New classes of large claim size distributions: 
Introduction, properties and applications

vorgelegt von
Diplom-Wirtschaftsmathematiker
Sergej Beck
aus Andronowka

Von der Fakultät II – Mathematik und Naturwissenschaften 
der Technischen Universität Berlin
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften
– Dr. rer. nat. –

genehmigte Dissertation

Promotionsausschuss:
Vorsitzender: Prof. Dr. Jörg Liesen
Berichter: Prof. Dr. Jochen Blath
Berichter: Prof. Dr. Sergey Foss
Berichter: Prof. Dr. Michael Scheutzow


Berlin 2014

D83
Acknowledgements

I would like to express my deep gratitude to Prof. Dr. Michael Scheutzow and Prof. Dr. Jochen Blath for their patient guidance and useful critiques of this research. I am grateful to Prof. Dr. Sergey Foss for his valuable suggestions after reviewing a first version of this manuscript.

I would also like to extend my thanks to the “Verein zur Förderung der Versicherungswissenschaften an den Berliner Universitäten e.V.” for their help in providing me the resources for the research.

Finally, I wish to thank the Institute of Mathematics, TU Berlin, and the Berlin Mathematical School for their assistance.
1 Contents

1.1 Abstract

Large claim size distributions play an important role in many areas of probability theory and related fields – in particular insurance and finance. They often describe ‘extreme events’ and are typically ‘heavy-tailed’. In this work, we introduce and investigate new large claim distribution classes.

The approach of the definition of our classes is based on ‘the catastrophe principle’. This principle states that a surprisingly large sum of a fixed number of contributors is dominated by just one contributor. This principle is often used in connection with extreme events and some established distribution classes in the literature contain distributions which obey this principle. However, the definitions of these classes may be too restrictive and there is no established distribution class which is solely defined to describe the catastrophe principle. Our approach to define a new distribution class is trying to fill this gap. The aim of this dissertation is to derive the structure, the classification and properties of our new distribution classes.
# Contents

1 Contents .......................................................... 3
   1.1 Abstract ..................................................... 3

2 Introduction ......................................................... 6
   2.1 Notation ....................................................... 11
      2.1.1 General .................................................. 11
      2.1.2 Distribution ............................................. 11
      2.1.3 Operations .............................................. 12
      2.1.4 Landau notation ....................................... 12
      2.1.5 Closure .................................................. 13

3 Distribution classes ............................................... 14
   3.1 Heavy-tailed distribution classes .............................. 14
      3.1.1 Definition .............................................. 14
      3.1.2 Classification of heavy-tailed classes ................. 16
      3.1.3 Properties of $S$ ........................................ 18
      3.1.4 Properties of the class $L$ ............................. 19
   3.2 The classes $OS$ and $OL$ .................................... 22
      3.2.1 Definition .............................................. 22
      3.2.2 Classification .......................................... 22
      3.2.3 Properties of $OS$ ..................................... 23
   3.3 Convolution equivalence classes $S(\gamma)$ and $L(\gamma)$ ....... 25
      3.3.1 Definition .............................................. 25

4 New large claim classes ............................................ 27
   4.1 The large claim class $A(\mathcal{E})$ .......................... 27
      4.1.1 Definition .............................................. 27
      4.1.2 Structure of $A(\mathcal{E})^{(n)}$ .......................... 29
      4.1.3 Properties of $A(\mathcal{E})$ and examples ............... 30
      4.1.4 Proofs .................................................. 33
   4.2 The large claim class $A(\mathcal{E})$ on the entire real line $\mathbb{R}$ 40
   4.3 The large claim class $A$ ..................................... 43
      4.3.1 Structure .............................................. 43
      4.3.2 Classification, examples and properties of $A$ ....... 44
      4.3.3 Proofs .................................................. 49
   4.4 The large claim class $J$ ..................................... 54
      4.4.1 Structure .............................................. 54
Problems in modelling ‘extreme events’ arise in many areas where such events can have significant consequences. Earthquakes, terrorism, industrial accidents, and hurricanes are examples of such extreme events, just to name a few. The damage of such an extreme event, including loss of lives, can cause an immense economic loss of resources. Although such losses occur rarely, for an insurance company they may be greater than the combined sum of all other losses from that year. The table 2.1 on the following page lists the most expensive events for insurance companies to date.

For modelling an extreme event, we need information about the probabilities – for example of the occurrence or the height of the damage. The tail of the distribution characterizes the risk of a claim crossing a certain level of cost and provides the corresponding probabilities. Also, the distribution of a claim can be classified according to the asymptotic behaviour of the tail. All distributions may be divided into two classes: heavy or light-tailed.

The distribution is heavy if the probability of crossing a (growing) level provided by the tail fails to be bounded by a decreasing exponential function. Otherwise the distribution is classified as light-tailed. In an insurance context, this is an important classification since sums of random variables with heavy-tailed distributions seem to follow ‘the catastrophe principle’, which states that a surprisingly large sum is dominated by just few or one contributor. In contrast to the sums of light-tailed distributions which tend to disobey this principle.
2 Introduction

<table>
<thead>
<tr>
<th>Event</th>
<th>Year</th>
<th>Insured losses</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Hurricane Katrina</td>
<td>2005</td>
<td>$72.3 billion</td>
</tr>
<tr>
<td>2. Tōhoku earthquake and tsunami</td>
<td>2011</td>
<td>$35 billion</td>
</tr>
<tr>
<td>3. Hurricane Andrew</td>
<td>1992</td>
<td>$25 billion</td>
</tr>
<tr>
<td>4. 9/11 Terrorist Attacks</td>
<td>2001</td>
<td>$23.1 billion</td>
</tr>
<tr>
<td>5. Northridge Earthquake</td>
<td>1994</td>
<td>$20.6 billion</td>
</tr>
<tr>
<td>6. Hurricane Ike</td>
<td>2008</td>
<td>$20.5 billion</td>
</tr>
<tr>
<td>7. Hurricane Ivan</td>
<td>2004</td>
<td>$14.9 billion</td>
</tr>
<tr>
<td>8. Hurricane Wilma</td>
<td>2005</td>
<td>$14.0 billion</td>
</tr>
<tr>
<td>9. Hurricane Rita</td>
<td>1989</td>
<td>$11.3 billion</td>
</tr>
<tr>
<td>10. Typhoon Mireille/No 19</td>
<td>1991</td>
<td>$9.0 billion</td>
</tr>
</tbody>
</table>


In practice, realistic claim size distributions of extreme events are very often heavy-tailed. However, it is difficult to formulate general statements for all heavy-tailed distributions, especially regarding ruin probabilities, but such results can be achieved for certain important subclasses. The class of heavy-tailed distributions is denoted by $\mathcal{K}$ and the most important subclass is formed by the subexponential distributions.

The concept of subexponentiality, which was introduced by Chistyakov in [5, 1964], plays an important role in modelling extreme events. In the following decades, modelling extreme events through subexponential distributions attracted increasing attention.

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. nonnegative random variables with unbounded support, that is, $\mathbb{P}(X_1 > x) > 0$ for all $x \geq 0$. The subexponential approach is based on the idea that the tail of the sum $S_n := \sum_{i=1}^{n} X_i$ is asymptotically equivalent to the tail of the maximum of $X_1, \ldots, X_n$, as $x \to \infty$, i.e., distributions of identical and independent random variables on $(0, \infty)$ satisfying

$$
\lim_{x \to \infty} \frac{\mathbb{P}(\max(X_1, \ldots, X_n) > x, X_1 + \cdots + X_n > x)}{\mathbb{P}(X_1 + \cdots + X_n > x)} = \lim_{x \to \infty} \frac{\mathbb{P}(\max(X_1, \ldots, X_n) > x)}{\mathbb{P}(X_1 + \cdots + X_n > x)} = 1, \quad (2.1)
$$

for all $n \geq 2$, are called subexponential. Obviously, the sum of subexponential random variables is typically dominated by its largest element in the case of an extreme event. The class $\mathcal{S}$ of subexponential distributions has several important stability properties, and in particular allows an elegant characterization of the asymptotic
2 Introduction

behaviour of the ruin probability in the Cramér-Lundberg model (and in a weaker form also for more general renewal models), we refer the reader to [13, 18] for a detailed review.

From an intuitive point of view, one might ask whether condition (2.1)-(2.3) on the distribution might be too restrictive and could be weakened. For example, one could require that the maximum is sufficiently close to, but not quite at the same level as, the sum of the claim sizes \( x \), say greater than \( x - K \) for some constant \( K \).

This leads to the following new approach.

Again, let \( X_1, X_2, \ldots \) be a sequence of i.i.d. nonnegative random variables. We consider distributions such that for all \( n \geq 2 \),

\[
\lim_{K \to \infty} \liminf_{x \to \infty} \frac{P(\max(X_1, \ldots, X_n) > x - K, X_1 + \cdots + X_n > x)}{P(X_1 + \cdots + X_n > x)} = 1. \tag{2.4}
\]

The class of distributions based on the approach (2.4) is denoted by \( \mathcal{J} \) and will be considered in Section 4.4. The class \( \mathcal{J} \) has many properties and is closed under several operations (e.g. such as minimum, mixture or convolution powers).

The basic idea behind the approach (2.4) is the domination of the sum \( S_n \) by the maximum of the random variables \( X_1, X_2, \ldots, X_n \), as \( x \to \infty \). However, at first sight, rather surprisingly, it turns out that our definition also admits some light-tailed distributions to \( \mathcal{J} \). Thus, on the one side the class \( \mathcal{J} \) does not form a subclass of \( \mathcal{K} \) and on the other side our class is obviously appropriate to model extreme events.

Further, we generalize the concept of (2.4) one more time. Regarding the rationale behind the approach (2.4), one might wonder whether one should also consider distribution functions \( F \) with the property that the corresponding i.i.d. random variables satisfy, for all \( n \geq 2 \),

\[
\lim_{x \to \infty} \frac{P(\max(X_1, \ldots, X_n) > g(x), X_1 + \cdots + X_n > x)}{P(X_1 + \cdots + X_n > x)} = 1, \tag{2.5}
\]

where \( g(x) \) is an unbounded, nonnegative and nondecreasing function. By choosing the function \( g \) we are able to vary the intensity of the domination of the sum by the largest element – for example, with \( g(x) = \frac{9}{10} x \) we ensure that the largest claim has to account for more than 90% of the aggregate claim. In the following, we will use the term ‘large claim class’ for classes, which are appropriate to model extreme events in the sense of (2.5).

Let \( \mathcal{G} \) be the set of all unbounded and nondecreasing functions \( g : \mathbb{R}_+ \to \mathbb{R}_+ \).

There is no restriction in the approach to define a new distribution class, i.e. we can request the condition (2.5) for any nonempty subset \( \mathcal{E} \subseteq \mathcal{G} \). The large claim distribution class, which is generated by the subset \( \mathcal{E} \subseteq \mathcal{G} \), is denoted by \( \mathcal{A}(\mathcal{E}) \).

Obviously, the constructed class is highly dependent on the chosen set of functions and by changing the set \( \mathcal{E} \) we can obtain a different distribution class. Of course, such an approach immediately raises a variety of questions. In this work, we derive systematically the structure, the classification and properties of the distribution classes generated by different sets of functions \( \mathcal{E} \subseteq \mathcal{G} \) and the approach of (2.5).
In Chapter 3, we start with an overview of already established distribution classes. We introduce the most popular heavy-tailed classes such as the subexponential, long-tailed and dominatedly varying distributions. We collect their properties and give some examples of the introduced heavy-tailed classes. Next, we introduce generalized versions of subexponential and long-tailed distributions denoted by $O-$subexponential and $O-$long-tailed distributions, the corresponding classes are denoted by $OS$ and $OL$. Both classes are not restricted to heavy-tailed distributions and contain some light-tailed distributions. We collect again properties of the large claim classes $OS$ and $OL$ and compare them to the properties of the heavy-tailed classes.

In Chapter 4, we introduce our distribution class in its most general form by requesting condition (2.5) for a not further specified (but fixed) subset of functions $E \subseteq G$. It turns out, that our general approach allows us to examine the structure and some of the closure properties of the general distribution class $A(E)$ independent of the choice of $E$. An extension of the approach of the definition by considering distributions on the entire real line of $\mathbb{R}$ is presented in Section 4.2.

In the subsequent sections of Chapter 4, we will examine large claim distributions classes of type $A(E)$, which are generated by specific subsets $E \subseteq G$ of functions. We start with the large claim class $A := A(H)$, where $H := \{g \in G : g(x) = \varepsilon x, \varepsilon > 0\}$. We will provide several equivalent formulations of $A$ and re-examine its structure, provide classification and examples.

The rest of the Chapter 4 is devoted to the description of the large claim class $J$, which we already introduced in (2.4). It turns out that we can generate the class $J$ by requesting the condition (2.5) for all unbounded and nondecreasing functions, i.e. the following statement holds

$$J = A(G).$$

Obviously, the large claim class $J$ is the smallest class of the type $A(E)$, $E \subseteq G$. Furthermore, we establish the relations between the class $J$ and other heavy-tailed and large claim classes. Finally, we compare the closure properties of the class $J$ to the properties of the classes $S$ and $OS$. In the last section of Chapter 4, we introduce a further extension of our approach to define a distribution class by considering a local version of our distribution class $J$.

In Chapter 5, we use the results of Chapter 4 to obtain applications of our distribution classes. It is well known that the concept of subexponentiality turned out to be very useful in the asymptotic analysis of risk probabilities in ruin models and random walks. We give a short review of obtained results for class $S$ and demonstrate that some results (in a weaker form) can be extended to the general class of distributions $A(E)$, $E \subseteq G$. A famous result of ruin theory is stated in Veraverbeke’s Theorem, which establishes the connection between important quantities of a random walk, see [35]. We state a version of the famous Veraverbeke’s Theorem for our smallest class $J$ and compare the results to the versions of the classes $S$ and $OS$. Finally, we consider the relation between the asymptotic tail behaviour of infinitely divisible laws and their Lévy measures.
2 Introduction

The results about the class $\mathcal{J}$ in Sections 4.4 and 5.2.6 are based on a paper ‘A new class of large claim size distributions: Definition, properties, and ruin theory’, [2, 2013]. The paper was a joint work with Prof. Dr. Jochen Blath and Prof. Dr. Michael Scheutzow.
2 Introduction

2.1 Notation

Throughout this dissertation the following notation is used.

2.1.1 General

- Capital letters typically stand for random variables. Thus $X$ and $Y$ are random variables. Suppose that $X$, $Y$ are defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

- The set of natural numbers, including zero, is denoted by $\mathbb{N} = \{0, 1, 2, \ldots\}$.

- A random variable with a subindex is a member of a sequence of independent, identically distributed random variables. Hence, $X_i$ is the $i$-th random variable in a countable sequence $(X_k)_{k \in \mathbb{N}}$.

- A random variable with an upper index is an i.i.d copy of the random variable. Thus, $X^{(i)}$ is the $i$-th i.i.d. copy of $X$.

- We denote by $S_n$ the sum of the first $n$ random variables, that is $S_n = \sum_{i=1}^{n} X_i$.

- We denote by $x_{k,n}$ the $k$-th largest element of a family of reals $(x_i)_{i=1, \ldots, n}$ for some $n \in \mathbb{N}$. Hence, $X_{k,n}$ denotes the (pointwise) $k$-th largest element of the finite sequence $X_1, \ldots, X_n$. In particular, $X_{1,n}$ resp. $X_{n,n}$ denotes the (pointwise) maximum and the (pointwise) minimum of the random variables $X_1, \ldots, X_n$.

- We say that a random variable is proper if it is finite a.s. Otherwise, it is defective.

- Let $\mathcal{G}$ denote the set of nonnegative, unbounded and nondecreasing functions $g : \mathbb{R}_+ \to \mathbb{R}_+$.

2.1.2 Distribution

- We write $F(x) = \mathbb{P}(X \leq x), \ x \in \mathbb{R}$, for the distribution function of a random variable $X$.

- We will use the same symbol $F$ to denote the distribution of the random variable $X$, because there is one-to-one correspondence between distributions and distribution functions.

- We define for any distribution function $F$ on $\mathbb{R}$ the tail function $\overline{F}(x) = 1 - F(x)$, $x \in \mathbb{R}$.

- We further say that the distribution function $F$ on $\mathbb{R}$ has right-unbounded support if $\overline{F}(x) > 0$ for all $x \geq 0$.

- Let $\mathcal{F}$ be the set of all distributions of random variables on $[0, \infty)$ with right-unbounded support.
2 Introduction

• Let $\mathcal{FR}$ be the set of all distribution of random variables on $(-\infty, \infty)$ with right-unbounded support.

• We write $F \ast G(x) := \int_{-\infty}^{x} F(x - y) dG(y)$ for the convolution of two distribution functions $F, G \in \mathcal{FR}$. Furthermore, we write $F^{2*}$ for the two-fold convolution of $F$ with itself, and, in general, $n \geq 0$, $F^{n*}$ for the $n$-fold convolution where $F^{1*} := F$, and $F^{0*}$ is the distribution corresponding to the Dirac measure at $0$.

• We write $F^+$ for the distribution of the random variable $X^+ := 1_{\{X \geq 0\}} X$.

2.1.3 Operations

Let $X, Y$ be two random variables with distributions $F, G \in \mathcal{F}$.

• We denote $X \vee Y$ the (pointwise) maximum of $X$ and $Y$. The distribution of the maximum is denoted by $F_{X \vee Y}$.

• We denote $X \wedge Y$ the (pointwise) minimum of $X$ and $Y$. The distribution of the minimum is denoted by $F_{X \wedge Y}$.

• We call a random variable $Z$ a mixture of $X$ and $Y$ with parameter $p \in (0, 1)$, if its distribution function is given by

$$pF(x) + (1 - p)G(x), \quad x \geq 0.$$ 

The distribution of the mixture is denoted by $F_Z$.

2.1.4 Landau notation

We will use the Landau asymptotic notation in all relevant cases of this dissertation. Let $f, g$ be two positive functions on $(-\infty, \infty)$.

• We write $f(x) = O(g(x))$ if $\limsup_{x \to \infty} \frac{f(x)}{g(x)} < \infty$.

• We write $f(x) = o(g(x))$ if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$.

• We say that $f$ and $g$ are strongly asymptotically equivalent (as $x \to \infty$), denoted by $f \sim g$, if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$.

• We say that $f$ and $g$ are weakly asymptotically equivalent (as $x \to \infty$), denoted by $f \asymp g$ if $f(x) = O(g(x))$ and $g(x) = O(f(x))$, i.e.

$$0 < \liminf_{x \to \infty} \frac{f(x)}{g(x)} \leq \limsup_{x \to \infty} \frac{f(x)}{g(x)} < \infty.$$
2 Introduction

2.1.5 Closure

- We say that the distribution class $C$ is closed under convolution power if $F \in C$ implies $F \ast F \in C$.
- We say that the distribution class $C$ is closed under convolution if $F, G \in C$ implies $F \ast G \in C$.
- We say that the distribution class $C$ is closed under convolution roots if $F^n \ast \in C$ for some $n \in \mathbb{N}$ implies $F \in C$.
- We say that the distribution class $C$ is closed under (weak) tail-equivalence, if $F \in C$ and $G \sim F$ (i.e., $G \simeq F$) imply that $G \in C$. 
3 Distribution classes

3.1 Heavy-tailed distribution classes

Heavy-tailed distributions have been the focus of study of many researchers in applied probability - for instance, in risk theory, insurance mathematics, financial mathematics, queueing theory, telecommunications and computing, to name but a few areas.

3.1.1 Definition

**Definition 1.** (Heavy-tailed distribution)

A distribution $F \in \mathcal{F}_\mathbb{R}$ is called *(right) heavy-tailed* if it has no positive exponential moments, i.e.

$$
\int_{-\infty}^{\infty} e^{\lambda x} dF(x) = \infty \quad \text{for all } \lambda > 0.
$$

Otherwise $F$ is said to be *light-tailed*. We denote the class of heavy-tailed distributions by $\mathcal{K}$. Following the definition of [42, Yang, 2011] we write $F \in \mathcal{D} \mathcal{K}$ if

$$
\lim_{x \to \infty} e^{\lambda x} F(x) = \infty \quad \text{for all } \lambda > 0.
$$

The following proposition states a well known property of heavy-tailed distributions and a proof can be found in [18, Theorem 2.6].

**Proposition 2.** $F \in \mathcal{K}$ if and only if for all $\lambda > 0$ the following holds:

$$
\limsup_{x \to \infty} F(x) \exp(\lambda x) = \infty. \quad (3.1)
$$
3 Distribution classes

Remark 3. From proposition above we obtain the inclusion $\mathcal{D} \subseteq \mathcal{K}$. An example of a distribution $F \in \mathcal{K} \setminus \mathcal{D}$ is given by a distribution which has no limit in (3.1), see e.g. [33, Proposition 1.2].

The class of heavy-tailed random variables $\mathcal{K}$ has a very rich structure, and the identification and discussion of relevant subclasses is still an area of active research, see e.g. [18] for a recent account. The most important subclasses of $\mathcal{K}$ will be defined below. We start with regularly varying distributions. Regular variation appears naturally in various fields of applied probability and distributions with regularly varying tails have many applications in risk theory, queuing theory and extreme value theory. A standard reference for the subject is [4], where the reader may find many applications.

Definition 4. (Regularly varying distributions)
A distribution $F \in \mathcal{F}$ is said to be regularly varying of index $\alpha \geq 0$ if
$$\lim_{x \to \infty} \frac{F(tx)}{F(x)} = t^{-\alpha}, \quad t > 0.$$ 

The class of regularly varying distributions of index $\alpha \geq 0$ is denoted by $\mathcal{R}_{-\alpha}$. The class of all regularly varying distributions is denoted by $\mathcal{R}$.
$$\mathcal{R} = \{ F \in \mathcal{F} : F \in \mathcal{R}_{-\alpha} \text{ for some } \alpha \geq 0 \} .$$

Next, we turn to subexponential distributions, which were first introduced and studied by Chistyakov, [5], in connection with their application to branching processes. Later subexponential distributions have been used in a wide variety of applications in probability theory, for instance in renewal theory and the theory of infinitely divisible distributions. We refer the reader to [12, 13, 18, 28, 34, 14].

Despite the fact that the name ‘subexponential’ suggests the property (3.1), the definition of a subexponential distribution is more restrictive.

Definition 5. (Subexponential distributions)
A distribution $F \in \mathcal{F}$ is called subexponential if for all $n \geq 2$,
$$\lim_{x \to \infty} \frac{F_n(x)}{F(x)} = n. \quad (3.2)$$

The class of subexponential distributions is denoted by $\mathcal{S}$.

The next proposition states some sufficient conditions for the class of subexponential distributions $\mathcal{S}$.

Proposition 6. (Conditions for subexponentiality)
Let $F \in \mathcal{F}$.

a) If the condition (3.2) holds for some $n \geq 2$, then
$$F \in \mathcal{S}.$$
3 Distribution classes

b) If \( \limsup_{x \to \infty} \frac{F(x)}{F(x)} \leq 2 \), then

\[ F \in S. \]

For the proof we refer the reader to [13, Embrechts et al., Theorem A3.20 c)] and [13, Embrechts et al., Lemma 1.3.4].

The next important and well-studied distribution class is closely related to the subexponential distributions.

Definition 7. (Long-tailed distributions)

A distribution \( F \in \mathcal{F} \) is called long-tailed if \( F(x) > 0 \) for all \( x \in \mathbb{R} \), and for any fixed \( y \in \mathbb{R} \setminus \{0\} \),

\[ \lim_{x \to \infty} \frac{F(x + y)}{F(x)} = 1. \]

The class of long-tailed distributions is denoted by \( \mathcal{L} \).

Remark 8. Note that we define subexponential distributions only for distributions on \([0, \infty)\) with right-unbounded support, in contrast to the class \( \mathcal{L} \), whose definition involve distributions on the whole line of \( \mathbb{R} \). We refer the reader to Section 4.2 for more details on this topic.

The classes \( S \) and \( \mathcal{L} \) are characterized by a variety of properties and applications. In the next sections, we will consider the properties and the relation between \( \mathcal{L} \) and \( S \). Finally, we introduce the class of dominatedly varying distributions.

Definition 9. (Dominatedly varying distributions)

A distribution \( F \in \mathcal{F} \) has a dominatedly varying tail, if

\[ \limsup_{x \to \infty} \frac{F(xu)}{F(x)} < \infty \]

for all (or equivalently for some) \( 0 < u < 1 \).

The class of distributions with dominatedly varying tail is typically denoted by \( \mathcal{D} \) and in some sources by \( \mathcal{OR} \) (compare with the definition of the class \( \mathcal{R} \)).

3.1.2 Classification of heavy-tailed classes

The purpose of this section is to summarize interrelationships between the most important heavy-tailed classes, which were introduced in the previous section. We also give some prominent examples of heavy-tailed distributions. Most of the inter-relationships between heavy-tailed classes \( S, D, L, K \) are well known.

Proposition 10. (Classification of heavy-tailed classes)

a) \( \mathcal{R} \subset S \subset \mathcal{L} \subset \mathcal{DK} \subset \mathcal{K} \);

b) \( \mathcal{D} \subset \mathcal{DK} \subset \mathcal{K} \);

c) \( \mathcal{D} \cap \mathcal{L} \subset S \);

d) \( \mathcal{D} \not\subset S \) and \( S \not\subset \mathcal{D} \).

16
3 Distribution classes

We refer the reader to Embrechts et al., [13, Section 1.4.1, pages 49-53], for most of these inclusions (the remaining ones are easy to check).

Example 11. Examples of heavy-tailed distributions.

a) In practice, many commonly used heavy-tailed distributions belong to the subexponential class. Hence, we give some prominent examples of subexponential distributions by providing their distribution functions or densities. The class of subexponential distributions $S$ includes the Pareto distribution

$$
F(x) = \begin{cases} 
\left( \frac{x}{x_m} \right)^\alpha & x \geq x_m, \quad \alpha, x_m > 0; \\
0 & x < x_m,
\end{cases}
$$

the Weibull distribution,

$$
F(x) = 1 - \exp(-cx^\beta), \ x \geq 0, \ 0 < \beta < 1, c > 0;
$$

the lognormal distribution,

$$
f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right), \ x \geq 0, \ \mu \in \mathbb{R}, \sigma > 0;
$$

and the loggamma distribution,

$$
f(x) = \frac{\alpha^\beta}{\Gamma(\beta)} (\ln x)^{\beta-1}x^{-\alpha-1}, \ x \geq 0, \ \alpha, \beta > 0,
$$

where $\Gamma$ is the gamma function.

b) There is no simple example for a distribution in $\mathcal{L} \setminus S$. The first example was given by Embrechts and Goldie in [10, page 255]. Recently, another example was given by Lin and Wang in [26, Example 2.1]. We will use the ideas from both examples to obtain distributions of our new large claim class. We also refer the reader to [18, Chapter 3.7], the authors show how to construct a distribution which is long-tailed but not subexponential.

c) A distribution $G \in \mathcal{D} \setminus \mathcal{L}$ is given by a ‘Peter and Paul’ distribution. Let the tail of $G$ be such that,

$$
\overline{G}(x) = 2^{-k}, \ x \in [2^k, 2^{k+1}), \ k \in \mathbb{N}.
$$

It immediately follows that for all $k \in \mathbb{N}$ and $x \geq 1$

$$
\frac{\overline{G}(2^k - 1)}{\overline{G}(2^k)} = 2, \\
\frac{\overline{G}(x)}{\overline{G}(2x)} = 2.
$$

Hence, $G \notin \mathcal{L}$ and $G \in \mathcal{D}$.
Figure (3.1) summarizes the interrelationships of heavy-tailed classes.

![Figure 3.1](image) Classification of heavy-tailed classes, source: [13, Figure 1.4.1].

### 3.1.3 Properties of \( S \)

Due to the many properties, well-studied structure, and many applications, the class of subexponential distributions is the most important subclass of heavy-tailed distributions. Their properties have been investigated by several authors including Embrechts and Goldie, [10], Foss et al., [18] and Yakymiv, [41]. In this section, we collect the properties of \( S \), which are later compared to the properties of other classes.

The closure properties under certain operations are of special interest in view of applications in risk theory, queueing theory, and infinite divisibility theory. First, we consider the closure of the class \( S \) under strong tail-equivalence, root convolution, and convolution power.

**Proposition 12. (Closure properties of the class \( S \))**

a) If \( F \in S \), \( G \in F \) and \( G \sim F \), then

\[ G \in S. \]

b) If \( F \in S \), then for all \( n \geq 1 \)

\[ F^{n*} \in S. \]

c) If \( F^{n*} \in S \) for \( n \geq 2 \), then

\[ F \in S. \]

For the proofs of the collected properties in the proposition above, we refer the reader to Embrechts et al., [13, Section A3.2, Lemmata A3.15 and A3.18].
3 Distribution classes

**Example 13.** It is well known that the sum of two independent subexponential random variables is not necessarily subexponential. A counterexample for nonclosure under convolution of the class $S$ is given in [25, Section 3].

The next result describes the question of the closure of the class $S$ under the operations maximum, minimum and mixture of two independent random variables.

**Proposition 14.** (Closure of the class $S$ under mixture, maximum and minimum)

Let $X, Y$ be two independent random variables with distributions $F, G \in \mathcal{F}$. If $F, G \in S$, then the following are equivalent:

a) Let $F_Z$ be the distribution of the mixture of $X$ and $Y$ with parameter $p \in (0, 1)$.

b) If $F, G \in S$, then

\[ F_{X \wedge Y} \in S. \]

The equivalence of i) and ii) in part a) of the proposition above was first proven in [41, Theorem 1], for a proof of the equivalence of ii) and iii) we refer the reader to [10, Theorem 2]. The last part of Proposition 14 was proven in [19, Theorem 1].

Finally, we study the ratio of

\[ \frac{P(S_n > x)}{P(X_1 > x)} \quad (3.3) \]

for all $n \in \mathbb{N}$ and $x \geq 0$. In the literature, an upper bound of (3.3) is named Kesten’s estimate or Kesten’s bound. In the subexponential case, we have the following Kesten’s bound.

**Lemma 15.** (Kesten’s bound for the class $S$)

If $F \in S$ then, given $\varepsilon > 0$, there exists $c > 0$ such that for all $n \geq 1$ and $x \geq 0$:

\[ \frac{F^{\varepsilon n}(x)}{F(x)} \leq c(1 + \varepsilon)^n. \]

Kesten’s bound is a frequently used tool in proofs and applications and, we also refer the reader to [31, Shneer, 2004] for estimates of the ratio (3.3) for other heavy-tailed distribution classes. A proof of the lemma above can be found in [13, Lemma 1.3.5].

3.1.4 Properties of the class $\mathcal{L}$

In this section, we consider the class of long-tailed distributions $\mathcal{L}$. As mentioned already, $\mathcal{L}$ is a very useful subclass of $\mathcal{K}$ and closely related to the class $S$. Due to their prominent role in ruin theory and many other areas, long-tailed distributions have been widely studied and are well examined. For a standard reference on long-tailed distributions, we refer the reader to Foss et al., [18]. Some of their properties, which will be used and compared in this work to the properties of other distribution classes, are collected in this section.
3 Distribution classes

The class of long-tailed distributions is closed under strong tail-equivalence. This property follows immediately from the definition of the class $\mathcal{L}$.

**Proposition 16.** (Closure of the class $\mathcal{L}$ under strong tail-equivalence)

If $F \in \mathcal{L}$, $G \in \mathcal{F}_R$ and $G \sim F$, then

$$G \in \mathcal{L}.$$  

**Example 17.** Neither $\mathcal{L}$ nor $\mathcal{S}$ are closed under weak tail-equivalence. Indeed, let

$$F(x) = 1 - \frac{x_m}{x}, \quad x \geq x_m,$$

be the distribution function of a Pareto distribution with parameters $\alpha = 1$, and $x_m > 0$, so that $F$ is subexponential and long-tailed. Let $G$ be the ‘Peter and Paul’ distribution, such that the tail of $G$ is as follows,

$$G(x) = 2^{-k}, \quad x \in [2^k, 2^{k+1}), \quad k \in \mathbb{N}.$$ 

Then the tails of $F$ and $G$ are weakly tail-equivalent, i.e. $F \asymp G$, but $G \notin \mathcal{L}$ and hence $G \notin \mathcal{S}$.

Next, we consider the convolution properties of $\mathcal{L}$. In contrast to the class $\mathcal{S}$ the long-tailed distributions are closed under convolution.

**Proposition 18.** (Convolution property of $\mathcal{L}$)

If $F, G \in \mathcal{L}$, then

$$G * F \in \mathcal{L}.$$ 

For a proof of the above proposition, we refer the reader to Foss et al., [18, Corollary 2.42, page 31].

We know already that the subexponential distributions form a subclass of $\mathcal{L}$. Hence, one may wonder under which conditions a long-tailed distribution becomes subexponential. The next proposition describes the relation between the classes $\mathcal{L}$ and $\mathcal{S}$.

**Proposition 19.** (Relation between the classes $\mathcal{S}$ and $\mathcal{L}$)

Assume $F \in \mathcal{F}$. Let $X_1$ and $X_2$ be two i.i.d random variables with a distribution $F$. Then the following assertions are equivalent:

a) $F \in \mathcal{S}$;

b) $F \in \mathcal{L}$ and there exists a function $h$ such that $h(x) \to \infty$, $h(x) < x/2$, and $\overline{F}(x-y) \sim \overline{F}(x)$ as $x \to \infty$ uniformly in $|y| \leq h(x)$,

$$P(X_1 + X_2 > x, \min(X_1, X_2) > h(x)) = o(\overline{F}(x)) \text{ as } x \to \infty; \quad (3.4)$$

c) $F \in \mathcal{L}$ and the relation (3.4) holds for every function $h$ such that $h(x) \to \infty.$
Proposition 19 was formulated and proved by Asmussen et al. in [1, Proposition 2] in a more general form which included local versions of subexponential and long-tailed distributions. In Section 4.5, we will consider local version of our distributions and give the original version of the proposition.

Remark 20. Note that we can change part c) of the proposition above without consequences, if we request the condition (3.4) not for all functions $h$ such that $h(x) \to \infty$, as $x \to \infty$, but rather for all unbounded, nonnegative and nondecreasing functions, i.e. for all $h \in \mathcal{G}$. This modified condition will be crucial for the definition of our new large claim classes. It turns out that the condition (3.4) for all $h \in \mathcal{G}$ generates our smallest distribution class, which will be considered in Section 4.4.
3 Distribution classes

3.2 The classes \(\mathcal{OS}\) and \(\mathcal{OL}\)

3.2.1 Definition

In this section, we introduce distribution classes, which are obtained by a generalization of the definitions of the heavy-tailed classes \(S\) and \(L\). Both generalizations are obtained by a weakening of the corresponding definition.

**Definition 21.** (\(O\)–Subexponential distributions)

A distribution \(F \in \mathcal{F}_\mathbb{R}\) is called \(O\)–subexponential if

\[
\limsup_{x \to \infty} \frac{F^{2^x}(x)}{F(x)} < \infty.
\]  

(3.5)

The class of \(O\)–subexponential distribution is denoted by \(\mathcal{OS}\).

In the following, the expression (3.5) will be abbreviated as follows. Let \(G \in \mathcal{F}_\mathbb{R}\), then

\[
c_G := \limsup_{x \to \infty} \frac{G^{2^x}(x)}{G(x)}.
\]

The class \(\mathcal{OS}\) was introduced and investigated by Klüppelberg in [21, 1990], where a distribution in \(\mathcal{OS}\) was called weak idempotent. Later, Shimura and Watanabe called it the class \(\mathcal{OS}\) in [30, 38]. In a similar way, it is possible to generalize the class of long-tailed distributions \(L\).

**Definition 22.** (The distribution class \(\mathcal{OL}\))

A distribution \(F \in \mathcal{F}_\mathbb{R}\) belongs to the class of generalized long-tailed distributions \(\mathcal{OL}\) if

\[
\limsup_{x \to \infty} \frac{F(x - 1)}{F(x)} < \infty.
\]

Shimura and Watanabe systematically investigated the class \(\mathcal{OS}\) and presented some more properties in [30, 2005]. Below, we collect some properties of \(\mathcal{OS}\) and compare them to the class of subexponential distributions \(S\).

3.2.2 Classification

In this section, we summarize interrelationships of the classes \(\mathcal{OS}\), \(\mathcal{OL}\) and heavy-tailed classes. Both classes are not restricted to heavy-tailed distributions and contain light-tailed distributions.

**Proposition 23.** (Classification of \(\mathcal{OS}\) and \(\mathcal{OL}\))

\(a\) \(S \subset \mathcal{OS} \subset \mathcal{OL}\);

\(b\) \(\mathcal{OS} \notin \mathcal{K}\);

\(c\) \(L \notin \mathcal{OS}\);

\(d\) \(L \cap \mathcal{OS} \cap \mathcal{F} \neq \emptyset\).
For the proof of part a) of the proposition above, we refer the reader to [30, Proposition 2.1].

**Example 24.** Distributions $F \in (\mathcal{L} \cap \mathcal{OS} \cap \mathcal{F}) \backslash \mathcal{S}$, $F \in \mathcal{OS} \backslash \mathcal{K}$ and $F \in \mathcal{L} \backslash \mathcal{OS}$.

An example of a distribution $F \in (\mathcal{L} \cap \mathcal{OS} \cap \mathcal{F}) \backslash \mathcal{S}$ was given by Lin and Wang in [26, Example 2.1] and we will expand on this example, using the proof’s idea to show other properties of the distribution in Section 4.1, see Example 51 on page 32. For a distribution $F \in \mathcal{L} \backslash \mathcal{OS}$ we refer the reader to [26, Appendix] and for a distribution $F \in \mathcal{OS} \backslash \mathcal{K}$ to Example 33 on page 25.

Although the class $\mathcal{OL}$ is not a subclass of $\mathcal{K}$, we can state that $O$–long-tailed distributions are ‘weakly heavy-tailed’ (in the sense of the next proposition).

**Proposition 25.** If $F \in \mathcal{OL}$, then $\lim_{x \to \infty} \exp(\lambda x) \overline{F}(x) = \infty$ for some $\lambda > 0$.

For the proof, we refer the reader to [30, Proposition 2.2].

### 3.2.3 Properties of $\mathcal{OS}$

The class $\mathcal{OS}$ has similar properties to $\mathcal{S}$, and some of them are collected in [30, 26, 21]. However, there are important differences between $\mathcal{OS}$ and $\mathcal{S}$ as well. The most important difference for us, is the weak tail-equivalence closure of $\mathcal{OS}$. In contrast to $\mathcal{L}$ and $\mathcal{S}$, the class $\mathcal{OS}$ is closed under weak tail-equivalence. We state this result in the next proposition.

**Proposition 26.** (Closure of the class $\mathcal{OS}$ under weak tail-equivalence) If $F \in \mathcal{OS}$, $G \in \mathcal{FR}$ and $G \sim \overline{F}$, then

$$G \in \mathcal{OS}.$$  

For a proof, we refer the reader to Remark 129 on page 99.

Next, we consider the convolution properties of $\mathcal{OS}$. Again in contrast to $\mathcal{S}$, the class $\mathcal{OS}$ is closed under convolution without additional requirements.

**Proposition 27.** (Convolution properties of the class $\mathcal{OS}$) If $F, G \in \mathcal{OS}$, then

$$G * F \in \mathcal{OS}.$$  

A proof of the proposition above, for example, can be found in [40, Lemma 3.1 (iii)].

**Example 28.** The class $\mathcal{OS}$ is not closed under root convolution. A counterexample is given by Watanabe and Shimura in [30, Proposition 1.1 (iv)]. This property is highly desirable and has many applications. For that reason, the nonclosure of $\mathcal{OS}$ under root convolutions can be a crucial disadvantage in comparison to other distribution classes.
Next, we compare the closure properties of $\mathcal{OS}$ under the operations maximum, minimum and mixture.

If $F, G \in \mathcal{OS}$, then we cannot conclude that $F \lor Y \in \mathcal{OS}$. A counterexample is given by Lin in [26, Proposition 3.1], i.e. Lin provides distributions $F, G \in \mathcal{OS}$ such that

$$F \lor Y \notin \mathcal{OS}.$$  \hspace{1cm} (3.6)

By combining the convolution closure of $\mathcal{OS}$ and (3.6), we see that the implication $F \ast G \in \mathcal{OS} \Rightarrow F \lor Y \in \mathcal{OS}$ also cannot hold in the general case. Hence, in contrast to the class $S$ a perfect equivalence result (Proposition 14) cannot hold for $\mathcal{OS}$. However, the class $\mathcal{OS} \cap \mathcal{F}$ is closed under the operation minimum.

**Proposition 29.** (Maximum, minimum and mixture in $\mathcal{OS}$)

Let $X, Y$ be two independent random variables with distributions $F, G \in \mathcal{F}$.

a) Let $F_Z$ be the distribution of the mixture of $X$ and $Y$ with parameter $p \in (0, 1)$.

If $F, G \in \mathcal{OS}$, then the following are equivalent:

i) $F \lor Y \in \mathcal{OS}$,

ii) $F_Z \in \mathcal{OS}$.

b) If $F, G \in \mathcal{OS}$, then

$$F \land Y \in \mathcal{OS}.$$  \hspace{1cm} (3.7)

**Proof:** a) Recall that a random variable $Z$ is a mixture of $X$ and $Y$ with parameter $p \in (0, 1)$, if its distribution function is given by

$$F_Z(x) = pF(x) + (1 - p)G(x), \hspace{0.5cm} x \geq 0.$$  \hspace{1cm} (3.7)

Thus, part a) follows from (3.7) and the closure of $\mathcal{OS}$ under weak tail-equivalence.

b) We refer the reader to [26, Lemma 3.1]. \hfill $\Box$

Finally, we give the Kesten’s bound for the class $\mathcal{OS}$. In comparison to the class $S$ the bound is weaker, but still a useful tool in many proofs and applications.

**Lemma 30.** (Kesten’s bound for $\mathcal{OS} \cap \mathcal{F}$)

If $F \in \mathcal{OS} \cap \mathcal{F}$ then, for every $\varepsilon > 0$, there exists $c > 0$ such that for all $n \geq 2$ and $x \geq 0$:

$$\frac{F_n^\varepsilon(x)}{F(x)} \leq c(cF + \varepsilon - 1)^n.$$  \hspace{1cm}

For a proof we refer the reader to [38, Lemma 6.3 (ii)]. Kesten’s bound for $\mathcal{OS}$ (for distributions on the whole real line of $\mathbb{R}$) can be found in [38, Lemma 6.3].
3 Distribution classes

3.3 Convolution equivalence classes \(S(\gamma)\) and \(L(\gamma)\)

In this section, we consider two important light-tailed distribution classes which will provide us with some important examples for this work.

3.3.1 Definition

**Definition 31.** (Distribution classes \(S(\gamma)\) and \(L(\gamma)\))

A distribution \(F \in \mathcal{F}\) belongs to \(S(\gamma), \gamma \geq 0\), if for any fixed \(y \in \mathbb{R}\)

\[
\lim_{x \to \infty} \frac{F(x+y)}{F(x)} = \exp(-\gamma y) \quad (3.8)
\]

and for some constant \(c \in (0, \infty)\)

\[
\lim_{x \to \infty} \frac{F^2(x)}{F(x)} = 2c < \infty. \quad (3.9)
\]

A distribution \(F \in \mathcal{F}\) belongs to \(L(\gamma), \gamma \geq 0\), if and only if \(F\) satisfies the condition (3.8).

The classes above were introduced independently by Chistyakov [5] and Chover, Ney and Wainger [7, 6], see also [11]. The class \(S(\gamma)\) is also called the convolution equivalence class, and applications in classical ruin theory and infinitely divisible laws can be found in [35, 12]. If \(\gamma = 0\) then we obtain \(S(\gamma) = S\) resp. \(L(\gamma) = L \cap \mathcal{F}\) the class of subexponential resp. long-tailed distributions.

**Remark 32.** It has been proved in [11, Lemma 2.1] that for any \(F \in S(\gamma), \gamma \geq 0\), the constant \(c\) in (3.9) can be calculated by

\[
c = \mathbb{E} \exp(\gamma X).
\]

**Example 33.** Consider the distribution \(F \in \mathcal{F}\) with the density

\[
f(x) = e^{-x} \frac{C}{1+x^2}, \quad x \geq 0,
\]

for \(C > 0\) such that \(\int_0^\infty f(x)dx = 1\). Note that there seems to be no closed-form expression for \(C\), but it can be evaluated numerically to \(C \approx 1.609\). By direct calculation we obtain \(F \in S(1)\).

Note that \(f(x)\) is obtained from the subexponential density \(2/(\pi(1+x^2))\) by multiplication with a negative exponential and a suitable constant. This is a typical way to construct distributions of the class \(S(\gamma), \gamma > 0\), see [11, Section 3, page 271] for more details.

From the definition of the classes \(S(\gamma)\) and \(L(\gamma)\) we immediately obtain the following relations.
3 Distribution classes

Proposition. (Relation between \( S(\gamma) \), \( L(\gamma) \) and classes \( OL, OS \))

a) \( \bigcup_{\gamma \geq 0} S(\gamma) \subset OS \);

b) \( \bigcup_{\gamma \geq 0} L(\gamma) \subset OL \).

The following proposition generalizes Proposition 19.

Proposition 34. Let \( \gamma \geq 0 \) and \( F \in L(\gamma) \). Let \( X_1 \) and \( X_2 \) be two i.i.d random variables with a distribution \( F \).

Then \( F \in S(\gamma) \) if and only if

\[
P(X_1 + X_2 > x, \min(X_1, X_2) > h(x)) = o(F(x)) \text{ as } x \to \infty
\]

for all \( h \in \mathcal{G} \).

Proof. The case \( \gamma = 0 \) was proven in [1, Proposition 2].

There exists a function \( h \in \mathcal{G} \) such that

\[
\sup_{|y| \leq h(x)} \left| \frac{\mathcal{F}(x - y)}{\exp(\gamma y)} - \mathcal{F}(x) \right| = o(F(x)), \tag{3.10}
\]

as \( x \to \infty \), uniformly in \( y \leq h(x) \). The existence of such a function \( h \) was proven in [18, Lemma 2.19] in the case \( \gamma = 0 \) and the construction in the case \( \gamma > 0 \) is similar.

Let \( g \in \mathcal{G} \) be such that \( g(x) \leq \min(h(x), \frac{x}{2}) \), \( x \geq 0 \). Then we obtain

\[
P(S_2 > x) = P(S_2 > x, X_1 \leq g(x)) + P(S_2 > x, X_2 \leq g(x)) + P(S_2 > x, X_1 > g(x), X_2 > g(x)) \tag{3.11}
\]

From (3.10) we obtain

\[
\frac{P(S_2 > x, X_1 \leq g(x))}{P(X_1 > x)} = \frac{\int_{0}^{g(x)} \frac{\mathcal{F}(x - y)}{\exp(\gamma y)}dF(y)}{\mathcal{F}(x)} \\
\sim \int_{0}^{g(x)} \exp(\gamma y)dF(y) \\
\sim E[\exp(\gamma X_1)]. \tag{3.12}
\]

By combining (3.11) and (3.12) we obtain the assertion. \( \Box \)
4 New large claim classes

In this chapter, we introduce new large claim classes, which are appropriate for modelling extreme events. In the following sections, we systematically study these new classes by providing their classification, properties and examples. First, we consider our new approach in the most general form and become increasingly concrete in Sections 4.3 and 4.4, until we arrive at our smallest distribution class.

Every step of concretization consists in a strengthening of the requirements on the considered distribution class. Thus, on the one side we shrink the distribution class, on the other side we can develop a deeper insight into the considered class.

4.1 The large claim class $\mathcal{A}(\mathcal{E})$

4.1.1 Definition

Denote the set of all nonnegative, unbounded and nondecreasing functions by $\mathcal{G}$ and recall that the distribution $F$ of i.i.d. nonnegative random variables $X_1, X_2, \ldots$ is subexponential, if $F$ is long-tailed and the following assertion holds for all $g \in \mathcal{G}$:

$$\lim_{x \to \infty} P(S_2 > x, X_{2,2} > g(x)) = 0,$$

(4.1)

see Remark 20 on page 21 for details.

We use the condition (4.1) as a starting point for an approach to define a new large claim class. First, we weaken this condition and consider the following property for all $g \in \mathcal{G}$:

$$\lim_{x \to \infty} \frac{P(S_2 > x, X_{2,2} > g(x))}{P(X_1 > x)} = \lim_{x \to \infty} \frac{P(X_{2,2} > g(x)|S_2 > x)}{P(S_2 > x)} = 0.$$  

(4.2)
4 New large claim classes

This means that given the event that the sum exceeds \( x \), the minimum of \( X_1 \) and \( X_2 \) has to be lower than a certain level \( g(x) \), which is growing unboundedly and nondecreasingly as \( x \to \infty \). Obviously, the largest element of \( X_1 \) and \( X_2 \) has to be larger than \( x - g(x) \). By choosing \( g \) such that the difference \( x - g(x) \) is growing faster than \( g(x) \) as \( x \to \infty \), e.g. \( g(x) = \varepsilon x, \varepsilon \in (0, \frac{1}{2}) \), we ensure that for large \( x \) the sum is dominated by the maximum.

Finally, we extend the approach to \( n \) random variables and weaken the condition once again by the reduction of the set of functions, i.e. by requesting the condition (4.2) only for a nonempty subset \( \mathcal{E} \) of \( \mathcal{G} \). Said in different terms, given the event that the sum \( S_n \) is larger than \( x \), the probability of \( X_{2,n} \) crossing the level \( g(x) \) tends to zero as \( x \to \infty \). By requesting a slow-growing level \( g \in \mathcal{E} \), we can ensure the domination of the sum by the largest element. Finally, we arrive at the following definition.

**Definition 35.** (Large claim class \( A(\mathcal{E})^{(n)} \))

Let \( \mathcal{E} \subseteq \mathcal{G} \) be not empty, then we define

\[
A(\mathcal{E})^{(n)} = \left\{ F \in \mathcal{F} : \lim_{x \to \infty} P(X_{2,n} > g(x) | S_n > x) = 0 \ \forall g \in \mathcal{E} \right\}.
\]

As a consequence, we obtain a way to construct different large claim classes.

**Example 36.** The catastrophe principle in \( A(\mathcal{E})^{(n)} \).

We model the aggregate claim from an insurance portfolio of \( n \) claims. In this context, the random variable \( X_1 \) with distribution \( F \in A(\mathcal{E})^{(n)} \) represents the height of the individual claim from the portfolio and the sum \( S_n \) represents the aggregate claim amount.

We construct \( A(\mathcal{E})^{(n)} \) as follows. Let \( n = 11 \) and \( \mathcal{E} = \{g\} \), where \( g(x) = \frac{1}{1000} x \). Then, for large \( x \), the maximum claim \( X_{1,11} \) tends to contribute at least 99% to the aggregate claim \( S_{11} \).

At first sight, the definition of \( A(\mathcal{E})^{(n)} \) may raise several questions. Hence, we will systematically study the classification, properties and applications of the class \( A(\mathcal{E})^{(n)}, \mathcal{E} \subseteq \mathcal{G} \), in the next sections. The following proposition collects some simple statements, which follow immediately from the definition of the class \( A(\mathcal{E})^{(n)} \).

**Proposition 37.** Let \( \mathcal{E}_1, \mathcal{E}_2 \subseteq \mathcal{G} \) be not empty. Then the following holds:

a) If \( \mathcal{E}_1 \subseteq \mathcal{E}_2 \), then \( A(\mathcal{E}_1)^{(n)} \supseteq A(\mathcal{E}_2)^{(n)} \).

b) Let \( g \in \mathcal{G} \) and define \( \mathcal{E}_2 := \{g\} \cup \mathcal{E}_1 \). If there exists \( f \in \mathcal{E}_1 \) such that \( f(x) \leq g(x) \) for all \( x \geq 0 \), then the following holds:

\[
A(\mathcal{E}_2)^{(n)} = A(\mathcal{E}_1)^{(n)}.
\]

c) If for all \( g \in \mathcal{E} \) holds \( g(x) \geq x, \forall x \geq 0 \), then for all \( n \geq 2 \):

\[
A(\mathcal{E})^{(n)} = \mathcal{F}.
\]
4 New large claim classes

Obviously, the class $A(\mathcal{E})^{(n)}$ is highly dependent on the set of functions $\mathcal{E}$. However, we can state some properties and consider the structure of $A(\mathcal{E})^{(n)}$ without specifying the subset $\mathcal{E} \subseteq \mathcal{G}$. The only request is the nonemptiness of the subset $\mathcal{E} \subseteq \mathcal{G}$.

4.1.2 Structure of $A(\mathcal{E})^{(n)}$

A better understanding of the structure provides us with additional ways of description and is useful to derive further properties of the class. Thus, we examine the structure of the class $A(\mathcal{E})^{(n)}$ independent of the chosen set of functions $\mathcal{E}$, which is fixed and nonempty.

Proposition 38. (Structural properties of $A(\mathcal{E})^{(n)}$)

Let $\mathcal{E} \subseteq \mathcal{G}$ be not empty. We write:

$$A_1(\mathcal{E})^{(n)} := \left\{ F \in \mathcal{F} : \lim_{K \to \infty} \inf_{x \geq 0} P(X_{2,n} \leq g(x) \lor K|S_n > x) = 1 \forall g \in \mathcal{E} \right\}.$$

a) Then the sets $A_1(\mathcal{E})^{(n)}$ and $A(\mathcal{E})^{(n)}$ are equal, i.e. for all $n \in \mathbb{N}$:

$$A(\mathcal{E})^{(n)} = A_1(\mathcal{E})^{(n)}.$$

b) For all $n \geq 2$:

$$A(\mathcal{E})^{(n)} \subseteq A(\mathcal{E})^{(n+1)}. \quad (4.3)$$

Remark 39. We are not able to answer the question: does the equality $A(\mathcal{E})^{(n)} = A(\mathcal{E})^{(n+1)}$ hold for all $\mathcal{E} \subseteq \mathcal{G}$?

However, under additional conditions on $\mathcal{E}$, e.g. if $\mathcal{E} = \mathcal{G}$ (see Proposition 72) we can give a positive answer to this question. The next proposition states a condition under which equality holds in (4.3).

Proposition 40. (Structural properties of $A(\mathcal{E})^{(n)}$ in $\mathcal{O} \mathcal{S}$)

Let $\mathcal{E} \subseteq \mathcal{G}$ be not empty. We write:

a) If $F^{n*} \in \mathcal{O} \mathcal{S}$, then:

$$F \in A(\mathcal{E})^{(n)} \iff F \in A(\mathcal{E})^{(n+1)}.$$

b) If $F^{2*} \in \mathcal{O} \mathcal{S}$, then for all $n \geq 2$:

$$F \in A(\mathcal{E})^{(n)} \iff F \in A(\mathcal{E})^{(n+1)}.$$

In the following, we will consider the class

$$A(\mathcal{E}) := A(\mathcal{E})^{(2)} = \bigcap_{n=2}^{\infty} A(\mathcal{E})^{(n)}.$$
4 New large claim classes

4.1.3 Properties of $A(\mathcal{E})$ and examples

In this section, we collect some important properties of the distribution class $A(\mathcal{E})$ for a nonempty and fixed set of functions $\mathcal{E} \subseteq \mathcal{G}$. We start with the most important property of $A(\mathcal{E})$: the closure under weak tail-equivalence. The closure holds for all $A(\mathcal{E})$, $\mathcal{E} \subseteq \mathcal{G}$, i.e. this property is independent of the subset $\mathcal{E}$.

**Proposition 41.** (Closure of the class $A(\mathcal{E})$ under weak tail-equivalence)

Let $\mathcal{E} \subseteq \mathcal{G}$ be not empty. If $F \in A(\mathcal{E})$ and $F \preceq G$, then $G \in A(\mathcal{E})$.

**Remark 42.** From the weak tail-equivalence closure of $A(\mathcal{E})$ we obtain an obvious observation. We cannot choose the set of functions $\mathcal{E}$ such that $A(\mathcal{E})$ equals the class of subexponential or long-tailed distributions, or another distribution class, which is not closed under weak tail-equivalence.

Next, we turn to the convolution properties of the distribution class $A(\mathcal{E})$, $\mathcal{E} \subseteq \mathcal{G}$. The next proposition states a condition on $\mathcal{E}$ such that the class $A(\mathcal{E})$ is closed under convolution power.

**Proposition 43.** (Closure of $A(\mathcal{E})$ under convolution power)

Let $\mathcal{E} \subseteq \mathcal{G}$ be not empty and let $E_n := \{\frac{1}{n}g : g \in \mathcal{E}\}$. If $F \in A(\mathcal{E}) \cap A(E_n)$, then

$$F^{n*} \in A(\mathcal{E}).$$

**Remark 44.** Note that if $\mathcal{E} = \mathcal{G}$, then $E_n = \mathcal{G}$, $n \geq 1$, and thus $A(\mathcal{E}) = A(E_n)$, $n \geq 1$.

By using the weak tail-equivalence of $A(\mathcal{E})$, we obtain the closure of $A(\mathcal{E})$ under convolution power in the distribution class $OS$.

**Proposition 45.** (Closure of $A(\mathcal{E})$ under convolution power in $OS$)

Let $\mathcal{E} \subseteq \mathcal{G}$ be not empty. If $F \in A(\mathcal{E})$ and $F \in OS$, then for all $n \geq 2$:

$$F^{n*} \in A(\mathcal{E}).$$

On the one side, we can not state the closure of $A(\mathcal{E})$ under (general) convolution for all $\mathcal{E} \subseteq \mathcal{G}$, a counterexample for nonclosure of $A(\mathcal{E})$, $\mathcal{E} = \mathcal{G}$, is given in in Section 4.4, Example 86 on page 59. On the other side, we can not exclude the existence of a subset $\mathcal{E} \subset \mathcal{G}$ such that the class $A(\mathcal{E})$ is closed under convolution. However, we can generalize the result of Proposition 43 to obtain closure under convolution of distributions with weakly equivalent tails.

**Proposition 46.** (Convolution closure of $A(\mathcal{E})$ under weakly equivalent tails)

Let $\mathcal{E} \subseteq \mathcal{G}$ be not empty and let $E_2 := \{\frac{1}{2}g : g \in \mathcal{E}\}$. If $F \in A(\mathcal{E}) \cap A(E_2)$ and $G \sim F$, then

$$F*G \in A(\mathcal{E}).$$
We turn to the next property of \(A(\mathcal{E})\), its closure under convolution roots. Recall that we say that a distribution class \(C\) is closed under convolution roots if \(F_{n^*} \in C\) for some \(n \in \mathbb{N}\) implies \(F \in C\). Further, we say that a distribution class \(C\) is closed under convolution roots in the class \(C^*\) if \(F_{n^*} \in C\) for some \(n \in \mathbb{N}\) and \(F \in C^*\) implies \(F \in C\).

The convolution root closure is crucial for many applications and most heavy-tailed classes are closed under root convolution. However, root convolution closure does not apply to the most important class \(OS\). By combining the properties of \(OS\) and the closure of the class \(A(\mathcal{E})\) under tail-equivalence, we can obtain the following result:

**Proposition 47.** (Closure of the class \(A(\mathcal{E})\) in \(OS\) under root convolution)

Let \(\mathcal{E} \subseteq \mathcal{G}\) be not empty, \(F \in \mathcal{F}\) and \(F_{n^*} \in A(\mathcal{E})\). Then the following assertions hold:

a) If \(F \in OS\), then \(F_{n^*} \asymp F\) and hence \(F \in A(\mathcal{E})\).

b) If \(F_{2^*} \in OS\), then \(F \in A(\mathcal{E})\).

**Remark 48.** Note that the condition \(F_{2^*} \in OS\) in Proposition 47 b) is weaker than the condition \(F \in OS\) in part a) of Proposition 47, again, because \(OS\) is not closed under convolution roots.

Finally, we consider the closure of the class \(A(\mathcal{E})\) under the operations maximum, minimum and mixture. Let \(Z\) be the of the mixture of \(X\) and \(Y\) with parameter \(p \in (0, 1)\). By using the tail-equivalence of the tails of distributions of \(X \vee Y\) and \(Z\) (i.e. \(F_{X \vee Y} \asymp F_Z\)) and Proposition 41, we can state immediately the following result.

**Proposition 49.** Let \(\mathcal{E} \subseteq \mathcal{G}\) be not empty, \(X, Y\) two independent random variables with distributions \(F, G \in \mathcal{F}\) and \(F_Z\) the distribution of the mixture of \(X\) and \(Y\) with parameter \(p \in (0, 1)\). Then

\[F_{X \vee Y} \in A(\mathcal{E}) \iff F_Z \in A(\mathcal{E}).\]

We obtain the following result for the intersection \(OS \cap A(\mathcal{E})\).

**Proposition 50.** (Maximum, minimum and convolution closure of \(A(\mathcal{E})\) in \(OS\))

Let \(\mathcal{E} \subseteq \mathcal{G}\) be not empty and \(X, Y\) be two independent random variables with distributions \(F, G \in \mathcal{F}\).

a) If \(F, G \in A(\mathcal{E})\), \(F \ast G \in OS \cap A(\mathcal{E})\), then

\[F_{X \vee Y} \in A(\mathcal{E}).\]

b) If \(F_{X \vee Y} \in OS \cap A(\mathcal{E})\), then

\[F \ast G \in A(\mathcal{E}).\]

c) If \(F, G \in OS \cap A(\mathcal{E})\), then

\[F_{X \wedge Y} \in OS \cap A(\mathcal{E}).\]
4 New large claim classes

Example 51. Distribution $F \in \mathcal{OS}\setminus\mathcal{A}(\mathcal{E})$.

This example was given by Wang et al. in [26, Proposition 3.1]. The authors construct a distribution such that $F \in \mathcal{OS}\cap\mathcal{L}\setminus\mathcal{S}$. Let $g(x) = \frac{1}{2}x$ and $\mathcal{E} \subset \mathcal{G}$ such that $g \in \mathcal{E}$. Then we show that $F \in \mathcal{OS}\cap\mathcal{L}\setminus\mathcal{A}(\mathcal{E})$ holds as well.

The construction of the distribution $F$ is as follows. Choose a constant $x_1 > 1$ and for every positive integer $n$, define $x_{n+1} := (2x_n)^2$. Obviously, $x_{n+1} > 2x_n$ and $x_n \to \infty$ as $n \to \infty$. Choose any fixed $\kappa \in (0, 1)$ and define

$$F(x) = \begin{cases} x_n^{-\kappa} + \frac{(2x_n)^{-2\kappa} - x_n^{-\kappa}}{x_n} (x - x_n) & \text{for } x \in [x_n, 2x_n), \\ \frac{1}{2} x_n^{-2\kappa} & \text{for } x \in [2x_n, x_{n+1}). \end{cases}$$

Then following holds for all $n \geq 1$:

$$\frac{F^2(x_n)}{F(2x_n)} = \frac{(2x_{n-1})^{-4\kappa}}{(2x_n)^{-2\kappa}} = \frac{x_n^{-2\kappa}}{(2x_n)^{-2\kappa}} = 2^{2\kappa} > 0. \quad (4.4)$$

From $F \in \mathcal{OS}$ there exists $C < \infty$ such that for all $n \in \mathbb{N}$:

$$\frac{\mathbb{P}(S_2 > 2x_n)}{\mathbb{P}(X_1 > 2x_n)} < C.$$

Suppose $F \in \mathcal{A}(\mathcal{E})$. Then, we obtain:

$$\lim_{n \to \infty} \frac{F^2(x_n)}{F(2x_n)} = \lim_{n \to \infty} \frac{\mathbb{P}(X_1 > x_n)\mathbb{P}(X_2 > x_n)}{\mathbb{P}(X_1 > 2x_n)} = \lim_{n \to \infty} \frac{\mathbb{P}(X_1 > x_n, X_2 > x_n, S_2 > 2x_n)}{\mathbb{P}(X_1 > 2x_n)} = \lim_{n \to \infty} \frac{\mathbb{P}(X_{1,2} > x_n, S_2 > 2x_n)}{\mathbb{P}(S_2 > 2x_n)} \leq C \lim_{n \to \infty} \frac{\mathbb{P}(X_{1,2} > x_n, S_2 > 2x_n)}{\mathbb{P}(S_2 > 2x_n)} = 0.$$

We have a contradiction to (4.4). Thus, we obtain $F \notin \mathcal{A}(\mathcal{E})$.

It is not possible to classify precisely the class $\mathcal{A}(\mathcal{E})$ for a general set of functions $\mathcal{E}$. However, further examples and classification will be given for some certain distribution classes of the type $\mathcal{A}(\mathcal{E})$, $\mathcal{E} \subseteq \mathcal{G}$. For example, if

$$\mathcal{H} := \{g \in \mathcal{G} | \exists \varepsilon > 0, g(x) = \varepsilon x\}$$

then we can refine many statements for the class $\mathcal{A}(\mathcal{H})$. Hence, we will consider the properties, the structure and the classification of the class $\mathcal{A}(\mathcal{H})$ in the next sections.
4 New large claim classes

4.1.4 Proofs

Throughout the proofs we will use the following notation. Denote by $X_1, X_2, X_3...$ i.i.d. random variables with common distribution $F \in \mathcal{F}$, and by $Y_1, Y_2, Y_3...$ i.i.d. random variables with common distribution $G \in \mathcal{F}$. Further, let

$$S_n := \sum_{k=1}^{n} X_k, \quad \text{and} \quad \hat{S}_n := \sum_{k=1}^{n} Y_k.$$

First, we prove the structural properties of $A(\mathcal{E})^{(n)}$, Proposition 38.

**Proof.**

a) The inclusion $A_1(\mathcal{E})^{(n)} \subset A(\mathcal{E})^{(n)}$ is obvious. We prove $A_1(\mathcal{E})^{(n)} \supset A(\mathcal{E})^{(n)}$.

Suppose $F \in A(\mathcal{E})^{(n)}$. Let $g \in \mathcal{E}$ and $\varepsilon > 0$.

By definition we know that there is a constant $x_0$ such that for all $x \geq x_0$:

$$P(X_{2,n} \leq g(x)|S_n > x) \geq 1 - \varepsilon. \quad (4.5)$$

Let $\delta > 0$. There is $K_0$ such that $P(X_{2,n} \leq g(x) \vee K) \geq 1 - \delta$ for all $K \geq K_0$ and $x \leq x_0$. Hence, we obtain for $x \leq x_0$ and $K \geq K_0$:

$$P(X_{2,n} \leq g(x) \vee K|S_n > x) \geq \frac{P(X_{2,n} \leq g(x) \vee K) + P(S_n > x) - 1}{P(S_n > x)} \geq 1 - \frac{\delta}{P(S_n > x_0)}. \quad (4.6)$$

where we used in (4.6) the inequality

$$P(A_1, A_2) \geq P(A_1) + P(A_2) - 1.$$

By (4.5) and (4.7), we see that $F \in A_1(\mathcal{E})^{(n)}$, since $\delta > 0$ and $\varepsilon > 0$ are arbitrary.

b) We show $A_1(\mathcal{E})^{(n)} \subset A_1(\mathcal{E})^{(n+1)}$, then we can deduce from part a) that $A(\mathcal{E})^{(n)} \subset A(\mathcal{E})^{(n+1)}$ holds as well.

Suppose $F \in A_1(\mathcal{E})^{(n)}$. Let $g \in \mathcal{E}$ and $\varepsilon > 0$. By definition we know that there exists a constant $K_0 > 0$ such that for all $K \geq K_0$ and $x \geq 0$:

$$P(X_{2,n} \leq g(x) \vee K, S_n > x) \geq \left(1 - \frac{\varepsilon}{n+1}\right)P(S_n > x) = \left(1 - \frac{\varepsilon}{n+1}\right)F^{n+1}(x). \quad (4.8)$$

Hence, we obtain for $x \geq 0$:

$$P(X_{2,n} \leq g(x) \vee K|S_{n+1} > x) \geq \frac{\int_{0}^{\infty} P(X_{2,n} \leq g(x) \vee K, S_n > x - t) dF(t)}{F^{(n+1)}(x)} \geq \frac{\int_{0}^{\infty} P(X_{2,n} \leq g(x - t) \vee K, S_n > x - t) dF(t)}{F^{(n+1)}(x)} \geq \frac{(1 - \frac{\varepsilon}{n+1}) \int_{0}^{\infty} F^{n+1}(x - t) dF(t)}{F^{(n+1)}(x)} = 1 - \frac{\varepsilon}{n+1}.$$
4 New large claim classes

Denote by $X_{2,1,i-1,i+1,...,n,n+1}$ the second largest element (pointwise) out of $n$ random variables, which are given by $X_1, X_1, X_{i-1}, X_{i+1}, ..., X_{n+1}$, $i = 1, ... , n$, and by $X_{2,2, ..., n,n+1}$ the second largest element (pointwise) out of $X_2, ..., X_{n+1}$.

Thus, we see that for all $x \geq 0$:

$$
\mathbb{P}(X_{2,n+1} \leq g(x) \lor K | S_{n+1} > x) = \mathbb{P}(X_{2,(1,...,n)} \leq g(x) \lor K, X_{2,(1,...,n-1,n+1)} \leq g(x) \lor K, ..., X_{2,(2, ..., n,n+1)} \leq g(x) \lor K | S_{n+1} > x)
\geq (n+1) \mathbb{P}(X_{2,n} \leq g(x) \lor K | S_{n+1} > x) - n
\geq 1 - \varepsilon,
$$

where we used the inequality

$$
\mathbb{P}(\bigcap_{i=1}^{n+1} A_i) \geq \sum_{i=1}^{n+1} \mathbb{P}(A_i) - n.
$$

We obtain $F \in A_1(\mathcal{E})^{(n+1)}$. Thus, by part a) we have $\mathcal{A}(\mathcal{E})^{(n)} \subset A(\mathcal{E})^{(n+1)}$.

Next, we prove the structural properties of the class $\mathcal{A}(\mathcal{E})^{(n)}$ in the class $\mathcal{OS}$, Proposition 40.

Proof. a) We prove the following assertion: If $F^{n*} \in \mathcal{OS}$, then $F \in A(\mathcal{E})^{(n+1)} \Rightarrow F \in A(\mathcal{E})^{(n)}$.

Suppose $F \in A(\mathcal{E})^{(n+1)}$, $F^{n*} \in \mathcal{OS}$ and $F \notin A(\mathcal{E})^{(n)}$.

Then there exists $g \in \mathcal{E}$ such that

$$
\lim_{x \to \infty} \sup \frac{\mathbb{P}(X_{2,n} > g(x), S_n > x)}{\mathbb{P}(S_n > x)} > 0,
$$

and

$$
\lim_{x \to \infty} \frac{\mathbb{P}(X_{2,n+1} > g(x), S_{n+1} > x)}{\mathbb{P}(S_{n+1} > x)} = 0.
$$

Thus, we have:

$$
\begin{align*}
\infty &= \lim_{x \to \infty} \sup \frac{\mathbb{P}(X_{2,n} > g(x), S_n > x)}{\mathbb{P}(X_{2,n+1} > g(x), S_{n+1} > x)} \\
&= \lim_{x \to \infty} \sup \frac{\mathbb{P}(X_{2,n} > g(x), S_n > x)}{\mathbb{P}(S_{n+1} > x)} \cdot \frac{\mathbb{P}(X_{2,n+1} > g(x), S_{n+1} > x)}{\mathbb{P}(S_n > x)} \\
&\leq \lim_{x \to \infty} \sup \frac{\mathbb{P}(S_{n+1} > x)}{\mathbb{P}(S_n > x)} \quad (4.9)
\end{align*}
$$

From $F^{n*} \in \mathcal{OS}$ $\Leftrightarrow \lim_{x \to \infty} \frac{\mathbb{P}(S_{n+1} > x)}{\mathbb{P}(S_n > x)} < \infty$ and (4.10) we obtain a contradiction.

The reverse direction $F \in A(\mathcal{E})^{(n+1)} \Leftrightarrow F \in A(\mathcal{E})^{(n)}$ follows from $A(\mathcal{E})^{(n)} \subset A(\mathcal{E})^{(n+1)}$, see part b) of Proposition 38. The proof is complete.

b) We prove: If $F^{2*} \in \mathcal{OS}$, then $F \in A(\mathcal{E})^{(n+1)} \Leftrightarrow F \in A(\mathcal{E})^{(n)}$ for all $n \geq 2$.

Suppose $F^{2*} \in \mathcal{OS}$. From Lemma 130 and the closure of $\mathcal{OS}$ under weak tail-equivalence we obtain $F^{n*} \in \mathcal{OS}$ for all $n \geq 2$. Thus, the assertion follows from part a).
Now, we prove the closure of the class $\mathcal{A}(\mathcal{E})$ under weak tail-equivalence, Proposition 41.

**Proof.** We prove the following property of the class $\mathcal{A}(\mathcal{E})$: If $F \in \mathcal{A}(\mathcal{E})$ and $\overline{F} \asymp \overline{G}$, then $G \in \mathcal{A}(\mathcal{E})$.

Suppose $F \in \mathcal{A}(\mathcal{E})$, $\overline{F} \asymp \overline{G}$ and $G \notin \mathcal{A}(\mathcal{E})$. There exists $h \in \mathcal{E}$ such that

$$\limsup_{x \to \infty} \frac{\mathbb{P}(Y_{2,2} > h(x), \hat{S}_2 > x)}{\mathbb{P}(\hat{S}_2 > x)} > 0.$$  \hfill (4.11)

By definition we have for all $g \in \mathcal{E}$:

$$\lim_{x \to \infty} \frac{\mathbb{P}(X_{2,2} > g(x), S_2 > x)}{\mathbb{P}(S_2 > x)} = 0.$$  \hfill (4.12)

Thus, by combining (4.11) and (4.12) we obtain:

$$\limsup_{x \to \infty} \frac{\mathbb{P}(Y_{2,2} > g(x), \hat{S}_2 > x)}{\mathbb{P}(\hat{S}_2 > x)} \frac{\mathbb{P}(S_2 > x)}{\mathbb{P}(X_{2,2} > g(x), S_2 > x)} = \limsup_{x \to \infty} \frac{\mathbb{P}(X_{2,2} > g(x), S_2 > x)}{\mathbb{P}(S_2 > x)} \limsup_{x \to \infty} \frac{\mathbb{P}(S_2 > x)}{\mathbb{P}(X_{2,2} > g(x), S_2 > x)}$$ \hfill (4.13)

From $\overline{F} \asymp \overline{G}$ and Lemma 128 we obtain $\overline{F^{2^e}} \asymp \overline{G^{2^e}}$, i.e. $\limsup_{x \to \infty} \frac{\mathbb{P}(S_2 > x)}{\mathbb{P}(X_{2,2} > g(x), S_2 > x)} < \infty$. Hence, by Lemma 131 we know that

$$\limsup_{x \to \infty} \frac{\mathbb{P}(Y_{2,2} > g(x), \hat{S}_2 > x)}{\mathbb{P}(X_{2,2} > g(x), S_2 > x)} \leq \left( \limsup_{x \to \infty} \frac{\overline{G}(x)}{\mathbb{F}(x)} \right)^2 < \infty.$$  

Thus, we get a contradiction to (4.13). We conclude $G \in \mathcal{A}(\mathcal{E})$. $\square$

Next, we prove the closure under convolution power of $\mathcal{A}(\mathcal{E})$, Proposition 43.

**Proof.** Recall the notation

$$\mathcal{E}_n := \{ \frac{1}{n} g : g \in \mathcal{E} \}.$$

We prove the following assertion: $F \in \mathcal{A}(\mathcal{E}) \cap \mathcal{A}(\mathcal{E}_n) \Rightarrow F^{n^*} \in \mathcal{A}(\mathcal{E})$.

Suppose $F \in \mathcal{A}(\mathcal{E}) \cap \mathcal{A}(\mathcal{E}_n)$. From $F \in \mathcal{A}(\mathcal{E}_n) := \mathcal{A}(\mathcal{E}_n)^{(2)}$ and part b) of Proposition 38 we know that $F \in \mathcal{A}(\mathcal{E}_n)^{(2)} \subset \mathcal{A}(\mathcal{E}_n)^{(2n)}$.

Hence, we can deduce that $\lim_{x \to \infty} \mathbb{P}(X_{2,2n} \leq \frac{1}{n} g(x) | S_{2n} > x) = 1$ holds for all $g \in \mathcal{E}$.

Denote by $S_n^{(1)}$ and $S_n^{(2)}$ two independent copies of $S_n$ and $S_{2n} = S_n^{(1)} + S_n^{(2)}$. Thus,
we have for all $g \in \mathcal{E}$:

$$\liminf_{x \to \infty} P(S_n^1 \land S_n^2 \leq g(x)|S_{2n} > x) = \liminf_{x \to \infty} \frac{P(S_n^1 \land S_n^2 \leq g(x), S_{2n} > x)}{P(S_{2n} > x)} \geq \liminf_{x \to \infty} \frac{P(X_{2,2n} \leq \frac{1}{n}g(x), S_{2n} > x)}{P(S_{2n} > x)} = \lim_{x \to \infty} \frac{P(X_{2,2n} \leq \frac{1}{n}g(x), S_{2n} > x)}{P(S_{2n} > x)} = 1.$$ 

Hence, we obtain $F^{n*} \in \mathcal{A}(\mathcal{E})^{(2)} = \mathcal{A}(\mathcal{E})$. 

Now, we prove the closure under convolution power of $\mathcal{A}(\mathcal{E})$ in the class $\mathcal{OS}$, Proposition 45.

**Proof.** We prove for all $n \geq 2$: $F \in \mathcal{A}(\mathcal{E}) \cap \mathcal{OS} \Rightarrow F^{n*} \in \mathcal{A}(\mathcal{E})$.

Suppose $F \in \mathcal{A}(\mathcal{E}) \cap \mathcal{OS}$. We obtain $\overline{F} \sim F^{n*}$ from $F \in \mathcal{OS}$ and hence, by using the closure of $\mathcal{A}(\mathcal{E})$ under weak tail-equivalence (Proposition 41), we see that $F^{n*} \in \mathcal{A}(\mathcal{E})$. 

Next, we prove the closure of $\mathcal{A}(\mathcal{E})$ under convolution of distributions with weakly equivalent tails, Proposition 46.

**Proof.** We prove the following property: If $F \in \mathcal{A}(\mathcal{E}) \cap \mathcal{A}(\mathcal{E}_2)$ and $\overline{F} \sim \overline{G}$, then $F \ast G \in \mathcal{A}(\mathcal{E})$.

Suppose $F \in \mathcal{A}(\mathcal{E}) \cap \mathcal{A}(\mathcal{E}_2)$, $\overline{F} \sim \overline{G}$ and $F \ast G \notin \mathcal{A}(\mathcal{E})$. Then, there exists an $h \in \mathcal{G}$ such that

$$\limsup_{x \to \infty} \frac{P((X_1 + Y_1) \land (X_2 + Y_2) > h(x), \hat{S}_2 + S_2 > x)}{P(S_2 + S_2 > x)} > 0. \quad (4.14)$$

From $F \in \mathcal{A}(\mathcal{E}) \cap \mathcal{A}(\mathcal{E}_2)$ and Proposition 43 we have $F^{2*} \in \mathcal{A}(\mathcal{E})$ and it follows by definition that

$$\limsup_{x \to \infty} \frac{P((X_1 + X_2) \land (X_3 + X_4) > h(x), S_4 > x)}{P(S_4 > x)} = 0. \quad (4.15)$$

Combining (4.14) and (4.15) yields

$$\limsup_{x \to \infty} \frac{P((X_1 + Y_1) \land (X_2 + Y_2) > h(x), \hat{S}_2 + S_2 > x)}{P((X_1 + X_2) \land (X_3 + X_4) > h(x), S_4 > x)} \frac{P(S_4 > x)}{P(S_2 + S_2 > x)} = \infty. \quad (4.16)$$
4 New large claim classes

By Lemma 128 we obtain \( (F^{2*}) \ast (G^{2*}) \preceq F^{2*} \), i.e. \( \limsup_{x \to \infty} \frac{\mathbb{P}(S_1 > x)}{\mathbb{P}(S_2 + S_2 > x)} < \infty \). Then, by Lemma 131 we obtain

\[
\limsup_{x \to \infty} \frac{\mathbb{P}(X_1 + Y_1) \land (X_2 + Y_2) > h(x), \hat{S}_2 + S_2 > x)}{\mathbb{P}((X_1 + X_2) \land (X_3 + X_4) > h(x), S_1 > x)} \leq \left( \limsup_{x \to \infty} \frac{F \ast G(x)}{F \ast F(x)} \right)^2 < \infty.
\]

Thus, by (4.16) we get a contradiction. \( \square \)

Next, we prove the root convolution properties of \( A(\mathcal{E}) \), Proposition 47.

**Proof.** First we prove part b).

We show closure of \( A(\mathcal{E}) \) under root convolution in the class \( OS \), i.e. if \( F^{n*} \in A(\mathcal{E}) \) and \( F^{2*} \in OS \), then \( F \in A(\mathcal{E}) \).

Suppose \( F^{n*} \in A(\mathcal{E}) \) and \( F^{2*} \in OS \). From Lemma 130 we obtain \( F^{2*} \preceq F^{n*} \). From the closure of the class \( A(\mathcal{E}) \) under weak tail-equivalence we obtain \( F^{2*} \in A(\mathcal{E}) \).

From \( F^{2*} \in OS \) we know that there exists a constant \( c \) such that

\[
\liminf_{x \to \infty} \frac{\mathbb{P}(S_2 > x)}{\mathbb{P}(S_4 > x)} > c > 0.
\]

We obtain by definition of \( A(\mathcal{E}) \) and \( F^{2*} \in A(\mathcal{E}) \) for all \( h \in \mathcal{E} \)

\[
0 = \limsup_{x \to \infty} \mathbb{P}\left( S_2^{(1)} \land S_2^{(2)} > h(x) \mid S_4 > x \right)
\geq c \limsup_{x \to \infty} \mathbb{P}(X_1 \land X_2 > h(x) \mid S_2 > x).
\]

Thus, we have \( F \in A(\mathcal{E}) \).

Finally, we prove part a). Suppose \( F^{n*} \in A(\mathcal{E}) \) and \( F \in OS \). From Lemma 130 we obtain \( F \preceq F^{n*} \). Thus, \( F \in A(\mathcal{E}) \) follows from the closure of the class \( A(\mathcal{E}) \) under weak tail-equivalence. \( \square \)

Finally, we prove the closure of \( A(\mathcal{E}) \) under the operations maximum, minimum and mixture, Proposition 50.

**Proof.** a) Abbreviate \( V_i := X_i \lor Y_i \) for \( i \in \{1, 2, 3, 4\} \) and denote by \( F_{X \lor Y} \) the distribution of \( X \lor Y \).

Suppose \( F, G \in A(\mathcal{E}), F \ast G \in OS \cap A(\mathcal{E}) \). We show that \( F_{X \lor Y} \in A(\mathcal{E}) \).

Then, for every \( g \in \mathcal{E} \) and \( x \geq 0 \),

\[
\frac{\mathbb{P}(V_1 \land V_2 > g(x), V_1 + V_2 > x)}{\mathbb{P}(V_1 + V_2 > x)} \leq \frac{\mathbb{P}(X_1 \land X_2 > g(x), X_1 + X_2 > x)}{\mathbb{P}(V_1 + V_2 > x)}
+ \frac{\mathbb{P}(Y_1 \land Y_2 > g(x), Y_1 + Y_2 > x)}{\mathbb{P}(V_1 + V_2 > x)}
+ 2 \frac{\mathbb{P}(X_1 \land Y_2 > g(x), X_1 + Y_2 > x)}{\mathbb{P}(V_1 + V_2 > x)}. \tag{4.18}
\]
4 New large claim classes

From $F \in A(\mathcal{E})$ we obtain for the first term on the right-hand side of (4.18):

$$\lim_{x \to \infty} \frac{\mathbb{P}(X_1 \land X_2 > g(x), S_2 > x)}{\mathbb{P}(V_1 + V_2 > x)} \leq \lim_{x \to \infty} \frac{\mathbb{P}(X_1 \land X_2 > g(x), S_2 > x)}{\mathbb{P}(S_2 > x)} = 0.$$ 

Analogously, for the second term:

$$\lim_{x \to \infty} \frac{\mathbb{P}(Y_1 \land Y_2 > g(x), Y_1 + Y_2 > x)}{\mathbb{P}(V_1 + V_2 > x)} = 0.$$ 

From $F \ast G \in OS \cap A(\mathcal{E})$ we obtain for the third term on the right-hand side of (4.18):

$$\lim_{x \to \infty} 2 \frac{\mathbb{P}(X_1 \land Y_2 > g(x), X_1 + Y_2 > x)}{\mathbb{P}(V_1 + V_2 > x)} \leq 2 \lim_{x \to \infty} \left( \frac{\mathbb{P}(X_1 \land Y_2 > g(x), S_2 + \hat{S}_2 > x)}{\mathbb{P}(S_2 + \hat{S}_2 > x)} \frac{\mathbb{P}(S_2 > x)}{\mathbb{P}(X_1 + Y_1 > x)} \right) = 0.$$ 

Combined, we arrive at

$$\lim_{x \to \infty} \mathbb{P}\left(\begin{array}{c} (X_1 \lor Y_1) \land (X_2 \lor Y_2) > g(x) \mid (X_1 \lor Y_1) + (X_2 \lor Y_2) > x \end{array}\right) = \lim_{x \to \infty} \frac{\mathbb{P}(V_1 \land V_2 > g(x), V_1 + V_2 > x)}{\mathbb{P}(V_1 + V_2 > x)} = 0$$

for all $g \in \mathcal{E}$, i.e. $F_{X \lor Y} \in A(\mathcal{E})$.

b) For the implication $F_{X \lor Y} \in OS \cap A(\mathcal{E}) \Rightarrow F \ast G \in A(\mathcal{E})$ abbreviate $W_i := X_i + Y_i$ for $i \in \{1, 2\}$. From $F_{X \lor Y} \in OS \cap A(\mathcal{E})$ and $F_{X \lor Y}^2 \in OS$ we obtain for all $g \in \mathcal{E}$

$$\lim_{x \to \infty} \frac{\mathbb{P}(W_1 \land W_2 > g(x), W_1 + W_2 > x)}{\mathbb{P}(W_1 + W_2 > x)} \leq \lim_{x \to \infty} \left( \frac{\mathbb{P}(W_1 + W_2 > g(x), W_1 + \cdots + W_4 > x) \mathbb{P}(V_1 + \cdots + V_4 > x)}{\mathbb{P}(V_1 + \cdots + V_4 > x) \mathbb{P}(V_1 + V_2 > x)} \right) = 0.$$ 

Hence, $F \ast G \in A(\mathcal{E})$.

c) We prove the last part of the proposition. The proof is analogous to the proof of the same assertion for the class $OS$, see [26, Lemma 3.1].

Suppose $F, G \in OS \cap A(\mathcal{E})$, then we prove the following assertion

$$F_{X \land Y} \in OS \cap A(\mathcal{E}).$$

Let $L_i := X_i \land Y_i$ with distribution function $F_L$ for $i \in \{1, 2\}$. The distribution functions of $X_i$ and $Y_i$ are denoted, respectively, by $F_{X_i}$ and $F_{Y_i}$ for $i \in \{1, 2\}$. For all
4 New large claim classes

g \in \mathcal{E} \text{ we have}
\begin{align*}
&\frac{\mathbb{P}(L_1 \land L_2 > g(x), L_1 + L_2 > x)}{\mathbb{P}(L_1 + L_2 > x)} \\
&\leq \frac{\int_{g(x)}^{\infty} \mathbb{P}(X_1 > (x - y) \lor g(x)) \mathbb{P}(Y_1 > (x - y) \lor g(x)) dF_{L_2}(y)}{\mathbb{P}(L_1 > x)} \\
&\leq \frac{\int_{g(x)}^{\infty} \mathbb{P}(X_1 > (x - y) \lor g(x)) \mathbb{P}(Y_1 > (x - y) \lor g(x)) \mathbb{P}(Y_2 \geq y) dF_{Y_2}(y)}{\mathbb{P}(X_1 > x) \mathbb{P}(Y_1 > x)} \\
&+ \frac{\int_{g(x)}^{\infty} \mathbb{P}(X_1 > (x - y) \lor g(x)) \mathbb{P}(Y_1 > (x - y) \lor g(x)) \mathbb{P}(X_2 \geq y) dF_{X_2}(y)}{\mathbb{P}(X_1 > x) \mathbb{P}(Y_1 > x)}.
\end{align*}

Using the inequality
\[ \mathbb{P}(Y_1 > (x - y) \lor g(x)) \mathbb{P}(Y_2 \geq y) \leq \mathbb{P}(Y_1 + Y_2 > x) \]
we obtain
\begin{align*}
\limsup_{x \to \infty} \frac{\mathbb{P}(L_1 \land L_2 > g(x), L_1 + L_2 > x)}{\mathbb{P}(L_1 + L_2 > x)} \\
&\leq \limsup_{x \to \infty} \frac{\int_{g(x)}^{\infty} \mathbb{P}(X_1 > (x - y) \lor g(x)) dF_{X_2}(y)}{\mathbb{P}(X_1 > x) \mathbb{P}(Y_1 > x)} \\
&+ \limsup_{x \to \infty} \frac{\mathbb{P}(X_1 + X_2 > x) \mathbb{P}(X_1 > x) \mathbb{P}(X_2 \geq y) dF_{X_2}(y)}{\mathbb{P}(X_1 > x) \mathbb{P}(Y_1 > x)} \\
&= \limsup_{x \to \infty} \frac{\mathbb{P}(S_2 > x) \mathbb{P}(S_2 > x, X_1 \land X_2 > g(x)) \mathbb{P}(S_2 > x)}{\mathbb{P}(Y_1 > x) \mathbb{P}(S_2 > x) \mathbb{P}(X_1 > x)} \\
&+ \limsup_{x \to \infty} \frac{\mathbb{P}(S_2 > x) \mathbb{P}(S_2 > x, Y_1 \land Y_2 > g(x)) \mathbb{P}(S_2 > x)}{\mathbb{P}(X_1 > x) \mathbb{P}(Y_1 > x) \mathbb{P}(S_2 > x)} \\
&= 0,
\end{align*}
since \( F, G \in \mathcal{C} \). Thus, \( F_{X \lor Y} \in \mathcal{A}(\mathcal{E}) \) and \( F_{X \lor Y} \in \mathcal{C} \) follows from [26, Lemma 3.1].

The proof is complete. \qed
4 New large claim classes

4.2 The large claim class $A(\mathcal{E})$ on the entire real line $\mathbb{R}$

This section is devoted to the extension of the definition of $A(\mathcal{E})$ to distributions on the entire real line $\mathbb{R}$. Hereafter, the distribution $F$ in this section is assumed to belong to a random variable on the entire real line $\mathbb{R}$.

The straight extension of the definition of a distribution class is not always reasonable. For example, if we consider distributions on the entire real line $\mathbb{R}$, then it turns out, that the condition
\[
\lim_{x \to \infty} \frac{F_{2\nu}(x)}{F(x)} = 2
\]
(4.19)
does not imply that the distribution is subexponential on $\mathbb{R}^+$. An example is given by Foss et al. in [18, Example 3.3], the authors construct a light-tailed distribution on the entire real line $\mathbb{R}$ such that the condition (4.19) is satisfied.

Before we extend the definition of $A(\mathcal{E})$ we state the results and the definition of the class of subexponential distributions on the entire real line $\mathbb{R}$. Denote by $F_+$ the distribution of the random variable $X^+ := \mathbb{1}_{\{X \geq 0\}}X$. Recall that we denote the set of all distributions of random variables on $(-\infty, \infty)$ with unbounded support by $\mathcal{F}_\mathbb{R}$.

**Definition 52.** (Distribution class $S_\mathbb{R}$)

Let $F \in \mathcal{F}_\mathbb{R}$. We say that a distribution is subexponential on the entire real line $\mathbb{R}$, if $F$ is long-tailed and
\[
\lim_{x \to \infty} \frac{F_{2\nu}(x)}{F(x)} = 2.
\]

We denote the class of subexponential distributions on the entire real line $\mathbb{R}$ by $S_\mathbb{R}$.

In contrast to the definition of the class $S$, an additional condition is required, namely, the distribution has to be long-tailed, to define the class $S_\mathbb{R}$. The next proposition clarifies the necessity of the additional condition in the definition of the class $S_\mathbb{R}$. The formulation follows [18, Lemma 3.4, page 42].

**Proposition 53.** (Distribution class $S_\mathbb{R}$)

Let $F$ be a distribution on $\mathbb{R}$ and $X$ be a random variable with distribution $F$. Let $G(B) := \mathbb{P}(X \in B|X \geq 0)$, $B \in \mathcal{B}(\mathbb{R}_+)$, be the conditional distribution. Then the following assertions are equivalent:

a) $F \in S_\mathbb{R}$, i.e. $F$ is long-tailed and
\[
\lim_{x \to \infty} \frac{F_{2\nu}(x)}{F(x)} = 2.
\]

b) The distribution $F^+$ of $X^+$ is subexponential, i.e. $F^+ \in S$.

c) The conditional distribution $G$ is subexponential, i.e. $G \in S$.

Next, we extend our definition of $A(\mathcal{E})$ to all distributions on the entire real line $\mathbb{R}$. 

40
4 New large claim classes

**Definition 54.** (The distribution class $A_{\mathbb{R}}(E)$)

Let $E \subseteq G$ be not empty.

$$A_{\mathbb{R}}(E) := \left\{ F \in F_{\mathbb{R}} : \lim_{x \to \infty} \mathbb{P}\{X_{2,2} > g(x) \mid S_2 > x\} = 0 \text{ for all } g \in E \right\}.$$

One might wonder which properties are still holding in the class $A_{\mathbb{R}}(E)$. First, we state the closure of $A_{\mathbb{R}}(E)$ under tail-equivalence.

**Proposition 55.** (Closure of the class $A_{\mathbb{R}}(E)$ under weak tail-equivalence)

Let $E \subseteq G$ be not empty. If $F \in A_{\mathbb{R}}(E)$ and $F \approx G$, then $G \in A_{\mathbb{R}}(E)$.

**Proof.** We used Lemmata 128 and 131 to prove the closure of $A(E), E \subseteq G$, under weak tail-equivalence, see Proposition 41. Note that both lemmata hold for distributions on the entire real line $\mathbb{R}$. Thus, we can use the same proof to show the closure of $A_{\mathbb{R}}(E)$. Denote by $X_1$ and $X_2$ i.i.d. random variables with common distribution $F \in F_{\mathbb{R}}$, and by $Y_1$ and $Y_2$ i.i.d. random variables with common distribution $G \in F_{\mathbb{R}}$. Further, let $S_2 := \sum_{k=1}^{2} X_k$, and $\hat{S}_2 := \sum_{k=1}^{2} Y_k$.

Suppose $F \in A_{\mathbb{R}}(E), F \approx G$ and $G \notin A_{\mathbb{R}}(E)$. There exists $h \in E$ such that

$$\limsup_{x \to \infty} \mathbb{P}(Y_{2,2} > h(x) \mid \hat{S}_2 > x) > 0. \quad (4.20)$$

By using the definition of $A_{\mathbb{R}}(E)$, we obtain for all $h \in E$:

$$\limsup_{x \to \infty} \frac{\mathbb{P}(Y_{2,2} > h(x) \mid \hat{S}_2 > x)}{\mathbb{P}(X_{2,2} > h(x) \mid S_2 > x)} = \infty. \quad (4.21)$$

From $F \approx G$ and Lemma 128 we obtain $F^2 \approx G^2$, i.e. $\limsup_{x \to \infty} \frac{\mathbb{P}(S_2 > x)}{\mathbb{P}(\hat{S}_2 > x)} < \infty$. Hence, by Lemma 131 we know that

$$\limsup_{x \to \infty} \frac{\mathbb{P}(Y_{2,2} > h(x) \mid \hat{S}_2 > x)}{\mathbb{P}(X_{2,2} > h(x) \mid S_2 > x)} \leq \left( \limsup_{x \to \infty} \frac{\sqrt{\mathbb{P}}(x)}{\mathbb{P}(x)} \right)^2 < \infty.$$

Thus, we get a contradiction to (4.21). We obtain $G \in A_{\mathbb{R}}(E)$. \qed

**Proposition 56.** Let $E \subseteq G$ be not empty and be $F^+$ the distribution of $X^+$. Moreover, let $G(B) := \mathbb{P}(X \in B \mid X \geq 0), B \in B(\mathbb{R}_+)$, be the conditional distribution. The following assertions are equivalent:

a) $F \in A_{\mathbb{R}}(E)$.

b) $F^+ \in A(E)$.

c) $G \in A(E)$.

**Proof.** The assertion follows from the closure of $A_{\mathbb{R}}(E)$ under weak tail-equivalence and $F^+(x) = F(x)$ for all $x > 0$. \qed
4 New large claim classes

Remark 57. By using the closure property of \( \mathcal{A}_R(\mathcal{E}) \) under weak tail-equivalence and the proposition above, we can transfer most properties from the class \( \mathcal{A}(\mathcal{E}) \) to the class \( \mathcal{A}_R(\mathcal{E}) \), e.g. closure under root convolution in \( \mathcal{O}_S \) and closure under convolution powers in \( \mathcal{O}_S \).
4 New large claim classes

4.3 The large claim class \( \mathcal{A} \)

In this section, we consider the class \( \mathcal{A}(\mathcal{H}) \), where \( \mathcal{H} \) is the set of all functions \( g(x) = \varepsilon x, \varepsilon > 0 \), i.e.
\[
\mathcal{H} := \{ g \in \mathcal{G} : g(x) = \varepsilon x, \varepsilon > 0 \}.
\]

We obtain a refinement of our statements for structure, classification and properties in comparison to the general class \( \mathcal{A}(\mathcal{E}) \), where the set of functions \( \mathcal{E} \subseteq \mathcal{G} \) is fixed but not further specified.

In comparison to the general class \( \mathcal{A}(\mathcal{E})^{(n)} \), \( \mathcal{E} \subseteq \mathcal{G} \), see Proposition 38, we can find more ways to describe the distribution class \( \mathcal{A}(\mathcal{H})^{(n)} \). We establish relations between the class \( \mathcal{A}(\mathcal{H})^{(n)} \) and other distribution classes. Finally, we provide the closure of the class \( \mathcal{A}(\mathcal{H})^{(n)} \) under convolution powers and weakly tail-equivalent distributions without any further requirements.

From the definition of the class \( \mathcal{A}(\mathcal{E})^{(n)} \) it is obvious that by extending the set of functions \( \mathcal{E} \) we obtain a smaller distribution class, i.e. if \( \mathcal{E}_1 \subseteq \mathcal{E}_2 \), then \( \mathcal{A}(\mathcal{E}_2)^{(n)} \subseteq \mathcal{A}(\mathcal{E}_1)^{(n)} \). In this section, we see that the reduction of the distribution class can provide accurate statements.

4.3.1 Structure

Recall the definition (see Definition 35) of the distribution class \( \mathcal{A}(\mathcal{H})^{(n)} \):
\[
\mathcal{A}(\mathcal{H})^{(n)} := \left\{ F \in \mathcal{F} : \lim_{x \to \infty} \mathbb{P}(X_{2,n} \leq g(x) | S_n > x) = 1 \forall g \in \mathcal{H} \right\}.
\]

The next proposition states several equivalent formulations of the class \( \mathcal{A}(\mathcal{H})^{(n)} \), which provide us with a better overall understanding.

**Proposition 58. (Structure of \( \mathcal{A}(\mathcal{H})^{(n)} \))**

We write
\[
\begin{align*}
\mathcal{A}_1(\mathcal{H})^{(n)} &:= \left\{ F \in \mathcal{F} : \lim_{K \to \infty} \inf_{x \geq 0} \mathbb{P}(X_{2,n} \leq g(x) \lor K | S_n > x) = 1 \forall g \in \mathcal{H} \right\}, \\
\mathcal{A}_2(\mathcal{H})^{(n)} &:= \left\{ F \in \mathcal{F} : \lim_{x \to \infty} \mathbb{P}(X_{2,n} \leq \varepsilon x | S_n > x) = 1 \forall \varepsilon > 0 \right\}, \\
\mathcal{A}_3(\mathcal{H})^{(n)} &:= \left\{ F \in \mathcal{F} : \lim_{x \to \infty} \mathbb{P}(X_{2,n} \leq \varepsilon S_n | S_n > x) = 1 \forall \varepsilon > 0 \right\}, \\
\mathcal{A}_4(\mathcal{H})^{(n)} &:= \left\{ F \in \mathcal{F} : \lim_{x \to \infty} \mathbb{P}(X_{1,n} > (1 - \varepsilon) S_n | S_n > x) = 1 \forall \varepsilon > 0 \right\}, \\
\mathcal{A}_5(\mathcal{H})^{(n)} &:= \left\{ F \in \mathcal{F} : \lim_{x \to \infty} \mathbb{P}(X_{1,n} > (1 - \varepsilon)x | S_n > x) = 1 \forall \varepsilon > 0 \right\}.
\end{align*}
\]

Then
\[
\mathcal{A}(\mathcal{H})^{(n)} = \mathcal{A}_1(\mathcal{H})^{(n)} = \mathcal{A}_2(\mathcal{H})^{(n)} = \mathcal{A}_3(\mathcal{H})^{(n)} = \mathcal{A}_4(\mathcal{H})^{(n)} = \mathcal{A}_5(\mathcal{H})^{(n)}.
\]
4 New large claim classes

The proposition above was first proven in a master’s thesis [32, 2007], advised by Prof. Dr. Michael Scheutzow.

Further, it is possible to transfer all structural properties from the general class $A(\mathcal{E})^{(n)}$, $\mathcal{E} \subseteq \mathcal{G}$, to the class $A(\mathcal{H})^{(n)}$. Hence, we obtain from Propositions 38 and 40 the following statement.

**Proposition 59.** a) For all $n \geq 2$:

$$A(\mathcal{H})^{(n)} \subset A(\mathcal{H})^{(n+1)}.$$  

b) If $F^{2^k} \in \mathcal{OS}$, then

$$F \in A(\mathcal{H})^{(n+1)} \Leftrightarrow F \in A(\mathcal{H})^{(n)}.$$  

c) If $F^{2^k} \in \mathcal{OS}$, then for all $n \geq 2$:

$$F \in A(\mathcal{H})^{(n+1)} \Leftrightarrow F \in A(\mathcal{H})^{(n)}.$$  

From here, we will consider only the class $A(\mathcal{H})^{(2)}$ and use the following abbreviation

$$A := A(\mathcal{H})^{(2)}.$$  

4.3.2 Classification, examples and properties of $A$

In this section, we provide the relations between the class $A$ and other distribution classes, including some examples to illustrate these relations. Finally, we state the closure of the class $A$ under convolution powers and weakly tail-equivalent distributions.

‘Peter and Paul’ distributions, which are concentrated on the set $\{2^n, n \in \mathbb{N}\}$, often serve as examples. The next proposition states a condition under which a ‘Peter and Paul’ distribution belongs to the class $A$ and the reverse statement holds.

**Proposition 60.** (A and ‘Peter and Paul’ distributions)

Let $X_1$ and $X_2$ be two i.i.d random variables with distribution $F \in \mathcal{F}$ and $\mathbb{P}(X_i = 2^j) = p_j$, $i = 1, 2$, $j \in \mathbb{N}$ and $\sum_{j=1}^{\infty} p_j = 1$. Then the following assertion holds:

$$\lim_{n \to \infty} \frac{p_{n+1}}{p_n p_{n-q}} = \infty, \forall q \in \mathbb{N} \Leftrightarrow F \in A.$$  

The direction $\lim_{n \to \infty} \frac{p_{n+1}}{p_n p_{n-q}} = \infty, \forall q \in \mathbb{N} \Rightarrow F \in A$ was first proven in a master’s thesis, [32], and the proof of the reverse direction was given in a further master’s thesis, [23], both master’s theses were advised by Prof. Dr. Michael Scheutzow.

Now, we consider the relations between $A$ and the classes $\mathcal{OS}$ and $\mathcal{OL}$. Neither $\mathcal{OS}$ nor $\mathcal{OL}$ can cover $A$ completely, there still exist distributions, which belong to $A$, but not to $\mathcal{OL}$. The reverse inclusion also does not hold, i.e. some distributions of the class $\mathcal{OS}$ do not belong to $A$. The next proposition summarizes these relations.
Proposition 61. (The relations between \( A \) and the classes \( OS, OL \))

The following statements hold:

a) \( A \not\subseteq OL \).

b) \( OS \not\subseteq A \).

Example 62. Distributions \( F \in A \setminus OL, F \in OS \setminus A \)

An example for a distribution \( F \in OS \setminus A \) was already given in Example 51. Now, we construct a 'Peter and Paul' distribution \( F \in A \setminus OL \). Let the tail of \( F \) be such that,

\[
F(x) = \frac{c}{k^k}, \quad x \in [2^k, 2^{k+1}), \quad k \in \mathbb{N} \setminus \{0\},
\]

where \( c := \left( \sum_{k=1}^{\infty} \frac{1}{k^k} \right)^{-1} \) is a normalizing constant. It is easy to see that \( F \notin OL \) and from Proposition 60 we obtain that the corresponding distribution belongs to the class \( A \).

Next, we want to demonstrate that for a certain set \( E \subset G \) the set of distributions \( A(E) \setminus A(G) \) is not empty.

Example 63. Distribution \( F \in A(E) \setminus A \).

We define the sequence \( h_n, n \geq 0 \), recursively by:

\[
h_0 := 1, \quad h_1 := 1, \quad h_{n+1} := \frac{h_n^3}{2}, \quad n \geq 0.
\]

We further define the sequence \( (a_n), n \geq 1 \), such that \( a_0 = 0, a_1 \geq 1, a_{n+1} = \frac{5}{3}a_n, \) \( n \geq 1 \). Then define the tail as follows:

\[
F(x) = h_n,
\]

for \( x \in [a_n, a_{n+1}), n \in \mathbb{N} \).

Next, we prove that \( F \notin A \). It is easy to see that

\[
\mathbb{P}(X_1, X_2 \in [a_n, a_{n+1}]) = (h_n - h_{n+2})^2 \sim h_n^2,
\]

\[
\mathbb{P}(X_{1,2} \geq a_{n+1}) \sim 2h_{n+1} = h_n^\frac{3}{2}.
\]

Therefore,

\[
\mathbb{P}(X_{1,2} > a_{n+1} | S_2 \geq 2a_n) \leq \frac{\mathbb{P}(X_{1,2} \geq a_{n+1})}{\mathbb{P}(X_1, X_2 \in [a_n, a_{n+1}])} \sim \frac{h_n^\frac{3}{2}}{h_n^2} \to 0,
\]

as \( n \to \infty \), and hence \( F \notin A \).

Let \( g(x) = \frac{7}{10}x \) and \( E \) such that for all \( h \in E \) we have \( g(x) \leq h(x), x \geq 0 \). Let \( (x_n)_{n \in \mathbb{N}} \) be an arbitrary sequence such that \( x_n \in [a_n, a_{n+1}] \). We see that

\[
\mathbb{P}(X_{2,2} > \frac{7}{10}x_n | S_2 > x_n) \leq \frac{\mathbb{P}(X_{2,2} > \frac{7}{10}x_n)}{\mathbb{P}(X_1 > x_n)} \leq \frac{\mathbb{P}(X_{2,2} \geq a_n)}{\mathbb{P}(X_1 > a_n)} \sim \frac{h_n^2}{h_{n+1}^2} \to 0
\]

as \( n \to \infty \), hence \( F \in A(E) \).
4 New large claim classes

We already know that the classes $OS$ and $OL$ cut across $A$, but are not able to completely cover the class $A$. A similar situation occurs with the class of heavy-tailed distributions $K$. The class $A$ contains some light-tailed distributions and some heavy-tailed distributions do not belong to the class $A$.

**Proposition 64. (The relations between $A$ and heavy-tailed classes)**

The following statements hold:

a) $A \nsubseteq K$.

b) $S \subset A$.

c) $L \nsubseteq A$.

**Remark 65.** Note that the inclusion $S \subset A$ will be proven in Section 4.4 by providing a proof of the stronger inclusion $S \subset A(G) \subset A$, see Proposition 76. Furthermore, we will provide an example of a light-tailed distribution, which belongs to the class $A(G) \subset A$, in Section 4.4, see Example 81 on page 57.

An example for a distribution $F \in L \setminus A$ was already given in Example 51. We introduce another way to construct distributions in the class $L \setminus A$. The following example is inspired by an example of a distribution $F \in L \setminus S$ in [10, Section 3] and was already given in a master's thesis, [32, 2007].

**Example 66. Distribution $F \in L \setminus A$.**

We define the sequence $h_n, n \geq 0$, recursively by:

$$h_1 := 1, \quad h_{n+1} := \frac{h_n^2}{4}, \quad n \geq 0.$$  

Clearly, $\lim_{n \to \infty} h_n = 0$. Further, the sequences $(a_n), (b_n)$ and $(c_n), n \geq 1,$ are defined such that

(i) $a_1 \geq 0$,

(ii) $c_n(b_n - a_n) = -\log \frac{h_n^2}{4}, \quad n \geq 1$,

(iii) $\lim_{n \to \infty} c_n = 0$,

(iv) $a_n < b_n \leq \frac{2}{3} a_n \leq a_{n+1}, \quad n \geq 1$.

It is easy to see that such a sequence exists by first fixing $c_n$, the difference $b_n - a_n$ such that (ii) and (iii) hold, then choosing $a_n$ and $b_n$ such that (i) and (iv) hold. Now define the tail of $F$ as follows:

$$F(x) = \begin{cases} 1 & \text{for } x \in [0, a_1], \\ h_n & \text{for } x \in [b_{n-1}, a_n], \\ h_n \exp\{-c_n(x - a_n)\} & \text{for } x \in [a_n, b_n]. \end{cases}$$

Clearly, $F$ is the tail of a (continuous) distribution in $F$ and we see that the property (iii) implies that $F \in L$.

Next, we show that $F \not\in A$: Let $X_1, X_2$ be independent with distribution $F$. Then

$$\mathbb{P}\{X_1, X_2 \in [a_n, b_n]\} = (h_n - h_{n+1})^2 \sim h_n^2,$$

$$\mathbb{P}\{X_{1,2} > b_n\} \sim 2h_{n+1} = \frac{h_n^2}{2}.$$
4 New large claim classes

Therefore,

\[ \Pr \{ X_{1,2} > b_n | S_n > 2a_n \} \leq \frac{\Pr \{ X_{1,2} > b_n \}}{\Pr \{ S_2 > 2a_n \}} \]
\[ \sim \frac{h_n^2}{2 \Pr \{ S_2 > 2a_n \}} \]
\[ \sim \frac{\Pr \{ X_1, X_2 \in [a_n, b_n] \}}{2 \Pr \{ S_2 > 2a_n \}} \]
\[ \leq \frac{1}{2}, \]

i.e. \( F \notin \mathcal{A} \).

Figure (4.1) summarizes the relations between \( \mathcal{A} \) and other distribution classes.

![Figure 4.1: Classification of heavy-tailed classes](image)

Remark 67. Note that we don’t give an example for a distribution in the intersection \( (\mathcal{A} \cap \mathcal{OL}) \setminus \mathcal{OS} \). This intersection is probably not empty, however, this question remains open.
Finally, we state the closure of \( \mathcal{A} \) under convolution power and distributions with weakly tail-equivalent tails. Recall the notation from Proposition 43

\[ \mathcal{E}_n := \left\{ \frac{1}{n} g : g \in \mathcal{E} \right\}. \]

If \( \mathcal{E} = \mathcal{H} \) then it is obvious that \( \mathcal{E}_n = \mathcal{H} \) holds for all \( n \in \mathbb{N} \). Thus, we obtain the closure properties of \( \mathcal{A} \) under convolution power and distributions with weakly tail-equivalent tails from Propositions 43 and 46.

**Proposition 68.** (Closure of \( \mathcal{A} \) under convolution power)

a) If \( F \in \mathcal{A} \), then \( F^n \in \mathcal{A} \).

b) If \( F \in \mathcal{A} \) and \( \overline{G} \sim \overline{F} \), then \( F \ast G \in \mathcal{A} \).

**Remark 69.** Finally, we summarize some open questions on the class \( \mathcal{A} \).

- Does the relation \( \mathcal{A}(\mathcal{H})^{(n)} = \mathcal{A}(\mathcal{H})^{(n+1)} \) hold for all \( n \geq 2 \)?

- Some relations between \( \mathcal{A} \) and classes \( \mathcal{L} \), \( \mathcal{OL} \) and \( \mathcal{OS} \) are not established. Examples of distributions in intersections \( (\mathcal{A} \cap \mathcal{OL}) \backslash \mathcal{OS} \), \( (\mathcal{A} \cap \mathcal{L}) \backslash \mathcal{OS} \) are not yet provided.

- Closure of the class \( \mathcal{A} \) under convolution and root convolution without further requirements.
4 New large claim classes

4.3.3 Proofs

First, we prove the structural properties of the class $\mathcal{A}(\mathcal{H})^{(n)}$, Proposition 58.

Proof. a) From part a) of Proposition 38 with $\mathcal{E} = \mathcal{H}$ we obtain $\mathcal{A}(\mathcal{H})^{(n)} = \mathcal{A}_1(\mathcal{H})^{(n)}$. $\mathcal{A}(\mathcal{H})^{(n)} = \mathcal{A}_2(\mathcal{H})^{(n)}$ is obvious.

The inclusions $\mathcal{A}_2(\mathcal{H})^{(n)} \subseteq \mathcal{A}_3(\mathcal{H})^{(n)} \subseteq \mathcal{A}_4(\mathcal{H})^{(n)} \subseteq \mathcal{A}_5(\mathcal{H})^{(n)}$ follows immediately from

$$\{X_{2,n} < \frac{\varepsilon x}{n-1}, S_n > x\} \subseteq \{X_{2,n} < \frac{\varepsilon S_n}{n-1}, S_n > x\} \subseteq \{X_{1,n} > (1-\varepsilon)S_n, S_n > x\} \subseteq \{X_{1,n} > (1-\varepsilon)x, S_n > x\}.$$

Let $F \in \mathcal{A}_5(\mathcal{H})^{(n)}$ and assume, by way of contradiction, that $F \notin \mathcal{A}_2(\mathcal{H})^{(n)}$. Then there exist $\varepsilon \in (0, 1)$, $\delta > 0$ and a sequence $x_i \to \infty$ such that

$$\mathbb{P}(X_{2,n} > \varepsilon x_i | S_n > x_i) \geq \delta.$$

Choose $\gamma \in (0, \varepsilon)$ such that $(1-\gamma)(1+\varepsilon - \gamma) \geq 1$ and $\bar{x} > 0$ such that

$$\mathbb{P}(X_{1,n} > (1-\gamma)x | S_n > x) \geq 1 - \frac{\delta}{3} \text{ for all } x \geq \bar{x}.$$

Defining

$$Q_i(\cdot) := \mathbb{P}(\cdot | S_n > x_i),$$

we get

$$Q_i(X_{1,n} > (1-\gamma)x_i, X_{2,n} > \varepsilon x_i) \geq Q_i(X_{1,n} > (1-\gamma)x_i) + Q_i(X_{2,n} > \varepsilon x_i) - 1 \geq \delta - \frac{\delta}{3} = \frac{2}{3} \delta,$$

for $x_i \geq \bar{x}$.

Further, for $x_i \geq \bar{x}$, we have

$$Q_i(X_{1,n} > x_i, X_{2,n} > \varepsilon x_i) \geq Q_i(X_{1,n} > (1-\gamma)(1+\varepsilon - \gamma)x_i, X_{2,n} > \varepsilon x_i) \geq Q_i(S_n > (1+\varepsilon - \gamma)x_i, X_{2,n} > \varepsilon x_i)$$

$$- Q_i(S_n > (1+\varepsilon - \gamma)x_i, X_{1,n} \leq (1-\gamma)(1+\varepsilon - \gamma)x_i) \geq Q_i(X_{1,n} > (1-\gamma)x_i, X_{2,n} > \varepsilon x_i)$$

$$- \mathbb{P}(X_{1,n} \leq (1-\gamma)(1+\varepsilon - \gamma)x_i | S_n > (1+\varepsilon - \gamma)x_i) \geq \frac{2}{3} \delta - \frac{\delta}{3} = \frac{\delta}{3},$$

where the second $\geq$ follows from the general inequality

$Q_i(A \cap B) \leq Q_i(A \cap C) + Q_i(B \cap C^c)$. This clearly contradicts Lemma 132. Therefore, $F \in \mathcal{A}_2(\mathcal{H})^{(n)}$ and $\mathcal{A}_5(\mathcal{H})^{(n)} \subset \mathcal{A}_2(\mathcal{H})^{(n)}$. The proof is complete. \hfill \Box
Finally, we prove Proposition 60.

**Proof.** First, we show the direction: \( \lim_{n \to \infty} \frac{\sum_{i=n+1}^{\infty} p_i}{p_n p_{n-q}} = \infty \) \( \forall q \in \mathbb{N} \Rightarrow F \in A. \)

Let \( \varepsilon > 0 \) and \( (x_k)_{k \in \mathbb{N}} \) be a fixed, but arbitrary sequence such that \( x_k \in [2^k, 2^{k+1}), \) \( k \geq 0. \)

The proof splits in two cases.

1. Case: \( x_k \in [2^k, (1+\varepsilon)2^k), k \geq 0. \)

   From \( S_2 > x_k \) we have \( X_{1,2} > x_k/2 \geq 2^k/2 = 2^{k-1}. \)

   Thus, \( X_{1,2} \geq 2^k > x_k - \varepsilon 2^k \geq (1- \varepsilon)x_k. \)

   Denote \( H_x := \bigcup_{k \in \mathbb{N}} [2^k, (1+ \varepsilon)2^k). \) We obtain for all \( \varepsilon > 0: \)

   \[ \lim_{x \to \infty, x \in H_x} P(X_{1,2} > (1- \varepsilon)x | S_2 > x) = 1. \] (4.23)

2. Case: \( x_k \in [(1+\varepsilon)2^k, 2^{k+1}), k \geq 0. \)

   Note that there exists an integer-value constant \( \tilde{q} < \infty \) such that for all \( k \in \mathbb{N} \) we have

   \[ ((1+\varepsilon)2^k, 2^{k+1}) \subseteq \bigcup_{r \leq \tilde{q}} [2^k + 2^{k-1-r}, 2^k + 2^{k-r}). \] (4.24)

   From (4.24) we can define a sequence of integer values \( (q_k)_{k \in \mathbb{N}} \) such that \( x_k \in [2^k + 2^{k-1-q_k}, 2^k + 2^{k-q_k}) \) and \( q_k \leq \tilde{q} \) for all \( k \in \mathbb{N}. \)

   The random variables \( X_1 \) and \( X_2 \) take value in the set \( \{2^j : j \in \mathbb{N}\}, \)

   thus \( S_2 \notin (2^k + 2^{k-1-q_k}, 2^k + 2^{k-q_k}) \) for all \( k \in \mathbb{N}. \) We obtain for all \( k \in \mathbb{N}: \)

   \[ \{S_2 > x\} = \{S_2 > 2^k + 2^{k-1-q_k}\} \]

   \[ = \{X_{1,2} > 2^{k+1}\} \cup \{X_{1,2} = 2^k, X_{2,2} > 2^{k-1-q_k}\}. \]

   Next, we prove that the probability of the event \( \{X_{1,2} = 2^k, X_{2,2} > 2^{k-1-q_k}\} \) is decreasing asymptotically faster than the probability of the event \( \{X_{1,2} \geq 2^{k+1}\} \) as \( k \to \infty. \) This fact will allow us to conclude the assertion. We obtain for both probabilities

   \[ P(X_{1,2} \geq 2^{k+1}) \sim 2 \sum_{i=k+1}^{\infty} p_i, \] (4.25)

   as \( k \to \infty, \) and

   \[ P(X_{1,2} = 2^k, X_{2,2} > 2^{k-1-q_k}) = P(X_1 = 2^k)P(2^k > X_2 > 2^{k-1-q_k}) \] (4.26)

   \[ + P(X_2 = 2^k)P(2^k > X_1 > 2^{k-1-q_k}) \] (4.27)

   \[ + P(X_1 = 2^k)P(X_2 = 2^k) \] (4.28)

   \[ = 2p_k \sum_{i=k-q_k}^{k-1} p_i + p_k^2. \] (4.29)
By combining (4.25) and (4.29), we obtain

\[
\frac{\mathbb{P}(X_{1,2} \geq 2^{k+1})}{\mathbb{P}(X_{1,2} = 2^k, X_{2,2} > 2^{k-1-qk})} \sim \frac{2^{\infty} \sum_{i=k+1}^\infty p_i}{2p_k \sum_{i=k-qk}^{k-1} p_i + p_k^2} \frac{2^{\infty} \sum_{i=k+1}^\infty p_i}{2p_k \sum_{i=k-qk}^{k-1} p_i - p_k^2} \geq \frac{2^{\infty} \sum_{i=k+1}^\infty p_i}{2p_k \sum_{i=k-qk}^{k-1} p_i},
\]

as \( k \to \infty \).

We show that the right side of (4.30) increases to infinity as \( k \to \infty \). Recall \( \bar{q} \) from (4.24) and denote for all \( k \in \mathbb{N} \)

\[
\bar{p}_{k-\bar{q}} := \max_{r=0,\ldots,\bar{q}} p_{k-r}.
\]

Then, we obtain from (4.30)

\[
\frac{\mathbb{P}(X_{1,2} \geq 2^{k+1})}{\mathbb{P}(X_{1,2} = 2^k, X_{2,2} > 2^{k-1-qk})} \geq \frac{\sum_{i=k+1}^\infty p_i}{p_k \sum_{i=k-qk}^{k} p_i} \frac{\sum_{i=k+1}^\infty p_i}{(\bar{q} + 1)p_k \bar{p}_{k-\bar{q}}} \to \infty,
\]

as \( k \to \infty \), where we used \( \lim_{n \to \infty} \frac{\sum_{i=1}^n p_i}{p_m \sum_{i=m-r}^{m-1} p_i} = \infty \) for all \( r \leq \bar{q} < \infty \) and \( q_k \leq \bar{q} \) for all \( k \in \mathbb{N} \).

Denote for \( k \in \mathbb{N} \):

\[
\alpha_k := \frac{\mathbb{P}(X_{1,2} \geq 2^{k+1} \mid S_2 > x_k)}{\mathbb{P}(X_{1,2} \geq 2^{k+1}, S_2 > x_k)} \frac{\mathbb{P}(X_{1,2} = 2^k, X_{2,2} > 2^{k-1-qk} \mid S_2 > x_k)}{\mathbb{P}(X_{1,2} = 2^k, S_2 > x_k)} \frac{\mathbb{P}(X_{1,2} \geq 2^{k+1})}{\mathbb{P}(X_{1,2} = 2^k, X_{2,2} > 2^{k-1-qk})}
\]

Then we obtain by (4.31)

\[
\alpha_k \to \infty,
\]
4 New large claim classes

as \( k \to \infty \).

We have

\[
1 = \mathbb{P}(S_2 > x_k | S_2 > x_k) \\
= \mathbb{P}(X_{1,2} \geq 2^{k+1} | S_2 > x_k) + \mathbb{P}(X_{1,2} = 2^k, X_{2,2} > 2^{k-1} - q_k | S_2 > x_k) \\
= \mathbb{P}(X_{1,2} \geq 2^{k+1} | S_2 > x_k) + \frac{\mathbb{P}(X_{1,2} \geq 2^{k+1} | S_2 > x_k)}{\alpha_k} \\
= (1 + \frac{1}{\alpha_k}) \mathbb{P}(X_{1,2} \geq 2^{k+1} | S_2 > x_k)
\]

and thus

\[
\lim_{k \to \infty} \mathbb{P}(X_{1,2} \geq 2^{k+1} | S_2 > x_k) = 1.
\]

If \( k \to \infty \), then \( x_k \to \infty \) holds as well. Denote \( H^\varepsilon = \bigcup_{k \in \mathbb{N}} [(1 + \varepsilon)2^k, 2^{k+1}) \). Since the choice of the sequence \( (x_k)_{k \in \mathbb{N}} \) was arbitrary, we obtain

\[
\lim_{x \to \infty, x \in H^\varepsilon} \mathbb{P}(X_{1,2} > (1 - \varepsilon)x | S_2 > x) = 1 \quad \forall \varepsilon > 0. \tag{4.32}
\]

By combining (4.23) and (4.32), we obtain the assertion:

\[
\lim_{x \to \infty} \mathbb{P}(X_{1,2} > (1 - \varepsilon)x | S_2 > x) = 1 \quad \forall \varepsilon > 0.
\]

From Proposition 58 and (4.22) we obtain the assertion:

\[
\lim_{n \to \infty} \sum_{i=n+1}^{\infty} \frac{p_i}{p_n p_{n-q}} = \infty \quad \forall q \in \mathbb{N} \Rightarrow F \in \mathcal{A}.
\]

Now, we prove the direction: \( \lim_{n \to \infty} \sum_{i=n+1}^{\infty} \frac{p_i}{p_n p_{n-q}} = \infty \quad \forall q \in \mathbb{N} \iff F \in \mathcal{A} \).

Suppose \( F \in \mathcal{A} \) and let \( q \in \mathbb{N} \) be arbitrary but fixed. From \( F \in \mathcal{A} \), Proposition 58 and (4.22) we obtain

\[
\lim_{n \to \infty} \mathbb{P}(X_{1,2} > (1 - \varepsilon)x_n | S_2 > x_n) = 1 \quad \forall \varepsilon > 0,
\]

for all sequences \( (x_n)_{n \in \mathbb{N}} \) with \( x_n \leq x_{n+1} \to \infty \), as \( n \to \infty \).

Select a sequence \( (x_n)_{n \in \mathbb{N}} \) such that \( x_n := 2^n + 2^n - 1 - q, n \in \mathbb{N} \)
and \( \varepsilon := \frac{2^n - 1 - q}{2^n + 2^n - 1 - q} = \frac{1}{1 + 2^n} \in (0, 1) \).

Then we obtain \( (1 - \varepsilon)x_n = x_n - 2^n - 1 - q = 2^n \) and thus

\[
\{X_{1,2} > (1 - \varepsilon)x_n \} = \{X_{1,2} > 2^n \} = \{X_{1,2} \geq 2^n + 1\}.
\]

It yields that

\[
\lim_{n \to \infty} \mathbb{P}(X_{1,2} \geq 2^{n+1} | S_2 > 2^n + 2^n - 1 - q) = 1.
\]

Further, we have

\[
\mathbb{P}(X_{1,2} \geq 2^{n+1}, S_2 > 2^{n} + 2^n - 1 - q) = \mathbb{P}(X_{1,2} \geq 2^{n+1}) \sim 2 \sum_{i=n+1}^{\infty} p_i \tag{4.33}
\]
4 New large claim classes

as \( n \to \infty \). Analogous to the lines (4.26)-(4.29), we obtain

\[
\mathbb{P}(X_{1,2} = 2^n, X_{2,2} > 2^{n-1-q}) \sim 2p_n \sum_{i=n-q}^{n-1} p_i + p_n^2.
\] (4.34)

By combining (4.33) and (4.34), we obtain

\[
\mathbb{P}(S_2 > 2^n + 2^{n-1-q}) = \mathbb{P}(X_{1,2} \geq 2^{n+1}) + \mathbb{P}(X_{1,2} = 2^n, X_{2,2} > 2^{n-1-q})
\]

\[
\sim 2 \sum_{i=n+1}^{\infty} p_i + 2p_n \sum_{i=n-q}^{n-1} p_i + p_n^2.
\]

Thus, we have

\[
\frac{\sum_{i=n+1}^{\infty} p_i}{\sum_{i=n+1}^{\infty} p_i + p_np_{n-q}} \geq \frac{2 \sum_{i=n+1}^{\infty} p_i}{2 \sum_{i=n+1}^{\infty} p_i + 2p_n \sum_{i=n-q}^{n-1} p_i + p_n^2}
\]

\[
\sim \mathbb{P}(X_{1,2} \geq 2^{n+1} \mid S_2 > 2^n + 2^{n-1-q})
\]

\[
\to 1,
\]

as \( n \to \infty \). We can deduce that \( p_np_{n-q} \) decreases asymptotically faster than \( \sum_{i=n+1}^{\infty} p_i \), i.e.

\[
\lim_{n \to \infty} \frac{\sum_{i=n+1}^{\infty} p_i}{p_np_{n-q}} = \infty.
\]

Since \( q \) was chosen arbitrarily, the assertion follows. \( \square \)
4 New large claim classes

4.4 The large claim class $\mathcal{J}$

In this section, we consider the most studied distribution class of the type $\mathcal{A}(\mathcal{E})$, $\mathcal{E} \subseteq \mathcal{G}$, which we obtain by extending the set $\mathcal{E}$ to the largest possible set of functions $\mathcal{G}$ (all nonnegative, unbounded and nondecreasing functions). We denote the distribution class by

$$\mathcal{J}^{(n)} := \mathcal{A}(\mathcal{G})^{(n)} = \left\{ F \in \mathcal{F} : \lim_{x \to \infty} \inf_{K \to \infty} \mathbb{P}(X_{2,n} > g(x)|S_n > x) = 0 \forall g \in \mathcal{G} \right\}. \quad (4.35)$$

On the one side, by choosing the largest set of functions, i.e. $\mathcal{E} = \mathcal{G}$, we obtain the smallest distribution class of the type $\mathcal{A}(\mathcal{E})$, $\mathcal{E} \subseteq \mathcal{G}$, on the other side, we can provide the most precise classification and refine the structural properties of the distribution class $\mathcal{J}^{(n)}$.

In comparison to the class $\mathcal{A}$, we improve most of our statements one more time and add some new properties. We will provide several equivalent formulations of (4.35) above and examine the structural properties. $\mathcal{J}^{(n)}$ is the smallest large claim class, which we study in this work, however, it still contains all subexponential and some light-tailed distributions. Although we will see that $\mathcal{J}^{(n)}$ is not closed under convolution, we will find below that we have closure under convolution powers and weakly tail-equivalent distributions. Further, we have closure under convolution roots, in contrast to $OS$ (see Example 28 on page 23) – this property is highly desirable as we will see in the next chapter.

4.4.1 Structure

Our study about the class $\mathcal{J}$ also begins with structural properties.

**Proposition 70. (Characterization of $\mathcal{J}^{(n)}$)**

Write for $n \geq 2$,

$$\mathcal{J}_1^{(n)} := \left\{ F \in \mathcal{F} : \lim_{K \to \infty} \lim_{x \to \infty} \inf \mathbb{P}(X_{2,n} \leq K|S_n > x) = 1 \right\},$$

$$\mathcal{J}_2^{(n)} := \left\{ F \in \mathcal{F} : \lim_{K \to \infty} \inf_{x \geq 0} \mathbb{P}(X_{2,n} \leq K|S_n > x) = 1 \right\},$$

$$\mathcal{J}_3^{(n)} := \left\{ F \in \mathcal{F} : \lim_{K \to \infty} \lim_{x \to \infty} \inf \mathbb{P}(X_{1,n} > x - K|S_n > x) = 1 \right\},$$

$$\mathcal{J}_4^{(n)} := \left\{ F \in \mathcal{F} : \lim_{K \to \infty} \lim_{x \to \infty} \inf \mathbb{P}(X_{1,n} > S_n - K|S_n > x) = 1 \right\}.$$

Then all these sets agree with the class $\mathcal{J}^{(n)}$:

$$\mathcal{J}_1^{(n)} = \mathcal{J}_2^{(n)} = \mathcal{J}_3^{(n)} = \mathcal{J}_4^{(n)} = \mathcal{J}^{(n)}.$$
4 New large claim classes

Remark 71. a) The proposition above was first proven in a master’s thesis [32, 2007], advised by Prof. Dr. Michael Scheutzow.

b) Note that the condition required for $J^{(n)}$ is precisely the tightness of the conditional laws of $X_{2,n}$, given $S_n > x$.

Recall the question which was stated in Remark 39: does the equality $A(E)^{(n)} = A(E)^{(n+1)}$ hold for all $E \subseteq G$? We can answer this question in the case of the class $J^{(n)}$. The class $J^{(n)}$ is independent of $n$, the number of random variables, which are considered in the definition. This property is stated in the next proposition.

**Proposition 72.** The following assertion holds for all $n \in \mathbb{N}$:

$$J^{(n)} = J^{(n+1)}.$$

Since all sets $J^{(n)}$, $n \geq 2$, agree with $J^{(2)}$ we will use the following abbreviation

$$J := J^{(2)}.$$

### 4.4.2 Classification

In Section 4.3, we give a condition under which a ‘Peter and Paul’ distribution belongs to the class $A$ and the reverse statement holds. Now, we can refine this result for the class $J$ and consider more general cases of discrete random variables.

**Proposition 73.** ($J$ and discrete random variables)

Let $X_1$ and $X_2$ be two i.i.d random variables with distribution $F \in \mathcal{F}$ and with $\mathbb{P}(X_i = x_n) = p_n$, $i = 1, 2$, $n \in \mathbb{N}$, where $\sum_{j=1}^{\infty} p_j = 1$ and $(x_n)_{n \in \mathbb{N}}$ is a sequence with $x_n \to \infty$, as $n \to \infty$, and $\frac{x_{n+1}}{x_n} \geq 1 + \varepsilon$ for some $\varepsilon > 0$ and all $n \in \mathbb{N}$. Then the following assertion holds:

$$\liminf_{n \to \infty} \frac{\sum_{k=n+1}^{\infty} p_k}{p_n} > 0 \iff F \in J.$$

The proof of the proposition above was first given in a master’s thesis [23, 2008], advised by Prof. Dr. Michael Scheutzow.

Next, we consider the relations between $J$ and other classes. The large claim class $J$ is the smallest class, which can be defined by our approach. Further, all distributions of the class $J$ are $O$–subexponential and belong to the class $A$. However, the reverse implications do not hold and an example for a distribution $F \in OS \setminus J$ is already given in Example 51 on page 32. We will also provide below an example for a distribution $F \in A \setminus J$.

**Proposition 74.** ($J$ and large claim distribution classes $A(E)$, $OS$)

a) $J \subseteq OS$;

b) $J \subseteq A(E), E \subseteq G$. In particular, $J \subseteq A$. 

55
Example 75. Distribution $F \in \mathcal{A}\setminus \mathcal{J}$.

Consider the ‘Peter and Paul’ distribution from Example 11. Let the tail of $F$ be such that

$$F(x) = \frac{c}{k^k}, \quad x \in [2^k, 2^{k+1}), \quad k \in \mathbb{N},$$

where $c := \left(\sum_{k=1}^{\infty} \frac{1}{k^k}\right)^{-1}$ is a normalizing constant. We already know that $F \in \mathcal{A}$ and from Proposition 73 we obtain $F \not\in \mathcal{J}$.

Next, we turn to the relations between $\mathcal{J}$ and the most important heavy-tailed classes $\mathcal{S}$, $\mathcal{L}$ and $\mathcal{D}$. The distributions in the intersection of the classes $\mathcal{L}$ and $\mathcal{J}$ are subexponential and dominatedly varying distributions always belong to the class $\mathcal{J}$. However, the definition of $\mathcal{J}$ does not restrict the class to heavy-tailed distributions and some light-tailed distributions belong to $\mathcal{J} \subset \mathcal{A}$. These results, amongst other, are stated in the next proposition.

Proposition 76. ($\mathcal{J}$ and heavy-tailed classes)

a) $\mathcal{J} \cap \mathcal{L} = \mathcal{S}$;

b) $\mathcal{S} \subset \mathcal{J}$;

c) $\mathcal{D} \subset \mathcal{J}$;

d) $\mathcal{J} \not\subset \mathcal{K}$;

Remark 77. a) The proof of the parts a), b) and c) of proposition above was first given in a master’s thesis [32, 2008]. An example of a distribution $F \in \mathcal{J}\setminus \mathcal{K}$ was first presented in a master’s thesis [16, 2008], advised by Prof. Dr. Michael Scheutzow.

b) As already mentioned in Remark 20, the condition (3.4) of Proposition 19 under which a long-tailed distribution becomes subexponential can serve as the definition of the class $\mathcal{J}$. Note that this condition and the definition of the class $\mathcal{J}$, see (4.35), do not match exactly. However, by combining the statements $\mathcal{J} \cap \mathcal{L} = \mathcal{S}$ and $\mathcal{J} \subset \mathbb{OS}$ from the propositions above, we obtain the equivalence of both approaches as the definition of the class $\mathcal{J}$.

Proposition 78. $F \in \mathcal{J}$ if and only if

$$\lim_{x \to \infty} \frac{\mathbb{P}(S_2 > x, X_{2,2} > g(x))}{\mathbb{P}(X_1 > x)} = 0$$

for all $g \in \mathcal{G}$.

An example of a light-tailed distribution in $\mathcal{J}$, proofs of parts b) and d) of Proposition 76 follow immediately from the next proposition.

Proposition 79. ($\mathcal{J}$ and light-tailed classes)

Let $\gamma \geq 0$. Then the following relation holds

$$\mathcal{J} \cap \mathcal{L}(\gamma) = \mathcal{S}(\gamma).$$

Remark 80. The proposition above generalizes the result of part b) of Proposition 76 and was obtained due to a suggestion of Prof. Dr. Sergey Foss.
Example 81. Distribution $F \in S(\gamma)$.

Consider the distribution $F \in \mathcal{F}$ with density

$$f(x) = C e^{-x} g(x), \quad x \geq 0,$$

where $g(x)$ is a subexponential density and $C > 0$ such that $\int_0^\infty f(x) \, dx = 1$. For example, if $g(x) = \frac{1}{2} \frac{x}{1+x^2}$ then we know from Example 33 that $F \in S(\gamma)$ with an index $\gamma = 1$. From proposition above we know that $F \in \mathcal{J}$. However, we want to provide another way to see that the distribution $F$ belongs to the class $\mathcal{J}$.

Since $g(x)$ is a subexponential density, we know by definition that

$$\lim_{x \to \infty} \frac{g^{2x}(x)}{g(x)} = \lim_{x \to \infty} \frac{\int_0^x g(y) g(x-y) \, dy}{g(x)} = 2 \quad (4.36)$$

and

$$g(x+t) \sim g(x),$$

as $x \to \infty$, for any fixed $t > 0$, see [18, Section 4.2] for a detailed review of subexponential densities. Further, we know that $g(x-t) \sim g(x)$, as $x \to \infty$, uniformly over $t$ in compact intervals, see [18, (2.18), page 18] for more details. Let $h \in \mathcal{G}$.

Thus, we see that

$$\lim_{z \to \infty} \lim_{x \to \infty} \frac{\int_{h(z)}^x g(y) g(x-y) \, dy}{g(x)} = 2. \quad (4.37)$$

Now, we rewrite the term of (4.36)

$$\lim_{x \to \infty} \frac{\int_0^x g(y) g(x-y) \, dy}{g(x)} = \lim_{z \to \infty} \lim_{x \to \infty} \left(2 \frac{\int_0^{h(z)} g(y) g(x-y) \, dy}{g(x)} + \frac{\int_{h(z)}^{x-h(z)} g(y) g(x-y) \, dy}{g(x)} \right).$$

Thus, by combining (4.36) and (4.37), we obtain for all $h \in \mathcal{G}$:

$$\lim_{z \to \infty} \lim_{x \to \infty} \frac{\int_{h(z)}^{x-h(z)} g(y) g(x-y) \, dy}{g(x)} = 0.$$

Using the last fact and (4.36), we obtain for the density $f$ and for all $h \in \mathcal{G}$:

$$\lim_{z \to \infty} \lim_{x \to \infty} \frac{\int_{h(z)}^{x-h(z)} f(y) f(x-y) \, dy}{\int_0^x f(y) f(x-y) \, dy} = \lim_{z \to \infty} \lim_{x \to \infty} \frac{\int_{h(z)}^{x-h(z)} g(y) g(x-y) \, dy}{\int_0^x g(y) g(x-y) \, dy}$$

$$= \lim_{z \to \infty} \lim_{x \to \infty} \frac{\int_{h(z)}^{x-h(z)} g(y) g(x-y) \, dy g(x)}{\int_0^x g(y) g(x-y) \, dy g(x)} = 0.$$
Thus, it follows that for all \( h \in \mathcal{G} \)

\[
\limsup_{z \to \infty} \frac{\int_{h(z)}^{\infty} f(x-y) \, dy \, dx}{\int_{0}^{\infty} f(y) f(x-y) \, dy \, dx} = \limsup_{z \to \infty} \frac{\mathbb{P}(X_{2,2} > h(z), S_2 > z)}{\mathbb{P}(S_2 > z)} = 0
\]

and hence \( F \in \mathcal{J} \).

The next example provides a distribution in the class \( \mathcal{J} \), which is not subexponential, but in contrast to the previous example, belongs to the class of heavy-tailed distributions \( \mathcal{K} \).

**Example 82.** Distribution \( F \in \mathcal{J} \cap \mathcal{K} \setminus \mathcal{S} \).

We consider the 'Peter and Paul' distribution \( F \) from part c) of Example 11, which is dominatedly varying and not long-tailed, i.e. \( F \in \mathcal{D} \) and \( F \notin \mathcal{L} \). From Proposition 76 a) and c) we obtain \( F \in \mathcal{J} \cap \mathcal{K} \setminus \mathcal{S} \).

Finally, figure (4.2) summarizes the relations between \( \mathcal{J} \) and other classes.

**Remark 83.** Note that there is no class \( \mathcal{A} \) in Figure 4.2, because some interrelations of the classes \( \mathcal{A} \) and \( \mathcal{J} \), \( \mathcal{OS} \), \( \mathcal{OL} \) remain undecided:

- Does the class \( \mathcal{J} \) form a proper subclass of \( \mathcal{A} \cap \mathcal{OS} \)?
- \( (\mathcal{L} \cap \mathcal{A}) \setminus \mathcal{OS} \neq \emptyset \)?
- \( (\mathcal{OL} \cap \mathcal{A}) \setminus \mathcal{OS} \neq \emptyset \)?
4 New large claim classes

4.4.3 Further properties of \( J \)

In this section, we collect the properties of the class \( J \). Closure under tail-equivalence holds in all classes of the type \( A(\mathcal{E}) \), \( \mathcal{E} \subseteq \mathcal{G} \). Hence, it holds in the class \( J \) as well.

**Proposition 84.** (Closure of the class \( J \) under weak tail-equivalence)

If \( F \in J \) and \( G \sim F \), then \( G \in J \).

The inclusion \( J \subseteq OS \) (Proposition 74) allows us to transfer all convolution properties from the intersection \( A(\mathcal{E}) \cap OS \) to the class \( J \). Thus, we obtain from Propositions 45 and 46 the closure of the class \( J \) under convolution power and convolution of two distributions with equivalent tails. However, the general convolution is not transferable from the class \( OS \) to the class \( J \) and a counterexample will be given below.

**Proposition 85.** (Convolution properties of \( J \))

a) If \( F \in J \), then \( F \sim F^{n*} \) and hence \( F^{n*} \in J \).

b) If \( F \in J \) and \( G \sim F \), then \( F*G \in J \).

**Example 86.** \( J \) is not closed under convolution closure.

Since \( S \subseteq L \), in the counterexample from [25, Section 3] and Proposition 18 (convolution closure of \( L \)) we know that there exist two distributions \( F_1, F_2 \) such that \( F_1, F_2 \in S \) and \( F_1*F_2 \in L \) but \( F_1*F_2 \notin S \). Since we have \( J \cap L = S \) from part a) of Proposition 76, \( F_1, F_2 \in J \) but \( F_1*F_2 \notin J \).

The statement of Proposition 14 (closure of \( S \) under mixture, maximum and minimum) remains true when \( S \) is replaced by \( J \).

**Proposition 87.** (Closure of the class \( S \) under mixture, maximum and minimum)

Let \( X, Y \) be two independent random variables with distributions \( F, G \in \mathcal{F} \).

a) Let \( F_Z \) be the distribution of the mixture of \( X \) and \( Y \) with parameter \( p \in (0, 1) \). If \( F, G \in J \), then the following are equivalent:

\[
i) F_{X\lor Y} \in J; \quad ii) F*G \in J; \quad iii) F_Z \in J. \quad (4.38)
\]

b) If \( F, G \in J \), then \( F_{X\land Y} \in J \).

Again, the inclusion \( J \subseteq OS \) allow us to derive the closure of the class \( J \) under root convolution. This property is crucial in some applications and gains a leverage against the class \( OS \), which is not closed under root convolution.
4 New large claim classes

**Proposition 88.** (Closure of the class $J$ under root convolution)

If $F^{**} \in J$, then $F \asymp F^{**}$ and hence

$$F \in J.$$  

Finally, we want to compare the closure properties of the classes $S$, $OS$ and $J$ and collect the properties of these classes in the following table. There is obviously no class, which is perfect in the sense of satisfaction of all closure properties. However, the class $J$ provide us with a certain sense of robustness which allow us to demonstrate several applications in the next chapter.

<table>
<thead>
<tr>
<th>Closure property</th>
<th>Class $S$</th>
<th>Class $OS$</th>
<th>Class $J$</th>
<th>Class $A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convolution power</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Convolution</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
<td>?</td>
</tr>
<tr>
<td>(Strong) Tail-equivalence</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>(Weak) Tail-equivalence</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Root convolution</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>?</td>
</tr>
</tbody>
</table>

Table 4.1: Closure properties of the distribution classes $S$, $OS$ and $J$
4.4.4 Proofs

First, we prove the structural properties of the class $J$, Proposition 70.

**Proof.** We show $J^{(n)} = J_1^{(n)} = J_2^{(n)} = J_3^{(n)} = J_4^{(n)}$. The inclusions $J^{(n)} \supseteq J_1^{(n)} \supseteq J_2^{(n)}$ and $J_1^{(n)} \subseteq J_3^{(n)}$ are obvious. The inclusion $J_1^{(n)} \subseteq J_4^{(n)}$ follows immediately from

$$\{ X_{2,n} < \frac{K}{n-1}, S_n > x \} \subseteq \{ X_{1,n} > S_n - K, S_n > x \},$$

for all $F \in F, K > 0$ and $x > 0$. It remains to show $J^{(n)} \subseteq J_1^{(n)} \subseteq J_2^{(n)}$ and $J_1^{(n)} \supseteq J_3^{(n)}$.

First, we prove the inclusion $J^{(n)} \subseteq J_1^{(n)}$. Suppose $F \in J^{(n)}$ and $F \notin J_1^{(n)}$, then there is $\tau > 0$ such that for any $m \geq 1$:

$$\lim_{x \to \infty} \mathbb{P}(X_{2,n} \leq m | S_n > x) \leq 1 - \tau.$$

For every $m \geq 1$, we choose an unbounded and strictly increasing sequence $(x^m_k)_{k \in \mathbb{N}}$ with $\lim_{m \to \infty} x^m_k = \infty$ such that for all $k \in \mathbb{N}$:

$$\mathbb{P}(X_{2,n} \leq m | S_n > x^m_k) \leq 1 - \frac{\tau}{2}.$$

Hence, we obtain

$$\limsup_{m \to \infty} \mathbb{P}(X_{2,n} \leq m | S_n > x^m_k) \leq 1 - \frac{\tau}{2},$$

contradicting the fact that

$$\lim_{x \to \infty} \mathbb{P}(X_{2,n} > g(x) | S_n > x) = 0 \text{ for all } g \in G,$$

so $J^{(n)} \subseteq J_1^{(n)}$.

Next, we show the inclusion $J_1^{(n)} \subseteq J_2^{(n)}$. Suppose $F \in J_1^{(n)}$. By definition we know that for every $\varepsilon > 0$ there are constants $x_0$ and $K_0$ such that for all $x \geq x_0$ and $K \geq K_0$:

$$\mathbb{P}(X_{2,n} \leq K | S_n > x) \geq 1 - \varepsilon. \quad (4.39)$$

Let $\delta > 0$. Increasing $K_0$ if necessary, we can assume that $\mathbb{P}(X_{2,n} \leq K) \geq 1 - \delta$ for all $K \geq K_0$. Hence, we obtain for $x \leq x_0$ and $K \geq K_0$:

$$\mathbb{P}(X_{2,n} \leq K | S_n > x) \geq \frac{\mathbb{P}(X_{2,n} \leq K) + \mathbb{P}(S_n > x) - 1}{\mathbb{P}(S_n > x)}$$

$$\geq 1 - \frac{\delta}{1 - \mathbb{P}(S_n > x_0)}. \quad (4.40)$$

By (4.39) and (4.40), we see that $F \in J_2^{(n)}$, since $\delta > 0$ and $\varepsilon > 0$ are arbitrary.
Finally, we show $J_1^{(n)} \supseteq J_3^{(n)}$. Suppose $F \in J_3^{(n)}$ and $F \notin J_1^{(n)}$. Then, there exists some $\delta > 0$ such that for any $m \geq 1$:

$$\lim_{x \to \infty} \inf P(X_{2,n} \leq m|S_n > x) \leq 1 - 2\delta.$$ 

For every $m \geq 1$, we choose an unbounded and strictly increasing sequence $(x_k^m)_{k \in \mathbb{N}}$ such that for all $k \in \mathbb{N}$:

$$P(X_{2,n} \leq m|S_n > x_k^m) \leq 1 - \delta.$$ 

Since $F \in J_3^{(n)}$ we know there exist $c > 0$ and $\bar{x} > 0$ such that for all $x \geq \bar{x}$:

$$P(X_{1,n} > x - c|S_n > x) \geq 1 - \frac{\delta}{3}.$$ 

Hence, we obtain for any $m \geq 1$ and all $k \geq 1$, $x_k^m \geq \bar{x}$:

$$P(X_{1,n} > x_k^m - c, X_{2,n} > m|S_n > x_k^m) = P(X_{1,n} > x_k^m - c|S_n > x_k^m) - P(X_{1,n} > x_k^m - c, X_{2,n} \leq m|S_n > x_k^m) 
\geq P(X_{1,n} > x_k^m - c|S_n > x_k^m) - P(X_{2,n} \leq m|S_n > x_k^m) 
\geq 1 - \frac{\delta}{3} - 1 + \delta = \frac{2}{3}\delta.$$ 

We get

$$\lim_{m \to \infty} \limsup_{x \to \infty} P(X_{1,n} > x - c, X_{2,n} > m|S_n > x) > 0,$$

which contradicts Lemma 132.

In the proof of Proposition 72 we will use part a) of Proposition 74, for that reason we give the proof of Proposition 74 a) already here.

**Proof.** We prove part a) of Proposition 74.

We show that the following assertion holds for all $n \geq 2$:

$$J^{(n)} \subseteq OS.$$  \hspace{1cm} (4.41)

Let $n \geq 3$. Suppose that $F \in J^{(n)} = J_1^{(n)}$. Then,

$$1 = \lim_{K \to \infty} \liminf_{x \to \infty} P(X_{2,n} < \frac{K}{n-1}|S_n > x) 
\leq \lim_{K \to \infty} \liminf_{x \to \infty} P(X_{1,n} > x|S_n > x + K) 
\leq \lim_{K \to \infty} \liminf_{x \to \infty} \frac{nP(X_n > x)}{P(S_{n-1} > x)P(X_n > K)} 
\leq n \lim_{K \to \infty} \left( \frac{1}{P(X_n > K)} \liminf_{x \to \infty} \frac{P(X_n > x)}{P(S_{n-1} > x)} \right).$$

Hence, we have $\liminf_{x \to \infty} \frac{P(X_{1,n} > x)}{P(S_{n-1} > x)} > 0$ and thus $\limsup_{x \to \infty} \frac{P(S_{n-1} > x)}{P(X_{1,n} > x)} < \infty$.

In the case $n = 2$ we use the inclusion $J^{(3)} = A(G)^{(3)} \supseteq A(G)^{(2)} = J^{(2)}$, which was already stated in part b) of Proposition 38.
4 New large claim classes

Now, we can prove Proposition 72.

Proof. We can deduce \( \mathcal{J}^{(n+1)} \supseteq \mathcal{J}^{(n)} \) from part b) of Proposition 38 with \( \mathcal{E} = \mathcal{G} \). It remains to prove the inclusion \( \mathcal{J}^{(n+1)} \subseteq \mathcal{J}^{(n)} \).

Suppose \( F \in \mathcal{J}^{(n+1)} \). From Proposition 40 b) with \( \mathcal{E} = \mathcal{G} \) and \( \mathcal{J}^{(n)} \subseteq \mathcal{OS} \), \( n \in \mathbb{N} \), (see the proof above, (4.41)) we obtain the assertion. \( \square \)

Next, we prove Proposition 73.

Proof. First, we show the direction: \( \lim_{n \to \infty} \sum_{k=n+1}^{\infty} \frac{p_k}{p_n} > 0 \Rightarrow F \in \mathcal{J} \).

We prove that the following holds
\[
\lim_{K \to \infty} \lim_{x \to \infty} \mathbb{P}(X_{1,2} \geq x - K | S_2 > x) = 1.
\]

Let \( \tilde{k} \in \mathbb{N} \) be such that
\[
\frac{x_n}{x_{n-\tilde{k}-1}} = \prod_{i=n-\tilde{k}}^{n-1} \frac{x_{i+1}}{x_i} \geq (1 + \varepsilon)^{\tilde{k}} \geq 2,
\]
for all \( n > \tilde{k} \).

Let \( (y_n)_{n \in \mathbb{N}} \) be an arbitrary, but fixed sequence such that \( y_n \to \infty \), as \( n \to \infty \).

Let \( K > 0 \) and without loss of generality we assume
\[
(x_{n+1} - x_n) \land (x_n - x_{n-1}) > K,
\]
for all \( n > \tilde{k} \).

The proof splits into two cases.

1. Case: \( y_n \in [x_n, x_n + K) \) for all \( n \in \mathbb{N} \).

From \( S_2 > y_n \) we have \( X_{1,2} > y_n / 2 \geq x_n / 2 = x_{n-\tilde{k}-1} \) and thus, \( X_{1,2} \geq x_{n-\tilde{k}} \) for all \( n > \tilde{k} \). Hence, we obtain for all \( n > \tilde{k} \)
\[
\{ S_2 > y_n \} = \{ X_{1,2} \geq x_n \} \cup \bigcup_{i=1, \ldots, \tilde{k}} \{ X_{1,2} = x_{n-i}, X_{2,2} > y_n - x_{n-i} \}.
\]

Further, we have for all \( n > \tilde{k} \):
\[
\mathbb{P}(X_{1,2} \geq x_n, S_2 > y_n) \geq \mathbb{P}(X_{1,2} \geq y_n) \geq \mathbb{P}(X_{1,2} \geq x_{n+1}).
\]

Denote \( J_n := \{ i \in \{ 1, \ldots, \tilde{k} \} : \mathbb{P}(X_{1,2} = x_{n-i}, X_{2,2} > y_n - x_{n-i}) > 0 \} \).
Then we obtain for all \( n > \tilde{k} \) and for all \( i \in J_n \):

\[
P(X_{1,2} = x_{n-i}, X_{2,2} > y_n - x_{n-i}) = P(X_1 = x_{n-i})P(x_{n-i} > X_2 > y_n - x_{n-i}) + P(X_2 = x_{n-i})P(x_{n-i} > X_1 > y_n - x_{n-i}) + P(X_1 = x_{n-i})P(X_2 = x_{n-i})
\]

\[
= 2p_{n-i}(P(X_1 > y_n - x_{n-i}) - P(X_1 > y_n - x_{n-i}) + p_{n-i}^2)
\]

\[
= 2p_{n-i}(P(X_1 > y_n - x_{n-i}) - \sum_{j=n-i}^{\infty} p_j + p_{n-i}^2)
\]

\[
\leq 2p_{n-i}(P(X_1 > y_n - x_{n-i}) - p_{n-i} + p_{n-i}^2)
\]

\[
\leq 2p_{n-i}(P(X_1 > y_n - x_{n-i})
\]

\[
\leq 2p_{n-i}(P(X_1 > K),
\]

where we used in the last step the inequality

\[
y_n - x_{n-i} \geq x_n - x_{n-i} > x_n - x_{n-1} > K.
\]

Let \( i \in J_n \) and \( n > \tilde{k} \), then we obtain

\[
\frac{P(X_{1,2} \geq x_n | S_2 > y_n)}{P(X_{1,2} = x_{n-i}, X_{2,2} > y_n - x_{n-i} | S_2 > y_n)} = \frac{P(X_{1,2} = x_{n-i}, X_{2,2} > y_n - x_{n-i}, S_2 > y_n)}{P(X_{1,2} \geq x_n)} \geq \frac{P(X_{1,2} = x_{n-i}, X_{2,2} > y_n - x_{n-i}, S_2 > y_n)}{2p_{n-i}(P(X_1 > K)}
\]

\[
\geq \frac{\sum_{j=n-i}^{\infty} p_j}{p_{n-i}P(X_1 > K)}.
\]

Denote

\[
\alpha_{n,K}^{(i)} := \frac{\sum_{j=n-i+1}^{\infty} p_j}{p_{n-i}P(X_1 > K)}.
\]

From Lemma 133 and \( \lim\inf_{n \to \infty} \frac{\sum_{j=n-i+1}^{\infty} p_j}{p_n} > 0 \), we obtain

\[
\alpha_{n,K}^{(i)} \to \infty,
\]

as first \( n \to \infty \) and then \( K \to \infty \). Recall the restriction (4.42), since \( x_{n+1} - x_n \wedge (x_n - x_{n-1}) \to \infty \), as \( n \to \infty \), we can increase \( K \to \infty \) without a contradiction.
2. Case: $y_i$ we deduce analogously to the 1. Case for all $i$ where we used in the last step the inequality

$$\lim_{n \to \infty} \lim_{K \to \infty} \mathbb{P}(X_{1,2} \geq x_n | S_2 > y_n) = 1.$$  \hspace{1cm} (4.47)

By letting $n \to \infty$ and then $K \to \infty$ in (4.47) we deduce that

$$\mathbb{P}(X_{1,2} \geq x_n | S_2 > y_n) \to 1.$$  \hspace{1cm} (4.48)

Denote

$$I_K := \bigcup_{n \in \mathbb{N}} [x_n, x_n + K].$$

Since the sequence $(y_n)_{n \in \mathbb{N}}$ is arbitrary and (4.48) holds as $n \to \infty$, we obtain

$$\lim_{K \to \infty} \lim_{n \to \infty} \mathbb{P}(X_{1,2} \geq x - K | S_2 > x) = 1.$$  \hspace{1cm} (4.49)

2. Case: $y_n \in [x_n + K, x_{n+1})$ for all $n \in \mathbb{N}$.

From $S_2 > y_n$ we have $X_{1,2} > y_n/2 \geq x_n/2 = x_{n-k}$. Thus, $X_{1,2} \geq x_{n-k}$ for all $n > \tilde{k}$.

Denote $J_n := \{i \in \{0, ..., \tilde{k}\} : \mathbb{P}(X_{1,2} = x_n-i, X_{2,2} > y_n-x_n-i) > 0\}$.

Then we obtain

$$\{S_2 > y_n\} = \{X_{1,2} \geq x_{n+1}\} \cup \bigcup_{i \in J_n} \{X_{1,2} = x_n-i, X_{2,2} > y_n-x_n-i\}.$$

We obtain analogously to the 1. Case (see lines (4.43)-(4.44)) for all $i \in J_n$:

$$\mathbb{P}(X_{1,2} = x_n-i, X_{2,2} > y_n-x_n-i) \leq 2p_{n-i}(\mathbb{P}(X_1 > y_n-x_n-i) \leq 2p_{n-i}(\mathbb{P}(X_1 > K),$$

where we used in the last step the inequality

$$y_n-x_n-i \geq x_n + K - x_n-i > x_n + K - x_n = K.$$

We deduce analogously to the 1. Case for all $i \in J_n$:

$$\frac{\mathbb{P}(X_{1,2} \geq x_{n+1} | S_2 > y_n)}{\mathbb{P}(X_{1,2} = x_n-i, X_{2,2} > y_n-x_n-i | S_2 > y_n)} \geq \frac{\mathbb{P}(X_{1,2} \geq x_n)}{2p_{n-i}(\mathbb{P}(X_1 > K)} \geq \sum_{j=n+1}^{\infty} p_j \geq \frac{\mathbb{P}(X_1 > K)}{p_{n-i}\mathbb{P}(X_1 > K)}.$$
4 New large claim classes

Denote

\[ \beta^{(i)}_{n,K} := \frac{\sum_{j=n+1}^{\infty} p_j}{p_{n-i} \mathbb{P}(X_1 > K)}. \]

From Lemma 133 and \( \lim \inf \to \infty \sum_{j=n+1}^{\infty} \frac{p_j}{p_n} > 0 \), we obtain

\[ \beta^{(i)}_{n,K} \to \infty, \]

as first \( n \to \infty \) and then \( K \to \infty \).

Thus, we have

\[ 1 = \mathbb{P}(S_2 > x_n | S_2 > y_n) = \mathbb{P}(X_{1,2} \geq x_{n+1} | S_2 > y_n) + \sum_{i \in J_n} \mathbb{P}(X_{1,2} = x_{n-i}, X_{2,2} > y_n - x_{n-i} | S_2 > y_n) \]

\[ = \left( 1 + \sum_{i \in J_n} \frac{1}{\beta^{(i)}_{n,K}} \right) \mathbb{P}(X_{1,2} \geq x_{n+1} | S_2 > y_n). \tag{4.50} \]

By letting \( n \to \infty \) and then \( K \to \infty \) in (4.50) we deduce that

\[ \mathbb{P}(X_{1,2} \geq x_{n+1} | S_2 > y_n) \to 1. \tag{4.51} \]

Denote

\[ I_K^c := \bigcup_{n \in \mathbb{N}} [x_n + K, x_{n+1}). \]

Since the sequence \((y_n)_{n \in \mathbb{N}}\) is arbitrary and (4.51) holds as \( n \to \infty \), we obtain

\[ \lim_{K \to \infty} \lim_{x \to \infty} \mathbb{P}(X_{1,2} \geq x - K | S_2 > x) = 1. \tag{4.52} \]

By combining (4.49) and (4.52), we obtain the assertion

\[ \lim_{K \to \infty} \lim_{x \to \infty} \mathbb{P}(X_{1,2} \geq x - K | S_2 > x) = 1. \]

Proof of " \( \Rightarrow \) " direction is complete.

Now, we show the direction: \( F \in \mathcal{J} \Rightarrow \lim \inf_{n \to \infty} \frac{\sum_{k=n+1}^{\infty} p_k}{p_n} > 0. \)

We argue by contradiction. Suppose \( F \in \mathcal{J} \) and \( \lim \inf_{n \to \infty} \frac{\sum_{k=n+1}^{\infty} p_k}{p_n} = 0. \)

Then we can define for \( n \in \mathbb{N} \):

\[ m(n) := \sup \{ m \in \mathbb{N} : x_m < \epsilon x_n, \ m + 2 < n, \ \sum_{j=m+1}^{n-1} p_j \geq \frac{\sum_{j=n+1}^{\infty} p_j}{p_n} \}, \tag{4.53} \]

66
with the usual convention \( \sup \{ \emptyset \} = -\infty \). Obviously, there exists \( n_0 \) such that \( m(n) > 0, m(n+1) \geq m(n) \) for all \( n \geq n_0 \) and \( m(n) \to \infty \), as \( n \to \infty \).

Define a sequence \( (K_n)_{n \geq n_0} \) such that \( K_n := x_{m(n)} \) for all \( n \geq n_0 \). From the definition of \( m(n) \) we know that \( K_n = x_{m(n)} < \varepsilon x_n < x_{n+1} \) for all \( n \geq n_0 \).

Let \( (y_n)_{n \in \mathbb{N}} \) be a sequence such that \( y_n \to \infty \), as \( n \to \infty \), and \( y_n \in [x_n + K_n, x_{n+1}) \) for all \( n \geq n_0 \).

Recall that \( \tilde{k} \in \mathbb{N} \) is a constant such that for all \( n \geq \tilde{k} \):
\[
\frac{x_n}{x_{n-k-1}} = \prod_{i=n-k-1}^{n-1} \frac{x_{i+1}}{x_i} \geq (1 + \varepsilon)^k \geq 2.
\]

Let \( n \) be large enough such that \( n \geq \tilde{k} \) and \( n \geq n_0 \).

From \( S_2 > y_n \) we have \( X_{1,2} > y_n/2 \geq y_n/2 = x_{n-k-1} \). Thus, \( X_{1,2} \geq x_{n-k} \).

Denote \( J_n := \{ i \in \{0, \ldots, k\} : \mathbb{P}(X_{1,2} = x_{n-i}, X_{2,2} > y_n - x_{n-i}) > 0 \} \).

Then we obtain for all \( n \geq n_0 \lor \tilde{k} \):
\[
\{ S_2 > y_n \} = \{ X_{1,2} \geq x_{n+1} \} \cup \bigcup_{i \in J_n} \{ X_{1,2} = x_{n-i}, X_{2,2} > y_n - x_{n-i} \}
\]

and thus,
\[
\mathbb{P}(S_2 > y_n) = \mathbb{P}(X_{1,2} \geq x_{n+1}) + \mathbb{P}(X_{1,2} = x_{n-i}, X_{2,2} > y_n - x_{n-i}) \\
\geq \mathbb{P}(X_{1,2} \geq x_{n+1}) + \mathbb{P}(X_{1,2} = x_n, X_{2,2} > y_n - x_n) \\
= 2 \sum_{j=n+1}^{\infty} p_j - \left( \sum_{j=n+1}^{\infty} p_j \right)^2 + 2p_n \mathbb{P}(x_n > x_1 > y_n - x_n) + p_n^2.
\]

We consider the probability of the event \( \{ X_{2,2} \leq K_n, S_2 > x \} \). We obtain for all \( n \geq n_0 \lor \tilde{k} \):
\[
\mathbb{P}(X_{2,2} \leq K_n, S_2 > y_n) = \mathbb{P}(X_{1,2} \geq x_{n+1}, X_{2,2} \leq K_n) \\
+ \mathbb{P}(X_{1,2} = x_{n-i}, X_{2,2} > y_n - x_{n-i}, X_{2,2} \leq K_n) \\
= \mathbb{P}(X_{1,2} \geq x_{n+1}, X_{2,2} \leq K_n) \\
+ \mathbb{P}(X_{1,2} = x_{n-i}, y_n - x_{n-i} < X_{2,2} \leq K_n).
\]

We consider the terms separately:
\[
\mathbb{P}(X_{1,2} \geq x_{n+1}, X_{2,2} \leq K_n) = \mathbb{P}(X_{1,2} \geq x_{n+1}, X_{2,2} \leq x_{m(n)}) \\
= \mathbb{P}(X_1 \geq x_{n+1}) \mathbb{P}(X_2 \leq m(n)) + \mathbb{P}(X_2 \geq x_{n+1}) \mathbb{P}(X_1 \leq x_{m(n)}) \\
= 2 \sum_{k=0}^{m(n)} \sum_{j=n+1}^{\infty} p_k p_j
\]

and for \( i \in J_n \)
\[
\mathbb{P}(X_{1,2} = x_{n-i}, y_n - x_{n-i} < X_{2,2} \leq K_n) = 0.
\]
where we used \( y_n - x_{n-i} > y_n - x_n \geq x_n + K_n - x_n = K_n \).

Altogether, we arrive at

\[
\mathbb{P}(X_{2,2} \leq K_n | S_2 > y_n) = \frac{\mathbb{P}(X_{2,2} \leq K_n, S_2 > y_n)}{\mathbb{P}(S_2 > y_n)} \\
\leq \frac{2 \sum_{k=0}^{m(n)} p_k \sum_{j=n+1}^{\infty} p_j}{2 \sum_{j=n+1}^{\infty} p_j - \left( \sum_{j=n+1}^{\infty} p_j \right)^2 + 2p_n \mathbb{P}(x_n > X_1 > y_n - x_n) + p_n^2} \\
\sim \frac{m(n)}{2 \sum_{j=n+1}^{\infty} p_j + 2p_n \mathbb{P}(x_n > X_1 > y_n - x_n) + p_n^2} \\
= \frac{m(n)}{\sum_{k=0}^{m(n)} p_k \sum_{j=n+1}^{\infty} p_j + 2p_n \mathbb{P}(x_n > X_1 > y_n - x_n) + p_n^2},
\]

as \( n \to \infty \).

Denote \( I := \bigcup_{n \in \mathbb{N}} [x_n + K_n, x_{n+1}) \).

From \( F \in \mathcal{F} \) we know that the following holds

\[
\lim_{n \to \infty} \liminf_{x \to \infty, x \in I} \mathbb{P}(X_{2,2} \leq K_n | S_2 > x) = 1.
\]

(4.55)

By combining \( \lim_{n \to \infty} \sum_{k=0}^{m(n)} p_k = 1 \), (4.54) and (4.55), we obtain

\[
\frac{2 \sum_{j=n+1}^{\infty} p_j}{2 \sum_{j=n+1}^{\infty} p_j + 2p_n \mathbb{P}(x_n > X_1 > y_n - x_n) + p_n^2} \to 1,
\]

as \( n \to \infty \). Thus, we have

\[
\frac{2 \sum_{j=n+1}^{\infty} p_j}{2p_n \mathbb{P}(x_n > X_1 > y_n - x_n) + p_n^2} \to \infty,
\]

(4.56)

as \( n \to \infty \).

The assertion (4.56) holds for all sequences \( (y_n)_{n \in \mathbb{N}} \) such that \( y_n \in [x_n + K_n, x_{n+1}) \), \( n \geq n_0 \lor \hat{k} \), as \( n \to \infty \). In particular, if \( y_n = x_n + K_n = x_n + x_{m(n)} \), \( n \geq n_0 \lor \hat{k} \), then we obtain

\[
\frac{2 \sum_{j=n+1}^{\infty} p_j}{2p_n \mathbb{P}(x_n > X_1 > x_{m(n)}) + p_n^2} \to \infty,
\]

(4.57)
as \( n \to \infty \). Next, we reformulate the left side of (4.57):

\[
\frac{2 \sum_{j=n+1}^{\infty} p_j}{2p_n\mathbb{P}(x_n > X_1 > x_{m(n)}) + p_n^2} = \frac{2 \sum_{j=n+1}^{\infty} p_j}{2p_n(\mathbb{P}(X_1 > x_{m(n)}) - \mathbb{P}(X_1 > x_n)) + p_n^2} = \frac{2 \sum_{j=n+1}^{\infty} p_j}{2p_n \sum_{j=m(n)+1}^{n-1} p_j + p_n^2} \leq \frac{2 \sum_{j=n+1}^{\infty} p_j}{2p_n \sum_{j=m(n)+1}^{n-1} p_j}.
\]

Thus, we obtain

\[
\frac{\sum_{j=n+1}^{\infty} p_j}{p_n \sum_{j=m(n)+1}^{n-1} p_j} \to \infty, \quad (4.58)
\]

as \( n \to \infty \).

From the definition of \( m(n) \) we obtain

\[
\infty = \lim_{n \to \infty} \frac{\sum_{j=n+1}^{\infty} p_j}{p_n \sum_{j=m(n)+1}^{n-1} p_j} \leq \lim_{n \to \infty} \frac{p_n \sum_{j=m(n)+1}^{n-1} p_j}{p_n \sum_{j=m(n)+1}^{n-1} p_j} = 1.
\]

Thus, we have a contradiction and conclude the assertion

\[
\liminf_{n \to \infty} \frac{\sum_{j=n+1}^{\infty} p_j}{p_n} > 0.
\]

The proof is complete.

Next, we prove the relations between the classes \( \mathcal{J} \) and other classes, Proposition 74.

\textbf{Proof.} a) \( \mathcal{J}^{(n)} \subseteq \mathcal{O} \mathcal{S} \), \( n \geq 2 \), was proven above, see (4.41) on page 62.

b) \( \mathcal{J} \subseteq \mathcal{A}(\mathcal{E}), \mathcal{E} \subseteq \mathcal{G} \), is obvious from the definition. A distribution \( F \in \mathcal{A} \setminus \mathcal{J} \) is given in Example 75.
Now, we turn to the proof of the relations between the classes $J$ and other heavy-tailed classes, Proposition 76.

**Proof.** a) Suppose $F \in J \cap L$. From $F \in J \subset OS$ we obtain

$$\lim_{x \to \infty} \frac{\mathbb{P}(S_2 > x, X_{2,2} > g(x))}{\mathbb{P}(S_2 > x)} = c \lim_{x \to \infty} \frac{\mathbb{P}(S_2 > x, X_{2,2} > g(x))}{\mathbb{P}(X_1 > x)} = 0,$$

for all $g \in \mathcal{G}$, where $c$ is a constant such that $\liminf_{x \to \infty} \frac{\mathbb{P}(X_1 > x)}{\mathbb{P}(S_2 > x)} > c > 0$.

From Proposition 19 and Remark 20 on page 21 we know that if $F \in L$ and

$$\lim_{x \to \infty} \frac{\mathbb{P}(S_2 > x, X_{2,2} > g(x))}{\mathbb{P}(X_1 > x)} = 0$$

for all $g \in \mathcal{G}$, then $F \in S$. Thus, the inclusion $J \cap L \subset S$ is complete. The inclusion $S \subset J$ is obvious and $S \subset L$ is well known. The proof is complete.

c) We show $D \subseteq J$.

Let $F \in D$ and

$$\gamma := \sup_{x \geq 0} \frac{F(x/2)}{F(x)}.$$

Let $\varepsilon > 0$. There exists $K_0 > 0$ such that for all $K \geq K_0$

$$\mathbb{P}(X_2 > K) \gamma < \varepsilon.$$

For $K \geq K_0$ and $x$ such that $x \geq 2K$, we get

$$\mathbb{P}(X_1 \wedge X_2 > K | S_2 > x) \leq 2 \mathbb{P}(X_1 > \frac{x}{2}, X_2 > K | S_2 > x)$$

$$\leq 2 \mathbb{P}(X_1 > \frac{x}{2}) \frac{\mathbb{P}(X_2 > K)}{\mathbb{P}(S_2 > x)}$$

$$\leq 2 \mathbb{P}(X_2 > K) \gamma < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the assertion follows. 

Next, we turn to the proof of the relations between the classes $J$ and other light-tailed classes, Proposition 79.

**Proof.** Let $\gamma > 0$. Suppose $F \in J \cap L(\gamma)$. From $F \in J \subset OS$ we obtain

$$\lim_{x \to \infty} \frac{\mathbb{P}(S_2 > x, X_{2,2} > g(x))}{\mathbb{P}(X_1 > x)} \leq C \lim_{x \to \infty} \frac{\mathbb{P}(S_2 > x, X_{2,2} > g(x))}{\mathbb{P}(S_2 > x)} = 0,$$

for all $g \in \mathcal{G}$, where $C$ is a constant such that $\limsup_{x \to \infty} \frac{\mathbb{P}(S_2 > x)}{\mathbb{P}(X_1 > x)} < C < \infty$. From Proposition 34 we obtain $F \in S(\gamma)$.

Suppose $F \in S(\gamma)$. From $F \in S(\gamma)$ and Proposition 34 we obtain

$$0 = \lim_{x \to \infty} \frac{\mathbb{P}(S_2 > x, X_{2,2} > g(x))}{\mathbb{P}(X_1 > x)} \leq \lim_{x \to \infty} \frac{\mathbb{P}(S_2 > x, X_{2,2} > g(x))}{\mathbb{P}(S_2 > x)}$$

for all $g \in \mathcal{G}$. Thus $F \in J$. 

\[70\]
4 New large claim classes

Next, we prove the convolution properties of the class \( J \), Proposition 85.

**Proof.** a) We prove closure under convolution powers of \( J \), i.e. if \( F \in J \) then \( F^{\circ \circ} \in J \). Suppose \( F^{\circ \circ} \in J \). We show \( F^{\circ \circ} \simeq F^{(n+1)^*} \) and hence \( F^{(n+1)^*} \in J \). From \( S_n \in J \subset OS \) we obtain

\[
\limsup_{x \to \infty} \frac{\mathbb{P}(S_{n+1} > x)}{\mathbb{P}(S_n > x)} \leq c F^{\circ \circ} \limsup_{x \to \infty} \frac{\mathbb{P}(S_{n+1} > x)}{\mathbb{P}(S_n > x)} \leq c F^{\circ \circ}.
\]

b) Follows from Proposition 46 with \( \mathcal{E} = \mathcal{G} \).

Now, we prove the closure of the class \( J \) under root convolution, Proposition 88.

**Proof.** We show root convolution closure for \( J \), i.e. if \( F^{\circ \circ} \in J \) then \( F \in J \). Let \( n = 2^m, m \in \mathbb{N} \). Suppose \( F^{2^{m^*}} \in J \). Since \( J \subset OS \) we have \( F^{2^{m^*}} \in OS \) and hence there exists a constant \( c_{2m} \) such that

\[
\liminf_{x \to \infty} \frac{\mathbb{P}(S_{2^{m-1}} > x)}{\mathbb{P}(S_{2^m} > x)} > c_{2m} > 0.
\]

We obtain by definition for all \( h \in \mathcal{G} \)

\[
0 = \limsup_{x \to \infty} \mathbb{P}(S_{2^m} > h(x) | S_{2^m} + S_{2^m} > x) \geq c_{2m} \limsup_{x \to \infty} \mathbb{P}(S_{2^{m-1}} + S_{2^{m-1}} > h(x) | S_{2^{m-1}} > x).
\]

Thus, we have \( F^{2^{m-1}} \in J \). We repeat the argument leading to (4.59) for \( (m - 1) \) times and arrive at

\[
0 \geq c_{2m} \limsup_{x \to \infty} \mathbb{P}(S_{2^{m-1}} + S_{2^{m-1}} > h(x) | S_{2^{m-1}} + S_{2^{m-1}} > x) \geq c_{2m} \cdot c_{2m-1} \limsup_{x \to \infty} \mathbb{P}(S_{2^{m-2}} + S_{2^{m-2}} > h(x) | S_{2^{m-2}} + S_{2^{m-2}} > x) \geq \ldots \geq c_{2m} \cdots c_2 \limsup_{x \to \infty} \mathbb{P}(X_1 + X_2 > h(x) | S_2 > x),
\]

which gives \( F \in J \). In case \( n \neq 2^m \) for all \( m \in \mathbb{N} \), we take \( \tilde{m} := \min \{ m \in \mathbb{N} : n < 2^m \} \). Denote by \( k := 2^{\tilde{m}} \). By the argument in the proof of a) we know that \( F^{n^*} \in J \Rightarrow F^{k^*} \in J \). From (4.62) we obtain \( F \in J \).

Finally, we prove closure of the class \( J \) under the operations minimum, maximum and mixture.

**Proof.** a) The equivalence of i) and iii) follows from Proposition 49 with \( \mathcal{E} = \mathcal{G} \), \( J \subset OS \) and the closure of \( OS \) under convolution.

The direction ii) \( \Rightarrow i \) follows from Proposition 50 a) with \( \mathcal{E} = \mathcal{G} \) and \( J \subset OS \).

The direction i) \( \Rightarrow ii \) follows from Proposition 50 b) with \( \mathcal{E} = \mathcal{G} \).

b) The assertion follows from part c) of Proposition 50 a) with \( \mathcal{E} = \mathcal{G} \) and \( J \subset OS \). \( \square \)
4.5 Local version of $J$

Recently, Asmussen, Foss and Korshunov introduced and systematically investigated local versions of the distribution classes $S$ and $L$, see [1] for details. The authors applied the obtained properties to risk theory, the compound distributions, infinitely divisible laws and branching theory. Many papers followed in refining these concepts, as an example we refer the reader to several papers, [1, 3, 36, 39, 37].

The purpose of this section is to demonstrate that it is reasonable and promising to define a local version of our distribution class $J$. We present the approach of definition of the local version of $J$, which is similar to the nonlocal case, and state a basic property of the new class, the closure under weak tail-equivalence.

First, we give the definition of the local version of subexponential and long-tailed distributions. For fixed $0 < T \leq \infty$, set $\Delta = (0, T]$ and $x + \Delta = (x, x + T]$. If $T = \infty$ then let $x + \Delta = (x, \infty)$.

**Definition 89.** $(\Delta-$long-tailed distributions)

We say that a distribution $F \in \mathcal{F}$ is $\Delta-$long-tailed, if $F(x + \Delta) > 0$ for all sufficiently large $x$ and for all fixed $t \in \mathbb{R}$:

$$\lim_{x \to \infty} \frac{F(x + t + \Delta)}{F(x + \Delta)} = 1.$$ 

We denote the class of $\Delta-$long-tailed distributions by $L_\Delta$.

Next, we define the local version of the class $S$.

**Definition 90.** $(\Delta-$subexponential distributions)

We say that a distribution $F \in \mathcal{F}$ is $\Delta-$subexponential, if $F \in L_\Delta$ and:

$$\lim_{x \to \infty} \frac{\mathbb{P}(S_2 \in x + \Delta)}{\mathbb{P}(X_1 \in x + \Delta)} = 2.$$ 

The class of $\Delta-$subexponential distributions is denoted by $S_\Delta$.

Now, we turn to one basic result for distributions of the classes $L_\Delta$ and $S_\Delta$, which was proved by Asmussen et al. in [1, Proposition 2]. We already stated this result for the case $T = \infty$ in Proposition 19. Here, we cite the general version, which includes the local case for all $T > 0$.

**Proposition 91.** *(Relation between $S$ and $L$ in the local case)*

Assume $F \in L_\Delta$ for some $\Delta$. Let $X_1$ and $X_2$ be two i.i.d random variables with common distribution $F$. The following assertions are equivalent:

a) $F \in S_\Delta$;

b) there exists a function $h$ such that $h(x) \to \infty$, $h(x) < x/2$, and $F(x - y + \Delta) \sim F(x + \Delta)$ as $x \to \infty$ uniformly in $|y| \leq h(x)$,

$$\mathbb{P}(X_1 + X_2 \in x + \Delta, X_{2,2} > h(x)) = o(\mathbb{P}(X_1 \in x + \Delta)) \text{ as } x \to \infty; \quad (4.63)$$

c) the relation (4.63) holds for every function $h$ such that $h(x) \to \infty$. 

72
The approach of the definition of our distribution class in the local case is analogous to the definition of the distribution class \( J \). In both cases the relation (4.63) can serve as an approach for the definition. Denote the set of all nonnegative, unbounded and nondecreasing functions by \( G \).

**Definition 92. (Distribution class \( J_\Delta \))**

Let \( T > 0 \), then we write

\[
J_\Delta = \left\{ F \in \mathcal{F} : \lim_{x \to \infty} \frac{\mathbb{P}(X_{2,2} > g(x), S_2 \in x + \Delta)}{\mathbb{P}(X_1 \in x + \Delta)} = 0 \quad \forall g \in G \right\}.
\]

**Remark 93.** \( S_\Delta \subset S \) and \( J_\Delta \subset J \).

Asmussen et al. stated in [1, Remark 2] that the inclusion \( S_\Delta \subset S \) holds for all \( T > 0 \). By using the same idea of [1, Remark 2], we can state the same assertion for the class \( J_\Delta \). If \( F \in J_\Delta \) for some \( \Delta \), then \( F \in J_{n\Delta} \) for any \( n \in \{2, 3, \ldots, \infty\} \) and \( F \in J \).

For any \( n \in \{2, 3, \ldots, \infty\} \) and \( g \in G \) we have:

\[
\lim_{x \to \infty} \frac{\mathbb{P}(X_{2,2} > g(x), S_2 \in x + n\Delta)}{\mathbb{P}(X_1 \in x + n\Delta)} = \frac{\sum_{k=0}^{n-1} \lim_{x \to \infty} \frac{\mathbb{P}(X_{2,2} > g(x), S_2 \in x + kT + \Delta)}{\mathbb{P}(X_1 \in x + kT + \Delta)}}{\mathbb{P}(X_1 \in x + n\Delta)} \leq \frac{\sum_{k=0}^{n-1} \lim_{x \to \infty} \frac{\mathbb{P}(X_{2,2} > g(x), S_2 \in x + kT + \Delta)}{\mathbb{P}(X_1 \in x + kT + \Delta)}}{\mathbb{P}(X_1 \in x + n\Delta)} = 0.
\]

It is obvious from the definition that the inclusion \( S_\Delta \subset J_\Delta \) holds. However, we need an example to show that the class \( S_\Delta \) is a proper subclass of \( J_\Delta \), i.e. \( S_\Delta \subsetneq J_\Delta \).

**Example 94.** Distribution \( F \in J_\Delta \setminus S_\Delta \).

Consider the distribution \( F \in \mathcal{F} \) with density

\[
f(x) = Ce^{-x}g(x), \quad x \geq 0,
\]

where \( g(x) \) is a subexponential density and \( C > 0 \) such that \( \int_0^\infty f(x)dx = 1 \). From Example 81 on page 57 we know already that for all \( h \in G \)

\[
\lim_{z \to \infty} \lim_{x \to \infty} \frac{\int_{h(z)}^{x-h(z)} g(y)g(x-y)dy}{g(x)} = 0.
\]

Using the last fact, we obtain for the density \( f \) and for all \( h \in G \):

\[
\lim_{z \to \infty} \lim_{x \to \infty} \frac{\int_{h(z)}^{x-h(z)} f(y)f(x-y)dy}{f(x)} = \lim_{z \to \infty} \lim_{x \to \infty} \frac{C^2e^{-x} \int_{h(z)}^{x-h(z)} g(y)g(x-y)dy}{Ce^{-x}g(x)} = \lim_{z \to \infty} \lim_{x \to \infty} \frac{C \int_{h(z)}^{x-h(z)} g(y)g(x-y)dy}{g(x)} = 0.
\]
4 New large claim classes

Let $T > 0$. Thus, it follows that for all $h \in G$:

$$\limsup_{z \to \infty} \int_{z}^{z+T} \int_{h(z)}^{z+T} f(y) f(x-y) dy \, dx = \limsup_{z \to \infty} \frac{\mathbb{P}(X_{2,2} > h(z), S_2 \in (z, z+T])}{\mathbb{P}(X_1 \in (z, z+T])}$$

$$= \limsup_{z \to \infty} \frac{\mathbb{P}(X_{2,2} > h(z), S_2 \in \Delta + z)}{\mathbb{P}(X_1 \in \Delta + z)}$$

$$= 0$$

and hence $F \in J_\Delta$, where $\Delta = (0, T]$. $F$ is light-tailed, hence $F \notin S$ and $F \notin S_\Delta$.

Finally, we generalize the closure of the class $J$ under weak tail-equivalence, which was stated in Proposition 84.

**Proposition 95. (Closure of $\Delta-$classes under weak tail-equivalence)**

a) If $F \in S_\Delta$ for some $\Delta$ and $F(x + \Delta) \asymp G(x + \Delta)$, then $G \in S_\Delta$.

b) If $F \in J_\Delta$ for some $\Delta$ and $F(x + \Delta) \asymp G(x + \Delta)$, then $G \in J_\Delta$.

**Proof.** a) The proof of part a) can be found [18, Theorem 4.22].

b) The proof of part b) is analogous to the proof of part a).

Let $X_1$ and $X_2$ be i.i.d. random variables random variables with a common distribution $F \in F$, and $Y_1$ and $Y_2$ i.i.d. random variables with a common distribution $G \in F$.

Suppose $F \in J_\Delta$ and $F(x + \Delta) \asymp G(x + \Delta)$. We prove that the following holds

$$\lim_{x \to \infty} \frac{\mathbb{P}(Y_{2,2} > g(x), Y_1 + Y_2 \in x + \Delta)}{\mathbb{P}(Y_1 \in x + \Delta)} = 0,$$

for all $g \in G$.

From $F(x + \Delta) \asymp G(x + \Delta)$ we know that there exist constants $c$ and $C$ such that for $\forall x \geq 0$:

$$0 < c < \liminf_{x \to \infty} \frac{G(x + \Delta)}{F(x + \Delta)} \leq \limsup_{x \to \infty} \frac{G(x + \Delta)}{F(x + \Delta)} < C < \infty. \quad (4.64)$$

Hence, we obtain for all $g \in G$:

$$\mathbb{P}(Y_{2,2} > g(x), Y_1 + Y_2 \in x + \Delta) \leq \mathbb{P}(Y_1 \in (g(x), x - g(x) + T), Y_1 + Y_2 \in x + \Delta)_{x-g(x)+T}$$

$$= \int_{g(x)}^{x-g(x)+T} G(x-y+\Delta) G(dy)$$

$$\leq C \int_{g(x)}^{x-g(x)+T} F(x-y+\Delta) G(dy)$$

$$\leq C \mathbb{P}(Y_1 > g(x), X_2 > g(x) - T, Y_1 + X_2 \in x + \Delta)$$

$$\leq C \mathbb{P}(Y_1 \wedge X_2 > g(x) - T, Y_1 + X_2 \in x + \Delta).$$

74
4 New large claim classes

By repeating the last steps we obtain

\[ P(Y_{2,2} > g(x), Y_1 + Y_2 \in x + \Delta) \leq C^2 P(X_{2,2} > g(x) - 2T, S_2 \in x + \Delta). \quad (4.65) \]

By (4.64) and (4.65) we conclude the assertion:

\[ \lim_{x \to \infty} \frac{P(Y_{2,2} > g(x), Y_1 + Y_2 \in x + \Delta)}{P(Y_1 \in x + \Delta)} \leq \frac{C^2}{c} \lim_{x \to \infty} \frac{P(X_{2,2} > g(x) - 2T, S_2 \in x + \Delta)}{P(X_1 \in x + \Delta)} = 0, \]

for all \( g \in \mathcal{G}. \) \qed
4 New large claim classes

4.6 Closure properties under random sums

Let \( \{X_i, i \geq 1\} \) be independent, identically distributed random variables with common distribution \( F \in \mathcal{F} \) and let \( N \) be a nonnegative integer-value random variable with masses \( p_n = \mathbb{P}(N = n), n \in \mathbb{N} \), which is independent of \( \{X_i, i \geq 1\} \).

**Definition 96.** (Random sum)

We say

\[
S_N = \sum_{i=1}^{N} X_i
\]

is a random sum generated by \( \{X_i, i \geq 1\} \) and \( N \). The distribution of \( S_N \) is called the compound distribution and is denoted by \( F_N \). The distribution of \( N \) is referred to as the primary distribution and the distribution of \( X_1 \) as the underlying distribution of the random sum.

In this section, we consider the closure properties of compound distributions where the underlying distribution \( F \) is heavy-tailed or belongs to a large claim distribution class.

Compound distributions are of considerable interest because they arise in many probability models. It is of central importance in actuarial mathematics, for instance, a compound distribution may be used to model the aggregate claim from an insurance portfolio for a given period of time. In this context, the subordinating random variable \( N \) represents the number of claims from the portfolio and \( X_1 \) represents the individual claim amount. Finally, the random sum \( S_N \) represents the aggregate claim amount.

Next, we will need some additional notation. Denote by \( N^{(1)} \) and \( N^{(2)} \) two independent copies of \( N \), and write

\[
(p \ast p)_n := \mathbb{P}(N^{(1)} + N^{(2)} = n), \quad n \geq 0.
\]

By using the probability weights of \( N \), one can express the compound distribution of \( S_N \) as

\[
F_N(x) = \sum_{k=0}^{\infty} p_k F^k(x).
\]

The asymptotic behaviour of the tail of a compound distribution obviously depends on the masses \( (p_n)_{n \in \mathbb{N}} \) and the asymptotic behaviour of the underlying distribution \( F \) of the random sum \( S_N \). Under suitable conditions on the masses \( (p_n)_{n \in \mathbb{N}} \), or the underlying distribution \( F \), we can obtain conditions under which the fact that distribution \( F \) belongs to a distribution class \( S, \mathcal{O}S, \mathcal{A}(E), \mathcal{E} \subseteq \mathcal{G} \), or \( \mathcal{J} \) implies that the compound distribution \( F_N \) belongs to the same class. The reverse direction \( F \in \mathcal{C} \iff F_N \in \mathcal{C} \), where \( \mathcal{C} \) is a distribution class, and the asymptotic behaviour of tails of distributions \( F \) and \( F_N \) are also of interest. These properties are crucial for applications, which will be presented in the next chapter.
New large claim classes

We give some classical results for the classes \( R \) and \( S \) in Theorems 97 and 98, which we will compare with our results for the classes \( OS \) and \( A(E) \), \( E \subseteq G \). First, we consider the case of regularly varying compound distributions. The formulation follows [15].

**Theorem 97. (Random sums in the class \( R \))**

With the above notation, let \( G \) be the distribution of \( N \). Then the following assertions hold:

a) Assume \( F \in R \), \( \mathbb{E}[N] < \infty \) and \( G(x) = o(F(x)) \). Then
\[
F_N(x) \sim \mathbb{E}[N] F(x).
\]

b) Assume distribution \( G \) is regularly varying with index \( \alpha \geq 0 \). If \( \alpha = 1 \), assume that \( \mathbb{E}[N] < \infty \). Moreover, let \( \mu := \mathbb{E}[X_1] < \infty \) and \( F(x) = o(G(x)) \). Then
\[
F_N(x) \sim \mathbb{P}(N > \frac{x}{\mu}) \sim (\mu)^\alpha \mathbb{P}(N > x).
\]

c) Assume distribution \( F_N \) is regularly varying with index \( \alpha \geq 0 \). Suppose \( \mu := \mathbb{E}[X] < \infty \) and \( F(x) = o(F_N(x)) \). In the case \( \alpha = 1 \) and \( \mathbb{E}[S_K] = \infty \), assume that \( xF(x) = o(F_N(x)) \). Then \( N \) is regularly varying with index \( \alpha \) and
\[
F_N(x) \sim (\mu)^\alpha G(x).
\]

For a proof of Theorem 97, we refer the reader to [15, Propositions 4.1, 4.3, 4.9]. Next, we turn to subexponential compound distributions.

**Theorem 98. (Random sums in the class \( S \))**

With the above notation, suppose that \( \sum_{k=0}^{\infty} p_k (1 + \varepsilon)^k < \infty \) for some \( \varepsilon > 0 \). Then the following assertions hold:

a) If \( F \in S \), then \( F_N \in S \) and
\[
F_N(x) \sim \mathbb{E}[N] F(x).
\]

b) Conversely, if (4.66) holds and there exists \( l \geq 2 \) such that \( p_l > 0 \), then \( F \in S \).

c) If \( p_n = (1-\alpha)\alpha^n, \alpha < 1 \), for all \( n \geq 0 \), then the following assertions are equivalent:

i) \( F \in S \);
ii) \( F_N \in S \);
iii) \( \lim_{x \to \infty} \frac{F_N(x)}{F(x)} = \alpha/(1 - \alpha) \).

We refer the reader to [13, Embrechts et al., Theorem A3.20 c)] for a proof and to [24] for results on random sums in the case of heavy-tailed distribution classes \( L, L \cap D \) and \( D \).
4 New large claim classes

Remark 99. Our overview is far from being complete and there are various extensions of Theorem 97 in cases where \( N \) may have a heavier tail, see [9, Theorem 1].

Under a suitable decay condition on the masses \((p_n)_{n \in \mathbb{N}}\), the following result for the class \( \mathcal{OS} \) was proven by Shimura and Watanabe in [38, Proposition 3.1].

**Theorem 100.** *(Random sums in the class \( \mathcal{OS} \))*

With the above notation, let \( p_0 + p_1 < 1 \) and \( \sup \{ x \geq 1 : \sum_{k=1}^{\infty} p_k x^k < \infty \} = \infty \).

Moreover, let \( F \in \mathcal{F} \).

a) The following are equivalent:
   i) \( F \in \mathcal{OS} \);
   ii) \( \overline{F}_N \Leftrightarrow \overline{F} \).

b) If \( \sup \{ k : p_k > 0 \} = \infty \), then the following are equivalent:
   i) \( F_N \in \mathcal{OS} \);
   ii) \( F_{n*} \in \mathcal{OS} \) for some \( n \geq 1 \);
   iii) \( F_{n*} \Leftrightarrow \overline{F}_N \) for some \( n \geq 1 \).

We can refine the conditions on the probabilities \((p_n)_{n \in \mathbb{N}}\) and the underlying distribution \( F \) of the random sum \( S_N \) by considering the directions \( F \in \mathcal{OS} \Rightarrow F_N \in \mathcal{OS} \) and \( F \in \mathcal{OS} \Leftrightarrow F_N \in \mathcal{OS} \) separately.

First, we consider the direction \( F \in \mathcal{OS} \Rightarrow F_N \in \mathcal{OS} \) and then transfer the obtained result to the class \( \mathcal{A}(\mathcal{E}), \mathcal{E} \subseteq \mathcal{G} \). Note that the condition on the masses \((p_n)_{n \in \mathbb{N}}\) is weaker than in the Theorem 100.

**Proposition 101.** With the above notation, suppose \( F \in \mathcal{OS} \cap \mathcal{F} \) and \( \sum_{k=1}^{\infty} p_k (c_F + \varepsilon - 1)^k < \infty \) for some \( \varepsilon > 0 \), then the following assertions hold:

a) \( \overline{F}_N \Leftrightarrow \overline{F} \) and hence \( F_N \in \mathcal{OS} \).

b) If additionally \( F \in \mathcal{A}(\mathcal{E}), \mathcal{E} \subseteq \mathcal{G} \), then \( \overline{F}_N \Leftrightarrow \overline{F} \) and hence \( F_N \in \mathcal{A}(\mathcal{E}) \).

**Remark 102.** a) Note that one cannot infer \( F_N \in \mathcal{J} \) or \( \overline{F} \Leftrightarrow \overline{F}_N \) from \( F \in \mathcal{J} \) without additional conditions on \( N \). This is true even if \( N \) is a geometric random variable, say, with a parameter \( p \in (0, 1) \). Indeed, a counterexample is given by the distribution \( F \in \mathcal{J} \) from Example 81 with \( g(x) = \frac{1}{2} \frac{1}{1+x^2} \) and geometric \( N \) that has a parameter \( p \) close enough to 1. To see this, consider a sequence of i.i.d. random variables \( X_1, X_2, X_3, \ldots \) with density \( f(x) = C \frac{e^{-x}}{1+x^2} \). Let \( \alpha > 0 \) be such that \( \alpha \mathbb{E} X_1 > 1 \). For all \( m \in \mathbb{N} \) we have

\[
\mathbb{P}(X_1 + \cdots + X_N > m) \geq \mathbb{P}(N \geq |\alpha m|) \mathbb{P}(X_1 + \cdots + X_{|\alpha m|} > m)
\]
4 New large claim classes

and

\[ P(X_1 > m) = \int_m^\infty \frac{Ce^{-x}}{1 + x^2} \, dx \leq Ce^{-m}. \]

If \( p \) is close enough to 1 we obtain by our choice of \( \alpha \) and the law of large numbers,

\[ \frac{P(X_1 + \cdots + X_N > m)}{P(X_1 > m)} \geq \frac{e^{pm}}{C} P(X_1 + \cdots + X_{\lceil am \rceil} > m) \to \infty, \]

for \( m \to \infty \), so \( F \not\sim F_N \) does not hold. It follows from part b) of Proposition 101 above that \( F_N \notin J \).

b) The assertion \( F_N \in J \Rightarrow F \in J \) also does not hold without additional conditions on \( F_N \). For example, let be \( N \) a nonnegative integer-value random variable with distribution \( G \). Assume \( G \in R \subset J \) and \( \{X_i, i \geq 1\} \) are i.i.d. random variables with common distribution \( F \in F \), \( \mathbb{E}[X_1] = 1 \) and \( F \notin J \). Then we obtain from part b) of Theorem 97:

\[ P(S_N > x) \sim P(N > x). \]

From the closure of \( J \) under tail-equivalence we deduce that \( F_N \in J \). Thus, we have \( F_N \in J \) and \( F \notin J \).

Now, we turn to the closure of random sums under convolution root, i.e. we consider the direction \( F_N \in C \Rightarrow F \in C \), where \( C \) is a distribution class. The condition on the masses \( (p_n)_{n \in \mathbb{N}} \) is again weaker than in Theorem 100.

**Proposition 103.** Let \( F \in F \). With the above notation, suppose \( F_N \in OS \) and

\[ \sum_{k=1}^\infty p_k(c_{F_N} + \varepsilon - 1)^k < \infty \text{ for some } \varepsilon > 0, \]

then the following assertions hold:

a) \( F^{m*} \asymp F_N \) for some \( m \geq 1 \) and hence

\[ F^{m*} \in OS. \]

b) If additionally \( F_N \in A(\mathcal{E}), \mathcal{E} \subseteq G \), then \( F^{m*} \asymp F_N \) for some \( m \geq 1 \) and hence

\[ F^{m*} \in A(\mathcal{E}). \]

c) If additionally \( F_N \in J \), then \( F \asymp F_N \) and hence

\[ F \in J. \]

A related result for the random sums in \( J \) and \( OS \) can be obtained under a condition on the primary distribution.

**Proposition 104.** Let \( F \in F \). With the above notation, suppose \( F_N \in OS \) and

\[ \liminf_{n \to \infty} \frac{P(N_1 + N_2 > n)}{P(N_1 > n)} > \limsup_{x \to \infty} \frac{F_N^{2*}(x)}{F_N(x)} = \limsup_{x \to \infty} \frac{P(S_{N_1} + S_{N_2} > x)}{P(S_{N_1} > x)}, \]

79
then the following assertions hold:

a) $F_{m*} \asymp F_N$ for some $m \geq 1$ and hence
\[ F_{m*} \in OS. \]

b) If $F_N \in A(E)$, then $F_{m*} \asymp F_N$ for some $m \geq 1$ and hence
\[ F_{m*} \in A(E). \]

c) If $F_N \in J$, then $F \asymp F_N$ and hence
\[ F \in J. \]

In [38, Remark 1.6] it is pointed out that $\lim_{n \to \infty} \frac{p_{n+1}}{p_n} = 0$ implies $\lim_{n \to \infty} \frac{(p*p)_n}{p_n} = \infty$. From there, we also recall some examples for $N$.

**Example 105.** The following distributions satisfy the condition $\liminf_{n \to \infty} \frac{(p*p)_n}{p_n} = \infty$:

1) Poisson distribution:
\[ p_n = \frac{c^n}{n!} e^{-c}, c > 0. \]

2) Geometric distribution:
\[ p_n = (1 - \lambda)^n \lambda, \lambda \in (0, 1). \]

3) Negative Binomial distribution:
\[ p_n = \binom{r + n - 1}{r - 1} p^n (1 - p)^r, p \in (0, 1), r > 0. \]
4 New large claim classes

4.6.1 Proofs

We begin with the direction $F \in \mathcal{C} \Rightarrow F_N \in \mathcal{C}$, where $\mathcal{C}$ is a distribution class, Proposition 101.

Proof. a) Suppose $F \in \mathcal{OS} \cap \mathcal{F}$ and $\sum_{k=1}^{\infty} p_k(c_F + \varepsilon - 1)^k < \infty$ for some $\varepsilon > 0$. From Lemma 30 (Kesten's bound) and $F \in \mathcal{J} \subset \mathcal{OS}$ we obtain for some suitable $c_1 \in (0, \infty)$ and all $x \geq 0$,

$$F_N(x) = \sum_{k=1}^{\infty} p_k F_k(x) \leq \sum_{k=1}^{\infty} p_k c_1 (c_F + \varepsilon - 1)^k F(x) = F(x) \sum_{k=1}^{\infty} p_k c_1 (c_F + \varepsilon - 1)^k.$$  

Hence, we see that $\limsup_{x \to \infty} F_N(x)/F(x) < \infty$. For the lower bound pick some $k \geq 1$ with $p_k > 0$. Then, for all $x \geq 0$,

$$F_N(x) \geq p_k \mathbb{P}(S_k > x) \geq p_k F(x).$$

We obtain $F_N \asymp F$ and therefore $F_N \in \mathcal{J}$.

b) We obtain $F_N \in \mathcal{A}(\mathcal{E})$ from part a), $F_N \asymp F$ and the closure of $\mathcal{A}(\mathcal{E})$ under weak tail-equivalence, Proposition 41.

Next, we provide the proof of the results on the direction $F_N \in \mathcal{C} \Rightarrow F \in \mathcal{C}$, where $\mathcal{C}$ is again a distribution class, Proposition 103.

Proof. a) Suppose $F_N \in \mathcal{OS}$, $F \in \mathcal{F}$ and that $\sum_{k=1}^{\infty} p_k(c_{F_N} + \varepsilon - 1)^k < \infty$ for some $\varepsilon > 0$. Again, we need to prove that there exists $m \in \mathbb{N}$ such that $F_N \asymp F_m$. To this end, by means of contradiction, suppose that for every integer $n \geq 2$,

$$\liminf_{x \to \infty} \frac{F_m(x)}{F_N(x)} = 0.$$  

Our proof then splits into two cases.

Case 1: $p_0 = 0$. For every $n \geq 1$, we choose an unbounded and strictly increasing sequence $(x^k_n)_{k \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$

$$\lim_{m \to \infty} \frac{F_m(x^k_n)}{F_N(x^k_n)} = 0 \quad \text{and in particular} \quad \lim_{m \to \infty} \frac{F_m(x^k_n)}{F_N(x^k_n)} = 0.$$  

From Lemma 30 (Kesten’s bound) and $p_0 = 0$ we deduce that, for some suitable $c_2 \in (0, \infty)$, for all $n, m \in \mathbb{N}$

$$\frac{F_m(x^k_n)}{F_N(x^k_n)} \leq \frac{F_m(x^k_n)}{F_N(x^k_n)} \leq c_2(c_{F_N} + \varepsilon - 1)^n.$$
4 New large claim classes

Since, by assumption, the right-hand side is summable in $n$, we can use the dominated convergence theorem to arrive at the desired contradiction:

\[
1 = \lim_{m \to \infty} \frac{F_{N}(x_m^m)}{F_{N}(x_m^m)} = \lim_{m \to \infty} \sum_{k=1}^{\infty} p_k \frac{F_{N}(x_m^m)}{F_{N}(x_m^m)} = \sum_{k=1}^{\infty} p_k \lim_{m \to \infty} \frac{F_{k}(x_m^m)}{F_{N}(x_m^m)} \leq \sum_{k=1}^{\infty} p_k \lim_{m \to \infty} \frac{F_{N}(x_m^m)}{F_{N}(x_m^m)} = 0.
\]

**Case 2:** $p_0 > 0$. This can be reduced to Case 1 by switching to the re-weighted random variable $\hat{N}$ with probabilities

\[
\hat{p}_n := \mathbb{P}(\hat{N} = n) := \frac{p_n}{1 - p_0},
\]

for $n > 0$ and $\hat{p}_0 = \mathbb{P}(\hat{N} = 0) := 0$. Thanks to Case 1 we have that $F_{\hat{N}} \in \mathcal{J}$. Further, we observe that

\[
\lim_{x \to \infty} \frac{F_{N}(x)}{F_{\hat{N}}(x)} = \lim_{x \to \infty} \frac{\sum_{n=0}^{\infty} p_n F_{n}(x)}{\sum_{n=0}^{\infty} \hat{p}_n F_{n}(x)} = \frac{1}{1 - p_0}.
\]

From the closure of the class $\mathcal{OS}$ under weak tail-equivalence and $F_{N} \asymp F_{\hat{N}}$ we conclude that $F_{N} \in \mathcal{OS}$.

b) We obtain $F_{m}^{*} \in \mathcal{A}(\mathcal{E})$ from part a), $F_{N} \asymp F_{m}^{*}$ and the closure of $\mathcal{A}(\mathcal{E})$ under weak tail-equivalence, Proposition 41.

c) The assertion follows from part b), the closure of $\mathcal{J}$ under weak tail-equivalence and root convolution, Propositions 84 and 88.

Next, we prove Proposition 104 using the arguments of the proof of Theorem 1.5 of Watanabe [38].

**Proof.** a) To infer that $F_{N} \asymp F_{m}^{*}$ for some $m \in \mathbb{N}$ we again argue by contradiction. So suppose that for every integer $m \geq 2$

\[
\liminf_{x \to \infty} \frac{F_{m}^{*}(x)}{F_{N}(x)} = 0.
\]

Let $c_{F_{N}} := \limsup_{x \to \infty} \frac{\mathbb{P}(S_{N_{1}} + S_{N_{2}} > x)}{\mathbb{P}(N_{1} > \delta)}$. From $F_{N} \in \mathcal{OS}$ we know that $c_{F_{N}} < \infty$ and from our assumption $\liminf_{n \to \infty} \frac{\mathbb{P}(N_{1} + N_{2} > n)}{\mathbb{P}(N_{1} > n)} > c_{F_{N}}$ we infer that there exists a $\delta > 0$ and an integer $m_0 = m_0(\delta)$ such that, for every $k \geq m_0 + 1$,

\[
\frac{\mathbb{P}(N_{1} + N_{2} > k)}{\mathbb{P}(N_{1} > k)} > c_{F_{N}} + \delta.
\]
4 New large claim classes

Let \((x_n)_{n \in \mathbb{N}}\) be a strictly increasing sequence with \(\lim_{n \to \infty} x_n = \infty\) such that

\[
\lim_{n \to \infty} \frac{F^{m_0}\ast(x_n)}{F_N(x_n)} = 0. \tag{4.68}
\]

Since \(F^{m_0}\ast(x) \geq F^k\ast(x)\) for \(1 \leq k \leq m_0\), we have

\[
\lim_{n \to \infty} \frac{F^k\ast(x_n)}{F_N(x_n)} = 0
\]

for \(1 \leq k \leq m_0\). As in [38], define \(I_j(n)\) and \(J_j(n)\) for \(j = 1, 2\) as

\[
J_1(n) = \sum_{k=0}^{m_0} (p \ast p)_k F^k\ast(x_n), \quad I_1(n) = \sum_{k=0}^{m_0} p_k F^k\ast(x_n),
\]

\[
J_2(n) = \sum_{k=m_0+1}^{\infty} (p \ast p)_k F^k\ast(x_n), \quad I_2(n) = \sum_{k=m_0+1}^{\infty} p_k F^k\ast(x_n).
\]

We see from (4.68) that

\[
\lim_{n \to \infty} \frac{I_1(n)}{F_N(x_n)} = \lim_{n \to \infty} \frac{J_1(n)}{F_N(x_n)} = 0, \tag{4.69}
\]

and since \(F^2_N = \sum_{k=0}^{\infty} (p \ast p)_k F^k\ast\), (4.67) and (4.69) give

\[
c_{F_N} \geq \limsup_{n \to \infty} \frac{F^2_N(x_n)}{F_N(x_n)} = \limsup_{n \to \infty} \frac{(J_1(n) + J_2(n))/F_N(x_n)}{(I_1(n) + I_2(n))/F_N(x_n)} = \limsup_{n \to \infty} \frac{J_2(n)}{I_2(n)}.
\]

To arrive at the desired contradiction, define \(h_{m_0+1}(x_n) := F^{(m_0+1)\ast}(x_n)\) and

\[
h_j(x_n) := \frac{F^j\ast(x_n)}{F^{(j-1)\ast}(x_n)}\]

for \(j > m_0 + 1\). We obtain

\[
\limsup_{n \to \infty} \frac{J_2(n)}{I_2(n)} = \limsup_{n \to \infty} \frac{\sum_{k=m_0+1}^{\infty} (p \ast p)_k \sum_{j=m_0+1}^{\infty} h_j(x_n)}{\sum_{k=m_0+1}^{\infty} p_k \sum_{j=m_0+1}^{\infty} h_j(x_n)}
\]

\[
= \limsup_{n \to \infty} \frac{\sum_{j=m_0+1}^{\infty} h_j(x_n) \sum_{k=j}^{\infty} \mathbb{P}(N_1 + N_2 = k)}{\sum_{j=m_0+1}^{\infty} h_j(x_n) \sum_{k=j}^{\infty} \mathbb{P}(N_1 = k)}
\]

\[
= \limsup_{n \to \infty} \frac{\sum_{j=m_0+1}^{\infty} h_j(x_n) \mathbb{P}(N_1 + N_2 > j - 1)}{\sum_{j=m_0+1}^{\infty} h_j(x_n) \mathbb{P}(N_1 > j - 1)}
\]

\[
> c_{F_N} + \delta.
\]

83
4 New large claim classes

This is a contradiction. Since $F_{m^*(x)} \leq F_N(x)^{\frac{1}{p_m}}$ with $p_m > 0$ for sufficiently large integers $m$, it follows that $F_N \asymp F_{m^*}$.

b) The assertion follows from part a) and the closure of $A(\mathcal{E})$ under weak tail-equivalence.

c) The assertion follows from part a), the closure of $\mathcal{J}$ under weak tail-equivalence and root convolution.

\[ \Box \]
In the previous chapter, we proved (as well as stated) various closure and structural properties for our new large claim classes. These properties shall, in subsequent chapters, be used in order to obtain results for certain risk models and other applications, see Theorems 119 and 127. We also give an overview of important asymptotic results for the Cramér-Lundberg model and for the random walks.

5.1 Sparre-Andersen and Cramér-Lundberg Model

Let $X_1, X_2, \ldots$ be a family of strictly positive i.i.d. random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with distribution $F$ and finite expectation $\mu > 0$. Let $N = \{N(t), t \geq 0\}$ be a renewal process with i.i.d. strictly positive waiting times $W_1, W_2, \ldots$. We assume that the $W_i$ have distribution $F_W$, are independent of the $X_i$, and with a finite expectation $1/\lambda$, for some $\lambda > 0$. We then define the total claim amount process as

$$S(t) := \sum_{i=1}^{N(t)} X_i, \quad t \geq 0.$$ 

Let $T_n := W_1 + \cdots + W_n, n \geq 1$ be the arrival times of the claims, where we set $T_0 := 0$. By $c > 0$ we denote the premium rate and by $u \geq 0$ the initial capital. Finally, we define the risk process, for $u \geq 0$, by

$$Z(t) := u + ct - S(t), \quad t \geq 0.$$ 

If the above claim arrival process $N$ is a Poisson process, we are in the classical Cramér-Lundberg model, otherwise, we are in the more general Sparre-Andersen
5 Applications

model. \( Z(t) \) is said to satisfy the net profit condition (NPC), if

\[
c > \lambda \mu. \tag{5.1}
\]

By

\[
\delta := \inf\{k \geq 1 : Z(T_k) < 0\}
\]

we denote the ruin time (with the usual convention that \( \inf\{\emptyset\} = \infty \)), and by

\[
\Psi(u) := \mathbb{P}\{\delta < \infty | Z(0) = u\}
\]

we denote the ruin probability. This quantity is the central object of study in ruin theory. We will be interested in obtaining asymptotic results for \( \Psi(u) \) for large \( u \).

It turns out that the asymptotic behaviour of the ruin probability \( \Psi \) in the classical Cramér-Lundberg heavily depends on the second tail distribution of \( F \).

**Definition 106.** (Second tail distribution)

Let \( F \in \mathcal{F} \) with finite expectation \( \mu \), then the second tail distribution is given by

\[
F^s(x) = \begin{cases} 
0, & x \leq 0, \\
\frac{1}{\mu} \int_0^x F(y)dy, & x > 0.
\end{cases}
\]

By assuming that the second tail distribution \( F^s \) is subexponential we can state an important result about the asymptotic behaviour of the ruin probability \( \Psi \).

**Theorem 107.** (Asymptotic behaviour of \( \Psi \) in the Cramér-Lundberg model)

With above notation, consider the Cramér-Lundberg model. Suppose the NPC condition is satisfied. Then the following assertions are equivalent:

i) \( F^s \in \mathcal{S} \);

ii) \( (1 - \Psi) \in \mathcal{S} \);

iii) \( \lim_{x \to \infty} \Psi(x) / F^s(x) = \lambda \mu / (c - \lambda \mu) \).

The key of the proof of Theorem 107 is the fact that we can rewrite the ruin probability in the Cramér-Lundberg model as the tail of a compound geometric distribution. The following equation, which is often referred in literature to as the Pollaczek-Khinchine formula for the ruin probability, holds in the Cramér-Lundberg model under the assumption of the NPC condition (5.1):

\[
\Psi(x) = \frac{\rho}{1 + \rho} \sum_{n=0}^{\infty} (1 + \rho)^{-n} \frac{F^s(x)^n}{F^s(x)}, \tag{5.2}
\]

where \( \rho := \frac{c}{\lambda \mu} - 1 \). For a proof of (5.2) we refer to [27, Proposition 4.2.12].

By dividing (5.2) on both sides with \( F^s(x) \) and using Kesten’s bound (Proposition 12) we can interchange limits and sums. Thus, we obtain

\[
\lim_{x \to \infty} \frac{\Psi(x)}{F^s(x)} = \lim_{x \to \infty} \frac{\rho}{1 + \rho} \sum_{n=0}^{\infty} (1 + \rho)^{-n} \frac{(F^s(x)^n}{F^s(x)}
\]

\[
= \frac{\rho}{1 + \rho} \sum_{n=0}^{\infty} n(1 + \rho)^{-n}
\]

\[
= \frac{1}{\rho}.
\]

86
5 Applications

Subsequently, we obtain the equivalence of i)-iii) in Theorem 107 from the properties of the geometric compound distribution in $S$, see Theorem 98.

Equipped with the properties of Chapter 4, we can also state results about the asymptotic behaviour of the ruin probability in a large claim case. By using the Kesten’s bound and the closure under weak tail-equivalence, we can apply the same idea of the proof of Theorem 107 to the large claim classes. We state the results for the class $J$ in the following theorem.

**Theorem 108. (Asymptotic behaviour of $\Psi$ in the large claim case)**

With above notation, consider the Cramér-Lundberg model. Suppose the NPC condition is satisfied.

a) If $(1-\Psi) \in J$, then $\Psi(x) \asymp \mathcal{F}(x)$ and hence $\mathcal{F}(x) \in J$.

b) Recall the notation $c_{F^s} := \limsup_{x \to \infty} \left( \frac{F^s(x)}{x} \right)^2$. If $(c_{F^s} + \varepsilon - 1)/(1 + \rho) < 1$ for some $\varepsilon > 0$, then $\Psi(x) \asymp \mathcal{F}(x)$ and hence $(1-\Psi) \in J$.

**Proof.** We start with part a). Suppose $(1-\Psi) \in J$. By (5.2) and part c) of Proposition 104 we obtain $\Psi(x) \asymp \mathcal{F}(x)$. Thus, by the tail-closure property of the class $J$ (Proposition 84) we know that $\mathcal{F}(x) \in J$.

Now, we prove b). Suppose $(c_{F^s} + \varepsilon - 1)/(1 + \rho) < 1$ for some $\varepsilon > 0$, then again by (5.2) and Proposition 101 b) with $\mathcal{E} = \mathcal{G}$, we obtain $(1-\Psi) \in J$.

Finally, we reformulate the classical ruin problem into a question about the maximum of an associated random walk with negative drift. We follow the exposition of [42]. Let

$$
\bar{X}_k := X_k - cW_k = -(Z(T_k) - Z(T_{k-1})), \quad k \geq 1.
$$

Note that the $\bar{X}_k$ are i.i.d. with values in $\mathbb{R}$. We denote their distribution by $F_{\bar{X}}$. We define the random walk $\{\bar{S}_n : n \geq 0\}$ by

$$
\bar{S}_n := \bar{X}_1 + \cdots + \bar{X}_n, \quad n \geq 1, \quad \bar{S}_0 := 0, \quad n \geq 1.
$$

By the strong law of large numbers, we have $\Psi(u) \equiv 1$ if $\mathbb{E}[\bar{X}_1] \geq 0$ (unless $\bar{X}_1 \equiv 0$). Otherwise, we say that the net profit condition holds, and we denote $\alpha := \mathbb{E}[\bar{X}_1] < 0$. Under the net profit condition, this is a discrete-time random walk with negative drift. For the ruin probability, we obtain

$$
\Psi(u) = \mathbb{P} \left\{ \sup_{n \geq 0} \bar{S}_n > u \right\}, \quad u \geq 0.
$$

Hence, we have expressed the probability of ruin in the Cramér-Lundberg model in terms of the distribution of the supremum of a random walk with negative drift, which is the object of investigation in the next section.
5 Applications

5.2 Supremum of the Random Walk

5.2.1 Notation of the Random Walk

Let \( \{X_i, i \geq 1\} \) be independent, identically distributed random variables on \((-\infty, \infty)\) with a common distribution \(F\) and finite expectation \(a < 0\). Define the random walk \(\{S_n : n \geq 0\}\) by \(S_0 = 0, S_n = \sum_{i=1}^{n} X_i, n \geq 1\). Let

\[
M := \sup_{n \geq 0} S_n
\]

be the supremum of the random walk with distribution \(F_M\). The asymptotics for the supremum of the random walk is an important object in studies of a random walk and has well known applications. The tail of \(F_M\) is known as the ruin probability in classical ruin theory and another application comes from queueing systems: the distribution of \(M\) coincides with the stationary distribution of the queue-length in simple queueing systems.

In the Cramér-Lundberg model the Pollaczek–Khinchine formula allows us to rewrite the risk probability \(\Psi\) as a compound distribution with the integrated tail distribution \(F_I\) as the underlying distribution, see (5.2). Hence, we are interested in obtaining a representation formula for the supremum of the random walk \(F_M\) and asymptotic results for \(F_M(u)\) for large \(u\).

It turns out that we also can rewrite the supremum of the random walk \(F_M\) as a geometric compound distribution. In contrast to the Pollaczek–Khinchine formula we have to use another underlying distribution. For that reason we will need some additional notation.

It follows from the Strong Law of Large Numbers that \(S_n \to -\infty\) as \(x \to \infty\), with probability 1, then the supremum of the random walk \(M\) is finite with a probability 1. Denote the first strictly ascending ladder epoch by

\[
\tau(1) = \inf\{n \geq 1 : S_n > 0\} \leq \infty,
\]

with the usual convention \(\inf\{\emptyset\} = \infty\). The random variable \(S_\tau\) is called the first strictly ascending ladder height, here with the convention \(S_\infty = -\infty\). Since \(M\) is almost surely finite the random variables \(\tau(1)\) and \(S_\tau(1)\) are defective.

The following (subsequent) strictly ascending ladder epochs are defined by induction. In the case the \(k\)-th strictly ascending ladder epoch \(\tau(k)\) is finite, then we define the \((k+1)\)-th strictly ascending ladder epoch \(\tau(k+1)\) by

\[
\tau(k+1) := \begin{cases} 
\inf\{k > \tau(k) : S_k > S_{\tau(k)}\}, & \tau_k < \infty, \\
\infty, & \tau_k = \infty.
\end{cases}
\]

The random variable \(S_{\tau(n)}\) is called the \(n\)-th strictly ascending ladder height. Given \(\tau(k) < \infty\) and \(\tau(k+1) = \infty\), then it is easy to see that the \(k\)-th ladder height equals the supremum of the random walk, i.e.

\[
M = \sum_{j=1}^{k-1} S_{\tau(j+1)} - S_{\tau(j)} = S_{\tau(k)}. \quad (5.3)
\]
5 Applications

Given \( \tau(1) < \infty \), the random variables \( X_{\tau(1)+i}, i = 1, 2, \ldots \) are mutually independent and do not depend on random variables \( X_1, \ldots, X_{\tau(1)} \). Again, given \( \tau(n) < \infty \), the random variables \( X_{\tau(n)+i}, i = 1, 2, \ldots \) are mutually independent and do not depend on random variables \( X_1, \ldots, X_{\tau(n)} \) and stopping times \( \tau(1), \ldots, \tau(n) \). Observe that the random variables \( (\tau(1), S_{\tau(1)}), (\tau(k+1) - \tau(k), S_{\tau(k+1)} - S_{\tau(k)}), k = 1, 2, \ldots, n-1 \), are mutually independent and identically distributed, see [20, Section 2.9, page 70] for more details on this i.i.d property. Let

\[
\overline{G}(x) := \mathbb{P}(S_{\tau(1)} > x | \tau(1) < \infty)
\]

and \( \{X_k, k \geq 1\} \) be a sequence of i.i.d. random variables such that

\[
\mathbb{P}(X_k \leq x) := \mathbb{P}(S_{\tau(k+1)} - S_{\tau(k)} \leq x | \tau(k+1) < \infty) = \mathbb{P}(S_{\tau(1)} \leq x | \tau(1) < \infty) = \overline{G}(x),
\]

for \( k \geq 1 \). We call \( G \) the conditional ascending ladder height distribution. Thus, given \( \tau(n) < \infty \), we obtain

\[
S_{\tau(n)} \overset{d}{=} \overline{X}_1 + \ldots + \overline{X}_n. \tag{5.4}
\]

Let

\[
p := \mathbb{P}(\tau(1) = \infty) = \mathbb{P}(M = 0),
\]

then the probability of the event \( \{\tau(1) < \infty\} \) is \((1 - p)\). Given \( \tau(1) < \infty \), it is easy to see that the probability of the event \( \{\tau(2) < \infty\} \) is again \((1 - p)\). By repeating the same step \( n \) times we obtain:

\[
\mathbb{P}(\tau(n) < \infty | \tau(n-1) < \infty) = 1 - p.
\]

Hence, the probability for the event \( \{\tau(n) < \infty\} \) is \((1 - p)^n\) and we can define a random variable

\[
u := \min\{n \geq 1 : \tau(n) = \infty\},
\]

which obviously has a geometric distribution with a parameter \( p \).

5.2.2 Representation formula for the supremum of the Random Walk

By combining (5.3), (5.4) and the definition of \( \nu \), it is easy to see that the random sum \( \overline{S}_{\nu-1} := \overline{X}_1 + \ldots + \overline{X}_{\nu-1} \), with the usual convention \( \overline{S}_0 = 0 \), coincides in distribution with the supremum \( M \) of the random walk:

\[
M \overset{d}{=} \overline{X}_1 + \ldots + \overline{X}_{\nu-1} = \overline{S}_{\nu-1}.
\]

Since we have

\[
\mathbb{P}(\nu = 1) = \mathbb{P}(\tau(1) = \infty) = p,
\]

89
we obtain the following representation
\[ P(M \leq x) = P(M \leq x, \nu = 1) + P(M \leq x, \nu > 1) \]
\[ = P(S_0 \leq x, \nu = 1) + P(M \leq x, \nu > 1) \]
\[ = P(M = 1) + P(M > 1)P(X_1 + \ldots + X_{\nu-1} \leq x | \nu > 1) \]
\[ = p + (1 - p)P(X_1 + \ldots + X_{\nu-1} \leq x) \]
\[ = p + (1 - p) \sum_{i=1}^{\infty} P(\nu = i)P(X_1 + \ldots + X_i \leq x) \]
\[ = p + \sum_{i=1}^{\infty} (1 - p)^{i-1}G_{i*}(x) \]
\[ = \sum_{i=0}^{\infty} (1 - p)^{i}G_{i*}(x). \]

Finally, we can state for the tail of the distribution of \( M \):
\[ F_M(x) = \frac{P(M > x)}{G(x)} = \frac{\sum_{i=0}^{\infty} (1 - p)^{i}G_{i*}(x)}{G(x)}. \]

5.2.3 Asymptotics for the supremum of the Random Walk

By using the Pollaczek–Khinchine formula and the tail closure property of the heavy-tailed class \( S \), we can state results about the asymptotics of the ruin probability in the Cramér-Lundberg model. Hence, by using the representation of \( F_M \) as a geometric random sum and the closure under tail-equivalence, we can establish similar results for the supremum of the random walk.

**Theorem 109.** (Asymptotic behaviour of \( F_M \) in the subexponential case)

With the notation from Section 5.2.1, assume that \( a < 0 \). Then the following assertions are equivalent:

i) \( G \in S \);

ii) \( F_M \in S \);

iii) \( \lim_{x \to \infty} \frac{F_M(x)}{G(x)} = \frac{(1 - p)}{p} \).

**Proof.** Since we can rewrite the tail of \( F_M \) as the tail of a compound distribution
\[ F_M(x) = \frac{\sum_{i=0}^{\infty} (1 - p)^{i}G_{i*}(x)}{G(x)} \]
and by observing that \( 0 < (1 - p) < 1 \), the equivalence of the assertions i) – iii) follows from part b) of Theorem 98.

Again, by using the representation (5.5) of the supremum of the random walk, the closure of \( A(\mathcal{E}) \), \( \mathcal{E} \subseteq \mathcal{G} \), under weak tail-equivalence and our results on random sums, see Proposition 101, we can establish similar results for the supremum of the random walk.
Theorem 110. (Asymptotic behaviour of $F_M$ in the large claim case)

With the notation from Section 5.2.1, assume that $a < 0$. Let $E \subseteq G$ be not empty.

a) If $p(c_G + \varepsilon - 1) < 1$ for some $\varepsilon > 0$ and $G \in A(E) \cap OS$, then $G \asymp F_M$ and hence

$$F_M \in A(E) \cap OS.$$ 

b) If $F_M \in A(E) \cap OS$, then $G^{m*} \asymp F_M$ for some $m \geq 1$ and hence

$$F^{m*} \in A(E) \cap OS.$$ 

Proof. We start with a). Suppose $G \in A(E) \cap OS$. By (5.5) and Proposition 104 c) we obtain $G \asymp F_M$. Thus, by the closure of the class $A(E)$ (Proposition 41) and the class $OS$ (Proposition 26) under weak tail-equivalence, we know that $F_M \in A(E) \cap OS$.

Next, we prove b). Suppose $(c_{F_I} + \varepsilon - 1)/(1 + \rho) < 1$ for some $\varepsilon > 0$, then again by (5.5) and Proposition 101 b) with $E = G$, we obtain $\Psi \in J$. \hfill \square

5.2.4 Further asymptotic results of the Random Walk

We can obtain another approach to describe the event $\{M > x\}$ if we consider the asymptotic behaviour of the integrated tail of the distribution $F$. First, we give the heuristic idea behind the connection between the tail of the supremum of the random walk $M$ and the integrated tail of the distribution $F$.

With notation from Section 5.2.1, consider the events $A_n := \{S_n \approx an, X_{n+1} > -an + x\}$, $n \geq 1$, which describe the random walk behaviour as typical until step $n$, and subsequently the random walk makes an impact such that $S_{n+1}$ crosses the level $x$. The crossing of level $x$ can happen in every step and the events $A_n$, $n \geq 1$, tend to be the mutually exclusive. Thus, we sum up in $n$.

$$\mathbb{P}(M > x) \approx \mathbb{P}\left( \bigcup_{n=1}^{\infty} A_n \right) \quad (5.6)$$

$$\approx \sum_{n=0}^{\infty} \mathbb{P}(S_n \approx an, X_{n+1} > -an + x) \quad (5.7)$$

$$\approx \sum_{n=0}^{\infty} \mathbb{P}(X_{n+1} > -an + x) \quad (5.8)$$

$$= \sum_{n=0}^{\infty} \mathbb{F}(-an + x) \quad (5.9)$$

$$\approx -\frac{1}{a} \int_{x}^{\infty} \mathbb{F}(y)dy. \quad (5.10)$$

Thus, we established heuristically the tail-equivalence between the tails of supremum of the random walk $M$ and the integrated tail distribution of $F$. However, for presenting further asymptotic results of $F_M$ we have to adjust the definition of $F^s$ to distributions on the whole real line of $\mathbb{R}$. 

91
5 Applications

Definition 111. (Integrated tail function)

Let $F \in \mathcal{F}_R$ and $\int_{0}^{\infty} F(y)dy < \infty$. Then we define the integrated tail function as

$$F_I(x) := 1 - \min \left\{ 1, \int_{x}^{\infty} F(y)dy \right\}.$$ 

The following general result gives us a lower bound for $F_M(x)$ without additional requirements on $F, F_I$ or $F_M$. The formulation follows [18].

Theorem 112. (Lower bound for $F_M$)

With the notation from Section 5.2.1, assume that $a < 0$. Then for any $x > 0$

$$\mathbb{P}(M > x) \geq \frac{\int_{x}^{\infty} F(y)dy}{\int_{x}^{\infty} F(y)dy - a}. \tag{5.11}$$

and, in particular,

$$\liminf_{x \to \infty} \frac{\mathbb{P}(M > x)}{F_I(x)} = \liminf_{x \to \infty} \frac{F_M(x)}{F_I(x)} \geq -\frac{1}{a}.$$ 

The statement of the Theorem 112 approves the heuristic approach from the lines (5.6)-(5.10). The proof of the Theorem 112 with additional requirement $F \in \mathcal{L}$ can be found in [43, Theorem 1], the general case was proven by Foss et. al in [18, Theorem 5.1, page 98]. The question that may arise is, in which cases are we able to give an upper bound for the ratio:

$$\frac{F_M(x)}{F_I(x)}, \ x \geq 0.$$ 

This is a serious open question. However, we can consider asymptotic results and in the subexponential case the following result was proven by Korshunov in [22, Theorem 1] and Veraverbeke in [35, Theorem 2, 1977].

Theorem 113. (Asymptotics of $F_M$ and $F_I$ in the subexponential case)

With the notation from Section 5.2.1, assume that $a < 0$.

$F_I \in S$ if and only if

$$\lim_{x \to \infty} \frac{F_M(x)}{F_I(x)} = -\frac{1}{a},$$
5 Applications

5.2.5 The integrated tail and conditional ascending ladder height

In the heavy-tailed case, we have already provided the asymptotic relation between the distribution of the supremum of the random walk $F_M$, the integrated tail function $F_I$, and the conditional ascending ladder height distribution $G$ respectively. The connection between $F_I$ and $G$ remains to be established.

Recall the notation from Section 5.2.1, then we have the following connection between the tail of the conditional ascending ladder height and the tail of distribution of the first strictly ascending ladder height $S_{\tau(1)}$:

$$G(x)(1 - p) = \mathbb{P}(S_{\tau(1)} > x),$$

where $p = \mathbb{P}(\tau(1) = \infty)$.

The following theorem states a lower bound for the tail of $G$ without additional requirements. The formulation of both theorems below follows [18].

**Theorem 114. (Lower bound for $G$)**

With the notation from Section 5.2.1, assume that $a < 0$. Then for any $x > 0$

$$G(x)(1 - p) = \mathbb{P}(S_{\tau(1)} > x) \geq -\frac{p}{a} \int_{x}^{\infty} F(y) dy. \quad (5.12)$$

In the case of an additional requirement on $F_I$, the tails of $G$ and $F_I$ are strongly tail-equivalent.

**Theorem 115. (Asymptotics of $G$ and $F_I$ in the heavy-tailed case)**

With the notation from Section 5.2.1, assume that $a < 0$ and $F_I \in \mathcal{L}$, then the following holds:

$$\lim_{x \to \infty} \frac{G(x)(1 - p)}{F_I(x)} = \lim_{x \to \infty} \frac{\mathbb{P}(S_{\tau(1)} > x)}{F_I(x)} = -\frac{p}{a}.$$

For a proof of Theorem 114 and Theorem 115, we refer the reader to [18, Lemma 5.9, page 110]. Finally, we state the most general result for the class $\mathcal{OL}$.

**Proposition 116. (Asymptotics of $G$ and $F_I$ in the class $\mathcal{OL}$)**

With the notation from Section 5.2.1, if $a < 0$ and $F_I \in \mathcal{OL}$ then

$$G \asymp F_I.$$ 

For a proof, we refer to [42, Lemma 2.2]. By using the closure property of $\mathcal{A}(\mathcal{E})$, $\mathcal{E} \subseteq \mathcal{G}$, under weak tail-equivalence, we can transfer the statement of proposition above. Hence, we obtain the following result.

**Proposition 117. (Asymptotics of $G$ and $F_I$ in the class $\mathcal{A}(\mathcal{E})$)**

With the notation from Section 5.2.1, let $\mathcal{E} \subseteq \mathcal{G}$ be not empty. If $a < 0$ and $F_I \in \mathcal{OL}$, then

$$G \in \mathcal{A}(\mathcal{E}) \iff F_I \in \mathcal{A}(\mathcal{E}).$$
5.2.6 Veraverbeke’s theorem

The famous Veraverbeke’s theorem collects the most important statements about the asymptotics of the tails of distributions $F_M$, $G$ and $F_I$ of the random walk in the subexponential case. Some of these statements are already introduced in previous sections. However, Veraverbeke’s theorem complements the missing links of the interrelations between the investigated quantities $F_M$, $G$ and $F_I$.

**Theorem 118. (Veraverbeke’s theorem)**

With the notation from Section 5.2.1, assume that $a < 0$. The following assertions are equivalent:

1) $F_I \in S$;
2) $G \in S$;
3) $F_M \in S$;
4) $F_M(x) \sim -\frac{1}{a} F_I(x)$.

Note that this theorem states the full equivalences between 1)–4). The implication 3) $\Rightarrow$ 4) and the equivalences between 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3) were proved by Veraverbeke in [35, Theorem 2, 1977]. The implication 4) $\Rightarrow$ 3) was proved by Korshunov in [22, Theorem 1, 1997]. Embrechts and Veraverbeke applied the results of Theorem 118 to the Sparre-Andersen model in [14]. Furthermore, Veraverbeke’s theorem has well-known applications to queueing theory.

Our goal is to extend the classical Veraverbeke’s theorem from the class $S$ to the class $J$. We start with the relation between $G$ and $F_M$ in the large claim case. We already know that we can describe $F_M$ as a geometric compound distribution:

$$F_M(x) = \sum_{i=0}^{\infty} p(1-p)^i G^{i\ast}(x),$$

see (5.5). Recall the notation

$$c_G = \limsup_{x \to \infty} \frac{G^{2\ast}(x)}{G(x)}.$$

By using our results for random sums from Section 4.6, we can obtain the following result:

**Theorem 119. (Veraverbeke’s theorem - Large claim case $J$)**

With the notation from Section 5.2.1, assume $a < 0$, $F_I \in OL$ and that one of the following conditions holds:

(i) $p(c_G + \varepsilon - 1) < 1$ for some $\varepsilon > 0$;
(ii) $F_M \in OS$;
(iii) $F_I \in J \cap DK$.

Then the following assertions are equivalent:

1) $F_I \in J$;
2) $G \in J$;
3) $F_M \in J$.

Each one of 1), 2) or 3) combined with each one of i), ii) or iii) implies

4) $F_M \asymp G \asymp F_I$. 

94
5 Applications

Corollary 120. With the notation from Section 5.2.1.
If $a < 0$ and $F_I \asymp H$ for some $H \in \mathcal{S}$, then $F_M \asymp G \asymp F_I$.

Theorem 119 is inspired by and should be compared with the recent partial generalization of Theorem 118 to the even larger class $OS$ by Yang and Wang (2011) in [42, Theorem 1.2, Theorem 1.3], which is, however considerably weaker.

Theorem 121. (Veraverbeke’s theorem for $OS$)
With the notation from Section 5.2.1, assume $a < 0$. If $F_I \in \mathcal{OL}$, then
a) $\limsup_{x \to \infty} \frac{F_I(x)}{F_M(x)} < \infty$;
b) (i) $F_I \in OS$ and (ii) $G \in OS$ are equivalent;
c) (iii) $F_M \asymp G \asymp F_I$ yields (i) or (ii).

If $F_I \in \mathcal{L}$ and $(c_{F_I}^{+} - 1) < a$, then (i) or (ii) yields (iii). In this case, (i) ((ii) or (iii)) implies $F_M \in OS$.

Remark 122. Note that the weak asymptotic tail-equivalence (iii) requires $F_I \in \mathcal{L}$ as opposed to the situation in Theorem 119. Further, part a) gives only a lower asymptotic bound for $F_M$ in terms of $F_I$.

The result of Theorem 119 establishes weak tail-equivalence between the quantities $F_M$, $G$ and $F_I$. However, since $\mathcal{J}$ contains light-tailed distributions (Example 81) which induce the Cramér case (Cramér-Lundberg and the additional condition from [13, Theorem 1.2.2] are fulfilled) and subexponential distributions our new class $\mathcal{J}$ does not provide the information about the path to ruin. Both scenarios, ‘one big jump’ in the case of subexponential distributions and ‘incremental ruin’ in the Cramér case (see also [13, Section 8.3.2] for a detailed review), are included.
5.2.6.1 Proof of Theorem 119

We prepare the proof by recalling a result due to Yang and Wang (2008) for reference, [42, Theorem 1.4], which in fact, is a simple and direct adaption of [17, Proposition 2.1].

**Theorem 123.** Let \( F \) be a distribution function on \((-\infty, \infty)\) such that \( F \) is integrable and \( F_I \in OS \cap DK \). Further, let \( \alpha \) and \( \beta \) be two fixed positive constants. Consider any sequence \( \{X_i : i \geq 1\} \) of independent random variables such that, for each \( i \geq 1 \), the distribution \( F_i \) of \( X_i \) satisfies the conditions

\[
F_i(x) \leq F(x), \text{ for all } x \in (-\infty, \infty), \quad \text{and} \quad \int_{-\infty}^{\infty} (y \vee -\beta) dF_i(y) \leq -\alpha.
\]

Then there exists a positive constant \( r \), depending only on \( F, \alpha \) and \( \beta \), such that for all sequences \( \{X_i : i \geq 1\} \) as above,

\[
\overline{F_M}(x) \leq r \overline{F_I}(x)
\]

for all \( x \in (-\infty, \infty) \).

Now the proof of the Veraverbeke’s theorem for class \( J \) can simply be reduced to the previously stated results.

**Proof.** Since by assumption \( F_I \in O\mathcal{L} \) and \( a := \mathbb{E}[X_k] < 0 \) we obtain from Proposition 116 that \( \overline{F_I} \sim \overline{G} \). Hence, the equivalence of 1) and 2) follows from the weak tail-equivalence closure of the class \( J \) (Proposition 84).

Now additionally assume \( p(c_G + \varepsilon - 1) < 1 \) holds for some \( \varepsilon > 0 \) (condition \( i \)). As we already know we can write \( F_M \) as a random sum

\[
\overline{F_M}(x) = (1 - p) \sum_{n=1}^{\infty} p^n \overline{G^{\mathbb{N}}}(x).
\]

(5.13)

Hence, we obtain \( \overline{F_M} \sim \overline{G} \) by application of Proposition 104. Now, applying the weak tail-equivalence closure of the class \( J \) we conclude the equivalence of 2) and 3).

Next, assume additionally that \( F_M \in OS \) (condition \( ii \)) holds. Again, by using the expression (5.13) and Proposition 104 b), we obtain \( \overline{F_M} \sim \overline{G} \), and hence the equivalence of 2) and 3).

Finally, under condition \( F_I \in J \cap DK \) we can use Theorem 123. By choosing \( F_i = F \) it is easy to see that we can find appropriate constants \( \alpha, \beta \) such that

\[
\int_{-\infty}^{\infty} (y \vee -\beta) dF(y) \leq -\alpha
\]

holds. Hence, there exists a constant \( r \) such that \( \overline{F_M}(x) \leq r \overline{F_I}(x) \) for all \( x \in (-\infty, \infty) \). By a) of Theorem 121 we obtain \( \overline{F_M} \sim \overline{F_I} \) and hence \( F_M, G \in J \). Now, \( \overline{F_M} \sim \overline{F_I} \) follows from \( \overline{F_I} \sim \overline{G} \) and \( \overline{F_M} \sim \overline{G} \).

\[ \square \]
5 Applications

5.3 Infinitely divisible distributions

In previous sections, we considered tails of distributions in ruin theory. The study of tails is also important in relation to infinitely divisible distributions and their Lévy measures.

Following [30], we denote by $\mathcal{ID}_+$ the class of all infinitely divisible distributions $\mu$ on $[0, \infty)$ with Laplace transform

$$\hat{\mu}(s) = \exp \left\{ \int_0^\infty (e^{-st} - 1)\nu(dt) \right\},$$

where the Lévy measure $\nu$ satisfies $\nu(t) > 0$ for every $t > 0$, and

$$\int_0^\infty (1 \wedge t)\nu(dt) < \infty.$$

Define the normalized Lévy measure $\nu_1$ as $\nu_1 = 1_{\{x>1\}}\nu/\nu(1, \infty)$.

The relation between the asymptotic tail behaviour of infinitely divisible laws and their Lévy measures has been studied by many authors in probability theory. Important applications can be found in queueing theory and fluctuation theory. We refer the reader to [8, 12, 30].

Let $\mathcal{C}$ be a distribution class. The question about the relation between $\nu$ and $\mu$ is that, in which distribution classes the tails of $\nu$ and $\mu$ are (weakly) tail-equivalent and the following assertion holds:

$$\nu \in \mathcal{C} \iff \mu \in \mathcal{C}.$$

Embrechts et al. proved in [12, Theorem 1] the classical result in the subexponential case.

**Theorem 124.** Let $\mu$ be a distribution in $\mathcal{ID}_+$ with Lévy measure $\nu$. Then the following assertions are equivalent:

1) $\mu \in S$;
2) $\nu_1 \in S$;
3) $\hat{\nu} \sim \nu$.

In the light-tailed case, Sgibnev extended in [29, Theorem 1] the above result to the class $S(\gamma)$.

**Theorem 125.** Let $\gamma \geq 0$ and $\mu$ be a distribution in $\mathcal{ID}_+$ with Lévy measure $\nu$. Then the following assertions are equivalent:

1) $\mu \in S(\gamma)$;
2) $\nu_1 \in S(\gamma)$;
3) $\mu \in L(\gamma)$, $\hat{\mu}(-\gamma) < \infty$ and $\hat{\nu}(x) \sim \hat{\mu}(-\gamma)\nu(x)$, where $\hat{\mu}(-\gamma) = \int_0^\infty \exp(\gamma t)\mu(dt) < \infty$.  

97
Shimura and Watanabe partially extended the result of Embrechts from the class $S$ to the class $OS$ in [30, Theorem 1.1]:

**Theorem 126.** Let $\mu$ be a distribution in $ID_+$ with Lévy measure $\nu$.

a) The following are equivalent:
1) $\nu_1 \in OS$;
2) $\nu \asymp \nu_1$.

b) The following are equivalent:
1) $\mu \in OS$;
2) $\nu_1^{n*} \in OS$ for some $n \geq 1$;
3) $\mu \asymp \nu_1^{n*}$ for some $n \geq 1$.

c) If $\nu_1$ is in $OS$, then $\mu$ is in $OS$. The converse does not hold.

Since the class $J$ is closed under convolution roots, one expects to be able to improve the result for $OS$ to class significantly. Indeed this is possible.

**Theorem 127.** Let $\mu$ be a distribution in $ID_+$ with Lévy measure $\nu$.

a) Then the following assertions are equivalent:
1) $\mu \in J$;
2) $\nu_1 \in J$.

b) If 1) or 2) holds then $\mu \asymp \nu_1$.

**Proof.** a) From Theorem 126 b), $J \subseteq OS$, Proposition 84 and $\mu \in J$ we infer $\nu \asymp \nu_1^{n*}$ and $\nu_1^{n*} \in J$ for some $n \geq 1$. The equivalence $\mu \in J \Leftrightarrow \nu_1 \in J$ follows immediately from Proposition 88 (closure of $J$ under root convolution).

b) If 1) holds the assertion follows from 126 b), Propositions 84 and 88.

If 2) holds then the assertion follows from Theorem 126 a) and $J \subseteq OS$. \qed
6 Appendix

In the following, we will provide some tools which are used throughout the dissertation. Denote by \( X, X_1, X_2, \ldots \) i.i.d. random variables with common distribution \( F \in \mathcal{F}_R \), and by \( Y, Y_1, Y_2, \ldots \) i.i.d. random variables with common distribution \( G \in \mathcal{F}_R \). Further, let

\[
S_n := \sum_{k=1}^{n} X_k, \quad \text{and} \quad \hat{S}_n := \sum_{k=1}^{n} Y_k.
\]

**Lemma 128.** Let \( F, G, H, I \in \mathcal{F}_R \). Suppose \( F \asymp G \) and \( H \asymp I \). Then

\[
F \ast H \asymp G \ast I.
\]

**Proof.** We obtain

\[
F \ast H(x) = \int_{\mathbb{R}} F(x - y) dH(y)
\]

\[
\asymp \int_{\mathbb{R}} G(x - y) dH(y) = G \ast H
\]

\[
= \int_{\mathbb{R}} H(x - y) dG(y) \asymp G \ast I.
\]

**Remark 129.** Recall the definition of \( O \)–subexponential distributions, see Definition 21 on page 22. From Lemma 128 we immediately obtain the closure of the class \( OS \) under weak tail-equivalence. If \( F \asymp G \), then \( F \ast F \asymp G \ast G \) and hence \( G \ast G \asymp G \).

**Lemma 130.** If \( F^{n*} \in OS \cap \mathcal{F} \), then \( F^{(n+1)*} \asymp F^{n*} \).

**Proof.** From \( F^{n*} \in OS \cap \mathcal{F} \) we know that there exists a constant \( c \) such that

\[
\liminf_{x \to \infty} \frac{F^{n*}(x)}{F^{2n*}(x)} > c > 0.
\]
Thus, we obtain
\[
\liminf_{x \to \infty} \frac{F^{n*}(x)}{F^{(n+1)*}(x)} = \liminf_{x \to \infty} \frac{F^{2n*}(x)}{F^{(n+1)*}(x)} > c \liminf_{x \to \infty} \frac{F^{2n*}(x)}{F^{(n+1)*}(x)} > 0.
\]

\[\square\]

Lemma 131. Let \( F, G \in \mathcal{F}_\mathbb{R} \). Suppose \( g \in \mathcal{G} \). Then:
\[
\limsup_{x \to \infty} \frac{\mathbb{P}(S_2 > x, X_1 \wedge X_2 > g(x))}{\mathbb{P}(S_2 > x, Y_1 \wedge Y_2 > g(x))} \leq \left( \limsup_{x \to \infty} \frac{F(x)}{G(x)} \right)^2.
\]

A proof can be found in [18], Lemma 2.36.

Lemma 132. For each \( F \in \mathcal{F} \), \( c \geq 0 \) and \( n \geq 2 \) we have
\[
\lim_{K \to \infty} \limsup_{x \to \infty} \mathbb{P}(X_{1,n} > x - c, X_{2,n} > K | S_n > x) = 0.
\]

Proof. For \( x - c \geq K \geq c \) we have
\[
\frac{\mathbb{P}(X_{1,n} > x - c, X_{2,n} > K, S_n > x)}{\mathbb{P}(S_n > x)} \leq \frac{\mathbb{P}(X_{1,n} > x - c, X_{2,n} > K)}{\mathbb{P}(X_{1,n} > x - c, X_{2,n} > c)} = \frac{1 - (1 - F(x-c))^n - nF(x-c)(F(K))^{n-1}}{1 - (1 - F(x-c))^n - nF(x-c)(F(c))^{n-1}} = \frac{nF(x-c)(1 - F(K)^{n-1} + o(F(x-c)))}{nF(x-c)(1 - F(c)^{n-1} + o(F(x-c)))} = \frac{1 - F(K)^{n-1} + o(1)}{1 - F(c)^{n-1} + o(1)}
\]
and the result follows by passing to the limit. \[\square\]

Lemma 133. Let \( (p_n)_{n \in \mathbb{N}} \subset \mathbb{R} \) be a sequence with \( p_n > 0, \forall n \in \mathbb{N} \) and \( \sum_{n=0}^{\infty} p_n = 1 \). Then following holds
\[
\liminf_{n \to \infty} \frac{\sum_{k=n+1}^{\infty} p_k}{p_n} > 0 \iff \liminf_{n \to \infty} \frac{\sum_{k=n+j}^{\infty} p_k}{p_n} > 0,
\]
for all \( 1 \leq j \).

Proof. Let \( j \geq 1 \). We prove the direction
\[
\liminf_{n \to \infty} \frac{\sum_{k=n+1}^{\infty} p_k}{p_n} > 0 \Rightarrow \liminf_{n \to \infty} \frac{\sum_{k=n+j}^{\infty} p_k}{p_n} > 0,
\]

the reverse direction is obvious. It suffices to prove the following assertion

$$\liminf_{n \to \infty} \frac{\sum_{k=n+j}^{\infty} p_k}{p_n} > 0 \Rightarrow \liminf_{n \to \infty} \frac{\sum_{k=n+j+1}^{\infty} p_k}{p_n} > 0.$$

Suppose \(\liminf_{n \to \infty} \frac{\sum_{k=n+j}^{\infty} p_k}{p_n} > 0\) and \(\liminf_{n \to \infty} \frac{\sum_{k=n+j+1}^{\infty} p_k}{p_n} = 0\). Thus, there exists a sequence \((m_n)_{n \in \mathbb{N}}\), \(m_n \to \infty\) as \(n \to \infty\), such that

$$\liminf_{n \to \infty} \frac{\sum_{k=m_n+j}^{\infty} p_k}{p_{m_n}} = 0. \tag{6.1}$$

From \(\liminf_{n \to \infty} \frac{\sum_{k=m_n+j}^{\infty} p_k}{p_{m_n}} \geq \liminf_{n \to \infty} \frac{\sum_{k=n+j}^{\infty} p_k}{p_n} > 0\) and (6.1) we obtain

$$\frac{\sum_{k=m_n+j+1}^{\infty} p_k}{\sum_{k=m_n+j+1}^{\infty} p_k + pm_n + j} = \frac{\sum_{k=m_n+j+1}^{\infty} p_k}{\sum_{k=m_n+j}^{\infty} p_k} \to 0,$$

as \(n \to \infty\). It follows that

$$\frac{\sum_{k=m_n+j+1}^{\infty} p_k}{p_{m_n} + j} \to 0,$$

as \(n \to \infty\). Thus, we obtain a contradiction to

$$\liminf_{n \to \infty} \frac{\sum_{k=n+j+1}^{\infty} p_k}{p_n} > 0$$

and the assertion follows. \(\square\)
Bibliography


Bibliography


