The Hermite-Biehler method and functions generating the Pólya frequency sequences

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Abstract

Under the Hermite-Biehler method we understand the approach to problems of stability, which exploits a deep relation between Hurwitz-stable functions and mappings of the upper half of the complex plane into itself (*i.e.* $\mathcal{R}$-functions). This method dates back to works by Hermite, Biehler, Hurwitz; in the first half of the XXth century it was extended to entire functions by Grommer and Kreĭn. Recent papers (from 1970 to the present date) highlighted another property related to Hurwitz-stability: the total nonnegativity of the corresponding Hurwitz matrix, that is nonnegativity of all its minors. Since each Hurwitz matrix is built from two Toeplitz matrices, this also involves the class $\mathcal{PF}$ of functions corresponding to totally nonnegative Toeplitz matrices (*i.e.* generating functions of Pólya frequency sequences). The present work investigates connections between $\mathcal{R}$-functions, $\mathcal{PF}$-functions and the localization of zeros (*e.g.* Hurwitz stability).

The first problem we deal with is to prove that zeros of one remarkable family of polynomials are interlacing. This family originates from the Jacobi tau method for the Sturm-Liouville eigenvalue problem, and the interlacing property guarantees that the spectra of approximations are real. At that, the Hurwitz stability comes from a specially composed differential equation, and then the Hermite-Biehler theorem implies the interlacing property for pairs of polynomials. The next topic is a complete description of functions generating the infinite totally nonnegative Hurwitz matrices. Our results exploit a connection between a factorization of the Hurwitz-type matrices and the expansion of $\mathcal{R}$-functions into Stieltjes continued fractions. Further we study solutions to the equation $z^p R(z^k) = \alpha$ with nonzero complex α, integer p, k and $R \in \mathcal{PF}$ by relating it to the class $\mathcal{R}$. Such equations appear from manipulations akin to the Hermite-Biehler method when we prove that functions of the form $\sum_{n=0}^{\infty}(\pm i)^{n(n-1)/2} a_n z^n$ have simple zeros distinct in absolute value under a certain condition on the coefficients $a_n \geqslant 0$. Lastly, we study the generalized Nevanlinna classes $\mathcal{N}_\varkappa^+$ and their connection to $\mathcal{PF}$-functions. The present work elaborates the criterion given by Krein and Langer and applies it to counting the number of zeros and poles of $\mathcal{PF}$-functions.

Keywords: Pólya frequency sequences · $\mathcal{R}$-functions · Generalized Nevanlinna classes · Hermite-Biehler theorem · Hurwitz stability · Localization of zeros · Sokal conjectures · Conjectures by Csordas, Charalambides and Waleffe

Zusammenfassung

Unter der Hermite-Biehler-Methode verstehen wir den Ansatz für Probleme der Stabilität, der eine tiefe Beziehung zwischen Hurwitz-stabilen Funktionen und Abbildungen der oberen Hälfte der komplexen Zahlenebene in sich selbst (d. h. $\mathcal{R}$-Funktionen) nutzt. Diese Methode stammt von Hermite, Biehler und Hurwitz; in der ersten Hälfte des zwanzigsten Jahrhunderts wurde es von Grommer und Kreĭn auf ganze Funktionen erweitert. Neuere Arbeiten (ab 1970) heben eine andere Eigenschaft in Verbindung mit Hurwitz-Stabilität hervor: die totale Nichtnegativität der entsprechenden Hurwitz-Matrix (also Nichtnegativität aller ihrer Minoren). Da jede Hurwitz-Matrix aus zwei Toeplitz-Matrizen besteht, führt dies zur Betrachtung der Klasse $\mathcal{PF}$ der Funktionen, die total nichtnegativen Matrizen entsprechen (d. h. die Klasse der erzeugenden Funktionen von Pólya-Frequenzfolgen). Die vorliegende Arbeit untersucht Verbindungen zwischen $\mathcal{R}$-Funktionen, $\mathcal{PF}$-Funktionen und die Lokalisierung von Nullstellen (z.B. Hurwitz-Stabilität).

Erster Untersuchungsgegenstand dieser Arbeit ist eine Folge von Polynomen, die aus der Jacobi-Tau-Methode für das Sturm-Liouville-Eigenwertproblem stammt. Wir untersuchen, wann die Nullstellen eines Polynoms sich mit den Nullstellen des nächsten Polynoms der Folge abwechseln. Diese Eigenschaft garantiert, dass die Spektren der entsprechenden Annäherungen real sind. Dabei kommt die Stabilität von einer eigens aufgestellten Differentialgleichung zum Einsatz und der Satz von Hermite und Biehler impliziert anschließend die erwünschte Eigenschaft für Paare von Polynomen. Das nächste Thema ist eine vollständige Beschreibung der Funktionen, die unendliche Matrizen vom Hurwitz-Typ erzeugen. Unsere Ergebnisse basieren auf einer Verbindung zwischen einer Faktorisierung solcher Matrizen und der Stieltjesschen Kettenbruchentwicklung der $\mathcal{R}$-Funktionen. Weiterhin untersuchen wir Lösungen der Gleichung $z^p R(z^k) = \alpha$ mit komplexen $\alpha \neq 0$, ganzen p, k und $R \in \mathcal{PF}$, indem wir eine Verbindung zur Klasse $\mathcal{R}$ herstellen. Solche Gleichungen ergeben sich aus Manipulationen ähnlich der Hermite-Biehler-Methode, wenn wir beweisen, dass Funktionen der Form $\sum_{n=0}^{\infty}(\pm i)^{n(n-1)/2} a_n z^n$ nur einfache Nullstellen mit unterschiedlichen Absolutwerten unter einer bestimmten Bedingung an die Koeffizienten $a_n \geqslant 0$ haben. Letztendlich untersuchen wir die verallgemeinerten Nevanlinna-Klassen $\mathcal{N}_\varkappa^+$ und ihre Verbindung zu $\mathcal{PF}$-Funktionen. Die vorliegende Arbeit präzisiert das Kriterium von Kreĭn und Langer, und wendet es zur Zählung von Nullstellen und Polen von $\mathcal{PF}$-Funktionen an.

Schlagwörter: Pólya-Frequenzfolgen · $\mathcal{R}$-Funktionen, Verallgemeinerte Nevanlinna-Klassen · Satz von Hermite und Biehler · Hurwitz-Stabilität · Lokalisierung der Nullstellen · Sokal-Vermutungen · Vermutungen von Csordas, Charalambides und Waleffe

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Chapter 1.

Introduction

1.1 Polynomial stability, interlacing zeros and the Hermite-Biehler theorem

A polynomial is called (Hurwitz) *stable* if all of its roots have negative real parts. Accordingly, a polynomial is called *quasi-stable* if it has no roots with positive real parts. This terminology came from the theory of dynamical systems. Continuous dynamical systems can be described with differential equations, for example $\ddot{\phi}(t) = \mu \sin \phi(t)$ for the simple pendulum. A solution of a differential equation is (Lyapunov) stable (*i.e.* small perturbations vanish with time) if the characteristic polynomial of the corresponding linearised equation is (Hurwitz) stable. Although the modern theory of stability was only created by Lyapunov in 1892, the linearisation method was already used much longer ago as a base for empirical laws of physics such as Hooke's law.

To answer whether a polynomial is stable or not became vital for the engineering of the XIXth century. This was evident to Maxwell: in [Ma1868] he proposed a criterion for polynomials of degree at most three and stated the problem of getting conditions for a polynomial of a higher degree. The answer (an algorithm) was given by Routh [Ro1877] (see its modern presentation in [Gan59, Chapter XV § 3]); nowadays it is known as the Routh scheme. Sturm's theorem and the Cauchy indices play the basic role for this scheme. The Routh algorithm also gives a solution to the generalized Hurwitz problem, *i.e.* helps to determine the number of zeros in the right half of the complex plane.

Nevertheless, Hermite [He1856] revealed a connection between the number of zeros of a complex polynomial in a half-plane and the signature of a certain quadratic form in 1856, twelve years earlier than Maxwell stated his problem. Hermite's results were inconvenient for applications until Hurwitz [Hu1895] expressed them (for real polynomials) as a condition on the coefficients of polynomials (now it is knows as the Hurwitz criterion). The further development is connected with the names of Liénard and Chipart: they, in particular, indicated [LiCh14] a relation between consecutive determinants of the Hurwitz matrix. Schur [Schu21] got an elementary derivation of the stability conditions and

extended them to complex polynomials. With a proper extension of the notion of stability, the Hurwitz criterion extends to entire (we touch upon this extension in Section 1.2.2, see also [Kre38, CM49]), rational (see [BT2011]) and further towards meromorphic functions.

Let a polynomial $f(z)$ be represented as $f(z) = g(iz) + ih(iz)$, where $g(z)$ and $h(z)$ only have real coefficients. The works [He1856] and [Bi1879] display that the stability of the polynomial $f(z)$ can be studied by finding a special relation between zeros of $g(z)$ and $h(z)$. We say that zeros of two polynomials $g(x)$ and $h(x)$ *interlace* (strictly) if all zeros of $g(z)$ and $h(z)$ are real, between each two consecutive zeros of $g(z)$ there is exactly one zero of $h(z)$, and conversely, between each two consecutive zeros of $g(z)$ there is exactly one zero of $h(z)$ (see Definition 2.1 on Page 19). The argument principle implies that all zeros of $g(z)$ and $h(z)$ are real and interlacing exactly when all zeros of $f(z)$ lie on one side of the imaginary axis. Moreover, they belong to the left or right half of the complex plane depending on the sign of the expression $g(z)h'(z) - g'(z)h(z)$ on the real line. Currently this fact is known as the Hermite-Biehler theorem. Its version for real polynomials can be formulated as the following.

Theorem H-B. *A real polynomial $f(z) := p(z^2)+zq(z^2)$ is stable if and only if $p(0)\cdot q(0) > 0$ and all zeros of $p(z)$ and $zq(z)$ are nonpositive and strictly interlacing.*

This statement is close to the one given in [Gan59, p. 228]. The replacement of $q(z)$ with $zq(z)$ provides the desired order of zeros: that is the zero of $p(z)$ and $q(z)$ closest to the origin belongs to the former polynomial.

It is worth noting that pairs of polynomials with interlacing zeros can help in constructing objects with certain properties other then Hurwitz stability. For example, two such pairs with positive coefficients can be combined into one polynomial with negative zeros (see [GoLu83, Lemma 3.4]). In Chapter 2 we consider a situation when this technique proved to be helpful. It occurs in connection with the spectral properties of a polynomial approximation of the differential eigenvalue problem $\ddot{u}(t) = \lambda u(t)$ with various boundary conditions. At that, the Hurwitz stability is derived from a special differential equation, and then the Hermite-Biehler theorem implies the interlacing property for pairs of polynomials. This bright approach to this issue was applied in [GoLu83] and significantly elaborated in [CCW2005]. The question in our study, however, consists in proving that other pairs of polynomials have interlacing zeros as well.

One of the most interesting types of polynomials with real interlacing zeros is presented by the polynomials orthogonal on the real line (with respect to some measure). Their deep theory emerged in works by Chebyshev and became a cornerstone of harmonic analysis as well as of functional analysis. Orthogonality is understood in the sense of the measured space L_2. Polynomials orthogonal on the real line satisfy (see *e.g.* [Sze67, p. 42]) the

so-called three-term recurrence:

$$P_n(z) = (A_n z + B_n)P_{n-1}(z) - C_n P_{n-2}(z), \quad \text{where } A_n, C_n > 0,\ n = 1, 2, \ldots,$$
$$P_0(z) \equiv \text{const} > 0, \quad P_{-1}(z) \equiv 0.$$

These relations define a corresponding sequence of orthogonal polynomials uniquely, and imply that it is a Sturm sequence. As a result, zeros of orthogonal polynomials with consecutive degrees must be interlacing (see *e.g.* [Sze67, p. 46]). (It is also very remarkable that some orthogonal polynomials on the real line satisfy a certain differential equation of the second order; such polynomials are called *classical*.)

A charming fact about (quasi-) stable polynomials was derived by Asner [Asn70]. It appears that all minors of the Hurwitz matrix corresponding to a quasi-stable polynomial are nonnegative. Twelve years later, Asner's result was extended by Kemperman [Kem82] to infinite Hurwitz matrices. It gave an idea to Holtz and Tyaglov that this relation has a deeper nature. They proved a criterion [HT2012] implying that a polynomial is quasi-stable if and only if its infinite Hurwitz matrix is *totally nonnegative* (*i.e.* all its minors are nonnegative). At the same time, their work left open questions, in particular:

- Is there an extension of this fact beyond the frames of polynomials, for example to entire functions?
- What can we say about more general power series with totally nonnegative infinite Hurwitz matrices?
- Does the class $\mathcal{PF}$ of functions generating Pólya frequency sequences (see Definition 1.9 herein) play a special role as a generalization of polynomials with negative zeros?
- Given a Hurwitz-stable real polynomial $f(z) = p(z^2) + zq(z^2)$, the ratio $\frac{q(z)}{p(z)}$ is an $\mathcal{R}$-function (*i.e.* maps the upper half of the complex plane, see Definition 1.1). Which connections do we have between the classes $\mathcal{R}$ and $\mathcal{PF}$?

We answer the first three questions in Chapter 3. Both Chapters 4 and 5 are devoted to studying further connections between the classes mentioned in the last question.

The recent work [HKK2015] suggests a generalization of the above result from [HT2012]: for a polynomial $f(z) = \sum_{i=0}^{n} a_i z^{n-i}$, where $a_i = 0$ as soon as $i < 0$ or $i > n$, the authors suggest considering the matrices $\mathcal{H}_M(f) = (a_{Mj-i})_{i,j=0}^{\infty}$ of order $M = 2, 3, 4, \ldots$ in addition to the "standard" Hurwitz matrix corresponding to $M = 2$ and to the Toeplitz matrix corresponding to $M = 1$. At that, for $M > 2$ a criterion analogous to the case $M = 2$ holds: total nonnegativity of $\mathcal{H}_M(f)$ is equivalent to that all zeros of the polynomial $f(z)$ lie outside the sector $\left\{z \in \mathbb{C} : |\arg(z)| < \frac{\pi}{M}\right\}$.

The so-called *robust stability* has a significant importance for engineering applications. One of its typical problems is to determine whether a polynomial $f(z)$ with coefficients

depending on parameters remains stable for all values of the parameters running over some set. The notion of *robustness* stands here for a certain reserve against a failure in the case of *e.g.* numerical inaccuracy or wrong measurements. Unfortunately, this topic is beyond the scope of the present thesis. Here we only mention one of the basic result in this field [Kha78]: all polynomials $f(z) = z^n + \sum_{i=1}^{n} a_i z^{n-i}$, where $\alpha_i \leqslant a_i \leqslant \beta_i$ for $i = 1, \dots, n$, are stable if and only if the four polynomials

$$\begin{aligned}
&\alpha_n + \alpha_{n-1}z + \beta_{n-2}z^2 + \beta_{n-3}z^3 + \alpha_{n-4}z^4 + \dots + z^n,\\
&\beta_n + \alpha_{n-1}z + \alpha_{n-2}z^2 + \beta_{n-3}z^3 + \beta_{n-4}z^4 + \dots + z^n,\\
&\beta_n + \beta_{n-1}z + \alpha_{n-2}z^2 + \alpha_{n-3}z^3 + \beta_{n-4}z^4 + \dots + z^n,\\
&\alpha_n + \beta_{n-1}z + \beta_{n-2}z^2 + \alpha_{n-3}z^3 + \alpha_{n-4}z^4 + \dots + z^n
\end{aligned}$$

are stable.

Another related problem is to find simple *sufficient* conditions on a polynomial which guarantee its stability. Among a great number of such results, let us only mention the relevant work [DP2005]. Its authors obtain an interesting sufficient condition of *almost strict total positivity* of matrices[1] and apply it to the Hurwitz matrix. As a consequence, a polynomial $f(z) = \sum_{i=0}^{n} a_i z^{n-i}$ with positive coefficients must be stable if

$$a_i a_{i+1} \geqslant \tilde{c} a_{i-1} a_{i+2}, \quad \text{where} \quad i = 1, 2, \dots n-2 \quad \text{and} \quad \tilde{c} \approx 4.0796.$$

This result is sharpened in [KV2008]: the polynomial $f(z)$ is stable provided that for $n > 5$

$$a_i a_{i+1} \geqslant \tilde{c} a_{i-1} a_{i+2}, \quad \text{where} \quad i = 1, 2, \dots n-2, \quad \tilde{c} \approx 2.1479$$

and the value of $\tilde{c}$ cannot be decreased; for $n = 5$ the inequalities are strict; for $n = 4$ the sharp constant is 2 and the inequalities are also strict. At that, the Hermite-Biehler theorem plays a key role for [KV2008].

The present thesis dwells on a few relations between $\mathcal{PF}$-functions, $\mathcal{R}$-functions and the localization of zeros (*e.g.* Hurwitz stability). The interlacing property of zeros of two polynomials (and, more generally, functions) plays one of the central roles here. This property provides bridges between various problems of analysis. On the one hand, there is a connection between totally nonnegative Hurwitz-type matrices and the interlacing property of polynomials. On the other hand, the ratio of interlacing polynomials is a rational function mapping the upper half of the complex plane into itself or into the lower half-plane. Applying the long division algorithm yields a connection to continued

[1] That is, the non-singular totally nonnegative matrices, such that a minor formed by its consecutive rows and consecutive columns is positive if and only if all of its diagonal entries are positive.

fractions and families of orthogonal polynomials. In turn, they arise in relation to the Stieltjes and Hamburger moment problems, approximation theory and so forth. Although all mentioned topics are well-investigated, there remains some *terra incognita* yet.

1.2 Function classes and their historical background

1.2.1 Functions mapping the upper half-plane into itself and their generalizations

Definition 1.1. We denote by $\mathcal{R}$ ($\mathcal{R}^{-1}$ resp.) the class of all functions $F(z)$ analytic in the complement of the real axis, which satisfy $F(\overline{z}) = \overline{F(z)}$ and

$$\frac{\operatorname{Im} F(z)}{\operatorname{Im} z} \geqslant 0 \qquad \left(\text{or } \frac{\operatorname{Im} F(z)}{\operatorname{Im} z} \leqslant 0 \text{ for } F \in \mathcal{R}^{-1} \text{ resp.}\right).$$

Note that it is a straightforward consequence of the definition that $\mathcal{R}$- and $\mathcal{R}^{-1}$-functions are *real*, *i.e.* they map the real line into itself (excepting their singular points). Furthermore, our definition includes real constants (as in [KaKr68]) into both classes $\mathcal{R}$ and $\mathcal{R}^{-1}$, although sometimes they are excluded in the literature (*e.g.* [Wig51]).

In various applications, $\mathcal{R}$ is also known as the *Pick* class or the *Nevanlinna* class.

Definition 1.2. Denote by $\mathcal{S}$ the subclass of $\mathcal{R}$-functions that are regular and nonnegative over the nonnegative reals. (When an $\mathcal{S}$-function is meromorphic, it can only have negative poles and nonpositive zeros.)

Continued fractions (especially of the Stieltjes and Jacobi types) served as one of the main tools for studying $\mathcal{R}$- and $\mathcal{S}$-functions for many years. It is difficult to overestimate the role of exact expansions of some special functions (*e.g.* the Hypergeometric series) into continued fractions for modern mathematics. The truncated fractions automatically give the best rational approximation, which was widely used for calculations in the pre-computer era. Already in Euler times there was a significant number of relations known [Kh2008], however the most striking results in this direction were obtained between the second half of XIXth century and the first half of XXth century. Such mathematicians as Chebyshev, Hermite, Laguerre, Markov and Sonin contributed to the studies of orthogonal polynomials arising from continued fractions. Stieltjes, in his work [St1894] gave a solution to the so-called Stieltjes *moment problem*, originating from the mechanics (as clearly seen from its terminology). This achievement was very fruitful for both theory and applications because of its relation to the theory of self-adjoint operators. For example, an interesting mechanical interpretation, which is given *e.g.* in [Akh65, pp. 234–236], served as a starting point in many constructions by Mark Kreĭn. In the Stieltjes moment problem, *masses* are distributed over the positive semi-axis. The corresponding moment problem with masses

on the whole real axis was solved by Hamburger [Ham20, Ham21a, Ham21b] with the help of continued fractions. Besides that, the work [St1894] in fact introduced the analytic theory of continued fractions. This theory was further elaborated by Grommer [Gro14], Hamburger [Ham20, Ham21a], Perron [Per57], Wall [Wall48] and others. Highly developed analytic theory of continued fractions serves well for treating a question of convergence in Chapter 3 of the present thesis.

It should be mentioned, that from the 1920s (see *e.g.* the work [Rie23] by Riesz) studies on the moment problem start to diverge from the mentioned tight connection with continued fractions. It reflects the common shift of mathematics towards stronger penetration of functional analysis. In the preface to his famous book on orthogonal polynomials [Sze67], Gabor Szegő wrote the following:

> "Despite the close relationship between continued fractions and the problem of moments, and notwithstanding recent important advances in this latter subject, continued fractions have been gradually abandoned as a starting point for the theory of orthogonal polynomials. In their place, the orthogonal property itself has been taken as basic, and it is this point of view which has been adopted in the following exposition of the subject."

Accordingly, the modern point of view offers a better understanding of the nature of orthogonal polynomials in itself. This contrasts with Stieltjes' approach, which involved continued fractions as the starting point, the tool and the main subject. There were other issues with the classical point of view. On the one hand, continued fractions can behave very intricately[2] and be less suited to multidimensional applications. Furthermore, rational approximations can be obtained from the Padé tables (which are more flexible[3]). On the other hand, modern functional-analytic methods give tools that are handy for dealing with both one- and multidimensional problems. They fit better to other chapters of functional analysis and introduce a convenient framework for involving matrix analysis.

Nevertheless, at the end of the day both approaches enriched each other, and continued fractions are still in use in literature about the moment problem: see *e.g.* the monograph [Akh65] by Akhiezer about the moment problem or the more modern study [KrLa79,

[2] One of the issues is that the question of convergence for analytic continued fractions can be very hard. In some cases, such a natural functional-analytic tool as compactness of sets of functions give a beautiful and clear answer. See *e.g.* [Ham20, § 7], where Hamburger applies Helly's selection theorem. Another good example is a compact operator constructed by Kreĭn, which is used (among other things) for answering whether a function represented by a certain continued fraction is meromorphic or not.

[3] The Padé tables present a more general framework for constructing rational approximations. Their basic theory in connection with continued fractions can be found in *e.g.* [Per57, Chapter V] and [Wall48, Chapter XX]. For a detailed survey of the modern convergence theory of Padé approximants see [ABMS2011].

KrLa81] by Kreĭn and Langer. It is largely because continued fractions can offer a decomposition of complicated mappings into the series of transparent linear-fractional transformations. In the case of the above (and many other) moment problems, this decomposition reflects real relations of the physical world.

The moment problem itself can be stated as the question: when for a given sequence $s_0, s_1, \cdots \in \mathbb{R}$ of *moments* does there exist a non-decreasing function $\sigma(x)$ such that $\int_0^{+\infty} x^n d\sigma(x) = s_n$? In Hamburger's moment problem, the integral extends into the whole real line. Stieltjes [St1894] writes a continued fraction of the form

$$\cfrac{1}{c_1 z + \cfrac{1}{c_2 + \cfrac{1}{c_3 z + \ddots}}} \sim \frac{s_0}{z} - \frac{s_1}{z^2} + \frac{s_3}{z^3} - \frac{s_4}{z^4} + \cdots,$$

and studies the case when $c_1, c_2, \ldots > 0$. It is remarkable that the above continuous fraction can converge even if the corresponding power series is divergent. Then Stieltjes introduces the integral

$$\int_0^{+\infty} \frac{d\sigma(x)}{x+z} \sim \frac{s_0}{z} - \frac{s_1}{z^2} + \frac{s_2}{z^3} - \frac{s_3}{z^4} + \cdots,$$

which has the same power series expansion. The involved function $\sigma(x)$ gives a solution to the moment problem whenever all coefficients $c_1, c_2, \ldots$ are positive (this constitutes a condition on the Hankel determinants composed of $s_0, s_1, \ldots$). Moreover, Stieltjes proved that this solution of the moment problem is unique (and the correspondent continued fraction converges) exactly when the series $\sum_n c_n$ is divergent. If the solution is not unique, the corresponding moment problem is called undetermined and there exists a continuum of its solutions. It is worth noting that in both determined and undetermined cases the formal power series

$$\frac{s_0}{z} - \frac{s_1}{z^2} + \frac{s_3}{z^3} - \frac{s_4}{z^4} + \cdots$$

is ([Ham20, pp. 268–269]) the asymptotic expansion for z close to $+\infty \cdot i$ of the Stieltjes continued fraction, as well as of the integral $\int_0^{+\infty} \frac{d\sigma(x)}{x+z}$. The class $\mathcal{S}$ is named after Stieltjes because of its close connection to the above moment problem. When we define the class $\mathcal{S}$ as in Definition 1.2, its entries can be represented in the form (*cf.* [KaKr68, § 5])

$$\gamma + \int_0^{+\infty} \frac{d\sigma(x)}{x + \frac{1}{z}} = \gamma + z \int_0^{+\infty} \frac{d\sigma(x)}{zx+1}$$

with a real constant $\gamma \geqslant 0$.

Stieltjes' (and Hamburger's) investigations of the undetermined case was elaborated by Nevanlinna (see [Nev22]). However, Nevanlinna's study itself was motivated by the following result. The work by Pick [Pick16] paved the way for including Hamburger's moment problem into a wider range of approximation problems (see [Akh65, Chapter 3]). Pick introduces the necessary and sufficient condition on values $\phi_0, \phi_1, \dots, \phi_k \in \mathbb{C}_+$ and pairwise distinct values $z_0, z_1, \dots, z_k \in \mathbb{C}_+$ that there exists an $\mathcal{R}$-function ϕ satisfying $\phi(z_0) = \phi_0,\ \phi(z_1) = \phi_1, \dots, \phi(z_k) = \phi_k$: the Hermitian form

$$\sum_{n,m=0}^{k} \frac{\phi_m - \overline{\phi_n}}{z_m - \overline{z}_n} \xi_m \overline{\xi}_n$$

must be nonnegative definite. Later, Nevanlinna [Nev29] extended this result to infinite sequences $(z_n)_{n=0}^{\infty}$ and $(\phi_n)_{n=0}^{\infty}$: it turns out that these Hermitian forms of all orders $k = 1, 2, \dots$ are nonnegative definite if and only if there exist a unique $\mathcal{R}$-function $\phi(z)$ satisfying $\phi(z_n) = \phi_n$ for all n.

The literature on the moment problem and on the Nevanlinna-Pick approximation is abundant, there is a large number of extensions, various reformulations *etc.*. The book [Akh65] can be recommended as a good introduction to advanced topics of moment problems, it highlights many corresponding questions like orthogonal polynomials, $\mathcal{R}$-functions and the Nevanlinna-Pick approximation. The paper [Berg94] reviews some of the more recent advances in describing the set of solution of indeterminate moment problems.

Numerous applications of $\mathcal{R}$-functions in spectral theory of operators prompted a fertile generalization of the class $\mathcal{R}$. The function classes $\mathcal{N}_\varkappa$ with $\varkappa = 0, 1, \dots$ were introduced in the prominent paper [KrLa77] by M. Krein and H. Langer.

Definition 1.3. A function $\phi(z)$ belongs to $\mathcal{N}_\varkappa$ whenever it is meromorphic in $\mathbb{C}_+$, for any set of non-real points $z_1, z_2, \dots, z_k$ the Hermitian form

$$\sum_{n,m=0}^{k} \frac{\phi(z_m) - \overline{\phi(z_n)}}{z_m - \overline{z}_n} \xi_m \overline{\xi}_n$$

has at most $\varkappa$ negative squares and for some set of points there are exactly $\varkappa$ negative squares.

It is convenient (and generally accepted) to define $\mathcal{N}_\varkappa$-functions in the lower half of the complex plane by complex conjugation, *i.e.* $\overline{\phi(z)} = \phi(\overline{z})$. As a result, the values in the two half-planes become an analytic continuation of each other through those real

points where real boundary limits exist. A significant particular case is presented by the classes $\mathcal{N}_\varkappa^+$. They contain all $\mathcal{N}_\varkappa$-functions $\varphi(z)$ such that $z\varphi(z)$ belongs to $\mathcal{R}$. As a foremost generalisation of the Stieltjes functions $\mathcal{N}_0^+$ (which is our $\mathcal{S}$ up to the change of variable $z \mapsto -\frac{1}{z}$), the classes $\mathcal{N}_\varkappa^+$ still have a significant potential of finding more applications.

The integral representations of $\mathcal{S}$- and $\mathcal{R}$-functions (see the formula (5.5) for the class $\mathcal{R}$ on Page 108) allowed using various complex-analytic methods which revealed deepest connections between various types of moment problems (*e.g.* on the unit circle). Further investigations in this direction significantly influenced the modern complex analysis itself, introducing many new objects (like the Hardy spaces). Nowadays, the method of integral representations became a powerful tool for studying functions of the classes $\mathcal{R}$, $\mathcal{S}$ and others, and found its way into a significant part of related works. One of the applications of this method appears in Chapter 5 of the present thesis: we prove a criterion for a function to be of the generalized Nevanlinna class $\mathcal{N}_\varkappa^+$.

1.2.2 Extension of Hurwitz stability to entire functions. Class $\mathcal{HB}$ and its generalizations

Definition 1.4. We use the notation $\overline{f}(z) := \overline{f(\overline{z})}$. Under the *real* and *imaginary parts* of an entire function $f(z)$ we understand the real entire functions $\frac{1}{2}(f(z) + \overline{f}(z))$ and $\frac{1}{2i}(f(z) - \overline{f}(z))$, respectively.

Extending the notion of stability to entire functions faces some difficulties. For example, if we state that an entire function is "stable" whenever it has no zeros with nonnegative real part, then both non-constant functions e^z and e^{-z} are "stable" simultaneously; this behaviour is somewhat unnatural since it is impossible for polynomials. Another problem in that case is that the conditions equivalent to stability, which are suitable for the polynomial case, do not work for general entire functions.

A reasonable extension to the notion of stability can be given by the generalization of the Hurwitz criterion. The generalization was first given by Grommer [Gro14, § 16, Satz IV] for functions of genus zero,[4] however he overlooked the condition on common zeros of odd and even parts. This flaw was addressed by Kreĭn: paper [Kre38] provides general representation of functions that have the infinite Hurwitz matrices (both real and complex versions) with positive leading principal minors. In other words, Kreĭn's work introduces some "natural" extension to the notion of stability, and points out one

[4]For the definition of genus and for other notions from the theory of entire functions see *e.g.* [CM49, Chapter III] or [Lev64, Chapter I].

serious drawback of this version of the Hurwitz criterion: it provides the result up to a factor which is an arbitrary real entire function (an even real entire function in the case of the Hurwitz criterion for real functions). The particular case of this result is stated as Theorem 3.32 on Page 63 of the present thesis. More about the role of Kreĭn's relevant investigations can be found in the review [Ost94].

Even though the extended Hurwitz criterion needs infinitely many conditions for implying the stability, in some practicable situations they can be equivalent to a finite set of conditions. One of the important examples is presented by quasipolynomials: this class of functions is of great interest for applications, and exploring its properties was a motivating factor for generalizing the Routh-Hurwitz theory from polynomials. The correspondent results were mainly obtained in [Pon42], see also [CM49, Chapter VII].

Another tool from the polynomial case is the Hermite-Biehler theorem. For entire functions of genus 0, conditions of this theorem are necessary and sufficient (both real and complex cases, see [Che41] and [Che42], respectively). For *real* entire functions of genus 1 the real version of those conditions remains necessary and sufficient for being in a narrower class, which does not contain entire functions like e^{-z} and $\frac{1}{z\Gamma(z)}$, see [Che41] and [Lev64, pp. 321, 334].

For complex entire functions of genus 1 both necessity and sufficiency parts of the unmodified Hermite-Biehler theorem fail to hold. On the one hand, the reality and interlacing property of zeros of the real and imaginary parts do not guarantee that all zeros lie in $\mathbb{C}_+$ or in $\mathbb{C}_-$. For example, the function $ze^z - i(z^2 - 1)$ has infinitely many zeros in both upper and lower halves of the complex plane. At the same time, its real and imaginary parts have real interlacing zeros. On the other hand, if all zeros of an entire function lie in the upper half of the complex plane then zeros of its real and imaginary parts do not need to be interlacing or real. Indeed, the zero of the function $(2z - i)\, e^{iz}$ belongs to $\mathbb{C}_+$, however the zeros of the real part $2z\cos z - \sin z$ and the imaginary part $\cos z + 2z\sin z$ do not interlace in the interval $(-\pi, \pi)$; furthermore, the imaginary part has two purely imaginary zeros. There is no doubt that we can construct functions of higher genus with worse behaviour.

These issues were resolved in works by Chebotarev, Pontrjagin, Levin, Meĭman, Neĭmark; see the summary of their results in [CM49, Lev64]. The key point here is the segregation of functions satisfying the Hermite-Biehler theorem from functions with "pathological" behaviour (caused, for example, by an excessively rapid growth). The segregation can be done through demanding some "additional" stability from the stable functions. This gives rise to the notion of *strong stability*, see [Pos81].

Definition 1.5. Let $g(z)$ and $h(z)$ be two real entire functions. The function $F(z) = g(z) + ih(z)$ is called strongly stable when all zeros of $F_\delta(z) = g(z) + (i+\delta)h(z)$ lie in the left half of the complex plane for all complex δ small enough in absolute value.

The strongly stable functions include all stable polynomials and functions which are close to them in some sense (for example, e^z); but they exclude such functions as e^{-z} and $\frac{1}{z\Gamma(z)}$. Strong stability can be understood as a kind of the "infinitesimal" robust stability.

Instead of studying the stability of a function $F(z)$ it is more convenient to study localization of zeros of the function $F(iz)$ with respect to the real line.

Definition 1.6. The Hermite-Biehler class $\mathcal{HB}$ is the set of all entire functions F with no zeros outside $\mathbb{C}_+$, which satisfy $|F(z)| < |F(\overline{z})|$ for all $z \in \mathbb{C}_+$.

A better understanding of this definition gives the following theorem, connecting the class $\mathcal{HB}$ with the set of strongly stable functions.

Theorem 1.7 (Chebotarev, [Che42]). *Given two real entire functions $g(z)$ and $h(z)$, the function $F(z) = g(z) + ih(z)$ belongs to the class $\mathcal{HB}$ if and only if $g(z) + (i+\delta)h(z)$ has no zeros in $\mathbb{C}_-$ for all complex δ small enough in absolute value.*

In fact, a function $F(z)$ belongs to the class $\mathcal{HB}$ as soon as $\frac{g(z)}{h(z)}$ avoids at least three values from the lower half-plane $\mathbb{C}_-$ when z runs over $\mathbb{C}_-$, see [Che42].

Unsurprisingly, strongly stable functions are functions satisfying the extended Hurwitz criterion up to multiplication by an arbitrary even entire function. As we noted above, the fraction of two interlacing polynomials is a rational function mapping the upper half of the complex plane into itself or into the lower half-plane. The class $\mathcal{HB}$ is exactly the class of functions such that the ratios of their real and imaginary parts are the meromorphic $\mathcal{R}$-functions. This deep connection provided an extension of the Hurwitz criterion. Moreover, this connection discloses the origin of the multiplication by an arbitrary entire function: this factor does not affect the ratios of real and imaginary parts.

A generalization of the class $\mathcal{HB}$ is suggested by Kaltenbäck and Woracek [KaWo99]. To get it, they replace the class $\mathcal{R}$ by one of generalized Nevanlinna classes $\mathcal{N}_\varkappa$. (Note that the definition of $\mathcal{HB}$ in [KaWo99] differs from ours in the sign of imaginary part, see Remark 2.2 in [PW2007a].) The work [KaWo99] brought up these generalized $\mathcal{HB}$-functions (the classes are denoted by $\mathcal{HB}_\varkappa$) in connection with indefinite versions of the de Branges spaces. The papers [PW2007a, PW2007b] investigate a relationship between symmetric and semibounded $\mathcal{HB}_\varkappa$-functions and give some of its applications.

Another interesting connection to the classes $\mathcal{N}_\varkappa$ is presented in [KW2003]. The authors show that, for $F \in \mathcal{HB}$, the properly selected branch of $-\dfrac{\log F(z)}{z}$ is an entry in $\mathcal{N}_\varkappa$ exactly when $F(z)$ belongs to the generalized Pólya class of order $\varkappa$ (and thus its order belongs to $[2\varkappa, 2\varkappa+2]$). Moreover, each $\mathcal{HB}$-function of finite order belongs to some generalized Pólya class.

1.2.3 Pólya frequency sequences and their generating functions

A doubly infinite sequence $(\rho_n)_{n=-\infty}^{\infty}$ is called *totally positive* if all of the minors of the (four-way infinite) Toeplitz matrix

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \rho_0 & \rho_1 & \rho_2 & \rho_3 & \rho_4 & \cdots \\ \cdots & \rho_{-1} & \rho_0 & \rho_1 & \rho_2 & \rho_3 & \cdots \\ \cdots & \rho_{-2} & \rho_{-1} & \rho_0 & \rho_1 & \rho_2 & \cdots \\ \cdots & \rho_{-3} & \rho_{-2} & \rho_{-1} & \rho_0 & \rho_1 & \cdots \\ \cdots & \rho_{-4} & \rho_{-3} & \rho_{-2} & \rho_{-1} & \rho_0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} =: T(f), \quad \text{where} \quad f(z) := \sum_{n=-\infty}^{\infty} \rho_n z^n.$$

are nonnegative (*i.e.* the matrix is *totally nonnegative*). The paper [Edr53] answers the question of convergence of the correspondent power series $f(z)$.

Theorem 1.8 (Edrei [Edr53], see also [Kar68, Section 8]). *Unless $\rho_n = \rho_0^{1-n}\rho_1^n$ for every $n \in \mathbb{Z}$, the series $f(z)$ converges in some annulus to a function with the following representation*

$$Cz^p e^{Az+\frac{A_0}{z}} \cdot \frac{\prod_{\nu>0}\left(1+\frac{z}{a_\nu}\right)}{\prod_{\mu>0}\left(1-\frac{z}{b_\mu}\right)} \cdot \frac{\prod_{\nu>0}\left(1+\frac{z^{-1}}{c_\nu}\right)}{\prod_{\mu>0}\left(1-\frac{z^{-1}}{d_\mu}\right)},$$

*where the products converge absolutely, integer p and coefficients satisfy $A, A_0, C \geqslant 0$, $a_\nu, b_\mu, c_\nu, d_\mu > 0$ for all ν, μ. The converse is also true: every function of this form generates (*i.e. *its Laurent coefficients give) a doubly infinite totally positive sequence.*

In the case of $\cdots = \rho_{-2} = \rho_{-1} = 0$, we assume the sequence to be terminating on the left of ρ_0 and call it *totally positive*. These sequences were studied in [AESW51, ASW52, Edr52]: they are generated by functions of the form

$$Cz^p e^{Az} \cdot \frac{\prod_{\nu>0}\left(1+\frac{z}{a_\nu}\right)}{\prod_{\mu>0}\left(1-\frac{z}{b_\mu}\right)},$$

where additionally p is nonnegative (the corresponding case of Theorem 1.8 is formulated

as Theorem 3.5 herein). The term *Pólya frequency sequence* is often used as a synonym for *totally positive sequence* (for example, in the book [Kar68]).

Definition 1.9. The class $\widetilde{\mathcal{PF}}$ we define as consisting of all functions generating (doubly infinite) totally positive sequences. The subclass of $\widetilde{\mathcal{PF}}$-functions smooth at the origin we denote by $\mathcal{PF}$.

Although akin designations for these classes occur in literature (*e.g.* [Al2001, Zh2004]), they are not perfect. These designations can be misleading since there is *another* closely related class of the so-called "Pólya frequency functions". In the book [Kar68, Chapter 8 § 4], for example, the designation "PF_r-function" stands for an entry in the latter class of order r. Nevertheless, we do not touch upon Pólya frequency functions, so our notation is acceptable for the present study.

It is a straightforward consequence of the definition that the Laurent series of a $\widetilde{\mathcal{PF}}$-function cannot contain gaps: that is, if $\rho_n \neq 0$ and $\rho_j = 0$ for some integers $n < j$ (or $n > j$), then $\rho_{j+1} = \rho_{j+2} = \cdots = 0$ (or $\rho_{j-1} = \rho_{j-2} = \cdots = 0$, respectively). The set of entire $\mathcal{PF}$-functions is the Laguerre-Pólya class $\mathcal{L} - \mathcal{P}^+$: the closure of polynomials with negative zeros with respect to the local uniform convergence. As a consequence, $\mathcal{PF}$- and $\widetilde{\mathcal{PF}}$-functions appear in solutions to various natural extensions of polynomial problems. One such extension is obtained in Chapter 3 of the present thesis.

Observe that if two functions $f(z)$ and $g(z)$ generate totally positive sequences, then both $f(z)g(z)$ and $\frac{1}{f(-z)}$ also do. Here, the doubly infinite case is admissible, although the convolution of corresponding series is a doubly infinite totally positive sequence only if the series for both $f(z)$ and $g(z)$ converge in some common annulus. Other properties of such functions are less obvious. Given $f \in \widetilde{\mathcal{PF}}$ it is possible to distinguish whether the minors of the Toeplitz matrix $T(f)$ are strictly positive or not based on the coefficients of its Weierstraß representation [Kar68, Chapter 8 § 10]. A bijection between $\mathcal{PF}$-functions $f(z)$ and their critical points up to the normalization $f'(z) = 1$ is constructed in [Er2004]. At that, the critical points are all real and satisfy a certain growth estimate. Numerators and denominators of Padé approximants to $\mathcal{PF}$-functions convergence separately; this remarkable property is shown in [ArEd70].

An important generalization is given by the functions generating Pólya frequency sequences of a *finite* order r, where $r = 1, 2, \ldots$ (also known as multiply positive sequences) introduced by Fekete [FePo12]. They are defined analogously to $\mathcal{PF}$-functions with an exception that the corresponding Toeplitz matrix satisfies a weaker condition that all of its minors of order $\leqslant r$ are nonnegative. Note that these functions are holomorphic in some neighbourhood of the origin provided that $r \geqslant 2$ [Kar68, p. 394]. The work [Scho55a]

reveals a connection of such functions with Obraschkoff's extension of Descartes' rule of signs. The finite Pólya frequency sequences of order r, as well as polynomials generating them, were extensively studied in [Scho55b]. In particular, such a polynomial of degree n has no zeros in the sector

$$|\operatorname{Arg} z| < \frac{r\pi}{n+r-1},$$

and this estimate is sharp. For the case of entire functions the work [KaOs89] shows that the corresponding class is rather wide: independently of r, their zeros can be any countable subset of $\mathbb{C} \setminus (0, +\infty)$ which has no local condensation points and is symmetric with respect to the real line. An analogous fact [Al2001] holds true for singularities when we abandon the supposition that functions of these classes are entire. The excessive wideness of functions generating Pólya frequency sequences of finite order motivated considering power series with sections and tiles satisfying this property. For the case $r \geqslant 3$ this is done in [OsZh98, OsZh99], where the authors obtain grow estimates; for the the case of sections, in [Zh2004] the reducibility of doubly (*i.e.* two-way) infinite sequences to pairs of one-way infinite sequences is proved.

1.3 Brief overview of forthcoming chapters

In the next chapter we present new results obtained in the paper [DB2016]. They are concerned with two conjectures by Csordas et al. regarding the interlacing property of zeros of special polynomials. These polynomials came from the Jacobi tau methods for the Sturm-Liouville eigenvalue problem. Their coefficients are the successive even derivatives of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ evaluated at the point one. The first conjecture states that the polynomials constructed from $P_n^{(\alpha,\beta)}$ and $P_{n-1}^{(\alpha,\beta)}$ are interlacing when $-1 < \alpha < 1$ and $-1 < \beta$. We prove it in a range of parameters wider than that given earlier by Charalambides and Waleffe. We also show that within narrower bounds another conjecture holds: it asserts that the polynomials constructed from $P_n^{(\alpha,\beta)}$ and $P_{n-2}^{(\alpha,\beta)}$ are also interlacing.

Chapter 3 gives a complete description of functions generating the infinite totally nonnegative Hurwitz matrices. In particular, we generalize the well-known result by Asner and Kemperman on the total nonnegativity of the Hurwitz matrices of real stable polynomials. An alternative criterion for entire functions to generate a Pólya frequency sequence is also obtained. The content of the chapter is based on the work [Dy2014]. The results exploit a connection between a factorization of totally nonnegative matrices of the Hurwitz type and the expansion of Stieltjes meromorphic functions into Stieltjes

continued fractions (*i.e.* regular C-fractions with positive coefficients).

Chapter 4 studies solutions of the equation $z^p R(z^k) = \alpha$ with nonzero complex α, integer p, k and $R(z)$ generating a (possibly doubly infinite) totally positive sequence. It is shown that the zeros of $z^p R(z^k) - \alpha$ are simple (or at most double in the case $\operatorname{Im} \alpha^k = 0$) and split evenly among the sectors $\left\{\frac{j}{k}\pi \leqslant \operatorname{Arg} z \leqslant \frac{j+1}{k}\pi\right\}$, $j = 0, \dots, 2k-1$. The initial result with the setting $k = 2$ was obtained in [Dy2013], while [Dy2016] generalizes it to other integer k and studies its relation to the class $\mathcal{R}$. Our approach rests on the fact that $z(\ln z^{p/k} R(z))'$ is an $\mathcal{R}$-function.

This result guarantees the same localization to zeros of entire functions

$$f(z^k) + z^p g(z^k) \quad \text{and} \quad g(z^k) + z^p f(z^k)$$

provided that $f(z)$ and $g(-z)$ have genus 0 and only negative zeros. As an application, we deduce that functions of the form $\sum_{n=0}^{\infty} (\pm i)^{n(n-1)/2} a_n z^n$ have simple zeros distinct in absolute value under a certain condition on the coefficients $a_n \geqslant 0$. This includes the "disturbed exponential" function corresponding to $a_n = q^{n(n-1)/2}/n!$ when $0 < q \leqslant 1$, as well as the partial theta function corresponding to $a_n = q^{n(n-1)/2}$ when $0 < q \leqslant q_* \approx 0.7457224107$.

In the last chapter, we give an elementary proof of the necessary and sufficient condition for a univariate function to belong the class $\mathcal{N}_\varkappa^+$. Although this class was introduced mainly to deal with the indefinite version of the Stieltjes moment problem (and corresponding π-Hermitian operators), it can be useful far beyond the original scope. In the last section we point out one connection between the classes $\mathcal{PF}$ and $\mathcal{N}_\varkappa^+$, which serves as an application of the obtained criterion. Our result elaborates the criterion given by Krein and Langer in their joint paper of 1977: they overlooked one attainable case. The correct condition was stated by Langer and Winkler in 1998, although they provided no proper reasoning. The proof, which is presented here, follows the work [Dy2015].

As an application of the criterion, Chapter 5 exposes a part of unpublished joint studies with Mikhail Tyaglov. We show that, in particular, the number of zeros (when it is finite) which has a $\widetilde{\mathcal{PF}}$ function determines the class $\mathcal{N}_\varkappa^+$ containing its logarithmic derivative. An analogous result holds for poles after the change of variable $z \mapsto -z$ in the logarithmic derivative.

Chapter 2.

Interlacing zeros of certain family of polynomials

2.1 Two conjectures stated by Csordas, Charalambides and Waleffe

The first topic of the present thesis is devoted to properties of zeros of polynomials which originate from mathematical physics. The orthogonal polynomials proved to be a very helpful tool for the discretization of linear differential operators. The main feature of the tau methods is the adoption of a polynomial basis which does not automatically satisfy the boundary conditions. This induces a problem at the boundary (*i.e.* the points ± 1 in the Jacobi case). The family of polynomials studied here is connected this way to the eigenproblem $u''(x) = \lambda u(x)$ on the interval $x \in (-1, 1)$ with various homogeneous boundary conditions (for the details see [CCW2005, CW2008]). We place our main emphasis on the analytic properties of the considered family itself, leaving aside the corresponding properties of the original differential operators. More information on the tau methods can be found in *e.g.* [CHQZ91, § 10.4.2].

The *Jacobi polynomials* (see their definition and basic properties in *e.g.* [Sze67, Ch. IV])

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} {}_2F_1\left[\begin{matrix} -n, & n+\alpha+\beta+1 \\ \alpha+1 & \end{matrix}; \frac{1-x}{2}\right], \quad n = 1, 2, \ldots$$

appear regularly in applications as classical orthogonal polynomials. They are more general than those of Chebyshev, Legendre and Gegenbauer. The Jacobi polynomials are orthogonal with respect to the measure $w_{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$ on the interval $(-1, 1)$ whenever both the parameters α and β are greater than -1:

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(x) w_{\alpha,\beta}(x)\, dx = 0 \quad \text{if} \quad k \neq n.$$

The usual normalization supposes that $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \frac{(\alpha+1)_n}{n!}$, where we applied the so-called *Pochhammer symbol* or the *rising factorial* defined as

$$(\alpha+1)_n := (\alpha+1) \cdot (\alpha+2) \cdots (\alpha+n).$$

In this notation we have

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (n+\alpha+\beta+1)_k}{k!\,(\alpha+1)_k} \left(\frac{1-x}{2}\right)^k, \quad n = 1, 2, \ldots. \tag{2.1}$$

Definition 2.1 (see [CM49, p. 17])**.** We say that the zeros of the polynomials $g(x)$ and $h(x)$ *interlace* (or *interlace strictly*) if the following conditions hold simultaneously:

- all zeros of $g(x)$ and $h(x)$ are simple, real and distinct (*i.e.* the polynomials are coprime),
- between each two consecutive zeros of $g(x)$ there is exactly one zero of the polynomial $h(x)$, and
- between each two consecutive zeros of $h(x)$ there is exactly one zero of the polynomial $g(x)$.

We say that the zeros of the polynomials $g(x)$ and $h(x)$ *interlace non-strictly* if their zeros are real and become strictly interlacing after dividing both polynomials by the greatest common divisor $\gcd(g, h)$. Roughly speaking, the zeros of two polynomials interlace non-strictly if they can meet but never pass through each other when changing continuously from a strictly interlacing state.

Definition 2.2. A pair $(g(x), h(x))$ is called *real* if for any real numbers A, B the combination $Ag(x) + Bh(x)$ has only real zeros. This is equivalent to the non-strict interlacing property of $g(x)$ and $h(x)$, which is shown in *e.g.* [CM49, Chapter I].

Remark 2.3. The phrases *"$g(x)$ and $h(x)$ interlace"*, *"$g(x)$ and $h(x)$ possess the interlacing property"*, *"$g(x)$ interlaces $h(x)$"*, *"$g(x)$ and $h(x)$ have interlacing zeros"* and *"the zeros of $g(x)$ and $h(x)$ are interlacing"* we use synonymously.

It is well-known that the orthogonal polynomials on the real line have real interlacing zeros (due to the so-called three-term recurrence; see *e.g.* [Sze67, pp. 42–47, Sections 3.2–3.3]). That is, in particular, the zeros of $P_n^{(\alpha,\beta)}$ and $P_{n-1}^{(\alpha,\beta)}$ interlace for all integers $n > 1$. In the current chapter we study zeros of polynomials that do not satisfy the three-term recurrence. More specifically, we consider

$$\begin{aligned} \phi_n^{(\alpha,\beta)}(\mu) &:= \sum_{k=0}^{[n/2]} \frac{d^{2k}}{dx^{2k}} P_n^{(\alpha,\beta)}(x)\bigg|_{x=1} \cdot \mu^k \\ &= \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{[n/2]} \frac{(-n)_{2k}(n+\alpha+\beta+1)_{2k}}{(\alpha+1)_{2k}} \left(\frac{\mu}{4}\right)^k, \quad n = 1, 2, \ldots, \end{aligned} \tag{2.2}$$

where the notation $[a]$ stands for the integer part of the number a.

Theorem 2.4 (Csordas, Charalambides and Waleffe [CCW2005]). *For every positive integer $n \geqslant 2$ the polynomial $\phi_n^{(\alpha,\beta)}(\mu)$, $-1 < \alpha < 1$, $-1 < \beta$, has only real negative zeros.*

The proof of this theorem given in [CCW2005] rests on the Hermite-Biehler theory (see Theorem H-B on Page 4).

Remark 2.5. In fact, the authors have shown that $\phi_n^{(\alpha,\beta)}(\mu)$ interlaces $\phi_{n-1}^{(\alpha+1,\beta+1)}(\mu)$. As a result, *the theorem remains valid when $1 \leqslant \alpha < 2$ and $0 < \beta$*. Note that the interlacing property here is strict, so it implies the simplicity of the zeros. Furthermore, *the polynomial $\phi_n^{(\alpha,\beta)}(\mu)$ has only simple negative zeros for $-1 < \alpha < 0$ and $-2 < \beta$,* as well. This follows as a straightforward consequence of Theorem 2.13* and Lemma 2.11 of the present study.

Based on Theorem 2.4 the authors of [CCW2005] conjectured that these polynomials also have the following property.

Conjecture 2.A ([CCW2005, p. 3559]). *For $-1 < \alpha < 1$, $-1 < \beta$ and $n \geqslant 4$ the zeros of the polynomials $\phi_n^{(\alpha,\beta)}$ and $\phi_{n-1}^{(\alpha,\beta)}$ interlace.*

In particular, this assertion would imply that the spectra of polynomial approximations to the corresponding differential operator are negative and simple (see [CW2008]). In [CW2008], Conjecture 2.A was proved for $-1 < \alpha, \beta < 0$ and $0 < \alpha, \beta < 1$: see Theorem 2.13 below. In fact, the upper bound on β is redundant. Theorem 2.13* with a shorter proof states that the conjecture holds true for

$$-1 < \alpha < 0,\ -1 < \beta \quad \text{or} \quad 0 \leqslant \alpha < 1,\ 0 < \beta \quad \text{or} \quad 1 \leqslant \alpha < 2,\ 1 < \beta. \tag{2.3}$$

The limitation of the present approach is highlighted in Remark 2.15: the proof of Theorem 2.13* relies on Theorem 2.14, which cannot be extended to the full range $-1 < \beta < 0 < \alpha < 1$. Accordingly, further studies on the validity of Conjecture 2.A need significant changes in methods.

Additionally, we study another assertion about the same polynomials.

Conjecture 2.B ([CCW2005, p. 3559]). *For $\phi_n^{(\alpha,\beta)}(\mu)$ as in Theorem 2.4 and for all $n \geqslant 5$, the zeros of the polynomials $\phi_n^{(\alpha,\beta)}(\mu)$ and $\phi_{n-2}^{(\alpha,\beta)}(\mu)$ interlace.*

Originally, this conjecture was stated for $-1 < \alpha < 1$ and $\beta > -1$. However, numerical calculations show that it fails for some values satisfying $-1 < \beta < 0 < \alpha < 1$. Our (partial) solution to Conjecture 2.B is given in Theorem 2.21: it holds true for $-1 < \alpha < 0 < \beta$ or $0 < \alpha < 1 < \beta$. We approach by extending the idea of [CW2008] to another pair of auxiliary polynomials. Certainly, there exists a relation between Conjecture 2.A and Conjecture 2.B as discussed in Section 2.5. In particular, Lemma 2.22 says that if the polynomials $\phi_n^{(\alpha,\beta)}(\mu)$, $\phi_{n-1}^{(\alpha,\beta)}(\mu)$ and $\phi_{n-2}^{(\alpha,\beta)}(\mu)$ are pairwise interlacing, then $\phi_n^{(\alpha,\beta-1)}(\mu)$ interlaces $\phi_{n-1}^{(\alpha,\beta-1)}(\mu)$. Involving relations between the

families $\left(\phi_n^{(\alpha,\beta)}(\mu)\right)_n$ and $\left(\phi_n^{(\alpha+1,\beta+1)}(\mu)\right)_n$ allows us to deduce the interlacing property of $\phi_n^{(\alpha+1,\beta)}(\mu)$ and $\phi_{n-1}^{(\alpha+1,\beta)}(\mu)$ in Lemma 2.23.

Vieta's formulae imply that the sum of all zeros of $\phi_n^{(\alpha,\beta)}(\mu)$ tends to $-\frac{1}{2}$ for even n and to $-\frac{1}{6}$ for odd n as $n \to \infty$. Thus, the assertion of Conjecture 2.B gives that the zero points of $\phi_n^{(\alpha,\beta)}(\mu)$ converge monotonically in n outside of any fixed interval containing the origin. If the assertions of both conjectures hold, then the fraction $\phi_{2n-1}^{(\alpha,\beta)}(\mu)/\phi_{2n}^{(\alpha,\beta)}(\mu)$ maps the upper half of the complex plane into itself and converges to a function meromorphic outside of any disk centred at the origin. This situation resembles how the quotients of orthogonal polynomials of the first and second kinds (*e.g.* [Akh65, p. 8]) behave.

Section 2.2 introduces connections between polynomials $\phi_n^{(\alpha,\beta)}(\mu)$ with different n, α and β. These connections allow us to extend and clarify the result [CW2008] in Section 2.3 (see Theorem 2.13*). We show that Conjecture 2.A holds true under the conditions (2.3). Section 2.4 contains the proof of Conjecture 2.B for $-1 < \alpha < 0 < \beta$ and $0 < \alpha < 1 < \beta$ (see Theorem 2.21). In the last section we show that the studied conjectures are actually related.

2.2 Basic relations between the involved polynomials $\phi_n^{(\alpha,\beta)}$ for various α and β

Being connected with the Jacobi polynomials, the family $\left(\phi_n^{(\alpha,\beta)}\right)_n$, where $n = 2, 3, \ldots$, inherits some of their properties. The formulae induced by the corresponding relations for the Jacobi case include (we omit the argument μ of $\phi_n^{(\alpha,\beta)}$ for brevity's sake):

$$(2n+\alpha+\beta)\phi_n^{(\alpha,\beta-1)} = (n+\alpha+\beta)\phi_n^{(\alpha,\beta)} + (n+\alpha)\phi_{n-1}^{(\alpha,\beta)}, \tag{2.4}$$

$$(2n+\alpha+\beta)\phi_n^{(\alpha-1,\beta)} = (n+\alpha+\beta)\phi_n^{(\alpha,\beta)} - (n+\beta)\phi_{n-1}^{(\alpha,\beta)}, \tag{2.5}$$

$$(n+\alpha+\beta)\phi_n^{(\alpha,\beta)} \overset{(2.4)+(2.5)}{=\!=\!=} (n+\beta)\phi_n^{(\alpha,\beta-1)} + (n+\alpha)\phi_n^{(\alpha-1,\beta)}, \tag{2.6}$$

$$\phi_{n-1}^{(\alpha,\beta)} \overset{(2.4)-(2.5)}{=\!=\!=} \phi_n^{(\alpha,\beta-1)} - \phi_n^{(\alpha-1,\beta)}. \tag{2.7}$$

The latter two identities contain labels of equations above the equality sign: we use the convention that this explains how the equalities can be obtained (up to some proper coefficients). For example, $2n+\alpha+\beta$ times (2.6) is the sum of $n+\beta$ times (2.4) and $n+\alpha$ times (2.5), whereas $2n+\alpha+\beta$ times (2.7) is the difference (2.4)−(2.5). The identities (2.4) and (2.5) can be checked by applying the formulae (see *e.g.* [PBM92, p. 737])

$$(2n+\alpha+\beta)P_n^{(\alpha,\beta-1)}(x) = (n+\alpha+\beta)P_n^{(\alpha,\beta)}(x) + (n+\alpha)P_{n-1}^{(\alpha,\beta)}(x), \tag{2.8}$$

$$(2n+\alpha+\beta)P_n^{(\alpha-1,\beta)}(x) = (n+\alpha+\beta)P_n^{(\alpha,\beta)}(x) - (n+\beta)P_{n-1}^{(\alpha,\beta)}(x), \tag{2.9}$$

respectively, to the left-hand side of the definition (2.2). The relations involving derivatives (not surprisingly) differ from those for the Jacobi polynomials:

$$(n+\alpha)\phi_{n-1}^{(\alpha,\beta+1)} = n\phi_n^{(\alpha,\beta)} - 2\mu\left(\phi_n^{(\alpha,\beta)}\right)', \tag{2.10}$$

$$\phi_n^{(\alpha,\beta+1)} \overset{\text{(2.10) and (2.4)}}{=\!=\!=} \phi_n^{(\alpha,\beta)} + \frac{2\mu}{n+\alpha+\beta+1}\left(\phi_n^{(\alpha,\beta)}\right)', \tag{2.11}$$

$$(n+\alpha)\phi_n^{(\alpha-1,\beta+1)} \overset{\text{(2.5), then (2.11)–(2.10)}}{=\!=\!=} \alpha\phi_n^{(\alpha,\beta)} + 2\mu\left(\phi_n^{(\alpha,\beta)}\right)', \tag{2.12}$$

$$\frac{n+\beta}{n+\alpha+\beta}2\mu\left(\phi_n^{(\alpha,\beta-1)}\right)' \overset{\text{(2.11) and (2.6)}}{=\!=\!=} (n+\alpha)\phi_n^{(\alpha-1,\beta)} - \alpha\phi_n^{(\alpha,\beta)}, \tag{2.13}$$

$$2\mu\left(\phi_n^{(\alpha,\beta-1)}\right)' \overset{\text{(2.10) and (2.7)}}{=\!=\!=} n\phi_n^{(\alpha-1,\beta)} - \alpha\phi_{n-1}^{(\alpha,\beta)}. \tag{2.14}$$

So we see that the presented identities are not independent in the sense that they can be obtained as combinations of others with various α and β. We derive the formula (2.10) with the help of the right-hand side of (2.2).

$$\begin{aligned}
&n\phi_n^{(\alpha,\beta)} - 2\mu\left(\phi_n^{(\alpha,\beta)}\right)' \\
&\quad= \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{[n/2]}\frac{(-n)_{2k}(n+\alpha+\beta+1)_{2k}}{(\alpha+1)_{2k}}\left(\frac{\mu}{4}\right)^k(n-2k) \\
&\quad= (n+\alpha)\frac{(\alpha+1)_{n-1}}{(n-1)!} \\
&\qquad\times\sum_{k=0}^{[n/2]}(-n+2k)\frac{(-n)(-n+1)\cdots(-n+2k-1)(n+\alpha+\beta+1)_{2k}}{(-n)(\alpha+1)_{2k}}\left(\frac{\mu}{4}\right)^k \\
&\quad= (n+\alpha)\phi_{n-1}^{(\alpha,\beta+1)}.
\end{aligned}$$

Certainly, we can combine formulae further obtaining *e.g.*

$$\begin{aligned}
n\phi_n^{(\alpha,\beta)} - 2\mu\left(\phi_n^{(\alpha,\beta)}\right)' &\overset{(2.10)}{=\!=\!=} (n+\alpha)\phi_{n-1}^{(\alpha,\beta+1)} \\
&\overset{(2.11)}{=\!=\!=} (n+\alpha)\phi_{n-1}^{(\alpha,\beta)} + \frac{n+\alpha}{n+\alpha+\beta}2\mu\left(\phi_{n-1}^{(\alpha,\beta)}\right)',
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
n\phi_n^{(\alpha,\beta)} - (n+\alpha)\phi_{n-1}^{(\alpha,\beta)} &\overset{\text{(2.11)–(2.10)}}{=\!=\!=} \frac{2n+\alpha+\beta}{n+\alpha+\beta}2\mu\left(\phi_n^{(\alpha,\beta-1)}\right)' \\
&\overset{\frac{d}{d\mu}(2.4)}{=\!=\!=} 2\mu\left(\phi_n^{(\alpha,\beta)}\right)' + \frac{n+\alpha}{n+\alpha+\beta}2\mu\left(\phi_{n-1}^{(\alpha,\beta)}\right)'.
\end{aligned} \tag{2.16}$$

The key role further plays the following combination of (2.10) and (2.11), which is valid for an arbitrary real A,

$$\frac{n+\alpha+\beta}{n+\alpha}\phi_n^{(\alpha,\beta)} + A\phi_{n-1}^{(\alpha,\beta)} = \frac{(1+A)n+\alpha+\beta}{n+\alpha}\phi_n^{(\alpha,\beta-1)} + 2\mu\frac{1-A}{n+\alpha}\left(\phi_n^{(\alpha,\beta-1)}\right)'. \tag{2.17}$$

The next identity stands apart and can be checked explicitly with the help of (2.2)

$$\phi_n^{(\alpha,\beta)}(\mu) - \phi_n^{(\alpha,\beta)}(0) = \tfrac{1}{4}(n+\alpha+\beta+1)_2 \cdot \mu\phi_{n-2}^{(\alpha+2,\beta+2)}(\mu).$$

It reflects the standard formula for the derivative of the Jacobi polynomial (*e.g.* [PBM92, p. 737]):

$$\left(P_n^{(\alpha,\beta)}(x)\right)^{(m)} = 2^{-m}(n+\alpha+\beta+1)_m\, P_{n-m}^{(\alpha+m,\beta+m)}(x). \tag{2.18}$$

Remark 2.6. Note that the equalities (2.4)–(2.18) are of formal nature, and therefore their validity requires no orthogonality from the Jacobi polynomials. That is, these equalities hold true if all coefficients are defined, not only for $\alpha, \beta > -1$.

Remark 2.7. A polynomial with only real zeros interlaces its derivative by Rolle's theorem.[1] Consequently, they both interlace any real combination of them.[2] So if in one of the formulae (2.10)–(2.12) the first term on the right-hand side has only real zeros, then all three involved polynomials are pairwise interlacing. For example, if $\phi_n^{(\alpha,\beta)}$ has only real zeros, then $\phi_{n-1}^{(\alpha,\beta+1)}$, $\phi_n^{(\alpha,\beta)}$ and $2\mu\left(\phi_n^{(\alpha,\beta)}\right)'$ are pairwise interlacing which is provided by (2.10).

Remark 2.8. The identities (2.6) and (2.7) show that the interlacing property of the polynomials $\phi_n^{(\alpha,\beta)}(\mu)$ and $\phi_{n-1}^{(\alpha,\beta)}(\mu)$ can also be expressed as the interlacing property of $\phi_n^{(\alpha,\beta-1)}(\mu)$ and $\phi_n^{(\alpha-1,\beta)}(\mu)$. Analogously, from the relations (2.13) and (2.14) it can be seen that this is also equivalent to the interlacing property of $\left(\phi_n^{(\alpha,\beta-1)}(\mu)\right)'$ and $\phi_n^{(\alpha-1,\beta)}(\mu)$.

Lemma 2.9. *If $-1 < \alpha < 1$, $-1 < \beta$ or if $1 \leqslant \alpha < 2$, $0 < \beta$, then the polynomials $\left(\phi_n^{(\alpha,\beta)}(\mu)\right)'$, $\left(\phi_{n-1}^{(\alpha,\beta)}(\mu)\right)'$ and $\left(\phi_n^{(\alpha,\beta-1)}(\mu)\right)'$ are pairwise interlacing.*

Proof. By Theorem 2.4 the polynomials $\phi_n^{(\alpha,\beta)}(\mu)$, $\phi_{n-1}^{(\alpha,\beta)}(\mu)$ have only (simple) negative zeros. Then the relation (2.10) shows that the polynomial $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$ with negative zeros interlaces (strictly) both $\phi_n^{(\alpha,\beta)}(\mu)$ and $\mu\left(\phi_n^{(\alpha,\beta)}(\mu)\right)'$. Moreover, the sign of $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$ at the origin and at the rightmost zero of $\phi_n^{(\alpha,\beta)}(\mu)$ is the same, and therefore $\left(\phi_n^{(\alpha,\beta)}(\mu), \mu\phi_{n-1}^{(\alpha,\beta+1)}(\mu)\right)$ is a real coprime pair. A similar consideration of the relation (2.11) gives that the pair $\left(\mu\phi_{n-1}^{(\alpha,\beta)}(\mu), \phi_{n-1}^{(\alpha,\beta+1)}(\mu)\right)$ is also real and coprime.

In particular, each interval between two consequent zeros of $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$ contains exactly one zero of $\phi_n^{(\alpha,\beta)}(\mu)$ as well as of $\phi_{n-1}^{(\alpha,\beta)}(\mu)$. Taking the signs of the last two polynomials at the ends of these intervals into account shows that the difference on the left-hand side of (2.16), and hence $\left(\phi_n^{(\alpha,\beta-1)}(\mu)\right)'$, changes its sign between consecutive zeros

[1] Non-strictly whenever the polynomial has a multiple zero.

[2] See the definition of a real pair on the page 19.

of $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$. Since $\deg\left(\phi_n^{(\alpha,\beta-1)}\right)' \leqslant \deg \phi_{n-1}^{(\alpha,\beta+1)}$, the polynomial $\mu\left(\phi_n^{(\alpha,\beta-1)}(\mu)\right)'$ necessarily interlaces $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$. Then the right-hand side of (2.16) shows that

$$\left(\phi_n^{(\alpha,\beta)}(\mu)\right)' + \frac{n+\alpha}{n+\alpha+\beta}\left(\phi_{n-1}^{(\alpha,\beta)}(\mu)\right)' \tag{2.19}$$

and $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$ have interlacing zeros. At the same time, by differentiating the equality (2.16) we obtain that

$$n\left(\phi_n^{(\alpha,\beta)}(\mu)\right)' - (n+\alpha)\left(\phi_{n-1}^{(\alpha,\beta)}(\mu)\right)' \tag{2.20}$$

is proportional to $\left(\mu\left(\phi_n^{(\alpha,\beta-1)}(\mu)\right)'\right)'$ and, hence, interlaces $\mu\left(\phi_n^{(\alpha,\beta-1)}(\mu)\right)'$. Put in other words, the polynomials (2.19) and (2.20) are interlacing. With appropriate factors, their sum gives $\left(\phi_n^{(\alpha,\beta)}(\mu)\right)'$ and their difference gives $\left(\phi_{n-1}^{(\alpha,\beta)}(\mu)\right)'$. This yields the lemma. □

2.3 Results on Conjecture 2.A

The Hermite-Biehler theorem (Theorem H-B on Page 4) plays a crucial role in [CCW2005] for proving Theorem 2.4, as well as for the present study.

Some bounds on the parameters α and β are necessary even for Theorem 2.4 (*i.e.* are satisfied if all zeros of $\phi_n^{(\alpha,\beta)}(\mu)$ are negative for every $n \geqslant 2$). The restriction $\alpha > -1$ corresponds to positivity of the coefficients (and to negativity of all zeros). The parameter α is bounded from above by $3.37228\ldots$. Indeed, if the polynomial $\phi_n^{(\alpha,\beta)}(\mu) =: \sum_{k=0}^{[n/2]} b_k \mu^k$ with positive leading coefficient has only negative zeros, then necessarily $b_1 > 0$ and one of Newton's inequalities implies

$$\frac{b_1^2}{b_0 b_2} \geqslant \frac{2\left[\frac{n}{2}\right]}{\left[\frac{n}{2}\right]-1} > 2. \tag{2.21}$$

At the same time, in our case we have

$$\frac{b_1^2}{b_0 b_2} = \frac{n(n-1)(\alpha+3)_2(n+\alpha+\beta+1)_2}{(n-2)(n-3)(\alpha+1)_2(n+\alpha+\beta+3)_2} \xrightarrow{n\to\infty} \frac{(\alpha+3)(\alpha+4)}{(\alpha+1)(\alpha+2)},$$

so the condition (2.21) fails to be true (along with the assertion of Theorem 2.4) for every n big enough and every β when $\alpha > \frac{1+\sqrt{33}}{2} = 3.37228\ldots$ or $\alpha < \frac{1-\sqrt{33}}{2} = -2.37228\ldots$. Computer experiments show that the polynomials $\phi_n^{(\alpha,\beta)}$ with *positive coefficients* can have zeros outside the real axis for a (large enough) negative β. So, some lower constraint on the parameter β is also required.

Definition 2.10. Denote the ith zero of a polynomial p with respect to the distance from the origin by $\mathbf{zr}_i(p)$. Put $\mathbf{zr}_i(p)$ equal to $-\infty$ if $\deg p < i$ and to zero if $i = 0$ (it is convenient since all coefficients of the polynomials we deal with are nonnegative).

Lemma 2.11. *Let $\alpha > -1$ and $n+\alpha+\beta > 0$, $n = 4, 5, \ldots$. The zeros of the polynomials $\phi_n^{(\alpha,\beta)}$ and $\phi_{n-1}^{(\alpha,\beta)}$ are negative and interlace non-strictly (strictly) if and only if the polynomial $\phi_n^{(\alpha,\beta-1)}$ has only real zeros (only simple real zeros, respectively). Moreover, if $\phi_n^{(\alpha,\beta-1)}$ has only real zeros, then $\mathbf{zr}_1\left(\phi_{n-1}^{(\alpha,\beta)}\right) \leqslant \mathbf{zr}_1\left(\phi_n^{(\alpha,\beta)}\right)$.*

Proof. The relation (2.17) with $A = 1$ and $A = -\frac{n+\alpha+\beta}{n}$ implies that each common zero of the polynomials $\phi_{n-1}^{(\alpha,\beta)}$ and $\phi_n^{(\alpha,\beta)}$ is a multiple zero of $\phi_n^{(\alpha,\beta-1)}$. The converse result is given by (2.10) and (2.11).

Suppose that $\phi_n^{(\alpha,\beta-1)}$ has only real zeros. The coefficients of this polynomial are positive under the assumptions of the lemma, and hence all of its zeros are negative. By Rolle's theorem, the pair $\left(\phi_n^{(\alpha,\beta-1)}, \mu\left(\phi_n^{(\alpha,\beta-1)}\right)'\right)$ is real. Therefore, the polynomials $\phi_{n-1}^{(\alpha,\beta)}$ and $\phi_n^{(\alpha,\beta)}$ have only real zeros by the formulae (2.10) and (2.11), respectively. The zeros are negative automatically since the coefficients of polynomials are positive. Moreover, we have that the first zero of $\phi_n^{(\alpha,\beta)}$ is closer to the origin than that of $\phi_{n-1}^{(\alpha,\beta)}$. The relation (2.17) holds for all real A, which yields that the polynomials $\phi_{n-1}^{(\alpha,\beta)}$ and $\phi_n^{(\alpha,\beta)}$ form a real pair and thus have (non-strictly) interlacing zeros.

Let $\phi_{n-1}^{(\alpha,\beta)}$ and $\phi_n^{(\alpha,\beta)}$ have negative interlacing zeros. Then any of their real combinations only has real zeros. This is true for $\phi_n^{(\alpha,\beta-1)}$ according to the identity (2.4). □

Corollary 2.12. *For $-1 < \alpha < 1$, $\beta > 0$ or $1 \leqslant \alpha < 2$, $\beta > 1$ the zeros of the polynomials $\phi_n^{(\alpha,\beta)}$ and $\phi_{n-1}^{(\alpha,\beta)}$, $n = 4, 5, \ldots$, interlace. (Furthermore, the polynomials $\phi_n^{(\alpha,\beta)}$ and $\mu\phi_{n-1}^{(\alpha,\beta)}$ interlace.)*

Proof. This corollary is provided by Theorem 2.4 (see also Remark 2.5) and Lemma 2.11. □

Theorem 2.13 ([CW2008, Theorem 3.10]). *Conjecture* 2.A *holds true for $-1 < \alpha, \beta < 0$ and for $0 < \alpha, \beta < 1$.*

This theorem relies on Theorem 3.8 and Theorem 3.9 of the same work and on Theorem 2.4. In fact, the original proof (which we extend in the next section to treat Conjecture 2.B) does not need any upper bound on β. It becomes more evident on recalling that the region of positive β is covered by Corollary 2.12 (*i.e.* is a straightforward consequence of Lemma 2.11 and Theorem 2.4).

Theorem 2.13*. *Conjecture* 2.A *holds true for $-1 < \alpha < 0$, $-1 < \beta$ or $0 \leqslant \alpha < 1$, $0 < \beta$ or $1 \leqslant \alpha < 2$, $1 < \beta$.*

Proof. Corollary 2.12 suits the case of positive α. For the region with negative α it is enough to prove only Theorem 2.14 (which is stated below) and Theorem 3.9 is not needed. Indeed, according to Theorem 2.14 and Theorem H-B we have that all zeros of $\phi_n^{(\alpha,\beta-1)}$ are simple and real for all n, so Lemma 2.11 is applicable. □

Theorem 2.14 (Equivalent to [CW2008, Theorem 3.8]). *If $-1 < \alpha < 0$, $-1 < \beta$ and $n = 4, 5, \ldots$, then the zeros of the polynomial $\Phi_n(1;\mu)$, where*

$$\Phi_n(x;\mu) := \sum_{k=0}^{n} \frac{d^k}{dx^k} P_n^{(\alpha,\beta-1)}(x)\mu^k,$$

lie in the open left half of the complex plane.

Remark 2.15. It is worth noting that this theorem cannot be extended to the full range $-1 < \beta < 0 < \alpha < 1$. According to computer experiments, Conjecture 2.A seems to hold in this range, while Theorem 2.14 fails *e.g.* for $n = 12$ when $\beta = -0.8$ and $\alpha \gtrapprox 0.97842$, or when $\beta = -0.9$ and $\alpha \gtrapprox 0.97140$. The reason is that proving Conjecture 2.A only requires negative simple zeros of the polynomial $\phi_n^{(\alpha,\beta-1)}(\mu)$, Theorem 2.14 nevertheless asserts additional properties of $\phi_{n-1}^{(\alpha+1,\beta)}(\mu)$ as given by the Hermite-Biehler theorem.

Proof. This proof is akin to [CW2008, Theorem 3.8] but uses other relations for the Jacobi polynomials. The polynomial $\Phi_n(x;\mu)$ satisfies the differential equation (here we consider μ as a parameter)

$$\Phi_n(x;\mu) = P_n^{(\alpha,\beta-1)}(x) + \mu\frac{d\Phi_n(x;\mu)}{dx}.$$

Let $\Phi_n := \Phi_n(x;\mu)$ for brevity and let $\overline{\frac{d\Phi_n}{dx}}$ denote a complex conjugate of $\frac{d\Phi_n}{dx}$. Multiplying by $\overline{\frac{d\Phi_n}{dx}} w_{\alpha,\beta+1}$, where $w_{\alpha,\beta+1} := w_{\alpha,\beta+1}(x) = (1-x)^\alpha(1+x)^{\beta+1}$, and integration over the interval $(-1,1)$ gives us

$$\int_{-1}^{1} \Phi_n \overline{\frac{d\Phi_n}{dx}} w_{\alpha,\beta+1}\,dx = \int_{-1}^{1} P_n^{(\alpha,\beta-1)} \overline{\frac{d\Phi_n}{dx}} w_{\alpha,\beta+1}\,dx + \mu\int_{-1}^{1}\left|\frac{d\Phi_n}{dx}\right|^2 w_{\alpha,\beta+1}\,dx.$$

Select μ so that $\Phi_n(1;\mu) = 0$. To estimate the real part of μ we add the last equation to its complex conjugate and obtain

$$\begin{aligned}\int_{-1}^{1} &\frac{d\left(|\Phi_n|^2\right)}{dx} w_{\alpha,\beta+1}\,dx \\ &= \int_{-1}^{1} P_n^{(\alpha,\beta-1)} \cdot 2\operatorname{Re}\frac{d\Phi_n}{dx} \cdot w_{\alpha,\beta+1}\,dx + 2\operatorname{Re}\mu\int_{-1}^{1}\left|\frac{d\Phi_n}{dx}\right|^2 w_{\alpha,\beta+1}\,dx.\end{aligned} \tag{2.22}$$

Since $w_{\alpha,\beta+1}$ increases on $(-1,1)$ and $\lim_{x\to -1+}\Phi_n w_{\alpha,\beta+1} = \lim_{x\to 1-}\Phi_n w_{\alpha,\beta+1} = 0$, the left-hand side satisfies

$$\int_{-1}^{1}\frac{d\left(|\Phi_n|^2\right)}{dx}w_{\alpha,\beta+1}\,dx = -\int_{-1}^{1}|\Phi_n|^2 w'_{\alpha,\beta+1}\,dx < 0.$$

Applying the relation (2.8) to the polynomial $P_n^{(\alpha,\beta-1)}$ three times gives us

$$\begin{aligned}P_n^{(\alpha,\beta-1)} &= \frac{n+\alpha+\beta}{2n+\alpha+\beta}P_n^{(\alpha,\beta)} + \frac{n+\alpha}{2n+\alpha+\beta}P_{n-1}^{(\alpha,\beta)}\\ &= \frac{(n+\alpha+\beta)_2}{(2n+\alpha+\beta)_2}P_n^{(\alpha,\beta+1)} + \frac{(n+\alpha+\beta)(n+\alpha)}{(2n+\alpha+\beta)_2}P_{n-1}^{(\alpha,\beta+1)}\\ &\quad + \frac{(n+\alpha)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)_2}P_{n-1}^{(\alpha,\beta+1)} + \frac{(n+\alpha-1)_2}{(2n+\alpha+\beta-1)_2}P_{n-2}^{(\alpha,\beta+1)},\end{aligned}$$

that is,

$$\begin{aligned}P_n^{(\alpha,\beta-1)} &= \frac{(n+\alpha+\beta)_2}{(2n+\alpha+\beta)_2}P_n^{(\alpha,\beta+1)} + \frac{2(n+\alpha)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta+1)}P_{n-1}^{(\alpha,\beta+1)}\\ &\quad + \frac{(n+\alpha-1)_2}{(2n+\alpha+\beta-1)_2}P_{n-2}^{(\alpha,\beta+1)}.\end{aligned}$$

By the definition of Φ_n and the formula (2.18),

$$\operatorname{Re}\frac{d\Phi_n}{dx} = 2^{-1}(n+\alpha+\beta)P_{n-1}^{(\alpha+1,\beta)} + \operatorname{Re}\mu\cdot 2^{-2}(n+\alpha+\beta)_2 P_{n-2}^{(\alpha+2,\beta+1)} + \psi,$$

where ψ is a polynomial of degree at most $n-3$. The difference (2.8) − (2.9) induces the identity $P_{n-1}^{(\alpha+1,\beta)} = P_{n-1}^{(\alpha,\beta+1)} + P_{n-2}^{(\alpha+1,\beta+1)}$, so we finally have

$$\int_{-1}^{1}P_n^{(\alpha,\beta-1)}\cdot 2\operatorname{Re}\frac{d\Phi_n}{dx}\cdot w_{\alpha,\beta+1}\,dx = \eta + \zeta\operatorname{Re}\mu, \quad \text{where} \quad \eta,\zeta > 0.$$

Now the terms of the relation (2.22) are estimated, and it yields $0 > \operatorname{Re}\mu$. □

2.4 Results on Conjecture 2.B

We just have shown that the result on Conjecture 2.A in [CW2008] can be obtained in a shorter way if we consider polynomials with shifted parameter values (we used $\phi_n^{(\alpha,\beta-1)}$) instead of real combinations of the polynomials $\phi_n^{(\alpha,\beta)}$ and $\phi_{n-1}^{(\alpha,\beta)}$. At the same time, to verify Conjecture 2.B we can combine both these ideas.

For any fixed $n > 3$ consider the intermediary polynomial

$$f(x;\mu) := \sum_{k=0}^{n}\mu^k\left(A\frac{d^k}{dx^k}P_n^{(\alpha,\beta)}(x) + \mu\frac{d^k}{dx^k}P_{n-1}^{(\alpha,\beta)}(x)\right).$$

Lemma 2.16. *The polynomial $f(1;\mu)$ is Hurwitz-stable on condition that $-1 < \alpha < 1$ and $\beta, A > 0$.*

Proof. From the definition of $f(x,\mu)$ the differential equation

$$\mu\frac{d}{dx}f(x;\mu) + AP_n^{(\alpha,\beta)}(x) + \mu P_{n-1}^{(\alpha,\beta)}(x) = f(x;\mu)$$

follows. Multiplication by $\overline{f(x;\mu)}w_{\alpha-1,\beta}(x)$ gives us

$$\overline{f}\frac{df}{dx}w_{\alpha-1,\beta} + AP_n^{(\alpha,\beta)}(x)\frac{\overline{f}}{\mu}w_{\alpha-1,\beta} + P_{n-1}^{(\alpha,\beta)}(x)\overline{f}w_{\alpha-1,\beta} = \frac{1}{\mu}|f|^2 w_{\alpha-1,\beta},$$

where we put $f := f(x;\mu)$ and $w_{\alpha-1,\beta} := w_{\alpha-1,\beta}(x) = (1-x)^{\alpha-1}(1+x)^{\beta}$ for brevity. Adding to this equality its complex conjugate and integrating yields

$$\int_{-1}^{1}\frac{d\left(|f|^2\right)}{dx}w_{\alpha-1,\beta}\,dx + A\int_{-1}^{1}P_n^{(\alpha,\beta)}(x)\left(\frac{\overline{f}}{\mu}+\frac{f}{\overline{\mu}}\right)w_{\alpha-1,\beta}\,dx$$
$$+\int_{-1}^{1}P_{n-1}^{(\alpha,\beta)}(x)\left(\overline{f}+f\right)w_{\alpha-1,\beta}\,dx = \left(\frac{1}{\mu}+\frac{1}{\overline{\mu}}\right)\int_{-1}^{1}|f|^2 w_{\alpha-1,\beta}\,dx. \quad (2.23)$$

Take μ so that $f(1;\mu) = 0$. Then the polynomial $f(x;\mu)$ can be represented as

$$f(x;\mu) = (1-x)\sum_{k=0}^{n-1}c_k P_k^{(\alpha,\beta)}(x)$$

with some complex constants c_k depending on μ (generally speaking). Observe that $c_{n-1} < 0$: denoting the leading coefficient in x by $\mathbf{lc}$, we obtain

$$c_{n-1} = \frac{\mathbf{lc}\left(f(x;\mu)\right)}{\mathbf{lc}\left(-xP_{n-1}^{(\alpha,\beta)}(x)\right)} = -\frac{A\cdot\mathbf{lc}\left(P_n^{(\alpha,\beta)}(x)\right)}{\mathbf{lc}\left(P_{n-1}^{(\alpha,\beta)}(x)\right)}$$
$$\overset{(2.1)}{=\!=\!=} -A\frac{(n+\alpha+\beta+1)_n\cdot(n-1)!\,2^{n-1}}{n!\,2^n\cdot(n+\alpha+\beta)_{n-1}} = -A\frac{(2n+\alpha+\beta-1)_2}{2n(n+\alpha+\beta)} < 0.$$

Then we have

$$\begin{aligned}\int_{-1}^{1}P_n^{(\alpha,\beta)}(x)fw_{\alpha-1,\beta}\,dx &= \int_{-1}^{1}P_n^{(\alpha,\beta)}(x)\sum_{k=0}^{n-1}c_k P_k^{(\alpha,\beta)}(x)w_{\alpha,\beta}\,dx = 0,\\ \int_{-1}^{1}P_{n-1}^{(\alpha,\beta)}(x)fw_{\alpha-1,\beta}\,dx &= c_{n-1}\int_{-1}^{1}\left(P_{n-1}^{(\alpha,\beta)}(x)\right)^2 w_{\alpha,\beta}\,dx < 0\end{aligned} \quad (2.24)$$

as a consequence of orthogonality of the Jacobi polynomials. Note that if $-1 < \alpha < 1$

and $\beta > 0$, then $w'_{\alpha-1,\beta} = (\beta(1-x) - (\alpha-1)(1+x)) \cdot w_{\alpha-2,\beta-1} > 0$ and

$$\lim_{x\to -1+} |f|^2 w_{\alpha-1,\beta} = \lim_{x\to 1-} |f|^2 w_{\alpha-1,\beta} = 0.$$

Therefore, integrating by parts yields

$$\int_{-1}^{1} \frac{d\left(|f|^2\right)}{dx} w_{\alpha-1,\beta}\, dx = -\int_{-1}^{1} \left(|f|^2\right) w'_{\alpha-1,\beta}\, dx < 0. \tag{2.25}$$

By the formulae (2.24)–(2.25), the left-hand side of the equation (2.23) is negative, thus necessarily $\operatorname{Re}\mu < 0$. □

Consider also another intermediary polynomial for a fixed $n > 3$, namely

$$g(x;\mu) := \sum_{k=0}^{n} \mu^k \left(\mu \frac{d^k}{dx^k} P_n^{(\alpha,\beta)}(x) + A \frac{d^k}{dx^k} P_{n-1}^{(\alpha,\beta)}(x)\right).$$

Lemma 2.17. *The polynomial $g(1;\mu)$ is Hurwitz-stable if $-1 < \alpha < 0$ and $\beta, A > 0$.*

Proof. This proof is analogous to the proofs of Theorem 2.14 and Lemma 2.16. From the definition of $g(x,\mu)$ we have

$$\mu \frac{dg(x;\mu)}{dx} + \mu P_n^{(\alpha,\beta)}(x) + A P_{n-1}^{(\alpha,\beta)}(x) = g(x;\mu)$$

which gives us (we put $g := g(x;\mu)$ and $g' := \frac{dg(x;\mu)}{dx}$ for brevity's sake)

$$\mu g' \overline{g'} w_{\alpha,\beta} + \mu P_n^{(\alpha,\beta)}(x) \overline{g'} w_{\alpha,\beta} + A P_{n-1}^{(\alpha,\beta)}(x) \overline{g'} w_{\alpha,\beta} = g \overline{g'} w_{\alpha,\beta}$$

after multiplication by $\overline{g'} w_{\alpha,\beta}$. Adding to the equation its complex conjugate and integrating yields

$$\begin{aligned}(\mu + \overline{\mu}) \int_{-1}^{1} |g'|^2 w_{\alpha,\beta}\, dx + \int_{-1}^{1} P_n^{(\alpha,\beta)}(x) \left(\mu \overline{g'} + \overline{\mu} g'\right) w_{\alpha,\beta}\, dx \\ + A \int_{-1}^{1} P_{n-1}^{(\alpha,\beta)}(x) \left(\overline{g'} + g'\right) w_{\alpha,\beta}\, dx = \int_{-1}^{1} \left(g\overline{g'} + \overline{g} g'\right) w_{\alpha,\beta}\, dx.\end{aligned} \tag{2.26}$$

Observe that g is a polynomial of degree n in x and its leading coefficient is given by $P_n^{(\alpha,\beta)}(x) \cdot \mu$. Consequently, substituting the explicit expression for $\mathbf{lc}\left(P_n^{(\alpha,\beta)}(x)\right)$ from the formula (2.1) gives

$$\begin{aligned}\mathbf{lc}(g') &= n\, \mathbf{lc}\left(P_n^{(\alpha,\beta)}(x)\right) \mu = \frac{(2n+\alpha+\beta-1)_2}{2(n+\alpha+\beta)} \cdot \frac{(n+\alpha+\beta)_{n-1}}{(n-1)!\, 2^{n-1}} \mu \\ &= \frac{(2n+\alpha+\beta-1)_2}{2(n+\alpha+\beta)} \mathbf{lc}\left(P_{n-1}^{(\alpha,\beta)}(x)\right) \mu.\end{aligned}$$

This allows us to calculate the third summand on the left-hand side of (2.26):

$$\begin{aligned}
&\int_{-1}^{1} P_{n-1}^{(\alpha,\beta)}(x)\left(\overline{g'}+g'\right) w_{\alpha,\beta}\,dx \\
&\quad= \int_{-1}^{1} P_{n-1}^{(\alpha,\beta)}(x)\frac{(2n+\alpha+\beta-1)_2}{2(n+\alpha+\beta)}\,\mathbf{lc}\left(P_{n-1}^{(\alpha,\beta)}(x)\right)(\overline{\mu}+\mu)\,x^{n-1}w_{\alpha,\beta}\,dx \qquad (2.27)\\
&\quad= (\overline{\mu}+\mu)\,\frac{(2n+\alpha+\beta-1)_2}{2(n+\alpha+\beta)}\int_{-1}^{1}\left(P_{n-1}^{(\alpha,\beta)}(x)\right)^2 w_{\alpha,\beta}\,dx.
\end{aligned}$$

Additionally, we have

$$\int_{-1}^{1} P_{n}^{(\alpha,\beta)}(x)g'w_{\alpha,\beta}\,dx = 0. \qquad (2.28)$$

Take μ so that $g(1;\mu)=0$. Then $w'_{\alpha,\beta} = (\beta(1-x)-\alpha(1+x))\cdot w_{\alpha-1,\beta-1} > 0$ and

$$\lim_{x\to -1+} |g|^2 w_{\alpha,\beta} = \lim_{x\to 1-} |g|^2 w_{\alpha,\beta} = 0$$

since $-1<\alpha<0$ and $\beta>0$. Integrating by parts we obtain

$$\int_{-1}^{1}\left(g\overline{g'}+\overline{g}g'\right) w_{\alpha,\beta}\,dx = -\int_{-1}^{1}|g|^2 w'_{\alpha,\beta}\,dx < 0. \qquad (2.29)$$

Now let us bring together the relations (2.26)–(2.29):

$$(\mu+\overline{\mu})\int_{-1}^{1}\left(|g'|^2 + A\frac{(2n+\alpha+\beta-1)_2}{2(n+\alpha+\beta)}\left(P_{n-1}^{(\alpha,\beta)}(x)\right)^2\right) w_{\alpha,\beta}\,dx = -\int_{-1}^{1}|g|^2 w'_{\alpha,\beta}\,dx,$$

hence $2\operatorname{Re}\mu = \mu+\overline{\mu} < 0$. That is, any zero μ of the polynomial $g(1;\mu)$ resides in the left half of the complex plane. □

Corollary 2.18. *For any positive A, the zeros of the polynomials*

$$\begin{aligned}
&2A\phi_n^{(\alpha,\beta)}(\mu) + (n+\alpha+\beta)\,\mu\phi_{n-2}^{(\alpha+1,\beta+1)}(\mu) \quad \textit{and}\\
&A\,(n+\alpha+\beta+1)\,\phi_{n-1}^{(\alpha+1,\beta+1)}(\mu) + 2\phi_{n-1}^{(\alpha,\beta)}(\mu)
\end{aligned} \qquad (2.30)$$

are interlacing provided that $-1<\alpha<1$ and $\beta>0$. If in addition $-1<\alpha<0$, the zeros of the polynomials

$$\begin{aligned}
&(n+\alpha+\beta+1)\,\mu\phi_{n-1}^{(\alpha+1,\beta+1)}(\mu) + 2A\phi_{n-1}^{(\alpha,\beta)}(\mu) \quad \textit{and}\\
&2\phi_n^{(\alpha,\beta)}(\mu) + A\,(n+\alpha+\beta)\,\phi_{n-2}^{(\alpha+1,\beta+1)}(\mu)
\end{aligned} \qquad (2.31)$$

are also interlacing.

Proof. To get the assertion we apply the relation (2.18) to the even and odd parts of the

polynomials $f(x;\mu)$ and $g(x;\mu)$. The even part of $f(x;\mu)$ is

$$\begin{aligned}\frac{f(x;\mu)+f(x;-\mu)}{2} &= A\sum_{k=0}^{[n/2]}\mu^{2k}\frac{d^{2k}}{dx^{2k}}P_n^{(\alpha,\beta)}(x)+\mu^2\sum_{k=0}^{[n/2]}\mu^{2k}\frac{d^{2k+1}}{dx^{2k+1}}P_{n-1}^{(\alpha,\beta)}(x)\\ &= A\sum_{k=0}^{[n/2]}\mu^{2k}\frac{d^{2k}}{dx^{2k}}P_n^{(\alpha,\beta)}(x)+\frac{n+\alpha+\beta}{2}\mu^2\sum_{k=0}^{[n/2]-1}\mu^{2k}\frac{d^{2k}}{dx^{2k}}P_{n-2}^{(\alpha+1,\beta+1)}(x).\end{aligned}$$

Analogously, for the odd part we have

$$\begin{aligned}\frac{f(x;\mu)-f(x;-\mu)}{2} &= A\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\mu^{2k+1}\frac{d^{2k+1}}{dx^{2k+1}}P_n^{(\alpha,\beta)}(x)+\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\mu^{2k+1}\frac{d^{2k}}{dx^{2k}}P_{n-1}^{(\alpha,\beta)}(x)\\ &= A\frac{n+\alpha+\beta+1}{2}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\mu^{2k+1}\frac{d^{2k}}{dx^{2k}}P_{n-1}^{(\alpha+1,\beta+1)}(x)+\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\mu^{2k+1}\frac{d^{2k}}{dx^{2k}}P_{n-1}^{(\alpha,\beta)}(x).\end{aligned}$$

The same manipulations with $g(x;\mu)$ give

$$\begin{aligned}\frac{g(x;\mu)+g(x;-\mu)}{2} &= \sum_{k=0}^{\left[\frac{n-1}{2}\right]}\mu^{2k+2}\frac{d^{2k+1}}{dx^{2k+1}}P_n^{(\alpha,\beta)}(x)+A\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\mu^{2k}\frac{d^{2k}}{dx^{2k}}P_{n-1}^{(\alpha,\beta)}(x)\\ &= \frac{n+\alpha+\beta+1}{2}\mu^2\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\mu^{2k}\frac{d^{2k}}{dx^{2k}}P_{n-1}^{(\alpha+1,\beta+1)}(x)+A\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\mu^{2k}\frac{d^{2k}}{dx^{2k}}P_{n-1}^{(\alpha,\beta)}(x)\end{aligned}$$

and

$$\begin{aligned}\frac{g(x;\mu)-g(x;-\mu)}{2} &= \sum_{k=0}^{[n/2]}\mu^{2k+1}\frac{d^{2k}}{dx^{2k}}P_n^{(\alpha,\beta)}(x)+A\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\mu^{2k+1}\frac{d^{2k+1}}{dx^{2k+1}}P_{n-1}^{(\alpha,\beta)}(x)\\ &= \mu\left(\sum_{k=0}^{[n/2]}\mu^{2k}\frac{d^{2k}}{dx^{2k}}P_n^{(\alpha,\beta)}(x)+A\frac{n+\alpha+\beta}{2}\sum_{k=0}^{[n/2]-1}\mu^{2k}\frac{d^{2k}}{dx^{2k}}P_{n-2}^{(\alpha+1,\beta+1)}(x)\right).\end{aligned}$$

The polynomials $f(1;\mu)$ and $g(1;\mu)$ are stable by Lemma 2.16 and Lemma 2.17, respectively. Thus, the pairs of polynomials mentioned in (2.30) and (2.31) have interlacing zeros by Theorem H-B. □

Lemma 2.19 (see *e.g.* [Wag2000, Lemma 3.4] or [CW2008, Lemma 3.5]). *Let real polynomials $p(x)$ and $q(x)$ such that $p(0), q(0) > 0$ have only negative zeros. Then $(p(x), xq(x))$ is a real pair if and only if the combinations*

$$\begin{aligned}&r_1(x) := r_1(x;A,B) := Ap(x)+Bxq(x) \quad \textit{and}\\ &r_2(x) := r_2(x;C,D) := Cp(x)+Dq(x)\end{aligned} \tag{2.32}$$

are nonzero outside the real line for all $A, B, C, D > 0$.

Recall that $p(x)$ and $xq(x)$ is a real pair whenever they interlace (non-strictly if $p(x)$ and $xq(x)$ have a common zero). For p and q as in this lemma we thus have $\deg q \leqslant \deg p \leqslant 1 + \deg q$ automatically. For completeness, let us derive Lemma 2.19.

Proof. Without loss of generality, assume in the proof that the polynomials p and q have no common zeros: if not, the zeros are real and we can factor them out of r_1 and r_2. The presence of common zeros prevents p and q from being strictly interlacing.

Let $(p(x), xq(x))$ be a real pair. The polynomials $p(x)$ and $xq(x)$ interlace exactly when between each pair of consecutive zeros $\mathbf{zr}_i(p), \mathbf{zr}_{i-1}(p)$ the polynomial $q(x)$ has exactly one zero, $i = 2, \dots, \deg p$. That is, the interlacing property of these polynomials is equivalent to

$$R(z) := \frac{q(z)}{p(z)} = \gamma + \sum_{i=1}^{n} \frac{A_i}{z - \mathbf{zr}_i(p)} \quad \text{implying}$$

$$zR(z) = \frac{zq(z)}{p(z)} = \gamma z + \sum_{i=1}^{n} A_i + \sum_{i=1}^{n} \frac{\mathbf{zr}_i(p) A_i}{z - \mathbf{zr}_i(p)},$$

where $\gamma \geqslant 0$ and the residues $A_i = \frac{q(\mathbf{zr}_i(p))}{p'(\mathbf{zr}_i(p))}$ are positive for all i. The straightforward check shows that

$$\operatorname{sign} \operatorname{Im} R(z) = -\operatorname{sign} \operatorname{Im} (zR(z)) = -\operatorname{sign} \operatorname{Im} z$$

for any $z \in \mathbb{C}$ such that $p(z) \neq 0$. Consequently, the combinations (2.32) have zeros only on the real line due to the relations $r_1(z; A, B) = 0 \implies zR(z) = -A/B \in \mathbb{R}$ and $r_2(z; C, D) = 0 \implies R(z) = -C/D \in \mathbb{R}$.

Conversely, let for any fixed $A, B, C, D > 0$ the polynomials $r_1(x; A, B)$ and $r_2(x; C, D)$ have only real roots. The zeros of $r_1(x)$, $r_2(x)$ are all negative since all their coefficients are positive. Moreover, $p(x)$ and $q(x)$ are coprime, and therefore

$$\begin{aligned} r_1(x_*; A, B) = 0 \quad &\implies \quad \operatorname{sign} p(x_*) = \operatorname{sign} q(x_*) = \operatorname{sign} r_2(x_*; C, D) \neq 0 \\ &\qquad \text{and} \quad r_1(x_*; \widetilde{A}, \widetilde{B}) \neq 0, \\ r_2(x_\#; C, D) = 0 \quad &\implies \quad \operatorname{sign} p(x_\#) = -\operatorname{sign} q(x_\#) = \operatorname{sign} r_1(x_\#; A, B) \neq 0 \\ &\qquad \text{and} \quad r_2(x_\#; \widetilde{C}, \widetilde{D}) \neq 0, \end{aligned}$$

where $B\widetilde{A} \neq A\widetilde{B}$ and $D\widetilde{C} \neq C\widetilde{D}$.

The roots of a polynomial depend continuously on its coefficients. Therefore, when the ratio B/A comes close to zero (or to infinity), the roots of $r_1(x)$ tend to the roots

of $p(x)$ (or to the roots of $xq(x)$, respectively). For any $A, B > 0$ and $i = 1, 2, \dots$ the zero $\mathbf{zr}_i(r_1)$ (here we count each possible multiple zero only once) can never coincide with a root of $p(x)$ or $xq(x)$ thus remaining in the interval

$$\bigcup_{A/B>0} \mathbf{zr}_i(r_1) = \left(\min\{\mathbf{zr}_i(p), \mathbf{zr}_{i-1}(q)\}, \max\{\mathbf{zr}_i(p), \mathbf{zr}_{i-1}(q)\} \right) =: I_i.$$

Indeed, on the one hand $\mathbf{zr}_i(r_1)$ belongs to I_i, since it can approach corresponding zeros of $p(x)$ and $xq(x)$ only from one direction (the opposite direction is either used by $r_2(x)$ or positive). On the other hand, each of the intervals I_i contains at most one zero of $r_1(x; A, B)$, because the polynomials $r_1(x; A, B)$ and $r_1(x; \widetilde{A}, \widetilde{B})$ remain coprime unless $B/A = \widetilde{B}/\widetilde{A}$. Hence, multiple zeros keep their multiplicities; in particular, the multiplicities of $\mathbf{zr}_{i-1}(q)$ and $\mathbf{zr}_i(p)$ are the same. For the polynomial $r_2(x)$ we analogously obtain

$$\bigcup_{C/D>0} \mathbf{zr}_i(r_2) = \left(\min\{\mathbf{zr}_i(q), \mathbf{zr}_i(p)\}, \max\{\mathbf{zr}_i(q), \mathbf{zr}_i(p)\} \right) =: J_i$$

and the multiplicities of $\mathbf{zr}_i(q)$ and $\mathbf{zr}_i(p)$ are the same. The product $xq(x)$ has a simple zero $\mathbf{zr}_0(q) = 0$, which implies that all zeros of $p(x)$ and $xq(x)$ are simple.

When $k := \deg p - \deg q - 1 \neq 0$ there exist $|k|$ surplus roots of $r_1(x)$ which disappear as $r_1(x)$ becomes proportional to $p(x)$ or $xq(x)$. Being negative, these roots must tend to $-\infty$. Since they can never meet a root of $p(x)$ or $q(x)$, they run the whole ray $\left(-\infty, \mathbf{zr}_{\deg p}(p)\right)$ if $k > 0$ and $\left(-\infty, \mathbf{zr}_{\deg q}(q)\right)$ if $k < 0$. This implies $|k| \leqslant 1$ since $r_1(x; A, B)$ and $r_1(x; \widetilde{A}, \widetilde{B})$ remain coprime for $B/A \neq \widetilde{B}/\widetilde{A}$. Considering $r_2(x)$ additionally yields $|\deg p - \deg q| \leqslant 1$, therefore $\deg q \leqslant \deg p \leqslant \deg q + 1$. For each $x < 0$ there exists $i = 1, \dots, \deg q + 1$ such that $x \in [\mathbf{zr}_i(q), \mathbf{zr}_{i-1}(q)) \subset \overline{J_i} \cup I_i$. Since J_i and I_i are disjoint, the only option is $\mathbf{zr}_i(q) \leqslant \mathbf{zr}_i(p) \leqslant \mathbf{zr}_{i-1}(q)$. This implies that the polynomial $p(x)$ interlace $xq(x)$ since $\deg p \leqslant \deg q + 1$. □

The next corollary complements the interlacing property of the polynomials $\phi_n^{(\alpha,\beta)}(\mu)$ and $\phi_{n-1}^{(\alpha+1,\beta+1)}(\mu)$ (see Remark 2.5).

Corollary 2.20. *If $-1 < \alpha < 0$ and $\beta > 0$ the pairs $\left(\phi_n^{(\alpha,\beta)}(\mu), \mu\phi_{n-2}^{(\alpha+1,\beta+1)}(\mu)\right)$ and $\left(\phi_n^{(\alpha,\beta)}(\mu), \mu\phi_n^{(\alpha+1,\beta+1)}(\mu)\right)$ possess the (strict) interlacing property.*

Proof. By Theorem 2.4, all involved polynomials have only real nonpositive zeros. Corollary 2.18 adds that the polynomials in (2.30)–(2.31) have (strictly) interlacing zeros. Therefore, Lemma 2.19 assures the asserted fact. □

Theorem 2.21. *If $-1 < \alpha < 0 < \beta$ and $n = 5, 6, \dots$, then the polynomial $\phi_n^{(\alpha+1,\beta+1)}(\mu)$ interlaces $\phi_{n-2}^{(\alpha+1,\beta+1)}(\mu)$, and the polynomial $\phi_n^{(\alpha,\beta)}(\mu)$ interlaces $\phi_{n-2}^{(\alpha,\beta)}(\mu)$.*

Proof. According to Corollary 2.12, we have

$$\begin{aligned}
&\mathbf{zr}_i\left(\phi_{n-2}^{(\alpha+1,\beta+1)}\right) < \mathbf{zr}_i\left(\phi_{n-1}^{(\alpha+1,\beta+1)}\right) < \mathbf{zr}_{i-1}\left(\phi_{n-2}^{(\alpha+1,\beta+1)}\right),\\
&\mathbf{zr}_i\left(\phi_{n-1}^{(\alpha+1,\beta+1)}\right) < \mathbf{zr}_i\left(\phi_{n}^{(\alpha+1,\beta+1)}\right) < \mathbf{zr}_{i-1}\left(\phi_{n-1}^{(\alpha+1,\beta+1)}\right)
\end{aligned} \tag{2.33}$$

for any positive integer $i \leqslant n/2$. From Corollary 2.20 we obtain that

$$\mathbf{zr}_i\left(\phi_{n-2}^{(\alpha+1,\beta+1)}\right) < \mathbf{zr}_i\left(\phi_{n}^{(\alpha,\beta)}\right) < \mathbf{zr}_{i-1}\left(\phi_{n-2}^{(\alpha+1,\beta+1)}\right), \tag{2.34}$$

$$\mathbf{zr}_i\left(\phi_{n}^{(\alpha+1,\beta+1)}\right) < \mathbf{zr}_i\left(\phi_{n}^{(\alpha,\beta)}\right) < \mathbf{zr}_{i-1}\left(\phi_{n}^{(\alpha+1,\beta+1)}\right). \tag{2.35}$$

Bringing together the right inequality in (2.34) and the left inequalities in (2.35) and (2.33), we obtain

$$\begin{aligned}
\mathbf{zr}_i\left(\phi_{n-2}^{(\alpha+1,\beta+1)}\right) &\overset{(2.33)}{<} \mathbf{zr}_i\left(\phi_{n-1}^{(\alpha+1,\beta+1)}\right) \overset{(2.33)}{<} \mathbf{zr}_i\left(\phi_{n}^{(\alpha+1,\beta+1)}\right)\\
&\overset{(2.35)}{<} \mathbf{zr}_i\left(\phi_{n}^{(\alpha,\beta)}\right) \overset{(2.34)}{<} \mathbf{zr}_{i-1}\left(\phi_{n-2}^{(\alpha+1,\beta+1)}\right)
\end{aligned}$$

for all positive integers $i \leqslant n/2$. This relation implies that the zeros of the polynomials $\phi_n^{(\alpha+1,\beta+1)}(\mu)$ and $\phi_{n-2}^{(\alpha+1,\beta+1)}(\mu)$ interlace.

By Corollary 2.12 we obtain (*cf.* (2.33))

$$\mathbf{zr}_i\left(\phi_{n-2}^{(\alpha,\beta)}\right) < \mathbf{zr}_i\left(\phi_{n-1}^{(\alpha,\beta)}\right) < \mathbf{zr}_i\left(\phi_{n}^{(\alpha,\beta)}\right), \quad i = 1, 2, \ldots.$$

This chain can be continued with the left inequality in (2.35) and the right inequality in (2.34) so that

$$\begin{aligned}
\mathbf{zr}_i\left(\phi_{n-2}^{(\alpha+1,\beta+1)}(\mu)\right) &\overset{(2.35)}{<} \mathbf{zr}_i\left(\phi_{n-2}^{(\alpha,\beta)}\right) < \mathbf{zr}_i\left(\phi_{n-1}^{(\alpha,\beta)}\right)\\
&< \mathbf{zr}_i\left(\phi_{n}^{(\alpha,\beta)}\right) \overset{(2.34)}{<} \mathbf{zr}_{i-1}\left(\phi_{n-2}^{(\alpha+1,\beta+1)}\right)
\end{aligned}$$

for each positive integer. □

2.5 Conclusion: relations between Conjecture 2.A and Conjecture 2.B

Here we obtain two instructive facts giving an idea about the limits of the current approach. In the present note, we used a modification of the method applied in [CW2008], so it has the same deficiency: the parameters α and β are constrained to provide the positivity of $w'_{\alpha,\beta}(x)$ and the convergence of integrals. However, these sufficient conditions seem to be quite far from being necessary.

The following two lemmata coupled with Theorem 2.21 give no new parameter range for Conjecture 2.A to hold. At the same time, the comparison to Lemma 2.11 clearly shows that this conjecture is less restrictive than Conjecture 2.B.

Lemma 2.22. *If the polynomials $\phi_n^{(\alpha,\beta)}(\mu)$, $\phi_{n-1}^{(\alpha,\beta)}(\mu)$ and $\phi_{n-2}^{(\alpha,\beta)}(\mu)$ are pairwise interlacing in such a way that*

$$\mathbf{zr}_1\left(\phi_{n-2}^{(\alpha,\beta)}\right) < \mathbf{zr}_1\left(\phi_{n-1}^{(\alpha,\beta)}\right) < \mathbf{zr}_1\left(\phi_n^{(\alpha,\beta)}\right),$$

then $\phi_n^{(\alpha,\beta-1)}(\mu)$ interlaces $\phi_{n-1}^{(\alpha,\beta-1)}(\mu)$.

Proof. By the formula (2.4), $\phi_n^{(\alpha,\beta-1)}(\mu)$ and $\phi_{n-1}^{(\alpha,\beta-1)}(\mu)$ have only real zeros. Furthermore,

$$\begin{aligned}\mathbf{zr}_{i+1}\left(\phi_n^{(\alpha,\beta)}\right) < \mathbf{zr}_i\left(\phi_{n-2}^{(\alpha,\beta)}\right) < \mathbf{zr}_i\left(\phi_{n-1}^{(\alpha,\beta-1)}\right)\\ < \mathbf{zr}_i\left(\phi_{n-1}^{(\alpha,\beta)}\right) < \mathbf{zr}_i\left(\phi_n^{(\alpha,\beta-1)}\right) < \mathbf{zr}_i\left(\phi_n^{(\alpha,\beta)}\right)\end{aligned}$$

for integer $i = 1,\ldots,[\frac{n-1}{2}]$. This implies the interlacing property for the polynomials $\phi_n^{(\alpha,\beta-1)}(\mu)$ and $\phi_{n-1}^{(\alpha,\beta-1)}(\mu)$. □

As an intermediate result (Corollary 2.20) we had the interlacing property of $\phi_n^{(\alpha,\beta)}(\mu)$ and $\phi_n^{(\alpha+1,\beta+1)}(\mu)$ when the parameters satisfy $-1 < \alpha < 0 < \beta$. Such a fact allows us to get a relationship complementing Lemma 2.22.

Lemma 2.23. *Let the polynomial pairs*

$$\left(\phi_n^{(\alpha,\beta)}(\mu), \phi_n^{(\alpha+1,\beta+1)}(\mu)\right) \quad \textit{and} \quad \left(\phi_n^{(\alpha,\beta)}(\mu), \phi_{n-1}^{(\alpha+1,\beta+1)}(\mu)\right)$$

be interlacing in such a way that

$$\mathbf{zr}_1\left(\phi_{n-1}^{(\alpha+1,\beta+1)}\right) < \mathbf{zr}_1\left(\phi_n^{(\alpha,\beta)}\right), \quad \mathbf{zr}_1\left(\phi_n^{(\alpha+1,\beta+1)}\right) < \mathbf{zr}_1\left(\phi_n^{(\alpha,\beta)}\right). \tag{2.36}$$

Then $\phi_n^{(\alpha+1,\beta)}(\mu)$ interlaces $\phi_{n-1}^{(\alpha+1,\beta)}(\mu)$.

Proof. The identities (2.4) and (2.5) give

$$\begin{aligned}(n+\alpha+\beta+2)\phi_n^{(\alpha+1,\beta+1)} + (n+\alpha+1)\phi_{n-1}^{(\alpha+1,\beta+1)} =\\ (2n+\alpha+\beta+2)\phi_n^{(\alpha+1,\beta)},\end{aligned} \tag{2.37}$$

$$(n+\alpha+\beta+1)\phi_n^{(\alpha+1,\beta)} - (2n+\alpha+\beta+1)\phi_n^{(\alpha,\beta)} = (n+\beta)\phi_{n-1}^{(\alpha+1,\beta)}. \tag{2.38}$$

From the inequalities (2.36) we obtain that each interval $\left(\mathbf{zr}_{i+1}\left(\phi_n^{(\alpha,\beta)}\right), \mathbf{zr}_i\left(\phi_n^{(\alpha,\beta)}\right)\right)$, $i = 1,\ldots,[n/2]-1$, contains the points $\mathbf{zr}_i\left(\phi_{n-1}^{(\alpha+1,\beta+1)}\right)$ and $\mathbf{zr}_i\left(\phi_n^{(\alpha+1,\beta+1)}\right)$ and no other zeros of the polynomials $\phi_{n-1}^{(\alpha+1,\beta+1)}$ and $\phi_n^{(\alpha+1,\beta+1)}$. Thus, the left-hand side of (2.37) also has exactly one zero on each of the intervals. As a result, $\phi_n^{(\alpha+1,\beta)}$ interlaces $\phi_n^{(\alpha,\beta)}$, so the zeros of their difference appearing in (2.38) and hence of $\phi_{n-1}^{(\alpha+1,\beta)}$ are interlacing with the zeros of $\phi_n^{(\alpha+1,\beta)}$. □

Chapter 3.

Total nonnegativity of infinite Hurwitz matrices of entire and meromorphic functions

3.1 Series corresponding to totally nonnegative Hurwitz, Hurwitz-type and Toeplitz matrices

In this chapter we demonstrate a connection of meromorphic $\mathcal{S}$-functions (see Definition 1.2) with total nonnegativity of corresponding Hurwitz-type matrices (Theorem 3.8). As an application we further develop the following results from [Asn70, Kem82, Ho2003, HT2012]. Asner [Asn70] established that the (finite) Hurwitz matrix of a real quasi-stable polynomial is totally nonnegative (although there are polynomials with totally nonnegative Hurwitz matrices which are not quasi-stable). Kemperman [Kem82] showed that quasi-stable polynomials have totally nonnegative *infinite* Hurwitz matrices. It turns out that the replacement of finite Hurwitz matrices with infinite Hurwitz matrices allows proving the converse: *a polynomial is quasi-stable if its infinite Hurwitz matrix is totally nonnegative*. The key to this is given in [Ho2003]: a special matrix factorization, which was successfully applied in [HT2012] to a closely related (in fact, more general) problem. Moreover, when a theorem involves an infinite Hurwitz matrix, it is natural to suggest that it can be generalized to entire functions or power series. The first goal of the current chapter is to obtain the following fact.

Theorem 3.1. *Given a power series $f(z) = z^j \sum_{k=0}^{\infty} f_k z^k$ in the complex variable z, where $f_0 > 0$ and j is a nonnegative integer, the infinite Hurwitz matrix*

$$\mathcal{H}_f = \begin{pmatrix} f_0 & f_2 & f_4 & f_6 & f_8 & \cdots \\ 0 & f_1 & f_3 & f_5 & f_7 & \cdots \\ 0 & f_0 & f_2 & f_4 & f_6 & \cdots \\ 0 & 0 & f_1 & f_3 & f_5 & \cdots \\ 0 & 0 & f_0 & f_2 & f_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is totally nonnegative if and only if the series f converges to a function of the form

$$f(z) = Cz^j e^{\gamma_1 z+\gamma_2 z^2} \frac{\prod_\mu \left(1+\frac{z}{x_\mu}\right) \prod_\nu \left(1+\frac{z}{\alpha_\nu}\right)\left(1+\frac{z}{\overline{\alpha_\nu}}\right)}{\prod_\lambda \left(1+\frac{z}{y_\lambda}\right)\left(1-\frac{z}{y_\lambda}\right)}, \tag{3.1}$$

where $C, \gamma_1, \gamma_2 \geqslant 0$, $x_\mu, y_\lambda > 0$, $\operatorname{Re}\alpha_\nu \geqslant 0$, $\operatorname{Im}\alpha_\nu > 0$ and

$$\textstyle\sum_\mu \frac{1}{x_\mu} + \sum_\nu \operatorname{Re}(\frac{1}{\alpha_\nu}) + \sum_\nu \frac{1}{|\alpha_\nu|^2} + \sum_\lambda \frac{1}{y_\lambda^2} < \infty.$$

Remark 3.2. Stating in this chapter that a power series converges, by default we assume it to be convergent in some neighbourhood of the origin. Moreover, where it creates no uncertainties we use the same designation for the series and for a function it converges to.

Remark 3.3. It is possible that $\{x_\mu\}_\mu \cap \{y_\lambda\}_\lambda \neq \varnothing$ in the expression (3.1). If so, the coinciding negative zeros and poles of the function $f(z)$ cancel each other out, while its positive poles remain untouched. For example, although the series $\sum_{k=0}^\infty z^k$ satisfies Theorem 3.1, it converges to the function $\frac{1}{1-z}$ with a unique positive pole. The number of such cancellations may be infinite, however it cannot affect the convergence of involved infinite products.

From the perspective of further generalizations of Theorem 3.1, it can be interesting to study the case of doubly infinite power series $f(z)$. Another related problem is to extend the criterion [HKK2015] (see its brief introduction on Page 5 herein) to entire functions or power series.

Our second goal is achieved by Theorem 3.4, which is an extension of [HT2012, Theorem 4.29].

Theorem 3.4. *A power series $f(z) = \sum_{k=0}^\infty f_k z^k$ with $f_0 > 0$ converges to an entire function of the form*

$$f(z) = f_0\, e^{\gamma z} \prod_\nu \left(1+\frac{z}{\alpha_\nu}\right), \tag{3.2}$$

where $\gamma \geqslant 0$, $\alpha_\nu > 0$ for all ν and $\sum_\nu \frac{1}{\alpha_\nu} < \infty$, if and only if the infinite matrix

$$\mathcal{D}_f = \begin{pmatrix} f_0 & f_1 & f_2 & f_3 & f_4 & \dots \\ 0 & f_1 & 2f_2 & 3f_3 & 4f_4 & \dots \\ 0 & f_0 & f_1 & f_2 & f_3 & \dots \\ 0 & 0 & f_1 & 2f_2 & 3f_3 & \dots \\ 0 & 0 & f_0 & f_1 & f_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is totally nonnegative.

This theorem complements the following well-known criterion established by Aissen, Edrei, Schoenberg and Whitney (its generalization is stated as Theorem 1.8 herein).

Theorem 3.5 ([AESW51, ASW52, Edr52], see also [Kar68, Section 8 § 5]). *Given a formal power series $f(z) = \sum_{k=0}^{\infty} f_k z^k$, $f_0 > 0$, the Toeplitz matrix*

$$T(f) = \begin{pmatrix} f_0 & f_1 & f_2 & f_3 & f_4 & \dots \\ 0 & f_0 & f_1 & f_2 & f_3 & \dots \\ 0 & 0 & f_0 & f_1 & f_2 & \dots \\ 0 & 0 & 0 & f_0 & f_1 & \dots \\ 0 & 0 & 0 & 0 & f_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.3}$$

is totally nonnegative if and only if $f(z)$ converges to a meromorphic function of the form:

$$f(z) = f_0\, e^{\gamma z} \frac{\prod_\nu \left(1 + \frac{z}{\alpha_\nu}\right)}{\prod_\mu \left(1 - \frac{z}{\beta_\mu}\right)}, \tag{3.4}$$

where $\gamma \geqslant 0$, $\alpha_\nu, \beta_\mu > 0$ for all μ, ν and $\sum_\nu \frac{1}{\alpha_\nu} + \sum_\mu \frac{1}{\beta_\mu} < \infty$.

If we require the series $f(z)$ to represent an entire function under the assumptions of Theorem 3.5, we obtain that it has the form (3.2), *i.e.* that this function belongs to the Laguerre-Pólya class (see Section 1.2.3). We prove Theorems 3.1 and 3.4 in Section 3.4.

Consider the *infinite Hurwitz-type matrix* (*i.e.* the *matrix of the Hurwitz type*)

$$H(p,q) = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \dots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ 0 & b_0 & b_1 & b_2 & b_3 & b_4 & \dots \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.5}$$

where $p(z) = \sum_{k=0}^{\infty} a_k z^k$ and $q(z) = \sum_{k=0}^{\infty} b_k z^k$ are formal power series. Given two arbitrary constants c and β, we also consider the matrix

$$J(c,\beta) = \begin{pmatrix} c & \beta & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & c & \beta & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & c & \beta & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.6}$$

Matrices of this type will appear in our factorizations below.

Finally, for an infinite matrix $A = (a_{ij})_{i,j=1}^{\infty}$ and a fixed number ρ, $0 < \rho \leqslant 1$, we consider the matrix norm

$$\|A\|_\rho := \sup_{i\geqslant 1} \sum_{j=1}^{\infty} \rho^{j-1}|a_{ij}|.$$

Remark 3.6. Convergence in this norm implies entry-wise convergence. Moreover, the norm $\|A\|_\rho$ of a matrix A coincides with the norm of the operator

$$A_\rho : x \mapsto A \cdot \operatorname{diag}(1, \rho, \rho^2, \dots) \cdot x,$$

acting on the space l_∞ of bounded sequences.

Remark 3.7. Let functions $g(z)$, $p(z)$, $q(z)$ and $g_k(z)$, $p_k(z)$, $q_k(z)$, $k = 1, 2, \dots$, be holomorphic on $\overline{D}_\rho := \{z \in \mathbb{C} : |z| \leqslant \rho\}$. Then, by Abel's theorem, the condition

$$\lim_{k\to\infty} \|T(g_k) - T(g)\|_\rho = 0$$

is equivalent to the uniform convergence of $g_k(z)$ to $g(z)$ on $\overline{D}_\rho$, and the condition

$$\lim_{k\to\infty} \|H(p_k, q_k) - H(p, q)\|_\rho = 0$$

is equivalent to the uniform convergence of $p_k(z)$ to $p(z)$ and $q_k(z)$ to $q(z)$ on $\overline{D}_\rho$.

In the present chapter we deal only with *meromorphic* $\mathcal{R}$- and $\mathcal{S}$-functions (the classes $\mathcal{R}$ and $\mathcal{S}$ are introduced in Definitions 1.1 and 1.2), and therefore sometimes do not mention this property explicitly for brevity's sake. The next result concerning properties of $\mathcal{S}$-functions is valuable on its own, besides proving Theorems 3.1 and 3.4.

Theorem 3.8. *Let the ratio $F(z) = \frac{q(z)}{p(z)}$ of power series $p(z) = \sum_{k=0}^{\infty} a_k z^k$ and $q(z) = \sum_{k=0}^{\infty} b_k z^k$ be normalized by the equality $p(0) = a_0 = 1$. Then the following conditions are equivalent:*

(i) *The ratio $F(z)$ is a meromorphic $\mathcal{S}$-function; its numerator $q(z)$ and denominator $p(z)$ are entire functions of genus 0 up to a common meromorphic factor $g(z)$ of the form* (3.4)*, $g(0) = 1$.*

(ii) *The infinite Hurwitz-type matrix $H(p, q)$ defined by* (3.5) *is totally nonnegative.*

(iii) *The matrix $H(p, q)$ possesses the infinite factorization*

$$H(p, q) = \lim_{j\to\infty} \left(J(b_0, \beta_0)\, J(1, \beta_1) \cdots J(1, \beta_j)\right) H(1, 1)\, T(g) \tag{3.7}$$

converging in $\|\cdot\|_\rho$*-norm for some* ρ, $0 < \rho \leqslant 1$. *The coefficients in* (3.7) *satisfy* $b_0 \geqslant 0$, $\beta_0, \beta_1, \dots, \beta_{\omega-1} > 0$, $\beta_\omega = \beta_{\omega+1} = \dots = 0$ *and* $\sum_{j=0}^{\infty} \beta_j < \infty$, *where* ω *can be a nonnegative integer or* $+\infty$. *At that,* $T(g)$ *denotes a totally nonnegative Toeplitz matrix of the form* (3.3) *with ones on its main diagonal.*

Furthermore, $T(g)$ *from the condition* (iii) *is the Toeplitz matrix of the function* $g(z)$ *appearing in the condition* (i).

Remark 3.9. Note that

$$J(c,0)H(1,1) = H(1,c). \tag{3.8}$$

If ω is a finite number in (iii), then $\beta_\omega = \beta_{\omega+1} = \dots = 0$ implying that

$$J(b_0,\beta_0)\cdots J(1,\beta_{\omega-1})\, H(1,1) = J(b_0,\beta_0)\cdots J(1,\beta_{\omega-1})\, J(1,\beta_\omega)\, H(1,1) = \cdots.$$

Consequently, in this case the factorization (3.7) can be expressed as follows

$$H(p,q) = \begin{cases} J(b_0,0)\, H(1,1)\, T(g) = H(1,b_0)\, T(g) & \text{if } \omega = 0; \\ J(b_0,\beta_0)\, J(1,\beta_1)\cdots J(1,\beta_{\omega-1})\, H(1,1)\, T(g) & \text{if } 0 < \omega < \infty. \end{cases} \tag{3.9}$$

Remark 3.10. The number ρ in Theorem 3.8 can be anywhere in $(0,1] \cap (0,\rho_0)$, here ρ_0 denotes the radius of convergence of $g(z)$ (which is positive by Theorem 3.5).

If we require $p(z)$ and $q(z)$ to be entire functions in Theorem 3.8, then the function $g(z)$ has the form (3.2) or $g(z) \equiv 1$ and (3.7) converges in $\|\cdot\|_1$.

Remark 3.11. In the case $q(0) = b_0 = 0$ it can be convenient to "trim" the matrix $H(p,q)$ by removing its first row and its trivial first column. This corresponds to replacing $J(0,\beta_0)$ in the factorization (3.7) by its diagonal analogue $\operatorname{diag}(1,\beta_0,1,\beta_0,\dots)$.

Remark 3.12. Since entire functions of genus 0 have unique Weierstraß' representations, it makes sense to consider the *greater common divisor* of a subset of this class (see *e.g.* [Con78]). Accordingly, two entire functions p and q of genus 0 are *coprime* whenever they have no common zeros, that is $\gcd(p,q) \equiv 1$.

Consider the continued fraction

$$b_0 + \cfrac{\beta_0 z}{1 + \cfrac{\beta_1 z}{\ddots + \cfrac{\beta_{\omega-1} z}{1}}} =: b_0 + \mathop{\mathbf{K}}_{0\leqslant j\leqslant \omega-1} \left(\frac{\beta_j z}{1}\right), \tag{3.10}$$

where $b_0 \geqslant 0$, $\beta_0, \beta_1, \dots, \beta_{\omega-1} > 0$ and $0 \leqslant \omega \leqslant \infty$.

This formula combines both finite (terminating) and infinite continued fractions: in the infinite case we assume $\omega = \infty$. We also allow $\omega = 0$ which is denoting $\mathop{\mathbf{K}}_{0 \leqslant j \leqslant \omega-1} \left(\frac{\beta_j z}{1}\right) = 0$. The following Corollary (see its proof in Subsection 3.2.2) connects the factorization (3.7) with continued fractions of this type.

Corollary 3.13. *Let $F(z) = \frac{q(z)}{p(z)}$ be a meromorphic $\mathcal{S}$-function, where the entire functions $p(z)$ and $q(z)$ are of genus 0. Then it can be expanded into a locally uniformly convergent continued fraction of the form* (3.10) *with exactly the* <u>*same*</u> *coefficients b_0 and $(\beta_j)_{j=0}^{\omega-1}$, $\sum_{j=0}^{\omega-1} \beta_j < \infty$, as in the factorization* (3.7) *of the matrix $H(p, q)$. No other continued fractions of the form*

$$F(z) = c_0 + \mathop{\mathbf{K}}_{1 \leqslant j \leqslant \omega} \left(\frac{c_j z^{r_j}}{1}\right), \qquad 0 \leqslant \omega \leqslant \infty,$$

where $c_j \neq 0$ and $r_j \in \mathbb{Z}_{>0}$ for $j = 1, \ldots, \omega$, can correspond to the Taylor series of $F(z)$.

Here the locally uniform convergence to a meromorphic function is understood as the uniform convergence on compact sets containing no poles of the limiting functions. In other words, this is the locally uniform convergence with respect to the distance between points on the Riemann sphere.

Remark 3.14. Corollary 3.13 implies that *each pair $(p(z), q(z))$ satisfying Theorem* 3.8 *determines a unique factorization of the form* (3.7).

Let $p(z)$ and $q(z)$ be real polynomials. Denote

$$u(z) := \sum_{k=0}^{n} a_k z^{n-k} = z^n p\left(\frac{1}{z}\right) \text{ and } v(z) := \sum_{k=0}^{n} b_k z^{n-k} = z^n q\left(\frac{1}{z}\right),$$

where $n = \max\{\deg p, \deg q\}$. In this case it is more common to work with the matrix $\widetilde{H}(u, v) := H(p, q)$ instead of $H(v, u)$.

In fact, Theorem 3.8 extends the following result by Holtz and Tyaglov to meromorphic functions. In [HT2012, a combination of Theorems 1.46 and 3.44, Corollaries 3.41–3.42] they established that the matrix $\widetilde{H}(u, v)$ is totally nonnegative if and only if it can be factored as follows

$$\widetilde{H}(u, v) = J(c_0, 1) \cdots J(c_j, 1)\, H(1, 0)\, T(g), \quad c_1, \ldots, c_j > 0, \tag{3.11}$$

where $T(g)$ is totally nonnegative and $g = \gcd(u, v)$. Note that the factorization (3.11) corresponds to (3.9) after the substitutions $b_0 = c_0$, $\beta_0 = (c_1)^{-1}$ and $\beta_{i-1} = (c_{i-1} c_i)^{-1}$ for $i = 2, \ldots, j$. Moreover, by Theorem 3.44 from [HT2012] the matrix $\widetilde{H}(u, v)$ is totally

nonnegative if and only if $v(z)$ and $u(z)$ have no positive zeros and $\frac{v}{u} \in \mathcal{R}^{-1}$. Since

$$\frac{p(z)}{q(z)} = v\left(\frac{1}{z}\right)/u\left(\frac{1}{z}\right),$$

we obtain the polynomial analogue of Theorem 3.8.

Earlier, Holtz (see [Ho2003]) found that the infinite Hurwitz matrix of a stable polynomial (*i.e.* a polynomial with no roots with nonnegative real part) has the factorization (3.11) with $T(g)$ equal to the identity matrix. Additionally, each of the factors $J(c_j, 1)$ corresponds to a step of the Routh scheme. These factorizations coincide because the problems considered in [Ho2003] and [HT2012] are closely connected (see the discussion in Section 1.1 or, for example, the monographs by Gantmacher [Gan59, Ch. XV] and Wall [Wall48, Chapters IX and X]). In order to deduce Theorem 3.1 from Theorem 3.8, we are using the same underlying connection.

3.2 Basic facts on $\mathcal{S}$-functions

3.2.1 $\mathcal{S}$-functions in connection with Hurwitz-type matrices

Consider power series

$$p(z) = \sum_{k=0}^{\infty} a_k z^k, \quad a_0 = 1, \quad \text{and} \quad q(z) = \sum_{k=0}^{\infty} b_k z^k, \quad b_0 \geqslant 0. \tag{3.12}$$

Let us introduce the following notations

$$p_0(z) := p(z), \;\; p_{-1}(z) := q(z), \;\; H := H(p, q) \;\text{ and }\; H_0 := H(p_0, p_{-1}) = H.$$

Denote the minor of a matrix A with rows $i_1, i_2, \ldots, i_k$ and columns $j_1, j_2, \ldots, j_k$ by

$$A\begin{pmatrix} i_1 & i_2 & \ldots & i_k \\ j_1 & j_2 & \ldots & j_k \end{pmatrix}.$$

In addition, we set

$$A^{(k)} := A\begin{pmatrix} 2 & 3 & \ldots & k \\ 2 & 3 & \ldots & k \end{pmatrix}. \tag{3.13}$$

If the number $\beta_0 = b_1 - b_0\, a_1 = H_0^{(3)}$ is nonzero, we define

$$p_1(z) := \frac{q(z) - b_0\, p(z)}{\beta_0 z} \quad \text{and} \quad H_1 := H(p_1, p_0).$$

Now we can perform the same manipulations with the pair $p_1(z)$, $p_0(z)$. That is, we can make the next step of the following algorithm. (In fact, this is the *Euclidean algorithm*, though infinite in general.)

At the jth step, $j = 0, 1, 2 \dots$, the series $p_j(z)$ and $p_{j-1}(z)$ are already defined, as well as the matrix $H_j = H(p_j, p_{j-1})$. We set

$$\beta_j := H_j^{(3)}, \tag{3.14}$$

and, if β_j is nonzero, we set

$$p_{j+1}(z) := \frac{p_{j-1}(z) - p_{j-1}(0)\, p_j(z)}{\beta_j z} \qquad (\, p_{j-1}(0) = 1 \text{ when } j \geqslant 1\,) \tag{3.15}$$

so that $H_{j+1} := H(p_{j+1}, p_j)$. These steps can be repeated unless $\beta_j = 0$. In Corollary 3.21 we will show that $\beta_j > 0$ whenever $F_j(z) = \frac{p_{j-1}(z)}{p_j(z)}$ represents a non-constant meromorphic $\mathcal{S}$-function. To do this we need some auxiliary facts.

Suppose that $\beta_i \neq 0$, $i = 0, 1, \dots, j$ for some nonnegative j, such that the power series $p_{j-1}(z)$, $p_j(z)$ and $p_{j+1}(z)$ are defined according to the recurrence formula (3.15).

Lemma 3.15. *The identity*[1]

$$H_j \begin{pmatrix} 2 & 3 & \dots & k & k+1 \\ 2 & 3 & \dots & k & i+1 \end{pmatrix} = \beta_j^{\left[\frac{k}{2}\right]} H_{j+1} \begin{pmatrix} 2 & 3 & \dots & k-1 & k \\ 2 & 3 & \dots & k-1 & i \end{pmatrix},$$

holds for all $k = 2, 3, \dots$ *and* $i = k, k+1, \dots$.

Proof. Without loss of generality we consider the case $j = 0$, since for higher values of j the relations (3.14)–(3.15) are analogous. In the case $k = 2m$

$$\beta_0^m H_1 \begin{pmatrix} 2 & 3 & \dots & 2m-1 & 2m \\ 2 & 3 & \dots & 2m-1 & i \end{pmatrix} \overset{a_0=1}{=\!=\!=} \beta_0^m H_1 \begin{pmatrix} 1 & 2 & \dots & 2m-1 & 2m \\ 1 & 2 & \dots & 2m-1 & i \end{pmatrix} =$$

$$\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{2m-2} & a_{i-1} \\ 0 & b_1 - b_0 a_1 & b_2 - b_0 a_2 & \dots & b_{2m-2} - b_0 a_{2m-2} & b_{i-1} - b_0 a_{i-1} \\ 0 & a_0 & a_1 & \dots & a_{2m-3} & a_{i-2} \\ 0 & 0 & b_1 - b_0 a_1 & \dots & b_{2m-3} - b_0 a_{2m-3} & b_{i-2} - b_0 a_{i-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{m-1} - b_0 a_{m-1} & b_{i-m} - b_0 a_{i-m} \end{vmatrix},$$

[1]The notation $[a]$ stands for the integer part of a.

so the addition of b_0 times the $(2\nu-1)$th row to the (2ν)th row for $\nu = 1, \dots, m$ yields

$$\beta_0^m H_1\begin{pmatrix} 2 & 3 & \dots & 2m-1 & 2m \\ 2 & 3 & \dots & 2m-1 & i \end{pmatrix} = \begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{2m-2} & a_{i-1} \\ b_0 & b_1 & b_2 & \dots & b_{2m-2} & b_{i-1} \\ 0 & a_0 & a_1 & \dots & a_{2m-3} & a_{i-2} \\ 0 & b_0 & b_1 & \dots & b_{2m-3} & b_{i-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{m-1} & b_{i-m} \end{vmatrix}$$
$$= H_0\begin{pmatrix} 2 & 3 & \dots & 2m & 2m+1 \\ 2 & 3 & \dots & 2m & i+1 \end{pmatrix}.$$

For $k = 2m+1$ the transformation is analogous. □

In particular, if we suppose that $\beta_0, \beta_1, \dots, \beta_{k-1} > 0$ for $k \geqslant 3$, this lemma implies the following chain of equalities

$$\begin{aligned} H_j^{(k)} &= \beta_j^{\left[\frac{k-1}{2}\right]} H_{j+1}^{(k-1)} \\ &= \beta_j^{\left[\frac{k-1}{2}\right]} \beta_{j+1}^{\left[\frac{k-2}{2}\right]} H_{j+2}^{(k-2)} = \dots = H_{j+k-3}^{(3)} \prod_{i=1}^{k-3} \beta_{i+j-1}^{\left[\frac{k-i}{2}\right]} = \prod_{i=1}^{k-2} \beta_{i+j-1}^{\left[\frac{k-i}{2}\right]}. \end{aligned} \tag{3.16}$$

The next theorem was established by Chebotarev, see [Che28] and [CM49, Ch. V § 1]; see also the proof of M. Schiffer and V. Bargmann in [Wig51, II.8]. At the same time, it can be derived as a particular case from Nevanlinna's results, see [Kre38, Theorem 8].

Theorem 3.16 ([Che28, CM49, Wig51, Kre38]). *A real meromorphic function $F(z)$ regular at the origin is an $\mathcal{R}$-function if and only if it has the form*

$$\begin{gathered} F(z) = B_0 + B_1 z + \sum_{1 \leqslant \nu \leqslant \omega} \left(\frac{A_\nu}{z+\sigma_\nu} - \frac{A_\nu}{\sigma_\nu} \right), \\ \text{where } \sum_{1 \leqslant \nu \leqslant \omega} \frac{|A_\nu|}{\sigma_\nu^2} < \infty,\ B_1 > 0 \text{ and } A_\nu < 0,\ \sigma_\nu \in \mathbb{R} \text{ for } \nu = 1, 2 \dots, \omega. \end{gathered} \tag{3.17}$$

The proof of this theorem relies on the following fact which we will use later.

Lemma 3.17 (see *e.g.* [CM49, Ch. VI § 8]). *Let entire functions $q(z)$ and $p(z)$ have no common zeros and be such that $F = \frac{q}{p} \in \mathcal{R}$ is not a constant. Then $F'(z) > 0$ on the real line (where $p(z) \neq 0$), the zeros of $p(z)$ and $q(z)$ are real, simple and interlacing.*

The interlacing property means that between each two consequent zeros of $p(z)$ there exists a unique root of $q(z)$ and *vice versa* (see Definition 2.1). The proof from [CM49]

is based on the behaviour of meromorphic $\mathcal{R}$-functions in neighbourhoods of its zeros and poles. For completeness, we infer this lemma here from the partial fraction expansion (3.17).

Proof. Let $F(z) = \frac{q(z)}{p(z)}$ have the form (3.17). If z is not real, then

$$\frac{\operatorname{Im} F(z)}{\operatorname{Im} z} = B_1 + \sum_{1 \leqslant \nu \leqslant \omega} \frac{-A_\nu}{|z + \sigma_\nu|^2} > 0.$$

Therefore, $F(z)$ (as well as $q(z)$) has no zeros outside the real axis.

Now from (3.17) it follows that $F(z)$ is real and can only have simple poles. Since

$$F'(z) = B_1 + \sum_{1 \leqslant \nu \leqslant \omega} \frac{-A_\nu}{(z + \sigma_\nu)^2} > 0, \quad z \in \mathbb{R} \setminus \{\sigma_\nu\}_{1 \leqslant \nu \leqslant \omega},$$

the function $F(z)$ grows between any of its two subsequent poles σ_ν and $\sigma_{\nu+1}$ from $-\infty$ to $+\infty$. So there is a unique $z_* \in (\sigma_\nu, \sigma_{\nu+1})$ such that $F(z_*) = q(z_*) = 0$; moreover, this zero is simple. For the same reason, exactly one zero of $p(z)$ exists between any two subsequent zeros of $q(z)$. □

The next theorem is a consequence of Grommer's theorem (see [Gro14, § 14, Satz III]) and Theorem 3.16. It can be proved by applying a transformation from the work of Hurwitz[2] [Hu1895] (see also [Kre38, § 6.1], [CM49, Ch. I § 7], [HT2012, Theorem 1.5], [Gan59]) to the matrices of the Hankel forms corresponding to $F(z)$.

Theorem 3.18 (*e.g.* [CM49, Ch. V § 3]). *A meromorphic function $F(z) = \frac{q(z)}{p(z)}$, where $p(z)$ and $q(z)$ are of the form* (3.12)*, is an $\mathcal{R}$-function if and only if*

$$H^{(2m+1)} > 0, \ m = 1, 2, \ldots, l, \ H^{(2l+3)} = H^{(2l+5)} = \cdots = 0,$$

where $0 \leqslant l \leqslant \infty$. Moreover, l is finite if and only if $F(z)$ is a rational function with exactly l poles, counting a pole at infinity (if exists).

Let $\beta_i \neq 0$, $i = 0, 1, \ldots, j$ for some nonnegative j, and the power series $p_{j-1}(z)$, $p_j(z)$ and $p_{j+1}(z)$ be defined by the recurrence formula (3.15). Suppose that the ratio $F_j(z) = \frac{p_{j-1}(z)}{p_j(z)}$ of two formal power series converges to *a meromorphic function*. Then there exist entire functions $\widetilde{p}_{j-1}(z)$ and $\widetilde{p}_j(z)$ with no common zeros such that

$$F_j(z) = \frac{\widetilde{p}_{j-1}(z)}{\widetilde{p}_j(z)}, \quad \widetilde{p}_{j-1}(0) = p_{j-1}(0) \quad \text{and} \quad \widetilde{p}_j(0) = p_j(0) = 1.$$

[2]The same transformation also allows us to express the formulae (12)–(13) from [Per57, p.121] as the relation (3.16).

Define the power series $g(z) := \frac{p_j(z)}{\widetilde{p}_j(z)}$ satisfying $g(0) = 1$. Then

$$\widetilde{p}_j(z) = \frac{p_j(z)}{g(z)} \quad \text{and} \quad \widetilde{p}_{j-1}(z) = \frac{p_{j-1}(z)}{g(z)}.$$

If, in addition, F_j is a non-constant $\mathcal{R}$-function, then by Theorem 3.18 the inequality $\beta_j = H_j^{(3)} > 0$ is satisfied. So, from (3.15) we find the formal relation

$$F_{j+1}(z) := \frac{p_j(z)}{p_{j+1}(z)} = \frac{\beta_j z}{F_j(z) - p_{j-1}(0)}.$$

Lemma 3.19. *The ratio $\frac{p_{j+1}(z)}{g(z)}$ converges to the entire function*

$$\widetilde{p}_{j+1}(z) := \frac{\widetilde{p}_{j-1}(z) - \widetilde{p}_{j-1}(0)\,\widetilde{p}_j(z)}{\beta_j z}. \tag{3.18}$$

The pairs $(\widetilde{p}_j(z), \widetilde{p}_{j+1}(z))$ and $(\widetilde{p}_{j-1}(z), \widetilde{p}_{j+1}(z))$ have no common zeros.

Proof. Dividing (3.15) by $g(z)$ gives us the relation (3.18). As a consequence, we obtain the expression

$$\widetilde{p}_{j-1}(z) = \beta_j z\,\widetilde{p}_{j+1}(z) + \widetilde{p}_{j-1}(0)\,\widetilde{p}_j(z).$$

Each common zero of any two summands in this equation must be a zero of the third summand. Since the functions $\widetilde{p}_{j-1}(z)$ and $\widetilde{p}_j(z)$ have no common zeros, the pairs $(\widetilde{p}_{j-1}(z), \widetilde{p}_{j+1}(z))$ and $(\widetilde{p}_j(z), \widetilde{p}_{j+1}(z))$ also have no common zeros. □

This lemma implies that the series $F_{j+1}(z)$ represents the meromorphic function

$$F_{j+1}(z) = \frac{\widetilde{p}_j(z)}{\widetilde{p}_{j+1}(z)}.$$

Lemma 3.20 (*cf.* [Akh65, pp. 127–128] or [KaKr68]). *If the meromorphic function $F_j(z)$ is not a constant, then $F_j \in \mathcal{S}$ if and only if $F_j, F_{j+1} \in \mathcal{R}$ and $F_j(0) \geqslant 0$.*

Proof. Let $F_j \in \mathcal{S}$, then Theorem 3.16 gives that it has the form

$$F_j(z) = B_0 + B_1 z + \sum_{1 \leqslant \nu \leqslant \omega} \left(\frac{A_\nu}{z + \sigma_\nu} - \frac{A_\nu}{\sigma_\nu} \right) = B_0 + B_1 z + z \sum_{1 \leqslant \nu \leqslant \omega} \frac{(-A_\nu/\sigma_\nu)}{z + \sigma_\nu},$$

$$\text{where} \quad \sum_{1 \leqslant \nu \leqslant \omega} \frac{|A_\nu|}{\sigma_\nu^2} < \infty, \quad B_0 \geqslant 0, \quad B_1 > 0$$

and $A_\nu < 0$, $\sigma_\nu > 0$ for $\nu = 1, 2, \ldots, \omega$. It is sufficient to show that $F_{j+1}(z)$ is a well-defined $\mathcal{R}$-function.

Consider the function

$$G_j(z) := \frac{F_j(-z) - F_j(0)}{-z} = B_1 + \sum_{1 \leqslant \nu \leqslant \omega} \frac{(-A_\nu/\sigma_\nu)}{-z + \sigma_\nu} = B_1 + \sum_{1 \leqslant \nu \leqslant \omega} \frac{(A_\nu/\sigma_\nu)}{z - \sigma_\nu}.$$

It has the form (3.17) and, hence, is a meromorphic $\mathcal{R}$-function by Theorem 3.16. The mappings $z \mapsto \frac{1}{z}$ and $z \mapsto -z$ are in the class $\mathcal{R}^{-1}$ (*i.e.* they map the upper half of the complex plane into the lower half of the complex plane). Since $G_j \in \mathcal{R}$, we have $G_j(-z)$ is in the class $\mathcal{R}^{-1}$ and $\frac{1}{G_j(-z)}$ is in $\mathcal{R}$. Therefore, the ratio

$$\frac{\beta_j}{G_j(-z)} = \frac{\beta_j z}{F_j(z) - p_{j-1}(0)} = F_{j+1}(z)$$

is also an $\mathcal{R}$-function since $\beta_j = H_j^{(3)} > 0$ by Theorem 3.18.

Conversely, let $F_j, F_{j+1} \in \mathcal{R}$ and $F_j(0) \geqslant 0$. By Theorem 3.18, the inequality $\beta_j > 0$ holds, thus $F_{j+1}(z) \not\equiv 0$ and the meromorphic function

$$G_j(z) := \frac{\beta_j}{F_{j+1}(-z)} = \frac{F_j(-z) - F_j(0)}{-z}$$

is an $\mathcal{R}$-function. On the one hand, Theorem 3.16 gives us

$$F_j(z) = B_0 + B_1 z + z \sum_{1 \leqslant \nu \leqslant \omega} \frac{(-A_\nu/\sigma_\nu)}{z + \sigma_\nu}, \quad \text{where} \quad \sum_{1 \leqslant \nu \leqslant \omega} \frac{|A_\nu|}{\sigma_\nu^2} < \infty,$$

$B_1 > 0$ and $A_\nu < 0$, $\sigma_\nu \in \mathbb{R}$ for $\nu = 1, 2 \ldots, \omega$, such that

$$G_j(z) = \frac{F_j(-z) - F_j(0)}{-z} = B_1 + \sum_{1 \leqslant \nu \leqslant \omega} \frac{(A_\nu/\sigma_\nu)}{z - \sigma_\nu}.$$

On the other hand, Theorem 3.16 states that each $\mathcal{R}$-function has negative residues at its poles. Therefore, $\frac{A_\nu}{\sigma_\nu} < 0$ for all ν since $G_j \in \mathcal{R}$. In other words, the poles $-\sigma_\nu$, $\nu = 1, 2 \ldots, \omega$, of the function F_j are negative. Consequently, $F_j \in \mathcal{S}$. □

Corollary 3.21. *If $F_j \in \mathcal{S}$ for some $j \geqslant 0$, then $\beta_j \geqslant 0$. The inequality $\beta_j > 0$ implies $F_{j+1} \in \mathcal{S}$, while the equality $\beta_j = 0$ implies that $F_j(z)$ is constant.*

Proof. If $F_j(z)$ is a constant then $\beta_j = H_j^{(3)} = 0$ by Theorem 3.18. Let now $F_j(z)$ be a non-constant $\mathcal{S}$-function. Applying Lemma 3.20 to it gives us $F_{j+1} \in \mathcal{R}$. By Theorem 3.18, $\beta_{j+1} = 0$ if $F_{j+1}(z)$ is a constant, and $\beta_{j+1} > 0$ if $F_{j+1}(z)$ is not constant. Moreover, we have $F_{j+1}(0) = \frac{p_j(0)}{p_{j+1}(0)} = 1 > 0$. Thus, the corollary holds in the cases of constant $F_j(z)$ or $F_{j+1}(z)$.

Suppose that $\beta_j, \beta_{j+1} > 0$. Then Lemma 3.15 implies that

$$H_j^{(2m+3)} = \beta_j^{\left[\frac{2m+2}{2}\right]} \beta_{j+1}^{\left[\frac{2m+1}{2}\right]} H_{j+2}^{(2m+1)} = \beta_j^{m+1} \beta_{j+1}^{m} H_{j+2}^{(2m+1)}, \quad m = 1, 2, \dots. \tag{3.19}$$

That is, for each positive integer m the sign of $H_{j+2}^{(2m+1)}$ coincides with the sign of $H_j^{(2m+3)}$. Since $F_j \in \mathcal{R}$, Theorem 3.18 yields $F_{j+2} \in \mathcal{R}$. That is, $F_{j+1} \in \mathcal{S}$ by Lemma 3.20. □

Theorem 3.22. *A meromorphic function $F(z) = \frac{q(z)}{p(z)}$, where $p(z)$ and $q(z)$ are series of the form* (3.12)*, is an $\mathcal{S}$-function if and only if there exists $\varkappa$, $2 \leqslant \varkappa \leqslant \infty$, such that*

$$H^{(k)} > 0, \ k = 2, 3, \dots, \varkappa, \quad \text{and} \quad H^{(\varkappa+1)} = H^{(\varkappa+2)} = \dots = 0. \tag{3.20}$$

Moreover, $\varkappa$ is finite if and only if $F(z)$ is a rational function with exactly $\left[\frac{\varkappa-1}{2}\right]$ poles, counting a pole at infinity (if exists).

Proof. By definition, $H_0^{(2)} = H^{(2)} = 1 > 0$. Denote $p_0(z) := p(z)$ and $p_{-1}(z) := q(z)$ such that $F(z) = F_0(z)$. Suppose that $F_0 \in \mathcal{S}$. From the recurrence formulae (3.14)–(3.15) we obtain the sequences $(p_j)_{j=-1}^m$ and $(\beta_j)_{j=0}^m$, where $\beta_j \neq 0$ for $0 \leqslant j < m$ and $0 \leqslant m \leqslant \infty$. Whenever $m < \infty$ we also have $\beta_m = 0$. Then $\beta_j > 0$ for all $j = 0, 1, \dots, m-1$ by Corollary 3.21. Furthermore, the identity (3.16) gives us

$$H^{(j)} = H_0^{(j)} = \prod_{i=1}^{j-2} \beta_{i-1}^{\left[\frac{j-i}{2}\right]} > 0, \quad j = 3, 4, \dots, m+2. \tag{3.21}$$

Let $m < \infty$, then $F_m(z)$ is a constant by Corollary 3.21. Therefore, we have $H_m^{(3)} = H_m^{(4)} = \dots = 0$ since all these minors contain proportional rows. By the identity (3.16), this is equivalent to $H^{(m+3)} = H^{(m+4)} = \dots = 0$.

So we obtained that $F \in \mathcal{S}$ implies (3.20) for $\varkappa = m+2$. The number of poles the function $F(z)$ can be determined from Theorem 3.18.

Now suppose that the conditions (3.20) hold. If $\varkappa = 2$ then $H^{(3)} = H^{(4)} = \dots = 0$ and hence, by Theorem 3.18, the function $F(z) \equiv q(0) \geqslant 0$ is a constant element of $\mathcal{S}$. In the case of $3 \leqslant \varkappa \leqslant \infty$ we have $\beta_0 > 0$, so by Lemma 3.15,

$$H_1^{(2m+1)} = \beta_0^{-\left[\frac{2m+1}{2}\right]} H^{(2m+2)} > 0, \qquad m = 1, 2, \dots, \left[\frac{\varkappa}{2}\right] - 1, \qquad \text{and}$$
$$H_1^{(2m+1)} = \beta_0^{-\left[\frac{2m+1}{2}\right]} H^{(2m+2)} = 0, \qquad m \geqslant \left[\frac{\varkappa}{2}\right].$$

Consequently, the functions $F(z)$ and $F_1(z)$ are $\mathcal{R}$-functions by Theorem 3.18, and Lemma 3.20 yields $F \in \mathcal{S}$. □

3.2.2 $\mathcal{S}$-functions as continued fractions

A continued fraction of the form

$$F(z) = c_0 + \mathop{\mathbf{K}}_{1\leqslant j\leqslant\omega}\left(\frac{c_j z^{r_j}}{1}\right), \quad \text{where}$$
$$c_j \neq 0 \quad \text{and} \quad r_j \in \mathbb{Z}_{>0} \quad \text{for} \quad j = 1,\dots,\omega, \quad 0 \leqslant \omega \leqslant \infty, \tag{3.22}$$

is called a *(general) C-fraction*. The special case of (3.22) that corresponds to $r_j = 1$ for all $j = 1,\dots,\omega$ is called a *regular C-fraction*. Continued fractions of the form (3.22) are able to represent power series uniquely, that is the following fact is true.

Theorem 3.23 ([LeSc39], see also [Per57, § 21, Sätze 3.2–3.5, 3.24]). *Each (formal) power series $F(z) = \sum_{k=0}^{\infty} s_k z^k$ corresponds to a fraction of the form* (3.22)*. This correspondence is set by the following sequence of relations*

$$F_0(z) = F(z), \quad c_0 = F(0), \quad F_i(z) = \frac{c_i z^{r_i}}{F_{i-1}(z) - F_{i-1}(0)}, \quad i = 1,2,\dots,\omega, \tag{3.23}$$

where $\omega \leqslant \infty$ is such that $F_{i-1}(z) \not\equiv F_{i-1}(0)$ for $i-1 < \omega$ and $F_\omega(z) \equiv F_\omega(0)$. The exponents r_i are positive integers chosen together with the complex constants c_i in such a way that $F_i(0) = 1$.

If two C-fractions (finite or infinite) of the form (3.22) *correspond to the same power series, then they coincide. A C-fraction is finite if and only if it corresponds to a rational function (and, hence, represents that function).*

Moreover, if an infinite continued fraction of the form (3.22) *converges uniformly in a closed region T containing the origin in its interior, it represents a regular analytic non-rational function of z throughout the interior of T. Further, the corresponding power series converges to the same function in and on the boundary of the largest circle which can be drawn with its center at the origin, lying wholly within T.*

Suppose that $F(z) = \frac{q(z)}{p(z)}$ is a meromorphic $\mathcal{S}$-function. We again denote $F_0(z) := F(z)$, $p_0(z) := p(z)$ and $p_{-1}(z) := q(z)$ and use the recurrence formulae (3.14)–(3.15) to obtain the sequences $(p_j)_{j=-1}^{\omega}$ and $(\beta_j)_{j=0}^{\omega}$, where $\beta_j \neq 0$ for all $j < \omega$ and $0 \leqslant \omega \leqslant \infty$. In the case $\omega < \infty$ we also have $\beta_\omega = 0$.

For each $j = 0, 1, \dots \omega - 1$, we apply Corollary 3.21, obtaining $\beta_j > 0$ and $F_j \in \mathcal{S}$. If ω is a finite number, then F_ω is a constant. From the relation (3.15) we have

$$F_j(z) = \frac{p_{j-1}(z)}{p_j(z)} = p_{j-1}(0) + \frac{\beta_j z}{F_{j+1}(z)}, \quad j = 0,1,\dots,\omega-1. \tag{3.24}$$

These formulae can be combined into the continued fractions

$$F(z) = F_0(z) = b_0 + \mathop{\mathbf{K}}_{0\leqslant j\leqslant \omega-1} \left(\frac{\beta_j z}{1}\right) \quad \text{and} \tag{3.25}$$

$$F_j(z) = 1 + \mathop{\mathbf{K}}_{j\leqslant k\leqslant \omega-1} \left(\frac{\beta_k z}{1}\right), \quad j = 1, 2, \ldots, \omega - 1, \tag{3.26}$$

where $\beta_k > 0$ for $k = 0, 1, \ldots, \omega - 1$. These are regular C-fractions, and the relations (3.24) set the correspondence between $F_j(z)$ and $F_{j+1}(z)$ satisfying (3.23). That is, the continued fractions in (3.25) and (3.26) correspond to $F(z)$ and $F_j(z)$ for all j, respectively, by Theorem 3.23. In particular, they are finite if and only if $F(z)$ is rational.

Furthermore, there is a power series $g(z)$ such that $\widetilde{p}_j(z) := \frac{p_j(z)}{g(z)}$ are entire functions for $j = -1, 0, \ldots, \omega$, and such that $\widetilde{p}_{j-1}(z)$ and $\widetilde{p}_j(z)$ have no common zeros for $j \geqslant 0$ (see Lemma 3.19). Note that the relations (3.24), (3.25) and (3.26) do not change, if we replace all the series $p_j(z)$ by the functions $\widetilde{p}_j(z)$.

It is convenient to study the continued fractions (3.25) and (3.26), using the following theorem.

Theorem 3.24 (Stieltjes, [St1894, n°s 68–69]; see also [Per57, Wall48][3]). *Let $b_0 \geqslant 0$, β_0, $\beta_1, \ldots, \beta_{\omega-1} > 0$ and $\beta_\omega = 0$ for $0 \leqslant \omega \leqslant \infty$. The sum $\sum_{j=0}^{\omega} \beta_j$ is finite if and only if the continued fraction* (3.25) *converges to a meromorphic $\mathcal{S}$-function and its partial numerators and denominators converge to coprime*[4] *entire functions of genus* 0. *That is, if the jth convergent (approximant) to $F(z)$ is denoted by $\frac{Q_j(z)}{P_j(z)}$, then for $j \to \infty$ we have*

$$P_j(z) \to p(z), \quad Q_j(z) \to q(z) \quad \textit{and}$$
$$b_0 + \mathop{\mathbf{K}}_{0\leqslant k\leqslant j-1} \left(\frac{\beta_k z}{1}\right) = \frac{Q_j(z)}{P_j(z)} \to \frac{q(z)}{p(z)} = F(z),$$

where $p(z)$ and $q(z)$ are coprime entire functions of genus 0. *The convergence is uniform on compact subsets of $\mathbb{C}$ containing no poles of the function $F(z)$.*

To apply this theorem we need to distinguish the case of $\sum_{j=0}^{\omega} \beta_j < \infty$ for $\omega = \infty$.

Lemma 3.25. *Let the functions $\widetilde{p}_1(z)$ and $\widetilde{p}_0(z)$ be entire of genus* 0 *and coprime. Let their ratio $F_1 := \frac{\widetilde{p}_0}{\widetilde{p}_1} \in \mathcal{S}$ be not rational. Then $\widetilde{p}_j(z) \to 1$ as $j \to \infty$ uniformly on compact subsets of $\mathbb{C}$, and $\sum_{j=1}^{\infty} \beta_j < \infty$.*

Proof. According to (3.18), all the functions $\widetilde{p}_j(z)$, $j = 0, 1, \ldots$, are entire of genus 0 and $\widetilde{p}_j(0) = 1$. Moreover, $\widetilde{p}_0, \ldots, \widetilde{p}_j$ are coprime (by Lemma 3.19) and hence they have only negative zeros (since $F_j = \frac{\widetilde{p}_{j-1}}{\widetilde{p}_j} \in \mathcal{S}$ by Corollary 3.21). Therefore, the following

[3]The separate convergence of numerators and denominators was shown by Śleszyński in [Sl1888].

[4]This fact was obtained by Maillet in [Mai08]; see also [Per57, p.150].

representation is valid for $j = 0, 1, \ldots$

$$\widetilde{p}_j(z) = \sum_{k=0}^{\infty} a_k^{(j)} z^k = \prod_{\nu=1}^{\infty} \left(1 + \frac{z}{\sigma_\nu^{(j)}}\right),$$

where $0 < \sigma_1^{(j)} \leqslant \sigma_2^{(j)} \leqslant \ldots$ and $\sum_{\nu=1}^{\infty} \frac{1}{\sigma_\nu^{(j)}} < \infty$. The coefficients $a_k^{(j)}$, $k = 1, 2, \ldots$, are equal to

$$a_k^{(j)} = \sum_{i_1=1}^{\infty} \sum_{\substack{i_2=1 \\ i_2 \notin \{i_1\}}}^{\infty} \cdots \sum_{\substack{i_k=1 \\ i_k \notin \{i_1, i_2, \ldots, i_{k-1}\}}}^{\infty} \frac{1}{\sigma_{i_1}^{(j)} \sigma_{i_2}^{(j)} \cdots \sigma_{i_k}^{(j)}}. \tag{3.27}$$

Note that these sums are convergent since,[5] for $k = 2, 3, 4, \ldots$,

$$a_k^{(j)} < \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \frac{1}{\sigma_{i_1}^{(j)}} \frac{1}{\sigma_{i_2}^{(j)}} \cdots \frac{1}{\sigma_{i_k}^{(j)}} = \left(a_1^{(j)}\right)^k. \tag{3.28}$$

By Lemma 3.17 the zeros of $\widetilde{p}_j(z)$ and $\widetilde{p}_{j-1}(z)$ (which are negative) must be simple and interlacing. In addition, $F_j'(z) > 0$ for real z implying that $0 < \sigma_1^{(j-1)} < \sigma_1^{(j)}$. Hence,

$$0 < \sigma_1^{(j-1)} < \sigma_1^{(j)} < \sigma_2^{(j-1)} < \sigma_2^{(j)} < \sigma_3^{(j-1)} < \sigma_3^{(j)} < \ldots. \tag{3.29}$$

Now we estimate the expression (3.27) using the inequalities (3.29) and obtain that $0 \leqslant a_k^{(j)} < a_k^{(j-1)}$ for $j = 1, 2, \ldots,$ *i.e.* the sequence of positive numbers $a_k^{(j)}$ decreases in j for fixed k. Therefore, there exists a finite $\lim_{j\to\infty} a_k^{(j)} \geqslant 0$ dependent on k.

At the same time, the equality (3.18) implies that the first Taylor coefficient $a_1^{(j)}$ for any j satisfies

$$a_1^{(j-1)} = \beta_j + a_1^{(j)}.$$

[5] In fact, even an estimate stronger than (3.28) is valid (*cf.* [Sl1888, p. 105]). For each tuple of distinct numbers $(i_1, i_2, \ldots, i_k)$ there is only one summand $\left(\sigma_{i_1}^{(j)} \sigma_{i_2}^{(j)} \cdots \sigma_{i_k}^{(j)}\right)^{-1}$ on the right-hand side of (3.27). At the same time, the sum

$$\sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \left(\sigma_{i_1}^{(j)} \sigma_{i_2}^{(j)} \cdots \sigma_{i_k}^{(j)}\right)^{-1}$$

contains exactly $k!$ such summands. Therefore,

$$a_k^{(j)} < \frac{1}{k!} \left(a_1^{(j)}\right)^k \text{ for } k = 2, 3, 4, \ldots.$$

Consequently,

$$a_1^{(0)} = \sum_{i=1}^{j} \beta_i + a_1^{(j)} \quad \text{and} \quad \sum_{j=1}^{\infty} \beta_j = a_1^{(0)} - \lim_{j\to\infty} a_1^{(j)}. \tag{3.30}$$

Therefore, the series $\sum_{j=0}^{\infty} \beta_j$ converges.

As a consequence, for an arbitrary positive number R there exists an integer j_0 depending on R, such that $\beta_j R < \frac{1}{4}$ for all $j \geqslant j_0$. By virtue of Worpitzky's test (as it stated in [Wall48, p. 45], see also [Wo1865]) the continued fraction

$$\widetilde{F}_{j_0}(z) = 1 + \mathop{\mathbf{K}}_{j=j_0}^{\infty} \left(\frac{\beta_j z}{1} \right)$$

converges to a holomorphic function uniformly in the disk $|z| < R$. By Theorem 3.23, this holomorphic function coincides with the sum of the power series F_{j_0} (since F_{j_0} corresponds to $\widetilde{F}_{j_0}(z)$, see (3.26)). Therefore, $\widetilde{p}_{j_0}$ has no zeros in this disk, that is $R < \sigma_1^{(j_0)} < \sigma_1^{(j)}$, $j \geqslant j_0$. Letting R tend to infinity, we obtain $\lim_{j\to\infty} \sigma_1^{(j)} = \infty$. According to (3.27) we have

$$a_1^{(j)} = \sum_{i=1}^{\infty} \frac{1}{\sigma_i^{(j)}}$$

and each term in this series monotonically tends to zero as $j \to \infty$. For any $\varepsilon > 0$ there exists N such that

$$\sum_{i=N+1}^{\infty} \frac{1}{\sigma_i^{(j_0)}} < \varepsilon, \text{ which for } j \geqslant j_0 \text{ gives } \sum_{i=N+1}^{\infty} \frac{1}{\sigma_i^{(j)}} \leqslant \sum_{i=N+1}^{\infty} \frac{1}{\sigma_i^{(j_0)}} < \varepsilon.$$

On the other hand, $\lim\limits_{j\to\infty} \sum\limits_{i=1}^{N} \dfrac{1}{\sigma_i^{(j)}} \leqslant \lim\limits_{j\to\infty} \dfrac{N}{\sigma_1^{(j)}} = 0$, so the coefficient $a_1^{(j)}$ also vanishes. Now from (3.28) we obtain that $\widetilde{p}_j(z) \to 1$ as $j \to \infty$ uniformly on compact subsets of $\mathbb{C}$. □

Corollary 3.26. *Under the assumptions of Lemma 3.25 there exists a positive number M independent of j such that*

$$\|H(\widetilde{p}_{j+1}, \widetilde{p}_j)\|_1 < M \quad \textit{for} \quad j = 0, 1, \ldots,$$

where the matrix $H(\widetilde{p}_{j+1}, \widetilde{p}_j)$ is defined by (3.5)*, and*

$$\|H(\widetilde{p}_{j+1}, \widetilde{p}_j) - H(1,1)\|_1 \xrightarrow{j\to\infty} 0.$$

Proof. Since $a_k^{(j)} \geqslant a_k^{(j+1)} \geqslant 0$ for all $j, k = 0, 1, \dots$, we have

$$\|H(\widetilde{p}_{j+1}, \widetilde{p}_j)\|_1 = \max\left\{\sum_{k=0}^{\infty} a_k^{(j+1)}, \sum_{k=0}^{\infty} a_k^{(j)}\right\} = \sum_{k=0}^{\infty} a_k^{(j)} \leqslant \sum_{k=0}^{\infty} a_k^{(0)} = \widetilde{p}_0(1) < \infty.$$

Analogously,

$$\|H(\widetilde{p}_{j+1}, \widetilde{p}_j) - H(1,1)\|_1 = \max\left\{\sum_{k=1}^{\infty} a_k^{(j+1)}, \sum_{k=1}^{\infty} a_k^{(j)}\right\} = \sum_{k=1}^{\infty} a_k^{(j)} \overset{a_0^{(j)}=1}{=\!=\!=} \widetilde{p}_j(1) - 1,$$

which is vanishing as $j \to \infty$ by Lemma 3.25. □

Proof of Corollary 3.13. If $F = \frac{p}{q} \in \mathcal{S}$ then $F(z)$ can be formally developed into the continued fraction (3.25) with the coefficients $b_0 = F(0)$ and $(\beta_j)_{j=0}^{\omega-1}$ given by (3.14). By Lemma 3.25, the coefficients of this continued fraction satisfy the condition $\sum_{j=0}^{\omega-1} \beta_j < \infty$. Consequently, Theorem 3.24 implies that (3.25) converges uniformly on compact sets containing no poles of its limiting function. Now, to finish the proof we apply Theorem 3.23: the continued fraction (3.25) corresponds to and converges to $F(z)$, and there is no other continued fraction of the form (3.22) corresponding to $F(z)$. □

3.3 Proof of Theorem 3.8

Suppose that meromorphic functions p and q are regular at the origin and have the Taylor expansion (3.12). Consider the Hurwitz-type matrix $H(p, q)$, defined by (3.5).

Lemma 3.27. *If $\frac{q}{p} \in \mathcal{S}$ and there exists a meromorphic function $g(z)$, $g(0) = 1$, such that the ratios $\widetilde{p}(z) := \frac{p(z)}{g(z)}$ and $\widetilde{q}(z) := \frac{q(z)}{g(z)}$ are entire of genus 0 and coprime, then the matrix $H(p, q)$ can be factored as in* (3.7)*, where the numbers β_j, $j = 0, 1, \dots$, are given by* (3.14) *(possibly followed by zeros). Moreover,*

$$\left\|H(p,q) - J(b_0, \beta_0)\, J(1, \beta_1) \cdots J(1, \beta_j)\, H(1,1)\, T(g)\right\|_\rho \xrightarrow{j\to\infty} 0,$$

where ρ, $0 < \rho \leqslant 1$, is such that $g(z)$ has no poles in the disk $|z| \leqslant \rho$.

Proof. For any two matrices $A = (a_{kl})_{k,l=1}^{\infty}$ and $B = (b_{kl})_{k,l=1}^{\infty}$ such that $\|A\|_1 < \infty$ and $\|B\|_\rho < \infty$ by definition of the involved norms we have that

$$\|A\|_1 \|B\|_\rho = \sup_{1\leqslant k<\infty} \sum_{l=1}^{\infty} |a_{kl}| \sup_{1\leqslant m<\infty} \sum_{j=1}^{\infty} |b_{mj}| \rho^{j-1} \geqslant \sup_{1\leqslant k<\infty} \sum_{l=1}^{\infty} |a_{kl}| \sum_{j=1}^{\infty} |b_{lj}| \rho^{j-1}.$$

Changing the order of summation on the right-hand side turns this relation into

$$\infty > \|A\|_1 \|B\|_\rho \geqslant \sup_{1\leqslant k<\infty} \sum_{j=1}^{\infty} \left| \sum_{l=1}^{\infty} a_{kl} b_{lj} \right| \rho^{j-1} = \|AB\|_\rho, \tag{3.31}$$

which implies the existence of the product AB. Now we note that the decomposition

$$H(p,q) = H(\widetilde{p},\widetilde{q})\, T(g) \tag{3.32}$$

is valid. It can be checked by the straightforward multiplication.

Denote $p_0(z) := \widetilde{p}(z)$ and $p_{-1}(z) := \widetilde{q}(z)$. With the help of the Euclidean algorithm (3.14)–(3.15) we construct the (longest possible) sequence $(p_j)_{j=-1}^{\omega}$ of entire functions, $0 \leqslant \omega \leqslant \infty$. By Corollary 3.21, the corresponding numbers $(\beta_j)_{j=0}^{\omega}$ satisfy $\beta_j > 0$ for all $j = 0, 1, \dots \omega - 1$. In the case of finite ω two last entries in $(p_j)_{j=-1}^{\omega}$ are actually constant: $p_{\omega-1}(z) \equiv p_{\omega-1}(0)$, $p_\omega(z) \equiv 1$, and additionally $\beta_\omega = 0$; we extend this equality by $\beta_{\omega+1} = \beta_{\omega+2} = \dots = 0$.

The identity (3.32) implies the factorization (3.9) when $q(z) \equiv q(0)p(z)$ (which is corresponding to $\omega = 0$). Suppose that $\omega > 0$. We expand $p_i(z)$ and $p_{i-1}(z)$, $i = 0, 1, \dots, j < \omega$, as follows

$$p_i(z) = \sum_{k=0}^{\infty} c_k z^k, \quad c_0 = 1, \quad \text{and} \quad p_{i-1}(z) = \sum_{k=0}^{\infty} d_k z^k,$$

so that the matrix $H_{i+1} := H(p_{i+1}, p_i)$ takes the form

$$H_{i+1} = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & \dots \\ 0 & \frac{1}{\beta_i}(d_1 - d_0 c_1) & \frac{1}{\beta_i}(d_2 - d_0 c_2) & \frac{1}{\beta_i}(d_3 - d_0 c_3) & \dots \\ 0 & c_0 & c_1 & c_2 & \dots \\ 0 & 0 & \frac{1}{\beta_i}(d_1 - d_0 c_1) & \frac{1}{\beta_i}(d_2 - d_0 c_2) & \dots \\ 0 & 0 & c_0 & c_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The multiplication of this matrix from the left by

$$J(d_0, \beta_i) = \begin{pmatrix} d_0 & \beta_i & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & d_0 & \beta_i & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & d_0 & \beta_i & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

gives us the matrix H_i. So, on putting i successively equal to $0, 1, \ldots, j$ and applying (3.32) we find that

$$H(p,q) = J(b_0, \beta_0)\, J(1, \beta_1) \cdots J(1, \beta_j)\, H(p_{j+1}, p_j)\, T(g). \tag{3.33}$$

The finite product on the left-hand side of (3.33) is well defined and associative, because the matrices $J(1, \cdot)$ and $H(1, 1)$ have at most two nonzero entries in each row and column. If ω is finite and $j = \omega - 1$, then the equality (3.33) coincides with (3.9). Therefore, since $\|T(g)\|_\rho = g(\rho) < \infty$, from (3.8), (3.31) and Corollary 3.26 we obtain the assertion of the theorem for $\omega < \infty$.

Suppose that $\omega = \infty$ and let us prove that the difference between the product in (3.33) and the right-hand side of (3.7) converges to zero as $j \to \infty$. Indeed, the sum $\sum_{j=0}^{\infty} \beta_j$ is finite by Lemma 3.25. Hence, there exists an index $j_0 \geqslant 1$ such that

$$\sum_{j=j_0}^{\infty} \beta_j < \frac{1}{2}. \tag{3.34}$$

Let

$$V := J(b_0, \beta_0)\, J(1, \beta_1) \cdots J(1, \beta_{j_0-1}) \ \text{ and } \ U_j := J(1, \beta_{j_0}) \cdots J(1, \beta_j)$$

for $j = j_0, j_0 + 1, \ldots$. Then we can express the equality (3.33) as follows

$$H(p,q) = V\, U_j\, H(p_{j+1}, p_j)\, T(g).$$

The matrix $U_j = J(1, \beta_{j_0}) \cdots J(1, \beta_j)$ is upper triangular and has no negative entries since it is a product of upper triangular matrices with nonnegative entries. The diagonal elements of U_j are the products of corresponding diagonal elements of $J(1, \beta_{j_0}), \ldots, J(1, \beta_j)$ and, thus, are equal to 1 on the odd rows and 0 on the even ones.

More specifically, denote the entries of U_j by $u_{kl}^{(j)}$ so that

$$U_j = \left(u_{kl}^{(j)}\right)_{k,l=1}^{\infty}.$$

For $j \geqslant j_0$ and $k, m = 1, 2, \ldots$ the equality $U_{j+1} = U_j\, J(1, \beta_{j+1})$ implies that

$$u_{k,1}^{(j+1)} = u_{k,1}^{(j)}, \qquad u_{k,2m}^{(j+1)} = u_{k,2m-1}^{(j)}\, \beta_{j+1}, \qquad u_{k,2m+1}^{(j+1)} = u_{k,2m}^{(j)} + u_{k,2m+1}^{(j)}.$$

The following entries for all $j \geqslant j_0$ must be zero

$$u_{k,2m-1}^{(j)} = u_{k,2m}^{(j)} = 0, \quad m = 1, 2, \ldots, \quad k = 2m, 2m+1, \ldots.$$

Observe that $U_{j_0} = J(1, \beta_{j_0})$, so the nonzero entries of U_{j_0} are only

$$u^{(j_0)}_{2m-1,2m} = \beta_j,\ u^{(j_0)}_{2m,2m+1} = 1 \text{ and } u^{(j_0)}_{2m-1,2m-1} = 1, \text{ where } m = 1, 2, \ldots.$$

Consequently, the estimate

$$u^{(j)}_{k,2m} + u^{(j)}_{k,2m+1} \leqslant \left(\sum_{i=j_0}^{j} \beta_i\right)^{m-\left[\frac{k}{2}\right]}, \quad m = 1, 2, \ldots, \quad k = 1, 2, \ldots, 2m \tag{3.35}$$

is valid for $j = j_0$. Now we perform the inductive step to prove (3.35) for $j \geqslant j_0$. Suppose that (3.35) holds for some $j \geqslant j_0$, then for $k \leqslant 2m$ we have (assuming $u^{(j)}_{k,2m-2} = 0$ when $m = 1$)

$$\begin{aligned} u^{(j+1)}_{k,2m} + u^{(j+1)}_{k,2m+1} &= u^{(j)}_{k,2m-1}\,\beta_{j+1} + \left(u^{(j)}_{k,2m} + u^{(j)}_{k,2m+1}\right) \\ &\leqslant \left(u^{(j)}_{k,2m-2} + u^{(j)}_{k,2m-1}\right)\beta_{j+1} + \left(u^{(j)}_{k,2m} + u^{(j)}_{k,2m+1}\right) \\ &\leqslant \left(\beta_{j+1} + \sum_{i=j_0}^{j} \beta_i\right)\left(\sum_{i=j_0}^{j} \beta_i\right)^{m-1-\left[\frac{k}{2}\right]} \leqslant \left(\sum_{i=j_0}^{j+1} \beta_i\right)^{m-\left[\frac{k}{2}\right]}. \end{aligned}$$

By induction, the conditions (3.35) hold for all $j \geqslant j_0$. Therefore, by (3.34),

$$u^{(j)}_{k,2m} + u^{(j)}_{k,2m+1} \leqslant \left(\textstyle\sum_{i=j_0}^{\infty} \beta_i\right)^{m-\left[\frac{k}{2}\right]} \leqslant 2^{-m+\left[\frac{k}{2}\right]},$$

where $m = 1, 2, \ldots$ and $k = 1, 2, \ldots, 2m$. As a consequence,

$$\|U_j\|_1 = \sup_{1\leqslant k<\infty}\left(u^{(j)}_{k,1} + \sum_{m=1}^{\infty}\left(u^{(j)}_{k,2m} + u^{(j)}_{k,2m+1}\right)\right) \leqslant \sum_{m=0}^{\infty} 2^{-m} = 2.$$

Since $u^{(j+1)}_{k,1} - u^{(j)}_{k,1} = 0$ and $u^{(j+1)}_{k,2m-1} - u^{(j)}_{k,2m-1} = u^{(j)}_{k,2m-2}$, $m > 1$, we have

$$\begin{aligned} \|U_{j+1} - U_j\|_1 &= \sup_{1\leqslant k<\infty} \sum_{m=1}^{\infty}\left(\left|u^{(j+1)}_{k,2m-1} - u^{(j)}_{k,2m-1}\right| + \left|u^{(j+1)}_{k,2m} - u^{(j)}_{k,2m}\right|\right) \\ &\leqslant \sup_{1\leqslant k<\infty}\left(\sum_{m=2}^{\infty} u^{(j)}_{k,2m-2} + \sum_{m=1}^{\infty} u^{(j)}_{k,2m} + \sum_{m=1}^{\infty} u^{(j+1)}_{k,2m}\right) \\ &= \sup_{1\leqslant k<\infty}\left(2\sum_{m=1}^{\infty} u^{(j)}_{k,2m} + \sum_{m=1}^{\infty} u^{(j+1)}_{k,2m}\right) \\ &= \sup_{1\leqslant k<\infty}\left(2\beta_j \sum_{m=1}^{\infty} u^{(j-1)}_{k,2m-1} + \beta_{j+1}\sum_{m=1}^{\infty} u^{(j)}_{k,2m-1}\right) \\ &\leqslant 2\beta_j\|U_{j-1}\|_1 + \beta_{j+1}\|U_j\|_1 \leqslant 4\beta_j + 2\beta_{j+1} \xrightarrow{j\to\infty} 0. \end{aligned}$$

That is, $(U_j)_{j=j_0}^{\infty}$ is a Cauchy sequence, hence it converges to its entry-wise limit U_*. So we have shown that $\|U_j\|_1$ is bounded uniformly in j and there exists a matrix U_* such that $\|U_j - U_*\|_1 \xrightarrow{j\to\infty} 0$.

The estimate (3.31) yields that

$$\|V\|_1 \leqslant \|J(b_0, \beta_0)\|_1 \, \|J(1, \beta_1)\|_1 \cdots \|J(1, \beta_{j_0-1})\|_1 < \infty$$

and

$$\begin{aligned} &\Big\|V\, U_j\, H_{j+1}\, T(g) - V\, U_*\, H(1,1)\, T(g)\Big\|_\rho \\ &\qquad \leqslant \|V\|_1 \Big\|U_j\, H_{j+1} - U_*\, H_{j+1} + U_*\, H_{j+1} - U_*\, H(1,1)\Big\|_1 \, \|T(g)\|_\rho \\ &\qquad \leqslant \|V\|_1 \, \|U_j - U_*\|_1 \, \|H_{j+1}\|_1 \, \|T(g)\|_\rho \\ &\qquad\quad + \|V\|_1 \, \|U_*\|_1 \, \|H_{j+1} - H(1,1)\|_1 \, \|T(g)\|_\rho. \end{aligned} \tag{3.36}$$

We have all ingredients to finish the prove of the lemma. Indeed, the expansion of $g(z)$ into a power series at the origin converges for $|z| \leqslant \rho$ absolutely, hence $\|T(g)\|_\rho < \infty$. According to Corollary 3.26,

$$\|H_{j+1}\|_1 = \|H(p_{j+1}, p_j)\|_1 < M \quad \text{and} \quad \|H(p_{j+1}, p_j) - H(1,1)\|_1 \xrightarrow{j\to\infty} 0,$$

so the right-hand side of (3.36) vanishes and, therefore,

$$\begin{aligned} H(p,q) &= VU_* H(1,1)\, T(g), \quad \text{where} \\ VU_* &= \lim_{j\to\infty} \big(J(b_0, \beta_0)\, J(1, \beta_1) \cdots J(1, \beta_j)\big). \end{aligned}$$

□

Remark 3.28. Applying this lemma to the ratio $\frac{p_{j_0-1}(z)}{p_{j_0}(z)}$ we can explicitly determine the matrix U_*. Since

$$u_{k,2m}^{(j+1)} = \beta_{j+1} u_{k,2m-1}^{(j)} \xrightarrow{j\to\infty} 0 \text{ for } m = 1, 2, \ldots \text{ and } k = 2m, 2m+1, \ldots,$$

from $H_{j_0} = U_* H(1,1)$ we get

$$U_* = H_{j_0}\, H^{\mathsf{T}}(0,1),$$

where A^{T} stands for the transpose of a matrix A.

Now consider meromorphic functions $p(z) =: p_0(z)$ and $q(z) =: p_{-1}(z)$, where $q(z) \not\equiv 0$ and $q(0) \geqslant 0$, with the power series expansions given by (3.12). Suppose

that $\beta_0, \beta_1, \dots, \beta_{j-1} \neq 0$. Then we can define the functions $p_1(z), p_2(z), \dots, p_j(z)$ via the formulae (3.15).

Lemma 3.29. *If the Hurwitz-type matrix $H_j := H(p_j, p_{j-1})$ satisfies the conditions*

$$H_j \begin{pmatrix} 2 & 3 & \dots & k-1 & k \\ 2 & 3 & \dots & k-1 & i \end{pmatrix} \geqslant 0, \quad k = 2, 3, \dots \text{ and } i = k, k+1, \dots, \tag{3.37}$$

$$\text{then} \quad p_{j-1}(0)\, p_j(z) \equiv p_{j-1}(z) \iff \beta_j := H_j \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} = 0.$$

Proof. Without loss of generality we assume $j = 0$. Suppose that $b_0\, p_0(z) \equiv p_{-1}(z)$. Then the minor $\beta_0 = H_0 \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$ has two proportional rows, and therefore it is zero.

The converse we prove by contradiction. Let $\beta_0 = b_1 - a_1 b_0 = 0$ and $p_0(z) \not\equiv b_0\, p_{-1}(z)$. Then there exists an integer $i > 1$ such that $b_i \neq b_0 a_i$ (since $p_{-1}(z) \not\equiv 0$). Therefore, according to (3.37) we have

$$H_0 \begin{pmatrix} 2 & 3 \\ 2 & i+1 \end{pmatrix} = \begin{vmatrix} a_0 & a_i \\ b_0 & b_i \end{vmatrix} > 0.$$

Consequently,

$$H_0 \begin{pmatrix} 2 & 3 & 4 \\ 2 & 3 & i+2 \end{pmatrix} = \begin{vmatrix} a_0 & a_1 & a_i \\ b_0 & b_1 & b_i \\ 0 & a_0 & a_{i-1} \end{vmatrix} = -a_0 \begin{vmatrix} a_0 & a_i \\ b_0 & b_i \end{vmatrix} < 0,$$

which contradicts the conditions (3.37). □

Corollary 3.30. *Let meromorphic functions $p(z)$ and $q(z)$ be such that $p(0) = 1$ and $q(0) \geqslant 0$. If the matrix $H(p, q)$ satisfies the conditions* (3.37)*, then $F := \frac{q}{p} \in \mathcal{S}$.*

Proof. For $q(z) \equiv 0$ this corollary is obvious. Suppose that $q(z) \not\equiv 0$. Set $p_0(z) := p(z)$ and $p_{-1}(z) := q(z)$ such that $F(z) = F_0(z)$.

Suppose that for some $j \geqslant 0$ we have constructed the sequences $p_{-1}(z), \dots, p_j(z)$ and $\beta_0, \dots, \beta_{j-1} > 0$. By Lemma 3.15 the matrix H_j satisfies (3.37). Therefore, according to Lemma 3.29, we have two mutually exclusive possibilities: $\beta_j > 0$ and $p_{j-1}(0) \cdot p_j(z) = p_{j-1}(z)$. Consider the latter case. We have $H_j^{(2)} = p_j(0) = 1$ and $H_j^{(3)} = H_j^{(4)} = \dots = 0$ with the notation (3.13). Additionally, the numbers β_i are positive for all $i = 0, 1, \dots, j-1$. By virtue of Lemma 3.15 the minors $H_0^{(k)}$ are positive for $k = 2, \dots, j+2$, and zero for $k > j+2$. So by Theorem 3.22 the considered function $F(z)$ is a rational $\mathcal{S}$-function.

If $\beta_j > 0$ we can define the function $p_{j+1}(z)$ by (3.15). According to Lemma 3.15 the matrix H_{j+1} satisfies (3.37). So we can make the next step of this algorithm. If this process is infinite, by Lemma 3.15 all the principal minors $H_0^{(k)}$ are positive. Hence $F \in \mathcal{S}$ by Theorem 3.22. □

Proof of Theorem 3.8.
The scheme of the proof is as follows: (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).

(i) $\Longrightarrow$ (iii): If the condition (i) holds then Lemma 3.27 provides us with the factorization (3.7). In addition, the meromorphic function $g(z)$ satisfying (i) determines the Toeplitz matrix $T(g)$, which is totally nonnegative by Theorem 3.5.

(iii) $\Longrightarrow$ (ii): The factors in (3.7) are totally nonnegative and have at most a finite number of nonzero entries in each column. Therefore, the Cauchy-Binet formula is applicable and it implies that the finite products of the form

$$J(b_0, \beta_0)\, J(1, \beta_1) \cdots J(1, \beta_j)\, H(1,1)\, T(g), \text{ where } 0 \leqslant j < \infty,$$

are totally nonnegative. As it is stated in (iii), these products converge as $j \to \infty$ to $H(p,q)$ in $\|\cdot\|_\rho$-norm and, hence, converge to $H(p,q)$ entry-wise. At the same time, the minors depend continuously on the matrix entries, so the entry-wise limit of totally nonnegative matrices is totally nonnegative itself. As a consequence, the matrix $H(p,q)$ is totally nonnegative if the condition (iii) holds.

(ii) $\Longrightarrow$ (i): The matrices $T(p)$ and $T(q)$ are submatrices of $H(p,q)$. Thus, if (ii) is true, by Theorem 3.5 both series $p(z)$ and $q(z)$ converge to meromorphic functions of the form (3.4). Therefore, $F(z)$ is a meromorphic function as well.

Now, Corollary 3.30 yields $F \in \mathcal{S}$. So according to Lemma 3.17, the zeros (we denote their number by $\omega_1 \leqslant \infty$) and poles (we denote their number by $\omega_2 \leqslant \infty$) of $F(z)$ are real, simple and interlacing. Moreover, $F(x) > F(0) \geqslant 0$ for $x > 0$, hence $F(z)$ has only the zeros $-\tau_1, -\tau_2, \ldots, -\tau_{\omega_1}$ and poles $-\sigma_1, -\sigma_2, \ldots, -\sigma_{\omega_2}$, satisfying the following condition (*cf.* (3.29))

$$0 \leqslant \tau_1 < \sigma_1 < \tau_2 < \sigma_2 < \tau_3 < \ldots. \tag{3.38}$$

Note that (3.38) implies the inequality $\omega_1 - 1 \leqslant \omega_2 \leqslant \omega_1$.

However, the functions $p(z)$ and $q(z)$ have the form (3.4), in particular they have no nonpositive poles. Therefore, all the numbers $-\tau_1, -\tau_2, \ldots, -\tau_{\omega_1}$ are among the zeros of $q(z)$, while $-\sigma_1, -\sigma_2, \ldots, -\sigma_{\omega_2}$ are among the zeros of $p(z)$. As a result,

$$q(z) = e^{\gamma_1 z}\widetilde{q}(z)\frac{\prod_\nu \left(1 + \frac{z}{\alpha_\nu}\right)}{\prod_\mu \left(1 - \frac{z}{\beta_\mu}\right)} \quad \text{and} \quad p(z) = e^{\gamma_2 z}\widetilde{p}(z)\frac{\prod_\nu \left(1 + \frac{z}{\alpha_\nu}\right)}{\prod_\mu \left(1 - \frac{z}{\beta_\mu}\right)},$$

where γ_1, γ_2, $(\alpha_\nu)_\nu$ and $(\beta_\mu)_\mu$ are appropriately chosen positive numbers,

$$\widetilde{q}(z) := b_0 z^j \prod_{\nu=j+1}^{\omega_1} \left(1 + \frac{z}{\tau_\nu}\right) \text{ for } j = 1 - \operatorname{sign} b_0 \text{ and } \widetilde{p}(z) := \prod_{\mu=1}^{\omega_2} \left(1 + \frac{z}{\sigma_\mu}\right).$$

After cancellations in the fraction $\dfrac{q(z)}{p(z)}$ we obtain $F(z) = e^{\gamma z}\dfrac{\widetilde{q}(z)}{\widetilde{p}(z)}$, where $\gamma := \gamma_1 - \gamma_2$.

Let us show that $\gamma = 0$. Set

$$G(z) := \tfrac{1}{b_0} e^{-\gamma z} F(z).$$

Recall that the numbers of zeros ω_1 and ω_2 are finite simultaneously. For $\omega_1, \omega_2 < \infty$ we can express the rational function $G(z)$ as a sum of partial fractions and ascertain that it agrees with the expansion (3.17). So in this case $G \in \mathcal{S}$.

Suppose that ω_1 and ω_2 are infinite. Then the inclusion $G \in \mathcal{S}$ follows form [CM49, Ch. IV § 10, Lemma 1]. Let us reproduce the proof here for completeness. The condition (3.38) implies that for $\operatorname{Im} z > 0$

$$\pi > \operatorname{Arg}(\tau_1 + z) > \operatorname{Arg}(\sigma_1 + z) > \operatorname{Arg}(\tau_2 + z) > \operatorname{Arg}(\sigma_2 + z) > \cdots > 0,$$

where $\operatorname{Arg} : \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$ denotes the principal value of argument. Whenever $\nu > 1$ or $\tau_1 \neq 0$, we obtain

$$0 < \operatorname{Arg}\left(1 + \frac{z}{\tau_\nu}\right) - \operatorname{Arg}\left(1 + \frac{z}{\sigma_\nu}\right) < \operatorname{Arg}\left(1 + \frac{z}{\tau_\nu}\right) - \operatorname{Arg}\left(1 + \frac{z}{\tau_{\nu+1}}\right) < \pi. \quad (3.39)$$

Therefore, if $\tau_1 > 0$,

$$0 < \sum_{\nu=1}^{\infty} \operatorname{Arg} \frac{1 + \frac{z}{\tau_\nu}}{1 + \frac{z}{\sigma_\nu}} = \sum_{\nu=1}^{\infty} \left(\operatorname{Arg}\left(1 + \frac{z}{\tau_\nu}\right) - \operatorname{Arg}\left(1 + \frac{z}{\sigma_\nu}\right)\right) <$$
$$\operatorname{Arg}\left(1 + \frac{z}{\tau_1}\right) - \lim_{\nu\to\infty} \operatorname{Arg}\left(1 + \frac{z}{\tau_\nu}\right) < \pi. \quad (3.40)$$

The values

$$\operatorname{Arg} \prod_{\nu=1}^{\infty} \frac{1 + \frac{z}{\tau_\nu}}{1 + \frac{z}{\sigma_\nu}} \in (-\pi, \pi] \quad \text{and} \quad \sum_{\nu=1}^{\infty} \operatorname{Arg} \frac{1 + \frac{z}{\tau_\nu}}{1 + \frac{z}{\sigma_\nu}},$$

when the sum is convergent, can differ only by $2\pi k$ for some integer k; the inequality (3.40) shows that, actually, $k = 0$. Therefore,

$$0 < \operatorname{Arg} G(z) = \operatorname{Arg} \prod_{\nu=1}^{\infty} \frac{1 + \frac{z}{\tau_\nu}}{1 + \frac{z}{\sigma_\nu}} < \pi \quad \text{when} \quad \operatorname{Im} z > 0, \quad (3.41)$$

i.e. $G \in \mathcal{S}$ since $G(z)$ is real. If $\tau_1 = 0$ we just replace all instances of $\left(1 + \frac{z}{\tau_1}\right)$ in the inequalities (3.39)–(3.41) with z and obtain the same.

Now if we assume $\gamma \neq 0$, then $\operatorname{Arg} F\left(\frac{\pi i}{\gamma}\right) = \operatorname{Arg}\left(-G\left(\frac{\pi i}{\gamma}\right)\right)$ and $G \in \mathcal{S}$, which contradicts the inclusion $F \in \mathcal{S}$. Thus $\gamma = 0$. □

3.4 Proofs of Theorems 3.1 and 3.4

First, let us prove the following auxiliary lemma.

Lemma 3.31. *Let $p(z)$, $q(z)$ be the formal power series $p(z) = \sum_{k=0}^{\infty} a_k z^k$ and $q(z) = \sum_{k=0}^{\infty} b_k z^k$ such that $a_0 = 0$ and $b_0 > 0$. The matrix $H(p, q)$ defined by* (3.5) *is totally nonnegative if and only if $p(z) \equiv 0$ and $q(z)$ converges to a function of the form*

$$C\, e^{\gamma z} \frac{\prod_\nu \left(1 + \frac{z}{\alpha_\nu}\right)}{\prod_\mu \left(1 - \frac{z}{\beta_\mu}\right)}, \tag{3.42}$$

where $\gamma \geqslant 0$, $\alpha_\nu, \beta_\mu > 0$ for all μ, ν and $\sum_\nu \frac{1}{\alpha_\nu} + \sum_\mu \frac{1}{\beta_\mu} < \infty$.

Proof. If $a_0 = 0$ then the total nonnegativity of $H = H(p, q)$ implies

$$0 \leqslant a_i = -\frac{1}{b_0} \begin{vmatrix} a_0 & a_i \\ b_0 & b_i \end{vmatrix} = -\frac{1}{b_0} H \begin{pmatrix} 2 & 3 \\ 2 & i+1 \end{pmatrix} \leqslant 0 \quad \forall i = 1, 2, 3, \ldots,$$

so $p(z) \equiv 0$. The Toeplitz matrix $T(q)$ defined by (3.3) is totally nonnegative as a submatrix of $H(p, q)$ (also *cf.* (3.32)). Therefore, by Theorem 3.5, $q(z)$ is of the form (3.42).

Conversely, if $p(z) \equiv 0$ then any nonzero minor of $H(p, q)$ is equal to a minor of $T(q)$. According to Theorem 3.5 the matrix $T(q)$ is totally nonnegative, hence $H(p, q)$ is totally nonnegative as well. □

Proof of Theorem 3.4. Suppose that the matrix $\mathcal{D}_f = H(f', f)$ is totally nonnegative (here $f'(z)$ denotes the formal derivative of $f(z)$). Let us show that the series $f(z)$ converges to the function

$$f(z) = f_0\, e^{\gamma z} \prod_{1 \leqslant \nu \leqslant \omega} \left(1 + \frac{z}{\alpha_\nu}\right), \tag{3.43}$$

where $\gamma \geqslant 0$, $\alpha_\nu > 0$ for all ν and $\sum_{1 \leqslant \nu \leqslant \omega} \frac{1}{\alpha_\nu} < \infty$ for some ω, $0 \leqslant \omega \leqslant \infty$. If $f_1 = 0$, by Lemma 3.31 we have $f(z) \equiv f_0 > 0$, *i.e.* (3.43) is satisfied.

Consider the case $f_1 \neq 0$. Theorem 3.8 implies that $f(z)$ and $f'(z)$ converge in a neighbourhood of the origin. Moreover, for some meromorphic function $g(z)$ of the form (3.42), the functions $\widetilde{f}(z) := \frac{f(z)}{g(z)}$ and $h(z) := \frac{f'(z)}{g(z)}$ are entire of genus 0, coprime

and have only negative zeros. In particular, the poles of $g(z)$ are positive (if any). Let us show that $g(z)$ has no poles. Observe that the logarithmic derivative of $g(z)$ on the right-hand side of the expression

$$h(z) = \frac{\widetilde{f}(z)g'(z) + g(z)\widetilde{f}'(z)}{g(z)} = \widetilde{f}'(z) + \widetilde{f}(z)\frac{g'(z)}{g(z)}$$

is multiplied by a function with no positive zeros. Therefore, each pole of $g(z)$ must be a pole of $h(z)$. But $h(z)$ is an entire function, thus $g(z)$ is entire of the form (3.43). That is, $f(z) = \widetilde{f}(z)g(z)$ can be represented as in (3.43).

Conversely, let $f(z)$ admit the representation (3.43). If $f(z)$ is a constant then by Lemma 3.31 the matrix $\mathcal{D}_f$ is totally nonnegative. Suppose now that $f(z)$ is not a constant and consider its logarithmic derivative

$$F(z) = \frac{f'(z)}{f(z)} = \gamma + \sum_{1 \leqslant \nu \leqslant \omega} \frac{1}{z + \alpha_\nu}, \qquad 0 \leqslant \omega \leqslant \infty. \tag{3.44}$$

Each summand on the right-hand side of (3.44) is in $\mathcal{R}^{-1}$, so $F \in \mathcal{R}^{-1}$. The function $f(z)$ is non-constant, hence $F(z) \not\equiv 0$. Therefore, the function $\frac{1}{F(z)}$ is an $\mathcal{R}$-function. Moreover, $F(z)$ (and hence $\frac{1}{F(z)}$) has only negative poles and zeros. Consequently, $\frac{1}{F(z)}$ is an $\mathcal{S}$-function.

Since $f(z)$ has the form (3.43), each common zero of $f(z)$ and $f'(z)$ is negative. In addition the functions $e^{-\gamma z}f(z)$ and $e^{-\gamma z}f'(z)$ are of genus 0. So the function $\frac{f(z)}{f'(z)} = \frac{1}{F(z)}$ satisfies (i) in Theorem 3.8. Consequently, the matrix

$$\mathcal{D}_f = H(f', f) = f_1\, H\left(\frac{f'}{f_1}, \frac{f}{f_1}\right), \quad \text{where} \quad f_1 = f'(0),$$

is totally nonnegative. □

Let $f(z) = \sum_{k=0}^{\infty} f_k z^k$, $f_0 > 0$, be a real entire function. Define its infinite Hurwitz matrix

$$\mathcal{H}_f = \begin{pmatrix} f_0 & f_2 & f_4 & f_6 & f_8 & f_{10} & \dots \\ 0 & f_1 & f_3 & f_5 & f_7 & f_9 & \dots \\ 0 & f_0 & f_2 & f_4 & f_6 & f_8 & \dots \\ 0 & 0 & f_1 & f_3 & f_5 & f_7 & \dots \\ 0 & 0 & f_0 & f_2 & f_4 & f_6 & \dots \\ 0 & 0 & 0 & f_1 & f_3 & f_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.45}$$

For the minors of $\mathcal{H}_f$ we use the same notation as in Section 3.2, such that

$$\mathcal{H}_f^{(k)} = \mathcal{H}_f\begin{pmatrix} 2 & 3 & \dots & k \\ 2 & 3 & \dots & k \end{pmatrix}.$$

Grommer in [Gro14, § 16, Satz IV] extended the Hurwitz criterion [Hu1895] to entire functions. However, he overlooked the condition on common zeros of odd and even parts, which was addressed by Kreĭn in [Kre38]. We only need the following particular case of this extension.

Theorem 3.32 ([Kre38, Theorem 12], [CM49, Ch. V § 4]). *Let a real entire function $f(z)$, $f(0) > 0$, be of genus 1 or 0. Suppose that its even part $(f(z) + f(-z))/2$ and its odd part $(f(z) - f(-z))/2$ have no common zeros.*

Then the function f can be represented as

$$f(z) = Ce^{\gamma z} \prod_{1\leqslant\mu\leqslant\omega_1} \left(1 + \frac{z}{x_\mu}\right) \prod_{1\leqslant\nu\leqslant\omega_2} \left(1 + \frac{z}{\alpha_\nu}\right)\left(1 + \frac{z}{\overline{\alpha}_\nu}\right), \tag{3.46}$$

where $0 \leqslant \omega_1, \omega_2 \leqslant \infty$, $\gamma \geqslant 0$ and $C > 0$, and its zeros satisfy the conditions

$$x_\mu > 0 \text{ for } 1 \leqslant \mu \leqslant \omega_1, \quad \operatorname{Re}\alpha_\nu > 0,\ \operatorname{Im}\alpha_\nu > 0 \text{ for } 1 \leqslant \nu \leqslant \omega_2,$$
$$\sum_{1\leqslant\mu\leqslant\omega_1} \frac{1}{x_\mu} < \infty \quad \text{and} \quad \sum_{1\leqslant\nu\leqslant\omega_2} \operatorname{Re}\frac{1}{\alpha_\nu} < \infty,$$

if and only if

$$\mathcal{H}_f^{(2)}(f), \mathcal{H}_f^{(3)}(f), \dots, \mathcal{H}_f^{(\omega_1+2\omega_2+1)}(f) > 0,$$
$$\mathcal{H}_f^{(\omega_1+2\omega_2+2)}(f) = \mathcal{H}_f^{(\omega_1+2\omega_2+3)}(f) = \dots = 0.$$

Note that the restriction on the genus of $f(z)$ implies the additional condition

$$\sum_{1\leqslant\nu\leqslant\omega_2} \frac{1}{|\alpha_\nu|^2} < \infty.$$

Based on Theorems 3.8 and 3.32 we deduce the following fact.

Theorem 3.1. *Given a power series $f(z) = z^j \sum_{k=0}^{\infty} f_k z^k$, where $f_0 > 0$ and j is a nonnegative integer, the infinite matrix $\mathcal{H}_f$ defined by* (3.45) *is totally nonnegative if and only if the series $f(z)$ converges to a function of the form*

$$f(z) = Cz^j e^{\gamma_1 z + \gamma_2 z^2} \frac{\prod_{1\leqslant\mu\leqslant\omega_1}\left(1 + \frac{z}{x_\mu}\right)\prod_{1\leqslant\nu\leqslant\omega_2}\left(1 + \frac{z}{\alpha_\nu}\right)\left(1 + \frac{z}{\overline{\alpha}_\nu}\right)}{\prod_{1\leqslant\lambda\leqslant\omega_3}\left(1 + \frac{z}{y_\lambda}\right)\left(1 - \frac{z}{y_\lambda}\right)}, \tag{3.47}$$

for some ω_1, ω_2 *and* ω_3, $0 \leqslant \omega_1, \omega_2, \omega_3 \leqslant \infty$. *Here* $C > 0$,

$$\gamma_1, \gamma_2 \geqslant 0,\ x_\nu, y_\lambda > 0,\ \operatorname{Re}\alpha_\nu \geqslant 0 \text{ and } \operatorname{Im}\alpha_\nu > 0 \text{ for all } \mu, \nu, \lambda, \tag{3.48}$$

$$\sum_{1\leqslant\mu\leqslant\omega_1} \frac{1}{x_\mu} + \sum_{1\leqslant\nu\leqslant\omega_2} \operatorname{Re}\left(\frac{1}{\alpha_\nu}\right) + \sum_{1\leqslant\nu\leqslant\omega_2} \frac{1}{|\alpha_\nu|^2} + \sum_{1\leqslant\lambda\leqslant\omega_3} \frac{1}{y_\lambda^2} < \infty. \tag{3.49}$$

Proof. Suppose that $f(z)$ is represented as (3.47). We can express it as

$$f(z) = Cz^j g(z^2) h(z), \quad \text{where}$$

$$h(z) := e^{\gamma_1 z} \prod_{1\leqslant\mu\leqslant\omega_1} \left(1 + \frac{z}{x_\mu}\right) \prod_{\substack{\operatorname{Re}\alpha_\nu>0 \\ 1\leqslant\nu\leqslant\omega_2}} \left(1 + \frac{z}{\alpha_\nu}\right)\left(1 + \frac{z}{\overline{\alpha}_\nu}\right) \quad \text{and} \tag{3.50}$$

$$g(z^2) := e^{\gamma_2 z^2} \prod_{\substack{\operatorname{Re}\alpha_\nu=0 \\ 1\leqslant\nu\leqslant\omega_2}} \left(1 + \frac{z^2}{i^2\alpha_\nu^2}\right) \Big/ \prod_{1\leqslant\lambda\leqslant\omega_3} \left(1 - \frac{z^2}{y_\lambda^2}\right). \tag{3.51}$$

Note that $g(z)$ has the form (3.42), so by Theorem 3.5 its Toeplitz matrix $T(g)$ is totally nonnegative.

Split $h(z)$ into the odd part $zh_o(z^2)$ and the even part $h_e(z^2)$ so that

$$h(z) = h_e(z^2) + zh_o(z^2).$$

The function $h(z)$ is of genus not exceeding 1 as well as $h_e(z^2)$ and $h_o(z^2)$. This implies that the genus of $h_o(z)$ and $h_e(z)$ is 0. Indeed, for example, the function $h_e(z^2)$ with zeros $\pm\delta_n$, $n = 1, 2, \ldots$, can be represented as the Weierstraß product

$$h_e(z^2) = e^{cz} \prod_n \left(1 - \frac{z}{\delta_n}\right) e^{\frac{z}{\delta_n}} \left(1 + \frac{z}{\delta_n}\right) e^{-\frac{z}{\delta_n}} = e^{cz} \prod_n \left(1 - \frac{z^2}{\delta_n^2}\right).$$

And since it depends only on z^2, we necessarily have $c = 0$ in this representation, which implies

$$h_e(z) = \prod_n \left(1 - \frac{z}{\delta_n^2}\right).$$

Let us show $h_o(z)$ and $h_e(z)$ are coprime. Denote $r := \gcd(h_o, h_e)$ if $h_o(z) \not\equiv 0$. If $h_o(z) \equiv 0$ we set $r(z) := h_e(z)$. Assume that $r(z) \not\equiv 1$. So it has zeros, since its genus is zero and $r(0) = h_e(0) = 1$. However, if $r(z_0) = 0$ then $h(\sqrt{z_0}) = h(-\sqrt{z_0}) = 0$. Since one of the points $\pm\sqrt{z_0} \neq 0$ is in the closed right half of the complex plane (independently of the branch of the square root) we get a contradiction to (3.50).

If $h_o(z) \equiv 0$ then $h_e(z) \equiv 1$ and $H(h_o, h_e) = H(0,1)$ is totally nonnegative. This implies the total nonnegativity of the matrix

$$\mathcal{H}_f = C\,H(gh_o, gh_e) = C\,H(h_o, h_e)\,T(g).$$

If $h_o(z) \not\equiv 0$, then the function $h(z)$ has the form (3.50) and its odd and even parts are coprime. That is, $h(z)$ satisfies the conditions of Theorem 3.32. Therefore,

$$\mathcal{H}_h^{(2)} = H^{(2)}(h_o, h_e), \quad \mathcal{H}_h^{(3)} = H^{(3)}(h_o, h_e), \quad \mathcal{H}_h^{(4)} = H^{(4)}(h_o, h_e), \ldots$$

is a positive sequence possibly followed by zeros. Thus, by Theorem 3.22, we obtain $\frac{h_e}{h_o} \in \mathcal{S}$. Then Theorem 3.8 applied to the function $\frac{h_e}{h_o}$ gives us the total nonnegativity of the matrix $H(h_o, h_e)$, and consequently of the matrix $\mathcal{H}_f$.

Let us prove the converse statement. Suppose that the Hurwitz matrix $\mathcal{H}_f$ is totally nonnegative. We can split the series $f_0^{-1} z^{-j} f(z) = \sum_{k=0}^{\infty} \frac{f_k}{f_0} z^k$ into the even part $q(z^2)$ and the odd part $zp(z^2)$ so that $f(z)$ can be expressed as follows

$$f(z) = f_0 z^j \left(q(z^2) + zp(z^2)\right).$$

Therefore, $\mathcal{H}_f = f_0 H(p, q)$ and the matrix $H(p, q)$ is totally nonnegative.

Suppose that $f_1 > 0$. Then, by Theorem 3.8, there exists a meromorphic function $g(z)$ of the form (3.42) such that $\widetilde{p}(z) = \frac{p(z)}{g(z)}$ and $\widetilde{q}(z) = \frac{q(z)}{g(z)}$ are coprime entire functions of genus 0. Moreover, the ratio $\frac{\widetilde{q}(z)}{\widetilde{p}(z)}$ is an $\mathcal{S}$-function. Let

$$\widetilde{f}(z) := \widetilde{q}(z^2) + z\widetilde{p}(z^2).$$

By Theorem 3.22 the minors $\mathcal{H}_{\widetilde{f}}^{(2)} = H^{(2)}(\widetilde{p}, \widetilde{q}), \mathcal{H}_{\widetilde{f}}^{(3)} = H^{(3)}(\widetilde{p}, \widetilde{q}), \ldots$ form a positive sequence possibly followed by zeros. Since $\widetilde{q}(z)$ and $\widetilde{p}(z)$ are coprime, the function $\widetilde{f}(z)$ has the form (3.46) by Theorem 3.32.

If $f_1 = 0$ then according to Lemma 3.31, $p(z) \equiv 0$ and $q(z)$ has the form (3.42). Here we set $g(z) := q(z)$ so that $\widetilde{f}(z) \equiv 1$.

Now consider both cases $f_1 = 0$ and $f_1 > 0$. We showed that $\widetilde{f}(z)$ can be represented as in (3.46), while $g(z)$ can be represented as in (3.42). That is, after the appropriate renaming of zeros and poles, the function $\widetilde{f}(z)$ has the form (3.50), the function $g(z^2)$ has the form (3.51) and the conditions (3.48)–(3.49) are satisfied. Thereby

$$f(z) = f_0 z^j g(z^2) \widetilde{f}(z)$$

can be represented as in (3.47). □

Chapter 4.

One helpful property of $\mathcal{PF}$-functions

The present chapter studies quite a general equation of the form $z^p R(z^k) = \alpha$, however the simple case $k = 2$ considered in Sections 4.8 and 4.9 has the most interesting applications. Corollary 4.35 introduces sufficient conditions on a function of the form $\sum_{n=0}^{\infty} i^{\pm\frac{n(n-1)}{2}} f_n z^n$, where $f_0 \neq 0$ and $f_n \geqslant 0$ for all n, which assure the simplicity of its zeros. It turns out that such functions as

$$\mathcal{F}(z; \pm iq) = \sum_{n=0}^{\infty} \frac{1}{n!} (\pm iq)^{\frac{n(n-1)}{2}} z^n, \quad \text{where} \quad 0 < q \leqslant 1, \quad \text{and}$$

$$\Theta_0(z; \pm iq) = \sum_{n=0}^{\infty} (\pm iq)^{\frac{n(n-1)}{2}} z^n, \qquad \text{where} \quad 0 < q \leqslant q_* \approx 0.7457224107,$$

have zeros which are simple and distinct in absolute value. The former function $\mathcal{F}(z; q)$ gives a solution to the functional-differential problem

$$\mathcal{F}'(z) = \mathcal{F}(qz), \quad \mathcal{F}(0) = 1,$$

while the latter is the partial theta function satisfying

$$\Theta_0(z; q) = 1 + z\Theta_0(zq; q).$$

The partial theta function participates in a number of beautiful Ramanujan-type relations ([AB2009, Chapter 6], [War2003]), it is related to q-series and some types of modular forms. Both $\mathcal{F}$ and Θ_0 appear in problems of statistics and combinatorics (see *e.g.* [So2009, So2012]) and their zeros are the subjects of conjectures by Alan Sokal. The details can be found in Section 4.9.

Nevertheless, general statements offer a better insight into the problem, give an opportunity to determine factors on which the result depends and to find possible generalizations. Their main drawback is an excessive amount of specific cases in Sections 4.4, 4.5 and 4.6. To give a survey of our results, we briefly introduce definitions of α-sets and α-points.

4.1 Properties of solutions to $z^p R(z^k) = \alpha$ for $R \in \mathcal{PF}$: summary of results

Herein, it is convenient to use the notion of α-point. Given a complex number α, the *α-set* of a function $f(z)$ is the set $\{z \in \mathbb{C} : f(z) = \alpha\}$ and points of this set are called *α-points*. A non-constant meromorphic function can clearly have only isolated α-points. We say that an α-point z_* of a function f *has multiplicity* $n \in \mathbb{Z}_{>0}$ whenever $f'(z_*) = \cdots = f^{(n-1)}(z_*) = 0 \neq f^{(n)}(z_*)$. The α-point is *simple* if its multiplicity equals one.

The current chapter aims at describing the behaviour of α-points of functions which can be represented[1] as $z^p R(z^k)$, where p is an integer, k is a positive integer and $R(z)$ is not constant and belongs to the class $\mathcal{PF}$ (see Definition 1.9). We confine ourselves to the case when $\gcd(|p|, k) = 1$: other cases can be treated by introducing the variable $\eta := z^{\gcd(|p|,k)}$. As a main tool, we use a relation of such functions to the so-called $\mathcal{R}$-functions, this class is introduced in Definition 1.1.

We start with describing the α-set of the expression $z^B R(z)$ in the closed upper half of the complex plane $\overline{\mathbb{C}_+} := \{z \in \mathbb{C} : \operatorname{Im} z \geqslant 0\}$, where $R(z)$ is as above, B is real and $\alpha \in \mathbb{C} \setminus \{0\}$. This is done in Theorem 4.11: if the equation $z^B R(z) = \alpha$ has solutions in $\mathbb{C}_+$, then the α-points are simple and distinct in absolute value. The α-points on the real line (excepting the origin) may be either simple or double. For real constants a and $b_1 \neq b_2$ Theorem 4.13 shows that solutions to $z^B R(z) = ae^{ib_1}$ and to $z^B R(z) = ae^{ib_2}$ alternate when ordered in absolute value (under the additional condition that none of them fall onto the real line). The corresponding properties of α-points in the whole complex plane are described in Theorem 4.15 and Remark 4.16. Our approach is based on Lemma 4.1: a function $\psi(z)$ is univalent in the upper half of the complex plane provided that $z\psi'(z)$ is an $\mathcal{R}$-function. In fact, this Lemma is an "appropriate" reformulation of classical results, however we need a construction from its proof. Section 4.3 then considers the properties of $\psi(z)$ on the real line under the additional assumption that $\psi(z)$ is meromorphic in $\overline{\mathbb{C}_+} \setminus \{0\}$. It is interesting to note that Theorem 4.11 can be interpreted as a wide generalization of the main theorem in [Bi1880].

The main result of the present chapter — a description of α-points of $z^p R(z^k)$ — is presented in Theorems 4.21, 4.23, 4.24, 4.25 and 4.26. To derive these theorems we track

[1] Functions of this form are the kth root transforms of $z^p R^k(z)$. In the particular case when $R(z)$ and $R'(z)$ are holomorphic and nonzero at $z = 0$, the function $zR^k(z)$ is univalent in some disk centred at the origin. Then, $zR(z^k)$ will be a univalent *function with k-fold symmetry* in this disk in the sense that the image of $zR(z^k)$ will be k-fold rotationally symmetric (see *e.g.* [Dur83, § 2.1] for the details). The term "functions with k-fold symmetry" could be good under the posed narrower conditions, however we study a more general case assuming no such regularity at the origin and allowing any integer p satisfying $\gcd(|p|, k) = 1$.

the solutions to

$$z^{p/k} R(z) = \alpha \cdot \exp\left(i\frac{2\pi n}{k}\right) \quad \text{and} \quad z^{p/k} R(z) = \overline{\alpha} \cdot \exp\left(i\frac{2\pi n}{k}\right), \quad n \in \mathbb{Z},$$

under the change of variable $z \mapsto z^k$. If we split the complex plane into $2k$ sectors

$$Q_j = \left\{\frac{n}{k}\pi < \operatorname{Arg} z < \frac{n+1}{k}\pi\right\}, \quad j = 0, \ldots, 2k-1,$$

then Theorem 4.21 states that for $\operatorname{Im} \alpha^k \neq 0$ all α-points are inner points of the sectors, simple, and those in distinct sectors strictly interlace with respect to their absolute value. Put in other words, if α-points of $z^p R(z^k)$ are denoted by z_i so that $\cdots \leqslant |z_{-1}| \leqslant |z_0| \leqslant |z_1| \leqslant \cdots$, then $\cdots < |z_{-1}| < |z_0| < |z_1| < \cdots$ and $z_i \in Q_n$ implies that $z_{i+1}, \ldots, z_{i+2k-1} \notin Q_n$ and[2] that $z_{i+2k} \in Q_n$. In fact, there is a formula for m such that $z_{i+1} \in Q_m$, which is trivial for $p = \pm 1$ or $k = 2$. Theorem 4.23 provides analogous properties in the case $\operatorname{Im} \alpha^k = 0$. In particular, it asserts that there are at most two α-points sharing the same absolute value, which are simple unless they occur at a sector boundary where they may collapse into a double α-point.

In turn, Theorem 4.24, Theorem 4.25 and Theorem 4.26 answer the question which sector contains the α-point that is minimal in absolute value for a meromorphic function $R(z)$. This automatically extends to the α-point that is the maximal in absolute value when $R\left(\frac{1}{z}\right)$ is meromorphic.

Theorems 4.21 and 4.23–4.26 describe zeros of entire functions of the form

$$f(z^k) + z^j g(z^k) \quad \text{or} \quad g(z^k) + z^j f(z^k), \quad j, k \in \mathbb{Z}_{>0}, \tag{4.1}$$

where (complex) entire functions $f(z)$ and $g(-z)$ are of genus 0 and have only negative zeros. Since $f(z^k)/f(0)$ and $g(z^k)/g(0)$ become real functions, the correspondence is provided by

$$f(z^k) + z^{-p} g(z^k) = z^{-p}\left(g(z^k) + z^p f(z^k)\right) = 0 \iff z^p \frac{f(z^k)/f(0)}{g(z^k)/g(0)} = -\frac{g(0)}{f(0)} \tag{4.2}$$

on setting $p := \pm j$. We can allow $f(z)$ and $g(-z)$ to be any functions of the class $\mathcal{PF}$ up to constant complex factors. Then the functions of the form (4.1) can be identified by the condition on their Maclaurin or Laurent coefficients. See Section 4.7 for further details.

Sections 4.8 and 4.9 apply the above results in the setting $k = 2$, which are summarized in Theorems 4.30 and 4.31. For a (complex) entire function H of the complex variable z consider its decomposition into odd and even parts such that $H(z) = f(z^2) + zg(z^2)$.

[2] As soon as the α-set of the function $z^p R(z^k)$ actually contains the point z_{i+2k}: Theorem 4.21 asserts nothing on existence of α-points, as well as Theorem 4.23.

Theorem 4.30 from Section 4.8 answers the following question: how are the zeros of the function $H(z)$ distributed if the ratio $\frac{f(z)}{g(z)}$ has only negative zeros and positive poles? The case when the ratio $\frac{f(z)}{g(z)}$ has only negative poles and positive zeros is treated by Theorem 4.31. The question appears to be connected to the Hermite-Biehler theorem (see Theorem H-B on Page 4). If the function $H(z)$ is a real Hurwitz-stable polynomial (or a strongly stable function, see Definition 1.5), then $f(z)$ and $g(z)$ only have simple negative interlacing (see Definition 2.1) zeros. Furthermore, if $H(z)$ is a polynomial and we additionally allow the ratio $\frac{f(z)}{g(z)}$ to have positive zeros and poles, then we will obtain the "generalized Hurwitz" polynomials[3] as introduced in [Ty2010]. In the same paper [Ty2010, Subsection 4.6], its author takes real stable polynomials $H(z) = f(z^2) + zg(z^2)$ and changes the sign of zeros of $f(z)$ or $g(z)$ obtaining "strange" polynomials with interesting behaviour. Item (ii) of our Theorem 4.30 and Item (v) of our Theorem 4.31 explain the nature of their "strangeness".

There are related questions which are not considered in the current work and can become the subject of forthcoming studies. One of them is: given a function $z^p R(z^k)$ with $R \in \mathcal{PF}$ to obtain more precise estimates on arguments and absolute values of its α-points based on some specific properties of $R(z)$. In this way, we can find such estimates for α-points lying close to the origin, which are not covered by the standard theory of value distribution. Another question (possibly related to the first one) is to make further progress toward proving the conjectures stated in Section 4.9.

4.2 Connection between $\mathcal{R}$-functions and univalent functions

Let us use the notation "$\arg$" for the multivalued argument function and "Arg" for the principal branch of argument, $-\pi < \operatorname{Arg} z \leqslant \pi$ for any z. We are starting from the following useful observation.[4]

Lemma 4.1. *Let ϕ be a function holomorphic in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ with values in $\mathbb{C}_+$ and let ψ be a fixed holomorphic branch of $\int \frac{\phi(z)}{z}\, dz$. Then the function ψ is univalent in $\mathbb{C}_+$. Moreover, if for some $z_1, z_2 \in \mathbb{C}_+$ we have*

$$\operatorname{Re}\psi(z_1) = \operatorname{Re}\psi(z_2) =: a \quad \text{and} \quad \operatorname{Im}\psi(z_1) =: b < \widetilde{b} := \operatorname{Im}\psi(z_2), \tag{4.3}$$

then $|z_1| < |z_2|$.

[3]These polynomials are related to the classes $\mathcal{HB}_\varkappa$ discussed in Section 1.2.2: see [Ty2010, p. 3]

[4]Many akin facts are well known. For example, considering functions $\Phi(\zeta) := \phi(e^{-\zeta})$ gives the problem from [Wol34] but in a strip. However, we place this lemma here since we need the relation between $|z_1|$ and $|z_2|$ satisfying (4.3) rather than the univalence itself.

Proof. First let us approximate the upper half-plane $\mathbb{C}_+$ by the set

$$\mathbb{C}_\delta := \{z \in \mathbb{C} : \ \delta < \operatorname{Arg} z < \pi - \delta, \ |z| > \delta\}, \quad \delta > 0.$$

For $z = re^{i\theta}$ we have $\dfrac{\partial z}{\partial r} = \dfrac{z}{|z|}$ and $\dfrac{\partial z}{\partial \theta} = iz$, so

$$r\frac{\partial}{\partial r}\operatorname{Im}\psi(z) = \operatorname{Im}\left(\frac{zr}{r}\psi'(z)\right) = \operatorname{Im}\phi(z) = -\frac{\partial}{\partial \theta}\operatorname{Re}\psi(z),$$

which is the matter of the Cauchy-Riemann equation. The lemma's hypothesis $\operatorname{Im}\phi(z) > 0$ for $z \in \overline{\mathbb{C}_\delta}$ yields that

$$\frac{\partial}{\partial r}\operatorname{Im}\psi(z) > 0 \quad \text{and} \tag{4.4}$$

$$\frac{\partial}{\partial \theta}\operatorname{Re}\psi(z) < 0. \tag{4.5}$$

The latter inequality implies that for each $r > 0$ there can be at most one value of $\theta \in [\delta, \pi - \delta]$ such that $\operatorname{Re}\psi(re^{i\theta}) = a$. Moreover, the set $\Gamma_\delta := \{z \in \mathbb{C}_\delta : \operatorname{Re}\psi(z) = a\}$ only consists of analytic arcs because $\operatorname{Re}\psi$ is a function harmonic in $\overline{\mathbb{C}_\delta}$. In other words, we obtained the following.

(a) For each $r > 0$ there is at most one point $z \in \Gamma_\delta$ satisfying $|z| = r$. That is, every arc of Γ in polar coordinates (r, θ) can be set by a function $\theta(r)$.

Furthermore, for every $R > \delta$ the domain $D := \{z \in \mathbb{C}_\delta : |z| < R\}$ contains at most a finite number of the arcs. Suppose that it contains an infinite number of them, then the ray $\{re^{i\theta} : r > 0\}$ for an appropriate fixed $\theta \in [\delta, \pi - \delta]$ meets $\overline{\Gamma_\delta}$ at an infinite number of points of D (since each arc has two ends on the boundary of D). The function $\operatorname{Re}\psi(re^{i\theta})$ is analytic in $r > 0$ as a function of two variables θ and r with θ fixed. Consequently, $\operatorname{Re}\psi(re^{i\theta})$ must be constant on that ray, because it attains the same value in points of a sequence converging to an internal point of its domain of analyticity. So, we have a contradiction unless Γ_δ is equal to $\{re^{i\theta} : r > 0\}$. However, in the case $\Gamma_\delta = \{re^{i\theta} : r > 0\}$ the curve Γ_δ contains only one arc.

Denote by $\gamma_1, \gamma_2, \ldots$ the connected components of Γ_δ according to their distance to the origin, so that[5] $\operatorname{dist}(0, \gamma_1) \leqslant \operatorname{dist}(0, \gamma_2) \leqslant \cdots$. To count all arcs in this manner is possible because D contains only a finite number of them for any $R > \delta$. It is enough to justify two additional statements, which together with (a) imply the lemma.

(b) On each arc γ_i, $i = 1, 2, \ldots$, the value of $\operatorname{Im}\psi$ increases (strictly) for increasing $|z|$.

[5]Here $\operatorname{dist}(0, \gamma_i) := \inf_{z \in \gamma_i} |z|$ is the distance between the origin and the component γ_i, $i = 1, 2, \ldots$.

(c) If we pass from γ_i to γ_{i+1} (due to (a) it corresponds to the grow of $|z|$), then $\operatorname{Im}\psi$ cannot decrease. (In fact, we will show that these arcs can be connected by a line segment of $\partial\mathbb{C}_\delta$ where $\operatorname{Im}\psi$ increases.)

To wit, the assertions (a)–(c) provide that any distinct points of $\mathbb{C}_\delta$ giving the same $\operatorname{Re}\psi$ give distinct $\operatorname{Im}\psi$ such that the conditions (4.3) imply $|z_1| < |z_2|$. In particular, this yields the univalence of ψ in $\mathbb{C}_\delta$. Furthermore, since δ is an arbitrary positive number, the lemma will hold in the whole open half-plane $\mathbb{C}_+$.

For the arc γ_i, $i = 1, 2, \dots$, consider its natural parameter τ. Orienting the arc according to the growth of r, we obtain $\frac{\partial\tau}{\partial r} > 0$. In addition, let us consider a coordinate ν changing in a direction orthogonal to τ, *i.e.* such that (τ, ν) form an orthogonal coordinate system. Then, with the help of the inequality (4.4) and one of the Cauchy-Riemann equations, we deduce that[6]

$$\begin{aligned} 0 < \frac{\partial \operatorname{Im}\psi(z)}{\partial r} &= \frac{\partial \operatorname{Im}\psi(z)}{\partial\tau}\frac{\partial\tau}{\partial r} + \frac{\partial \operatorname{Im}\psi(z)}{\partial\nu}\frac{\partial\nu}{\partial r} \\ &= \frac{\partial \operatorname{Im}\psi(z)}{\partial\tau}\frac{\partial\tau}{\partial r} \pm \frac{\partial \operatorname{Re}\psi(z)}{\partial\tau}\frac{\partial\nu}{\partial r} = \frac{\partial \operatorname{Im}\psi(z)}{\partial\tau}\frac{\partial\tau}{\partial r} \end{aligned}$$

for $z \in \Gamma_\delta$. Therefore, it is true that $z_1, z_2 \in \gamma_i$ and $|z_1| < |z_2|$ imply $\operatorname{Im}\psi(z_1) < \operatorname{Im}\psi(z_2)$, which is equivalent to (b).

Now, given two consecutive arcs γ_i and γ_{i+1} consider the arguments θ_1 and θ_2 of their adjacent points, *i.e.*

$$\theta_1 := \lim_{\substack{|z|\to r_1\\ z\in\gamma_i}} \operatorname{Arg} z \text{ and } \theta_2 := \lim_{\substack{|z|\to r_2\\ z\in\gamma_{i+1}}} \operatorname{Arg} z, \text{ where } r_1 = \sup_{z\in\gamma_i}|z|,\ r_2 = \inf_{z\in\gamma_{i+1}}|z|.$$

The arguments can be either $\pi - \delta$ or δ, because the arcs are regular and hence can only end at the boundary of $\mathbb{C}_\delta$. Observe that $\theta_1 = \theta_2$. Indeed, let for example $\theta_1 = \pi - \delta$. Then (4.5) yields $\operatorname{Re}\psi(z) > a$ as $|z| = r_1$, $z \in \mathbb{C}_\delta$. However, $\theta_2 = \delta$ in its turn would imply $\operatorname{Re}\psi(z) < a$ when $|z| = r_2$. So, in the "semi-annulus" $\{z \in \mathbb{C}_\delta : r_1 < |z| < r_2\}$ there would be such z that $\operatorname{Re}\psi(z) = a$, *i.e.* $z \in \Gamma_\delta$ which contradicts the fact that γ_i and γ_{i+1} are consecutive arcs of Γ_δ.

Since $\theta_1 = \theta_2$, the ray $\Theta := \left\{re^{i\theta_1}, r > \delta\right\}$ meets both arcs γ_i and γ_{i+1} in the limiting points $r_1e^{i\theta_1}$ and $r_2e^{i\theta_1}$, respectively. As a consequence, we obtain that $\operatorname{Im}\psi(r_1e^{i\theta_1}) < \operatorname{Im}\psi(r_2e^{i\theta_1})$ since $\operatorname{Im}\psi$ grows everywhere on Θ by the condition (4.4). Then (b) implies that $\sup_{z\in\gamma_i}\operatorname{Im}\psi(z) \leqslant \inf_{z\in\gamma_{i+1}}\operatorname{Im}\psi(z)$. Thus, the condition (c) is satisfied as well. □

[6] In fact we have more: $\partial \operatorname{Im}\psi(z)/\partial\nu = 0$ implies that the gradient of $\operatorname{Im}\psi$ on γ_i is tangential to γ_i.

4.3 Properties of α-points on the real line

Lemma 4.2. *Under the conditions of Lemma 4.1, let the function ϕ admit an analytic continuation through the interval $(x_1, x_2) \subset \mathbb{R} \setminus \{0\}$. Then the function ψ defined as in Lemma 4.1 has no α-points with multiplicity more than two in (x_1, x_2).*

Proof. The assertion of this lemma is exactly that $\phi(z) = z\psi'(z)$ has no multiple zeros in (x_1, x_2). However, if ϕ could have a double zero x_0, then $\operatorname{Im}\phi(z)$ in the semi-disk $\{z \in \mathbb{C}_+ : |z - x_0| < \varepsilon \ll 1\}$ must have values of both signs (since $\phi(z)$ is close to $(z - x_0)^2$ for such z). In its turn, this contradicts $\phi(\mathbb{C}_+) \subset \mathbb{C}_+$. □

Further in this section, we restrict the $\mathcal{R}$-functions ϕ_1, ϕ_2 to be meromorphic in $\mathbb{C}$ and real on the real line (where finite), *i.e.* to have the (absolutely convergent) Mittag-Lefler representation[7]

$$B + Az - \frac{A_0}{z} - \sum_{\nu \neq 0} \frac{zA_\nu/a_\nu}{z - a_\nu}, \tag{4.6}$$

where $B, a_\nu \in \mathbb{R}$; $a_\nu \neq 0$; $A, A_0 \geqslant 0$ and $A_\nu > 0$ for all $\nu \neq 0$, such that $\phi_1, \phi_2 \not\equiv$ const. For our purposes, we need functions of more general form. Let us take a non-constant function $\phi(z)$ with the representation $\phi_1(z) - \phi_2(1/z)$, where $\phi_1(z)$ and $\phi_2(z)$ are as given by (4.6). Note that both mappings $z \mapsto \frac{1}{z}$ and $z \mapsto -z$ map the upper half of the complex plane $\mathbb{C}_+$ into the lower half-plane. Therefore, ϕ is necessarily an $\mathcal{R}$-function.

Remark 4.3. If $z\psi'(z)$ has the form (4.6), then $\psi(z)$ can be represented as

$$\psi(z) = \int \frac{z\psi'(z)}{z} dz = C + B\ln z + Az + \frac{A_0}{z} - \sum_\nu \frac{A_\nu}{a_\nu} \ln\left(1 - \frac{z}{a_\nu}\right)$$

for some complex constant C. This implies the equality

$$\operatorname{Re}\psi(z) = \operatorname{Re} C + B\ln|z| + \left(A + \frac{A_0}{|z|^2}\right)\operatorname{Re} z - \sum_\nu \frac{A_\nu}{a_\nu} \ln\left|1 - \frac{z}{a_\nu}\right|. \tag{4.7}$$

Remark 4.4. If $z\psi'(z) = \phi(z) = \phi_1(z) - \phi_2(1/z)$, then we introduce two auxiliary functions ψ_1 and ψ_2 (single-valued in $\overline{\mathbb{C}_+}$ where regular) so that $z\psi_1'(z) =: \phi_1(z)$ and $\psi(z) - \psi_1(z) =: \psi_2(z)$. These settings then imply $z\psi_2'(z) = -\phi_2(\frac{1}{z}) = z^2\left(\frac{1}{z}\right)' \cdot \phi_2\left(\frac{1}{z}\right)$, that is $\psi_2\left(\frac{1}{z}\right) = \int \frac{\phi_2(z)}{z} dz$. Both $\phi_1(z)$ and $\phi_2(z)$ satisfy (4.6), therefore

$$\operatorname{Re}\psi(z) = \operatorname{Re}\psi_1(z) + \operatorname{Re}\psi_2(z),$$

[7] Non-constant real meromorphic functions of this form (and only of this form) map $\mathbb{C}_+$ into $\mathbb{C}_+$, see Theorem 3.16.

where both $\operatorname{Re}\psi_1(z)$ and $\operatorname{Re}\psi_2(1/z)$ have the form (4.7). In particular, the function ψ has a logarithmic singularity in each pole $x_* \neq 0$ of ϕ, and $\operatorname{Re}\psi(z) \to +\infty \cdot x_*$ when $z \to x_*$. The notation $+\infty \cdot x_*$ stands for $+\infty$ if $x_* > 0$, and for $-\infty$ if $x_* < 0$.

Lemma 4.5. *If $x\psi'(x) = \phi(x) = \phi_1(x) - \phi_2(1/x)$, where $x \in \mathbb{R}$ and $\phi_1(x)$, $\phi_2(x)$ have the form* (4.6)*, then the following assertions are true.*

(a) *The function $\operatorname{Im}\psi(x)$ can change its value only at the origin and in poles of ϕ.*

(b) *Between every two consecutive negative poles $x_2 < x_1$ of ϕ, there is exactly one local maximum of $\operatorname{Re}\psi$.*

(c) *Between every two consecutive positive poles $x_1 < x_2$ of ϕ, there is exactly one local minimum of $\operatorname{Re}\psi$.*

(d) *In* (b) *and* (c)*, x_1 can be set to zero provided that ϕ is regular between 0 and x_2, and $\lim_{t\to 0+} |\phi(tx_2)| = \infty$. In this case we have $\operatorname{Re}\psi(tx_2) \to +\infty \cdot x_2$ as $t \to 0+$.*

Proof. Take a real $x \neq 0$ such that both functions $\phi_1(x)$ and $-\phi_2(1/x)$ are regular. Since their values are real on the real line, the condition

$$x\frac{\partial \operatorname{Im}\psi(x)}{\partial x} = r\frac{\partial \operatorname{Im}\psi(x)}{\partial r} = \operatorname{Im}\phi(x) = \operatorname{Im}\phi_1(x) - \operatorname{Im}\phi_2(1/x) = 0$$

is satisfied. So the assertion (a) is true.

The function $x\frac{\partial \operatorname{Re}\psi(x)}{\partial x} = \operatorname{Re}\phi(x) = \phi_1(x) - \phi_2(1/x)$ strictly increases from $-\infty$ to $+\infty$ between the points x_1 and x_2, and hence it changes its sign exactly once in the interval $(\min(x_1, x_2), \max(x_1, x_2))$. That is, $\operatorname{sign} x \cdot \operatorname{Re}\psi(x)$ changes from decreasing to increasing on this interval, which is giving us the assertions (b) and (c) for both zero and nonzero x_1.

Suppose that the function ϕ is regular between 0 and x_2 and $\lim_{t\to 0+} |\phi(tx_2)|$ is infinite. Then ϕ increases in this interval, so $\lim_{t\to 0+} \phi(tx_2) = -\infty \cdot x_2$. Therefore, $-\psi'(tx_2) = -\frac{\phi(tx_2)}{tx_2} > \frac{1}{t}$ for small enough $t > 0$ and

$$\begin{aligned}\operatorname{Re}\psi(tx_2) &= \operatorname{Re}\psi\left(\tfrac{1}{2}x_2\right) + \int_{\frac{1}{2}x_2}^{tx_2} \frac{\phi(x)}{x}dx \\ &= \operatorname{Re}\psi\left(\tfrac{1}{2}x_2\right) + x_2\int_t^{\frac{1}{2}} \left(-\frac{\phi(sx_2)}{sx_2}\right) ds \to +\infty \cdot x_2 \quad \text{as} \quad t \to 0+,\end{aligned}$$

which is (d). □

Lemma 4.6. *In addition to the conditions of Lemma 4.5, suppose that ϕ is a regular function in the interval $\mathfrak{I} = (\min\{0, x_2\}, \max\{0, x_2\}) \subset \mathbb{R}$, x_2 is a pole of ϕ and the limit $\mathfrak{B} := \lim_{t\to 0+} \phi(tx_2)$ is finite.*[8]

[8]This limit exists since the function ϕ increases in $\mathfrak{I}$.

(a) *If* $\mathfrak{B}x_2 > 0$*, then* $\operatorname{Re}\psi(x)$ *is an increasing function in* $\mathfrak{I}$ *such that* $\operatorname{Re}\psi(\mathfrak{I}) = \mathbb{R}$*, and furthermore,* $\operatorname{Re}\psi(z) \neq \operatorname{Re}\psi(x)$ *on condition that* $|z| \leqslant |x|$ *with* $x \in \mathfrak{I}$ *and* $z \in \overline{\mathbb{C}_+} \setminus \{x\}$.

(b) *If* $\mathfrak{B}x_2 < 0$*, then* $\operatorname{Re}\psi(x)$ *has exactly one local extremum in* $\mathfrak{I}$ *and tends to* $+\infty \cdot x_2$ *as* x *approaches* 0 *or* x_2.

(c) *If* $\mathfrak{B} = 0$ *then* $\operatorname{Re}\psi(x)$ *is an increasing function in* $\mathfrak{I}$ *and the inequality* $\operatorname{Re}\psi(z) \neq \operatorname{Re}\psi(x)$ *holds provided that* $|z| \leqslant |x|$ *with* $z \in \overline{\mathbb{C}_+} \setminus \{x\}$, $x \in \mathfrak{I}$. *Moreover,* $\lim_{t\to 0+} \frac{\phi(tx_2)}{tx_2}$ *is positive or* $+\infty$. *If additionally* $\operatorname{Re}\psi(tx_2)$ *is unbounded as* $t \to 0+$*, then* $\operatorname{Re}\psi(\mathfrak{I}) = \mathbb{R}$.

Proof. In the interval $\mathfrak{I}$, the function $x\frac{\partial \operatorname{Re}\psi(x)}{\partial x} = \phi(x)$ strictly increases, and hence changes its sign at most once. Therefore, $\operatorname{Re}\psi(x)$ has at most one local extremum: maximum for $x_2 < 0$ and minimum for $x_2 > 0$. Suppose that $0 < |\mathfrak{B}| < \infty$. Then the equality $x\frac{\partial \operatorname{Re}\psi(x)}{\partial x} = \phi(x)$ yields the following relation

$$\operatorname{Re}\psi(tx_2) = \operatorname{Re}\psi\left(\tfrac{1}{2}x_2\right) + \int_{\frac{1}{2}x_2}^{tx_2} \frac{\phi(x) - \mathfrak{B}}{x} dx + \mathfrak{B}\ln\frac{tx_2}{\frac{1}{2}x_2} \sim \mathfrak{B}\ln t \xrightarrow{t\to 0+} -\infty\cdot\mathfrak{B}.$$

On account of $\operatorname{Re}\psi(x) \to +\infty \cdot x_2$ when $x \to x_2$ (see Remark 4.4) this relation implies the assertion (b) and that $\operatorname{Re}\psi$ increases in $\mathfrak{I}$ from $-\infty$ to $+\infty$ if $\mathfrak{B}x_2 > 0$. Therefore, to obtain (a) it is enough to use the inequality

$$\operatorname{Re}\psi(-|z|) < \operatorname{Re}\psi(z) < \operatorname{Re}\psi(|z|), \quad \text{where} \quad \operatorname{Im} z > 0, \tag{4.8}$$

which is a consequence of (4.5). Indeed, if for example $x_2 < 0$, then we have $\operatorname{Re}\psi(x) \leqslant \operatorname{Re}\psi(-|z|) < \operatorname{Re}\psi(z)$ for each $x \in \mathfrak{I}$ satisfying $|x| \geqslant |z|$.

Since $\phi(x)$ is increasing, the condition $\mathfrak{B} = 0$ implies $\frac{\phi(x)}{x} > 0$ in the interval $\mathfrak{I}$, *i.e.* that $\operatorname{Re}\psi$ is growing independently of the sign of x_2. The inequality $\lim_{t\to 0+} \frac{\phi(tx_2)}{tx_2} \neq 0$ is provided by the fact that $\mathcal{R}$-functions cannot vanish faster then linearly.[9] Furthermore, $\operatorname{Re}\psi$ runs through the whole $\mathbb{R}$ on condition that it is unbounded near the origin, as asserted in (c). If $|z| \leqslant |x|$ with $z \in \overline{\mathbb{C}_+} \setminus \{x\}$ and $x \in \mathfrak{I}$, then the inequality (4.8) provides $\operatorname{Re}\psi(z) \neq \operatorname{Re}\psi(x)$. □

Remark 4.7. In Lemma 4.5 and Lemma 4.6, the value of x_2 can be taken equal to $+\infty$ or $-\infty$ at the cost of some of the conclusions. With such a choice, the condition $\operatorname{Re}\psi(x) \to +\infty \cdot x_2$ as $x \to x_2$ may be violated. This, in turn, implies that the function $\operatorname{Re}\psi(x)$ in (b), (c) and (d) of Lemma 4.5 and (b) of Lemma 4.6 may lose the extremum and become monotonic. In cases (a) and (c) of Lemma 4.6, $\operatorname{Re}\psi(\mathfrak{I})$ becomes only a semi-infinite interval of the real line, instead of the equality $\operatorname{Im}\psi(\mathfrak{I}) = \mathbb{R}$.

[9] This fact follows from the integral representation (5.5). It can be also deduced from the expression (4.6) when $\mathcal{R}$-functions have the form considered in this lemma.

4.4 Location of α-points in the closed upper half-plane

Lemma 4.8. *Let functions $\phi_1(z)$, $\phi_2(z)$ be of the form* (4.6) *and let $\psi(z)$ be a holomorphic branch of $\int \left(\phi_1(z)-\phi_2(\frac{1}{z})\right)\frac{dz}{z}$. If two points $z_1, z_2 \in \overline{\mathbb{C}_+}$ that are regular for ψ satisfy $|z_1| < |z_2|$ and $\operatorname{Re}\psi(z_1) = \operatorname{Re}\psi(z_2) =: a$, then*

(a) $\operatorname{Im}\psi(z_1) \leqslant \operatorname{Im}\psi(z_2)$;

(b) *for each $\varrho \in (\operatorname{Im}\psi(z_1), \operatorname{Im}\psi(z_2))$ there exists $z \in \overline{\mathbb{C}_+}$ such that $|z_1| < |z| < |z_2|$ and $\psi(z) = a + i\varrho$;*

(c) *z_1 and z_2 can be connected by a piecewise analytic curve of a finite length, on which ψ is smooth and $\operatorname{Im}\psi(z)$ is a non-decreasing function of $|z|$; the curve is a subinterval of $\mathbb{R}$ if and only if equality holds in* (a)*;*

(d) *furthermore, equality holds in* (a) *if and only if $z_1, z_2 \in \mathbb{R}$, $z_1 \cdot z_2 > 0$ and $\psi(z) \neq a$ for all $z \in \overline{\mathbb{C}_+}$ such that $|z_1| < |z| < |z_2|$.*

Proof. Recall that the function $\phi(z) = \phi_1(z) - \phi_2(1/z)$ maps $\mathbb{C}_+ \to \mathbb{C}_+$, *i.e.* satisfies Lemma 4.1. Thus if $z_3, z_4 \in \mathbb{C}_+$ and $\operatorname{Re}(z_3) = \operatorname{Re}(z_4)$, then the condition $\operatorname{Im}\psi(z_3) > \operatorname{Im}\psi(z_4)$ induces $|z_3| > |z_4|$, and $\operatorname{Im}\psi(z_3) < \operatorname{Im}\psi(z_4)$ induces $|z_3| < |z_4|$. As a consequence, the assertion (a) holds provided that both z_1, z_2 are not real.

The real part of ψ goes to $\pm\infty$ on approaching a (nonzero) pole of ϕ, as stated in Remark 4.4. Consequently, it is impossible for a pole of ϕ to be a limiting point of the set

$$\Gamma := \{z \in \mathbb{C}_+ : \operatorname{Re}\psi(z) = a,\ |z_1| < |z| < |z_2|\},$$

so the function ψ is regular in a neighbourhood of Γ. (Recall that $z_1 = 0$ is allowed by the lemma's condition only if ψ is regular at the origin.)

Analogously to Γ_δ from the proof of Lemma 4.1, points of Γ form an analytic curve possibly containing multiple disconnected components — analytic arcs. Due to (4.5), for each $r > 0$ there exist at most one value of $\theta \in (0, \pi)$ such that $re^{i\theta} \in \Gamma$. That is, if some z_3, z_4 satisfy $|z_3| < |z_4|$, $z_3 \in \gamma_1$ and $z_4 \in \gamma_2$, where γ_1 and γ_2 are arbitrary distinct arcs of Γ, then necessarily $\sup_{z\in\gamma_1} |z| =: r_1 \leqslant r_2 := \inf_{z\in\gamma_2} |z|$.

Suppose that the arcs γ_1 and γ_2 are consecutive, *i.e.* that $\Gamma \cap \{z : r_1 < |z| < r_2\} = \varnothing$. Within this setting, the limits

$$\zeta_1 := \lim_{\substack{|z|\to r_1\\ z\in\gamma_1}} z \quad \text{and} \quad \zeta_2 := \lim_{\substack{|z|\to r_2\\ z\in\gamma_2}} z$$

exist and real. Moreover, they have the same sign: for example, $\zeta_1 < 0 < \zeta_2$ implies that $\operatorname{Re}\psi(ir_2) < \operatorname{Re}\psi(\zeta_2) = a = \operatorname{Re}\psi(\zeta_1) < \operatorname{Re}\psi(ir_1)$ according to (4.5) and hence $\operatorname{Re}\psi(ir_*) = a$ for some $r_* \in (r_1, r_2)$ by continuity, which contradicts

to $\Gamma \cap \{z \in \mathbb{C}_+ : r_1 < |z| < r_2\} = \varnothing$. The supposition $\zeta_2 < 0 < \zeta_1$ implies a contradiction in a similar way. Consequently, one (and only one) of the inequalities $\operatorname{Re}\psi(z) < a$ and $\operatorname{Re}\psi(z) > a$ holds for all $z \in \mathbb{C}_+$ satisfying $r_1 < |z| < r_2$; the former inequality $\operatorname{Re}\psi(z) < a$ corresponds to the positive sign of ζ_1, ζ_2, while the latter corresponds to the negative sign.

Now, each nonzero singularity x_* of $\psi(z)$ is a pole of $\phi(z)$, and Lemma 4.5 states that $\operatorname{Re}\psi(x_*) \to +\infty \cdot x_*$. That is, for any $z \in \mathbb{C}_+$ close enough to x_* we have $\operatorname{Re}\psi(z) > a$ when $x_* > 0$ and $\operatorname{Re}\psi(z) < a$ when $x_* < 0$. At the same time, we have seen that $\operatorname{Re}\psi(z) < a$ if $0 < \zeta_1 \leqslant |z| \leqslant \zeta_2$ and $z \in \mathbb{C}_+$, so the condition $0 < \zeta_1 < x_* < \zeta_2$ cannot be satisfied. Analogously, the inequality $\operatorname{Re}\psi(z) > a$ holds provided that $\zeta_2 \leqslant -|z| \leqslant \zeta_1 < 0$ and $z \in \mathbb{C}_+$, so the condition $\zeta_2 < x_* < \zeta_1 < 0$ cannot be satisfied. As a consequence, the function ϕ is regular in the interval $\mathfrak{I} := [\min\{\zeta_1, \zeta_2\}, \max\{\zeta_1, \zeta_2\}]$. The function $\operatorname{Re}\psi$ is non-constant and has at most one extremum inside $\mathfrak{I}$ by Lemma 4.5, satisfies $\operatorname{Re}\psi(\zeta_1) = \operatorname{Re}\psi(\zeta_2) = a$, so the equality $\operatorname{Re}\psi(z) = a$ is impossible in $\mathfrak{I} \setminus \{\zeta_1, \zeta_2\}$. As a summary, we obtain that one of the inequalities $\operatorname{Re}\psi(z) < a$ or $\operatorname{Re}\psi(z) > a$ holds for all $z \in \overline{\mathbb{C}_+}$ such that $r_1 < |z| < r_2$.

By Lemma 4.5, $\operatorname{Im}\psi(z)$ is constant in $\mathfrak{I}$ (this fact implies the equality in (a) for $z_1 = \zeta_1 \neq 0$ and $z_2 = \zeta_2$). We obtain that, on $\gamma_1 \cup \mathfrak{I} \cup \gamma_2$, the function $\psi(z)$ is regular and $\operatorname{Im}\psi(z)$ is continuous and non-decreasing as $|z|$ grows. In particular, $\operatorname{Im}\psi(z)$ attains all intermediate values. This reasoning is applicable for each pair of consecutive arcs constituting the set Γ. That is, any two points $z_1, z_2 \in \overline{\mathbb{C}_+} \setminus \{0\}$ with $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ can be connected by a piecewise analytic curve containing intervals of the real line and all arcs of Γ. It remains to check the case when $z_1 = 0$. In this case, $\psi(z)$ is regular at the origin, and thus it is strictly increasing in some real interval enclosing z_1 (due to $\mathfrak{B} = \lim_{z\to 0} z(\psi(z))' = 0$, see the assertion (c) of Lemma 4.6). Then (4.8) shows that z_1 is the end of some arc from Γ. Choosing this arc as γ_1 allows us to apply the previous part of the proof. At that, $\operatorname{Im}\psi(z_1) < \operatorname{Im}\psi(z_2)$.

Note that, on the one hand, poles of $\phi(z) = \phi_1(z) - \phi_2(1/z)$ can concentrate only at the origin since both $\phi_1(z)$ and $\phi_2(z)$ are meromorphic. On the other hand, each interval between poles contains at most two ends of arcs from Γ. Therefore, the number of arcs in $\Gamma \cap \{z : |z_1| < |z| < |z_2|\}$ is finite. Each of the arcs has a finite length since ψ is smooth in a neighbourhood of Γ, so the length of the curve connecting z_1 with z_2 is finite. This implies the assertions (b) and (c) of the lemma. Furthermore, we necessarily have $\operatorname{Im}\psi(z_1) < \operatorname{Im}\psi(z_2)$ unless this piecewise analytic curve is a segment of the real line. In other words, the assertions (a) and (d) are proved. □

Lemma 4.9. *Suppose that $f(z)$ is holomorphic at z_0, $g(z)$ is holomorphic at $f_0 = f(z_0)$ such that $g'(f_0) \neq 0$, and that n is a positive integer number. Then $f'(z_0) = f''(z_0) = \cdots = f^{(n)}(z_0) = 0$ if and only if*

$$\frac{dg(f(z))}{dz}\Big|_{z=z_0} = \frac{d^2g(f(z))}{dz^2}\Big|_{z=z_0} = \cdots = \frac{d^ng(f(z))}{dz^n}\Big|_{z=z_0} = 0. \tag{4.9}$$

Analogously, if a function $f(z)$ is holomorphic at z_0 such that $f'(z_0) \neq 0$ and $g(z)$ is holomorphic at $f_0 = f(z_0)$, then the condition (4.9) *is equivalent to $g'(f_0) = g''(f_0) = \cdots = g^{(n)}(f_0) = 0$.*

Proof. Both facts follows from solving equations provided by the chain rule sequentially:

$$\begin{aligned}
\frac{dg(f(z))}{dz}\Big|_{z=z_0} &= g'(f_0)f'(z_0),\\
\frac{d^2g(f(z))}{dz^2}\Big|_{z=z_0} &= g''(f_0)(f'(z_0))^2 + g'(f_0)f''(z_0),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{d^ng(f(z))}{dz^n}\Big|_{z=z_0} &= g^{(n)}(f_0)(f'(z_0))^n + \cdots + g'(f_0)f^{(n)}(z_0).
\end{aligned}$$

□

The machinery presented in the previous sections is suitable for studying functions of the form

$$V(z) = e^{Az+C+\frac{A_0}{z}} z^B \frac{\prod_{\nu>0}\left(1+\frac{z}{a_\nu}\right)^{\kappa_\nu}}{\prod_{\mu>0}\left(1-\frac{z}{b_\mu}\right)^{\lambda_\mu}} \quad \text{(a branch regular in } \mathbb{C}_+\text{)}, \tag{4.10}$$

where $C \in \mathbb{C}$, $B \in \mathbb{R}$, $A, A_0 \geqslant 0$, and $a_\nu, \kappa_\nu, b_\mu, \lambda_\mu$ are positive reals for all ν, μ. Along with functions as in (4.10), we study functions of the more general form

$$W(z) = e^{Az+C+\frac{A_0}{z}} z^B \frac{\prod_{\nu>0}\left(1+\frac{z}{a_\nu}\right)^{\kappa_\nu}}{\prod_{\mu>0}\left(1-\frac{z}{b_\mu}\right)^{\lambda_\mu}} \frac{\prod_{\nu>0}\left(1+\frac{1}{zc_\nu}\right)^{\widetilde{\kappa}_\nu}}{\prod_{\mu>0}\left(1-\frac{1}{zd_\mu}\right)^{\widetilde{\lambda}_\mu}} = V_1(z)\, V_2\left(\frac{1}{z}\right), \tag{4.11}$$

where both V_1 and V_2 admit the representation (4.10).

Remark 4.10. Let $V(z)$, $V_1(z)$ and $V_2(z)$ be as in (4.10) and $W(z) = V_1(z)\, V_2\left(\frac{1}{z}\right)$. The straightforward fact is that both V and W are regular and nonzero outside the real line. Moreover, the expression $z(\ln V(z))'$ is a meromorphic function of the form (4.6); the function $z(\ln W(z))' = z(\ln V_1)'(z) - \frac{1}{z}\cdot(\ln V_2)'\left(\frac{1}{z}\right)$ is meromorphic for $z \neq 0$ and maps $\mathbb{C}_+ \to \mathbb{C}_+$. (Lemma 5.2 and Corollary 5.3 consider particular cases in details.)

Consequently, they have an analytic continuation in a neighbourhood of each real α-point (excluding the origin) for $\alpha \neq 0$. This allows us to determine the multiplicity of such α-points.

Theorem 4.11. *If a function W defined in $\overline{\mathbb{C}_+}$ has the form* (4.11) *such that $W(z) \not\equiv e^C z^B$, then for any $\alpha \in \mathbb{C} \setminus \{0\}$ the α-points of $W(z)$ lying in $\overline{\mathbb{C}_+}$ (if they exist) are at most double and distinct in absolute value from other solutions to $|W(z)| = |\alpha|$. The α-points inside $\mathbb{C}_+$ must be simple.*

Remark 4.12. If $W(z)$ is regular and nonzero for $z = 0$, then it has the form (4.10) with $A_0 = B = 0$. Therefore, the equality $W(0) = \alpha \neq 0$ implies that $\ln W(z)$ is regular at the origin and

$$(\ln W)'(0) = A + \sum_{\nu>0} \frac{\kappa_\nu}{a_\nu} + \sum_{\mu>0} \frac{\lambda_\mu}{b_\mu} > 0,$$

so the α-point $z = 0$ can only be simple. (Since $\mathfrak{B} = \lim_{z\to 0} z(\ln W(z))' = 0$, applying Lemma 4.6 (c) also yields the simplicity of $z = 0$.)

Proof of Theorem 4.11. Let $\psi(z)$ be a branch of $\ln W(z)$ continuous in $\mathbb{C}_+$, then $z\psi'(z) = \phi_1(z) - \phi_2(1/z)$ with $\phi_1(z)$ and $\phi_2(z)$ of the form (4.6) (*cf.* Remark 4.10). The function $z\psi'(z)$ is a non-constant $\mathcal{R}$-function, so $\operatorname{Im} z\psi'(z) > 0$ for every $z \in \mathbb{C}_+$. In particular, $\frac{W'(z)}{W(z)} = \psi'(z) \neq 0$, that is $W'(z) \neq 0$ and thus all non-real α-points of $W(z)$ are simple.

The inequality (4.5) for $r > 0$ implies $\operatorname{Re}\psi(re^{i\theta_1}) > \operatorname{Re}\psi(re^{i\theta_2})$, that is

$$|W(re^{i\theta_1})| > |W(re^{i\theta_2})| \quad \text{on condition that} \quad 0 \leqslant \theta_1 < \theta_2 \leqslant \pi. \tag{4.12}$$

Consequently, if $W(z) = \alpha$ then $|W(|z|e^{i\theta})| \neq |W(z)| = |\alpha|$ for all $\theta \in [0, \pi] \setminus \{\operatorname{Arg} z\}$.

Each α-point of W is an $(\operatorname{Ln}\alpha + 2i\pi n)$-point of ψ, where $\operatorname{Ln}\alpha$ denotes the principal value of $\ln\alpha$ and n is some integer dependent on the α-point. At that, each α-point of W has the same multiplicity as the coinciding $(\operatorname{Ln}\alpha + 2i\pi n)$-point of ψ by Lemma 4.9. The multiplicities of real $(\operatorname{Ln}\alpha + 2i\pi n)$-points of ψ are at most 2 by Lemma 4.2. So all α-points of W on the real line are at most double. □

Theorem 4.13. *Under the assumptions of Theorem 4.11, if $|z_1| < |z_2|$, $W(z_1) = \alpha$ and $W(z_2) = \alpha e^{i\theta}$ with a real $\theta > 0$, then for every $\varrho \in (0, \theta)$ there exists $z_* \in C_{12} := \{z \in \overline{\mathbb{C}_+} : |z_1| < |z| < |z_2|\}$ such that $W(z_*) = \alpha e^{i\varrho}$, unless simultaneously $\theta = 0 \pmod{2\pi k}$, both z_1 and z_2 are real of the same sign, $W(z)$ is regular in $(\min\{z_1, z_2\}, \max\{z_1, z_2\})$ and $|W(z)| \neq |\alpha|$ in the semi-annulus C_{12}.*

Proof. This is a straightforward corollary of Lemma 4.8 for $\psi(z)$ being a branch of $\ln W(z)$. Like in the proof of Theorem 4.11, we only need to account that the exponential function maps $\alpha + 2i\pi n$ for all integers n to the same point e^{α}. □

If the α-set of W is not empty, then α-points of W are assumed to be enumerated according to the growth of their absolute values, *i.e.* $\cdots \leqslant |z_0| \leqslant |z_1| \leqslant |z_2| \leqslant \cdots$ and $W(z) = \alpha \iff z \in \bigcup_k z_k$. At that, each multiple α-point we count only once.

Theorem 4.14. *For an α-point $z_i \in \mathbb{R}$ of the function W, only the following possibilities exist:*

(a) *The point z_i belongs to an interval between two consecutive positive poles or negative zeros of W. If z_i is double, then the interval contains no other α-points of W. If z_i is simple, then the interval contains exactly one another α-point: either z_{i-1} or z_{i+1}.*

(b) *The point z_i belongs to an interval between the origin and the maximal negative zero, or between the origin and the minimal positive pole. Then exactly one another α-point (if z_i is simple) or no other α-points (if z_i is double) lies on the same interval provided that $A_0 > 0$ or $Bz_i < 0$ in* (4.11). *If $A_0 = 0$ and $Bz_i \geqslant 0$, then z_i is the α-point minimal in absolute value (moreover, it is the minimal solution to $|W(z)| = |\alpha|$) and the same interval contains no other α-points.*

(c) *The point z_i lies on a ray of the real line, in which W has no poles or zeros. Then this ray contains at most one another α-point of W. If $A_0 = 0$, $Bz_i \geqslant 0$ and one end of this ray is the origin, then z_i is the only α-point on the ray and its absolute value is minimal among all solutions to $|W(z)| = |\alpha|$.*

In the cases (a)–(c)*, the number and multiplicities of α-points of W in the corresponding interval are equal to the number and multiplicities of $|\alpha|$-points of $|W|$.*

Proof. Let us denote by $\psi(z)$ some branch of the function $\ln W(z)$ which is continuous in $\mathbb{C}_+$, then $z\psi'(z) = \phi(z) := \phi_1(z) - \phi_2(1/z)$ with $\phi_1(z)$ and $\phi_2(z)$ of the form (4.6):

$$\phi(z) = B + Az - \sum_{\nu>0} \frac{-z\kappa_\nu}{z + a_\nu} - \sum_{\mu>0} \frac{z\lambda_\mu}{z - b_\mu} - \frac{A_0}{z} - \sum_{\nu>0} \frac{\widetilde{\kappa}_\nu}{zc_\nu + 1} - \sum_{\mu>0} \frac{\widetilde{\lambda}_\mu}{zd_\mu - 1} \tag{4.13}$$

(*cf.* Remark 4.10 and (5.33)). Consequently, in each continuous interval $\mathfrak{I}$ of $\{z \in \mathbb{R} : z \neq 0,\ W(z) \neq 0,\ W(z) \neq \infty\}$, the function $\operatorname{Im}\psi(z)$ is constant (Lemma 4.5 (a)). Furthermore, $\operatorname{Re}\psi(z)$ has exactly one extremum between each pair of consecutive positive poles or negative zeros of $W(z)$ by Lemma 4.5 (b)–(c), that is no, or one double, or two simple $(\ln|\alpha|)$-points. Each α-point of $W(z)$ is a $(\operatorname{Ln}\alpha + 2i\pi n)$-point of $\psi(z)$ with some integer n, and their multiplicities are the same by Lemma 4.9. The equality $(\operatorname{Im}\psi(z))' = 0$ for $z \in \mathfrak{I}$ then implies that all $(\operatorname{Ln}\alpha + 2i\pi n)$-points of $\psi(z)$ with the above-mentioned n and $(\ln|\alpha|)$-points of $\operatorname{Re}\psi(z)$ coincide with multiplicities in $\mathfrak{I}$. As a summary, we

obtain (a). Moreover, the number of α-points of W and their multiplicities in $\Im$ are therefore equal to the number and multiplicities of $|\alpha|$-points of $|W|$.

The assertion (b) follows from Lemma 4.5 (d) and from Lemma 4.6. Indeed, if x_2 denotes the maximal negative zero or the minimal positive pole, then $\operatorname{sign} z_i = \operatorname{sign} x_2$ and the limit determining the properties of z_i is

$$\mathfrak{B} = \lim_{t\to 0+} \phi(tx_2) = \begin{cases} -\lim_{t\to 0+} \frac{A_0}{tx_2} = -\infty \cdot x_2, & \text{if } A_0 > 0 \\ B, & \text{otherwise.} \end{cases}$$

Similarly, the assertion (c) is a corollary of Remark 4.7. □

In the sequel we consider only the case of $C = 0$: otherwise, the equality $W(z) = \alpha$ can be replaced with $W(z)e^{-C} = \alpha e^{-C}$.

Theorem 4.15. *Let $W(z)$ be a function of the form* (4.11) *distinct from z^B, such that κ_ν, $\widetilde{\kappa}_\nu$, λ_μ, $\widetilde{\lambda}_\mu$ are positive integers and $C = 0$. Choose the branch of z^B which is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$ and positive for $z > 0$. Given a complex number $\alpha \notin \mathbb{R}$ such that $\alpha e^{\pm iB\pi} \notin \mathbb{R}$, each α-point of $W(z)$ in $\mathbb{C} \setminus \mathbb{R}$ is simple and distinct in absolute value from other α-points. If z_i, z_{i+1} are two consecutive points of the α-set, then* $\operatorname{Im} z_i \cdot \operatorname{Im} z_{i+1} < 0$.

Moreover, the equations $W(x) = \alpha$ and $W_\pm(-x) := \lim_{y\to\pm 0} W(-x+iy) = \alpha$ have no solution for $x > 0$.

Note that in the case of integer B, the conditions $\alpha e^{\pm iB\pi} \notin \mathbb{R}$ and $\alpha \notin \mathbb{R}$ of this theorem are equivalent; furthermore, the function $W(x)$ is defined for $x < 0$ and equal to $W_-(x) = W_+(x)$.

Proof. On the one hand, for $x > 0$ the functions $W(x)$, $e^{-iB\pi}W_+(-x)$ and $e^{iB\pi}W_-(-x)$ are real. On the other hand, both α and $\alpha e^{\pm iB\pi}$ are non-real. Therefore, there is no solution to $W(x) = \alpha$ and to $W_\pm(-x) = \alpha$ when $x > 0$. Since $W(\overline{z}) = \overline{W(z)}$, we can find the solutions to $W(z) = \alpha$ in the rest of the complex plane $\mathbb{C} \setminus \mathbb{R}$ from the equations $W(z) = \alpha$ and $W(z) = \overline{\alpha}$ in the upper half-plane.

Now assume that z varies in $\mathbb{C}_+$. Theorem 4.11 implies that all α-points (as well as all $\overline{\alpha}$-points) of the function $W(z)$ are simple and distinct in absolute value. Furthermore, according to the remark following (4.12) absolute values of α-points and of $\overline{\alpha}$-points cannot coincide (due to $\alpha \neq \overline{\alpha}$). On account of $\overline{\alpha} = \alpha e^{-2i \arg \alpha}$, Theorem 4.13 (with the setting $\theta = 2\pi$) induces that if we have two solutions z_i, z_{i+k} to $W(z) = \alpha$ with some positive integer k, then there is a solution z_* to $W(z) = \overline{\alpha}$ such that $|z_i| < |z_*| < |z_{i+k}|$. Conversely, between each pair of $\overline{\alpha}$-points there is an α-point by the same theorem. That is, the absolute values of α- and $\overline{\alpha}$-points in $\mathbb{C}_+$ interlace each other. This fact provides the theorem, because $W(z) = \overline{\alpha}$ is equivalent to $W(\overline{z}) = \alpha$. □

Remark 4.16. If in Theorem 4.15 we take the number $\alpha \neq 0$ real, then the equations $W(z) = \alpha$ and $W(\overline{z}) = \alpha$ are satisfied simultaneously. As a result, each α-point of $W(z)$ in $\mathbb{C} \setminus \mathbb{R}$ is simple and there is a unique α-point with the matching absolute value (which is the complex conjugate). For an α-point z_i on the real line (such points are positive unless z^B is real for $z < 0$, *i.e.* unless B is integer) there are only the possibilities (a)–(c) of Theorem 4.14. The α-set of W for $\alpha e^{\pm iB\pi} \in \mathbb{R}$ and $B \notin \mathbb{Z}$ can be studied similarly; the main distinction is that W is not continuous on the negative semi-axis, so the corresponding result will be concerned with the limiting values W_+ or W_-.

Remark 4.17. Functions of the form (4.10) are in the class $\mathcal{PF}$ exactly when the exponents κ_ν, λ_μ are positive integers, $B \in \mathbb{Z}_{\geqslant 0}$, $C \in \mathbb{R}$ and $A_0 = 0$. The expression (4.11) determines a $\widetilde{\mathcal{PF}}$-function whenever $\kappa_\nu, \lambda_\mu, \widetilde{\kappa}_\nu, \widetilde{\lambda}_\mu \in \mathbb{Z}_{>0}$ and $B \in \mathbb{Z}$. See the definition of $\mathcal{PF}$ and $\widetilde{\mathcal{PF}}$ in Section 1.2.3. Properties of such functions relevant to the current chapter are given in Sections 5.2 and 5.3 of the last chapter.

Hereinafter we concentrate on the case $B = \frac{p}{k}$ of (4.11) with positive integers κ_ν, $\widetilde{\kappa}_\nu$, λ_μ, $\widetilde{\lambda}_\mu$, integer $k \geqslant 2$ and $p \neq 0$. We assume that $\gcd(|p|, k) = 1$, *i.e.* the fraction $\frac{p}{k}$ is irreducible. The kth root is a k-valued holomorphic function in the punctured plane $\mathbb{C}\setminus\{0\}$. So, let $\sqrt[k]{\cdot}$ denote its branch that is holomorphic in $\overline{\mathbb{C}_+} \setminus \{0\}$ and maps the positive semi-axis into itself. Then

$$R(w) = \left(\sqrt[k]{w}\right)^p e^{Aw + A_0 w^{-1}} \frac{\prod_{\nu>0}\left(1 + \frac{w}{a_\nu}\right)}{\prod_{\mu>0}\left(1 - \frac{w}{b_\mu}\right)} \frac{\prod_{\nu>0}\left(1 + \frac{1}{wc_\nu}\right)}{\prod_{\mu>0}\left(1 - \frac{1}{wd_\mu}\right)}, \tag{4.14}$$

where the coefficients satisfy $A, A_0 \geqslant 0$ and $a_\nu, b_\mu, c_\nu, d_\mu > 0$ for all ν, μ, is a single-valued meromorphic function in $\overline{\mathbb{C}_+} \setminus \{0\}$ regular for $\operatorname{Im} w \neq 0$.

4.5 Composition with kth power function

In the current section we assume that a function $G \not\equiv z^p$ has the representation

$$G(z) := e^{Az^k + A_0 z^{-k}} z^p \frac{\prod_{\nu>0}\left(1 + \frac{z^k}{a_\nu}\right)}{\prod_{\mu>0}\left(1 - \frac{z^k}{b_\mu}\right)} \frac{\prod_{\nu>0}\left(1 + \frac{z^{-k}}{c_\nu}\right)}{\prod_{\mu>0}\left(1 - \frac{z^{-k}}{d_\mu}\right)} \tag{4.15}$$

for some integers $k \geqslant 2$ and p, $\gcd(|p|, k) = 1$, in which the coefficients satisfy $A, A_0 \geqslant 0$ and $a_\nu, b_\mu, c_\nu, d_\mu > 0$ for all ν, μ. As we noted above, the case when $|p|$ and k are not coprime does not need any additional study: it can be treated by introducing the new variable $\eta := z^{1/\gcd(|p|,k)}$. Furthermore, the location of zeros and poles of $G(z)$ is clear from the expression (4.15), so we concentrate on the equation $G(z) = \alpha$ where $\alpha \in \mathbb{C}\setminus\{0\}$.

For the sake of brevity denote $e_m := \exp\left(i\frac{m}{k}\pi\right)$. The condition $\gcd(|p|, k) = 1$ implies that

- $(e_{mp})_{m=-k}^{k-1}$ is a cyclic group of order $2k$ generated by e_p when p is odd (thus $e_{mp} = e_n$ for $n \in \mathbb{Z}$ if and only if $mp \equiv n \pmod{2k}$);
- $(e_{mp})_{m=0}^{k-1}$ and $(e_{mp+1})_{m=0}^{k-1}$ are two disjoint cyclic groups of order k generated by e_p when p is even (the former group contains $e_0 = 1$ and the latter one contains $e_k = -1$).

Denote the sectors of the complex plane with the central angle $\frac{\pi}{k}$ by

$$Q_s := \left\{z \in \mathbb{C} \setminus \{0\} : 0 < \operatorname{Arg} z e_{-s} < \tfrac{\pi}{k}\right\} \quad \text{and}$$
$$\widetilde{Q}_s := \left\{z \in \mathbb{C} \setminus \{0\} : 0 \leqslant \operatorname{Arg} z e_{-s} < \tfrac{\pi}{k}\right\},$$

where $s \in \mathbb{Z}$, so that they are numbered in an anticlockwise direction and $Q_s = Q_{2k+s}$, $\widetilde{Q}_s = \widetilde{Q}_{2k+s}$.

The substitution $z \mapsto \widetilde{z}e_{2m}$ turns $G(z) = \alpha$ with a fixed α into the equivalent equation $G(\widetilde{z}e_{2m}) = G(\widetilde{z})e_{2pm} = \alpha$ where $m \in \mathbb{Z}$, which gives us the following remark. (Note that we suppress the trivial case $G(z)$ identically equal to z^p with no special attention).

Remark 4.18. Let $G(z)$ and $R(w)$ be as in (4.15) and (4.14), respectively, $\alpha \neq 0$ and $w \in \mathbb{C}_+ \cup (0, +\infty)$. Substituting $z = \sqrt[k]{w}e_{2m}$ into (4.15) shows that if

$$R(w) = \alpha e_{-2pm}, \quad \text{where} \quad m = 0, \dots, k-1, \tag{4.16}$$

then $z = \sqrt[k]{w}e_{2m} \in \widetilde{Q}_{2m}$ solves the equation $G(z) = \alpha$. Analogously, if the equality

$$R(w) = \overline{\alpha}e_{2pm}, \quad \text{where} \quad m = 0, \dots, k-1, \tag{4.17}$$

holds for some m, then $z = \overline{\sqrt[k]{w}}e_{2m} \in \widetilde{Q}_{2m-1}$ solves $G(z) = \alpha$. Conversely, for each solution of $G(z) = \alpha$ there exists an integer m (unique under the condition $0 \leqslant m < k$) such that $R(z^k) = \alpha e_{-2pm}$ provided that $z^k \in \mathbb{C}_+ \cup (0, +\infty)$, or such that $R(\overline{z}^k) = \overline{\alpha}e_{2pm}$ provided that $z^k \notin \mathbb{C}_+ \cup [0, +\infty)$. In this sense, the equation $G(z) = \alpha$ can be replaced with the relation

$$R(w) \in \Omega, \quad \text{where} \quad \Omega := \{\alpha e_{-2pm}\}_{m=0}^{k-1} \cup \{\overline{\alpha}e_{2pm}\}_{m=0}^{k-1} \tag{4.18}$$

for $w \in \overline{\mathbb{C}_+}$, and then all α-points of $G(z)$ can be determined from the solutions to (4.18).

Remark 4.19. The relation (4.18) shows that the equation $G(z) = \alpha$ has different properties depending whether $\operatorname{Im}\alpha^k$ is zero or not. The case of $\overline{\alpha} \in \{\alpha e_{-2pm}\}_{m=0}^{k-1}$ coincides[10] with $\operatorname{Im}\alpha e_{ps} = 0$ for some $s = 0, \dots, k-1$, and thus to $\operatorname{Im}\alpha^k = 0$. If it occurs,

[10] On the one hand, the condition that $\overline{\alpha} = \alpha e_{-2p\widetilde{s}}$ for some integer $\widetilde{s} = 0, \dots, k-1$ coincides with $\overline{\alpha e_{-p\widetilde{s}}} = \alpha e_{-p\widetilde{s}}$ and therefore to $\operatorname{Im}\alpha e_{-p\widetilde{s}} = 0$. On the other hand, changing the order gives $\overline{\alpha} \in \{\alpha e_{-2pm}\}_{m=0}^{k-1} = \{\alpha e_{2p(k-m)}\}_{m=0}^{k-1} = \{\alpha e_{2pm}\}_{m=0}^{k-1}$, which is $\overline{\alpha} = \alpha e_{2ps}$ for some integer $s = 0, \dots, k-1$; the last expression can be written as $\overline{\alpha e_{ps}} = \alpha e_{ps}$, or equivalently $\operatorname{Im}\alpha e_{ps} = 0$.

then the equivalent equation $G(\zeta e_{-s})e_{ps} = \alpha e_{ps}$ in $\zeta \in \mathbb{C}$ has real coefficients and, hence, solutions symmetric with respect to the real line. Consequently, each solution to $G(z) = \alpha$ has the reflected point $\overline{z}e_{-2s}$ with the same absolute value as a pair such that $G(\overline{z}e_{-2s}) = G(z) = \alpha$ (unless $\overline{z}e_{-2s} = z$). In the case of $\overline{\alpha} \notin \{\alpha e_{-2pm}\}_{m=0}^{k-1}$, which is equivalent to $\operatorname{Im} \alpha^k \neq 0$, the relation (4.18) has no real solutions, and solutions to (4.16) and (4.17) have distinct absolute values, as is shown in Theorem 4.11. Accordingly, then all solutions of $G(z) = \alpha$ are distinct in absolute value.

We examine these cases in detail in Theorem 4.23 and Theorem 4.21, respectively.

Definition 4.20. Denote by Ξ the set of absolute values of all solutions to $G(z) = \alpha$ with G of the form (4.15), that is

$$\Xi := \{\xi > 0 : \exists \theta \in (-\pi, \pi] \text{ such that } G(\xi e^{i\theta}) = \alpha\}.$$

Let $\cdots < \xi_i < \xi_{i+1} < \cdots$ be the entries of Ξ, such that $\Xi = \{\xi_n\}_{n\in I}$, and let $\ldots, z_i, z_{i+1}, \ldots$ be the corresponding α-points or, more precisely, $|z_i| = \xi_i$ and $G(z_i) = \alpha$ for all $i \in I$ (that is, z_i stands for any of the α-points which correspond to the value of ξ_i). The corresponding index set $I = \{n \in \mathbb{Z} : \omega_1 < n < \omega_2\}$ is a finite or infinite interval of integers, $-\infty \leqslant \omega_1 < \omega_2 \leqslant +\infty$.

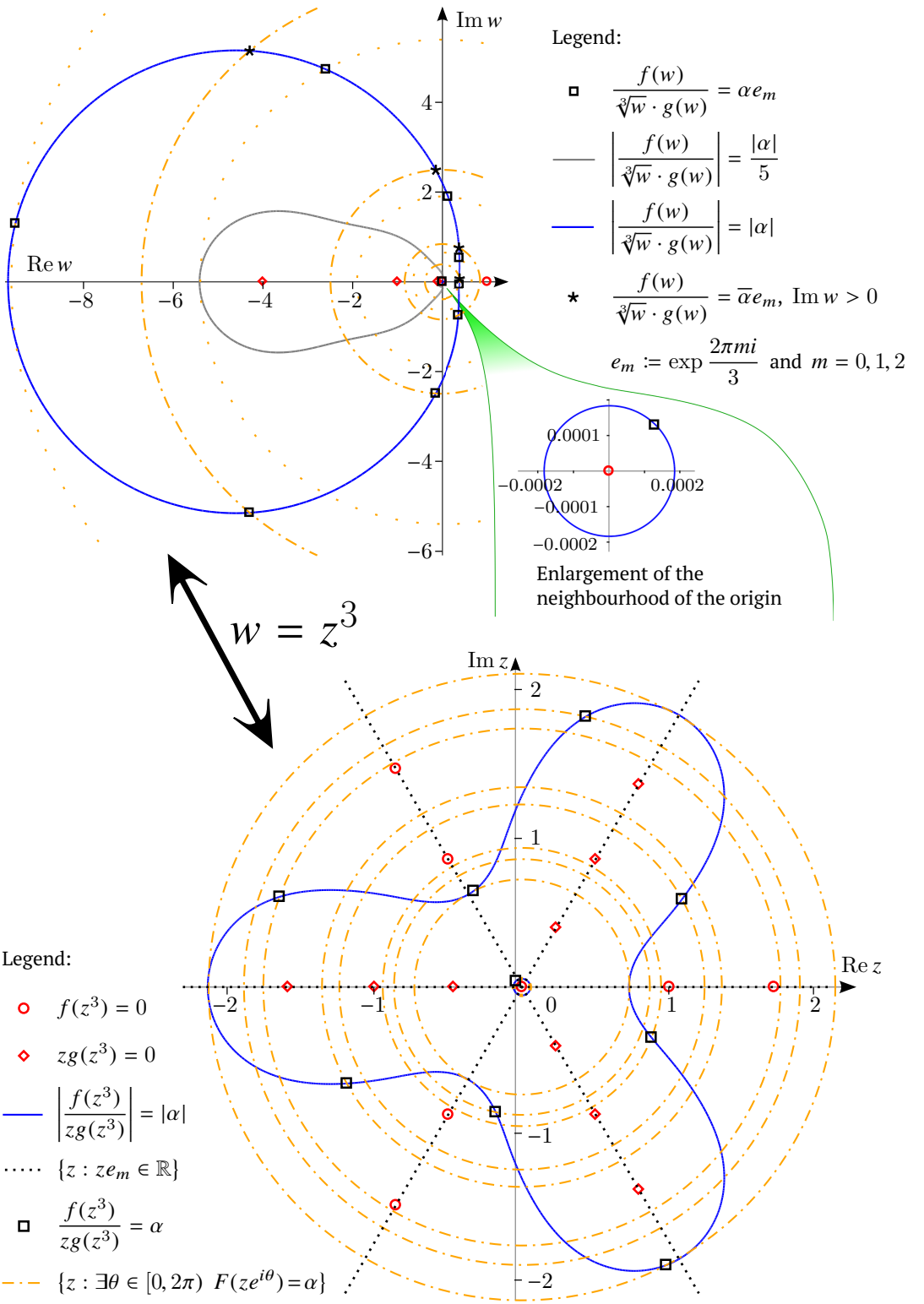
For brevity's sake, we omit the index set I and write $|z_i| \in \Xi$ to specify that the integer $i \in I$ and thus z_i is an actual α-point of G. Accordingly, $|z_i| \notin \Xi$ means that $i \notin I$, which implies that $I \subsetneqq Z$ and the set $\Xi = \{|z_n|\}_{n\in I}$ is not a doubly infinite sequence. If $\omega_1 > -\infty$ then it is convenient to put $\omega_1 = -1$, so that z_0 becomes one of the α-points of G minimal in absolute value.

Theorem 4.21. *If $\operatorname{Im} \alpha^k \neq 0$ and $G(z)$ has the form* (4.15)*, then the α-set of $G(z)$ satisfies the following two properties.*

(a) *Each α-point z_i is simple, satisfies $\operatorname{Im} z_i^k \neq 0$ and is distinct in absolute value from other α-points of G (*i.e. $G(z) = \alpha$ *and* $|z| = |z_i| \Longrightarrow z = z_i$*).*

(b) *For each two consecutive α-points z_i, z_{i+1}, the inclusions $\alpha \in Q_{2q-\varkappa}$ and $z_i \in Q_{2m-\sigma}$ with $q, m \in \mathbb{Z}$ and $\varkappa, \sigma \in \{0, 1\}$ imply that $z_{i+1} \in Q_{2l-1+\sigma}$, where l is an integer solution of $p(l + m) \equiv 2q + 1 - \varkappa - \sigma \pmod{k}$.*

Proof. Note that each element of Ω raised to the kth power equals α^k or $\overline{\alpha}^k$. The expression (4.14) yields that $\operatorname{Im} R^k(w) = 0 \neq \operatorname{Im} \alpha^k$ and, hence, $R(w) \notin \Omega$ provided that $w \in \mathbb{R}$. Consequently, all solutions to (4.18) lie in the open upper half-plane $\mathbb{C}_+$. That is $G(z) \neq \alpha$ for $\operatorname{Im} z^k = 0$. The function $R(w)$ satisfies the conditions of Theorem 4.11, thus solutions to $R(w) \in \Omega$ in $\mathbb{C}_+$ are simple and (since the equality $|R(w)| = |\alpha|$ is necessary for $R(w) \in \Omega$) distinct in absolute value. Therefore, all α-points of G are simple by Lemma 4.9 and distinct in absolute value: if $G(z) = \alpha$ and $|z| = |z_i|$ for some integer i, then $z = z_i$.



The α-set of the function $F(z) = \dfrac{f(z^3)}{zg(z^3)} = \dfrac{(z^3+0.1)(z^3+1)(z^3+4)}{z(z^3-1)(z^3-5)}$ for $\alpha = -1-i$ is presented in the bottom graph. The plot of the corresponding intermediary function $R(w)$ is in the top graph. The α-points of $F(z)$ coincide with zeros of the polynomial $z^9 + (1+i)z^7 + 5.1z^6 - 6(1+i)z^4 + 4.5z^3 + 5(1+i)z + 0.4$.

Figure 4.1 – Illustration to Theorems 4.21, 4.23, 4.24, 4.25 and 4.26.

Now let $|z_i|, |z_{i+1}| \in \Xi$. There exist integers q, m, l and $\varkappa, \sigma, \tau = 0, 1$ such that $\alpha \in Q_{2q-\varkappa}$, $z_i \in Q_{2m-\sigma}$ and $z_{i+1} \in Q_{2l-\tau}$. Without loss of generality we assume $0 \leqslant q, m, l \leqslant k-1$. At that, z_i corresponds to a solution w_i of (4.16) or (4.17) when $\sigma = 0$ or $\sigma = 1$, respectively. Analogously, z_{i+1} corresponds to a solution w_{i+1} of (4.16) or (4.17) depending on whether τ is zero or not. Figure 4.1 illustrates the correspondence between α-points of $G(z)$ and solutions of (4.16)–(4.18).

First, suppose that $\operatorname{Im} \alpha^k > 0$, *i.e.* $\varkappa = 0$ and $\alpha \in Q_{2q}$. Then the points $\alpha e_{-2pm} \in Q_{2q-2pm}$ of the set Ω occur exactly once in each sector Q_j with the even indices $j = 0, 2, \ldots, 2k-2$ when m runs over the integers $0, \ldots, k-1$. Analogously, the points $\overline{\alpha} e_{2pm} \in Q_{-2q-1+2pm}$ of the set Ω occur exactly once in each sector Q_j with the odd indices $j = 1, 3, \ldots, 2k-1$ when $m = 0, \ldots, k-1$. Consequently, $\sigma = 0$ induces the equation $R(w_i) = \alpha e_{-2pm} \in Q_{2q-2pm}$, while $\sigma = 1$ induces $R(w_i) = \overline{\alpha} e_{-2pm} \in Q_{-2q-1+2pm}$. Combining these equalities together gives

$$R(w_i) \in \Omega \cap Q_{(-1)^\sigma((2q+\sigma)-2pm)}. \tag{4.19}$$

The same reasoning for w_{i+1} provides us with the condition

$$R(w_{i+1}) \in \Omega \cap Q_{(-1)^\tau((2q+\tau)-2pl)}. \tag{4.20}$$

Since $R(w_{i+1}) = R(w_i)e^{i\theta}$ with an appropriate real θ, for each $\rho \in (0, \theta)$ there exists w_* satisfying $|w_i| < |w_*| < |w_{i+1}|$ and $R(w_*) = R(w_i)e^{i\varrho}$ by Theorem 4.13. However, z_i and z_{i+1} are consecutive α-points, so $R(w_*)$ cannot belong to Ω for any $\rho \in (0, \theta)$. At the same time, Ω has exactly one point in each sector of the complex plane, and we necessarily have $R(w_{i+1}) \in \Omega \cap Q_{(-1)^\sigma((2q+\sigma)-2pm)+1}$ from (4.19). Thus,

$$(-1)^\tau((2q+\tau)-2pl) \equiv (-1)^\sigma((2q+\sigma)-2pm)+1 \pmod{2k}$$

on account of the relation (4.20). Checking the parity immediately gives $\tau = 1-\sigma$. As a result,

$$\begin{aligned} \sigma = 0 \implies \quad & (2q+1)-2pl \equiv -(2q-2pm+1) = 2q+1-2(1+2q-pm) \quad \text{and} \\ \sigma = 1 \implies \quad & 2q-2pl \equiv -(2q+1-2pm)+1 = 2q-2(2q-pm) \end{aligned}$$

modulo $2k$. These relations imply that $2pl \equiv 2((1-\sigma)+2q-pm) \pmod{2k}$, or equivalently $p(l+m) \equiv 2q+1-\sigma \pmod{k}$.

Now let $\operatorname{Im} \alpha^k < 0$ — that is to say, $\varkappa = 1$ and $\alpha \in Q_{2q-1}$, so consequently $\alpha e_{-2pm} \in Q_{2q-1-2pm}$ and $\overline{\alpha} e_{2pm} \in Q_{-2q+2pm}$. It implies that

$$\begin{aligned} R(w_i) &\in \Omega \cap Q_{(-1)^\sigma(2q-(1-\sigma)-2pm)} \quad \text{and} \\ R(w_{i+1}) &\in \Omega \cap Q_{(-1)^\tau(2q-(1-\tau)-2pl)} = \Omega \cap Q_{(-1)^\sigma(2q-(1-\sigma)-2pm)+1} \end{aligned}$$

analogously to the case of positive $\operatorname{Im}\alpha^k$. Due to the parity, we have $\tau = 1-\sigma$, and thus

$$\begin{aligned} \sigma = 0 \implies & -(2q-2pl) \equiv 2q-1-2pm+1 = 2q-2pm \pmod{2k} \\ \text{and} \quad \sigma = 1 \implies & 2q-1-2pl \equiv -(2q-2pm)+1 = -2q+2pm+1 \pmod{2k}. \end{aligned}$$

The last two equations are equivalent to $2pl \equiv 4q-2\sigma-2pm \pmod{2k}$, which coincides with $p(l+m) \equiv 2q-\sigma \pmod{k}$. □

Remark 4.22. The rays of the line $\{z \in \mathbb{C} : \operatorname{Im} ze_s = 0\}$, which is given by $z = \overline{z}e_{-2s}$, can be expressed via the sectors Q_i of the complex plane by the formula

$$\overline{Q}_{2m} \cap \overline{Q}_{-2s-2m-1} \setminus \{0\} = \begin{cases} \{z \in \mathbb{C} : ze_s > 0\}, & \text{if } m \equiv -\left\lceil \frac{s}{2} \right\rceil \pmod{k}, \\ \{z \in \mathbb{C} : ze_s < 0\}, & \text{if } m \equiv -\left\lceil \frac{s-k}{2} \right\rceil \pmod{k}, \\ \varnothing & \text{otherwise;} \end{cases} \tag{4.21}$$

the notation $\lceil a \rceil$ stands for the minimal integer which is greater or equal to a real number a.

Theorem 4.23. *Let $\operatorname{Im}\alpha^k = 0$, $\alpha \neq 0$ and the integers s, l, m be such that $\operatorname{Im}\alpha e_{ps} = 0$ and $p(m-l) \equiv 1 \pmod{k}$, then:*

(a) *A point z satisfies the conditions $G(z) = \alpha$ and $|z| = |z_i|$ if and only if $z \in \{z_i, z_i^*\}$, where $z_i^* := \overline{z}_i e_{-2s}$.*

(b) *The inclusion $z_i \in Q_{2m} \cup Q_{-2s-2m-1}$ for some integer m implies that both $z_i^* \neq z_i$ are simple α-points and $z_{i+1} \in \overline{Q}_{2l} \cup \overline{Q}_{-2s-2l-1}$ (when $|z_{i+1}| \in \Xi$).*

(c) *The conditions $z_i^* = z_i$ and $\arg z_i = \arg z_{i+1}$ imply that both z_i, z_{i+1} are simple, $\arg z_i \neq \arg z_{i-1}$ provided that $|z_{i-1}| \in \Xi$ and $\arg z_{i+1} \neq \arg z_{i+2}$ provided that $|z_{i+2}| \in \Xi$.*

(d) *If $z_i^* = z_i$ and $\arg z_i \neq \arg z_{i+1}$, then z_i is simple or double (which corresponds to, respectively, $\arg z_i = \arg z_{i-1}$ or $\arg z_i \neq \arg z_{i-1}$ on condition that $|z_{i-1}| \in \Xi$). Furthermore, $z_i \in \overline{Q}_{2m} \cap \overline{Q}_{-2s-2m-1}$ with m given by (4.21) implies $z_{i+1} \in \overline{Q}_{2l} \cup \overline{Q}_{-2s-2l-1}$.*

(e) *If $z_i^* = z_i$ and $|z_{i+1}| \notin \Xi$, then the multiplicity of z_i is at most 2.*

In other words, if $\operatorname{Im} z_i e_s \neq 0$, then z_i is simple, $\operatorname{Im} z_i^k \neq 0$ and the reflected point $z_i^* = \overline{z}_i e_{-2s}$ also solves $G(z) = \alpha$; no other α-points share the same absolute value. Furthermore, $z_i \in Q_{2m}$ and $z_i^* \in Q_{-2s-2m-1}$ for some integer m (probably after exchanging $z_i \leftrightarrow z_i^*$).

If $\operatorname{Im} z_i e_s = 0$ *i.e.* $z_i \in \overline{Q}_{2m} \cap \overline{Q}_{-2s-2m-1}$ for some m satisfying (4.21), then Theorem 4.23 asserts that z_i is simple or double, and there are no other solutions of $G(z) = \alpha$

sharing the same absolute value. If z_i is not the first or the last α-point (with respect to the absolute value), then either z_i is double or exactly one another α-point adjacent to z_i has the same argument (in fact, it belongs to the same interval between two consequent singularities of $\ln G$).

Proof. The equality $G(z_i) = \alpha$ is equivalent to $G(\overline{z}_i e_{-2s}) = \alpha$ since

$$G(\overline{z}_i e_{-2s}) = \overline{G(z_i e_{2s})} = \overline{\alpha e_{2ps}} = \overline{\alpha e_{ps}} e_{-ps} = \alpha e_{ps} e_{-ps} = \alpha.$$

Consequently, $G(z_i) = \alpha$ if and only if $G(z_i^*) = \alpha$, where $z_i^* = \overline{z}_i e_{-2s}$. The points z_i and z_i^* coincide exactly when $z_i e_s$ is a real number (*cf.* Remark 4.19).

Choose the integer m satisfying $z_i \in \widetilde{Q}_{2m} \cup \widetilde{Q}_{-2m-2s-1}$, which implies the same inclusion for z_i^*. We constrain ourselves to the case $z_i \in \overline{Q}_{2m}$ and thus $z_i^* \in \overline{Q}_{-2m-2s-1}$: this causes no loss of generality since z_i and z_i^* are interchangeable with each other. The closed sector $\overline{Q}_{2m}$ replaces $\widetilde{Q}_{2m}$ due to the possibility $z_i = z_i^* \in \overline{Q}_{2m} \cap \overline{Q}_{-2m-2s-1}$ (*cf.* Remark 4.22). Note that the point $w_i := z_i^k = (z_i e_{-2m})^k \in \overline{\mathbb{C}_+}$ satisfies

$$R(w_i) = R\left((z_i e_{-2m})^k\right) = G(z_i e_{-2m}) = \alpha e_{-2pm}, \tag{4.22}$$

where the second equality is valid since $z_i e_{-2m} \in \overline{Q}_0$ and thus $\sqrt[k]{(z_i e_{-2m})^k} = z_i e_{-2m}$ (*cf.* Remark 4.18). Conversely, if $R(w_i) = \alpha e_{-2pm}$ then both $z_i = \sqrt[k]{w_i} e_{2m}$ and $z_i^* = \overline{\sqrt[k]{w_i}} e_{-2m-2s}$ are α-points of G.

The function $R(w)$ has the form (4.14), and hence satisfies the conditions of Theorem 4.11. Therefore, solutions of $R(w) \in \Omega$ in the closed upper half-plane $\overline{\mathbb{C}_+}$ are distinct in absolute value; those in $\mathbb{C}_+$ are additionally simple, and those on the real line are simple or double. In particular, if $R(w) \in \Omega$ and $|w| = |w_i|$ then $w = w_i$, which implies the assertion (a). Moreover, by Lemma 4.9 the multiplicities of z_i, z_i^* are equal to one in the assertion (b), and are at most two in the assertions (c)–(e). The assertion (e) is therefore proved, because it only states that the multiplicity does not exceed two.

Now let $|z_{i+1}| \in \Xi$, which means that there is at least one α-point, z_{i+1}, with absolute value greater than $|z_i|$. Then, by analogy with z_i, the points z_{i+1} and $\overline{z}_{i+1} e_{-2s}$ are the only solutions of the equation $G(z) = \alpha$ which satisfy $|z| = |z_{i+1}|$. Furthermore, we can assume that $z_{i+1} \in \overline{Q}_{2l}$ for some integer l without loss of generality. Then $w_{i+1} := z_{i+1}^k \in \overline{\mathbb{C}_+}$ implies $z_{i+1} = \sqrt[k]{w_{i+1}} e_{2l}$ and, similarly to (4.22), the equality $R(w_{i+1}) = \alpha e_{-2pl}$.

Observe that the points $w_i, w_{i+1} \in \overline{\mathbb{C}_+}$ satisfy the conditions $|w_i| < |w_{i+1}|$, $R(w_i) = \alpha e_{-2pm}$ and $R(w_{i+1}) = \alpha e_{-2pl} = \alpha e_{-2pm+2\delta}$ for an appropriate integer δ. Moreover, the quantity $\alpha e_{2pm} e^{i\varrho}$ cannot belong to Ω for all $\varrho \in (0, \frac{2\delta\pi}{k})$: otherwise there exists w_* satisfying $|w_i| < |w_*| < |w_{i+1}|$ and $R(w_*) \in \Omega$ by Theorem 4.13, which contradicts to the

fact that z_i and z_{i+1} are two consecutive α-points. As stated in Theorem 4.13, this is only possible in two cases: if $\delta = 1$ or if simultaneously: $\delta = 0,\ \operatorname{Arg} w_i = \operatorname{Arg} w_{i+1} \in \{0, \pi\}$ and $|R(w)| \neq |\alpha|$ provided that $|w_i| < |w| < |w_{i+1}|$. In the former case, we necessarily obtain the equation $-2pl \equiv -2pm + 2\delta \pmod{2k}$ with respect to the unknown l, that is $p(m-l) \equiv \delta = 1 \pmod{k}$. This proves the assertion (b), because in the correspondent case $z_i \in Q_{2m} \cup Q_{-2s-2m-1}$ we have that $\operatorname{Arg} w_i \notin \{0, \pi\}$, and the simplicity of z_i, z_i^* is shown above.

To obtain the remaining assertions (c)–(d), we assume that $z_i^* = z_i$ and thus $w_i \in \mathbb{R}$. Let $\mathfrak{I} \subset \{w \in \mathbb{R} : w \neq 0,\ R(w) \neq 0,\ R(w) \neq \infty\}$ be the maximal continuous subinterval containing w_i. Theorem 4.14 applied to $R(w)$ yields that

- the condition that w_i is double implies $\mathfrak{I} \not\ni w_{i-1}, w_{i+1}$;
- if w_i is simple and $\mathfrak{I} \ni w_{i+1}$, then $\mathfrak{I} \not\ni w_{i-1}$;
- if w_i is simple and $\mathfrak{I} \not\ni w_{i+1}$, then $\mathfrak{I} \ni w_{i-1}$ unless $\sqrt[k]{|w_{i-1}|} \notin \Xi$.

Let us show that $w_i \in \mathfrak{I} \not\ni w_{i+1}$ and $w_i \cdot w_{i+1} > 0$ together imply $\delta = 1$, and therefore $\arg z_i \neq \arg z_{i+1}$. Indeed, since $\mathfrak{I} \not\ni w_{i+1}$ the function $R(w)$ has a singularity in the interval between w_i and w_{i+1}, so Theorem 4.13 gives $\delta = 1$. Accordingly, $z_i \in \overline{Q}_{2m},\ z_{i+1} \in \overline{Q}_{2l}$ with $l \not\equiv m \pmod{k}$, and hence $\arg z_i \neq \arg z_{i+1}$. In other words, we obtained that if $\arg z_i = \arg z_{i+1}$, then necessarily $w_i, w_{i+1} \in \mathfrak{I}$, and furthermore z_i and z_{i+1} are simple α-points by Lemma 4.9. The equality $\arg z_{i+1} = \arg z_{i+2}$ (or $\arg z_i = \arg z_{i-1}$) analogously yields that both z_{i+1}, z_{i+2} (or z_i, z_{i-1}) are simple and both w_{i+1}, w_{i+2} (or w_i, w_{i-1}, respectively) belong to the same subinterval of $\{w \in \mathbb{R} : w \neq 0,\ R(w) \neq 0,\ R(w) \neq \infty\}$. Consequently, the assertion (c) is true since at most two of the points $w_{i-1}, w_i, w_{i+1}, w_{i+2}$ can lie in $\mathfrak{I}$. Recall that if w_i is double, then $\mathfrak{I} \not\ni w_{i-1}, w_{i+1}$; this fact implies the assertion (d) on account of Lemma 4.9 with the above proof of (c). □

4.6 Location of the α-point that is minimal or maximal in absolute value

Let a function F have the form

$$F(z) := z^p e^{Az^k} \frac{\prod_{1 \leqslant \nu \leqslant \omega_1} \left(1 + \frac{z^k}{a_\nu}\right)}{\prod_{1 \leqslant \mu \leqslant \omega_2} \left(1 - \frac{z^k}{b_\mu}\right)}, \qquad F(z) \not\equiv z^p, \tag{4.23}$$

where k and p are integer such that $k \geqslant 2$ and $\gcd(|p|, k) = 1,\ 0 \leqslant \omega_1, \omega_2 \leqslant +\infty$, $A \geqslant 0$ and $a_\nu, b_\mu > 0$ for all ν, μ. Such functions are the particular case of (4.15) and, therefore, satisfy conditions of Theorem 4.21 and Theorem 4.23. The next two theorems reveal another property of the α-set of F. Assuming that the α-set is nonempty, they

answer which of the sectors contains the α-point (or α-points) of the function F that is minimal in absolute value.

Theorem 4.24. *Consider a complex number $\alpha \neq 0$ and a function F of the form* (4.23) *with $p > 0$. Let $q = 0, \dots, k-1$ and $\varkappa = 0, 1$ be chosen so that $\alpha \in \widetilde{Q}_{2q-\varkappa}$, and the integer m be such that $pm \equiv q \pmod{k}$.*

If $\alpha^k \not< 0$, then the α-point z_0 of $F(z)$ closest to the origin is simple and distinct in argument and absolute value from the succeeding α-point (or points). Moreover, $\alpha \in Q_{2q-\varkappa}$ implies $z_0 \in Q_{2m-\varkappa}$. If $\alpha e_{-2q} > 0$, then $z_0 e_{-2m} > 0$.

If $\alpha^k < 0$, that is $\alpha e_{-2q+1} > 0$, then the two zeros of $F(z) - \alpha$ closest to the origin (counting double zeros as two) are equal in absolute value or in argument. In the latter case, both zeros belong to the ray $\{z e_{-2m+1} > 0\}$. In the former case, one of them belongs to Q_{2m-1} and another belongs to $Q_{2\widetilde{m}} = Q_{-2m-2s}$ where $\widetilde{m}$ satisfies $p\widetilde{m} \equiv q - 1 \pmod{k}$ and s is introduced in Remark 4.19.

Proof. Let z_0 denote the solution of the equation $F(z) = \alpha$ that is minimal in absolute value. Consider the corresponding point $w_0 \in \overline{\mathbb{C}_+}$ determined by $w_0 = z_0^k$ if $\operatorname{Im} z_0^k \geqslant 0$ and by $w_0 = \overline{z}_0^k$ if $\operatorname{Im} z_0^k \leqslant 0$. Recall that (see Remark 4.18) the equality $F(z_0) = \alpha$ is equivalent to $R(w_0) \in \Omega$, where

$$\Omega = \{\alpha e_{-2pm}\}_{m=0}^{k-1} \cup \{\overline{\alpha} e_{2pm}\}_{m=0}^{k-1} = \{\alpha e_{2m}\}_{m=0}^{k-1} \cup \{\overline{\alpha} e_{2m}\}_{m=0}^{k-1}$$

and the function $R(w) = F\left(\sqrt[k]{w}\right)$ is defined in $\overline{\mathbb{C}_+} \setminus \{0\}$ by the equality (4.14), or more specifically

$$R(w) = \left(\sqrt[k]{w}\right)^p e^{Aw} \frac{\prod_{1 \leqslant \nu \leqslant \omega_1} \left(1 + \frac{w}{a_\nu}\right)}{\prod_{1 \leqslant \mu \leqslant \omega_2} \left(1 - \frac{w}{b_\mu}\right)}. \tag{4.24}$$

Denote by w_* the point of the set $\{w \in \overline{\mathbb{C}_+} : |R(w)| = |\alpha|\}$ which is the closest to the origin. The assertions (b) and (c) of Theorem 4.14 imply the inequality $0 < w_* < b_1$ since $B = \frac{p}{k} > 0$. (Moreover, the point w_* necessarily exists when $F(z)$ has poles.) The function $R(w)$ has the form (4.24), that is $R(w_*) > 0$ and hence $R(w_*) = |\alpha|$. Putting $z_* := \sqrt[k]{w_*} e_{2m}$ we obtain $F(z_*) = |\alpha| e_{2pm}$. As suggested by the theorem's statement, the integer m satisfies $pm \equiv q \pmod{k}$. Consequently, if $\alpha e_{-2q} = |\alpha| > 0$ then the point $z_0 := z_*$ satisfying the inequality $z_0 e_{-2m} > 0$ is the zero of $F(z) - \alpha$ that we are looking for; it is simple by Lemma 4.9. (The example is given in Figure 4.2a, $\alpha = e^{i2\pi/3}$.)

Recall that the α-point z_0 is minimal in absolute value, therefore $R(w) \notin \Omega$ on condition that $|w| < |w_0|$. Put

$$\theta := \begin{cases} \operatorname{Arg} R(w_0) & \text{if } \operatorname{Arg} R(w_0) \geqslant 0, \\ \operatorname{Arg} R(w_0) + 2\pi & \text{otherwise,} \end{cases}$$

so that $\operatorname{Arg} R(w_*) = \operatorname{Arg} |\alpha| = 0 \leqslant \theta < 2\pi$; then for each $\varrho \in (0, \theta)$ there exists $\widetilde{w} \in \overline{\mathbb{C}_+}$ by Theorem 4.13 such that $|w_*| < |\widetilde{w}| < |w_0|$ and $R(\widetilde{w}) = |\alpha| e^{i\varrho}$. Consequently, for each $\varrho \in (0, \theta)$ the condition $|\alpha| e^{i\varrho} \notin \Omega$ holds true when $\theta > 0$.

Suppose now that $\alpha \in Q_{2q}$, which is equivalent to $0 < \operatorname{Arg}(\alpha e_{-2q}) < \frac{\pi}{k}$. Since the set Ω contains no other points of Q_{2q}, the inequality in the expression $R(w_0) = |\alpha| e^{i\theta} \neq \alpha e_{-2q}$ implies $\frac{\pi}{k} < \theta < 2\pi$, leading us to the contradiction $|\alpha| e^{i\varrho} \in \Omega$ with $\varrho = \operatorname{Arg}(\alpha e_{-2q}) \in (0, \theta)$. Therefore, the equality $R(w_0) = \alpha e_{-2q}$ must be true. In other words, we have $R(w_0) = \alpha e_{-2pm}$, and hence $z_0 = \sqrt[k]{w_0} e_{2m} \in Q_{2m}$ is the required α-point.

Analogously, suppose that $\alpha \in Q_{2q-1}$, that is $\overline{\alpha} \in Q_{-2q}$ and $0 < \operatorname{Arg}(\overline{\alpha} e_{2q}) < \frac{\pi}{k}$. Then the equality $R(w_0) = |\alpha| e^{i\theta} = \overline{\alpha} e_{2q}$ is satisfied, because the opposite condition $|\alpha| e^{i\theta} \neq \overline{\alpha} e_{2q}$ implies $\operatorname{Arg}(\overline{\alpha} e_{2q}) < \frac{\pi}{k} < \theta < 2\pi$, which is impossible by Theorem 4.13. Consequently, we obtain $R(w_0) = \overline{\alpha} e_{2q} = \overline{\alpha} e_{2pm}$ and, as stated in Remark 4.18, $z_0 = \overline{\sqrt[k]{w_0}} e_{2m} \in Q_{2m-1}$ (for the illustration see Figure 4.2a with $\alpha = e^{i\pi/2}$). Combining two last cases gives the implication $\alpha \in Q_{2q-\varkappa} \implies z_0 \in Q_{2m-\varkappa}$, while the simplicity of z_0 follows from Theorem 4.21.

The last case is $\alpha e_{-2q+1} > 0$, or equivalently[11] $\operatorname{Arg}(\overline{\alpha} e_{2q}) = \frac{\pi}{k} = \operatorname{Arg}(\alpha e_{2q+2ps})$. Just like in the previous case, we have $R(w_0) = \overline{\alpha} e_{2q} = \overline{\alpha} e_{2pm}$, and therefore the equality $z_0 = \overline{\sqrt[k]{w_0}} e_{2m} \in \widetilde{Q}_{2m-1}$ determines the α-point with the smallest absolute value. Unless $z_0 e_{-2m+1} > 0$, Theorem 4.23 yields that there exists exactly one another α-point of F with the same absolute value, namely $z_0^* := \overline{z}_0 e_{-2s} \in \widetilde{Q}_{-2m-2s}$, and that both z_0, z_0^* are simple. This situation appears in Figure 4.3, $\alpha = \frac{i}{5}$, and Figure 4.2a, $\alpha = e^{i\pi/3}$. The case of $z_0 e_{-2m+1} > 0$, that is $w_0 < 0$, needs a special attention. Let $-a_1$ be the maximal negative zero of $R(w)$. The interval $(-a_1, 0)$ contains one double (namely w_0) or two simple (w_0 and w_1) solutions to $R(w) \in \Omega$ as provided by (b) of Theorem 4.14. In the latter case, $R(w) \notin \Omega$ for all w satisfying $|w_0| < |w| < |w_1|$, which is given by Theorem 4.13. Lemma 4.9 then implies that these solutions determine the corresponding properties of the double α-point z_0 or, respectively, of the simple pair z_0, z_1 with $z_1 e_{-2m+1} > 0$ (as it is shown in Figure 4.3 for $\alpha = i$). When $R(w)$ has no zeros, the result is the same provided that F has at least two (or one double) α-points: see Theorem 4.14 (c) and Remark 4.7. □

[11] The right-hand side follows from $\alpha e_{2q+2ps} = \alpha e_{ps} \cdot e_{2q+ps} = \overline{\alpha e_{ps}} \cdot \overline{e_{-2q-ps}} = \overline{\alpha e_{-2q}} = \overline{\alpha} e_{2q}$.

Figure 4.2a

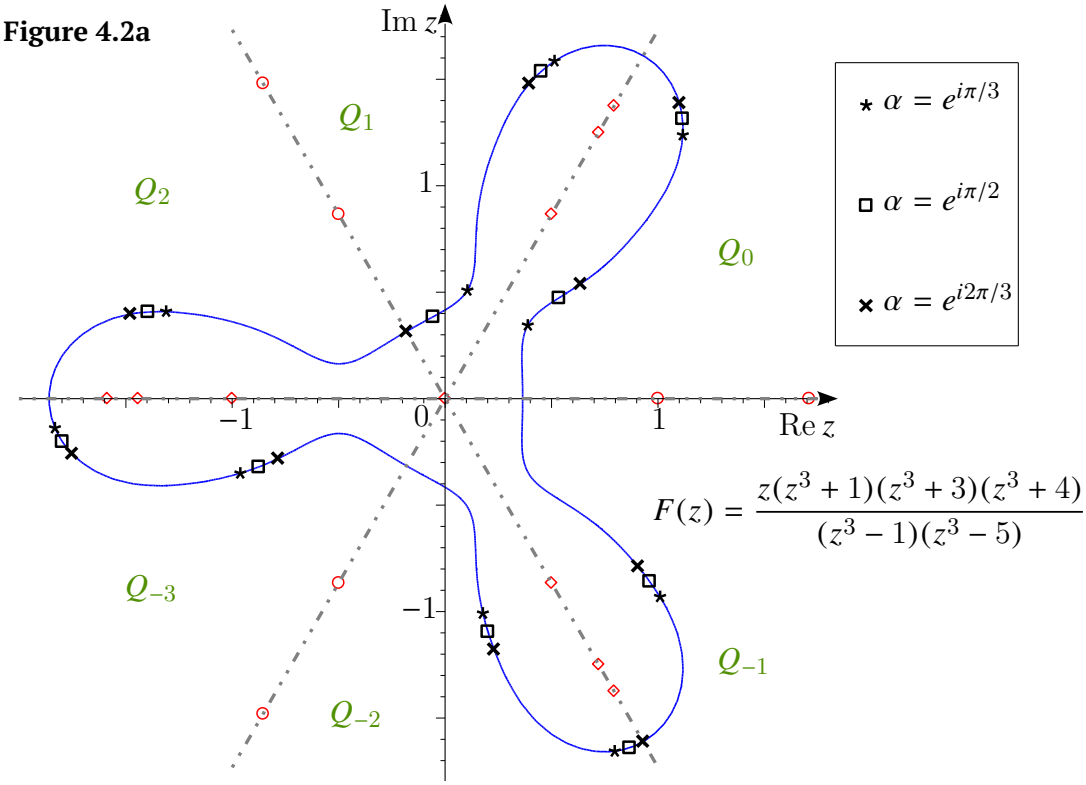


Figure 4.2b

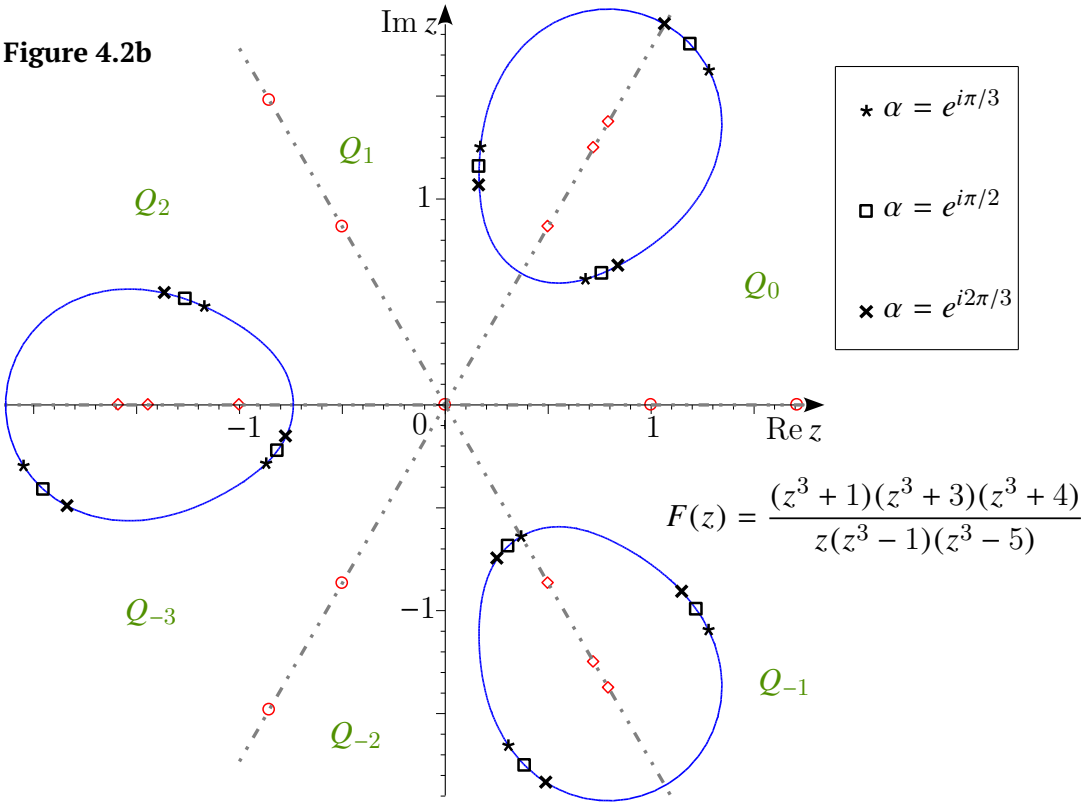


Figure 4.2 – The solutions to $F(z) = \alpha$ with $k = 3$, $p = \pm 1$ for different values of α. (The isoline $|F(z)| = 1$ and zeros of the numerator and denominator of F have the same marks as in Figure 4.3, see the next page.)

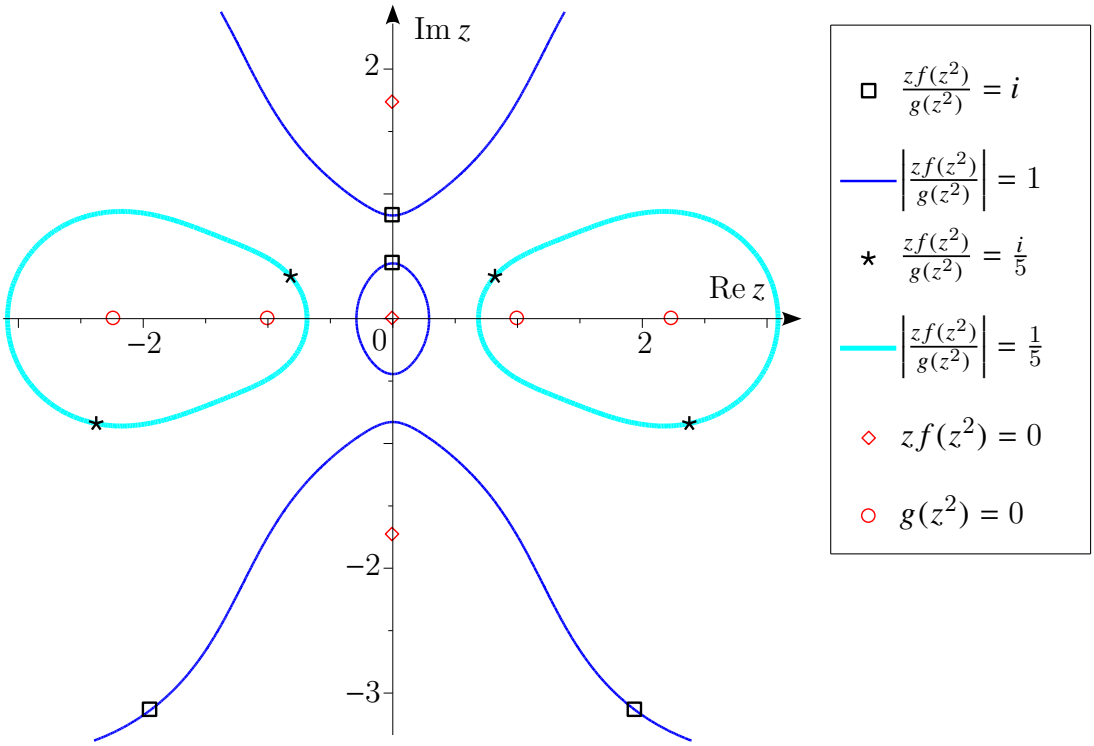


Figure 4.3 – The solutions to $\frac{zf(z^2)}{g(z^2)} = \alpha$, where $f(z) = z + 3$, $g(z) = (z-1)(z-5)$ and α is equal to i or $\frac{i}{5}$.

Theorem 4.25. *Let $\alpha^k < 0$ under the conditions of Theorem 4.24, and let the two zeros of $F(z) - \alpha$ closest to the origin (counting double zeros as two) be equal in argument. Then $p = 1$.*

Proof. The case $\alpha = i$ in Figure 4.3 illustrates that these conditions are consistent. In the proof we use the notation involved in the proof of Theorem 4.24. The assertion of Theorem 4.25 can be stated as $z_0 e_{-2m+1} > 0 \implies p = 1$, because all other situations are impossible (see the statement of Theorem 4.24).

Let $z_0 e_{-2m+1} > 0$, which induces the inequality $w_0 < 0$. On the one hand, in this case $R(w_0) = |\alpha| e^{i\frac{\pi}{k}}$ and $R(w_*) = |\alpha|$ (see the proof of Theorem 4.24). Denote by $\psi(w)$ a branch of $\ln R(w)$ which is continuous in $\mathbb{C}_+$ and real at w_*, then $\operatorname{Im}\psi(w_*) = 0$ and $\operatorname{Im}\psi(w_0) = \frac{\pi}{k} + 2\pi n$ for some integer n. Item (b) of Lemma 4.8 yields that $n = 0$, since $R(w) \notin \Omega$ for all[12] $w_* < |w| < -w_0$. That is to say,

$$\frac{\pi}{k} = \operatorname{Im}\psi(w_0) - \operatorname{Im}\psi(w_*) = \operatorname{Im}\int_{w_*}^{w_0} \frac{R'(w)}{R(w)} dw, \tag{4.25}$$

where the integration is over any contour wholly lying in $\mathbb{C}_+$.

[12] If $R(w) \in \Omega$ for some $w \in \mathbb{C}_+$ satisfying $|w| < |w_0|$, then z_0 cannot be the α-point of $F(z)$ minimal in absolute value; see the proof of Theorem 4.24 for the details.

On the other hand, the function $R^k(w) = w^p e^{Akw} \dfrac{\prod_{\nu=1}^{\omega_1}\left(1+\frac{w}{a_\nu}\right)^k}{\prod_{\mu=1}^{\omega_2}\left(1-\frac{w}{b_\mu}\right)^k}$ is meromorphic in $\mathbb{C}$. The domain $D = \{w \in \mathbb{C} : |w| < |w_0|,\ |R^k(w)| < |\alpha|^k\}$ is not empty since $p > 0$. Its boundary $\overline{D} \setminus D$ is the analytic curve $\{w \in \mathbb{C} : |w| \leqslant |w_0|,\ |R^k(w)| = |\alpha|^k\}$, because $\left|R(|w_0|e^{i\varrho})\right| > |R(w_0)| = |\alpha|$ for any real $\varrho \in (-\pi, \pi)$ due to $\overline{R(w)} = R(\overline{w})$ and (4.12); this curve is closed but not necessarily connected. By definition, the closure of D cannot contain any pole of $R^k(w)$, so this function is holomorphic in $\overline{D}$. Cauchy's argument principle thus states, that

$$\oint_{\overline{D}\setminus D} \frac{R'(w)}{R(w)}dw = \frac{1}{k}\oint_{\overline{D}\setminus D} \frac{(R^k(w))'}{R^k(w)}dw = 2i\frac{Z}{k}\pi, \tag{4.26}$$

where Z is the number of zeros of $R^k(w)$ inside D counting multiple zeros so many times which multiplicities they have. Since $\overline{R(w)} = R(\overline{w})$, the contour $\overline{D} \setminus D$ is symmetric with respect to the real line. Consequently, the left-hand side of (4.26) can be modified in the following way:

$$\oint_{\overline{D}\setminus D} \frac{R'(w)}{R(w)}dw = \int_\gamma \frac{R'(w)}{R(w)}dw - \int_\gamma \frac{R'(\overline{w})}{R(\overline{w})}d\overline{w} = 2\int_\gamma \frac{R'(w)}{R(w)}dw,$$

where the contour γ can be any contour lying wholly in $\mathbb{C}_+$, which starts at w_* and ends at w_0. On account of (4.26), we therefore have the following expression

$$\operatorname{Im}\int_\gamma \frac{R'(w)}{R(w)}dw = \frac{Z}{k}\pi$$

contradicting to (4.25) unless $Z = 1$. However, p is the multiplicity of the zero of $R^k(w)$ at the origin, so $1 = Z \geqslant p \geqslant 1$. □

Observe that the change of variable $z \mapsto \zeta e_{-1}$ implies $z^k \mapsto -\zeta^k$. Hence, the function

$$\widetilde{F}(\zeta) := \frac{e_{-p}}{F(\zeta e_{-1})} = \zeta^{-p} e^{A\zeta^k} \frac{\prod_{\mu=1}^{\omega_2}\left(1+\frac{\zeta^k}{b_\mu}\right)}{\prod_{\nu=1}^{\omega_1}\left(1-\frac{\zeta^k}{a_\nu}\right)}$$

has the form (4.23) with a positive power of ζ as the first factor provided that $p < 0$. Moreover,

$$\begin{aligned} F(z) = \alpha &\iff \widetilde{F}(\zeta) = \frac{e_{-p}}{\alpha} =: \widetilde{\alpha}, \\ \alpha \in Q_{2q-\varkappa} &\iff \widetilde{\alpha} \in Q_{-2q+\varkappa-p-1} \quad \text{and} \\ \alpha e_{-2q+\varkappa} > 0 &\iff \widetilde{\alpha} e_{2q-\varkappa+p} > 0. \end{aligned} \tag{4.27}$$

This way the case of $p < 0$ can be reduced to the situation studied in the last two theorems. Unfortunately, the notation convenient in Theorem 4.24 suits this case worse inducing more complicated relations.

Theorem 4.26. *Suppose that all conditions of Theorem 4.24 hold excepting that $p < 0$.*

If $\alpha \in Q_{2q-\varkappa}$, then the α-point z_0 of $F(z)$ closest to the origin is simple and distinct in argument and absolute value from the succeeding α-point. Furthermore, $z_0 \in Q_{2m-\sigma}$ where $\sigma := \varkappa$ for even p, $\sigma := 1 - \varkappa$ for odd p, and the integer m satisfies[13] *$pm \equiv q - (-1)^\sigma \left\lceil \frac{p}{2} \right\rceil \pmod{k}$.*

If $\alpha e_{-2q+\varkappa} > 0$, where p and $\varkappa$ have the same parity, then the α-point z_0 of $F(z)$ closest to the origin is simple and distinct in argument and absolute value from the succeeding α-point (or points). Moreover, $z_0 e_{-2m+1} > 0$ for $pm \equiv q + \left\lceil \frac{p-1}{2} \right\rceil \pmod{k}$.

If $\alpha e_{-2q+\varkappa} > 0$, where p and $\varkappa$ have distinct parity, then the two zeros of $F(z) - \alpha$ closest to the origin (counting double zeros as two) are equal in absolute value or in argument. In the latter case, which is only possible when $p = -1$, both the zeros belong to the ray $\{z e_{-2m} > 0\}$. In the former case, one of them belongs to Q_{2m} and another belongs to $Q_{-2s-2m-1}$. Here m solves $pm \equiv q - \left\lceil \frac{p+1}{2} \right\rceil \pmod{k}$ and s is as in Remark 4.19.

Proof. With the notation

$$\widetilde{q} := -q + \left\lceil \frac{\varkappa - p - 1}{2} \right\rceil \quad \text{and} \quad \widetilde{\varkappa} := 2\widetilde{q} + 2q - \varkappa + p + 1, \tag{4.28}$$

the relations (4.27) immediately yield

$$\alpha \in Q_{2q-\varkappa} \iff \widetilde{\alpha} \in Q_{2\widetilde{q}-\widetilde{\varkappa}} \xrightarrow{\text{Theorem 4.24}} \zeta_0 \in Q_{2\widetilde{m}-\widetilde{\varkappa}} \iff z_0 \in Q_{2\widetilde{m}-\widetilde{\varkappa}-1},$$

where $\widetilde{m}$ satisfies $(-p) \cdot \widetilde{m} \equiv \widetilde{q} \pmod{k}$ and ζ_0 is the solution to $\widetilde{F}(\zeta) = \widetilde{\alpha}$ minimal in absolute value. That is, modulo k we have

$$p\widetilde{m} \equiv q - \left\lceil \frac{\varkappa - p - 1}{2} \right\rceil = \begin{cases} q + \frac{p}{2}, & \text{if } p \text{ is even} \\ q + \frac{p+1}{2} - \varkappa, & \text{if } p \text{ is odd} \end{cases} \tag{4.29}$$

Let m denote such an integer that $z_0 \in Q_{2m-\sigma}$ for some $\sigma \in \{0, 1\}$. Then necessarily $2m - \sigma \equiv 2\widetilde{m} - \widetilde{\varkappa} - 1 \pmod{2k}$, which is satisfied by $m = \widetilde{m} - \widetilde{\varkappa}$ and $\sigma = 1 - \widetilde{\varkappa}$. At that, the second of the expressions (4.28) yields $\widetilde{\varkappa} = 1 - \varkappa$ if p is even and $\widetilde{\varkappa} = \varkappa$ if p is odd. The relation (4.29) within these settings becomes

$$pm \equiv \begin{cases} q + \frac{p}{2} - p\widetilde{\varkappa}, & \text{if } p \text{ is even;} \\ q + \frac{p+1}{2} - \widetilde{\varkappa}(p+1), & \text{if } p \text{ is odd} \end{cases} = \begin{cases} q + (-1)^{\widetilde{\varkappa}} \frac{p}{2}, & \text{if } p \text{ is even;} \\ q + (-1)^{\widetilde{\varkappa}} \frac{p+1}{2}, & \text{if } p \text{ is odd} \end{cases}$$

[13] Recall that $\left\lceil \frac{p}{2} \right\rceil$ stands for the minimal integer greater than or equal to $\frac{p}{2}$. Here $\left\| \left\lceil \frac{p}{2} \right\rceil \right\| \leqslant \left| \frac{p}{2} \right|$ since $p < 0$.

modulo k. Since $p < 0$, the last equality implies

$$pm \equiv q + (-1)^{\widetilde{\varkappa}} \left\lceil \frac{p}{2} \right\rceil = q - (-1)^{\sigma} \left\lceil \frac{p}{2} \right\rceil \pmod{k}.$$

However, this relation coincides with the relation for m suggested by the theorem's statement. For the corresponding illustration see Figure 4.2b, $\alpha = e^{i\pi/2}$.

When $\widetilde{\alpha}$ satisfies $\widetilde{\alpha} e_{-2\widetilde{q}} > 0$, from the relation (4.27) we have $-2\widetilde{q} \equiv 2q - \varkappa + p \pmod{2k}$, which determines the pair $q, \varkappa$ satisfying the inequality $\alpha e_{-2q+\varkappa} > 0$ instead of (4.28). In particular, p and $\varkappa$ have the same parity. So, $z_0 e_{-2m+1} = \zeta_0 e_{-2m} > 0$ for

$$pm = -(-p) \cdot m \equiv -\widetilde{q} \equiv q + \frac{p - \varkappa}{2} = q + \left\lceil \frac{p-1}{2} \right\rceil \pmod{k}.$$

The corresponding plot can be found in Figure 4.2b, $\alpha = e^{i\pi/3}$.

When $\widetilde{\alpha} e_{-2\widetilde{q}+1} > 0$, the relation $-2\widetilde{q} + 1 \equiv 2q - \varkappa + p \pmod{2k}$ provides another pair $q, \varkappa$ making the inequality $\alpha e_{-2q+\varkappa} > 0$ true. This gives us that $z_0 \in Q_{2\widetilde{m}-2}$ or $z_0 e_{-2\widetilde{m}+2} > 0$ (the latter is only possible for $p = -1$ by Theorem 4.25) whenever

$$p\widetilde{m} \equiv -\widetilde{q} \equiv q + \frac{p - 1 - \varkappa}{2} \pmod{k}.$$

The change $m := \widetilde{m} - 1$ gives $z_0 \in Q_{2m}$ or $z_0 e_{-2m} > 0$ whenever

$$pm \equiv q - p + \frac{p - 1 - \varkappa}{2} = q - \frac{p + 1 + \varkappa}{2} = q - \left\lceil \frac{p+1}{2} \right\rceil \pmod{k}.$$

For $z_0 \in Q_{2m}$, the integer s defined as in Remark 4.19 provides the expression $\overline{z}_0 e_{-2s}$ for the α-point of $F(z)$ which is equidistant with z_0 from the origin. See the relevant example in Figure 4.2b, $\alpha = e^{i2\pi/3}$. □

Remark 4.27. Suppose that a function $H(z)$ has the form (4.23). Then the last three theorems give a straightforward conclusion concerning the solution of the equation $H(1/z) = \alpha$ with the *maximal* absolute value. It is of special interest when $H(z)$ is rational: then both $H(z)$ and $H(1/z)$ can be represented as in (4.23).

4.7 Zeros of entire functions

Let the positive integers j and k be coprime and $k \geqslant 2$. Theorems 4.21, 4.23–4.26 admit a transition to describing the zeros of functions of the forms

$$H_1(z) := f(z^k) + z^j g(z^k) \quad \text{and} \quad H_2(z) := g(z^k) + z^j f(z^k),$$

where the functions $f(z)$ and $g(-z)$ are entire, have genus 0 and *only negative zeros*. At least one of the functions f and g needs to be non-constant to exclude the trivial case.

Note that both functions $f(z^k)/f(0)$ and $g(z^k)/g(0)$ must be real. They have no common zeros, therefore $f(z^k) \neq 0,\ g(z^k) \neq 0$ and $z \neq 0$ when $H_1(z) = 0$ or $H_2(z) = 0$.

To adapt the facts stated in Sections 4.5 and 4.6 for studying zeros of the functions H_1 and H_2, put

$$F_1(z) := z^{-j}\frac{f(z^k)/f(0)}{g(z^k)/g(0)}, \quad F_2(z) := z^{j}\frac{f(z^k)/f(0)}{g(z^k)/g(0)} \quad \text{and} \quad \alpha := -\frac{g(0)}{f(0)}. \tag{4.30}$$

Then the following identities hold:

$$H_1(z) = \left(1 - \frac{F_1(z)}{\alpha}\right) z^j g(z^k) \quad \text{and} \quad H_2(z) = \left(1 - \frac{F_2(z)}{\alpha}\right) g(z^k). \tag{4.31}$$

Recall that $z^j g(z^k)$ and $H_i(z)$ have no common zeros for $i = 1, 2$. Therefore, the equalities (4.31) imply that

$$F_1(z) = \alpha \iff H_1(z) = 0 \quad \text{and} \quad F_2(z) = \alpha \iff H_2(z) = 0.$$

That is, the zero set of $H_i(z)$ *coincides* with the α-set of $F_i(z)$ for $i = 1, 2$. Moreover, (4.31) give that each α-point z_* of the function $F_i(z)$ is the zero of $H_i(z)$ with *the same multiplicity*.

Since the functions $F_i(z)$ have the form (4.23), the zeros of $H_i(z)$ for $i = 1, 2$ are located as is asserted about α-points of $F_i(z)$ by Theorems 4.21, 4.23–4.26 with $\alpha = -\frac{g(0)}{f(0)}$ and $p = (-1)^i \cdot j$.

Remark 4.28. Some extensions of the fact proposed in the current section are possible; here we give two examples. However, it is unclear whether studying such functions is well-motivated.

1. Assume that $f(z)$ and $g(z)$ are functions regular and nonzero at the origin, and that $\frac{f(z)}{g(-z)}$ does not coincide with z^p up to a constant (to suppress the trivial case). From the comparison of the formulae (4.30) with (4.15) and (4.23) it is seen that $f(z)/f(0)$ and $g(-z)/g(0)$ can be allowed to have the form

$$e^{Az} \cdot \frac{\prod_{\nu>0}\left(1 + \frac{z}{a_\nu}\right)}{\prod_{\mu>0}\left(1 - \frac{z}{b_\mu}\right)}, \text{ where } A \geqslant 0 \text{ and } a_\nu, b_\mu > 0 \text{ for all } \mu, \nu. \tag{4.32}$$

Put in other words, if $f(0), g(0) \in \mathbb{C} \setminus \{0\}$ then $f(z)/f(0)$ and $g(-z)/g(0)$ can generate any *totally positive sequences* (see Definition 1.9) which start with 1. Indeed, after multiplying H_i by the denominators of $f(z^k)$ and $g(z^k)$ we obtain the entire function $\widetilde{H}_i$ with the same zeros as H_i. Then it is enough to note that the exponential factor originating from those of $f(z^k)$ and $g(z^k)$ is allowed in the representation (4.23). So, the result of the current section extends to such functions without any changes.

2. Let $f(z)$ and $g(-z)$ be nontrivial functions of the class $\widetilde{\mathcal{PF}}$ up to a complex constant factor, *i.e.* let them be of the form

$$Cz^p e^{Az+\frac{A_0}{z}} \cdot \frac{\prod_{\nu>0}\left(1+\frac{z}{a_\nu}\right)}{\prod_{\mu>0}\left(1-\frac{z}{b_\mu}\right)} \cdot \frac{\prod_{\nu>0}\left(1+\frac{z^{-1}}{c_\nu}\right)}{\prod_{\mu>0}\left(1-\frac{z^{-1}}{d_\mu}\right)}, \tag{4.33}$$

where the products converge absolutely, $p \in \mathbb{Z}$, $A, A_0 \geqslant 0$ and $a_\nu, b_\mu, c_\nu, d_\mu > 0$ for all ν, μ, and the complex constant C is nonzero. In addition let $f(z) \not\equiv \text{const} \cdot z^p g(-z)$. With the help of analogous manipulations we still can obtain a transition of Theorems 4.21 and 4.23: it is enough additionally to factor some power of z out of H_i (this cannot change the zero set excepting the origin).

Remark 4.29. Allowing $f(z)$ and $g(-z)$ to be arbitrary functions of the forms (4.32) or (4.33) with $C \in \mathbb{C} \setminus \{0\}$ can be useful in the following sense. Consider the power series

$$f(z) = \sum_{n=-\infty}^{\infty} f_n z^n \quad \text{and} \quad g(z) = \sum_{n=-\infty}^{\infty} g_n z^n$$

such that $f_n \neq f_0^{1-n} f_1^n$ and $g_m \neq g_0^{1-m} g_1^m$ for some $n, m \in \mathbb{Z}$. Then by Theorem 1.8, the series converge and the functions $f(z)/f_0$ and $g(-z)/g_0$ generate totally positive sequences (possibly doubly infinite) if and only if the Toeplitz matrices $(f_{n-i}/f_0)_{i,n=-\infty}^{\infty}$ and $((-1)^{n-i} g_{n-i}/g_0)_{i,n=-\infty}^{\infty}$ have all their minors nonnegative. However, then

$$H_1(z) = \sum_{n=-\infty}^{\infty} (f_n + z^j g_n) z^{kn} \quad \text{and} \quad H_2(z) = \sum_{n=-\infty}^{\infty} (g_n + z^j f_n) z^{kn}$$

are the Laurent series. This gives us the conditions in terms of *the Laurent coefficients* of $H_1(z)$ and $H_2(z)$, which provide that the zeros of $H_1(z)$ and $H_2(z)$ are localized according to Theorems 4.21 and 4.23 (and to Theorems 4.24–4.26 when the series $f(z)$, $g(z)$ are not doubly infinite, that is the limiting functions are meromorphic).

4.8 Conclusions for the case $k = 2$

Note that in the particular case of $p = \pm 1$ the relations modulo k from Theorems 4.21, 4.23–4.26 have obvious solutions. The setting $k = 2$ (implying that p is odd) also provides us with simple (and very useful) solutions. Let us restate the facts of Sections 4.5–4.6 for this particular situation.

Denote $p = 2j + 1$. The congruence modulo k (a linear Diophantine equation) from Theorem 4.21 becomes $l \equiv 1 + \varkappa + \sigma + m \pmod 2$. If $\alpha \in \mathbb{R}$ (or $i\alpha \in \mathbb{R}$), then the constant s in Theorems 4.23–4.26 equals 0 (or 1, respectively). The congruence

from Theorem 4.23 turns into $l = 1 + m \pmod 2$. The equation from Theorem 4.24 becomes $m = q \pmod 2$, and those from Theorem 4.26 become

$$m \equiv \begin{cases} q + j + 1 \pmod 2, & \text{if} \quad \alpha \in Q_{2q-\varkappa} \quad \text{or} \quad \alpha e_{-2q} > 0; \\ q + j \pmod 2, & \text{if} \quad \alpha e_{-2q+1} > 0. \end{cases}$$

Let $N = (z_n)_{n=1}^{\omega}$ be the set of all α-points of $F(z)$, where $|z_{n-1}| \leqslant |z_n|$ for all n and each α-point counts so many times which multiplicity it has. Then we have the following two theorems as a summary.

Theorem 4.30 (*cf.* [Dy2013]). *Let a function $F(z)$ have the form* (4.23), $p = 2j + 1$, $j < 0$ *and $k = 2$. then the α-points $N = (z_n)_{n=1}^{\omega}$ of $F(z)$ for $\alpha \neq 0$ are distributed as follows:*

(i) *If $\operatorname{Im} \alpha^2 \neq 0$, then all α-points are simple and satisfying $0 < |z_1| < |z_2| < \cdots$, $\operatorname{Im} z_n^2 \neq 0$ for every integer $n > 0$, and $z_n \in Q_l$ implies that $z_{n+1} \in Q_{l+\operatorname{sign}(\operatorname{Im} \alpha^2)}$. Moreover, $(-1)^j \operatorname{Im} \alpha \operatorname{Im} z_1 > 0$ and $\operatorname{Im} \alpha^2 \operatorname{Re} z_1 \operatorname{Im} z_1 < 0$.*

(ii) *If $\operatorname{Im} \alpha = 0$, then $0 < |z_1| \leqslant |z_2| < |z_3| \leqslant |z_4| < |z_5| \leqslant \cdots$, there are no purely imaginary α-points, real α-points can be simple or double, other α-points are simple. Moreover, for each positive integer n the following five conditions hold*

$$\begin{aligned} &|z_{2n-1}| = |z_{2n}| \implies z_{2n-1} = \overline{z}_{2n}, \\ &|z_{2n-1}| < |z_{2n}| \implies \operatorname{Arg} z_{2n-1} = \operatorname{Arg} z_{2n} \in \{0, \pi\}, \\ &|z_1| < |z_2| \implies j = -1, \\ &\operatorname{Re} z_{2n} \operatorname{Re} z_{2n+1} < 0 \quad \textit{and} \quad (-1)^j \alpha \operatorname{Re} z_1 < 0. \end{aligned}$$

(iii) *If $\operatorname{Re} \alpha = 0$, then $0 < |z_1| < |z_2| \leqslant |z_3| < |z_4| \leqslant |z_5| < \cdots$, there are no real α-points, purely imaginary α-points can be simple or double, other α-points are simple. Moreover, for each positive integer n the following five conditions hold*

$$\begin{aligned} &|z_{2n}| = |z_{2n+1}| \implies z_{2n} = -\overline{z}_{2n+1}, \\ &|z_{2n}| < |z_{2n+1}| \implies \operatorname{Arg} z_{2n} = \operatorname{Arg} z_{2n+1} \in \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}, \\ &\operatorname{Im} z_{2n-1} \operatorname{Im} z_{2n} < 0, \quad (-1)^j \operatorname{Im} \alpha \operatorname{Im} z_1 > 0 \quad \textit{and} \quad \operatorname{Re} z_1 = 0. \end{aligned}$$

Theorem 4.31. *Let a function $F(z)$ have the form* (4.23), $p = 2j + 1$, $j \geqslant 0$ *and $k = 2$. Then the α-points $N = (z_n)_{n=1}^{\omega}$ of $F(z)$ for $\alpha \neq 0$ are distributed as follows:*

(iv) *If $\operatorname{Im} \alpha^2 \neq 0$, then all α-points are simple and satisfying $0 < |z_1| < |z_2| < \cdots$, $\operatorname{Im} z_n^2 \neq 0$ for every integer $n > 0$, and $z_n \in Q_l$ implies that $z_{n+1} \in Q_{l+\operatorname{sign}(\operatorname{Im} \alpha^2)}$. Moreover, $\operatorname{Im} \alpha \operatorname{Im} z_1 > 0$ and $\operatorname{Re} \alpha \operatorname{Re} z_1 > 0$.*

(v) *If* $\operatorname{Im}\alpha = 0$, *then* $0 < |z_1| < |z_2| \leqslant |z_3| < |z_4| \leqslant |z_5| < \cdots$, *there are no purely imaginary* α*-points, real* α*-points can be simple or double, other* α*-points are simple. Moreover, for each positive integer n the following five conditions hold*

$$\begin{aligned}
&|z_{2n}| = |z_{2n+1}| \implies z_{2n} = \overline{z}_{2n+1},\\
&|z_{2n}| < |z_{2n+1}| \implies \operatorname{Arg} z_{2n} = \operatorname{Arg} z_{2n+1} \in \{0, \pi\},\\
&\operatorname{Re} z_{2n-1} \operatorname{Re} z_{2n} < 0, \quad \operatorname{Re} z_1 = 0 \quad \textit{and} \quad \alpha z_1 > 0.
\end{aligned}$$

(vi) *If* $\operatorname{Re}\alpha = 0$, *then* $0 < |z_1| \leqslant |z_2| < |z_3| \leqslant |z_4| < |z_5| \leqslant \cdots$, *there are no real* α*-points, purely imaginary* α*-points can be simple or double, other* α*-points are simple. Moreover, for each positive integer n the following five conditions hold*

$$\begin{aligned}
&|z_{2n-1}| = |z_{2n}| \implies z_{2n-1} = -\overline{z}_{2n},\\
&|z_{2n-1}| < |z_{2n}| \implies \operatorname{Arg} z_{2n-1} = \operatorname{Arg} z_{2n} \in \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\},\\
&|z_1| < |z_2| \implies j = 0,\\
&\operatorname{Im} z_{2n} \operatorname{Im} z_{2n+1} < 0 \quad \textit{and} \quad \operatorname{Im}\alpha \operatorname{Im} z_1 > 0.
\end{aligned}$$

Remark 4.32. Two last theorems have analogous statements if, instead of $F(z)$ satisfying (4.23), we take a function $G(z)$ of the more general form (4.15). Then, generally speaking, we cannot assert where the α-point that is the least in absolute value occurs (it may be absent at all).

Remark 4.33. Note that in all cases (i)–(vi) the α-point split evenly among the quadrants of complex plane. That is, if the α-set of a function satisfies (i) or (iv), then for any $r > 0$ the number of α-points in the finite sector $\{z \in Q_n : |z| < r\}$ can differ from the number of α-points in $\{z \in Q_j : |z| < r\}$ at most by 1 (here $n, j = 1, \ldots, 4$). The cases appearing in (ii), (iii), (v) and (vi) are the "degenerated" those of (i) and (iv) with possible ingress of some α-points onto the real or imaginary axes.

Let us turn to zeros of entire functions by applying the idea of Section 4.7. An entire function $H(z) = \sum_{n=0}^{\infty} f_n z^n$, $f_0 \neq 0$, splits up into the even and odd parts according to

$$\begin{aligned}
&H(z) = f(z^2) + zg(-z^2), \quad \text{where}\\
&f(z) = \sum_{n=0}^{\infty} f_{2n} z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} f_{2n+1} z^n.
\end{aligned} \tag{4.34}$$

Since

$$H(z) = 0 \iff \frac{f(z^2)/f_0}{zg(-z^2)/f_1} = -\frac{f_1}{f_0},$$

zeros of $H(z)$ are distributed as stated in Theorem 4.30 if both $f(z)/f_0$ and $g(z)/f_1$ have only negative zeros and the genus equal to 0 up to factors of the form e^{cz}, $c \geqslant 0$. Similarly, zeros of $H(z)$ are distributed as stated in Theorem 4.31 provided that both $f(z)/f_0$ and $g(z)/f_1$ have only positive zeros and the genus equal to 0 up to factors of the form e^{-cz}, where $c \geqslant 0$.

A real entire function $\widetilde{H}(z) = f(z^2)+zg(z^2)$ is *strongly stable* if $f(z^2)+(1+\eta)zg(z^2)$ has no zeros in the closed right half of the complex plane for all complex η which are small enough. This is the "proper" extension to entire functions of the polynomial stability. Recall that $\widetilde{H}(z)$ is strongly stable whenever $\widetilde{H}(iz)$ belongs to the class $\mathcal{HB}$, which is introduced in section 1.2.2.

Remark 4.34. *If $\widetilde{H}(z)$ is a strongly stable function of genus at most 1, then zeros of the function $H(z)$, which is defined by* (4.34)*, are distributed as stated in Theorem* 4.30. Indeed, Theorem H-B (its conditions remain necessary and sufficient for strongly stable functions of genus $\leqslant 1$, this is the class discussed on Page 12) then implies that $f(g)$ and $g(z)$ have genus 0 and their zeros are negative, simple and interlacing. At that, the interlacing property of $f(z)$ and $g(z)$ remains redundant.

Observe that if a complex number μ satisfies $\mu^4 = -1$ (*i.e.* μ is a primitive 8th root of unity), then we have the identity

$$\begin{aligned}\sum_{n=0}^{\infty} f_n \mu^{n(n-1)}(\mu^{-1}z)^n &= \sum_{n=0}^{\infty} f_n \mu^{n(n-2)} z^n \\ &= \sum_{n=0}^{\infty} f_{2n}\mu^{4n(n-1)} z^{2n} + \sum_{l=0}^{\infty} f_{2l+1}\mu^{4l^2-1} z^{2l+1} \\ &= \sum_{n=0}^{\infty} f_{2n} 1^{\frac{n(n-1)}{2}} z^{2n} + \sum_{l=0}^{\infty} f_{2l+1}(-1)^{l^2}\mu^{-1} z^{2l+1} \\ &= \sum_{n=0}^{\infty} f_{2n} z^{2n} + \overline{\mu}\sum_{l=0}^{\infty}(-1)^l f_{2l+1} z^{2l+1}.\end{aligned}$$

Consequently, the following fact is true.

Corollary 4.35. *Consider the functions*

$$h(z) = \sum_{n=0}^{\infty} i^{\frac{n(n-1)}{2}} f_n z^n \quad \text{and} \quad \overline{h}(z) = \sum_{n=0}^{\infty} i^{-\frac{n(n-1)}{2}} f_n z^n,$$

where $f(z) = \sum_{n=0}^{\infty} f_{2n} z^n$ and $g(z) = \sum_{n=0}^{\infty} f_{2n+1} z^n$ are entire functions of genus 0 and have only negative zeros. For $\mu = \sqrt{i}$, zeros of the function $h(\overline{\mu}z)$ distributed as claimed in Theorem 4.30 for $\alpha = \overline{\mu} f_1/f_0$ and zeros of the function $\overline{h}(\mu z)$ distributed as claimed in Theorem 4.30 for $\alpha = \mu f_1/f_0$.

4.9 Two problems by A. Sokal

"Disturbed exponential" function. Alan Sokal [So2009] put forward the hypothesis that

Conjecture 4.A. *The entire function*

$$\mathcal{F}(z;q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} z^n}{n!},$$

where q is a complex number, $0 < |q| \leq 1$, can have only simple zeros.

The stronger version of the conjecture claims that

Conjecture 4.B. *The function $\mathcal{F}(z;q)$ for $q \in \mathbb{C}$, $0 < |q| \leq 1$, can have only simple zeros with distinct absolute values.*

The following facts on $\mathcal{F}(z;q)$ are known. The function $\mathcal{F}$ is the unique solution to the following Cauchy problem

$$\mathcal{F}'(z) = \mathcal{F}(qz), \quad \mathcal{F}(0) = 1,$$

which can be checked by substitution. Moreover, when $|q| = 1$ this function has the exponential type 1, for q lower in absolute value the function $\mathcal{F}$ is of zero genus.

The case of positive q was studied extensively. It is known that all zeros of $\mathcal{F}$ are negative (see [La1898, pp. 35, 177], [PoSc14, pp. 90, 104] and [MFB71]), simple and satisfy Conjecture 4.B as well as certain further conditions [Liu98, La2000]. Conjecture 4.B holds true for negative q as well, see *e.g.* [Dy2013, pp. 11–12, 17–18]. The properties of $\mathcal{F}(z;q)$ for complex q were studied in [Ala14, Val38, EO2007]. According to [So2009], Conjecture 4.B is true if $|q| < 1$ and the zeros of $\mathcal{F}(z;q)$ are big enough in absolute value (A. Eremenko) as well as for small $|q|$.

Let us prove that Conjecture 4.B also holds true for purely imaginary values of the parameter. As we pointed out, for positive $q \leqslant 1$ the function $\mathcal{F}(z;q) = f(z^2) + zg(z^2)$ has only negative zeros. In particular, it is stable;[14] the Hermite-Biehler theorem implies that the zeros of $f(z)$ and $g(z)$ are negative and interlacing. Therefore, by Corollary 4.35 the zeros of $\mathcal{F}(z;\pm iq)$ with $0 < q \leqslant 1$ are simple and their absolute values are distinct.

Up to the change of variable $z \mapsto zN^{-1}$, the polynomials

$$P_N(z;q) = \sum_{n=0}^{N} \binom{N}{n} z^n q^{\frac{n(n-1)}{2}}$$

[14] See Theorem H-B for polynomials; for $0 < q < 1$ its conditions are necessary for stability since the genus of $\mathcal{F}(z;q)$ is 0, see the discussion on Page 12. The setting $q = 1$ gives $f(z) = \cosh\sqrt{z}$ and $g(z) = \frac{\sinh\sqrt{z}}{\sqrt{z}}$, so zeros of $f(z)$ and $g(z)$ are negative and interlacing.

are the Jensen polynomials for the function $\mathcal{F}(z;q)$, see [Lev64, p. 343]; they approximate this function in the sense that

$$P_N\left(\frac{z}{N};q\right) = 1+z+\sum_{n=2}^{N}\frac{q^{\frac{n(n-1)}{2}}z^n}{n!}\left(1-\frac{1}{N}\right)\cdot\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{n-1}{N}\right)\xrightarrow{N\to\infty}\mathcal{F}(z;q).$$

The polynomial version of this conjecture has the following form.

Conjecture 4.C. *For all $N>0$ the polynomial $P_N(z;q)$ where $|q|<1$, can have only simple roots, separated in absolute value by at least the factor $|q|$.*

The original statement (which is equivalent to the one given here) is concerned with the family of polynomials $\left\{P_N\left(zw^{N-1};w^{-2}\right)\right\}_{N\in\mathbb{Z}_{>0}}$, where $w^{-2}=q$. Observe that 4.C $\implies$ 4.B $\implies$ 4.A.

The approach for $\mathcal{F}(z;q)$ extends to the polynomials $P_N(z;q)$ without changes. Their zeros are negative for positive q provided that the polynomials coincide with the action of the *multiplier sequence*[15] $(q^{n(n-1)/2})_{n=0}^{\infty}$ on the polynomial $(z+1)^N$. This justifies the assertion of Conjecture 4.C for purely imaginary q without bounds on the ratio of subsequent (by the absolute value) roots.

Partial theta function. An analogous problem by A. Sokal appears in [KS2013]. The partial theta function

$$\Theta_0(z;q) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}}z^n$$

has only negative zeros whenever $0<q\leqslant\widetilde{q}\approx 0.3092493386$, which is shown in [KS2013] (see also [Ko2013]). Splitting it into the even part $f(z^2)$ and the odd part $zg(z^2)$ gives

$$\Theta_0(z;q) = f(z^2)+zg(z^2),$$
$$f(z) = \sum_{n=0}^{\infty} q^{n(2n-1)}z^n = \sum_{n=0}^{\infty} q^{n(2n-2)}(qz)^n = \sum_{n=0}^{\infty}\left(q^4\right)^{\frac{n(n-1)}{2}}(qz)^n = \Theta_0(qz;q^4)$$

and

$$g(z) = \sum_{n=0}^{\infty} q^{n(2n+1)}z^n = \sum_{n=0}^{\infty} q^{n(2n-2)}(q^3z)^n = \Theta_0(q^3z;q^4).$$

Thus, both $f(z)$ and $g(z)$ have only negative zeros whenever $0<q^4\leqslant\widetilde{q}$. Therefore, by Corollary 4.35 all zeros of $\Theta_0(z;iq)$ are simple and distinct in absolute value if $0<q^4\leqslant\widetilde{q}$,

[15] The definition and properties of multiplier sequences can be found in *e.g.* [PoSc14], [Obr63, Ch. II] and [Lev64, Ch. VIII Sec. 3]. The fact that $(q^{n(n-1)/2})_{n=0}^{\infty}$ is a multiplier sequence (of the first kind) for $0<q\leqslant 1$ was first shown by Laguerre [La1898, p. 35]. The more modern proof follows from Satz 10.1 of [Obr63, p. 42] applied to the function $\Phi(z):=\exp\left(\frac{1}{2}z(z-1)\cdot\ln q\right)$.

that is if $0 < q \leqslant q_* \approx 0.7457224107$. This is a partial positive answer to the following question.

Problem 4.D (see [KS2013, p. 832]). *Is it true that all zeros of $\Theta_0(z;q)$ remain simple within the open disk $|q| < \widetilde{q}$?*

With the help of exactly the same manipulations we could deduce that, for example, the (Jacobi) theta function

$$\Theta(z;iq) = \sum_{n=-\infty}^{\infty} (iq)^{\frac{n(n-1)}{2}} z^n, \quad 0 < q < 1,$$

also has its zeros simple, distinct in absolute value and residing in the quadrants of the complex plane rotated by $\pi/4$ (according to the Remark 4.32). However, this is redundant (although yet instructive) because the exact information is provided by the Jacobi triple product formula (see *e.g.* [HaWr75, Theorem 352]) which is valid for any complex $z \neq 0$ and $|q| < 1$:

$$\sum_{n=-\infty}^{\infty} q^{\frac{n(n-1)}{2}} z^n = \prod_{j=1}^{\infty} (1-q^j)(1+zq^{j-1}) \left(1 + \frac{q^j}{z}\right).$$

Chapter 5.

Functions of classes $\mathcal{N}_\varkappa^+$ and their application to counting distinct zeros and poles of $\mathcal{PF}$-functions

5.1 Representation of $\mathcal{N}_\varkappa^+$-functions

Let us recall Definitions 1.3: a function $\varphi(z)$ belongs to $\mathcal{N}_\varkappa$ whenever it is meromorphic in $\mathbb{C}_+$, the Hermitian form

$$h_\varphi(\xi_1,\dots,\xi_k|z_1,\dots,z_k) := \sum_{n,m=0}^{k} \frac{\varphi(z_m)-\overline{\varphi(z_n)}}{z_m-\overline{z}_n}\xi_m\overline{\xi}_n \tag{5.1}$$

has at most $\varkappa$ negative squares for any set of non-real points $z_1, z_2, \dots, z_k$ and, for some set of points, there are exactly $\varkappa$ negative squares. In this chapter we prove the necessary and sufficient condition for a function to belong to the class $\mathcal{N}_\varkappa^+$, where

Definition 5.1. The class $\mathcal{N}_\varkappa^+$ is a subclass of $\mathcal{N}_\varkappa$ of functions $\varphi(z)$ such that $z\varphi(z)$ belongs to $\mathcal{R}$ (that is, maps the upper half of the complex plane into itself).

Our approach rests on the asymptotic analysis of the corresponding Hermitian forms. As a main tool, we use the basic Nevanlinna-Pick theory for the half-plane within the framework presented, for example, in [Akh65, Chapter 3] or [Don74, Chapters II–III]. It is shown that, roughly speaking, being in $\mathcal{N}_\varkappa^+$ differs from being in the Stieltjes class $\mathcal{N}_0^+$ in having $\varkappa$ simple negative poles, one of which can reach the origin and merge there into another singularity.

Precisely, *a function $\varphi(z)$ belongs to the class $\mathcal{N}_\varkappa^+$ if and only if it has one of the forms*

$$\varphi(z) = c_0 + \sum_{j=1}^{\varkappa} \frac{\gamma_j}{\alpha_j - z} + \int_0^\infty \frac{d\nu(t)}{t-z}; \tag{5.A}$$

$$\varphi(z) = c_0 + \frac{c_1}{z} - \frac{c_2}{z^2} + \sum_{j=1}^{\varkappa-1} \frac{\gamma_j}{\alpha_j - z} + \int_0^\infty \frac{d\nu(t)}{t-z}, \tag{5.B}$$

where $\max\{c_1, c_2\} > 0, \ \ \nu(0+) = 0;$

$$\varphi(z) = c_0 + \frac{c_1}{z} - \frac{c_2}{z^2} + \sum_{j=1}^{\varkappa-1} \frac{\gamma_j}{\alpha_j - z} + \frac{1}{z}\int_0^\infty \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\sigma(t), \tag{5.C}$$

$$\textit{where} \int_0^1 \frac{d\sigma(t)}{t} = \infty, \quad \sigma(0+) = 0, \quad \int_0^\infty \frac{d\sigma(t)}{1+t^2} < \infty.$$

Here $c_0, c_2 \geqslant 0$, $c_1 \in \mathbb{R}$, $\gamma_j, \alpha_j < 0$ *for* $j = 1, 2, \ldots, \varkappa$ *and* $\nu(t), \sigma(t)$ *are nondecreasing left-continuous functions such that* $\nu(0) = \sigma(0) = 0$ *and* $\int_0^\infty \frac{d\nu(t)}{1+t} < \infty$. The function $\sigma(t)$ is intentionally denoted by a distinct letter to emphasize that its constraints at infinity are weaker. For our purposes, it is more convenient to formulate this criterion as Theorem 5.7: to give the corresponding formulae for functions $\Phi(z)$ such that $\varphi(z) := \frac{1}{z}\Phi(z)$ belong to the class $\mathcal{N}_\varkappa^+$. At that, the case (5.A) corresponds to the representation (5.7), and the cases (5.B) and (5.C) correspond to the representation (5.6).

This result corrects Theorem 3.8 of [KrLa77]: the authors put the condition $\int_0^\infty \frac{d\sigma(t)}{1+t} < +\infty$ in the case (5.C). As a result, Theorem 3.8 fails to address $\mathcal{N}_\varkappa^+$-functions like

$$\psi(z) = \frac{1}{z} \cdot \left(\frac{1}{\sqrt{z}} \cot \frac{1}{\sqrt{z}} - \sqrt{z} \cot \sqrt{z}\right)$$

with $\varkappa = 1$. More than likely, this mistake is just an oversight: for proving the representations (5.A)–(5.B) the authors use, in fact, the measure $\frac{d\sigma(t)}{t}$ as $d\nu(t)$; then they put $\nu(t)$ instead of $\sigma(t)$ in (5.C).

It is worth noting that the flaw in [KrLa77, Theorem 3.8] does not affect dependent results which assume that the function $z\varphi(z)$ has an asymptotic expansion of the form $\sum_{n=0}^\infty s_n z^n$ as $z \to i \cdot 0+$ or of the form $\sum_{n=0}^\infty s_n z^{-n}$ as $z \to +\infty \cdot i$. Indeed, in the former case $\varphi(z)$ can be expressed as in (5.A) or (5.B), and in the latter case our representation (5.C) reduces to the item 3. of Theorem 3.8 in [KrLa77]. For example, the important result [KrLa81, § 5], which is mentioned below, holds. On the other hand, the function $z\psi(z)$, where $\psi(z)$ is given above, has no such asymptotic expansions.

Lemma 5.3 from [LaWi98, p. 421] (see Theorem 5.7 herein) has a proper statement, and the function $\psi(z)$ is allowed as an entry of $\mathcal{N}_1^+$. At the same time, *the proof in* [LaWi98] *relies on the aforementioned Theorem 3.8 from* [KrLa77], and the relevant piece of the proof is omitted as "similar" to another part. Our proof does not depend on results of the works [KrLa77, LaWi98]. Notwithstanding that the major part of the current chapter is devoted to the proof of the above criterion, we are mainly interested in its applications presented in Sections 5.2 and 5.3.

5.2 Number of zeros and poles of functions from $\mathcal{PF}$ and $\widetilde{\mathcal{PF}}$

Let us recall that functions of the class $\mathcal{PF}$ can be represented as the product

$$Cz^p e^{Az} \cdot \frac{\prod_{\nu>0}\left(1+\frac{z}{a_\nu}\right)^{k_\nu}}{\prod_{\mu>0}\left(1-\frac{z}{b_\mu}\right)^{l_\mu}} \tag{5.2}$$

where $C, A \geqslant 0,\ p \in \mathbb{Z}_{\geqslant 0}$, the multiplicity of the zero $-a_\nu$ is k_ν, the order of the pole b_μ equals l_μ and $a_\nu, b_\mu > 0$ for all ν, μ (thus, each zero or pole appears in (5.2) only once). In turn, entries of the more general class $\widetilde{\mathcal{PF}}$ have the form

$$Cz^p e^{Az+\frac{A_0}{z}} \cdot \frac{\prod_{\nu>0}\left(1+\frac{z}{a_\nu}\right)^{k_\nu}}{\prod_{\mu>0}\left(1-\frac{z}{b_\mu}\right)^{l_\mu}} \cdot \frac{\prod_{\nu>0}\left(1+\frac{z^{-1}}{c_\nu}\right)^{\widetilde{k}_\nu}}{\prod_{\mu>0}\left(1-\frac{z^{-1}}{d_\mu}\right)^{\widetilde{l}_\mu}}, \tag{5.3}$$

where additionally $c_\nu, d_\mu > 0$ and $\widetilde{k}_\nu, \widetilde{l}_\mu$ denote, respectively, the multiplicity of the zero $-c_\nu$ and the order of the pole d_μ for all ν, μ. In the expressions (5.2) and (5.3) each of the products can be void, finite or infinite — convergent uniformly in z on compact sets which contain no poles. Both these classes are defined in Section 1.2.3.

The following relation was suggested by Mikhail Tyaglov to the author of these lines:

Lemma 5.2. *For a non-constant meromorphic function $R(z)$ to belong to the class $\mathcal{PF}$ up to an arbitrary constant multiplier it is necessary and sufficient that the fraction*

$$F(z) = \frac{zR'(z)}{R(z)} = z(\ln R(z))'$$

is an $\mathcal{R}$-function.

The arbitrary constant multiplier arises since $F(z)$ does not depend on it. This lemma admits the following straightforward extension to $\widetilde{\mathcal{PF}}$-functions.

Corollary 5.3. *A function $R(z)$ meromorphic outside the origin generates a doubly infinite totally positive sequence up to an arbitrary constant multiplier if and only if the fraction $F(z) = \frac{zR'(z)}{R(z)} = z(\ln R(z))'$ is an $\mathcal{R}$-function.*

We prove Lemma 5.2 and Corollary 5.3 in Section 5.6. These facts can be useful for counting distinct zeros and poles of $\mathcal{PF}$-functions, because the number of positive and negative poles of $\mathcal{R}$-functions can be determined with the help of the developed machinery of C-fractions and Hamburger's moment problem [Ham20].

Taking into account the notion of the classes $\mathcal{N}_\varkappa^+$ allows us to go slightly further. The following fact is an application of the above criterion, or more specifically, of Theorem 5.7.

Theorem 5.4. *A $\widetilde{\mathcal{PF}}$-function $R(z)$ has exactly $\varkappa < \infty$ distinct zeros if and only if its logarithmic derivative $G(z) = \dfrac{R'(z)}{R(z)}$ belongs to $\mathcal{N}_\varkappa^+$. Analogously, a $\widetilde{\mathcal{PF}}$-function $R(z)$ has exactly $\lambda < \infty$ distinct poles if and only if $G(-z)$ belongs to $\mathcal{N}_\lambda^+$. At that, if $R(z)$ has an essential singularity at the origin, then it counts as both a zero and a pole.*

If the logarithmic derivative $G(z)$ of $R(z)$ admits the asymptotic expansion

$$G(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \frac{s_1}{z^3} - \cdots \quad \text{as} \quad z \to +\infty \cdot i,$$

then Theorem 5.4 allows one to apply to $G \in \mathcal{N}_\varkappa^+$ the method involved in [KrLa79, KrLa81] for studying the number of zeros and poles of $R(z)$. In particular, the number $\varkappa$ can be determined from the number of the sign changes in a sequence of Hankel determinants composed of the numbers $s_0, s_1, \ldots$. This question deserves more elaboration, however is not in frames of the present thesis.

5.3 Logarithms of $\mathcal{PF}$-functions as illustration to Lemma 4.1

Let a function $R(z)$ have the form (5.2) and let $R(0) = 1$. Denote by $Q(z)$ the branch of the multivalued function $\ln R(z)$, which is analytic near the origin and satisfies $Q(0) = 0$. For the principal branch of the logarithm we use the notation $\operatorname{Ln} z$. Then

$$Q(z) = Az + \sum_{\nu>0} k_\nu \operatorname{Ln}\left(1 + \frac{z}{a_\nu}\right) - \sum_{\mu>0} l_\mu \operatorname{Ln}\left(1 - \frac{z}{b_\mu}\right). \tag{5.4}$$

The sums in this equality converge simultaneously with the products in (5.2) whenever the products do not converge to zero. That is, the representation (5.4) is locally uniformly convergent outside the real line and in a neighbourhood of the origin. Then the function $Q(z)$ satisfies the following interesting property.

Lemma 5.5. *If $R(z) \in \mathcal{PF}$ is given by* (5.2) *and $R(0) = 1$, then the function $Q(z)$ defined as in* (5.4) *is a univalent $\mathcal{R}$-function.*

Proof of Lemma 5.5. Indeed, the function $\operatorname{Ln}(1 + z/\alpha)$ maps the upper half of the complex plane into itself if $\alpha > 0$ and into the lower half if $\alpha < 0$: for $\operatorname{Im} z > 0$ we have

$$\operatorname{sign} \operatorname{Im} \operatorname{Ln}\left(1 + \frac{z}{\alpha}\right) = \operatorname{sign} \frac{\operatorname{Im} z}{\alpha} = \operatorname{sign} \alpha.$$

As a sum of $\mathcal{R}$-functions, the right-hand side of the expression (5.4) is an $\mathcal{R}$-function, holomorphic in the upper half-plane $\mathbb{C}_+$. Accordingly, $Q(z)$ is an $\mathcal{R}$-function as well.

Now let us assume that Lemma 5.2 is true. Then it yields that $z\dfrac{R'(z)}{R(z)} = zQ'(z)$ also maps the upper half of the complex plane into itself. Therefore, the univalence follows from Lemma 4.1. □

Note that the imaginary part of $Q(z)$ defined this way has a limiting value

$$\lim_{y\to 0+} \operatorname{Im} Q(x+iy), \quad \text{for real } x, y.$$

As can be seen from the expression (5.4), this limit equals to a piecewise constant function of x which decreases when $x < 0$, equal to zero close to $x = 0$ and increases when $x > 0$. This is the *density* of measure from the integral (Poisson) representation (5.5) of $Q(z)$ up to the factor $\frac{1}{\pi}$ (a consequence of the Stieltjes-Perron inversion formula, *e.g.* [Akh65, p. 124]).

5.4 Representation of $z\varphi(z)$ with $\varphi \in \mathcal{N}_\varkappa^+$

Each $\mathcal{R}$-function Φ has the following integral representation (see *e.g.* [Akh65, p. 92], [Don74, p. 20]):

$$\Phi(z) = bz + a + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\sigma(t) \tag{5.5}$$

where a is real, $b \geqslant 0$ and $\sigma(t)$ is a real non-decreasing function satisfying $\int_{-\infty}^{\infty} \frac{d\sigma(t)}{1+t^2} < \infty$. The converse is also true: all functions of the form (5.5) belong to $\mathcal{R}$.

To be definite, we assume that the function $\sigma(t)$ is *left-continuous*, that is $\sigma(t) = \sigma(t-)$ for all $t \in \mathbb{R}$. Accordingly, the notation for integrals with respect to $d\sigma(t)$ is as in the formula $\int_\alpha^\beta f(t)\, d\sigma(t) := \int_{[\alpha,\beta)} f(t)\, d\sigma(t)$ for arbitrarily taken real numbers α, β and function $f(t)$.

Remark 5.6. A function Φ given by the formula (5.5) is holomorphic outside the real line. Furthermore, it has an analytic continuation through the intervals outside the support of $d\sigma$. The function $\varphi(z) := \Phi(z)/z$ has the same singularities with the exception of the origin (generally speaking). We can additionally note that,

$$\text{if } z_1 < z_2 < t \text{ or } t < z_1 < z_2, \text{ then } \frac{1}{t-z_2} - \frac{1}{t-z_1} = \frac{z_2 - z_1}{(t-z_1)(t-z_2)} > 0.$$

Consequently, given a real interval (α, β) that has no common points with the support of $d\sigma$, the condition $\alpha < z_1 < z_2 < \beta$ implies $\Phi(z_1) < \Phi(z_2)$ unless $\Phi(z) \equiv a$, which is seen from the representation (5.5). (This fact is also seen immediately from the definition of the class $\mathcal{R}$: see [Don74, p. 18].) Put in other words, the function $\Phi(z)$ increases in the interval (α, β) unless it is identically constant (*cf.* Lemma 3.17).

Theorem 5.7 (Coincides with Lemma 5.3 from [LaWi98, p. 421]). *Let a function $\Phi \in \mathcal{R}$. The function $\varphi(z) := \Phi(z)/z$ belongs to $\mathcal{N}_\varkappa$ if and only if the representation* (5.5) *of $\Phi(z)$ is either of the form*

$$\Phi(z) = bz + a + \sum_{n=1}^{\varkappa-1} \frac{\sigma_n}{\lambda_n - z} + \int_0^\infty \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\sigma(t) \tag{5.6a}$$

where $\lambda_n < 0$, $n = 1, 2, \dots, \varkappa - 1$ and

$$0 < \Phi(0-) = a + \sum_{n=1}^{\varkappa-1} \frac{\sigma_n}{\lambda_n} + \int_0^\infty \frac{d\sigma(t)}{t+t^3} \leqslant \infty, \tag{5.6b}$$

or of the form

$$\Phi(z) = bz + a + \sum_{n=1}^{\varkappa} \frac{\sigma_n}{\lambda_n - z} + \int_0^\infty \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\sigma(t), \tag{5.7a}$$

where $\lambda_n < 0$, $n = 1, 2, \dots, \varkappa$ and

$$\Phi(0-) = a + \sum_{n=1}^{\varkappa} \frac{\sigma_n}{\lambda_n} + \int_0^\infty \frac{d\sigma(t)}{t+t^3} \leqslant 0. \tag{5.7b}$$

Note that $\mathcal{R}$-functions of the forms (5.6a) and (5.7a) have the corresponding limit $\Phi(0-)$ defined, because (when non-constant) they grow monotonically (see Remark 5.6) outside the support of corresponding measure $d\sigma(t)$. Moreover, all numbers σ_n in (5.6)–(5.7) are positive due to the condition $\Phi \in \mathcal{R}$.

5.5 Proof of Theorem 5.7

Definition 5.8. A real point λ is called a *point of increase* of a function $\sigma(t)$ if $\sigma(\lambda + \varepsilon) > \sigma(\lambda - \varepsilon)$ for every $\varepsilon > 0$ small enough. In particular, the set of all points of increase of a non-decreasing function $\sigma(t)$ is the support of $d\sigma(t)$.

In each punctured neighbourhood of the point of increase λ there exists λ' such that the limit

$$\sigma'(\lambda') = \lim_{\varepsilon \to 0} \frac{\sigma(\lambda' + \varepsilon) - \sigma(\lambda' - \varepsilon)}{2\varepsilon}$$

is positive or nonexistent. Indeed, otherwise $\sigma'(t) \leqslant 0$ in the closed interval $-\varepsilon \leqslant t \leqslant \varepsilon$ for some ε small enough, thus integrating $\sigma'(t)$ over this interval leads us to a contradiction. Consequently, if we know that the function $\sigma(t)$ has at least $\varkappa$ negative points of increase, then we always can select $\varkappa$ points of increase $\lambda_\varkappa < \lambda_{\varkappa-1} < \dots < \lambda_1 < 0$, in which the derivative σ' is nonexistent or positive. Given such a set of points,

denote

$$\delta := \frac{1}{3}\min\Big\{-\lambda_1, \min_{1\leqslant n\leqslant \varkappa-1}(\lambda_n-\lambda_{n+1})\Big\}, \tag{5.8}$$

put $U_n := (\lambda_n-\delta, \lambda_n+\delta)$, where $n = 1,\dots,\varkappa$, and $U_0 := \mathbb{R}\setminus\Big(\bigcup_{n=1}^{\varkappa} U_n\Big)$.

Lemma 5.9. *Consider a function $\Phi(z) = z\varphi(z)$ of the form* (5.5)*. Let the function $\sigma(t)$ have at least $\varkappa$ negative points of increase $\lambda_\varkappa < \dots < \lambda_1$ in which the derivative σ' is nonexistent or positive. Then the Hermitian form $h_\varphi(\xi_1,\dots,\xi_\varkappa|\lambda_1+i\eta,\dots,\lambda_\varkappa+i\eta)$ defined in* (5.1) *has $\varkappa$ negative squares for some small values of $\eta > 0$.*

Proof. Since

$$\frac{1}{t-z} = \frac{z+t-z}{t(t-z)} = \frac{z}{t(t-z)} + \frac{1}{t} \quad \text{and} \tag{5.9a}$$

$$\frac{1}{t} - \frac{t}{1+t^2} = \frac{1+t^2-t^2}{t(1+t^2)} = \frac{1}{t(1+t^2)}, \tag{5.9b}$$

from the expression (5.5) we obtain

$$\begin{aligned}\varphi(z) &= b + \frac{a}{z} + \frac{1}{z}\int_{-\infty}^{\infty}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)d\sigma(t)\\ &= b + \frac{a}{z} + \int_{-\infty}^{\infty}\left(\frac{1}{t-z} + \frac{1}{z(1+t^2)}\right)\frac{d\sigma(t)}{t} = \sum_{n=0}^{\varkappa}\varphi_n(z),\end{aligned} \tag{5.10}$$

where

$$\varphi_n(z) = \int_{U_n}\frac{1}{t-z}\cdot\frac{d\sigma(t)}{t}, \quad n = 1,\dots,\varkappa,$$

are the terms dominant on the intervals U_n, and

$$\varphi_0(z) := b + \frac{\widetilde{a}}{z} + \frac{1}{z}\int_{U_0}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)d\sigma(t) \quad \text{with} \quad \widetilde{a} := a + \sum_{n=1}^{\varkappa}\int_{U_n}\frac{d\sigma(t)}{t+t^3}$$

contains the remainder term. Let us additionally assume $z_n := \lambda_n + i\eta$.

On the one hand, each function φ_n is holomorphic outside U_n; therefore when $m, k \neq n$ we have that the limits

$$\lim_{\eta\to 0}\frac{\varphi_n(z_m)-\varphi_n(\overline{z}_k)}{z_m-\overline{z}_k} = \begin{cases}\dfrac{\varphi_n(\lambda_m)-\varphi_n(\lambda_k)}{\lambda_m-\lambda_k}, & \text{if } k\neq m,\\ \varphi_n'(\lambda_m), & \text{if } k = m\end{cases} \tag{5.11}$$

are finite as $n = 0,\dots,\varkappa$. On the other hand, with the notation $\widetilde{\sigma}(t) = \int_{-\lambda_\varkappa-\delta}^{t}|s|^{-1}d\sigma(s)$,

that is $\widetilde{\sigma}(t) = -\int_{-\lambda_\varkappa-\delta}^{t} s^{-1} d\sigma(s)$ when $t < 0$, for $n \neq 0$ we have

$$\rho_n^2(\eta) := -\frac{\varphi_n(z_n) - \varphi_n(\overline{z}_n)}{z_n - \overline{z}_n} = \int_{U_n} \frac{t - \overline{z}_n - t + z_n}{(z_n - \overline{z}_n)(t - z_n)(t - \overline{z}_n)} d\widetilde{\sigma}(t) = \int_{U_n} \frac{d\widetilde{\sigma}(t)}{|t - z_n|^2}$$
$$\geqslant \int_{\lambda_n-\eta}^{\lambda_n+\eta} \frac{d\widetilde{\sigma}(t)}{(t-\lambda_n)^2 + \eta^2} \geqslant \int_{\lambda_n-\eta}^{\lambda_n+\eta} \frac{d\widetilde{\sigma}(t)}{2\eta^2} = \frac{1}{\eta} \cdot \frac{\widetilde{\sigma}(\lambda_n+\eta) - \widetilde{\sigma}(\lambda_n-\eta)}{2\eta}.$$

Consequently, $\limsup_{\eta\to 0+}\left(\eta \cdot \rho_n^2(\eta)\right)$ is positive or $+\infty$ because λ_n is a point of increase of $\sigma(t)$. Moreover, we fix $\rho_n(\eta) > 0$ for definiteness. In terms of big O notation, there exists a sequence of positive numbers $\eta_1, \eta_2, \ldots$ tending to zero such that

$$\frac{1}{\rho_n(\eta_k)} = O\left(\sqrt{\eta_k}\right) \quad \text{as} \quad k \to +\infty \quad \text{and} \quad n = 1, \ldots, \varkappa. \tag{5.12}$$

According to (5.11), we additionally have

$$-\frac{\varphi(z_n) - \varphi(\overline{z}_n)}{z_n - \overline{z}_n} = -\frac{\varphi_n(z_n) - \varphi_n(\overline{z}_n)}{z_n - \overline{z}_n} - \sum_{m\neq n} \frac{\varphi_m(z_n) - \varphi_m(\overline{z}_n)}{z_n - \overline{z}_n} = \rho_n^2(\eta) + O(1) \tag{5.13}$$

when η is assumed to be small. Furthermore,

$$-\frac{\varphi_n(z_n) - \varphi_n(\overline{z}_m)}{z_n - \overline{z}_m} = \int_{U_n} \frac{t - \overline{z}_m - t + z_n}{(z_n - \overline{z}_m)(t - z_n)(t - \overline{z}_m)} d\widetilde{\sigma}(t) = \int_{U_n} \frac{d\widetilde{\sigma}(t)}{(t - z_n)(t - \overline{z}_m)},$$

which implies (with the help of the elementary inequality $2\alpha\beta \leqslant \frac{\alpha^2}{c} + c\beta^2$ valid for any positive numbers)

$$\begin{aligned}
\left|\frac{\varphi_n(z_n) - \varphi_n(\overline{z}_m)}{z_n - \overline{z}_m}\right| &\leqslant \int_{U_n} \frac{d\widetilde{\sigma}(t)}{|t - z_n| \cdot |t - \overline{z}_m|} \\
&\leqslant \int_{U_n} \frac{d\widetilde{\sigma}(t)}{2\rho_n(\eta)|t - z_n|^2} + \int_{U_n} \frac{\rho_n(\eta)\, d\widetilde{\sigma}(t)}{2|t - z_m|^2} \\
&= \frac{1}{2\rho_n(\eta)} \rho_n^2(\eta) + \frac{\rho_n(\eta)}{2} \int_{U_n} \frac{d\widetilde{\sigma}(t)}{|t - z_m|^2} \\
&\leqslant C(n, m, \delta)\rho_n(\eta),
\end{aligned} \tag{5.14}$$

because the distance between U_n and z_m is more than δ. The factor $C(n, m, \delta) > 0$ in (5.14) is independent of η. The finiteness of (5.11) gives

$$\frac{\varphi(z_n) - \varphi(\overline{z}_m)}{z_n - \overline{z}_m} = \frac{\varphi_n(z_n) - \varphi_n(\overline{z}_m)}{z_n - \overline{z}_m} + \frac{\varphi_m(z_n) - \varphi_m(\overline{z}_m)}{z_n - \overline{z}_m} + O(1) \quad \text{as} \quad \eta \to 0+.$$

This can be combined with the estimate (5.14), thus giving us for small η

$$\left|\frac{\varphi(z_n) - \varphi(\overline{z}_m)}{(z_n - \overline{z}_m)\rho_n(\eta)\rho_m(\eta)}\right| \leqslant \frac{C(n, m, \delta)}{\rho_m(\eta)} + \frac{C(m, n, \delta)}{\rho_n(\eta)} + O\left(\frac{1}{\rho_n(\eta)\rho_m(\eta)}\right). \tag{5.15}$$

The relations (5.13) and (5.15) allow us to make the final step in the proof. The substitution $\xi_n \mapsto \zeta_n/\rho_n(\eta)$ gives us

$$\begin{aligned} h_\varphi\left(\frac{\zeta_1}{\rho_1(\eta)},\dots,\frac{\zeta_\varkappa}{\rho_\varkappa(\eta)}\Big|z_1,\dots,z_\varkappa\right) &= \sum_{n,m=1}^{\varkappa}\frac{\varphi(z_n)-\varphi(\overline{z}_m)}{z_n-\overline{z}_m}\cdot\frac{\zeta_n\overline{\zeta}_m}{\rho_n(\eta)\rho_m(\eta)} \\ &= R(\eta)-\sum_{n=1}^{\varkappa}|\zeta_n|^2, \quad \text{where} \end{aligned} \tag{5.16}$$

$$|R(\eta)| = \sum_{n=1}^{\varkappa}\left(|\zeta_n|^2 O\Big(\frac{1}{\rho_n^2(\eta)}\Big)+\sum_{m\neq n}\zeta_n\overline{\zeta}_m O\Big(\frac{1}{\rho_m(\eta)}+\frac{1}{\rho_n(\eta)}\Big)\right). \tag{5.17}$$

According to (5.12), in each neighbourhood of zero we can choose $\eta \in \{\eta_k\}_{k=1}^{\infty}$, such that the inequality $\frac{1}{\rho_n(\eta)} \leqslant M_n\sqrt{\eta}$ holds true for a fixed number $M_n > 0$ dependent only on φ and n. For such choice of η, the estimate (5.17) implies $|R(\eta)| \leqslant M\sqrt{\eta}\cdot\sum_{n=1}^{\varkappa}|\zeta_n|^2$ with some fixed M. Therefore, the sign of the Hermitian form (5.16) will be determined by the last term $-\sum_{n=1}^{\varkappa}|\zeta_n|^2$ alone for every set of complex numbers $\{\zeta_1,\dots,\zeta_\varkappa\}$ as soon as $\eta \in \{\eta_k\}_{k=1}^{\infty}$ is small enough. □

Lemma 5.10. *Under the conditions of Lemma 5.9 assume that $\Phi(z)$ is regular in the interval $(-\varepsilon,0)$ and $0 < \Phi(0-) \leqslant \infty$. Then the Hermitian form $h_\varphi(\xi_0,\dots,\xi_\varkappa|z_0,\dots,z_\varkappa)$, where $z_m = \lambda_m + i\eta$ for $m = 1,\dots,\varkappa$ and $z_0 = -\sqrt{\eta}+i\mu$, is negative definite when the numbers $\eta > 0$ and $\mu > 0$ are chosen appropriately.*

Proof. Split the function $\varphi(z)$ into two parts $\psi_0(z)$ and $\psi_1(z)$ such that $\varphi(z) = \psi_0(z) + \psi_1(z)$ and

$$\psi_1(z) := b + \frac{1}{z}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)}\left(\frac{1}{t-z}-\frac{t}{1+t^2}-\frac{1}{t+t^3}\right)d\sigma(t).$$

The integral here is analytic for $|z| < \varepsilon$ and vanishes at the origin (see (5.9b)). The function $\psi_1(z)$ therefore is also analytic for $|z| < \varepsilon$. The part $\psi_0(z)$ has the form

$$\begin{aligned} \psi_0(z) &:= \frac{a}{z}+\frac{1}{z}\int_{(-\varepsilon,\varepsilon)}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)d\sigma(t)+\frac{1}{z}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)}\frac{d\sigma(t)}{t+t^3} \\ &= \frac{A}{z}+\frac{1}{z}\int_0^{\varepsilon}\frac{d\sigma(t)}{t-z}, \end{aligned}$$

where we put

$$A := a + \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)}\frac{d\sigma(t)}{t+t^3}-\int_0^{\varepsilon}\frac{t\,d\sigma(t)}{1+t^2}, \quad \textit{i.e. } A \text{ is a finite real constant.}$$

The integral over $(-\varepsilon,0)$ is zero in the representation of ψ_0, because the function $\Phi(z)$ is regular in this interval, and thus $\sigma(t)$ is constant for $-\varepsilon < t < 0$.

First assume that x varies on $(-\varepsilon, 0)$ close enough to 0, so that $\Phi(x) > 3M > 0$. On the one hand, one of the Cauchy-Riemann equations and the condition $\Phi'(x) \geqslant 0$ (see Remark 5.6) imply

$$\frac{\partial \operatorname{Im} \varphi(x+iy)}{\partial y}\bigg|_{y=0} = \frac{d}{dx}\varphi(x) = \frac{d}{dx}\frac{\Phi(x)}{x} = \frac{\Phi'(x)x - \Phi(x)}{x^2} < -\frac{3M}{x^2},$$

Given x we can chose $\mu \in (0, -x)$ such that

$$\left|\frac{\partial \operatorname{Im} \varphi(x+iy)}{\partial y}\bigg|_{y=0} - \frac{\varphi(x+i\mu) - \varphi(x-i\mu)}{2i\mu}\right| \leqslant \frac{M}{x^2} \tag{5.18}$$

relying on the fact that $\varphi(x+iy)$ is smooth for real y. The last two inequalities together imply that

$$\frac{\varphi(z_0) - \varphi(\overline{z}_0)}{z_0 - \overline{z}_0} = \frac{\varphi(x+i\mu) - \varphi(x-i\mu)}{(x+i\mu) - (x-i\mu)} < -\frac{3M}{x^2} + \frac{M}{x^2} = -\frac{2M}{x^2}$$

for $z_0 = x + i\mu$. Therefore,

$$\begin{aligned} \rho_0^2(x^2) &:= -\frac{\psi_0(z_0) - \psi_0(\overline{z}_0)}{z_0 - \overline{z}_0} \\ &= -\frac{\varphi(z_0) - \varphi(\overline{z}_0)}{z_0 - \overline{z}_0} + \frac{\psi_1(z_0) - \psi_1(\overline{z}_0)}{z_0 - \overline{z}_0} \geqslant \frac{M}{x^2} \end{aligned} \tag{5.19}$$

for small enough $|x|$ on account of the smoothness of $\psi_1(z)$. We assume $\rho_0(x^2) > 0$ for definiteness.

On the other hand, the definition of $\psi_0(z)$ implies that

$$\begin{aligned} &\frac{\psi_0(z_0) - \psi_0(\overline{z}_m)}{z_0 - \overline{z}_m} \\ &\quad = \frac{A/z_0 - A/\overline{z}_m}{z_0 - \overline{z}_m} + \frac{1}{z_0\overline{z}_m(z_0 - \overline{z}_m)}\int_0^\varepsilon \left(\frac{\overline{z}_m}{t - z_0} - \frac{z_0}{t - \overline{z}_m}\right) d\sigma(t) \\ &\quad = \frac{1}{z_0\overline{z}_m}\left(-A + \int_0^\varepsilon \frac{\overline{z}_m(t - \overline{z}_m) - z_0(t - z_0)}{(t - \overline{z}_m)(t - z_0)(z_0 - \overline{z}_m)} d\sigma(t)\right) \\ &\quad = -\frac{1}{z_0\overline{z}_m}\left(A + \int_0^\varepsilon \frac{t - z_0 - \overline{z}_m}{(t - \overline{z}_m)(t - z_0)} d\sigma(t)\right), \quad m = 0, \dots, \varkappa. \end{aligned} \tag{5.20}$$

The inequalities $\left|\frac{-z_0}{t - z_0}\right| \leqslant 1$ and, hence, $\left|\frac{-z_0}{(t-\overline{z}_m)(t-z_0)}\right| \leqslant \frac{1}{|z_m|}$ are valid for all $t \geqslant 0$. Let us apply the latter to estimating the absolute value of the expression (5.20):

$$\begin{aligned} \left|\frac{\psi_0(z_0) - \psi_0(\overline{z}_m)}{z_0 - \overline{z}_m}\right| &\leqslant \frac{1}{|z_0 z_m|}\left(|A| + \int_0^\varepsilon \left|\frac{-z_0}{(t - \overline{z}_m)(t - z_0)} + \frac{1}{(t - z_0)}\right| d\sigma(t)\right) \\ &\leqslant \frac{|A|}{|z_0 z_m|} + \frac{1}{|z_0 z_m^2|}\int_0^\varepsilon d\sigma(t) + \frac{1}{|z_0 z_m|}\int_0^\varepsilon \frac{d\sigma(t)}{|t - z_0|}. \end{aligned} \tag{5.21}$$

In particular, putting $m = 0$ in (5.21) gives us

$$\begin{aligned}\rho_0^2(x^2) &= \left|\frac{\psi_0(z_0) - \psi_0(\overline{z}_0)}{z_0 - \overline{z}_0}\right| \leqslant \frac{2}{|z_0^3|}\int_0^\varepsilon d\sigma(t) + \frac{|A|}{|z_0^2|} \\ &\leqslant 2\frac{\sigma(\varepsilon-) - \sigma(0)}{|x|^3} + \frac{|A|}{x^2},\end{aligned} \tag{5.22}$$

which complements

$$\left|\frac{\varphi(z_0) - \varphi(\overline{z}_0)}{z_0 - \overline{z}_0} + \rho_0^2(x^2)\right| = \left|\frac{\psi_1(z_0) - \psi_1(\overline{z}_0)}{z_0 - \overline{z}_0}\right| = O\,(1) \quad \text{as} \quad x \to 0, \tag{5.23}$$

where $O\,(1)$ on the right-hand side does not depend on $\mu \in (0, \varepsilon)$. Now recall that $\operatorname{Re} z_\varkappa = \lambda_\varkappa < \cdots < \operatorname{Re} z_1 = \lambda_1 < -\varepsilon$. Since $|t - z_0| = t - x + \mu < t - 2x$ provided that $t \geqslant 0$ and $\mu < |x|$, from (5.21) we obtain

$$\begin{aligned}&\left|\frac{\psi_0(z_0) - \psi_0(\overline{z}_m)}{z_0 - \overline{z}_m}\right| \\ &\quad\leqslant \frac{1}{|z_0 z_m^2|}\int_0^\varepsilon d\sigma(t) + \frac{|A|}{|z_0 z_m|} + \frac{1}{|z_0 z_m|}\int_0^\varepsilon \frac{|t - z_0|}{|t - z_0|^2} d\sigma(t) \\ &\quad\leqslant \frac{1}{|z_0 z_m^2|}\int_0^\varepsilon d\sigma(t) + \frac{2|A|}{|z_0 z_m|} + \frac{|z_0|}{|z_m|}\left(\frac{A}{|z_0|^2} + \frac{1}{|z_0|^2}\int_0^\varepsilon \frac{t - 2x}{|t - z_0|^2} d\sigma(t)\right) \\ &\quad\overset{(5.20)}{=\!=} \frac{1}{|z_0 z_m^2|}\int_0^\varepsilon d\sigma(t) + \frac{2|A|}{|z_0 z_m|} + \frac{|z_0|}{|z_m|}\left(-\frac{\psi_0(z_0) - \psi_0(\overline{z}_0)}{z_0 - \overline{z}_0}\right) \\ &\quad\leqslant \frac{\sigma(\varepsilon-) - \sigma(0)}{|x|\lambda_m^2} + \frac{2|A|}{|x\lambda_m|} + \frac{2|x|}{|\lambda_m|}\rho_0^2(x^2) = O\left(\frac{1}{x}\right) + O\left(x\rho_0^2(x^2)\right),\end{aligned} \tag{5.24}$$

where x tends to zero and $m = 1, \ldots, \varkappa$.

To implement the same technique as in the proof of Lemma 5.9 it is enough to put $x := -\sqrt{\eta}$ and to study the order of summands in the form $h_\varphi(\xi_0, \ldots, \xi_\varkappa | z_0, \ldots, z_\varkappa)$. In addition to $x > -\varepsilon$, we suppose that $x > -\delta$, where the positive number δ is defined in (5.8); consequently $\eta < \min\{\varepsilon^2, \delta^2\}$. We regard η as tending to zero, so the conditions (5.19) and (5.22)–(5.23) imply that

$$\frac{1}{\rho_0(\eta)} = O\,(\sqrt{\eta}), \quad \rho_0(\eta) = O\left(\eta^{-\frac{3}{4}}\right), \quad \text{and thus} \tag{5.25}$$

$$\frac{\varphi(z_0) - \varphi(\overline{z}_0)}{(z_0 - \overline{z}_0)\rho_0^2(\eta)} = -1 + O\,(\eta)\,. \tag{5.26}$$

Now we make use of the same notation as in the proof of Lemma 5.9. If $m, n \neq 0$ and $m \neq n$, then the estimates (5.13) and (5.15) concerning $\varphi(z_1), \ldots, \varphi(z_\varkappa)$ are valid. Since the distance between U_n and z_0 is more than δ, the inequality (5.14) is satisfied on

condition that $m = 0 \neq n$. Then (5.14) and (5.24) give us the following:

$$\begin{aligned}\left|\frac{\varphi(z_0)-\varphi(\overline{z}_n)}{z_0-\overline{z}_n}\right| &\leqslant \left|\frac{\varphi_n(z_0)-\varphi_n(\overline{z}_n)}{z_0-\overline{z}_n}\right| + \left|\frac{\psi_0(z_0)-\psi_0(\overline{z}_n)}{z_0-\overline{z}_n}\right| \\ &\quad + \left|\frac{(\varphi(z)-\varphi_n(z_0)-\psi_0(z_0))-(\varphi(\overline{z}_n)-\varphi_n(\overline{z}_n)-\psi_0(\overline{z}_n))}{z_0-\overline{z}_n}\right| \\ &\overset{(5.14)}{\leqslant} C(n,0,\delta)\,\rho_n(\eta) + \left|\frac{\psi_0(z_0)-\psi_0(\overline{z}_n)}{z_0-\overline{z}_n}\right| + O\,(1) \\ &\overset{(5.24)}{=\!=} C(n,0,\delta)\,\rho_n(\eta) + O\Big(\eta^{-\frac{1}{2}}\Big) + O\left(\sqrt{\eta}\rho_0^2(\eta)\right) + O\,(1)\end{aligned}$$

as $\eta \to 0+$. Assume that η is taken from the sequence $\{\eta_k\}_{k=1}^\infty$ corresponding to (5.12) and that the choice of $\mu \in (0, \sqrt{\eta})$ satisfies the condition (5.18). Then

$$\begin{aligned}\left|\frac{\varphi(z_n)-\varphi(\overline{z}_0)}{(z_n-\overline{z}_0)\rho_n(\eta)\rho_0(\eta)}\right| &= \left|\frac{\varphi(z_0)-\varphi(\overline{z}_n)}{(z_0-\overline{z}_n)\rho_0(\eta)\rho_n(\eta)}\right| \\ &\overset{(5.25)}{=\!=} O\,(\sqrt{\eta}) + \left(O\Big(\eta^{-\frac{1}{2}+\frac{1}{2}}\Big) + O\Big(\eta^{\frac{1}{2}-\frac{3}{4}}\Big) + O\Big(\eta^{\frac{1}{2}}\Big)\right)\frac{1}{\rho_n(\eta)} \\ &\overset{(5.12)}{=\!=} O\,(\sqrt{\eta}) + O\Big(\eta^{-\frac{1}{4}}\Big)O\,(\sqrt{\eta}) = O\,(\sqrt[4]{\eta})\,.\end{aligned}$$

This estimate together with (5.26), (5.12), (5.13) and (5.15) yields that

$$\begin{aligned}h\left(\frac{\zeta_0}{\rho_0(\eta)},\ldots,\frac{\zeta_\varkappa}{\rho_\varkappa(\eta)}\Big|z_0,\ldots,z_\varkappa\right) &= \sum_{n,m=1}^{\varkappa}\frac{\varphi(z_n)-\varphi(\overline{z}_m)}{(z_n-\overline{z}_m)\rho_n(\eta)\rho_m(\eta)}\,\zeta_n\overline{\zeta}_m \\ &= -\sum_{m=0}^{\varkappa}|\zeta_m|^2 + O\,(\sqrt[4]{\eta})\sum_{n,m=0}^{\varkappa}\zeta_m\overline{\zeta}_n,\end{aligned}$$

where $O\left(\sqrt[4]{\eta}\right)$ does not depend on $\zeta_0,\ldots,\zeta_\varkappa$. That is, this Hermitian form is negative definite provided that the value of $\eta \in \{\eta_k\}_{k=1}^\infty$ is small enough. □

Proof of Theorem 5.7. Suppose that $\Phi(z) = z\varphi(z)$ can be represented as in (5.7). Then for specially chosen numbers $z_1,\ldots,z_\varkappa \notin \mathbb{R}$ the Hermitian form $h_\varphi(\xi_1,\ldots,\xi_\varkappa|z_1,\ldots,z_\varkappa)$ has $\varkappa$ negative squares by Lemma 5.9. Let us show that this is the greatest possible number of negative squares in the form $h[\varphi] := h_\varphi(\xi_1,\ldots,\xi_k|z_1,\ldots,z_k)$.

Denote $\widetilde{\sigma}(t) = \int_0^t s^{-1}d\sigma(s)$. Since the integral

$$0 \leqslant \int_0^\infty \frac{d\sigma(t)}{t(1+t^2)} = \int_0^\infty \frac{d\widetilde{\sigma}(t)}{1+t^2} \overset{(5.7b)}{\leqslant} -a - \sum_{i=1}^{\varkappa}\frac{\sigma_i}{\lambda_i} < \infty$$

is finite, we can split the last term of (5.7a) divided by z into two parts to obtain (*cf.* (5.10))

$$\varphi(z) = b + \frac{a}{z} + \sum_{i=1}^{\varkappa}\left(\frac{\sigma_i/\lambda_i}{\lambda_i - z} + \frac{\sigma_i/\lambda_i}{z}\right) + \int_0^\infty \frac{d\sigma(t)}{t(t-z)} + \frac{1}{z}\int_0^\infty \frac{d\sigma(t)}{t(1+t^2)},$$

that is $\varphi(z) = \widetilde{\varphi}(z) + \varphi_0(z)$, where

$$\widetilde{\varphi}(z) := \sum_{i=1}^{\varkappa} \frac{\sigma_i/\lambda_i}{\lambda_i - z} \quad \text{and} \quad \varphi_0(z) := b + \frac{\Phi(0-)}{z} + \int_0^\infty \frac{d\widetilde{\sigma}(t)}{t-z}.$$

The functions $\varphi_0(z)$ and $-\widetilde{\varphi}(z)$ have the form (5.5), *i.e.* belong to the class $\mathcal{R}$. For $\mathcal{R}$-functions and any set of numbers $\{z_1, \dots, z_k\}$ the Hermitian form (5.1) is nonnegative definite. That is, the conditions

$$\begin{aligned} h[\varphi_0] &:= h_{\varphi_0}(\xi_1, \dots, \xi_k | z_1, \dots, z_k) \geqslant 0 \quad \text{and} \\ h[\widetilde{\varphi}] &:= h_{\widetilde{\varphi}}(\xi_1, \dots, \xi_k | z_1, \dots, z_k) \leqslant 0 \end{aligned}$$

holds true. Moreover, since $\widetilde{\varphi}(z)$ is a rational function with $\varkappa$ poles, which is bounded at infinity, the rank of $h[\widetilde{\varphi}]$ can be at most $\varkappa$ (see Theorem 3.3.3 and its proof in [Akh65, pp. 105–108] or Theorem 1 in [Don74, p. 34]). Therefore, the form $h[\varphi] = h_\varphi(\xi_1, \dots, \xi_k | z_1, \dots, z_k)$ has at most $\varkappa$ negative squares as a sum of $h[\varphi_0]$ and $h[\widetilde{\varphi}]$. (This become evident after the reduction of the Hermitian form $h[\widetilde{\varphi}]$ to principal axes since $h[\varphi_0]$ is nonnegative definite irrespectively of coordinates.)

Suppose that $\Phi(z)$ can be expressed as in (5.6), and let $\varepsilon_0 > 0$ be such that $\Phi(-\varepsilon) > 0$ provided that $0 < \varepsilon < \varepsilon_0$. In particular, it implies $\max_i \lambda_i < -\varepsilon_0$ since $\Phi(\max_i \lambda_i +) < 0$. Lemma 5.10 provides a set of points $\{z_0, \dots, z_{\varkappa-1}\}$ such that the corresponding Hermitian form $h[\varphi]$ has $\varkappa$ negative squares. Let us prove that $h[\varphi]$ has at most $\varkappa$ squares negative. Consider the function

$$\begin{aligned} \varphi_\varepsilon(z) &:= \frac{\Phi(z) - \Phi(-\varepsilon)}{z+\varepsilon} + \frac{\Phi(-\varepsilon)}{z+\varepsilon} \\ &\overset{(5.6a)}{=\!=} b + \frac{\Phi(-\varepsilon)}{z+\varepsilon} + \sum_{i=1}^{\varkappa-1} \frac{\sigma_i/(\lambda_i+\varepsilon)}{\lambda_i - z} + \int_0^\infty \frac{1}{t-z} \cdot \frac{d\sigma(t)}{t+\varepsilon}. \end{aligned}$$

Denote $\Phi_\varepsilon(z) := z\varphi_\varepsilon(z)$ and $A_i := \dfrac{\sigma_i \lambda_i}{\lambda_i + \varepsilon}$, then

$$\begin{aligned} \Phi_\varepsilon(z) &= bz + \Phi(-\varepsilon) - \frac{\varepsilon\Phi(-\varepsilon)}{z+\varepsilon} + \sum_{i=1}^{\varkappa-1} \frac{A_i z/\lambda_i}{\lambda_i - z} + \int_0^\infty \frac{z}{t-z} \cdot \frac{d\sigma(t)}{t+\varepsilon} \\ &\overset{(5.9a)}{=\!=} bz + \Phi(-\varepsilon) - \frac{\varepsilon\Phi(-\varepsilon)}{z+\varepsilon} + \sum_{i=1}^{\varkappa-1} \frac{A_i}{\lambda_i - z} - \sum_{i=1}^{\varkappa-1} \frac{A_i}{\lambda_i} + \int_0^\infty \left(\frac{1}{t-z} - \frac{1}{t}\right) \frac{t\,d\sigma(t)}{t+\varepsilon} \\ &\overset{(5.9b)}{=\!=} bz + \left[\Phi(-\varepsilon) - \sum_{i=1}^{\varkappa-1} \frac{A_i}{\lambda_i} - \int_0^\infty \frac{d\sigma(t)}{(t+\varepsilon)(1+t^2)}\right] \\ &\qquad - \frac{\varepsilon\Phi(-\varepsilon)}{z+\varepsilon} + \sum_{i=1}^{\varkappa-1} \frac{A_i}{\lambda_i - z} + \int_0^\infty \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) \frac{t\,d\sigma(t)}{t+\varepsilon}, \end{aligned}$$

i.e. $\Phi_\varepsilon \in \mathcal{R}$. Moreover, $\Phi_\varepsilon(z)$ is an increasing function when $-\varepsilon < z < 0$ (see Remark 5.6) which implies

$$\Phi_\varepsilon(0-) = \lim_{z\to 0-} \int_0^\infty \frac{z}{t-z} \cdot \frac{d\sigma(t)}{t+\varepsilon} \leqslant 0,$$

since the integrand is negative. That is, the function $\Phi_\varepsilon(z)$ has the form (5.7). As it is shown above, we have $\varphi_\varepsilon \in \mathcal{N}_\varkappa^+$ for each ε between 0 and ε_0.

Given a fixed set of points $\{z_1, \dots, z_k\}$ there exists some positive number $\varepsilon_1 < \varepsilon_0$, such that for all $0 < \varepsilon < \varepsilon_1$ the form $h[\varphi_\varepsilon] := h_{\varphi_\varepsilon}(\xi_1, \dots, \xi_k | z_1, \dots, z_k)$ has at least the same number of negative squares as the form $h[\varphi]$. (Indeed: the characteristic numbers of $h[\varphi]$ depend continuously on its coefficients.) Suppose that the Hermitian form $h[\varphi]$ has more than $\varkappa$ negative squares. Then $h[\varphi_\varepsilon]$ must have more than $\varkappa$ negative squares as well, which is impossible. Thus, the form $h[\varphi]$ has at most $\varkappa$ negative squares.

Suppose that $\varphi \in \mathcal{N}_\varkappa^+$. Then the function $\Phi(z) = z\varphi(z)$ can be represented as in (5.5) and the form $h[\varphi]$ for any set of numbers $\{z_1, \dots, z_k\}$ has at most $\varkappa$ negative squares (as stated in the definition of $\mathcal{N}_\varkappa^+$). By Lemma 5.9, the function $\sigma(t)$ appearing in (5.5) can have at most $\varkappa$ negative points of increase. These points are isolated, and therefore (see Definition 5.8) for negative t the function $\sigma(t)$ is a step function with at most $\varkappa$ steps. That is, all negative singular points of $\Phi(z)$ are simple poles; they have negative residues since $\Phi \in \mathcal{R}$, *i.e.* $\sigma_i > 0$ for all i. Here we have two mutually exclusive options: $\Phi(0-) \leqslant 0$, then $\Phi(z)$ has the form (5.7) corresponding to some $\varkappa_0 \leqslant \varkappa$, and $0 < \Phi(0-) \leqslant \infty$, *i.e.* $\Phi(z)$ has the form (5.6) corresponding to $\varkappa_0 \leqslant \varkappa+1$. The sufficiency (first) part of the current proof shows that $\varphi \in \mathcal{N}_{\varkappa_0}^+$ in both cases. Since the classes $\mathcal{N}_{\varkappa_0}^+$ and $\mathcal{N}_\varkappa^+$ are disjoint by definition, we necessarily have $\varkappa_0 = \varkappa$. □

5.6 Proofs of Lemma 5.2 and Theorem 5.4

Proof of Lemma 5.2. First suppose that, for a non-constant meromorphic function $R(z)$, the expression $F(z) = z(\ln R(z))'$ is an $\mathcal{R}$-function and prove that this condition is sufficient for the inclusion $R \in \mathcal{PF}$ up to a constant multiplier. Being a fraction of two meromorphic functions $F(z)$ is meromorphic itself. As a meromorphic $\mathcal{R}$-function, it has the representation (see Theorem 3.16):

$$F(z) = B_0 - \frac{A_0}{z} + Az - \sum_{\nu>0}\left(\frac{A_\nu}{z+a_\nu} - \frac{A_\nu}{a_\nu}\right) - \sum_{\mu>0}\left(\frac{B_\mu}{z-b_\mu} + \frac{B_\mu}{b_\mu}\right), \tag{5.27}$$

where $(-a_\nu)_{\nu>0}$ and $(b_\mu)_{\mu>0}$ denote negative and positive poles of $F(z)$ respectively, $A_\nu > 0$ and $B_\mu > 0$ for all $\mu, \nu \geqslant 1$, $B_0 \in \mathbb{R}$, $A, A_0 \geqslant 0$ and $\sum_\nu A_\nu a_\nu^{-2} + B_\mu b_\mu^{-2} < +\infty$.

Since

$$F(z) = B_0 + Az - \frac{A_0}{z} - \sum_{\nu>0}\left(\frac{A_\nu}{z+a_\nu} - \frac{\frac{A_\nu z}{a_\nu} + A_\nu}{z+a_\nu}\right) - \sum_{\mu>0}\left(\frac{B_\mu}{z-b_\mu} + \frac{\frac{B_\mu z}{b_\mu} - B_\mu}{z-b_\mu}\right)$$
$$= B_0 + Az - \frac{A_0}{z} + \sum_{\nu>0}\frac{A_\nu z/a_\nu}{z+a_\nu} - \sum_{\mu>0}\frac{B_\mu z/b_\mu}{z-b_\mu},$$

we have

$$\frac{R'(z)}{R(z)} = \frac{F(z)}{z} = A + \frac{B_0}{z} + \frac{A_0}{z^2} + \sum_{\nu>0}\frac{A_\nu/a_\nu}{z+a_\nu} - \sum_{\mu>0}\frac{B_\mu/b_\mu}{z-b_\mu}. \tag{5.28}$$

On the left-hand side of this equality stands the logarithmic derivative of a meromorphic function, so $A_0 = 0$ and the residues $k_\nu := A_\nu/a_\nu$ and $l_\mu := B_\mu/b_\mu$ can be only integers. By definition, all residues are negative and satisfy

$$\sum_{\nu>0} k_\nu/a_\nu + \sum_{\mu>0} l_\mu/b_\mu = \sum_{\nu>0} k_\nu a_\nu a_\nu^{-2} + \sum_{\mu>0} l_\mu b_\mu b_\mu^{-2} < +\infty. \tag{5.29}$$

Moreover, since R is regular at the origin, $B_0 \in \mathbb{Z}_{\geqslant 0}$ and

$$\frac{R'(z)}{R(z)} = \frac{F(z)}{z} = A + \frac{p}{z} + \sum_{\nu>0}\frac{k_\nu}{z+a_\nu} - \sum_{\mu>0}\frac{l_\mu}{z-b_\mu}, \tag{5.30}$$

where we put $p := B_0$. Since

$$\frac{(R(z)z^{-p})'}{R(z)z^{-p}} = \frac{R'(z)z^{-p} - pR(z)z^{-p-1}}{R(z)z^{-p}} = \frac{R'(z)}{R(z)} - \frac{p}{z},$$

on supposing the integration contour to be entirely in the upper or in the lower half of the complex plane for non-real z we obtain

$$\frac{R(z)}{z^p} = \exp\ln\frac{R(z)}{z^p} = \exp\left(\int_0^z\left(\frac{F(z)}{z} - \frac{p}{z}\right)dz - \ln C\right).$$

Substituting (5.30) into the right-hand side yields

$$R(z) = Cz^p e^{Az}\cdot\frac{\prod_{\nu>0}\left(1+\frac{z}{a_\nu}\right)^{k_\nu}}{\prod_{\mu>0}\left(1-\frac{z}{b_\mu}\right)^{l_\mu}}, \tag{5.31}$$

and hence $\frac{1}{C}R \in \mathcal{PF}$.

To prove the necessity let $R(z)$ have the form (5.31) and the condition (5.29) holds. Therefore, the logarithmic derivative of $R(z)$ is represented by (5.30), that is by (5.28)

with the notation $A_\nu := k_\nu a_\nu$ and $B_\mu := l_\mu b_\mu$. Consequently, $F(z)$ can be expressed as in (5.27), which implies $F \in \mathcal{R}$. □

Proof of Corollary 5.3. To derive this fact from Lemma 5.2 it is enough to note that the condition $R \in \widetilde{\mathcal{PF}}$ is equivalent to the equality

$$R(z) = R_1(z)\, R_2\left(\frac{1}{z}\right),$$

for some $R_1(z)$ and $R_2(z)$ of the form (5.2). As a consequence,

$$\frac{z(R_1(z)\, R_2(1/z))'}{R_1(z)\, R_2(1/z)} = \frac{zR_1'(z)}{R_1(z)} - \frac{R_2'(1/z)}{zR_2(1/z)} = F_1(z) - F_2\left(\frac{1}{z}\right). \tag{5.32}$$

Since $-F_2\left(\frac{1}{z}\right)$ and $F_2(z)$ are $\mathcal{R}$-functions simultaneously, the corollary is proved. □

Proof of Theorem 5.4. For a function $R(z)$ represented by (5.3) we have that

$$zG(z) = p + Az - \frac{A_0}{z} - \sum_{\nu>0} \frac{-zk_\nu}{z + a_\nu} - \sum_{\nu>0} \frac{\widetilde{k}_\nu}{zc_\nu + 1} - \sum_{\mu>0} \frac{zl_\mu}{z - b_\mu} - \sum_{\mu>0} \frac{\widetilde{l}_\mu}{zd_\mu - 1}. \tag{5.33}$$

If $R(z)$ has $\varkappa_0 < +\infty$ negative zeros, then the fourth and fifth terms on the right-hand side together have exactly $\varkappa_0$ summands. At that, each of the summands provides a simple negative pole with negative residue to the function $G(z)$, and furthermore $G(z)$ has no other negative poles. Only the third and last terms on the right-hand side of (5.33) are responsible for the essential singularity of $R(z)$ which can occur at the origin: other terms are bounded.

If there is an infinite number of summands the last term, then for an arbitrary positive integer n

$$T(z) := -\frac{A_0}{z} - \sum_{\mu>0} \frac{\widetilde{l}_\mu}{zd_\mu - 1} > \sum_{\mu=1}^{n} \frac{1}{(-z)d_\mu + 1}$$

provided that $z < 0$. Then, assuming $d_1 < d_2 < d_3 < \cdots$ without loss of generality,

$$T\left(-\frac{1}{d_n}\right) > \sum_{\mu=1}^{n} \frac{d_n}{d_\mu + d_n} > \sum_{\mu=1}^{n} \frac{d_n}{2d_n} = \frac{n}{2} \xrightarrow{n\to+\infty} +\infty. \tag{5.34}$$

Analogously, the condition $A_0 > 0$ implies

$$T\left(-\frac{1}{d_n}\right) > A_0 d_n \xrightarrow{n\to+\infty} +\infty. \tag{5.35}$$

On accounting that $zG(z) - T(z)$ is bounded at the origin, and that $T(z)$ is an increasing function, we conclude $\lim_{z\to 0-} zG(z) = \lim_{z\to 0-} T(z) = +\infty$. Therefore, if $R(z)$ has an essential singularity at the origin, then Theorem 5.7 yields that $G \in \mathcal{N}^+_{\varkappa_0+1}$.

If the last sum on the right-hand side of (5.33) contains at most a finite number of summands and if $A_0 = 0$, then

$$j := \lim_{z\to 0} zG(z) = p - \sum_{\nu>0} \widetilde{k}_\nu + \sum_{\mu>0} \widetilde{l}_\mu$$

is a finite integer. Moreover, $j > 0$ means $R(0) = 0$, while $j < 0$ means $R(0) = \infty$. Theorem 5.7 therefore gives $G \in \mathcal{N}^+_{\varkappa_0+1}$ in the former case, and $G \in \mathcal{N}^+_{\varkappa_0}$ in the case $j \leqslant 0$.

Conversely, let $G \in \mathcal{N}^+_\varkappa$. Then $zG(z)$ has one of the forms (5.6) and (5.7) by Theorem 5.7, and therefore it has at most $\varkappa$ negative poles. Then $zG(z)$ can be represented as in (5.33), where the fourth and fifth terms on the right-hand side together have at most $\varkappa < \infty$ summands. Accordingly, the function $zG(z) - T(z)$ is smooth and thus bounded for z close enough to 0. So if $zG(z)$ has a pole or an essential singularity at the origin, then $\lim_{z\to 0-} zG(z) = +\infty$ due to the estimates (5.34) and (5.35). If in opposite $zG(z)$ is regular at the origin, this limit is equal to the residue of $G(z) = (\ln R(z))'$ at the origin. That is, $0 < \lim_{z\to 0-} zG(z) < \infty$ is true exactly when $R(0) = 0$, the inequality $\lim_{z\to 0-} zG(z) < 0$ is equivalent to that $R(z)$ has a pole at $z = 0$, and the condition $\lim_{z\to 0-} zG(z) = 0$ denotes that $R(z)$ is regular and nonzero at the origin.

Summarizing, we obtain that $zG(z)$ can be expressed as in (5.6) when $R(z)$ has a zero or an essential singularity at the origin. Otherwise, $zG(z)$ has the form (5.7), which means that $R(z)$ is regular and nonzero or has a pole at $z = 0$. This statement implies the first part of the theorem.

Now, the function $R(z)$ has λ distinct poles if and only if $R_0(z) := \frac{1}{R(-z)}$ has λ distinct zeros (counting the possible essential singularities at 0). Applying the first part of the present theorem to $R_0 \in \widetilde{\mathcal{PF}}$ yields that the above is equivalent to that the function $G_0(z) := (\ln R_0(z))'$ belongs to the class $\mathcal{N}^+_\lambda$. The identity

$$G_0(z) = \left(\ln \frac{1}{R(-z)}\right)' = \left(\frac{1}{R(-z)}\right)' R(-z) = -\frac{(R(-z))'}{R(-z)} = \frac{R'(-z)}{R(-z)} = G(-z)$$

concludes the proof of the Theorem. □

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