Some relations between stochastic and deterministic differential equations

vorgelegt von
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von der Fakultät II - Mathematik und Naturwissenschaften
der Technischen Universität Berlin
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften
Dr.rer.nat.

genehmigte Dissertation

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Berlin 2016
to my family
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Introduction

Extensive research on stochastic differential equation has been done ever since the foundation was laid by Kiyoshi Itô in the mid-forties. Even though our interest in these equations is purely theoretical, stochastic differential equations are still of paramount importance as they are used to model the growth of a population, share prices at the stock market, interacting particles etc. We investigate some relations between stochastic differential equations and their deterministic counterparts.

A central object in this thesis is the connection between a stochastic differential equation

\[ dX_t = b(X_t)dt + \sigma dW_t, \quad X_0 = x \]  

(SDE)

and a (deterministic) ordinary differential equation

\[ \partial_t X_t = b(X_t), \quad X_0 = x. \]  

(ODE)

For both, (SDE) and (ODE), there exist conditions under which uniqueness and existence for global solutions is known to hold true. A very classical one is that the drift $b$ is globally Lipschitz continuous. However, dropping this assumption leads to interesting questions. For instance, if uniqueness fails to hold in (ODE), can the addition of noise in (SDE), i.e. $\sigma > 0$, restore it? In fact, this can happen. A lot of research has been done in this direction, see for example [Fla11] and references therein. In contrast, we focus on the existence of global solutions from now on. More precisely, assume that the drift $b$ is designed in such a way that uniqueness is ensured but explosion in finite time occurs for the ODE, i.e.

\[ \epsilon(x) := \lim_{N \to \infty} \inf \{t \geq 0: |X_t| \geq N\} < \infty, \]

which means that the solution only exists locally up to time $\epsilon(x)$. Can the noise in (SDE) guarantee the existence of a (global) solution, for which $\epsilon(x) = \infty$ almost surely?
Pioneer work was done by Scheutzow in the nineties. In [Sch95] he shows that in the one-dimensional case the noise cannot turn an explosive ODE into a non-explosive SDE. This is achieved by proving an integral inequality. In [Sch93] he provides an ODE in $\mathbb{R}^2$, which explodes for every initial condition $x \in \mathbb{R}^2$, but the corresponding SDE, with any $\sigma > 0$, is non-explosive almost surely for every initial condition. Furthermore, the Markov process associated to this SDE admits an invariant probability measure. Hence, the noise “stabilizes” the equation.

To complete the picture let us mention, that [Sch93] provides a second example with the opposite feature, i.e. there exists an ODE which does not explode in finite time, but the corresponding SDE does for every initial condition with probability one. Therefore, the noise can also destabilize.

Coming back to stabilization by noise, Scheutzow proved the existence of an invariant measure by specifying and validating a Lyapunov function, i.e. a function $V \in C^2(\mathbb{R}^2; [0, \infty))$, such that

$$\inf_{|x| > R} V(x) \to \infty \quad \text{and} \quad \sup_{|x| > R} \mathcal{L}V(x) \to -\infty \quad \text{as} \quad R \to \infty,$$

where $\mathcal{L}$ is the generator associated to the SDE and is given by

$$\mathcal{L}f(x) = b(x) \cdot \nabla f(x) + \frac{\sigma^2}{2} \Delta f(x)$$

for every $f$ belonging to a suitable class of functions. The existence of an invariant probability measure follows from [Kha11, Theorem 3.5 and 3.7].

In recent works [BHW12], [AKM12], [HM15a] and [HM15b] other authors investigate different equations which exhibit stabilization by noise or noise-induced stabilization as they call it. Their proof also uses Lyapunov functions but is more constructive than [Sch93], which essentially relies on intuition.

We point out the latest work of D. Herzog and J. Mattingly on this topic, [HM15a], [HM15b]. They deal with a large family of SDEs in the complex plane of the form

$$dZ_t = (Z_t^n + F(Z_t)) \, dt + \sigma dW_t, \quad Z_0 = z, \quad (1)$$

where $n \geq 2$, $F$ is a polynomial in $z, \bar{z}$ of degree less or equal to $n - 1$, $\sigma \geq 0$ and $W$ a complex Brownian motion. They established the existence of an invariant probability measure by constructing a generalized Lyapunov function if $\sigma > 0$.

A question raised by M. Scheutzow is whether explosive ODEs can be stabilized in such a way that there exists a random attractor. We call such a phenomenon noise-induced strong stabilization. Indeed, the existence of a
random attractor implies the existence of an invariant probability measure, hence it is a stronger kind of stabilization. In Chapter 2 we see that noise-induced strong stabilization is strictly stronger than noise-induced stabilization. We even show explosion in finite time for the local stochastic flow generated by equation (1). This may appear surprising, since in [HM15a, HM15b] the existence of an invariant measure was shown (for \( \sigma > 0 \)) and in particular

\[
\forall z \in \mathbb{C}: \ P(e(z) = \infty) = 1.
\]

We prove in Theorem 2.3.1 that this does not hold uniformly in the initial condition, i.e.

\[
\mathbb{P} (\forall z \in \mathbb{C}: e(z) = \infty) = 0.
\]

Continuity of the solution with respect to the initial condition is essential for the proof, which is purely deterministic. It also reveals that the result holds true, when the noise \( W \) is replaced by any process with continuous paths, see Remark 2.5.4.

Chapter 3 is devoted to find an example for noise-induced strong stabilization. The SDE at hand given on \([0, \infty) \times [0, \pi]\)

\[
\begin{align*}
\text{dr}_t &= \left(-r_t^w \cos^2(\phi_t) + r_t^v\right) dt, \\
\text{d}\phi_t &= -r_t^\gamma \cos^2(\phi_t) dt + \sigma dW_t
\end{align*}
\]

depends on the parameters \( w, v, \gamma > 0 \). We give a parameter regime, for which the deterministic equation (2), i.e. \( \sigma = 0 \), explodes in finite time for every initial condition of the form \( r_0 > 0, \phi_0 \in [0, \pi] \), see Proposition 3.2.1.

In case \( \sigma > 0 \), we manage to prove the existence of a global weak set attractor, see Theorem 3.3.1 for a specific parameter regime

\[
w > v > 1, \quad v - \frac{2}{3} \gamma < 1, \quad w - \gamma > 1,
\]

which overlaps with the one, which yields explosion in the deterministic case. The proof relies mostly on three things, an exit time estimate, a thorough analysis of the behavior of trajectories away from the origin and a Markov-like argument. We conclude Chapter 3 with some conjectures and open questions concerning other parameter regimes. Up to now, we are not aware of another example of noise-induced strong stabilization.

This family of equations was developed together with Jonathan Mattingly at Duke University and the joint research greatly contributed to improve the content of Section 3.3.1 and Appendix A. The conjectures at the end stem...
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from heuristic ideas presented by J. Mattingly, which we could not make rigorous up to now.

An interesting question posed to us by M. Röckner is what happens in the gradient case. More precisely, can we construct a potential \( V : \mathbb{R}^2 \to \mathbb{R} \), such that the corresponding ODE with \( b = -\nabla V \) is strongly stabilized by noise? We treat this question in Chapter 4 but, unfortunately, we can only show noise-induced stabilization. Even though we are convinced that the example we construct also allows for strong stabilization, we are not able to prove this rigorously. The integrability condition

\[
\int_{\mathbb{R}^2} e^{-\frac{1}{\sigma^2}V(x)} \, dx < \infty
\]

is a strong indicator for the existence of an invariant measure. Indeed, we manage to construct a potential, which admits explosion in finite time if \( \sigma = 0 \) and satisfies this integrability condition for \( \sigma > 0 \). By [BKRS15, Theorem 5.3.9] we obtain a Lyapunov function and therefore existence of an invariant probability measure. However, this is not very helpful in showing the existence of an attractor. We try to conduct a chaining argument, but it fails due to strong expansion of balls in a small area of the state space. We conclude by elucidating the expansion problem and briefly present an alternative potential with similar features.

A second major phenomenon we investigate in this thesis are macroscopic limit results for interacting particle systems. A system of particles consists initially of \( N \in \mathbb{N} \) particles. A particle “\( a \)” proliferates (splits into two particles) at rate \( \lambda^{a,N} \) and moves in \( \mathbb{R}^d \) according to

\[
\begin{align*}
\text{d}X_t^{a,N} = -\frac{1}{N} \sum_a \nabla V \left( X_t^{a,N} - X_t^{\tilde{a},N} \right) \, dt + \sigma \text{d}\mathbf{B}^a_t,
\end{align*}
\]

for a detailed description see Section 1.4. The particles can interact in two different ways. First, in the spatial component through the gradient \( \nabla V \) in equation (3) and second, through the branching, since the rate \( \lambda^{a,N} \) is of the form

\[
\lambda_t^{a,N} = F_N \left( S_t^N, X_t^{a,N} \right),
\]

where \( S^N \) is the empirical measure. The functionals \( F_N \) we have in mind, typically mollify the empirical measure, i.e. \( F_N(\mu, x) = F(\theta_N * \mu, x) \), where \( \theta_N(x) = N^\beta \theta(N^\beta/dx) \) for a “nice” function \( \theta \) and \( \beta \in [0, 1] \). The number \( \beta \in [0, 1] \) is often called the range of interaction. The case \( \beta = 0 \) corresponds to long range interaction, i.e. the rate \( \lambda^{a,N} \) feels the presence of all particles.
within a radius that does not shrink as $N \to \infty$. $\beta = 1$ is short range interaction, where only particles in a neighborhood of radius $N^{-1/d}$ matter. The intermediate case $\beta \in (0, 1)$ is also called moderate interaction. Note that one can also alter the range of interaction in equation (3) by changing $V$ to $V_N(x) = N^{d/2}V(x)$, but in the present thesis we only deal with the case of long range interaction in the spatial component, see Chapter 5. A macroscopic limit result is concerned with the question of convergence of the empirical measure. In the cases we are interested in, the empirical measure converges, as $N \to \infty$, to a partial differential equation of the form

$$\partial_t u_t = \frac{\sigma^2}{2} \Delta u_t + \text{div} \left( u_t (\nabla V * u_t) \right) + F(u_t)u_t.$$

Since the limiting equation is deterministic, our limit results can be seen as a law of large numbers for the empirical measure. It is referred to as a macroscopic model or macroscopic limit, because it models the behavior of all particles, or rather their density, at once.

There is a vast literature on macroscopic limit results, but the ones we were particularly inspired by and relate to Chapter 5 and 6 are [Oel85, Oel89, Ste00, Phi07]. The first deals with the case of intermediate range, precisely $\beta \in (0, \frac{d}{d-2})$, in the spatial component but features no branching mechanism. The non-local term $\text{div} \left( u_t (\nabla V * u_t) \right)$ changes in the corresponding PDE to $\text{div} \left( u_t \nabla u_t \right) = \Delta (u_t^2)$. The model in [Oel89] incorporates birth, death and migration among different types of particles. The interaction range in the branching mechanisms and the spatial component are both intermediate, but rely on stricter assumptions. [Ste00] is very much written in the spirit of [Oel89], but applied to derive a different equation, i.e. the chemo taxis equation.

In all of these three papers a standard approach is used. First, one shows tightness of the family of empirical measures or of a mollified version of it. Second, one proves that all limits along subsequences fulfill the macroscopic limit equation. Finally, show uniqueness of the limiting equation to obtain convergence not just along subsequences. This is also the procedure we follow in Chapter 5 and 6. The last mentioned paper, [Phi07], is different as it uses McKean-Vlasov equations to handle in-between steps. These are obtained by taking the limits $N$-to-infinity, interaction range $\beta$ in the spatial component to zero and noise coefficient $\sigma$ to zero one at a time. He provides quantitative estimates for each of these convergence results to be able to take all the limits simultaneously. Due to the lack of branching mechanisms, the limiting PDE is the porous medium equation, $\partial_t u = \Delta u^2$.

For completeness, we also mention [Var91] and [Uch00], which treat the case of short range interaction in the spatial component without proliferation.
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The nature of this problem as well as the methods they use are very different. Further, the PDE they obtain is $\partial_t u = \Delta P(u)$, where $P$ is called the pressure and is only known implicitly.

Motivated by mathematical oncology we investigate a particle system with long range interaction in the spatial component, see equation (3), with proliferation in Chapter 5. We prove convergence of the mollified empirical measure towards the solution of (PDE) under certain assumptions relating $F_N$ and $F$. Tightness of the mollified empirical measure is shown via energy estimates. Another distinguishing feature of our approach is the usage of tools from stochastic partial differential equations, which was F. Flandoli’s idea.

At last, in Chapter 6, we derive the Fisher-Kolmogorov-Petrowskii-Piskunov (FKPP) equation, i.e. $\partial_t u = \Delta u + u(1 - u)$, from a system of proliferating Brownian particles. Although this case has already been treated, for example in Chapter 5 for the one-dimensional case or in [Oel89], we present an approach based on semigroup theory, which is new in the context of particle systems of this type. The idea is due to C. Olivera. Other novelties of our work include the uniform convergence of the mollified empirical measure in the space variable and the weaker assumption on the interaction range $\beta < 1/2$. A more detailed explanation about differences compared to [Oel89] can be found in the Subsection 6.1.1.
Acknowledgment

My sincere gratitude goes to Prof. Michael Scheutzow, not just for introducing me to the topic or for spending countless hours of very helpful discussions but for his honesty, patience and continuous support throughout the years of my PhD and even before.

Further, I thank Prof. Peter Imkeller for refereeing this thesis and Prof. Jörg Liesen for accepting to be the chairman of my defense.

I also want to thank Franco Flandoli for agreeing without hesitation to work on a common project and for hosting me in Pisa. It was very pleasant to work with him and the other members of the stochastic group in Pisa.

I am very grateful to Prof. Jonathan Mattingly, who hosted me at Duke University and who shared many of his ideas which were essential to this thesis.

I am very happy that I have spend time together with some people in the graduate school, Maite and Sebastian the sofa thieves; Alberto, Adrian, Atul and Giovanni from the not-so-cool office next door; Sara the commuter; Eva, Jennifer and Moritz from the dark-side corridor; Benedikt and Massimo my dear office mates and Giuseppe the colleague. Thanks to all of you for the enjoyable moments we had. Special thanks go to Giuseppe for proofreading parts of this thesis.

A big thanks goes to my parents, Antje and Marian, and my sister, Nina, for their support and the cheerful moments they share with me.

Financial support by the Berlin Mathematical School and the RTG 1845 “Stochastic Analysis with Application in Biology, Finance and Physics” as much as the mathematical environment their provide in Berlin is gratefully acknowledged.
Introduction
Chapter 1

Notation, basic definitions and properties

We present some basic definitions and notation necessary to understand the forthcoming chapters. The first part relates to the Chapters 2 to 4, while the content from Section 1.4 on refers to the Chapters 5 and 6.

1.1 Stochastic differential equations

Let \((W_t)_{t \geq 0}\) be a \(d\)-dimensional Brownian motion on a suitable filtrated probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\). Fix the function \(b: \mathbb{R}^d \to \mathbb{R}^d\) and \(\sigma \in [0, \infty)\).

We often work with stochastic differential equations, short SDEs, of the following form

\[
dX_t = b(X_t)dt + \sigma dW_t \quad X_0 = x \in \mathbb{R}^d.
\]  

Equation (1.1) can only be understood formally. However, we call a process \((X_t)_{t \geq 0}\) a solution to equation (1.1), if it is adapted, has almost surely continuous paths and satisfies the integral equation almost surely, i.e.

\[
X_t = x + \int_0^t b(X_s)ds + \sigma W_t \quad \text{for all } t \geq 0 \quad \mathbb{P}\text{-a.s.}.
\]

In particular, the right-hand side is well-defined.

At times we want to emphasize the dependence on the initial value \(x \in \mathbb{R}^d\) and alter the starting time to \(s \geq 0\). In this case we denote the solution by \(\phi_{s,t}(x)\), which solves a more general integral equation,

\[
\phi_{s,t}(x) = x + \int_s^t b(\phi_{u,t}(x))du + \sigma (W_t - W_s) \quad \text{for all } t \geq s \geq 0 \quad \mathbb{P}\text{-a.s.}
\]
Chapter 1. Notation, basic definitions and properties

If \( \sigma = 0 \) any solution to (1.1) is deterministic, in this case we call the equation ordinary differential equation, short ODE, and interpret it in the stronger differential sense
\[
\partial_t X_t = b(X_t), \quad X_0 = x \in \mathbb{R}^d.
\]
Existence and uniqueness of solutions to these equations depend on the coefficients \( b \) and \( \sigma \). In the subsequent chapters of this thesis we work with coefficients that guarantee uniqueness, but existence is less obvious. In fact, solutions might only exist up to a finite time.

1.1.1 Explosion in finite time

Let \( b \) be continuously differentiable, then it follows, see for example [Kha11, pages 71-75], that there exists a unique solution to (1.1) up to a possibly finite time \( \epsilon(x, \omega) \), which is given by
\[
\epsilon(x) = \lim_{N \to \infty} \inf \{ t \geq 0 : |X_t| \geq N \}.
\]
If \( \epsilon(x) < \infty \) almost surely, we say that the solution or the SDE (1.1) explodes in finite time or just explodes or blows-up. \( \epsilon \) is the explosion time and may also depend on \( s \geq 0 \) if the starting time is set to \( s \).
If, in case \( \sigma > 0 \), equation (1.1) has a solution for all times for every initial condition, i.e. \( \mathbb{P}(\epsilon(x) = \infty) = 1 \) for each \( x \in \mathbb{R}^d \), we call the SDE weakly complete or complete. If this holds true uniform in the initial condition, i.e. \( \mathbb{P}(\epsilon(x) = \infty \text{ for all } x \in \mathbb{R}^d) = 1 \), we call the SDE strongly complete.
Often we refer to a solution for a fixed initial condition as one-point motion.

Note that there is a difference in considering all one-point motions (for each initial condition) separately or simultaneously. In the latter case we talk about “flows”.

1.2 Local stochastic flows

We introduce local stochastic flows on \( \mathbb{R}^d \).

**Definition 1.2.1**

Let \( \epsilon(s, x) \), \( s \geq 0 \), \( x \in \mathbb{R}^d \) be a random field with values in \( (s, \infty) \), such that \( \epsilon(s, x) \) is lower semicontinuous in \( s \) and \( x \). Set \( D_{s,t}(\omega) := \{ x \in \mathbb{R}^d : \epsilon(s, x, \omega) > t \} \) and let \( \phi_{s,t}(x, \omega), x \in \mathbb{R}^d, 0 \leq s \leq t < \epsilon(s, x) \) be a continuous \( \mathbb{R}^d \)-valued random field defined on the random domain of parameters \( (s, t, x) \) for which \( x \in D_{s,t}(\omega) \). Denote the range of \( \phi_{s,t}(\cdot, \omega) \) on \( D_{s,t}(\omega) \) by \( R_{s,t}(\omega) \). \( \phi \) (or \( \phi_{s,t} \)) is called a (maximal) local stochastic flow, if for almost all \( \omega \in \Omega \)
1.3 Random dynamical system and attractors

i) \( \phi_{s,s}(\cdot, \omega) = \text{Id}_{\mathbb{R}^d} \) for all \( s \geq 0 \),

ii) \( \phi_{s,t}(\cdot, \omega): \mathbb{D}_{s,t}(\omega) \rightarrow \mathbb{R}_{s,t}(\omega) \) is a homeomorphism for all \( 0 \leq s < t \) and the inverse is continuous in \((s,t,x)\),

iii) \( \phi_{s,u}(\cdot, \omega) = \phi_{t,u}(\phi_{s,t}(\cdot, \omega), \omega) \) holds on \( \mathbb{D}_{s,u}(\omega) \) for all \( 0 \leq s \leq t \leq u \),

iv) \( \limsup_{t \uparrow e(s,x,\omega)} |\phi_{s,t}(x, \omega)| = \infty \) for all \( s \geq 0, x \in \mathbb{R}^d \) and \( e(s,x,\omega) < \infty \)

holds true.

A (maximal) local stochastic flow is called *stochastic flow* if for all \( 0 \leq s \leq t \) and \( \omega \in \Omega \) \( \mathbb{D}_{s,t}(\omega) = \mathbb{R}_{s,t}(\omega) = \mathbb{R}^d \).

**Remark 1.2.2**

A local stochastic flow is already given by the properties i) to iii). However, the maximality property iv) ensures uniqueness of a (maximal) local stochastic flow. In the following, whenever we say “local stochastic flow”, we refer to the maximal one.

The connection between SDEs and stochastic flows is given by equation (1.2). More precisely, according to [Kun90, Theorem 4.7.1], if \( b \) is continuously differentiable, then there exists a local stochastic flow \( \phi_{s,t}(x) \) for all \( x \in \mathbb{R}^d \) and \( 0 \leq t \leq e(s,x) \), such that for each \( s, x \) it is the (maximal) solution to (1.2) and \( e(s,x) \) is the explosion time.

### 1.3 Random dynamical system and attractors

Random attractors are defined for random dynamical system and their definition goes back to [CF94]. We only present a few definitions and results, which are necessary to comprehend the following chapters. For a more detailed description of random dynamical systems we refer to [Arn98] and references therein.

**Definition 1.3.1**

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space satisfying the usual conditions.

a) A family of mappings \( \theta_t: \Omega \rightarrow \Omega, \ t \in \mathbb{R} \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \) is called a measureable dynamical system (MDS) if

- \((\omega, t) \mapsto \theta_t(\omega) \) is \((\mathcal{F} \otimes \mathcal{B}(\mathbb{R})) - \mathcal{F} \) -measurable,
- \( \theta_0 = \text{Id}_\Omega \),
b) A random dynamical system (RDS) over a MDS \((\theta_t)_{t \in \mathbb{R}}\) is a map 
\[ \varphi: \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d \quad (t, \omega, x) \mapsto \varphi(t, \omega, x) \]
satisfying 
- \(\varphi\) is \((\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)) - \mathcal{B}(\mathbb{R}^d)\) -measurable and
- for all \(\omega \in \Omega\) and \(s, t \in \mathbb{R}\)
  \[ \varphi(0, \omega) = \text{Id}_{\mathbb{R}^d}, \]
  \[ \varphi(t + s, \omega) = \varphi(t, \theta_s(\omega)) \circ \varphi(s, \omega). \]

**Definition 1.3.2** (Attractor)
Let \(\varphi\) be an RDS on \(\mathbb{R}^d\) over an MDS \(\theta\).

a) A random subset \(A(\omega) \subset \mathbb{R}^d\) is called random compact set if \(A(\omega)\) is non-empty and compact for all \(\omega \in \Omega\) and the mapping \(\omega \mapsto \inf_{y \in A(\omega)} |x - y|\) is measurable for each \(x \in \mathbb{R}^d\).

b) A random compact set \(A\) is called a global weak (strong) set attractor for \(\varphi\) if \(A\) is strictly invariant, i.e.
\[ \varphi(t, \omega, A(\omega)) = A(\theta_t(\omega)) \]
for all \(\omega \in \Omega\) and \(t \in \mathbb{R}\) and attracts all deterministic compact sets \(B\), i.e.
\[ \lim_{t \to \infty} \sup_{x \in B} \inf_{y \in A(\omega)} |\varphi(t, \theta_{-t}(\omega), x) - y| = 0 \]
in probability (almost surely) for all compact sets \(B\).

There is a handy criterion for the existence of a global weak set attractor. We copy it directly from [Dim06, Theorem 8.2.2].

**Theorem 1.3.3**
Let \(\varphi\) be an RDS over an MDS. Then the following statements are equivalent.

i) There is an RDS \(\tilde{\varphi}\) over the same MDS indistinguishable from \(\varphi\), i.e. \(\mathbb{P}(\varphi(t, \omega) = \tilde{\varphi}(t, \omega) \text{ for all } t \in \mathbb{R}) = 1\), which has a global weak set attractor.

ii) For every \(\varepsilon > 0\) there exists \(R_0 > 0\), such that for all \(R > 0\) there is a \(t_0 > 0\), such that for all \(t \geq t_0\)
\[ \mathbb{P}(\varphi(t, \omega)(B_0(R)) \subset B_0(R_0)) \geq 1 - \varepsilon. \]
1.4 Interacting particle systems

We interpret an interacting particle system as a system of SDEs, in which each particle is driven by an SDE of special kind.

The system, defined on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\), is composed of particles at positions \(X^{a,N}_t \in \mathbb{R}^d\). We label particles by a multi-index \(a = (k, i_1, ..., i_n)\) with \(i_1, ..., i_n \in \{1, 2\}\) and \(k = 1, ..., N\), where \(N \in \mathbb{N}\) is the number of particles at time \(t = 0\). The particles already alive at time \(t = 0\) are those with label \(a = (k)\), \(k = 1, ..., N\). Their descendants require an additional labeling. Set \((a, -) := (k, i_1, ..., i_{n-1})\) if \(a = (k, i_1, ..., i_n)\), we may say that \(a\) is a descendant of \((a, -)\). Each particle “lives” only during a random time interval: particle \(a\) lives on \(I^{a,N}_t = [T^{a,N}_0, T^{a,N}_1) \subset [0, \infty)\) where \(T^{a,N}_0, T^{a,N}_1\) are \(\mathcal{F}_t\)-stopping times. It was born at time \(T^{a,N}_0\) (or it exists from \(t = 0\) if \(T^{a,N}_0 = 0\)) and “dies” at time \(T^{a,N}_1\) when it is replaced by two independent particles (this is a proliferation event). The number of living particles can only increase. We set \(X^{a,N}_{T^{a,N}_1} := \lim_{t \uparrow T^{a,N}_1} X^{a,N}_t\) and impose \(T^{a,N}_0 = T^{(a,-),N}_{T^{a,N}_1}\) and \(X^{a,N}_{T^{a,N}_0} = X^{(a,-),N}_{T^{(a,-),N}_1}\). Denote by \(A_N\) the set of all labels \(a\) and by \(A_N^t\) the set of particle labels alive at time \(t\), namely the set of \(a \in A_N\) such that \(t \in I^{a,N}_t\). The empirical measure at time \(t\) is defined as

\[
S^N_t = \frac{1}{N} \sum_{a \in A^t_N} \delta_{X^{a,N}_t}.
\]

In the random time interval \(I^{a,N}_t\), particle \(a\) interacts with all other living particles through a potential \(V \in C^2_b(\mathbb{R}^d)\). The dynamics of \(X^{a,N}_t\) is described by the gradient system

\[
dX^{a,N}_t = -\frac{1}{N} \sum_{a \in A^t_N} \nabla V \left( X^{a,N}_t - X^{(a,-),N}_{T^{(a,-),N}_1} \right) dt + \sigma dB^{a}_t,
\]

where \(B^{a}\) are independent standard \(d\)-dimensional Brownian motions in \(\mathbb{R}^d\) and \(\sigma > 0\).

Each particle proliferates at a random rate \(\lambda^{a,N}_t\) which, in most applications, depends on the density of particles in its neighborhood. More precisely, we have standard Poisson processes \(N^{0,a}\), independent between themselves and with respect to the Brownian motions and initial conditions \(X^{(k),N}_0\). We change their times randomly by setting \(N^{a,N}_t := N^{0,a}_{\Lambda^{a,N}_t}\) where \(\Lambda^{a,N}_t = \int_0^t 1_{s \in I^{a,N}_t} \lambda^{a,N}_s ds\). Further, we assume that \(\lambda^{a,N}_t\) is measurable and bounded. The process \(N^{a,N}_t\) is a jump process with rate \(\lambda^{a,N}_t\) and the stopping time \(T^{a,N}_1\) is defined as the time at which \(N^{a,N}_t\) jumps from 0 to 1 (and then remains equal to 1).
1.4.1 Further notation

Let $\mu$ be a measure on $\mathbb{R}^d$ and $f: \mathbb{R}^d \to \mathbb{R}$ in $L^1(\mu)$. We define

$$\langle f, \mu \rangle := \langle \mu, f \rangle := \int_{\mathbb{R}^d} f(x) \mu(dx).$$

If $\mu$ has a density $g$ with respect to the Lebesgue measure, we may write $\langle f, g \rangle$ instead of $\langle f, \mu \rangle$. Further, if $f$ and $g$ are $\mathbb{R}^d$-valued, we take the scalar product in $\mathbb{R}^d$ inside the integral

$$\langle f, g \rangle := \int_{\mathbb{R}^d} f(x) \cdot g(x) dx.$$

If $\mu$ has a density $h$ with respect to another measure $\nu$, we may write $\langle f, h\nu \rangle$ instead of $\langle f, \mu \rangle$.

We denote the convolution between $f$ and $\mu$ by

$$(f \ast \mu)(x) := \langle f(x - \cdot), \mu \rangle = \int_{\mathbb{R}^d} f(x - y) \mu(dy).$$

Recall the empirical measure from (1.3)

$$S_N^t = \frac{1}{N} \sum_{a \in A_{t}} \delta_{X_{a,N}^t}.$$

The total relative mass is denoted by

$$\left[ S_N^t \right] := S_N^t (\mathbb{R}^d) = \langle S_N^t, 1 \rangle = \frac{\text{Card}(A_{t}^N)}{N}.$$

Since, in our models, the number of particles may only increase, we have the inequality

$$\left[ S_N^t \right] \leq \left[ S_N^T \right]$$

for all $t \in [0, T]$, (1.5)

that we use very often. The quantity $\left[ S_N^T \right]$ has, moreover, an exponential moment, uniformly in $N$.

**Lemma 1.4.1**

Assume that $\lambda_{t}^{a,N}$ is bounded, i.e. there exists a constant $K \in (0, \infty)$ such that for all $t \in [0, T], N \in \mathbb{N}, a \in A_{t}^N, \omega \in \Omega$ it holds $\lambda_{t}^{a,N} \leq K$. There exists a $\gamma > 0$ such that $\sup_N \mathbb{E} \left[ e^{\gamma [S_N^T]} \right] < \infty$.

The proof can also be found in [FL15, Lemma 3.11]. The assumption of this lemma is true for all the applications we consider in this thesis.
1.4. Interacting particle systems

Proof. On the same probability space \((\Omega, \mathcal{F}, P)\) one can construct two processes, \(\{X^a_t; a \in A^N\}\), which is equal in law to our particle system, and \(\{Y^a_t; a \in A^N\}\), such that \(\langle S(X)_t^N, 1 \rangle \leq \langle S(Y)_t^N, 1 \rangle\) a.s. (we distinguish the objects of the different particle systems by adding “(X)” or “(Y)”), where the branching rate of \(\{Y^a_t; a \in A^N\}\) is constant, i.e. \(\lambda(Y)^a_{t,N} \equiv K\) for all \(a \in A(Y)^t_N\) and \(t \in I(Y)^a_N\). Further, one has the representation

\[
\langle S(Y)_T^N, 1 \rangle = \frac{1}{N} \sum_{k=1}^{N} N_T^{(k)},
\]

where \(N_T^{(k)}\) is a Yule process with parameter \(K\) and start in 1 that describes the number of descendant of \(a(Y) = (k)\) which are alive at time \(T\). Furthermore, \(N_T^{(k)}\) are identically distributed and independent random variables for \(k = 1, \ldots, N\). Together with Jensen’s inequality it follows

\[
\sup_{N \in \mathbb{N}} \mathbb{E}\left[e^{\gamma \langle S(Y)_t^N, 1 \rangle}\right] = \sup_{N \in \mathbb{N}} \left(\mathbb{E}\left[e^{\frac{\gamma N}{T}N_T^{(1)}}\right]\right)^N \leq \mathbb{E}\left[e^{\gamma N_T^{(1)}}\right].
\]

Because \(N_T^{(1)}\) is geometrically distributed with parameter \(K\), the right-hand side is finite if \(\gamma < \frac{1}{1-e^{-K T}}\). \(\square\)

Let \(\theta \in L^2(\mathbb{R}^d)\) be a smooth (in \(C^\infty\)) probability density. Define \(\theta_N(x) = \epsilon_N^d \theta(\epsilon_N^{-1}x)\) for some sequence \((\epsilon_N)_{N \in \mathbb{N}}\), that convergences to 0 as \(N \to \infty\). We call \((\theta_N)_{N \in \mathbb{N}}\) a classical family of mollifiers. We introduce the mollified empirical measure \(h^N_t\) defined as

\[
h^N_t(x) = \left(\theta_N * S^N_t\right)(x).
\]

There is a correspondence between the weak convergence of \(S^N_t\) and \(h^N_t\). It is based on the identity

\[
\langle S^N_t, \phi \rangle = \langle S^N_t, \phi \rangle - \langle \theta_N * S^N_t, \phi \rangle + \langle \theta_N * S^N_t, \phi \rangle = \langle S^N_t, \phi - \theta_N * \phi \rangle + \langle h^N_t, \phi \rangle,
\]

where \(\theta_N(x) := \theta_N(-x)\). Note that \(C_b\left(\mathbb{R}^d\right)\) is the family of bounded and continuous functions.

**Lemma 1.4.2**

For every \(\phi \in C_b\left(\mathbb{R}^d\right)\) and every \(t \in [0, T]\) the following two statements are equivalent:

i) \(\langle S^N_t, \phi \rangle\) converges to \(\langle f, \phi \rangle\) in probability,

ii) \(\langle h^N_t, \phi \rangle\) converges to \(\langle f, \phi \rangle\) in probability,

where \(f\) is an arbitrary function in \(L^1(\mathbb{R}^d)\).
Chapter 1. Notation, basic definitions and properties

Proof. The quantities $|\langle S_N^t, \phi \rangle - \langle u_t, \phi \rangle|$ and $|\langle h_N^t, \phi \rangle - \langle u_t, \phi \rangle|$ differ at most by

$$\left|\langle S_N^t, \phi - \theta \ast \phi \rangle\right|,$$

which is bounded by

$$\left\|\phi - \theta \ast \phi\right\|_{\infty} \left[S_N^t\right].$$

Applying Markov’s inequality and Lemma 1.4.1, there is a constant $C > 0$ such that for every $\varepsilon > 0$

$$\mathbb{P}\left(\left\|\phi - \theta \ast \phi\right\|_{\infty} \left[S_N^t\right] > \varepsilon\right) \leq \frac{C \left\|\phi - \theta \ast \phi\right\|_{\infty}}{\varepsilon}.$$

Since $\left\|\phi - \theta \ast \phi\right\|_{\infty} \to 0$ as $N \to \infty$ for every $\phi \in C_b\left(\mathbb{R}^d\right)$, we deduce that $\left|\langle S_N^t, \phi - \theta \ast \phi \rangle\right|$ converges to zero in probability. The proof is complete.

We repeatedly use the identity

$$\int_{\mathbb{R}^d} h_N^t(x)dx = \left[S_N^t\right],$$

which follows from Fubini’s theorem. Other inequalities we often use are

$$\left|\left(\theta \ast \left(f S_N^t\right)\right)(x)\right| \leq \left\|f\right\|_{\infty} h_N^t(x), \quad \left|\left(f \ast S_N^t\right)(x)\right| \leq \left\|f\right\|_{\infty} \left[S_N^t\right]$$

(1.7)

for every bounded measurable $f : \mathbb{R}^d \to \mathbb{R}$ and

$$\frac{1}{N} \int_{\mathbb{R}^d} |\nabla \theta_N(x)|^2 dx \leq C, \quad \frac{1}{N} \int_{\mathbb{R}^d} |\theta_N(x)|^2 dx \leq C \varepsilon_N^2,$$

(1.8)

which hold with a suitable constant $C > 0$. The bound on the first term follows for compactly supported $\theta$ from from

$$\frac{1}{N} \int_{\mathbb{R}^d} |\nabla \theta_N(x)|^2 dx = \frac{\varepsilon^{-d-2}}{N} \int_{\mathbb{R}^d} \varepsilon_N^{-d} |\nabla \theta\left(\varepsilon_N^{-1} x\right)|^2 dx = \frac{\varepsilon^{-d-2}}{N} \int_{\mathbb{R}^d} |\nabla \theta\left(x\right)|^2 dx$$

and the assumption $\sup_N \varepsilon_N^{-d-2}/N < \infty$. For the bound on the second term it is sufficient to assume $\sup_N \varepsilon_N^{-d}/N < \infty$ and $\theta \in L^2(\mathbb{R}^d)$.
Chapter 2

Blow-up of a stable stochastic differential equation

This chapter is based on joint work with M. Scheutzow, see \[LS15\]. We present a family of two-dimensional ODEs which exhibits explosion in finite time. However, considered as an SDE with additive white noise, it is known to be complete. Furthermore, the associated Markov process even admits an invariant probability measure. Theorem 2.3.1 shows, that the SDE is almost surely not strongly complete, i.e. there exist (random) initial conditions for which the solution explodes in finite time.

First, we introduce the equations followed by the main theorem and its proof. Remark 2.5.4 generalizes our observations to the case, in which the additive white noise is replaced by any other continuous driver.

2.1 Introduction

Consider the complex-valued (Itô-type) stochastic differential equation

\[ dZ_t = (Z^n_t + F(Z_t)) \, dt + \sigma dB_t, \tag{2.1} \]

where \( n \geq 2, \sigma \geq 0, F \in \mathcal{O}(|z|^{n-1}) \) as \( |z| \to \infty \) is locally Lipschitz and \( B = W^{(1)} + iW^{(2)} \) is a complex Brownian motion on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) satisfying the usual conditions.

Under the additional assumption that \( F \) is a polynomial of \( z \) and \( \bar{z} \) of degree at most \( n - 1 \) it is known (see \[HM15a\] and \[HM15b\]) that for every fixed initial condition \((X_0, Y_0) = (x_0, y_0)\) the one-point motion, i.e. the process which solves this equation and starts in \((x_0, y_0)\), exhibits non-explosion almost surely if \( \sigma > 0 \) and, moreover, the associated Markov process admits a (unique) invariant probability measure. This is a remarkable fact, since
there is explosion in finite time for some initial conditions in the deterministic case (i.e. \( \sigma = 0 \)). This is obvious in the particular case \( F = 0 \) (take an initial condition on the positive real line) and will follow from our main result for general \( F \). Turning an explosive ODE into a non-explosive SDE with an invariant distribution by adding additive noise, so-called noise-induced stabilization, was also studied in [Sch93] and more recently in [BHW12, AKM12]. Now, we would like to know if the noise induces an even stronger kind of stability, namely the existence of a random attractor. In this chapter, we show that the corresponding local stochastic flow explodes (or blows up) almost surely and therefore there cannot be a random attractor (for the definition and basic properties of random attractors, see [CF94]). If \( F \) is a polynomial in \( z, \bar{z} \), then SDE (2.1) is complete. Since the local stochastic flow associated to (2.1) explodes, it is not strongly complete. So far there are only few examples which are known to be complete but not strongly complete, see for instance [Elw78, LS11].

2.2 Change of coordinates

We encountered the driving equation in a simpler form, i.e.

\[
dZ_t = Z_t^2 \, dt + \sigma dB_t,
\]

in [AKM12]. Despite the simple form, it has all the interesting properties, which are as difficult to prove as in the more general case. Their work was generalized by David Herzog and Jonathan Mattingly to arbitrary order and additional lower order perturbation term in [HM15a, HM15b]. However, in their work they almost exclusively worked in polar coordinates. We do something similar but for our purpose it is more convenient to transform equation (2.1) into Cartesian coordinates. The rest of this chapter deals only with equation (2.2) below.

Denote the real and imaginary part of \( F \) by \( F_1 \) and \( F_2 \), i.e. \( F = F_1 + iF_2 \). Further, there are functions \( \hat{F}_1, \hat{F}_2: \mathbb{R}^2 \to \mathbb{R} \), such that \( F_j(x + iy) = \hat{F}_j(x, y) \), \( j = 1, 2 \). If we rewrite \( Z_t = X_t + iY_t \), SDE (2.1) is equivalent to

\[
\begin{align*}
dX_t &= \left( \sum_{j=0}^{\left\lfloor n/2 \right\rfloor} (-1)^j \binom{n}{2j} X_t^{n-2j} Y_t^{2j} + \hat{F}_1(X_t, Y_t) \right) \, dt + \sigma dW_t^{(1)}, \\
dY_t &= \left( \sum_{j=0}^{\left\lfloor n/2 \right\rfloor} (-1)^j \binom{n}{2j+1} X_t^{n-2j-1} Y_t^{2j+1} + \hat{F}_2(X_t, Y_t) \right) \, dt + \sigma dW_t^{(2)}.
\end{align*}
\]

(2.2)
2.3. Main result

Abbreviate

\[ b_1(x, y) := \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n}{2j} x^{n-2j} y^{2j}, \]
\[ b_2(x, y) := \sum_{j=0}^{\lfloor n-1/2 \rfloor} (-1)^j \binom{n}{2j+1} x^{n-2j-1} y^{2j+1}. \]

At first sight these drift terms look quite unhandy, but the following lemma yields convenient expressions.

Lemma 2.2.1
For \( x > 0, y \in \mathbb{R} \) we have

\[ b_1(x, y) = (x^2 + y^2)^{\frac{n}{2}} \cos \left( n \arctan \left( \frac{y}{x} \right) \right), \]
\[ b_2(x, y) = (x^2 + y^2)^{\frac{n}{2}} \sin \left( n \arctan \left( \frac{y}{x} \right) \right). \]

Proof. Write \( z \) in Cartesian and polar coordinates, i.e. \( z = x + iy = re^{i\phi} \). For \( x > 0 \) polar coordinates can be expressed in terms of Cartesian coordinates via \( r = \sqrt{x^2 + y^2}, \phi = \arctan(y/x) \). Therefore,

\[ z^n = (x + iy)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j}(iy)^j \]
\[ = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n}{2j} x^{n-2j} y^{2j} + i \sum_{j=0}^{\lfloor n-1/2 \rfloor} (-1)^j \binom{n}{2j+1} x^{n-2j-1} y^{2j+1}; \]
\[ z^n = r^n e^{in\phi} = r^n \cos(n\phi) + ir^n \sin(n\phi) \]
\[ = (x^2 + y^2)^{\frac{n}{2}} \cos \left( n \arctan \left( \frac{y}{x} \right) \right) + i (x^2 + y^2)^{\frac{n}{2}} \sin \left( n \arctan \left( \frac{y}{x} \right) \right). \]

The lemma follows by comparing the real and imaginary parts of both expressions. \( \square \)

2.3 Main result

According to [Kun90, Theorem 4.7.1] there exists a local stochastic flow \( \phi_{s,t}(x, \omega), x \in \mathbb{R}^2, 0 \leq s \leq t < e(s, x) \), which is the maximal solution to equation (2.2) starting at time \( s \) in \( x \), where \( e(s, x) \) is the explosion time. In particular it is a maximal local stochastic flow. We denote its maximal domain by \( D_z(\omega) := [0, e(0, z, \omega)) \).
Chapter 2. Blow-up of a stable SDE

In the following, we write \( \phi_t \) instead of \( \phi_{0,t} \) and denote the \( i \)th component of \( \phi_t \) by \( \phi^{(i)}_t \), \( i = 1, 2 \). We use \( \phi^{(1)}_t(z) \) and \( X_t \) respectively \( \phi^{(2)}(z) \) and \( Y_t \) interchangeably, whenever the initial condition \( z \) is not of importance or clear from the context. In the same manner we sometimes write \( D \) instead of \( D_z(\omega) \).

Our main result is the explosion (or blow up or lack of strong completeness) of the local stochastic flow \( \phi \).

**Theorem 2.3.1**

Let \( \phi \) be the local stochastic flow associated to (2.2), then there exists \( T \in (0, \infty) \) such that

\[
\lim_{x_0 \to \infty} P \left( \sup_{z \in I} \sup_{t \leq T} \phi^{(1)}_t(z) = \infty \right) = 1,
\]

where the initial set is given by \( I := \{x_0\} \times [-\tan \left( \frac{\pi}{2^n} \right) x_0, \tan \left( \frac{\pi}{2^n} \right) x_0] \).

**Remark 2.3.2**

The theorem shows that we have almost sure blow-up:

\[
P \left( \exists z \in \mathbb{R}^2: \sup_{t \leq T} \phi^{(1)}_t(z) = \infty \right) = \lim_{x_0 \to \infty} P \left( \exists z \in I: \sup_{t \leq T} \phi^{(1)}_t(z) = \infty \right) = 1.
\]

To justify the second step we show that the sets

\[
A := \left\{ \exists z \in I: \sup_{t \leq T} \phi^{(1)}_t(z) = \infty \right\} \quad B := \left\{ \sup_{z \in I} \sup_{t \leq T} \phi^{(1)}_t(z) = \infty \right\}
\]

are equal. Take \( \omega \in A^c \), then \((t, z) \mapsto \phi^{(1)}_t(z, \omega)\) is continuous on \([0, T] \times I\). Therefore, \( \left( \sup_{t \leq T} \phi^{(1)}_t(\cdot, \omega) \right) \) is a compact set, because \( I \) is compact. It follows

\[
\sup_{z \in I} \sup_{t \leq T} \phi^{(1)}_t(z, \omega) = \max_{z \in I} \sup_{t \leq T} \phi^{(1)}_t(z, \omega) < \infty.
\]

Hence, \( \omega \in B^c \) and therefore \( B \subset A \). \( A \subset B \) is obvious.

### 2.4 Heuristic idea

For the rest of this section fix \( \alpha \in \left( 0, \tan \left( \frac{\pi}{2n} \right) \right) \), and define the cone

\[
C := \{(x, y) \in \mathbb{R}^2: x \geq x^*, |y| \leq \alpha x \},
\]
where we will choose $x^* > 0$ sufficiently large later on (depending only on $n$ and $F$).

We know that for every initial condition in $\mathcal{C}$, the solution of the SDE will almost surely eventually leave $\mathcal{C}$. Some trajectories leave this region via the upper boundary and some via the lower boundary. Due to the continuity of the map $z \mapsto \phi_t(z)$, one may hope to be able to show that there will be (random) initial conditions in between these two kinds of points for which the trajectories will actually remain inside $\mathcal{C}$ forever. In the following section we see that if such trajectories exist, then they explode within time $T$ (which is small provided the initial condition has a large $x$-component) provided the noise in the $x$-direction is not too large up to time $T$.

![Figure 2.1: Bounds away from the x-axis](image)

It then remains to show that there actually exist trajectories which stay inside $\mathcal{C}$ forever (until they blow up). We sketch the idea of the proof in case $W^{(1)} \equiv 0$: Figure 1 shows the image of the set of initial conditions $\{(x_0, y), |y| \leq \tan \left( \frac{\pi}{2n} \right) x_0 \}$ under the map $\phi_t$ for some $x_0 > x^* > 0$ and some $t > 0$. The idea of the proof is to show that, for large $x_0$, it is very unlikely that any trajectory whose $y$-coordinate happens to be above level $\alpha x_0/2$ at some time will hit the level $y = \alpha x_0/4$ before leaving the cone $\mathcal{C}$ through its upper boundary (Lemma 2.5.1). This will then allow us to show the existence of points which stay inside $\mathcal{C}$ forever (until explosion).
2.5 Auxiliary results and proof of Theorem 2.3.1

First, we establish a lower bound for the $x$-component as long as the trajectory stays inside the cone $C$. Then we formalize what is shown in Figure 2.1. Define

$$
\tau(z) := \inf \{ t \in D_z : \phi_t^{(2)}(z) \geq \alpha \phi_t^{(1)}(z) \}, \\
\overline{\tau}(z) := \inf \{ t \in D_z : \phi_t^{(2)}(z) \leq -\alpha \phi_t^{(1)}(z) \}, \\
\tau(z) := \tau(z) \land \overline{\tau}(z)
$$

with the usual convention $\inf \emptyset = \infty$.

2.5.1 Lower bound

Note that we have a lower bound $\varepsilon > 0$ of the following term uniformly for all $(x, y) \in C$

$$\cos \left( n \arctan \left( \frac{y}{x} \right) \right) \geq \varepsilon > 0.$$ 

Because of $F \in O(|z|^{n-1}) \subset o(|z|^n)$ as $|z| \to \infty$, there exists $x^* > 0$, such that

$$\frac{\left| \hat{F}_1(x, y) \right|}{(x^2 + y^2)^{\frac{n}{2}}} \leq \frac{\varepsilon}{2}$$

holds for all $x \geq x^*$, $y \in \mathbb{R}$. Fix $c > 0$, $x_0 \geq x^* + c$ and $z \in \mathcal{Z} = \{ x_0 \} \times [-\tan \left( \frac{\pi}{2n} \right) x_0, \tan \left( \frac{\pi}{2n} \right) x_0]$. Then on the event

$$\{ \tau(z) > T \} \cap \left\{ \sup_{t \in [0, T]} |W_t^{(1)}| \leq c \right\},$$

we have for all $t \in [0, T] \cap D$ (recall $D$ is the maximal domain on which $X_t$ is defined)

$$X_t = x_0 + \int_0^t b_1(X_s, Y_s) + \hat{F}_1(X_s, Y_s) ds + \sigma W_t^{(1)}$$

$$\geq x_0 - c + \int_0^t \left( X_s^2 + Y_s^2 \right)^{\frac{n}{2}} \left( \cos \left( n \arctan \left( \frac{Y_s}{X_s} \right) \right) - \frac{\left| \hat{F}_1(X_s, Y_s) \right|}{(X_s^2 + Y_s^2)^{\frac{n}{2}}} \right) ds$$

$$\geq x_0 - c + \frac{\varepsilon}{2} \int_0^t X_s^n ds.$$
Applying a (reversed) Gronwall type argument (similar to \cite[page 83f]{Bih56}), we see that for all \( t \in [0, T] \cap D \)

\[
X_t \geq \frac{x_0 - c}{\left(1 - \frac{1}{2}(n-1)(x_0 - c)^{n-1}t\right)^{\frac{1}{n-1}}}.
\]

We define

\[
T := \frac{1}{\frac{1}{2}(n-1)(x_0 - c)^{n-1}},
\]

which is an upper bound for the explosion time, i.e. \( X \) blows up up to time \( T \) on the set \( \{ \tau(z) > T \} \cap \{ \sup_{t \in [0,T]} \sigma|W_t^{(1)}| \leq c \} \). Observe that the heuristic ideas remain valid on \( \{ \sup_{t \in [0,T]} \sigma|W_t^{(1)}| \leq c \} \) when replacing \( x_0 \) by \( x_0 - c \): if the event \( \{ \sup_{t \in [0,T]} \sigma|W_t^{(1)}| \leq c \} \) occurs, then any trajectory starting in \( \mathfrak{F} \) which does not leave the cone \( C \) up to \( T \) blows up before (or at) time \( T \).

### 2.5.2 Bounds away from the \( x \)-axis

It remains to show, that the \( Y \) coordinate behaves as predicted in the heuristics section. This is indeed the case, when the Brownian motion \( W^{(2)} \) does not fluctuate too much, see Lemma 2.5.1. Throughout the rest of this chapter, \( c > 0 \) will be fixed and \( x_0 > x^* + c \) is a number which will later be sent to infinity.

Because of \( F \in \mathcal{O}(|z|^{n-1}) \) there is a \( C > 0 \) such that for all \((x,y) \in \mathbb{R}^2 \) with \(|(x,y)| \geq x^* \), where \( x^* > 0 \) is sufficiently large,

\[
\frac{|\dot{F}_2(x,y)|}{|(x,y)|^{n-1}} \leq C
\]

holds true. Further, we define \( x_1 := x_0 - c \) and \( T := \frac{1}{\frac{1}{2}(n-1)x_1^{n-1}} \) as before. Observe that \( x_1 \) tends to \( \infty \) and \( T \) tends to 0 as \( x_0 \to \infty \).

#### Lemma 2.5.1

For \( x^* \) sufficiently large, the following holds. Let \((X_t,Y_t)_{t \in [0,T]} \) solve equation \((2.2)\) with initial condition \((X_0,Y_0) = z \in \{x_0\} \times [-\tan \left(\frac{\pi}{2n}\right)x_0, \tan \left(\frac{\pi}{2n}\right)x_0]\). Define \( \nu^+ := \inf\{t \geq 0 : Y_t \geq \frac{\alpha}{2}x_1\} \). Then for all \( t \in [\nu^+, \tau(z)] \cap D \), where, again, \( D \) is the maximal domain on which \( X_t \) is defined, we have

\[
Y_t \geq \frac{\alpha}{4}x_1 \quad \text{on} \quad \left\{ \sup_{t \in [0,T]} \sigma|W_t^{(2)}| \leq \frac{\alpha}{8}x_1 \right\} \cap \left\{ \inf_{t \in [0,T] \cap D} X_t \geq x_1 \right\} := B.
\]
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Proof. Define \( \tau := \inf \{ t > \nu^+: Y_t \leq \frac{\alpha}{4}x_1 \} \land \tau(z) \). We show \( \tau = \tau(z) \) on \( B \), which proves the statement. For \( t \geq 0 \), such that \( \nu^+ + t \in D \) we have

\[
Y_{(\nu^++t)\land \tau} = Y_{\nu^+} + \sigma(W_{(\nu^++t)\land \tau}^{(2)} - W_{\nu^+}^{(2)}) + \int_{\nu^+}^{(\nu^++t)\land \tau} b_2(X_s, Y_s) + \hat{F}_2(X_s, Y_s) \, ds
\]

\[
\geq \frac{\alpha}{4} x_1 + \int_{\nu^+}^{(\nu^++t)\land \tau} |(X_s, Y_s)|^{n-1} \times \left[ \sqrt{X_s^2 + Y_s^2 \sin \left(n \arctan \left( \frac{Y_s}{X_s} \right) \right)} - \frac{|\hat{F}_2(X_s, Y_s)|}{|(X_s, Y_s)|^{n-1}} \right] \, ds
\]

\[
\geq \frac{\alpha}{4} x_1 + \int_{\nu^+}^{(\nu^++t)\land \tau} |(X_s, Y_s)|^{n-1} \times \left[ \sqrt{X_s^2 + Y_s^2 \sin \left(n \arctan \left( \frac{Y_s}{X_s} \right) \right) - C} \right] \, ds
\]

\[
\geq \frac{\alpha}{4} x_1.
\]

We justify the last step by showing \( I \geq 0 \). First, note that there exists a \( b_n > 0 \), such that for all \( z \in [0, b_n) \)

\[
\sin(n \arctan(z)) \geq z.
\]

Second, define \( a_n := \sin(n \arctan(b_n)) \). Recall, that for \( s \in [\nu^+, \tau] \), we have

\[
\alpha X_s \geq Y_s \geq \frac{\alpha}{4} x_1,
\]

which implies

\[
\sin \left( n \arctan \left( \frac{Y_s}{X_s} \right) \right) \geq a_n \land \frac{Y_s}{X_s}.
\]

Finally,

\[
I \geq \left( a_n \land \frac{Y_s}{X_s} \right) \sqrt{X_s^2 + Y_s^2} - C \geq (a_nX_s) \land Y_s - C \geq (a_nx_1) \land \frac{\alpha}{4} x_1 - C
\]

is non-negative if we choose \( x^* \) (and therefore also \( x_1) \) sufficiently large. \( \square \)

**Remark 2.5.2**

If \( \nu^+ \) is replaced by \( \nu^- := \inf \{ t \geq 0: Y_t \leq -\frac{\alpha}{2}x_1 \} \) then we obtain in the same way

\[
Y_t \leq -\frac{\alpha}{4} x_1
\]

for \( t \in [\nu^-, \tau(z)] \cap D \).
The previous lemma and remark are a formal description of what was explained in Section 2.4, see also Figure 2.1. It will be very useful to show the existence of points which stay inside $C$ until explosion (Lemma 2.5.3). Recall that $\tau(z)$ is the exit time of $C$ for $z \in \mathcal{I}$.

**Lemma 2.5.3**

\[
\left\{ \sup_{z \in \mathcal{I}} \tau(z) > T \right\} \supset \left\{ \sup_{t \in [0,T]} \sigma |W_t^{(2)}| \leq \frac{\alpha}{8} x_1 \right\} \cap \left\{ \sup_{t \in [0,T]} \sigma |W_t^{(1)}| \leq c \right\}.
\]

**Proof.** Define the random sets

\[
R := \{ z \in \mathcal{I} : \tau(z) \leq \tau(z) \wedge T \},
\]
\[
B := \{ z \in \mathcal{I} : \tau(z) \leq \tau(z) \wedge T \},
\]
\[
G := \mathcal{I} \setminus (R \cup B).
\]

Note that $R$ and $B$ are disjoint and

\[
\left\{ \sup_{z \in \mathcal{I}} \tau(z) > T \right\} = \{ G \neq \emptyset \}.
\]

For ease of notation we define

\[
B_1 := \left\{ \sup_{t \in [0,T]} \sigma |W_t^{(1)}| \leq c \right\}, \quad B_2 := \left\{ \sup_{t \in [0,T]} \sigma |W_t^{(2)}| \leq \frac{\alpha}{8} x_1 \right\}.
\]

Let $\omega \in B_1 \cap B_2$. Since $\omega \in B_1$ there is a minimal drift in the $x$-component for all trajectories starting in $\mathcal{I}$ as long as they stay inside $C$. Furthermore, there is a lower bound in the $x$-coordinate for those trajectories, namely $x_1 = x_0 - c$.

Note that some points of $\mathcal{I}$ are not contained in the cone $C$, for example $(x_0, \tan \left( \frac{\pi}{2n} \right) x_0) \in R(\omega)$ and $(x_0, -\tan \left( \frac{\pi}{2n} \right) x_0) \in B(\omega)$ and therefore $R(\omega)$ and $B(\omega)$ are non-empty.

Assume now that $\omega \in \{ G = \emptyset \}$ which is equivalent to $\omega \in \{ \mathcal{I} = R \cup B \}$. We show that $R(\omega)$ and $B(\omega)$ are (non-empty) closed subsets of $\mathcal{I}$, whose disjoint union is equal to the connected set $\mathcal{I}$, which is a contradiction.

Take a converging sequence $z_n \to z$ with $z_n \in R(\omega)$ for all $n \in \mathbb{N}$ and assume that $z \in B(\omega)$. Then, thanks to the continuity of $\phi_t(z,\omega)$ in $(t,z)$, there is a (random) $n \in \mathbb{N}$ such that

\[
\sup_{t \in [0,\tau(z)]} \left| \phi_t^{(2)}(z,\omega) - \phi_t^{(2)}(z_n,\omega) \right| \leq \frac{\alpha}{3} x_1.
\]
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Due to Lemma 2.5.1 we can conclude that $\phi_t^{(2)}(z, \omega)$ was never above $\alpha x_1/2$ before time $\tau(z)$, and therefore $\phi_t^{(2)}(z_n, \omega)$ was never above $5\alpha x_1/6$. Because of $z \in B(\omega)$, there is a time $\tau(z)(\omega) < T$ such that $\phi_{\tau(z)}^{(2)}(z, \omega) \leq -\alpha x_1$, which means that $\phi_{\tau(z)}^{(2)}(z_n, \omega) \leq -2\alpha x_1/3$. Again, due to Lemma 2.5.1, $z_n$ cannot be in $R(\omega)$. Since this is a contradiction, we have $z \notin B(\omega)$ and therefore $z \in R(\omega)$. Thus, $R(\omega)$ is closed and, by symmetry, so is $B(\omega)$. Therefore the proof of the lemma is complete. □

2.5.3 Proof of Theorem 2.3.1

Note that with the lower bound on the $x$-component (see (2.3)) we have the following inclusion

$$\left\{ \sup_{z \in I} \sup_{t \leq T} \phi_t^{(1)}(z) = \infty \right\} \supset \left\{ \sup_{z \in I} \tau(z) > T \right\} \cap \left\{ \sup_{t \in [0, T]} \sigma|W_t^{(1)}| \leq c \right\} =: A.$$ 

We show that the probability of $A$ already tends to 1 as $x_0 \to \infty$.

$$\mathbb{P}(A) \geq \mathbb{P} \left( A, \sup_{t \in [0, T]} \sigma|W_t^{(2)}| \leq \frac{\alpha}{8} x_1 \right) \quad (2.4)$$

Lemma 2.5.3 allows us to omit the event $\left\{ \sup_{z \in I} \tau(z) > T \right\}$, so the right-hand side of (2.4) equals

$$= \mathbb{P} \left( \sup_{t \in [0, T]} \sigma|W_t^{(1)}| \leq c, \sup_{t \in [0, T]} \sigma|W_t^{(2)}| \leq \frac{\alpha}{8} x_1 \right)$$

$$\geq 1 - \mathbb{P} \left( \sup_{t \in [0, T]} \sigma|W_t^{(1)}| > c \right) - \mathbb{P} \left( \sup_{t \in [0, T]} \sigma|W_t^{(2)}| > \frac{\alpha}{8} x_1 \right)$$

which converges to 1 as $x_0 \to \infty$ (which implies $x_1 \to \infty$ and $T \to 0$). This completes the proof.

Remark 2.5.4

We never used any specific properties of the Brownian motions $W^{(1)}, W^{(2)}$, apart from the fact that both are processes with continuous paths. Note that in this case the SDE (2.2) written in integral form can be solved pathwise for each $\omega \in \Omega$ and the local maximal solutions depend continuously upon the initial condition, therefore all arguments above remain valid in this case. Depending on the nature of the noise, the equation may or may not be complete.

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Chapter 3

Circular model

We consider a 2-dimensional SDE in polar coordinates depending on various parameters in this chapter, see equation (3.1). It was developed during the author’s stay at Duke University, Durham, North Carolina, together with Jonathan Mattingly and the joint research greatly contributed to the Section 3.3.1 and Appendix A.

Considered in Cartesian coordinates the solutions spin around the origin, hence the name circular model. We show that if the parameters belong to a specific regime then the deterministic system will explode in finite time (Proposition 3.2.1), but the stochastic flow corresponding to the stochastic equation admits a weak random attractor (Theorem 3.3.1). As a byproduct the proof of the main theorem provides an estimate on the tails of the invariant distribution. We conclude with some conjectures about other parameter regimes, in which there might exist an attractor as well. These conjectures stem from heuristic ideas presented by J. Mattingly, which we could not make rigorous so far.

3.1 Introduction

As we know, the addition of noise can stabilize an explosive ODE, such that it becomes a non-explosive SDE. For examples see [Sch93], [AKM12], [BHW12], [HM15a] and [HM15b]. This phenomenon is often called noise-induced stability or noise-induced stabilization, if in addition, the corresponding Markov process also possesses an invariant probability measure.

We investigate whether the noise induces an even stronger kind of stability, namely the existence of a random attractor. We call such a phenomenon noise-induced strong stabilization. The existence of a random attractor implies non-explosion and the existence of an invariant distribution, but not
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vice versa, see [Och99].
In the previous chapter we showed that a certain family of SDEs, which exhibits noise-induced stabilization as shown in [HM15a] and [HM15b], is not strongly complete, i.e. there exist (random) initial conditions for which the solutions explode in finite time. Hence, there is no random attractor.
Here, we give a positive answer to the question whether noise-induced strong stabilization is possible or not. There are general criteria for the existence of random attractors like monotonicity (see [Sch08]) or a drift inwards far away from the origin (see [DS11]). Because of the blow-up in the deterministic case neither setting is fitting for our problem.

The circular model is given by the 2-dimensional SDE in polar coordinates

$$
\begin{align*}
\dot{r}_t &= \left( -r_t^w \cos^2(\phi_t) + r_t^v \right) dt, \\
\dot{\phi}_t &= -r_t^\gamma \cos^2(\phi_t) dt + \sigma dW_t,
\end{align*}
$$

\tag{3.1}

where \( w, v, \gamma > 0, \sigma \geq 0 \) and \((W_t)_{t \geq 0}\) is a standard one-dimensional Brownian motion. Since the equation is \( \pi \)-periodic, we work with periodic boundary condition for the angular component \((\phi_t)_{t \geq 0}\). Hence, the state space reduces to \( [0, \infty) \times [0, \pi] \).

Explosion, existence of an invariant distribution or existence of a random attractor highly depend on the radial process \((r_t)_{t \geq 0}\). Therefore, it is advantageous to work with polar coordinates providing an equation for the radial component.

Our aim is to find values for \( w, v, \gamma > 0 \), such that in the deterministic case, i.e. \( \sigma = 0 \), the solution explodes in finite time for some or almost all initial conditions, while in the stochastic case, i.e. \( \sigma > 0 \), the random dynamical system corresponding to \((3.1)\) admits a random attractor.

3.2 Preliminaries

3.2.1 Explosion in the deterministic case

Consider the deterministic equation on \([0, \infty) \times [0, \pi]\)

$$
\begin{align*}
\dot{r}_t &= -r_t^w \cos^2(\phi_t) + r_t^v, \\
\dot{\phi}_t &= -r_t^\gamma \cos^2(\phi_t).
\end{align*}
$$

\tag{3.2}

For initial conditions of the form \( r_0 > 0, \phi_0 = \pi/2 \) there is explosion in finite time if and only if \( v > 1 \). A natural question is whether the solution for any initial condition of the form \( r_0 > 0, \phi_0 \in [0, \pi] \) explodes in finite time.
First, for every initial condition \( r_0 > 0, \phi_0 \in [0, \pi] \) there exists a finite time \( T = T(r_0, \phi_0) \), such that

\[
(r_T, \phi_T) \in \mathcal{E} := \{(r, \phi) : r^{w-v} \cos^2(\phi) \leq 1, \phi \geq \frac{\pi}{2}\}.
\]

Further, observe that \( \mathcal{E} \) is a trapping region, i.e. \((r_T, \phi_T) \in \mathcal{E} \) implies \((r_t, \phi_t) \in \mathcal{E} \) for all \( t \geq T \) up to the explosion time \( \varepsilon(r_0, \phi_0) \). Without loss of generality we assume \((r_0, \phi_0) \in \mathcal{E} \). Then, due to \((r_t, \phi_t) \in \mathcal{E} \) for all \( t < \varepsilon(r_0, \phi_0) \), it holds

\[
r_t \nearrow \infty \quad \text{and} \quad \phi_t \searrow \frac{\pi}{2} \quad \text{as} \quad t \nearrow \varepsilon(r_0, \phi_0).
\]

The following proposition provides a sufficient condition for explosion in finite time, i.e. \( \varepsilon(r_0, \phi_0) < \infty \) for all \( r_0 > 0, \phi_0 \in [0, \pi] \).

**Proposition 3.2.1**

For parameter choice \( 2\gamma > w - v, v > 1 \) solutions to equation (3.2) explode in finite time for all initial conditions of the form \( r_0 > 0, \phi_0 \in [0, \pi] \).

**Proof.** To show explosion in finite time in the case \( w \leq v \) is not difficult, but we omit it, since it is not of relevance for the rest of this chapter.

Assume \( w > v \) and, as above, \((r_0, \phi_0) \in \mathcal{E} \). Define \( r_k = r_0 2^k \) and \( \phi_k \in (\pi/2, \pi] \), such that \( r_k^{w-v} \cos^2(\phi_k) = 1/2 \), i.e. \( \phi_k = \arccos\left(\sqrt{\frac{1}{2}r_k^{w-v}}\right) \). Due to (3.3) the solution \((r_t, \phi_t)\) will enter the box \( B_k := [r_k, \infty) \times [\pi/2, \phi_k] \) in finite time for any \( k \in \mathbb{N}_0 \). We denote the time the process \((r_t, \phi_t)\) needs to enter \([r_k, \infty) \times [\pi/2, \phi_k+1]\) when started in \( B_k \) by \( t_{k,k+1} \). We further denote the time the process \((r_t, \phi_t)\) needs to enter \( B_{k+1} \) when started in \([r_k, \infty) \times [\pi/2, \phi_k+1]\) by \( s_{k,k+1} \). To complete the proof we need to show that \( t_{k,k+1} \) and \( s_{k,k+1} \) are summable over \( k \in \mathbb{N}_0 \).

The times \( s_{k,k+1} \) are summable because either the solution is already in \( B_{k+1} \), then \( s_{k,k+1} = 0 \), or it is inside \([r_k, r_{k+1}] \times [\pi/2, \phi_k+1]\), which is a subset of

\[
\left\{(r, \phi) : r^{w-v} \cos^2(\phi) \leq \frac{1}{2}, \phi \geq \frac{\pi}{2}\right\}.
\]

Note, that the drift in this region is strong enough to have explosion in finite time, hence the times \( s_{k,k+1} \) are summable. Moreover, we can estimate the times \( t_{k,k+1} \) by solving the auxiliary equation

\[
\dot{\psi}_t = -r_k^2 \cos^2(\psi_t), \quad \psi_0 = \phi_k.
\]

This process is an upper bound for the angular process \( \phi \), when the solution \((r_t, \phi_t)\) started in \( B_k \). Hence, the time at which \( \psi \) reaches \( \phi_{k+1} \) serves as an
upper bound for $t_{k,k+1}$. This time can be computed explicitly, i.e.

$$r_k^{-\gamma} \left( \sqrt{2r_{k+1}^{w-v} - 1} - \sqrt{2r_k^{w-v} - 1} \right).$$

Note, that this upper bound is summable in $k \in \mathbb{N}_0$ because $2\gamma > w - v$ and $r_k$ grows exponentially in $k$. Therefore, also the times $t_{k,k+1}$ are summable.

### 3.2.2 Stochastic flows

Note that one can transform equation (3.1) into Cartesian coordinates, which we do not recommend due to two reasons. First, the drift terms turn out to be clunky and, second, the transformation back to polar coordinates is not uniquely defined at the origin. Hence, unfortunately, we cannot use the standard theory for existence of local stochastic flows, which is developed for the state space $\mathbb{R}^d$, that easily.

We argue as follows. Consider equation (3.1) on $\mathbb{R}^2$, where $r_t$ is replaced by its absolute value. This allows us to apply the standard theory to deduce existence of local stochastic flows, see below. In fact, this is not true if any of the parameters is less than 1, since this may lead to non-uniqueness of local solutions. However, for other reasons we have to assume $w > v > 1$ and $\gamma \geq 1$ will be assumed from now on to avoid this technical issue.

Now, we restrict ourselves with initial conditions of the form $r_0 \geq 0, \phi_0 \in \mathbb{R}$. It is obvious that the radial process $r$ never becomes negative. The angular process $\phi$ drifts to $-\infty$, but it exists at least locally. Due to the $\pi$-periodicity of the drift, we can identify the state space for $\phi$ by $[0, \pi]$ as long as there is no explosion in finite time. Further, the drift in the angular coordinate is bounded from above by the radial component. Hence, if the radial component does not blow-up then neither does the angular process. In Theorem 3.3.1 we show that there is no explosion in finite time.

Therefore, in the following we work with the equation (3.1) (with $r_t$ instead of $|r_t|$) and its state space $[0, \infty) \times [0, \pi]$ and still apply the standard theory as displayed in Chapter 1.

According to [Kun90, Theorem 4.7.1] there exists a local stochastic flow $(r_{s,t}(z), \phi_{s,t}(z)), z \in [0, \infty) \times [0, \pi], 0 \leq s \leq t < e(s, z)$, which is the maximal solution to equation (3.1) starting at time $s$ in $z$, where $e(s, z)$ is the explosion time. In particular, it is a maximal local stochastic flow. We denote its maximal domain by $D_z(\omega) := [0, e(0, z, \omega))$.

In the following, we denote $r_{0,t}(z)$ or $\phi_{0,t}(z)$ by $r_t(z)$ or respectively by $\phi_t(z)$. We abuse notation by omitting the initial value “(z)”, whenever it is not of importance or clear from the context.

Random attractors are defined for random dynamical systems (RDS), but
we show the existence of a weak attractor by using the following theorem, which is a reformulation of the Theorem 1.3.3.

**Theorem 3.2.2**

There exists a global weak set attractor for the RDS associated to (3.1) if and only if for every $\varepsilon > 0$ there exists $R_0 > 0$, such that for all $R > 0$ there is a $t_0 > 0$, such that for all $t \geq t_0$

$$
P\left( \sup_{z \in [0,R] \times [0,\pi]} r_t(z) \leq R_0 \right) \geq 1 - \varepsilon.
$$

We justify this reformulation. We show, that the local flow $(r_{s,t}(z), \phi_{s,t}(z))$ is in fact global, see Theorem 3.3.1. Therefore, the local RDS $\varphi$ generated by the SDE (3.1) see [Arn98, Theorem 2.3.36, page 95], is global as well, again see [Arn98, page 95]. Furthermore, the RDS $\varphi$, possibly living on another probability space, can be generated such that the distributions of $\{\varphi(t-s,\cdot, \theta_s(\cdot)) : -\infty < s \leq t < \infty\}$ and $\{(r_{s,t}(\cdot, \cdot), \phi_{s,t}(\cdot, \cdot)) : -\infty < s \leq t < \infty\}$ coincide, due to a perfection technique, see [AS95].

### 3.3 Main result

The main result of this chapter states the existence of a global weak random set attractor for a certain parameter regime.

**Theorem 3.3.1**

For any $\sigma > 0$ and parameter choice $w > v > 1$, $\frac{2}{3} \gamma + 1 > v$, $w - 1 > \gamma$ the random dynamical system associated to (3.1) admits a global weak random set attractor. In particular, there exists a stochastic flow solving (3.1) for all time, i.e. $\mathbb{D}_z(\omega) = [0,\infty)$ for all $z \in [0,\infty) \times [0,\pi]$ almost surely.

We split the proof in several parts, each treating a different aspect of the problem. The first provides a quantitative estimate for the expected time to cross a neighborhood around $\pi/2$. In there the drift in the radial component leads to the explosion in the deterministic case. In the second part, we show that the drift inwards, away from this “dangerous” neighborhood around $\pi/2$, compensates the growth with high probability. Finally, we conclude with a Markov-like argument to extend the local existence to all times.

#### 3.3.1 Crossing the critical region

We give an estimate on the time it takes to cross the critical region defined as $\{(r, \phi) : r^w \cos^2(\phi) \leq r^v\}$. First, introduce the one-dimensional auxiliary
angular process $\tilde{\phi}^R(\psi) = \tilde{\phi}$ by freezing the radial component, i.e.
\[
\begin{align*}
\tilde{\phi}_t &= -R^2 \cos^2(\tilde{\phi}_t) dt + \sigma dW_t, \\
\tilde{\phi}_0 &= \psi,
\end{align*}
\]
where $R > 0$ is fixed for the moment and $\psi \in [0, \pi]$. Before we elucidate the relation to the real angular process, we estimate the expected time it takes to cross $\pi/2$ for $\tilde{\phi}$. Let $0 \leq a < \pi/2 < b < \infty$ and define
\[
\begin{align*}
\tilde{\nu}_{a,b}(\psi) &:= \nu_{a,b} := \inf\{t > 0: \tilde{\phi}_t(\psi) \notin (a, b)\}, \\
\tilde{\nu}_a(\psi) &:= \nu_a := \lim_{b \to \infty} \tilde{\nu}_{a,b} = \inf\{t > 0: \tilde{\phi}_t(\psi) = a\},
\end{align*}
\]
for $\psi \in [0, \pi]$. Set $u_{a,b}(\phi) = \mathbb{E}\tilde{\nu}_{a,b}(\phi)$, then $u_{a,b}$ can be determined by solving the one-dimensional Poisson equation, see [Bas98, p. 45],
\[
\begin{align*}
-R^2 \cos^2(\phi) u'(\phi) + \frac{\sigma^2}{2} u''(\phi) &= -1, \\
u(a) = u(b) &= 0.
\end{align*}
\tag{3.4}
\]
To be able to write down the solution to this equation we introduce some auxiliary function and variables
\[
\begin{align*}
A(\phi) &:= \int_0^\phi \cos^2(\beta) d\beta = \frac{1}{2} \phi + \frac{1}{4} \sin(2\phi), \\
K &:= \frac{2}{\sigma^2} R^2, \\
f(\phi) &:= \int_a^\phi e^{KA(\beta)} d\beta, \\
g(\phi) &:= \frac{2}{\sigma^2} \int_a^{\phi} \int_a^\beta e^{KA(\beta) - A(z)} dz d\beta.
\end{align*}
\]
Equation (3.4) can easily be solved by a reduction of order method and its solution is
\[
u_{a,b}(\phi) = \frac{g(b)}{f(b)} f(\phi) - g(\phi).
\]
Since we want to know the first time at which the process passes $a$, we pass to the limit $b \to \infty$. First note the following
\[
\begin{align*}
\frac{g(b)}{f(b)} &= \frac{2}{\sigma^2} \int_a^b e^{-KA(z)} dz - \frac{2}{\sigma^2} \int_a^b \int_a^z e^{-K(A(z) - A(\beta))} d\beta dz d\beta \\
&\quad \xrightarrow{b \to \infty} \frac{2}{\sigma^2} \int_a^\infty e^{-KA(z)} dz.
\end{align*}
\]
Therefore,
\[
\lim_{b \to \infty} u_{a,b}(\phi) = \lim_{b \to \infty} \int_a^\phi e^{KA(\beta)} d\beta \int_a^\infty e^{-KA(z)} dz - \frac{2}{\sigma^2} \int_a^\phi \int_\beta^\infty e^{K(A(\beta) - A(z))} d(z)d\beta
\]
\[
= \int_a^\phi \int_\beta^\infty e^{-K(A(z) - A(\beta))} d(z)d\beta =: u_a(\phi)
\]
The theorem of monotone convergence yields
\[
u_a = \lim_{b \to \infty} \nu_{a,b} = \lim_{b \to \infty} \mathbb{E} \tilde{\nu}_{a,b} = \mathbb{E} \lim_{b \to \infty} \tilde{\nu}_{a,b} = \mathbb{E} \tilde{\nu}_a.
\]
Due to monotonicity we have
\[
\mathbb{E} \sup_{\phi \in [a,\pi]} \tilde{\nu}_a(\phi) = \mathbb{E} \tilde{\nu}_a(\pi) = u_a(\pi) = \frac{2}{\sigma^2} \int_a^\pi \int_\beta^\infty e^{-K(A(z) - A(\beta))} d(z)d\beta.
\]
It turns out to be advantageous to know how fast \(u_a(\pi)\) converges to zero in \(R\) or equivalently in \(K\). The following theorem gives a good bound on the convergence speed.

**Lemma 3.3.2**
There exists a constant \(C\), such that for all \(K > 1\)
\[
\int_0^\pi \int_\beta^\infty e^{-K(A(z) - A(\beta))} d(z)d\beta \leq CK^{-\frac{2}{3}}
\]
The proof is quite lengthy and not enlightening. It can be found in Appendix A.

Now, fix an initial value \(z = (r_0, \psi)\) with \(r_0 > R\) and \(\psi \in [0, \pi]\). We compare the auxiliary process \(\tilde{\phi}^R(\psi)\) with the real angular process \(\phi_t(z)\). Therefore, introduce the stopping times
\[
\nu_a(z) := \inf\{t > 0: \phi_t(z) \leq a\},
\]
\[
\nu^R(z) := \inf\{t > 0: \phi_t(z) \leq R\}.
\]
It holds for \(t \leq \nu^R(z)\)
\[
\phi_t = \psi + \int_0^t r_s^2 \cos^2(\phi_s) ds + \sigma W_t
\]
\[
\leq \psi - R^2 \int_0^t \cos^2(\phi_s) ds + \sigma W_t
\]
The following lemma yields the comparison between the auxiliary process \(\tilde{\phi}\) and the real angular process \(\phi\).
Lemma 3.3.3
It holds
\[ \phi_t(z) \leq \tilde{\phi}_t(\psi) \quad \text{for all } t \leq \nu^R(z) \ \text{a.s..} \]

In particular,
\[ \nu_a(z) \wedge \nu^R(z) \leq \tilde{\nu}_a(\psi). \]

Proof. Define
\[ \varrho := \inf \left\{ t \geq 0 : \tilde{\phi}_t < \phi_t \right\} \]
and assume \( \varrho < \nu^R(z) \). It follows
\[ 0 > \frac{\partial}{\partial t} \left( \tilde{\phi}_t - \phi_t \right) \bigg|_{t=\varrho} = -\cos^2(\phi_\varrho) \left( R^\gamma - r^\gamma_\varrho \right) \geq 0. \]
This is a contraction, from which the lemma follows. \( \Box \)

The following lemma follows easily from the considerations above.

Lemma 3.3.4
There exists a constant \( C = C(\sigma) \) such that for all \( r_0 > R > 1 \), \( a \in [0, \pi/2) \) we have
\[ E \left[ \sup_{z \in \{r_0\} \times [0, \pi]} \nu_a(z) \wedge \nu^R(z) \right] \leq CR^{-\frac{2}{3}}. \]

3.3.2 Step down
Unfortunately, crossing the critical region quickly is not enough, since trajectories will reenter inevitably. Therefore, we have to use the strong negative drift \(-r_t^w \cos^2(\phi_t)\) outside the critical region to compensate the growth.

We fix \( 1 > \varepsilon > \tilde{\varepsilon} > 0 \) (small), \( \sigma > 0, T \geq 1, k \in \mathbb{N} \) and introduce some notation
\[ r_k := \rho \cdot 2^k, \quad \rho > 0, \]
\[ B_k := [r_k, \infty) \times [\pi/2 - \varepsilon, \pi), \]
\[ I(k) := \{r_k\} \times [0, \pi], \]
\[ \tilde{c} := \cos^2 \left( \frac{\pi}{2} - \varepsilon + \tilde{\varepsilon} \right) > 0, \]
\[ \tau_{k+1} := \inf \left\{ t \geq 0 : \sup_{z \in I(k)} r_t(z) \geq r_{k+1} \right\}, \]
\[ \tau_{k-1} := \inf \left\{ t \geq 0 : \sup_{z \in I(k)} r_t(z) \leq r_{k-1} \right\}, \]
\[ 34 \]
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\[ \tau_k(z) := \inf \{ t \geq 0 : r_t(z) \leq r_k \}, \]
\[ \tau_k(z) := \inf \{ t \geq 0 : (r_t(z), \phi_t(z)) \notin B_k \} = \nu_{r/2-\varepsilon}(z) \land \nu^{r_k}(z), \]
\[ \text{BBM}_k := \left\{ \sup_{0 \leq s \leq t \leq T} \sigma(W_t - W_s) - \frac{\tilde{c}}{2} \gamma r_{k-2}(t-s) \leq \tilde{\varepsilon} \right\}, \]
\[ \text{BBM}_k := \left\{ \inf_{0 \leq s \leq t \leq T} \sigma(W_t - W_s) + r_{k+1}(t-s) \geq -d \right\}, \quad d > 0, \]
\[ \text{BBM}_k := \text{BBM}_k \cap \overline{\text{BBM}_k}. \]

At the moment several variables are not specified, for instance \( \rho \). Many estimates only hold for big values of \( r \). Thus, \( \rho \) will be picked large enough to guarantee that all estimates hold. Other variables, on which \( \rho \) depends, like \( \tilde{\varepsilon}, \varepsilon, d, \sigma \) and \( T \) are not of much importance and are therefore kept generic but fixed.

In the following we want to estimate the probability of \( \{ \tau_{k-1} \geq \tau_{k+1} \} \).

\( \tilde{c} \) or rather \( \bar{c} \) serves as a lower bound for \( \cos^2(\phi) \) outside of the critical region.

We refer to \( G := \{ (\phi, r) : \cos^2(\phi) \geq \bar{c}, r \geq \rho \} \) as the good region. Inside the good region the drift in the \( r \)-component is negative and furthermore this region is insensitive to noise.

\( \text{BBM}_k \) will guarantee that one-point motions which leave the critical region will not directly reenter, whereas \( \text{BBM}_k \) makes sure that the one-point motions do not move through the good region too fast, see proof of Theorem \ref{thm:3.3.5}, in particular equation (3.5).

**Velocity in \( r_t \)**

In the following we derive an estimate for the least amount of time it takes to go from a level \( r > 0 \) to a higher level \( R > r \). Because of

\[ \text{dr}_t = \left( -r_t^w \cos^2(\phi_t) + r_t^v \right) dt \leq r_t^v dt \]

one can conclude, via a comparison argument, that the process \( \left( \tilde{r}_t \right)_{t \geq 0} \) solving the (deterministic) equation

\[ \frac{\partial}{\partial t} \tilde{r}_t = \tilde{r}_t^v, \quad \tilde{r}_0 = r \]

is an upper bound for the radial component \( \left( r_t \right)_{t \geq 0} \) starting anywhere in \([0, r]\).

This ordinary one-dimensional differential equation can be solved explicitly

\[ \tilde{r}_t = \frac{r}{(1 - (v-1)r^{v-1}t)^{1/v-1}}. \]
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Hence,
\[
d(r, R) := \frac{1}{v-1} \left( \frac{1}{r^{v-1}} - \frac{1}{R^{v-1}} \right), \quad R \geq r > 0
\]
is a lower bound for the time the radial component \((r_t)_{t \geq 0}\) needs go from \(r\) to \(R\).
Assume a trajectory stays in the good region \(\{ (\phi, r) : \cos^2(\phi) \geq \tilde{c}, r \geq \rho \}\)
for a sufficiently long time period. Further, take \(\rho\) such that \(\rho^{v-w} \leq \tilde{c}/2\).
Because of
\[
\frac{d}{dt} \hat{r}_t = \left( -r_t^w \cos^2(\phi_t) + r_t^v \right) dt \leq (-\tilde{c}r_t^w + r_t^v) dt \leq -\frac{\tilde{c}}{2} r_t^w dt
\]
the process \((\hat{r}_t)_{t \geq 0}\) solving the (deterministic) equation
\[
\frac{\partial}{\partial t} \hat{r}_t = -\frac{\tilde{c}}{2} \hat{r}_t^w, \quad \hat{r}_0 = R
\]
is an upper bound for the radial component \((r_t)_{t \geq 0}\) starting anywhere in \([0, R]\). Also this ordinary one-dimensional differential equation can be solved explicitly
\[
\hat{r}_t = \frac{R}{\left( 1 + (w-1)\frac{\tilde{c}}{2} R^{w-1} t \right)^{\frac{1}{w-1}}}.
\]
Therefore,
\[
d_{\tilde{c}, w}(R, r) := \frac{2}{\tilde{c}(w-1)} \left( \frac{1}{r^{w-1}} - \frac{1}{R^{w-1}} \right), \quad R \geq r > \rho
\]
is an upper bound for the time the radial component \((r_t)_{t \geq 0}\) starting somewhere in \([r, R]\) needs go to \(r\) inside the good region \(G\).

Proof of the step down

In the following we abbreviate
\[
A_k := \{ \tau_{k-1} \leq \theta_k \leq \tau_{k+1} \}, \quad B_k := \left\{ \sup_{z \in I(k)} \tau_{k-2}(z) \leq d(r_k, r_{k+1}) \right\},
\]
\[
D_k := \left\{ \sup_{z \in I(k)} \tau_{k-2}(z) > \theta_k \right\},
\]
where \(\theta_k = d(r_k, r_{k+1}) + d_{\tilde{c}/2, w}(r_{k+1}, r_{k-2})\).
In the following we show that the probability of $A_k$ is converging to 1 as $k$ tends to infinity. On $A_k$ all points start at the level $r_k$ and are below $r_{k-1}$ before time $\theta_k$ while none ever exceeded the level $r_{k+1}$ before. We call this “step down”. We estimate the probability of $A_k$ in terms of the probability of $B_k$ and BBM$_k$, of which we can compute explicit bounds. $D_k$ is just an auxiliary event, which cannot occur at the same time as $B_k$ and BBM$_k$, as we show below.

On the event $\{\sup_z \tau_{k-2}(z) \leq d(r_k, r_{k+1})\}$ all one-point motions leave the unbounded box $B_{k-2}$ (see figure 3.1) before they could pass $r_{k+1}$, since the time to do so is at least $d(r_k, r_{k+1})$. Leaving this box means being below $r_{k-2}$ or being strictly inside the good region. Together with BBM$_k$ trajectories in the good region will necessarily go under $r_{k-2}$. Afterwards there is not enough time to go above $r_{k-1}$ again, hence all points are simultaneously below $r_{k-1}$ at or before the time $\theta_k$.

**Theorem 3.3.5**

*For every sigma > 0, w > v > 1, $\frac{2}{3}\gamma + 1 > v$, w − 1 > $\gamma$ there is an R > 0, such that for $\rho \geq R$ the following holds true*

i) $B_k \cap \text{BBM}_k \cap D_k = \emptyset$ for all $k \geq 2$, 

---

**Figure 3.1: Step Down**

![Diagram illustrating 'step down' concept with levels $r_k$, $r_{k-1}$, $r_{k+1}$, and box boundaries $B_{k-2}$ and $B_k$.](image-url)
Chapter 3. Circular model

\[ ii) B_k \cap BBM_k = B_k \cap BBM_k \cap D_k^c \subset A_k \text{ for all } k \geq 2 \]

\[ iii) \mathbb{P}(B_k^c), \mathbb{P}(BBM_k^c) \to 0 \text{ as } k \to \infty. \]

**Proof.** First, we choose \( \rho \) sufficiently large, such that \( \theta_1 \leq T_1 \).

i) On \( \{ \sup_{z \in I(k)} \tau_{k-2}(z) \leq d(r_k, r_{k+1}) \} \cap BBM_k \cap \{ \sup_{z \in I(k)} \tau_{k-2}(z) > \theta_k \} \), there is a \( z \in I(k) \), such that

\[ \tau_{k-2}(z) \leq d(r_k, r_{k+1}) \quad \text{and} \quad \tau_{k-2}(z) > \theta_k, \]

which implies

\[ \phi_{\tau_{k-2}(z)}(z) = \frac{\pi}{2} - \varepsilon \quad \text{and} \quad r_{\tau_{k-2}(z)}(z) \in [r_{k-2}, r_{k+1}]. \]

Now we show that on the set \( BBM_k \) the process \( (\phi(z), r(z)) \) will spend sufficient time in the good region \( G \) that it will necessarily go below \( r_{k-2} \) before time \( \theta_k \). Since this is a contradiction to \( \tau_{k-2}(z) > \theta_k \), \( B_k \cap BBM_k \cap D_k \) is empty.

Define \( \vartheta := \inf \{ t \geq \tau_{k-2}(z) : \cos^2(\phi_t(z)) \leq \bar{c} \} \land \tau_{k-2}(z) \). Let \( t \geq \tau_{k-2}(z) \), then we can conclude

\[ \phi_{t \wedge \vartheta}(z) = \phi_{\tau_{k-2}(z)}(z) - \int_{\tau_{k-2}(z)}^{t \wedge \vartheta} (\gamma(z)) \cos^2(\phi_s(z))ds + \sigma \left( W_{t \wedge \vartheta} - W_{\tau_{k-2}(z)} \right) \]

\[ \geq \frac{\pi}{2} - \varepsilon - r_{k+1}^2(t \wedge \vartheta - \tau_{k-2}(z)) + \sigma \left( W_{t \wedge \vartheta} - W_{\tau_{k-2}(z)} \right) \]

\[ \geq \frac{\pi}{2} - \varepsilon - d - 2r_{k+1}^2(t \wedge \vartheta - \tau_{k-2}(z)) \]

and

\[ \phi_{t \wedge \vartheta}(z) \leq \frac{\pi}{2} - \varepsilon - r_{k-2}^2 \bar{c}(t \wedge \vartheta - \tau_{k-2}(z)) + \sigma \left( W_{t \wedge \vartheta} - W_{\tau_{k-2}(z)} \right) \]

\[ \leq \frac{\pi}{2} - \varepsilon + \bar{c} - r_{k-2}^2 \bar{c} \frac{\pi}{2} \] 

(3.5)

Note that the \( \phi(z) \) process is \( \pi \)-periodic, but the upper and lower bounds in (3.5) are not. The bounds here only serve to estimate the time of \( \phi(z) \) spent in \( G \). Taking the minimum with \( \vartheta \) prevents any interference with the boundaries 0 and \( \pi \) of the angular process \( \phi(z) \). Thus, we identify \( \{ \theta : \cos^2(\theta) \geq \bar{c} \} \) with the interval \( [-\pi/2 + \varepsilon - \bar{c}, \pi/2 - \varepsilon + \bar{c}] \).

As we see, the upper bound makes sure that the process never leaves the interval at the right boundary. The lower bound hits \( -\pi/2 + \varepsilon - \bar{c} \) at time

\[ t_0 = \tau_{k-2}(z) + \frac{\pi - 2\varepsilon + \bar{c} - d}{2r_{k+1}^2}. \]

least amount of time spent in good region.
3.3. Main result

We know from the subsection before that the time it takes in the good region to go from \( r_{k+1} \) down to \( r_{k-2} \) is at most

\[
d_{\frac{4}{3},w}(r_{k+1}, r_{k-2}) := \frac{2}{\varepsilon(w - 1)} \left( \frac{1}{r_{k-1}^{w-1}} - \frac{1}{r_{k+1}^{w-1}} \right).
\]

Because of \( w - 1 > \gamma \) we can pick \( \rho \) big enough, such that

\[
d_{\frac{4}{3},w}(r_{k+1}, r_{k-2}) \leq \frac{\pi - 2\varepsilon + \bar{\varepsilon} - d}{2r_{k+1}^2}
\]

holds. On the other hand, this means

\[
\tau_{k-2}(z) \leq \tau_{k-2}(z) + d_{\frac{4}{3},w}(r_{k+1}, r_{k-2}) \leq d(r_k, r_{k+1}) + d_{\frac{4}{3},w}(r_{k+1}, r_{k-2}) = \theta_k,
\]

which contradicts, as aforementioned, \( \tau_{k-2}(z) > \theta_k \).

ii) The first equality is just a reformulation of i).

On the event \( \{ \sup_{z} \tau_{k-2}(z) \leq d(r_k, r_{k+1}) \} \cap \text{BBM} \cap \{ \sup_{z} \tau_{k-2}(z) \leq \theta_k \} \) each trajectory goes below \( r_{k-2} \) before time \( \theta_k \). From the calculation of the last part, we even know that none of these exceeded \( r_{k+1} \) until \( \theta_k \). On the other hand, for any one-point motion to go from \( r_k \) to \( r_{k-2} \) and then over \( r_{k-1} \) it takes at least

\[
d_{1,w}(r_k, r_{k-2}) + d(r_{k-2}, r_{k-1}) = \tilde{\theta}_k
\]

time units. Because of \( w \geq v \) this time is bigger than \( \theta_k \), i.e.

\[
\theta_k = d(r_k, r_{k+1}) + d_{\frac{4}{3},w}(r_{k+1}, r_{k-2}) \leq d_{1,w}(r_k, r_{k-2}) + d(r_{k-2}, r_{k-1}) = \tilde{\theta}_k.
\]

To see this it is sufficient to show that \( d(r_{k-2}, r_{k-1}) - d(r_k, r_{k+1}) \) is positive, since it decays slower in \( \rho \) than \( d_{\frac{4}{3},w}(r_{k+1}, r_{k-2}) - d_{1,w}(r_k, r_{k-2}) \).

\[
d(r_{k-2}, r_{k-1}) - d(r_k, r_{k+1}) = \frac{1}{v - 1} \left( r_{k-2}^{1-v} - r_{k-1}^{1-v} - r_k^{1-v} + r_{k+1}^{1-v} \right)
\]
\[
= \frac{1}{v - 1} r_k^{1-v} \left( 2^{2(1-v)} - 2^{-(1-v)} - 1 + 2^{1-v} \right)
\]
\[
= \frac{1}{v - 1} r_k^{1-v} \left( 4^{v-1} - 2^{v-1} - 1 + 2^{1-v} \right).
\]

The function \( x \mapsto 4^x - 2^x - 1 + 2^{-x} \) is increasing on \([0, \infty)\) and \(0\) at \(0\). Since \( v > 1 \) we are done.

Thus, at time \( \theta_k \) the whole flow is below \( r_{k-1} \) and was never above \( r_{k+1} \), i.e.

\[
\tau_{k-1} \leq \theta_k < \tau_{k+1}.
\]
iii) Note that \( \tau_k(z) \) coincides with \( \nu_{\pi/2-\varepsilon}(z) \wedge \nu^r(z) \) defined in Section 3.3.1. Apply Chebyshev’s inequality

\[
\mathbb{P}(B_{k}^c) = \mathbb{P}\left( \sup_{z \in I(k)} \tau_{k-2}(z) > d(r_k, r_{k+1}) \right) \\
\leq \mathbb{E}\left( \sup_{z \in I(k)} \nu_{\pi/2-\varepsilon}(z) \wedge \nu^r(z) \right) / d(r_k, r_{k+1}) \\
\leq (v - 1) C(\sigma) \frac{r_k^{\frac{2}{3} - \gamma}}{r_k^{\frac{1}{3} - \frac{1}{\gamma} + 1}} \\
= (v - 1) C(\sigma) \frac{1}{4^{1-v}} \frac{r_k^{\frac{2}{3} - \gamma} - r_{k+1}^{\frac{2}{3} - \gamma}}{r_k^{\frac{1}{3} - \frac{1}{\gamma} + 1}} \\
= \tilde{C}(v, \gamma, \sigma) r_k^{\frac{2}{3} - \gamma - v - \frac{1}{3}}.
\]

This bound tends to 0 as \( k \to \infty \) because of \( v < 2\gamma/3 + 1 \).

To estimate \( \mathbb{P}(\text{BBM}_k^c) \) consider the following lemma first.

**Lemma 3.3.6**

For every \( \sigma > 0 \) there is an \( R > 0 \) such that for all \( r \geq R \) and for all \( \varepsilon > 0 \)

\[
\mathbb{P}\left( \sup_{0 \leq s \leq t \leq T} (\sigma(W_t - W_s) - r(t - s)) > \varepsilon \right) \leq \frac{4 \sqrt{2}}{\sigma^2} e^{\frac{\varepsilon^2}{8}} r^2 e^{-\frac{\sigma^2 r^2}{4}}\]

holds true.

**Proof.** Define

\[
T_\varepsilon := \inf\left\{ t \geq 0 : \sup_{0 \leq s \leq t} X_s - X_t > \varepsilon \right\} \quad \text{with} \quad X_t = \sigma W_t + rt.
\]

It follows

\[
\mathbb{P}\left( \sup_{0 \leq s \leq t \leq T} (\sigma(W_t - W_s) - r(t - s)) \leq \varepsilon \right) \\
= \mathbb{P}\left( \sup_{0 \leq s \leq t \leq T} (\sigma(W_s - W_t) + r(t - s)) \leq \varepsilon \right) \\
= \mathbb{P}\left( \sup_{0 \leq s \leq t \leq T} (X_s - X_t) \leq \varepsilon \right) = \mathbb{P}\left( \sup_{0 \leq t \leq T} \left( \sup_{0 \leq s \leq t} X_s - X_t \right) \leq \varepsilon \right) \\
= \mathbb{P}(T_\varepsilon > T).
\]
In [Tay75] the author could explicitly compute the Laplace transform of $T_\varepsilon$, i.e. for $\beta > 0$

$$Ee^{-\beta T_\varepsilon} = \frac{\delta e^{-\Gamma \varepsilon}}{\delta \cosh(\delta \varepsilon) - \Gamma \sinh(\delta \varepsilon)},$$

where $\delta = \sqrt{\left(\frac{r}{\sigma^2}\right)^2 + \frac{2\beta}{\sigma^2}}$ and $\Gamma = \frac{r}{\sigma^2}$.

We establish a more handy upper bound for this Laplace transform. It holds in the special case $\beta = 1$

$$e^{-T_\varepsilon} = \frac{\delta e^{-\Gamma \varepsilon}}{\delta \cosh(\delta \varepsilon) - \Gamma \sinh(\delta \varepsilon)} = \frac{2}{\left(1 - \frac{\Gamma}{\delta}\right)} e^{(\delta + \Gamma)\varepsilon} + \frac{2}{\left(1 + \frac{\Gamma}{\delta}\right)} e^{(-\delta + \Gamma)\varepsilon} \leq \frac{2}{\left(1 - \frac{\Gamma}{\delta}\right)} e^{(-\delta + \Gamma)\varepsilon} \leq 2 \sqrt{\frac{\Gamma^2 + \frac{2}{\sigma^2}}{\Gamma^2 + \frac{2}{\sigma^2} - \Gamma}} e^{-2\varepsilon \Gamma}.$$

(3.6)

Now, we pick $R$, such that the following holds true for all $r \geq R$

$$\sqrt{\frac{\Gamma^2 + \frac{2}{\sigma^2}}{\Gamma^2 + \frac{2}{\sigma^2} - \Gamma}} \geq \frac{1}{2\sigma^2 \Gamma} \quad \text{and} \quad \Gamma^2 \geq \frac{2}{\sigma^2}.$$

We continue equation (3.6)

$$2 \sqrt{\frac{\Gamma^2 + \frac{2}{\sigma^2}}{\Gamma^2 + \frac{2}{\sigma^2} - \Gamma}} e^{2\varepsilon \Gamma} \leq 4 \sqrt{2\Gamma^2 \sigma^2 e^{-2\varepsilon \Gamma}} = \frac{4 \sqrt{2}}{\sigma^2} r^2 e^{-\frac{\varepsilon}{\sigma^2 r}}.$$

We conclude by using Chebyshev’s inequality

$$P\left(\sup_{0 \leq s \leq t \leq T} (\sigma(W_t - W_s) - r(t - s)) > \varepsilon\right) = P(T_\varepsilon \leq T) = P(e^{-T_\varepsilon} \geq e^{-T}) \leq e^T E e^{-T_\varepsilon} \leq \frac{4 \sqrt{2}}{\sigma^2} r^2 e^{-\frac{\varepsilon}{\sigma^2 r}}.$$

□

To be able to apply this lemma to the events $\text{BBM}_k$ and $\text{BBM}_k$, we choose $\rho \geq R$, where $R$ is the number from Lemma 3.3.6. It follows

$$P(\text{BBM}_k) \leq \frac{4 \sqrt{2}}{\sigma^2} r^2 \left(\frac{c \gamma}{2^k \gamma_{k-2}}\right)^2 e^{-\frac{\varepsilon \gamma}{\sigma^2 r \gamma_{k-2}}}.$$
As one easily sees $BBM_k$ is of the same form as $\overline{BBM}_k$ after a multiplication with $-1$, i.e.

$$P(BBM_k) = P\left(\sup_{0 \leq s \leq t \leq T} \left(\sigma(W_s - W_t) - r_{k+1}^\gamma(t-s)\right) \leq d\right).$$

Therefore,

$$P(\overline{BBM}^c_k) \leq 4\sqrt{2} \sigma^2 e^{\frac{\gamma}{\sigma^2}} e^{-\frac{2d}{\sigma^2} r_{k+1}^\gamma}.$$ 

The inequality $P(\overline{BBM}^c_k) \leq P(BBM^c_k) + P(\overline{BBM}^c_k)$ gives us a vanishing upper bound for $k \to \infty$. □

The following corollary is a consequence of Theorem 3.3.5 ii) and iii).

**Corollary 3.3.7**

For $w > v > 1$, $\frac{2}{3} \gamma + 1 > v$, $w - 1 > \gamma$ it holds for each $\sigma > 0$

$$P(A_k) \to 1 \text{ as } k \to \infty.$$

### 3.3.3 Markov-like argument

An attractor attracts all trajectories starting form an arbitrary bounded set, but in the previous section the initial set was of the form $\{r_k\} \times [0, \pi]$. Fortunately, we do not lose any generality, because for any bounded set $B$ with $\sup_{x \in B} |x| \leq r$ the following holds true for all $\omega \in \Omega$ and $t \geq s \geq 0$ (such that $e(s, z, \omega) > t$ for all $z \in B$)

$$\sup_{z \in B} r_{s,t}(z) \leq \sup_{z \in \{r\} \times [0, \pi]} r_{s,t}(z). \quad (3.7)$$

**Dominating jump process**

In the following we abuse notation and write, for example, $r_{K_n}$ for the level $r_k = \rho 2^k$ if $K_n = k$. This notation overlaps with $r_t = r_{0,t}$, which describes the radial component of the (local) stochastic flow. However, in the latter case the time index is $[0, \infty)$-valued and it may have an additional argument “($z$)” to indicate the initial value.

Recall $I(k) = \{r_k\} \times [0, \pi]$ for $k \in \mathbb{N}$. Define the jump process $(K_t)_{t \geq 0}$
starting in $k \in \mathbb{N}$ inductively.

$$
\tau^0 := 0, \quad K_0 := k,
$$

$$
\tau^n := \inf \left\{ t > \tau^{n-1} : \rho < \sup_{z \in I(K_{\tau^{n-1}})} r_{\tau^{n-1},t}(z) \leq r_{K_{\tau^{n-1}}-1} \right\},
$$

$$
\tau^n := \inf \left\{ t > \tau^{n-1} : \sup_{z \in I(K_{\tau^{n-1}})} r_{\tau^{n-1},t}(z) \geq r_{K_{\tau^{n-1}}+1} \right\} \land \left( \theta_{K_{\tau^{n-1}}} + \tau^{n-1} \right),
$$

$$
\tau^n := \tau^n \land \tau^n,
$$

$$
K_t = \begin{cases} 
K_{\tau^{n-1}} - 1 & \text{if } t \in [\tau^n, \tau^{n+1}), \; \tau^n \leq \tau^n, \\
K_{\tau^{n-1}} + 1 & \text{if } t \in [\tau^n, \tau^{n+1}), \; \tau^n > \tau^n.
\end{cases}
$$

So far $K_t$ is only defined for $t < \tau := \lim_{n \to \infty} \tau^n$, where $\tau$ might be finite with positive probability. We will deal with this issue in section “Semi-Markov property”.

“$\rho <$” in the definition of $\tau^n$ guarantees that $(K_t)_{t \geq 0}$ never takes values below 1. Further, taking the minimum in the definition of $\tau^n$ with $\theta_{K_{\tau^{n-1}}} + \tau^{n-1}$ provides an uniform upper bound for $\tau^n - \tau^{n-1}$, namely $\theta_1$.

**Lemma 3.3.8**

For any initial value $k \in \mathbb{N}$ the following holds

a) $$
\sup_{z \in I(k)} r_{\tau^n}(z) \leq r_{K_{\tau^n}} \quad \text{for all } n \in \mathbb{N} \cup \{0\},
$$

b) $$
\sup_{z \in I(k)} r_{t}(z) \leq r_{K_{t+1}} \quad \text{for all } t < \tau.
$$

**Proof.** We prove a) via induction. The case $n = 0$ is trivial. Take a number $n \in \mathbb{N}$ and assume a) holds for all natural numbers less than $n$.

First case: $\tau^n \leq \tau^n$

In this case we have $K_{\tau^n} = K_{\tau^{n-1}} - 1$, it follows

$$
\sup_{z \in I(k)} r_{\tau^n}(z) = \sup_{z \in I(k)} r_{\tau^{n-1},\tau_{n}} \left( r_{\tau^{n-1}}(z) \right) \leq \sup_{z \in I(K_{\tau^{n-1}})} r_{\tau^{n-1}}(z) \leq r_{K_{\tau^{n-1}}-1} = r_{K_{\tau^n}},
$$

1st case

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Second case: $\tau^n > \tau^n$
In this case we have $K_{\tau^n} = K_{\tau^n-1} + 1$, it follows

$$
\sup_{z \in I(k)} r_{\tau^n}(z) \leq \sup_{z \in I(K_{\tau^n-1})} r_{\tau^n-1,\tau^n}(z)
$$

2nd case

$$
\leq r_{K_{\tau^n-1}+1} = r_{K_{\tau^n}}.
$$

b) follows easily from a). Take $t \in [\tau^n, \tau^{n+1})$.
We have $K_t = K_{\tau^n}$, it follows

$$
\sup_{z \in I(k)} r_{t}(z) = \sup_{z \in I(k)} r_{\tau^n,t}(r_{\tau^n}(z)) \overset{\text{a), (3.7)}}{\leq} \sup_{z \in I(K_{\tau^n})} r_{\tau^n,t}(z)
$$

$$
= r_{K_{\tau^n}+1} = r_{K_t+1}.
$$

\[\square\]

**Semi-Markov property**

In this subsection we recall parts of the theory of Semi-Markov processes and Markov renewal processes. For a more detailed description we refer to [Çın75, p. 313ff].

Let, for each $n \in \mathbb{N}$, $X_n$ be a random variable taking values in a countable set $E$ and $T_n$ an $\mathbb{R}^+$-valued random variable, such that $0 = T_0 \leq T_1 \leq T_2 \leq \ldots$.

**Definition 3.3.9**

We call the stochastic process $(X_n, T_n)_{n \in \mathbb{N}}$ a **Markov renewal process with state space $E$**, if

$$
P(X_{n+1} = j, T_{n+1} - T_n \leq t | X_0, \ldots, X_n; T_0, \ldots T_n) = P(X_{n+1} = j, T_{n+1} - T_n \leq t | X_n)
$$

holds for all $n \in \mathbb{N}$, $j \in E$ and $t \in \mathbb{R}^+$. The process $Z_t = X_n$ for $t \in [T_n, T_{n+1})$ is called **Semi-Markov process** and $(X_n)_{n \in \mathbb{N}}$ its **embedded Markov chain**.

**Remark 3.3.10**

A Markov renewal process is called **time-homogeneous**, if in addition

$$
P(X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i) = P(X_1 = j, T_1 \leq t | X_0 = i)
$$

holds for any $n \in \mathbb{N}$, $i, j \in E$ and $t \in \mathbb{R}^+$.

In the following we only work with time-homogeneous processes.
3.3. Main result

We denote for \( i, j \in E \)
\[
\mathbb{P}_i := \mathbb{P}(X_0 = i), \\
m(i) := \mathbb{E}_i T_1, \text{ where } \mathbb{E}_i \text{ is the expectation under } \mathbb{P}_i, \\
P_{i,j} := \mathbb{P}_i (X_1 = j) = \lim_{t \to \infty} \mathbb{P}_i (X_1 = j, T_1 \leq t), \\
P := (P_{i,j})_{i,j \in E}.
\]

**Definition 3.3.11**
A Semi-Markov process is called **irreducible** or **recurrent** if the corresponding embedded Markov chain is irreducible or recurrent. Further, let \( S_{i_1}, S_{i_2}, \ldots \) be a sub-sequence of \( T_1, \ldots \), such that \( S_{i_1} < S_{i_2} < \ldots \) and \( K_{S_n} = j \) for every \( n \in \mathbb{N} \). \( (Z_t)_{t \geq 0} \) is called **periodic with period** \( \delta \) if \( S_{i_1}, S_{i_2} - S_{i_1}, S_{i_3} - S_{i_2}, \ldots \) take values in a discrete set \( \{0, \delta, 2\delta, \ldots \} \) where \( \delta > 0 \) is the largest such number. If there is no such \( \delta > 0 \), \( (Z_t)_{t \geq 0} \) is called **aperiodic**.

**Theorem 3.3.12**
If the Semi-Markov process is irreducible, recurrent and aperiodic, \( \nu \) is a non-trivial solution to \( v = vP \) and \( m(k) < \infty \) for all \( k \in E \), then for any \( i \in E \)
\[
\lim_{t \to \infty} \mathbb{P}_i (Z_t = j) = \frac{1}{\nu m} \nu(j)m(j),
\]
where \( \nu m := \sum_{j \in E} \nu(j)m(j) < \infty \).

The proof can be found in [Çın75, p. 342].

Analogous to the notation we define \( T_n := \tau^n \) and \( X_n := K_{\tau^n} \). By definition, \( (X, T) \) is a Markov renewal process with state space \( \mathbb{N} \) and \( (K_t)_{t \geq 0} \) is the corresponding Semi-Markov process. Further, the embedded Markov chain \( (X_n)_{n \in \mathbb{N}} \) has the transition probabilities
\[
P_{i,j} = \begin{cases} 
1 - p_i & \text{if } j = (i - 1) \lor 1, \\
p_i & \text{if } j = i + 1, \\
0 & \text{else},
\end{cases}
\]
where \( p_i := \mathbb{P}_i (X_1 = i + 1), i \geq 1 \). Note \( p_1 = 1 \). Without loss of generality we assume \( p_i > 0 \), which implies that \( (X_n)_{n \in \mathbb{N}} \) is irreducible and therefore also \( (K_t)_{t \geq 0} \).

**Lemma 3.3.13**
For \( w > v > 1, \frac{2}{3} \gamma + 1 > v, w - 1 > \gamma \) it holds for each \( \sigma > 0 \)
\[
p_i \leq 1 - \mathbb{P}(A_i) \to 0 \text{ as } i \to \infty.
\]
Proof. Note the difference between the stopping times $\tau_i, \bar{\tau}_i$ from the definition of the dominating jump process $(K_t)_{t \geq 0}$ and $\tau_i, \bar{\tau}_i$ from Section 3.3.2. Take $k = i$ as the initial value for the jump process $(K_t)_{t \geq 0}$.

$$p_i = \mathbb{P}_i(X_1 = i + 1) = \mathbb{P}_i(\tau^1 > \bar{\tau}^1)$$
$$= 1 - \mathbb{P}(\tau_{i-1} \leq \bar{\tau}_{i+1} \wedge \theta_i)$$
$$\leq 1 - \mathbb{P}(A_i) \xrightarrow{3.3.7} 0 \text{ as } i \to \infty.$$

□

The Markov chain $(X_n)_{n \in \mathbb{N}}$ is recurrent and has an invariant probability distribution $\nu$ if

$$\sum_{i=2}^{\infty} \prod_{j=1}^{i-1} \frac{p_j}{1 - p_{j+1}} < \infty.$$

This series is indeed finite by Cauchy’s ratio test and Lemma 3.3.13. It also holds $\tau = \infty$ a.s., see [Çın75, Prop. 3.16, p. 327]. This implies that the (local) flow $(r_{s,t}(z), \phi_{s,t}(z))$ is in fact global.

Finally, recall that $T_1 = \tau^1$ and $\tau^1 \leq \theta_i$ $\mathbb{P}_i$-a.s., which yields

$$m(i) = \mathbb{E}_i T_1 \leq \theta_i \leq \theta_1.$$

Hence, $\nu m = \sum_{i=1}^{\infty} m(i) \nu(i) \leq \theta_1 < \infty$ and thereby the measure $\mu$ defined via $\mu(i) = \frac{1}{\nu m} \nu(i) m(i)$ is a probability measure.

Now we only need to ensure that the Semi-Markov process is aperiodic in order to be able to apply Theorem 3.3.12. For any $\delta > 0$ there is a $n \in \mathbb{N}$ such that $\theta_n < \delta$. Hence, the period of $n$ must be less than $\delta$. But since periodicity is a class property the Semi-Markov process $(K_t)_{t \geq 0}$ is aperiodic.

### 3.3.4 Weak attractor

Now, we prove the existence of a global weak set attractor. Recall the equivalent criterion from Theorem 3.2.2.

**Proof of Theorem 3.3.1**

Let $\varepsilon > 0$, $R > 0$ and choose $k(R) = k \in \mathbb{N}$, such that $R \leq r_k$. Lemma 3.3.8 yields

$$\mathbb{P} \left( \sup_{z \in [0,R] \times [0,\pi]} r_t(z) \leq R_0 \right) \geq \mathbb{P}_k \left( r_{K_{i+1}} \leq R_0 \right) = \mathbb{P}_k \left( K_t \leq \log_2 \left( \frac{R_0}{\rho} \right) - 1 \right)$$

(3.8)
Further, pick $R_0(\varepsilon) = R_0 > 0$, such that
\[ \mu\left(\{1, 2, \ldots, \lceil\log_2(R_0) - \log_2(\rho)\rceil - 1\}\right) \geq 1 - \frac{\varepsilon}{2}. \tag{3.9} \]

Theorem 3.3.12 says that for each $A \subseteq \mathbb{N}$ there is a $t_0(\varepsilon, k) = t_0 > 0$, such that for all $t \geq t_0$
\[ |\mathbb{P}_k(K_t \in A) - \mu(A)| \leq \frac{\varepsilon}{2}. \tag{3.10} \]

Plugging in (3.10) and (3.9) into (3.8) yields
\[ \mathbb{P}\left(\sup_{z \in [0,R] \times [0,\pi]} r_t(z) \leq R_0 \right) \geq \mathbb{P}_k\left(K_t \leq \log_k\left(\frac{R_0}{\rho}\right) - 1\right) \geq \mu\left(\{1, 2, \ldots, \lceil\log_2(R_0) - \log_2(\rho)\rceil - 1\}\right) - \frac{\varepsilon}{2} \geq 1 - \varepsilon. \]

### 3.3.5 Tail estimate

This sections aims to estimate the tails of the diameter of the attractor. Since the attractor supports any invariant measure (see [Och99, Theorem 2]), it also serves as an upper bound for the their tails. Thanks to the previous analysis we have an upper bound for the diameter of the attractor, namely $(r_{K_{t+1}})_{t \geq 0}$ started in its invariant distribution.

**Corollary 3.3.14**

For parameters $w > v > 1, \frac{2}{3}\gamma + 1 > v, w - 1 > \gamma$ and for each $\sigma > 0$, we have for sufficiently large $\rho$ that for all $x \geq 1$
\[ \mathbb{P}(r_{K_{t+1}} > \rho x) \lesssim 2^{-\frac{1}{2}(\lceil\log_2(x)\rceil - 1)^2(1+\frac{2}{3}\gamma-v)} \]
holds true.

**Proof.** It suffices to estimate the tails of the invariant probability measure $\mu$ of $(K_t)_{t \geq 0}$. Due to Theorem 3.3.12 we know that this invariant measure is of the following form
\[ \mu(n) = C^{-1}\nu(n)m(n), \quad n \in \mathbb{N}, \]
where $C > 0$ is a normalization constant, $\nu$ the invariant measure of the embedded Markov chain $(X_n)_{n \in \mathbb{N}}$ and $m(n)$ the waiting time in state $n$ as above.

First, we handle the invariant measure of the embedded Markov chain. It
contributes the most to the decay of the tails of $\mu$.

Recall the transition matrix of the embedded Markov chain

$$P_{i,j} = \begin{cases} 1 - p_i & \text{if } j = (i - 1) \lor 1, \\ p_i & \text{if } j = i + 1, \\ 0 & \text{else,} \end{cases}$$

where $p_i$ is converging to 0 as $i \to \infty$. Furthermore, we can extract an upper bound for the decay from the proof of Theorem 3.3.5

$$p_i \lesssim r_i^{v-\frac{3}{2} \gamma - 1} =: r_i^\alpha \rho^{\alpha 2\omega}.$$  

Recall that according to our parameter regime $\alpha < 0$. A simple computation yields (where $Z$ is a normalization constant)

$$\nu(n) = Z^{-1} \prod_{i=1}^{n-1} \frac{p_i}{1 - p_{i+1}} \lesssim \rho^{(n-1)\alpha} 2^{\frac{1}{2} n(n-1)\alpha}.$$  

Combining this with the inequality

$$m(n) \leq \theta_n \lesssim r_n^{-(v-1)} = \rho^{-(v-1)2^{-n(v-1)},}$$

yields

$$\mu(n) \lesssim \rho^{(n-1)\alpha} 2^{\frac{1}{2} n^2\alpha - n(v-1+\frac{\omega}{2})} \lesssim 2^{\frac{1}{2} n^2\alpha}.$$  

Finally, we can estimate the tails of the dominating jump process, let $x > 1$

$$\mathbb{P}(r_{K_{i+1}} > \rho x) = \mathbb{P}(K_t > \log_2(x) - 1) = \sum_{k=\lfloor \log_2(x) - 1 \rfloor}^{\infty} \mu(k) \approx \mu(\lfloor \log_2(x) - 1 \rfloor) \lesssim \rho^{(\lfloor \log_2(x)-2 \rfloor)\alpha} 2^{\frac{1}{2} ((\lfloor \log_2(x)-1 \rfloor)^2 \alpha - (\lfloor \log_2(x)-1 \rfloor)(v-1+\frac{\omega}{2}))} \lesssim 2^{\frac{1}{2} ((\lfloor \log_2(x)-1 \rfloor)^2 \alpha)}.$$  

\[\square\]

### 3.3.6 Conclusion

We managed to show the existence of a weak attractor for the parameter choices $w > v > 1, \frac{2}{3} \gamma + 1 > v, w - 1 > \gamma, v > 1$ is clearly necessary, otherwise there is no explosion in the deterministic case. Also $w > v$ is necessary, otherwise the drift in the radial component is non-negative and therefore there is no compensation.

Recall that the condition $2\gamma > w - v$ implies explosion in finite time for all initial conditions of the form $r_0 > 0, \phi \in [0, \pi]$ in the deterministic case. Note that these conditions are not contradictory to the ones above, choose for example $\gamma = v = 2, w = 4$.  

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3.3. Main result

Conjectures

In the following we only present ideas based on heuristics, which have not been made rigorous yet. We obtain these heuristics from a scaling analysis of the terms in the infinitesimal generator of the driving equation (3.1). Which term dominates the others depends, of course, on the spatial position, therefore we divide the state space in critical and good region as above. The behavior in these regions changes with the parameters. Thus, we go through some parameter regimes and state our conjectures.

Case: $w - 1 > \gamma$:
This condition yields a strong drift inwards in the good region, see the proof of Theorem [3.3.5]. In fact, we think that after a (typical) trajectory (started in $r_0$) traversed through the good region, it enters the critical region on the level which is independent of $r_0$. In other words, in the good region points come down from infinity in finite time. The condition $2\gamma/3 + 1 > v$ makes sure that we do not explode in finite time in the critical region. If $2\gamma/3 + 1 = v$, we still think that there is no explosion in the critical region, but we conjecture that the tails of the invariant measure decay only polynomially.

Case: $w - 1 = \gamma$:
The decay in the good region is proportional to the starting value, i.e. if started in $r_0$ then the level when hitting the critical region is approximately $r_0 e^{-\pi}$. We claim that this is sufficient to compensate the growth in the critical region if $2\gamma/3 + 1 > v$. But if $2\gamma/3 + 1 = v$ this is more delicate as the growth in the critical region is also proportional to the starting value. In contrast to the good region the proportional growth is random and furthermore, for large $\sigma > 0$ it should be smaller than $e^{-\pi}$. Hence, there should an invariant probability measure provided $\sigma$ is sufficiently large.

Case: $w - 1 < \gamma$:
We think that after traversing the good region trajectories arrive at level $r_0 - c$ for some constant $c > 0$. If $2\gamma/3 + 1 > v$ there might still be an invariant measure if in addition $w - v > \gamma/3$ holds. If $2\gamma/3 + 1 = v$ there is no invariant measure, trajectories tend to “spiral” outwards.

No matter the condition on $w$ and $\gamma$, if $2\gamma/3 + 1 < v$ we think there is explosion in finite time in the critical region.
Chapter 3. Circular model
Chapter 4
Gradient case

In this chapter we want to find an example for noise-induced strong stabilization in the gradient case. That means that the drift coefficient $b$ is of the form $b = -\nabla V$ for a so-called potential $V: \mathbb{R}^2 \to \mathbb{R}$. On one hand this restricts the set of possible drift functions from which we can choose, on the other hand it provides more structure, for example there is a nice integrability condition (see below), which is a strong indicator for the existence of an invariant probability measure.

The content of this chapter is based on work together with Michael Scheutzow. Unfortunately, we did not manage to show a last missing piece to prove the existence of a random attractor. Nevertheless, we present the current state of research and point out some of the difficulties.

4.1 Construction of the potential

We want to construct a function $V: \mathbb{R}^2 \to \mathbb{R}$, such that

$$d(X_t, Y_t) = -\nabla V (X_t, Y_t) \, dt + \sigma d\left(W_t^{(1)}, W_t^{(2)}\right), \quad (X_0, Y_0) = (x', y'), \quad (4.1)$$

explodes for all initial condition $(x', y') \in \mathbb{R}^2$, if $\sigma = 0$. Whereas, in case of $\sigma > 0$, the system corresponding to (4.1) is non-explosive a.s. for every initial condition and there even exists a random attractor.

While constructing the potential $V$ it is advantageous to incorporate the following properties.

i) $V$ is two times continuously differentiable.

ii) There is a region, which narrows when moving away from the origin, say along the $x$-axis, in which $V$ decays sufficiently fast, say of order $-x^3$. 

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iii) $V$ satisfies the integrability condition

$$\int_{\mathbb{R}^2} e^{-\frac{2}{\sigma^2} V(x)} \, dx < \infty. \quad (4.2)$$

Condition i) implies that the drift coefficient $b = -\nabla V$ is locally Lipschitz, which generates a local stochastic flow with the same line of argumentation as in the previous chapters. The second condition is a bit vague, but very important. The region in which $V$ decays very fast is the same in which $b$ grows of order $x^2$, which is sufficient to have explosion in the deterministic case provided the solutions stay long enough in this region. That this region narrows as solutions move away from the origin helps the noise to “kick” the solutions away from the $x$-axis. The integrability condition provides a good insight on how fast the region should shrink. Furthermore, the integrability condition is necessary for the existence of an invariant probability measure. More precisely, if there exists an invariant probability measure $\mu$, then

$$\mu(A) = \left( \int_{\mathbb{R}^2} e^{-\frac{2}{\sigma^2} V(x)} \, dx \right)^{-1} \int_A e^{-\frac{2}{\sigma^2} V(x)} \, dx \quad \text{for all } A \in \mathcal{B}(\mathbb{R}^2) \quad (4.3)$$

holds true. This follows from the fact that $x \mapsto e^{-\frac{2}{\sigma^2} V(x)}$ is a non-negative solution to the stationary Fokker-Planck-Kolmogorov equation (explanation see below) and [Ebe14, Theorem 4.2].

We restrict the construction of $V$ on the half-plane $\{(x, y) : x \geq x_0\}$ for some $x_0 > 0$. Outside this plane we choose $V$ to be “nice”, such that i) and iii) are not violated and such that the drift points towards this half-plane.

Consider in the following $x \geq x_0, y \in [-1, 1]$. Choose, according to ii),

$$V(x, 0) = -x^3$$

to guarantee explosion in the deterministic case along the positive $x$-axis. We choose $V$ to be symmetric in $y$, i.e. $V(\cdot, -y) = V(\cdot, y)$, therefore it is sufficient to focus on the upper half of the strip. Fix $y \geq 0$, we define

$$V(x, y) = \begin{cases} -x^3 & \text{for } y \leq z^{-1}(x), \\ Q_n(x, y) + 1_{y \geq 1}(y - 1)^3 & \text{for } y \in [z^{-1}(x), \infty), \end{cases}$$

where $Q_n$ is a polynomial of degree $n \geq 4$ in $x$. Further, $z^{-1} : [x_0, \infty) \to (0, 1]$ is the inverse function of $z : (0, 1] \to [x_0, \infty)$, which is designed such that the integrability condition is satisfied. The symmetry of $V$ implies the symmetry of $z$, i.e. $z(-y) = z(y)$.

The coefficients of the polynomial $x \mapsto Q_n(x, y)$ are going to depend on $z$
and therefore on \(y\).

Since we need \(V\) to be \(C^2\) we have to match the derivatives of \(-x^3\) and \(Q_n\) at \(x = z(y)\). We make the following Taylor-like ansatz for \(Q_n\) (where \(z = z(y)\))

\[
Q_n(x, y) = (x - z)^n + b(x - z)^2 + c(x - z) + d,
\]

where we pick \(b, c, d\) in such a way that \(x \mapsto V(x, y)\) is \(C^2\) at \(z\), i.e.

\[
d = -z^3, \quad c = -3z^2, \quad 2b = -6z.
\]

Note that it is necessary that the leading order coefficient of \(Q_n(\cdot, y)\) is positive (here \(= 1\)), otherwise there is explosion also away from the \(x\)-axis and no hope of satisfying the integrability condition. So far the function \(V\) is designed, such that its partial derivatives in \(x\) exists up to second order. Now, we check if the same holds true for derivatives in \(y\). First,

\[
\partial_y Q_n(x, y) = \left(-n(x - z(y))^{n-1} - 3(x - z(y))^2\right) z'(y)
\]

\[
\partial_{yy} Q_n(x, y) = \left((n-1)(x - z(y))^{n-2} + 6(x - z(y))\right) (z'(y))^2
\]

\[
+ \left(-n(x - z(y))^{n-1} - 3(x - z(y))^2\right) z''(y)
\]

\[
\partial_x \partial_y Q_n(x, y) = \left(-n(n-1)(x - z(y))^{n-2} - 6(x - z(y))\right) z'(y).
\]

One easily sees that at \(x = z\), i.e. we choose \(y\) such that \(z(y) = x\), we have \(\partial_y Q_n(z(y), y) = \partial_{yy} Q_n(z(y), y) = \partial_{yx} Q_n(z(y), y) = 0\). Note that the derivatives \(\partial_y z, \partial_{yy} z\) do not matter for the differentiability of \(V\).

**Corollary 4.1.1**

If \(y \mapsto z(y) \in C^2(\mathbb{R}\{0\})\) then \(V \in C^2(\mathbb{R}^2)\).

### 4.2 Integrability condition

#### 4.2.1 Critical region

The region, in which the integrability condition might fail, is around the positive \(x\)-axis. Therefore, we focus on the area \(\{(x, y): x \geq x_0, y \in [0, 1]\}\). At the same time, we also deal with the area \(\{(x, y): x \geq x_0, y \in [-1, 0]\}\), thanks to the symmetry of \(V\).

First, note there is a \(K > 1\) such that for \(x \geq Kz(y)\)

\[
-Q_n(x, y) \leq -\frac{1}{2} x^n.
\]
Chapter 4. Gradient case

It follows
\[
\int_0^1 \int_{x_0}^\infty e^{-\frac{2}{\sigma^2} V(x,y)} \, dx \, dy \leq \left( \int_0^1 \int_{x_0}^{Kz} e^{\frac{2}{\sigma^2} x^3} \, dx \, dy \right) + \left( \int_0^1 \int_{Kz}^\infty e^{-\frac{2}{\sigma^2} Q_n(x,y)} \, dx \, dy \right).
\]

For the second term we obtain for some constant \( C_II \in (0, \infty) \)
\[
II \leq \int_0^1 \int_{Kz}^\infty e^{-\frac{2}{\sigma^2} x^3} \, dx \, dy \leq \sqrt{\sigma^2} \int_0^1 \int_0^\infty e^{-t^3} \, dt \, dy = \sqrt{\sigma^2} C_II < \infty.
\]

For the first term we make a very rough estimate
\[
I = \int_0^1 \int_{x_0}^{Kz} e^{\frac{2}{\sigma^2} x^3} \, dx \, dy \leq \int_0^1 (Kz - x_0) e^{\frac{2}{\sigma^2} K^3 z^3} \, dy \leq \int_0^1 e^{\frac{z^4}{2}} \, dy,
\]
where the last inequality holds true uniformly in \( \sigma \) and \( K \) provided \( x_0 \) is sufficiently large. The function \( z \) needs to fulfill certain properties

i) \( z(y) \to \infty \) as \( y \to 0 \),

ii) \( z \) is (strictly) monotone decreasing and \( z(1) = x_0 \),

iii) \( z \) is two times differentiable,

iv) the convergence in i) is such that
\[
\int_0^1 e^{z(y)^4} \, dy < \infty.
\]

**Remark 4.2.1**

We try to avoid to assume a specific form for the function \( z \) to be flexible when needed. Hence, we keep \( z \) as generic as possible. However, we have some feasible choices in mind, for example

a) \( z(y) = x_0 + \sqrt[\frac{1}{2}]{-\alpha \log |y|} \) for some \( \alpha \in (0, 1) \),

b) \( z(y) = \sqrt[\frac{1}{2}]{-x_0^\frac{3}{2} - \alpha \log |y|} \) for some \( \alpha \in (0, 1) \),

c) \( z(y) = x_0 + \sqrt[\frac{1}{2}]{-\alpha \log |\frac{1}{2}|} \) for some \( \alpha \in (0, 1) \),

d) \( z(y) = \log (e^{x_0} - \log |y|) \).
4.3. Fokker-Planck-Kolmogorov equation and Lyapunov functions

Definition a) is not suitable, since \( z'(1) = -\infty \). Also, in the other cases the derivative is \(-\infty\) at a certain point (which is bigger than one). Nevertheless, the definition of \( z \) is mostly relevant close to the \( x \)-axis, hence we could cut it off or smooth it for \(|y| > 1\).

Later we see that it might be helpful to change the 4th root in a)-c) to an \( m \)th root, where \( m > n \).

Note that this way \( x_0 \) depends on \( \sigma \), but is part of the definition of \( z \), hence of \( V \) as well. It is rather unnatural to have the potential depending on the noise coefficient. Of course one can fix \( \sigma = 1 \).

Due to property iv) we have \( I < \infty \).

### 4.2.2 Remaining area

Here, we deal with the region \([x_0, \infty) \times [1, \infty)\). As before, there is a \( K > 1 \) such that for \( x \geq Kz(y) \)

\[-Q_n(x, y) \leq -\frac{1}{2}x^n.\]

It follows

\[
\int_1^\infty \int_{x_0}^\infty e^{-\frac{2}{\sigma^2}V(x,y)} \, dx \, dy \leq \int_1^\infty \int_{x_0}^x e^{-\frac{2}{\sigma^2}(Q_n(x,y)+(y-1)^3)} \, dx \, dy
\]

\[
= I \left[ \int_1^\infty \int_{x_0}^x e^{-\frac{2}{\sigma^2}(Q_n(x,y)+(y-1)^3)} \, dx \, dy \right].
\]

For the first term there is a constant \( C = C(K, x_0) \) that depends only on \( K \) and \( x_0 \), such that

\[
I = \int_1^\infty e^{-\frac{2}{\sigma^2}(y-1)^3} \int_{x_0}^x e^{-\frac{2}{\sigma^2}Q_n(x,y)} \, dx \, dy \leq C \int_1^\infty e^{-\frac{2}{\sigma^2}(y-1)^3} \, dy < \infty.
\]

For the second term we have

\[
II \leq \int_1^\infty e^{-\frac{2}{\sigma^2}(y-1)^3} \, dy \int_{Kx_0}^\infty e^{-\frac{1}{\sigma^2}x^n} \, dx < \infty.
\]

### 4.3 Fokker-Planck-Kolmogorov equation and Lyapunov functions

In this section we briefly discuss known results on the Fokker-Planck-Kolmogorov equation and Lyapunov functions. From the integrability condition
from the previous section we conclude the existence of solutions to (4.1) and the existence of a unique invariant probability measure, see below. The operator
\[ Lf = \langle b, \nabla f \rangle + \frac{\sigma^2}{2} \Delta f \]
is called generator of the SDE (4.1) acting on a suitable class of functions \( f \). Denote by \( L^* \) the adjoint operator, i.e.
\[ L^* f = -\text{div} (fb) + \frac{\sigma^2}{2} \Delta f. \]
We call the equation \( L^* f = 0 \) stationary Fokker-Planck-Kolmogorov equation. Note that in the literature this equation is often formulated in a weak sense. In the special case at hand \( (b = -\nabla V) \), the function \( e^{-\frac{\sigma^2}{2} V} \) is a positive solution to the stationary Fokker-Planck-Kolmogorov equation. It can be normalized to the probability measure \( \mu \), see (4.3). This is not enough to conclude that \( \mu \) is the invariant measure, since the drift \( b \) has an area, in which it grows superlinear. However, there is a nice criterion for this situation in a recent book [BKRS15]. We cite a less general version of [BKRS15, Theorem 5.3.9] to fit our needs.

**Theorem 4.3.1**

Let \( \mu \) be as above. If
\[ b_i \in L^{2+}_\text{loc} (\mathbb{R}^2) \quad \text{and} \quad \frac{|b(x)|}{1 + |x|} \in L^{1}(\mu), \]
then there is a function \( V \in W^{2+}_{\text{loc}} (\mathbb{R}^2) \) such that \( V(x) \to \infty \) and \( LV(x) \to -\infty \) as \( |x| \to \infty \).

The class \( W^{2+}_{\text{loc}} (\mathbb{R}^2) \) consists of all functions \( f \), such that the restriction of \( f \) to each ball \( U \) belongs to the Sobolev space \( W^{p_U, 2} (U) \) for some \( p_U > 2 \). \( L^{2+}_{\text{loc}} (\mathbb{R}^2) \) is defined accordingly. The function \( V \) is a so-called Lyapunov function.

Such a Lyapunov function implies that a solution to (4.1) exists for all times almost surely for every \( x \in \mathbb{R}^2 \), see [Kha11, Theorem 3.5, page 75]. Furthermore, we can conclude that there exists an invariant probability measure, [Kha11, Theorem 3.7, page 80]. With the insight prior to (4.3) we know that \( \mu \) is this unique invariant probability measure. It remains to verify the assumptions of Theorem 4.3.1. The first assumption is clear, since \( b \) is locally bounded. Concerning the second assumption, we prove a stronger statement, \( |b(x)| \in L^{1}(\mu) \). We use
4.4 Uniform control in the initial condition

similar estimate as in the previous section.

\[ \int_0^1 \int_{x_0}^\infty |b(x, y)| \mu(dx, dy) = \int_0^1 \int_{x_0}^\infty |b(x, y)| e^{-\frac{2}{\sigma^2} V(x, y)} dx dy \]

\[ \leq \int_0^1 \int_{x_0}^{Kz} 3x^2 e^{-\frac{2}{\sigma^2} x^3} dx dy + \int_0^1 \int_{Kz}^\infty n x^{n-1} e^{-\frac{1}{\sigma^2} x^n} dx dy \]

\[ \leq \int_0^1 \int_{x_0}^{Kz} 3x^2 e^{-\frac{2}{\sigma^2} x^3} dx dy + \int_0^1 \int_{Kz}^\infty n x^{n-1} e^{-\frac{1}{\sigma^2} x^n} dx dy \]

\[ = \frac{\sigma^2}{2} \int_0^1 e^{\frac{1}{\sigma^2} (Kz)^3} dy + \frac{1}{2} \sigma^2 < \infty. \]

The remaining area can be treated analogously.

We have established existence of solutions of (4.1) together with an invariant measure. Unfortunately, this does not help to show existence of an attractor or even strong completeness.

4.4 Uniform control in the initial condition

Our aim in the upcoming subsections is to gain control on the solutions such that a step down argument as in the previous chapter can be conducted. Unfortunately, the system at hand has not the same monotonicity properties as the circular model (see Subsection 3.3.1). Nevertheless, we do the following. We show that it is very likely that each solution with initial condition in \( \{x_k\} \times \mathbb{R} \) will go left of \( \{x_{k-2}\} \times \mathbb{R} \) before a certain time \( K_k \).

We show that the same holds true for all solutions with initial condition in \( \{x_k\} \times [-y^*, y^*]^c \), for some \( y^* > 0 \), simultaneously, see Subsection 4.4.2. To conclude that also all solutions with initial condition in \( \{x_k\} \times [-y^*, y^*] \) perform the step down simultaneously we apply a chaining technique. We cover the line \( \{x_k\} \times [-y^*, y^*] \) with finitely many small balls and try to estimate their expansion. However, this technique does not work, because we cannot control the expansion sufficiently well. We illustrate the problem in more detail in Subsection 4.4.3.

We need to know how much time solutions spend close to the \( x \)-axis, therefore we freeze the \( x \)-component and calculate expected exit times similar to the previous chapter.

4.4.1 Frozen \( x \)-component

To obtain a better insight we take a closer look on the behavior of the \( y \)-coordinate close to the \( x \)-axis for \( x \)-coordinate frozen at \( x^* > x_0 \), i.e. we
consider
\[d\tilde{Y}_t = b_2(x^*, \tilde{Y}_t)dt + \sigma dW_t^{(2)} \quad \tilde{Y}_0 \in [-1, 1],\]
where \(b_2 = -\partial_y V\), i.e.
\[b_2(x^*, y) = \left\{ \begin{array}{ll} 0, & |y| \leq z^{-1}(x^*) \\ \left(n(x^* - z(y))^{n-1} + 3(x^* - z(y))^2\right) z'(y), & |y| \in (z^{-1}(x^*), 1] \end{array} \right..\]

Since \(x^*\) is fix, we abbreviate \(b_2(x^*, y) = b_2(y)\) in the following. Fix the interval \([a, b] \subset [-1, 1]\), where \(b\) is the upper bound of the considered interval and is not to be mistaken with the drift \(b = -\nabla V\). After this subsection the interval bound will not appear anymore.

In order to estimate the expected stopping time for leaving an interval \([a, b]\), we solve the following ODE
\[b_2(y)u'(y) + \frac{\sigma^2}{2} u''(y) = -1 \quad u(a) = u(b) = 0.\]

Introduce some auxiliary functions
\[A(y) = \int_0^y b_2(u)du = -\mathbb{1}_{\{|y| \geq z^{-1}(x^*)\}} \left((x^* - z(y))^n + (x^* - z(y))^3\right),\]
\[f(y) = \int_0^y e^{-\frac{y}{2}A(u)}du,\]
\[g(y) = \int_0^y \int_0^u e^{\frac{x}{2}(A(r) - A(u))}drdu.\]

It follows
\[\frac{\sigma^2}{2} u(y) = g(b) - g(a) \left(f(y) - f(a)\right) - \left(g(y) - g(a)\right).\]

If we pick \(a = -b\), then thanks to the symmetry of \(g, u\) simplifies to
\[\frac{\sigma^2}{2} u(y) = g(a) - g(y) = g(b) - g(y).\]

Obviously \(g(b) \to \infty\) as \(x^* \to \infty\), unless we let \(b\) decay in \(x^*\). We determine the rate at which \(b\) should decay in \(x^*\), but first we need to do some preliminary calculations. Take \(b > u > r > z^{-1}(x^*)\)
\[A(r) - A(u) = (x^* - z(u))^n - (x^* - z(r))^n + (x^* - z(u))^3 - (x^* - z(r))^3,\]
\[g(b) = \frac{1}{2} \left(z^{-1}(x^*)\right)^2 + z^{-1}(x^*) \int_{z^{-1}(x^*)}^b e^{-\frac{x}{2}A(u)}du\]
\[+ \int_{z^{-1}(x^*)}^b \int_{z^{-1}(x^*)}^u e^{\frac{x}{2}(A(r) - A(u))}drdu.\]
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Furthermore,

\[ I \leq z^{-1}(x^*) \int_{z^{-1}(x^*)}^{b} e^{\frac{2u}{\sigma^2}} [(x^* - z(u))^n + (x^* - z(u))^3] \, du \]
\[ \leq z^{-1}(x^*) \left( b - z^{-1}(x^*) \right) e^{\frac{2}{\sigma^2}} [(x^* - z(b))^n + (x^* - z(b))^3], \]
\[ II \leq \int_{z^{-1}(x^*)}^{b} \int_{z^{-1}(x^*)}^{u} e^{\frac{2v}{\sigma^2}} [(x^* - z(u))^n - (x^* - z(r))^n + (x^* - z(u))^3 - (x^* - z(r))^3] \, dr \, du \]
\[ \leq \frac{1}{2} \left( b - z^{-1}(x^*) \right)^2 e^{\frac{2}{\sigma^2}} [(x^* - z(b))^n + (x^* - z(b))^3]. \]

Note that \( z^{-1} \) decays rapidly, in fact for some of the examples above (with \( m \)th root) we have

\[ z^{-1}(x) \approx e^{-\frac{1}{\alpha}x^m}. \]

If we choose \( b = z^{-1}(\beta x^*) \) for some \( \beta \in (0, 1) \), such that \( \beta x^* > x_0 \) and \( m > n \), we obtain

\[ \max \{ I, II \} \lesssim e^{-C(x^*)^m}. \]

Two things are apparent here. First, \( b = z^{-1}(\beta x^*) \) is very small, even converges to zero. Hence, the one-point motion typically do not move too far away from the \( x \)-axis. On the other hand, they move sufficiently far away, such that the drift in \( x \)-direction is negative, see Lemma 4.4.1. Second, the time to overcome the level \( b = z^{-1}(\beta x^*) \) is very short and the time to cross the \( x \)-axis afterwards as well. Hence, a typical trajectory highly fluctuates around the \( x \)-axis.

4.4.2 Step down

As aforementioned, in this subsection we conduct a similar step down argument as in the previous chapter.

Starting close to \( x \)-axis

Take the levels \( x_k := 2^k x_0, k \geq 1 \). We want to show that it is very likely to step down from a starting level \( x_k \) to \( x_{k-1} \) while never surpassing the higher level \( x_{k+1} \). The upcoming estimates and arguments hold only far away from the origin. Therefore, we often say “for \( x_0 \) sufficiently large”, which means that there exists an \( x' > 0 \) such that the corresponding statement holds true for all \( x_0 \geq x' \).
Chapter 4. Gradient case

Let \((X_t, Y_t)_{t \geq 0}\) be the solution to equation (4.1) with initial condition \((x, y) \in \{x_k\} \times \mathbb{R}\) for a \(k \geq 2\), we introduce certain objects.

\[
\tau_k := \inf \{ t \geq 0 : X_t > x_k \}, \quad \iota_k := \inf \{ t \geq 0 : X_t < x_k \},
\]

\[
\nu_t(dy) := \lambda \left( \{ s \in [0, t] : Y_s \in dy \} \right) \quad \text{occupation measure of } Y,
\]

\[
\tilde{\nu}_t(dy) := \lambda \left( \{ s \in [0, t] : \tilde{Y}_s \in dy \} \right) \quad \text{occupation measure of } \tilde{Y},
\]

\[
M_k := \left[ -z^{-1}(\beta x_k), z^{-1}(\beta x_k) \right]^c, \quad \beta \in \left( \frac{1}{2}, 1 \right),
\]

\[
K_k := \frac{1}{18x_k}.
\]

The quantity \(K_k\) serves as the time up to which we consider the solution. We show (in Lemma 4.4.4) that solutions cannot grow to much in \(x\)-direction up to time \(K_k\) with high probability. In \(M_k\) the negative drift in \(x\)-direction is very strong, see the next lemma, and we aim to estimate the time spent in there with \(\nu_t\) or \(\tilde{\nu}_t\). First, recall

\[
b_1(x, y) = -\partial_x V(x, y) = -n (x - z(y))^{n-1} + 6z(y) (x - z(y)) + 3z(y)^2
\]

\[= -n (x - z(y))^{n-1} + 6z(y)x - 3z(y)^2,
\]

if \(|y| \geq z^{-1}(x)\), 0 otherwise.

The following lemma provides an estimate for the negative drift in the \(x\)-direction in \(M_k\).

**Lemma 4.4.1**

For \(x_0\) sufficiently large and \(k \geq 3\) it holds

\[
\sup_{x \in [x_{k-2}, x_{k+1}], y \in M_{k-2}} b_1(x, y) \leq -\frac{n(1 - \beta)^{n-1}}{2} x_k^{n-1}.
\]

**Proof.** Note that for all \(y \in M_{k-2}\) it holds (due to symmetry of \(z\)) \(z(y) \leq \beta x_k\). It follows

\[
\sup_{x \in [x_{k-2}, x_{k+1}], y \in M_{k-2}} b_1(x, y) \leq \sup_{x \in [x_{k-2}, x_{k+1}]} -n (x - \beta x_{k-2})^{n-1} + 6\beta x_{k-2}x
\]

\[\leq -nx_k^{n-1}(1 - \beta)^{n-1} + 6\beta x_{k-2}x_{k+1}
\]

\[\leq -nx_k^{n-1}(1 - \beta)^{n-1} + 24\beta x_{k-2}^2
\]

\[\leq -\beta n x_k^{n-1}(1 - \beta)^{n-1}.
\]

\(\square\)
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Furthermore, the following events will be of importance.

\[ \text{BBM}_k := \left\{ \sup_{t \in [0, K_k]} \sigma W_t^{(1)} \leq \frac{x_k}{2} \right\}, \quad U_k := \left\{ K_k \leq \inf_{y \in \mathbb{R}} \tau_{k+1} \right\}, \]

\[ L_k := \{ K_k \leq \tau_{k-2} \}, \quad O_k := \left\{ \frac{\nu_{K_k}(M_{k-2})}{K_k} \geq \alpha \right\}, \quad \alpha \in (0, 1). \]

The set \( \text{BBM}_k \) bounds the Brownian motion and allows for a rather explicit bound on its probability. It is also a subset of \( U_k \), see Lemma 4.4.4, hence we can estimate the probability of \( U_k \) as well. On \( U_k \) there is no trajectory that exceeds \( x_{k+1} \) before time \( K_k \), that holds even uniformly in the initial condition.

The aim is to estimate the probability of \( L_k \) by expressing it in terms of the other events, see Remark 4.4.3. We formulate the step down lemma.

**Lemma 4.4.2**

For \( x_0 > 0 \) sufficiently large it holds for any \( k \geq 3 \) and \( \sigma > 0 \)

\[ L_k \cap \text{BBM}_k \cap U_k \cap O_k = \emptyset. \]

**Remark 4.4.3**

Note, it follows \( L_k \supset \cap \text{BBM}_k \cap U_k \cap O_k \) and later we show

\[ \mathbb{P}(\text{BBM}_k \cap U_k \cap O_k) \rightarrow 1 \text{ as } k \rightarrow \infty. \]

This implies

\[ \mathbb{P}(\tau_k < K_k \leq \tau_{k+1}) = \mathbb{P}(L_k \cap U_k) \geq \mathbb{P}(\text{BBM}_k \cap U_k \cap O_k) \rightarrow 1 \text{ as } k \rightarrow \infty. \]

**Proof.** Take \( \omega \in L_k \cap \text{BBM}_k \cap U_k \cap O_k \) and show \( X_{K_k}(\omega) \leq x_{k-2} \), which contradicts \( \omega \in L_k \). To ease the notation we omit \( \omega \) in the following.

\[ X_{K_k} = x_k + \int_0^{K_k} b_1(X_s, Y_s)ds + \sigma W_{K_k}^{(1)} \leq x_k + \frac{x_k}{2} + \int_0^{K_k} b_1(X_s, Y_s)ds \leq \frac{3x_k}{2} + \int_0^{K_k} \sup_{x \in [x_{k-2}, x_{k+1}]} b_1(x, Y_s)ds \]

\[ = \frac{3x_k}{2} + K_k \int_{\mathbb{R}, x \in [x_{k-2}, x_{k+1}]} b_1(x, y) \frac{\nu_{K_k}(dy)}{K_k} \]

\[ \leq \frac{3x_k}{2} + K_k \sup_{x \in [x_{k-2}, x_{k+1}], y \in M_{k-2}} b_1(x, y) \frac{\nu_{K_k}(M_{k-2})}{K_k} + 3x_{k+1} K_k \frac{\nu_{K_k}(M_{k-2})}{K_k} \]

\[ \leq \frac{3x_k}{2} - K_k \frac{n}{2} a_{k-2}^{n-1} (1 - \beta)^{n-1} \alpha + 3x_{k+1} K_k (1 - \alpha) \]

\[ \leq x_{k-2} \]
where the last inequality holds true if $x_{k-2}$ is sufficiently large, which is true, because $x_{k-2} > x_0$.

\[\square\]

**Lemma 4.4.4**

It holds $\mathbb{P}(\text{BBM}_k) \to 1$ as $k \to \infty$ and BBM$_k \subseteq U_k$ for all $k \geq 1$.

**Proof.** The first assertion is clear since $\sup_{k \geq 1} K_k < \infty$ and $x_k \to \infty$ as $k \to \infty$.

Concerning the second claim, we have for $\omega \in \text{BBM}_k$ (we omit $\omega$ in the calculations)

\[
X_t = x_k + \int_0^t b_1(X_s, Y_s)ds + \sigma W_t^{(1)} \leq x_k + \frac{1}{2}x_k + 3 \int_0^t X_s^2 ds.
\]

By a Gronwall-type argument, see [Bih56, page 83f], we obtain

\[
X_t \leq \frac{\frac{3}{2}x_k}{1 - 3 \cdot \frac{3}{2}x_k t},
\]

hence

\[
\sup_{t \in [0, K_k]} X_t \leq \sup_{t \in [0, K_k]} \frac{\frac{3}{2}x_k}{1 - \frac{9}{2}x_k t} = \frac{\frac{3}{2}x_k}{1 - \frac{9}{2}x_k K_k} = 2x_k = x_{k+1}.
\]

Obviously, this implies $\omega \in U_k$. \[\square\]

This lemma even implies that on BBM$_k$ trajectories that are left of $x_{k-2}$ cannot pass $x_{k-1}$ before time $K_k$.

It remains to show that also $\mathbb{P}(O_k, U_k)$ tends to 1. We have not proved this rigorously, but we are convinced that this holds true. We outline the idea.

First, show that the process $|Y|$ stochastically dominates the auxiliary process $|\tilde{Y}|$, where both processes that in the same initial value and are driven by the same Brownian motion.

Second, show that any auxiliary process $|\tilde{Y}|$ stochastically dominates the auxiliary process $|\tilde{Y}|$ which starts in 0.

Finally, compute $\mathbb{P}(\tilde{O}_k)$, where the occupation is the one of $\tilde{Y}$ started in 0. Due to the many crossings of the x-axis of $\tilde{Y}$ it is plausible that even at the relatively small time $K_k$ the process $\tilde{Y}$ is close to its stationary state. Hence, the occupation measure is close to its expectation, the speed measure.
4.4. Uniform control in the initial condition

Starting far away from the $x$-axis

We consider initial conditions $(x_k, y)$ for $y \geq y^*$, where $y^*$ is very large, in fact $y^* = \sqrt{2nx_{k+1}^{n-1}}$. In the previous subsection we dealt with the more difficult case, but here we show the step down uniformly in the initial condition, i.e. we show

$$\mathbb{P} \left( \sup_{(x_0,y_0) \in \{x_k\} \times [y^*,\infty)} X_{K_k} \leq x_{k-1} \right) \to 1 \text{ as } k \to \infty.$$ 

First, we show that the processes stay above $\{ (x,y) : y = 1 \}$ sufficiently long. The following lemma provides a sufficient estimate on the drift of $Y$ away from the $x$-axis.

**Lemma 4.4.5**

Assume that $z(y) \searrow 0$, $z'(y) \nearrow 0$ as $y \to \infty$ as well as $z'(1) \geq -1$. Then for $y \geq 1$ and $x$ sufficiently large, we have

$$b_2(x,y) \geq -2nx^{n-1} - 3y^2.$$ 

Furthermore, for $x > 0$ the ODE

$$dy_t = (-x - 3y_t^2) dt, \quad y_0 = y$$

has a unique solution

$$y_t = \sqrt{\frac{x}{3}} \tan \left( \arctan \left( \sqrt{\frac{3}{x}} y \right) - \sqrt{3}xt \right)$$

up to a finite time.

**Proof.** It follows for $y \geq 1$ and large $x$

$$b_2(x,y) = \left( n (x - z(y))^{n-1} + 3 (x - z(y))^2 \right) z'(y) - 3(y - 1)^2$$

$$\geq - \left( nx^{n-1} + 3x^2 \right) z'(1) - 3y^2 \geq 2nx^{n-1} - 3y^2.$$ 

Concerning the second assertion of the lemma, there exists a unique solution up to a finite time due the locally Lipschitz coefficients. A simple computation verifies that the displayed function is indeed the solution. \hfill \Box

Take a solution to (4.1) with initial condition $(x_0, y_0) \in \{x_k\} \times [y^*, \infty)$, define

$$BBM_k := \left\{ \inf_{t \in [0,K_k]} \sigma W_t^{(2)} \geq -\frac{y^*}{2} \right\}, \quad \rho_k := \inf \left\{ t \geq 0 : \inf_{y_0 \geq y^*} Y_t < 1 \right\},$$

$$\hat{t}_k := \frac{\arctan \left( \sqrt{3} \right)}{\sqrt{6nx_{k+1}^{n-1}}}.$$
In the following lemma we show that on BBMY$_k$ no solution goes below 1 before time $\hat{t}_k$. This will be necessary to use the strong negative drift in the $x$-direction and therefore show the step down.

**Lemma 4.4.6**

We assume that the assumptions on $z$ of Lemma 4.4.5 hold. Then, $\{\rho_k > \hat{t}_k\} \supset BBMY_k \cap U_k$ holds for each $k \geq 3$ and sufficiently large $x_0$.

**Proof.** By Lemma 4.4.5 it holds on $BBMY_k \cap U_k$ for $t \leq \rho_k$

$$Y_t = (y_0 + \int_0^t b_2(X_s, Y_s)ds + \sigma W^{(2)}_t \geq \frac{y^*}{2} - \int_0^t 2nx^{n-1}_{k+1} + 3Y^2_sds.$$

Hence, by a reverse Gronwall like argument (similar to [Bih56] page 83f), we have for $t \leq \rho_k$

$$Y_t \geq \sqrt{\frac{2nx^{n-1}_{k+1}}{3}} \tan \left( \arctan \left( \sqrt{\frac{3}{2nx^{n-1}_{k+1}}} y^* \right) - \sqrt{6nx^{n-1}_{k+1}} t \right).$$

Recall $y^* = \sqrt{\frac{2nx^{n-1}_{k+1}}{3}}$ and compute the time $t^*$ at which the bound on the right-hand side hits 1

$$t^* = \sqrt{\frac{3}{6nx^{n-1}_{k+1}}} \left( \arctan \left( \sqrt{\frac{3}{2nx^{n-1}_{k+1}}} y^* \right) - \arctan \left( \sqrt{\frac{3}{2nx^{n-1}_{k+1}}} \right) \right) \geq \arctan(\sqrt{3}) = \hat{t}_k.$$

Hence, $\{\rho_k > \hat{t}_k\} \supset BBMY_k \cap U_k$. \hfill \Box

Recall the stopping times from above

$$\tau_k(x, y) := \inf\{t \geq 0 : X_t(x, y) > x_k\},$$

$$\underline{\tau}_k(x, y) := \inf\{t \geq 0 : X_t(x, y) < x_k\}.$$

Now, we can prove the main claim of this subsection.

**Lemma 4.4.7**

Again, we assume that the assumptions on $z$ of Lemma 4.4.5 hold true. Then, for $x_0$ sufficiently large and for all $k \geq 3$ we have

$$\left\{ \sup_{y_0 \in [y^*, \infty)} \tau_{k-2}(x_k, y_0) \leq K_k \right\} \supset BBM_k \cap BBMY_k.$$
4.4. Uniform control in the initial condition

**Proof.** We use a similar argument as in the previous subsection. Assume

\[ \omega \in \left\{ \sup_{y_0 \in [y^*, \infty)} \tau_{k-2}(x_k, y_0) > K_k \right\} \cap \text{BBM}_k \cap \text{BBMY}_k. \]

In the following we omit \( \omega \). There is an initial condition, which we generically denote by \((x_k, y_0)\), such that its corresponding solution never went below \( x_{k-2} \) until time \( K_k \). Furthermore, we know from Lemma 4.4.4 that the same solution never exceeded \( x_{k+1} \) until time \( K_k \). With Lemma 4.4.6 we estimate

\[ X_{\hat{t}_k} = x_k + \int_0^{\hat{t}_k} b_1(X_s, Y_s)ds + \sigma W_t^{(1)} \leq \frac{3}{2} x_k + \hat{t}_k \sup_{x \in [x_{k-2} \cdot x_{k+1}], y \geq 1} b_1(x, y) \]

\[ \leq \frac{3}{2} x_k - \hat{t}_k n^{-\frac{1}{2}} x_{n-1} = \frac{3}{2} x_k - \frac{\arctan \left( \frac{\sqrt{3}}{\sqrt{n}} \right) \sqrt{n} x_{n-1}}{2 \sqrt{6}} \leq x_{k-2}. \]

Again, the last inequality holds because \( x_0 < x_{k-2} \) is sufficiently large. Since \( \hat{t}_k < K_k \), this contradicts \( \tau_{k-2}(x_k, y_0) > K_k \), thus completes the proof. \( \square \)

4.4.3 Expansion of small balls

In the previous subsections we managed to gain control on the time the one-point motions need to do the step down. Further, the level to which these points step down is low enough to allow for a certain error, when dealing with multiple one-point motions simultaneously. A fitting technique is the so-called chaining, see [Sch09] for a detailed description. The basic idea is to make sure that small balls do not expand too much and conclude that points starting close to each other (i.e. in the same ball) are not too far after the considered time period. This way we can cover to whole initial set, in our case the finite line \( \{x_{k-1}\} \times (-y^*, y^*) \), and show that it is very likely that all points step down simultaneously and not just every point by itself.

Conceptually, we want to conduct the following lines of argumentation. Take two solutions \( Z = (X, Y) \) and \( Z' = (X', Y') \) to equation (4.1) with corresponding initial condition \( z = (x, y) \) and \( z' = (x', y') \) and consider the evolution of their difference

\[ |Z_t - Z'_t| \leq |z - z'| + \int_0^t |b(X_s, Y_s) - b(X'_s, Y'_s)|ds \]

\[ \leq |z - z'| + \sup_{x \geq x_0, y \in \mathbb{R}} \| Db(x, y) \| \int_0^t |Z_s - Z'_s|ds. \]

Hence, by Gronwall

\[ |Z_t - Z'_t| \leq |z - z'| e^{\sup_{x \geq x_0, y \in \mathbb{R}} \| Db(x, y) \|}. \]
Chapter 4. Gradient case

In our case, for any feasible choice of \( z \), \( \| Db(x, y) \| \) grows too much in \( x \), in fact super-exponential, \( \approx e^{Ck^m} \). Hence, the number of balls we need to cover the line \( \{ x_k \} \times [-\sqrt{2nx_k^{n-1}}, \sqrt{2nx_k^{n-1}}] \) is very large. According to our estimations each of these balls performs the step down with probability

\[
\approx P(BBM_k) = P \left( \sup_{t \in [0, K_k]} \sigma W_i^{(1)} \leq \frac{x_k}{2} \right) \geq 1 - \frac{2}{x_k} \sqrt{\frac{2K_k}{\pi}} e^{-\frac{x_k^2}{4K_k}}
\]

\[
= 1 - \frac{2}{x_k} \sqrt{\frac{1}{9x_k \pi}} e^{-\frac{18x_k^2}{8}}.
\]

Even though the step down is very likely for every ball, there are simply too many of them (more than \( \exp(\exp(Cx_k^{m}/x_k)) \)) to step down simultaneously with high probability.

Of course, this Ansatz is a bit crude. But the main problem stems from the drift in \( y \)-direction, which allows expansion with a high rate \( \approx e^{Ck^m} \).

We might not care about expansion in \( y \)-direction, since after a step down we restart with the infinite line \( \{ x_{k-1} \} \times (-\infty, \infty) \). But the difference in the \( y \)-component of two one-point motions influences the drift in \( x \)-direction, which can also lead to a big expansion in \( x \)-direction.

To this point we are not sure whether such a big expansion in the \( y \)-direction really happens, since in most of the state space the drift in \( y \)-direction is non-expanding or even contracting.

**Alternative model**

Another choice for a potential with similar properties is

\[ V(x, y) = -x^3 + A(x)y^2 \]

for a rapidly growing \( A \). For example, \( A(x) = e^{x^4} \) implies the integrability condition. Observe that a \( \hat{Y} \) process with frozen \( x \)-component, as above, contracts everywhere. However, this is not necessarily true for the non-frozen process. Further, a step down argument in the sense of the previous section is not possible anymore.
Chapter 5

Mean field limit with proliferation

The content of this chapter is taken from the paper [FL15]. It was developed during the author’s stay in Pisa in close cooperation with Franco Flandoli. An interacting particle system with long range interaction in the spatial component is considered. Particles, in addition to the interaction, proliferate with a rate depending on the empirical measure. We prove convergence of the empirical measure to the solution of a parabolic equation with non-local nonlinear transport term and proliferation term of logistic type, see Theorem 5.1.1 and equation (5.9). To prove this convergence, we start by showing tightness, Section 5.3, of the mollified empirical measure. Hence, we deduce convergences along a subsequence of any subsequence. Second, we show that any such limit is a solution of a certain PDE (see Section 5.5), which has a unique solution (Section 5.4). Therefore, the sequence itself is converging to the PDE.

The idea to use methods from stochastic partial differential equations (see end of Section 5.3) and the proof of uniqueness of the limiting PDE are due to F. Flandoli.

5.1 Introduction and main result

Our starting motivation for this research came from Mathematical Oncology, where cells interact by random dynamics and proliferate. One would like to discover appropriate macroscopic limits (PDEs), because, for instance, a cell-level simulation of a tumor is too demanding, it involves a number of particles of the order of $10^9$.

A more detailed description of the motivation from Mathematical Oncology
Chapter 5. Mean field limit with proliferation

is given in Section 5.1.3.

### 5.1.1 Microscopic model

We consider a particle system, as described Section 1.4, where the dynamics of $X_{t}^{a,N}$ is given by the gradient system

$$
\frac{dX_{t}^{a,N}}{dt} = -\frac{1}{N} \sum_{a \in A_{t}^{N}} \nabla V \left( X_{t}^{a,N} - X_{t}^{\tilde{a},N} \right) dt + \sigma dB_{t}^{a},
$$

(5.1)

where $V \in C^{2}(\mathbb{R}^{d})$ with compact support. Further, we set the proliferation rate to

$$
\lambda_{t}^{a,N} = F_{N} \left( S_{t}^{N}, X_{t}^{a,N} \right),
$$

where the properties of the functionals $F_{N} : M_{+} \left( \mathbb{R}^{d} \right) \times \mathbb{R}^{d} \to [0, \infty)$ will be specified below ($M_{+} \left( \mathbb{R}^{d} \right)$ is the set of Borel finite positive measures on $\mathbb{R}^{d}$).

### 5.1.2 Macroscopic limit

Denote by $W^{1,2}_{+} \left( \mathbb{R}^{d} \right)$ the subset of the Sobolev space $W^{1,2} \left( \mathbb{R}^{d} \right)$ consisting of all non-negative functions and by $C_{b}^{\beta} \left( \mathbb{R}^{d} \right)$, $\beta \in (0,1)$, the space of bounded, $\beta$-Hölder continuous functions. The $\beta$-Hölder seminorm of $f$ will be denoted by $[f]_{\beta} := \sup_{x \neq y} \left( |f(x) - f(y)| / |x - y|^{\beta} \right)$. We say that a map $F : W^{1,2}_{+} \left( \mathbb{R}^{d} \right) \to C_{b}^{\beta} \left( \mathbb{R}^{d} \right)$ satisfies the mild Lipschitz conditions if it is Lipschitz in the $C_{b} \left( \mathbb{R}^{d} \right)$-norm and has linear growth in the $C_{b}^{\beta} \left( \mathbb{R}^{d} \right)$-norm, namely for every $u, v \in W^{1,2}_{+} \left( \mathbb{R}^{d} \right)$

$$
\| F(u) - F(v) \|_{\infty} \leq L_{F} \| u - v \|_{W^{1,2}},
$$

(5.2)

$$
[F(u)]_{\beta} \leq C \left( \| u \|_{W^{1,2}} + 1 \right).
$$

(5.3)

The macroscopic limit result below requires a number of natural assumptions that we list now, plus the more critical assumption (5.7), that we emphasize in the statement of the theorem.

Let $(\theta_{N})_{N \in \mathbb{N}}$ be a classical family of mollifiers with compactly supported $\theta$. Recall the mollified empirical measure $h_{t}^{N}$ defined as

$$
h_{t}^{N} (x) = (\theta_{N} * S_{t}^{N}) (x).
$$

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For technical reasons we assume the following rate of convergence of the mollifiers

\[
\sup_{N \in \mathbb{N}} \frac{\epsilon_N^{d-2}}{N} < \infty. \tag{5.4}
\]

At least in the case of example (5.10)-(5.11) below, the role of \( \theta_N \) is auxiliary, it does not appear in the model, hence the restriction does not have a biological relevance. There is a technical condition

\[
\lim_{R \to \infty} \sup_{N \in \mathbb{N}} \mathbb{P} \left( \| h_0^N \|_{L^2(\mathbb{R}^d)}^2 > R \right) = 0, \tag{5.5}
\]

for which we state a simple and relevant sufficient condition in the Appendix C.

Sometimes, the term \([S_T^N]\) could spoil properties associated to adaptedness in the inequalities, therefore we also introduce, for every \( R > 0 \), the stopping time

\[
\tau_N^R := \inf \left\{ r \geq 0 : [S_T^N] > R \text{ or } \| h_0^N \|_{L^2(\mathbb{R}^d)} > R \right\} \tag{5.6}
\]

with the usual convention \( \inf \emptyset = \infty \). We state our main theorem of this chapter.

**Theorem 5.1.1**

Assume that (5.4), (5.5) hold and that \( \langle S_0^N, \phi \rangle \) converges in probability to \( \langle u_0, \phi \rangle \), as \( N \to \infty \), for every \( \phi \in C_c^\infty (\mathbb{R}^d) \), where \( u_0 \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \) and \( u_0 \geq 0 \). Further, assume that, for some \( \beta \in (0,1) \), there is a map \( F : W^{1,2}_c(\mathbb{R}^d) \to C^\beta(\mathbb{R}^d) \) which satisfies the mild Lipschitz condition (5.2)-(5.3) above and there is a sequence of positive real numbers \( (\alpha_N)_{N \in \mathbb{N}} \) converging to zero such that

\[
\| F (\theta_N \ast \mu) - F_N (\mu, \cdot) \|_\infty \leq \alpha_N (1 + \langle \mu, 1 \rangle). \tag{5.7}
\]

Assume moreover that there exists a constant \( C_F > 0 \) such that

\[
| F_N (\mu, x) | \leq C_F, \quad \| F (u) \|_\infty \leq C_F \tag{5.8}
\]

for every \( \mu \in M_+(\mathbb{R}^d) \), \( x \in \mathbb{R}^d \), \( u \in W^{1,2}_c(\mathbb{R}^d) \), \( N \in \mathbb{N} \). Then the process \( h_0^N(x) \) converges in probability, as \( N \to \infty \), to the unique weak solution of the PDE

\[
\partial_t u_t = \text{div} ( (\nabla V * u_t) u_t ) + \frac{\sigma^2}{2} \Delta u_t + F (u_t) u_t, \quad u|_{t=0} = u_0. \tag{5.9}
\]

The topology of convergence of \( h^N \) to \( u \) is
Chapter 5. Mean field limit with proliferation

- the weak star topology of $L^\infty(0,T;L^2(\mathbb{R}^d))$,
- the weak topology of $L^2(0,T;W^{1,2}(\mathbb{R}^d))$ and
- the strong topology of $L^2_{\text{loc}}([0,T] \times \mathbb{R}^d)$.

The definition of a weak solution of the PDE (5.9) is given below in Section 5.4.

Having in mind Fisher-Kolmogorov-Petrovskii-Piskunov equation, see (6.1), and in general the concept of logistic growth, the most natural example of the functional $F(u)$ is $F(u)(x) = (1 - u(x))$, which means that the proliferation rate decreases, when it approaches the threshold $u = 1$ (the value 1 is conventional). More precisely, due to the term $\text{div}((\nabla V * u_t)u_t)$, the solution $u_t$ may overcome any threshold, therefore we correct the classical logistic term and consider

$$F(u)(x) = (1 - u(x))^+, \tag{5.10}$$

which inhibits proliferation completely when the threshold is exceeded. Our theory covers this case only in dimension $d = 1$, where $W^{1,2}_+(\mathbb{R}^d) \subset C^\beta_b(\mathbb{R}^d)$ for some $\beta \in (0,1)$ due to Sobolev’s embedding theorem. In this case we take

$$F_N(\mu, x) = (1 - (\theta_N * \mu)(x))^+ \tag{5.11}$$

and assumption (5.7) is obviously satisfied; the others are elementary. In dimension $d > 1$ this example is not covered but we can treat the case

$$F(u)(x) = (1 - (W * u)(x))^+, \quad (5.10)$$

where $W$ is a Lipschitz continuous, compactly supported probability density. In this case we simply take

$$F_N(\mu, x) = (1 - (W * \mu)(x))^+. \tag{5.11}$$

The validity of assumptions (5.2), (5.3), (5.7) and (5.8) in this case is shown in Lemma 5.2.1 below.

A variety of macroscopic limit results has been proved in the literature, related to the present one are [Oel85, Oel89, Ste00, Phi07, MB15, Mét86, NO88] and references therein, from which we have taken several elements of inspiration. However, a result of mean field type with proliferation, in the sense described here, is not treated in the previous references. We also mention a different class of macroscopic limit results,
5.1. Introduction and main result

for instance [Var91], [Uch00], which require very different techniques. In the more applied literature, two examples of works related to our problem are [BS10], [BV05].

A feature of our approach, shared by some of the previously quoted references, is that we do not use only results of tightness of measure-valued processes but of processes. A more distinguished feature, probably shared only by [Mét86] (which however is very different), is that we use typical tools of the theory of Stochastic Partial Differential Equations, for instance the tightness criterion used for stochastic Navier-Stokes equations in [FG95], [BBNP14], see also [BM13], [BMO15].

5.1.3 Motivations from Mathematical Oncology

Although the aim of this chapter is mostly theoretical, we have been inspired by lectures about the emerging field of Mathematical Oncology. In this area, roughly speaking, models are classified as macroscopic, when described by partial differential equations, or microscopic, when described by stochastic ordinary differential equations or, even more often, cellular automata and other discrete stochastic models - in addition there are multiscale models with mixture of the previous two cases. Macroscopic models look at the tumor on the tissue level, microscopic ones at the cellular level. The link between the two descriptions is of interest for various reasons, in particular because a precise justification of the macroscopic models is difficult from general arguments based on fluid dynamics or mechanical models, since a biological tissue is something different. At the cellular level it is easier to be more realistic and thus results on the macroscopic limit of cellular systems is a way to justify or improve the macroscopic models. It also yields different interpretations of the constants appearing in these models - characterizing these constants is a major problem for real applications. After these general comments, which clarify why we investigate the macroscopic limit, note that most models used nowadays in Mathematical Oncology are more complex than our one, since they involve different cell types - e.g. normoxic and hypoxic tumor cells, or cells with different genetic mutations, or cells of the extracellular matrix - molecular fields like oxygen and growth factors, and possibly objects related to the angiogenic cascade. See for instance [HGM+09] as an example of a complex model. Our model, chosen as a starting point, captures only a few features of such complexity. First, the interaction between cells which may incorporate for instance a certain degree of repulsion resulting from the fact that cells cannot press each other too much - see the term called “crowding effect” in [HGM+09], which is different from our one but corresponding to a similar mechanism. Secondly, cancer
cell proliferation in each model.

Concerning proliferation in detail, very often in the literature of Mathematical Oncology it is taken of logistic form $u(1 - u)$ (which corresponds to $F(u)(x) = (1 - u(x))^+$ above). Such choice of $F$ simply corresponds to the fact that cells proliferate better when the neighbor is not crowded. However, other forms of $F$ may be interesting as well. A phenomenon observed in vitro is that certain isolated cancer cells, even if embedded in a liquid that is rich of nutrients, do not proliferate. They need to adhere to other cells to proliferate. A functional $F$ which charges - in the sense that decreases the proliferation rate - not only the excessive presence of cells in the neighbor but also the opposite case, an excessive isolation, may be more realistic. Cells which separate from the main cloud by random motion will continue their travel to meet blood vessels and lead to metastasis, but along the trip they do not proliferate as often as the cells close to the main tumor body. This, although vague, could be a motivation for investigating a general proliferation mechanism of the form $\lambda_a^N = F_N(S^N_t, X^{a,N}_t)$ above; more modeling work is necessary and is part of our research program.

5.2 Preparation

We show that the assumptions (5.2), (5.3), (5.7) and (5.8) hold for $F, F_N$ in (5.10) and (5.11).

Lemma 5.2.1
If $F(u)(x) = (1 - (W * u)(x))^+$, $F_N(\mu, x) = (1 - (W * \mu)(x))^+$, with $W$ a Lipschitz continuous, compactly supported probability density, assumptions (5.2), (5.3), (5.7) and (5.8) are fulfilled.

Proof. First, it is obvious that $F$ maps $W^{1,2}_+ (\mathbb{R}^d)$ (in fact even $L^2_+ (\mathbb{R}^d)$) into $C^\beta_0 (\mathbb{R}^d)$ for any $\beta \in (0, 1)$, thus (5.8) holds true (the range are functions with values in $[0, 1]$). The map $u \mapsto W * u$ is Lipschitz continuous, being linearly bounded, from $W^{1,2}_+ (\mathbb{R}^d)$ (in fact $L^2_+ (\mathbb{R}^d)$) to $C^\beta_0 (\mathbb{R}^d)$, hence (5.2) and (5.3) are true for $F$ by composition with the Lipschitz bounded function $r \mapsto (1 - r)^+$ (we also use the fact that $W * u \geq 0$). Finally,

$$
\left|(1 - (W * \theta_N * \mu)(x))^+ - (1 - (W * \mu)(x))^+\right| \\
\leq \left|(W * \theta_N * \mu)(x) - (W * \mu)(x)\right| \\
\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \theta_N(z - y) |W(x - z) - W(x - y)| \mu(dy) dz
$$
\[
\phi(X^a_t) = \phi(X^a_{T^a_1}) 1_{t \geq T^a_1} - \phi(X^a_{T^a_1}) 1_{t \geq T^a_1} + \int_0^t 1_{s \in I^a} \nabla \phi(X^a_s) \cdot dX^a_s + \frac{\sigma^2}{2} \int_0^t 1_{s \in I^a} \Delta \phi(X^a_s) ds.
\]

With a few computations, one can see that the empirical measure \(S^N_t\) satisfies

\[
d \langle S^N_t, \phi \rangle = - \left\langle \left( \nabla V * S^N_t \right), \nabla \phi \right\rangle dt + \frac{\sigma^2}{2} \left\langle S^N_t, \Delta \phi \right\rangle dt + \left\langle F_N \left( S^N_t, \cdot \right), S^N_t, \phi \right\rangle dt + dM^{1, \phi, N}_t + dM^{2, \phi, N}_t
\]

for every \(\phi \in C^2_b(\mathbb{R}^d)\) and where

\[
M^{1, \phi, N}_t := \frac{\sigma}{N} \sum_{a \in A^N} \int_0^t 1_{s \in I^a} \nabla \phi(X^a_s) \cdot dB^a_s,
\]

\[
M^{2, \phi, N}_t := \frac{1}{N} \sum_{a \in A^N} \phi(X^a_{T^a_1}) 1_{t \geq T^a_1} - \frac{1}{N} \sum_{a \in A^N} \int_0^t \phi(X^a_s) \lambda_s^a ds.
\]

We deduce that \(h^N_t(x)\) satisfies

\[
dh^N_t(x) = \left( \text{div} \left( \theta_N * \left( \left( \nabla V * S^N_t \right) S^N_t \right)(x) \right) + \frac{\sigma^2}{2} \Delta h^N_t(x)
\]

\[
+ \left( \theta_N * \left( F_N \left( S^N_t, \cdot \right) S^N_t \right) \right)(x) \right) dt + dM^{1,N}_t(x) + dM^{2,N}_t(x),
\]

where

\[
M^{1,N}_t(x) := - \frac{\sigma}{N} \sum_{a \in A^N} \int_0^t 1_{s \in I^a} \nabla \theta_N(x - X^a_s) \cdot dB^a_s,
\]

\[
M^{2,N}_t(x) := \frac{1}{N} \sum_{a \in A^N} \theta_N(x - X^a_{T^a_1}) 1_{t \geq T^a_1} - \frac{1}{N} \sum_{a \in A^N} \theta_N(x - X^a_s) \lambda_s^a ds.
\]
5.3 Tightness

The estimates necessary for the tightness are presented in this section and combined at the end to obtain tightness in the topologies mentioned in the Theorem 5.1.1. We only use (5.8) as an assumption on $F_N$. In the following we need a lemma, also known as generalized Itô formula, see for example [HWY92, page 245].

Definition 5.3.1

We define, for a function $f : \mathbb{R} \to \mathbb{R}$,

$$f(t-) := \lim_{s \uparrow t} f(s), \quad \text{the left limit of } f \text{ at } t,$$

$$Jf(t) := f(t) - f(t-), \quad \text{the jump size of } f \text{ at } t,$$

if the limit exists. Further, let $(X_t)_{t \geq 0}$ be a stochastic process. We define its quadratic variation, when the limit exists and is independent of the partitions,

$$[X]_t = \mathbb{P} \lim_{k \to \infty} \sum_{j=0}^{n-1} \left( X_{t_{j+1} \land t} - X_{t_j \land t} \right)^2,$$

where the maximal distance of two consequent sites in the partition $\{0 = t_0^k < t_1^k < \cdots < t_n^k = T\}$ converges to 0 as $k \to \infty$. We denote the continuous part of the quadratic variation by $[X]^c$, i.e.

$$[X]^c_t = [X]_t - \sum_{s \leq t} JX_s^2.$$

We are aware that we used the brackets "[ , ]" already to denote the $\beta$-Hölder seminorm, but we are convinced that the reader will not be confused in the following.

Lemma 5.3.2 (Generalized Itô formula)

Let $X$ be a one-dimensional semimartingale such that $X_t, X_{t-}$ take values in an open set $U \subset \mathbb{R}$ and $f : U \to \mathbb{R}$ twice continuously differentiable. Then $f(X)$ is a semimartingale and

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d[X]_s^c + \sum_{s \leq t} (Jf(X_s) - f'(X_{s-}) JX_s).$$

We use the generalized Itô formula to derive the following energy estimate.
Lemma 5.3.3
It holds
\[
\frac{1}{2} \left\| h_t^N \right\|_{L^2}^2 + \frac{\sigma^2}{2} \int_0^t \left\| \nabla h_t^N \right\|_{L^2}^2 \, dr
= \frac{1}{2} \left\| h_0^N \right\|_{L^2}^2 - \int_0^t \langle \theta_N * \left( (\nabla V * S_r^N) S_r^N \right) \nabla h_r^N \rangle \, dr
+ \int_0^t \langle \theta_N * \left( F_N \left( S_r^N, \cdot \right) S_r^N \right) h_r^N \rangle \, dr
+ \frac{1}{2} \left\| [M^1, N]_t \right\|_{L^1}
+ \int_{\mathbb{R}^d} \int_0^t h_r^N(x) d \left( M_r^1(x) + M_r^2(x) \right) \, dx
+ \frac{1}{2} \int_{\mathbb{R}^d} \frac{1}{N^2} \sum_{a \in A_N} \left( \theta_N(x - X_t^{a}) \right)^2 1_{t \geq T_t} \, dx.
\]

Proof. For every \( x \in \mathbb{R}^d \), we apply the generalized Itô formula to \( (h^N(x))^2 \) and integrate over \( x \in \mathbb{R}^d \):
\[
\left\| h_t^N \right\|^2_{L^2} = \left\| h_0^N \right\|^2_{L^2} + 2 \int_{\mathbb{R}^d} \int_0^t h_r^N(x) dh_r^N(x) \, dx + \int_{\mathbb{R}^d} h^N(x) \, dx
+ \int_{\mathbb{R}^d} \sum_{a \in A_N} \left( J \left( h_r^N(x) \right)^2 - 2 h_{r, a}^N(x) h_r^N(x) \right) \, dx
= \left\| h_0^N \right\|^2_{L^2} + \int_{\mathbb{R}^d} \left[ M^{1, N}(x) \right]_t \, dx + \sigma^2 \int_0^t \int_{\mathbb{R}^d} h_r^N(x) \Delta h_r^N(x) \, dx dr
+ 2 \int_0^t \int_{\mathbb{R}^d} h_r^N(x) \, dx \text{div} \left( \theta_N * \left( (\nabla V * S_r^N) S_r^N \right) \right) (x) \, dx dr
+ 2 \int_0^t \int_{\mathbb{R}^d} h_r^N(x) \theta_N * \left( F_N \left( S_r^N, \cdot \right) S_r^N \right) (x) \, dx dr
+ 2 \int_{\mathbb{R}^d} \int_0^t h_r^N(x) d \left( M_r^1(x) + M_r^2(x) \right) \, dx
+ \int_{\mathbb{R}^d} \sum_{a \in A_N} \left( J \left( h_r^N(x) \right)^2 - 2 h_{r, a}^N(x) h_r^N(x) \right) \, dx,
\]
where we have used the fact that \( [h^N(x)]_t^c = [M^{1, N}(x)]_t \) and that \( h_{r, a}^N(x) = h_r^N(x) \) for a.e. \( r \) (hence we may replace \( h_{r, a}^N(x) \) by \( h_r^N(x) \) in ordinary Lebesgue integrals). It remains to understand the last line of the previous formula. At every jump time \( r \), we have
\[
J \left( h_r^N(x) \right)^2 - 2 h_{r, a}^N(x) h_r^N(x)
= \left( h_r^N(x)^2 - h_{r, a}^N(x)^2 \right) - 2 h_{r, a}^N(x) \left( h_r^N(x) - h_{r, a}^N(x) \right)
= h_r^N(x)^2 + h_{r, a}^N(x)^2 - 2 h_{r, a}^N(x) h_r^N(x).
\]
We generically denote by applicable, which we unite in Corollary 5.3.7 below.

Moreover, $Jh^N_{T_1}(x) = \frac{1}{N} \theta_N(x - X^N_{T_1})$, hence

$$
\sum_{r \leq t} \left( Jh^N_r(x) \right)^2 = \sum_{a \in A^N} \left( Jh^N_{T_1}(x) \right)^2 1_{t \geq T_1} = \frac{1}{N^2} \sum_{a \in A^N} \left( \theta_N(x - X^N_{T_1}) \right)^2 1_{t \geq T_1}.
$$

In the up-coming sequence of lemmata, we estimate all the terms on the right-hand side of Lemma 5.3.3, in such a way, that Gronwall’s lemma is applicable, which we unite in Corollary 5.3.7 below.

We generically denote by $C > 0$ any constant depending only on $\sigma$, $T$, $\|\nabla V\|_{L^\infty}$, $C_F$, $\sup_N \epsilon_N^{d-2}/N$, moments of $[S^N_T]$ (see Lemma 1.4.1), $\|\theta\|^2_{L^2}$ and $\|\nabla \theta\|^2_{L^2}$ (recall that $\theta$ has compact support).

**Lemma 5.3.4**

There exists a constant $C > 0$ such that

$$
\left| \int_0^t \langle \theta_N \ast \left( (\nabla V \ast S^N_r) S^N_r \right), \nabla h^N_r \rangle \, dr \right| \leq \frac{\sigma^2}{4} \int_0^t \left\| \nabla h^N_r \right\|^2_{L^2} \, dr + C \left[ S^N_T \right]^2 \int_0^t \left\| h^N_r \right\|^2_{L^2} \, dr,
$$

and

$$
\left| \int_0^t \langle \theta_N \ast \left( F_N (S^N_r, \cdot) S^N_r \right), h^N_r \rangle \, dr \right| \leq C \int_0^t \left\| h^N_r \right\|^2_{L^2} \, dr.
$$

For the second inequality to hold true we need to assume (5.8).

**Proof.** By Hölder’s inequality we have

$$
\left| \int_0^t \langle \theta_N \ast \left( (\nabla V \ast S^N_r) S^N_r \right), \nabla h^N_r \rangle \, dr \right| \leq \frac{\sigma^2}{4} \int_0^t \left\| \nabla h^N_r \right\|^2_{L^2} \, dr + \frac{C}{\sigma^2} \int_0^t \left\| \theta_N \ast \left( (\nabla V \ast S^N_r) S^N_r \right) \right\|^2_{L^2} \, dr
$$

and then we handle the second term by means of the bound (using also (1.5))

$$
\left| \theta_N \ast \left( (\nabla V \ast S^N_r) S^N_r \right) (x) \right| \leq \| \nabla V \|_{L^\infty} \left[ S^N_T \right] h^N_r (x),
$$

which follows from (1.7). The left-hand side of the second inequality of the lemma is bounded from above by

$$
\leq \int_0^t \left\| \theta_N \ast \left( F_N (S^N_r, \cdot) S^N_r \right) \right\| \, dr \leq C_F \int_0^t \left\| h^N_r \right\|^2_{L^2} \, dr,
$$

because $\left| \theta_N \ast \left( F_N (S^N_r, \cdot) S^N_r \right) (x) \right| \leq C_F h^N_r (x)$ by (1.7) and (5.8). □
Lemma 5.3.5
Under the assumption (5.4) there exists a constant $C > 0$, such that

$$
E \left[ \sup_{t \in [0, T]} \left\| [M_{t}^{1,N}] \right\|_{L^1} \right] \leq C,
$$

$$
E \left[ \sup_{t \in [0, T]} \int_{\mathbb{R}^d} \frac{1}{N^2} \sum_{a \in A^N} \left( \theta_N(x - X_T^a) \right)^2 1_{\tau \geq T^a_1} dx \right] \leq C.
$$

Proof. Since

$$
\left[ M_{t}^{1,N}(x) \right]_t = \frac{\sigma^2}{N^2} \sum_{a \in A^N} \int_0^t 1_{s \in I^a} |\nabla \theta_N(x - X^a_s)|^2 ds,
$$

we have (using the change of variable $x \mapsto x - X^a_s$ in the Lebesgue integral over $\mathbb{R}^d$)

$$
\int_{\mathbb{R}^d} M_{t}^{1,N}(x) dx = \frac{\sigma^2}{N^2} \sum_{a \in A^N} \int_0^t 1_{s \in I^a} \left( \int_{\mathbb{R}^d} |\nabla \theta_N(x - X^a_s)|^2 dx \right) ds
$$

$$
= \frac{\sigma^2}{N^2} \sum_{a \in A^N} \int_0^t 1_{s \in I^a} \left( \int_{\mathbb{R}^d} |\nabla \theta_N(x)|^2 dx \right) ds
$$

$$
= \frac{\sigma^2}{N} \int_{\mathbb{R}^d} |\nabla \theta_N(x)|^2 dx \int_0^t \left[ S^N_s \right] ds.
$$

We conclude the first estimate of the lemma using Lemma 1.4.1 and (1.8). Similarly, since

$$
\int_{\mathbb{R}^d} \frac{1}{N^2} \sum_{a \in A^N} \left( \theta_N(x - X_T^a) \right)^2 1_{\tau \geq T^a_1} dx \leq \frac{1}{N} \left[ S^N_t \right] \int_{\mathbb{R}^d} \theta_N^2(x) dx
$$

we deduce the second estimate of the lemma. \hfill \Box

Recall the definition of $\tau_R^N$ in (5.6) for $R > 0$

$$
\tau_R^N := \inf \left\{ r > 0 : \left[ S^N_r \right] > R \text{ or } \left\| h_0^N \right\|_{L^2(\mathbb{R}^d)} > R \right\}
$$

Lemma 5.3.6
Under the assumption (5.4) there exists a constant $C > 0$ such that for every $R > 0$

$$
E \left[ \sup_{t \in [0, T]} \left| \int_{\mathbb{R}^d} \int_0^{\tau \geq \tau_R^N} h_s^N(x) dx \left( M_{s}^{1,N}(x) + M_{s}^{2,N}(x) \right) dx \right| \right] \leq 2 + C \mathbb{E} \left[ \int_0^T \left\| h_{s, \tau_R^N}^N \right\|_{L^2}^2 ds \right]
$$

holds true.
Proof. Obviously the expected value on the left-hand side above is bounded by

\[
\leq 2 + \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_{\mathbb{R}^d} \int_0^{T \wedge \tau^*_R} h_s^N(x) dM_s^{1,N}(x) dx \right|^2 \right] \\
+ \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_{\mathbb{R}^d} \int_0^{T \wedge \tau^*_R} h_{s-}^N(x) dM_s^{2,N}(x) dx \right|^2 \right]
\]

(we have also replaced \(h_{s-}^N(x)\) by \(h_s^N(x)\) in the integral with respect to the continuous martingale \(M_s^{1,N}\)). Concerning the \(M_s^{1,N}\)-term, we have

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_{\mathbb{R}^d} \int_0^{T \wedge \tau^*_R} h_s^N(x) dM_s^{1,N}(x) dx \right|^2 \right] = \sigma^2 \mathbb{E} \left[ \sum_{a \in A^N} \int_0^{T \wedge \tau^*_R} 1_{s \in I^a}(X_s^a) \cdot dB_s^a \right]^2,
\]

(we have used the stochastic Fubini Theorem) where

\[
g_s^N(y) := \int_{\mathbb{R}^d} h_s^N(x) \nabla \theta_N(x - y) dx.
\]

By Itô Isometry,

\[
\leq C \frac{\sigma^2}{N^2} \mathbb{E} \left[ \sum_{a \in A^N} \int_0^{T \wedge \tau^*_R} 1_{s \in I^a}(X_s^a) \cdot dB_s^a \right]^2
\]

\[
= C \frac{\sigma^2}{N^2} \mathbb{E} \left[ \sum_{a \in A^N} \int_0^{T \wedge \tau^*_R} \left| g_s^N(X_s^a) \right|^2 ds \right].
\]

But \(\frac{1}{N} \left| g_s^N(y) \right|^2 \leq C \left\| h_s^N \right\|_{L^2}, \) by (1.8), hence

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_{\mathbb{R}^d} \int_0^{T \wedge \tau^*_R} h_s^N(x) dM_s^{1,N}(x) dx \right|^2 \right] \leq C \sigma^2 \mathbb{E} \left[ \int_0^{T \wedge \tau^*_R} \left| S_s^N \right| \left\| h_s^N \right\|^2_{L^2} ds \right].
\]
Concerning the $M_{2,2}^N$-term, we have

$$
E \left[ \sup_{t \in [0,T]} \left| \int_{\mathbb{R}^d} \int_0^{T \wedge \tau_N^R} h_{a-}^N(x) dM_{s,t}^N(x) dx \right|^2 \right]
$$

$$
= \frac{1}{N^2} E \left[ \sup_{t \in [0,T]} \left| \sum_{a \in AN} \int_{\mathbb{R}^d} \int_0^{T \wedge \tau_N^R} h_{a-}^N(x) \theta_N(x - X_{a-}^s) d (\mathcal{N}_s^a - \Lambda_s^a) dx \right|^2 \right]
$$

$$
= \frac{1}{N^2} E \left[ \sup_{t \in [0,T]} \left| \sum_{a \in AN} \int_0^{T \wedge \tau_N^R} \tilde{g}_{a-}^N(X_{a-}^s) \cdot d (\mathcal{N}_s^a - \Lambda_s^a) \right|^2 \right]
$$

where $\tilde{g}_s^N(y) := \int_{\mathbb{R}^d} h_s^N(x) \theta_N(x - y) dx$. The process $\sum_{a \in AN} \int_0^{T \wedge \tau_N^R} \tilde{g}_{a-}^N(X_{a-}^s) \cdot d (\mathcal{N}_s^a - \Lambda_s^a)$ is a martingale with respect to the filtration $\mathcal{G}_t := \mathcal{F}_{A_t}$, hence the last expression is bounded by

$$
\leq \frac{C}{N^2} E \left[ \left| \sum_{a \in AN} \int_0^{T \wedge \tau_N^R} \tilde{g}_{a-}^N(X_{a-}^s) \cdot d (\mathcal{N}_s^a - \Lambda_s^a) \right|^2 \right].
$$

Since the jumps of $\mathcal{N}_s^a$ and $\mathcal{N}_s^{a'}$, for $a \neq a'$, never occur at the same time, we have

$$
E \left[ \left( \int_0^{T \wedge \tau_N^R} \tilde{g}_{a-}^N(X_{a-}^s) d (\mathcal{N}_s^a - \Lambda_s^a) \right) \left( \int_0^{T \wedge \tau_N^R} \tilde{g}_{a'}^N(X_{a'}^s) d (\mathcal{N}_s^{a'} - \Lambda_s^{a'}) \right) \right] = 0.
$$

Hence, the last expression is equal to

$$
= \frac{C}{N^2} \sum_{a \in A} E \left[ \left| \int_0^{T \wedge \tau_N^R} \tilde{g}_{a-}^N(X_{a-}^s) \cdot d (\mathcal{N}_s^a - \Lambda_s^a) \right|^2 \right].
$$

(5.12)

It is known that

$$
E \left[ \left| \int_0^{T \wedge \tau_N^R} \tilde{g}_{a-}^N(X_{a-}^s) d (\mathcal{N}_s^a - \Lambda_s^a) \right|^2 \right] = E \left[ \int_0^{T \wedge \tau_N^R} \left| \tilde{g}_{a-}^N(X_{a-}^s) \right|^2 d\Lambda_s^a \right]
$$

$$
= E \left[ \int_0^{T \wedge \tau_N^R} \left| \tilde{g}_s^N(X_s^a) \right|^2 1_{s \in I_a} F_N \left( S_s^N, X_s^a \right) ds \right]
$$

$$
\leq C_F E \left[ \int_0^{T \wedge \tau_N^R} \left| \tilde{g}_s^N(X_s^a) \right|^2 1_{s \in I_a} ds \right]
$$

and therefore (5.12) is bounded from above by

$$
\leq \frac{C}{N} E \left[ \int_0^{T \wedge \tau_N^R} \frac{1}{N} \sum_{a \in A} \left| \tilde{g}_s^N(X_s^a) \right|^2 1_{s \in I_a} ds \right]
$$

$$
= \frac{C}{N} E \left[ \int_0^{T \wedge \tau_N^R} \int_{\mathbb{R}^d} \left| \tilde{g}_s^N(x) \right|^2 S_s^N (dx) ds \right].
$$

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As above for $g^N_s(y)$, using (1.8) we have $\frac{1}{N} \left| g^N_s(y) \right|^2 \leq C \left\| h^N_s \right\|^2_{L^2}$. Finally, this yields

$$\leq C E \left[ \int_0^{T \wedge \tau^N_R} \left\| h^N_s \right\|^2_{L^2} \left[ S^N_s \right] ds \right]$$

The result of the lemma follows by using the fact that $\left[ S^N_s \right] \leq R$, since the integral in $s$ runs only up to $T \wedge \tau^N_R$. \hfill \Box

**Corollary 5.3.7**

Assume (5.4), (5.5) and (5.8), then we have

$$\lim_{R_1 \to \infty} \sup_{N \in \mathbb{N}} \mathbb{P} \left( \sup_{t \in [0,T]} \left\| h^N_t \right\|^2_{L^2} + \int_0^T \left\| \nabla h^N_r \right\|^2_{L^2} dr > R_1 \right) = 0.$$

**Proof.** Step 1. From Lemmata 5.3.3 and 5.3.4 we have for each $R > 0$

$$\frac{1}{2} \left\| h^N_{t \wedge \tau^N_R} \right\|^2_{L^2} + \frac{\sigma^2}{4} \int_0^{t \wedge \tau^N_R} \left\| \nabla h^N_r \right\|^2_{L^2} dr \leq \frac{1}{2} \left\| h^N_0 \right\|^2_{L^2} + C \int_0^{t \wedge \tau^N_R} \left[ S^N_r \right]^2 \left\| h^N_r \right\|^2_{L^2} dr + C \int_0^{t \wedge \tau^N_R} \left\| h^N_r \right\|^2_{L^2} dr + a_N + b_{N,R},$$

where

$$a_N := \frac{1}{2} \sup_{t \in [0,T]} \left\| M^1,N \right\|_{L^1} + \frac{1}{2} \sup_{t \in [0,T]} \int_{\mathbb{R}^d} \frac{1}{N^2} \sum_{a \in A^N} \left( \theta_N(x - X^a_{T_t^N}) \right)^2 1_{t \geq T_t^N} dx,$$

$$b_{N,R} := \sup_{t \in [0,T]} \left| \int_{\mathbb{R}^d} \int_0^{t \wedge \tau^N_R} h^N_{a-}(x) d \left( M^1,N_s(x) + M^2,N_s(x) \right) dx \right|.$$  

Hence, writing $\chi^N_R = \left\| h^N_0 \right\|^2_{L^2} \leq R$, from Lemmata 5.3.5 and 5.3.6 it follows

$$\frac{1}{2} E \left[ \chi^N_R \sup_{t \in [0,A]} \left\| h^N_{t \wedge \tau^N_R} \right\|^2_{L^2} \right] \leq R^2/2 + C + C \left( R^2 + R + 1 \right) \int_0^T E \left[ \chi^N_R \sup_{t \in [0,A]} \left\| h^N_{t \wedge \tau^N_R} \right\|^2_{L^2} \right] ds.$$

By Gronwall’s lemma, we obtain

$$E \left[ \chi^N_R \sup_{t \in [0,T]} \left\| h^N_{t \wedge \tau^N_R} \right\|^2_{L^2} \right] \leq C_0(R),$$

where $C_0(R)$ is a constant depending only on $R$. 

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where $C_0 (R) := (R^2 + 2C) \exp (2C (R^2 + R + 1))$. Moreover, for the same reasons,

\[
\frac{\sigma^2}{4} \mathbb{E} \left[ \chi_R^N \int_0^{t \land \tau_R^N} \| \nabla h_t^N \|_{L^2}^2 \, dr \right] \\
\leq R^2/2 + C + C \left( R^2 + R + 1 \right) \int_0^t \mathbb{E} \left[ \chi_R^N \sup_{r \in [0,s]} \| h_r^N \|_{L^2}^2 \right] \, ds \leq C_1 (R),
\]

where $C_1 (R) = R^2/2 + C + C \left( R^2 + R + 1 \right) TC_0 (R)$ and we have used the previous bound in the last term.

**Step 2.** For every $R_1 > 0$, the probability $\mathbb{P} \left( \sup_{t \in [0,T]} \| h_t^N \|_{L^2}^2 \geq R_1 \right)$ is bounded from above by

\[
\begin{align*}
\leq & \mathbb{P} \left( \sup_{t \in [0,T]} \| h_t^N \|_{L^2}^2 \geq R_1, \chi_R^N = 1 \right) + \mathbb{P} \left( \| h_0^N \|_{L^2}^2 > R \right) \\
\leq & \mathbb{P} \left( \chi_R^N \sup_{t \in [0,T]} \| h_t^N \|_{L^2}^2 \geq R_1 \right) + \mathbb{P} \left( \| h_0^N \|_{L^2}^2 > R \right) \\
\leq & \mathbb{P} \left( \chi_R^N \sup_{t \in [0,T]} \| h_t^N \|_{L^2}^2 \geq R_1, \tau_R \geq T \right) + \mathbb{P} \left( \tau_R < T \right) + \mathbb{P} \left( \| h_0^N \|_{L^2}^2 > R \right) \\
\leq & \frac{1}{R_1} \mathbb{E} \left[ \chi_R^N \sup_{t \in [0,T]} \| h_t^N \|_{L^2}^2 \geq R_1 \right] + \frac{1}{R} \mathbb{E} \left[ \left[ S_T^N \right] \right] + \mathbb{P} \left( \| h_0^N \|_{L^2}^2 > R \right) \\
\leq & \frac{C_0 (R)}{R_1} + \frac{C}{R} + \mathbb{P} \left( \| h_0^N \|_{L^2}^2 > R \right).
\end{align*}
\]

Now, we want to send $R \to \infty$ as $R_1 \to \infty$ in the following way. Take $R$ as a function of $R_1$ (denoted by $R_1 \mapsto R(R_1)$), such that $\lim_{R_1 \to \infty} R(R_1) = +\infty$ and $\lim_{R_1 \to \infty} C_0 \left( R(R_1) \right) / R_1 = 0$, where the function $C_0(R)$ has been defined in Step 1. This implies, together with assumption (5.5),

\[
\lim_{R_1 \to \infty} \sup_{N \in \mathbb{N}} \mathbb{P} \left( \sup_{t \in [0,T]} \| h_t^N \|_{L^2}^2 \geq R_1 \right) = 0.
\]

This proves half of the claim of the corollary.
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Step 3. For every $R_1 > 0$, by similar arguments

$$
P \left( \int_0^T \|\nabla h_r^N\|_{L^2}^2 dr \geq R_1 \right)$$

$$\leq P \left( \chi_N \frac{\sigma^2}{4} \int_0^{T \wedge \tau_R} \|\nabla h_r^N\|_{L^2}^2 dr > \frac{\sigma^2}{4} R_1 \right) + P (\tau_R < T) + P \left( \|h_0^N\|_{L^2}^2 > R \right)$$

$$\leq \frac{4}{\sigma^2 R_1} \mathbb{E} \left[ \chi_N \frac{\sigma^2}{4} \int_0^{T \wedge \tau_R} \|\nabla h_r^N\|_{L^2}^2 dr \right] + \frac{1}{R} \mathbb{E} \left[ \left[ S_r^N \right] \right] + P \left( \|h_0^N\|_{L^2}^2 > R \right)$$

$$\leq \frac{16 C_1 (R)}{\sigma^4 R_1} + \frac{C}{R} + P \left( \|h_0^N\|_{L^2}^2 > R \right).$$

As above, we conclude that the second half of the claim of the corollary holds true. □

In order to show tightness of the family of the functions $\left( h_r^N \right)_{N \in \mathbb{N}}$, in addition to the previous bound which shows regularity in space, we also need some regularity in time. See the compactness criteria below.

Lemma 5.3.8

Assume (5.4), (5.5) and (5.8), given any $\alpha \in (0, 1/2)$, it holds

$$\lim_{R \to \infty} \sup_{N \in \mathbb{N}} P \left( \int_0^T \int_0^T \frac{\|h_t^N - h_s^N\|_{W^{-1,2}}^2}{|t - s|^{1 + 2\alpha}} ds \, dt > R \right) = 0.$$

Proof. Step 1. We need to estimate $\|h_t^N - h_s^N\|_{W^{-1,2}}^2$ in such a way that it cancels with the singularity in the denominator at $t = s$. First, we have

$$\|h_t^N - h_s^N\|_{W^{-1,2}}^2 \leq C \left( \int_s^t \text{div} \left( \theta_N * \left( \nabla V * S_r^N \right) S_r^N \right) dr \right)^2_{W^{-1,2}}$$

$$+ C \left( \int_s^t \frac{\sigma^2}{2} \Delta h_r^N dr \right)^2_{W^{-1,2}}$$

$$+ C \left( \int_s^t \theta_N * \left( F_N \left( S_r^N, \cdot \right) S_r^N \right) dr \right)^2_{W^{-1,2}}$$

$$+ C \left( M_t^{1,N} - M_s^{1,N} \right)^2_{W^{-1,2}} + C \left( M_t^{2,N} - M_s^{2,N} \right)^2_{W^{-1,2}}$$
and thus by Hölder’s inequality

\[
\leq C (t-s) \int_s^t \| \text{div} \left( \theta_N \ast \left( \left( \nabla V \ast S_r^N \right) S_r^N \right) \right) \|^2_{W^{-1,2}} \, dr \\
+ C (t-s) \int_s^t \left\| \frac{\sigma^2}{2} \Delta h_r^N \right\|^2_{W^{-1,2}} \, dr \\
+ C (t-s) \int_s^t \left\| h_N \ast \left( F_N \left( S_r^N, \cdot \right) S_r^N \right) \right\|^2_{W^{-1,2}} \, dr \\
+ C \left\| M_t^{1,N} - M_s^{1,N} \right\|^2_{W^{-1,2}} + C \left\| M_t^{2,N} - M_s^{2,N} \right\|^2_{L^2}.
\]

Notice that \( L^2 \subset W^{-1,2} \) with continuous embedding, namely there exists a constant \( C > 0 \) such that \( \| f \|_{W^{-1,2}} \leq C \| f \|_{L^2} \) for all \( f \in L^2 \). The linear operator \( \text{div} \) is bounded from \( L^2 \) to \( W^{-1,2} \), namely \( \| \text{div} f \|_{W^{-1,2}} \leq C \| f \|_{L^2} \) and the operator \( \Delta \) is bounded from \( W^{1,2} \) to \( W^{-1,2} \), namely \( \| \Delta f \|_{W^{-1,2}} \leq C \| f \|_{W^{1,2}} \). Therefore (we denote by \( C > 0 \) any constant independent of \( N, h_r^N, t, s \))

\[
\leq C (t-s) \int_s^t \left\| h_N \ast \left( \left( \nabla V \ast S_r^N \right) S_r^N \right) \right\|^2_{L^2} \, dr + C (t-s) \int_s^t \left\| h_r^N \right\|^2_{W^{-1,2}} \, dr \\
+ C (t-s) \int_s^t \left\| h_N \ast \left( F_N \left( S_r^N, \cdot \right) S_r^N \right) \right\|^2_{L^2} \, dr \\
+ C \left\| M_t^{1,N} - M_s^{1,N} \right\|^2_{L^2} + C \left\| M_t^{2,N} - M_s^{2,N} \right\|^2_{L^2}.
\]

and now using (1.7), assumption (5.8) and “\( \int_s^t \leq \int_0^T \)” in all terms,

\[
\left\| h_t^N - h_s^N \right\|^2_{W^{-1,2}} \leq C (t-s) \int_0^T \left( \left[ S_r^N \right]^2 + 1 \right) \left\| h_r^N \right\|^2_{L^2} \, dr \\
+ C (t-s) \int_0^T \left\| h_r^N \right\|^2_{W^{-1,2}} \, dr \\
+ C \left\| M_t^{1,N} - M_s^{1,N} \right\|^2_{L^2} + C \left\| M_t^{2,N} - M_s^{2,N} \right\|^2_{L^2} \\
\leq C (t-s) \left( \left[ S_r^N \right]^2 + 1 \right) \sup_{r \in [0,T]} \left\| h_r^N \right\|^2_{L^2} \\
+ C (t-s) \int_0^T \left\| h_r^N \right\|^2_{W^{-1,2}} \, dr \\
+ C \sum_{i=1,2} \left\| M_t^{i,N} - M_s^{i,N} \right\|^2_{L^2}.
\]

Accordingly, we split the estimate of \( \mathbb{P} \left( \int_0^T \int_0^T \left\| h_t^N - h_s^N \right\|^2_{W^{-1,2}} \, ds \, dt > R \right) \) in four more elementary estimates, that now we handle separately. The final result is a consequence of them.
The number $C_\alpha = \int_0^T \int_0^T \frac{1}{|t-s|^{1+2\alpha}} \, ds \, dt$ is finite, hence the first term is bounded by (renaming the constant $C$)

$$\mathbb{P} \left( \int_0^T \int_0^T C (t-s) \left( \left[ S_T^N \right] + 1 \right) \sup_{r \in [0,T]} \| h_r^N \|_{L^2} \, ds \, dt > R \right)$$

$$= \mathbb{P} \left( \left( \left[ S_T^N \right] + 1 \right) \sup_{r \in [0,T]} \| h_r^N \|_{L^2}^2 > R/C \right)$$

$$\leq \mathbb{P} \left( \left[ S_T^N \right] + 1 > \sqrt{R/C} \right) + \mathbb{P} \left( \sup_{r \in [0,T]} \| h_r^N \|_{L^2} > \sqrt{R/C} \right)$$

and both these terms are, uniformly in $N$, small for large $R$, due to Lemma 1.4.1 and the estimates proved in the tightness part. The second term, i.e. $C (t-s) \int_0^T \| h_r^N \|_{W^{1,2}}^2 \, dr$, is similar.

**Step 2.** Concerning the martingale terms, we prove that

$$\mathbb{E} \left\| M_{t,N}^{i,N} - M_{s,N}^{i,N} \right\|_{L^2}^2 \leq C |t-s|$$

for some constant $C > 0$, $i = 1, 2$. By Chebyshev’s inequality it follows that

$$\lim_{R \to \infty} \sup_{N \in \mathbb{N}} \mathbb{P} \left( \int_0^T \int_0^T \frac{\left\| M_{t,N}^{i,N} - M_{s,N}^{i,N} \right\|_{L^2}^2}{|t-s|^{1+2\alpha}} \, ds \, dt > R \right) = 0$$

and the proof will be complete. For notational convenience, we abbreviate, for $i = 1, 2$,

$$M_{t,N}^{i,N}(x) = \frac{1}{N} \sum_{a \in A_N} M_t^{i,a}(x).$$

Note that for every $x \in \mathbb{R}^d$ the processes $M_1^{1,a}(x)$ and $M_2^{1,a}(x)$ are martingales. It follows, with computations similar to those of Lemma 5.3.6 for $t \geq s$

$$\mathbb{E} \left\| M_{t,N}^{1,N} - M_{s,N}^{1,N} \right\|_{L^2}^2 = \int_{\mathbb{R}^d} \frac{1}{N^2} \sum_{a \in A_N} \mathbb{E} \left[ \left( M_t^{1,a}(x) - M_s^{1,a}(x) \right)^2 \right] \, dx$$

$$= \int_{\mathbb{R}^d} \frac{1}{N^2} \sum_{a \in A_N} \mathbb{E} \left[ \int_s^t 1_{r \in I_a} \| \nabla \theta_N (x - X_r^a) \|_{L^2}^2 \, dr \right] \, dx$$

$$= \frac{1}{N} \| \nabla \theta_N \|_{L^2}^2 \mathbb{E} \int_s^t \frac{1}{N} \sum_{a \in A_N} 1_{r \in I_a} \, dr \leq C (t-s),$$
where in the last inequality we have used (1.8) and Lemma 1.4.1. Similarly, for the second martingale,

\[
E \left\| M_{t}^{2,N} - M_{s}^{2,N} \right\|_{L^2}^2 = \int_{\mathbb{R}^d} \frac{1}{N^2} \sum_{a \in A^N} E \left[ M_{t,a}^{2,a}(x)^2 - M_{s,a}^{2,a}(x)^2 \right] \, dx \\
= \int_{\mathbb{R}^d} \frac{1}{N^2} \sum_{a \in A^N} E \left[ \int_{s}^{t} 1_{r \in I_s} \theta_N (x - X_{r}^a)^2 \lambda_r^a \, dr \right] \, dx \\
\leq C_F \frac{1}{N} \left\| \theta_N \right\|_{L^2}^2 \int_{s}^{t} \frac{1}{N} \sum_{a \in A^N} 1_{r \in I_s} \, dr \leq C (t-s).
\]

\[ \square \]

A version of Aubin-Lions’ lemma, see \[Lio69\], \[FG95\], \[BBNP14\], states that when \( E_0 \subset E \subset E_1 \) are three Banach spaces with continuous dense embedding, \( E_0, E_1 \) reflexive, with \( E_0 \) compactly embedded into \( E \), given \( p, q \in (1, \infty) \) and \( \alpha \in (0, 1) \), then the space \( L^q (0, T; E_0) \cap W^{\alpha,p} (0, T; E_1) \) is compactly embedded into \( L^q (0, T; E) \). We use this lemma with \( E = L^2 (D) \), \( E_0 = W^{1,2} (D) \) and \( E_1 = W^{-1,2} (\mathbb{R}^d) \), where \( D \) is a regular bounded domain, and with \( p = q = 2, \alpha \in (0, 1/2) \). The lemma states the following compact embedding

\[
L^2 (0, T; W^{1,2} (D)) \cap W^{\alpha,2} (0, T; W^{-1,2} (\mathbb{R}^d)) \subset \subset L^2 (0, T; L^2 (D)).
\]

Note that for \( \alpha p > 1 \), \( W^{\alpha,p} (0, T; E_1) \) is embedded into \( C ([0, T]; E_1) \), hence it is not suitable for our purposes since we have to deal with discontinuous processes. However, for \( \alpha p < 1 \) the space \( W^{\alpha,p} (0, T; E_1) \) includes piecewise constant functions, as one can easily check. Therefore it is a suitable space for càdlàg processes.

Now, consider the space

\[
Y_0 := L^\infty (0, T; L^2 (\mathbb{R}^d)) \cap L^2 (0, T; W^{1,2} (\mathbb{R}^d)) \cap W^{\alpha,2} (0, T; W^{-1,2} (\mathbb{R}^d)).
\]

Using the Fréchet topology on \( L^2_{loc} ([0, T] \times \mathbb{R}^d) \) induced by the metric

\[
d (f, g) = \sum_{n=1}^{\infty} 2^{-n} \left( 1 \wedge \int_{0}^{T} \int_{B(0,n)} |(f - g)(t, x)|^2 \, dx \, dt \right),
\]

where \( B(0, n) = \{ x \in \mathbb{R}^d : |x| < n \} \) we conclude that \( L^2 (0, T; W^{1,2} (\mathbb{R}^d)) \cap W^{\alpha,2} (0, T; W^{-1,2} (\mathbb{R}^d)) \) is compactly embedded into \( L^2_{loc} ([0, T] \times \mathbb{R}^d) \) (the proof is elementary, using that if a set is compact in \( L^2 (0, T; L^2 (B(0, n))) \) for every \( n \) then it is compact in \( L^2_{loc} ([0, T] \times \mathbb{R}^d) \) with this topology; see 85
a similar result in [BM13]). We denote the spaces \( L^\infty (0, T; L^2 (\mathbb{R}^d)) \) and \( L^2 (0, T; W^{1,2} (\mathbb{R}^d)) \) endowed respectively with the weak star and weak topology by \( L^\infty_{ws} (0, T; L^2 (\mathbb{R}^d)) \) and \( L^2_w (0, T; W^{1,2} (\mathbb{R}^d)) \). We have that \( Y_0 \) is compactly embedded into
\[
Y := L^\infty_{ws} (0, T; L^2 (\mathbb{R}^d)) \cap L^2_w (0, T; W^{1,2} (\mathbb{R}^d)) \cap L^2_{loc} ([0, T] \times \mathbb{R}^d). \tag{5.13}
\]

Denote by \( (Q^N)_{N \in \mathbb{N}} \) the laws of \( (h^N)_{N \in \mathbb{N}} \) on \( Y_0 \). From the “boundedness in probability” of the family \( (Q^N)_{N \in \mathbb{N}} \) in \( Y_0 \) as stated in Corollary 5.3.7 and Lemma 5.3.8, it follows that the family \( (Q^N)_{N \in \mathbb{N}} \) is tight in \( Y \). Hence, by Prohorov’s theorem, from every subsequence of \( (Q^N)_{N \in \mathbb{N}} \) it is possible to extract a further subsequence which converges weakly to a probability measure \( Q \) on \( Y \). We shall prove that every such limit measure \( Q \) is a Dirac measure \( Q = \delta_u \) concentrated to the same element \( u \in Y \); hence the whole sequence \( (Q^N)_{N \in \mathbb{N}} \) converges to \( \delta_u \); and also the processes \( (h^N)_{N \in \mathbb{N}} \) converge in probability to \( u \).

5.4 Uniqueness of the limit PDE

Recall that \( V \in C^2_c (\mathbb{R}^d) \). Further, let \( F \) satisfy the assumptions of Theorem 5.1.1 above.

**Definition 5.4.1**

Given \( u_0 \in L^2 (\mathbb{R}^d) \cap L^1 (\mathbb{R}^d) \), \( u_0 \geq 0 \), we call a function \( u \geq 0 \) a weak solution of equation (5.9) if it belongs to the class
\[
L^\infty (0, T; L^2 (\mathbb{R}^d)) \cap L^2 (0, T; W^{1,2} (\mathbb{R}^d))
\]
and satisfies
\[
\langle u_t, \phi \rangle = \langle u_0, \phi \rangle - \int_0^t \langle (\nabla V * u_r) u_r, \nabla \phi \rangle \, dr - \frac{\sigma^2}{2} \int_0^t \langle \nabla u_r, \nabla \phi \rangle \, dr + \int_0^t \langle F (u_r) u_r, \phi \rangle \, dr \tag{5.14}
\]
for a.e. \( t \in [0, T] \) and for all \( \phi \in W^{1,2} (\mathbb{R}^d) \).

Notice that, due to \( u \in L^\infty (0, T; L^2 (\mathbb{R}^d)) \) and the fact that \( \nabla V \) is bounded and compactly supported, we have that \( (\nabla V * u_r) \) is bounded, hence
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the first integral in the weak equation is well defined. Moreover, since $F$ is uniformly bounded (see (5.8)), the last term is also well defined.

The following remark addresses the fact that equation (5.14) provides more regularity in time than what we ask for in the definition. This way equation (5.14) makes sense for all $t \in [0, T]$.

**Remark 5.4.2**

If $u$ is a solution in the sense of the definition, then there exists a (unique) representative of $u$ of class $C_w \left([0, T]; L^2 \left(\mathbb{R}^d \right) \right)$. Indeed, given $\phi \in W^{1,2} \left(\mathbb{R}^d \right)$, from identity (5.14) we deduce that the a.e. defined function $t \mapsto \langle u_t, \phi \rangle$ is a.s. equal to a continuous function $g_\phi$. Let $\pi : [0, T] \to L^2 \left(\mathbb{R}^d \right)$ be a bounded measurable representative (bounded in $L^2 \left(\mathbb{R}^d \right)$ by $C$). Let $\{\phi_n\} \subset W^{1,2} \left(\mathbb{R}^d \right)$ be dense in $L^2 \left(\mathbb{R}^d \right)$. Let $\Upsilon \subset [0, T]$ be a set of Lebesgue measure $T$ such that $\langle u(t), \phi_n \rangle = g_{\phi_n}(t)$ for all $t \in \Upsilon$. Then, for $\phi \in L^2 \left(\mathbb{R}^d \right)$ and $t, s \in \Upsilon$,

$$|\langle \pi(t) - \pi(s), \phi \rangle| \leq |g_{\phi_n}(t) - g_{\phi_n}(s)| + 2C \|\phi_n - \phi\|_{L^2}.$$ 

From this it follows that $t \mapsto \langle \pi(t), \phi \rangle$ is uniformly continuous on $\Upsilon$ hence uniquely extendible to a continuous function $t \mapsto L_t(\phi)$ on $[0, T]$. From this it is easy to extract a re-definition of $\pi(t)$ for $t \not\in \Upsilon$ so that $t \mapsto \langle \pi(t), \phi \rangle$ is continuous on $[0, T]$.

Finally, it is not difficult to show that identity (5.14) holds for all $t \in [0, T]$ for the representative of class $C_w \left([0, T]; L^2 \left(\mathbb{R}^d \right) \right)$.

**Theorem 5.4.3**

*Under the assumptions on $V$ and $F$, mentioned at the beginning of the section, there is at most one weak solution of equation (5.9).*

**Proof. Step 1.** If $u^{(i)}$, $i = 1, 2$ are two solutions and $v_t = u^{(1)}_t - u^{(2)}_t$, from the equation (in weak form)

$$\partial_t v_t = \frac{\sigma^2}{2} \Delta v_t + \text{div} \left( \left( \nabla V * v_t \right) u^{(1)}_t \right) + \text{div} \left( \left( \nabla V * u^{(2)}_t \right) v_t \right) + F\left( u^{(1)}_t \right) u^{(1)}_t - F\left( u^{(2)}_t \right) u^{(2)}_t$$

and the property $\partial_t v \in L^2 \left(0, T; W^{-1,2} \left(\mathbb{R}^d \right) \right)$ (see Step 2 below), we have

$$\frac{1}{2} \|v_t\|_{L^2}^2 + \frac{\sigma^2}{2} \int_0^t \|\nabla v_s\|_{L^2}^2 \, ds = - \int_0^t \langle (\nabla V * v_s) u^{(1)}_s, \nabla v_s \rangle \, ds - \int_0^t \langle (\nabla V * u^{(2)}_s) v_s, \nabla v_s \rangle \, ds$$
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\[ + \int_0^t \left\langle F \left( u_s^{(1)} \right) u_s^{(1)} - F \left( u_s^{(2)} \right) u_s^{(2)}, v_s \right\rangle ds. \]

Since
\[ \| \nabla V \ast v_s \|_\infty = \sup_x \left| \int \nabla V (x - y) v_s (y) \, dy \right| \leq \| \nabla V \|_{L^2} \| v_s \|_{L^2}, \]
\[ \| \nabla V \ast u_s^{(2)} \|_\infty \leq \| \nabla V \|_{L^2} \| u_s^{(2)} \|_{L^2}, \]

the terms on the second line can be bounded by

\[ \leq \frac{\sigma^2}{4} \int_0^t \| \nabla v_s \|_{L^2}^2 \, ds + C \int_0^t \| (\nabla V \ast v_s) u_s^{(1)} \|_{L^2}^2 \, ds \]
\[ + C \int_0^t \| (\nabla V \ast u_s^{(2)}) v_s \|_{L^2}^2 \, ds \]
\[ \leq \frac{\sigma^2}{4} \int_0^t \| \nabla v_s \|_{L^2}^2 \, ds + C \int_0^t \| \nabla V \|_{L^2}^2 \int_0^t \| v_s \|_{L^2}^2 \left( \| u_s^{(1)} \|_{L^2}^2 + \| u_s^{(2)} \|_{L^2}^2 \right) \, ds. \]

The terms on the last line, using assumptions (5.2)-(5.8), can be bounded by

\[ \int_0^t \left\langle F \left( u_s^{(1)} \right) u_s^{(1)} - F \left( u_s^{(2)} \right) u_s^{(2)}, v_s \right\rangle \, ds \]
\[ \leq \int_0^t \left\langle F \left( u_s^{(1)} \right) - F \left( u_s^{(2)} \right) \right\| u_s^{(1)} \|_{L^2}, \| v_s \|_{L^2} \right\rangle \, ds + \int_0^t \left\langle F \left( u_s^{(1)} \right) \right\| v_s \|_{L^2} \right\rangle \, ds \]
\[ \leq \int_0^t L_F \| u - u \|_{W^{1,2}} \| u_s^{(2)} \|_{L^2} \| v_s \|_{L^2} \, ds + \int_0^t C_F \| v_s \|_{L^2}^2 \, ds \]
\[ \leq \frac{\sigma^2}{8} \int_0^t \| \nabla v_s \|_{L^2}^2 \, ds + C \int_0^t \| v_s \|_{L^2}^2 \left( 1 + \| u_s^{(2)} \|_{L^2}^2 \right) \, ds. \]

It is then sufficient to apply Gronwall’s lemma to deduce \( v = 0 \).

**Step 2.** To complete the proof, check \( v \in W^{1,2} \left( 0, T; W^{-1,2} \left( \mathbb{R}^d \right) \right) \), to be able to apply the chain rule. To this end, we check that any weak solution \( u \) of equation (5.9) given by the definition has this regularity property.

The term \( u_0 \) is in \( W^{1,2} \left( 0, T; W^{-1,2} \left( \mathbb{R}^d \right) \right) \). The function \( r \mapsto (\nabla V \ast u_r) u_r \) is of class \( L^2 \left( 0, T; L^2 \left( \mathbb{R}^d \right) \right) \), since \( \nabla V \ast u_r \) is bounded, as remarked after the definition; hence the function \( r \mapsto \text{div} \left( (\nabla V \ast u_r) u_r \right) \) is of class \( L^2 \left( 0, T; W^{-1,2} \left( \mathbb{R}^d \right) \right) \) and thus the function \( t \mapsto \int_0^t \text{div} \left( (\nabla V \ast u_r) u_r \right) \, dr \) is of class \( W^{1,2} \left( 0, T; W^{-1,2} \left( \mathbb{R}^d \right) \right) \). \( r \mapsto \Delta u_r \) is of class \( L^2 \left( 0, T; W^{-1,2} \left( \mathbb{R}^d \right) \right) \) because \( u \in L^2 \left( 0, T; W^{1,2} \left( \mathbb{R}^d \right) \right) \) and thus the function \( t \mapsto \int_0^t \Delta u_r \, dr \) is of class \( W^{1,2} \left( 0, T; W^{-1,2} \left( \mathbb{R}^d \right) \right) \). Finally the function \( r \mapsto F \left( u_r \right) u_r \) is of class \( L^2 \left( 0, T; L^2 \left( \mathbb{R}^d \right) \right) \), since \( F \left( u_r \right) \) is bounded, as remarked after the definition;
hence the $t \mapsto \int_0^t \div ((\nabla V * u_r) u_r) \, dr$ is of class $W^{1,2} \left( 0, T; W^{-1,2} \left( \mathbb{R}^d \right) \right)$. It follows that the function

$$
t \mapsto u_0 + \frac{1}{2} \int_0^t \Delta u_r \, dr + \int_0^t F(u_r) \, u_r \, dr$$

is of class $W^{1,2} \left( 0, T; W^{-1,2} \left( \mathbb{R}^d \right) \right)$. But it easily coincides with the function $t \mapsto u(t)$ (see also Remark 5.4.2).

### 5.5 Passage to the limit

Given $\chi : [0,T] \to \mathbb{R}$ of class $C^1$ with $\chi_T = 0$ and given $\psi \in W^{1,2} \left( \mathbb{R}^d \right)$, define $\phi_t := \chi_t \psi$. Using the identity satisfied by $h_t^N$ against the test function $\phi_t$ we have

$$
0 = \int_0^T \int_{\mathbb{R}^d} \left( \frac{\partial \phi_t}{\partial t} h_t^N - \frac{\sigma^2}{2} \nabla \phi_t \cdot \nabla h_t^N \right) \, dx \, dt
- \int_0^T \int_{\mathbb{R}^d} \theta_N \ast \left( (\nabla V \ast S_t^N) S_t^N \right) \cdot \nabla \phi_t \, dx \, dt
+ \int_0^T \int_{\mathbb{R}^d} \theta_N \ast \left( F_N \left( S_t^N, \cdot \right) S_t^N \right) \phi_t \, dx \, dt
+ \left( h_0^N, \phi_0 \right)
+ \int_{\mathbb{R}^d} \int_0^T \phi_t \, dM_t^{1,N} \, dx + \int_{\mathbb{R}^d} \int_0^T \phi_t \, dM_t^{2,N} \, dx.
$$

Denote by $Q^N$ the law of $h_t^N$ and assume a subsequence $Q^{N_k}$ weakly converges, in the topology of the space $Y$ defined by (5.13), to a probability measure $Q$. Given $\phi_t := \chi_t \psi$ as above, the functional

$$u \mapsto \Psi_\phi(u) := \langle u_0, \phi_0 \rangle + \int_0^T \int_{\mathbb{R}^d} \left( \frac{\partial \phi_t}{\partial t} u_t - \frac{\sigma^2}{2} \nabla \phi_t \cdot \nabla u_t
- \nabla \phi_t \cdot (\nabla V * u_t) u_t + F(u_t) u_t \phi_t \right) \, dx \, dt$$

is continuous on $Y$. Note that if $\Psi_\phi(u) = 0$ for all $\phi$ as above, then $u$ satisfies equation (5.14). The details of this claim are treated in “Step 2” in the proof of the following Lemma.

It is known (part of the so-called Portmanteau Theorem) that weak convergence of $Q^{N_k}$ to $Q$ implies $Q(A) \leq \liminf_{k \to \infty} Q^{N_k}(A)$ if $A$ is an open set; and the set $\{ u \in Y : \Psi_\phi(u) > \epsilon \}$ is open since $\Psi_\phi$ is continuous. Hence, for every $\epsilon > 0,$

\[
Q \left( u : |\Psi_\phi(u)| > \epsilon \right) \leq \liminf_{k \to \infty} Q^{N_k} \left( u : |\Psi_\phi(u)| > \epsilon \right)
= \liminf_{k \to \infty} P \left( |\Psi_\phi(h_t^{N_k})| > \epsilon \right),
\]
We shall prove that this lim inf is zero. It follows that $Q(u : |Ψ_ϕ(u)| > ε) = 0$. Since this holds for every $ε > 0$, we deduce $Q(u : Ψ_ϕ(u) = 0) = 1$.

**Lemma 5.5.1**
Under the assumptions of Theorem 5.1.1, we have that $Q$ is supported on the set of weak solutions of equation (5.9).

**Proof.** It remains to prove that $\lim_{k \to \infty} P \left( |Ψ_ϕ(h_N^k)| > ε \right) = 0$, that the assertion of the lemma follows from the fact that $Q(u ∈ X : Ψ_ϕ(u) = 0) = 1$ for every $ϕ$ of the form $ϕ_t = χ_tψ$ as above, and that $Q$ is concentrated on non-negative functions. We prove these claims in three different steps.

**Step 1.** We show that $\lim_{k \to \infty} P \left( |Ψ_ϕ(h_N^k)| > ε \right) = 0$. We write $N$ instead of $N_k$ to simplify the notation. It holds

$$Ψ_ϕ(h_N^k) = \langle u_0, φ_0 \rangle + \int_0^T \int_{\mathbb{R}^d} \left( \frac{∂φ_t}{∂t} + \frac{σ^2}{2} Δφ_t - ∇φ_t · (ΔV * h_t N) \right)$$

$$+ F(h_t N) φ_t h_t N dx dt.$$

Using the identity satisfied by $h_t N$ and $φ_t$, see (5.15), we have

$$Ψ_ϕ(h_N^k) = \int_0^T \int_{\mathbb{R}^d} \left[ \theta_N * \left( (ΔV * S_t^N) S_t^N \right) - (ΔV * h_t N) h_t N \right] · Δφ_t dx dt$$

$$+ \int_0^T \int_{\mathbb{R}^d} \left[ F(h_t N) h_t N - \theta_N * \left( F_N(S_t^N, \cdot) \right) S_t^N \right] φ_t dx dt$$

$$+ \langle u_0 - h_0 N, φ_0 \rangle - \int_{\mathbb{R}^d} \int_0^T φ_t dM^1_{t, N} dx - \int_{\mathbb{R}^d} \int_0^T φ_t dM^2_{t, N} dx$$

and further

$$\langle u_0 - h_0 N, φ_0 \rangle = \langle u_0 - S_0 N, φ_0 \rangle + \langle S_0 N, φ_0 - θ_N * φ_0 \rangle.$$

From Lemmata 5.5.2 and 5.5.3 below it follows

$$|Ψ_ϕ(h_N)| \leq C(ε_N + α_N) \left[ S_T^N \right] \left[ S_T^N \right] \epsilon_N \left( \int_0^T \left| h_t N \right|_{W^{1,2}}^2 dt + 1 \right)$$

$$+ \left| \langle u_0 - S_0 N, φ_0 \rangle \right| + \left| θ_N * φ_0 - φ_0 \right| \left[ S_0 N \right]$$

$$+ \left| \int_{\mathbb{R}^d} \int_0^T φ_t dM^1_{t, N} dx \right| + \left| \int_{\mathbb{R}^d} \int_0^T φ_t dM^2_{t, N} dx \right|,$$

where the constant $C$ depends only on the quantities described below before Lemma 5.5.2. In order to prove $\lim_{N \to \infty} P \left( |Ψ_ϕ(h_N)| > ε \right) = 0$, it is
5.5. Passage to the limit

sufficient to prove the same result for each one of the previous terms. We have \( \lim_{N \to \infty} \mathbb{P} \left( C (\epsilon_N + \alpha_N) \left[ S_t^N \right]^2 > \varepsilon \right) = 0 \) from Chebyshev’s inequality and Lemma 1.4.1. The same applies to the terms \( \left\| \theta_N \ast \phi_0 - \phi_0 \right\| \left[ S_0^N \right] \) and \( C \left[ S_t^N \right] \epsilon_N^\beta \). The term \( \left\langle u_0 - S_0^N, \phi_0 \right\rangle \) is obvious by the assumption of convergence in probability on \( S_0^N \). The two martingale terms can be treated again by Chebyshev’s inequality and Lemma 5.5.4 below. Finally

\[
\mathbb{P} \left( C \left[ S_t^N \right] \epsilon_N^\beta \int_0^T \left\| h_t^N \right\|_{W^{1,2}}^2 \, dt > \varepsilon \right) = \mathbb{P} \left( \left[ S_t^N \right] \int_0^T \left\| h_t^N \right\|_{W^{1,2}}^2 \, dt > \varepsilon \epsilon_N^{-\beta} / C \right) \\
\leq \mathbb{P} \left( \left[ S_t^N \right] > \sqrt{\varepsilon \epsilon_N^{-\beta} / C} \right) + \mathbb{P} \left( \int_0^T \left\| h_t^N \right\|_{W^{1,2}}^2 \, dt > \sqrt{\varepsilon \epsilon_N^{-\beta} / C} \right).
\]

The first term goes to zero by Chebyshev’s inequality and Lemma 1.4.1. The second one goes to zero as \( N \to \infty \) by Corollary 5.3.1.

**Step 2.** We prove that the assertion of the lemma follows from the fact that \( Q (u \in Y : \Psi_{\phi,\alpha} (u) = 0) = 1 \) for every \( \phi \) of the form \( \phi_t = \chi_t \psi \) as above. Let \( \{ \chi^n \} \) be a sequence of functions \( \chi^n : [0, T] \to \mathbb{R} \) of class \( C^{1} \), with \( \chi^0_T = 0 \), which is dense in \( L^2 (0, T; \mathbb{R}) \). Let \( \{ \psi^m \} \) be a dense sequence in \( W^{1,2} (\mathbb{R}^d) \).

Set \( \phi_t^{n,m} = \chi_t^n \psi^m \); we have \( Q (u \in Y : \Psi_{\phi^{n,m},\alpha} (u) = 0, \forall n, m \in \mathbb{N}) = 1 \). Let us prove that the set \( A := \{ u \in Y : \Psi_{\phi^{n,m},\alpha} (u) = 0, \forall n, m \in \mathbb{N} \} \) is contained in the set of weak solutions.

If \( u \in A \), then \( \Psi_{\chi^n,\psi^m} (u) = 0 \) for every \( m \in \mathbb{N} \) and every Lipschitz continuous \( \chi : [0, T] \to \mathbb{R} \) with \( \chi_T = 0 \). This claim follows by approximation with the sequence \( \chi^n \) and is not difficult. We omit the details.

Given \( u \in Y \), there exists a set \( \Upsilon \subset [0, T) \) of full measure, such that

\[
\lim_{n \to \infty} \int_{t_0}^{t_0 + \frac{1}{n}} \int_{\mathbb{R}^d} \psi^m u_t(x) \, dx \, dt = \int_{\mathbb{R}^d} \psi^m u_{t_0} \, dx.
\]

for every \( t_0 \in \Upsilon \) and every \( m \in \mathbb{N} \). Given \( t_0 \in \Upsilon \), take the new sequence \( \chi^n_t \) defined (at least for \( n \) large enough) as \( \chi^n_t = 0 \) for \( t > t_0 + \frac{1}{n} \), \( \chi^n_t = -1 \) for \( t < t_0 \), \( \chi^n_t = -1 + n (t - t_0) \) for \( t \in \left[ t_0, t_0 + \frac{1}{n} \right] \). We have

\[
\int_0^T \int_{\mathbb{R}^d} \frac{\partial \phi_t^{n,m}}{\partial t} u_t(x) \, dx \, dt = n \int_{t_0}^{t_0 + \frac{1}{n}} \int_{\mathbb{R}^d} \psi^m u_t(x) \, dx \, dt \to \int_{\mathbb{R}^d} \psi^m u_{t_0} \, dx.
\]

We deduce, from \( \Psi_{\chi^n,\psi^m} (u) = 0, \forall n, m \in \mathbb{N} \), that identity (5.14) holds at time \( t_0 \), for all \( \phi = \psi^m, m \in \mathbb{N} \). Therefore it is true, for each \( \phi = \psi^m \), a.s. in time. By density of \( \{ \psi^m \} \) and the regularity properties of \( u \) it is easy to deduce that (5.14) holds for every \( \phi \in W^{1,2} (\mathbb{R}^d) \).
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Step 3. By the Portmanteau Theorem, the weak convergence of \( Q^N \) to \( Q \) implies that \( Q(A) \geq \limsup_{k \to \infty} Q^{N_k}(A) \) if \( A \) is a closed set. Note that \( \{u: u \geq 0\} \) is closed in \( Y \), hence

\[
Q(u: u \geq 0) \geq \limsup_{k \to \infty} Q^{N_k}(u: u \geq 0) = \limsup_{k \to \infty} \mathbb{P}(h^{N_k} \geq 0) = 1.
\]

Thus, \( Q \) is concentrated on non-negative functions, which completes the proof.

In the next three lemmata we denote by \( C \) any constant depending only on \( \sigma^2, T, C_F, \|D^2V\|_\infty, \|\theta\|_{L^2}, \|\nabla \theta\|_{L^2}, \sup_N \epsilon_N^{d-2}/N, \|\phi\|_\infty, \|\nabla \phi\|_\infty, \mathbb{E} \left[\|S^N_T\|\right] \) and the diameter of the support of \( \theta \). Note that \( D^2V \) is the Jacobi matrix of \( \nabla V \) and the norm on \( \mathbb{R}^{d \times d} \) is the induced operator norm. Lemma 5.5.2 below treats the convergence of the divergence terms.

**Lemma 5.5.2**
There exists a constant \( C > 0 \) such that

\[
\left| \int_0^T \int_{\mathbb{R}^d} \left[ \theta_N * \left( (\nabla V * S^N_t) S^N_t \right) - (\nabla V * h^N_t) h^N_t \right] \cdot \nabla \phi_t \, dx \, dt \right| \leq C \epsilon_N \left[ S^N_T \right]^2
\]

holds true.

**Proof.** The left-hand side is bounded from above by

\[
\leq \|\nabla \phi\|_\infty \int_0^T \int_{\mathbb{R}^d} \left[ \theta_N * \left( (\nabla V * S^N_t) S^N_t \right) - (\nabla V * S^N_t) h^N_t \right] \cdot \nabla \phi_t \, dx \, dt
\]

\[
+ \|\nabla \phi\| \left| \int_0^T \int_{\mathbb{R}^d} \left( (\nabla V * S^N_t) h^N_t - (\nabla V * h^N_t) h^N_t \right) \, dx \, dt \right|.
\]

We separately bound the two terms. The first one is a sort of commutation lemma estimate. We have

\[
\left| \theta_N * \left( (\nabla V * S^N_t) S^N_t \right) (x) - (\nabla V * S^N_t) \left( \theta_N * S^N_t \right) (x) \right|
\]

\[
\leq \int_{\mathbb{R}^d} \theta_N (x - y) \left| (\nabla V * S^N_t) (y) - (\nabla V * S^N_t) (x) \right| S^N_t (dy)
\]

\[
\leq \|D^2V\|_\infty \left[ S^N_T \right] \int_{\mathbb{R}^d} \theta_N (x - y) |x - y| S^N_t (dy).
\]

Since \( \theta \) is non-negative and compactly supported with diameter \( K \), we have

\[
\int_{\mathbb{R}^d} \theta_N (x - y) |x - y| S^N_t (dy) \leq K \epsilon_N \int_{\mathbb{R}^d} \theta_N (x - y) S^N_t (dy) = K \epsilon_N h^N_t (x).
\]
Summarizing, the first term above is bounded by
\[
\leq \|\nabla \phi\|_{\infty} \left\| D^2 V \right\|_{\infty} K \epsilon_n \int_0^T \mathbb{S}_r^N \int_{\mathbb{R}^d} h_t^N (x) \, dx \, dr
\]
\[
\leq \|D^2 V\|_{\infty} \|\nabla \phi\|_{\infty} K \epsilon_n T \left[ \mathbb{S}_T^N \right]^2,
\]
where in the last inequality we used (1.6). The second term above is bounded from above by
\[
\leq \|\nabla \phi\|_{\infty} \int_0^t \int_{\mathbb{R}^d} |\nabla V * S_r^N - \nabla V * h_r^N| \, dx \, dr
\]
\[
\leq \|D^2 V\|_{\infty} \|\nabla \phi\|_{\infty} K \epsilon_n \int_0^t \mathbb{S}_r^N \int_{\mathbb{R}^d} h_r^N \, dx \, dr
\]
\[
\leq \|D^2 V\|_{\infty} \|\nabla \phi\|_{\infty} K \epsilon_n T \left[ \mathbb{S}_T^N \right]^2,
\]
where we used the fact that
\[
\left| (\nabla V * S_r^N) (x) - (\nabla V * h_r^N) (x) \right|
\]
\[
= \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \nabla V (x - y) - \nabla V (x - z) \right| \theta_N (z - y) \, d \mathbb{S}_r^N (dy) \right|
\]
\[
\leq \|D^2 V\|_{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |y - z| \theta_N (z - y) \, d \mathbb{S}_r^N (dy) \leq \|D^2 V\|_{\infty} K \epsilon_n \left[ \mathbb{S}_r^N \right].
\]

\[
Lemma 5.5.3 \text{ proves the convergence of the proliferation terms.}
\]

**Lemma 5.5.3**
Under the assumptions (5.3) and (5.7) there exists a constant \( C > 0 \) such that
\[
\left| \int_0^T \int_{\mathbb{R}^d} \left[ F \left( h_t^N, h_t^N - \theta_N * \left( F_N \left( \mathbb{S}_t^N, \cdot \right) \right) \right) \right] \phi_t \, dx \, dt \right|
\]
\[
\leq C \left[ \mathbb{S}_T^N \right]^2 \epsilon_N^2 \left( \int_0^T \| h_t^N \|_{W^{1,2}}^2 \, dt + 1 \right) + C \alpha_N \left[ \mathbb{S}_T^N \right]^2.
\]

**Proof.** Using the assumptions (5.3) and (5.7) yields for a finite measure \( \mu \)
\[
|F (\theta_N * \mu) (x) - F_N (\mu, y)|
\]
\[
\leq |F (\theta_N * \mu) (x) - F (\theta_N * \mu) (y)| + |F (\theta_N * \mu) (y) - F_N (\mu, y)|
\]
\[
\leq C \left( \| \theta_N * \mu \|_{W^{1,2}} + 1 \right) |x - y|^\beta + \alpha_N \left[ \mu \right],
\]

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for a constant $C > 0$. Recall that $\theta$ is compactly supported, hence
\[
\left| \left[ F(\theta_N \ast \mu \ast \mu - \theta_N \ast (F_N(\mu, \cdot) \mu)) (x) \right] \right| \\
\leq \int_{\mathbb{R}^d} \theta_N (x-y) |F(\theta_N \ast \mu) (x) - F_N(\mu,y)| \mu(dy) \\
\leq \int_{\mathbb{R}^d} \theta_N (x-y) \left( C (\|\theta_N \ast \mu\|_{W^{1,2}} + 1) |x-y|^\beta + \alpha_N [\mu] \right) \mu(dy) \\
\leq \left( C (\|\theta_N \ast \mu\|_{W^{1,2}} + 1) \epsilon_N^\beta + \alpha_N [\mu] \right) (\theta_N \ast \mu) (x).
\]
Therefore the left-hand side in the statement of the lemma is bounded from above by
\[
\leq \|\phi\|_\infty \int_0^T (C (\|h_t^N\|_{W^{1,2}} + 1) \epsilon_N^\beta + \alpha_N [S_t^N]) \int_{\mathbb{R}^d} h_t^N (x) dx dt \\
\leq \|\phi\|_\infty \int_0^T \left( C (\|h_t^N\|_{W^{1,2}} + 1) \epsilon_N^\beta + \alpha_N [S_t^N] \right) [S_t^N] dt.
\]

□

Finally, Lemma 5.5.4 provides the needed control on the martingale terms. **Lemma 5.5.4**  
Under (5.4) there exists a constant $C > 0$, such that for $i = 1, 2$ we have
\[
\mathbb{E} \left[ \left| \int_{\mathbb{R}^d} \int_0^T \phi_t(x) dM_t^{1,N}(x) dx \right|^2 \right] \leq C \epsilon_N^2.
\]

**Proof.** The proof is very similar to the one of Lemma 5.3.6 but taking advantage of the smoothness of $\phi$ which did not appear there. We only sketch the computations. Denote
\[
g_t^N(y) := -\int_{\mathbb{R}^d} \phi_t(x) \nabla \theta_N (x - y) dx = \int_{\mathbb{R}^d} \nabla \phi_t(x) \theta_N (x - y) dx.
\]
For the first martingale term we have
\[
\mathbb{E} \left[ \left| \int_{\mathbb{R}^d} \int_0^T \phi_t(x) dM_t^{1,N}(x) dx \right|^2 \right] = \frac{\sigma^2}{N^2} \sum_{a \in A^N} \mathbb{E} \left[ \int_0^T 1_{t \in I_a} \left| g_t^N(X_t^a) \right|^2 dt \right] \\
\leq \frac{\sigma^2}{N} \|\theta_N\|_{L^2} \|\nabla \phi\|_{\infty} \mathbb{E} \int_0^T [S_t^N] dt.
\]
To complete the estimate for the first martingale term use (1.8) and Lemma 1.4.1. Set
\[
\tilde{g}_t^N(y) := -\int_{\mathbb{R}^d} \phi_t(x) \theta_N (x - y) dx,
\]

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for the second martingale term we have
\[
E \left[ \left| \int_{\mathbb{R}^d} \int_0^T \phi_t(x) dM_t^{X_t^N} dx \right|^2 \right] = \frac{1}{N^2} \sum_{a \in A^N} E \left[ \int_0^T 1_{t \in I^a} \left| \tilde{g}_t^N (X_t^a) \right|^2 \lambda_t^a dt \right] \\
\leq \frac{C_F}{N} \| \theta_N \|^2_{L^2} \| \phi \|^2_{L^\infty} E \int_0^T [S_t^N] dt
\]
and we conclude by the same argument. This completes the proof. \qed

With the preceding three lemmata, which are used in the proof of Lemma 5.5.1, the proof of Theorem 5.1.1 is complete.
Chapter 5. Mean field limit with proliferation
Chapter 6

Uniform convergence of proliferating particles to the FKPP equation

This chapter is essentially the paper [FLO16], which is similar to the previous one. However, the techniques used in this chapter differ from the ones before. We consider a particle system, in which the particles do not interact through a potential, but still through their proliferation events. The macroscopic limit PDE is the FKPP equation, details see below. The main difference compared to the previous chapter is a semigroup approach, which is new in the framework of these systems. The idea to use this approach and in particular the proofs of Lemma 6.3.1, 6.3.2 and 6.4.1 are due to C. Olivera and F. Flandoli. Works by Karl Oelschläger remark that they also cover the FKPP equation. We state our point of view to stress the difference to our work, see Section 6.1.1.

6.1 Introduction

Consider the so-called Fisher-Kolmogorov-Petrowskii-Piskunov, short FKPP, equation - with all constants equal to 1, which is always possible by suitable rescalings

$$\frac{\partial u}{\partial t} = \Delta u + u (1 - u), \quad u|_{t=0} = u_0. \quad (6.1)$$

This is a paradigm of equations arising in biology and other fields. For instance, in the mathematical description of cancer growth, although being too simplified to capture several features of true tumors, it may serve to
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explore mathematical features of diffusion and proliferation. In such applications, it describes a density of cancer cells which diffuse and proliferate with proliferation modulated by the density itself, such that, starting with an initial density \(0 \leq u_0 \leq 1\), the growth due to proliferation cannot exceed the threshold 1. Having in mind this example, it is natural to expect that this equation is the macroscopic limit of a system of microscopic particles. We prove a result in this direction, where the particle system is the one described in Section 6.1.2 below. The proliferation rate of particle “a” is given by the random time-dependent rate

\[
\lambda^a,N_t = \left(1 - \left(\theta_N * S_t^N\right)\left(X_t^a,N\right)\right)^+, \tag{6.2}
\]

where the notation is the one introduced for particle systems in Section 1.4, i.e. \(N\) is the number of initial particles, \(X_t^a,N\) is the position of particle “a”, \(S_t^N\) is the empirical measure, \(\theta_N\) is a classical family of mollifiers - and \(\theta_N * S_t^N\) is a smoothed version of the empirical measure. Formula (6.2) quantifies the fact that proliferation is slower when the empirical measure is more concentrated, and stops above a threshold. Since there is no reason why the mollified empirical measure \(\theta_N * S_t^N\) is smaller than one, we have to cut with the positive part in (6.2). Hence, initially the limit PDE will have the proliferation term \(u (1 - u)^+\), which is meaningful also for \(u > 1\), but by a uniqueness result, the term reduces to \(u (1 - u)\) when \(0 \leq u_0 \leq 1\).

The final result is natural and expected, but there is a technical difficulty which, in our opinion, is not sufficiently clarified in the literature. The proof of convergence of the particle system to the PDE relies on the tightness of the empirical measure and a passage to the limit in the identity satisfied by the empirical measure. This identity includes the nonlinear term

\[
\left\langle \left(1 - \theta_N * S_t^N\right)^+ S_t^N, \phi \right\rangle,
\]

where \(\phi\) is a smooth test function. Since \(S_t^N\) converges only weakly, it is required that \(\theta_N * S_t^N\) converges uniformly, in the space variable, in order to pass to the limit. Maybe in special cases one can perform special tricks but the question of uniform convergence is a natural one in this problem and it is also of independent interest, hence we investigate when it holds true.

Following the proposal of K. Oelschläger [Oel85], [Oel89], we assume

\[
\theta_N (x) = N^\beta \theta \left(N^{\beta/d} x\right). \tag{6.3}
\]

Recall that the case \(\beta = 0\) is the mean field one (long range interaction), the case \(\beta = 1\) corresponds to local (like nearest neighbor) interactions, while the
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case $0 < \beta < 1$ corresponds to an intermediate regime, called “moderate” by [Oel85]. Our main result is that uniform convergence of $\theta_N * S^N_t$ to $u$ holds under the condition

$$\beta < \frac{1}{2}.$$  

In addition to our main result, Theorem 6.1.1, see also Appendix B, where we show that this condition arises with other proofs of uniform convergence. We believe this condition is strict for the uniform convergence. There is a possibility that it is not strict for the convergence of the particle system to the FKPP equation.

6.1.1 Comparison with related problems and results

First, we clarify that the problem treated here is more correct and difficult than a two-step approach which does not clarify the true relation between the particle system and the PDE, although it gives a plausible indication of the link. The two-step approach freezes first the parameter in the mollifier, namely it treats particles proliferating with rate

$$\lambda_{t,a,N_0,N} = \left(1 - \left(\theta_{N_0} * S^{N_0,N}_t\right) \left(X_{t,a,N_0,N}^N\right)\right)^+$$

and proves that $S^{N_0,N}_t$ weakly converges as $N \to \infty$, to the solution $u_{N_0}$ of the following equation with non-local proliferation

$$\frac{\partial u_{N_0}}{\partial t} = \Delta u_{N_0} + u_{N_0} \left(1 - \theta_{N_0} * u_{N_0}\right)^+.$$  \hspace{1cm} (6.4)

The second step consists in proving that $u_{N_0}$ converges to the solution $u$ of the FKPP equation. The link between the particle system $X_{a,N_0,N}$ and the solution $u$ of the FKPP equation is only conjectured by this approach. In principle the conjecture could be wrong. Take a system of particle interactions with short range couplings, where the two-step approach leads to the porous media equation with the non-linearity $\Delta u^2$ (see [Phi07]). But a direct link between the particle system and the limit PDE (the so-called hydrodynamic limit problem) leads to a non-linearity of the form $\Delta f(u)$ where $f(u)$ is not necessarily $u^2$ (see [Var91], [Uch00]). For a proof of the mean field result of convergence of $S^{N_0,N}_t$ to $u_{N_0}$ as $N \to \infty$, see for instance [CM07], [FL15]. The issue of uniform convergence of $\theta_N * S^N_t$ to $u$ does not arise and weak convergence of the measures $S^{N_0,N}_t$ is sufficient.

Going back to the problem with the rates (6.2), K. Oelschläger papers [Oel85], [Oel89] have been our main source of inspiration. Our attempt in the present work is to clarify a result of convergence in the case of diffusion.
and proliferation under assumptions comparable to those of \[Oel85\], \[Oel89\], but possibly with some additional degree of generality and with a new proof. Concerning the generality, the remarks about extensions in \[Oel89\] page 575, are crucial. Hence, we review some details to explain our view.

We recall that \[Oel85\] does not contain proliferation terms, while \[Oel89\] is very general and includes them among several other aspects, FKPP equation is covered by \[Oel89\]. Assumption (2.4) of \[Oel89\], applied to the FKPP equation, can be summarized by the condition 
\[
\beta < \frac{d}{(d+1)(d+2)},
\]
which is considerably more restrictive than our condition \(\beta < \frac{1}{2}\). Therefore, our result would be the most general in the literature, because of a larger range of \(\beta\) and no need to assume that \(V\) has the special form \(V = W * W\) in terms of another mollifier \(W\) (assumed in \[Oel85\], \[Oel89\]).

We have extended the assumption \(\beta < \frac{d}{(d+1)(d+2)}\) and removed the restriction \(V = W * W\) of \[Oel89\] and, hopefully, we have given a modern proof which clarifies certain issues of the tightness and the convergence problem. Concerning extensions of the range of \(\beta\), maybe there are other directions, as remarked in \[Oel89\], page 575; our specific extension is however motivated not only by the generality but also by the property of uniform convergence (not proved in \[Oel89\]), which seems relevant in itself.

Other interesting works related to the problem of particle approximation of FKPP equation are \[MB15\], \[MRC87\], \[Méte86\], \[NO88\], \[Ste00\] and \[BS10\], \[BV05\] from the more applied literature. For the FKPP limit of discrete lattice systems, even the more difficult question of the hydrodynamic limit has been solved, see \[DMFL86\] with completely local interaction, but the analogous problem for diffusions is more difficult and has not been done.

To solve the problem of uniform convergence, we propose a new approach using semigroup theory. Traces of this approach can be found in \[Méte86\] and \[CM07\], but have been used for other purposes. In the work \[FL15\] it is remarked that uniform convergence can be obtained as a byproduct of energy inequalities and Sobolev convergence, under the assumption \(\beta < \frac{d}{d+2}\), but only in dimension \(d = 1\), where the condition is more restrictive than \(\beta < 1/2\).

The approach extends to other models, in particular with interactions in the spatial component. With the same technique, under appropriate assumptions on the convolution kernels \(\theta_N\) below, we may recover a result, under different assumptions, of \[Oel85\], where the macroscopic PDE has the form
\[
\frac{\partial u}{\partial t} = \Delta u + \text{div} (uF(u)) + u(1-u), \quad u|_{t=0} = u_0
\]
and \(F\) is a local nonlinear function, not a non-local operator as in mean field theories.
We insist on the fact that our proliferation rate is natural from the viewpoint of Biology. It is very different from the constant rate used in the probabilistic formulae used by McKean and others to represent solutions of the FKPP equations; these formula have several reasons of interest but do not have a biological meaning - constant proliferation rate would lead to exponential blow-up of the number of particles. Constant rates do not pose the difficulties described above in taking the limit in the nonlinear term. Approximation by finite systems of these representation formula therefore pose different problems. For this and other directions, different from our one, see [McK75], [RT04] and references therein.

6.1.2 The microscopic model

We summarize the particle system described so far (see Section 1.4 for the basic definitions).

During its lifetime the position of \( a \in \Lambda^N, X_{t}^{a,N} \in \mathbb{R}^d \), is given by

\[
dX_{t}^{a,N} = \sqrt{2}dB_{t}^{a}.
\]

(6.5)

Particles do not interact through an interaction kernel, i.e. \( V = \text{const} \), but through their branching mechanism. The proliferation rate \( \lambda_{t}^{a,N} \) is given by

\[
\lambda_{t}^{a,N} = \left( 1 - \left( \theta_N * S_t^N \right) \left( X_{t}^{a,N} \right) \right)^{+},
\]

where

\[
\theta_N(x) = \epsilon^{-d} \theta \left( \epsilon^{-1} x \right) \text{ with } \epsilon_N = N^{-\frac{d}{2}},
\]

(6.6)

is a family of mollifiers.

6.1.3 Assumptions and main result

Throughout this chapter we assume

\[
\beta \in \left( 0, \frac{1}{2} \right)
\]

(6.7)

and that \( \theta : \mathbb{R}^d \to \mathbb{R} \) is a probability density of class

\[
\theta \in W^{\alpha_0,2} \left( \mathbb{R}^d \right) \text{ for some } \alpha_0 \in \left( \frac{d}{2}, \frac{d(1-\beta)}{2\beta} \right)
\]

(6.8)

(notice that, for \( \beta > 0 \), the inequality \( \frac{d}{2} < \frac{d(1-\beta)}{2\beta} \) is equivalent to \( \beta < \frac{1}{2} \). The weaker assumption \( \beta = 1 \) corresponds to nearest-neighbor (or contact)
interaction and it is just the natural scaling to avoid that the kernel is more
centered than the typical space around a single particle, when the parti-
cles are uniformly distributed. The case \( \beta = 0 \) corresponds to mean field
interaction. The explanation for condition (6.7) is given at the beginning of
Section 6.4. At the biological level it means that the modulation of prolifer-
atation by the local density of cells is not completely local, but has a certain
range of action, which is less than long range as a mean field model.

Recall the mollified empirical measure \( h^N_t \) defined as

\[
h^N_t(x) = \left( \theta_N * S^N_t \right)(x).
\]

Assume that for some \( \rho_0 \geq \alpha_0 - 1 \)

\[
\sup_{N \in \mathbb{N}} \mathbb{E} \left[ \int_{\mathbb{R}^d} |(I - A)^{\rho_0/2} h^N_0(x)|^2 \, dx \right] < \infty. \tag{6.9}
\]

When the initial positions \( X_0^{(i)}, i = 1, ..., N, \) are independent identically
distributed with common probability density \( u_0 \in W^{\rho_0,2}(\mathbb{R}^d) \), with \( \rho_0 \leq \alpha_0 \),
this condition is satisfied, see Appendix C below.

**Theorem 6.1.1**

Assume that \( S^N_0 \) converges weakly to \( u_0(x) \, dx \), as \( N \to \infty \), in probability,
where \( u_0 \in L^1(\mathbb{R}^d) \), uniformly continuous and

\[
0 \leq u_0(x) \leq 1 \text{ for all } x \in \mathbb{R}^d,
\]

which implies \( u_0 \in L^2(\mathbb{R}^d) \). Further, we assume (6.7), (6.8) and (6.9). Then
for every \( \alpha \in (d/2, \alpha_0) \) the process \( h^N \) converges in probability, in the

- weak star topology of \( L^\infty(0, T; L^2(\mathbb{R}^d)) \),
- the weak topology of \( L^2(0, T; W^{\alpha,2}(\mathbb{R}^d)) \) and
- the strong topology of \( L^2(0, T; W^{\alpha,2}_{\text{loc}}(\mathbb{R}^d)) \)

as \( N \to \infty \), to the unique weak solution of the PDE (6.1).

Note that the topology of convergence of \( h^N \) to \( u \) includes the convergence
in \( L^2(0, T; C(D)) \) for every regular bounded domain \( D \subset \mathbb{R}^d \). The notion of
weak solution of the PDE (6.1) is given by Definition 6.6.1.
6.2 Preparation

6.2.1 Analytic Semigroup and Sobolev Spaces

For every $\alpha \in \mathbb{R}$, the Sobolev spaces $W^{\alpha,2}(\mathbb{R}^d)$ are well defined, see [Tri78] for the material recalled here. For positive $\alpha$ the restriction of $f \in W^{\alpha,2}(\mathbb{R}^d)$ to $B(0,R)$, a ball with radius $r > 0$, is in $W^{\alpha,2}(B(0,R))$. The family of operators, for $t \geq 0$,

$$\left( e^{tA} f \right) (x) = \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}} f(y) \, dy$$

defines an analytic semigroup in each space $W^{\alpha,2}(\mathbb{R}^d)$. With little abuse of notation we write $e^{tA}$ for each value of $\alpha$. The infinitesimal generator, say in $L^2(\mathbb{R}^d)$, is the operator $A : D(A) \subset L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defined as $Af = \Delta f$. Fractional powers $(I - A)^{\beta}$ are well defined for every $\beta \in \mathbb{R}$ and $\|(I - A)^{\beta} f\|_{L^2(\mathbb{R}^d)}$ is equivalent to the norm in $W^{\alpha,2}(\mathbb{R}^d)$. Recall also that (see [Paz83]), for every $\beta > 0$, and given $T > 0$, there is a constant $C_{\beta,T}$ such that

$$\left\| (I - A)^{\beta} e^{tA} \right\|_{L^2 \to L^2} \leq \frac{C_{\beta,T}}{t^\beta}$$

for $t \in (0,T]$.

6.2.2 Equation for the empirical measure and its mild formulation

Starting from this section, we drop the superscript $N$ in $X_{i,N}^a$, $I_{a,N}$, $T_{i,N}^a$, $\lambda_{i,N}$, $\mathcal{N}_{i,N}^a$ to simplify notations. Let $\delta$ denote a point outside $\mathbb{R}^d$, the so-called grave state, where we assume the processes $X_i^a$ live when $t \notin I_a$. Hence, whenever a particle proliferates and therefore dies, it stays forever in the grave state $\delta$. In the sequel, the test functions $\phi$ are assumed to be defined over $\mathbb{R}^d \cup \{\delta\}$ and be such that $\phi(\delta) = 0$. Using Itô formula over random time intervals, one can show that $\phi(X_t^a)$, with $\phi \in C^2(\mathbb{R}^d)$, satisfies

$$\phi(X_t^a) = \phi(X_{T_0}^a) 1_{t \geq T_0}^a - \phi(X_{T_1}^a) 1_{t \geq T_1}^a + \sqrt{2} \int_0^t 1_{s \in I_a} \nabla \phi(X_s^a) \cdot dB_s^a$$

$$+ \int_0^t 1_{s \in I_a} \Delta \phi(X_s^a) \, ds.$$
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With a few computations, one can see that the empirical measure $S_t^N$ satisfies

\[ d\langle S_t^N, \phi \rangle = \langle S_t^N, \Delta \phi \rangle \, dt + \left\langle \left( 1 - h_t^N \right)^+ S_t^N, \phi \right\rangle \, dt + dM_t^{1,\phi,N} + dM_t^{2,\phi,N} \]

for every $\phi \in C_b^2 (\mathbb{R}^d)$ and where

\[ M_t^{1,\phi,N} := \sqrt{\frac{2}{N}} \sum_{a \in \Lambda_N} \int_0^t 1_{s \in I_a} \nabla \phi (X_s^a) \cdot dB_s^a, \]

\[ M_t^{2,\phi,N} := \frac{1}{N} \sum_{a \in \Lambda_N} \rho (X_{T_a}^a) 1_{t \geq T_a} - \frac{1}{N} \sum_{a \in \Lambda_N} \int_0^t \phi (X_s^a) \lambda_s^a \, ds. \]

We deduce that $h_t^N (x)$ satisfies

\[ dh_t^N (x) = \Delta h_t^N (x) \, dt + \left( \theta_N * \left( \left( 1 - h_t^N \right)^+ S_t^N \right) \right) (x) \, dt \]
\[ + dM_t^{1,N} (x) + dM_t^{2,N} (x), \]

where

\[ M_t^{1,N} (x) := -\sqrt{\frac{2}{N}} \sum_{a \in \Lambda_N} \int_0^t 1_{s \in I_a} \nabla \theta_N (x - X_s^a) \cdot dB_s^a, \]

\[ M_t^{2,N} (x) := \frac{1}{N} \sum_{a \in \Lambda_N} \theta_N (x - X_{T_a}^a) 1_{t \geq T_a} - \frac{1}{N} \int_0^t \sum_{a \in \Lambda_N} \theta_N (x - X_s^a) \lambda_s^a \, ds \]
\[ = \frac{1}{N} \sum_{a \in \Lambda_N} \int_0^t \theta_N (x - X_s^a) \, d (N_s^a - A_s^a). \]

Following a standard procedure, used for instance by [DPZ92], we may rewrite this equation in mild form:

\[ h_t^N = e^{tA} h_0^N + \int_0^t e^{(t-s)A} \left( \theta_N * \left( \left( 1 - h_s^N \right)^+ S_s^N \right) \right) \, ds \]
\[ + \int_0^t e^{(t-s)A} dM_s^{1,N} + \int_0^t e^{(t-s)A} dM_s^{2,N}. \]

(6.11)

This opens the possibility of a semigroup approach, which is a main novelty of this paper.

6.3 Main estimates on martingale terms

Let $\alpha \in (d/2, \alpha_0)$, as in the statement of Theorem 6.1.1.
Lemma 6.3.1
There exists a constant $C' > 0$ such that for all $N \in \mathbb{N}$, $t \in [0, T]$, small $h > 0$
\[
\left\| \int_0^t (I - A)^{\frac{\alpha}{2}} e^{(t+h-s)A} dM_s^{1,N} \right\|_{L^2(\Omega \times \mathbb{R}^d)} \leq C'.
\]

Proof.
\[
\frac{N^2}{2} \left\| \int_0^t (I - A)^{\frac{\alpha}{2}} e^{(t+h-s)A} dM_s^{1,N} \right\|^2_{L^2(\Omega \times \mathbb{R}^d)}
= \int_{\mathbb{R}^d} \mathbb{E} \left[ \sum_{a \in \Lambda^N} \int_0^t \left( (I - A)^{\frac{\alpha}{2}} e^{(t+h-s)A} 1_{s \in I^a} \nabla \theta_N(\cdot - X_s^a) \right) (x) \cdot dE_s^a \right]^2 \, dx
= \int_{\mathbb{R}^d} \mathbb{E} \left[ \sum_{a \in \Lambda^N} \int_0^t \left( (I - A)^{\frac{\alpha}{2}} e^{(t+h-s)A} 1_{s \in I^a} \nabla \theta_N(\cdot - X_s^a) \right) (x)^2 \, ds \right] \, dx
= \mathbb{E} \left[ \sum_{a \in \Lambda^N} \int_0^t 1_{s \in I^a} \left( \int_{\mathbb{R}^d} \left( (I - A)^{\frac{\alpha}{2}} e^{(t+h-s)A} \nabla \theta_N(\cdot - X_s^a) \right) (x)^2 \, dx \right) ds \right].
\]
We have
\[
\left( (I - A)^{\frac{\alpha}{2}} e^{(t+h-s)A} \nabla \theta_N(\cdot - X_s^a) \right) (x) = \left( (I - A)^{\frac{\alpha}{2}} e^{(t+h-s)A} \nabla \theta_N \right) (x - X_s^a).
\]
Then, by change of variable,
\[
\int_{\mathbb{R}^d} \left| \left( (I - A)^{\frac{\alpha}{2}} e^{(t+h-s)A} \nabla \theta_N(\cdot - X_s^a) \right) (x) \right|^2 \, dx
= \int_{\mathbb{R}^d} \left| \left( (I - A)^{\frac{\alpha}{2}} e^{(t+h-s)A} \nabla \theta_N \right) (x) \right|^2 \, dx.
\]
Therefore, being $\frac{1}{N} \sum_{a \in \Lambda^N} 1_{s \in I^a} = \left[ S_s^N \right] \leq \left[ S_T^N \right]$, \[
\left\| \int_0^t (I - A)^{\frac{\alpha}{2}} e^{(t+h-s)A} dM_s^{1,N} \right\|^2_{L^2(\Omega \times \mathbb{R}^d)}
= \frac{2}{N} \mathbb{E} \left[ \int_0^t \left( \frac{1}{N} \sum_{a \in \Lambda^N} 1_{s \in I^a} \right) \left( \int_{\mathbb{R}^d} \left| \left( (I - A)^{\frac{\alpha}{2}} e^{(t+h-s)A} \nabla \theta_N \right) (x) \right|^2 \, dx \right) ds \right]
\leq \frac{2}{N} \mathbb{E} \left( \left[ S_T^N \right] \right) \int_0^t \left\| (I - A)^{\frac{\alpha}{2}} e^{(t+h-s)A} \nabla \theta_N \right\|^2_{L^2} \, ds.
\]
From assumption \[6.8\] and the condition $\alpha \in (d/2, \alpha_0)$, we have $\frac{\alpha}{d} (2\alpha + d) < 1$, hence there exists a small $\varepsilon > 0$ such that $\frac{\alpha}{d} (2\alpha + \varepsilon + d) \leq 1$ and at the
same time $\alpha + \frac{\varepsilon}{2} \leq \alpha_0$. Denoting by $C > 0$ any constant independent of $N$ and recalling that $\epsilon_N = N^{-\frac{\sigma}{2}}$, we have

$$
\leq \frac{C}{N} \int_0^t \left\| (I - A)^{(1-\varepsilon/2)/2} e^{(t-s)A} \right\|^2_{L^2 \rightarrow L^2} \left\| \nabla (I - A)^{-1/2} \right\|^2_{L^2 \rightarrow L^2} 
\times \left\| (I - A)^{(\alpha + \varepsilon/2)/2} e^{hA} \theta_N \right\|^2_{L^2} ds
$$

$$
\leq \frac{C}{N} \parallel \theta_N \parallel^2_{W^{\sigma+\varepsilon/2,2}} \int_0^t \frac{1}{(t-s)^{1-\varepsilon/2}} ds \leq C \epsilon_N^{-2} \alpha^{-\varepsilon - d} N \leq C,
$$

where we have used Lemma 6.6.7 below.

\[ \square \]

**Lemma 6.3.2**
There exists a constant $C > 0$ such that for all $N \in \mathbb{N}$, $t \in [0, T]$, small $h > 0$

$$
\left\| \int_0^t (I - A)^{\alpha/2} e^{(t-h-s)A} dM_{s}^{2,N} \right\|_{L^2(\Omega \times \mathbb{R}^d)} \leq C.
$$

**Proof.** Since

$$
M_{t}^{2,N} = \frac{1}{N} \sum_{a \in \Lambda^N} \int_0^t \theta_N (x - X_{s-a}^a) d \langle \mathcal{N}_s^a - \Lambda_s^a \rangle,
$$

we have

$$
\left\| \int_0^t (I - A)^{\alpha/2} e^{(t-h-s)A} dM_{s}^{2,N} \right\|^2_{L^2(\Omega \times \mathbb{R}^d)} = \frac{1}{N^2} \int_{\mathbb{R}^d} E \left[ \left\| \sum_{a \in \Lambda^N} \int_0^t (I - A)^{\alpha/2} e^{(t-h-s)A} 1_{s \in I^n} \nabla \theta_N (\cdot - X_{s-a}^a) \right\|^2 \right] dx.
$$

Write $g_{t,s,h}^{a,N} (X_{s-a}^a)$ for $(I - A)^{\alpha/2} e^{(t-h-s)A} 1_{s \in I^n} \nabla \theta_N (\cdot - X_{s-a}^a)$ $(x)$. Since the jumps of $\mathcal{N}_s^a$ and $\mathcal{N}_s^{a'}$, for $a \neq a'$, never occur at the same time, we have

$$
E \left[ \left( \int_0^t g_{t,s,h}^{a,N} (X_{s-a}^a) d \langle \mathcal{N}_s^a - \Lambda_s^a \rangle \right) \left( \int_0^t g_{t,s,h}^{a',N} (X_{s-a}^{a'}) d \langle \mathcal{N}_s^{a'} - \Lambda_s^{a'} \rangle \right) \right] = 0.
$$

Hence, the last expression is equal to

$$
= \frac{1}{N^2} \sum_{a \in \Lambda^N} \int_{\mathbb{R}^d} E \left[ \left\| \int_0^t g_{t,s,h}^{a,N} (X_{s-a}) \cdot d \langle \mathcal{N}_s^a - \Lambda_s^a \rangle \right\|^2 \right] dx.
$$
6.4 Main estimate on $h_t^N$

It is known that

$$E \left[ \left\| \int_0^t g_{t,s,h}^{a,N} (X^a_s) \, d(N^a_s - \Lambda^a_s) \right\|^2 \right] = E \left[ \int_0^t \left| g_{t,s,h}^{a,N} (X^a_s) \right|^2 \, d\Lambda^a_s \right].$$

Therefore, the last expression simplifies to

$$= \frac{1}{N^2} \sum_{a \in \Lambda^N} \int_{\mathbb{R}^d} E \left[ \int_0^t \left| g_{t,s,h}^{a,N} (X^a_s) \right|^2 \, \lambda^a_s \, ds \right] \, dx$$

$$= \frac{1}{N^2} \sum_{a \in \Lambda^N} E \left[ \int_0^t \left( \int_{\mathbb{R}^d} \left| (I - A)^{\alpha/2} e^{(t+h-s)A} 1_{s \in I^a} \times \nabla \theta_N (\cdot - X^a_s) \right| (x) \, dx \right) \, \lambda^a_s \, ds \right].$$

Taking into account the boundedness of $\lambda^a_s$,

$$= \frac{1}{N^2} \sum_{a \in \Lambda^N} E \left[ \int_0^t \left( \int_{\mathbb{R}^d} \left| (I - A)^{\alpha/2} e^{(t+h-s)A} 1_{s \in I^a} \nabla \theta_N \right| (x) \, dx \right) \, \lambda^a_s \, ds \right]$$

$$\leq \frac{1}{N} E \left[ \int_0^t \left( \frac{1}{N} \sum_{a \in \Lambda^N} 1_{s \in I^a} \right) \left\| (I - A)^{\alpha/2} e^{(t+h-s)A} \nabla \theta_N \right\|_{L^2} \, ds \right]$$

$$\leq \frac{1}{N} E \left( \left[ S^N_T \right] \right) \int_0^t \left\| (I - A)^{\alpha/2} e^{(t+h-s)A} \nabla \theta_N \right\|_{L^2}^2 \, ds.$$

This is the same expression as in the previous proof, which is bounded by a constant, uniformly in $N$ and $t$.

\[ \square \]

6.4 Main estimate on $h_t^N$

As described above, we need an estimate on $h_t^N$ in a Hölder norm (in space), which we gain by Sobolev’s embedding theorem. Since we work in an $L^2$-setting (computations not reported here in the $L^p$-setting do not help since they reintroduce difficulties from other sides), we have

$$W^{\alpha,2} (\mathbb{R}^d) \subset C_\epsilon^\alpha (\mathbb{R}^d) \quad \text{if} \quad 2 (\alpha - \varepsilon) \geq d.$$  

This is the reason for the restriction on $\alpha$, namely $2 \alpha > d$. Recall that $\alpha_0$ and $\rho_0$ were introduced in (6.8) and (6.9) respectively.

**Lemma 6.4.1**

Assume $\alpha \in (d/2, \alpha_0)$. Then there exist constants $C, C' > 0$ such that for all $N \in \mathbb{N}$, $t \in (0, T]$

$$\left\| h_t^N \right\|_{L^2(\Omega; W^{\alpha,2}(\mathbb{R}^d))} \leq C E \left[ \left\| (I - A)^{\alpha/2} h_t^N \right\|_{L^2(\mathbb{R}^d)}^2 \right]^{1/2} \leq C' \left( 1 + \frac{1}{t^{(\alpha - \rho_0) \vee 0}} \right).$$
Proof. The first inequality follows from the fact that the two norms
$$\|\cdot\|_{W^{\alpha,2}(\mathbb{R}^d)} \text{ and } \|(I-A)^{\frac{\alpha}{2}}\cdot\|_{L^2(\mathbb{R}^d)}$$
are equivalent. From the mild formulation (6.11) we have
$$\|\| (I-A)^{\frac{\alpha}{2}} e^{(t+h)A} h_0^N \|_{L^2(\Omega \times \mathbb{R}^d)}$$
\[
\leq \|\| (I-A)^{\frac{\alpha}{2}} e^{(t+h)A} h_0^N \|_{L^2(\Omega \times \mathbb{R}^d)}
+ \int_0^t \|\| (I-A)^{\frac{\alpha}{2}} e^{(t+h-s)A} \left( \theta_N * \left( (1-h_s^N)^+ S_s^N \right) \right) \|_{L^2(\Omega \times \mathbb{R}^d)}
+ \int_0^t (I-A)^{\frac{\alpha}{2}} e^{(t+h-s)A} dM_s^1, N \|_{L^2(\Omega \times \mathbb{R}^d)}
+ \int_0^t (I-A)^{\frac{\alpha}{2}} e^{(t+h-s)A} dM_s^2, N \|_{L^2(\Omega \times \mathbb{R}^d)}^2.
\]

The last two terms are bounded by a constant by Lemmata 6.3.1 and 6.3.2
For the first term, where \( C > 0 \) is a constant that may change from instance to instance, we have
$$\|\| (I-A)^{\alpha/2} e^{(t+h)A} h_0^N \|_{L^2(\Omega \times \mathbb{R}^d)}$$
\[
\leq \|\| (I-A)^{(\alpha-\rho)/2} e^{(t+h)A} \|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \|\| (I-A)^{\rho/2} h_0^N \|_{L^2(\Omega \times \mathbb{R}^d)}
\leq \frac{C}{(t+h)^{(\alpha-\rho)/2}}.
\]

And the second one,
$$\int_0^t \|\| (I-A)^{\frac{\alpha}{2}} e^{(t+h-s)A} \left( \theta_N * \left( (1-h_s^N)^+ S_s^N \right) \right) \|_{L^2(\Omega \times \mathbb{R}^d)}
\leq \int_0^t \|\| e^{(t-s)A} \|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)}
\times\|\| (I-A)^{\frac{\alpha}{2}} e^{hA} \left( \theta_N * \left( (1-h_s^N)^+ S_s^N \right) \right) \|_{L^2(\Omega \times \mathbb{R}^d)}
ds.$$

Since the operator \( f \mapsto (I-A)^{\frac{\alpha}{2}} e^{hA} f \) is positive on \( L^2(\mathbb{R}^d) \), see Lemma 6.6.8 it holds \( (I-A)^{\frac{\alpha}{2}} e^{hA} f \leq (I-A)^{\frac{\alpha}{2}} e^{hA} g \) if \( f \leq g \). Because of
$$0 \leq \left( \theta_N * \left( (1-h_s^N)^+ S_s^N \right) \right) (x) \leq h_s^N (x),$$
we deduce
\[ 0 \leq (I - A)^{\frac{\alpha}{2}} e^{hA} \left( \theta_N * \left( (1 - h_s^N)^+ S_s^N \right) \right) \leq (I - A)^{\frac{\alpha}{2}} e^{hA} h_s^N. \]

Hence,
\[ \int_0^t \left\| (I - A)^{\frac{\alpha}{2}} e^{(t+s)A} \left( \theta_N * \left( (1 - h_s^N)^+ S_s^N \right) \right) \right\|_{L^2(\Omega \times \mathbb{R}^d)} ds \leq C \int_0^t \left\| (I - A)^{\frac{\alpha}{2}} e^{hA} h_s^N \right\|_{L^2(\Omega \times \mathbb{R}^d)} ds. \]

Until now we have proved
\[ \left\| (I - A)^{\alpha/2} e^{hA} h_t^N \right\|_{L^2(\Omega \times \mathbb{R}^d)} \leq \frac{C}{(t + h)^{(\alpha - \rho)/2}} + C \int_0^t \left\| (I - A)^{\alpha/2} e^{hA} h_s^N \right\|_{L^2(\Omega \times \mathbb{R}^d)} ds + C. \]

By Gronwall’s lemma we deduce
\[ \left\| (I - A)^{\alpha/2} e^{hA} h_t^N \right\|_{L^2(\Omega \times \mathbb{R}^d)} \leq \frac{C}{(t + h)^{(\alpha - \rho)/2}} + C. \]

Take the limit \( h \to 0 \) to complete the proof. \( \square \)

**Remark 6.4.2**

The result is true also for \( \alpha = 0 \), i.e.
\[ \sup_{t \in [0, T]} \mathbb{E} \left[ \left\| h_t^N \right\|_{L^2(\mathbb{R}^d)}^2 \right] \leq C. \] (6.12)

### 6.5 Other estimates on \( h_t^N \)

In order to show tightness of the family of the functions \( (h_t^N)_N \), in addition to the previous bound which shows regularity in space, we also need some regularity in time. We proceed as in the previous chapter. The following lemma and its proof are very close to Lemma 5.3.8. But due to some, however small, differences, we give a complete proof again.

**Lemma 6.5.1**

Given any \( \gamma \in (0, 1/2) \), it holds
\[ \lim_{R \to \infty} \sup_{N \in \mathbb{N}} \mathbb{P} \left( \int_0^T \int_0^T \frac{\left\| h_t^N - h_s^N \right\|_{W^{2,2}}^2}{|t - s|^{1+2\gamma}} ds dt > R \right) = 0. \]
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Proof. Step 1. We need to estimate $\|h_t^N - h_s^N\|^2_{W^{2,2}}$ in such a way that it cancels with the singularity in the denominator at $t = s$. Notice that $L^2 \subset W^{-2,2}$ with continuous embedding, namely there exists a constant $C > 0$ such that $\|f\|_{W^{-2,2}} \leq C \|f\|_{L^2}$ for all $f \in L^2$; similarly for $W^{-1,2} \subset W^{-2,2}$. Moreover, the linear operator $\Delta$ is bounded from $L^2$ to $W^{-2,2}$. Therefore (we denote by $C > 0$ any constant independent of $N, h^N, t, s$)

$$
\|h_t^N - h_s^N\|^2_{W^{2,2}} \leq C \left\| \int_s^t \Delta h_r^N \, dr \right\|_{W^{2,2}}^2 + C \left\| \int_s^t \theta_N * \left( (1 - \theta_N * S_r^N) S_r^N \right) \, dr \right\|_{W^{2,2}}^2 + C \left\| M_{t}^{1,N} - M_{s}^{1,N} \right\|^2_{W^{2,2}} \left( \text{bounded by (renaming the constant } C) \right)
$$

hence, by Hölder’s inequality

$$
\leq C (t - s) \int_s^t \|h_r^N\|^2_{L^2} \, dr + C (t - s) \int_s^t \left\| \theta_N * \left( (1 - \theta_N * S_r^N) S_r^N \right) \right\|^2_{L^2} \, dr + C \left\| M_{t}^{1,N} - M_{s}^{1,N} \right\|^2_{W^{-1,2}} + C \left\| M_{t}^{2,N} - M_{s}^{2,N} \right\|^2_{L^2}.
$$

Using (1.7)

$$
\leq C (t - s) \int_s^t \|h_r^N\|^2_{L^2} \, dr + C \left\| M_{t}^{1,N} - M_{s}^{1,N} \right\|^2_{W^{-1,2}} + C \left\| M_{t}^{2,N} - M_{s}^{2,N} \right\|^2_{L^2}.
$$

Accordingly, we split the estimate of $\mathbb{P} \left( \int_0^T \int_0^T \|h_r^N - h_s^N\|^2_{W^{-2,2}} \, ds \, dt > R \right)$ in three more elementary estimates, that we handle separately. The final result will be a consequence of them.

The number $C_{\gamma} = \int_0^T \int_0^T \frac{1}{|r-s|^{1+2\gamma}} \, ds \, dt$ is finite, hence the first term is bounded by (renaming the constant $C$)

$$
\mathbb{P} \left( \int_0^T \int_0^T C (t - s) \left( \left[ S_T^N \right] + 1 \right) \sup_{r \in [0,T]} \left\| h_r^N \right\|^2_{L^2} \, ds \, dt > R \right) = \mathbb{P} \left( \left( \left[ S_T^N \right] + 1 \right) \sup_{r \in [0,T]} \left\| h_r^N \right\|^2_{L^2} > R/C \right)
$$

$$
\leq \mathbb{P} \left( \left[ S_T^N \right] + 1 > \sqrt{R/C} \right) + \mathbb{P} \left( \sup_{r \in [0,T]} \left\| h_r^N \right\|^2_{L^2} > \sqrt{R/C} \right)
$$

and both these terms are, uniformly in $N$, small for large $R$, due to Lemma 1.4.1 and estimate (6.12).
6.5. Other estimates on $h_t^N$

**Step 2.** Concerning the martingale terms, we now prove that

$$
\mathbb{E} \left\| M_t^{1,N} - M_s^{1,N} \right\|_{W^{-1,2}}^2 \leq C |t - s|
$$

and

$$
\mathbb{E} \left\| M_t^{2,N} - M_s^{2,N} \right\|_{L^2} \leq C |t - s|
$$

for some constant $C > 0$. By Chebyshev’s inequality it follows that

$$
\lim_{R \to \infty} \sup_{N \in \mathbb{N}} \mathbb{P} \left( \int_0^T \int_0^T \left\| M_t^{1,N} - M_s^{1,N} \right\|_{W^{-1,2}}^2 \frac{dt}{|t - s|^{1+2\gamma}} ds > R \right) = 0,
$$

$$
\lim_{R \to \infty} \sup_{N \in \mathbb{N}} \mathbb{P} \left( \int_0^T \int_0^T \left\| M_t^{2,N} - M_s^{2,N} \right\|_{L^2}^2 \frac{dt}{|t - s|^{1+2\gamma}} ds > R \right) = 0
$$

and the proof will be complete. For notational convenience, we abbreviate, for $i = 1, 2$,

$$
M_t^{i,N}(x) = \frac{1}{N} \sum_{a \in A^N} M_t^{i,a}(x).
$$

Note that for every $x \in \mathbb{R}^d$ the processes $M^{1,N}(x)$ and $M^{2,N}(x)$ are martingales. It follows, with computations similar to those of Lemma 6.3.1 for $t \geq s$

$$
\mathbb{E} \left\| M_t^{1,N} - M_s^{1,N} \right\|_{W^{-1,2}}^2 = \frac{1}{N^2} \sum_{a \in A^N} \mathbb{E} \left( \int_s^t 1_{r \in I^a} (I - \Delta)^{-\frac{1}{2}} \nabla \theta_N (x - X_r^a)^2 dr \right) dx
$$

$$
= \frac{1}{N} \left\| (I - \Delta)^{-\frac{1}{2}} \nabla \theta_N \right\|_{L^2}^2 \mathbb{E} \int_s^t \frac{1}{N} \sum_{a \in A^N} 1_{r \in I^a} dr
\leq \frac{1}{N} \left\| \theta_N \right\|_{L^2}^2 (t - s) \leq C (t - s).
$$

Similarly, for the second martingale, in analogy with Lemma 6.3.2

$$
\mathbb{E} \left\| M_t^{2,N} - M_s^{2,N} \right\|_{L^2}^2 = \frac{1}{N^2} \sum_{a \in A^N} \mathbb{E} \left[ M_t^{2,a}(x)^2 - M_s^{2,a}(x)^2 \right] dx
$$

$$
= \frac{1}{N^2} \sum_{a \in A^N} \mathbb{E} \left[ \int_s^t 1_{r \in I^a} \theta_N (x - X_r^a)^2 \lambda_r^a dr \right] dx
\leq C_F \frac{1}{N} \left\| \theta_N \right\|_{L^2} \mathbb{E} \int_s^t \frac{1}{N} \sum_{a \in A^N} 1_{r \in I^a} dr \leq C (t - s).
$$

$\Box$


6.6 Passage to the limit

6.6.1 Criterion of compactness

Again, the content of this subsection is very similar to the tightness argument at the end of Section 5.3. Due to the small differences, we cannot just cite the argument from the previous chapter. We give an adapted version here.

A version of Aubin-Lions lemma, see [Lio69], [FG95], [BBNP14], states that when \( E_0 \subset E \subset E_1 \) are three Banach spaces with continuous dense embedding, \( E_0, E_1 \) reflexive, with \( E_0 \) compactly embedded into \( E \), given \( p, q \in (1, \infty) \) and \( \gamma \in (0, 1) \), the space \( L^q(0, T; E_0) \cap W^{\gamma, p} (0, T; E_1) \) is compactly embedded into \( L^q(0, T; E) \).

Given the number \( \alpha_0 \) in assumption (6.8), we take any pair \( \alpha' < \alpha \) in the interval \( (d/2, \alpha_0) \). We use Aubin-Lions lemma with \( E = W^{\alpha', 2}(D) \), \( E_0 = W^{\alpha, 2}(D) \), \( 0 < \gamma < \frac{1}{2} \) and \( E_1 = W^{-2, 2}(\mathbb{R}^d) \), where \( D \) is any regular bounded domain. The lemma states the compact embedding

\[
L^2 \left( 0, T; W^{\alpha, 2}(D) \right) \cap W^{\gamma, 2} \left( 0, T; W^{-2, 2}(\mathbb{R}^d) \right) \subset L^2 \left( 0, T; W^{\alpha', 2}(D) \right).
\]

Notice that for \( \gamma p > 1 \), the \( W^{\gamma, p}(0, T; E_1) \) is embedded into \( C([0, T]; E_1) \), hence it is not suitable for our purposes since we deal with discontinuous processes. However, for \( \gamma p < 1 \) the space \( W^{\gamma, p}(0, T; E_1) \) includes piecewise constant functions, as one can easily check. Therefore, it is a suitable space for càdlàg processes.

Now, consider the space

\[
Y_0 := L^\infty \left( 0, T; L^2(\mathbb{R}^d) \right) \cap L^2 \left( 0, T; W^{\alpha, 2}(\mathbb{R}^d) \right) \cap W^{\gamma, 2} \left( 0, T; W^{-2, 2}(\mathbb{R}^d) \right).
\]

Using the Fréchet topology on \( L^2 \left( 0, T; W^{\alpha', 2}_{\text{loc}}(\mathbb{R}^d) \right) \) induced by the metric

\[
d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \left( 1 \wedge \int_0^T \| (f - g)(t, \cdot) \|_{W^{\alpha', 2}(B(0, n))}^p \, dt \right)
\]

one has that \( L^2 \left( 0, T; W^{\alpha, 2}(\mathbb{R}^d) \right) \cap W^{\gamma, 2} \left( 0, T; W^{-2, 2}(\mathbb{R}^d) \right) \) is compactly embedded into \( L^2 \left( 0, T; W^{\alpha', 2}_{\text{loc}}(\mathbb{R}^d) \right) \) (the proof is elementary, using the fact that if a set is compact in \( L^2 \left( 0, T; W^{\alpha', 2}_{\text{loc}}(B(0, n)) \right) \) for every \( n \) then it is compact in \( L^2 \left( 0, T; W^{\alpha', 2}_{\text{loc}}(\mathbb{R}^d) \right) \) with this topology; see a similar result in [BM13]). Denoting by \( L^\infty_w \left( 0, T; L^2(\mathbb{R}^d) \right) \) and \( L^2 \left( 0, T; W^{\alpha, 2}(\mathbb{R}^d) \right) \) the spaces \( L^\infty \left( 0, T; L^2(\mathbb{R}^d) \right) \) and \( L^2 \left( 0, T; W^{\alpha, 2}(\mathbb{R}^d) \right) \) endowed respectively
with the weak star and weak topology, we have that \( Y_0 \) is compactly embedded into

\[
Y := L^\infty_w (0, T; L^2 (\mathbb{R}^d)) \cap L^2_w (0, T; W^{\alpha, 2}(\mathbb{R}^d)) \cap L^2 (0, T; W^{\alpha', 2}_{loc}(\mathbb{R}^d)).
\]

(6.13)

Notice that

\[
L^2 (0, T; W^{\alpha', 2}_{loc}(\mathbb{R}^d)) \subset L^2 (0, T; C (D))
\]

for every regular bounded domain \( D \subset \mathbb{R}^d \).

Denote by \( (Q^N)_{N \in \mathbb{N}} \) the laws of \( (h^N)_{N \in \mathbb{N}} \) on \( Y_0 \). From the “boundedness in probability” of the family \( (Q^N)_{N \in \mathbb{N}} \) in \( Y_0 \) as stated in Lemma 6.4.1 (notice that square integrability in time of \( \|h^N_t\|_{L^2(\Omega; W^{\alpha, 2}(\mathbb{R}^d))} \) comes from the assumption \( \alpha_0 - \rho_0 \leq 1 \) which implies \( \alpha - \rho_0 < 1 \) and Lemma 6.5.1, it follows that the family \( (Q^N)_{N \in \mathbb{N}} \) is tight in \( Y \), hence relatively compact by Prohorov’s theorem. From every subsequence of \( (Q^N)_{N \in \mathbb{N}} \) it is possible to extract a further subsequence which converges weakly to a probability measure \( Q \) on \( Y \). We shall prove that every such limit measure \( Q \) is a Dirac measure, \( Q = \delta_u \), concentrated on the same element \( u \in Y \), hence the whole sequence \( (Q^N)_{N \in \mathbb{N}} \) converges to \( \delta_u \); and also the processes \( (h^N)_{N \in \mathbb{N}} \) converge in probability to \( u \).

Finally, since \( \alpha' < \alpha \) are arbitrary in the interval \( (d/2, \alpha_0) \), in Theorem \( 6.1.1 \) we have stated the weak convergence in \( L^2 (0, T; W^{\alpha, 2}(\mathbb{R}^d)) \) and the strong convergence in \( L^2 (0, T; W^{\alpha, 2}_{loc}(\mathbb{R}^d)) \) with the same symbol \( \alpha \in (d/2, \alpha_0) \).

### 6.6.2 Convergence

We consider also the auxiliary equation

\[
\frac{\partial u}{\partial t} = \Delta u + u (1 - u)^+ \quad \text{for a.e. } (t, x) \in [0, T] \times \mathbb{R}^d.
\]

(6.14)

**Definition 6.6.1**

Given \( u_0 : \mathbb{R}^d \to \mathbb{R} \) measurable, with \( 0 \leq u_0 (x) \leq 1 \) (resp. \( u_0 (x) \geq 0 \)), we call a measurable function \( u : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) a weak solution of equation (6.1) (resp. of equation (6.14)), if

\[
0 \leq u_t (x) \leq 1
\]

(resp. \( u_t (x) \geq 0 \)) for a.e. \( (t, x) \in [0, T] \times \mathbb{R}^d \) and

\[
\langle u_t, \phi \rangle = \langle u_0, \phi \rangle + \int_0^t \langle u_r, \Delta \phi \rangle \, dr + \int_0^t \langle (1 - u_r) u_r, \phi \rangle \, dr
\]

(6.15)
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(resp. with the term \((1 - u_r)^+\) in place of \((1 - u_r)\)) for all \(\phi \in C_c^\infty \left( \mathbb{R}^d \right)\) and a.e. \(t \in [0, T]\).

**Remark 6.6.2**
If \(u : [0, T] \times \mathbb{R}^d \to \mathbb{R}\) is a measurable function, with \(0 \leq u_t (x) \leq 1\) (resp. \(u_t (x) \geq 0\)), such that
\[
0 = \int_0^T \int_{\mathbb{R}^d} \left( \frac{\partial \phi_t}{\partial t} + \Delta \phi_t + (1 - u_t) \phi_t \right) u_t dx dt + \langle u_0, \phi_0 \rangle
\]
(resp. with the term \((1 - u_r)^+\) in place of \((1 - u_r)\)) for all time-dependent test functions \(\phi_t\), of class \(C_c^\infty \left( [0, T] \times \mathbb{R}^d \right)\), then one can prove (by taking test functions \(\phi_t (x)\) of the form \(\eta^t \cdot \phi (x)\) with \(\eta^t\) converging to \(1\)·≤ \(t\)) that, for every time-independent test function \(\phi \in C_c^\infty \left( \mathbb{R}^d \right)\) we have that identity (6.15) (resp. with the term \((1 - u_r)^+\) in place of \((1 - u_r)\)) holds.

We denote the limit of a converging subsequence of \((Q^N)_{N \in \mathbb{N}}\) by \(Q\).

**Lemma 6.6.3**
Under the assumptions of Theorem 6.1.1 \(Q\) is supported on the set of weak solutions of equation (6.14).

**Proof.** **Step 1.** We apply Remark 6.6.2. For each \(\phi \in C_c^\infty \left( [0, T] \times \mathbb{R}^d \right)\), we introduce two functionals
\[
u \mapsto \Psi_\phi (u) := \int_0^T \int_{\mathbb{R}^d} \left( \frac{\partial \phi_t}{\partial t} + \Delta \phi_t + (1 - u_t)^+ \phi_t \right) u_t dx dt + \langle u_0, \phi_0 \rangle
\]
\[
u \mapsto \Psi_\phi^+ (u) := \int_0^T \int_{\mathbb{R}^d} u_t \phi dx dt.
\]
They are continuous on \(Y\): since \(\phi\) is bounded measurable and compactly supported and we have at most the quadratic term \(u_t^2\) under the integral signs, the topology of \(L^2 \left( \mathbb{R}^d \right)\), weaker than the topology of \(Y\), is sufficient to prove continuity. Recall that \(Q^N\) is the law of \(\eta^N_t\). We take a weakly converging subsequence \(Q^{N_k}\), in the topology of the space \(Y\) defined by (6.13), to a probability measure \(Q\). By the Portmanteau Theorem, for every \(\epsilon > 0\),
\[
Q \left( u : |\Psi_\phi (u)| > \epsilon \right) \leq \liminf_{k \to \infty} Q^{N_k} \left( u : |\Psi_\phi (u)| > \epsilon \right) = \liminf_{k \to \infty} \mathbb{P} \left( |\Psi_\phi \left( \eta^N_t \right)| > \epsilon \right).
\]
To show that $Q(u : |\Psi_\phi(u)| > \epsilon) = 0$ we prove in Step 2 below that this \(\liminf\) is zero. This implies \(Q(u : \Psi_\phi(u) = 0) = 1\), since the former statement holds true for every $\epsilon > 0$. By a classical argument of density of a countable set of test functions, we deduce

$$Q \left( \Psi_\phi(u) = 0 \text{ for all } \phi \in C^\infty_c([0, T] \times \mathbb{R}^d) \right) = 1.$$ 

Similarly, if $\phi_t \geq 0$, $\phi \in C^\infty_c([0, T] \times \mathbb{R}^d)$, we apply the same argument to $\Psi^+_\phi$ and obtain

$$Q \left( u : \int_0^T \int_{\mathbb{R}^d} u_t \phi_t dx \, dt < 0 \right) \leq \liminf_{k \to \infty} \mathbb{P} \left( \int_0^T \int_{\mathbb{R}^d} h_t^N \phi_t dx \, dt < 0 \right) = 0,$$

hence, $Q$ is supported on functions $u$ such that $u_t(x) \geq 0$ for a.e. $(t, x) \in [0, T] \times \mathbb{R}^d$. These two properties prove that $Q$ is supported on the set of weak solutions of equation (6.14).

**Step 2.** It remains to prove that $\liminf_{k \to \infty} \mathbb{P} \left( |\Psi_\phi(h_t^N)| > \epsilon \right) = 0$. We write $N$ instead of $N_k$ to simplify the notation. We have

$$\Psi_\phi(h_t^N) = \int_0^T \int_{\mathbb{R}^d} \left( \frac{\partial \phi_t}{\partial t} + \Delta \phi_t + (1 - h_t^N)^+ \phi_t \right) h_t^N \, dx \, dt + \langle u_0, \phi_0 \rangle.$$

By Itô formula, for every $\phi_t \in C^\infty_c([0, T] \times \mathbb{R}^d)$, one has

$$0 = \int_0^T \int_{\mathbb{R}^d} \left( \frac{\partial \phi_t}{\partial t} + \Delta \phi_t \right) h_t^N \, dx \, dt + \int_0^T \int_{\mathbb{R}^d} \theta_N * \left( (1 - h_t^N)^+ S_t^N \right) \phi_t \, dx \, dt$$

$$+ \langle h_0^N, \phi_0 \rangle + \int_{\mathbb{R}^d} \int_0^T \phi_t dM_{t,1}^N \, dx + \int_{\mathbb{R}^d} \int_0^T \phi_t dM_{t,2}^N \, dx.$$

Hence,

$$\Psi_\phi(h_t^N) = \int_0^T \int_{\mathbb{R}^d} \left[ (1 - h_t^N)^+ h_t^N - \theta_N * \left( (1 - h_t^N)^+ S_t^N \right) \right] \phi_t \, dx \, dt$$

$$- \int_{\mathbb{R}^d} \int_0^T \phi_t dM_{t,1}^N \, dx - \int_{\mathbb{R}^d} \int_0^T \phi_t dM_{t,2}^N \, dx$$

$$- \langle h_0^N, \phi_0 \rangle + \langle u_0, \phi_0 \rangle.$$

In order to prove $\lim_{N \to \infty} \mathbb{P} \left( |\Psi_\phi(h_t^N)| > \epsilon \right) = 0$, it is sufficient to prove the same result for each of the previous terms. Lemma [6.6.4] deals with the first term and the two martingale terms can be treated by Chebyshev’s inequality and Lemma [6.6.5] below. The terms

$$\langle u_0, \phi_0 \rangle - \langle h_0^N, \phi_0 \rangle = \langle u_0, \phi_0 \rangle - \langle S_0^N, \theta_N * \phi_0 \rangle$$

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converges to zero in probability by the assumption that $S_0^N$ converges weakly to $u_0(x) \, dx$, as $N \to \infty$, in probability. \hfill \Box

\textbf{Lemma 6.6.4}

It holds that

$$\int_0^T \int_{\mathbb{R}^d} \left[ (1 - h_t^N)^+ h_t^N - \theta_N * \left( (1 - h_t^N)^+ S_t^N \right) \right] \phi_t \, dx \, dt \to 0 \quad \text{as} \quad N \to \infty$$

in probability.

\textit{Proof.} We split the inner integral into

$$\left| \left\langle \theta_N * (S_t^N (1 - h_t^N)^+) - h_t^N (1 - h_t^N)^+, \phi_t \right\rangle \right| \leq \left| \left\langle \theta_N * (S_t^N (1 - h_t^N)^+) - S_t^N (1 - h_t^N)^+, \phi_t \right\rangle \right|$$

$$+ \left| \left\langle S_t^N (1 - h_t)^+ - S_t^N (1 - h_t)^+, \phi_t \right\rangle \right|$$

$$+ \left| \left\langle h_t^N (1 - h_t)^+ - h_t^N (1 - h_t)^+, \phi_t \right\rangle \right|$$

$$= I_t^N + I_{\perp t}^N + III_t^N + IV_t^N,$$

where $h$ denotes the almost sure limit of $(h^N)_{N \in \mathbb{N}}$ given by Skorokhod’s representation theorem. To prove Lemma \[6.6.4\] it is sufficient to show that each term on the right-hand side integrated in time converges in probability to zero. In order to prove that, it is sufficient to show that the expectation converge to zero for every $t \in [0, T]$, because

$$\mathbb{P} \left( \int_0^T I_t^N \, dt > \varepsilon \right) \leq \frac{1}{\varepsilon} \mathbb{E} \int_0^T I_t^N \, dt = \frac{1}{\varepsilon} \int_0^T \mathbb{E} I_t^N \, dt \to 0.$$

Note, there is a compact set $K$, such that $K \supset \cup_{t \in [0, T]} \text{supp}(\phi_t)$. For ease of notation we omit the subscript $t$ in the following.

First,

$$I_t^N = \left| \left\langle \theta_N * (S_t^N (1 - h_t^N)^+) - S_t^N (1 - h_t^N)^+, \varphi \right\rangle \right|$$

$$= \left| \left\langle S_t^N (1 - h_t)^+, \theta_N * \varphi - \varphi \right\rangle \right|$$

$$\leq \left[ S_t^N \right] \| \theta_N * \varphi - \varphi \|_\infty.$$
Hence,

\[
\mathbb{E} I^N \leq \mathbb{E} \left[ S_t^N \right] \| \theta_N \ast \varphi - \varphi \|_\infty \leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[ S_t^N \right] \to 0.
\]

Second, observe that

\[
II^N = \left\langle S_t^N \left( 1 - h_t^N \right)^+ - S_t^N \left( 1 - h_t \right)^+ , \varphi \right\rangle
\leq \| \varphi \|_\infty \left[ S_T^N \right] \sup_{x \in K} \left| \left( 1 - h_t^N (x) \right)^+ - \left( 1 - h_t (x) \right)^+ \right|
\leq \| \varphi \|_\infty \left[ S_T^N \right] \sup_{x \in K} \left| h_t^N (x) - h_t (x) \right|,
\]

and by Sobolev’s embedding theorem and Lemma 6.4.1 we have

\[
\sup_{x \in K} \left| h_t^N (x) - h_t (x) \right| \to 0.
\]

It follows

\[
\mathbb{E} II^N \leq \| \varphi \|_\infty \mathbb{E} \left( \left[ S_T^N \right] \sup_{x \in K} \left| h_t^N (x) - h_t (x) \right| \right)
\leq \| \varphi \|_\infty \mathbb{E} \left( \left[ S_T^N \right]^2 \right) \mathbb{E} \left( \sup_{x \in K} \left| h_t^N (x) - h_t (x) \right|^2 \right)
\leq \| \varphi \|_\infty \sup_{N \in \mathbb{N}} \mathbb{E} \left( \left[ S_T^N \right]^2 \right) \mathbb{E} \left( \sup_{x \in K} \left| h_t^N (x) - h_t (x) \right|^2 \right) \to 0.
\]

The third term converges to zero pointwise due to the weak convergence of \( S_t^N \) and \( h_t^N \).

Finally, the last term also converges pointwise. From Section 6.6.1 we have

\[
\int_K \left| h_t^N (x) - h_t (x) \right|^2 \, dx \to 0.
\]

Therefore,

\[
\left| \left\langle h_t^N \left( 1 - h_t^N \right)^+ - h_t^N \left( 1 - h_t \right)^+ , \varphi \right\rangle \right|
\leq \left( \int_K \left| h_t^N \right|^2 \, dx \right)^{\frac{1}{2}} \left( \int_K \left| \left( 1 - h_t^N \right)^+ - \left( 1 - h_t \right)^+ \right|^2 \, dx \right)^{\frac{1}{2}}
\leq \left( \int_K \left| h_t^N \right|^2 \, dx \right)^{\frac{1}{2}} \left( \int_K \left| h_t^N - h_t \right|^2 \, dx \right)^{\frac{1}{2}} \to 0.
\]

□
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In the next lemma we denote by $C$ any constant depending only on $T$, $\|\theta\|_{L^2}^2$, $\sup_N \bar{\varepsilon}^d / N$, $\|\phi\|_\infty$, $\|\nabla \phi\|_\infty$, $\mathbb{E}\left[\left|S_T^N\right|\right]$.

**Lemma 6.6.5**

For $i = 1, 2$

$$\mathbb{E}\left[\left|\int_{\mathbb{R}^d} \int_0^T \phi_t(x) dM_t^i(x) dx\right|^2\right] \leq CN^{\beta-1}.$$ 

**Proof.** The computations in this proof are less detailed, because they are similar to the ones of Lemma 6.3.1 and 6.3.2. Set

$$g_t^N(y) := -\int_{\mathbb{R}^d} \phi_t(x) \nabla \theta_N(x - y) dx = \int_{\mathbb{R}^d} \nabla \phi_t(x) \theta_N(x - y) dx.$$ 

For the first martingale term we have

$$\mathbb{E}\left[\left|\int_{\mathbb{R}^d} \int_0^T \phi_t(x) dM_t^i(x) dx\right|^2\right] = \frac{2}{N^2} \sum_{a \in A^N} \mathbb{E}\left[\int_0^T 1_{t \in I_a} \left|g_t^N(X_t^n)\right|^2 dt\right] \leq \frac{2}{N} \|\theta_N\|_{L^2}^2 \|\phi\|_\infty^2 \mathbb{E}\int_0^T \left|S_t^N\right| dt.$$ 

The assertion for $i = 1$ follows from Lemma 1.4.1 and

$$\frac{1}{N} \|\theta_N\|_{L^2}^2 \leq C e^{-d} = CN^{\beta-1}.$$ 

Set

$$\tilde{g}_t^N(y) := -\int_{\mathbb{R}^d} \phi_t(x) \theta_N(x - y) dx,$$ 

then for the second martingale term we have

$$\mathbb{E}\left[\left|\int_{\mathbb{R}^d} \int_0^T \phi_t(x) dM_t^i(x) dx\right|^2\right] = \frac{1}{N^2} \sum_{a \in A^N} \mathbb{E}\left[\int_0^T 1_{t \in I_a} \left|\tilde{g}_t^N(X_t^n)\right|^2 \lambda_t^a dt\right] \leq \frac{1}{N} \|\theta_N\|_{L^2}^2 \|\phi\|_\infty^2 \mathbb{E}\int_0^T \left[S_t^N\right] dt.$$ 

and we conclude by the same argument. This completes the proof. \hfill \Box

### 6.6.3 Auxiliary results

**Theorem 6.6.6**

There is at most one weak solution of equation (6.14). The unique solution has the additional property $u_t(x) \leq 1$, hence it is also the unique solution of (6.1).
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Proof. Let \( u^1, u^2 \) be two weak solutions of the equation (6.14) with the same initial condition \( u_0 \). Let \( \{\rho_\varepsilon(x)\}_\varepsilon \) be a classical family of symmetric mollifiers.

For any \( \varepsilon > 0 \) and \( x \in \mathbb{R}^d \) we can use \( \rho_\varepsilon(x - \cdot) \) as test function in the equation (6.15). Set \( u^i_\varepsilon(t, x) = u^i(t, x) \ast_x \rho_\varepsilon(x) \) for \( i = 1, 2 \). Then we have

\[
 u^i_\varepsilon(t, x) = (u_0 \ast \rho_\varepsilon)(x) + \int_0^t \Delta u^i_\varepsilon(s, x) \, ds + \int_0^t (\rho_\varepsilon \ast (1 - u^i_\varepsilon)^+ u^i_\varepsilon)(s, x) \, ds.
\]

By writing this identity in mild form we obtain (we write \( u^i(t, \cdot) \) for the function \( u^i(t, \cdot) \) and \( S(t) \) for \( e^{tA} \))

\[
 u^i_\varepsilon(t) = S(t)(u_0 \ast \rho_\varepsilon) + \int_0^t S(t - s) \left( \rho_\varepsilon \ast \left( (1 - u^i(s))^+ u^i(s) \right) \right) \, ds.
\]

Write \( g(u) \) for the function \( u \to (1 - u)^+ u \) from \([0, \infty)\) into \([0, \infty)\). The function \( U = u^1 - u^2 \) satisfies

\[
 \rho_\varepsilon \ast U(t) = \int_0^t S(t - s) \left( \rho_\varepsilon \ast \left[ g \left( u^1(s) \right) - g \left( u^2(s) \right) \right] \right) \, ds.
\]

Taking the limit as \( \varepsilon \to 0 \) we have

\[
 U(t) = \int_0^t S(t - s) \left[ g \left( u^1(s) \right) - g \left( u^2(s) \right) \right] \, ds.
\]

Hence,

\[
 \|U(t)\|_\infty \leq \int_0^t \|g \left( u^1(s) \right) - g \left( u^2(s) \right)\|_\infty \, ds.
\]

Notice that the function \( g \) is globally Lipschitz, with Lipschitz constant 1 (compute the derivative). It follows

\[
 \|U(t)\|_\infty \leq \int_0^t \|U(s)\|_\infty \, ds.
\]

By Gronwall’s lemma we conclude \( U = 0 \).

It is a classical result that equation (6.1) has a unique weak solution, with the property \( u_t \in [0, 1] \), if \( u_0 \) is bounded, uniformly continuous and of class \( L^2 \) (see [Smo94], Chapter 14, Section A). Hence, this solution is also a solution of equation (6.14) and coincides with the unique weak solution of that equation. \( \square \)

Lemma 6.6.7

For every \( \alpha \geq 0 \) and every family of mollifiers \( (\theta_N)_{N \in \mathbb{N}} \) with \( \theta \in W^{\alpha, 2}(\mathbb{R}^d) \) there exists a constant \( C \geq 0 \) such that

\[
 \|\theta_N\|_{W^{\alpha, 2}(\mathbb{R}^d)} \leq C\varepsilon_N^{-\alpha} \varepsilon_N^{-d/2} \|\theta\|_{W^{\alpha, 2}(\mathbb{R}^d)}.
\]
Proof. First, we compute the Fourier transform of $\theta_N$, i.e. for all $\lambda \in \mathbb{R}^d$

$$\hat{\theta}_N(\lambda) = \int_{\mathbb{R}^d} e^{i \lambda \cdot x} \theta_N(x) \, dx = \epsilon_N^{-d} \int_{\mathbb{R}^d} e^{i \lambda \cdot \epsilon_N^{-1} x} \, dx$$

$$= \int_{\mathbb{R}^d} e^{i \epsilon_N \lambda \cdot y} \, dy = \hat{\theta}(\epsilon_N \lambda).$$

Second, recall that the norms

$$f \mapsto \|f\|_{W^\alpha,2(\mathbb{R}^d)}^2 \quad \text{and} \quad f \mapsto \int_{\mathbb{R}^d} (1 + |\lambda|^2) \alpha |\hat{f}(\lambda)|^2 \, d\lambda$$

are equivalent. Therefore, there is a constant $C \geq 0$, which may change from instance to instance, such that

$$\|\theta_N\|_{W^\alpha,2(\mathbb{R}^d)}^2 \leq C \int_{\mathbb{R}^d} (1 + |\lambda|^2) \alpha |\hat{\theta}_N(\lambda)|^2 \, d\lambda = C \int_{\mathbb{R}^d} (1 + |\lambda|^2) \alpha |\hat{\theta}(\epsilon_N \lambda)|^2 \, d\lambda$$

$$= C \epsilon_N^{-d} \int_{\mathbb{R}^d} \left(1 + |\epsilon_N^{-1} \eta|^2\right)^\alpha |\hat{\theta}(\eta)|^2 \, d\eta$$

$$= C \epsilon_N^{-d} \epsilon_N^{-2\alpha} \int_{\mathbb{R}^d} \left(\epsilon_N^2 + |\eta|^2\right)^\alpha |\hat{\theta}(\eta)|^2 \, d\eta$$

$$\leq C \epsilon_N^{-d} \epsilon_N^{-2\alpha} \int_{\mathbb{R}^d} (1 + |\eta|^2)^\alpha |\hat{\theta}(\eta)|^2 \, d\eta$$

$$\leq C \epsilon_N^{-d} \epsilon_N^{-2\alpha} \|\theta\|_{H^\alpha,2(\mathbb{R}^d)}^2.$$
or equivalently
\[
\sum_{i=1}^{n} \text{Re} \hat{f}(0) \xi_i \bar{\xi}_j + \sum_{i<j} \left(1 + |\lambda_i - \lambda_j|^2\right)^{c/2} e^{-t|\lambda_i - \lambda_j|^2} \\
\times \left(\text{Re} \hat{f}(\lambda_i - \lambda_j) \xi_i \bar{\xi}_j + \text{Re} \hat{f}(\lambda_j - \lambda_i) \xi_j \bar{\xi}_i\right) \geq 0.
\]

Corresponding to any couple \((i, j) \in \{1, \ldots, n\}^2\), let \(\tilde{\xi}_1, \ldots, \tilde{\xi}_n \in \mathbb{C}\) be such that \(\tilde{\xi}_i = \xi_i, \tilde{\xi}_j = \xi_j, \) and \(\tilde{\xi}_k = 0\) for \(k \notin \{i, j\}\). We know that
\[
\sum_{ij=1}^{n} \text{Re} \hat{f}(\lambda_i - \lambda_j) \tilde{\xi}_i \tilde{\xi}_j \geq 0,
\]

hence, if \(i = j\)
\[
\text{Re} \hat{f}(0) \xi_i \bar{\xi}_i \geq 0
\]

while for \(i \neq j\)
\[
\text{Re} \hat{f}(\lambda_i - \lambda_j) \xi_i \bar{\xi}_j + \text{Re} \hat{f}(\lambda_j - \lambda_i) \xi_j \bar{\xi}_i \geq 0.
\]

Using these two facts completes the proof. \(\square\)
Chapter 6. Uniform convergence to the FKPP equation
Appendices
Appendix A

Convergence rate of a double integral depending on a parameter

We prove Lemma 3.3.2. Recall $A(\phi) = \phi/2 + \sin(2\phi)/4$.

Lemma
There exists a constant $C$, such that for all $K \geq 1$

$$
\int_0^\pi \int_0^\infty e^{-K(A(z)-A(\beta))} \, dz \, d\beta \leq CK^{-\frac{3}{4}}
$$

holds true.

Proof. First, note that $A$ is strictly increasing, since its first derivative is positive almost everywhere. When $K$ is large we expect the main contribution to come from the integration area where $\beta$ is around $\pi/2$ and $z$ slightly bigger. This is because $A$ grows the slowest at $\pi/2$.

We will handle this area at the end, since it is the most delicate.

First, consider

$$
\int_0^\pi \int_0^\infty e^{-K(A(z)-A(\beta))} \, dz \, d\beta \leq \int_0^\pi \int_{\beta+1}^\infty e^{-\frac{1}{2}K(\beta-1)} \, dz \, d\beta
$$

$$
= \frac{2}{K} \int_0^\pi \left[ -e^{-\frac{1}{2}K(\beta-1)} \right]_{\beta+1}^{\infty} \, d\beta = \frac{2}{K} \pi.
$$

Fix $\delta \in (0, 1)$. In the next step we show that also $\int_0^\pi \int_{\beta+\delta}^{\beta+1} \ldots$ is of lower order, i.e. decays faster in $K$ than $K^{-\frac{3}{4}}$. There exists a constant $C_\delta > 0$, such that for each pair $\beta \in [0, \pi]$ and $z \geq \beta + \delta$ we have $A(z) - A(\beta) \geq C_\delta$. 

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Therefore,
\[
\int_0^\pi \int_{\beta+\delta}^{\beta+1} e^{-K(A(z)-A(\beta))} dzd\beta \leq (b-a)(1-\delta)e^{-C\delta K}.
\]

As expected, the main contribution comes from the area, where \(z\) is almost equal to \(\beta\). Now we will restrict the relevant area for \(\beta\) to a neighborhood of \(\pi/2\).

Fix \(\hat{\delta} \in (\delta, 1)\) and let \(\beta \in [0, \pi/2 - \hat{\delta}]\) and \(z \in [\beta, \beta + \delta]\). We use the mean value theorem to derive the following estimate
\[
A(z) - A(\beta) = A'(\xi)(z - \beta) \geq \min_{\xi \in [\beta, \beta+\delta]} \cos(\xi)^2 (z - \beta)
\]
\[
\geq \min_{\xi \in [0, \pi/2 - \delta + \hat{\delta}]} \cos(\xi)^2 (z - \beta).
\]

With this in hand we easily conclude
\[
\int_0^{\pi/2 - \hat{\delta}} \int_{\beta}^{\beta+\delta} e^{-K(A(z)-A(\beta))} dzd\beta \leq \int_0^{\pi/2 - \hat{\delta}} \int_{\beta}^{\beta+\delta} e^{-KC\delta(z-\beta)} dzd\beta
\]
\[
= \left(\frac{\pi}{2} - \hat{\delta}\right) \left(1 - e^{KC\delta}\right) \frac{1}{KC\delta}.
\]

With the same argument and \(\tilde{\delta} \in (0, 1)\) we have
\[
\int_{\pi/2 - \tilde{\delta}}^\pi \int_{\beta}^{\beta+\delta} e^{-K(A(z)-A(\beta))} dzd\beta \leq \left(\pi - \frac{\pi}{2} - \tilde{\delta}\right) \left(1 - e^{KC\delta}\right) \frac{1}{KC\delta}.
\]

The remaining area is \(\beta \in (\pi/2 - \hat{\delta}, \pi/2 + \tilde{\delta})\), \(z \in (\beta, \beta+\delta)\). But first of all, note that the following equation holds true for all \(\beta, z\)
\[
A(z) - A(\beta) = \frac{1}{2}(z - \beta) + \frac{1}{4}(\sin(2z) - \sin(2\beta))
\]
\[
= \cos^2(\beta)(z - \beta) + \frac{1}{4} \cos(2\beta)(\sin(2(z - \beta)) - 2(z - \beta)) - 2(z - \beta)) \quad (A.1)
\]
\[
+ \frac{1}{4} \sin(2\beta)(\cos(2(z - \beta)) - 1).
\]

This equality can be proven by using sum-identities of sine and cosine. Nevertheless, we came up with this by Taylor expanding \(A(z) - A(\beta)\) in \(\beta\). It is natural to work with Taylor expansions here, since we can control the error with \(\delta, \hat{\delta}, \tilde{\delta}\). We will find estimates for the last two terms on the right hand
side of (A.1), since they are not easy to handle. Introduce the new variable $x = z - \beta \in [0, \delta)$ and consider

$$\sin(2(z - \beta)) - 2(z - \beta) = -2x + \sin(2x) \quad \text{Taylor}$$

$$= -2x + 2x - \frac{7}{6}x^3 - \frac{x^3}{6} + \frac{4x^5}{15} + \sum_{k=3}^{\infty} \frac{(-1)^k(2x)^{2k+1}}{(2k+1)!} \leq -\frac{7}{6}x^3. \quad \leq 0 \text{ if } x \leq \sqrt{\frac{5}{2}}$$

For the last term in (A.1) we distinguish two cases. First, $\beta \in \left[\frac{\pi}{2}, \frac{\pi}{2} + \tilde{\delta}\right)$ implies

$$\frac{1}{4} \sin(2\beta)(\cos(2x) - 1) \geq 0.$$

And if $\beta \in (\frac{\pi}{2} - \hat{\delta}, \frac{\pi}{2}]$ we have

$$\frac{1}{4} \sin(2\beta)(\cos(2x) - 1) = -\frac{1}{2} \sin(2\beta) \sin^2(x) \geq -\frac{1}{2} \sin(2\beta)x^2.$$

To summarize

$$A(z) - A(\beta) \geq \begin{cases} 
\cos^2(\beta)x - \frac{7}{24}\cos(2\beta)x^3 & \beta \in \left[\frac{\pi}{4}, \frac{\pi}{2} + \tilde{\delta}\right) \\
\cos^2(\beta)x - \frac{1}{2}\sin(2\beta)x^2 - \frac{7}{24}\cos(2\beta)x^3 & \beta \in \left(\frac{\pi}{4} - \hat{\delta}, \frac{\pi}{4}\right]
\end{cases}.$$

We will also estimate the coefficients $\cos^2(\beta), -\frac{1}{2} \sin(2\beta), -\frac{7}{24} \cos(2\beta)$, abusing the fact that $\beta$ is arbitrary close to $\pi/2$. For all $\beta \in (\frac{\pi}{2} - \hat{\delta}, \frac{\pi}{2} + \tilde{\delta})$ with $\hat{\delta}, \tilde{\delta}$ small enough, we have

$$\cos^2(\beta) \geq \frac{18}{19} \left(\beta - \frac{\pi}{2}\right)^2 \quad \text{and} \quad -\frac{7}{24} \cos(2\beta) \geq \frac{5}{18}.$$

And for all $\beta \in (\frac{\pi}{2} - \hat{\delta}, \frac{\pi}{2}]$ with $\hat{\delta}$ small enough

$$-\frac{1}{2} \sin(2\beta) \geq \beta - \frac{\pi}{2}.$$

Introducing another variable $\alpha = \pi/2 - \beta \in (-\hat{\delta}, \hat{\delta})$ and putting everything together yields

$$A(z) - A(\beta) \geq \begin{cases} 
\frac{18}{19} \alpha^2x + \frac{5}{18}x^3 & \alpha \in (-\hat{\delta}, 0] \\
\frac{18}{19} \alpha^2x - \alpha x^2 + \frac{5}{18}x^3 & \alpha \in [0, \hat{\delta})
\end{cases}.$$
Appendix A. Convergence rate of a double integral

Back to our integrals applying this estimate

$$\int_{\pi/2}^{\pi/2+\delta} \int_{\beta}^{\beta+\delta} e^{-K(A(z)-A(\beta))} dz d\beta$$

$$\leq \int_{\pi/2}^{\pi/2+\delta} \int_{\beta}^{\beta+\delta} e^{-K(\cos^2(\beta)(z-\beta) - \frac{1}{2\pi} \cos(2\beta)(z-\beta)^3)} dz d\beta$$

$$\leq \int_{-\delta}^{0} \int_{0}^{\delta} e^{-K\left(\frac{18}{19}\alpha^2 x + \frac{7}{24} x^3\right)} dx d\alpha$$

$$= \int_{0}^{\delta} e^{-\frac{7}{18}\pi x^3} \int_{-\delta}^{0} e^{-\frac{1}{2}\left(\alpha \sqrt{\frac{36Kx}{19}}\right)^2} dx d\alpha$$

$$\leq \sqrt{\frac{19\pi}{18}} K^{-\frac{1}{2}} \int_{0}^{\delta} \frac{1}{\sqrt{x}} e^{-\frac{7}{18}\pi x^3} \left[\int_{-\infty}^{\infty} \sqrt{\frac{36Kx}{19}} e^{-\frac{1}{4}\left(\alpha \sqrt{\frac{36Kx}{19}}\right)^2} d\alpha dx\right]$$

$$y=K\frac{1}{2}x$$

$$\leq \sqrt{\frac{19\pi}{18}} K^{-\frac{1}{2}} \int_{0}^{\delta} \frac{1}{\sqrt{y}} e^{-\frac{7}{18}\pi y^3} dy$$

$$= \sqrt{\frac{19\pi}{18}} K^{-\frac{2}{3}} \int_{0}^{\infty} \frac{1}{\sqrt{y}} e^{-\frac{7}{18}\pi y^3} dy.$$
And finally the last part

\[
\int_{\pi/2 - \delta}^{\pi/2} \int_{\beta}^{\beta + \delta} e^{-K(A(z) - A(\beta))} dz d\beta \\
\leq \int_{\pi/2 - \delta}^{\pi/2} \int_{\beta}^{\beta + \delta} e^{-K(\cos^2(\beta)(z - \beta) - \frac{1}{2} \sin(2\beta)(z - \beta)^2 - \frac{7}{24} \cos(2\beta)(z - \beta)^3)} dz d\beta \\
\leq \int_{0}^{\delta} \int_{0}^{\delta} e^{-K(\frac{1}{18} \alpha^2 x - \alpha x^2 + \frac{7}{24} x^3)} dx d\alpha \\
= \int_{0}^{\delta} \int_{0}^{\delta} e^{-K\left(\left(\sqrt{\frac{18}{19} \alpha - \frac{x^2}{18}}\right)^2 - \frac{3}{18} \alpha x^2 + \frac{7}{24} x^3\right)} dx d\alpha \\
= \int_{0}^{\delta} e^{-\frac{7}{18} K x^3} \int_{0}^{\delta} e^{-\frac{1}{3} \alpha \left(\sqrt{2Kx^\frac{18}{19} (\alpha - \frac{x^2}{18})}\right)^2} d\alpha dx \\
\leq \sqrt{\frac{19\pi}{18}} K^{-\frac{1}{2}} \int_{0}^{\delta} \frac{1}{\sqrt{x}} e^{-\frac{7}{18} K x^3} \int_{-\infty}^{\infty} \sqrt{\frac{36K}{38\pi}} e^{-\frac{1}{2} \left(\sqrt{2Kx^\frac{18}{19} (\alpha - \frac{x^2}{18})}\right)^2} d\alpha dx \\
\leq \sqrt{\frac{19\pi}{18}} K^{-\frac{1}{2}} K^{-\frac{1}{2}} \int_{0}^{\infty} \frac{1}{\sqrt{yK^{-\frac{1}{2}}}} e^{-\frac{1}{2} y^\frac{1}{2}} dy \\
= \sqrt{\frac{19\pi}{18}} K^{-\frac{3}{2}} \int_{0}^{\infty} \frac{1}{\sqrt{y}} e^{-\frac{1}{2} y^\frac{1}{2}} dy. 
\]

\[\square\]
Appendix A. Convergence rate of a double integral
Appendix B

On the condition $\beta < \frac{1}{2}$

Since we deduced the threshold $\beta < 1/2$ from an approach based on Sobolev’s embedding theorem in the spaces $W^{\alpha,2}$, it is natural to ask what happens if we use $W^{\alpha,p}$-topologies (which allow one to use much smaller $\alpha$, taking advantage of large $p$) or even Hölder topologies. We have done partial computations in these directions and the threshold $\beta < 1/2$ is the same in all approaches we have outlined. We show just a partial computation in Hölder norms.

The restriction (in all approaches) seems to come from the estimate of the Brownian martingale. Recall it is given by

$$
\tilde{M}^{1,N}_t(x) := \frac{1}{N} \sum_{a \in \Lambda_N} \int_0^t \left( e^{(t-s)A} \nabla \theta_N \right) (x - X_a^s) 1_{s \in I_a} \cdot dB_a^s.
$$

In order to investigate its Hölder properties, let us invoke Kolmogorov regularity criterion. Hence, we estimate, by the Burkholder-Davis-Gundy inequality,

$$
\mathbb{E} \left[ \left| \tilde{M}^{1,N}_t(x) - \tilde{M}^{1,N}_t(x') \right|^p \right] 
= \frac{1}{N^p} \mathbb{E} \left[ \left| \sum_{a \in \Lambda_N} \int_0^t \left( \left( e^{(t-s)A} \nabla \theta_N \right) (x - X_a^s) \right) \right|^{p/2} 
- \left( e^{(t-s)A} \nabla \theta_N \right) (x' - X_a^s) 1_{s \in I_a} \cdot dB_a^s \right]^{p/2} 
\leq \frac{C}{N^p} \mathbb{E} \left[ \left| \sum_{a \in \Lambda_N} \int_0^t \left( \left( e^{(t-s)A} \nabla \theta_N \right) (x - X_a^s) \right) \right|^{p/2} 
- \left( e^{(t-s)A} \nabla \theta_N \right) (x' - X_a^s) \right|^{p/2} 1_{s \in I_a} ds \right]^{p/2}.
$$
Appendix B. On the condition $\beta < \frac{1}{2}$

\[
= \frac{C}{N^p} \mathbb{E} \left[ \int_0^t \sum_{a \in \Lambda_N^s} \left( (e^{(t-s)}A) \nabla \theta_N \right) (x - X_s^a) \right.

- \left( (e^{(t-s)}A) \nabla \theta_N \right) (x' - X_s^a) \left| ds \right|^{p/2} \bigg]

\leq \frac{C}{N^p} \mathbb{E} \left[ \int_0^t \sum_{a \in \Lambda_N^s} \left[ (e^{(t-s)}A) \nabla \theta_N \right]^2 \alpha \left| x - x' \right|^{2\alpha} \left| ds \right|^{p/2} \bigg] \left( \int_0^t \left[ (e^{(t-s)}A) \nabla \theta_N \right]^2 \alpha \left| ds \right|^{p/2} \bigg) \left| x - x' \right|^{\alpha p}

\leq C \mathbb{E} \left[ \left[ (e^{(t-s)}A) \nabla \theta_N \right]^2 \alpha \left| ds \right|^{p/2} \right] \left( \frac{1}{N} \int_0^t \left[ (e^{(t-s)}A) \nabla \theta_N \right]^2 \alpha \left| ds \right|^{p/2} \bigg) \left| x - x' \right|^{\alpha p}.

To apply Kolmogorov criterion we need $\alpha p > d$. If we choose $\alpha > 0$ such that

\[
\frac{1}{N} \int_0^t \left[ (e^{(t-s)}A) \nabla \theta_N \right]^2 \alpha \left| ds \right| \leq C,
\]

then we can take $p$ so large that $\alpha p > d$. Hence, we may choose $\alpha$ as small as we want. A rough computation (we do not give details) gives us

\[
\left[ (e^{(t-s)}A) \nabla \theta_N \right]^2 \alpha \leq \left( \frac{C}{(t-s)^{\frac{d+1}{2}} + \frac{\alpha}{2}} \right) \left( I - A \right)^{-\frac{1}{2} + \alpha} \nabla \theta_N \alpha \left| ds \right|^{p/2} \bigg) \left| x - x' \right|^{\alpha p},
\]

hence,

\[
\frac{1}{N} \int_0^t \left[ (e^{(t-s)}A) \nabla \theta_N \right]^2 \alpha \left| ds \right| \leq \frac{C}{N} \left( I - A \right)^{-\frac{1}{2} + \alpha} \nabla \theta_N \alpha \left| ds \right| \leq \frac{C}{N} \left( I - A \right)^{-\alpha} \theta_N \alpha \left| ds \right| \leq \frac{C}{N} \left( I - A \right)^{-\alpha} \theta_N \alpha \left| ds \right| \leq \frac{C}{N} N^{2\beta + 2\alpha \beta / d}.
\]

Since $\alpha$ can be taken arbitrarily small, the condition is $\beta < 1/2$. 

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Appendix C

Relevant and sufficient condition for (6.9)

Recall condition (6.9)

\[
\sup_{N \in \mathbb{N}} \mathbb{E} \left[ \int_{\mathbb{R}^d} \left| (I - A)^{\rho_0/2} h_0^N(x) \right|^2 \, dx \right] < \infty
\]

for some \( \rho_0 \geq 0 \). The special case \( \rho_0 = 0 \) coincides with assumption (5.5).

The following Proposition gives an easy sufficient condition, of obvious interest in applications, for both assumptions on the initial condition.

**Proposition**

Assume that \( X_0^i, \ i = 1, ..., N \), are independent identically distributed random variables with common probability density \( u_0 \in L^2(\mathbb{R}^d) \). Then

\[
\sup_{N \in \mathbb{N}} \mathbb{E} \int_{\mathbb{R}^d} \left| \left( \theta_N * S_0^N \right)(x) \right|^2 \, dx < \infty.
\]

If we further assume \( u_0 \in W^{\rho_0,2}(\mathbb{R}^d) \), where \( \rho_0 \in [0, \alpha_0] \) and \( \alpha_0 \) satisfies assumption (6.8), then

\[
\sup_{N \in \mathbb{N}} \mathbb{E} \left\| \theta_N * S_0^N \right\|_{W^{\rho_0,2}(\mathbb{R}^d)}^2 < \infty.
\]

**Proof.** Albeit being a special case we start by proving the first assertion.

**Step 1.** By the i.i.d. property

\[
\mathbb{E} \int_{\mathbb{R}^d} \left| \left( \theta_N * S_0^N \right)(x) \right|^2 \, dx = \frac{1}{N^2} \int_{\mathbb{R}^d} \mathbb{E} \left[ \left( \sum_{i=1}^{N} \theta_N(x - X_0^i) \right)^2 \right] \, dx
\]

\[
= \frac{1}{N} \int_{\mathbb{R}^d} \mathbb{E} \left[ \left| \theta_N(x - X_0^i) \right|^2 \right] \, dx + \frac{N(N - 1)}{N^2} \int_{\mathbb{R}^d} \mathbb{E} \left[ \left| \theta_N(x - X_0^1) \right|^2 \right] \, dx.
\]
Appendix C. Relevant and sufficient condition for (6.9)

For the last term notice that
\[ E \left[ \theta_N \left( x - X_1^i \right) \right] = (\theta_N * u_0) (x) \]
hence,
\[
\frac{N(N-1)}{N^2} \int_{\mathbb{R}^d} E \left[ \theta_N \left( x - X_1^i \right) \right]^2 \, dx \leq \| \theta_N * u_0 \|_{L^2}^2 \leq C,
\]
because \( \theta_N * u_0 \to u_0 \) in \( L^2(\mathbb{R}^d) \). About the first term, we have
\[
\frac{1}{N} \int_{\mathbb{R}^d} E \left[ \left| \theta_N \left( x - X_1^i \right) \right|^2 \right] \, dx = \frac{1}{N} \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} \left| \theta_N \left( x - X_1^i \right) \right|^2 \, dx \right] \, dx \\
= \frac{1}{N} \left( \int_{\mathbb{R}^d} |\theta_N (x)|^2 \, dx \right) \leq C
\]
by (1.8). Hence \( E \int_{\mathbb{R}^d} \left| \left( \theta_N * S_0^N \right) (x) \right|^2 \, dx \leq C \). If \( \rho_0 \) is an integer, the proof can be easily modified. Let us treat the general case in the next step.

**Step 2.** Similarly to a property already used in the proof of Lemma 6.3.1, one has the following translation invariance property:
\[
\left( (I - A)^{\rho_0/2} \theta_N \left( \cdot - X_0^i \right) \right) (x) = \left( (I - A)^{\rho_0/2} \theta_N \right) \left( x - X_0^i \right).
\]
Therefore
\[
E \int_{\mathbb{R}^d} \left| \left( (I - A)^{\rho_0/2} \left( \theta_N * S_0^N \right) \right) (x) \right|^2 \, dx \\
= \frac{1}{N^2} \int_{\mathbb{R}^d} E \left[ \sum_{i=1}^{N} \left( (I - A)^{\rho_0/2} \theta_N \left( \cdot - X_0^i \right) \right) (x) \right]^2 \, dx \\
= \frac{1}{N^2} \int_{\mathbb{R}^d} E \left[ \sum_{i=1}^{N} \left( (I - A)^{\rho_0/2} \theta_N \left( x - X_0^i \right) \right) \right]^2 \, dx \\
= \frac{1}{N} \int_{\mathbb{R}^d} E \left[ \left( (I - A)^{\rho_0/2} \theta_N \right) \left( x - X_0^1 \right) \right]^2 \, dx \\
+ \frac{N(N-1)}{N^2} \int_{\mathbb{R}^d} E \left[ \left( (I - A)^{\rho_0/2} \theta_N \right) \left( x - X_0^1 \right) \right]^2 \, dx.
\]
For the last term we have (using the fact that the operator \( (I - A)^{\rho_0/2} \) is
self-adjoint in $L^2 \left( \mathbb{R}^d \right)$

$$
\mathbb{E} \left[ \left( (I - A)^{\rho_0/2} \theta_N \right) (x - X_0^1) \right] = \int \left( (I - A)^{\rho_0/2} \theta_N \right) (x - y) u_0 (y) \, dy
$$

$$
= \left< (I - A)^{\rho_0/2} \theta_N, u_0 (x - \cdot) \right>
$$

$$
= \left< \theta_N, (I - A)^{\rho_0/2} u_0 (x - \cdot) \right>
$$

$$
= \int \theta_N (z) \left( (I - A)^{\rho_0/2} u_0 (x - \cdot) \right) (z) \, dz
$$

$$
= \int \theta_N (z) \left( (I - A)^{\rho_0/2} u_0 \right) (x - z) \, dz
$$

$$
= \left< \theta_N * (I - A)^{\rho_0/2} u_0, \theta_N \right>
$$

where we have used again a translation invariance property. Hence,

$$
\frac{N (N - 1)}{N^2} \int_{\mathbb{R}^d} \mathbb{E} \left[ \left( (I - A)^{\rho_0/2} \theta_N \right) (x - X_0^1) \right]^2 \, dx
$$

$$
\leq \left\| \theta_N * (I - A)^{\rho_0/2} u_0 \right\|_{L^2}^2 \leq C \left\| (I - A)^{\rho_0/2} u_0 \right\|_{L^2}^2 \leq C
$$

because the convolutions with $\theta_N$ are equibounded in $L^2 \left( \mathbb{R}^d \right)$. For the first term, we have

$$
\frac{1}{N} \int_{\mathbb{R}^d} \mathbb{E} \left[ \left( (I - A)^{\rho_0/2} \theta_N \right) (x - X_0^1) \right]^2 \, dx
$$

$$
= \frac{1}{N} \mathbb{E} \left[ \int_{\mathbb{R}^d} \left( (I - A)^{\rho_0/2} \theta_N \right) (x - X_0^1) \right]^2 \, dx
$$

$$
= \frac{1}{N} \mathbb{E} \left[ \int_{\mathbb{R}^d} \left| \left( (I - A)^{\rho_0/2} \theta_N \right) (x) \right|^2 \, dx \right]
$$

$$
\leq \frac{1}{N} C \epsilon_N^{-2 \rho_0} \epsilon_N^{-d} \left\| \theta \right\|_{W^{2 \rho_0, 2} \left( \mathbb{R}^d \right)} \leq \frac{1}{N} C N^{\frac{d}{2} (2 \rho_0 + d)}
$$

$$
\leq \frac{1}{N} C N^{\frac{d}{2} (2 \alpha_0 + d)} \leq \frac{1}{N} C N^{\frac{d}{2} \left( \frac{d (1 - \beta)}{2 \beta} + d \right)} \leq C
$$

where we have used Lemma 6.6.7 below, $\epsilon_N = N^{\frac{d}{2}}$, $\rho_0 \leq \alpha_0$ and the condition $\alpha_0 \leq \frac{d (1 - \beta)}{2 \beta}$ imposed in assumption (6.8). \qed
Appendix C. Relevant and sufficient condition for (6.9)
Bibliography


Bibliography


