$ is an acyclic graph. System (S2) is

1. consensus stable for all values of $\beta$ if $G_x$ consists of one reach.
2. not consensus stable for any value of $\beta$ if $G_x$ consists of more than one reach.
3. asymptotically consensus stable for all values of $\beta$ if it is consensus stable and additionally $G_x$ consists of one reach.

**Proof:** By Theorem 5.6, system (S2) is consensus stable if and only if $G_x \cup G_\dot{x}$ consists of one reach and all non-zero eigenvalues of $L$ have a negative real part. It is asymptotically consensus stable if additionally $G_x$ consists of one reach.

(1) First we show that if $G_x \cup G_\dot{x}$ is acyclic and $G_\dot{x}$ consists of one reach, then (S2) is consensus stable for all values of $\beta$. To prove this statement we first need to show that the condition that $G_\dot{x}$ consists of one reach and $G_x \cup G_\dot{x}$ is acyclic implies that $G_x \cup G_\dot{x}$ consists of one reach and all non-zero eigenvalues of $L$ have a negative real part for all values of $\beta$.

The first part of this statement is trivially proved by the fact that if $G_\dot{x}$ consists of one reach, then $G_x \cup G_\dot{x}$ also consists of one reach. It remains to show that if $G_x$ consists of one reach and $G_x \cup G_\dot{x}$ is acyclic, then all non-zero eigenvalues of $L$ have a negative real part.

Let $L$ denote the Laplacian of the union graph $G_x \cup G_\dot{x}$. If $G_x \cup G_\dot{x}$ is acyclic, then, with an appropriate node numbering, $L$ is upper triangular. This implies that both $L_x$ and $L_\dot{x}$ are simultaneously upper triangular. The $i$-th diagonal entry of the Laplacian matrices contains the in-degree of the $i$-th node. Then, the quadratic matrix polynomial $P(\lambda) = \lambda^2 I + \lambda \beta L_\dot{x} + L_x$ is also upper triangular. Therefore, its determinant is given by

$$
\det P(\lambda) = \prod_{i=1}^{n} (\lambda^2 + \lambda \beta d_{\dot{x}_i} + d_{x_i}),
$$

where $d_{x_i} \geq 0$ and $d_{\dot{x}_i} \geq 0$ are the in-degrees of node $v_i$, $i = 1, \ldots, n$, in the graphs $G_x$ and $G_\dot{x}$, respectively. Thus, the $2n$ eigenvalues of $P(\lambda)$ are determined by $n$ quadratic equations

$$
\lambda^2 + \lambda \beta d_{\dot{x}_i} + d_{x_i} = 0, \quad i = 1, \ldots, n,
$$

that correspond to the $n$ nodes $v_1, \ldots, v_n$. The following four kinds of quadratic equation can occur depending on the incoming edges of $v_i$:

A. Node $v_i$ has no incoming edges: $d_{x_i} = d_{\dot{x}_i} = 0$. Then the corresponding $i$-th quadratic equation (7.17) simplifies to $\lambda^2 = 0$, which implies that there is a zero eigenvalue of $P(\lambda)$.
b. Node $v_i$ receives only position information: $d_{x_i} = 0$, $d_{\dot{x}_i} \neq 0$. Then we obtain from (7.17) that $\lambda^2 + d_{x_i} = 0$ must hold. Thus, $\lambda_{1/2} = \pm \sqrt{d_{x_i}}$ is a purely imaginary eigenvalue of $P(\lambda)$.

c. Node $v_i$ receives only velocity information: $d_{x_i} = 0$, $d_{\dot{x}_i} \neq 0$. Then, we obtain from (7.17) that $\lambda^2 + \lambda \beta d_{x_i} = 0$ must hold. Thus, there are two solutions $\lambda_1 = 0$ and $\lambda_2 = -\beta d_{x_i}$ that are both eigenvalues of $P(\lambda)$.

d. Node $v_i$ receives both position and velocity information: $d_{x_i} \neq 0$, $d_{\dot{x}_i} \neq 0$. Then, (7.17) has the solutions $\lambda_{1/2} = -0.5\beta d_{x_i} \pm 0.5\sqrt{\beta^2 d_{x_i}^2 - 4d_{x_i}}$. As both $d_{x_i}$, $d_{\dot{x}_i}$ are positive and real, $P(\lambda)$ has two corresponding eigenvalues with a negative real part.

If $G_x \cup G_{\dot{x}}$ is acyclic and $L$ can be written in upper triangular form, then $v_n$ satisfies case (a), i.e. $d_{x,n} = d_{\dot{x},n} = 0$. If $G_x$ consists of one reach, then all the other nodes have at least one incoming edge, i.e. case (b) never occurs. All eigenvalues obtained in case (c) and (d) are either zero or have a negative real part. This implies that algorithm (S2) is consensus stable.

(2) Next, we show that if $G_x$ does not consist of one reach and $G_x \cup G_{\dot{x}}$ is acyclic, then the system is not consensus stable for any value of $\beta$. If $G_x$ consists of more than one reach but the union graph is acyclic, then one of the following two cases occurs:

- $G_x \cup G_{\dot{x}}$ consists of more than one reach. In this case, the result follows directly form Theorem 5.6.
- $G_x \cup G_{\dot{x}}$ consists of one reach. In this case, there is at least one node that receives position, but not velocity information, i.e. case b of the above discussion occurs. Then it follows that $L$ has a purely imaginary eigenvalue pair. By Theorem 5.6 it follows that the system is not consensus stable.

(3) The last statement follows directly from Theorem 5.6.

This concludes the proof.

The defining property of acyclic graphs is that they contain no circles. Information that enters the system is reached down from the root node through the system and updated with the current states of the nodes on the way, until it reaches the leaf nodes. Thus, acyclic graphs model a hierarchic system. It is interesting to see that in this case, it is necessary for consensus stability of the system that each agent, except for the one corresponding with the root node, obtains some velocity information.

Consider now a general double integrator consensus system. Let us denote the nodes of $G_x \cup G_{\dot{x}}$ that are not part of a cycle by $V_a$. Then $V_a$ induces an acyclic subgraph of $G_x \cup G_{\dot{x}}$. We can make a statement about stability of the overall system by looking at these agents.

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4 I.e., nodes that have no outgoing edges.
Figure 7.14: Graph containing an acyclic subgraph that is not part of a circle. Its Laplacian can be written in form (7.18). The nodes belonging to $V_a$ are circled.

**Lemma 7.11** Consider the double integrator consensus system (S2) on page 35 with the corresponding system matrix $L = \begin{pmatrix} 0 & -L_x & -L_x \end{pmatrix}$, where $L_x, L_{\dot{x}}$ are Laplacians of the weighted directed graphs $G_x$ and $G_{\dot{x}}$, respectively, and $\beta > 0$. Let $V_a$ be the set of nodes that are not part of a cycle in $G_x \cup G_{\dot{x}}$. The system is

1. consensus stable only if its subsystem induced by $V_a$ is consensus stable.
2. asymptotically consensus stable only if its subsystem induced by $V_a$ is asymptotically consensus stable.
3. not consensus stable for any value of $\beta$ if its subsystem induced by $V_a$ is not consensus stable.

**Proof:** Suppose that the subgraph induced by $V_a$ consists of one reach. Then, there is a permutation matrix $P$ such that

$$PL_xP^T = \begin{pmatrix} L_{x_{11}} & 0 & 0 \\ L_{x_{21}} & L_{x_a} & 0 \\ L_{x_{31}} & L_{x_{32}} & L_{x_{33}} \end{pmatrix}, \quad PL_{\dot{x}}P^T = \begin{pmatrix} L_{\dot{x}_{11}} & 0 & 0 \\ L_{\dot{x}_{21}} & L_{\dot{x}_a} & 0 \\ L_{\dot{x}_{31}} & L_{\dot{x}_{32}} & L_{\dot{x}_{33}} \end{pmatrix}. \quad (7.18)$$

where $L_{x_a}, L_{\dot{x}_a}$ are lower triangular matrices of size $|V_a| \times |V_a|$. The structure of the corresponding graph is illustrated in Figure 7.14. It follows that

$$\text{spec}(L) = \text{spec}(\lambda^2I + \lambda\beta L_x + L_{\dot{x}}) = \text{spec}(\lambda^2I + \lambda\beta L_{x_{11}} + L_{x_{11}}) \cup \text{spec}(\lambda^2I + \lambda\beta L_{x_{33}} + L_{x_{33}}) \cup \text{spec}(\lambda^2I + \lambda\beta L_{x_a} + L_{x_a}) \quad (7.19)$$

holds. Therefore, the acyclic subsystem and the remaining parts of (S2) must satisfy the conditions of Theorem 5.6 individually. It follows directly that (asymptotic) consensus stability of the acyclic subsystem if necessary for the (asymptotic) stability of the whole system.
Thus, the statements (1) and (2) hold. Statement (3) then follows with Lemma 7.10.

If the subgraph induced by $V_a$ consists of several reaches, then the permutation in (7.18) will consist of more than three diagonal blocks. The number of diagonal blocks will then correspond to the number of reaches in the graph induced by $V_a$. The same argumentation applies. □

Thus, when considering consensus stability of the algorithm, all nodes that have no outgoing edges and obtain velocity information can be neglected and the system can be simplified accordingly. If the remaining system achieves consensus, then so do its acyclic sub-systems. Conversely, if a system is consensus stable, then adding consensus stable acyclic groups of nodes to it will not destabilise it as long as the joint connectivity of the position and velocity graphs does not change and condition 1 of Lemma 7.10 remains satisfied.

7.7 DISCUSSION

Remember that the necessary and sufficient conditions for the double integrator consensus system $(S_2)$ to be consensus stable are that the graph $G_x \cup \dot{G}_x$ consists of one reach and the matrix $\mathcal{L}$ has no eigenvalues with a positive real part and no purely imaginary eigenvalues. These conditions were stated in Theorem 5.6 on page 60. In Chapter 6 we additionally obtained Theorem 6.9 on page 76 under the condition that the communication networks are undirected. Theorem 6.9 expresses the conditions from Theorem 5.6 on the spectrum of $\mathcal{L}$ in terms of the graphs $G_x$ and $\dot{G}_x$ as well as their Laplacian matrices. In particular, it shows that consensus stability of the system does not depend on $\beta$ if the communication networks are undirected.

In this chapter, we obtained a similar result for the special case that the communication networks are directed but the graph $G_x \cup \dot{G}_x$ is acyclic in Section 7.6. In this case, consensus stability of the system depends only on whether or not the agents can obtain velocity information.

No such statements can be made for the more general cases considered in this chapter. We are, however, able to identify a class of systems that are not consensus stable for some values of $\beta$ in Lemma 7.6. A necessary condition for the system to belong to this class is that the graph $G_x$ contains at least one circle. Together with the results in Section 7.6 this leads us to the observation that circles in the graph $G_x \cup \dot{G}_x$ are the reason why some double integrator consensus systems are not consensus stable for small values of $\beta$ and some, like the one illustrated in Example 7.1, are not consensus stable for any value of $\beta$. Furthermore, if $G_x$ consists of one reach and $\dot{G}_x$ is empty, then by Lemma 7.8 the system $(S_2)$ is consensus stable. Thus, addi-
tion of edges to $G_x$ can be detrimental to the consensus stability of the system.

A similar observation can be made about the purely imaginary eigenvalues of $\mathcal{L}$. We give a sufficient condition for $\mathcal{L}$ to have purely imaginary eigenvalues independently of $\beta$ in Lemma 7.5. In fact, this condition is the same as was obtained for undirected graphs in Lemma 6.2. However, as can be seen for example in Figure 7.10, some systems may have additional purely imaginary eigenvalues for only some values of $\beta$.

In summary, we have obtained that the double integrator consensus system (S2) with heterogeneous directed communication topologies is a very complex system. Apart from some special cases, its consensus stability cannot be determined by looking only at the structure of the corresponding communication graphs. This stands in sharp contrast to the results obtained for (S2) under the assumption that the communication networks are undirected, as well as to the results for single integrator consensus systems with digraph communication topologies in Section 4.1.
In the first part of this thesis we introduced the single integrator consensus system \((S_1)\) on page 31, given by

\[
\dot{x}(t) = -Lx(t),
\]
\[
x(0) = x_0,
\]

and the double integrator consensus system \((S_2)\) on page 35, given by

\[
\ddot{x}(t) = -L_x x(t) - \beta L \dot{x}(t),
\]
\[
\dot{x}(0) = x_0, \quad \dot{x}(0) = \dot{x}_0,
\]

where \(L\) and \(L_x\) are \(n \times n\) Laplacians of the communication graphs \(G\) and \(G_x\), respectively. We have defined a corresponding consensus stability notion and presented necessary and sufficient conditions for the systems to be consensus stable. Apart from the initial conditions, these systems have no external inputs. As we have outlined in the previous chapters, their behaviour is determined by the eigenvalues of the corresponding Laplacian matrix \(L\) in the case of \((S_1)\) (see Lemma 4.4), and the eigenvalues of the quadratic matrix polynomial \(P(\lambda) = \lambda^2 I + \lambda \beta L_x + L_x\) in the case of \((S_2)\) (see Theorem 5.6).

We have claimed that consensus algorithms without external inputs are appealing because they operate distributedly and autonomously on large networks of agents. For example, when a consensus stable double integrator consensus algorithm is implemented on a vehicle platoon, its achievement will be that the vehicles arrive at a given formation and continue to move as a group.

While their effectiveness is undeniable, the power of the algorithms \((S_1)\) and \((S_2)\) is limited to providing cohesion to the agent network. To stay with the vehicle platoon example, one might desire that the vehicles arrive at a certain destination, move with a given speed, or change their formation. In this case, we need to modify the algorithms to make them accommodating to our goals. One particular solution is to introduce an external control input to the networked system, which will be done in this chapter.

This approach was first taken by Tanner, [93] for single integrator consensus systems. Tanner assumes that some agents, called leaders,
are controlled from the outside. The remaining agents are deemed followers and execute the protocol (S1). As the agents do not evaluate the information they receive, there is no way for an individual agent to decide whether or not its direct neighbours are following the same algorithm. This allows externally controlled agents to seamlessly attach themselves to the consensus network and to influence the group’s behaviour. These additional agents can be seen as external controllers that steer and correct the network. Then, if the system is controllable via this external input, the leader agents can steer it to the desired formation or consensus value.

In this chapter, we first introduce the single integrator leader-follower consensus system and review its controllability in Section 8.1. We then introduce the double integrator leader-follower consensus system with heterogeneous communication topologies in Section 8.2. Controllability of the double integrator leader-follower consensus system in heterogeneous networks is the main topic of this chapter, and it is studied in Section 8.3. Controllability conditions are given based both on the properties of the underlying communication networks and on linear-algebraic properties of a corresponding quadratic matrix polynomial. Following these generally applicable results, the special case that one of the communication graphs is empty is studied in Section 8.3.1, while the special case that the communication topologies are homogeneous is the topic of Section 8.3.2.

The graph theoretic definitions and results used throughout this chapter can be found in Chapter 3, the notation in Chapter 2 and linear algebraic terms and lemmas in Appendix A.1. Classic controllability notions and results are listed in Appendix A.2. Remember that to improve readability, the signs \(\rightarrow\) and \(\leftarrow\), as well as \(\rightarrow\) and \(\leftarrow\) in the margins indicate whether a result holds for weighted or unweighted digraphs or weighted or unweighted undirected graphs, respectively.

### 8.1 Single Integrator Leader-Follower Consensus System

In this section we define the leader-follower consensus system and introduce the consensus controllability problem for agents with single integrator dynamics. Consider again the system (S1) on page 31, given by

\[
\dot{x}(t) = -Lx(t) \\
x(0) = x_0,
\]

where \(L \in \mathbb{R}^{n \times n}\) is the Laplacian matrix of the communication graph \(G = (V, E, w)\) and \(x : \mathbb{R}^+ \to \mathbb{R}^n\) is the state vector of the agents. As before, the agents are labeled by \(1, \ldots, n\).
Without loss of generality, choose the last \( n_1 < n \) of the agents in the above system to be leaders and let the leaders obtain an external input. The remaining \( n_f := n - n_1 \) agents are called followers and continue to generate their control input via the consensus algorithm.

The distinction in leader and follower agents induces a partitioning of the Laplacian matrix \( L \) and the state vector \( x \) of the system (S1), given by

\[
L = \begin{pmatrix}
-A^f & -B^f \\
-C^f & -D^f
\end{pmatrix}, \quad x(t) = \begin{pmatrix} x_f(t) \\ x_1(t) \end{pmatrix},
\]

(8.1)

where

- \( A^f \in \mathbb{R}^{n_f \times n_f} \), \( B^f \in \mathbb{R}^{n_f \times n_l} \), \( C^f \in \mathbb{R}^{n_l \times n_f} \), \( D^f \in \mathbb{R}^{n_l \times n_l} \), and
- \( x_f : \mathbb{R}^+ \to \mathbb{R}^{n_f}, x_1 : \mathbb{R}^+ \to \mathbb{R}^{n_l} \).

If the leader agents receive the external control input \( u : \mathbb{R}^+ \to \mathbb{R}^{n_l} \), the corresponding control system is given by

\[
\begin{pmatrix}
\dot{x}_f \\
\dot{x}_1
\end{pmatrix} = \begin{pmatrix} A^f & B^f \\ C^f & D^f \end{pmatrix} \begin{pmatrix} x_f \\ x_1 \end{pmatrix} + \begin{pmatrix} 0_{n_f \times n_l} \\ I_{n_l} \end{pmatrix} u(t).
\]

(C1)

The following result states that the leader dynamics, represented by \((C^f, D^f)\), can be neglected when studying controllability of the above system.

**Lemma 8.1** System (C1) is controllable if and only if the matrix pair \((A^f, B^f)\) is controllable.

**Proof:** If \((A^f, B^f)\) is controllable, it follows directly from Lemma A.7 on page 136 of the appendix that (C1) is controllable.

If \((A^f, B^f)\) is not controllable, then by Lemma A.6 there is a vector \( v \in \mathbb{C}^n \) and a value \( \lambda_0 \in \mathbb{C} \) such that \( v^T (A^f - \lambda_0 I \ | \ B^f) = 0_{n_f + n_l} \) holds. Then

\[
\begin{pmatrix} v^T \\ 0_{n_f}^T \end{pmatrix} \begin{pmatrix} A^f - \lambda_0 I & B^f \\ C^f & D^f - \lambda_0 I \end{pmatrix} = 0_{n_f + n_l + n_1}
\]

(8.2)

holds. Therefore, by Lemma A.6, (C1) is not controllable. \( \square \)

Thus, in order to establish controllability of the system (C1), it is sufficient to consider controllability of the system

\[
\begin{align*}
\dot{x}_f(t) &= A^f x_f(t) + B^f u(t), \\
x_f(0) &= x_{f_0},
\end{align*}
\]

(C2)

\[1\] In recent publications followers have also been called floating agents, cf. Mesbahi and Egerstedt, [64] for a discussion. To date, “follower agents” remains the prevalent term in consensus literature.
where \( u : \mathbb{R}^+ \to \mathbb{R}^{n_l} \) is the “internal” control input derived from the states of the leader agents. The above system is from here on referred to as the single integrator leader-follower (LF) consensus system.

Attached to the single integrator LF consensus system is the leader-follower graph \( G^{lf} := (V^{lf}, E^{lf}, w^{lf}) \), that is obtained from the graph \( G = (V, E, w) \) by removing all edges between the leaders and from followers to leaders. Therefore, \( V^{lf} = V \) and \( E^{lf} \subseteq E \). The Laplacian of \( G^{lf} \) is given by

\[
L^{lf} := \begin{pmatrix}
I_{n_f} & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
-A^f & -B^f \\
-C^f & -D^f
\end{pmatrix}
\begin{pmatrix}
0_{n_f \times n_f} & 0_{n_f \times n_l}
\end{pmatrix}. \tag{8.3}
\]

We call the nodes of \( G^{lf} \) corresponding to leader and follower agents leader and follower nodes and use the sets \( V^f \) and \( V^l := V^{lf} \setminus V^f \) to denote them. The follower graph \( G^f := (V^f, E^f, w^f) \) is then the subgraph of \( G^{lf} \) induced by the follower nodes. The leader graph \( G^l := (V^l, \emptyset) \) is the subgraph induced by the leader nodes. The set of edges from leader to follower nodes is denoted by \( E^l := E^{lf} \setminus E^f \). An example of a leader-follower graph is given in Figure 8.1a on page 109.

By construction, a non-trivial LF graph is always directed. However, the follower graph may be directed or undirected. If \( G^f \) is undirected, then \( A^f \) is symmetric. Along with the usual graph-theoretic connectivity notions outlined in Section 3.4, we define leader-follower connectivity of \( G^{lf} \).

**Definition 8.2 (leader-connected, leader-follower connected)** A follower node is called leader-connected if there is a directed path to it from some leader and not leader-connected otherwise. The LF graph is called leader-follower (LF) connected if every follower node is leader-connected.

Undirected follower graphs are LF connected if every connected component of the follower graph has at least one node with an incoming edge from a leader. Directed follower graphs are LF connected if every iSCC\(^2\) of the follower graph has at least one node with an incoming edge from a leader. LF connectivity of the LF graph is directly connected with the rank of \( A^f \), as shown in the following lemma.

**Lemma 8.3** Let \( n_f \) be the number of follower agents in the system (C2), \( A^f \) the corresponding \( n_f \times n_f \) system matrix and \( G^{lf}, G^f \) the LF and the follower graph, respectively. It holds that \( \text{rank}(A^f) = n_f - p \) if and only if \( G^f \) has \( p \) iSCC that are not leader-connected. Particularly, \( \text{rank}(A^f) = n_f \) if and only if \( G^{lf} \) is LF connected.

---

\(^2\) Independent strongly connected component, i.e. a strongly connected subgraph of \( G^f \) that obtains no information from outside itself, cf. Section 3.4 on page 15 for details.
8.1 SINGLE INTEGRATOR LEADER-FOllower CONSensus SYSTEM

(a) Example of a leader-follower graph, node \( v_6 \) is the leader.

(b) Example of a leader-follower graph, the nodes \( v_8 \) and \( v_9 \) are the leaders. The edges \( E^f \) and \( E^l \) are dashed, the edges \( E^\dot{f} \) and \( E^\dot{l} \) are solid lines. A “spread out” version of this graph is shown in Figure 8.2b.

Figure 8.1: Single and double integrator leader-follower graphs.

Proof: In order to prove Lemma 8.3 we first derive an explicit formulation of \( \text{rank}(A^f) \).

Let \( L^f \) denote the Laplacian matrix of the follower graph \( G^f \). Suppose that \( G^f \) consists of \( k \) iSCC of the corresponding sizes \( m_1, \ldots, m_k \). Let the corresponding reaches of \( G^f \) have exclusive parts of size \( k_1, \ldots, k_k \), \( k_i \geq m_i \) and the common part of size \( c := n_f - \sum_{i=1}^{k} k_i \).

Then, assuming an appropriate node numbering, \( L^f \) can be decomposed as, cf. (3.18) on page 23:

\[
L^f = \begin{pmatrix}
L^f_{111} & 0 & 0 & \ldots & \ldots & 0 \\
L^f_{121} & L^f_{122} & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & L^f_{k11} & 0 & 0 \\
0 & 0 & L^f_{k21} & L^f_{k22} & 0 \\
L^f_{k1+1,11} & L^f_{k1+1,12} & \ldots & L^f_{k1+1,k_1} & L^f_{k1+1,k_2} & L^f_{k1+1,k+1}
\end{pmatrix}
\]  (8.4)

Here, for \( i = 1, \ldots, k \),

- the blocks \( L^f_{i11} \in \mathbb{R}^{m_i \times m_i} \) describe the edges between the nodes in the iSCC of the i-th reach,
- the blocks \( \{L^f_{i21} | L^f_{i22}\} \in \mathbb{R}^{(k_i - m_i) \times k_i} \) describe the edges incoming to the nodes that belong to the exclusive part of the i-th reach, but not to the iSCC,
- the blocks \( \{L^f_{k1+1,1i} | L^f_{k1+1,k}\} \in \mathbb{R}^{c \times k_i} \) describe the edges incoming from the exclusive part of the i-th reach to the common part of the reaches, and
- the block \( L^f_{k1+1,k+1} \in \mathbb{R}^{c \times c} \) describes the edges within the common part of the reaches.
By construction, the blocks $L^f_{12i}$, $i = 1, \ldots, k$, and $L^f_{k+1,k+1}$, if they exist, are irreducibly diagonally dominant and, thus, of full rank. It follows from Lemma 3.6 that $L^f$ has $k$ zero eigenvalues. The corresponding set of left eigenvectors is given by, cf. (3.19) on page 23,

$$\left\{ \begin{pmatrix} q_1 \\ q_{n_1-m_1} \\ \vdots \\ q_{n_f-k_1-m_2} \\ q_{k} \end{pmatrix}, \begin{pmatrix} q_{n_1} \\ q_2 \\ \vdots \\ q_{n_f-k} \\ q_{k} \end{pmatrix}, \ldots, \begin{pmatrix} q_{n_f-k} \\ q_{k} \end{pmatrix} \right\},$$

(8.5)

where $q_1 \in \mathbb{R}^{m_1}$ are positive vectors.

Remember that the Laplacian matrix $L^f$ of $G^f$ is given by $L^f = \begin{pmatrix} -A^f & -B^f \\ -n_1 \times n_f & \emptyset \end{pmatrix}$. It immediately follows that $-A^f = L^f + \text{diag}(\sum_{i=1}^{n_f} (B^f)_{ij})$. Therefore, $-A^f$ can be decomposed as $-A^f =

$$\begin{pmatrix} L_{11}^f + d(b_{11}^f) & 0 & \cdots & 0 \\ L_{21}^f & L_{22}^f + d(b_{12}^f) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ L_{k+1,1}^f & \cdots & L_{k,1}^f + d(b_{k1}^f) & 0 \end{pmatrix},$$

(8.6)

where $d(b_{1i}^f), d(b_{12}^f), i = 1, \ldots, k$, and $d(b_{k1}^f)$ denote the diagonal matrices with the row sums of the corresponding rows of $B^f$ on the diagonal. All the blocks $L^f_{12i}$, $i = 1, \ldots, k$, and $L^f_{k+1,k+1}$, are irreducibly diagonally dominant with a positive diagonal. All entries of $B^f$ are non-negative. It thus follows that the sums $L^f_{12i} + d(b_{1i}^f), i = 1, \ldots, k$ and $L^f_{k+1,k+1} + d(b_{k1}^f)$ are also irreducibly diagonally dominant matrices and thus have full rank. Therefore,

$$\text{rank}(A^f) = \sum_{i=1}^{k} (k_i - m_i) + c + \sum_{i=1}^{k} \text{rank}(L^f_{11} + d(b_{1i}^f)) = n_f - \sum_{i=1}^{k} m_i + \sum_{i=1}^{k} \text{rank}(L^f_{11} + d(b_{1i}^f)).$$

(8.7)

It follows that $\text{rank}(A^f) = n_f$ if and only if for $i = 1, \ldots, k$, $\text{rank}(L^f_{11} + d(b_{1i}^f)) = m_i$.

Now, if the $i$-th iSCC of $G^f$ has no incoming edges from a leader, then the corresponding matrix $d(b_{1i}^f)$ in (8.6) is zero. Therefore, $\text{rank}(L^f_{11} + d(b_{1i}^f)) = \text{rank}(L^f_{11}) = m_i - 1$. If $G^f$ has $p \leq k$ iSCC that are not leader-connected, then $\text{rank}(A^f) = n_f - p$ immediately follows.

On the other hand, if, for some $i \in \{1, \ldots, k\}$, the $i$-th iSCC of $G^f$ has an incoming edge from a leader, then there is at least one non-zero entry on the diagonal of $d(b_{1i}^f)$. As the corresponding block of $L^f_{11}$. is
the Laplacian of an iSCC, it is irreducible. Therefore, $L_{i1}^f + d(b_{i1}^f)$ is irreducibly diagonally dominant and, thus, of full rank.

Particularly, if all iSCC of $G^f$ are leader-connected, $\text{rank}(A^f) = n_f$ follows.

Example 8.4 Consider again the system given by the graph in Example 3.1 on page 17. A leader-follower system can be obtained from it by choosing up to three nodes as leaders. Clearly the obtained LF graph will not be LF connected unless $\{v_1, v_3\} \subseteq V^l$. The resulting system with $V^l = \{v_1, v_3\}$ then reads

$$\dot{x}_f(t) = \begin{pmatrix} -1 & 0 \\ 1 & -3 \end{pmatrix} x_f(t) + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} u(t), \quad x_f(0) = x_{f,0} (8.8)$$

and is controllable due to $B^f$ having full rank.

For $n_1 = 1$ and an unweighted undirected follower graph, Lemma 8.3 is identical to Proposition 10.5 of Mesbahi and Egerstedt, [64]. We will further need the following definition.

Definition 8.5 (leader-symmetric LF graph, Mesbahi and Egerstedt, [64], Proposition 10.13) An LF graph is called leader-symmetric if and only if there is a non-identity automorphism of $G^l$ such that the nodes in $V^l$ remain invariant under its action.

Example 8.6 The graph in Figure 8.1a, where $v_6$ is the leader node, is leader-symmetric. There is an automorphism acting on the nodes $v_1, v_2$.

The system (C2) was first suggested by Tanner, [93], and then further extended by Björkenstam et al., [14], Rahmani and Mesbahi, [79], Rahmani and Mesbahi, [80], Ji and Egerstedt, [51], Ji et al., [52], Rahmani et al., [81], Egerstedt et al., [31], Lou and Hong, [59]. The goal of research focused on LF consensus systems is to connect the matrix-based controllability conditions in Lemma 8.8 with the properties of the graph $G^l$. No graph-theoretic conditions that are both necessary and sufficient for controllability have been obtained to date, however a set of necessary conditions exists. The remainder of this section gives an overview of existing results for (C2) that we will use as a reference point for the double integrator LF consensus system in Section 8.3. It follows the recently published book by Mesbahi and Egerstedt, [64] and the paper by Lou and Hong, [59].

Lemma 8.7 (Mesbahi and Egerstedt, [64], Proposition 10.5) If the LF consensus system (C2) is controllable, then the LF graph $G^l$ is leader-follower connected.
This result is proved for unweighted graphs in [64]. However, its extension to weighted digraphs is straightforward using Lemma 8.3. Furthermore, the following linear-algebraic necessary and sufficient condition exists for undirected graphs.

\[ \text{Lemma 8.8 (Mesbahi and Egerstedt, [64], Proposition 10.2) The LF system } (C_2) \text{ is controllable if and only if none of the left eigenvectors of } A^f \text{ are simultaneously orthogonal to all columns of } B^f. \]

This result is proved for unweighted undirected graphs in [64]. However, its extension to weighted directed graphs is trivial using Lemma A.6. Additionally, the following result can be given for undirected graphs.

\[ \text{Lemma 8.9 (Mesbahi and Egerstedt, [64], Lemma 10.4) Suppose that } G \text{ is an undirected graph. The LF system } (C_2) \text{ is controllable if and only if } L(G) \text{ and } -A^f \text{ do not share an eigenvalue.} \]

If the graph G is directed, it follows from the proof in [64], that the above lemma gives only a sufficient condition for controllability. That is, if L(G) and -A^f do not share an eigenvalue, then (C_2) is controllable. Finally, we have the following result.

\[ \text{Lemma 8.10 The LF system } (C_2) \text{ is not controllable if either of the following conditions holds.} \]

1. \( A^f \) has an eigenvalue with geometric multiplicity greater than the number of linearly independent columns of \( B^f \) ([64], Proposition 10.3).3
2. The LF graph is leader-symmetric ([59], Corollary 3.10).
3. There exist non-trivial almost equitable partitions on \( G^{lf} \) and \( G^f \), say \( \pi \) and \( \pi_f \), such that all non-trivial cells of \( \pi \) are contained in \( \pi_f \) ([59], Theorem 3.4).

### 8.2 Double Integrator LF Consensus System

We can define the double integrator leader-follower consensus system corresponding to (S2) on page 35 in the same manner as for the single integrator system. We consider a network of \( n \) agents with double integrator dynamics running the consensus algorithm (S2), given by

\[
\begin{align*}
\ddot{x}(t) &= -L_x x(t) - \beta L_x \dot{x}(t), \\
x(0) &= x_0, \quad \dot{x}(0) = \dot{x}_0,
\end{align*}
\]

\[ (S2) \]

---

3 This is proved for unweighted graphs in [64], however its extension to weighted graphs is trivial.
where $\beta > 0$, $L_x$ is the Laplacian matrix of the position graph $G_x$ and $L_\dot{x}$ is the Laplacian matrix of the velocity graph $G_\dot{x}$. As in the previous section, we without loss of generality choose the last $n_1 < n$ of the agents in the above system to be leaders and call the remaining $n_f = n - n_1$ agents followers. The followers continue to obey the consensus algorithm, while the leaders obtain an additional external input. As the communication networks are heterogeneous, we assume that the leaders may communicate either their position, their velocity or both. The corresponding partitioning of the Laplacian matrices is then

$$L_x = \begin{pmatrix} -A_x^f & -B_x^f \\ -C_x^f & -D_x^f \end{pmatrix}, \quad \beta L_x = \begin{pmatrix} -A_x^f & -B_x^f \\ -C_x^f & -D_x^f \end{pmatrix}, \quad x(t) = \begin{pmatrix} x_f(t) \\ x_1(t) \end{pmatrix}. \quad (8.9)$$

Here, $A_x^f, A_\dot{x}^f \in \mathbb{R}^{n_f \times n_f}$, $B_x^f, B_\dot{x}^f \in \mathbb{R}^{n_f \times n_1}$, $C_x^f, C_\dot{x}^f \in \mathbb{R}^{n_f \times n_f}$, and $D_x^f, D_\dot{x}^f \in \mathbb{R}^{n_f \times n_1}$. Using the partitioning (8.9), we can rewrite the system (S2) as the following control system

$$\begin{pmatrix} \dot{x}_f(t) \\ \dot{x}_1(t) \end{pmatrix} = \begin{pmatrix} 0 & I_{n_f} & 0 & 0 \\ A_x^f & A_\dot{x}^f & B_x^f & B_\dot{x}^f \\ 0 & 0 & 0 & I_{n_1} \\ C_x^f & C_\dot{x}^f & D_x^f & D_\dot{x}^f \end{pmatrix} \begin{pmatrix} x_f(t) \\ \dot{x}_f(t) \\ x_1(t) \end{pmatrix} + \begin{pmatrix} 0_{2n_f \times 2n_1} \\ I_{2n_1} \end{pmatrix} u(t). \quad (C3)$$

where $x_f : \mathbb{R}^+ \to \mathbb{R}^{n_f}$, $x_1 : \mathbb{R}^+ \to \mathbb{R}^{n_1}$ are the positions of the leader and follower agents and $u : \mathbb{R}^+ \to \mathbb{R}^{2n_1}$ is the external input available to the leader agents. The following lemma shows that the leader dynamics can be neglected when considering controllability of (C3).

**Lemma 8.11 (Goldin, [39])** System (C3) is controllable if and only if the matrix pair $\left( \begin{pmatrix} 0 & I_{n_f} \\ A_x^f & A_\dot{x}^f \end{pmatrix}, \begin{pmatrix} 0_{n_f \times n_1} & 0_{n_f \times n_1} \\ B_x^f & B_\dot{x}^f \end{pmatrix} \right)$ is controllable.

**Proof:** If $\left( \begin{pmatrix} 0 & I_{n_f} \\ A_x^f & A_\dot{x}^f \end{pmatrix}, \begin{pmatrix} 0_{n_f \times n_1} & 0_{n_f \times n_1} \\ B_x^f & B_\dot{x}^f \end{pmatrix} \right)$ is controllable, then it follows from Lemma A.7 of the appendix that (C3) is controllable.

If the matrix pair is not controllable, then by Lemma A.6, there is a vector $v \in \mathbb{C}^{2n_f}$ and a value $\lambda_0 \in \mathbb{C}$ such that

$$v^T \begin{pmatrix} -\lambda_0 I_{n_f} & I_{n_f} & 0 & 0 \\ A_x^f & A_\dot{x}^f - \lambda_0 I_{n_f} & B_x^f & B_\dot{x}^f \end{pmatrix} = \Omega^T_{2n_f + 2n_1} \quad (8.10)$$

holds. But then it also holds that

$$\begin{pmatrix} v^T \\ \Omega^T_{2n_f} \end{pmatrix} \begin{pmatrix} -\lambda_0 I_{n_f} & I_{n_f} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ A_x^f & A_\dot{x}^f - \lambda_0 I_{n_f} & B_x^f & B_\dot{x}^f & 0 & 0 \\ 0 & 0 & -\lambda_0 I_{n_1} & I_{n_1} & 0 & 0 \\ C_x^f & C_\dot{x}^f & D_x^f & D_\dot{x}^f & 0 & I_{n_1} \end{pmatrix} = \Omega^T_{2n_f + 4n_1}. \quad (8.11)$$

Therefore, by Lemma A.6, (C3) is not controllable. □
Remark 8.12 Lemma 8.11 is based on the implicit assumption that the position and velocity of the leader can be treated as two independent control inputs. Depending on the system, this may not be the case and only position or velocity of the leader may be available as an independent control input. In this case, either the matrix $B^L_\chi$ or $B^f_\chi$ should be set to zero in the corresponding model. Lemma 8.11, as well as all results obtained hereafter remain applicable.

Let $B^{\text{2int}}_f := (B^L_\chi | B^f_\chi)$. Following Lemma 8.11, we henceforth investigate the controllability of the system

$$
\begin{aligned}
(\dot{x}_f(t), \dot{\dot{x}}_f(t), x_f(0), \dot{x}_f(0)) = (0, I, A^f_\chi A^L_\chi, x_f(t), \dot{x}_f(t)) + (0, B^{\text{2int}}_f, u(t),)
\end{aligned}
$$

(C4)

where $u(t) = (x^T_f(t), \dot{x}^T_f(t))^T$. This system can be rewritten as a double integrator LF system in a straightforward way:

$$
\dot{x}_f(t) = A^f_\chi x_f(t) + A^L_\chi \dot{x}_f(t) + B^{\text{2int}}_f u(t).
$$

(C5)

Attached to this system are the velocity LF graph $G^V_\chi = (V^f \cup V^L, E^V_\chi, w^V_\chi)$ and the position LF graph $G^P_\chi = (V^f \cup V^L, E^P_\chi, w^P_\chi)$, that are obtained from $G^V_\chi$ and $G^P_\chi$ by removing all the edges between the leader nodes and from follower to leader nodes. In complete analogy with the single integrator LF graph defined in the previous section, the corresponding Laplacians of $G^V_\chi$ and $G^P_\chi$ are given by

$$
L^V_\chi = \left( \begin{array}{cc} -A^f_\chi & -B^f_\chi \\ 0_{n_1 \times n_f} & 0_{n_1 \times n_1} \end{array} \right), \quad L^P_\chi = \left( \begin{array}{cc} -A^f_\chi & -B^f_\chi \\ 0_{n_1 \times n_f} & 0_{n_1 \times n_1} \end{array} \right).
$$

(8.12)

The double integrator LF graph $G^{\text{2int}}_\chi = (V^{\text{2int}}_\chi, E^{\text{2int}}_\chi, w^{\text{2int}}_\chi)$ is $G^V_\chi \cup G^P_\chi$ then consists of

- the induced velocity follower graph $G^V_\chi = (V^f, E^V_\chi, w^V_\chi)$,
- the induced position follower graph $G^P_\chi = (V^f, E^P_\chi, w^P_\chi)$,
- the induced leader graph $G^L = (V^L, \emptyset)$, and
- the directed edges from the leader nodes to the follower nodes,

$E^L_\chi$ and $E^L_\chi$ with the associated weight functions $w^L_\chi$ and $w^L_\chi$.

Note that unlike the graphs considered so far, $G^{\text{2int}}_\chi$ is allowed to have multiple edges whenever an edge between two nodes is contained in both $G^V_\chi$ and $G^P_\chi$. This will not be a problem, as we only use Laplacian matrices corresponding with the position and velocity LF graphs in the following and these graphs do not have multiple edges. An example of a leader-follower graph and the contained subgraphs is given in Figure 8.1b and Figure 8.2.
Definition 8.13 (second order leader-follower connected) The LF graph corresponding with the double integrator LF consensus problem (C5) is called second order leader-follower connected (SOLF connected) if either of the following holds:

- the position leader-follower graph $G^\lf_x$ is LF connected, or
- $G^\lf_x$ has a subgraph $G_{nc} := (V_{nc}, E_{nc}, w_{nc})$ that consists of $k$ not leader-connected reaches. However, the graph $G^\lf_x \cup (V^f \cup V^l, E^l_x, w^l_x)$ is LF connected, and there are at least $k$ nodes in $V_{nc}$ that are connected to linearly independent sets of leaders.

Note that the second condition implies that $n_l \geq k$. The condition that the nodes in $V_{nc}$ are connected to linearly independent sets of leaders is equivalent to saying that the submatrix of $B^f_x$ obtained by taking the rows corresponding to $V_{nc}$ has at least rank $k$.

Example 8.14 Both Figure 8.2a and Figure 8.2b show examples of SOLF connected LF graphs. The graph in Figure 8.2a is SOLF connected because $G^\lf_x$ is LF connected, while Figure 8.2b illustrates a graph that is SOLF connected even though $G^\lf_x$ is not LF connected.

If the LF graph is SOLF connected, then it is LF connected, but the converse does not necessarily hold. Therefore, unlike LF connectivity in Lemma 8.3 on page 108, SOLF connectivity does not imply that the matrices $A^f_x, A^l_x$ have full rank. Instead we have the following result.

Lemma 8.15 (Goldin, [39]) Let $A^f_x \in \mathbb{R}^{n_t \times n_t}, B^f_{2\text{int}} \in \mathbb{R}^{n_t \times n_l}$ be given by (C5) with the corresponding double integrator LF graph $G^\lf_{2\text{int}}$. The matrix $(A^f_x \mid B^f_{2\text{int}})$ has rank $n_t$ if and only if $G^\lf_{2\text{int}}$ is SOLF connected.

Proof: As before, we let $G^\lf_x$ and $G^f_x$ denote the position leader-follower and the position follower graph, respectively. First we show
that if the graph $G_{2\text{int}}^{lf}$ is SOLF connected, then the rank condition follows.

Suppose that the double integrator LF graph is SOLF connected, i.e. one of the conditions of Definition 8.13 is satisfied. If the first condition holds, i.e. $G_{x}^{lf}$ is LF connected, then by Lemma 8.3 on page 108, $A_{x}^{f}$ has rank $n_{f}$ and the result follows.

Now consider the second possible condition of Definition 8.13. If $G_{x}^{lf}$ is not LF connected, then there are $k$ iSCC of $G_{x}^{lf}$ that are not LF connected. Then, by Lemma 8.3, rank($A_{x}^{f}$) = $n_{f} - k$. Without loss of generality, let the not leader connected reaches of $G_{x}^{lf}$ consist of $k_{1}$ nodes. Clearly, $k_{1} \geq k$ nodes in the subgraph corresponding to $A_{x}^{f}$.

As we are assuming that $G_{x}^{lf}$ is SOLF connected, there are at least $k$ leaders and, therefore, $B_{x,nc}^{f}$ has at least $k$ columns. Furthermore, as there are at least $k$ nodes in the subgraph corresponding to $A_{x}^{f}$ that are connected to linearly independent sets of leaders, rank($B_{x,nc}^{f}$) = $k_{1}$ holds. Thus, this condition is satisfied.

Assume, without loss of generality, that $k = 1$. By Lemma 3.6 on page 22, $A_{x}^{f}$ has exactly one left eigenvector $v$ with the property that $v$ is non-negative. Furthermore, $(v)_{i} > 0$ only if the corresponding node $v_{i}$ belongs to the iSCC of the reach. Therefore, $w^{T} = v^{T}$ is the only solution candidate of (8.14).

All entries of $B_{x,nc}^{f}$ are either zero or $-1$. As all entries of $v$ are either positive or zero, it follows that $v^{T}B_{x,nc}^{f} = 0_{n_{f}}$ is satisfied only if all rows of $B_{x}^{f}$ corresponding to the iSCC of the reach are identically zero. As the graph $G_{2\text{int}}^{lf}$ is SOLF connected there is at least one edge
from a leader node to a node in the iSCC of \(A_{\text{nc}}^f\). Therefore there is at least one entry in one row of \(B_{x,\text{nc}}^f\) corresponding with to the iSCC that is not zero. Thus, no non-zero solution of (8.14) exists and condition b is also satisfied. Therefore, \(\text{rank}(A_{\text{nc}}^f | B_{x,\text{nc}}^f) = k_1\) follows.

Now we show that \((A_x^f | B_{\text{2int}}^f)\) has rank \(n_f\) only if \(G_{\text{2int}}^f\) is SOLF connected. If the graph is not SOLF connected, then it follows that \(G_{\text{2int}}^f\) is not LF connected. Furthermore, either of the following holds:

- There are either less than \(k\) nodes in the not leader-connected reaches of \(G_{\text{2int}}^f\) that are connected to linearly independent sets of followers and \(\text{rank}(B_{x,\text{nc}}^f) < k\) holds.
- The graph corresponding to \((A_x^f | B_{\text{2int}}^f)\) is not LF connected.

Both conditions immediately imply that \(\text{rank}(A_x^f | B_{\text{2int}}^f) < n_f\).

The notion of leader symmetry translates naturally from the single to the double integrator LF graph. Leader-symmetric single integrator LF graphs were defined in Definition 8.5.

Definition 8.16 (leader-symmetric double integrator LF graph) A double integrator LF graph is called leader-symmetric if and only if \(G_x^f\) and \(G_x^f\) share a non-identity automorphism such that the nodes in \(V_x\) remain invariant under its action.

In the following section we extend the necessary and sufficient conditions for controllability of (C2) on page 107, given in Lemma 8.7-8.10, to the system (C5) on page 114.

8.3 CONTROLLABILITY CONDITIONS FOR DOUBLE INTEGRATOR LF CONSSENSUS SYSTEMS

In a classic result, Arnold and Laub, [5], relate controllability of systems of the form (C5) on page 114 to properties of quadratic matrix polynomials.

Lemma 8.17 (Arnold and Laub, [5], Theorem 2) Let \(A_x^f, A_x^f \in \mathbb{R}^{n_x \times n_x}, B_{\text{2int}}^f \in \mathbb{R}^{n_x \times n_i}, x_f : \mathbb{R}^+ \to \mathbb{R}^{n_x} \) and \(u : \mathbb{R}^+ \to \mathbb{R}^{n_i}\). The system \(\dot{x}_f(t) = A_x^f x_f(t) + A_x^f x_f(t) + B_{\text{2int}}^f u(t)\) is controllable if and only if

\[
\text{rank}(\lambda^2 I - \lambda A_x^f - A_x^f | B_{\text{2int}}^f) = n_f \quad \text{for all} \quad \lambda \in \mathbb{C}. \quad (8.15)
\]

Remark 8.18 Since Arnold and Laub study mechanical systems, they make the in that context natural assumption that \(A_x^f\) is symmetric positive definite. However, their proof does not use this assumption. Thus, it is apparent that Lemma 8.17 holds for any matrices \(A_x^f, A_x^f\).
From this result we can infer matrix theoretic controllability conditions for the double integrator LF consensus system \((C_4)\) on page 114 in the spirit of the previous section. From here on let

\[
P^f(\lambda) := \lambda^2 I - \lambda A^f - A^f_x, \quad \lambda \in \mathbb{C}
\]

\[P(\lambda) := \lambda^2 I + \lambda \beta L_x + L_x, \quad \lambda \in \mathbb{C}.
\]

The following result is a natural extension of Lemma 8.7 to the double integrator consensus system.

\[\text{Lemma 8.19 (Goldin, [39])} \quad \text{If the double integrator LF consensus system \((C_5)\) is controllable, then the LF graph } G^\text{lf}_{2\text{int}} \text{ is SOLF connected.}
\]

**Proof:** Set \(\lambda = 0\) in equation (8.15), then the result follows from Lemma 8.15 and Lemma 8.17. \(\square\)

The double integrator version of Lemma 8.8 reads:

\[\text{Lemma 8.20 (Goldin, [39])} \quad \text{Let } P^f(\lambda), P(\lambda) \text{ be given by (8.16), and } B^\text{lf}_{2\text{int}} \text{ be the input matrix of the system } (C_5). \text{The double integrator LF system } (C_5) \text{ is controllable if and only if none of the left eigenvectors of } P^f(\lambda) \text{ are simultaneously orthogonal on all columns of } B^\text{lf}_{2\text{int}}.
\]

**Proof:** If system \((C_5)\) is not controllable, then by Lemma 8.17 \(\rank(\lambda^2 I - \lambda A^f_x - A^f | B^\text{lf}_{2\text{int}}) < n_f\) for all \(\lambda \in \mathbb{C}\). Thus, there exists a vector \(v \in \mathbb{C}^{n_f}\) and a value \(\lambda_0 \in \mathbb{C}\) such that

\[
v^T(\lambda_0^2 I - \lambda_0 A^f_x - A^f | B^\text{lf}_{2\text{int}}) = 0^T_{n_f},
\]

(8.17)

holds. Thus, \(v^T(\lambda_0^2 I - \lambda_0 A^f_x - A^f | B^\text{lf}_{2\text{int}}) = 0^T_{n_f},\) i.e. \((\lambda_0, v)\) is a left eigenpair of \(P^f(\lambda)\). Moreover, \(v^T B^\text{lf}_{2\text{int}} = 0^T_{n_f},\) i.e. that \(v\) is orthogonal on every column of \(B^\text{lf}_{2\text{int}}\).

If the system \((C_5)\) is controllable, then by Lemma 8.17 \(\rank(\lambda^2 I - \lambda A^f_x - A^f | B^\text{lf}_{2\text{int}}) = n_f\) holds for all \(\lambda \in \mathbb{C}\). Thus, there is no vector \(v \in \mathbb{C}^{n_f}\) such that

\[
v^T(\lambda_0^2 I - \lambda_0 A^f_x - A^f | B^\text{lf}_{2\text{int}}) = 0^T_{n_f},
\]

(8.18)

for some \(\lambda_0 \in \mathbb{C}\). Any \(v\) that satisfies \(v^T(\lambda_0^2 I - \lambda_0 A^f_x - A^f_x | B^\text{lf}_{2\text{int}}) = 0^T_{n_f}\) is a left eigenvector of \(P^f(\lambda)\). Therefore, no eigenvector of \(P^f(\lambda)\) is simultaneously orthogonal on all columns of \(B^\text{lf}_{2\text{int}}. \square\)

The following result gives a formulation of Lemma 8.9 for double integrator LF systems with directed communication topologies.

\[\text{Lemma 8.21} \quad \text{Let } P^f(\lambda) \text{ be given by (8.16a), and } B^\text{lf}_{2\text{int}} \text{ be the input matrix of the system } (C_5). \text{The double integrator LF system } (C_5) \text{ is controllable if } P^f(\lambda) \text{ and } P(\lambda) \text{ do not share an eigenvalue.}\]
Theorem 8.23 (Goldin, [39]) Let $P^f(\lambda)$ be given by (8.16a), and $B^f_{2\text{int}}$ be the input matrix of the system (C5). Then, the double integrator LF system (C5) is not controllable if either of the following holds.

1. $P^f(\lambda)$ has an eigenvalue with geometric multiplicity greater than the number of linearly independent columns of $B^f_{2\text{int}}$.
2. The double integrator LF graph is leader-symmetric.

Proof: (1) Suppose that $P^f(\lambda)$ has an eigenvalue $\lambda_0 \in \mathbb{C}$ with the geometric multiplicity $m_1 > \text{rank}(B^f_{2\text{int}})$. Then,

$$\text{rank}(P^f(\lambda_0)) = \text{rank}(\lambda_0^2I - \lambda_0 A^f_x - A^f_x) = n_f - m_1 < n_f - \text{rank}(B^f_{2\text{int}}).$$

Thus, the rank of $(\lambda_0^2I - \lambda_0 A^f_x - A^f_x \mid B^f_{2\text{int}})$ can be at most $n_f - m_1 + \text{rank}(B^f_{2\text{int}}) < n_f$ and the system is not controllable by Lemma 8.17.
(2) By Definition 8.16, if $G_{2 \text{int}}^f$ is leader-symmetric, then there exists a non-identity leader-invariant automorphism on $G_x^f$ and $G_x^l$. By Lemma 3.10 on page 25 there exist $J = \left( \begin{array}{cc} 1 & 0 \\ 0 & J_2 \end{array} \right)$, $J_1 \in \mathbb{R}^{n_f \times n_f}$, $J_2 \in \mathbb{R}^{n_l \times n_l}$, such that

$$J_1 A_x^f = A_x^f J_1, \quad J_1 A_x^l = A_x^f J_1, \quad J_1 B_x^f = B_x^f J_2, \quad J_1 B_x^l = B_x^l J_2 \quad (8.22)$$

holds. This implies that

$$J_1 P^f(\lambda) = P^f(\lambda) J_1.$$ \quad (8.23)

If $\lambda_0$ is an eigenvalue of $P(\lambda)^f$ with the left eigenvector $v$, then

$$v^T P^f(\lambda_0) J_1 = (v^T J_1) P^f(\lambda_0) = 0_{n_f}^T \quad (8.24)$$

holds, i.e. $J_1 v$ is also a left eigenvector of $P(\lambda)^f$ affording $\lambda_0$. Then, clearly, $v - J_1 v$ is also a left eigenvector of $P(\lambda)^f$ affording $\lambda_0$. Furthermore, as the non-zero rows of $B_x^f$ and $B_x^l$ correspond to fixed points of the automorphism, $J_2 = I_{n_l}$ must hold. Then,

$$(v^T - v^T J_1) (\lambda B_x^f + B_x^l) = v^T (\lambda B_x^f + B_x^l) - v^T J_1 (\lambda B_x^f + B_x^l) =$$

$$v^T (\lambda B_x^f + B_x^l) - v^T (\lambda B_x^f + B_x^l) J_2 = 0_{n_l}^T.$$ \quad (8.25)

That is, the vector $v - J_1 v$ is orthogonal on all columns of $B_{2 \text{int}}^f$ and by Lemma 8.20 the system is not controllable. \hfill \Box

Note that condition (3) of Lemma 8.10 on page 112 can also be restated for system (C5) on page 114. However, the condition that is obtained in the process implies that $G_x = G_x$ must hold, which is a special case considered in Section 8.3.2.

The following result implicitly shows that the intuitive extension of condition (3) to the system (C5) is wrong. If the pairs $G_x, G_x^f$ and $G_x, G_x^l$ both satisfy condition (3) and the composite graphs $G_x \cup G_x$, $G_x^f \cup G_x^l$ satisfy condition (3), no statement can be made about the controllability of the corresponding double integrator system. This is formally stated in the following.

$\Rightarrow$ \quad \textbf{Lemma 8.24} \quad The matrix pairs $(A_x^f, B_x^f), (A_x^l, B_x^l)$ being controllable is not necessary for the system (C5) on page 114 to be controllable.

\textbf{Proof:} Consider the following counter-example. Let

$$A_1 = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -4 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & -3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -4 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -6 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & -5 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & -5 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -3 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & -3 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & -3 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & -3 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -4 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T.$$


The corresponding LF graphs are shown in Figure 8.3. Both graphs contain the same almost equitable partition, given by \( \{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}, \{v_7, v_8\}, \{u\} \). The follower graphs contain almost equitable partitions with the same non-trivial cells. Thus, by Lemma 8.10 neither the matrix pair \((A_1, B_1)\) nor the matrix pair \((A_2, B_2)\) are controllable. However, letting \( A_f^x = A_2, A_f^\dot{x} = A_1, B_f^\text{int} = (B_1 | B_2) \) in (C5) on page 114 results in a controllable double integrator LF system. \( \square \)

Interestingly, letting \( A_f^x = A_1, A_f^\dot{x} = A_2 \) in Lemma 8.24 results in a double integrator LF system that is not controllable. Note that both the systems \((A_1, B_2)\) and \((A_2, B_1)\) are controllable individually.

The conditions stated so far hold for any matrices \( A_f^x, A_f^\dot{x}, B_f^\text{int} \). In particular they hold for any choice of graphs. In the following we will extend these conditions to some special cases.

### 8.3.1 One of the communication graphs is empty

First, let us assume that only position information is exchanged among the agents.

**Lemma 8.25** Consider the double integrator LF consensus algorithm (C5) on page 114. If no velocity information is available to the followers, i.e. \( A_f^x = 0_{n_f \times n_f} \) holds, then the system is controllable if and only if \( (A_f^x, B_f^x) \) is controllable.

**Proof:** First note that if \( A_f^x = 0_{n_f \times n_f} \) this implies that \( B_f^x = 0_{n_f \times n_f} \) due to our modelling choice. Furthermore, if \( A_f^x = 0_{n_f \times n_f} \) and \( B_f^x = 0_{n_f \times n_f} \), then the condition

\[
\text{rank}(\lambda^2 I - \lambda A_f^x - A_f^\dot{x} | B_f^\text{int}) = n_f \quad \text{for all } \lambda \in \mathbb{C}
\]  

of Lemma 8.17 is reduced to

\[
\text{rank}(\lambda^2 I - A_f^x | B_f^x) = n_f \quad \text{for all } \lambda \in \mathbb{C}.
\]
This is precisely the condition that \((A_x^f, B_x^f)\) has to be controllable. □

From Lemma 7.9 on page 97 we know that that the system \(\ddot{x}(t) = -L_x x(t)\) is not consensus stable. Now we have shown that if a set of leader agents is chosen that makes the resulting system \((A_x^f, B_x^f)\) controllable, then the system can be driven to a desired position and velocity, including position and velocity consensus, with only position measurements provided.

Next, suppose that only velocity information is available to the followers.

\[
\text{Lemma 8.26} \quad \text{Consider the double integrator LF consensus system } (C_5). \text{ If no position information is available to the followers, i.e. } A_x^f = 0_{n_f \times n_f} \text{ holds, then the system is controllable only if there are at least as many leaders in the network as there are followers. Moreover, the system velocity is controllable if and only if } (A_x^f, B_x^f) \text{ is controllable.}
\]

\[
\text{Proof:} \quad \text{First, note that } A_x^f = 0_{n_f \times n_f} \text{ implies that } B_x^f = 0_{n_f \times n_l} \text{ due to our modelling choice. If } A_x^f = 0_{n_f \times n_f} \text{ and } B_x^f = 0_{n_f \times n_l}, \text{ then the condition}
\]

\[
\text{rank}(\lambda^2 I - \lambda A_x^f - A_x^f | B_x^f_{\text{int}}) = n_f \quad \text{for all } \lambda \in \mathbb{C} \quad (8.29)
\]

of Lemma 8.17 is reduced to

\[
\text{rank}(\lambda(I - A_x^f)x | B_x^f) = n_f \quad \text{for all } \lambda \in \mathbb{C}. \quad (8.30)
\]

It is obvious that for \(\lambda = 0\), the condition (8.30) can be satisfied only if \(\text{rank}(B_x^f_{\text{int}}) = \text{rank}(B_x^f) \geq n_f\). This implies that \(B_x^f\) has \(n_f\) linearly independent columns. Therefore, there are at least \(n_f\) leaders.

Substituting \(y(t) := \dot{x}_f(t)\) in \((C_5)\) we obtain the system

\[
\dot{y}(t) = A_x^f y(t) + B_x^f u(t), \quad y(0) = \dot{x}_f(0), \quad (8.31)
\]

which is clearly controllable if and only if \((A_x^f, B_x^f)\) is controllable. □

The first statement of Lemma 8.26 is a corollary of Lemma 8.15, as \(n_1 = n_f\) is a necessary condition for the system to be SOLF connected if \(\text{rank}(A_x^f | B_x^f) = 0\). Furthermore, it follows from Lemma 8.26 that controllability of the velocity subsystem given by \(G_x^f\) does not imply that the double integrator system is controllable, the same way that consensus stability of the velocity system in Section 5.3 does not imply that the double integrator system is consensus stable.

### 8.3.2 Homogeneous Consensus Networks

Finally, we consider the case that the communication networks between the followers are homogeneous, i.e. \(G_x^f = G_x^l\) holds.
Lemma 8.27 (Goldin and Raisch, [41]) Consider the double integrator LF consensus system (C5). If $A^f_x = A^f_t$ holds, then the system is controllable if and only if $(A^f_x, B^f_{2\text{int}})$ is controllable.

**Proof:** If $A^f_x = A^f_t = A^f$, then the left and right eigenvectors of $P^f(\lambda)$ and $A^f$ coincide. It thus follows by Lemma 8.20 that no left eigenvectors of $A^f_x$ being orthogonal on all columns of $B^f_{2\text{int}}$ is a necessary and sufficient condition for the controllability of (C5) in this case. This, in turn, is equivalent to the controllability of $(A^f_x, B^f_{2\text{int}})$. \hfill \square

Note that Lemma 8.27 does not assume that $B^f_x = B^f_t$. The edges from followers to leaders may be different for the position and velocity graph. However, as $A^f_x = A^f_t$ holds, each of the follower nodes has the same total weight of incoming edges from both leader velocity and leader position nodes.

For $B^f_x = B^f_t$, the necessity of Lemma 8.27 has been stated independently by Jiang et al., [53] without proof.

### 8.4 Weight Controllability of LF Consensus Systems

Both almost equitable partitions and automorphisms describe the existence of symmetries in graphs. Automorphisms indicate that a graph is unaffected by certain rotations or reflections due to symmetries in its edge distribution. Almost equitable partitions mark clusters of nodes that are indistinguishable from outside the cluster. The complete graph, as shown in Figure 8.4, is a highly symmetric graph. As pointed out in Lemma 6.6 on page 74, it is not controllable if the input matrix is a Laplacian matrix. A corresponding result for leader-follower systems was formulated by Tanner, [93] and states that the unweighted complete graph on $n$ nodes is uncontrollable by less than $n - 1$ leaders. It is, however, a highly unintuitive and unsatisfactory result that a complete graph, essentially representing a centralised network, cannot be controlled by one leader agent.

Graph symmetries are properties of the corresponding graphs and only translate to properties of the corresponding control system via the Laplacian matrices. Though they cannot be removed at base level, they may be influenced at matrix level by weighting the edges of the
graph differently, thus negating the undesired symmetries. This can be implemented at each agent individually in the form of a simple gain.

Motivated by this, weight controllability was introduced in Goldin and Raisch, [43] and Goldin, [39]. An LF system is called weight controllable if it is controllable for almost all choices of the corresponding weight function. Thus, if a particular LF consensus system is not controllable with fixed weights, but weight controllable, then the weights in the algorithm can be adjusted in such a way that it becomes controllable.

Goldin and Raisch, [43] show that a single integrator LF consensus system is weight controllable if and only if it is leader-follower connected. Goldin, [39] studies weight controllability of double integrator LF consensus systems. It turns out that a SOLF-connected graph is weight controllable. These results show that for almost all LF systems connectivity implies controllability. Thus, although the fully connected LF consensus system used in the introduction to this chapter is not controllable with uniform weights, it is controllable for almost all non-uniform choices of weights. Further work on weight controllability is certainly needed. For example, it would be interesting to find a non-randomized weight assignment algorithm that guarantees controllability of the system it is applied to.
CONCLUSIONS

This thesis focuses on consensus algorithms for a system of agents with double integrator dynamics in continuous time. The main body is devoted to studying the corresponding double integrator consensus system with $n$ agents, given by

$$\ddot{x}(t) = -L_x x(t) - \beta L_x \dot{x}(t),$$

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0 \quad (S_2)$$

where $x : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ are the positions of the agents, $\dot{x} : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ are their velocities, and $L_x, L_\dot{x}$ are the $n \times n$ Laplacian matrices of the graphs $G_x, G_\dot{x}$ that describe the underlying communication topology between the agents. If $G_x \neq G_\dot{x}$, then the communication topology is called heterogeneous, otherwise it is called homogeneous.

While some results are available for double integrator consensus systems with homogeneous communication topologies, less attention had been devoted to the system $(S_2)$ with heterogeneous communication topologies prior to this work, despite its occurrence in real-life applications. For this reason, it is chosen as the main object of study, which focuses on both its (consensus) stability in Chapter 5-7 and its controllability in Chapter 8.

CONSENSUS STABILITY OF DOUBLE INTEGRATOR CONSENSUS SYSTEMS IN HETEROGENEOUS NETWORKS

The main findings in Chapter 5 are that consensus stability of $(S_2)$, formally defined in Chapter 4, is largely a property of graph connectivity and symmetry, as shown in Theorem 5.6. In fact, if the graphs $G_x$ and $G_\dot{x}$ are undirected, by Theorem 6.9 graph connectivity and absence of certain symmetries turn out to be necessary and sufficient conditions for consensus stability. This is shown in Chapter 6. In the general case, when $G_x$ and $G_\dot{x}$ are allowed to be directed graphs, complex phenomena arise, that are illustrated in Chapter 7. In this case, general statements can be made only for a subset of graphs. In particular, we completely characterise consensus stability of acyclic systems in Lemma 7.10.
FURTHER WORK ON CONSENSUS STABILITY may be devoted to extending the obtained results to systems with switching communication topologies, i.e. systems where the graphs $G_x = G_x(t)$, $G_x = G_x(t)$ are allowed to change over time, as well as systems with communication delays. In the case of switching systems, a viable approach would be to find a common Lyapunov function.

Another possible extension is to consider agents with higher order integrator dynamics. These were not studied in this thesis due to the fact that agents that can be feedback linearised to have single and double integrator dynamics are prevalent in engineering applications. Furthermore, not all methods used in this thesis can be directly extended to higher order consensus systems, because the corresponding matrix polynomial, and thus also the eigenproblem, will be of higher order as well. It is known that equations of order five and more cannot be straightforwardly solved analytically, and even for cubic equations the solution process is much more involved than for quadratic ones. Thus different methods are required in order to consider consensus stability of higher order systems. A Lyapunov function approach might be successful here.

CONTROLLABILITY OF DOUBLE INTEGRATOR CONSENSUS SYSTEMS IN HETEROGENEOUS NETWORKS

In Chapter 8, the system (S2) is rewritten as the leader-follower control system with $n_l$ leaders and $n_f$ followers, given by

$$\begin{align*}
\dot{x}_f(t) &= -A_f^x x_f(t) - A_f^\dot{x} \dot{x}_f(t) + B_{2\text{int}}^f u(t), \\
x_f(0) &= x_{f0}, \quad \dot{x}_f(0) = \dot{x}_{f0},
\end{align*}$$

(C4)

where $x_f : \mathbb{R}^+ \to \mathbb{R}^{n_f}$ is the vector of the follower agents’ positions, $\dot{x}_f : \mathbb{R}^+ \to \mathbb{R}^n$ the vector of their velocities, $u : \mathbb{R}^+ \to \mathbb{R}^{n_l}$ is the control input provided by leader agents, $A_f^x, A_f^\dot{x}$ are $n_f \times n_f$ real matrices that describe the consensus algorithm between the followers and $B_{2\text{int}}^f \in \{0, 1\}^{n_f \times n_f}$ describes the communication links from leaders to followers. The controllability of (C4) is investigated. For this the classic controllability notion, given in Section A.2, is adopted.

Like in the case of consensus stability, the controllability of the system with heterogeneous communication topologies has not been previously studied. Linear algebraic necessary and sufficient conditions for the controllability of the system (C4) are given in Lemma 8.20 and Lemma 8.21, while graph-theoretic sufficient conditions are presented in Theorem 8.23. Unlike consensus stability of system (S2), the findings in Chapter 8 can be straightforwardly linked to those on the corresponding widely studied single integrator leader-follower system in Theorem 8.20. Both share the property that certain symmetries in the graphs, like the existence of automorphisms and almost
equitable partitions, make the overall system uncontrollable. In many cases this is not intuitive, as we would expect the system with the given interconnection structure to be controllable generically.

Further work on controllability may include further study of particular systems and their controllability properties, with special focus on interconnection of controllable and uncontrollable position and velocity subgraphs. Unlike consensus stability of higher-order consensus systems, their controllability properties can be obtained straightforwardly with the same methods applied in this thesis. Additionally, necessary and sufficient conditions for leader-follower systems to be controllable generically are an interesting field for future study. Some attempts in this direction have already been made in Goldin and Raisch, [43], Goldin, [39]. However, a lot more open questions remain. For example, it would be interesting to find a non-randomized weight assignment algorithm that guarantees controllability of the system it is applied to.

In summary, the results obtained in this thesis extend available results on the consensus stability and controllability of double integrator consensus algorithms. The intricate connections between graph properties and network dynamics are illustrated, and the problems that arise from them are demonstrated.
APPENDIX

A.1 LINEAR ALGEBRA AND CALCULUS

This section contains an overview of the linear algebraic terms and results used in the present work. It mostly follows Wilkinson, [101]. Some of the notation used here has been previously listed in Chapter 2.

A.1.1 Eigenvalues, Eigenvectors, Jordan Matrix

A matrix $A \in \mathbb{R}^{n \times n}$ is invertible if it is quadratic and has full rank. The kernel of a matrix $A \in \mathbb{R}^{n \times n}$, denoted $\ker(A)$, are all vectors $v \in \mathbb{C}^n$ that satisfy $Av = 0_n$. A matrix has full rank if and only if $\ker(A) = \{0_n\}$.

Two matrices $A, B \in \mathbb{R}^{n \times n}$ are similar ([101] p. 6) if there exists an invertible matrix $C \in \mathbb{C}^{n \times n}$ such that $CAC^{-1} = B$. Every real $n \times n$ matrix $A$ is similar to its Jordan form ([101] p. 10), which is a block diagonal matrix

$$J = V^{-1}AV = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{pmatrix}, \quad (A.1)$$

where $p$ is the number of linearly independent eigenvectors of $A$. Each block $J_i$, $i \in \{1, \ldots, p\}$, corresponds either with the $i$-th real eigenvalue or the $i$-th complex eigenvalue pair of $A$. For a real eigenvalue $\lambda_i$ of $A$, $i \in \{1, \ldots, p\}$, the block $J_i$ has the form

$$J_i = \begin{pmatrix} \lambda_i & 1 \\ \lambda_i & \ddots \\ & \ddots & 1 \\ & & \lambda_i \end{pmatrix}, \quad (A.2)$$

where $J_i \in \mathbb{C}^{k_i \times k_i}$, $1 \leq k_i \leq n - p$ and $\sum_{i=1}^{p} k_i = n$. 

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The sizes $k_i$ of the Jordan blocks are the partial multiplicities of the eigenvalues $\lambda_i$, i.e. all corresponding Jordan block have size 1 if $\lambda_i$ is semi-simple. Otherwise $\lambda_i$ is called deficient. If $\lambda_i$ is deficient, then at least one corresponding Jordan block has size $k_i > 1$ and there exist $k_i - 1$ generalised eigenvectors corresponding to $\lambda_i$. The generalised eigenvectors are given by the Jordan chain $Av_i j = \lambda_i v_{i,j-1}$, $j \in \{1, \ldots, k_i\}$, where $v_{i,0}$ denotes the eigenvector of $A$ affording $\lambda_i$. The number $k_i$ is called the length of the corresponding Jordan chain.

The geometric multiplicity of the eigenvalue $\lambda_i$, denoted $\text{geo}(\lambda_i)$ is the number of Jordan blocks corresponding to it, while the sum of the sizes of all Jordan blocks corresponding to it is its algebraic multiplicity, denoted $\text{alg}(\lambda_i)$. An eigenvalue $\lambda_i$ is called simple if has exactly one Jordan block of size one corresponding to it, i.e. $\text{alg}(\lambda_i) = \text{geo}(\lambda_i) = 1$. Real eigenvalues of $A$ always have corresponding real eigenvectors, while imaginary eigenvalues may have complex eigenvectors ([101] p. 8-9).

The matrix $V$ in equation (A.1) can be chosen as the matrix of eigenvectors and generalized eigenvectors of $A$ ([101] p. 10). Denote the left eigenvectors and generalised eigenvectors of $A$ by $w_1$, $w_2$, ..., $w_n$ and the right eigenvectors and generalised eigenvectors $v_1, v_2, \ldots, v_n$, that are normalized such that $w_i^T v_i = 1$. It is a classic result ([101] p. 4 eq. (3.6)) that $w_i^T v_j = 0$ if $i \neq j$. Thus if

$$V = \begin{pmatrix} v_1 & v_2 & \ldots & v_n \end{pmatrix}, \quad \text{then} \quad V^{-1} = \begin{pmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_n^T \end{pmatrix}. \quad (A.3)$$

If all eigenvalues of $A$ are semi-simple, then its Jordan form is a diagonal matrix, and $A$ is called diagonalisable.

### A.1.2 Matrix Exponential

Let $A \in \mathbb{R}^{n \times n}$. The function

$$e^{At} := \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}, \quad t \in \mathbb{R}^+, \quad (A.4)$$

is called the matrix exponential. If $A$ has the Jordan decomposition $A = V\Delta V^{-1}$, where $\Delta = \text{diag}(\delta_1, \ldots, \delta_k)$ is its Jordan form, then

$$e^{At} = Ve^{\delta t}V^{-1}. \quad (A.5)$$
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Given a Jordan block $J_i \in \mathbb{C}^{n_i \times n_i}$, for $i = 1, \ldots, k$, that corresponds with the eigenvalue $\lambda_i \in \mathbb{C}$ of $A$, it holds that

$$e^{\beta_i t} = e^{\lambda_i t} \begin{pmatrix}
1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\
0 & 1 & t & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 1 & t \\
\end{pmatrix}.$$  \hspace{1cm} (A.6)

Furthermore, for a complex number $\gamma_i \in \mathbb{C}$, it holds that

$$e^{\gamma_i t} = \cos(\gamma t) + i \sin(\gamma t), \hspace{1cm} (A.7a)$$

$$e^{-\gamma_i t} = \cos(\gamma t) - i \sin(\gamma t). \hspace{1cm} (A.7b)$$

A.1.3 Symmetry, Definiteness and Numerical Range

A matrix $A \in \mathbb{R}^{n \times n}$ is called symmetric if $A = A^T$, it is called anti-symmetric ([101] p. 24) if $A = -A^T$ holds. Any real square matrix $A$ can be decomposed into a symmetric and an antisymmetric part ([78], P3), namely $A = \text{sym}(A) + \text{asym}(A)$, where

$$\text{sym}(A) = 0.5(A + A^T), \hspace{0.5cm} \text{asym}(A) = 0.5(A - A^T). \hspace{1cm} (A.8)$$

A matrix $A \in \mathbb{C}^{n \times n}$ is called hermitian ([101] p. 24) if $(A)_{ij} = \overline{(A)_{ji}}$, where $\overline{(*)}$ denotes the complex conjugate of a number.

Symmetric and hermitian matrices have a real spectrum and an orthonormal eigenvector basis, and are therefore diagonalisable. All eigenvalues of symmetric and hermitian matrices are real and the corresponding left and right eigenvectors coincide, i.e. $V^{-1} = V^T$ in (A.1) ([101] p. 24).

A symmetric or hermitian matrix is called positive definite if and only if all its eigenvalues are positive ([101] p. 27). Analogously, $A$ is positive semi-definite, if all its eigenvalues are non-negative. Negative definiteness and semi-definiteness are defined in the same fashion. If neither of the above applies, $A$ is called indefinite.

The numerical range of an $n \times n$ matrix is given by

$$\text{NR}(A) = \left\{ \frac{v^* A v}{v^* v} \mid v \in \mathbb{C}^n, v \neq 0_n \right\}. \hspace{1cm} (A.9)$$

$\text{NR}(A)$ is a compact and convex set that is symmetric to the real axis. It is a known result ([78], P8) that for a symmetric or hermitian matrix $A$, $\text{NR}(A)$ is the convex hull of the eigenvalues of $A$, i.e. a segment of the real axis. Furthermore, for a general real matrix $A$,

$$\text{re}(\text{NR}(A)) = \text{NR}(\text{sym}(A)), \hspace{0.5cm} \text{im}(\text{NR}(A)) = \text{NR}(\text{asym}(A)). \hspace{1cm} (A.10)$$
These ideas are illustrated in Figure A.1. It is important to bear in mind that if a general matrix has eigenvalues only in the right half-plane, it does not follow that its numerical range also lies only in the right half-plane. Further information on the numerical range of matrices can be found in the overview paper by Psarrakos and Tsatsomeros, [78].

A.1.4 Gershgorin Disc Theorem and Irreducibility

The Gershgorin disc theorem is a way to estimate the locations of the eigenvalues of a matrix. Let $A$ be a complex $n \times n$ matrix. For $i = 1, \ldots, n$, let $r_i = \sum_{j \neq i} |(A)_{ij}|$ be the sum of the absolute values of the non-diagonal entries in the $i$-th row of $A$. Let $D((A)_{ii}, r_i)$ be the closed disc centered at $(A)_{ii}$ with radius $r_i$. Such a disc is called a (row) Gershgorin disc, cf. Varga, [97].

**Lemma A.1 (Gershgorin disc theorem, [97])** Every eigenvalue of $A$ lies within at least one of the Gershgorin discs $D((A)_{ii}, r_i)$.

A matrix $A \in \mathbb{R}^{n \times n}$ is called weakly (row) diagonally dominant, if

$$|(A)_{ii}| \geq \sum_{j=1, j \neq i}^{n} |(A)_{ij}|,$$

for all $i = 1, \ldots, n$, (A.11)

it is called strongly (row) diagonally dominant if the above inequality is strict. Column Gershgorin disks and dominance are defined in the same way.

A permutation matrix $P$ is a square binary matrix that has exactly one entry 1 in every row and column and zero entries otherwise. It holds that $P^T = P^{-1}$. 

![Figure A.1: Illustration of the numerical range of a matrix $A$. The numerical range of its symmetric part is the indicated part of the real axis. The numerical range of its antisymmetric part is the indicated part of the imaginary axis. Locations of the eigenvalues of $A$ are indicated by crosses.](image.png)
A matrix $A$ is called reducible if there is a permutation matrix $P$ such that

$$P A P^T = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix},$$

where $A_1$ and $A_3$ are square matrices. Otherwise $A$ is called irreducible.

If an irreducible matrix is weakly diagonally dominant, but in at least one row (or column) is strictly diagonally dominant, then the matrix is called irreducibly diagonally dominant. Every irreducibly diagonally dominant matrix has the following property.

**Lemma A.2 (Varga, [97] Theorem 1.11)** For any $A \in \mathbb{C}^{n \times n}$ which is irreducibly diagonally dominant, $A$ is non-singular.

### A.1.5 Square Root of a Complex Number

The following representation of square roots of a complex number will be useful. The square roots of a complex number $z = a + bi$, $a, b \in \mathbb{R}$, are given by ([1], p. 17)

$$\sqrt{z} = \pm \left( \text{sign}(b) \sqrt{ \frac{|z| + a}{2} } + i \sqrt{ \frac{|z| - a}{2} } \right).$$

### A.1.6 Matrix Polynomials

A recent book on matrix polynomial theory is [38] by Gohberg et al. Here we summarise some of the definitions important for this paper. The function $P : \mathbb{C} \to \mathbb{C}^{n \times n}$, given by

$$P(\lambda) = I_n \lambda^m + \sum_{i=0}^{m-1} L_i \lambda^i, \quad \lambda \in \mathbb{C}, \quad m \in \mathbb{N}$$

(A.14)
is called a monic matrix polynomial of degree $m$, where $L_i$ are $n \times n$ real matrices. The eigenvalues of (A.14) are defined as solutions $\lambda_0 \in \mathbb{C}$ of $\det P(\lambda_0) = 0$ and the corresponding eigenvectors as solutions of $P(\lambda_0)v = 0 \in \mathbb{C}^n$.

All the eigenvalues of $P(\lambda)$ are real or arise in complex-conjugated pairs. If $L_i = L_i^T$, $i = 0, \ldots, m - 1$, then we speak of a self-adjoint matrix polynomial ([38], p. 253). If the matrix polynomial is self-adjoint, then $P(\lambda_0)v = \lambda_0 v$ and $v^*P(\lambda_0)v = \lambda_0 v^*$ for all its eigenvalues $\lambda_0$ and eigenvectors $v$, i.e. its left and the right eigenvectors coincide ([38], Theorem 10.1).

\footnote{In general, the matrices $L_i$ can be complex, however, QEP with complex matrix coefficients are not considered in this thesis.}
Every matrix polynomial admits a number of matrix pencil linearisations ([38] Theorem 1.1), where the \( n \times n \) matrix polynomial (A.14) is transformed to a \( nm \times nm \) matrix pencil \( (P_1 - \lambda I) \), which is linear in \( \lambda \). The pencil \( P_1 \) has the same spectral properties as \( P(\lambda) \). One of the most common linearisations involves the matrix

\[
P_1 = \begin{pmatrix}
0 & I_n & 0 & \ldots & 0 \\
0 & 0 & I_n & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-L_0 & -L_1 & \ldots & \ldots & -L_{m-1}
\end{pmatrix}.
\]  

(A.15)

A complex number \( \lambda_0 \) is an eigenvalue of \( P(\lambda) \) if and only if it is an eigenvalue of \( P_1 \).

Eigenproblems of matrix polynomials of degree two, \( P(\lambda) = \lambda^2 I + \lambda L_1 + L_0 \), are usually referred to as quadratic eigenvalue problems (QEP). QEPs arise in a number of engineering applications, including dynamic analysis of structural, mechanical, and acoustic systems, electrical circuit simulation, fluid mechanics, and microelectromechanical systems. For an extensive review of applications and solutions of the QEP see Tisseur and Meerbergen, [96].

A.2 CONTROL THEORY

The following introduction follows Khalil, [56] and Kailath, [54].

A linear time-invariant dynamical system in the state space form is given by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t), \\
x(0) &= x_0,
\end{align*}
\]  

(T1)

where \( x(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^n \) is the state vector, \( u(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^m \) is the input vector, \( y(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^p \) is the output vector, \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{p \times n} \) and \( D \in \mathbb{R}^{p \times m} \).

A.2.1 Solution of LTI Systems

We consider solutions of the system (T1) in Section 5.3. Using variation of parameters, the solution of (T1) is given by

\[
x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau,
\]  

(A.16)

where \( e^{At} \) is the matrix exponential defined in Section A.1.2.
A.2.2 Stability of LTI systems

In the following we give a brief overview over some existing stability definitions for LTI systems. It largely follows Chapters 3 and 4 of Khalil, [56]. Consider a general autonomous LTI system which satisfies

\[
\dot{x}(t) = Ax(t),
\]
\[
x(t_0) = x_0, \quad t \geq t_0
\]

where \( x : \mathbb{R}^+ \to \mathbb{R}^n \) is the state vector, \( t_0 \geq 0 \) is the initial time, and \( x_0 \in \mathbb{R}^n \). A point \( x_e \in \mathbb{R}^n \) is an equilibrium point of system (T2) if it satisfies

\[
Ax_e = 0.
\]  

\begin{align*}
\text{Definition A.3 (Khalil, [56], Definition 4.4, 4.5)} \quad 
\text{The equilibrium point } x_e \text{ of system (T2) is called} \\
\begin{itemize}
    \item \underline{stable}, if for each \( \epsilon > 0 \) there exists a positive constant \( \delta = \delta(\epsilon) > 0 \), such that \\
    \[ \|x(t_0) - x_e\| < \delta \implies \|x(t) - x_e\| < \epsilon, \text{ for all } t \geq t_0 \geq 0, \] (A.18)
    \item \underline{unstable} if not stable,
    \item \underline{asymptotically stable} if it is stable and there exists a positive constant \( c \), such that \\
    \[ \|x(t_0) - x_e\| < c \implies \lim_{t \to \infty} \|x(t) - x_e\| = 0, \] (A.19)
    \item \underline{globally exponentially stable} if there exist positive constants \( \alpha \) and \( \epsilon \) such that \\
    \[ \|x(t) - x_e\| \leq \epsilon \|x(t_0) - x_e\| e^{-\alpha(t-t_0)} \] (A.20)
\end{itemize}
\end{align*}

for all \( x(t_0) \in \mathbb{R}^n \) and for all \( t \geq t_0 \). The largest constant \( \alpha \) which may be utilised in (A.20) is called \textit{convergence rate}.

If a system is globally exponentially stable, then, in particular, it is asymptotically stable. The following well-known theorem relates stability of linear systems to the eigenvalues of the corresponding system matrices.

\begin{align*}
\text{Theorem A.4 (Khalil, [56], Theorem 4.5)} \quad 
\text{The equilibrium point } x_e = 0 \text{ of system (T2) on page 135 is stable if and only if all eigenvalues } \lambda_i \text{ of } A \text{ satisfy } \text{re} (\lambda_i) \leq 0 \text{ and for every eigenvalue with } \text{re} (\lambda_i) = 0 \text{ and algebraic multiplicity } \text{alg} (\lambda_i) \geq 2, \text{ rank} (A - \lambda_i I) = n - \text{alg} (\lambda_i), \text{ where } n \text{ is the dimension of } x.
\end{align*}

The equilibrium point \( x_e = 0 \) is (globally) asymptotically stable if and only if all eigenvalues of \( A \) satisfy \( \text{re} (\lambda_i) < 0 \).
A.2.3 Controllability of LTI Systems

Definition A.5 (controllability, Kailath, [54], p. 84) The system \((T_1)\) is said to be controllable, if there is a suitable input \(u(t)\) that will take the system to any desired state from any initial state in finite time.

The following condition describes controllability of LTI systems.

Lemma A.6 (Kailath, [54], Theorem 2.4-9) The following conditions are equivalent.

1. The system \((T_1)\) is controllable.
2. (PBH-Test) \(\text{rank}(A - \lambda I | B) = n\) for any complex number \(\lambda\).

Note that \(\text{rank}(A - \lambda I | B) = n\) is automatically satisfied for any \(\lambda\) that is not an eigenvalue of \(A\), because then \(\text{rank}(A - \lambda I) = n\) holds by definition of eigenvalues.

The following lemma simplifies the study of controllability for certain systems.

Lemma A.7 (Bhandarkar and Fahmy, [11]) Let the matrix pair \((A, B)\) have the form

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},
\]

(A.21)

where \(B_1\) has \(m_1\) rows and \(\text{rank}(B_1) = m_1\), and the matrix \(A\) is partitioned correspondingly. Then, the pair \((A, B)\) is controllable if the pair \((A_{22}, A_{21})\) is controllable.


[52] Zhijian Ji, Hai Lin, and Tong Heng Lee. A graph theory based characterization of controllability for multi-agent systems with


