Backward Stochastic Differential Equations with Jumps are Stable

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Στους αγαπημένους μου γονείς και στα αδέρφια μου

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Abstract

A backward stochastic differential equation is a stochastic differential equation whose terminal value is known, in contrast to a (forward) stochastic differential equation whose initial value is known, and whose solution has to be adapted to a given filtration. The main aim of this thesis is to provide the suitable framework for the stability of stochastic differential equations with jumps, hereinafter BSDEs or BSDE when we refer to a single object. With the term stability we understand the continuity of the operator that maps the standard data of a BSDE, a set which among others includes the terminal value of the BSDE and the filtration with respect to which the solution has to be adapted, to its solution. In other words, the stability property allows to obtain an approximation of the solution of the BSDE under interest, once we determine an approximation of the standard data of the BSDE under interest.

In this thesis we provide a general wellposedness result of multidimensional BSDEs with stochastic Lipschitz generator and which is driven by a possibly stochastically discontinuous square-integrable martingale. The time horizon can be infinite and as already implicitly has been stated, the right-continuous filtration is allowed to be stochastically discontinuous. Moreover, we provide a framework under which the stability property of BSDEs is verified. This framework allows for both continuous-time and discrete-time $\mathbb{L}^2-$type approximations, which can turn out to be particularly useful for the well-posedness of numerical schemes for BSDEs. These results are presented in the second and the fourth chapter of this thesis. In the third chapter the stability of martingale representations is obtained, a result which lies at the core of the stability property of BSDEs. The property of the stability of martingale representations is not only a useful tool for our current needs, but it is also an interesting result on its own. Roughly speaking, it amounts to the convergence of the spaces generated by a convergent sequence of stochastic integrators as well as of their corresponding orthogonal spaces.

Apart from these main results, a series of other results have been obtained, which either improve or complement classical ones. The most interesting of them is of purely analytic nature. It provides a characterisation of the weak-convergence of finite measures on the positive real-line by means of relatively compact sets of the Skorokhod space endowed with the J_1-topology. We remain in the Skorokhod space, where we refine a classical result on convergence of the jump-times of a J_1-convergent sequence. More precisely, we deal with the case of a multidimensional J_1-convergent sequence and we prove that the times that the heights of the jumps lie in a suitable fixed set form a convergent sequence in the extended positive real-line. We proceed with the theory of Young functions, where the contribution amounts to the following result. We prove that the conjugate Young function of the composition of a moderate Young function with $\mathbb{R}_+ \ni x \mapsto \frac{1}{2}x^2 \in \mathbb{R}_+$ is also a moderate Young function with further nice properties. Finally, a new inequality regarding generalised inverses complements a classical one.

Zusammenfassung

Eine rückwärts-stochastische Differentialgleichung (BSDE) ist eine stochastische Differentialgleichung, dessen Endwert gegeben ist. Dies steht im Gegensatz zu vorwärts-stochastischen Differentialgleichungen, bei denen der Anfangswert gegeben ist und die Lösung an eine Filtration angepasst werden muss. Ziel dieser Arbeit ist, die mathematischen Grundlagen für die Stabilität von stochastischen Differentialgleichungen mit Sprüngen zu formulieren. Unter Stabilität verstehen wir hier die Stetigkeit des Operators, welcher die Daten der BSDE – diese beinhalten u.a. den Endwert und die entsprechende Filtration – auf ihre Lösung abbildet. In anderen Worten ermöglicht die Stabilitätseigenschaft, eine Approximation der Lösung der BSDE zu erhalten, sobald wir eine Approximation der Daten der BSDE haben.

Die vorliegende Arbeit beinhaltet ein allgemeines Resultat zur Wohlgestelltheit von mehrdimensionalen BSEDs mit stochastischem Lipschitz Generator, die von einem möglicherweise stochastischen und unstetigen aber integrierbaren Martingal gesteuert wird. Wir erlauben einen unendlichen Zeithorizont sowie stochastische Unstetigkeiten in der rechtsstetigen Filtration. Darüber hinaus analysieren wir Bedingungen unter der die Stabilitätseigenschaft der BSDE erfüllt ist. Dies erlaubt sowohl zeitkontinuierliche als auch zeitdiskrete $\mathbb{L}^2-$Approximationen, welche speziell für die Wohl-gestelltheit numerischer Verfahren für BSDEs von Nutzen sein kann. Die entsprechenden Resultate befinden sich im zweiten und vierten Kapitel dieser Arbeit. Im dritten Kapitel wird die Stabilität der Martingal Darstellungen gezeigt. Diese Eigenschaft liefert nicht nur ein für diese Arbeit wichtiges Hilfsmittel sondern stellt an sich schon ein interessantes Ergebnis in Hinblick auf die Konvergenz von Räumen, die durch konvergente Folgen von stochastischen Integratoren generiert werden, dar.

Neben den erwähnten Hauptresultaten liefert diese Arbeit noch eine Reihe weiterer Ergebnisse. Das vielleicht Interessanteste ist rein algebraisch und liefert eine Charakterisierung der schwachen Konvergenz endlicher Maße auf der positiven reellen Achse relativ kompakter Mengen des Skorokhod Raums ausgestattet mit der J_1-Topologie. Zudem gibt es eine Verfeinerung eines klassischen Resultats über die Sprungzeiten von J_1-konvergenten Folgen. Schließlich wird noch bewiesen, dass die konjugierte Young Funktion der Komposition einer moderaten Young Funktion mit $\mathbb{R}_+ \ni x \mapsto \frac{1}{2}x^2 \in \mathbb{R}_+$ eine moderate Young Funktion mit guten Eingeschäften ist.

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Synopsis

The dissertation, which is based on joint work with Antonis Papapantoleon and Dylan Possamaï, is divided in four chapters. At the beginning of each chapter, except for the first one, we provide some introductory comments about the topic of the chapter, while at the end of each one, again the exception is the first chapter, we compare our results with the existing literature. In the first chapter we provide most of the notation as well as the machinery we are going to use. The experienced reader may skim through it in order to identify the notation, which in most of the cases is the one commonly used in Cohen and Elliott [20], Dellacherie and Meyer [26, 27], Jacod and Shiryaev [41] and He, Wang, and Yan [35]. We have tried to make the dissertation self-contained, so even the non-experienced reader may have direct access to the arguments we use. In the remaining results of this chapter either we provide sufficient reference for their proof or we provide the proof when we want to underline specific points.

However, in the first chapter there are also results which are new, to the best knowledge of the author. These are

- The right inequality in Lemma I.34.(vii), which turns out to be powerful enough for obtaining the *a priori* estimates in Chapter II, see Lemma II.16.
- Proposition I.51 which provides some nice properties of moderate Young functions, which we believe may prove fruitful for convergence results under an $\mathbb{L}^2-$setting.
- Subsection I.6.1, where we prove that the times the jump of a J_1-convergent sequence form a $|\cdot|-$convergent sequence. This result will be used in Chapter III in order to construct a family of J_1-convergent sequences of submartingales.

Moreover, Section I.7 can be regarded as a new contribution in the sense that it provides a new perspective. To be more precise, it is not the difficulty of the results we present in this section, but that we set the suitable framework under which we obtain a slight generalisation of the classical result [41, Lemma III.4.24]. This in turn, will permit the co-existence of an Itô stochastic integrator with jumps and an integer-valued random measure, which is in particular the case in numerical schemes.

In Chapter II we provide a general wellposedness result for multidimensional BSDEs with possibly unbounded random time horizon and driven by a general martingale in a filtration only assumed to satisfy the usual hypotheses, *i.e.* the filtration may be stochastically discontinuous. We show that for stochastic Lipschitz generators and unbounded, possibly infinite, time horizon, these equations admit a unique solution in appropriately weighted spaces. Our result allows in particular to obtain a wellposedness result for BSDEs driven by discrete-time approximations of general martingales. The current chapter relies heavily on [54] but in virtue of Section I.7 we obtain a generalisation which allows the integrator of the Itô stochastic integral to be a square-integrable martingale with non-trivial purely discontinuous part, as has been described above.

In Chapter III the core of the thesis is presented, the stability of martingale representations. In the first section we settle the framework and we state the main theorem. Then we present the preparatory results we need for the proof, which is finally presented in Section III.7.

Finally, in Chapter IV we present the result that justifies the title of the dissertation. Here we follow a different approach for presenting the proof. We provide the main arguments in Section IV.3 in order to reduce the complexity of the problem under consideration. Then, in the remaining sections of this chapter we prove that the pair of sufficient conditions we have determined are indeed satisfied.

Basic Notation and Terminology

$\emptyset$ denotes the empty set

$\mathbb{R}$ denotes the set of real numbers

$[a,b) = \{x \in \mathbb{R}, a \leq x < b\}$, similarly for $[a,b], (a,b], (a,b)$

$|x|$ denotes the absolute value of the real number x

$\mathbb{R}_+ = [0,\infty)$

$\overline{\mathbb{R}}_+ = [0,\infty]$

$\mathbb{N} = \{1,2,3,\dots\}$, *i.e.* the set of positive integers

p, q denote positive integers. More generally, p_i, q_i denote positive integers, for i in an arbitrary index-set

ℓ is a fixed positive integer

$\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$

$\mathbb{R}^p =$ the Euclidean $p-$dimensional space

$(\mathbb{R}^{p\times q}, +)$ is the group of $p \times q-$matrices with entries in $\mathbb{R}$

$(y_i)_{i\in I}$ denotes a family indexed by the set I

$x \wedge y = \min(x,y)$, for $x, y \in \mathbb{R}$

A^c denotes the complement of the set A

$\mathbb{1}_A$ denotes the indicator function of the set A

δ_α denotes the Dirac measure sitting at α

a.s. is the abbreviation for "almost surely"

$\sigma(\mathcal{A})$ denotes the $\sigma-$*algebra* generated by the class $\mathcal{A}$

$\mathcal{C} \otimes \mathcal{D}$ denotes the product $\sigma-$*algebra* of the classes $\mathcal{C}$ and $\mathcal{D}$

$\mathcal{B}(E)$ denotes the Borel $\sigma-$algebra of the metric space E

$s \uparrow t$ stands for $s \to t$ with $s < t$. Analogously, $s \downarrow t$ stands for $s \to t$ with $s > t$

$f(t-), \lim\limits_{s\uparrow t} f(s) := \lim\limits_{\substack{s\to t\\ s<t}} f(s)$, *i.e.* the left-limit of f at the point $t \in (0,\infty)$, if it exists

$f(t+), \lim\limits_{s\downarrow t} f(s) := \lim\limits_{\substack{s\to t\\ s>t}} f(s)$, *i.e.* the right-limit of f at the point $t \in \mathbb{R}_+$, if it exists

$\displaystyle\int_A f(z)\,\mu(\mathrm{d}z)$ denotes the Lebesgue–Stieltjes integral of f with respect to the measure μ over the set A

$\displaystyle\int_A f(z)\,\mathrm{d}z$ denotes the Lebesgue–Stieltjes integral of f with respect to the Lebesgue measure over the set A

CHAPTER I

Elements of Stochastic Calculus

In this chapter we outline the notation as well as the definitions and the results from stochastic calculus we will make use of. The main references will be the books of Cohen and Elliott [20], of Dellacherie and Meyer [26, 27], of He et al. [35] and of Jacod and Shiryaev [41]. For [26, 27] we will use the following convention: we will refer to the Statement number 2 of Chapter I by Statement I.2, p. 7, *i.e* [26, Definition I.2, p. 7][1] refers to the definition of a random variable. For [41], when we want to refer to the Statement number 3 in Section 1 of Chapter I, we will write Statement I.1.3, *i.e.* [41, Definition I.1.3] refers to the definition of the complete stochastic basis.

The basic notation has been provided on page v. However, before we proceed to the first section of the chapter we provide additional notation which completes the basic one. Since we deal with limit theorems with possibly multi-dimensional objects and in order to familiarise the reader with the notation we use, we will reserve some letters for specific purposes. More specifically, the letters k, l and m will serve as indexes of the elements of sequences and they will be assumed to lie in $\overline{\mathbb{N}}$. The letters i, j, p and q are reserved to denote arbitrary natural integers. In the case of p, q, the same holds true when they are indexed by a natural integer, *e.g.* $p_{37} \in \mathbb{N}$. The aforementioned letters will represent (mainly) the dimension of the spaces or the position of the elements of a multi-dimensional object. The calligraphic letter ℓ will be assumed to be a fixed positive integer, except for Chapter IV where we assume $\ell = 1$. It is going to denote exclusively the dimension of the martingales which will serve as "stochastic integrators". The capital letter E, whether it is indexed (*e.g.* E_1) or not, will denote either an Euclidean space or the group $(\mathbb{R}^{p\times q}, +)$. Therefore $(E, +)$ is always a group. We abuse notation and denote by 0 the *neutral element* of the (arbitrary) group $(E, +)$.

Let us fix a matrix $v \in \mathbb{R}^{p\times q}$,[2] its *transpose* will be denoted by $v^\top \in \mathbb{R}^{q\times p}$. The element at the $i-$th row and $j-$th column of v will be denoted by v^{ij}, for $1 \le i \le p$ and $1 \le j \le q$ and it will be called the $(i, j)-$*element of* v. However, the notation needs some care when we deal with sequences of elements of $\mathbb{R}^{p\times q}$, *e.g.* if $(v^k)_{k\in\overline{\mathbb{N}}} \subset \mathbb{R}^{p\times q}$ then we will denote by $v^{k,ij}$ the $(i, j)-$element of v^k, for every $1 \le i \le p$, $1 \le j \le q$ and for every $k \in \overline{\mathbb{N}}$. We will identify $\mathbb{R}^p$ with $\mathbb{R}^{1\times p}$, *i.e.* the arbitrary $x \in \mathbb{R}^p$ will be identified as a row-vector of length p. The $i-$*th element* of $x \in \mathbb{R}^p$ will be denoted by x^i, for $1 \le i \le p$. Moreover, if $(x^k)_{k\in\overline{\mathbb{N}}} \subset \mathbb{R}^p$ then $x^{k,i}$ denotes the $i-$th element of x^k, for every $1 \le i \le p$ and for every $k \in \overline{\mathbb{N}}$. The *trace* of a square matrix $w \in \mathbb{R}^{p\times p}$ is given by $\mathrm{Tr}[w] := \sum_{i=1}^p w^{ii}$. We endow the space $E := \mathbb{R}^{p\times q}$ with the norm $\|\cdot\|_2$, defined by $\|v\|_2^2 = \mathrm{Tr}[v^\top v]$ and we remind the reader that this norm is derived from the inner product defined for any $(v, w) \in \mathbb{R}^{p\times q} \times \mathbb{R}^{p\times q}$ by $\mathrm{Tr}[v^\top w]$. We will additionally endow E with the norm $\|\cdot\|_1$, which is defined by $\|v\|_1 := \sum_{i=1}^p \sum_{j=1}^q |v^{ij}|$. The associated to E Borel $\sigma-$algebra will be denoted by $\mathcal{B}(E)$. The space E is always finite-dimensional, therefore no confusion arises with respect to which topology the Borel $\sigma-$algebra is meant, since the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ generate the same topology. Finally, let (F, δ) be a complete metric space. If the sequence $(x^k)_{k\in\mathbb{N}} \subset F$ converges to $x^\infty \in F$ under the metric δ, then we will say that sequence $(x^k)_{k\in\overline{\mathbb{N}}}$ is a $\delta-$*convergent sequence* and we will denote it by $x^k \xrightarrow[k\to\infty]{\delta} x^\infty$[3]. If the space F is a (real) vector space and is endowed with a norm $\|\cdot\|$, then the metric induced by the norm $\|\cdot\|$ will be denoted by $\delta_{\|\cdot\|}$. In this case we will say that a sequence is $\|\cdot\|-$convergent or $\delta_{\|\cdot\|}-$convergent interchangeably.

[1] In [26] the page numbering consists of the number of the page and the letter of the chapter. In order to avoid the repetition of the chapter-letter we will write only the number of the page.

[2] Observe that we have not mentioned explicitly that $p, q \in \mathbb{N}$. That was the purpose of the previous paragraph. In order to avoid repetitions, no further reference will be made for the aforementioned letters.

[3] We will omit the index associated to the convergence, *i.e.* $k \to \infty$, whenever it is clear to which index the convergence refers.

The following notational abuse will considerably simplify some statements. For a given finite family of spaces $(E_i)_{i\in\{1,\dots,q\}}$ we endow its Cartesian product with the $\|\cdot\|_1-$norm

$$\prod_{i=1}^{q} E_i \ni e = (e_1,\dots,e_q) \xmapsto{\|\cdot\|_1} \sum_{i=1}^{q} \|e_i\|_1$$

as well as with the $\|\cdot\|_2-$norm

$$\prod_{i=1}^{q} E_i \ni e = (e_1,\dots,e_q) \xmapsto{\|\cdot\|_2} \left(\sum_{i=1}^{q} \|e_i\|_2^2\right)^{\frac{1}{2}}.$$

Let, now, $f:\mathbb{R}_+\to\mathbb{R}^{p\times q}$ be a function. The value of the function f at point s will be denoted by $f(s)$, while the value of its $(i,j)-$element at point s will be denoted by $f^{ij}(s)$. If $p=1$ the value of its $i-$element at point s will be denoted by $f^i(s)$. If f is real-valued, it will be said *non-decreasing* (resp. *non-increasing*, *increasing*, *decreasing*) if for every $0\le s<t$ holds $f(s)\le f(t)$ (resp. $f(s)\ge f(t)$,$f(s)<f(t)$, $f(s)>f(t)$).

Let us fix a function $f:(\mathbb{R}_+,\delta_{|\cdot|})\to(E,\delta_E)$. We define $f(0-):=f(0)$, $f(t-),\lim_{s\uparrow t} f(s):=\lim_{\substack{s\to t\\ s<t}} f(s)$ for $t>0$ and $f(t+),\lim_{s\downarrow t} f(s):=\lim_{\substack{s\to t\\ t<s}} f(s)$, whenever the limits exist and are elements of E. If for every $t\in\mathbb{R}_+$ the values $f(t-),f(t+)$ are well-defined and $f(t)=f(t+)$, resp. $f(t-)=f(t)$, then the function will be called *càdlàg*, resp. *càglàd*. The French abbreviation càdlàg stands for *continu à droite avec limite à gauche*, *i.e.* right-continuous with left limits, while càglàd stands for *continu à gauche avec limite à droite*, *i.e.* left-continuous with right limits. If f is càdlàg, then we associate to it the càglàd function $f_-:(\mathbb{R}_+,\delta_{|\cdot|})\to(E,\delta_E)$ defined by $f_-(0):=f(0)$ and $f_-(t):=f(t-)$, for $t\in(0,\infty)$, as well as the function $\Delta f:(\mathbb{R}_+,\delta_{|\cdot|})\to(E,\delta_E)$ defined by $\Delta f:=f-f_-$. If $\Delta f(t)\neq 0$, then the point t will be said to be a *point of discontinuity for f*. We close this paragraph with the following definition. If the limit $\lim_{s\to\infty} f(s)$ is well-defined, then we extend the function f on $\overline{\mathbb{R}}_+$ and we define *the value of f at the symbol* ∞ to be $f(\infty):=\lim_{s\to\infty} f(s)$.

We conclude this part with the definition of some useful functions. The *identity function* on $\mathbb{R}$, *i.e.* $\mathbb{R}\ni x\longmapsto x\in\mathbb{R}$, will be denoted by Id, while the identity function on $\mathbb{R}^\ell$ will be denoted by Id_ℓ. The *canonical $i-$projection* $\mathbb{R}^p\ni x\longmapsto x^i\in\mathbb{R}$ will be denoted by π^i, for $1\le i\le p$, where we suppress in the notation the indication of the domain. Finally, we define the functions $\mathbb{R}_+\ni x\xmapsto{\text{quad}}\frac{1}{2}x^2\in\mathbb{R}_+$ and $\mathbb{R}^\ell\ni x\xmapsto{\text{q}} x^\top x\in\mathbb{R}^{\ell\times\ell}$.

I.1. Preliminaries and notation

A *probability space* $(\Omega,\mathcal{G},\mathbb{P})$ is a triplet consisting of the sample space Ω, of the $\sigma-$algebra $\mathcal{G}$ on Ω, whose elements are called *events*, and the probability measure $\mathbb{P}$ defined on the measurable space $(\Omega,\mathcal{G})$. Every probability space we are going to use will be assumed to be *complete*, *i.e.* the $\sigma-$algebra $\mathcal{G}$ is complete. The latter means that all $\mathbb{P}-$negligible sets belong to $\mathcal{G}$. The arbitrary subset of $\Omega\times\mathbb{R}_+$ will be referred to as *random set*. A measurable function $\xi:(\Omega,\mathcal{G})\to(E,\mathcal{B}(E))$ will be called an ($E-$valued) *random variable*. A family $M=(M_t)_{t\in\mathbb{R}_+}$ of $E-$valued random variables will be called ($E-$valued) *stochastic process* or simply ($E-$valued) *process*. When the process M is considered as a mapping from the measurable space $(\Omega\times\mathbb{R}_+,\mathcal{G}\otimes\mathcal{B}(\mathbb{R}_+))$ into $(E,\mathcal{B}(E))$ we will say that M is $\mathcal{G}\otimes\mathcal{B}(\mathbb{R}_+)-$measurable. M_t denotes the value of the process at time t, while for fixed $\omega\in\Omega$ the function $\mathbb{R}_+\ni t\longmapsto M_t(\omega)\in E$ is called the *$\omega-$path* of the process M.

■ The arbitrary probability space $(\Omega,\mathcal{G},\mathbb{P})$ is fixed for the rest of the chapter.

For this section $E=\mathbb{R}^{p\times q}$. We will say that a property P holds $\mathbb{P}-$almost surely, abbreviated by $\mathbb{P}-a.s.$, if $\mathbb{P}(\{\omega\in\Omega, P(\omega)\text{ is true}\})=1$. A random set A will be called *evanescent* if the set $\{\omega\in\Omega,\exists t\in\mathbb{R}_+\text{ with }(\omega,t)\in A\}$ is $\mathbb{P}-$negligible. Observe that, since the probability space is complete, we are eligible for writing $\mathbb{P}(\{\omega\in\Omega,\exists t\in\mathbb{R}_+\text{ with }(\omega,t)\in A\})=0$. Two processes M^1,M^2 will be called *indistinguishable* if the random set $\{\omega\in\Omega,\exists t\in\mathbb{R}_+\text{ with }M^1_t(\omega)\neq M^2_t(\omega)\}$ is evanescent.

We introduce some more notational conventions. For the rest of the section we will denote by ζ a real-valued random variable and by ξ,ξ_1,ξ_2 *etc.* $E-$valued random variables. The property $\xi_1\le\xi_2$ stands for $\xi_1\le\xi_2$ $\mathbb{P}-a.s.$ element-wise, *i.e.* the elements of $\xi_2-\xi_1$ are non-negative real numbers $\mathbb{P}-a.s.$, with the obvious interpretation for $\xi_1\ge\xi_2$, $\xi_1<\xi_2$, $\xi_1>\xi_2$, $\xi_1=\xi_2$ and $\xi_1\neq\xi_2$. Moreover, we define $[\xi_1\le\xi_2]:=\{\omega\in\Omega,\xi_1(\omega)\le\xi_2(\omega)\}$ and $[\xi_1\ge\xi_2],[\xi_1<\xi_2],[\xi_1>\xi_2],[\xi_1=\xi_2],[\xi_1\neq\xi_2]$ analogously.

Expectations under $\mathbb{P}$ will be denoted by $\mathbb{E}[\cdot]$, *i.e.*

$$\mathbb{E}[\zeta] := \int_\Omega \zeta(\omega)\,\mathrm{d}\mathbb{P}(\omega) = \int_\Omega \zeta\,\mathrm{d}\mathbb{P} \quad \text{or} \quad \mathbb{E}[\xi] := \int_\Omega \xi(\omega)\,\mathrm{d}\mathbb{P}(\omega) = \int_\Omega \xi\,\mathrm{d}\mathbb{P},$$

where in the latter case the expectation is calculated element-wise, therefore $\mathbb{E}[\xi]$ is a $p \times q-$matrix. If $\mathbb{E}[|\zeta|] < \infty$ (resp. $\mathbb{E}\big[\,\|\xi\|_1\big] < \infty$, which is equivalent to $\mathbb{E}\big[\,\|\xi\|_2\big] < \infty$) then ζ (resp. ξ) will be called *integrable*. For $\mathcal{F}$ a sub$-\sigma-$algebra of $\mathcal{G}$ we define the spaces

$$\mathbb{L}^1(\mathcal{F};E) := \big\{\xi \text{ is an } \mathcal{F}-\text{measurable } E-\text{valued random variable}, \mathbb{E}\big[\|\xi\|_1\big] < \infty\big\}$$

and

$$\mathbb{L}^2(\mathcal{F};E) := \big\{\xi \text{ is an } \mathcal{F}-\text{measurable } E-\text{valued random variable}, \mathbb{E}\big[\|\xi\|_2^2\big] < \infty\big\},$$

where $\mathbb{P}-a.s.$ equal random variables belong to the same class of equivalence. For $\vartheta \in \{1,2\}$, the corresponding norm $\|\cdot\|_{\mathbb{L}^\vartheta(\mathcal{F};E)}$ is defined by $\|\cdot\|^\vartheta_{\mathbb{L}^\vartheta(\mathcal{F};E)} := \mathbb{E}\big[\|\cdot\|^\vartheta_\vartheta\big]$. The Riesz–Fischer Theorem, *e.g.* see Cohen and Elliott [20, Theorem 1.5.33], let us know that the normed spaces $(\mathbb{L}^1(\mathcal{F};E), \|\cdot\|_1)$ and $(\mathbb{L}^2(\mathcal{F};E), \|\cdot\|_2)$ are Banach. Choose a $\vartheta \in \{1,2\}$ and let $(\xi^k)_{k\in\overline{\mathbb{N}}}, (\eta_t)_{t\in\overline{\mathbb{R}}_+} \subset \mathbb{L}^\vartheta(\mathcal{F};E)$. If

$$\mathbb{E}\big[\|\xi^k - \xi^\infty\|^\vartheta_\vartheta\big] \xrightarrow[k\to\infty]{} 0, \text{ resp. } \mathbb{E}\big[\|\eta_t - \eta_\infty\|^\vartheta_\vartheta\big] \xrightarrow[t\uparrow\infty]{} 0,$$

then we will say interchangeably that the sequence $(\xi^k)_{k\in\mathbb{N}}$ converges to ξ^∞ in $\mathbb{L}^\vartheta(\mathcal{F};E)-$mean or that the sequence $(\xi^k)_{k\in\overline{\mathbb{N}}}$ is $\mathbb{L}^\vartheta(\mathcal{F};E)-$convergent, resp. the family $(\eta_t)_{t\in\overline{\mathbb{R}}_+}$ converges to η_∞ in $\mathbb{L}^\vartheta(\mathcal{F};E)-$mean or that the family $(\eta_t)_{t\in\overline{\mathbb{R}}_+}$ is $\mathbb{L}^\vartheta(\mathcal{F};E)-$convergent . We will denote the above convergence by

$$\xi^k \xrightarrow{\mathbb{L}^\vartheta(\mathcal{F};E)} \xi^\infty, \text{ resp. } \eta_t \xrightarrow{\mathbb{L}^\vartheta(\mathcal{F};E)} \eta_\infty.$$

The convergence in probability under a metric δ of the sequence $(\eta^k)_{k\in\overline{\mathbb{N}}}$ will be denoted by

$$\eta^k \xrightarrow{(\delta,\mathbb{P})} \eta^\infty.$$

The conditional expectation of ξ with respect to $\mathcal{F}$ will be denoted by the $E-$valued $\mathcal{F}-$measurable random variable $\mathbb{E}[\xi|\mathcal{F}]$, where $\mathbb{E}[\xi|\mathcal{F}]^{ij} := \mathbb{E}[\xi^{ij}|\mathcal{F}]$ for $1 \le i \le p, 1 \le j \le q$. Since we do not require any integrability property for the random variable ξ, we are implicitly making use of the generalised conditional expectation; see He et al. [35, Section I.4] or Jacod and Shiryaev [41, Definition 1.1].

Definition I.1. An $E-$valued stochastic process M will be called *càdlàg* (resp. *continuous*, *càglàd*) if

$$\mathbb{P}\big([\mathbb{R}_+ \ni t \longmapsto M_t \in E \text{ is càdlàg (resp. continuous, càglàd)}]\big) = 1.$$

When M is càdlàg, we define the $E-$valued stochastic processes $M_- := (M_{t-})_{t\in\mathbb{R}_+}$ and $\Delta M := (\Delta M_t)_{t\in\mathbb{R}_+}$, where

$$M_{0-}(\omega) := M_0(\omega), M_{t-}(\omega) := \lim_{s\uparrow t} M_s(\omega) \text{ and } \Delta M_t(\omega) := M_t(\omega) - M_{t-}(\omega)$$

for every $\omega \in \Omega$ such that $\lim_{s\uparrow t} M_s(\omega)$ is well-defined.

Definition I.2. A family $\mathbb{G} = (\mathcal{G}_t)_{t\in\mathbb{R}_+}$ of sub$-\sigma-$*algebrae* of $\mathcal{G}$ will be called *filtration* if it is

(i) increasing, *i.e.* $\mathcal{G}_s \subset \mathcal{G}_t$ for every $0 \le s \le t$,
(ii) right-continuous, *i.e.* $\mathcal{G}_t = \cap_{u>t}\mathcal{G}_u$ for every $t \in \mathbb{R}_+$, and
(iii) complete, *i.e.* the $\sigma-$algebra $\mathcal{G}_0$[4] contains all $\mathbb{P}-$negligible sets of $\mathcal{G}$.

We abuse the usual convention and we set $\mathcal{G}_\infty, \mathcal{G}_{\infty-} := \sigma\big(\{A \subset \mathcal{G}, \exists t \in \mathbb{R}_+ \text{ such that } A \in \mathcal{G}_t\}\big)$.

The reader familiar with the standard terminology of stochastic calculus will recognise that a filtration is defined in such a way that it satisfies the *usual conditions a priori*. This implies that whenever we make use of the *natural filtration of* M, which will be denoted by $\mathbb{F}^M$, we will refer to the *usual augmentation* of $\mathbb{F}^M_+ = (\mathcal{F}^M_{t+})_{t\in\mathbb{R}_+}$, where $\mathcal{F}_{t+}$ is defined by $\mathcal{F}^M_{t+} := \bigcap_{u>t}\sigma\big(\{M_s, s \le u\}\big)$ for every $t \in \mathbb{R}_+$.

Definition I.3. The probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with a filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in\mathbb{R}_+}$ will be referred to as the $\mathbb{G}-$*stochastic basis* and it will be denoted by $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$. When it is clear to which filtration we refer, we will simply write stochastic basis, *i.e.* we will omit the symbol of the filtration from the notation.

■ The probability space $(\Omega, \mathcal{G}, \mathbb{P})$ is endowed with an arbitrary filtration $\mathbb{G}$ for the rest of the chapter.

[4] In view of (i) the $\sigma-$algebra $\mathcal{G}_t$ contains all $\mathbb{P}-$negligible sets of $\mathcal{G}$, for every $t > 0$.

Definition I.4. Let M be an $E-$valued stochastic process.

(i) We will say that the process M is *adapted to the filtration* $\mathbb{G}$ if M_t is $\mathcal{G}_t-$measurable for every $t \in \mathbb{R}_+$. We will also use indifferently the term M *is* $\mathbb{G}-$*adapted*.

(ii) An $E-$valued process M is a $\mathbb{G}-$*martingale* (resp. $\mathbb{G}-$*submartingale*, $\mathbb{G}-$*supermartingale*) on the $\mathbb{G}-$stochastic basis if $M_t \in \mathbb{L}^1(\mathcal{G}_t; E)$[5] for every $t \in \mathbb{R}_+$ and

$$\mathbb{E}[M_t|\mathcal{G}_s] = M_s \text{ (resp. } \mathbb{E}[M_t|\mathcal{G}_s] \geq M_s,\ \mathbb{E}[M_t|\mathcal{G}_s] \leq M_s) \text{ for every } 0 \leq s \leq t.$$

(iii) We will say that the $E-$valued process M is *non-decreasing* (resp. *non-increasing*, *increasing*, *decreasing*) if $M_0 = 0\ \mathbb{P} - a.s.$ and its paths are càdlàg and non-decreasing, (resp. non-increasing, increasing, decreasing) $\mathbb{P}-$almost surely. In this case $\lim_{t\uparrow\infty} M_t$ exists $\mathbb{P} - a.s.$ and therefore we define $M_\infty(\omega) := \lim_{t\uparrow\infty} M_t(\omega)$ for every ω for which the limit is well-defined.

(iv) We will say that the $E-$valued process M is *of finite variation* if $M_0 = 0\ \mathbb{P} - a.s.$ and its paths are càdlàg with finite variation over each compact interval of $\mathbb{R}_+$ $\mathbb{P}-$almost surely. In this case, the *variation process* of M is also an $E-$valued process defined as $\mathrm{Var}(M)_t^{ij}(\omega) := \mathrm{Var}(M^{ij}(\omega))_t$, for every $1 \leq i \leq p$, $1 \leq j \leq q$ and denoted by $\mathrm{Var}(M) := \big(\mathrm{Var}(M)_t\big)_{t\in\mathbb{R}_+}$.

Definition I.5. Let X, Y be real-valued $\mathbb{G}-$martingales.

(i) If their product XY is a $\mathbb{G}-$martingale, then X and Y will be called (*mutually*) $\mathbb{G}-$*orthogonal* and this will be denoted by $X \perp\!\!\!\perp Y$.

(ii) The $\mathbb{G}-$martingale X will be called *purely discontinuous* $\mathbb{G}-$*martingale* if $X \perp\!\!\!\perp M$ for every continuous $\mathbb{G}-$martingale M.

Theorem I.6. *Any $E-$valued $\mathbb{G}-$martingale X admits a unique, up to indistinguishability, decomposition*

$$X = X_0 + X^c + X^d,$$

where $X_0^c = X_0^d = 0$, X^c is a continuous $\mathbb{G}-$martingale and X^d is a purely discontinuous $\mathbb{G}-$martingale.

Proof. Apply Jacod and Shiryaev [41, Theorem I.4.18] element-wise. □

Definition I.7. Let X be a $\mathbb{G}-$martingale. The unique continuous $\mathbb{G}-$martingale X^c, resp. purely discontinuous $\mathbb{G}-$martingale X^d, associated to X by Theorem I.6 will be called the *continuous part of* X, resp. the *purely discontinuous part of* X. The pair (X^c, X^d) will be called the *natural pair of* X *under* $\mathbb{G}$.

Given the new terms introduced in the previous theorem, it is a good point to introduce some further notation.

Notation I.8. Let $(X^k)_{k\in\overline{\mathbb{N}}}$ be a sequence of $\mathbb{R}^p-$valued martingales. Then, for $k \in \overline{\mathbb{N}}$, $1 \leq i \leq p$

- $X^{k,c,i}$ denotes the $i-$element of the continuous part of X^k.
- $X^{k,d,i}$ denotes the $i-$element of the purely discontinuous part of X^k.

Corollary I.9. *Let X and Y be two purely discontinuous $\mathbb{G}-$martingales having the same jumps,* i.e. *$\Delta X = \Delta Y$ up to indistinguishability. Then X and Y are indistinguishable.*

Proof. Apply Theorem I.6 to $X - Y$. □

We proceed with the definition of two $\sigma-$algebrae on $\Omega \times \mathbb{R}_+$ which are of utmost importance.

Definition I.10. (i) The $\mathbb{G}-$*optional* $\sigma-$*algebra* is the $\sigma-$algebra $\mathcal{O}^{\mathbb{G}}$ on $\Omega \times \mathbb{R}_+$ which is generated by all càdlàg and $\mathbb{G}-$adapted processes considered as mappings on $\Omega \times \mathbb{R}_+$. A process which is $\mathcal{O}^{\mathbb{G}}-$measurable will be called $\mathbb{G}-$optional process.

(ii) The $\mathbb{G}-$*predictable* $\sigma-$*algebra* is the $\sigma-$algebra $\mathcal{P}^{\mathbb{G}}$ on $\Omega \times \mathbb{R}_+$ which is generated by all left-continuous and $\mathbb{G}-$adapted processes considered as mappings on $\Omega \times \mathbb{R}_+$. A $\mathcal{P}^{\mathbb{G}}-$measurable process will be called $\mathbb{G}-$predictable process.

Notation I.11. Now we can provide part of the notation for the spaces we are going to use as well as some further convenient notation.

[5] Observe that the integrability condition implies that M is $\mathbb{G}-$adapted.

• $\mathcal{M}(\mathbb{G};E)$ will denote the space of all *uniformly integrable* $\mathbb{G}-$*martingales*,[6] *i.e.* the space of all $\mathbb{G}-$martingales X such that the family $(\|X_t\|_1)_{t\in\mathbb{R}_+}$ is uniformly integrable; see Definition I.24 for the notion of uniform integrability.

• $\mathcal{M}^c(\mathbb{G};E) := \{X \in \mathcal{M}(\mathbb{G};E), X \text{ is continuous}\}$ and

• $\mathcal{M}^d(\mathbb{G};E) := \{X \in \mathcal{M}(\mathbb{G};E), X^{ij} \text{ is purely discontinuous for } 1 \le i \le p, 1 \le j \le q\}$.

• $\mathcal{H}^2(\mathbb{G};E)$ will denote the space of *square integrable* $\mathbb{G}-$*martingales*, *i.e.* the space of $\mathbb{G}-$martingales X such that $\sup_{t\in\mathbb{R}_+} \mathbb{E}\big[\|X_t\|_2^2\big] < \infty$.

• $\mathcal{H}^{2,c}(\mathbb{G};E) := \mathcal{H}^2(\mathbb{G};E) \cap \mathcal{M}^c(\mathbb{G};E)$.[7]

• $\mathcal{H}^{2,d}(\mathbb{G};E) := \mathcal{H}^2(\mathbb{G};E) \cap \mathcal{M}^d(\mathbb{G};E)$.

• $\mathcal{V}^+(\mathbb{G};E)$ will denote the space of all $E-$valued, càdlàg, $\mathbb{G}-$adapted and non-decreasing processes.

• $\mathcal{V}(\mathbb{G};E)$ will denote the space of all $E-$valued, càdlàg, $\mathbb{G}-$adapted and of finite variation processes.

• $\mathcal{V}^+_{\text{pred}}(\mathbb{G};E) := \{V \in \mathcal{V}^+, V \text{ is } \mathbb{G}-\text{predictable}\}$

• $\mathcal{V}_{\text{pred}}(\mathbb{G};E) := \{V \in \mathcal{V}, V \text{ is } \mathbb{G}-\text{predictable}\}$.

• $\mathcal{A}^+(\mathbb{G};E) := \{A \in \mathcal{V}^+, A_\infty := \lim_{t\uparrow\infty} A_t \in \mathbb{L}^1(\mathcal{G};E)\}$.

• $\mathcal{A}(\mathbb{G};E) := \{A \in \mathcal{V}, \text{Var}(A)_\infty := \lim_{t\uparrow\infty} \text{Var}(A)_t \in \mathbb{L}^1(\mathcal{G};E)\}$.

• $\mathcal{A}^+_{\text{pred}}(\mathbb{G};E) := \mathcal{V}^+_{\text{pred}}(\mathbb{G};E) \cap \mathcal{A}^+(\mathbb{G};E)$.

• $\mathcal{A}_{\text{pred}}(\mathbb{G};E) := \mathcal{V}_{\text{pred}}(\mathbb{G};E) \cap \mathcal{A}(\mathbb{G};E)$.

• $\mathcal{S}_{\text{sp}}(\mathbb{G};E)$ will denote the space of all $\mathbb{G}-$adapted processes S which can be written in the form $S = S_0 + X + A$, where S_0 is E-valued and $\mathcal{G}_0-$measurable, $X \in \mathcal{M}(\mathbb{G};E)$ and $A \in \mathcal{V}_{\text{pred}}(\mathbb{G};E)$. The elements of $\mathcal{S}_{\text{sp}}(\mathbb{G};E)$ will be said $\mathbb{G}-$*special semimartingales*.

• If $\mathbb{R}_+ \ni t \overset{f}{\longmapsto} f(t) \in E$ is a function, we will abuse notation and we will denote in the same way the process $\Omega \times \mathbb{R}_+ \ni (\omega, t) \overset{f}{\longmapsto} f(t) \in E$.

• Given two filtrations $\mathbb{F} := (\mathcal{F}_t)_{t\in\mathbb{R}_+}$, $\mathbb{H} := (\mathcal{H}_t)_{t\in\mathbb{R}_+}$, we will say that $\mathbb{F}$ *is a sub-filtration of* $\mathbb{H}$ if for every $t \in \mathbb{R}_+$ holds $\mathcal{F}_t \subset \mathcal{H}_t$.

Definition I.12. The elements of the space $\mathcal{S}_{\text{sp}}(\mathbb{G};E)$ will be called $\mathbb{G}-$*special semimartingales*. The (unique up to indistinguishability) decomposition $S = S_0 + X + A$ such that $X \in \mathcal{M}(\mathbb{G};E)$ and $A \in \mathcal{V}_{\text{pred}}(\mathbb{G};E)$ is called the $\mathbb{G}-$*canonical decomposition of* S.

Definition I.13. (i) The mapping $\tau : \Omega \to \overline{\mathbb{R}}_+$ is called $\mathbb{G}-$*stopping time* if $\{\tau \le t\} \in \mathcal{G}_t$ for every $t \in \mathbb{R}_+$.

(ii) Let X be a $\mathbb{G}-$optional process and τ be a $\mathbb{G}-$stopping time. The process X^τ is defined by $X^\tau_t := X_{\tau\wedge t}$ and will be said to be the *stopped process* X *at time* τ.

(iii) Let τ be a $\mathbb{G}-$stopping time. We define $\mathcal{G}_\tau := \big\{A \in \mathcal{G}, A \cap \{\tau \le t\} \in \mathcal{G}_t \text{ for all } t \in \mathbb{R}_+\big\}$. Moreover, $\mathcal{G}_{\tau-} := \sigma\big(\mathcal{G}_0 \cup \big\{A \cap \{\tau < t\}, t \in \mathbb{R}_+ \text{ and } A \in \mathcal{G}_t\big\}\big)$.

(iv) Let τ_1, τ_2 be $\mathbb{G}-$stopping times. We define the *stochastic interval* by

$$]\!]\tau_1, \tau_2]\!] := \{(\omega, t) \in \Omega \times \mathbb{R}_+, \tau_1(\omega) < t \le \tau_2(\omega)\}$$

and analogously for $[\![\tau_1, \tau_2]\!], [\![\tau_1, \tau_2[\![,]\!]\tau_1, \tau_2[\![$. In particular, we will denote $[\![\tau_1, \tau_1]\!]$ by $[\![\tau_1]\!]$ and we will refer to it as the *graph of the stopping time* τ_1.

(v) Let ρ be a $\mathbb{G}-$stopping time. If $[\![0, \rho[\![\in \mathcal{P}^{\mathbb{G}}$, then ρ will be called $\mathbb{G}-$*predictable* (*stopping*) *time*.

(vi) Let σ be a $\mathbb{G}-$stopping time. If there exists a sequence $(\rho_k)_{k\in\mathbb{N}}$ of $\mathbb{G}-$predictable times such that $[\![\sigma]\!] \subset \cup_{k\in\mathbb{N}}[\![\rho_k]\!]$, then σ will be called $\mathbb{G}-$*accessible* (*stopping*) *time*.

(vii) Let τ be a $\mathbb{G}-$stopping time. If $\mathbb{P}([\tau = \rho] \cap [\tau < \infty]) = 0$ for every $\mathbb{G}-$predictable time ρ, then τ will be called $\mathbb{G}-$*totally inaccessible* (*stopping*) *time*.

Theorem I.14. *The* $\mathbb{G}-$*predictable* $\sigma-$*algebra is generated by any one of the following collections of random sets:*

(i) $A \times \{0\}$ *for* $A \in \mathcal{G}_0$, *and* $[\![0, \tau]\!]$ *where* τ *is a* $\mathbb{G}-$*stopping time.*

(ii) $A \times \{0\}$ *for* $A \in \mathcal{G}_0$, *and* $A \times (t, u]$ *where* $t < u$, $A \in \mathcal{G}_t$.

Definition I.15. (i) The filtration $\mathbb{G}$ is said to be *quasi-left-continuous* if each $\mathbb{G}-$accessible time is a $\mathbb{G}-$predictable time.

[6] In view of Definition I.2 and Dellacherie and Meyer [27, Theorem VI.4, p. 69] for every element of $\mathcal{M}(\mathbb{G};E)$ we can choose a càdlàg and $\mathbb{G}-$adapted modification. Therefore, we can assume, and we will do so, that all the elements of the space $\mathcal{M}(\mathbb{G};E)$ are càdlàg. The interested reader should consult [27, Section 1 of Chapter VI] for the general discussion regarding càdlàg modifications of supermartingales.

[7] It is a consequence of Theorem I.27 that $\mathcal{H}^2(\mathbb{G},;E) \subset \mathcal{M}(\mathbb{G};E)$.

(ii) A càdlàg $\mathbb{G}-$adapted process M is called $\mathbb{G}-$*quasi-left-continuous* if $\Delta M_\rho \mathbb{1}_{[\rho<\infty]} = 0$ $\mathbb{P}-a.s.$ for every $\mathbb{G}-$predictable time ρ.

The following characterisations will be useful.

Theorem I.16. *The filtration $\mathbb{G}$ is quasi-left-continuous if and only if every element of $\mathcal{M}(\mathbb{G};\mathbb{R})$ is $\mathbb{G}-$quasi-left-continuous.*

Proof. See He et al. [35, Theorem 5.36]. We inform the reader that our definition and [35, Definition 3.39] differ, but [35, Theorem 3.40] proves that they are equivalent. □

Definition I.17. A random set A is called *thin* if there exists a sequence of $\mathbb{G}-$stopping times $(\tau^k)_{k\in\mathbb{N}}$ such that $A = \cup_{k\in\mathbb{N}}[\![\tau^k]\!]$. If, moreover, the sequence $(\tau^k)_{k\in\mathbb{N}}$ satisfies $[\![\tau^m]\!]\cap[\![\tau^n]\!] = \emptyset$ for all $m \neq n$, then it will be called an *exhausting sequence* for the thin set A.

Proposition I.18. *If X is a càdlàg and $\mathbb{G}-$adapted process, the random set $[\Delta X \neq 0]$ is thin. An exhausting sequence for the set $[\Delta X \neq 0]$ is called a sequence that exhausts the jumps of X.*

Proof. See [41, Proposition I.1.32]. □

Proposition I.19. *Let X be a càdlàg and $\mathbb{G}-$adapted process. Then, the following are equivalent:*

(i) *X is $\mathbb{G}-$quasi-left-continuous.*

(ii) *There exists an exhausting sequence of $\mathbb{G}-$totally inaccessible stopping times that exhausts the jumps of X.*

Proof. See Jacod and Shiryaev [41, Proposition I.2.26]. □

We close this section with the following theorems.

Theorem I.20 (Existence of the $\mathbb{G}-$optional projection)**.** *Let M be a $\mathcal{G}\otimes\mathcal{B}(\mathbb{R}_+)-$measurable $E-$valued process such that for every $\mathbb{G}-$stopping time τ holds $\mathbb{E}\big[\|X_\tau\|_1 \mathbb{1}_{[\tau<\infty]}\big] < \infty$. Then there exists a unique $E-$valued $\mathbb{G}-$optional process, denoted by $\Pi_o^{\mathbb{G}}(M)$, such that for every $\mathbb{G}-$stopping time τ we have*

$$\mathbb{E}[M_\tau \mathbb{1}_{[\tau<\infty]}|\mathcal{G}_\tau] = \Pi_o^{\mathbb{G}}(M)_\tau \mathbb{1}_{[\tau<\infty]} \ \mathbb{P}-a.s..$$

*The process $\Pi_o^{\mathbb{G}}(M)$ will be called the $\mathbb{G}-$*optional projection *of M.*

Proof. See He et al. [35, Theorem 5.1]. □

Theorem I.21 (Existence of the $\mathbb{G}-$predictable projection)**.** *Let M be a $\mathcal{G}\otimes\mathcal{B}(\mathbb{R}_+)-$measurable $E-$valued process such that for every $\mathbb{G}-$predictable time ρ holds $\mathbb{E}\big[\|X_\rho\|_1 \mathbb{1}_{[\rho<\infty]}\big] < \infty$. Then there exists a unique $E-$valued $\mathbb{G}-$predictable process, denoted by $\Pi_p^{\mathbb{G}}(M)$, such that for every $\mathbb{G}-$predictable time ρ we have*

$$\mathbb{E}[M_\rho \mathbb{1}_{[\rho<\infty]}|\mathcal{G}_\rho] = \Pi_p^{\mathbb{G}}(M)_\rho \mathbb{1}_{[\rho<\infty]} \ \mathbb{P}-a.s..$$

*The process $\Pi_p^{\mathbb{G}}(M)$ will be called the $\mathbb{G}-$*predictable projection *of M.*

Proof. See He et al. [35, Theorem 5.2]. □

Theorem I.22. *Let M be a $\mathcal{G}\otimes\mathcal{B}(\mathbb{R}_+)-$measurable $E-$valued process and X be a $\mathbb{G}-$optional, resp. $\mathbb{G}-$predictable process. If the $\mathbb{G}-$optional projection, resp. $\mathbb{G}-$predictable projection, of M exists, then the $\mathbb{G}-$optional projection, resp. $\mathbb{G}-$predictable projection, of XM exists. In this case,*

$$\Pi_o^{\mathbb{G}}(XM) = X\,\Pi_o^{\mathbb{G}}(M), \ \textit{resp.} \ \Pi_p^{\mathbb{G}}(XM) = X\,\Pi_p^{\mathbb{G}}(M).$$

The following result provides a convenient property of the optional projection of a process. Recall that its wellposedness is verified by Theorem I.20. The reader may anticipate to the Subsection I.4.1 for the Lebesgue–Stieltjes integral of a process A.

Lemma I.23. *Let $A \in \mathcal{V}(\mathbb{G};\mathbb{R})$ and Y be a uniformly integrable and measurable process. Then for every $\mathbb{G}-$stopping time holds*

$$\mathbb{E}\bigg[\int_{(0,\tau]} Y_s \,\mathrm{d}A_s\bigg] = \mathbb{E}\bigg[\int_{(0,\tau]} \Pi_o^{\mathbb{G}}(Y)_s \,\mathrm{d}A_s\bigg]. \tag{I.1}$$

Proof. See Cohen and Elliott [20, Theorem 8.1.21]. □

I.2. Uniform integrability

We devote this section to the central concept of uniform integrability, which lies behind many important results of stochastic analysis. As we are mainly interested in the limit behaviour of sequences of martingales, we need to mention that it is the uniform integrability of the sequence which guarantees that the limit will remain a martingale. The precise statement is given at Proposition I.147. This classical result is going to be one of the cornerstones in Chapter III. Now we proceed to present the required machinery for proving that a family of random variables is uniformly integrable, whose main tool will be the class of moderate Young functions.

Definition I.24. Let $\mathcal{U} \subset \mathbb{L}^1(\mathcal{G};\mathbb{R})$. The set $\mathcal{U}$ is said to be a *uniformly integrable* subset of $\mathbb{L}^1(\mathcal{G};\mathbb{R})$ if

$$\lim_{c\to\infty} \sup_{\xi\in\mathcal{U}} \mathbb{E}[|\xi|\mathbb{1}_{[|\xi|>c]}] = 0.$$

Theorem I.25. *The set $\mathcal{U} \subset \mathbb{L}^1(\mathcal{G};\mathbb{R})$ is uniformly integrable if and only if the following conditions are both satisfied*

(i) $\sup_{\xi\in\mathcal{U}} \mathbb{E}[|\xi|] < \infty$ *and*
(ii) *for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, $\sup_{\xi\in\mathcal{U}} \mathbb{E}[|\xi|\mathbb{1}_A] < \varepsilon$ for all $A \in \mathcal{G}$ with $\mathbb{P}(A) < \delta$.*

Proof. See Dellacherie and Meyer [26, Theorem II.19, p. 22] or Cohen and Elliott [20, Theorem 2.5.4] or He et al. [35, Theorem 1.9]. □

Theorem I.26 (Dunford–Pettis Compactness Criterion)**.** *Let $\mathcal{U} \subset \mathbb{L}^1(\mathcal{G};\mathbb{R})$. Then the following are equivalent:*

(i) *$\mathcal{U}$ is uniformly integrable.*
(ii) *$\mathcal{U}$ is relatively compact in $\mathbb{L}^1(\mathcal{G};\mathbb{R})$ endowed with the weak*[8] *topology.*
(iii) *$\mathcal{U}$ is relatively weakly sequentially compact,* i.e. *every sequence $(\xi^k)_{k\in\mathbb{N}} \subset \mathcal{U}$ contains a subsequence $(\xi^{k_l})_{l\in\mathbb{N}}$ which converges weakly.*

Proof. See Dellacherie and Meyer [26, Theorem II.25, p. 27]. □

Theorem I.27 (de La Vallée Poussin)**.** *Let $\mathcal{U} \subset \mathbb{L}^1(\mathcal{G};\mathbb{R})$. Then $\mathcal{U}$ is uniformly integrable if and only if there exists a function $\Upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\lim_{x\to\infty} \frac{\Upsilon(x)}{x} = \infty \text{ and } \sup_{\xi\in\mathcal{U}} \mathbb{E}[\Upsilon(|\xi|)] < \infty.$$

Proof. See Dellacherie and Meyer [26, Theorem II.22, p. 24]. □

Remark I.28. Let $\mathcal{U} \subset \mathbb{L}^1(\mathcal{G})$ be a uniformly integrable set and Υ be the function associated to $\mathcal{U}$ by Theorem I.27. In Subsection I.2.2, more precisely in Corollary I.48, we will see that we can choose the function Υ to be additionally convex and moderate; see Notation I.41 for the latter. We will refer to Corollary I.48 as the *de La Vallée Poussin–Meyer criterion.*

In the following corollary we collect some convenient results regarding uniform integrability.

Corollary I.29. (i) *Let $(\xi_i)_{i\in I} \subset \mathbb{L}^1(\mathcal{G};\mathbb{R})$ be a uniformly integrable family and $\alpha \in \mathbb{R}$. Then $(\alpha\xi_i)_{i\in I}$ is uniformly integrable.*
(ii) *Let $(\xi_i)_{i\in I}, (\zeta_i)_{i\in I} \subset \mathbb{L}^1(\mathcal{G};\mathbb{R})$ with $|\zeta_i| \le |\xi_i|$ for every $i \in I$. If $(\xi_i)_{i\in I}$ is uniformly integrable, then so it is $(\zeta_i)_{i\in I}$.*
(iii) *Let $\xi \in \mathbb{L}^1(\mathcal{G};\mathbb{R})$ and $(\mathcal{F}_i)_{i\in I}$ be a family of sub$-\sigma-$algebrae of $\mathcal{G}$. Then the family $\big(\mathbb{E}[\xi|\mathcal{F}_i]\big)_{i\in I}$ is uniformly integrable.*
(iv) *Let $(\xi_i^k)_{i\in I} \subset \mathbb{L}^1(\mathcal{G};\mathbb{R})$ be uniformly integrable for $k = 1,\dots,p$. Then the family $\big(\sum_{k=1}^p \xi_i^k\big)_{i\in I}$ is uniformly integrable.*
(v) *Let $\mathcal{A}^k := (\xi_i^k)_{i\in I^k} \subset \mathbb{L}^1(\mathcal{G};\mathbb{R})$ be uniformly integrable for $k = 1,\dots,p$. Then the family $\cup_{k=1}^p \mathcal{A}^k$ is uniformly integrable.*

Proof. (i) It follows immediately by the definition.
(ii) It follows immediately by the definition.
(iii) See He et al. [35, Theorem 1.8].

[8] We use the probabilistic convention for the term weak. In functional analysis this is the weak*-topology, which is also denoted by $\sigma(\mathbb{L}^1, \mathbb{L}^\infty)$.

(iv) It is immediate from Theorem I.25. However, we provide the proof for the convenience of the reader.

For the condition (i) of the aforementioned theorem we have

$$\sup_{i\in I}\mathbb{E}\Big[\Big|\sum_{k=1}^{p}\xi_i^k\Big|\Big]\leq\sup_{i\in I}\mathbb{E}\Big[\sum_{k=1}^{p}|\xi_i^k|\Big]\leq\sum_{k=1}^{p}\sup_{i\in I}\mathbb{E}\big[|\xi_i^k|\big]<\infty,$$

where we have used that for every $k\in\{1,\dots,p\}$ the family $(\xi_i^k)_{i\in I}$ satisfies Theorem I.25, therefore $\sup_{i\in I}\mathbb{E}\big[|\xi_i^k|\big]<\infty$.

We proceed to prove condition (ii) of Theorem I.25. To this end, let $\varepsilon>0$. Then, for every $k\in\{1,\dots,p\}$, since the family $(\xi_i^k)_{i\in I}$ is uniformly integrable, there exists a $\delta^k(\varepsilon,p)>0$ such that $\sup_{i\in I}\mathbb{E}\big[|\xi_i^k|\mathbb{1}_A\big]<\frac{\varepsilon}{p}$ for all $A\in\mathcal{G}$ with $\mathbb{P}(A)<\delta^k(\varepsilon,p)$. Define $\delta(\varepsilon,p):=\min\{\delta^k(\varepsilon,p),k=1,\dots,p\}$. Then it is immediate that

$$\sup_{i\in I}\mathbb{E}\Big[\Big|\sum_{k=1}^{p}\xi_i^k\Big|\mathbb{1}_A\Big]\leq\sup_{i\in I}\mathbb{E}\Big[\sum_{k=1}^{p}|\xi_i^k|\mathbb{1}_A\Big]\leq\sum_{k=1}^{p}\sup_{i\in I}\mathbb{E}\big[|\xi_i^k|\mathbb{1}_A\big]<\sum_{k=1}^{p}\frac{\varepsilon}{p}=\varepsilon.$$

(v) One can argue as in the proof of (iv) by choosing the smallest δ among the δ's which are obtained by Theorem I.25. Alternatively, by the Dunford–Pettis compactness criterion and using the classical argument for proving that the finite union of sequentially compact sets is sequentially compact.

□

The following definition provides the analogous of Definition I.24 for the multi-dimensional case.

Definition I.30. Let $\mathcal{U}\subset\mathbb{L}^1(\mathcal{G};E)$, where $E=\prod_{j=1}^{p}E_i$. The set $\mathcal{U}$ is said to be *uniformly integrable* if the family $\big\{\|\xi\|_1,\xi\in\mathcal{U}\big\}$ is uniformly integrable.

The following lemma is nothing more that Corollary I.29.(ii),(iv).

Lemma I.31. *Let $E':=\prod_{j=1}^{p}E_i$ and $T:(E',\|\cdot\|_1)\to(\mathbb{R}^{\dim(E')},\|\cdot\|_1)$ be an isometry that sorts the elements of E in a row. Then, $\mathcal{U}\subset\mathbb{L}^1(\mathcal{G};E)$ is uniformly integrable if and only if $\big(T^i(\xi)\big)_{\xi\in\mathcal{U}}$ is uniformly integrable for every $i\in\{1,\dots,\dim(E')\}$, where $T^i(\xi)$ denotes the $i-$element of the vector $T(\xi)$.*

Proof. Let $\mathcal{U}\subset\mathbb{L}^1(\mathcal{G};E)$ be uniformly integrable, *i.e.* $\big\{\|\xi\|_1,\xi\in\mathcal{U}\big\}$ is uniformly integrable. Then, $|T^i(\xi)|\leq\|\xi\|_1$, for every $i\in\{1,\dots,\dim(E')\}$ and by Corollary I.29.(ii) we can conclude.

Conversely, let $\mathcal{U}\subset\mathbb{L}^1(\mathcal{G};E)$ be a family such that $\big(T^i(\xi)\big)_{\xi\in\mathcal{U}}$ is uniformly integrable for every $i\in\{1,\dots,\dim(E')\}$. Then, $\|\xi\|_1=\sum_{i=1}^{\dim(E')}|T^i(\xi)|$. Therefore, we can conclude by Corollary I.29.(iv). □

Theorem I.32 (Vitali's Convergence Theorem)**.** *Let $(\xi^k)_{k\in\mathbb{N}}\subset\mathbb{L}^1(\mathcal{G};E)$ be a sequence of $E-$valued random variables and ξ^∞ be an $E-$valued random variable such that*

$$\mathbb{P}\Big(\big[\delta_{\|\cdot\|_1}(\xi^k,\xi^\infty)>\varepsilon\big]\Big)\longrightarrow 0 \text{ for every } \varepsilon>0.$$

The following are equivalent:

(i) *The convergence $\xi^k\xrightarrow{\mathbb{L}^1(\mathcal{G};E)}\xi^\infty$.*
(ii) *The sequence $(\xi^k)_{k\in\mathbb{N}}$ is uniformly integrable.*

In either case, $\xi^\infty\in\mathbb{L}^1(\mathcal{G};E)$. Moreover, the following are equivalent

(i) *The convergence $\xi^k\xrightarrow{\mathbb{L}^2(\mathcal{G};E)}\xi^\infty$ holds.*
(ii) *The sequence $\big(\|\xi^k\|_2^2\big)_{k\in\mathbb{N}}$ is uniformly integrable.*

Proof. See Leadbetter, Cambanis, and Pipiras [45, Theorem 11.4.2]. □

I.2.1. Generalised Inverses. The concept of generalised inverse of an increasing function or process appears often in the literature either as the right derivative of the Young conjugate of a Young function, see Definition I.40.(i), or as time change, see *e.g.* [60, Lemma 0.4.8, Proposition 0.4.9]. We are interested in both cases.

Definition I.33. Let $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function. The *right-continuous generalised inverse of* χ is the function $\chi^{-1,r} : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ defined by

$$\chi^{-1,r}(s) := \begin{cases} \inf\{t \in \mathbb{R}_+, \chi(t) > s\}, & \text{if } \{t \in \mathbb{R}_+, \chi(t) > s\} \neq \emptyset, \\ \infty, & \text{if } \{t \in \mathbb{R}_+, \chi(t) > s\} = \emptyset \end{cases}$$

and the *left-continuous generalised inverse of* χ is the function $\chi^{-1,l} : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ defined by

$$\chi^{-1,l}(s) := \begin{cases} \inf\{t \in \mathbb{R}_+, \chi(t) \geq s\}, & \text{if } \{t \in \mathbb{R}_+, \chi(t) \geq s\} \neq \emptyset, \\ \infty, & \text{if } \{t \in \mathbb{R}_+, \chi(t) \geq s\} = \emptyset. \end{cases}$$

It is well-known, *e.g.* see Revuz and Yor [60, Lemma 0.4.8, p. 7], that the right-continuous generalised inverse of a non-decreasing and right-continuous function is also non-decreasing and right-continuous, which justifies the used term. In the following lemma the new result that it is provided by the author is the right inequality of (vii). Using this simple inequality we will be able to establish Theorem II.14. However, we present the other properties, since we will make use of them in the proof or later in Subsection I.2.2. The interested reader can find their proofs in a slightly more general framework in Embrechts and Hofert [32] and the references therein.

Lemma I.34. *Let $\chi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a càdlàg and increasing function with $\chi_0 = 0$.*

(i) *$\chi^{-1,l}$ is càglàd and increasing, while $\chi^{-1,r}$ is càdlàg and increasing.*
(ii) *$\chi^{-1,l}(s) = \chi^{-1,r}(s-)$ and $\chi^{-1,l}(s+) = \chi^{-1,r}(s)$.*
(iii) *$s \leq \chi(t)$ if and only if $\chi^{-1,l}(s) \leq t$ and $s < \chi(t)$ if and only if $\chi^{-1,r}(s) < t$.*
(iv) *$\chi(t) < s$ if and only if $t < \chi^{-1,l}(s)$ and $\chi(t) \leq s$ if and only if $t \leq \chi^{-1,r}(s)$.*
(v) *$\chi\left(\chi^{-1,r}(s)\right) \geq \chi\left(\chi^{-1,l}(s)\right) \geq s$, for $s \in \mathbb{R}_+$, and at most one of the inequalities can be strict.*
(vi) *For $s \in \chi(\mathbb{R}_+) := \{s \in \mathbb{R}_+, \exists t \in \mathbb{R}_+ \text{ such that } \chi(t) = s\}$, $\chi(\chi^{-1,l}(s)) = s$.*
(vii) *For s such that $\chi^{-1,l}(s) < \infty$, we have*

$$s \leq \chi\left(\chi^{-1,l}(s)\right) \leq s + \Delta\chi(\chi^{-1,l}(s)),$$

where $\Delta\chi(\chi^{-1,l}(s))$ is the jump of the function χ at the point $\chi^{-1,l}(s)$.

Proof. We need to prove that $\chi(\chi^{-1,l}(s)) - s \leq \Delta\chi(\chi^{-1,l}(s))$ for any s such that $\chi^{-1,l}(s) < \infty$. By (vi), when $s \in \chi(\mathbb{R}_+)$, we have since χ is increasing

$$\chi(\chi^{-1,l}(s)) - s = 0 \leq \Delta\chi(\chi^{-1,l}(s)).$$

Now if $s \notin \chi(\mathbb{R}_+)$ and $s > \chi_\infty := \lim_{t\to\infty} \chi(t)$, then $\chi^{-1,l}(s) = \infty$, so that this case is automatically excluded. Therefore, we now assume that $s \notin \chi(\mathbb{R}_+)$ and $s \leq \chi_\infty$. Since $s \notin \chi(\mathbb{R}_+)$, there exists some $t \in \mathbb{R}_+$ such that $s \in [\chi(t-), \chi(t))$. Then, we immediately have $\chi^{-1,l}(s) = t$. Hence

$$s + \Delta\chi(\chi^{-1,l}(s)) = s + \Delta\chi(t) \geq \chi(t) = \chi(\chi^{-1,l}(s)),$$

since $s \geq \chi(t-)$. □

Lemma I.35. *Let g be a non-decreasing sub-multiplicative function on $\mathbb{R}_+$, that is to say*

$$g(x + y) \leq \gamma g(x)g(y),$$

for some $\gamma > 0$ and for every $x, y \in \mathbb{R}_+$. Let A be a càdlàg and non-decreasing function with associated Borel measure μ_A. Then it holds that

$$\int_{(0,t]} g(A_s)\, \mu_A(\mathrm{d}s) \leq \gamma g\left(\max_{\{s,\, A_s^{-1,l} < \infty\}} \Delta A_{L_s}\right) \int_{(A_0, A_t]} g(s)\, \mathrm{d}s.$$

Corollary I.36. *Let $A \in \mathcal{V}^+(\mathbb{G};\mathbb{R})$ and g as in Lemma I.35 with the additional assumption that A has uniformly bounded jumps, say by K. Then there exists a universal constant $K' > 0$ such that*

$$\int_{(0,t]} g(A_s(\omega))\, \mu_{A(\omega)}(\mathrm{d}s) \leq K' \int_{(A_0(\omega), A_t(\omega)]} g(s)\, \mathrm{d}s \ \mathbb{P} - a.s..$$

The constant K' equals $\gamma g(K)$, where γ is the sub-multiplicativity constant of g.

I.2.2. Moderate Young Functions. The main aim of this subjection is to present a brief overview to moderate Young functions; see Definition I.37 and Notation I.41. The importance of this sub-class of Young functions is that the Burkholder–Davis–Gundy Inequality, see Theorem I.101, is valid for them. Moreover, we can make use of the Doob's Maximal Inequality for a function Ψ whose Young conjugate is moderate; see Theorem I.104 and Definition I.40.(i).

Definition I.37. A function $\Upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ is called *Young function* if it is non-decreasing, convex and satisfies

$$\Upsilon(0) = 0 \text{ and } \lim_{x\to\infty} \frac{\Upsilon(x)}{x} = \infty.$$

Moreover, the non-negative, non-decreasing and right-continuous function $\upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ for which

$$\Upsilon(x) = \int_{[0,x]} \upsilon(z)\,\mathrm{d}z$$

will be called the *right derivative of* Υ.

Lemma I.38. *Let $\Upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ be a Young function, then its right derivative exists and is unbounded.*

Proof. The existence and the properties of the right derivative of Υ, called υ, is a well-known result of convex analysis, *e.g.* see Rockafellar [61, Theorem 23.1, Theorem 24.1, Corollary 24.2.1]. Assume, now, that υ is bounded by a positive constant C. Then

$$\sup_{x\in\mathbb{R}_+} \frac{\Upsilon(x)}{x} = \sup_{x\in\mathbb{R}_+} \frac{1}{x}\int_{[0,x]} \upsilon(z)\,\mathrm{d}z \le \sup_{x\in\mathbb{R}_+} \frac{Cx}{x} = C < \infty,$$

which contradicts to the property $\lim_{x\to\mathbb{R}_+} \frac{1}{x}\Upsilon(x) = \infty$. □

For our convenience, let us collect all the Young functions in a set.

Notation I.39. $\mathcal{YF} := \{\Upsilon : \mathbb{R}_+ \to \mathbb{R}_+, \Upsilon \text{ is a Young function}\}$.

In view of the previous lemma it is immediate that the right-continuous generalised inverse of υ is real-valued and unbounded. These are properties which justify the following definition.

Definition I.40. (i) The *Young conjugate operator* $^\star : \mathcal{YF} \longrightarrow \mathcal{YF}$ is defined by

$$\mathcal{YF} \ni \Upsilon \longmapsto \Upsilon^\star(\cdot) := \int_{[0,\cdot]} \upsilon^{-1,r}(z)\,\mathrm{d}z \in \mathcal{YF},$$

where υ is the right derivative of the Young function Υ and $\upsilon^{-1,r}$ is the right-continuous generalised inverse of υ; see Definition I.33.

(ii) The *conjugate index operator* $^\star : [1,\infty] \longrightarrow [1,\infty]$ is defined by

$$[1,\infty] \ni \vartheta \longmapsto \vartheta^\star = \left\{\begin{array}{ll} \infty, & \text{if } \vartheta = 1 \\ \frac{\vartheta}{\vartheta-1}, & \text{if } \vartheta \in (1,\infty) \\ 1, & \text{if } \vartheta = \infty \end{array}\right\} \in [1,\infty].$$

Notation I.41. To a Young function Υ with right derivative υ we associate the constants

$$\underline{c}_\Upsilon := \inf_{x>0} \frac{x\upsilon(x)}{\Upsilon(x)} \text{ and } \overline{c}_\Upsilon := \sup_{x>0} \frac{x\upsilon(x)}{\Upsilon(x)}.$$

Remark I.42. Let $\Upsilon \in \mathcal{YF}$ with right derivative υ. Then for every $x \in \mathbb{R}_+$

$$\Upsilon(x) = \int_{[0,x]} \upsilon(z)\,\mathrm{d}z \le x\upsilon(x), \text{ which implies } \frac{x\upsilon(x)}{\Upsilon(x)} \ge 1.$$

Therefore $\overline{c}_\Upsilon, \underline{c}_\Upsilon \ge 1$.

Definition I.43. A Young function Υ is said to be *moderate* if $\overline{c}_\Upsilon < \infty$. The set of all moderate Young functions will be denoted by $\mathcal{YF}_{\mathrm{mod}}$.

Lemma I.44. *Let $\Upsilon \in \mathcal{YF}$. Then*

(i) *For every $x, y \in \mathbb{R}_+$ holds $xy \le \Upsilon(x) + \Upsilon^\star(y)$. This inequality is called* Young's Inequality.
(ii) *$\Upsilon \in \mathcal{YF}_{\mathrm{mod}}$ if and only if $\Upsilon(\lambda x) \le \lambda^{\overline{c}_\Upsilon}\Upsilon(x)$ for every $x \in \mathbb{R}_+$ and for every $\lambda \ge 1$.*
(iii) *$\Upsilon \in \mathcal{YF}_{\mathrm{mod}}$ if and only if there exists $C_2 > 0$ such that $\Upsilon(2x) \le C_2\Upsilon(x)$ for every $x \in \mathbb{R}_+$.*

(iv) *If* $\bar{c}_\Upsilon < \infty$, *then the function* $\mathbb{R}_+ \ni x \mapsto \Upsilon(x)/x^{\bar{c}_\Upsilon}$ *is non-increasing. Inversely, given the constant* $\underline{c}_\Upsilon$, *the function* $\mathbb{R}_+ \ni x \mapsto \Upsilon(x)/x^{\underline{c}_\Upsilon}$ *is non-decreasing.*

(v) $(\underline{c}_\Upsilon)^\star = \bar{c}_{\Upsilon^\star}$ *and* $(\bar{c}_\Upsilon)^\star = \underline{c}_{\Upsilon^\star}$.

Proof. (i) This is a well-known result, see *e.g.* Long [48, Theorem 3.1.1 (a)].

(ii) See [48, Theorem 3.1.1 (c)-(d)].

(iii) In view of (ii) it is only left to prove the sufficient direction, *i.e.* assuming that there exists $C_2 > 0$ such that $\Upsilon(2x) \le C_2\Upsilon(x)$ for every $x \in \mathbb{R}_+$, we need to prove that Υ is moderate. For this see He et al. [35, Definition 10.32, Lemma 10.33.2)].

(iv) See [48, Theorem 3.1.1 (e)]

(v) See [48, Theorem 3.1.1 (f)]. □

The previous lemma provides us with an easy criterion for proving that the Young conjugate of a Young function is moderate.

Corollary I.45. *Let* $\Upsilon \in \mathcal{YF}$, *then* $\Upsilon^\star \in \mathcal{YF}_{\mathrm{mod}}$ *if and only if* $\underline{c}_\Upsilon > 1$.

Proof. By the definition of the conjugate indices we have that $\bar{c}_{\Upsilon^\star} < \infty$ if and only if $\underline{c}_\Upsilon > 1$. □

Example I.46. For $\vartheta \in (1,\infty)$ we define $\mathbb{R}_+ \ni x \overset{\Upsilon_\vartheta}{\longmapsto} \frac{1}{\vartheta}x^\vartheta \in \mathbb{R}_+$. Let us now fix a $\vartheta \in (1,\infty)$. Then the derivative of Υ_ϑ is $\mathbb{R}_+ \ni x \overset{\upsilon_\vartheta}{\longmapsto} x^{\vartheta-1} \in \mathbb{R}_+$ and we can easily calculate that $\bar{c}_{\Upsilon_\vartheta} = \underline{c}_{\Upsilon_\vartheta} = \vartheta$. Moreover, υ_ϑ is continuous and increasing, therefore its right-continuous generalised inverse is the usual inverse function. Then, we have directly that $\upsilon_\vartheta^{-1,r}(x) = x^{\frac{1}{\vartheta-1}}$, for every $x \in \mathbb{R}_+$. Consequently,

$$(\Upsilon_\vartheta)^\star(x) = \int_{[0,x]} \upsilon_\vartheta^{-1,r}(z)\,\mathrm{d}z = \int_{[0,x]} z^{\frac{1}{\vartheta-1}}\,\mathrm{d}z = \frac{\vartheta-1}{\vartheta}x^{\frac{\vartheta}{\vartheta-1}} = \frac{1}{\vartheta^\star}x^{\vartheta^\star} = \Upsilon_{\vartheta^\star}(x), \text{ for every } x \in \mathbb{R}_+.$$

Observe, moreover, that $\Upsilon_\vartheta \in \mathcal{YF}_{\mathrm{mod}}$ for every $\vartheta \in (1,\infty)$.

It is the next lemma that enables us to always choose a moderate Young function when we apply Theorem I.27.

Lemma I.47 (Meyer)**.** *Let* $\mathcal{U} \subset \mathbb{L}^1(\mathcal{G})$ *be a uniformly integrable set. Then there exists* $\Phi \in \mathcal{YF}_{\mathrm{mod}}$ *such that the set* $\{\Phi(|X|), X \in \mathcal{U}\}$ *is uniformly integrable.*

Proof. See Meyer [52, Lemme, p. 770]. □

Now we can provide provide the de La Vallée Poussin–Meyer criterion. We urge the reader to compare with Theorem I.27.

Corollary I.48 (de La Vallée Poussin–Meyer Criterion)**.** *A set* $\mathcal{U} \subset \mathbb{L}^1(\mathcal{G})$ *is uniformly integrable if and only if there exists* $\Phi \in \mathcal{YF}_{\mathrm{mod}}$ *such that* $\sup_{X\in\mathcal{U}} \mathbb{E}[\Phi_{\mathcal{A}}(|X|)] < \infty$.

Proof. The sufficient direction is immediate, since a moderate Young function satisfies the requirements of Theorem I.27.

For the necessary direction, by Lemma I.47 there exists a moderate sequence $\mathcal{A}$ such that the set $\{\Phi_{\mathcal{A}}(|X|), X \in \mathcal{U}\}$ is uniformly integrable. Then, by Theorem I.25.(i) we can conclude. □

Let us return for a while to Theorem I.27 and have a closer look to its proof; see Dellacherie and Meyer [26, Theorem II.22, p. 24]. Following the notation of [26], we can see that the function G can be represented as a Lebesgue integral whose integrand g is a non-decreasing, piecewise-constant, unbounded and positive function, *i.e.* the function G can be assumed in particular convex. Moreover, the values of g form a non-decreasing and unbounded sequence $(g_n)_{n\in\mathbb{N}}$, which is suitably chosen such that the condition of Theorem I.27 is satisfied. The improvement of Meyer [52, Lemme, p. 1] (now we follow the notation of [52]) consists in constructing a piecewise constant integrand f such that $f(2t) \le 2f(t)$. This implies that

$$\int_{[0,2x]} f(z)\,\mathrm{d}z = \int_{[0,x]} 2f(2t)\,\mathrm{d}t \le 4\int_{[0,x]} f(t)\,\mathrm{d}t$$

i.e. the Young function with right derivative f is moderate; see Lemma I.44.(iii). The aforementioned property of f is equivalent to $f_{2n} \le 2f_n$ for every $n \in \mathbb{N}$, where f_n is the value of the piecewise constant f on the interval $[n, n+1)$.

We have made the above discussion in order to underline the importance of a sequence $(f^k)_{k\in\mathbb{N}}$ with the aforementioned property (among others[9]) and to justify the following notation.

[9]The sequence $(f_n)_{n\in\mathbb{N}\cup\{0\}}$ can be chosen such that $f_0 = 1$, $f_n \in \mathbb{N}$ for every $n \in \mathbb{N}\cup\{0\}$ and non-decreasing.

Notation I.49. (i) Let $\mathcal{U} \subset \mathbb{L}^1(\mathcal{G}; E)$ be a uniformly integrable set and $\Phi_{\mathcal{U}}$ be a moderate Young function obtained by the de La Vallée Poussin–Meyer criterion. Then, we can associate to the function $\Phi_{\mathcal{U}}$, hence also to the uniformly integrable set $\mathcal{U}$, a sequence $\mathcal{A}_{\mathcal{U}} := (\alpha^k)_{k\in\mathbb{N}}$ which satisfies

[Mod] For every $k \in \mathbb{N}$ holds $\alpha^k \in \mathbb{N}$, $\alpha^k \leq \alpha^{k+1}$, $\alpha^{2k} \leq 2\alpha^k$ and $\lim_{k\to\infty} \alpha^k = \infty$.

(ii) A sequence $\mathcal{A}$ which satisfies the properties [Mod] will be called *moderate sequence* and the associated moderate Young function will be

$$\Phi_{\mathcal{A}}(x) := \int_0^x \mathbb{1}_{[0,1)}(t) + \sum_{k=1}^{\infty} \alpha^k \mathbb{1}_{[k,k+1)}(t)\,\mathrm{d}t, \text{ for } x \in \mathbb{R}_+. \tag{I.2}$$

The right derivative of $\Phi_{\mathcal{A}}$ will de denoted by $\phi_{\mathcal{A}}$.

Remark I.50. It has been already checked that a moderate sequence $\mathcal{A}$ associates to a moderate Young function. However, the converse is not true, *e.g.* the right derivative of quad (recall that it has been defined on p. 2) is Id.

If we may anticipate into Chapter III, the coming Proposition I.51 is the tool that will allow us to prove Lemma III.39. The latter will be crucial for obtaining sufficient integrability for the weak limits of the orthogonal martingales. There we make use of the Burkholder–Davis–Gundy Inequality in the form of Theorem I.101 as well as of Doob's Inequality in the form of Theorem I.104. In order to apply both of the above inequalities for the *same* Young function Φ, we need Φ to be moderate with moderate Young conjugate and, additionally, Φ must be (suitably) dominated by a quadratic function. But since it is too early for providing more details around that point, let us explain the intuition behind Proposition I.51.

The function quad will play a crucial role in the following. Recall that in its definition we required the presence of the fraction $\frac{1}{2}$ so that $\mathrm{quad}^\star = \mathrm{quad}$. This is straightforward since the right derivative of quad is simply Id and we have immediately that $\mathrm{Id}^{-1,r} = \mathrm{Id}$. Let us fix for the next lines two arbitrary Young functions Φ_1, Φ_2 such that $\Phi_1(x) \leq \Phi_2(x)$ for large $x \in \mathbb{R}_+$. Then it is well-known, e.g. see Krasnosel'skii and Rutickii [43, Theorem 2.1, p. 14] or Rao and Ren [58, Proposition I.2], that for their Young conjugates $\Phi_1^\star, \Phi_2^\star$ holds $\Phi_2^\star(x) \leq \Phi_1^\star(x)$ for large $x \in \mathbb{R}_+$. Assume, moreover, that Φ_1 is moderate, then it is straightforward to prove that $\Phi_1 \circ \mathrm{quad}$ is also moderate and due to the growth of Φ_1 we can conclude that $\mathrm{quad}(x) \leq \Phi_1 \circ \mathrm{quad}(x)$ for large $x \in \mathbb{R}_+$, where $\Phi_1 \circ \mathrm{quad}(x) := \Phi_1(\mathrm{quad}(x))$, for $x \in \mathbb{R}_+$. Therefore, in view of the previous comments, we have that $(\Phi_1 \circ \mathrm{quad})^\star(x) \leq \mathrm{quad}(x)$. Proposition I.51 verifies that, if we choose Φ_1 to be a moderate Young function associated to a moderate sequence $\mathcal{A}$, then the Young function $(\Phi_1 \circ \mathrm{quad}(x))^\star$ has nice properties, namely it is moderate and, moreover, accepts a Young function Υ such that $\Upsilon \circ (\Phi_1 \circ \mathrm{quad})^\star = \mathrm{quad}$.

Proposition I.51. *Let $\mathcal{A} := (\alpha^k)_{k\in\mathbb{N}}$ be a moderate sequence and define $\Phi_{\mathcal{A},\mathrm{quad}} := \Phi_{\mathcal{A}} \circ \mathrm{quad}$. Let, moreover, $\Psi := (\Phi_{\mathcal{A},\mathrm{quad}})^\star$ and ψ be the right derivative of Ψ. Then*

(i) $\Phi_{\mathcal{A},\mathrm{quad}}, \Psi \in \mathcal{YF}_{\mathrm{mod}}$.

(ii) *ψ is continuous and can be written as a concatenation of linear and constant functions defined on intervals. The slopes of the linear parts consist a non–increasing and vanishing sequence of positive numbers.*

(iii) *It holds $\Psi(x) \leq \mathrm{quad}(x)$, where the equality holds on a compact neighbourhood of 0, and*

$$\lim_{x\uparrow\infty} (\mathrm{quad}(x) - \Psi(x)) = \infty.$$

(iv) *There exists $\Upsilon \in \mathcal{YF}$ such that $\Upsilon \circ \Psi = \mathrm{quad}$.*

Proof. (i) We will prove initially that $\Phi_{\mathcal{A},\mathrm{quad}}$ is a Young function. It is sufficient to prove that it can be written as a Lebesgue integral whose integrand is a càdlàg, increasing and unbounded function; see Definition I.37 and Lemma I.38.

For every $x \in \mathbb{R}_+$ holds by definition

$$\Phi_{\mathcal{A},\mathrm{quad}}(x) = \Phi_{\mathcal{A}}(\frac{1}{2}x^2) = \int_{[0,\frac{1}{2}x^2]} \phi_{\mathcal{A}}(z)\,\mathrm{d}z = \int_{[0,x]} t\phi_{\mathcal{A}}(\frac{1}{2}t^2)\,\mathrm{d}t. \tag{I.3}$$

We define $\phi_{\mathcal{A},\mathrm{quad}} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\phi_{\mathcal{A},\mathrm{quad}}(t) := t\phi_{\mathcal{A}}(\frac{1}{2}t^2) = t\mathbb{1}_{[0,\sqrt{2})}(t) + t\sum_{k=1}^{\infty} \alpha^k \mathbb{1}_{[\sqrt{2k},\sqrt{2k+2})}(t), \tag{I.4}$$

i.e. $\phi_{\mathcal{A},\mathrm{quad}}$ is càdlàg and piecewise-linear. Observe, moreover, that

□ $\Delta\phi_{\mathcal{A},\mathrm{quad}}(\sqrt{2}) = (\alpha^1 - 1)\sqrt{2} \geq 0$.

□ $\Delta\phi_{\mathcal{A},\mathrm{quad}}(\sqrt{2k+2}) = (\alpha^{k+1} - \alpha^k)\sqrt{2k+2} \geq 0$, for every $k \in \mathbb{N}$.

□ $\phi_{\mathcal{A},\mathrm{quad}}$ has increasing slopes; the value of the slope of the linear part defined on the interval $[\sqrt{2k}, \sqrt{2k+2})$ is determined by the value of the respective element $\alpha^k \geq 1$, for every $k \in \mathbb{N}$.

□ $\lim_{s\to\infty} \phi_{\mathcal{A},\mathrm{quad}}(s) = \infty$.

Therefore, $\Phi_{\mathcal{A},\mathrm{quad}}$ is a Young function and its conjugate Ψ is also a Young function.

We will prove now that both $\Phi_{\mathcal{A},\mathrm{quad}}$ and Ψ are moderate. We have directly that $\underline{c}_{\mathrm{quad}} = \overline{c}_{\mathrm{quad}} = 2$. Moreover, by the property $\alpha^{2k} \leq 2\alpha^k$ we have that $\Phi_{\mathcal{A}}$ is moderate, hence $\overline{c}_{\Phi_{\mathcal{A}}} < \infty$. Now we obtain

$$\underline{c}_{\Phi_{\mathcal{A},\mathrm{quad}}} = \inf_{x>0} \frac{x\phi_{\mathcal{A},\mathrm{quad}}(x)}{\Phi_{\mathcal{A},\mathrm{quad}}(x)} = \inf_{x>0} \frac{x^2\phi_{\mathcal{A}}(\frac{1}{2}x^2)}{\Phi_{\mathcal{A}}(\frac{1}{2}x^2)} = 2\inf_{x>0} \frac{\frac{1}{2}x^2\phi_{\mathcal{A}}(\frac{1}{2}x^2)}{\Phi_{\mathcal{A}}(\frac{1}{2}x^2)} = 2\inf_{u>0} \frac{u\phi_{\mathcal{A}}(u)}{\Phi_{\mathcal{A}}(u)} = 2\underline{c}_{\Phi_{\mathcal{A}}} \geq 2,$$

because for every Young function $\Upsilon \in \mathcal{YF}$ holds $\underline{c}_{\Upsilon} \geq 1$; see Remark I.42. For $\overline{c}_{\Phi_{\mathcal{A},\mathrm{quad}}}$ we have

$$\overline{c}_{\Phi_{\mathcal{A},\mathrm{quad}}} = \sup_{x>0} \frac{x\phi_{\mathcal{A},\mathrm{quad}}(x)}{\Phi_{\mathcal{A},\mathrm{quad}}(x)} = 2\sup_{u>0} \frac{u\phi_{\mathcal{A}}(u)}{\Phi_{\mathcal{A}}(u)} = 2\overline{c}_{\Phi_{\mathcal{A}}} < \infty. \tag{I.5}$$

Hence, $\Phi_{\mathcal{A},\mathrm{quad}}$ is a moderate Young function. Moreover, since $\underline{c}_{\Phi_{\mathcal{A},\mathrm{quad}}} > 1$, we have from Corollary I.45 that Ψ is moderate.

(ii) For the rest of the proof, *i.e.* for parts (ii)-(iv), we will simplify the notation and we will write ϕ for the function $\phi_{\mathcal{A},\mathrm{quad}}$.

Firstly, let us observe that ψ is real-valued, resp. unbounded, since ϕ is unbounded, resp. real-valued.[10] In order to determine the value $\psi(s)$ for $s \in (0,\infty)$ let us define two sequences of subsets of $\mathbb{R}_+$ $(C^k)_{k\in\mathbb{N}\cup\{0\}}$, $(J^k)_{k\in\mathbb{N}\cup\{0\}}$ by

$$C^k := \big[\phi(\sqrt{2k}), \lim_{t\uparrow\sqrt{2k+2}} \phi(t)\big) \text{ and } J^{k+1} := \big[\lim_{t\uparrow\sqrt{2k+2}} \phi(t), \phi(\sqrt{2k+2})\big), \text{ for } k \in \mathbb{N}\cup\{0\}. \tag{I.6}$$

Observe that

□ $C^k = \phi\big([\sqrt{2k}, \sqrt{2k+2})\big) \neq \emptyset$ for every $k \in \mathbb{N}\cup\{0\}$, because ϕ is continuous and increasing on $[\sqrt{2k}, \sqrt{2k+2})$.

□ $J^k = \emptyset$ if and only if ϕ is continuous at $\sqrt{2k+2}$, which is further equivalent to $\alpha^k = \alpha^{k+1}$.

For our convenience, let us define two sequences $(s^k)_{k\in\mathbb{N}\cup\{0\}}$, $(s_-^{k+1})_{k\in\mathbb{N}\cup\{0\}}$ of positive numbers as follows:

$$\begin{aligned} s^0 := 0,\ s^k &:= \phi(\sqrt{2k}) = \alpha^k\sqrt{2k}, \text{ for } k \in \mathbb{N} \text{ and} \\ s_-^{k+1} &:= \lim_{x\uparrow\sqrt{2k+2}} \phi(x) = \alpha^k\sqrt{2k+2}, \text{ for } k \in \mathbb{N}\cup\{0\}. \end{aligned} \tag{I.7}$$

The introduction of the last notation permits us to write $C^k = [s^k, s_-^{k+1})$ and $J^k = [s_-^{k+1}, s^{k+1})$, for $k \in \mathbb{N}\cup\{0\}$. Now we are ready to determine the values of ψ on $(0,\infty)$. The reader should keep in mind that the function ϕ is increasing and right-continuous.

• Let $s \in C^0 = \big[\phi(0), \phi(\sqrt{2}-)\big) = \phi\big([0,\sqrt{2})\big)$, then

$$\begin{aligned} \psi(s) &= \inf\{t \in \mathbb{R}_+, \phi(t) > s\} \overset{s\in\phi([0,\sqrt{2})}{=} \inf\big\{t \in [0,\sqrt{2}), \phi(t) > s\big\} \\ &\overset{(I.4)}{=} \inf\{t \in \mathbb{R}_+, \mathrm{Id}(t)\mathbb{1}_{[0,\sqrt{2})}(t) > s\} = s, \end{aligned}$$

where the second equality is valid because ϕ is continuous on $[0,\sqrt{2})$ with $\phi\big([0,\sqrt{2})\big) = [0, s_-^1)$. To sum up, $\psi\mathbb{1}_{[s^0,s_-^1)} = \mathrm{Id}\mathbb{1}_{[s^0,s_-^1)}$.

• Let $s \in J^1 = \big[\phi(\sqrt{2}-), \phi(\sqrt{2})\big) = [s_-^1, s^1)$. If $J^1 = \emptyset$, which amounts to $\alpha^1 = 1$, there is nothing to prove. On the other hand, if $J^1 \neq \emptyset$, then $\phi(\sqrt{2}-) \leq s < \phi(\sqrt{2})$ and consequently

$$\psi(s) = \inf\{t \in \mathbb{R}_+, \phi(t) > s\} = \sqrt{2} \text{ for every } s \in J^1.$$

To sum up, $\psi\mathbb{1}_{[s_-^1,s^1)} = \sqrt{2}\mathbb{1}_{[s_-^1,s^1)}$.

For the general case let us fix a $k \in \mathbb{N}$. We will distinguish between the cases $s \in C^k$ and $s \in J^k$. For the latter we can argue completely analogous to the case $s \in J^1$, but for the sake of completeness we will provide its proof.

[10] The reader who is not familiar with generalised inverses may find helpful Embrechts and Hofert [32], especially the comments after [32, Remark 2.2].

• Let $s \in C^k = \big[\phi(\sqrt{2k}), \phi(\sqrt{2k+2}-)\big) = \phi\big([\sqrt{2k}, \sqrt{2k+2})\big)$. Since the image of $[\sqrt{2k}, \sqrt{2k+2})$ through $\mathbb{R}_+ \ni t \overset{\chi}{\longmapsto} \alpha^k t \in \mathbb{R}_+$ is C^k, then ψ has to coincide with $\mathbb{R}_+ \ni s \overset{\chi^{-1,r}}{\longmapsto} \frac{1}{\alpha^k} s \in \mathbb{R}_+$ on $C^k = [s^k, s_-^{k+1})$. To sum up, $\psi(s)\mathbb{1}_{[s^k, s_-^{k+1})} = \frac{1}{\alpha^k}\mathrm{Id}\mathbb{1}_{[s^k, s_-^{k+1})}$.

• Let $s \in J^{k+1} = \big[\phi(\sqrt{2k+2}-), \phi(\sqrt{2k+2})\big) = [s_-^{k+1}, s^{k+1})$. If $J^{k+1} = \emptyset$, which amounts to $\alpha^{k+1} = \alpha^k$, there is nothing to prove. On the other hand, if $J^{k+1} \neq \emptyset$, then $\phi(\sqrt{2k+2}-) \leq s < \phi(\sqrt{2k+2})$ and consequently

$$\psi(s) = \inf\{t \in \mathbb{R}_+, \phi(t) > s\} = \sqrt{2k+2} \text{ for every } s \in J^{k+1}.$$

To sum up, $\psi\mathbb{1}_{[s_-^{k+1}, s^{k+1})} = \sqrt{2k+2}\mathbb{1}_{[s_-^{k+1}, s^{k+1})}$.

In total we have that the right derivative of Ψ can be written as a concatenation of linear and constant functions defined on intervals

$$\psi(s) = \mathrm{Id}\mathbb{1}_{[s^0, s_-^1)} + \sum_{k=1}^{\infty} \frac{1}{\alpha^k}\mathrm{Id}\mathbb{1}_{[s^k, s_-^{k+1})} + \sum_{k=0}^{\infty} \sqrt{2k+2}\mathbb{1}_{[s_-^{k+1}, s^{k+1})} \tag{I.8}$$

where the interval end-points are defined in (I.7). Recall now that $\alpha^k \geq 1$, for every $k \in \mathbb{N}$, therefore we have that the slopes of ψ are smaller than 1. Moreover, since $\alpha^k \leq \alpha^{k+1}$ and $\lim_{k\to\infty} \alpha^k = \infty$, we have that $\frac{1}{\alpha^{k+1}} \leq \frac{1}{\alpha^k}$ and $\lim_{k\to\infty} \frac{1}{\alpha^k} = 0$. Finally, as it can be easily checked, ψ is continuous. This causes no surprise, since ϕ is strictly increasing; see Embrechts and Hofert [32, Proposition 2.3.(7)].[11]

(iii) Let us consider the function $\zeta : \mathbb{R}_+ \to \mathbb{R}$ defined by $\zeta := \mathrm{Id} - \psi$, *i.e.* ζ is also continuous. Moreover, ζ is differentiable on a superset of $\mathbb{R}_+ \setminus \{(s^k)_{k\in\mathbb{N}\cup\{0\}} \cup (s_-^{k+1})_{k\in\mathbb{N}\cup\{0\}}\}$, which is clearly an open and dense subset of $\mathbb{R}_+$ since there is no accumulation point in the sequence $(s^k)_{k\in\mathbb{N}\cup\{0\}} \cup (s_-^{k+1})_{k\in\mathbb{N}\cup\{0\}}$. Obviously

$$\zeta'(x) = \begin{cases} 1 - \frac{1}{\alpha^k}, & \text{for } x \in (s^k, s_-^{k+1}) \text{ for } k \in \mathbb{N}\cup\{0\}, \\ 1, & \text{for } x \in (s_-^{k+1}, s^{k+1}) \text{ for } k \in \mathbb{N}\cup\{0\}, \text{ whenever } (s_-^{k+1}, s^{k+1}) \neq \emptyset. \end{cases}$$

Define $M := \min\{k \in \mathbb{N}, \alpha^k > 1\}$, which is a well-defined positive integer, since $\alpha^k \longrightarrow \infty$. Recall now that $\alpha^k \geq 1$ for $k \in \mathbb{N}$ and we can conclude that $\zeta'(x) > 0$ almost everywhere on $[s^M, \infty)$.

We prove now that quad and ψ coincide only on a compact neighbourhood of 0. By the definition of M we have that $\alpha^k = 1$ for $k \in \{1, \dots, M-1\}$ and $\alpha^M > 1$, therefore $\mathrm{Id}\mathbb{1}_{[0,s^M]} = \psi\mathbb{1}_{[0,s^M]}$ and $x > \psi(x)$ for every $x \in (s^M, \infty)$.

Finally, it is left to prove that $\lim_{x\uparrow\infty}(\mathrm{Id}(x) - \psi(x)) = \infty$. Recall that $(s_-^{k+1}, s^{k+1}) \neq \emptyset$ whenever k is such that $\alpha^k < \alpha^{k+1}$, *i.e.* there are infinitely many non-trivial intervals (s_-^{k+1}, s^{k+1}). But these intervals correspond to the intervals that ψ is constant. Since Id is increasing we can conclude that ζ is unbounded and therefore also that the required limit is valid.

(iv) For the following recall (I.7), (I.8) and the definition of M. Let us start with the introduction of the auxiliary function $\eta : \cup_{k=M}^{\infty} \big[\Psi(s_-^{k+1}), \Psi(s^{k+1})\big) \longrightarrow (0,1]$ defined by $\eta(z) := \frac{\Psi(s^{k+1}) - z}{\Psi(s^{k+1}) - \Psi(s_-^{k+1})}$.[12] Recall now that Ψ is continuous and increasing, which allows us to define the function $\upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\begin{aligned} \upsilon(z) = \mathbb{1}_{[0,\Psi(s^M))}(z) &+ \sum_{k=M}^{\infty} \alpha^k \mathbb{1}_{[\Psi(s^k), \Psi(s_-^{k+1}))}(z) \\ &+ \sum_{k=M}^{\infty} \big(\eta(z)\alpha^k + (1-\eta(z))\alpha^{k+1}\big)\, \mathbb{1}_{[\Psi(s_-^{k+1}), \Psi(s^{k+1}))}(z). \end{aligned} \tag{I.9}$$

We can directly check that υ is indeed well-defined, non–negative, non–decreasing and unbounded. Therefore, the function υ is the right derivative of a Young function, call Υ.

We intend to prove that $\Upsilon \circ \Psi = \mathrm{quad}$, which is equivalent to proving that the right-derivative of $\Upsilon \circ \Psi$ equals Id. The following simple calculations allow us to evaluate the right derivative of $\Upsilon \circ \Psi$ in terms of υ, ψ and Ψ. For $x \in \mathbb{R}_+ \to \mathbb{R}_+$ holds

$$\Upsilon \circ \Psi(x) = \int_{[0,\Psi(x)]} \upsilon(t)\,\mathrm{d}t = \int_{[0,x]} \psi(z)\upsilon\big(\Psi(z)\big)\,\mathrm{d}z.$$

[11] In [32] the presented results regard the left-continuous generalised inverse of a function. However, by Lemma I.34.(ii) we can conclude also for the right-continuous generalised inverse.

[12] In fact, $\eta(z)$ is the unique number in $[0,1)$ for which z can be written as convex combination of $\Psi(s_-^{k+1})$ and $\Psi(s^{k+1})$.

Now we can compare the right derivative of $\Upsilon \circ \Psi$, which is the function $\psi(\upsilon \circ \Psi) : \mathbb{R}_+ \to \mathbb{R}_+$, with the identity function Id. To this end, we will consider the behaviour of $\psi(\upsilon \circ \Psi)$ on the intervals $[0, s^M)$, $[s^k, s_-^{k+1})$, for $k \geq M$ and $[s_-^{k+1}, s^{k+1})$ for $k \geq M$, which consist a partition of $\mathbb{R}_+$. Before we proceed, let us evaluate the function υ at $\Psi(s)$ for $s \in \mathbb{R}_+$:

$$\begin{aligned}
\upsilon\big(\Psi(s)\big) &= \mathbb{1}_{[0,\Psi(s^M))}(\Psi(s)) + \sum_{k=M}^{\infty} \alpha^k \mathbb{1}_{[\Psi(s^k),\Psi(s_-^{k+1}))}(\Psi(s)) \\
&\qquad + \sum_{k=M}^{\infty} \left\{\eta\big(\Psi(s)\big)\alpha^k + \big[1-\eta\big(\Psi(s)\big)\big]\alpha^{k+1}\right\} \mathbb{1}_{[\Psi(s_-^{k+1}),\Psi(s^{k+1}))}(\Psi(s)) \\
&= \mathbb{1}_{[0,s^M)}(s) + \sum_{k=M}^{\infty} \alpha^k \mathbb{1}_{[s^k,s_-^{k+1})}(s) \\
&\qquad + \sum_{k=M}^{\infty} \left\{\eta\big(\Psi(s)\big)\alpha^k + \big[1-\eta\big(\Psi(s)\big)\big]\alpha^{k+1}\right\} \mathbb{1}_{[s_-^{k+1},s^{k+1})}(s) \qquad \text{(I.10)}
\end{aligned}$$

because Ψ is continuous and increasing.

- Let $s \in [0, s^M)$. At the end of the proof of (iii) we obtained that $\mathrm{Id}\mathbb{1}_{[0,s^M)} = \psi\mathbb{1}_{[0,s^M)}$. Therefore, we can conclude that

$$\psi(s)(\upsilon \circ \Psi)(s) = s\upsilon\big(\Psi(s)\big) = s.$$

- Let $s \in [s^k, s_-^{k+1})$, for some $k \geq M$. Then,

$$\psi(s)\upsilon(\Psi(s)) \underset{\text{(I.10)}}{\overset{\text{(I.8)}}{=}} \frac{1}{\alpha^k}\alpha^k s = s.$$

- Let $s \in [s_-^{k+1}, s^{k+1})$ for some $k \geq M$. Then, for the chosen s there exists (unique) $\mu_s \in [0, 1)$ such that

$$s = \mu_s s_-^{k+1} + (1-\mu_s)s^{k+1}. \qquad \text{(I.11)}$$

But, Ψ is linear on $[s_-^{k+1}, s^{k+1})$, *i.e.*

$$\Psi(s) = \mu_s \Psi(s_-^{k+1}) + (1-\mu_s)\Psi(s^{k+1}) \qquad \text{(I.12)}$$

and $\Psi(s) \in \big[\Psi(s_-^{k+1}), \Psi(s^{k+1})\big)$. Therefore, by definition of η

$$\begin{aligned}
\eta(\Psi(s)) &= \frac{\Psi(s^{k+1}) - \Psi(s)}{\Psi(s^{k+1}) - \Psi(s_-^{k+1})} \overset{\text{(I.12)}}{=} \frac{\Psi(s^{k+1}) - \mu_s\Psi(s_-^{k+1}) - (1-\mu_s)\Psi(s^{k+1})}{\Psi(s^{k+1}) - \Psi(s_-^{k+1})} \\
&= \frac{\cancel{\Psi(s^{k+1})} - \mu_s\Psi(s_-^{k+1}) - \cancel{\Psi(s^{k+1})} + \mu_s\Psi(s^{k+1})}{\Psi(s^{k+1}) - \Psi(s_-^{k+1})} = \mu_s \qquad \text{(I.13)}
\end{aligned}$$

and finally, in view of

$$\begin{aligned}
\psi(s)(\upsilon \circ \Psi)(s) = \sqrt{2k+2}\,\upsilon(\Psi(s)) &\overset{\text{(I.10)}}{=} \sqrt{2k+2}\left\{\eta\big(\Psi(s)\big)\alpha^k + \big[1-\eta\big(\Psi(s)\big)\big]\alpha^{k+1}\right\} \\
&\overset{\text{(I.13)}}{=} \sqrt{2k+2}(\mu_s\alpha^k + (1-\mu_s)\alpha^{k+1}) \\
&\overset{\text{(I.7)}}{=} \mu_s s_-^{k+1} + (1-\mu_s)s^{k+1} \overset{\text{(I.11)}}{=} s,
\end{aligned}$$

we can conclude. □

In the rest of this section we will state some classical results, which shed more light on the nice properties of a moderate Young function. Before we proceed we provide the necessary notation.

Notation I.52. Let $\Upsilon \in \mathcal{YF}$. In the following, ξ denotes a $\mathcal{G}-$measurable and real-valued variable.

- $\widetilde{\mathbb{L}}^{\Upsilon}(\mathcal{G}) := \{\xi,\ \mathbb{E}\big[\Upsilon(|\xi|)\big] < \infty\}$
- $\mathbb{L}^{\Upsilon}(\mathcal{G}) := \{\xi,\ \text{exists } \lambda > 0 \text{ such that } \mathbb{E}\left[\Upsilon(|\xi|/\lambda)\right] < \infty\}$.
- $\|\xi\|_{\Upsilon} := \inf\{\lambda > 0, \mathbb{E}\left[\Upsilon(|\xi|/\lambda)\right] \leq 1\}$, where $\inf \emptyset := \infty$.

Proposition I.53. *Let $\Upsilon \in \mathcal{YF}$.*

(i) *A sufficient condition for the set $\widetilde{\mathbb{L}}^{\Upsilon}$ to be vector subspace of $\mathbb{L}^1(\mathcal{G};\mathbb{R})$ is Υ to be moderate.*
(ii) *The set $\mathbb{L}^{\Upsilon}(\mathcal{G})$ is a vector subspace of $\mathbb{L}^1(\mathcal{G};\mathbb{R})$.*

(iii) *The functional* $\|\cdot\|_\Upsilon : \mathbb{L}^\Upsilon \longrightarrow \mathbb{R}_+$ *is a norm.*[13]
(iv) *The normed space* $(\mathbb{L}^\Upsilon(\mathcal{G}), \|\cdot\|_\Upsilon)$ *is Banach.*
(v) *If* $\Upsilon \in \mathcal{YF}_{\text{mod}}$*, then the vector spaces* $\widetilde{\mathbb{L}}^\Upsilon$ *and* $\mathbb{L}^\Upsilon$ *are the same.*

Proof. (i) See Rao and Ren [58, Subsection III.3.1 Theorem 2.(ii), p. 46] and [58, Subsection II.2.3, Definition 1] for the notation.
(ii) See [58, Subsection III.3.1 Proposition 6, p. 49].
(iii) See [58, Subsection III.3.1 Theorem 3, p. 54].
(iv) See [58, Subsection III.3.3 Theorem 10, p. 67].
(v) Let $\xi \in \widetilde{\mathbb{L}}^\Upsilon(\mathcal{G})$, *i.e.* $\mathbb{E}\big[\Upsilon(|\xi|)\big] < \infty$, which implies that $\|\xi\|_\Upsilon \leq 1 \vee \mathbb{E}\big[\Upsilon(|\xi|)\big]$, where we have used the convexity of Υ and the fact that $\Upsilon(0) = 0$. For the converse we need the Young function Υ to be moderate. Let $\xi \in \mathbb{L}^\Upsilon(\mathcal{G})$, then

$$\mathbb{E}\big[\Upsilon(|\xi|)\big] = \mathbb{E}\Big[\Upsilon\Big(\frac{\|\xi\|_\Upsilon\,|\xi|}{\|\xi\|_\Upsilon}\Big)\Big] \leq \begin{cases} \mathbb{E}\big[\Upsilon\big(|\xi|/\|\xi\|_\Upsilon\big)\big] & \text{if } \|\xi\|_\Upsilon \leq 1, \\ \overline{c}_\Upsilon^{\|\xi\|_\Upsilon}\,\mathbb{E}\big[\Upsilon\big(|\xi|/\|\xi\|_\Upsilon\big)\big] & \text{otherwise,} \end{cases} \tag{I.14}$$

where in the first case we used the convexity of the Young function Υ, while in the second Lemma I.44.(ii). Therefore, since by definition of $\|\xi\|_\Upsilon$ we have $\mathbb{E}\Big[\Upsilon\Big(\frac{|\xi|}{\|\xi\|_\Upsilon}\Big)\Big] \leq 1$, in either case $\xi \in \widetilde{\mathbb{L}}^\Upsilon(\mathcal{G})$. □

In view of the previous result the following definition is allowed.

Definition I.54. Let $\Upsilon \in \mathcal{YF}$. The set $\mathbb{L}^\Upsilon(\mathcal{G})$ will be called *Orlicz space*.

We close this section with the following very convenient result.

Theorem I.55. *Let* $S \in \mathcal{S}_{\text{sp}}(\mathbb{G};\mathbb{R})$ *with* $\mathbb{G}-$*canonical decomposition* $S = X + A$ *with* A *non-decreasing,* i.e. S *is a* $\mathbb{G}-$*submartingale, and* $\Phi \in \mathcal{YF}_{\text{mod}}$*. Then*

$$\mathbb{E}[\Phi(A_\infty)] \leq (2\overline{c}_\Phi)^{\overline{c}_\Phi}\,\mathbb{E}[\Phi(\sup_{s>0}|S_s|)].$$

Proof. See Lenglart, Lépingle, and Pratelli [47, Théorème 3.2.1)]. □

I.3. Uniformly integrable martingales

In this section the state space is $\mathbb{R}^{p\times q}$, except otherwise stated, and it is denoted by E. We are going to present results regarding the space of uniformly integrable $\mathbb{G}-$martingales[14] as well as results for its subspace containing all the square integrable $\mathbb{G}-$martingales. Recall that by Notation I.11 the former is denoted by $\mathcal{M}(\mathbb{G};E)$, while the latter by $\mathcal{H}^2(\mathbb{G};E)$.

We start with the lemma that verifies that indeed $\mathcal{H}^2(\mathbb{G};E) \subset \mathcal{M}(\mathbb{G};E)$.

Lemma I.56. *If* $X \in \mathcal{H}^2(\mathbb{G};E)$*, then* $X \in \mathcal{M}(\mathbb{G};E)$*.*

Proof. We apply Theorem I.27 for the function $\mathbb{R}_+ \ni x \overset{\Phi}{\longmapsto} x^2 \in \mathbb{R}_+$. Obviously, $\Phi \in \mathcal{Y}$. Moreover,

$$\sup_{t\in\mathbb{R}_+}\mathbb{E}\big[\Phi(\|X_t\|_1)\big] \leq \sup_{t\in\mathbb{R}_+}\mathbb{E}\big[\|X_t\|_1^2\big] \leq pq\sup_{t\in\mathbb{R}_+}\mathbb{E}\big[\|X_t\|_2^2\big] < \infty.$$

□

Theorem I.57. *Let* X *be an* $E-$*valued stochastic process and* $X_\infty(\omega) := \lim_{t\uparrow\infty} X_t(\omega)$*, for every* $\omega \in \Omega$*.*
(i) *If* $X \in \mathcal{M}(\mathbb{G};E)$*, then the random variable* X_∞ *is* $\mathbb{P}-$*almost surely well-defined and lies in* $\mathbb{L}^1(\mathcal{G}_\infty;E)$*. Moreover,*

$$X_t \xrightarrow{\mathbb{L}^1(\mathcal{G}_\infty;E)} X_\infty \quad \textit{and} \quad X_t = \mathbb{E}[X_\infty|\mathcal{G}_t]\ \mathbb{P}-a.s. \textit{ for every } t \in \mathbb{R}_+,$$

i.e. the $\mathbb{G}-$martingale X is (right) closed by its terminal value X_∞.
(ii) *If* $X \in \mathcal{M}(\mathbb{G};E)$*, then* $X \in \mathcal{H}^2(\mathbb{G};E)$ *if and only if* $\|X_\infty\|_{\mathbb{L}^2(\mathcal{G}_\infty;E)} < \infty$*. In this case,*

$$\sup_{t\in\mathbb{R}_+}\|X_t\|_{\mathbb{L}^2(\mathcal{G};E)} \leq \|X_\infty\|^2_{\mathbb{L}^2(\mathcal{G};E)} \leq pq\sup_{t\in\mathbb{R}_+}\|X_t\|_{\mathbb{L}^2(\mathcal{G};E)}.$$

[13] As usual we identify the $\mathbb{P}-a.s.$ equal random variables.
[14] Recall that the $\mathbb{G}-$stochastic basis is already given in Section I.1.

PROOF. (i) For every pair (i,j), for $1 \le i \le p$, $1 \le j \le q$, we apply Dellacherie and Meyer [27, Theorem VI.6]. More precisely, for every pair (i,j), for $1 \le i \le p$, $1 \le j \le q$, there exists $\Omega^{ij} \subset \Omega$ with $\mathbb{P}(\Omega^{ij}) = 1$ such that $X^{ij}_\infty(\omega)$ is well-defined for every $\omega \in \Omega^{ij}$,

$$X^{ij}_t \xrightarrow{\mathbb{L}^1(\mathcal{G}_\infty;\mathbb{R})} X^{ij}_\infty \text{ and } X^{ij}_t = \mathbb{E}[X^{ij}_\infty|\mathcal{G}_t]\ \mathbb{P}-a.s. \text{ for every } t \in \mathbb{R}_+.$$

By the above convergence we obtain

$$\|X_t - X_\infty\|_{\mathbb{L}^1(\mathcal{G}_\infty;E)} = \sum_{i=1}^p \sum_{j=1}^q \mathbb{E}\Big[\big|X^{ij}_t - X^{ij}_\infty\big|\Big] \xrightarrow[t\uparrow\infty]{} 0.$$

Define now $\Omega_E := \bigcap\{\Omega^{ij}, 1 \le i \le p \text{ and } 1 \le j \le q\}$. Then $\mathbb{P}(\Omega_E) = 1$ and $X^{ij}_\infty(\omega)$ is well-defined for every $1 \le i \le p$, $1 \le j \le q$ and for every $\omega \in \Omega_E$. Therefore, also the property $X_t = \mathbb{E}[X_\infty|\mathcal{G}_t]$ holds $\mathbb{P}-$almost surely .

(ii) Let $X \in \mathcal{H}^2(\mathbb{G};E)$, *i.e.* $\sup_{t\in\mathbb{R}_+} \mathbb{E}\big[\,\|X_t\|_2^2\big] < \infty$. We prove initially that $X^{ij}_\infty \in \mathbb{L}^2(\mathcal{G}_\infty;\mathbb{R})$ for $1 \le i \le p$, $1 \le j \le q$. The obvious domination

$$\sup_{t\in\mathbb{R}_+} \mathbb{E}\big[\,|X^{ij}_t|^2\big] \le \sup_{t\in\mathbb{R}_+} \mathbb{E}\big[\,\|X_t\|_2^2\big] < \infty$$

enables us to apply He et al. [35, Theorem 6.8.1] element-wise and, consequently, to obtain that

$$\mathbb{E}[\,|X^{ij}_\infty|^2] = \sup_{t\in\mathbb{R}_+} \mathbb{E}[\,|X^{ij}_t|^2] \ \text{ for } 1 \le i \le p, 1 \le j \le q. \tag{I.15}$$

But this is equivalent to $X_\infty \in \mathbb{L}^2(\mathcal{G};E)$, more precisely $X_\infty \in \mathbb{L}^2(\mathcal{G}_\infty;E)$.

Conversely, let $X_\infty \in \mathbb{L}^2(\mathcal{G}_\infty;E)$, *i.e.* $\mathbb{E}[\,|X^{ij}_\infty|^2] < \infty$, for $1 \le i \le p$, $1 \le j \le q$. By He et al. [35, Theorem 6.8.1] applied element-wise we obtain that Equality (I.15) holds. Therefore,

$$\sup_{t\in\mathbb{R}_+} \|X_t\|^2_{\mathbb{L}^2(\mathcal{G};E)} = \sup_{t\in\mathbb{R}_+} \sum_{i=1}^p \sum_{j=1}^q \mathbb{E}\big[|X^{ij}_t|^2\big] \overset{(I.15)}{\le} \sup_{t\in\mathbb{R}_+} \sum_{i=1}^p \sum_{j=1}^q \mathbb{E}\big[|X^{ij}_\infty|^2\big] = \|X_\infty\|^2_{\mathbb{L}^2(\mathcal{G};E)}. \tag{I.16}$$

Moreover,

$$\begin{aligned}
\mathbb{E}\big[\|X_\infty\|_2^2\big] &= \sum_{i=1}^p \sum_{j=1}^q \mathbb{E}\big[|X^{ij}_\infty|^2\big] \overset{(I.15)}{=} \sum_{i=1}^p \sum_{j=1}^q \sup_{t\in\mathbb{R}_+} \mathbb{E}\big[|X^{ij}_t|^2\big] \\
&= \sup\Big\{ \sum_{i=1}^p \sum_{j=1}^q \mathbb{E}\big[|X^{ij}_{t_{ij}}|^2\big], t_{ij} \in \mathbb{R}_+ \text{ for } 1 \le i \le p, 1 \le j \le q \Big\} \\
&\le \sum_{i=1}^p \sum_{j=1}^q \Big(\sup_{t\in\mathbb{R}_+} \mathbb{E}\big[|X^{ij}_t|^2\big]\Big) \le \sum_{i=1}^p \sum_{j=1}^q \Big(\sup_{t\in\mathbb{R}_+} \mathbb{E}\big[\|X_t\|_2^2\big]\Big) = pq \sup_{t\in\mathbb{R}_+} \mathbb{E}\big[\|X_t\|_2^2\big].
\end{aligned}$$

The above inequality and (I.16) allow us to conclude.

□

Remark I.58. In view of the previous theorem, for an $X \in \mathcal{M}(\mathbb{G};E)$ we can write $(X_t)_{t\in\mathbb{R}_+}$ or $(X_t)_{t\in\overline{\mathbb{R}}_+}$ interchangeably. Observe that, due to the "left-continuity" of X at the symbol ∞ the random variable $\sup_{t\in\overline{\mathbb{R}}_+} |X_t|$ is $\mathbb{P}-a.s.$ well-defined and $\mathbb{P}-a.s.$ equal to $\sup_{t\in\mathbb{R}_+} |X_t|$.

Definition I.59. An $E-$valued and $\mathcal{G}\otimes\mathcal{B}(\mathbb{R}_+)-$measurable process M is *of class* (D) *for the filtration* $\mathbb{G}$ if the family $\{\|M_\tau\|_1 \mathbb{1}_{[\tau<\infty]}, \tau \text{ is } \mathbb{G}-\text{stopping time}\}$ is uniformly integrable.

Theorem I.60 (Doob–Meyer Decomposition)**.** *If X is a real-valued, right-continuous $\mathbb{G}-$submartingale of class* (D)*, then there exists a unique, up to indistinguishability, $A \in \mathcal{V}^+_{\text{pred}}(\mathbb{G};\mathbb{R})$ with $A_0 = 0$ such that $X - A \in \mathcal{M}(\mathbb{G};\mathbb{R})$.*

Corollary I.61. *Let $X \in \mathcal{H}^2(\mathbb{G};\mathbb{R})$, then there exists a unique, up to indistinguishability, $\langle X\rangle \in \mathcal{V}^+_{\text{pred}}(\mathbb{G};\mathbb{R})$ with $\langle X\rangle_0 = 0$ such that $X^2 - \langle X\rangle \in \mathcal{M}(\mathbb{G};\mathbb{R})$.*

PROOF. It is immediate by Jensen Inequality that the process X^2 is a $\mathbb{G}-$submartingale. Then we conclude by Theorem I.60. □

Theorem I.62. *Let $X, Y \in \mathcal{H}^2(\mathbb{G};\mathbb{R})$. Then, there exists a unique up to indistinguishability process $\langle X, Y\rangle^{\mathbb{G}} \in \mathcal{A}_{\text{pred}}(\mathbb{G};\mathbb{R})$ such that $XY - \langle X, Y\rangle^{\mathbb{G}} \in \mathcal{M}(\mathbb{G};\mathbb{R})$. Moreover,*

$$\langle X, Y\rangle^{\mathbb{G}} = \frac{1}{4}\big(\langle X+Y, X+Y\rangle^{\mathbb{G}} - \langle X-Y, X-Y\rangle^{\mathbb{G}}\big).$$

Furthermore, $\langle X, X\rangle^{\mathbb{G}}$ is non-decreasing and it admits a continuous version if and only if X is $\mathbb{G}-$quasi-left-continuous.

Proof. See Jacod and Shiryaev [41, Theorem I.4.2]. □

The process $\langle X, Y\rangle^{\mathbb{G}}$ is called the *predictable quadratic variation* or the *angle bracket* of the pair (X, Y). When no confusion can arise we will omit the symbol of the filtration from the notation. Moreover, $\langle X, X\rangle$ will be also denoted as $\langle X\rangle$.

Notation I.63. When $X \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p), Y \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^q)$ we will denote by $\langle X, Y\rangle$ the element of $\mathcal{A}_{\text{pred}}(\mathbb{G}; E)$ for which $\langle X, Y\rangle^{ij} = \langle X^i, Y^j\rangle$.

Theorem I.64. *Let $A \in \mathcal{A}(\mathbb{G};\mathbb{R})$. There exists a process, which will be called the* compensator of A *and denoted by $A^{(p,\mathbb{G})}$, which is unique up to an evanescent set, and which is characterised by being an element of $\mathcal{A}_{\text{pred}}$ such that $A - A^{(p,\mathbb{G})} \in \mathcal{M}(\mathbb{G};\mathbb{R})$.*

Proof. See Jacod and Shiryaev [41, Theorem I.3.18]. □

Corollary I.65. *Let $X \in \mathcal{M}(\mathbb{G};\mathbb{R}) \cap \mathcal{V}_{\text{pred}}(\mathbb{G};\mathbb{R})$. Then X is indistinguishable from the zero process.*

Proof. See Jacod and Shiryaev [41, Corollary I.3.16]. □

I.4. Stochastic integration

This section is devoted to introducing the notation and the main results regarding integration with respect to a finite variation process, a square-integrable martingale and an integer-valued random measure. The reader may recall that we work with the $\mathbb{G}-$stochastic basis, where $\mathbb{G}$ is an arbitrary filtration, until the end of the chapter.

I.4.1. Stochastic integration with respect to an increasing process. This section is devoted to Lebesgue–Stieltjes integration either with respect to a finite variation function or a process of finite variation. In the latter case notation has been provided on page v. Nevertheless, we will introduce additional one which will be used interchangeably. Let a non–negative measure ϱ defined on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ and G be any finite dimensional space. We will denote the Lebesgue–Stieltjes integral of any measurable map $f : (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \longrightarrow (G, \mathcal{B}(G))$ by

$$\int_{(u,t]} f(s)\varrho(\mathrm{d}s) \quad \text{and} \quad \int_{(u,\infty)} f(s)\varrho(\mathrm{d}s), \text{ for any } u, t \in \mathbb{R}_+.$$

The above integrals as well as every Lebesgue–Stieltjes integral are to be understood in a component–wise sense. In case ϱ is a finite measure with associated distribution function $F^{\varrho}(\cdot) := \varrho([0, \cdot])$, we will indifferently denote the above integrals by

$$\int_{(u,t]} f(s)\mathrm{d}F^{\varrho}_s \quad \text{and} \quad \int_{(u,\infty)} f(s)\mathrm{d}F^{\varrho}_s, \text{ for any } u, t \in \mathbb{R}_+.$$

When there is no confusion as to which measure the distribution function F^{ϱ} is associated to, we will omit the upper index and we will write F. Conversely, if the distribution function is given, say F, then we will denote the associated Borel measure by μ_F.

When $\tilde{\varrho}$ is a signed measure with Jordan–Hahn decomposition $\tilde{\varrho} = \tilde{\varrho}^+ - \tilde{\varrho}^-$, then we define

$$\int_{(u,t]} f(s)\tilde{\varrho}(\mathrm{d}s) := \int_{(u,t]} f(s)\tilde{\varrho}^+(\mathrm{d}s) - \int_{(u,t]} f(s)\tilde{\varrho}^-(\mathrm{d}s) \text{ for any } u, t \in \mathbb{R}_+.$$

and analogously for

$$\int_{(u,\infty)} f(s)\tilde{\varrho}(\mathrm{d}s) := \int_{(u,\infty)} f(s)\tilde{\varrho}^+(\mathrm{d}s) - \int_{(u,\infty)} f(s)\tilde{\varrho}^-(\mathrm{d}s) \text{ for any } u \in \mathbb{R}_+.$$

If $V^{\tilde{\varrho}}$ is the associated to the signed measure $\tilde{\varrho}$ finite variation function we will denote interchangeably the above integrals by

$$\int_{(u,t]} f(s)\,\mathrm{d}V_s^{\tilde{\varrho}} \text{ and } \int_{(u,\infty)} f(s)\mathrm{d}V_s^{\tilde{\varrho}} \text{ for any } u,t\in\mathbb{R}_+.$$

More generally, for any measure $\bar{\varrho}$ on $(\mathbb{R}_+\times E,\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(E))$ and for any measurable map $g:(\mathbb{R}_+\times E,\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(E))\longrightarrow(G,\mathcal{B}(G))$ we will denote the Lebesgue–Stieltjes integral by

$$\int_{(u,t]\times A} g(s,x)\varrho(\mathrm{d}s,\mathrm{d}x) \text{ and } \int_{(u,\infty)\times A} g(s,x)\varrho(\mathrm{d}s,\mathrm{d}x), \text{ for any } t,u\in\mathbb{R}_+, A\in\mathcal{B}(E).$$

Conversely, if $A\in\mathcal{V}^+(\mathbb{G})$ is given, then we will denote the associated to $A(\omega)$ Borel measure by $\mu_{A(\omega)}$, for every $\omega\in\Omega$. Then, μ_A is $\mathbb{P}-$almost surely well-defined. Finally, the integration with respect to a finite variation process will be understood pathwise. More precisely, if $A\in\mathcal{V}(\mathbb{G};\mathbb{R})$ and H is an $E-$valued random function such that $H(\omega,\cdot)$ is Borel measurable, then we define the random integral, denoted by $\int_{(0,\cdot]} H_s\,\mathrm{d}A_s$, as follows:

$$\int_{(0,\cdot]} H_s\,\mathrm{d}A_s(\omega) := \begin{cases}\int_{(0,\cdot]} H_s(\omega)\,\mathrm{d}A_s(\omega) & \text{if } \int_{(0,\cdot]}\|H_s(\omega)\|_1\,\mathrm{d}\big[\mathrm{Var}(A)_s(\omega)\big]<\infty,\\ \infty & \text{otherwise.}\end{cases}$$

Clearly, if H is assumed $\mathbb{G}-$optional, then $\int_{(0,\cdot]} H_s\,\mathrm{d}A_s$ is a process.

I.4.2. The Itô stochastic integral. In this section we present the main results concerning Itô stochastic integration with respect to an $\ell-$dimensional square-integrable martingale. We follow Jacod and Shiryaev [41, Section III.6.a]. We assume that the real-valued case is familiar to the reader. If not, they may consult Jacod and Shiryaev [41, Section I.4d]. For the following, the arbitrary process $X\in\mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell)$ is fixed. In Section I.3 we have seen that the $\mathbb{R}^{\ell\times\ell}-$valued process $\langle X\rangle$ is well-defined and such that $\langle X\rangle\in\mathcal{A}(\mathbb{G};\mathbb{R}^{\ell\times\ell})$. Consider, now, a factorization of $\langle X\rangle$, that is to say, write

$$\langle X\rangle_\cdot = \int_{(0,\cdot]} \frac{\mathrm{d}\langle X\rangle_s}{\mathrm{d}F_s}\,\mathrm{d}F_s, \tag{I.17}$$

where F is an increasing $\mathbb{G}-$predictable and càdlàg process and $\left(\frac{\mathrm{d}\langle X\rangle_\cdot}{\mathrm{d}F_\cdot}\right)^{ij} := \frac{\mathrm{d}\langle X\rangle^{ij}_\cdot}{\mathrm{d}F_\cdot}$ for $1\le i,j\le\ell$. It can be proven by standard arguments that $\left(\frac{\mathrm{d}\langle X\rangle_\cdot}{\mathrm{d}F_s}\right)_{t\in\overline{\mathbb{R}}_+}$ is a $\mathbb{G}-$predictable process with $\frac{\mathrm{d}\langle X\rangle_t}{\mathrm{d}F_t}$ being a symmetric, non-negative definite element of $\mathbb{R}^{\ell\times\ell}$, for every $t\in\overline{\mathbb{R}}_+$.

Notation I.66. We will denote by

- $\mathbb{H}^2(\mathbb{G},X;\mathbb{R}^{p\times\ell}) := \Big\{Z:(\Omega\times\mathbb{R}_+,\mathcal{P}^{\mathbb{G}})\longrightarrow(\mathbb{R}^{p\times\ell},\mathcal{B}(\mathbb{R}^{p\times\ell})),\ \|Z\|_{\mathbb{H}^2(\mathbb{G},X;\mathbb{R}^\ell)}<\infty\Big\}$,

  where $\|Z\|^2_{\mathbb{H}^2(\mathbb{G},X;\mathbb{R}^{p\times\ell})} := \mathbb{E}\Big[\int_{(0,\infty)}\mathrm{Tr}\Big[Z_s\frac{\mathrm{d}\langle X\rangle_s}{\mathrm{d}F_s}Z_s^\top\Big]\mathrm{d}F_s\Big]$.

The following theorem verifies that the above space provides a suitable class of integrands for the Itô integral with respect to an $\ell-$dimensional X.

Theorem I.67. *For every $Z\in\mathbb{H}^2(\mathbb{G},X;\mathbb{R}^{p\times\ell})$ the* Itô stochastic integral of Z with respect to X under the filtration $\mathbb{G}$, *denoted indifferently by $Z\cdot X$ or by $\int_0^\cdot Z_s\mathrm{d}X_s$,*[15] *is a well-defined $\mathbb{R}^{p\times1}-$valued process. Moreover,*

(i) *If $Z\in\mathbb{H}^2(\mathbb{G},X;\mathbb{R}^{p\times\ell})$ and bounded then*

$$(Z\cdot X)^i = \sum_{j=1}^{\ell} Z^{ij}\cdot X^j \quad \textit{for every } i=1,\dots,p.$$

(ii) *For every $Z\in\mathbb{H}^2(\mathbb{G},X;\mathbb{R}^{p\times\ell})$, there exists a sequence $(Z^k)_{k\in\mathbb{N}}\subset\mathbb{H}^2(\mathbb{G},X;\mathbb{R}^{p\times\ell})$, where Z^k is bounded for every $k\in\mathbb{N}$, such that*

$$\big\|Z^k\cdot X_\infty - Z\cdot X_\infty\big\|_{\mathbb{L}^2(\mathcal{G};\mathbb{R}^p)} \xrightarrow[k\to\infty]{} 0.$$

(iii) *$(Z\cdot X)^\top\in\mathcal{H}^2(\mathbb{G};\mathbb{R}^p)$, for every $Z\in\mathbb{H}^2(\mathbb{G},X;\mathbb{R}^{p\times\ell})$.*

[15] We have abstained from using the transpose of X in the notation.

(iv) *If* $Z^1, Z^2 \in \mathbb{H}^2(\mathbb{G}, X; \mathbb{R}^\ell)$ *then*

$$\langle Z^1 \cdot X, Z^2 \cdot X\rangle_\cdot = \int_{(0,\cdot]} Z^1_s \frac{\mathrm{d}\langle X\rangle_s}{\mathrm{d}F_s}(Z^2_s)^\top \,\mathrm{d}F_s.$$

(v) *Let* $Y \in \mathcal{H}^2(\mathbb{G}; \mathbb{R})$ *and define the* $\mathbb{R}^\ell-$*valued* $\mathbb{G}-$*predictable process* $r^Y := (r^{Y1}, \dots, r^{Y\ell})$ *by*

$$\langle Y, X^i\rangle_\cdot = \int_{(0,\cdot]} r^{Yi}_s \,\mathrm{d}F_s, \text{ for every } i = 1, \dots, \ell.$$

Then, for $Z \in \mathbb{H}^2(\mathbb{G}, X; \mathbb{R}^\ell)$, $Z \cdot X$ *is the unique, up to indistinguishability, square-integrable martingale with* $Z \cdot X_0 = 0$ *such that*

$$\langle Z \cdot X, Y\rangle_\cdot = \int_{(0,\cdot]} Z_s(r^Y_s)^\top \mathrm{d}F_s, \text{ for every } Y \in \mathcal{H}^2(\mathbb{G}; \mathbb{R}).$$

(vi) *For* $Z \in \mathbb{H}^2(\mathbb{G}; \mathbb{R}^{p\times\ell})$, *the jump process of* $(Z \cdot X)^i$ *is indistinguishable from* $\sum_{n=1}^\ell Z^{in}\Delta X^n$. *In other words, the* $i-$*element of the purely discontinuous part of* $Z \cdot X$ *can be described by* $\sum_{n=1}^\ell Z^{in} \cdot X^{d,n}$. *This allows us to denote* $(Z \cdot X)^d$ *by* $Z \cdot X^d$.

(vii) *If* $X \in \mathcal{H}^{2,c}(\mathbb{G}; \mathbb{R}^\ell)$, *then for every* $Z \in \mathbb{H}^2(\mathbb{G}, X; \mathbb{R}^\ell)$ *holds* $Z \cdot X \in \mathcal{H}^{2,c}(\mathbb{G}; \mathbb{R})$.

Proof. We use Jacod and Shiryaev [41, Theorem III.6.4] to verify that $(Z \cdot X)^i$ is well-defined, for $i = 1, \dots, p$. For (ii) we use again element-wise [41, Theorem III.6.4 a)]. For (vi) see Jacod and Shiryaev [41, Proposition III6.9]. The other parts are the same as in [41, Theorem III.6.4 a)], but in a different order. □

Notation I.68. • To the class $\mathbb{H}^2(\mathbb{G}, X; \mathbb{R}^{p\times\ell})$ we associate the space $\mathcal{L}^2(\mathbb{G}, X; \mathbb{R}^p)$ where

$$\mathcal{L}^2(\mathbb{G}, X; \mathbb{R}^p) := \Big\{W \cdot X, W \in \mathbb{H}^2(\mathbb{G}, X; \mathbb{R}^{p\times\ell})\Big\}. \tag{I.18}$$

In other words, $\mathcal{L}^2(\mathbb{G}, X; \mathbb{R}^p)$ is the space of the $\mathbb{R}^p-$valued Itô stochastic integrals with respect to the pair $(\mathbb{G}, X)$.

• When $\mathbb{F}$ is a filtration such that $X \in \mathcal{H}^2(\mathbb{F}; \mathbb{R}^\ell)$, in other words when X is both $\mathbb{G}-$ and $\mathbb{F}-$ adapted, for a $Z \in \mathbb{H}^2(\mathbb{F}, X; \mathbb{R}^{p\times\ell})$, we will denote the Itô stochastic integral of Z with respect to X under the filtration $\mathbb{G}$, resp. under the filtration $\mathbb{F}$, by $Z \cdot X^{\mathbb{G}}$, resp. $Z \cdot X^{\mathbb{F}}$.

We close this subsection with the following well-known result.

Theorem I.69 (Galtchouk–Kunita–Watanabe Decomposition)**.** *Let* $X \in \mathcal{H}^2(\mathbb{G}; \mathbb{R}^\ell)$ *and* $Y \in \mathcal{H}^2(\mathbb{G}; \mathbb{R}^p)$. *Then, there exists* $Z \in \mathbb{H}^2(\mathbb{G}, X; \mathbb{R}^{p\times\ell})$ *such that*

$$Y^j = (Z \cdot X)^j + N^j \text{ for every } j = 1, \dots, p,$$

where $\langle N^j, X^i\rangle = 0$ *for* $i = 1, \dots, \ell$ *and* $j = 1, \dots, p$. *The stochastic integral* $Z \cdot X$ *does not depend upon the chosen version of* Z.

Proof. See Jacod [40, Chapitre IV, Section 2]. □

I.4.3. Stochastic integral with respect to a random measure. In this section we describe the construction of the stochastic integral with respect to a compensated integer-valued random measure and we present the results we are going to use in the next chapters. We will follow closely Jacod and Shiryaev [41, Section II.1]. However, Cohen and Elliott [20, Chapter 13] and He et al. [35, Chapter 11] will be also useful. The state space will be assumed to be E, except otherwise explicitly stated.

Definition I.70. A *random measure* μ on $\mathbb{R}_+ \times \mathbb{R}^\ell$ is a family $\mu = \big(\mu(\omega; \mathrm{d}t, \mathrm{d}x)\big)_{\omega\in\Omega}$ of measures on $\big(\mathbb{R}_+ \times \mathbb{R}^\ell, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^\ell)\big)$ satisfying $\mu(\omega; \{0\} \times \mathbb{R}^\ell) = 0$ identically.

We proceed to introduce some notation and definitions.

Notation I.71. Write

• $\widetilde{\Omega} := \Omega \times \mathbb{R}_+ \times \mathbb{R}^\ell$.

• $(\widetilde{\Omega}, \widetilde{\mathcal{P}}^{\mathbb{G}}) := (\widetilde{\Omega}, \mathcal{P}^{\mathbb{G}} \otimes \mathcal{B}(\mathbb{R}^\ell))$. A measurable function $U : (\widetilde{\Omega}, \widetilde{\mathcal{P}}) \longrightarrow (E, \mathcal{B}(E))$ will be said $\mathbb{G}-$*predictable function*

• $(\widetilde{\Omega}, \widetilde{\mathcal{O}}) := (\widetilde{\Omega}, \mathcal{O} \otimes \mathcal{B}(\mathbb{R}^\ell))$ A measurable function $U : (\widetilde{\Omega}, \widetilde{\mathcal{O}}) \longrightarrow (E, \mathcal{B}(E))$ will be said $\mathbb{G}-$*optional function*

Definition I.72. Given a random measure μ and a $\mathbb{G}-$optional function W we define the *stochastic integral of W with respect to the random measure μ* to be the $E-$valued process

$$U * \mu.(\omega) := \begin{cases} \int_{(0,\cdot]} W(\omega,t,x)\,\mu(\omega;\mathrm{d}t,\mathrm{d}x), & \text{if } \int_{(0,\cdot]} \left\|W(\omega,t,x)\right\|_1 \mu(\omega;\mathrm{d}t,\mathrm{d}x) < \infty, \\ \infty, & \text{otherwise.} \end{cases}$$

Observe that we have defined the integral $\omega-$wise. This is indeed eligible, because W is assumed to be $\mathbb{G}-$optional function. Therefore, $W(\omega,\cdot)$ is $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R}^\ell)-$measurable for each ω. Observe, moreover, that for a real-valued and $\mathbb{G}-$optional function W which is positive, then $W*\mu\in\mathcal{V}^+(\mathbb{G};\mathbb{R})$.

Remark I.73. This point is also a good chance to adopt a convenient abuse of notation. Let a function $h:\big(\mathbb{R}^\ell,\mathcal{B}(\mathbb{R}^\ell)\big)\longrightarrow\big(E,\mathcal{B}(E)\big)$. We will treat h also as the $\mathbb{G}-$predictable function $\widetilde{\Omega}\ni(\omega,t,x)\overset{h}{\longmapsto} h(x)\in E$. Analogously, if A is an $E-$valued $\mathbb{G}-$predictable process, resp. $\mathbb{G}-$optional process, we will treat it also as $\widetilde{\Omega}\ni(\omega,t,x)\longmapsto A_t(\omega)\in E$, which is a $\mathbb{G}-$predictable function, resp. a $\mathbb{G}-$optional function.

Definition I.74. A random measure is called $\mathbb{G}-$*optional*, resp. $\mathbb{G}-$*predictable*, if the process $W*\mu$ is $\mathbb{G}-$optional process, resp. $\mathbb{G}-$predictable process, for every $\mathbb{G}-$optional, resp. $\mathbb{G}-$predictable, function W.

We introduce now the set from which we will choose the random measures we are going to work with.

Definition I.75. A $\mathbb{G}-$optional random measure will be said $\mathbb{G}-$*predictably $\sigma-$integrable* or interchangeably $\widetilde{\mathcal{P}}^{\mathbb{G}}-\sigma-$finite if there exists a positive $\mathbb{G}-$predictable function V such that $V*\mu\in\mathcal{A}^+(\mathbb{G};\mathbb{R})$. The set of $\mathbb{G}-$optional $\mathbb{G}-$predictably $\sigma-$integrable random measures will be denoted by $\widetilde{\mathcal{A}}_\sigma(\mathbb{G})$.

Observe that an equivalent condition for a random measure to lie in $\widetilde{\mathcal{A}}_\sigma(\mathbb{G})$ is the existence of a partition of $\widetilde{\mathcal{P}}^{\mathbb{G}}$, say $(A^k)_{k\in\mathbb{N}}$, such that $\mathbb{1}_{A^k}*\mu\in\mathcal{A}^+(\mathbb{G};\mathbb{R})$ for every $k\in\mathbb{N}$.

Definition I.76. A random measure μ will be called *integer-valued $\mathbb{G}-$random measure* if it satisfies

(i) $\mu(\omega;\{t\}\times\mathbb{R}^\ell)\le 1$ identically.
(ii) For each $A\in\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R}^\ell)$, the random variable $\mu(\cdot;A)$ takes values in $\overline{\mathbb{N}}\cup\{0\}$.
(iii) $\mu\in\widetilde{\mathcal{A}}_\sigma(\mathbb{G})$.

In the remaining chapters we will be interested in random measures associated to the jumps of càdlàg processes. The next proposition verifies that to every $\mathbb{G}-$adapted, càdlàg and $\mathbb{R}^\ell-$valued process associates an integer-valued $\mathbb{G}-$random measure.

Proposition I.77. *Let X be a $\mathbb{G}-$adapted càdlàg $\mathbb{R}^\ell-$valued process. Then*

$$\mu^X(\omega;\mathrm{d}t,\mathrm{d}x) := \sum_{s>0}\mathbb{1}_{[\Delta X\neq 0]}(\omega,s)\delta_{(s,\Delta X_s(\omega))}(\mathrm{d}t,\mathrm{d}x)^{16}$$

defines an integer-valued $\mathbb{G}-$random measure on $\mathbb{R}\times\mathbb{R}^\ell$.

Proof. See Jacod and Shiryaev [41, Proposition II.1.16]. □

Actually, every integer-valued measure μ has an analogous representation. More precisely, [41, Proposition II1.14] verifies that for every integer-valued random measure μ there exists a thin, $\mathbb{G}-$optional set D and an $E-$valued optional process β such that

$$\mu(\omega;\mathrm{d}t,\mathrm{d}x) = \sum_{s\ge 0}\mathbb{1}_D(\omega,s)\delta_{(s,\beta_s(\omega))}(\mathrm{d}t,\mathrm{d}x).$$

Notation I.78. We will maintain the notation of the last proposition and we will denote *the integer-valued $\mathbb{G}-$random measure associated to a $\mathbb{G}-$adapted and càdlàg process X* by μ^X.

Now that we have defined the necessary objects, we can start presenting the desirable machinery. We start with the core of this section. The following theorem allows us to transform the process $W*\mu$, where W is a $\mathbb{G}-$optional and μ an integer-valued $\mathbb{G}-$random measure, into a $\mathbb{G}-$martingale.

[16] Recall that δ_z denotes the Dirac measure sitting at z.

Theorem I.79. *Let $\mu \in \widetilde{\mathcal{A}}_\sigma(\mathbb{G})$. There exists a random measure, called the* compensator of μ under $\mathbb{G}$ *and denoted by $\nu^{\mathbb{G}}$, which is unique up to a $\mathbb{P}-$null set, and which is characterised as being a $\mathbb{G}-$predictable random measure satisfying either of the two following equivalent properties:*

(i) *For every non-negative $\mathbb{G}-$predictable function W holds $\mathbb{E}[W * \mu_\infty] = \mathbb{E}[W * \nu^{\mathbb{G}}_\infty]$.*

(ii) *If W is a $\mathbb{G}-$predictable function such that $|W| * \mu \in \mathcal{A}^+(\mathbb{G};\mathbb{R})$, then $|W| * \nu^{\mathbb{G}} \in \mathcal{A}^+(\mathbb{G};\mathbb{R})$ and the process $W * \mu - W * \nu^{\mathbb{G}}$ is a uniformly integrable $\mathbb{G}-$martingale. In other words, $W * \nu^{\mathbb{G}}$ is* the compensator under the filtration $\mathbb{G}$ *of the process $W * \mu$.*

Moreover, there is a version of $\nu^{\mathbb{G}}$ such that $\nu^{\mathbb{G}}(\{s\} \times \mathbb{R}^\ell) \leq 1$ identically.

Proof. See Jacod and Shiryaev [41, Theorem II.1.8] and [41, Proposition II.1.17]. □

Remark I.80. (i) For every a random measure $\mu \in \widetilde{\mathcal{A}}_\sigma(\mathbb{G})$ we are going hereinafter to use the "nice" version of $\nu^{\mathbb{G}}$ for which the property $\nu^{\mathbb{G}}(\{s\} \times \mathbb{R}^\ell) \leq 1$ holds for every $s \in \mathbb{R}_+$.

(ii) Observe that for every càdlàg process X holds $\mathbb{1}_{\mathbb{R}_+ \times \{0\}} * \mu^X = 0$ $\mathbb{P}-$almost surely. This, in view of Theorem I.79, in particular implies that $\int_{(0,\infty)\times\{0\}} \nu^{(X,\mathbb{G})}(\mathrm{d}s, \mathrm{d}x) = 0$ $\mathbb{P}-$almost surely.

When it is clear to which filtration refers the measurability of an integer-valued random measure, then we will not use the symbol of the filtration in the description. However, this is not the case for the compensator of an integer-valued measure, where we will always prefer to denote the filtration. Since in this dissertation we will be interested only for integer-valued measures associated to some adapted process, we will present the rest results for these type of random measures. Therefore, we fix for the rest of the section an arbitrary $\mathbb{R}^\ell-$valued, $\mathbb{G}-$adapted and càdlàg process X with associated integer-valued measure μ^X.

Notation I.81. For the associated to X integer-valued random measure μ^X we adopt the following notation.

- The compensator of μ^X under the filtration $\mathbb{G}$ will be denoted by $\nu^{(X,\mathbb{G})}$.
- If $X \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell)$, then the integer-valued measure associated to the jumps of the martingale X will be denoted by μ^{X^d}. In other words, μ^{X^d} is the integer-valued measure associated to the purely discontinuous part of the $\mathbb{G}-$martingale X. In this case, $\mu^{X^d} \in \widetilde{\mathcal{A}}_\sigma(\mathbb{G})$, *i.e.* its compensator is well-defined. The compensator of μ^{X^d} under the filtration $\mathbb{G}$ will be denoted by $\nu^{(X^d,\mathbb{G})}$. The reason for this change in notation will be discussed at the end of the section.
- The $\mathbb{G}-$compensated integer-valued measure $\widetilde{\mu}^{(X,\mathbb{G})}$ is defined by
$$\widetilde{\mu}^{(X,\mathbb{G})}(\omega; \mathrm{d}t, \mathrm{d}x) := \mu^X(\omega; \mathrm{d}t, \mathrm{d}x) - \nu^{(X,\mathbb{G})}(\omega; \mathrm{d}t, \mathrm{d}x).$$
- For W an $E-$valued $\mathbb{G}-$predictable function and a $\mathbb{G}-$predictable time ρ we denote
$$\int_{\mathbb{R}^\ell} W(\rho, x)\mu^X(\{\rho\} \times \mathrm{d}x)(\omega) := W(\omega, \rho(\omega), \Delta X_{\rho(\omega)}(\omega))\mathbb{1}_{[\Delta X_\rho \neq 0, \|W(\omega,\rho(\omega),\Delta X_{\rho(\omega)}(\omega))\|_1 < \infty]} + \infty \mathbb{1}_{[\Delta X_\rho \neq 0, \|W(\omega,\rho(\omega),\Delta X_{\rho(\omega)}(\omega))\|_1 = \infty]}$$

and
$$\int_{\mathbb{R}^\ell} W(\rho, x)\nu^{(X,\mathbb{G})}(\{\rho\} \times \mathrm{d}x)(\omega) := \int_{\mathbb{R}_+ \times \mathbb{R}^\ell} W(\omega, \rho(\omega), x)\mathbb{1}_{[\![\rho]\!]}(\omega, t)\nu^{(X,\mathbb{G})}(\omega; \mathrm{d}t, \mathrm{d}x),$$

if $\left\| \int_{\mathbb{R}_+ \times \mathbb{R}^\ell} W(\omega, \rho(\omega), x)\mathbb{1}_{[\![\rho]\!]}(\omega, t)\nu^{(X,\mathbb{G})}(\omega; \mathrm{d}t, \mathrm{d}x) \right\|_1 < \infty$, otherwise we define it to be equal to ∞. Moreover,

$$\int_{\mathbb{R}^\ell} W(\rho, x)\widetilde{\mu}^{(X,\mathbb{G})}(\{\rho\} \times \mathrm{d}x)(\omega) := \int_{\mathbb{R}^\ell} W(\rho, x)\mu^X(\{\rho\} \times \mathrm{d}x)(\omega) - \int_{\mathbb{R}_+ \times \mathbb{R}^\ell} W(\omega, \rho(\omega), x)\mathbb{1}_{[\![\rho]\!]}(\omega, t)\nu^{(X,\mathbb{G})}(\omega; \mathrm{d}t, \mathrm{d}x).$$

The following remark justifies the forthcoming Definition I.83.

Remark I.82. By Jacod and Shiryaev [41, Lemma II.1.25] we have for every $\mathbb{G}-$predictable time ρ that
$$\int_{\mathbb{R}^\ell} W(\rho, x)\nu^{(X,\mathbb{G})}(\{\rho\} \times \mathrm{d}x) = \mathbb{E}\left[\int_{\mathbb{R}^\ell} W(\rho, x)\mu^X(\{\rho\} \times \mathrm{d}x) \middle| \mathcal{G}_{\rho-} \right] \text{ on } [\rho < \infty].$$

In other words, for every $\mathbb{G}-$predictable time ρ

$$\mathbb{E}\Big[\int_{\mathbb{R}^\ell} W(\rho,x)\widetilde{\mu}^{(X,\mathbb{G})}(\{\rho\}\times \mathrm{d}x)\Big|\mathcal{G}_{\rho-}\Big] = 0 \text{ on } [\rho<\infty].$$

If, moreover, for every $\mathbb{G}-$predictable time ρ

$$\mathbb{E}\Big[\Big\|\int_{\mathbb{R}^\ell} W(\rho,x)\widetilde{\mu}^{(X,\mathbb{G})}(\{\rho\}\times \mathrm{d}x)\Big\|_1 \mathbb{1}_{[\rho<\infty]}\Big] < \infty,$$

then the two above properties imply by Theorem I.21 that

$$\Pi_p^{\mathbb{G}}\Big(\int_{\mathbb{R}^\ell} W(\cdot,x)\widetilde{\mu}^{(X,\mathbb{G})}(\{\cdot\}\times \mathrm{d}x)\Big) = 0,$$

i.e. its $\mathbb{G}-$predictable projection is indistinguishable from the zero process. Now, by [41, Theorem I.4.56 a)], to every $\mathbb{G}-$optional process H with the following properties

(i) $H_0 = 0$,
(ii) $\Pi_p^{\mathbb{G}}(H) = 0$ and
(iii) $\sum_{s\leq\cdot} \|H_s\|_2^2 \in \mathcal{A}^+(\mathbb{G};\mathbb{R})$

we can associate a $Y \in \mathcal{H}^2(\mathbb{G};E)$ such that $\Delta Y = H$. Recalling Corollary I.9 we realise that we can actually associate a unique, up to indistinguishability, element of $\mathcal{H}^{2,d}(\mathbb{G};E)$, namely Y^d.

Definition I.83. Let $\mu^X \in \widetilde{\mathcal{A}}_\sigma(\mathbb{G})$. An $E-$valued $\mathbb{G}-$predictable function W is said to be stochastically integrable with respect to $\widetilde{\mu}^{(X,\mathbb{G})}$ if $W \in \mathbb{H}^2(\mathbb{G},\mu^X;E)$, where

$$\mathbb{H}^2(\mathbb{G},\mu^X;E) := \Big\{U : \big(\widetilde{\Omega},\widetilde{\mathcal{P}}^{\mathbb{G}}\big) \to \big(E,\mathcal{B}(E)\big), \sum_{s\leq\cdot}\Big\|\int_{\mathbb{R}^\ell} W(s,x)\widetilde{\mu}^{(X,\mathbb{G})}(\{s\}\times \mathrm{d}x)\Big\|_2^2 \in \mathcal{A}^+(\mathbb{G};\mathbb{R})\Big\}.$$

In this case, the *stochastic integral of W with respect to $\widetilde{\mu}^{(X,\mathbb{G})}$* is defined to be the element of $\mathcal{H}^{2,d}(\mathbb{G};E)$, which will be denoted by $W\star\widetilde{\mu}^{(X,\mathbb{G})}$, such that

$$\Delta(W\star\widetilde{\mu}^{(X,\mathbb{G})})_\cdot = \int_{\mathbb{R}^\ell} W(\cdot,x)\widetilde{\mu}^{(X,\mathbb{G})}(\{\cdot\}\times \mathrm{d}x)$$

up to indistinguishability.

Notation I.84. To the class $\mathbb{H}^2(\mathbb{G},\mu^X;E)$ we associate the space $\mathcal{K}^2(\mathbb{G},\mu^X;E)$ where

$$\mathcal{K}^2(\mathbb{G},\mu^X;E) := \Big\{W\star\widetilde{\mu}^{(X,\mathbb{G})}, W\in\mathbb{H}^2(\mathbb{G},\mu^X;E)\Big\}. \tag{I.19}$$

In other words, $\mathcal{K}^2(\mathbb{G},\mu^X;E)$ is the space of the $E-$valued stochastic integrals with respect to the pair $(\mathbb{G},\mu^X)$.

Proposition I.85. *Let W be a $\mathbb{G}-$predictable function such that $|W|*\mu^X \in \mathcal{A}^+(\mathbb{G};\mathbb{R})$ or equivalently $|W|*\nu^{(X,\mathbb{G})} \in \mathcal{A}^+(\mathbb{G};\mathbb{R})$. Then the stochastic integral $W\star\widetilde{\mu}^{(X,\mathbb{G})}$ is well-defined. Moreover, the following are true*

(i) *$W\star\widetilde{\mu}^{(X,\mathbb{G})}$ is a $\mathbb{G}-$martingale of finite variation.*
(ii) *$W\star\widetilde{\mu}^{(X,\mathbb{G})} = W*\mu^X - W*\nu^{(X,\mathbb{G})}$.*

Proof. See Jacod and Shiryaev [41, Proposition II.1.28]. □

Proposition I.86. *Let H be a bounded $\mathbb{G}-$predictable process and $W\in\mathbb{H}^2(\mathbb{G},\mu^X;E)$. Then $HW\in\mathbb{H}^2(\mathbb{G},\mu^X;E)$ and $H\cdot\big(W\star\widetilde{\mu}^{(X,\mathbb{G})}\big) = (HW)\star\widetilde{\mu}^{(X,\mathbb{G})}$.*

Proof. See [41, Proposition II.1.30]. □

Proposition I.87. *X is $\mathbb{G}-$quasi-left-continuous if and only if there exists a version of $\nu^{(X,\mathbb{G})}$ that satisfies $\int_{\mathbb{R}^\ell}\nu^{(X,\mathbb{G})}(\{t\}\times \mathrm{d}x) = 0$ for every $(\omega,t)\in\Omega\times\mathbb{R}_+$.*

Proof. See Jacod and Shiryaev [41, Corollary II.1.19]. □

Notation I.88. For every $\Gamma : \big(\widetilde{\Omega}, \mathcal{G} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^\ell)\big) \to \big(E, \mathcal{B}(E)\big)$ we define

$$M_{\mu^X}[\Gamma] := \mathbb{E}\big[\Gamma * \mu^X_\infty\big].$$

We will refer to M_{μ^X} as the *Doléans-Dade measure associated to the random measure* μ^X. We have abused the terminology for M_{μ^X}, however, when we restrict the domain of M_{μ^X} on the indicator functions of $\widetilde{\mathcal{P}}^{\mathbb{G}}-$measurable sets, then it is easy to check that M_{μ^X} is a $\sigma-$finite measure on $\big(\widetilde{\Omega}, \widetilde{\mathcal{P}}^{\mathbb{G}}\big)$. We will denote by $M_{\mu^X}|_{\widetilde{\mathcal{P}}^{\mathbb{G}}}$ the restriction of M_{μ^X} on the set of $\mathbb{G}-$predictable functions.

Definition I.89. The integer-valued measure μ^X is said to be a $\mathbb{G}-$*martingale* (*integer-valued random*) *measure* if $M_{\mu^X}|_{\widetilde{\mathcal{P}}^{\mathbb{G}}} = 0$.

Theorem I.90. *Let μ be an integer-valued random measure. The following are equivalent:*

(i) *μ is a $\mathbb{G}-$martingale integer-valued random measure.*
(ii) $\nu^{\mathbb{G}} = 0$.
(iii) *For every $E-$valued $\mathbb{G}-$predictable function W such that there exists an increasing sequence of $\mathbb{G}-$stopping times, say $(\tau^k)_{k\in\mathbb{N}}$, for which $\lim_{k\to\infty}\tau^k = \infty$ $\mathbb{P}-$almost surely and $(W*\mu)^{\tau^k} \in \mathcal{A}(\mathbb{G};E)$ for every $k \in \mathbb{N}$, the process $(W*\mu)^{\tau^k} \in \mathcal{M}(\mathbb{G};E)$ for every $k \in \mathbb{N}$.*

Proof. See Cohen and Elliott [20, Theorem 13.2.27]. □

Corollary I.91. *The integer-valued random measure $\widetilde{\mu}^{(X,\mathbb{G})}$ is a $\mathbb{G}-$martingale measure.*

Proof. Using Theorem I.79 we can see that the compensator of $\widetilde{\mu}^{(X,\mathbb{G})}$ is 0. Now, we can conclude by Theorem I.90. □

Definition I.92. Let $W : \big(\widetilde{\Omega}, \mathcal{G} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^\ell)\big) \longrightarrow \big(\mathbb{R}^p, \mathcal{B}(\mathbb{R}^p)\big)$ be such that $\|W\|_1\mu^X \in \widetilde{\mathcal{A}}_\sigma(\mathbb{G})$. Then we define the $\widetilde{\mathcal{P}}^{\mathbb{G}}-$*conditional projection of W on μ^X*, denoted by $M_{\mu^X}\big[W|\widetilde{\mathcal{P}}^{\mathbb{G}}\big]$, as the $\mathbb{R}^p-$valued $\mathbb{G}-$predictable function with

$$M_{\mu^X}\big[W|\widetilde{\mathcal{P}}^{\mathbb{G}}\big]^i := \frac{\mathrm{d}M_{W^i\mu^X}|_{\widetilde{\mathcal{P}}^{\mathbb{G}}}}{\mathrm{d}M_{\mu^X}|_{\widetilde{\mathcal{P}}^{\mathbb{G}}}} \text{ for every } i = 1, \dots, p.$$

Theorem I.93. *Let $W \in \mathbb{H}^2(\mathbb{G}, \mu^X; \mathbb{R})$ and N be a uniformly integrable real-valued $\mathbb{G}-$martingale such that $|\Delta N|\mu \in \widetilde{\mathcal{A}}_\sigma(\mathbb{G})$. Then, if $[W \star \widetilde{\mu}^{(X,\mathbb{G})}, N] \in \mathcal{A}(\mathbb{G};\mathbb{R})$, we have*

$$\langle W \star \widetilde{\mu}^{(X,\mathbb{G})}, N\rangle_\cdot = \big(W M_\mu\big[\Delta N\big|\widetilde{\mathcal{P}}^{\mathbb{G}}\big]\big) * \nu^{(X,\mathbb{G})}_\cdot.$$

Proof. See Cohen and Elliott [20, Theorem 13.3.16]. □

Theorem I.94. *Let $W \in \mathbb{H}^2(\mathbb{G}, \mu^{X^d}; \mathbb{R}^p)$. Then*

$$\langle U \star \widetilde{\mu}^{(X^d,\mathbb{G})}\rangle_\cdot = \int_{(0,\cdot]\times\mathbb{R}^\ell} \big(U(s,x)\big)^\top U(s,x)\, \nu^{(X^d,\mathbb{G})}(\mathrm{d}s, \mathrm{d}x) - \sum_{s\le\cdot}\left(\int_{\mathbb{R}^\ell} U(s,x)\,\nu(\{s\}\times\mathrm{d}x)\right)^\top \int_{\mathbb{R}^\ell} U(s,x)\,\nu(\{s\}\times\mathrm{d}x).$$

Proof. See He et al. [35, Theorem 11.21]. □

The purpose of the following discussion is to clarify a subtle point in the construction of the stochastic integral with respect to μ^Y, where Y is an $\mathbb{R}^\ell$-valued, $\mathbb{G}-$adapted and not $\mathbb{G}-$quasi-left-continuous process. Assume additionally that $\mathrm{Tr}[\mathrm{q}] * \mu^Y \in \mathcal{A}^+(\mathbb{G};\mathbb{R})$.[17] Then $\mu^Y \in \widetilde{A}_\sigma(\mathbb{G})$ and its compensator $\nu^{(Y,\mathbb{G})}$ are well-defined. Since we have assumed that Y is not $\mathbb{G}-$quasi-left-continuous, there exists a finite $\mathbb{G}-$stopping time ρ such that $\nu^{(Y,\mathbb{G})}(\{\rho\}\times\mathbb{R}^\ell) > 0$ $\mathbb{P}-$almost surely, for every version of $\nu^{(Y,\mathbb{G})}$. This assumption can be justified by Proposition I.87.

Let us proceed now with the observation that $\mathrm{Id}_\ell \in \mathbb{H}^2(\mathbb{G}, \mu^Y; \mathbb{R})$. Therefore, we can associate to the process Y the $\mathbb{G}-$martingale $\mathrm{Id}_\ell \star \widetilde{\mu}^{(Y,\mathbb{G})} \in \mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^\ell)$. Abusing notation we will denote $\mathrm{Id}_\ell \star \widetilde{\mu}^{(Y,\mathbb{G})}$ by Y^d, *i.e*

$$\Delta Y^d_t := \int_{\mathbb{R}^\ell} x\, \widetilde{\mu}^{(Y,\mathbb{G})}(\{t\}\times\mathrm{d}x) = \Delta Y_t - \int_{\mathbb{R}^\ell} x\, \nu^{(Y,\mathbb{G})}(\{t\}\times\mathrm{d}x). \tag{I.20}$$

[17] The function q has been defined on p. 2.

At the end of the current discussion we will see that, if $Y \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell)$, then its purely discontinuous part and $\mathrm{Id}_\ell \star \widetilde{\mu}^{(Y,\mathbb{G})}$ are indistinguishable, which does not lead to a notational conflict. Now that we have constructed the square-integrable $\mathbb{G}-$martingale Y^d, we have also the option to work with μ^{Y^d} with associated compensator $\nu^{(Y^d,\mathbb{G})}$. Then, we can write (I.20) as

$$\int_{\mathbb{R}^\ell} x\, \widetilde{\mu}^{(Y,\mathbb{G})}(\{t\}\times \mathrm{d}x) = \Delta Y^d = \int_{\mathbb{R}^\ell} x\, \mu^{Y^d}(\{t\}\times \mathrm{d}x). \tag{I.21}$$

A seemingly naive question that one may pose is whether $\widetilde{\mu}^{(Y,\mathbb{G})}$ and $\widetilde{\mu}^{(Y^d,\mathbb{G})}$ generate the same spaces of stochastic integrals. Of course, we expect that the compensators $\nu^{(Y,\mathbb{G})}$ and $\nu^{(Y^d,\mathbb{G})}$ do not coincide, since they correspond to different integer-valued measures μ^Y and μ^{Y^d}. Therefore, in general $\mathcal{K}^2(\mathbb{G},\mu^X;E) \neq \mathcal{K}^2(\mathbb{G},\mu^{X^d};E)$ for the arbitrary state space E. From a different perspective, we know by Corollary I.91 that $\widetilde{\mu}^{(Y,\mathbb{G})}$ is a $\mathbb{G}-$martingale measure, while for μ^{Y^d} this is not true. For example, if V is a positive real-valued $\mathbb{G}-$predictable function such that $V * \mu^{Y^d} \in \mathcal{A}^+(\mathbb{G};\mathbb{R})$, then we obtain a $\mathbb{G}-$submartingale.

So, we proceed by examining a $\mathbb{G}-$predictable function W whose value $W(\omega,s,x)$ is not linear on x. Since an analogous function will be used in Chapter III in order to construct suitable martingale sequences, it is a good opportunity to see in detail why we will finally prefer the $\mathbb{G}-$compensated integer-valued measure $\widetilde{\mu}^{(Y^d,\mathbb{G})}$. This will justify also the second bullet in Notation I.81, where we substitute the notation of the integer-valued measure associated to the process X with μ^{X^d}, if the process $X \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell)$.

Say $W(\omega,s,x) = \|x\|_2^2 \wedge 1$.[18] Then, the jump of $W \star \widetilde{\mu}^{(Y,\mathbb{G})}$ at ρ, which was fixed at the beginning of the discussion, is given by

$$\int_{\mathbb{R}^\ell} \left(\|x\|_2^2 \wedge 1\right) \mu^Y(\{\rho\}\times \mathrm{d}x) - \int_{\mathbb{R}^\ell} \left(\|x\|_2^2 \wedge 1\right) \nu^{(Y,\mathbb{G})}(\{\rho\}\times \mathrm{d}x) = \left\|\Delta Y_\rho\right\|_2^2 \wedge 1 - \mathbb{E}\Big[\left\|\Delta Y_\rho\right\|_2^2 \wedge 1 \Big| \mathcal{G}_{\rho-}\Big],$$

while the respective jump of $W \star \widetilde{\mu}^{(Y^d,\mathbb{G})}$ is given by

$$\begin{aligned}
&\int_{\mathbb{R}^\ell} \left(\|x\|_2^2 \wedge 1\right) \mu^{Y^d}(\{\rho\}\times \mathrm{d}x) - \int_{\mathbb{R}^\ell} \left(\|x\|_2^2 \wedge 1\right) \nu^{(Y^d,\mathbb{G})}(\{\rho\}\times \mathrm{d}x) = \left\|\Delta Y^d_\rho\right\|_2^2 \wedge 1 - \mathbb{E}\Big[\left\|\Delta Y^d_\rho\right\|_2^2 \wedge 1 \Big| \mathcal{G}_{\rho-}\Big] \\
&\overset{\text{(I.21)}}{=} \left\| \int_{\mathbb{R}^\ell} x\, \widetilde{\mu}^{(Y,\mathbb{G})}(\{\rho\}\times \mathrm{d}x)\right\|_2^2 \wedge 1 - \mathbb{E}\Big[\left\| \int_{\mathbb{R}^\ell} x\, \widetilde{\mu}^{(Y,\mathbb{G})}(\{\rho\}\times \mathrm{d}x)\right\|_2^2 \wedge 1 \Big| \mathcal{G}_{\rho-}\Big] \\
&= \left\|\Delta Y_\rho - \int_{\mathbb{R}^\ell} x\, \nu^{(Y,\mathbb{G})}(\{\rho\}\times \mathrm{d}x)\right\|_2^2 \wedge 1 - \mathbb{E}\Big[\left\|\Delta Y_\rho - \int_{\mathbb{R}^\ell} x\, \nu^{(Y,\mathbb{G})}(\{\rho\}\times \mathrm{d}x)\right\|_2^2 \wedge 1 \Big| \mathcal{G}_{\rho-}\Big].
\end{aligned}$$

It is clear now, that the spaces $\mathcal{K}^2(\mathbb{G},\mu^Y;E)$ and $\mathcal{K}^2(\mathbb{G},\mu^{Y^d};E)$ are not equal. However, when $Y \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell)$, then we have by Jacod and Shiryaev [41, Lemma I.2.27] that $\Pi^{\mathbb{G}}_p(Y) = Y_-$. In other words,

$$\int_{\mathbb{R}^\ell} x\, \nu^{(Y,\mathbb{G})}(\omega;\{t\}\times \mathrm{d}x) = 0 \text{ for every } (\omega,t)\times\Omega\times\mathbb{R}_+.$$

Therefore, the purely discontinuous part of the $\mathbb{G}-$martingale Y has jumps ΔY, which in view of the above property implies that

$$\Delta Y_t(\omega) = \int_{\mathbb{R}^\ell} x\, \mu^Y(\omega;\{t\}\times \mathrm{d}x).$$

This verifies our comment at the beginning of the discussion, that the abuse of notation (I.20) does not conflict with the case $Y \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell)$. Intuitively, one can say that having already a $\mathbb{G}-$martingale, we lose the information of the jump-compensating procedure which transforms a $\mathbb{G}-$adapted process to a $\mathbb{G}-$martingale.

The above discussion was made on the basis that Y is a $\mathbb{G}-$adapted process which is not $\mathbb{G}-$quasi-left-continuous. In view of Definition I.15 and Proposition I.87, this subtle difference vanishes when the process Y is $\mathbb{G}-$quasi-left-continuous, hence the compensator $\nu^{(Y,\mathbb{G})}$ has no jumps.

Henceforth, assuming that X is a $\mathbb{G}-$martingale, when we refer to the jump process ΔX^d we will mean the $\mathbb{R}^\ell-$valued process

$$\Delta X^d_\cdot := \left(\int_{\mathbb{R}^\ell} \pi^1(x)\widetilde{\mu}^{(X,\mathbb{G})}(\{\cdot\}\times \mathrm{d}x), \dots, \int_{\mathbb{R}^\ell} \pi^\ell(x)\widetilde{\mu}^{(X,\mathbb{G})}(\{\cdot\}\times \mathrm{d}x)\right), \tag{I.22}$$

[18] The fact that $W \in \mathbb{H}^2(\mathbb{G},\mu^{Y^d};\mathbb{R})$ is an outcome of the definition of Y^d and that $W \in \mathbb{H}^2(\mathbb{G},\mu^Y;\mathbb{R})$.

while, assuming that X is simply an $\mathbb{G}-$adapted process, when we refer to the jump process ΔX we will mean the $\mathbb{R}^\ell-$valued process

$$\Delta X_\cdot = \Big(\int_{\mathbb{R}^\ell} \pi^1(x)\mu^X(\{\cdot\}\times \mathrm{d}x), \dots, \int_{\mathbb{R}^\ell} \pi^\ell(x)\mu^X(\{\cdot\}\times \mathrm{d}x)\Big). \tag{I.23}$$

Theorem I.95. *Let $Y \in \mathcal{H}^2(\mathbb{G},\mathbb{R}^p)$ and assume that $X \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell)$. Then, there exists unique, up to indistinguishability, $U \in \mathbb{H}^2(\mathbb{G},\mu^{X^d};\mathbb{R}^p)$ and $N \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p)$ with $M_{\mu^{X^d}}\big[\Delta N\big|\widetilde{\mathcal{P}}^{\mathbb{G}}\big] = 0$ such that*

$$Y = U \star \widetilde{\mu}^{(X^d,\mathbb{G})} + N.$$

The $\mathbb{G}-$predictable function U is uniquely determined up to indistinguishability by the triplet $\big(\mathbb{G},\mu^{X^d},Y\big)$. If, moreover, X is $\mathbb{G}-$quasi-left-continuous, then U is indistinguishable from $M_{\mu^{X^d}}\big[\Delta Y\big|\widetilde{\mathcal{P}}^{\mathbb{G}}\big]$.

Proof. See Jacod and Shiryaev [41, Theorem III.4.20]. □

I.5. Selected results from stochastic calculus

In this section we collect some well-known results from stochastic calculus which we are going to use heavily. However, the Burkholder–Davis–Gundy Inequality and the Doob's $\mathbb{L}^\Upsilon-$Inequality will be stated with the help of Young functions, which are not the forms that are usually met in the literature. We will rely on the generality of these results in order to prove Theorem III.3 by utilising Proposition I.51.

For the current section we work in the $\mathbb{G}-$stochastic basis, where $\mathbb{G}$ is an arbitrary filtration.

Definition I.96. Let $X \in \mathcal{M}(\mathbb{G};\mathbb{R}^p)$ and $Y \in \mathcal{M}(\mathbb{G};\mathbb{R}^q)$. We define the (*optional*) *quadratic variation of the pair* (X,Y) is the $\mathbb{R}^{p\times p}-$valued $\mathbb{G}-$optional process $[X,Y]$ which is defined for every $t \in \mathbb{R}_+$ by

$$[X,Y]^{ij}_t := \langle X^{c,i}, Y^{c,j}\rangle_t + \sum_{0<s\le t} \Delta X^i_s \Delta Y^j_s \text{ for } i = 1,\dots,p \text{ and } j = 1,\dots,q.$$

Remark I.97. For $X,Y \in \mathcal{M}(\mathbb{G};\mathbb{R}^p)$, the quadratic variation satisfies the so called *polarisation identity*

$$[X,Y] = \frac{1}{4}\big([X+Y]-[X-Y]\big).$$

Proposition I.98. *Let S be a $\mathbb{G}-$adapted process such that it can be written in the form $S = M + A$, where $M \in \mathcal{M}(\mathbb{G};\mathbb{R})$ and $A \in \mathcal{V}(\mathbb{G};\mathbb{R})$. Then, the following are equivalent:*

(i) $S \in \mathcal{S}_{\mathrm{sp}}(\mathbb{G};\mathbb{R})$.
(ii) $\sup_{u\le\cdot}|S_u - S_0| \in \mathcal{A}^+(\mathbb{G};\mathbb{R})$.

Proof. See [41, Proposition I.4.23]. □

Proposition I.99. *Let $X,Y \in \mathcal{H}^2(\mathbb{G};\mathbb{R})$. Then $[X,Y] \in \mathcal{A}(\mathbb{G};\mathbb{R})$ and $[X,Y]-\langle X,Y\rangle \in \mathcal{M}(\mathbb{G};\mathbb{R})$.*

Proof. See [41, Proposition I.4.50]. □

Theorem I.100 (Kunita–Watanabe Inequality)**.** *Let $X,Y \in \mathcal{M}(\mathbb{G};\mathbb{R})$ and H,K be $\mathcal{B}\otimes\mathcal{G}-$measurable processes. Then for every $t \in \overline{\mathbb{R}}_+$[19] holds*

$$\int_{(0,\cdot]} |H_s|\,|K_s|\,\mathrm{dVar}\big([X,Y]\big)_s \le \Big(\int_{(0,\cdot]} |H_s|^2\,\mathrm{d}[X]_s\Big)^{\frac12}\Big(\int_{(0,\cdot]} |K_s|^2\,\mathrm{d}[Y]_s\Big)^{\frac12}\ \mathbb{P}-a.s. \tag{I.24}$$

and if $\langle X\rangle, \langle Y\rangle$ and $\langle X,Y\rangle$ all exist,

$$\int_{(0,\cdot]} |H_s|\,|K_s|\,\mathrm{dVar}\big(\langle X,Y\rangle\big)_s \le \Big(\int_{(0,\cdot]} |H_s|^2\,\mathrm{d}\langle X\rangle_s\Big)^{\frac12}\Big(\int_{(0,\cdot]} |K_s|^2\,\mathrm{d}\langle Y\rangle_s\Big)^{\frac12}\ \mathbb{P}-a.s.. \tag{I.25}$$

Moreover,

$$\mathbb{E}\Big[\int_{(0,\cdot]} |H_s|\,|K_s|\,\mathrm{dVar}\big([X,Y]\big)_s\Big] \le \mathbb{E}\Big[\int_{(0,\cdot]} |H_s|^2\,\mathrm{d}[X]_s\Big]^{\frac12}\mathbb{E}\Big[\int_{(0,\cdot]} |K_s|^2\,\mathrm{d}[Y]_s\Big]^{\frac12} \tag{I.26}$$

and if $\langle X\rangle, \langle Y\rangle$ and $\langle X,Y\rangle$ all exist,

$$\mathbb{E}\Big[\int_{(0,\cdot]} |H_s|\,|K_s|\,\mathrm{dVar}\big(\langle X,Y\rangle\big)_s\Big] \le \mathbb{E}\Big[\int_{(0,\cdot]} |H_s|^2\,\mathrm{d}\langle X\rangle_s\Big]^{\frac12}\mathbb{E}\Big[\int_{(0,\cdot]} |K_s|^2\,\mathrm{d}\langle Y\rangle_s\Big]^{\frac12}. \tag{I.27}$$

[19] In the case $t = \infty$ the domain of the Lebesgue–Stieltjes integral is written $(0,\infty)$.

PROOF. See Cohen and Elliott [20, Theorem 11.4.1]. □

Theorem I.101 (Burkholder–Davis–Gundy $\Phi-$Inequality)**.** *If X is a real-valued $\mathbb{G}-$martingale and $\Phi \in \mathcal{YF}_{\text{mod}}$, then*

$$\frac{1}{4\bar{c}_\Phi}\| \sup_{t\in\mathbb{R}_+} |X|\|_\Phi \leq \|[X]^{\frac{1}{2}}_\infty\|_\Phi \leq 6\bar{c}_\Phi\| \sup_{t\in\mathbb{R}_+} |X|\|_\Phi.$$

If at least one of the quantities $\| \sup_{t\in\mathbb{R}_+} |X|\|_\Phi$, $\|[X]^{\frac{1}{2}}_\infty\|_\Phi$, $\mathbb{E}[\Phi(\sup_{t\in\mathbb{R}_+} |X|)]$ and $\mathbb{E}[\Phi([X]^{\frac{1}{2}}_\infty)]$ is finite, then there exist positive constants c_Φ, C_Φ (depending only on Φ) such that

$$c_\Phi\mathbb{E}[\Phi(\sup_{t\in\mathbb{R}_+} |X|)] \leq \mathbb{E}[\Phi([X]^{\frac{1}{2}}_\infty)] \leq C_\Phi\mathbb{E}[\Phi(\sup_{t\in\mathbb{R}_+} |X|)].$$

PROOF. For the first inequality see Dellacherie and Meyer [27, Theorem VII.92, p. 287]. Observe that we can obtain the Burkholder–Davis–Gundy Inequality in a slightly more general framework, *i.e.* when Φ is a convex and moderate function such that $\Phi(0) = 0$.

For the second inequality, since Φ is moderate we have from Proposition I.53.(v) that $\mathbb{E}[\Phi([X]^{\frac{1}{2}}_\infty)] < \infty$ if and only if $\|[X]^{\frac{1}{2}}_\infty\|_\Phi < \infty$ and analogously $\mathbb{E}[\Phi(\sup_{t\in\mathbb{R}_+} |X|)] < \infty$ if and only if $\| \sup_{t\in\mathbb{R}_+} |X|\|_\Phi < \infty$. Then, for the constants c_Φ, C_Φ of second inequality see He et al. [35, Theorem 10.36]. □

We state also the Burkholder–Davis–Gundy Inequality in its mostly-known form because it will allow us to identify the space

$$\mathcal{H}^1(\mathbb{G};\mathbb{R}) := \big\{M \in \mathcal{M}(\mathbb{G};\mathbb{R}), \sup_{t\in\mathbb{R}_+} \big|M_t\big| \in \mathbb{L}^1(\mathcal{G};\mathbb{R})\big\}$$

with the space $\big\{M \in \mathcal{M}(\mathbb{G};\mathbb{R}), [M]^{\frac{1}{2}}_\infty \in \mathbb{L}^1(\mathcal{G};\mathbb{R})\big\}$. Obviously, $\mathcal{H}^1(\mathbb{G};\mathbb{R}) \subset \mathcal{M}(\mathbb{G};\mathbb{R})$. Moreover, the inclusion is strict; solve Revuz and Yor [60, Exercise 3.15 of Chapter II].

Lemma I.102. *Let $\Phi \in \mathcal{YF}$ and $X, Y \in \mathcal{M}(\mathbb{G};\mathbb{R})$ such that $\mathbb{E}\big[\Phi\big([X]^{\frac{1}{2}}_\infty\big)\big]$, $\mathbb{E}\big[\Phi^\star\big([Y]^{\frac{1}{2}}_\infty\big)\big] < \infty$. Then $[X, Y] \in \mathcal{A}(\mathbb{G};\mathbb{R})$ and, consequently, its compensator $[X, Y]^{p,\mathbb{G}}$ is well-defined.*

PROOF. The integrability conditions $\mathbb{E}\big[\Phi\big([X]^{\frac{1}{2}}_\infty\big)\big]$, $\mathbb{E}\big[\Phi^\star\big([Y]^{\frac{1}{2}}_\infty\big)\big] < \infty$ verify that $[X]^{\frac{1}{2}}_\infty$, $[Y]^{\frac{1}{2}}_\infty$ are finite $\mathbb{P}-$almost surely. By Kunita–Watanabe Inequality (I.24) and Young's Inequality, see Lemma I.44.(i), we have

$$\text{Var}\big([X, Y]\big)_\infty \leq [X]^{\frac{1}{2}}_\infty[X]^{\frac{1}{2}}_\infty \leq \Phi\big([X]^{\frac{1}{2}}_\infty\big) + \Phi^\star\big([Y]^{\frac{1}{2}}_\infty\big).$$

Then, we can conclude the existence of $[X, Y]^{p,\mathbb{G}}$ by Theorem I.64. □

Theorem I.103 (Burkholder–Davis–Gundy $\mathbb{L}^\vartheta-$Inequality)**.** *If X is a real-valued $\mathbb{G}-$martingale with $M_0 = 0$ and $\vartheta \geq 1$, then there exist constants c_ϑ, C_ϑ such that*

$$c_\vartheta\mathbb{E}\Big[(\sup_{t\in\mathbb{R}_+} |X|)^\vartheta\Big]^{\frac{1}{\vartheta}} \leq \mathbb{E}\Big[[X]^{\frac{\vartheta}{2}}_\infty\Big]^{\frac{1}{\vartheta}} \leq \bar{c}_\vartheta\mathbb{E}\Big[(\sup_{t\in\mathbb{R}_+} |X|)^\vartheta\Big]^{\frac{1}{\vartheta}}.$$

PROOF. See Cohen and Elliott [20, Theorem 11.5.5]. □

We conclude this section with the celebrated Doob's Inequality and a useful lemma whose main argument relies on Doob's $\mathbb{L}^\Upsilon-$Inequality. The latter will be useful in Chapter IV.

Theorem I.104 (Doob's $\mathbb{L}^\Upsilon-$Inequality)**.** *Let $\Upsilon \in \mathcal{YF}$ such that $\Upsilon^\star \in \mathcal{YF}_{\text{mod}}$ and X be*

(i) *either a real-valued $\mathbb{G}-$martingale*
(ii) *or a real-valued positive càdlàg $\mathbb{G}-$submartingale.*

Then

$$\| \sup_{t\in\mathbb{R}_+} |X_t|\|_\Upsilon \leq 2\,\bar{c}_{\Upsilon^\star}\|X_\infty\|_\Upsilon.$$

PROOF. If X is a positive càdlàg $\mathbb{G}-$submartingale see Dellacherie and Meyer [27, Paragraph VI.103, p. 169] and [27, Paragraph VI.97 p. 161],[27, Remark VI.21, p. 82] for the notation. If X is a $\mathbb{G}-$martingale,[20] then by Jensen's Inequality $|X|$ is a positive càdlàg $\mathbb{G}-$submartingale, which reduces to the former case. □

[20] Recall that we use its càdlàg modification

The classical form of Doob's Inequality is presented in the next corollary. Observe however that the constant of Corollary I.105 is improved comparing to the one we get from Theorem I.104. The interested reader may have the answer in Rao and Ren [58, Section 2.4].

Corollary I.105 (Doob's $\mathbb{L}^{\vartheta}-$Inequality)**.** *Let $\vartheta, \vartheta^{\star} \in (1,\infty)$ be such that $\frac{1}{\vartheta}+\frac{1}{\vartheta^{\star}}=1$ holds and X be*

(i) *either a real-valued $\mathbb{G}-$martingale*
(ii) *or a real-valued positive càdlàg $\mathbb{G}-$submartingale.*

Then

$$\Big(\mathbb{E}\Big[\sup_{t\in\mathbb{R}_+}|X_t|^{\vartheta}\Big]\Big)^{\frac{1}{\vartheta}} \leq \vartheta^{\star}\Big(\mathbb{E}\big[\,|X_{\infty}|^{\vartheta}\,\big]\Big)^{\frac{1}{\vartheta}}.$$

Proof. See Dellacherie and Meyer [27, Paragraph VI.101, p. 166]. □

Lemma I.106. *Let $(\mathbb{G}^k)_{k\in\overline{\mathbb{N}}}$ be a sequence of filtrations, where $\mathbb{G}^k := \big(\mathcal{G}^k_t\big)_{t\in\mathbb{R}_+}$, and $(\xi^k)_{k\in\overline{\mathbb{N}}}$ be a sequence of $\mathbb{R}^p-$valued random variables such that $\big(\|\xi^k\|_2^2\big)_{k\in\overline{\mathbb{N}}}$ is uniformly integrable. Then the sequence $\Big(\sup_{t\in\mathbb{R}_+}\big\|\mathbb{E}\big[\xi^k\big|\mathcal{G}^k_t\big]\big\|_2^2\Big)$ is uniformly integrable.*

Proof. Let $\Phi \in \mathcal{YF}_{\text{mod}}$ be a Young function for which the sequence $\big(\|\xi^k\|_2^2\big)_{k\in\overline{\mathbb{N}}}$ satisfies the de La Vallée Poussin–Meyer Criterion, *i.e.*

$$\sup_{k\in\overline{\mathbb{N}}}\mathbb{E}\Big[\Phi\big(\|\xi^k\|_2^2\big)\Big] < \infty. \tag{I.28}$$

Then $\Psi := \Phi\circ\text{quad} \in \mathcal{YF}_{\text{mod}}$ (see Proposition I.51.(i)). Using the fact that Φ is increasing, we can write (I.28) as

$$M := \sup_{k\in\overline{\mathbb{N}}}\mathbb{E}\Big[\Psi\big(\|\xi^k\|_2\big)\Big] < \infty. \tag{I.29}$$

The latter form will be more convenient for later use. Before we proceed to prove the claim of the lemma, we provide some helpful results. In order to ease notation, let us denote the Orlicz norm $\big\|\|\xi^k\|_2\big\|_{\Psi}$ by Ξ^k for $k\in\overline{\mathbb{N}}$. Observe that $\Xi^k<\infty$, for every $k\in\overline{\mathbb{N}}$ because of (I.29). We are going to prove that $\sup_{k\in\overline{\mathbb{N}}}\Xi^k<\infty$. To this end, observe that

- If $M\leq 1$, then $\sup_{k\in\overline{\mathbb{N}}}\Xi^k\leq 1$ by the definition of the Orlicz norm; see Notation I.52.
- If $M>1$, then using the convexity of Ψ and the fact that $\Psi(0)=0$ we obtain

$$\mathbb{E}\Big[\Psi\Big(\frac{\|\xi^k\|_2}{M}\Big)\Big] \leq \frac{\mathbb{E}\big[\Psi\big(\|\xi^k\|_2\big)\big]}{M} \overset{\text{(I.29)}}{\leq} 1 \text{ which implies } \Xi^k\leq M.$$

and consequently

$$\sup_{k\in\overline{\mathbb{N}}}\Xi^k \leq 1\vee M \overset{\text{(I.29)}}{<} \infty. \tag{I.30}$$

We proceed now to prove uniform integrability of $\big(\sup_{t\in\mathbb{R}_+}\big\|\mathbb{E}\big[\xi^k\big|\mathcal{G}^k_t\big]\big\|_2^2\big)_{k\in\overline{\mathbb{N}}}$. By de La Vallée Poussin Theorem, see Theorem I.27, it suffices to prove that $\sup_{k\in\overline{\mathbb{N}}}\mathbb{E}\Big[\Phi\Big(\sup_{t\in\mathbb{R}_+}\big\|\mathbb{E}[\xi^k|\mathcal{G}^k_t]\big\|_2^2\Big)\Big]<\infty$, or equivalently that $\sup_{k\in\overline{\mathbb{N}}}\mathbb{E}\Big[\Psi\big(\sqrt{2}S^k\big)\Big]<\infty$, where we have defined $S^k := \sup_{t\in\mathbb{R}_+}\big\|\mathbb{E}[\xi^k|\mathcal{G}^k_t]\big\|_2$. Analogously to (I.14) we can obtain

$$\mathbb{E}\Big[\Psi\big(\sqrt{2}S^k\big)\Big] \leq 1\vee \overline{c}_{\Psi}^{\|\sqrt{2}S^k\|_{\Psi}}. \tag{I.31}$$

Recall that $\Psi\in\mathcal{YF}_{\text{mod}}$ due to its definition $\Psi=\Phi\circ\text{quad}$. By Proposition I.51 we have that also $\Psi^{\star}\in\mathcal{YF}_{\text{mod}}$. Therefore, $\overline{c}_{\Psi}, \overline{c}_{\Psi^{\star}} = \underline{c}_{\Psi}\in(1,\infty)$. For the validity of the last argument recall Notation I.41, Definition I.43 and Corollary I.45. Consequently, we can conclude the required property if we prove that $\sup_{k\in\overline{\mathbb{N}}}\|S^k\|_{\Psi}<\infty$, where the $\sqrt{2}$ can be taken out, since $\|\cdot\|_{\Psi}$ is a norm. To this end we will utilise Doob's $\mathbb{L}^{\Upsilon}-$inequality. We can obtain now by standard properties of norms and the fact that $\|\cdot\|_2\leq\|\cdot\|_1$ (and consequently $S^k\leq\sup_{t\in\mathbb{R}_+}\big\|\mathbb{E}[\xi^k|\mathcal{G}^k_t]\big\|_1$) the following inequalities

$$\begin{aligned}\big\|S^k\big\|_{\Psi} &\leq \Big\|\sup_{t\in\mathbb{R}_+}\big\|\mathbb{E}[\xi^k|\mathcal{G}^k_t]\big\|_1\Big\|_{\Psi} \leq \sum_{j=1}^{p}\Big\|\sup_{t\in\mathbb{R}_+}\big|\mathbb{E}[\xi^{k,j}|\mathcal{G}^k_t]\big|\Big\|_{\Psi} \overset{\text{Doob's In.}}{\leq} 2\overline{c}_{\Psi^{\star}}\sum_{j=1}^{p}\big\|\xi^{k,i}\big\|_{\Psi}\\ &\leq 2p\overline{c}_{\Psi^{\star}}\big\|\|\xi^k\|_2\big\|_{\Psi} = 2p\overline{c}_{\Psi^{\star}}\Xi^k.\end{aligned}$$

The above inequality and (I.30) imply the desired

$$\sup_{k\in\overline{\mathbb{N}}}\left\|S^k\right\|_{\Psi}\leq 2p\overline{c}_{\Psi^\star}\sup_{k\in\overline{\mathbb{N}}}\Xi^k<\infty,$$

which, in view of Equation (I.31), implies also the finiteness of $\sup_{k\in\overline{\mathbb{N}}}\mathbb{E}\big[\Psi(\sqrt{2}S^k)\big]$. Therefore, the sequence $\Big(\sup_{t\in\mathbb{R}_+}\big\|\mathbb{E}\big[\xi^k\big|\mathcal{G}^k_t\big]\big\|_2^2\Big)$ is uniformly integrable since it satisfies the de La Vallée Poussin theorem for the Young function Φ. □

I.6. The Skorokhod space

In this section we present the Skorokhod space $\mathbb{D}(E)$ which is the natural path-space for semi-martingales adapted to some filtration that satisfies the usual conditions. The weaker of the topologies the Skorokhod space will be endowed with, will be the (Skorokhod) J_1-topology. The J_1-topology is metrisable and makes the space to be complete and separable, *i.e.* $\mathbb{D}(E)$ becomes Polish. The main reference of this section is Jacod and Shiryaev [41].

Definition I.107. (i) The function $\alpha:\mathbb{R}_+\longrightarrow E$ will be called *càdlàg* if α is right-continuous and for every $t\in\mathbb{R}$ the value $\alpha(t-)$ exists.

(ii) The $E-$*Skorokhod space* is the space of all càdlàg, $E-$valued functions defined on $\mathbb{R}_+$ and it will be denoted by $\mathbb{D}(E)$, *i.e.*

$$\mathbb{D}(E):=\{\alpha:\mathbb{R}_+\longrightarrow E,\alpha\text{ is càdlàg}\}.$$

The zero function will be denoted by 0.

(iii) The filtration $\mathbb{D}(E):=\big(\mathscr{D}_t(E)\big)_{t\in\mathbb{R}_+}$ is the right-continuous filtration obtained by the filtration generated by the canonical projections, *i.e.*

$$\mathscr{D}_t(E):=\bigcap_{u>t}\sigma\big(\{\alpha(s),\alpha\in\mathbb{D}(E)\text{ and }s\leq u\}\big)$$

Moreover, $\mathscr{D}(E):=\sigma\big(\{\alpha(s),\alpha\in\mathbb{D}(E)\text{ and }s\in\mathbb{R}_+\}\big)$.

We will mostly denote the elements of $\mathbb{D}(\mathbb{R})$ with small Greek letters, usually the first three one of the Greek alphabet, *i.e.* α,β and γ.

Definition I.108. (i) Let $\lambda:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ be continuous and increasing such that $\lambda(0)=0$ and $\lim_{t\to\infty}\lambda(t)=\infty$. Then λ will be called *time change*.

(ii) The set of all time changes will be denoted by Λ, *i.e.*

$$\Lambda:=\{\lambda:\mathbb{R}_+\longrightarrow\mathbb{R}_+,\lambda\text{ is continuous and increasing with }\lambda(0)=0,\lim_{t\to\infty}\lambda(t)=\infty\}.$$

(iii) To each $\alpha\in\mathbb{D}(E)$ and each interval $I\subset\mathbb{R}_+$ we associate the *moduli*

$$w(\alpha;I):=\sup_{s,t\in I}\|\alpha(s)-\alpha(t)\|_2$$

and

$$w'_N(\alpha,\theta):=\inf\Big\{\max_{i\leq r}w(\alpha;[t_{i-1},t_i)),0=t_0<t_1<\cdots<t_r=N\text{ with }\inf_{i<r}(t_i-t_{i-1})\geq\theta\Big\}.$$

Theorem I.109. *There is a metrisable topology on $\mathbb{D}(E)$, called the* Skorokhod J_1-topology*, for which the space is Polish. Moreover, the following are equivalent:*

(i) *The sequence $(\alpha^k)_{k\in\mathbb{N}}$ converges to α^∞ under the J_1-topology.*

(ii) *There exists a sequence $(\lambda^k)_{k\in\mathbb{N}}\subset\Lambda$ such that*

$$\sup_{s\in\mathbb{R}_+}|\lambda^k(s)-s|\xrightarrow[k\to\infty]{}0\quad\textit{and}\quad\sup_{s\leq N}\|\alpha^k(\lambda^k(s))-\alpha^\infty(s)\|_2\xrightarrow[k\to\infty]{}0,\ \textit{for all }N\in\mathbb{N}.\tag{I.32}$$

The metric under which $\mathbb{D}(E)$ becomes Polish will be denoted by $\delta_{\mathrm{J}_1(E)}$.[21] *Moreover, the associated Borel $\sigma-$algebra equals the $\sigma-$algebra $\mathscr{D}(E)$, which was defined Definition I.107.*(ii).

[21] We will omit the state space from the notation, when it is clear to which one we refer.

Proof. See Jacod and Shiryaev [41, Theorem VI.1.14 a)]. For the definition of δ_{J_1} see [41, Definition VI.1.26]. In [41, Lemma VI.1.30, Lemma VI.1.31, Corollary VI.1.32] it is proven that δ_{J_1} is indeed a metric. In [41, Lemma VI.1.33] it is proven that $(\mathbb{D}(E), \delta_{\mathrm{J}_1})$ is complete. The separability of the space is proven in [41, Corollary VI.1.43 a)].

For (i) $\Rightarrow$ (ii) see [41, Lemma VI.1.31], while for the implication (ii) $\Rightarrow$ (i) see [41, Lemma VI.1.44]. For the validity of the statement regarding the Borel $\sigma-$algebra and $\mathscr{D}(E)$ see [41, Lemma VI.1.45, Lemma VI.1.46]. □

Remark I.110. Recall the well-known fact that a càdlàg function α has at most countably many points of discontinuities; see Billingsley [7, Section 12, Lemma 1] for the case the domain is $[0,1]$ and then one can easily conclude for the case $\mathbb{R}_+$, as a countable union of finite sets is a countable set. Therefore, the function $\Delta\alpha$ is zero except from a set of Lebesgue measure zero.

Theorem I.111. *A set $D \subset \mathbb{D}(E)$ is* relatively compact for the Skorokhod J_1-topology *if and only if for every $N \in \mathbb{N}$ both of the following conditions hold:*

$$\sup_{\alpha\in D}\sup_{s\leq N}\|\alpha(s)\|_2 < \infty \text{ and } \lim_{\theta\downarrow 0}\sup_{\alpha\in D} w'_N(\alpha,\theta) = 0 \text{ for every } N \in \mathbb{N}.$$

Proof. See Jacod and Shiryaev [41, Theorem VI.1.14 b)]. More precisely, [41, Lemma VI.1.39] proves the necessity and [41, Corollary VI.1.43] provides the sufficiency. □

We will say that the sequence $(\alpha^k)_{k\in\overline{\mathbb{N}}}$ is a $\delta_{\mathrm{J}_1(E)}-$convergent sequence if it satisfies Theorem I.109 and we will denote its convergence also by $\alpha^k \xrightarrow{\mathrm{J}_1(E)} \alpha^\infty$. Apart from the J_1-topology we will equip the space $\mathbb{D}(E)$ with two more topologies, which are presented below.

Notation I.112. Let $\alpha, \beta \in \mathbb{D}(E)$.

- The *locally uniform topology* is the one associated with the metric δ_{lu}, where

$$\delta_{\mathrm{lu}}(\alpha,\beta) = \sum_{k\in\mathbb{N}} \frac{1}{2^k}\big(w(\alpha-\beta;[0,k]\wedge 1\big).$$

The convergence of a sequence $(\alpha^k)_{k\in\mathbb{N}}$ to α^∞ under the metric δ_{lu} will be denoted by $\alpha^k \xrightarrow{\mathrm{lu}} \alpha^\infty$.

- The *uniform topology* is associated with the norm $\|\cdot\|_\infty$, where$\|\alpha\|_\infty := \sup_{s\in\mathbb{R}_+}\|\alpha(s)\|_2$. The convergence of a sequence $(\alpha^k)_{k\in\mathbb{N}}$ to α^∞ under the metric $\delta_{\|\cdot\|_\infty}$ will be denoted by $\alpha^k \xrightarrow{\|\cdot\|_\infty} \alpha^\infty$.

The next proposition provides the relationship between $\delta_{\mathrm{J}_1(\mathbb{D}(E))}$ and δ_{lu}.

Proposition I.113. (i) *The J_1-topology is weaker than the locally uniform topology.*
(ii) *Let $\alpha^\infty \in \mathbb{D}(E)$ be a continuous function. Then*

$$\alpha^k \xrightarrow{\mathrm{J}_1(\mathbb{D}(E))} \alpha^\infty \text{ if and only if } \alpha^k \xrightarrow{\delta_{\mathrm{lu}}} \alpha^\infty.$$

Proof. See Jacod and Shiryaev [41, Proposition VI.1.17]. □

Remark I.114. For the metrics $\delta_{\mathrm{J}_1(\mathbb{D}(E))}$, δ_{lu} and $\delta_{\|\cdot\|_\infty}$ holds

$$\delta_{\mathrm{J}_1(\mathbb{D}(E))}(\alpha,0) = \delta_{\mathrm{lu}}(\alpha,0) \text{ for every } \alpha \in \mathbb{D}(E)$$

and

$$\delta_{\mathrm{J}_1(\mathbb{D}(E))}(\alpha,\beta) \leq \delta_{\mathrm{lu}}(\alpha,\beta) \leq \sup_{s\in\mathbb{R}_+}\|\alpha(s)-\beta(s)\|_2 \text{ for every } \alpha,\beta \in \mathbb{D}(E).$$

The following lemma is essentially [35, Example 15.11]. We provide an alternative proof in some special cases of the aforementioned result since we will need later to elaborate on similar cases.

Lemma I.115. *Let $(E, \|\cdot\|_2)$, $(x_i^k)_{k\in\overline{\mathbb{N}}} \subset E$ and $(t_i^k)_{k\in\overline{\mathbb{N}}} \subset \mathbb{R}_+$, for $i = 1,2$, such that $t_1^k < t_2^k$ for every $k \in \overline{\mathbb{N}}$.*

(i) *If $t_1^k \xrightarrow{|\cdot|} \infty$, then $x_1^k \mathbb{1}_{[t_1^k,\infty)} \xrightarrow{\mathrm{J}_1(E)} 0$.*
(ii) *If $x_1^k \xrightarrow{\|\cdot\|_2} 0$, then $x_1^k \mathbb{1}_{[t_1^k,\infty)} \xrightarrow{\mathrm{J}_1(E)} 0$.*
(iii) *If $t_1^k \xrightarrow{|\cdot|} t_1^\infty \in \mathbb{R}_+$ and $x_1^k \xrightarrow{\|\cdot\|_2} x_1^\infty$, then $x_1^k \mathbb{1}_{[t_1^k,\infty)} \xrightarrow{\mathrm{J}_1(E)} x_1^\infty \mathbb{1}_{[t_1^\infty,\infty)}$.*

(iv) *If* $t_1^k \xrightarrow{|\cdot|} t_1^\infty \in \mathbb{R}_+$, $t_2^k \xrightarrow{|\cdot|} \infty$ *and* $x_1^k \xrightarrow{\|\cdot\|_2} x_1^\infty$, *then*

$$x_1^k \mathbb{1}_{[t_1^k,\infty)} + x_2^k \mathbb{1}_{[t_2^k,\infty)} \xrightarrow{\mathrm{J}_1(E)} x_1^\infty \mathbb{1}_{[t_1^\infty,\infty)}.$$

(v) *If* $t_1^k \xrightarrow{|\cdot|} t_1^\infty$, $t_2^k \xrightarrow{|\cdot|} t_2^\infty$, *where* $t_1^\infty < t_2^\infty \in \mathbb{R}_+$, $x_1^k \xrightarrow{\|\cdot\|_2} x_1^\infty$ *and* $x_2^k \xrightarrow{\|\cdot\|_2} x_2^\infty$ *then*

$$x_1^k \mathbb{1}_{[t_1^k,\infty)} + x_2^k \mathbb{1}_{[t_2^k,\infty)} \xrightarrow{\mathrm{J}_1(E)} x_1^\infty \mathbb{1}_{[t_1^\infty,\infty)} + x_2^\infty \mathbb{1}_{[t_2^\infty,\infty)}.$$

Proof. For (i) and (ii), since the limit function is (trivially) continuous, we can use Proposition I.113 in order to justify that we can choose without loss of generality $\lambda^k = \mathrm{Id}$ for every $k \in \mathbb{N}$. In view of Theorem I.109 it is therefore only left to prove that

$$\sup_{s \le N} \|\alpha^k(s)\|_2 \longrightarrow 0 \text{ for every } N \in \mathbb{N}.$$

(i) Assume that $t_1^k \xrightarrow{|\cdot|} \infty$. Then for every $N \in \mathbb{N}$ there exists a $k_0(N) \in \mathbb{N}$ such that $t_1^k \ge N$ for every $k \ge k_0(N)$. Therefore, for every fixed $N \in \mathbb{N}$, it holds $\sup_{s \le N} \|\alpha^k(s)\|_2 = 0$ for every $k \ge k_0(N)$.

(ii) Assume now $x_1^k \xrightarrow{\|\cdot\|_1} 0$, *i.e.* for every $\varepsilon > 0$ there exists a $k_0(\varepsilon) \in \mathbb{N}$ such that $\|x^k\|_2 < \varepsilon$. Let us fix an $\varepsilon > 0$ and an $N \in \mathbb{N}$, then $\sup_{s \le N} \|x^k\|_2 < \varepsilon$ for every $k \ge k_0(\varepsilon)$, which allows us to conclude. Observe that the choice of k_0 is independent of N.

For (iii)-(v) see He et al. [35, Example 15.11], which deals with convergence of piecewise constant functions with possibly infinitely many jumps. For this reason recall conditions (i)-(iii) of He et al. [35, Remark 15.32] for the detailed properties of the jump times. □

Remark I.116. The alert reader may have observed the condition imposed in Lemma I.115.(v) for the jump-times t_1^∞ and t_2^∞, *i.e.* $t_1^\infty < t_2^\infty$. Let us examine what may fail in the case $t_1^\infty = t_2^\infty$. To this end we set the following framework: assume that $t_1^k \uparrow \bar{t}$ and $t_2^k \downarrow \bar{t}$, *i.e.* $t_1^k \to \bar{t}$ with $t_1^k \le \bar{t}$ and $t_2^k \to \bar{t}$ with $\bar{t} \le t_2^k$. Then, in view of Lemma I.115.(iii) we have that

$$\alpha_1^k := \mathbb{1}_{[t_1^k,\infty)} \xrightarrow{\mathrm{J}_1(\mathbb{R})} \mathbb{1}_{[\bar{t},\infty)} \text{ and } \alpha_2^k := -\mathbb{1}_{[t_2^k,\infty)} \xrightarrow{\mathrm{J}_1(\mathbb{R})} -\mathbb{1}_{[\bar{t},\infty)}. \tag{I.33}$$

On the other hand, the sequence $(\alpha_1^k + \alpha_2^k)_{k\in\mathbb{N}}$ does not converge under the J_1-topology, since $(\mathbb{1}_{[t_1^k,t_2^k)})_{k\in\mathbb{N}}$ fails to be a $\delta_{\mathrm{J}_1(\mathbb{R})}-$Cauchy sequence. For the last claim see the comments right after Billingsley [7, Section 12, Lemma 2, p. 126] in conjunction with [7, Example 12.2].

Let us explain a bit more this (counter-)example. By Theorem I.109 we have that there exists a sequence of time changes $(\lambda_i^k)_{k\in\mathbb{N}}$, for $i = 1, 2$, such that the respective convergence in (I.33) hold. However, the sum of the converging sequences is not converging, because we cannot find a sequence of time changes $(\lambda^k)_{k\in\mathbb{N}}$ such that it is *finally common* for the sequences $(\mathbb{1}_{[t_1^k,\infty)})_{k\in\mathbb{N}}$, $(\mathbb{1}_{[t_2^k,\infty)})_{k\in\mathbb{N}}$ and such that (I.33) holds.

After providing the main results for the Skorokhod J_1-topology, we need to clarify some details.

Remark I.117. (i) The space $(\mathbb{D}(E), \delta_{\mathrm{J}_1(E)})$ is not a topological vector space. This is direct by Remark I.116 which proves that the addition is not $\delta_{\mathrm{J}_1}-$continuous.

(ii) The space $\mathbb{D}(\mathbb{R}^{p\times q})$ can be identified with $\mathbb{D}(\mathbb{R})^{p\times q}$. Let us endow the space

$$\mathbb{D}(\mathbb{R})^{p\times q} := \{\alpha : \mathbb{R}_+ \longrightarrow \mathbb{R}^{p\times q}, \alpha^{ij} \text{ is càdlàg for every } 1 \le i \le p, 1 \le q\}.$$

with the product topology, which is compatible with the metric

$$\delta_\Pi(\alpha, \beta) := \sum_{i=1}^p \sum_{j=1}^q \delta_{\mathrm{J}_1(\mathbb{R})}(\alpha^{ij}, \beta^{ij}).$$

However, the topological spaces $(\mathbb{D}(\mathbb{R}^{p\times q}), \delta_{\mathrm{J}_1}(\mathbb{R}^{p\times q}))$ and $(\mathbb{D}(\mathbb{R})^{p\times q}, \delta_\Pi)$ do not coincide. Indeed, by Remark I.116 we have

$$\delta_\Pi\big((\alpha_1^k, \alpha_2^k), (\mathbb{1}_{[\bar{t},\infty)}, \mathbb{1}_{[\bar{t},\infty)})\big) \longrightarrow 0 \text{ but } \delta_{\mathrm{J}_1(\mathbb{R}^{p\times q})}\big((\alpha_1^k, \alpha_2^k), (\mathbb{1}_{[\bar{t},\infty)}, \mathbb{1}_{[\bar{t},\infty)})\big) \not\longrightarrow 0.$$

Now it may be clear to the reader why we have always indicated the state space in the notation of the metric. We will (almost) always indicate the state space in the notation for the sake of clarity.

(iii) For notational convenience we have stated the main results for the space $\mathbb{D}(E)$ and not for the space $\mathbb{D}(\mathbb{R}^d)$, where d is a natural integer, as in Jacod and Shiryaev [41, Chapter VI]. This can be done without loss of generality by defining an isometry from every finite dimensional space $(E, \|\cdot\|_2)$ to the space $(\mathbb{R}^p, \|\cdot\|_2)$, where p denotes the dimension of the space E.

The following are classical criteria for obtaining the joint convergence.

Proposition I.118. *Let $(\alpha^k)_{k\in\overline{\mathbb{N}}} \subset \mathbb{D}(E)$. The convergence $\alpha^k \xrightarrow{\mathrm{J}_1(E)} \alpha^\infty$ holds if and only if whenever $(t^k)_{k\in\overline{\mathbb{N}}} \subset \mathbb{R}_+$ be such that $t^k \longrightarrow t^\infty$ the following conditions hold:*

(i) $\left\|\alpha^k(t^k) - \alpha^\infty(t^\infty)\right\|_2 \wedge \left\|\alpha^k(t^k) - \alpha^\infty(t^\infty-)\right\|_2 \longrightarrow 0$.

(ii) *If $\left\|\alpha^k(t^k) - \alpha^\infty(t^\infty)\right\|_2 \longrightarrow 0$ and $(s^k)_{k\in\mathbb{N}} \subset \mathbb{R}_+$ be such that $t^k \leq s^k$ for every $k \in \mathbb{N}$ and $s^k \longrightarrow t^\infty$, then $\left\|\alpha^k(s^k) - \alpha^\infty(t^\infty)\right\|_2 \longrightarrow 0$.*

(iii) *If $\left\|\alpha^k(t^k) - \alpha^\infty(t^\infty-)\right\|_2 \longrightarrow 0$ and $(s^k)_{k\in\mathbb{N}} \subset \mathbb{R}_+$ be such that $s^k \leq t^k$ for every $k \in \mathbb{N}$ and $s^k \longrightarrow t^\infty$, then $\left\|\alpha^k(s^k) - \alpha^\infty(t^\infty-)\right\|_2 \longrightarrow 0$.*

Proof. See Ethier and Kurtz [33, Proposition 3.6.5]. □

Remark I.119. Assume $\alpha^k \xrightarrow{\mathrm{J}_1(E)} \alpha^\infty$ and $t^k \longrightarrow t^\infty$. Then by Proposition I.118.(i) we have that the only possible limit points of the sequences $\left(\alpha^k(t^k)\right)_{k\in\mathbb{N}}$, $\left(\alpha^k(t^k-)\right)_{k\in\mathbb{N}}$ are $\alpha^\infty(t^\infty)$ and $\alpha^\infty(t^\infty-)$.

Proposition I.120. *Let $(\alpha^k)_{k\in\overline{\mathbb{N}}} \subset \mathbb{D}(E_1)$ and $(\beta^k)_{k\in\overline{\mathbb{N}}} \subset \mathbb{D}(E_2)$. The convergence $(\alpha^k, \beta^k) \xrightarrow{\mathrm{J}_1(E_1\times E_2)} (\alpha^\infty, \beta^\infty)$ holds if and only if*

(i) $\alpha^k \xrightarrow{\mathrm{J}_1(E_1)} \alpha^\infty$,

(ii) $\beta^k \xrightarrow{\mathrm{J}_1(E_2)} \beta^\infty$ *and*

(iii) *for every $t \in \mathbb{R}_+$ there exists a sequence $(t^k)_{k\in\mathbb{N}}$ with $t^k \longrightarrow t$ and such that $\Delta\alpha^k(t^k) \xrightarrow{\|\cdot\|_2} \Delta\alpha^\infty(t)$ and $\Delta\beta^k(t^k) \xrightarrow{\|\cdot\|_2} \Delta\beta^\infty(t)$.*

Proof. For the implication (i) ⇒ (ii) apply Jacod and Shiryaev [41, Proposition VI.2.1 a)] and then conclude using Remark I.117.(i). For the implication (ii) ⇒ (i) apply [41, Proposition VI.2.2 b)] for the spaces $\mathbb{D}(E_1)$, $\mathbb{D}(E_2)$ and then conclude using Remark I.117.(i). □

Looking back to Remark I.116, it is clear that there exists no sequence $(t^k)_{k\in\mathbb{N}}$ that satisfies the condition (iii) of Proposition I.120 for the point $\bar{t}$.

The following proposition will be helpful to prove Corollary I.125.

Proposition I.121. *Let $(\alpha^k)_{k\in\overline{\mathbb{N}}}$ be a $\delta_{\mathrm{J}_1(E)}-$convergent sequence and $t \in \mathbb{R}_+$.*

(i) *There exists a sequence $(t^k)_{k\in\mathbb{N}}$ such that $t^k \longrightarrow t$, $\alpha^k(t^k) \longrightarrow \alpha^\infty(t)$ and $\alpha^k(t^k-) \longrightarrow \alpha^\infty(t-)$. Therefore, the convergence $\Delta\alpha^k(t^k) \xrightarrow{\|\cdot\|_2} \Delta\alpha^\infty(t)$ also holds.*

(ii) *Let $\Delta\alpha^\infty(t) \neq 0$ and $(t^k)_{k\in\mathbb{N}}$ be a sequence such that $t^k \longrightarrow t$ and $\Delta\alpha^k(t^k) \longrightarrow \Delta\alpha^\infty(t)$. Then, any other sequence $(s^k)_{k\in\mathbb{N}}$ for which holds $s^k \longrightarrow t$ and $\Delta\alpha^k(s^k) \xrightarrow{\|\cdot\|_2} \Delta\alpha^\infty(t)$ coincides with $(t^k)_{k\in\mathbb{N}}$ for k large enough.*

Proof. See Jacod and Shiryaev [41, Proposition VI.2.1]. □

The next classical result informs us that we can obtain the convergence of the sum of jointly convergent sequences.

Corollary I.122. *Let $\left((\alpha^k, \beta^k)\right)_{k\in\mathbb{N}}$ be a $\delta_{\mathrm{J}_1(E\times E)}-$convergent sequence. Then*

$$(\alpha^k, \beta^k, \alpha^k + \beta^k) \xrightarrow{\mathrm{J}_1(E\times E\times E)} (\alpha^\infty, \beta^\infty, \alpha^\infty + \beta^\infty).$$

Proof. We will prove it for $E = \mathbb{R}$. Let, therefore, $(\alpha^k, \beta^k)_{k\in\overline{\mathbb{N}}} \subset \mathbb{D}(\mathbb{R})$ and assume that

$$(\alpha^k, \beta^k) \xrightarrow{\mathrm{J}_1(\mathbb{R}^2)} (\alpha^\infty, \beta^\infty).$$

Choose a $t \in \mathbb{R}_+$ such that $(\Delta\alpha^\infty(t), \Delta\beta^\infty(t)) \neq 0$. Then, by Proposition I.121 there exists a sequence $(t^k)_{k\in\mathbb{N}}$ such that $t^k \longrightarrow t$ and

$$(\Delta\alpha^k(t^k), \Delta\beta^k(t^k)) \xrightarrow{\|\cdot\|_2} (\Delta\alpha^\infty(t), \Delta\beta^\infty(t)).$$

In particular, we can conclude that $\Delta\alpha^{k,1}(t^k) + \Delta\alpha^{k,2}(t^k) \xrightarrow{|\cdot|} \Delta\alpha^{\infty,1}(t) + \Delta\alpha^{\infty,2}(t)$. Now, we can conclude by Proposition I.120 that

$$(\alpha^k, \beta^k, \alpha^k + \beta^k) \longrightarrow (\alpha^\infty, \beta^\infty, \alpha^\infty + \beta^\infty).$$

□

Remark I.123. The result above does not depend on the dimensions of the state spaces, and can be generalized inductively to an arbitrary number of sequences, as long as a common converging sequence exists.

A convenient variation of the Proposition I.121 is provided in Coquet, Mémin, and Słomiński [23, Lemma 1], which we state also here.

Lemma I.124. *Let* $(\alpha^k)_{k\in\overline{\mathbb{N}}} \subset \mathbb{D}(\mathbb{R}^p)$. *The convergence* $\alpha^k \xrightarrow{\mathrm{J}_1(\mathbb{R}^p)} \alpha^\infty$ *holds if and only if the following hold*

(i) $\alpha^{k,i} \xrightarrow{\mathrm{J}_1(\mathbb{R})} \alpha^{\infty,i}$, *for* $i = 1, \dots, p$,

(ii) $\sum_{i=1}^{q} \alpha^{k,i} \xrightarrow{\mathrm{J}_1(\mathbb{R})} \sum_{i=1}^{q} \alpha^{\infty,i}$, *for* $q = 1, \dots, p$.

We can provide now a corollary which will be proven quite convenient, since it allows us to conclude the joint convergence of two sequences once we can find a common element between the elements of the two converging sequences.

Corollary I.125. *Let* $(\alpha^k)_{k\in\overline{\mathbb{N}}}$, *resp.* $(\beta^k)_{k\in\overline{\mathbb{N}}}$, *be a* $\delta_{\mathrm{J}_1(\mathbb{R}^{p_1})}-$*convergent, resp.* $\delta_{\mathrm{J}_1(\mathbb{R}^{p_2})}-$*convergent, sequence. If there exist* $1 \le i_1 \le p_1$, $1 \le i_2 \le p_2$ *such that* $\alpha^{k,i_1} = \beta^{k,i_2}$ *for every* $k \in \mathbb{N}$, *then* $(\alpha^k, \beta^k) \xrightarrow{\mathrm{J}_1(\mathbb{R}^{p_1}\times\mathbb{R}^{p_2})} (\alpha^\infty, \beta^\infty)$.

Proof. We will apply Proposition I.120. It is only left to verify that condition (iii) of the aforementioned proposition holds. To this end let us fix a $t \in \mathbb{R}_+$. By (i), resp. (ii), and Proposition I.121 we have that there exists a sequence $(t_1^k)_{k\in\mathbb{N}}$, resp. $(t_2^k)_{k\in\mathbb{N}}$, such that

$$\Delta\alpha^k(t_1^k) \longrightarrow \Delta\alpha^\infty(t), \text{ resp. } \Delta\beta^k(t_2^k) \longrightarrow \Delta\beta^\infty(t).$$

However, by (iii) and Proposition I.121.(ii) we have that $(t_1^k)_{k\in\mathbb{N}}$ and $(t_2^k)_{k\in\mathbb{N}}$ finally coincide. Therefore, we can choose one of the two time sequences, say $(t_1^k)_{k\in\mathbb{N}}$, in order to conclude

$$\Delta\alpha^k(t_1^k) \longrightarrow \Delta\alpha^\infty(t), \text{ resp. } \Delta\beta^k(t_1^k) \longrightarrow \Delta\beta^\infty(t),$$

Proposition I.120.(iii) is satisfied. □

We close this part with another convenient result.

Lemma I.126. *Let* $(\alpha^k)_{k\in\overline{\mathbb{N}}} \subset \mathbb{D}(E)$ *and* $f : (E, \|\cdot\|_2) \longrightarrow (\mathbb{R}^p, \|\cdot\|_2)$ *be continuous. If* $\alpha^k \xrightarrow{\mathrm{J}_1(E)} \alpha^\infty$ *then* $f(\alpha^k) \xrightarrow{\mathrm{J}_1(\mathbb{R}^p)} f(\alpha^\infty)$.

Proof. The continuity of f allows us to apply Proposition I.120. □

I.6.1. J_1-continuous functions. The aim of this sub–sub–section is the following: for a given $\delta_{\mathrm{J}_1(E)}-$converging sequence $(\alpha^k)_{k\in\overline{\mathbb{N}}}$ and a given (not necessarily continuous) function $g : E \longrightarrow \mathbb{R}$, we need to determine suitable conditions for the function g and a set $I \subset E$ such that

$$\sum_{0<t\le\cdot} g(\Delta\alpha^k(t))\mathbb{1}_I(\Delta\alpha^k(t)) \xrightarrow{\mathrm{J}_1(\mathbb{R})} \sum_{0<t\le\cdot} g(\Delta\alpha^\infty(t))\mathbb{1}_I(\Delta\alpha^\infty(t)). \tag{I.34}$$

This is Proposition I.134, which refines the classical result Jacod and Shiryaev [41, Corollary VI.2.8], and it will be crucial for constructing in Chapter III a family of J_1-convergent sequences of submartingales. In order to obtain Corollary I.133, we need to refine another classical result, namely [41, Proposition VI.2.7]; the refinement of the latter is Proposition I.132. For simplicity, we provide the results for $E = \mathbb{R}^q$.

Before we proceed we introduce the necessary notation and provide a simple example, which will clarify to a great extend the main idea of Proposition I.132.

Definition I.127. For $\beta \in \mathbb{D}(\mathbb{R}^q)$ and $\gamma \in \mathbb{D}(\mathbb{R})$ we introduce the sets

$$U(\beta) := \{u > 0, \exists t > 0 \text{ with } \|\Delta\beta(t)\|_2 = u\},$$
$$W(\gamma) := \{u \in \mathbb{R} \setminus \{0\}, \exists t > 0 \text{ with } \Delta\gamma(t) = u\}$$

and

$$\mathcal{I}(\gamma) := \{(v, w) \subset \mathbb{R}, v < w \text{ with } vw > 0 \text{ and } v, w \notin W(\gamma)\}.$$

For $\alpha \in \mathbb{D}(\mathbb{R}^q)$ we define the set

$$\mathcal{J}(\alpha) := \Big\{ \prod_{i=1}^{q} I_i, \text{ where } I_i \in \mathcal{I}(\alpha^i) \cup \{\mathbb{R}\} \text{ for every } i = 1, \dots, q \Big\} \setminus \{\mathbb{R}^q\}.$$

The set $W(\gamma)$, which is at most countable, collects the heights of the jumps of a real-valued function γ. The set $\mathcal{I}(\gamma)$ collects all the open intervals of $\mathbb{R} \setminus \{0\}$ with boundary points of the same sign, which, moreover, do *not* belong to $W(\gamma)$.

Remark I.128. For the rest of this subsection we will use additionally the notation α_s for the value of the arbitrary $\alpha \in \mathbb{D}(\mathbb{R}^q)$ at the point s.

Notation I.129. Let $\alpha \in \mathbb{D}(\mathbb{R}^m)$ and $I := \prod_{i=1}^m I_i \in \mathcal{J}(\alpha)$.

(i) We define the time points

$$t^0(\alpha, \theta) := 0, \ t^{n+1}(\alpha, \theta) := \inf\{t > t^n(\alpha, \theta), \left\|\Delta\alpha_t\right\|_2 > \theta\}, \ n \in \mathbb{N}.$$

If $\{t > t^n(\alpha, \theta), \left\|\Delta\alpha_t\right\|_2 > \theta\} = \emptyset$, then we set $t^{n+1}(\alpha, \theta) := \infty$.

(ii) To the set I we associate the set of indices

$$J_I := \big\{ i \in \{1, \dots, m\}, \ I_i \neq \mathbb{R} \big\}. \tag{I.35}$$

(iii) For the pair (α, I) we define the time points

$$s^0(\alpha, I) := 0, \ s^{n+1}(\alpha, I) := \inf\{s > s^n(\alpha, I), \Delta\alpha^i_s \in I_i \text{ for every } i \in J_I\}, \ n \in \mathbb{N}. \tag{I.36}$$

If $\{s > s^n(\alpha, I), \Delta\alpha^i_s \in I_i \text{ for every } i \in J_I\} = \emptyset$, then we set $s^{n+1}(\alpha, I) := \infty$.

The value of $s^n(\alpha, I)$ marks the $n-$th time at which the value of $\Delta\alpha$ lies in the set I and it is well-defined since $J_I \neq \emptyset$ for $I \in \mathcal{J}(\alpha)$.

Example I.130. Let $(t^k)_{k\in\overline{\mathbb{N}}} \subset \mathbb{R}_+$, $(x^k)_{k\in\overline{\mathbb{N}}} \subset \mathbb{R}$ be such that $t^k \longrightarrow t_\infty$ and $x^k \downarrow x^\infty \in \mathbb{R} \setminus \{0\}$. Define $\gamma^k_\cdot := x^k \mathbb{1}_{[t^k,\infty)}(\cdot)$, for every $k \in \overline{\mathbb{N}}$. By Lemma I.115.(iii), we have $\gamma^k \xrightarrow{\mathrm{J}_1(\mathbb{R})} \gamma^\infty$. On the other hand, for $I := (\frac{1}{2}x^\infty, \frac{3}{2}x^\infty)$ holds $s^1(\gamma^k, I) = t^k$ for all but finitely many k and $s^1(\gamma^\infty, I) = t^\infty$, *i.e.* $s^1(\gamma^k, I) \xrightarrow[k\to\infty]{} s^1(\gamma^\infty, I)$. Moreover, for $w > x^\infty$ we also have

$$\Delta\gamma^k_{t^k} \mathbb{1}_{(x^\infty,w)}(\Delta\gamma^k_{t^k}) = x^k \mathbb{1}_{(x^\infty,w)}(x^k) = x^k \ \text{ for all but finitely many } k \in \mathbb{N},$$

and

$$\Delta\gamma^\infty_{t_\infty} \mathbb{1}_{(x^\infty,w)}(\Delta\gamma^\infty_{t_\infty}) = x^\infty \mathbb{1}_{(x^\infty,w)}(x^\infty) = 0.$$

Therefore, for $\mathbb{R} \ni x \overset{g}{\longmapsto} x\mathbb{1}_{(x^\infty,w)}(x)$

$$g(\Delta\gamma^k_{s^1(\gamma^k,I)}) = \Delta\gamma^k_{t^k} \mathbb{1}_{(x^\infty,w)}(\Delta\gamma^k_{t^k}) \xrightarrow[n\to\infty]{} x^\infty \neq 0 = \Delta\gamma^\infty_{t_\infty} \mathbb{1}_{(x^\infty,w)}(\Delta\gamma^\infty_{t_\infty}) = g(\Delta\gamma^\infty_{s^1(\alpha^\infty)}),$$

and for this reason we *cannot* obtain the convergence

$$\Delta\gamma^k_{t^k} \mathbb{1}_{(x^\infty,w)}(\Delta\gamma^n_{t^k}) \mathbb{1}_{(x^\infty,w)}(\cdot) \xrightarrow{\mathrm{J}_1(\mathbb{R})} \Delta\gamma^\infty_{t_\infty} \mathbb{1}_{(x^\infty,w)}(\Delta\gamma^\infty_{t_\infty}) \mathbb{1}_{(x^\infty,w)}(\cdot).$$

We can remedy the problem appearing in the previous example by not allowing elements of $W(\gamma^\infty)$ to be endpoints of the interval appearing in the indicator. Recall that $W(\gamma)$ is at most countable, therefore we are allowed to choose values for the endpoints from a dense subset of $\mathbb{R} \setminus \{0\}$. More precisely, we will choose intervals from $\mathcal{I}(\gamma)$ so that only finitely many jumps occur on any every compact time-interval.

For the rest of this subsection we adopt a convenient abuse of notation and we will assume the extended positive real line $[0, \infty]$ endowed with a metric $\tilde{\delta}_{|\cdot|}$ which extends the usual metric of $\mathbb{R}_+$ and for which $\tilde{\delta}_{|\cdot|}(\infty, \infty) = 0$. Moreover, the convergence of a sequence $(t^k)_{k\in\mathbb{N}} \subset [0, \infty]$ to the symbol ∞ will be understood as follows: either $t^k = \infty$ for every $k \in \mathbb{N}$ or for the subsequence $(t^{k_l})_{l\in\mathbb{N}}$, where $(k_l)_{l\in\mathbb{N}} := \{k \in \mathbb{N}, t^{k_l} < \infty\}$, $t^{k_l} \longrightarrow \infty$ in the usual sense. In this case, we will denote the convergence of a sequence $(t^k)_{k\in\mathbb{N}} \subset [0, \infty]$ to the symbol ∞ as usually by $t^k \longrightarrow \infty$.

Proposition I.131. *Fix $q, n \in \mathbb{N}$.*

(i) *The function $\mathbb{D}(\mathbb{R}^q) \ni \alpha \longmapsto t^n(\alpha, \theta) \in \overline{\mathbb{R}}_+$ is continuous at each point α such that $\theta \notin U(\alpha)$.*

(ii) *If $t^n(\alpha, \theta) < \infty$, then the function $\mathbb{D}(\mathbb{R}^q) \ni \alpha \longmapsto \Delta\alpha_{t^n(\alpha,\theta)} \in \mathbb{R}^q$ is continuous at each point α such that $\theta \notin U(\alpha)$.*

PROOF. See [41, Proposition VI.2.7]. □

The following proposition refines the above result.

Proposition I.132. *Fix $q, n \in \mathbb{N}$.*

(i) *The function $\mathbb{D}(\mathbb{R}^q) \ni \alpha \longmapsto s^n(\alpha, I) \in \overline{\mathbb{R}}_+$ is continuous at each point $(\alpha, I) \in \mathbb{D}(\mathbb{R}^q) \times \mathcal{J}(\alpha)$.*

(ii) *If $s^n(\alpha, I) < \infty$, $I \in \mathcal{J}(\alpha)$, then the function $\mathbb{D}(\mathbb{R}^q) \ni \alpha \longmapsto \Delta\alpha_{s^n(\alpha,I)} \in \mathbb{R}^q$ is continuous.*

PROOF. Let $(\alpha^k)_{k\in\overline{\mathbb{N}}}$ be such that $\alpha^k \xrightarrow{\mathrm{J}_1(\mathbb{R}^q)} \alpha^\infty$ and $I := \prod_{i=1}^q I_i \in \mathcal{J}(\alpha^\infty)$. Observe that $J_I \neq \emptyset$, by definition of $\mathcal{J}(\alpha^\infty)$. We define $s^{k,n} := s^n(\alpha^k, I)$, for $k \in \overline{\mathbb{N}}$ and $n \in \mathbb{N}$.

(i) The convergence $s^{k,0} \xrightarrow[k\to\infty]{} s^{\infty,0}$ holds by definition. Assume that the convergence $s^{k,n} \xrightarrow[k\to\infty]{} s^{\infty,n}$ holds for some arbitrary $n \in \mathbb{N}$. We will prove that the convergence $s^{k,n+1} \xrightarrow[k\to\infty]{} s^{\infty,n+1}$ holds also.

For the following, fix a positive number θ_I^∞ such that $\theta_I^\infty \notin \cup_{i\in J_I}\{|u|, u \in W(\alpha^{\infty,i})\}$ and

$$\theta_I^\infty < \frac{1}{q} \min \cup_{i\in J_I}\{|v|, v \in \partial I_i\}.^{22}$$

Recalling the Notation I.129.(i), we have for every $i \in J_I$ that

$$\left\{t \in \mathbb{R}_+, |\Delta\alpha^{\infty,i}| > \theta_I^\infty\right\} = \left\{t^l\left(\alpha^{\infty,i}, \theta_I^\infty\right) < \infty, l \in \mathbb{N}\right\}. \tag{I.37}$$

In other words, the set $\left\{t^l\left(\alpha^{\infty,i}, \theta_I^\infty\right) < \infty, l \in \mathbb{N}\right\}$ is another way to write the set of times that $\alpha^{\infty,i}$ exhibits a jump of height greater than θ_I^∞.

Case 1: $\qquad s^{\infty,n+1} < \infty.$

By property (I.37), there exist unique $l^{i,n}, l^{i,n+1} \in \mathbb{N}$ with $l^{i,n} < l^{i,n+1}$ such that

$$t^{l^{i,n}}(\alpha^{\infty,i}, \theta_I^\infty) = s^{\infty,n} \text{ and } t^{l^{i,n+1}}(\alpha^{\infty,i}, \theta_I^\infty) = s^{\infty,n+1} \text{ for every } i \in J_I. \tag{I.38}$$

By Proposition I.131 and the above identities we obtain

$$t^{l^{i,n}}(\alpha^{k,i}, \theta_I^\infty) \xrightarrow[k\to\infty]{} s^{\infty,n} \text{ and } \Delta\alpha^{k,i}_{t^{l^{i,n}}(\alpha^{k,i},\theta_I^\infty)} \xrightarrow[k\to\infty]{} \Delta\alpha^{\infty,i}_{s^{\infty,n}} \text{ for every } i \in J_I, \tag{I.39}$$

as well as

$$t^{l^{i,n+1}}(\alpha^{k,i}, \theta_I^\infty) \xrightarrow[k\to\infty]{} s^{\infty,n+1} \text{ and } \Delta\alpha^{k,i}_{t^{l^{i,n+1}}(\alpha^{k,i},\theta_I^\infty)} \xrightarrow[k\to\infty]{} \Delta\alpha^{\infty,i}_{s^{\infty,n+1}} \text{ for every } i \in J_I. \tag{I.40}$$

Recall, now, the induction hypothesis, *i.e.* the convergence $s^{k,n} \xrightarrow[k\to\infty]{} s^{\infty,n}$ is true. The Proposition I.121.(ii) guarantees that every sequence $(r^k)_{k\in\mathbb{N}}$ which converges to the time point $s^{\infty,n}$ and it is such that $\Delta\alpha^{k,i}_{r^k} \xrightarrow[k\to\infty]{} \Delta\alpha^{\infty,i}_{s^{\infty,n}}$, for some $i = 1, \ldots, q$, then it finally coincides with $(s^{k,n})_{n\in\mathbb{N}}$. Therefore, in view of the Convergence (I.39) we can obtain that

$$\left(t^{l^{i,n}}(\alpha^{k,i}, \theta_I^\infty) - s^{k,n}\right)_{k\in\mathbb{N}} \in c_{00}(\mathbb{N}), \text{ for every } i \in J_I, \tag{I.41}$$

where $c_{00}(\mathbb{N}) := \{(x^m)_{m\in\mathbb{N}} \subset \mathbb{R}^{\mathbb{N}}, \exists m_0 \in \mathbb{N} \text{ such that } x^m = 0 \text{ for every } m \geq m_0\}$. Define for $i \in J_I$

$$k_0^{n,i} := \max\{k \in \mathbb{N}, t^{l^{i,n}}(\alpha^{k,i}, \theta_I^\infty) \neq s^{k,n}\} < \infty.$$

Since J_I is a finite set, the number

$$\bar{k}_0^n := \max\left\{k_0^{n,i}, i \in J_I\right\}$$

is well–defined and finite, therefore

$$s^{l^{i,n}}(\alpha^{k,i}, U) = s^{k,n}, \text{ for every } k > \bar{k}_0^n \text{ and for every } i \in J_I. \tag{I.42}$$

Now, in view of Convergence (I.40) we can conclude the induction step once we prove the analogous to (I.41) for $n+1$ in place of n, *i.e.*

$$(t^{l^{i,n+1}}(\alpha^{k,i}, \theta_I^\infty) - s^{k,n+1})_{k\in\mathbb{N}} \in c_{00}(\mathbb{N}), \text{ for every } i \in J_I. \tag{I.43}$$

At this point we distinguish two cases:

Case 1.1: $\qquad$ For every $i \in J_I$, holds $l^{i,n+1} = 1 + l^{i,n}$.

[22] Recall that ∂A denotes the $|\cdot|-$boundary of the set $A \subset \mathbb{R}$.

By Proposition I.121.(ii), Convergence (I.40) and the convergence $\alpha^k \xrightarrow{\mathrm{J}_1(\mathbb{R}^q)} \alpha^\infty$, we can conclude that

$$\big(t^{l^{i,n+1}}(\alpha^{k,i},\theta_I^\infty) - t^{l^{j,n+1}}(\alpha^{k,j},\theta_I^\infty)\big) \in c_{00}(\mathbb{N}), \text{ for every } i,j \in J_I. \tag{I.44}$$

Therefore, we can fix hereinafter an index from J_I and we will do so for $\mu := \min J_I$, *i.e.* μ is the minimum element of J_I. Define

$$k_0^{n+1,i} := \max\{k \in \mathbb{N}, t^{l^{i,n+1}}(\alpha^{k,i},\theta_I^\infty) \neq t^{l^{\mu,n+1}}(\alpha^{k,\mu},\theta_I^\infty)\} \text{ for } i \in J_I \setminus \{\mu\}.$$

By property (I.44), we obtain that $k_0^{n+1,i} < \infty$ for every $i \in J_I \setminus \{\mu\}$. Since $J_I \setminus \{\mu\}$ is a finite set, the number

$$\bar{k}_0^{n+1} := \max\big\{k_0^{n+1,i}, i \in J_I \setminus \{\mu\}\big\}$$

is well–defined and finite. Observe that

$$t^{1+l^{i,n}}(\alpha^{k,i},\theta_I^\infty) = t^{l^{i,n+1}}(\alpha^{k,i},\theta_I^\infty) = t^{l^{\mu,n+1}}(\alpha^{k,\mu},\theta_I^\infty), \text{ for } k > \bar{k}_0^{n+1} \text{ and for } i \in J_I, \tag{I.45}$$

where the first equality holds by the assumption of this sub-case. Moreover, by Convergence (I.40) we obtain that

$$1 = \prod_{i\in J_I} \mathbb{1}_{I_i}(\Delta\alpha^{k,i}_{t^{1+l^{i,n}}(\alpha^{k,i},\theta_I^\infty)}), \text{ for all but finitely many } k, \tag{I.46}$$

since $\Delta\alpha^{\infty,i}_{s^{\infty,n+1}}$ lies in the interior of the open interval I_i, for every $i \in J_I$.

For notational convenience, we will assume that the above convergence holds for $k > \bar{k}_0^{n+1}$. Therefore, for $k > \bar{k}_0^n \vee \bar{k}_0^{n+1}$

$$\begin{aligned}
s^{k,n+1} &= \inf\{s > s^{k,n}, \Delta\alpha_s^{k,i} \in I_i \text{ for every } i \in J_I\} = \inf \bigcap_{i\in J_I} \big\{s > s^{k,n}, \Delta\alpha_s^{k,i} \in I_i\big\} \\
&\overset{\text{(I.41)}}{\underset{k>\bar{k}_0^n}{=}} \inf \bigcap_{i\in J_I} \big\{s > t^{l^{i,n}}(\alpha^{k,i},\theta_I^\infty), \Delta\alpha_s^{k,i} \in I_i\big\} \\
&= \inf \bigcap_{i\in J_I} \Big\{s \in \Big(t^{l^{i,n}}(\alpha^{k,i},\theta_I^\infty), t^{l^{\mu,n+1}}(\alpha^{k,i},\theta_I^\infty)\Big], \Delta\alpha_s^{k,i} \in I_i\Big\} \\
&\qquad\qquad \wedge \inf \bigcap_{i\in J_I} \big\{s > t^{l^{\mu,n+1}}(\alpha^{k,i},\theta_I^\infty), \Delta\alpha_s^{k,i} \in I_i\big\} \\
&\overset{\text{(I.45)}}{\underset{k>\bar{k}_0^{n+1}}{=}} \inf \bigcap_{i\in J_I} \Big\{s \in \Big(t^{l^{i,n}}(\alpha^{k,i},\theta_I^\infty), t^{1+l^{i,n}}(\alpha^{k,i},\theta_I^\infty)\Big], \Delta\alpha_s^{k,i} \in I_i\Big\} \\
&\qquad\qquad \wedge \inf \bigcap_{i\in J_I} \big\{s > t^{1+l^{i,n}}(\alpha^{k,i},\theta_I^\infty), \Delta\alpha_s^{k,i} \in I_i\big\} \\
&= \bigcap_{i\in J_I} \big\{t^{1+l^{i,n}}(\alpha^{k,i},\theta_I^\infty)\big\} \wedge \inf \bigcap_{i\in J_I} \big\{s > t^{1+l^{i,n}}(\alpha^{k,i},\theta_I^\infty), \Delta\alpha_s^{k,i} \in I_i\big\}\big) \\
&= \bigcap_{i\in J_I} \big\{t^{1+l^{i,n}}(\alpha^{k,i},\theta_I^\infty)\big\} \overset{\text{(I.45)}}{=} t^{l^{\mu,n+1}}(\alpha^{k,i},\theta_I^\infty) \;=\; t^{l^{i,n+1}}(\alpha^{k,i},\theta_I^\infty),
\end{aligned}$$

i.e. Property (I.43) indeed holds.

Case 1.2**:** There exists $i \in J_I$ for which holds $l^{i,n+1} > 1 + l^{i,n}$.

Define $\bar{J}_I : \{i \in J_I, l^{i,n+1} > 1 + l^{i,n}\}$ and fix $\xi^i \in (l^{i,n}, l^{i,n+1}) \cap \mathbb{N}$, for every $i \in \bar{J}_I$. Recall that by Proposition I.131 holds

$$\lim_{k\to\infty} t^{\xi^i}(\alpha^{k,i},\theta_I^\infty) = t^{\xi^i}(\alpha^{\infty,i},\theta_I^\infty) \text{ and } \lim_{k\to\infty} \Delta\alpha^{k,i}_{t^{\xi^i}(\alpha^{k,i},\theta_I^\infty)} = \Delta\alpha^{\infty,i}_{t^{\xi^i}(\alpha^{\infty,i},\theta_I^\infty)}. \tag{I.47}$$

We can conclude that property (I.43) holds, if for every $\bar{s} \in (s^{\infty,n}, s^{\infty,n+1})$ such that

$$\lim_{k\to\infty} t^{\xi^i}(\alpha^{k,i},\theta_I^\infty) = \bar{s} \text{ for every } i \in \bar{J}_I \tag{I.48}$$

holds

$$0 = \prod_{i\in J_I\setminus\bar{J}_I} \mathbb{1}_{I_i}\big(\Delta\alpha_{\bar{s}}^{k,i}\big) \prod_{i\in\bar{J}_I} \mathbb{1}_{I_i}\big(\Delta\alpha^{k,i}_{t^{\xi^i}(\alpha^{k,i},\theta_I^\infty)}\big) \text{ for all but finitely many } k. \tag{I.49}$$

assume to the contrary that

$$1 = \prod_{i\in J_I\setminus \bar{J}_I} \mathbb{1}_{I_i}\big(\Delta\alpha^{k,i}_{\bar{s}}\big) \prod_{i\in \bar{J}_I} \mathbb{1}_{I_i}\big(\Delta\alpha^{k,i}_{t^{\xi^i}(\alpha^{k,i},\theta^\infty_I)}\big) \text{ for all but finitely many } k,$$

then in view of definition of $s^{\infty,n+1}$, we would also have $s^{\infty,n+1} = t^{\xi^i}(\alpha^{\infty,i},\theta^\infty_I)$ for every $i \in \bar{J}_I$. But, this contradicts property (I.38). The contradiction arises in view of

$$t^{\xi^i}(\alpha^{\infty,i},\theta^\infty_I) < t^{l^{i,n+1}}(\alpha^{k,i},\theta^\infty_I), \text{ since } \xi^i < l^{i,n+1}.$$

Case 2: $$s^{\infty,n+1} = \infty.$$

We distinguish two cases:

Case 2.1: $$s^{\infty,n} < \infty$$

Using the same arguments as the ones used in Property (I.38) for $s^{\infty,n}$, we can associate to $t^n(\alpha^{\infty,i},\theta_I)$ a unique natural number $l^{i,n}$ such that Convergence (I.39) holds. Moreover, by definition of $s^{\infty,n+1}$ we obtain that

$$\{s > s^{\infty,n}, \Delta\alpha^{\infty,i}_s \in I_i \text{ for every } i \in J_I\} = \emptyset,$$

or, equivalently,

$$\text{for every } s > s^{\infty,n} \text{ there exists an } i \in J_I \text{ such that } \Delta\alpha^{\infty,i} \notin I_i. \tag{I.50}$$

Assume, now, that

$$\liminf_{k\to\infty} s^{k,n+1} = \bar{s}, \text{ for some } \bar{s} \in (s^{k,n},\infty).$$

Therefore, there exists $(k_l)_{l\in\mathbb{N}}$ such that $s^{k_l,n+1} \xrightarrow[l\to\infty]{} \bar{s}$. Equivalently, $\Delta\alpha^{k_l,i}_{s^{k_l,n+1}} \in I_i$ for all but finitely many l for every $i \in J_I$. By convergence $\alpha^k \xrightarrow{\mathrm{J}_1(\mathbb{R}^q)} \alpha^\infty$, Proposition I.120.(iii) and convergence $s^{k_l,n+1} \xrightarrow[l\to\infty]{} \bar{s}$, we conclude that $\Delta\alpha^{\infty,i}_{\bar{s}} \in I_i$, for every $i \in J_I$. But this contradicts the assumption $s^{\infty,n+1} = \infty$ in view of its equivalent form (I.50).

Case 2.2: $$s^{\infty,n} = \infty.$$

By definition of $s^{k,n+1}$ holds

$$s^{k,n} < s^{k,n+1} \text{ whenever } s^{k,n} < \infty, \text{ for } k \in \overline{\mathbb{N}}, n \in \mathbb{N},$$

and

$$s^{k,n+1} = \infty \text{ whenever } s^{k,n} = \infty, \text{ for } k \in \overline{\mathbb{N}}, n \in \mathbb{N}.$$

Therefore, induction hypothesis $s^{k,n} \xrightarrow[k\to\infty]{} s^{\infty,n}$ and the above yield $s^{k,n+1} \xrightarrow[k\to\infty]{} \infty = s^{\infty,n+1}$.

(ii) Assume that there exist $n \in \mathbb{N}$ and $I \in \mathcal{J}(\alpha)$ such that $s^{\infty,n} < \infty$. By (i), we have that $s^{k,n} \xrightarrow[k\to\infty]{} s^{\infty,n}$, which in conjunction with convergence $\alpha^k \xrightarrow{\mathrm{J}_1(\mathbb{R}^q)} \alpha^\infty$ and Proposition I.120 implies that

$$\Delta\alpha^k_{s^{k,n}} \xrightarrow[k\to\infty]{} \Delta\alpha^\infty_{s^{\infty,n}}.$$

□

Corollary I.133. *Let* $\alpha^k \xrightarrow{\mathrm{J}_1(\mathbb{R}^q)} \alpha^\infty$ *and* $I \in \mathcal{J}(\alpha^\infty)$. *Define*

$$\hat{n} := \begin{cases} \max\{n \in \mathbb{N}, s^n(\alpha^\infty, I) < \infty\}, & \text{if } \{n \in \mathbb{N}, s^n(\alpha^\infty, I) < \infty\} \text{ is non–empty and finite,} \\ \infty, & \text{otherwise.} \end{cases}$$

Then for every function $g : \mathbb{R}^q \to \mathbb{R}$ *which is continuous on*

$$C := \prod_{i=1}^q A_i, \text{ where } A_i := \begin{cases} W(\alpha^{\infty,i}), & \text{if } i \in J_I, \\ W(\alpha^{\infty,i}) \cup \{0\}, & \text{if } i \in \{1,\dots,q\} \setminus J_I, \end{cases}$$

and for every $0 \le n \le \hat{n}$ *holds*

$$g(\Delta\alpha^k_{s^n(\alpha^k,I)}) \xrightarrow[k\to\infty]{} g(\Delta\alpha^\infty_{s^n(\alpha^\infty,I)}).$$

In particular, for the càdlàg functions

$$\beta^k_\cdot := g(\Delta\alpha^k_{s^n(\alpha^k,I)})\mathbb{1}_{[s^n(\alpha^k,I),\infty)}(\cdot), \quad \text{for } k \in \overline{\mathbb{N}},$$

the convergence $\beta^k \xrightarrow{\mathrm{J}_1(\mathbb{R})} \beta^\infty$ *holds.*

Proof. Fix an $n \in \mathbb{N}$ such that $n \leq \hat{n}$. By Proposition I.132.(ii) holds

$$\Delta\alpha^k_{s^n(\alpha^k,I)} \xrightarrow[k\to\infty]{} \Delta\alpha^\infty_{s^n(\alpha,\infty,I)}, \text{ where } \Delta\alpha^{\infty,i}_{s^n(\alpha^\infty,I)} \in \begin{cases} W(\alpha^{\infty,i}), & \text{if } i \in J_I, \\ W(\alpha^{\infty,i}) \cup \{0\}, & \text{if } i \in \{1,\dots,q\} \setminus J_I. \end{cases}$$

Therefore, by definition of the time $s^n(\alpha^\infty, I)$ and of the set C holds

$$g(\Delta\alpha^k_{s^n(\alpha^k,I)}) \xrightarrow[k\to\infty]{} g(\Delta\alpha^\infty_{s^n(\alpha^\infty,I)}). \tag{I.51}$$

By Proposition I.132.(i), the above convergence and Lemma I.115 we obtain the convergence

$$\beta^k \xrightarrow{\mathrm{J}_1(\mathbb{R})} \beta^\infty.$$

□

Proposition I.134. *Fix some subset* $I := \prod_{i=1}^q I_i$ *of* $\mathbb{R}^q$ *and a function* $g : \mathbb{R}^q \to \mathbb{R}$. *Define the map*

$$\mathbb{D}(\mathbb{R}^q) \ni \alpha \longmapsto \hat{\alpha}[g,I] := (\alpha^1,\dots,\alpha^q,\alpha^{g,I}) \in \mathbb{D}(\mathbb{R}^{q+1}),$$

where

$$\alpha^{g,I}_\cdot := \sum_{0<t\leq\cdot} g(\Delta\alpha_t)\mathbb{1}_I(\Delta\alpha_t). \tag{I.52}$$

Then, the map $\hat{\alpha}[g,I]$ *is* J_1-*continuous at each point* α *for which* $I \in \mathcal{J}(\alpha)$ *and for each function* g *which is continuous on the set*

$$C := \prod_{i=1}^q A_i, \text{ where } A_i := \begin{cases} W(\alpha^i), & \text{if } i \in J_I, \\ W(\alpha^i) \cup \{0\}, & \text{if } i \in \{1,\dots,q\} \setminus J_I. \end{cases} \tag{I.53}$$

Proof. The arguments are similar to those in the proof of [41, Corollary VI.2.8], but for the convenience of the reader we will present the proof below.

Let $(\alpha^k)_{k\in\overline{\mathbb{N}}} \subset \mathbb{D}(\mathbb{R}^q)$ be such that $\alpha^k \xrightarrow{\mathrm{J}_1(\mathbb{R}^q)} \alpha^\infty$, $I := \prod_{i=1}^q I_i \in \mathcal{J}(\alpha^\infty)$ and a function $g : \mathbb{R}^q \to \mathbb{R}$ which is continuous on the set

$$C := \prod_{i=1}^q A_i, \text{ where } A_i := \begin{cases} W(\alpha^{\infty,i}), & \text{if } i \in J_I, \\ W(\alpha^{\infty,i}) \cup \{0\}, & \text{if } i \in \{1,\dots,q\} \setminus J_I. \end{cases}$$

For notational convenience, denote $s^{k,p} := s^p(\alpha^k, I)$, for $k \in \overline{\mathbb{N}}$ and $p \in \mathbb{N}$, and $\mathbb{R}^q \ni x \xmapsto{g_I} g(x)\mathbb{1}_I(x)$. Recall that for every $i \in J_I$ holds $\partial I_i \cap W(\alpha^{\infty,i}) = \emptyset$, therefore g_I remains continuous on C.

Step 1: Let

$$\hat{p} := \begin{cases} 0, & \text{if } \{p \in \mathbb{N}, s^{\infty,p} < \infty\} = \emptyset, \\ \max\{p \in \mathbb{N}, s^{\infty,p} < \infty\}, & \text{if } \{p \in \mathbb{N}, s^{\infty,p} < \infty\} \text{ is non–empty and finite}, \\ \infty, & \text{if } \{p \in \mathbb{N}, s^{\infty,p} < \infty\} \text{ is non–empty and infinite}, \end{cases}$$

- If $\hat{p} = 0$, then by convergence $\alpha^k \xrightarrow{\mathrm{J}_1(\mathbb{R})} \alpha^\infty$ holds in particular $\alpha^k_0 \xrightarrow[k\to\infty]{} \alpha^\infty_0$. Since $s^0(\alpha^k, I) = s^0(\alpha^\infty, I) = 0$ by definition, we can conclude.
- If $\hat{p} < \infty$, using Corollary I.133 we get that for every $p \in \mathbb{N}$ with $p \leq \hat{p}$ it holds that

$$g_I(\Delta\alpha^k_{s^{k,p}})\mathbb{1}_{[s^{k,p},\infty)}(\cdot) \xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R})} g_I(\Delta\alpha^\infty_{s^{\infty,p}})\mathbb{1}_{[s^{\infty,p},\infty)}(\cdot).$$

By Proposition I.132 we obtain that for $p > \hat{p}$ it holds $s^{k,p} \xrightarrow[k\to\infty]{} \infty$. Therefore, by Lemma I.115.(i) we conclude that for every $p > \hat{p}$

$$g_I(\Delta\alpha^k_{s^{k,p}})\mathbb{1}_{[s^{k,p},\infty)}(\cdot) \xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R})} 0. \tag{I.54}$$

Using now the fact that $s^{\infty,p-1} < s^{\infty,p}$, for every $1 \leq p \leq \hat{p}$, in conjunction with Proposition I.120 we can prove by induction that, for every $1 \leq q \leq \hat{p}$, the following holds

$$\sum_{1\leq p\leq q} g_I(\Delta\alpha^k_{s^{k,p}})\mathbb{1}_{[s^{k,p},\infty)}(\cdot) \xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R})} \sum_{1\leq p\leq q} g_I(\Delta\alpha^\infty_{s^{\infty,p}})\mathbb{1}_{[s^{\infty,p},\infty)}(\cdot).$$

Analogously, we can prove by induction that for $q > \hat{p}$ holds

$$\sum_{\hat{p}<p\leq q} g_I(\Delta\alpha^k_{s^{k,p}})\mathbb{1}_{[s^{k,p},\infty)}(\cdot) \xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R})} 0,$$

by the continuity of the limit in Convergence (I.54) for every $p > \hat{p}$. In view of Proposition I.120 we can combine the two last convergences, to obtain the convergence of their sum. In particular, we can have for every $q \in \mathbb{N}$

$$\sum_{1\leq p\leq q} g_I(\Delta\alpha^k_{s^{k,p}})\mathbb{1}_{[s^{k,p},\infty)}(\cdot) \xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R})} \sum_{1\leq p\leq q} g_I(\Delta\alpha^\infty_{s^{\infty,p}})\mathbb{1}_{[s^{\infty,p},\infty)}(\cdot). \tag{I.55}$$

• If $\hat{p} = \infty$, we can use the arguments of the first part of the previous case (i.e. when $p \leq \hat{p}$) to prove by induction that for every $q \geq 1$ holds

$$\sum_{1\leq p\leq q} g_I(\Delta\alpha^k_{s^{k,p}})\mathbb{1}_{[s^{k,p},\infty)}(\cdot) \xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R})} \sum_{1\leq p\leq q} g_I(\Delta\alpha^\infty_{s^{\infty,p}})\mathbb{1}_{[s^{\infty,p},\infty)}(\cdot).$$

Step 2: Let now $N > 0$ and define $p_N := \min\{p \in \mathbb{N}, s^{\infty,p} > N\}$. By the definition of $(s^{\infty,p})_{p\in\mathbb{N}}$ and the fact that $\alpha^\infty \in \mathbb{D}(\mathbb{R}^q)$ we can easily conclude that $s^{\infty,p} \xrightarrow[p\to\infty]{} \infty$, hence p_N is well–defined. Then, by Proposition I.132 the convergence $s^{k,p_N} \xrightarrow[k\to\infty]{} s^{\infty,p_N}$ holds. Observe that we do not need to assume $s^{\infty,p_N} < \infty$ at this point. Using the last convergence, there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ holds $s^{k,p_N} > N$.

Define $\alpha^{k,g,I} := (\alpha^k)^{g,I}$ for $k \in \overline{\mathbb{N}}$, (see again (I.52)) and observe now that

$$\begin{aligned}\alpha^{\infty,g,I}_\cdot \mathbb{1}_{[0,s^{\infty,p_N})}(\cdot) = \sum_{0<t\leq\cdot} g_I(\Delta\alpha^\infty_t)\mathbb{1}_{[0,s^{\infty,p_N})}(\cdot) &= \sum_{0<t<s^{\infty,p_N}\wedge\cdot} g_I(\Delta\alpha^\infty_t)\\ &= \sum_{1\leq p\leq p_N-1} g_I(\Delta\alpha^\infty_{s^{\infty,p}})\mathbb{1}_{[s^{\infty,p},\infty)}(\cdot).\end{aligned}$$

Moreover, we have for $k \geq k_0$

$$\begin{aligned}\alpha^{k,g,I}_\cdot \mathbb{1}_{[0,s^{k,p_N})}(\cdot) = \sum_{0<t\leq\cdot} g_I(\Delta\alpha^k_t)\mathbb{1}_{[0,s^{k,p_N})}(\cdot) &= \sum_{0\leq t<s^{k,p_N}\wedge\cdot} g_I(\Delta\alpha^k_t)\\ &= \sum_{1\leq p\leq p_N-1} g_I(\Delta\alpha^k_{s^{k,p}})\mathbb{1}_{[s^{k,p},\infty)}(\cdot).\end{aligned}$$

By Convergence (I.55) and for the chosen p_N there exists a sequence $(\lambda^{N,k})_{k\in\mathbb{N}} \subset \Lambda$, for Λ as defined in Definition I.108.(ii), for which the sequence $\left(\sum_{1\leq p\leq p^N} g_I(\Delta\alpha^k_{s^{k,p}})\mathbb{1}_{[s^{k,p},\infty)}(\cdot)\right)_{k\in\overline{\mathbb{N}}}$ satisfies Theorem I.109. In view of the last equalities on $[0, N]$ and for $k \geq k_0$ we have that

$$\begin{aligned}&\sup_{s\in[0,N]} |\alpha^{k,g,I}_{\lambda^{N,k}(s)} - \alpha^{\infty,g,I}_s| \vee \sup_{s\in[0,N]} |\lambda^{N,k}(s) - s|\\ &\quad= \sup_{s\in[0,N]} \Big|\sum_{1\leq p\leq p_N-1} g_I(\Delta\alpha^k_{s^{k,p}})\mathbb{1}_{[s^{k,p},\infty)}(\lambda^{N,k}(s)) - \sum_{1\leq p\leq p_N-1} g_I(\Delta\alpha^\infty_{s^{\infty,p}})\mathbb{1}_{[s^{\infty,p},\infty)}(s)\Big|\\ &\qquad\vee \sup_{s\in[0,N]} |\lambda^{N,k}(s) - s| \xrightarrow[(\mathrm{I.55})]{n\to\infty} 0,\end{aligned}$$

Now that we have the family of sequences $\{(\lambda^{N,k})_{k\in\mathbb{N}}, N \in \mathbb{N}\}$, we can follow the same arguments as in the proof of [41, Lemma VI.1.31] in order to construct a sequence $(\lambda^k)_{k\in\mathbb{N}} \subset \Lambda$, for which the sequence $(\alpha^{k,g,I})_{k\in\mathbb{N}}$ satisfies Theorem I.109. Therefore we can conclude that

$$\alpha^{k,g,I} \xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R})} \alpha^{\infty,g,I}.$$

Now we have only to observe that $\alpha^{\infty,g,I}$ may have a jump at t only if $t = s^p(\alpha^\infty, I)$, for some $p \in \mathbb{N}$. Hence, by combining Proposition I.120 and Proposition I.132 we can conclude the joint convergence

$$(\alpha^{k,1}, \dots, \alpha^{k,q}, \alpha^{k,g,I}) \xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R}^{q+1})} (\alpha^{\infty,1}, \dots, \alpha^{\infty,q}, \alpha^{\infty,g,I}). \tag{I.56}$$

□

I.6.2. Application to càdlàg processes. In this section we collect some results regarding convergence of random variables taking their values in the Polish space $\mathbb{D}(E)$. We will follow closely Jacod and Shiryaev [41, Section VI.3]. For this section, we additionally endow the arbitrary probability space $(\Omega, \mathcal{G}, \mathbb{P})$ with an arbitrary sequence of filtrations $(\mathbb{G}^k)_{k\in\overline{\mathbb{N}}}$.

Definition I.135. Let $(M^k)_{k\in\overline{\mathbb{N}}}$ be an arbitrary sequence such that M^k is an $E-$valued càdlàg process, for every $k \in \overline{\mathbb{N}}$.

(i) The sequence $(M^k)_{k\in\mathbb{N}}$ converges *in probability under the* $\mathrm{J}_1(E)-$*topology* to M^∞ if

$$\mathbb{P}\Big(\delta_{\mathrm{J}_1(E)}(M^k, M^\infty) > \varepsilon\Big) \xrightarrow[k\to\infty]{} 0, \quad \text{for every } \varepsilon > 0,$$

and we denote it by $M^k \xrightarrow{(\mathrm{J}_1(E),\mathbb{P})} M^\infty$.

(ii) Let $\vartheta \in \{1, 2\}$. The sequence $(M^k)_{k\in\mathbb{N}}$ converges *in* $\mathbb{L}^\vartheta-$*mean under the* $\mathrm{J}_1(E)-$*topology* to M^∞ if

$$\mathbb{E}\Big[\big(\delta_{\mathrm{J}_1(E)}(M^k, M^\infty)\big)^\vartheta\Big] \xrightarrow[k\to\infty]{} 0,$$

and we denote it by $M^k \xrightarrow{(\mathrm{J}_1(E),L^\vartheta)} M^\infty$.

(iii) Analogously, we denote by $M^k \xrightarrow{(\mathrm{lu},\mathbb{P})} M^\infty$, resp. $M^k \xrightarrow{(\mathrm{lu},L^\vartheta)} M^\infty$, the convergence *in probability*, resp. *in* $\mathbb{L}^\vartheta-$*mean*, *under the locally uniform topology.*

(iv) Moreover, let $(N^{k,1})_{k\in\overline{\mathbb{N}}}$ be a sequence of E_1-valued and càdlàg processes and $(N^{k,2})_{k\in\overline{\mathbb{N}}}$ be a sequence of E_2-valued and càdlàg processes. For $\vartheta_1, \vartheta_2 \in \{1, 2\}$ we will write

$$(N^{k,1}, N^{k,2}) \xrightarrow{(\mathrm{J}_1(E_1\times E_2),\mathbb{L}^{\vartheta_1}\times\mathbb{L}^{\vartheta_2})} (N^{\infty,1}, N^{\infty,1}),$$

if the following convergence hold

$$(N^{k,1}, N^{k,2}) \xrightarrow{(\mathrm{J}_1(E_1\times E_2),\mathbb{P}))} (M^\infty, N^\infty), N^{k,1} \xrightarrow{(\mathrm{J}_1(E_1),\mathbb{L}^{\vartheta_1})} N^{\infty,1} \text{ and } N^{k,2} \xrightarrow{(\mathrm{J}_1(E_2),\mathbb{L}^{\vartheta_2})} N^{\infty,2}.$$

Definition I.136. Let $(\mathbf{\Gamma}, \delta_{\mathbf{\Gamma}})$ be a metric space.

(i) The *set of probability measures* defined on $\big(\mathbf{\Gamma}, \mathcal{B}(\mathbf{\Gamma})\big)$ is denoted by $\mathscr{P}(\mathbf{\Gamma})$.

(ii) Let $\mathscr{A} \subset \mathscr{P}(\mathbf{\Gamma})$. The set $\mathscr{A}$ will be called *tight* if for every $\varepsilon > 0$ there exists a compact set K in $\mathbf{\Gamma}$ such that $\mathbb{Q}(\mathbf{\Gamma} \setminus K) \le \varepsilon$ for every $\mathbb{Q} \in \mathscr{A}$.

(iii) The *law of a* $\mathbf{\Gamma}$*-valued random variable* Ξ, which will be denoted by $\mathcal{L}(\Xi)$, is the probability measure on $\big(\mathbf{\Gamma}, \mathcal{B}(\mathbf{\Gamma})\big)$ defined for every $A \in \mathcal{B}(\mathbf{\Gamma})$ by $\mathcal{L}(\Xi)(A) := \mathbb{P}\big(\{\omega \in \Omega, \Xi(\omega) \in A\}\big)$.

(iv) Let $(\Xi^k)_{k\in\overline{\mathbb{N}}}$ be a sequence of $\mathbf{\Gamma}-$valued random variables. We will say that Ξ^k *converges weakly to* Ξ^∞ if for every $f : \big(\mathbf{\Gamma}, \delta_{\mathbf{\Gamma}}\big) \to (\mathbb{R}, |\cdot|)$ continuous and bounded holds

$$\mathbb{E}\big[f(\Xi^k)\big] \longrightarrow \mathbb{E}\big[f(\Xi^\infty)\big].$$

We will denote the above convergence by $\Xi^k \xrightarrow{\mathcal{L}} \Xi^\infty$ or $\mathcal{L}(\Xi^k) \xrightarrow{\mathrm{w}} \mathcal{L}(\Xi^\infty)$ interchangeably. The random variable Ξ^∞ will be called the *weak limit (point) of the sequence* $(\Xi^k)_{k\in\mathbb{N}}$.

(v) Let $(\Xi^k)_{k\in\overline{\mathbb{N}}}$ be a sequence of $\mathbf{\Gamma}-$valued random variables. The sequence $(\Xi^k)_{k\in\overline{\mathbb{N}}}$ will be called *tight* if the sequence of the associated laws $\big(\mathcal{L}(X^k)\big)_{k\in\overline{\mathbb{N}}}$ is tight.

Remark I.137. The reader may recall the Dunford–Pettis Compactness Criterion, see Theorem I.26, which deals with relatively compact subsets of $\mathscr{P}(\mathbb{R})$.

Lemma I.138. *Let X be a real-valued and càdlàg process. The set*

$$V(X) := \big\{u \in \mathbb{R} \setminus \{0\}, \mathbb{P}\big([\Delta X_t = u, \text{ for some } t > 0]\big) > 0\big\}$$

is at most countable.

Proof. It is an immediate consequence of [41, Lemma VI.3.12]. □

We state the well-known Prokhorov's theorem in the case of Polish spaces.

Theorem I.139 (Prokhorov)**.** *Let $\mathbf{\Gamma}$ be a separable and complete metric space. The set $\mathscr{A} \subset \mathscr{P}(\mathbf{\Gamma})$ is relatively compact if and only if $\mathscr{A}$ is tight.*

Proof. See Prokhorov [57, Theorem 1.12]. □

Proposition I.140. *Let $(X^k)_{k\in\overline{\mathbb{N}}}$ be a sequence of $\mathbb{R}^q-$valued càdlàg processes such that $X^k \xrightarrow{\mathcal{L}} X^\infty$ in $\mathbb{D}(\mathbb{R}^q)$. Then, $(X^k_{t_1},\ldots,X^k_{t_n}) \xrightarrow{\mathcal{L}} (X^\infty_{t_1},\ldots,X^\infty_{t_n})$ in $\mathbb{R}^{qn}$, for every $t_i \in \left\{t \in \mathbb{R}_+, \mathbb{P}([\Delta X^\infty_t \neq 0]) > 0\right\}$ and for every $n \in \mathbb{N}$.*

Proof. See Jacod and Shiryaev [41, Proposition VI.3.14]. □

Theorem I.141. *Let $(X^k)_{k\in\overline{\mathbb{N}}}$ be a sequence of $E-$valued càdlàg processes. Then, the following are equivalent:*

(i) $X^k \xrightarrow{(\mathrm{J}_1(E),\mathbb{P})} X^\infty$.
(ii) *For every subsequence $(X^{k_l})_{l\in\mathbb{N}}$ there exists a further subsequence $(X^{k_{l_m}})_{m\in\mathbb{N}}$ such that*

$$X^{k_{l_m}} \xrightarrow{\mathrm{J}_1(E)} X^\infty \ \mathbb{P}-a.s..$$

Proof. See Dudley [28, Theorem 9.2.1], which can be applied in this case because the Skorokhod space is Polish under the J_1-topology. □

Definition I.142. A sequence $(X^k)_{k\in\mathbb{N}}$ of processes is called $C-$tight if it is tight and if all weak limit points of the sequence $\left(\mathcal{L}(X^k)\right)_{k\in\mathbb{N}}$ are laws of continuous processes.

Lemma I.143. *Let $(X^k)_{k\in\mathbb{N}}$ be an $\mathbb{R}^p-$valued $C-$tight sequence and $(Y^k)_{k\in\mathbb{N}}$ be an $\mathbb{R}^q-$valued tight, resp. $C-$tight, sequence. Then:*

(i) *If $p = q$, then $(X^k + Y^k)_{k\in\mathbb{N}}$ is tight, resp. $C-$tight.*
(ii) *$(X^k, Y^k)_{k\in\mathbb{N}}$ is an $\mathbb{R}^{p+q}-$valued tight, resp. $C-$tight, sequence.*

Proof. See Jacod and Shiryaev [41, Corollary VI.3.33]. □

Definition I.144. Let A and B two increasing processes. We say that B *strongly majorizes* A if the process $B - A$ is itself increasing.

In the following proposition the probability space is endowed with a sequence of filtrations $(\mathbb{G}^k)_{k\in\mathbb{N}}$.

Proposition I.145. *Let $(A^k)_{k\in\mathbb{N}}$, $(B^k)_{k\in\mathbb{N}}$ be sequences of càdlàg processes. Assume, moreover, that A^k, B^k are increasing and such that B^k strongly majorizes A^k for every $k \in \mathbb{N}$. If $(B^k)_{k\in\mathbb{N}}$ is tight, resp. $C-$tight, then A is also tight, resp. $C-$tight.*

Proof. See Jacod and Shiryaev [41, Proposition VI.3.35]. □

The following theorem provides a sufficient condition for a sequence of $\mathbb{R}^q-$valued square-integrable $\mathbb{G}^k-$martingales to be tight.

Theorem I.146. *Let $(X^k)_{k\in\mathbb{N}}$ be a sequence such that $X^k - X^k_0 \in \mathcal{H}^2(\mathbb{G}^k;\mathbb{R}^q)$ for every $k \in \mathbb{N}$. If*

(i) *the sequence $(X^k_0)_{k\in\mathbb{N}}$ is tight in $\mathbb{R}$ and*
(ii) *the sequence $\left(\mathrm{Tr}\left[\langle X^k\rangle\right]\right)_{k\in\mathbb{N}}$ is $C-$tight in $\mathbb{D}(\mathbb{R})$,*

then the sequence $(X^k)_{k\in\mathbb{N}}$ is tight in $\mathbb{D}(\mathbb{R}^q)$.

Proof. See Jacod and Shiryaev [41, Theorem VI.4.13]. □

The last mentioned theorem is proven in Rebolledo [59] and it is a consequence of *Aldous' Criterion for Tightness,*; for more details for the aforementioned criterion the reader may consult Aldous [2] and [41, Section VI.4a]. The role of Theorem I.146 in Chapter III will not be as evident as its importance may deserve, since it will be applied in a single point of the proof of a lemma among many lemmata; see Lemma III.31. For this reason, we need to specifically comment at this point, that it is an extremely convenient result, which enabled us to obtain the tightness of a sequence of *joint laws*.

We proceed, now, to another useful result. The message of the next theorem can be loosely described as follows: the weak limit of a sequence of martingales will be a uniformly integrable martingale with respect to its natural filtration as soon as the family of random variables obtained by the convergent sequence of martingales is uniformly integrable. It is well-known that the martingale property can be weakened to the submartingale property.

Proposition I.147. *Let $(H^k)_{k\in\overline{\mathbb{N}}}$ be a sequence of E_1-valued càdlàg processes, where H^k is $\mathbb{G}^k-$adapted for every $k \in \mathbb{N}$.*[23] *Assume, moreover, that $(\Theta^k)_{k\in\overline{\mathbb{N}}}$ is a sequence such that Θ^k is an E_2-valued càdlàg $\mathbb{G}^k-$martingale for $k \in \mathbb{N}$.*[24] *Furthermore, let Θ^∞ be a càdlàg adapted process defined on the canonical space $\big(\mathbb{D}(\mathbb{R}^q), \mathscr{D}(\mathbb{R}^q), \mathbb{D}(\mathbb{R}^q)\big)$ and let $D \subset \mathbb{R}_+$ be dense. Assume that:*

(i) *The family $\big\{\|\Theta^k_t\|_1, t \in \mathbb{R}_+$ and $k \in \mathbb{N}\big\}$ is uniformly integrable.*
(ii) $H^k \xrightarrow{\mathcal{L}} H^\infty$.
(iii) *For all $t \in D$, $\mathbb{D}(\mathbb{R}^q) \ni \alpha \longmapsto \Theta^\infty_t(\alpha)$ is $\mathcal{L}(H^\infty) - a.s.$ continuous and*
(iv) $\Theta^k_t - \Theta^\infty_t \circ H^k \xrightarrow{(|\cdot|,\mathbb{P})} 0$ *for all $t \in D$.*

Then the process $\Theta^\infty \circ H^\infty$ is a martingale with respect to the filtration generated by H^∞.

Proof. See Jacod and Shiryaev [41, Proposition IX.1.12]. The uniform integrability of the limit martingale is a consequence of Dunford–Pettis Compactness Criterion, see Theorem I.26. □

We will close the section with the introduction of another important notion for limit theorems and the presentation of some classical related results.

Notation I.148. For every $k \in \mathbb{N}$, $\mathscr{H}^k$ will denote the set of $\mathbb{G}^k-$*predictable elementary processes* bounded by 1. In other words,

$$\mathscr{H}^k := \Big\{H : (\Omega \times \mathbb{R}_+, \mathcal{P}^{\mathbb{G}^k}) \longrightarrow \big(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\big), H_t = h_0 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^p h_i \mathbb{1}_{(s_i,s_{i+1}]}(t) \text{ with}$$
$$p \in \mathbb{N}, 0 = s_0 < s_1 < \ldots, < s_{p+1} \text{ and } h_i \text{ is } \mathcal{G}^k_{s_i} - \text{measurable with } |h_i| \le 1\Big\}$$

Definition I.149. A sequence $(X^k)_{k\in\mathbb{N}}$ of $\mathbb{R}^q-$valued càdlàg processes, where X^k is $\mathbb{G}^k-$adapted for each $k \in \mathbb{N}$, is said to be *predictably uniformly tight*, in short P-UT, if for every $t > 0$ the family of random variables $\big\{\sum_{i=1}^q (H^{k,i} \cdot X^{k,i})_t, k \in \mathbb{N}$ and $H^{k,i} \in \mathscr{H}^k\big\}$ is tight in $\mathbb{R}$,[25] that is

$$\lim_{K\uparrow\infty} \sup \left\{\mathbb{P}\left(\left|\sum_{i=1}^q (H^{k,i} \cdot X^{k,i})_t\right| > K\right), H^{k,i} \in \mathscr{H}^k \text{ and } k \in \mathbb{N}\right\} = 0.$$

Remark I.150. It is immediate by the definition that, if the sequences $(X^k)_{k\in\mathbb{N}}$, $(Y^k)_{k\in\mathbb{N}}$ are P-UT then so is $(X^k + Y^k)_{k\in\mathbb{N}}$.

Proposition I.151. *Let $(V^k)_{k\in\mathbb{N}}$ be such that $V^k \in \mathcal{V}(\mathbb{G}^k; \mathbb{R}^p)$ for every $k \in \mathbb{N}$.*

(i) *If $\big(\mathrm{Var}(V^k)_t\big)_{k\in\mathbb{N}}$ is tight in $\mathbb{R}$ for every $t \in \mathbb{R}_+$, then $(V^k)_{k\in\mathbb{N}}$ is* P-UT.
(ii) *If, moreover, $V^k \in \mathcal{V}_{\mathrm{pred}}(\mathbb{G}^k; \mathbb{R}^p)$ for every $k \in \mathbb{N}$ and the sequence $\big(\mathrm{Var}(V^k)_t\big)_{k\in\mathbb{N}}$ is* P-UT, *then $\big(\mathrm{Var}(V^k)_t\big)_{k\in\mathbb{N}}$ is tight in $\mathbb{R}$ for every $t \in \mathbb{R}_+$.*

Proof. (i) It is immediate by the definition of the variation process and the fact that the elements of $\mathscr{H}^k$ are bounded by 1, for every $k \in \mathbb{N}$.
(ii) This is Jacod and Shiryaev [41, Proposition VI.6.12]

□

Theorem I.152. *Let the sequence $(S^k)_{k\in\mathbb{N}}$ of $\mathbb{R}^q-$valued càdlàg processes and assume it is* P-UT. *If $S^k \xrightarrow{\mathcal{L}} S^\infty$, then $(S^k, [S^k]) \xrightarrow{\mathcal{L}} (S^\infty, [S^\infty])$ in $\mathbb{D}(\mathbb{R}^q \times \mathbb{R}^{q\times q})$. If the former convergence holds in probability, then the later holds in probability as well.*

Proof. See Jacod and Shiryaev [41, Theorem VI.6.26]. □

Proposition I.153. *Let the sequence $(X^k)_{k\in\overline{\mathbb{N}}}$ of $\mathbb{R}^q-$valued processes be such that*

(i) *X^k is $\mathbb{G}^k-$martingale, for every $k \in \mathbb{N}$,*
(ii) $\sup_{k\in\mathbb{N}} \mathbb{E}\left[\sup_{0\le s\le t} \|X^k_s\|_2^2\right] < \infty$, *for every $t \in \mathbb{R}_+$ and*
(iii) $X^k \xrightarrow{\mathcal{L}} X^\infty$.

[23] We do not require the property $k = \infty$.
[24] As in 23, we do not require the property for $k = \infty$.
[25] In the literature, it is also used the term *bounded in probability*, with the obvious interpretation.

Then $(X^k)_{k\in\mathbb{N}}$ *is* P-UT *and thus* $(X^k,[X^k]) \xrightarrow{\mathcal{L}} (X^\infty,[X^\infty])$. *If the convergence in* (iii) *holds in probability, then the above convergence is also true in probability,* i.e. $(X^k,[X^k]) \xrightarrow{(\mathrm{J}_1(\mathbb{R}^q\times\mathbb{R}^{q\times q}),\mathbb{P})} (X^\infty,[X^\infty])$.

Proof. We will apply Jacod and Shiryaev [41, Corollary VI.6.30]. We need only to prove that $\sup_{k\in\mathbb{N}}\mathbb{E}\left[\sup_{s\le t}\|\Delta X^k_s\|_2\right]<\infty$, for every $t\in\mathbb{R}_+$. To this end let us fix a $t\in\mathbb{R}_+$. We observe that in view of the inequality

$$\sup_{k\in\mathbb{N}}\Big\|\sup_{s\le t}\|\Delta X^k_s\|_2\Big\|_{\mathbb{L}^1(\mathcal{G};\mathbb{R})}\le\sup_{k\in\mathbb{N}}\Big\|\sup_{s\le t}\|\Delta X^k_s\|_2\Big\|_{\mathbb{L}^2(\mathcal{G};\mathbb{R})}=\sup_{k\in\mathbb{N}}\mathbb{E}\Big[\sup_{s\le t}\|\Delta X^k_s\|_2^2\Big]^{\frac12}$$

it is sufficient to prove that the right-hand side is finite. For the equality we have used that the function $\mathbb{R}_+\ni x\mapsto x^2\in\mathbb{R}_+$ is continuous. Now, we have

$$\begin{aligned}\sup_{k\in\mathbb{N}}\mathbb{E}\left[\sup_{s\le t}\|\Delta X^k_s\|_2^2\right]&=\sup_{k\in\mathbb{N}}\mathbb{E}\left[\sup_{0\le s\le t}\sum_{i=1}^q|\Delta X^{k,i}_s|^2\right]\le\sup_{k\in\mathbb{N}}\mathbb{E}\left[\sup_{0\le s\le t}\left(2\sum_{i=1}^q|X^{k,i}_s|^2+2\sum_{i=1}^q|X^{k,i}_{s-}|^2\right)\right]\\&\le\sup_{k\in\mathbb{N}}\left\{2\sum_{i=1}^q\mathbb{E}\Big[\big(\sup_{0\le s\le t}|X^{k,i}_s|\big)^2\Big]+2\sum_{i=1}^q\mathbb{E}\Big[\big(\sup_{0\le s\le t}|X^{k,i}_{s-}|\big)^2\Big]\right\}\le4\sup_{k\in\mathbb{N}}\sum_{i=1}^q\mathbb{E}\Big[\big(\sup_{0\le s\le t}|X^{k,i}_s|\big)^2\Big]\\&=4\sup_{k\in\mathbb{N}}\mathbb{E}\Big[\sum_{i=1}^q\sup_{0\le s\le t}|X^{k,i}_s|^2\Big]\le4q\sup_{k\in\mathbb{N}}\mathbb{E}\Big[\sup_{0\le s\le t}\|X^{k,i}_s\|_2^2\Big]\overset{\text{(ii)}}{<}\infty,\end{aligned}$$

where in the third inequality we have used that the processes are càdlàg. therefore, $(X^k)_{k\in\mathbb{N}}$ is P-UT. For the convergence in probability, we apply [41, Theorem VI.6.22 (c)]. □

I.6.3. Weak convergence of filtrations. The purpose of this subsection is to introduce the notion for convergence of $\sigma-$algabrae and of filtrations we are going to make use of in the sequel. In the literature, such notions have been proposed by Aldous [3] and Hoover [38]. However, we are going to use that of Coquet et al. [23], which weakens the notion introduced by Hoover. Let us, now, proceed to the introduction of these notions.

Definition I.154. (i) A sequence of $\sigma-$*algebrae* $(\mathcal{G}^k)_{k\in\mathbb{N}}$ converges *weakly* to the $\sigma-$algebra $\mathcal{G}^\infty$ if, for every $\xi\in\mathbb{L}^1(\mathcal{G}^\infty;\mathbb{R})$, we have

$$\mathbb{E}[\xi|\mathcal{G}^k]\xrightarrow{\mathbb{P}}\mathbb{E}[\xi|\mathcal{G}^\infty].$$

We denote the weak convergence of $\sigma-$algebrae by $\mathcal{G}^k\xrightarrow{\mathrm{w}}\mathcal{G}^\infty$.

(ii) A sequence of filtrations $\big(\mathbb{G}^k:=(\mathcal{G}^k_t)_{t\in\mathbb{R}_+}\big)_{k\in\mathbb{N}}$ converges *weakly* to $\mathbb{G}^\infty:=(\mathcal{G}^\infty_t)_{t\in\mathbb{R}_+}$, if, for every $\xi\in\mathbb{L}^1(\mathcal{G}^\infty_\infty;\mathbb{R})$, we have

$$\mathbb{E}[\xi|\mathcal{G}^k_\cdot]\xrightarrow{(\mathrm{J}_1(\mathbb{R}),\mathbb{P})}\mathbb{E}[\xi|\mathcal{G}^\infty_\cdot].$$

We denote the weak convergence of the filtrations by $\mathbb{G}^k\xrightarrow{\mathrm{w}}\mathbb{G}^\infty$.

(iii) Consider the sequence $\big((M^k,\mathbb{G}^k)\big)_{k\in\overline{\mathbb{N}}}$, where M^k is an $E-$valued càdlàg process and $\mathbb{G}^k$ is a filtration, for any $k\in\overline{\mathbb{N}}$. The sequence $\big((M^k,\mathbb{G}^k)\big)_{k\in\mathbb{N}}$ converges *in the extended sense* to $(M^\infty,\mathbb{G}^\infty)$ if for every $\xi\in\mathbb{L}^1(\mathcal{G}^\infty_\infty;\mathbb{R})$,

$$\big(M^k,\mathbb{E}[\xi|\mathcal{G}^k_\cdot]\big)\xrightarrow{(\mathrm{J}_1(E\times\mathbb{R}),\mathbb{P})}\big(M^\infty,\mathbb{E}[\xi|\mathcal{G}^\infty_\cdot]\big).\tag{I.57}$$

We denote the convergence in the extended sense by $\big(M^k,\mathbb{G}^k\big)\xrightarrow{\text{ext}}(M^\infty,\mathbb{G}^\infty)$.

Remark I.155. For the definition of weak convergence of filtrations, we could have used only random variables ξ of the form $\mathbb{1}_A$, for $A\in\mathcal{G}^\infty_\infty$. Indeed, the two definitions are equivalent; see Coquet et al. [23, Remark 1.1)].

The following result, which is Hoover [38, Theorem 7.4], provides a sufficient condition for weak convergence of $\sigma-$*algebrae* which are generated by random variables.

Example I.156. Let $(\xi^k)_{k\in\overline{\mathbb{N}}}$ be a sequence of random variables such that $\xi^k\xrightarrow{\mathbb{P}}\xi^\infty$. Then the convergence $\sigma(\xi^k)\xrightarrow{\mathrm{w}}\sigma(\xi^\infty)$ holds, where $\sigma(\psi)$ denotes the $\sigma-$*algebra* generated by the random variable ψ.

In the next example, which is Coquet et al. [23, Proposition 2], a sufficient condition for the weak convergence of the natural filtrations of stochastic processes is provided.

Example I.157. Let M^k be a process with independent increments, for every $k \in \overline{\mathbb{N}}$. If $M^k \xrightarrow{(\mathrm{J}_1(\mathbb{R}^q),\mathbb{P})} M^\infty$, then $\mathbb{G}^{M^k} \xrightarrow{\mathrm{w}} \mathbb{G}^{M^\infty}$.

In the remainder of this section, we fix an arbitrary sequence of filtrations $(\mathbb{G}^k)_{k\in\overline{\mathbb{N}}}$ on $(\Omega, \mathcal{G}, \mathbb{P})$, with $\mathbb{G}^k := (\mathcal{G}^k_t)_{t\in\mathbb{R}_+}$, and an arbitrary sequence $(M^k)_{k\in\overline{\mathbb{N}}}$, where $M^k \in \mathcal{M}(\mathbb{G}^k;\mathbb{R}^q)$ for every $k \in \overline{\mathbb{N}}$. Recall that the random variables $M^k_\infty := \lim_{t\to\infty} M^k_t$ are well–defined $\mathbb{P}-a.s.$, and $M^k_\infty \in \mathbb{L}^1(\Omega, \mathcal{G}^k_\infty, \mathbb{P}; \mathbb{R}^q)$, for $k \in \overline{\mathbb{N}}$; see Theorem I.57.

Next, we would like to discuss how to deduce the extended convergence of martingales and filtrations from individual convergence results. Such properties have already been obtained by Mémin [53, Proposition 1.(iii)], where he refers to Coquet et al. [23, Proposition 7] for the proof. However, the authors in [23] proved the result under the additional assumption that the processes are adapted to their natural filtrations. Moreover, they consider a finite time horizon T, which gives the time point T a special role for the $\mathrm{J}_1(\mathbb{R})-$topology on $\mathbb{D}([0,T];\mathbb{R})$; see also Jacod and Shiryaev [41, Remark VI.1.10]. In addition, in Coquet et al. [23, Remark 1.2)] the convergence $M^k_\infty \xrightarrow{\mathbb{L}^1(\mathcal{G}^\infty_\infty;\mathbb{R}^q)} M^\infty_\infty$ is assumed, although it is not necessary (note that we have translated their results into our notation). This is restrictive, in the sense that they have to assume in addition the $\mathcal{G}^\infty_\infty-$measurability of M^k_∞, for each $k \in \mathbb{N}$.

We present below, for the sake of completeness, the statement and proof of the aforementioned resuts for the infinite time horizon case, under the condition $M^k_\infty \xrightarrow{\mathbb{L}^1(\mathcal{G};\mathbb{R}^q)} M^\infty_\infty$.

Proposition I.158. *Assume the convergence* $M^k_\infty \xrightarrow{\mathbb{L}^1(\Omega,\mathcal{G},\mathbb{P};\mathbb{R}^q)} M^\infty_\infty$ *holds. Then, the convergence* $\mathbb{G}^k \xrightarrow{\mathrm{w}} \mathbb{G}^\infty$ *is equivalent to the convergence* $(M^k, \mathbb{G}^k) \xrightarrow{\mathrm{ext}} (M^\infty, \mathbb{G}^\infty)$.

Proof. The first step is to show that the $\mathbb{L}^1$ convergence of $(M^k_\infty)_k$ together with the weak convergence of the filtrations implies the convergence of the martingales in the $\mathrm{J}_1(\mathbb{R}^q)-$topology. Let $\varepsilon > 0$ and $\mathbb{G}^k \xrightarrow{\mathrm{w}} \mathbb{G}^\infty$, then

$$\begin{aligned}
&\mathbb{P}\Big(d_{\mathrm{J}_1(\mathbb{R}^q)}(M^k, M^\infty) > \varepsilon\Big)\\
&\quad\le \mathbb{P}\Big(d_{\mathrm{J}_1(\mathbb{R}^q)}\big(M^k, \mathbb{E}[M^\infty_\infty|\mathcal{G}^k_\cdot]\big) > \frac{\varepsilon}{2}\Big) + \mathbb{P}\Big(d_{\mathrm{J}_1(\mathbb{R}^q)}\big(\mathbb{E}[M^\infty_\infty|\mathcal{G}^k_\cdot], M^\infty\big) > \frac{\varepsilon}{2}\Big)\\
&\quad\le \mathbb{P}\Big(\sup_{t\in[0,\infty)} \big|M^k_t - \mathbb{E}[M^\infty_\infty|\mathcal{G}^k_\cdot]\big| > \frac{\varepsilon}{2}\Big) + \mathbb{P}\Big(d_{\mathrm{J}_1(\mathbb{R}^q)}\big(\mathbb{E}[M^\infty_\infty|\mathcal{G}^k_\cdot], M^\infty\big) > \frac{\varepsilon}{2}\Big)\\
&\quad\le \frac{2}{\varepsilon}\mathbb{E}\big[|M^k_\infty - M^\infty_\infty|\big] + \mathbb{P}\Big(d_{\mathrm{J}_1(\mathbb{R}^q)}\big(\mathbb{E}[M^\infty_\infty|\mathcal{G}^k_\cdot], \mathbb{E}[M^\infty_\infty|\mathcal{G}^\infty_\cdot]\big) > \frac{\varepsilon}{2}\Big) \xrightarrow{k\to\infty} 0, \qquad \text{(I.58)}
\end{aligned}$$

where the first summand converges to 0 by assumption and the second one by the weak convergence of the filtrations. Let us point out that for the second inequality we have used that for $\alpha, \beta \in \mathbb{D}(\mathbb{R}^q)$ holds

$$d_{\mathrm{J}_1(\mathbb{R}^q)}(\alpha, \beta) \le d_{\mathrm{lu}}(\alpha, \beta) \le d_{\|\cdot\|_\infty}(\alpha, \beta),$$

by the definition of the metrics, while for the third inequality we used Doob's martingale inequality.

The next step is to apply Lemma I.124 to $(N^k)_{k\in\overline{\mathbb{N}}} := (M^k, \mathbb{E}[\xi|\mathcal{G}^k_\cdot])_{k\in\mathbb{N}}$, for $\xi \in \mathbb{L}^1(\mathcal{G}^\infty_\infty;\mathbb{R})$, in order to obtain the convergence in the extended sense. The $\mathrm{J}_1(\mathbb{R})-$convergence of each $(N^{k,i})_{k\in\overline{\mathbb{N}}}$, for $i = 1, \dots, q$, and of the partial sums $(\sum_{i=1}^p N^{k,i})_{k\in\overline{\mathbb{N}}}$, for $p = 1, \dots, q$ follows from the previous step and Lemma I.124. Moreover, the $\mathrm{J}_1(\mathbb{R})-$convergence of $(N^{k,q+1})_{k\in\overline{\mathbb{N}}}$ follows from the definition of the weak convergence of filtrations. Hence, we just have to show the $\mathrm{J}_1(\mathbb{R})-$convergence of $(\sum_{i=1}^{q+1} N^{k,i})_{k\in\overline{\mathbb{N}}}$.

By assumption, we have $\sum_{m=1}^q M^{k,m}_\infty + \xi \xrightarrow[k\to\infty]{\mathbb{L}^1(\mathcal{G};\mathbb{R})} \sum_{m=1}^q M^{\infty,m}_\infty + \xi$, and arguing as in (I.58) we obtain

$$\mathbb{E}\Big[\sum_{m=1}^q M^{k,m}_\infty + \xi\Big|\mathcal{G}^k_\cdot\Big] \xrightarrow[k\to\infty]{(\mathrm{J}_1(\mathbb{R}),\mathbb{P})} \mathbb{E}\Big[\sum_{m=1}^q M^{\infty,m}_\infty + \xi\Big|\mathcal{G}^\infty_\cdot\Big].$$

Moreover, by the linearity of conditional expectations, we get that

$$\sum_{m=1}^q \mathbb{E}[M^{k,m}_\infty|\mathcal{G}^k_\cdot] + \mathbb{E}[\xi|\mathcal{G}^k_\cdot] \xrightarrow[k\to\infty]{(\mathrm{J}_1(\mathbb{R}),\mathbb{P})} \sum_{m=1}^q \mathbb{E}[M^m_\infty|\mathcal{G}^\infty_\cdot] + \mathbb{E}[\xi|\mathcal{G}^\infty_\cdot].$$

The converse statement is trivial. □

Now we are going to present the cornerstones for the convergence in the extended sense, which are Mémin [53, Theorem 11, Corollary 12]. Here we state and prove them in the multidimensional case.

Theorem I.159. *Let $(S^k)_{k\in\overline{\mathbb{N}}}$ be a sequence of $\mathbb{R}^q-$valued $\mathbb{G}^k-$special semimartingales with $\mathbb{G}^k-$canonical decomposition $S^k = S_0^k + M^k + A^k$, for every $k \in \overline{\mathbb{N}}$. Assume that S^∞ is $\mathbb{G}^\infty-$quasi–left–continuous and the following properties hold*

(i) *The sequence $\left([S^{k,i}]_\infty^{1/2}\right)_{k\in\overline{\mathbb{N}}}$ is uniformly integrable, for every $i = 1, \dots, q$.*
(ii) *The sequence $(\|\mathrm{Var}(A^k)_\infty\|_1)_{k\in\mathbb{N}}$ is tight.*
(iii) *The extended convergence $(S^k, \mathbb{G}^k) \xrightarrow{\text{ext}} (S^\infty, \mathbb{G}^\infty)$ holds.*

Then

$$(S^k, M^k, A^k) \xrightarrow{\left(\mathrm{J}_1(\mathbb{R}^{3q}),\mathbb{P}\right)} (S^\infty, M^\infty, A^\infty).$$

Proof. By Mémin [53, Theorem 11], we obtain for every $i = 1, \dots, q$ the following convergence

$$(S^{k,i}, M^{k,i}, A^{k,i}) \xrightarrow{\left(\mathrm{J}_1(\mathbb{R}^{3}),\mathbb{P}\right)} (S^{\infty,i}, M^{\infty,i}, A^{\infty,i}).$$

Then, by assumption $S^k \xrightarrow{(\mathrm{J}_1(\mathbb{R}^q),\mathbb{P})} S^\infty$, and using Corollary I.125 and Remark I.123 we obtain the required result. □

Theorem I.160. *Let $M^k \in \mathcal{H}^2(\mathbb{G}^k, \infty; \mathbb{R}^q)$ for any $k \in \overline{\mathbb{N}}$ and M^∞ be $\mathbb{G}^\infty-$quasi–left–continuous. If the following convergence hold*

$$(M^k, \mathbb{G}^k) \xrightarrow{\text{ext}} (M^\infty, \mathbb{G}^\infty) \quad \textit{and} \quad M^k_\infty \xrightarrow{\mathbb{L}^2(\mathcal{G};\mathbb{R}^q)} M^\infty_\infty,$$

then

(i) $(M^k, [M^k], \langle M^k\rangle) \xrightarrow{\left(\mathrm{J}_1(\mathbb{R}^q\times\mathbb{R}^{q\times q}\times\mathbb{R}^{q\times q}),\mathbb{P}\right)} (M^\infty, [M^\infty], \langle M^\infty\rangle)$,
(ii) *for every $i = 1, \dots, q$, we have*

$$[M^{k,i}]_\infty \xrightarrow[k\to\infty]{\mathbb{L}^1(\mathcal{G};\mathbb{R})} [M^{\infty,i}]_\infty \quad \textit{and} \quad \langle M^{k,i}\rangle_\infty \xrightarrow[k\to\infty]{\mathbb{L}^1(\mathcal{G};\mathbb{R})} \langle M^{\infty,i}\rangle_\infty.$$

Proof. (i) By Mémin [53, Corollary 12], we obtain for every $i = 1, \dots, q$ the convergence

$$(M^{k,i}, [M^k]^{ii})^\top \xrightarrow{\left(\mathrm{J}_1(\mathbb{R}^2),\mathbb{P}\right)} (M^{\infty,i}, [M^\infty]^{ii})^\top,$$

which in conjunction with Corollary I.125 and the convergence $M^k \xrightarrow{(\mathrm{J}_1(\mathbb{R}^q),\mathbb{P})} M^\infty$ implies

$$\left(M^k, [M^k]^{11}, \dots, [M^k]^{qq}\right) \xrightarrow{\left(\mathrm{J}_1(\mathbb{R}^{2q}),\mathbb{P}\right)} \left(M^\infty, [M^\infty]^{11}, \dots, [M^\infty]^{qq}\right) \tag{I.59}$$

On the other hand, let $i, j \in \{1, \dots, q\}$ with $i \neq j$. Observe, now, that the sequence $(M^k)_{k\in\overline{\mathbb{N}}}$ satisfies the requirements of Proposition I.153. Therefore, the sequence is P-UT and by Theorem I.152, we obtain

$$(M^k, [M^k]) \xrightarrow{\left(\mathrm{J}_1(\mathbb{R}^q\times\mathbb{R}^{q\times q}),\mathbb{P}\right)} (M^\infty, [M^\infty]). \tag{I.60}$$

Let us now show that the predictable quadratic variations converge as well. By Mémin [53, Corollary 12], we have for $i = 1, \dots, q$

$$\langle M^k\rangle^{ii} \xrightarrow{(\mathrm{J}_1(\mathbb{R}),\mathbb{P})} \langle M^\infty\rangle^{ii}. \tag{I.61}$$

Moreover the convergence $M^k_\infty \xrightarrow{\left(\mathrm{J}_1(\mathbb{R}^2),\mathbb{L}^2\right)} M^\infty_\infty$ implies in particular, for every $i, j = 1, 2, \dots, q$ with $i \neq j$, that

$$M^{k,i}_\infty + M^{k,j}_\infty \xrightarrow{\left(\mathrm{J}_1(\mathbb{R}^2),\mathbb{L}^2\right)} M^{\infty,i}_\infty + M^{\infty,j}_\infty. \tag{I.62}$$

In view of (I.59) and (I.62), we can apply Mémin [53, Corollary 12] to $(M^{k,i} + M^{k,j})_{k\in\overline{\mathbb{N}}}$ and $(M^{k,i} - M^{k,j})_{k\in\overline{\mathbb{N}}}$, for every $i, j = 1, 2, \dots, q$ with $i \neq j$. Therefore we get that

$$\langle M^{k,i} + M^{k,j}\rangle \xrightarrow{(\mathrm{J}_1(\mathbb{R}),\mathbb{P})} \langle M^{\infty,i} + M^{\infty,j}\rangle,$$
$$\text{and} \quad \langle M^{k,i} - M^{k,j}\rangle \xrightarrow{(\mathrm{J}_1(\mathbb{R}),\mathbb{P})} \langle M^{\infty,i} - M^{\infty,j}\rangle.$$

Now recall that M^∞ is quasi–left–continuous, which implies that the processes $\langle M^{\infty,i}+M^{\infty,j}\rangle$ and $\langle M^{\infty,i}-M^{\infty,j}\rangle$ are continuous for every $i,j=1,2,\dots,q$ with $i\neq j$. Therefore, by Theorem I.62 and Proposition I.120 and the last results we obtain

$$\begin{aligned}\langle M^k\rangle^{ij}=\frac{1}{4}(\langle M^{k,i}+M^{k,j}\rangle-\langle M^{k,i}-M^{k,j}\rangle)&\xrightarrow{(\mathrm{J}_1(\mathbb{R}),\mathbb{P})}\\ &\frac{1}{4}(\langle M^{\infty,i}+M^{\infty,j}\rangle-\langle M^{\infty,i}-M^{\infty,j}\rangle)=\langle M^\infty\rangle^{ij}.\end{aligned}\tag{I.63}$$

Concluding, by (I.60), (I.61), (I.63) and due to the continuity of $\langle M^\infty\rangle$ we have

$$(M^k,[M^k],\langle M^k\rangle)\xrightarrow{\left(\mathrm{J}_1(\mathbb{R}^q\times\mathbb{R}^{q\times q}\times\mathbb{R}^{q\times q}),\mathbb{P}\right)}(M^\infty,[M^\infty],\langle M^\infty\rangle).$$

(ii) Let $i=1,\dots,q$. The sequence $(M^{k,i})_{k\in\overline{\mathbb{N}}}$ satisfies the conditions of Mémin [53, Corollary 12]. In the middle of the proof of the aforementioned corollary, we can find the required convergence. □

I.7. Martingale representation of square-integrable martingales

This section deals with the martingale representation of a square integrable martingale with respect to a pair of square-integrable martingales. In order to provide a better description of how we will understand a martingale representation let us provide the classical result [41, Lemma III.4.24]. We are going to translate the general framework of the aforementioned lemma into the square-integrable case. Now, given a pair X,Y where $X\in\mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell)$ and $Y\in\mathcal{H}^2(\mathbb{G};\mathbb{R})$, there exist a unique, up to indistinguishability, decomposition of the martingale Y with respect to the natural pair of X which satisfies the following:

$$Y=Z\cdot X^c+U\star\widetilde{\mu}^{(X^d,\mathbb{G})}+N,\tag{I.64}$$

where $Z\in\mathbb{H}^2(\mathbb{G},X^c;\mathbb{R}^\ell)$, $U\in\mathbb{H}^2(\mathbb{G},\mu^{X^d};\mathbb{R})$ and

$$\langle N^c,X^{c,i}\rangle=0\text{ for every }i=1,\dots,\ell\text{ and }M_{\mu^{X^d}}\left[\Delta N\middle|\widetilde{\mathcal{P}}^{\mathbb{G}}\right]=0.$$

Observe that the above properties imply that $N\perp\!\!\!\perp H\cdot X^c+W\star\widetilde{\mu}^{(X^d,\mathbb{G})}$ for every $H\in\mathbb{H}^2(\mathbb{G},X^c;\mathbb{R}^\ell)$ and $W\in\mathbb{H}^2(\mathbb{G},\mu^{X^d};\mathbb{R})$, a property which justifies the alternative term *orthogonal decomposition of Y with respect to X*. Before we proceed, let us provide the following definition for the representation property of a martingale X with respect to a filtration.

Definition I.161. Let $X\in\mathcal{H}^2(\mathbb{G};\mathbb{R})$. We will say that the martingale X possesses the $\mathbb{G}-$predictable representation property if for every real-valued $\mathbb{G}-$martingale Y holds

$$Y^c\in\mathcal{L}^2(\mathbb{G},X^c;\mathbb{R})\text{ and }Y^d\in\mathcal{K}^2(\mathbb{G},\mu^{X^d};\mathbb{R}).$$

In other words, there exist $Z\in\mathbb{H}^2(\mathbb{G},X^c;\mathbb{R}^{1\times\ell})$ and $U\in\mathbb{H}^2(\mathbb{G},\mu^{X^d};\mathbb{R})$ such that

$$Y=Y_0+Z\cdot X^c+U\star\widetilde{\mu}^{(X^d,\mathbb{G})}.$$

Having in mind that we want to develop a framework suitable also for discrete-time approximations, the presence of a continuous martingale in the representation rules out this possibility. Consequently, we would like to be able to construct an analogous orthogonal decomposition of Y with respect to a pair $(X^\circ,X^\natural)\in\mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell)\times\mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^\ell)$. Under this framework, the martingale X° will play the role of the integrator of the Itô stochastic integral and the purely discontinuous martingale $X^\natural$ will provide the integer-valued random measure. Since X° may also have a purely discontinuous part, it is natural to ask for the orthogonality of the Itô integral and the stochastic integral with respect to the integer-valued random measure that will appear in the decomposition. After these comments, we can provide the following definition.

Definition I.162. Let $(X^\circ,X^\natural)\in\mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell)\times\mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^\ell)$[26] and $Y\in\mathcal{H}^2(\mathbb{G};\mathbb{R}^p)$. The decomposition

$$Y=Y_0+Z\cdot X^\circ+U\star\widetilde{\mu}^{(X^\natural,\mathbb{G})}+N$$

will be called the *orthogonal decomposition of Y with respect to* $(\mathbb{G},X^\circ,\mu^{X^\natural})$ or the *representation of the martingale Y with respect to* $(\mathbb{G},X^\circ,\mu^{X^\natural})$ if

(i) $Z\in\mathbb{H}^2(\mathbb{G},X^\circ;\mathbb{R}^{p\times\ell})$ and $U\in\mathbb{H}^2(\mathbb{G},\mu^{X^\natural};\mathbb{R}^p)$,
(ii) $\langle Z\cdot X^\circ,U\star\widetilde{\mu}^{(X^\natural,\mathbb{G})}\rangle=0$ and

[26] Observe that the purely discontinuous part of $X^\natural$ is $X^\natural$ itself.

(iii) $N \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p)$ with $\langle N^j, X^{\circ,i}\rangle = 0$ for every $i = 1,\dots,\ell$, $j = 1,\dots,p$ and $M_{\mu^{X^\natural}}[\Delta N|\widetilde{\mathcal{P}}^{\mathbb{G}}] = 0$.

It is immediate that the decomposition (I.64) is an orthogonal decomposition according to Definition I.162. The Proposition I.165, which can be seen as a generalisation of Jacod and Shiryaev [41, Lemma III.4.24], provides a framework for the well-posedness of orthogonal decompositions. Moreover, the presented form allows for embedding discrete-time orthogonal decompositions into continuous-time orthogonal decompositions.

Notation I.163. For $X^\circ \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell)$, $X^\natural \in \mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^\ell)$ and for every $i = 1,\dots,\ell$

- $X^{\circ,i}$ denotes the $i-$element of X°.
- $X^{\circ,c,i}$ denotes the $i-$element of the continuous part of X°.
- $X^{\circ,d,i}$ denotes the $i-$element of the purely discontinuous part of X°.
- $X^{\natural,i}$ denotes the $i-$element of $X^\natural$.
- $\mathcal{H}^{2,\perp}(\mathbb{G}, X^\circ, \mu^{X^\natural};\mathbb{R}^p) := \big\{N \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p), N^i \perp\!\!\!\perp Y \text{ for every } Y \in \mathcal{L}^2(\mathbb{G}, X^\circ;\mathbb{R}) \cup \mathcal{K}^2(\mathbb{G}, \mu^{X^\natural};\mathbb{R})\big\}$.

Lemma I.164. *Let $(X^\circ, X^\natural) \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell) \times \mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^\ell)$ such that $M_{\mu^{X^\natural}}[\Delta X^\circ|\mathcal{P}^{\mathbb{G}}] = 0$. Then, for every $Y^\circ \in \mathcal{L}^2(\mathbb{G}, X^\circ;\mathbb{R})$, $Y^\natural \in \mathcal{K}^2(\mathbb{G}, \mu^{X^\natural};\mathbb{R})$ holds $Y^\circ \perp\!\!\!\perp Y^\natural$. In particular, $\langle X^\circ, X^\natural\rangle = 0$.*

Proof. For every $i = 1,\cdots,\ell$ and every $Y^\natural \in \mathcal{K}^2(\mathbb{G}, \mu^{X^\natural};\mathbb{R})$ holds

$$\langle X^{\circ,i}, Y^\natural\rangle = \langle X^{\circ,c,i}, Y^\natural\rangle + \langle X^{\circ,d,i}, Y^\natural\rangle = 0, \tag{I.65}$$

where the summand $\langle X^{\circ,c,i}, Y^\natural\rangle$ vanishes because $X^{\circ,c,i} \in \mathcal{H}^{2,c}(\mathbb{G};\mathbb{R})$ and $Y^\natural \in \mathcal{H}^{2,d}(\mathbb{G};\mathbb{R})$, while the summand $\langle X^{\circ,d,i}, Y^\natural\rangle$ equals to 0 in view of the assumption $M_{\mu^{X^\natural}}[\Delta X^\circ|\mathcal{P}^{\mathbb{G}}] = 0$ and Theorem I.93. The equality (I.65) already proves the second statement.

Assume now the factorisation

$$\langle X^\circ\rangle = \int_{(0,\cdot]} \frac{\mathrm{d}\langle X^\circ\rangle_s}{\mathrm{d}F_s}\mathrm{d}F_s.$$

In view of (I.65) we obtain $r^{Y^\natural} := \frac{\mathrm{d}\langle Y^\natural, X^{\circ,i}\rangle_s}{\mathrm{d}F_s} = 0$. Consequently, by Theorem I.67.(v) we have for every $Z \in \mathbb{H}^2(\mathbb{G}, X^\circ;\mathbb{R}^{1\times\ell})$ that

$$\langle Y^\natural, Z \cdot X^\circ\rangle = \int_{(0,\cdot]} Z(r^{Y^\natural})^\top \,\mathrm{d}F_s = 0.$$

□

Proposition I.165. *Let $(X^\circ, X^\natural) \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell) \times \mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^\ell)$ with $M_{\mu^{X^\natural}}[\Delta X^\circ|\mathcal{P}^{\mathbb{G}}] = 0$. For every $Y \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p)$ the orthogonal decomposition of Y with respect to $(\mathbb{G}, X^\circ, \mu^{X^\natural})$, say*

$$Y = Y_0 + Z \cdot X^\circ + U \star \widetilde{\mu}^{(X^\natural,\mathbb{G})} + N,$$

is well-defined and unique up to indistinguishability. In particular, $N \in \mathcal{H}^{2,\perp}(\mathbb{G}, X^\circ, \mu^{X^\natural};\mathbb{R}^p)$.

Proof. By the Galtchouk–Kunita–Watanabe Decomposition, see Theorem I.69, there exists $Z \in \mathbb{H}^2(\mathbb{G}, X^\circ;\mathbb{R}^{p\times\ell})$ such that

$$Y^j - Y_0^j = (Z \cdot X^\circ)^j + \overline{N}^j \tag{I.66}$$

with $\langle \overline{N}^j, X^{\circ,i}\rangle = 0$ for $i = 1,\dots,\ell$ and $j = 1,\dots,p$. Moreover, by Theorem I.95, there exists a unique $U \in \mathbb{H}^2(\mathbb{G}, \mu^{X^\natural};\mathbb{R}^p)$ and $N \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p)$ with $M_{\mu^{X^\natural}}[\Delta N|\mathcal{P}^{\mathbb{G}}] = 0$ such that

$$\overline{N}^j = U^j \star \widetilde{\mu}^{(X^\natural,\mathbb{G})} + N^j \text{ for every } j = 1,\dots,p. \tag{I.67}$$

In total, we have determined $Z \in \mathbb{H}^2(\mathbb{G}, X^\circ;\mathbb{R}^{p\times\ell})$, $U \in \mathbb{H}^2(\mathbb{G}, \mu^{X^\natural};\mathbb{R}^p)$ and $N \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p)$ such that

$$Y = Z \cdot X^\circ + U \star \widetilde{\mu}^{(X^\natural,\mathbb{G})} + N,$$

which can be written without loss of generality as

$$Y = Y_0 + Z \cdot X^\circ + U \star \widetilde{\mu}^{(X^\natural,\mathbb{G})} + N. \tag{I.68}$$

We have to verify that this decomposition satisfies the properties (ii)-(iii) of Definition I.162 and, moreover, that it does not depend on the way we have determined Z, U and N

We will prove initially that the $\mathbb{G}-$predictable function U is the one characterised by the triplet $(\mathbb{G}, \mu^{X^\natural}, Y)$. To this end, we are going to prove that $M_{\mu^{X^\natural}}\left[\Delta(Z\cdot X^\circ)\big|\widetilde{\mathcal{P}}^{\mathbb{G}}\right]=0$. By Theorem I.67.(v), we can write

$$\Delta(Z\cdot X^\circ)^{d,i}=\sum_{n=1}^{\ell} Z^{in}\Delta X^{\circ,d,n} \text{ for every } i=1,\dots,p.$$

Therefore, for every positive and bounded $\mathbb{G}-$predictable function W holds

$$\begin{aligned} M_{\mu^{X^\natural}}\left[W\Delta(Z\cdot X^\circ)^{d,i}\right] &= M_{\mu^{X^\natural}}\Big[W\sum_{n=1}^{\ell} Z^{in}\Delta X^{\circ,d,n}\Big]=\sum_{n=1}^{\ell} M_{\mu^{X^\natural}}\left[WZ^{in}\Delta X^{\circ,d,n}\right]\\ &=\sum_{n=1}^{\ell} M_{\mu^{X^\natural}}\Big[WZ^{in}M_{\mu^{X^\natural}}\left[\Delta X^{\circ,d,n}\big|\widetilde{\mathcal{P}}^{\mathbb{G}}\right]\Big]=0, \end{aligned}$$

where we used the assumption $M_{\mu^{X^\natural}}[\Delta X^\circ|\mathcal{P}^{\mathbb{G}}]=0$ and that Z is a $\mathbb{G}-$predictable process in order to conclude. The above, after using standard monotone class arguments, allows us to conclude the required property $M_{\mu^{X^\natural}}\left[\Delta(Z\cdot X^\circ)\big|\widetilde{\mathcal{P}}^{\mathbb{G}}\right]=0$. By (I.68), the equalities

$$M_{\mu^{X^\natural}}[\Delta(Z\cdot X^\circ)|\mathcal{P}^{\mathbb{G}}]=M_{\mu^{X^\natural}}[\Delta X^\circ|\mathcal{P}^{\mathbb{G}}]=0$$

and the linearity of the Doléans-Dade measure $M_{\mu^\natural}$ we obtain that the following hold $M_{\mu^{X^\natural}}-$almost everywhere

$$M_{\mu^{X^\natural}}\left[\Delta Y\big|\widetilde{\mathcal{P}}^{\mathbb{G}}\right]=M_{\mu^{X^\natural}}\left[\Delta\overline{N}\big|\widetilde{\mathcal{P}}^{\mathbb{G}}\right]=M_{\mu^{X^\natural}}\left[\Delta(U\star\widetilde{\mu}^{(X^\natural,\mathbb{G})})\big|\widetilde{\mathcal{P}}^{\mathbb{G}}\right].$$

Hence the $\mathbb{G}-$predictable function U is uniquely determined $M_{\mu^{X^\natural}}-$almost everywhere, see Theorem I.95.

We need to prove now that $\langle Z\cdot X^\circ, U\star\widetilde{\mu}^{(X^\natural,\mathbb{G})}\rangle=0$ as well as $\langle N, X^\circ\rangle=0$. But the former is immediate by Lemma I.164, since it verifies that

$$\langle Z\cdot X^\circ, U\star\widetilde{\mu}^{(X^\natural,\mathbb{G})}\rangle^{ij}=0 \text{ for every } i,j=1,\dots,p. \tag{I.69}$$

We proceed to prove the $\langle N, X^\circ\rangle=0$. To this end let is fix an $i=1,\cdots,\ell$ and a $j\in\{1,\dots,p\}$. Finally, by (I.66) and (I.69) we obtain for every $i=1,\dots,\ell$

$$\langle N^j, X^{\circ,i}\rangle \overset{(I.67)}{=} \langle\overline{N}^j, X^{\circ,i}\rangle-\langle U^j\star\widetilde{\mu}^{(X^\natural,\mathbb{G})}, X^{\circ,i}\rangle=0. \tag{I.70}$$

To sum up,

(i) $Z\in\mathbb{H}^2(\mathbb{G}, X^\circ;\mathbb{R}^{p\times\ell})$ and $U\in\mathbb{H}^2(\mathbb{G},\mu^{X^\natural};\mathbb{R}^p)$, by (I.66) and (I.67). Moreover, $Z\cdot X^\circ$ and $U\star\widetilde{\mu}^{(X^\natural,\mathbb{G})}$ are unique up to indistinguishability.

(ii) $\langle Z\cdot X^\circ, U\star\widetilde{\mu}^{(X^\natural,\mathbb{G})}\rangle=0$ by (I.69).

(iii) $\langle N^j, X^{\circ,i}\rangle=0$ for every $j=1,\dots,p$, $i=1,\dots,\ell$ and $M_{\mu^{X^\natural}}[\Delta N|\mathcal{P}^{\mathbb{G}}]=0$, by (I.70) and (I.67) respectively.

In view of the above properties, we can argue analogously to Lemma I.164 in order to prove that $\langle N, Y^\circ\rangle=0$ for every $Y^\circ\in\mathcal{L}^2(\mathbb{G}, X^\circ;\mathbb{R}^p)$ and that $\langle N, Y^\natural\rangle=0$ for every $Y^\natural\in\mathcal{K}^2(\mathbb{G},\mu^{X^\natural};\mathbb{R}^p)$. Therefore, $N\in\mathcal{H}^{2,\perp}(\mathbb{G}, X^\circ,\mu^{X^\natural};\mathbb{R}^p)$. □

In view of the framework of Proposition I.165, we will work

- In Chapter II with a process $\overline{X}$, where $\overline{X}:=(X^\circ, X^\natural)\in\mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell)\times\mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^\ell)$ such that $M_{\mu^{X^\natural}}\left[\Delta X^\circ\big|\widetilde{\mathcal{P}}^{\mathbb{G}}\right]=0$.
- In Section III.8 and Chapter IV we will work with a sequence $(\overline{X}^k)_{k\in\overline{\mathbb{N}}}$, where for every $k\in\overline{\mathbb{N}}$ $\overline{X}^k:=(X^{k,\circ}, X^{k,\natural})\in\mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell)\times\mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^\ell)$ such that $M_{\mu^{X^{k,\natural}}}\left[\Delta X^{k,\circ}\big|\widetilde{\mathcal{P}}^{\mathbb{G}^k}\right]=0$ and the probability space will be endowed with a sequence of filtrations $(\mathbb{G}^k)_{k\in\overline{\mathbb{N}}}$.

However, we will need also a process $C^{\overline{X}}\in\mathcal{V}^+_{\text{pred}}(\mathbb{G};\mathbb{R})$, resp. a sequence $(C^{\overline{X}^k})\subset\mathcal{V}^+_{\text{pred}}(\mathbb{G};\mathbb{R})$, which will be responsible, among others, for the factorisation of $\langle X^\circ\rangle$, resp. $(\langle X^{k,\circ}\rangle)_{k\in\overline{\mathbb{N}}}$.

In the following lemma, which is essentially Lemma I.166, we prove that for $\overline{X}:=(X^\circ, X^\natural)$ as in Lemma I.164 there exists a process $C^{\overline{X}}$ with the desired properties. For this reason, we collect in the assumption following the lemma the properties we will need for the pair $(\overline{X}, C^{\overline{X}})$ to satisfy.

Lemma I.166. *Let* $\overline{X} := (X^\circ, X^\natural) \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell) \times \mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^\ell)$ *with* $M_{\mu^{X^d}}[\Delta X^c|\widetilde{\mathcal{P}}^{\mathbb{G}}] = 0$. *Then, there exists* $C^{\overline{X}} \in \mathcal{V}^+_{\text{pred}}(\mathbb{G};\mathbb{R})$ *such that*

(i) *Each component of* $\langle X^\circ\rangle$ *is absolutely continuous with respect to* $C^{\overline{X}}$. *In other words, there exists a predictable, positive definite and symmetric* $\ell\times\ell-$*matrix* $\frac{\mathrm{d}\langle X^\circ\rangle}{\mathrm{d}C^{\overline{X}}}$ *such that for any* $1 \le i,j \le \ell$

$$\langle X^\circ\rangle^{ij}_\cdot = \int_{(0,\cdot]} \frac{\mathrm{d}\langle X^\circ\rangle^{ij}_s}{\mathrm{d}C^{\overline{X}}_s}\,\mathrm{d}C^{\overline{X}}_s.$$

(ii) *The* disintegration property *given* $C^{\overline{X}}$ *holds for the compensator* $\nu^{(X^\natural,\mathbb{G})}$, i.e. *there exists a transition kernel* $K^{\overline{X}} : (\Omega\times\mathbb{R}_+,\mathcal{P}) \longrightarrow \mathcal{R}(\mathbb{R}^\ell,\mathcal{B}(\mathbb{R}^\ell))$, *where* $\mathcal{R}\big(\mathbb{R}^\ell,\mathcal{B}(\mathbb{R}^\ell)\big)$ *is the space of Radon measures on* $\big(\mathbb{R}^\ell,\mathcal{B}(\mathbb{R}^\ell)\big)$ *such that*

$$\nu^{(X^\natural,\mathbb{G})}(\omega;\mathrm{d}t,\mathrm{d}x) = K^{\overline{X}}_t(\omega;\mathrm{d}x)\,\mathrm{d}C^{\overline{X}}_t.$$

Moreover, for every every pair $(\omega,t)\in\Omega\times\mathbb{R}_+$ *the transition kernel possesses the following properties*

$$K^{\overline{X}}_t(\omega;\{0\}) = 0,\ \int_{\mathbb{R}^\ell}(\|x\|_2^2\wedge 1)\,K^{\overline{X}}_t(\omega;\mathrm{d}x) \le 1 \text{ and } \Delta C^{\overline{X}}_{(\omega)}K^{\overline{X}}_t(\omega;\mathbb{R}^\ell) \le 1. \tag{I.71}$$

(iii) $C^{\overline{X}}$ *can be chosen to be continuous if and only if* $\overline{X}$ *is* $\mathbb{G}-$*quasi-left-continuous.*

Proof. We can follow exactly the same arguments as in Jacod and Shiryaev [41, Proposition II.2.9] for the process

$$C^{\overline{X}}_\cdot := \big\|\mathrm{Var}\big(\langle X^\circ\rangle\big)_\cdot\big\|_1 + (\|\mathrm{Id}_\ell\|_2^2\wedge 1) * \nu^{(X^\natural,\mathbb{G})}_\cdot.$$

We need to underline that under our framework the process $\langle X^\circ\rangle$ is not necessarily continuous. However, Lemma I.164 verifies that we can indeed follow the same arguments as in [41, Proposition II.2.9]. □

Assumption ($\mathbb{C}$). *We will say that the pair* $(\overline{X}, C^{\overline{X}})$ *satisfies* Assumption $\mathbb{C}$ under $\mathbb{G}$, *where* $\overline{X} := (X^\circ, X^\natural)$, *if:*

(i) $\overline{X} \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell)\times\mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^\ell)$ *with* $M_{\mu^{X^\natural}}[\Delta X^\circ|\mathcal{P}^{\mathbb{G}}] = 0$.

(ii) $C^{\overline{X}} \in \mathcal{V}^+_{\text{pred}}(\mathbb{G};\mathbb{R})$.

(iii) *The process* $C^{\overline{X}}$ *satisfies the properties* (i) - (iii) *of Lemma I.166.*

Remark I.167. Given an $X \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell)$ we have that its natural pair (X^c, X^d) satisfies trivially the condition $M_{\mu^{X^d}}[\Delta X^c|\widetilde{\mathcal{P}}^{\mathbb{G}}] = 0$, since $\Delta X^c = 0$ identically. Let us denote $\overline{X} := (X^c, X^d)$ and mention that there exist several possible choices for $C^{\overline{X}}$ such that the pair $(\overline{X}, C^{\overline{X}})$ satisfies Assumption ($\mathbb{C}$). In Jacod and Shiryaev [41, Proposition II.2.9], for example, the following process is used

$$\big\|\mathrm{Var}\big(\langle X^{c,i}, X^{c,j}\rangle\big)_\cdot\|_1 + (\|\mathrm{Id}_\ell\|_2^2\wedge 1) * \nu^{(X^d,\mathbb{G})}_\cdot, \tag{I.72}$$

while one could also take

$$\mathrm{Tr}[\langle X^c\rangle_\cdot] + \|\mathrm{Id}_\ell\|_2^2 * \nu^{(X^d,\mathbb{G})}_\cdot.$$

CHAPTER II

Backward Stochastic Differential Equations with Jumps

II.1. Introduction

This chapter, which follows closely Papapantoleon et al. [54], is devoted to obtaining a wellposedness result for multidimensional backward stochastic differential equations with jumps[1] driven by a square-integrable martingale in a filtration which may be stochastically discontinuous. Initially, let us explain roughly what is a BSDE.[2] To this end let us set the framework. The probability space $(\Omega, \mathcal{G}, \mathbb{P})$ is given and it will be assumed fixed throughout the chapter. Assume, now, that we are also given an arbitrary sextuple $(\mathbb{G}, T, X, \xi, C, f)$, where $\mathbb{G}$ is a filtration, T is a $\mathbb{G}-$stopping time which may be infinite, $X^T \in \mathcal{H}^2(\mathbb{G}; \mathbb{R}^\ell)$, $\xi \in \mathbb{L}^2(\mathcal{F}_T; \mathbb{R}^p)$, $C \in \mathcal{V}^+_{\text{pred}}(\mathbb{G}; \mathbb{R})$ and f is a stochastic operator which will be called the generator of the BSDE; the properties the data should satisfy will be made precise in **(F1)** - **(F5)** on p. 58. We are interested in proving the existence and the uniqueness of a quadruple (Y, Z, U, N) which satisfies on $[\![0, T]\!]$

$$Y_t = Y_0 - \int_{(0,t]} f(s, Y_s, Z_s, U(s, \cdot))\mathrm{d}C_s \\ - \int_0^t Z_s \mathrm{d}X^c_s - \int_0^t \int_{\mathbb{R}^\ell} U(s, x)\widetilde{\mu}(\mathrm{d}s, \mathrm{d}x) - \int_0^t \mathrm{d}N_s \text{ with } Y_T = \xi \quad \mathbb{P} - a.s.,$$

or equivalently

$$Y_t = \xi + \int_{(t,T]} f(s, Y_s, Z_s, U(s, \cdot))\mathrm{d}C_s - \int_t^T Z_s \mathrm{d}X^c_s - \int_t^T \int_{\mathbb{R}^\ell} U(s, x)\widetilde{\mu}(\mathrm{d}s, \mathrm{d}x) - \int_t^T \mathrm{d}N_s, \ \mathbb{P} - a.s.. \quad \text{(II.1)}$$

The quadruple (Y, Z, U, N) will be said to be *the solution of the BSDE* (II.1) or simply the solution of the BSDE.

In general, the existence or the uniqueness of the solution, hereinafter (E/U), is not guaranteed for an arbitrarily given sextuple as above. It turns out that for the (E/U) the properties of the Lebesgue–Stieltjes integrator C play a critical role. More precisely, when C is continuous and the data satisfy a suitable $\mathbb{L}^2-$type condition, then the *a priori* estimates of the BSDE can be obtained, see *e.g.* El Karoui and Huang [29] and El Karoui and Mazliak [30, Subsection 2.1.2] for the need of the integrability condition. However, when C is allowed to have jumps, *i.e.* when the driving martingale is not $\mathbb{G}-$quasi-left-continuous, the counter-example of Confortola, Fuhrman, and Jacod [22, Section 4.3] shows that there are cases that either the solution does not exist or there are infinitely many solutions. This interesting phenomenon is further analysed in Section II.2. Then, we proceed to show that a framework for (E/U) finally can be provided once the stochastic operator f satisfies a stochastic Lipschitz condition and the interplay between the stochastic bounds of f and the jumps of the integrator C can be tamed in the sense of Condition (II.18). This is stated in Theorem II.14, where the (E/U) is ensured in appropriately weighted spaces. Our result allows in particular to obtain a wellposedness result for BSDEs driven by discrete-time approximations of square-integrable martingales.

It would be an unforgivable omission if we had not mentioned that it was the seminal work of Pardoux and Peng [55] that established the theory of non-linear BSDEs as a notably active field of research. Since we will deal with a Lipschitz-type generator we intend to mention only the related works, although the theory of BSDEs has also been developed towards other directions. However, their abundance does not allow us to claim that the following list is comprehensive; to the contrary, we could regard it as rather superficial.

[1] Hereinafter, we will refer to a backward stochastic differential equation with jumps as BSDE. When we refer to many such equations the acronym alters to BSDEs.

[2] For a detailed introduction in the case the BSDE is driven by a Brownian motion, the reader may consult El Karoui, Peng, and Quenez [31].

We have initially to go back some decades, more precisely in 1973, when Davis and Varaiya [24] and Bismut [8, 9][3] have dealt with linear versions of BSDEs driven by a Brownian motion. Then we proceed to 1990, when the first systematic treatment of non-linear BSDEs was made by Pardoux and Peng [55] in the case the driving martingale is the Brownian motion and the generator is globally Lipschitz. More general results on BSDEs driven by a càdlàg martingale were obtained by Buckdahn [16], El Karoui et al. [31], as well as El Karoui and Huang [29] and Carbone, Ferrario, and Santacroce [17], where the underlying filtration is assumed to be quasi-left-continuous. To the best of our knowledge, the first articles that went beyond this assumption were developed in a very nice series of papers by Cohen and Elliott [19] and Cohen, Elliott, and Pearce [21], where the only assumption on the filtration is that the associated $\mathbb{L}^2-$space is separable, so that a very general martingale representation result due to Davis and Varaiya [25], involving countably many orthogonal martingales, holds. In these works, the martingales driving the BSDE are actually imposed by the filtration, and not chosen *a priori*, and the non-decreasing process C is not necessarily related to them, but has to be deterministic and can have jumps in general, though they have to be small for existence to hold (see [19, Theorem 5.1]). Similarly, Bandini [6] obtained wellposedness results in a context of a general filtration allowing for jumps, with a fixed driving martingale and associated random process C, which must have again small jumps, see [6, Equation (1.1)]. We mention again the work by Confortola et al. [22] which concentrates on the pure-jump general case and gives in particular counterexamples to existence. Finally, Bouchard, Possamaï, Tan, and Zhou [13] provide a general method to obtain *a priori* estimates in general filtrations when the martingale driving the equation has quadratic variation absolutely continuous with respect to the Lebesgue measure.

We describe, now, the structure of the chapter. Section II.2, which is mainly based on the counterexample of Confortola et al. [22, Section 4.3], intends to shed some light on the various situations in which a solution may or may not exist. In Section II.3 we provide the main notation we are going to use in the chapter, which will be used in Chapter IV as well. Then, in Section II.4 we state the theorem for (E/U) of (II.1). The *a priori* estimates for (II.1) are provided in Section II.5, while in the subsequent section the proof of the theorem is presented. In Section II.7 we compare our results with the related literature.

II.2. A simple counterexample and its analysis

As we have already mentioned, Confortola et al. [22, Section 4.3 Case (1)] provided a counterexample to the existence or the uniqueness of the solution of a BSDE in case the integrator C is not a continuous process. We are going to have a detailed look on their counterexample, and, based on it, we are going to discuss the well-posedness of variants of (II.1). We will see that the well-posedness of the BSDE is affected by seemingly slight changes in its form. To this end, let us provide the framework of the section,which will be set to be as simple as possible.

The probability space $(\Omega, \mathcal{G}, \mathbb{P})$ has been already given. We assume that there exists a $\Pi \in \mathcal{G}$ such that $p := \mathbb{P}(\Pi) \in (0,1)$. We define, now, for $r \in (0,\infty)$ the random variable ρ by

$$\rho(\omega) = \begin{cases} r, & \text{when } \omega \in \Pi, \\ \infty, & \text{for } \omega \in \Pi^c. \end{cases}$$

We proceed by defining the *one-point* process $X = \mathbb{1}_{[\rho,\infty)}$, whose natural filtration is denoted by $\mathbb{F}^X$. In other words, ρ is an $\mathbb{F}^X-$stopping time. By Jacod and Shiryaev [41, Lemma III.1.29] we have a characterisation of the elements of $\mathbb{F}^X$. Moreover, by the aforementioned lemma we have a characterisation of the $\mathbb{F}^X-$predictable functions, which in this special case are processes. We have, then, that U is $\mathbb{F}^X-$predictable if and only if is of the form

$$U(\omega,t) = U_0 + U_\infty \mathbb{1}_{(r,\infty]}(t)\mathbb{1}_\Pi(\omega), \tag{II.2}$$

where U_0 is $\mathcal{F}_0^X-$measurable and U_∞ is $\mathcal{F}_r^X-$measurable. The compensator of X under $\mathbb{F}^X$ is described by Jacod and Shiryaev [41, Theorem III.1.33]. Indeed, for $\mathbb{P}_\rho$ being the distribution of ρ on $\big([0,\infty], \mathcal{B}([0,\infty])\big)$, we calculate initially the distribution function F_ρ of ρ, which is given by

$$F_\rho(t) = \begin{cases} 0, & \text{for } t \in [0,r), \\ p, & \text{for } t \in [r,\infty), \\ 1, & \text{for } t = \infty \end{cases} \text{ and also } \mathbb{P}_\rho([t,\infty]) = \begin{cases} 1, & \text{for } t \in [0,r], \\ 1-p, & \text{for } t \in (r,\infty]. \end{cases}$$

[3]The published version of [24] states that the article was received on October 27, 1971. It is also present in bibliography of [9], though it is never referred to in the text.

Consequently, the càdlàg function

$$C_t := \int_{(0,t]} \frac{1}{\mathbb{P}_\rho\big([s,\infty]\big)} F_\rho(\mathrm{d}s) = \begin{cases} 0, & \text{for } t \in (0,r), \\ \frac{\Delta F_\rho(r)}{\mathbb{P}_\rho([r,\infty])} = p, & \text{for } t \in [r,\infty), \\ \frac{\Delta F_\rho(r)}{\mathbb{P}_\rho([r,\infty])} + \frac{\Delta F_\rho(\infty)}{\mathbb{P}_\rho(\{\infty\})} = p+1, & \text{for } t = \infty \end{cases}$$

restricted on $\mathbb{R}_+$ is the $\mathbb{F}^X$-compensator of the process X, *i.e.* the process $\widetilde{X} := X - C$ is an $\mathbb{F}^X-$martingale on $\mathbb{R}_+$.

We proceed to designate the rest of the data. We fix some generator $f : [0,T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, the terminal time $T \in (r,\infty)$ and the terminal condition ξ. Now we consider the following BSDEs

$$Y_t + \int_t^T U_s \mu^X(\mathrm{d}s) = \xi + \int_{(t,T]} f(s, Y_{s-}, U_s)\mathrm{d}C_s, \tag{II.3}$$

$$Y_t + \int_t^T U_s \widetilde{\mu}^X(\mathrm{d}s) = \xi + \int_{(t,T]} f(s, Y_{s-}, U_s)\mathrm{d}C_s, \tag{II.4}$$

$$Y_t + \int_t^T U_s \mu^X(\mathrm{d}s) = \xi + \int_{(t,T]} f(s, Y_s, U_s)\mathrm{d}C_s \tag{II.5}$$

and

$$Y_t + \int_t^T U_s \widetilde{\mu}^X(\mathrm{d}s) = \xi + \int_{(t,T]} f(s, Y_s, U_s)\mathrm{d}C_s. \tag{II.6}$$

The first case corresponds to the one Confortola et al. [22] considered, where they chose the generator to have the form $f(t,y,u) = \frac{1}{p}(y + g(u))$ for a deterministic function $g : \mathbb{R} \longrightarrow \mathbb{R}$. The authors proved that the BSDE (II.3) can admit either infinitely many solutions or none. We would like to examine apart from the aforementioned counterexample also the BSDEs (II.4) - (II.6), which can be categorised as follows. The BSDE (II.4) is a modification of (II.3), where the stochastic integral is assumed to be taken with respect to the compensated integer-valued random measure. The BSDEs (II.5) and (II.6) are modifications of the first two, in the sense that now the dependence of the Lebesgue–Stieltjes integral is on Y instead of Y_-.

Before we proceed, let us comment on the structure of the solution we are seeking. Given the structure of the filtration $\mathbb{F}^X$, the terminal condition ξ can always have a decomposition of the form

$$\xi(\omega) =: \xi^\Pi \mathbb{1}_\Pi(\omega) + \xi^{\Pi^c}\mathbb{1}_{\Pi^c}(\omega),\ (\xi^\Pi, \xi^{\Pi^c}) \in \mathbb{R} \times \mathbb{R}.$$

Once again because of the structure of $\mathbb{F}^X$ and of the form (II.2) for the $\mathbb{F}^X-$predictable functions, the possible solutions for the BSDEs above necessarily have the following form

$$Y_t(\omega) = Y_0 \mathbb{1}_{[0,r)}(t) + \xi^\Pi \mathbb{1}_{[r,\infty)}(t)\mathbb{1}_\Pi(\omega) + \xi^{\Pi^c}\mathbb{1}_{[r,\infty)}(t)\mathbb{1}_{\Pi^c}(\omega),$$

and

$$U_t(\omega) = U_0 + U_\infty \mathbb{1}_{(r,\infty]}(t)\mathbb{1}_\Pi(\omega),$$

for some $Y_0, U_0 \in \mathbb{R}$ and U_∞ is $\mathcal{F}_r^X-$measurable. However, only the value U_r is actually involved because of the form of X and C. Now we can distinguish between the following cases:

(C1) Consider the BSDE (II.3). Then, there exists a solution if and only if there exists a fixed point, called $Y_0^\star$, for the equation

$$\xi^\Pi + pf(r, x, \xi^\Pi - \xi^{\Pi^c}) = x.$$

The pair $(Y_0^\star, \xi^\Pi - \xi^{\Pi^c})$ is a solution of (II.3). The solution is unique if and only if the fixed point is unique. In case f is globally Lipschitz with respect to its second argument, *i.e.*

$$|f(t, y_1, u) - f(t, y_2, u)| \le L^y |y_1 - y_2|,$$

then, a sufficient condition for the existence and uniqueness of the solution is $L^y \Delta C_r < 1$.

(C2) Consider the BSDE (II.4), where, comparing to the previous case, the integrand of the Lebesgue–Stieltjes integral depends on Y instead of Y_-. Then, there exists a solution if and only if there exists a fixed point, called $Y_0^\star$, for the equation

$$\xi^\Pi + pf(r, x, \xi^\Pi - \xi^{\Pi^c}) + p(\xi^\Pi - \xi^{\Pi^c}) = x.$$

The pair $(Y_0^\star, \xi^\Pi - \xi^{\Pi^c})$ is a solution of (II.4). The solution is unique if and only if the fixed point is unique. In case f is globally Lipschitz with respect to the second argument as above, a sufficient condition for the existence and uniqueness of the solution is again $L^y \Delta C_r < 1$.

(C3) Consider now the BSDE (II.5). Then, there exists a solution if and only if there exists a fixed point, called $\upsilon^\star(r)$, for the equation

$$\xi^\Pi - \xi^{\Pi^c} - pf\big(r, \xi^{\Pi^c}, x\big) + pf(r, \xi^\Pi, x) = x.$$

The pair $(\xi^{\Pi^c} + pf\big(r, \xi^{\Pi^c} + p\upsilon^\star(r), \upsilon^\star(r)\big), \upsilon^\star(r))$ is a solution of (II.5). The solution is unique if and only if the fixed point is unique. In case f is globally Lipschitz with respect to its third argument, *i.e.*

$$|f(t, y, u_1) - f(t, y, u_2)| \leq L^u |u_1 - u_2|,$$

then, a sufficient condition for the existence and uniqueness of the solution is $2L^u \Delta C_r < 1$. This condition is not necessary: let $f'(t, y, u) = \frac{1}{p}(g(y) + u)$, where g is a deterministic function, then it holds that $L^u \Delta C_r = 1$; however, (II.5) admits a unique solution, which is given by the pair

$$\big(\xi^\Pi + g(\xi^\Pi), \xi^\Pi - \xi^{\Pi^c} + g(\xi^\Pi) - g(\xi^{\Pi^c})\big).$$

(C4) Finally, consider a BSDE similar to (II.5), where the stochastic integral is taken with respect to the compensated jump process. Then, there exists a solution if and only if there exists a fixed point, called $\upsilon^\star(r)$, for the equation

$$\xi^\Pi - \xi^{\Pi^c} - pf\big(r, \xi^{\Pi^c}, x\big) + pf(r, \xi^\Pi, x) = x.$$

The pair $(\xi^{\Pi^c} + pf\big(r, \xi^{\Pi^c}, \upsilon^\star(r)\big), \upsilon^\star(r))$ is a solution of (II.6). The solution is unique if and only if the fixed point is unique. In case f is globally Lipschitz with respect to its third argument as above, a sufficient condition for the existence and uniqueness of the solution is again $2L^u \Delta C_r < 1$. Once again this condition in not necessary; indeed, for f' as in (C3), $L^u \Delta C_r = 1$, while the unique solution of the BSDE (II.6) is the pair

$$\big((1-p)[\xi^\Pi + g(\xi^\Pi)] + p[\xi^{\Pi^c} + g(\xi^{\Pi^c})], \xi^\Pi - \xi^{\Pi^c} + g(\xi^\Pi) - g(\xi^{\Pi^c})\big).$$

Now, returning to the original counterexample of [22], we can observe that the sufficient condition $L^y \Delta C_r < 1$ is violated there, which explains why wellposedness issues can arise. However, an important observation here is that the structure of the generator plays a crucial role as well. Indeed, if we consider the same BSDE with the following generator $f(t, y, u) = \ell(y + g(u))$ with $\ell \neq \frac{1}{p}$, then the BSDE admits a unique solution.

Remark II.1. We observe that the dependence of the integrand on Y or Y_- is not always that innocuous. Indeed, the same BSDE might have a solution in the one formulation but not in the other. Observe furthermore that in the first situation the Lipschitz constant L^y appears in the condition for the existence and uniqueness of a solution, while in the second case the Lipschitz constant L^u appears. We will see that in our framework we can treat both cases simultaneously, hence naturally both Lipschitz constants will appear in our condition; see Condition **(F4)** and Conditions (II.19).

II.3. Suitable spaces and associated results

We start by endowing the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ with an arbitrary filtration $\mathbb{G}$. Moreover, we consider an arbitrary $\overline{X} := (X^\circ, X^\natural) \in \mathcal{H}^2(\mathbb{G}; \mathbb{R}^\ell) \times \mathcal{H}^{2,d}(\mathbb{G}; \mathbb{R}^\ell)$ and an arbitrary $C^{\overline{X}} \in \mathcal{V}^+(\mathbb{G}; \mathbb{R})$ such that the pair $(\overline{X}, C^{\overline{X}})$ satisfies Assumption $(\mathbb{C})$ under $\mathbb{G}$.

Notation II.2. To the pair $(\overline{X}, C^{\overline{X}})$ which satisfies Assumption $(\mathbb{C})$ under $\mathbb{G}$ (with associated transition kernel $K^{\overline{X}}$) we will associate the following:

- $c^{\overline{X}}$ denotes the square root of the Radon–Nikodým derivative $\mathrm{d}\langle X^\circ \rangle / \mathrm{d}C^{\overline{X}}$. In other words, $c^{\overline{X}}$ is the unique $\mathbb{G}-$predictable, positive definite, symmetric and $\mathbb{R}^{\ell \times \ell}-$valued process for which

$$(c^{\overline{X}})^\top c^{\overline{X}} = \frac{\mathrm{d}\langle X^\circ \rangle}{\mathrm{d}C^{\overline{X}}}. \tag{II.7}$$

- $\mu_\Delta^{\overline{X}}$ denotes the random measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ which is defined via

$$\mu_\Delta^{\overline{X}}(\omega; (0, \cdot]) := \sum_{0 < s \leq \cdot} \Big(\Delta C_s^{\overline{X}}(\omega)\Big)^2 \text{ and for which, moreover, holds } \frac{\mathrm{d}\mu_\Delta^{\overline{X}}}{\mathrm{d}C^{\overline{X}}}(\omega; \cdot) = \Delta C_\cdot^{\overline{X}}(\omega). \tag{II.8}$$

• For $W : \widetilde{\Omega} \longrightarrow \mathbb{R}^p$ such that $W(\omega, s, \cdot) : \big(\mathbb{R}^\ell, \mathcal{B}(\mathbb{R}^\ell)\big) \longrightarrow \big(\mathbb{R}^p, \mathcal{B}(\mathbb{R}^p)\big)$ is measurable for $\mathrm{d}\mathbb{P} \otimes \mathrm{d}C^{\overline{X}}-$almost every $(\omega, s) \in \Omega \times \mathbb{R}_+$, we abuse notation slightly and we denote

$$\widehat{K}_t^{\overline{X}}(W_s(\cdot))(\omega) := \int_{\mathbb{R}^\ell} W(\omega, s, x) K_t^{\overline{X}}(\omega; \mathrm{d}x), \ s, t \geq 0.$$

• For $W : \widetilde{\Omega} \longrightarrow \mathbb{R}^p$ such that $W(\omega, s, \cdot) : \big(\mathbb{R}^\ell, \mathcal{B}(\mathbb{R}^\ell)\big) \longrightarrow \big(\mathbb{R}^p, \mathcal{B}(\mathbb{R}^p)\big)$ is measurable for $\mathrm{d}\mathbb{P} \otimes \mathrm{d}C^{\overline{X}}-$almost every $(\omega, s) \in \Omega \times \mathbb{R}_+$ we denote

$$\Big(\|W_s(\cdot)\|_t^{\overline{X}}(\omega)\Big)^2 := \widehat{K}_t^{\overline{X}}(\|W_s(\cdot)\|_2^2)(\omega) - \Delta C_t^{\overline{X}}(\omega)\|\widehat{K}_t^{\overline{X}}(W_s(\cdot))(\omega)\|_2^2 \text{ for every } (\omega, t) \in \Omega \times \mathbb{R}_+. \quad \text{(II.9)}$$

Lemma II.3. *To the pair $(\overline{X}, C^{\overline{X}})$, which satisfies the Assumption $(\mathbb{C})$ under $\mathbb{G}$, we associate the process $\langle \overline{X} \rangle := \langle X^\circ + X^\natural \rangle$. Then,*

$$\langle \overline{X} \rangle_\cdot = \int_{(0,\cdot]} \Big[c_s^{\overline{X}} c_s^{\overline{X}} + \widehat{K}_s^{\overline{X}}(\mathrm{q}) - \Delta C_s^{\overline{X}} \widehat{K}_s^{\overline{X}}(\mathrm{Id}_\ell^\top) \widehat{K}_s^{\overline{X}}(\mathrm{Id}_\ell) \Big] \mathrm{d}C_s^{\overline{X}}.$$

Proof. Since the pair $(\overline{X}, C^{\overline{X}})$ satisfies the Assumption $(\mathbb{C})$ under $\mathbb{G}$, we have in particular that

$$\langle \overline{X} \rangle = \langle X^\circ \rangle + 2\langle X^\circ, X^\natural \rangle + \langle X^\natural \rangle \overset{\text{Lem. I.164}}{=} \langle X^\circ \rangle + \langle X^\natural \rangle. \quad \text{(II.10)}$$

Therefore, the process $\langle \overline{X} \rangle$ admits the following representation

$$\begin{aligned}
\langle \overline{X} \rangle_\cdot &:= \langle X^\circ \rangle_\cdot + \langle X^\natural \rangle_\cdot \\
&= \int_{(0,\cdot]} \frac{\mathrm{d}\langle X^\circ \rangle_s}{\mathrm{d}C_s^{\overline{X}}} \mathrm{d}C_s^{\overline{X}} + \mathrm{q} * \nu_\cdot^{(X^\natural, \mathbb{G})} - \sum_{0 < s \leq \cdot} \int_{\mathbb{R}^\ell} x^\top \nu^{(X^\natural, \mathbb{G})}(\{s\} \times \mathrm{d}x) \int_{\mathbb{R}^\ell} x \nu^{(X^\natural, \mathbb{G})}(\{s\} \times \mathrm{d}x) \\
&= \int_{(0,\cdot]} \Big[(c_s^{\overline{X}})^\top c_s^{\overline{X}} + \int_{\mathbb{R}^\ell} x^\top x \, K_s^{\overline{X}}(\mathrm{d}x) \Big] \mathrm{d}C_s^{\overline{X}} - \sum_{0 < s \leq \cdot} \int_{\mathbb{R}^\ell} x^\top K_s^{\overline{X}}(\mathrm{d}x) \Delta C_s^{\overline{X}} \int_{\mathbb{R}^\ell} x K_s^{\overline{X}}(\mathrm{d}x) \Delta C_s^{\overline{X}} \\
&= \int_{(0,\cdot]} \Big[(c_s^{\overline{X}})^\top c_s^{\overline{X}} + \widehat{K}_s^{\overline{X}}(\mathrm{q}) \Big] \mathrm{d}C_s^{\overline{X}} - \sum_{s \leq \cdot} \widehat{K}_s^{\overline{X}}(\mathrm{Id}_\ell^\top) \widehat{K}_s^{\overline{X}}(\mathrm{Id}_\ell) \Big(\Delta C_s^{\overline{X}} \Big)^2 \\
&= \int_{(0,\cdot]} \Big[(c_s^{\overline{X}})^\top c_s^{\overline{X}} + \widehat{K}_s^{\overline{X}}(\mathrm{q}) - \Delta C_s^{\overline{X}} \widehat{K}_s^{\overline{X}}(\mathrm{Id}_\ell^\top) \widehat{K}_s^{\overline{X}}(\mathrm{Id}_\ell) \Big] \mathrm{d}C_s^{\overline{X}}.
\end{aligned}$$

□

We proceed now by defining the spaces that will be necessary for our analysis, see also El Karoui and Huang [29]. Let $\beta \geq 0$, $A : (\Omega \times \mathbb{R}_+, \mathcal{G} \otimes \mathcal{B}(\mathbb{R}_+)) \longrightarrow \mathbb{R}_+$ be a càdlàg, increasing and measurable process and τ be a $\mathbb{G}-$stopping time.

Notation II.4. We define the following spaces:

- $\mathbb{L}^2_{A,\beta}(\mathcal{G}_\tau; \mathbb{R}^p) := \Big\{ \xi, \ \mathbb{R}^p-\text{valued}, \ \mathcal{G}_\tau-\text{measurable}, \ \|\xi\|^2_{\mathbb{L}^2_{A,\beta}(\mathcal{G}_\tau;\mathbb{R}^p)} := \mathbb{E}\left[\mathrm{e}^{\beta A_\tau} \|\xi\|_2^2 \right] < \infty \Big\}$,
- $\mathcal{H}^2_{A,\beta}(\mathbb{G}; \mathbb{R}^p) := \Big\{ M \in \mathcal{H}^2(\mathbb{G}; \mathbb{R}^p), \ \|M\|^2_{\mathcal{H}^2_{A,\beta}(\mathbb{G};\mathbb{R}^p)} := \mathbb{E}\Big[\int_{(0,\tau]} \mathrm{e}^{\beta A_t} \mathrm{d}\mathrm{Tr}\left[\langle M \rangle_t \right] \Big] < \infty \Big\}$,
- $\mathbb{H}^2_{A,\beta}(\mathbb{G}, \overline{X}; \mathbb{R}^p) := \Big\{ \phi$ is an $\mathbb{R}^p-$valued $\mathbb{G}-$optional semimartingale with càdlàg paths and
$$\|\phi\|^2_{\mathbb{H}^2_{A,\beta}(\mathbb{G},\overline{X};\mathbb{R}^p)} := \mathbb{E}\Big[\int_{(0,\tau]} \mathrm{e}^{\beta A_t} \|\phi_t\|^2 \mathrm{d}C_t^{\overline{X}} \Big] < \infty \Big\},$$
- $\mathcal{S}^2_{A,\beta}(\mathbb{G}, \overline{X}; \mathbb{R}^p) := \Big\{ \phi$ is an $\mathbb{R}^p-$valued $\mathbb{G}-$optional semimartingale with càdlàg paths and
$$\|\phi\|^2_{\mathcal{S}^2_{A,\beta}(\mathbb{G},\overline{X};\mathbb{R}^p)} := \mathbb{E}\Big[\sup_{t \in [\![0,\tau]\!]} \mathrm{e}^{\beta A_t} \|\phi_t\|^2 \Big] < \infty \Big\},$$
- $\mathbb{H}^2_{A,\beta}(\mathbb{G}, X^\circ; \mathbb{R}^{p\times\ell}) := \Big\{ Z \in \mathbb{H}^2(\mathbb{G}, X^\circ; \mathbb{R}^{p\times\ell}), \ \|Z\|^2_{\mathbb{H}^2_{A,\beta}(\mathbb{G},X^\circ;\mathbb{R}^{p\times\ell})} < \infty \Big\}$
with $\|Z\|^2_{\mathbb{H}^2_{A,\beta}(\mathbb{G},X^\circ;\mathbb{R}^{p\times\ell})} := \mathbb{E}\Big[\int_{(0,\tau]} \mathrm{e}^{\beta A_t} \mathrm{d}\mathrm{Tr}[\, \langle Z \cdot X^\circ \rangle_t] \Big]$,

- $\mathbb{H}^2_{A,\beta}(\mathbb{G},\mu^{X^\natural};\mathbb{R}^p) := \Big\{U \in \mathbb{H}^2(\mathbb{G},X^{\natural;\mathbb{R}^p}),\ \|U\|_{\mathbb{H}^2_{A,\beta}(\mathbb{G},\mu^{X^\natural};\mathbb{R}^p)} < \infty\Big\}$,

  with $\|U\|_{\mathbb{H}^2_{A,\beta}(\mathbb{G},\mu^{X^\natural};\mathbb{R}^p)} := \mathbb{E}\Big[\int_{(0,\tau]} \mathrm{e}^{\beta A_t}\mathrm{dTr}[\langle U\star\widetilde{\mu}^{(X^\natural,\mathbb{G})}\rangle_t]\Big]$,
- $\mathcal{H}^{2,\perp}_{A,\beta}(\mathbb{G},X^\circ,\mu^{X^\natural};\mathbb{R}^p) := \Big\{M \in \mathcal{H}^{2,\perp}(\mathbb{G},X^\circ,\mu^{X^\natural};\mathbb{R}^p),\ \|M\|^2_{\mathcal{H}^{2,\perp}_{A,\beta}(\mathbb{G},X^\circ,\mu^{X^\natural};\mathbb{R}^p)} < \infty\Big\}$

  with $\|M\|^2_{\mathcal{H}^{2,\perp}_{A,\beta}(\mathbb{G},X^\circ,\mu^{X^\natural};\mathbb{R}^p)} := \mathbb{E}\Big[\int_{(0,\tau]} \mathrm{e}^{\beta A_t}\mathrm{dTr}[\,\langle M\rangle_t]\Big]$.
- For $(Y,Z,U,N) \in \mathbb{H}^2_{A,\beta}(\mathbb{G},\overline{X};\mathbb{R}^p)\times\mathbb{H}^2_{A,\beta}(\mathbb{G},X^\circ;\mathbb{R}^{p\times\ell})\times\mathbb{H}^2_{A,\beta}(\mathbb{G},\mu^{X^\natural};\mathbb{R}^p)\times\mathcal{H}^{2,\perp}_{A,\beta}(\mathbb{G},X^\circ,\mu^{X^\natural};\mathbb{R}^p)$

and assuming that $A = \alpha^2\cdot C^{\overline{X}}$ for a measurable process $\alpha : (\Omega\times\mathbb{R}_+,\mathcal{G}\otimes\mathcal{B}(\mathbb{R}_+)) \longrightarrow \mathbb{R}$, we define

$$\|(Y,Z,U,N)\|^2_{A,\beta,\overline{X}} := \|\alpha Y\|^2_{\mathbb{H}^2_{A,\beta}(\mathbb{G},\overline{X};\mathbb{R}^p)} + \|Z\|^2_{\mathbb{H}^2_{A,\beta}(\mathbb{G},X^\circ;\mathbb{R}^{p\times\ell})} + \|U\|^2_{\mathbb{H}^2_{A,\beta}(\mathbb{G},\mu^{X^\natural};\mathbb{R}^p)} + \|N\|^2_{\mathcal{H}^{2,\perp}_{A,\beta}(\mathbb{G},X^\circ,\mu^{X^\natural};\mathbb{R}^p)}.$$

- For $(Y,Z,U,N) \in \mathcal{S}^2_{A,\beta}(\mathbb{G},\overline{X};\mathbb{R}^p)\times\mathbb{H}^2_{A,\beta}(\mathbb{G},X^\circ;\mathbb{R}^{p\times\ell})\times\mathbb{H}^2_{A,\beta}(\mathbb{G},\mu^{X^\natural};\mathbb{R}^p)\times\mathcal{H}^{2,\perp}_{A,\beta}(\mathbb{G},X^\circ,\mu^{X^\natural};\mathbb{R}^p)$

and assuming that $A = \alpha^2\cdot C^{\overline{X}}$ for a measurable process $\alpha : (\Omega\times\mathbb{R}_+,\mathcal{G}\otimes\mathcal{B}(\mathbb{R}_+)) \longrightarrow \mathbb{R}$, we define

$$\|(Y,Z,U,N)\|^2_{\star,A,\beta,\overline{X}} := \|Y\|^2_{\mathcal{S}^2_{A,\beta}(\mathbb{G},\overline{X};\mathbb{R}^p)} + \|Z\|^2_{\mathbb{H}^2_{A,\beta}(\mathbb{G},X^\circ;\mathbb{R}^{p\times\ell})} + \|U\|^2_{\mathbb{H}^2_{A,\beta}(\mathbb{G},\mu^{X^\natural};\mathbb{R}^p)} + \|N\|^2_{\mathcal{H}^{2,\perp}_{A,\beta}(\mathbb{G},X^\circ,\mu^{X^\natural};\mathbb{R}^p)}.$$

The next lemma will be useful for future computations and, in addition, justifies the definition of the norms on the spaces provided above.

Lemma II.5. *Let $(Z,U) \in \mathbb{H}^2_{A,\beta}(\mathbb{G},X^\circ;\mathbb{R}^{p\times\ell})\times\mathbb{H}^2_{A,\beta}(\mathbb{G},\mu^{X^\natural};\mathbb{R}^p)$. Then*

$$\|Z\|^2_{\mathbb{H}^2_{A,\beta}(\mathbb{G},X^\circ;\mathbb{R}^{p\times\ell})} = \mathbb{E}\left[\int_{(0,\tau]} \mathrm{e}^{\beta A_t}\|c_tZ_t\|_2^2\,\mathrm{d}C^{\overline{X}}_t\right] \tag{II.11}$$

and

$$\|U\|^2_{\mathbb{H}^2_{A,\beta}(\mathbb{G},\mu^{X^\natural};\mathbb{R}^p)} = \mathbb{E}\left[\int_{(0,\tau]} \mathrm{e}^{\beta A_t}\big(\,|||U_t(\cdot)|||^{\overline{X}}_t\,\big)^2\mathrm{d}C^{\overline{X}}_t\right]. \tag{II.12}$$

Furthermore $|||U_t(\cdot)|||^{\overline{X}}_t \geq 0$, for every $t\geq 0$ and

$$\|Z\cdot X^\circ + U\star\widetilde{\mu}^{(X^\natural,\mathbb{G})}\|^2_{\mathcal{H}^2_{A,\beta}(\mathbb{G};\mathbb{R}^p)} = \|Z\|_{\mathbb{H}^2_{A,\beta}(\mathbb{G},X^\circ;\mathbb{R}^{p\times\ell})} + \|U\|_{\mathbb{H}^2_{A,\beta}(\mathbb{G},\mu^{X^\natural};\mathbb{R}^p)}.$$

Proof. Let $Z := (Z^1,\dots,Z^p)^\top \in \mathbb{H}^2_{A,\beta}(\mathbb{G},X^\circ;\mathbb{R}^{p\times\ell})$, then using Theorem I.67, we get for $i,j = 1,\dots,p$

$$\langle Z\cdot X^\circ\rangle^{ij} = \int_{(0,\cdot]} Z^i_s\frac{\mathrm{d}\langle X^\circ\rangle_s}{\mathrm{d}C^{\overline{X}}_s}(Z^j_s)^\top\,\mathrm{d}C^{\overline{X}}_s = \int_{(0,\cdot]} Z^i_s c^{\overline{X}}_s(c^{\overline{X}}_s)^\top(Z^j_s)^\top\,\mathrm{d}C^{\overline{X}}_s, \tag{II.13}$$

for $c^{\overline{X}}$ as introduced in (II.7). The first result is then obvious in view of Notation I.66 and Notation II.4.

Using Assumption ($\mathbb{C}$) and following Notation II.2, we get for U a $\mathbb{G}-$predictable function taking values in $\mathbb{R}^p$ that

$$\int_{\mathbb{R}^\ell} U(\omega,t,x)\,\nu^{(X^\natural,\mathbb{G})}(\omega;\{t\}\times\mathrm{d}x) = \widehat{K}^{\overline{X}}_t(U_t(\cdot))(\omega)\Delta C^{\overline{X}}_t(\omega),\ t\geq 0,$$

and

$$\begin{aligned}\int_{\mathbb{R}^\ell} \|U(\omega,t,x)\|_2^2\,\nu^{(X^\natural,\mathbb{G})}(\omega;\{t\}\times\mathrm{d}x) &= \int_{\mathbb{R}^\ell}\|U(\omega,t,x)\|_2^2K^{\overline{X}}_t(\omega;\mathrm{d}x)\Delta C^{\overline{X}}_t(\omega)\\ &= \widehat{K}^{\overline{X}}_t(\|U_t(\cdot)\|_2^2)(\omega)\Delta C^{\overline{X}}_t(\omega),\quad t\geq 0.\end{aligned}$$

In particular, now, for $U \in \mathbb{H}^2_{A,\beta}(\mathbb{G},\mu^{X^\natural};\mathbb{R}^p)$ and similarly to the computations in Lemma II.3, we have

$$\langle U\star\widetilde{\mu}^{(X^\natural,\mathbb{G})}\rangle_\cdot = \int_{(0,\cdot]}\Big[\widehat{K}^{\overline{X}}_t\big((U_t^\top U_t)(\cdot)\big) - \Delta C^{\overline{X}}_s\,\widehat{K}^{\overline{X}}_t\big(U_t(\cdot)^\top\big)\widehat{K}^{\overline{X}}_t(U_t(\cdot))\Big]\,\mathrm{d}C^{\overline{X}}_s, \tag{II.14}$$

from which the second result is also clear in view of Notation II.4. Moreover, we have

$$\begin{aligned}\|Z\cdot X^\circ + U\star\widetilde{\mu}^{(X^\natural,\mathbb{G})}\|^2_{\mathcal{H}^2_{A,\beta}(\mathbb{G};\mathbb{R}^p)} &= \mathbb{E}\left[\int_{(0,\tau]} \mathrm{e}^{\beta A_t}\mathrm{dTr}\left[\langle Z\cdot X^\circ + U\star\widetilde{\mu}^{(X^\natural,\mathbb{G})}\rangle_t\right]\right]\\ &= \mathbb{E}\left[\int_{(0,\tau]} \mathrm{e}^{\beta A_t}\mathrm{dTr}\left[\langle Z\cdot X^\circ\rangle_t\right] + \int_{(0,\tau]} \mathrm{e}^{\beta A_t}\mathrm{dTr}\left[\langle U\star\widetilde{\mu}^{(X^\natural,\mathbb{G})}\rangle_t\right]\right],\end{aligned}$$

where the second equality holds because $\langle Z\cdot X^\circ, U\star\widetilde{\mu}^{(X^\natural,\mathbb{G})}\rangle = 0$. This is true by Lemma I.164, since the pair $(\overline{X}, C^{\overline{X}})$ satisfies Assumption $(\mathbb{C})$ under $\mathbb{G}$.

Notice finally that the process $\mathrm{Tr}[\langle U\star\widetilde{\mu}^{(X^\natural,\mathbb{G})}\rangle]$ is non-decreasing, and observe that

$$\Delta\mathrm{Tr}\left[\langle U\star\widetilde{\mu}^{(X^\natural,\mathbb{G})}\rangle_t\right] = \left(|||U_t(\cdot)|||_t^{\overline{X}}\right)^2\Delta C_t^{\overline{X}},\ t\geq 0. \tag{II.15}$$

Since $C^{\overline{X}}$ is non-decreasing, we can deduce that $|||U_t(\cdot)|||_t^{\overline{X}} \geq 0$. □

We conclude this section with the following convenient result. We provide initially the necessary notation.

Notation II.6. • For $\mathrm{d}\mathbb{P}\otimes\mathrm{d}C^{\overline{X}} - a.e.\ (\omega,t)\in\Omega\times\mathbb{R}_+$ we denote

$$\mathfrak{H}^{\overline{X}}_{\omega,t} := \overline{\left\{\mathcal{U}:(\mathbb{R}^\ell,\mathcal{B}(\mathbb{R}^\ell))\longrightarrow(\mathbb{R}^p,\mathcal{B}(\mathbb{R}^p)),\ |||\,\mathcal{U}(\cdot)|||_t^{\overline{X}}(\omega)<\infty\right\}}.$$

• $\mathfrak{H}^{\overline{X}} := \left\{U:[0,T]\times\Omega\times\mathbb{R}^\ell\longrightarrow\mathbb{R}^p,\ U_t(\cdot)\in\mathfrak{H}^{\overline{X}}_{\omega,t},\ \text{for } \mathrm{d}\mathbb{P}\otimes\mathrm{d}C^{\overline{X}} - a.e.\ (\omega,t)\in\Omega\times\mathbb{R}_+\right\}.$

Lemma II.7. *The space* $\left(\mathfrak{H}^{\overline{X}}_{\omega,t}, |||\cdot|||_t^{\overline{X}}(\omega)\right)$ *is Polish, for* $\mathrm{d}\mathbb{P}\otimes\mathrm{d}C^{\overline{X}} - a.e.\ (\omega,t)\in\Omega\times\mathbb{R}_+$.

Proof. The fact that $|||\cdot|||_t^{\overline{X}}(\omega)$ is indeed a norm is immediate from (II.15) and Kunita–Watanabe's inequality. Then, the space is clearly Polish since $K_t(\omega;\mathrm{d}x)$ is a regular measure, as it integrates $x\longmapsto\|x\|_2^2$ for $\mathrm{d}\mathbb{P}\otimes\mathrm{d}C^{\overline{X}} - a.e.\ (\omega,t)\in\Omega\times\mathbb{R}_+$. □

II.4. Framework and statement of the theorem

In this section we fix a filtration $\mathbb{G}$, a $\mathbb{G}-$stopping time T, a process $\overline{X} := (X^\circ, X^\natural)$ and a $C^{\overline{X}}\in\mathcal{V}^+(\mathbb{G};\mathbb{R})$ such that

■ the pair of stopped processes $(\overline{X}^T, (C^{\overline{X}})^T)$ satisfies Assumption $(\mathbb{C})$ under $\mathbb{G}$.

We fix also a positive integer n, which denotes the dimension of the state space of the solution we are seeking, that is the state space will be $\mathbb{R}^n$. In order to ease notation, since T is given, we will refer to the stopped processes $\overline{X}^T$, $(X^\circ)^T$, $(X^\natural)^T$,$(C^{\overline{X}})^T$ simply as $\overline{X}$, X°, $X^\natural$ and $C^{\overline{X}}$. Hence, the time interval on which we will be working throughout this section will always be the stochastic time interval $[\![0,T]\!]$. In addition, a non-decreasing process A will be fixed below, see (II.17). In order to further simplify notation, and since there is no danger of confusion, we will omit $\mathbb{G}, \overline{X}, A$ and the state spaces, from the spaces and the norms.

Notation II.8. For any $\beta\geq 0$ and $(\omega,t)\in\Omega\times\mathbb{R}_+$ we define:

- $\mathcal{P} := \mathcal{P}^{\mathbb{G}}$, $\widetilde{\mathcal{P}} := \widetilde{\mathcal{P}}^{\mathbb{G}}$.
- $\mathbb{L}^2_\beta := \mathbb{L}^2_\beta(\mathcal{G}_T;\mathbb{R}^n)$.
- $\mathbb{H}^2_\beta := \mathbb{H}^2_{A,\beta}(\mathbb{G},\overline{X};\mathbb{R}^n)$, $\mathbb{H}^{2,\circ}_\beta := \mathbb{H}^2_\beta(\mathbb{G},X^\circ;\mathbb{R}^{n\times\ell})$, $\mathbb{H}^{2,\natural}_\beta := \mathbb{H}^2_\beta(\mathbb{G},\mu^{X^\natural};\mathbb{R}^n)$.
- $\mathcal{S}^2_\beta := \mathcal{S}^2_{A,\beta}(\mathbb{G},\overline{X};\mathbb{R}^n)$, $\mathcal{H}^{2,\perp}_\beta := \mathcal{H}^{2,\perp}_{A,\beta}(\mathbb{G},X^\circ,\mu^{X^\natural};\mathbb{R}^n)$, $\mathcal{H}^2_\beta := \mathcal{H}^2_{A,\beta}(\mathbb{G};\mathbb{R}^n)$.
- $|||\cdot|||_t := |||\cdot|||_t^{\overline{X}}$, $\mathfrak{H}_{\omega,t} := \mathfrak{H}^{\overline{X}}_{\omega,t}$, $\mathfrak{H} := \mathfrak{H}^{\overline{X}}$.
- $\|\cdot\|_\beta := \|\cdot\|_{A,\beta,\overline{X}}$, $\|\cdot\|_{\star,\beta} := \|\cdot\|_{\star,A,\beta,\overline{X}}$.
- $C := C^{\overline{X}},\ c := c^{\overline{X}},\ \mu := \mu^{X^\natural},\ \nu := \nu^{(X^\natural,\mathbb{G})},\ \widetilde{\mu} := \widetilde{\mu}^{(X^\natural,\mathbb{G})}$.

When $\beta = 0$, we also suppress it from the notation of the previous spaces.

The data $(\mathbb{G}, T, \overline{X}, \xi, C, f)$ of the BSDE

$$Y_t = \xi + \int_{(t,T]} f(s,Y_s,Z_s,U(s,\cdot))\mathrm{d}C_s - \int_t^T Z_s\mathrm{d}X_s^c - \int_t^T\int_{\mathbb{R}^\ell} U(s,x)\widetilde{\mu}(\mathrm{d}s,\mathrm{d}x) - \int_t^T \mathrm{d}N_s,\ \mathbb{P}-a.s.. \tag{II.1}$$

should satisfy the following conditions:

(F1)[4] The process $\overline{X} = (X^\circ, X^\natural)$ belongs to $\mathcal{H}^2(\mathbb{G};\mathbb{R}^\ell) \times \mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^\ell)$ and the pair $(\overline{X}, C)$ satisfies Assumption $(\mathbb{C})$.

(F2) The terminal condition satisfies $\xi \in \mathbb{L}^2_{\hat\beta}$ for some $\hat\beta > 0$.

(F3) The generator[5] of the equation $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{n\times\ell} \times \mathfrak{H} \longrightarrow \mathbb{R}^n$ is such that for any $(y, z, u) \in \mathbb{R}^n \times \mathbb{R}^{n\times\ell} \times \mathfrak{H}$, the map

$$(\omega, t) \longmapsto f(\omega, t, y, z, u(\omega, t, \cdot)) \text{ is } \mathcal{F}_t \otimes \mathcal{B}([0, t])- \text{ measurable.}$$

Moreover, f satisfies a stochastic Lipschitz condition, that is to say there exist

$$r : (\Omega \times \mathbb{R}_+, \mathcal{P}) \longrightarrow (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \text{ and } \vartheta = (\theta^c, \theta^d) : (\Omega \times \mathbb{R}_+, \mathcal{P}) \longrightarrow (\mathbb{R}^2_+, \mathcal{B}(\mathbb{R}^2_+)),$$

such that, for $\mathrm{d}\mathbb{P} \otimes \mathrm{d}C - a.e.$ $(\omega, t) \in \Omega \times \mathbb{R}_+$

$$\begin{aligned}\|f(\omega, t, y, z, u(\omega, t, \cdot)) - f(\omega, t, y', z', u'(\omega, t, \cdot))\|_2^2 \\ \leq r_t(\omega)\|y - y'\|_2^2 + \theta^c_t(\omega)\|c_t(\omega)(z - z')\|_2^2 + \theta^d_t(\omega)\big(\|\!|u_t(\cdot) - u'_t(\cdot)|\!\|_t(\omega)\big)^2. \end{aligned} \tag{II.16}$$

(F4) Let[6] $\alpha^2_\cdot := \max\{\sqrt{r_\cdot}, \theta^c_\cdot, \theta^d_\cdot\}$ and define the increasing, $\mathbb{G}-$predictable and càdlàg process

$$A_\cdot := \int_{(0,\cdot]} \alpha^2_s \,\mathrm{d}C_s. \tag{II.17}$$

Then there exists $\Phi > 0$ such that

$$\Delta A_t(\omega) \leq \Phi, \text{ for } \mathrm{d}\mathbb{P} \otimes \mathrm{d}C - a.e.\ (\omega, t) \in \Omega \times \mathbb{R}_+. \tag{II.18}$$

(F5) We have for the same $\hat\beta$ as in **(F2)**

$$\mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\hat\beta A_t} \frac{\|f(t, 0, 0, \mathbf{0})\|_2^2}{\alpha_t^2}\, \mathrm{d}C_t\right] < \infty,$$

where $\mathbf{0}$ denotes the null application from $\mathbb{R}^\ell$ to $\mathbb{R}$.

Remark II.9. In the case where the integrator C of the Lebesgue–Stieltjes integral is a continuous process, we can choose between the integrands

$$\big(f(t, Y_t, Z_t, U(t, \cdot))\big)_{t\in[\![0,T]\!]} \quad \text{and} \quad \big(f(t, Y_{t-}, Z_t, U(t, \cdot))\big)_{t\in[\![0,T]\!]},$$

and we still obtain the same solution, as they coincide on a $\mathrm{d}\mathbb{P}\otimes\mathrm{d}C-$null set. However, in the case where the integrator C is càdlàg, the corresponding solutions may differ. In the formulation of the problem we have chosen the first one, while all our results can readily be adapted to the second case as well. Recall, however, that in Section II.2 we have seen that, in special cases, the conditions for existence and uniqueness of a solution in these two cases can differ significantly.

In classical results on BSDEs, the pair (ξ, f) is called *standard data*. In our case, the following definition generalises this term.

Definition II.10. We will say that the sextuple $(\mathbb{G}, T, \overline{X}, \xi, C, f)$ is the *standard data under* $\hat\beta$, whenever its elements satisfy Assumptions **(F1)**–**(F5)** for this specific $\hat\beta$.

Definition II.11. A *solution of the BSDE* (II.1) *with standard data* $(\mathbb{G}, T, \overline{X}, \xi, C, f)$ under $\hat\beta > 0$ is a quadruple of processes

$$(Y, Z, U, N) \in \mathbb{H}^2_\beta \times \mathbb{H}^{2,\circ}_\beta \times \mathbb{H}^{2,\natural}_\beta \times \mathcal{H}^{2,\perp}_\beta \quad \text{or} \quad (Y, Z, U, N) \in \mathcal{S}^2_\beta \times \mathbb{H}^{2,\circ}_\beta \times \mathbb{H}^{2,\natural}_\beta \times \mathcal{H}^{2,\perp}_\beta,$$

for some $\beta \leq \hat\beta$ such that, $\mathbb{P} - a.s.$, for any $t \in [\![0, T]\!]$,

$$Y_t = \xi + \int_{(t,T]} f(s, Y_s, Z_s, U(s, \cdot))\mathrm{d}C_s - \int_t^T Z_s^\top \mathrm{d}X^c_s - \int_t^T \int_{\mathbb{R}^n} U(s, x)\widetilde{\mu}(\mathrm{d}s, \mathrm{d}x) - \int_t^T \mathrm{d}N_s. \tag{II.1}$$

[4]We simply restate the black square above in order to have a convenient reference for it.

[5]This is also called *driver* of the BSDE.

[6]We assume, without loss of generality, that $\alpha_t > 0$, $\mathrm{d}\mathbb{P} \otimes \mathrm{d}C - a.e.$

Remark II.12. We emphasise that in (II.1), the stochastic integrals are well defined since $(Z,U,N) \in \mathbb{H}^{2,\circ}_\beta \times \mathbb{H}^{2,\natural}_\beta \times \mathcal{H}^{2,\perp}_\beta$. Let us verify that the integral

$$\int_{(0,\cdot]} f(s,Y_s,Z_s,U(s,\cdot))\mathrm{d}C_s$$

is also well-defined. First of all, we know by definition that for any $(y,z,u) \in \mathbb{R}^n \times \mathbb{R}^{n\times\ell} \times \mathfrak{H}$, there exists a $\mathrm{d}\mathbb{P}\otimes\mathrm{d}C-$null set $\mathcal{N}^{y,z,u}$ such that for any $(\omega,t) \notin \mathcal{N}^{y,z,u}$

$$f(\omega,t,y,z,u(\omega,t,\cdot)) \text{ is well defined and } u(\omega,t,\cdot) \in \mathfrak{H}_{\omega,t}.$$

Moreover, by Lemma II.7, we know also that for some $\mathrm{d}\mathbb{P}\otimes\mathrm{d}C-$null set $\widetilde{\mathcal{N}}$, we have for every $(\omega,t) \notin \widetilde{\mathcal{N}}$, that $\mathfrak{H}_{\omega,t}$ is Polish for the norm $|||\cdot|||_t(\omega)$, so that it admits a countable dense subset which we denote by $H_{\omega,t}$. Let us then define

$$H := \left\{u \in \mathfrak{H},\ u(\omega,t,\cdot) \in H_{\omega,t},\ \forall(\omega,t) \notin \widetilde{\mathcal{N}}\right\},\ \mathcal{N} := \bigcup \left\{\mathcal{N}^{y,z,u},\ (y,z,u) \in \mathbb{Q}^d \times \mathbb{Q}^{n\times\ell} \times H\right\},$$

where $\mathbb{Q}$ and $\mathbb{Q}^{n\times\ell}$ are the subsets of $\mathbb{R}$ and $\mathbb{R}^{n\times\ell}$ with rational components.

Then, since H is countable, $\mathcal{N}$ is still a $\mathrm{d}\mathbb{P}\otimes\mathrm{d}C-$null set. Then, it suffices to use **(F3)** to realize that for any $(\omega,t) \notin \mathcal{N}\cup\widetilde{\mathcal{N}}$, f is continuous in (y,z,u), and conclude that we can actually define $f(\omega,t,y,z,u(\omega,t,\cdot))$ outside a universal $\mathrm{d}\mathbb{P}\otimes\mathrm{d}C-$null set. This implies in particular that for any $(Y,Z,U) \in \mathbb{H}^2_\beta \times \mathbb{H}^{2,\circ}_\beta \times \mathbb{H}^{2,\natural}_\beta$

$$f(\omega,t,Y_t(\omega),Z_t(\omega),U(\omega,t,\cdot)) \text{ is defined for } \mathrm{d}\mathbb{P}\otimes\mathrm{d}C-a.e.\ (\omega,t) \in \Omega\times[\![0,T]\!].$$

Finally, it suffices to use **(F3)** and **(F5)** to conclude that

$$\int_{(0,T]} \|f(\omega,t,Y_t(\omega),Z_t(\omega),U(\omega,t,\cdot))\|_2 \mathrm{d}C_t(\omega) \text{ is finite } \mathbb{P}-a.s..$$

II.4.1. Existence and uniqueness: statement. We devote this subsection to the statement of our theorem. Before that, we need some preliminary results of a purely analytical nature, whose proofs are relegated to Appendix A.1.

Lemma II.13. *Fix $\beta,\Phi>0$ and consider the set $\mathcal{C}_\beta := \{(\gamma,\delta) \in (0,\beta]^2,\ \gamma<\delta\}$. We define the following quantity*

$$\Pi^\Phi(\gamma,\delta) := \frac{9}{\delta} + (2+9\delta)\frac{\mathrm{e}^{(\delta-\gamma)\Phi}}{\gamma(\delta-\gamma)}.$$

Then, the infimum of Π^Φ is given by

$$M^\Phi(\beta) := \inf_{(\gamma,\delta)\in\mathcal{C}_\beta} \Pi^\Phi(\gamma,\delta) = \frac{9}{\beta} + \frac{\Phi^2(2+9\beta)}{\sqrt{\beta^2\phi^2+4}-2}\exp\Big(\frac{\beta\Phi+2-\sqrt{\beta^2\Phi^2+4}}{2}\Big),$$

and is attained at the point $\big(\overline{\gamma}^\Phi(\beta),\beta\big)$ where

$$\overline{\gamma}^\Phi(\beta) := \frac{\beta\Phi-2+\sqrt{4+\beta^2\Phi^2}}{2\Phi}.$$

In addition, if we define

$$\Pi^\Phi_\star(\gamma,\delta) := \frac{8}{\gamma} + \frac{9}{\delta} + 9\delta\frac{\mathrm{e}^{(\delta-\gamma)\Phi}}{\gamma(\delta-\gamma)},$$

then the infimum of $\Pi^\Phi_\star$ is given by $M^\Phi_\star(\beta) := \inf_{(\gamma,\delta)\in\mathcal{C}_\beta}\Pi^\Phi_\star(\gamma,\delta) = \Pi^\Phi_\star(\overline{\gamma}^\Phi_\star(\beta),\beta)$, where $\overline{\gamma}^\Phi_\star(\beta)$ is the unique solution in $(\overline{\gamma}^\Phi_\star(\beta),\beta)$ of the equation with unknown x

$$8(\beta-x)^2 - 9\beta\mathrm{e}^{(\beta-x)\Phi}\big(\Phi x^2 - (\beta\Phi-2)x - \beta\big) = 0.$$

Moreover, it holds

$$\lim_{\beta\to\infty} M^\Phi(\beta) = \lim_{\beta\to\infty} M^\Phi_\star(\beta) = 9\mathrm{e}\Phi.$$

Theorem II.14. *Let $(\mathbb{G},T,\overline{X},\xi,C,f)$ be standard data under $\hat{\beta}$. If $M^\Phi(\hat{\beta}) < \frac{1}{2}$ (resp. $M^\Phi_\star(\hat{\beta}) < \frac{1}{2}$), then there exists a unique quadruple (Y,Z,U,N) which satisfies* (II.1) *and with $\|(Y,Z,U,N)\|_{\hat{\beta}} < \infty$ (resp. $\|(Y,Z,U,N)\|_{\star,\hat{\beta}} < \infty$).*

Remark II.15. Using the results of Lemma II.13 and Theorem II.14, it is immediate that as soon as

$$\Phi < \frac{1}{18\mathrm{e}}, \tag{II.19}$$

then there always exists a unique solution of the BSDE for $\hat{\beta}$ large enough.

Moreover, let us now argue why the above condition rules out the counterexamples of Section II.2 from our setting. The generator $f(t,y,u) = \frac{1}{p}(y+g(u))$ needs to be Lipschitz so that it fits in our framework, and to satisfy (II.16). Let us further assume that the function g is also Lipschitz, say with associated constant L^g. Then, using Young's Inequality, we can obtain

$$|f(t,y,u) - f(t,y',u')|^2 \leq \frac{1+\varepsilon}{p^2}|y-y'|^2 + \frac{1}{p^2}\big(1+\frac{(L^g)^2}{\varepsilon}\big)|u-u'|^2, \text{ for every } \varepsilon > 0.$$

Therefore, we have that $\alpha_r^2 = \max\big\{\sqrt{1+\varepsilon}/p, \big(1+\frac{(L^g)^2}{\varepsilon}\big)/p^2\big\}$, and it holds

$$\alpha_r^2 \Delta C_r = \max\big\{\sqrt{1+\varepsilon}, \big(1+\frac{(L^g)^2}{\varepsilon}\big)/p\big\} \geq \sqrt{1+\varepsilon} > \frac{1}{18\mathrm{e}} \text{ for every } \varepsilon > 0.$$

II.5. *A priori* estimates

The method of proof we will use follows and extends the one of El Karoui and Huang [29]. In [29], as well as in Pardoux and Peng [55], the result is obtained using fixed-point arguments and the so-called *a priori* estimates. However, we would like to underline that the proof of such estimates in our case is significantly harder, due to the fact that the process C is not necessarily continuous.

The following result can be seen as the *a priori* estimates for a BSDE whose generator does not depend on the solution. In order to keep notation as simple as possible, as well as to make clearer the link with the data of the problem we consider, we will reuse part of the notation of **(F1)**–**(F5)**, namely ξ, T, f, C, α and A, only for the next two lemmata.

Lemma II.16. *Let y be an $n-$dimensional $\mathbb{G}-$semimartingale of the form*

$$y_t = \xi + \int_{(t,T]} f_s \,\mathrm{d}C_s - \int_t^T \mathrm{d}\eta_s, \tag{II.20}$$

where T is a $\mathbb{G}-$stopping time, $\xi \in \mathbb{L}^2(\mathcal{G}_T;\mathbb{R}^n)$, f is an $n-$dimensional optional process, $C \in \mathcal{V}^+_{\mathrm{pred}}(\mathbb{G};\mathbb{R})$ and $\eta \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^n)$.

Let $A := \alpha^2 \cdot C$ for some predictable process α. Assume that there exists $\Phi > 0$ such that property (II.18) *holds for A. Suppose there exists $\beta \in \mathbb{R}_+$ such that*

$$\mathbb{E}\left[\mathrm{e}^{\beta A_T}\|\xi\|_2^2\right] + \mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\beta A_t}\frac{\|f_t\|_2^2}{\alpha_t^2}\,\mathrm{d}C_t\right] < \infty.$$

Then we have for any $(\gamma,\delta) \in (0,\beta]^2$, with $\gamma \neq \delta$,

$$\|\alpha y\|^2_{\mathbb{H}^2_\delta} \leq \frac{2\mathrm{e}^{\delta\Phi}}{\delta}\|\xi\|^2_{\mathbb{L}^2_\delta} + 2\Lambda^{\gamma,\delta,\Phi}\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}}, \quad \|y\|^2_{\mathcal{S}^2_\delta} \leq 8\|\xi\|^2_{\mathbb{L}^2_\delta} + \frac{8}{\gamma}\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}},$$

$$\|\eta\|^2_{\mathcal{H}^2_\delta} \leq 9\left(1+\mathrm{e}^{\delta\Phi}\right)\|\xi\|^2_{\mathbb{L}^2_\delta} + 9\left(\frac{1}{\gamma\vee\delta} + \delta\Lambda^{\gamma,\delta,\Phi}\right)\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}},$$

where we have defined

$$\Lambda^{\gamma,\delta,\Phi} := \frac{1\vee\mathrm{e}^{(\delta-\gamma)\Phi}}{\gamma|\delta-\gamma|}.$$

As a consequence, we have

$$\|\alpha y\|^2_{\mathbb{H}^2_\delta} + \|\eta\|^2_{\mathcal{H}^2_\delta} \leq \widetilde{\Pi}^{\delta,\Phi}\|\xi\|^2_{\mathbb{L}^2_\delta} + \Pi^\Phi(\gamma,\delta)\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}}, \tag{II.21}$$

$$\|y\|^2_{\mathcal{S}^2_\delta} + \|\eta\|^2_{\mathcal{H}^2_\delta} \leq \widetilde{\Pi}^{\delta,\Phi}_\star\|\xi\|^2_{\mathbb{L}^2_\delta} + \Pi^\Phi_\star(\gamma,\delta)\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}}, \tag{II.22}$$

where

$$\widetilde{\Pi}^{\delta,\Phi} := 9 + (9 + \frac{2}{\delta})\mathrm{e}^{\delta\Phi} \quad \textit{and} \quad \widetilde{\Pi}^{\delta,\Phi}_{\star} := 17 + 9\mathrm{e}^{\delta\Phi}.$$

Proof. Recall the identity

$$y_t = \xi + \int_{(t,T]} f_s \,\mathrm{d}C_s - \int_t^T \mathrm{d}\eta_s = \mathbb{E}\left[\xi + \int_{(t,T]} f_s \,\mathrm{d}C_s \middle| \mathcal{G}_t\right] \tag{II.23}$$

and introduce the anticipating function

$$F_t := \int_{(t,T]} f_s \,\mathrm{d}C_s. \tag{II.24}$$

For $\gamma \in \mathbb{R}_+$, we have by the Cauchy–Schwarz inequality,

$$\begin{aligned}\|F_t\|_2^2 &\le \int_{(t,T]} \mathrm{e}^{-\gamma A_s}\,\mathrm{d}A_s \int_{(t,T]} \mathrm{e}^{\gamma A_s}\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s \le \int_{(A_t,A_T]} \mathrm{e}^{-\gamma A_{L_s}}\,\mathrm{d}s \int_{(t,T]} \mathrm{e}^{\gamma A_s}\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s \\ &\le \int_{(A_t,A_T]} \mathrm{e}^{-\gamma s}\,\mathrm{d}s \int_{(t,T]} \mathrm{e}^{\gamma A_s}\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s \le \frac{1}{\gamma}\mathrm{e}^{-\gamma A_t}\int_{(t,T]} \mathrm{e}^{\gamma A_s}\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s,\end{aligned} \tag{II.25}$$

where for the third inequality we used Lemma I.34.(vii). For $t = 0$, since we assumed that

$$\mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\beta A_t}\frac{\|f_t\|_2^2}{\alpha_t^2}\,\mathrm{d}C_t\right] < \infty,$$

we have that $\mathbb{E}\left[|F_0|^2\right] < \infty$ is true for $0 < \gamma < \beta$. For $\delta \in \mathbb{R}_+$ and by integrating (II.25) with respect to $\mathrm{e}^{\delta A_t}\,\mathrm{d}A_t$ it follows

$$\begin{aligned}\int_{(0,T]} \mathrm{e}^{\delta A_t}\|F_t\|_2^2\,\mathrm{d}A_t &\overset{\text{(II.25)}}{\le} \frac{1}{\gamma}\int_{(0,T]} \mathrm{e}^{(\delta-\gamma)A_t}\int_{(t,T]} \mathrm{e}^{\gamma A_s}\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s\,\mathrm{d}A_t \\ &= \frac{1}{\gamma}\int_{(0,T]} \mathrm{e}^{\gamma A_s}\frac{\|f_s\|_2^2}{\alpha_s^2}\int_{(0,s)} \mathrm{e}^{(\delta-\gamma)A_t}\,\mathrm{d}A_t\,\mathrm{d}C_s \le \frac{1}{\gamma}\int_{(0,T]} \mathrm{e}^{\gamma A_s}\frac{\|f_s\|_2^2}{\alpha_s^2}\int_{(0,s]} \mathrm{e}^{(\delta-\gamma)A_t}\,\mathrm{d}A_t\,\mathrm{d}C_s,\end{aligned} \tag{II.26}$$

where we used Tonelli's Theorem in the equality. We can now distinguish between two cases:

- For $\delta > \gamma$, we apply Corollary I.36 for $g(x) = \mathrm{e}^{(\delta-\gamma)x}$, and inequality (II.26) becomes

$$\begin{aligned}\int_{(0,T]} \mathrm{e}^{\delta A_t}\|F_t\|_2^2\,\mathrm{d}A_t &\le \frac{\mathrm{e}^{(\delta-\gamma)\Phi}}{\gamma}\int_{(0,T]} \mathrm{e}^{\gamma A_s}\frac{\|f_s\|_2^2}{\alpha_s^2}\int_{(A_0,A_s]} \mathrm{e}^{(\delta-\gamma)t}\,\mathrm{d}t\,\mathrm{d}C_s \\ &\le \frac{\mathrm{e}^{(\delta-\gamma)\Phi}}{\gamma(\delta-\gamma)}\int_{(0,T]} \mathrm{e}^{\delta A_s}\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s,\end{aligned} \tag{II.27}$$

which is integrable if $\delta \le \beta$.

- For $\delta < \gamma$, inequality (II.26) can be rewritten as follows

$$\begin{aligned}\int_{(0,T]} \mathrm{e}^{\delta A_t}\|F_t\|_2^2\,\mathrm{d}A_t &\le \frac{1}{\gamma}\int_{(0,T]} \mathrm{e}^{\gamma A_s}\frac{\|f_s\|_2^2}{\alpha_s^2}\int_{(A_0,A_s]} \mathrm{e}^{(\delta-\gamma)A_{L_t}}\,\mathrm{d}t\,\mathrm{d}C_s \\ &\overset{\text{Lem. I.34.(vii)}}{\le} \frac{1}{\gamma}\int_{(0,T]} \mathrm{e}^{\gamma A_s}\frac{\|f_s\|_2^2}{\alpha_s^2}\int_{(A_0,A_s]} \mathrm{e}^{(\delta-\gamma)t}\,\mathrm{d}t\,\mathrm{d}C_s \\ &\le \frac{1}{\gamma|\delta-\gamma|}\int_{(0,T]} \mathrm{e}^{\gamma A_s}\frac{\|f_s\|_2^2}{\alpha_s^2}\left(\mathrm{e}^{(\delta-\gamma)A_0} - \mathrm{e}^{(\delta-\gamma)A_s}\right)\mathrm{d}C_s \\ &\le \frac{1}{\gamma|\delta-\gamma|}\int_{(0,T]} \mathrm{e}^{\gamma A_s}\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s,\end{aligned} \tag{II.28}$$

which is integrable if $\gamma \le \beta$.

To sum up, for $\gamma, \delta \in (0, \beta]$, $\gamma \ne \delta$, we have

$$\mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\delta A_t}\|F_t\|_2^2\,\mathrm{d}A_t\right] \le \Lambda^{\gamma,\delta,\Phi}\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}}. \tag{II.29}$$

For the estimate of $\|\alpha y\|_{\mathbb{H}^2_\delta}$ we first use the fact that

$$\begin{aligned}
\|\alpha y\|^2_{\mathbb{H}^2_\delta} &= \mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\delta A_t}\|y_t\|_2^2\,\mathrm{d}A_t\right] \le 2\mathbb{E}\left[\int_{(0,T]} \mathbb{E}\left[\mathrm{e}^{\delta A_t}\|\xi\|_2^2 + \mathrm{e}^{\delta A_t}\|F_t\|_2^2\middle|\,\mathcal{G}_t\right]\mathrm{d}A_t\right] \\
&= 2\mathbb{E}\left[\int_{(0,\infty)} \mathbb{E}\left[\mathrm{e}^{\delta A_t}\|\xi\|_2^2 + \mathrm{e}^{\delta A_t}\|F_t\|_2^2\middle|\,\mathcal{G}_t\right]\mathrm{d}A_t^T\right] \\
&\overset{\text{Lem. I.23}}{=} 2\mathbb{E}\left[\int_{(0,\infty)} \mathrm{e}^{\delta A_t}\|\xi\|_2^2 + \mathrm{e}^{\delta A_t}\|F_t\|_2^2\,\mathrm{d}A_t^T\right] = 2\mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\delta A_t}\|\xi\|_2^2 + \mathrm{e}^{\delta A_t}\|F_t\|_2^2\,\mathrm{d}A_t\right] \\
&\underset{\text{(II.29)}}{\overset{\text{Cor. I.36}}{\le}} \frac{2\mathrm{e}^{\delta\Phi}}{\delta}\|\xi\|^2_{\mathbb{L}^2_\delta} + 2\Lambda^{\gamma,\delta,\Phi}\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}}. \qquad \text{(II.30)}
\end{aligned}$$

In the second equality we have used that the processes $\|\xi\|_2^2\mathbb{1}_{\Omega\times[0,\infty]}(\cdot)$ and $\|F_\cdot\|_2^2$ are uniformly integrable, hence their optional projections are well defined. Indeed, using (II.25) and remembering that $\mathbb{E}[\|F_0\|_2^2] < \infty$, we can conclude the uniform integrability of $\|F_\cdot\|_2^2$. Then, by Theorem I.22 it holds that

$$\begin{aligned}
\Pi^{\mathbb{G}}_o\left(\mathrm{e}^{\delta A_\cdot}\|\xi\|_2^2 + \mathrm{e}^{\delta A_\cdot}\|F_\cdot\|_2^2\right)_t &= \mathrm{e}^{\delta A_t}\Pi^{\mathbb{G}}_o\left(\|\xi\|_2^2\mathbb{1}_{\Omega\times[0,\infty]}(\cdot)\right)_t + \mathrm{e}^{\delta A_t}\Pi^{\mathbb{G}}_o\left(\|F_\cdot\|_2^2\right)_t \\
&= \mathrm{e}^{\delta A_t}\mathbb{E}\left[\|\xi\|_2^2\middle|\,\mathcal{G}_t\right] + \mathrm{e}^{\delta A_t}\mathbb{E}\left[\|F_t\|_2^2\middle|\,\mathcal{G}_t\right] = \mathbb{E}\left[\mathrm{e}^{\delta A_t}\|\xi\|_2^2 + \mathrm{e}^{\delta A_t}\|F_t\|_2^2\middle|\,\mathcal{G}_t\right],
\end{aligned}$$

which justifies the use of Lemma I.23.

For the estimate of $\|y\|_{\mathcal{S}^2_\delta}$ we have

$$\begin{aligned}
\|y\|_{\mathcal{S}^2_\delta} &= \mathbb{E}\left[\sup_{0\le t\le T}\left(\mathrm{e}^{\frac{\delta}{2}A_t}\|y_t\|_2\right)^2\right] \le 2\mathbb{E}\left[\sup_{0\le t\le T}\mathbb{E}\left[\sqrt{\mathrm{e}^{\delta A_t}\|\xi\|_2^2 + \mathrm{e}^{\delta A_t}\|F_t\|_2^2}\middle|\,\mathcal{G}_t\right]^2\right] \\
&\le 2\mathbb{E}\left[\sup_{0\le t\le T}\mathbb{E}\left[\sqrt{\mathrm{e}^{\delta A_T}\|\xi\|_2^2 + \frac{1}{\gamma}\mathrm{e}^{(\delta-\gamma)A_t}\int_{(t,T]}\mathrm{e}^{\gamma A_s}\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_t}\middle|\,\mathcal{G}_t\right]^2\right] \\
&\le 2\mathbb{E}\left[\sup_{0\le t\le T}\mathbb{E}\left[\sqrt{\mathrm{e}^{\delta A_T}\|\xi\|_2^2 + \frac{1}{\gamma}\int_{(0,T]}\mathrm{e}^{(\gamma+(\delta-\gamma)^+)A_s}\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s}\middle|\,\mathcal{G}_t\right]^2\right] \\
&\le 8\mathbb{E}\left[\mathrm{e}^{\delta A_T}\|\xi\|_2^2 + \frac{1}{\gamma}\int_{(0,T]}\mathrm{e}^{(\gamma\vee\delta)A_s}\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s\right] \le 8\|\xi\|^2_{\mathbb{L}^2_\delta} + \frac{8}{\gamma}\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}} \qquad \text{(II.31)}
\end{aligned}$$

for $\gamma\vee\delta\le\beta$, where in the second and third inequalities we used the inequality $a+b\le\sqrt{2(a^2+b^2)}$ and (II.25) respectively.

What remains is to control $\|\eta\|_{\mathcal{H}^2_\delta}$. We remind once more the reader that $\int_t^T \mathrm{d}\eta_s = \xi - y_t + F_t$, hence

$$\mathbb{E}\left[\|\xi - y_t + F_t\|^2\middle|\,\mathcal{G}_t\right] = \mathbb{E}\left[\int_{(t,T]}\mathrm{dTr}[\langle\eta\rangle_s]\middle|\,\mathcal{G}_t\right]. \qquad \text{(II.32)}$$

In addition, we have

$$\begin{aligned}
\int_{(0,T]}\mathrm{e}^{\delta A_s}\,\mathrm{dTr}\langle\eta\rangle_s &= \int_{(0,T]}\int_{(A_0,A_s]}\delta\mathrm{e}^{\delta t}\,\mathrm{d}t\,\mathrm{dTr}[\langle\eta\rangle_s] + \mathrm{Tr}[\langle\eta\rangle_T] \\
&\overset{\text{Lem. I.34.(vii)}}{\le} \int_{(0,T]}\int_{(A_0,A_s]}\delta\mathrm{e}^{\delta A_{L_t}}\,\mathrm{d}t\,\mathrm{dTr}[\langle\eta\rangle_s] + \mathrm{Tr}[\langle\eta\rangle_T] \\
&= \delta\int_{(0,T]}\int_{(0,s]}\mathrm{e}^{\delta A_t}\,\mathrm{d}A_t\mathrm{dTr}[\langle\eta\rangle_s] + \mathrm{Tr}[\langle\eta\rangle_T] \\
&\le \delta\int_{(0,T]}\mathrm{e}^{\delta A_t}\int_{(t,T]}\mathrm{dTr}[\langle\eta\rangle_s]\mathrm{d}A_t + \mathrm{Tr}[\langle\eta\rangle_T], \qquad \text{(II.33)}
\end{aligned}$$

so that

$$\|\eta\|_{\mathcal{H}^2_\delta} \le \delta\mathbb{E}\left[\int_{(0,T]}\mathrm{e}^{\delta A_t}\int_{(t,T]}\mathrm{dTr}[\langle\eta\rangle_s]\,\mathrm{d}A_t\right] + \mathbb{E}\big[\mathrm{Tr}[\langle\eta\rangle_T]\big]. \qquad \text{(II.34)}$$

For the first summand on the right-hand-side of (II.34), we have

$$\begin{aligned}
\mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\delta A_t} \int_{(t,T]} \mathrm{dTr}[\langle\eta\rangle_s]\,\mathrm{d}A_t\right] &\overset{\text{Lemma I.23}}{=} \mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\delta A_t}\mathbb{E}\left[\int_{(t,T]} \mathrm{dTr}[\langle\eta\rangle_s]\,\middle|\,\mathcal{G}_t\right]\,\mathrm{d}A_t\right] \\
&\overset{\text{(II.32)}}{=} \mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\delta A_t}\mathbb{E}\left[\,\|\xi - y_t + F_t\|^2\,\middle|\,\mathcal{G}_t\right]\,\mathrm{d}A_t\right] \\
&\overset{\text{(II.32)}}{\leq} 3\mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\delta A_t}\mathbb{E}\left[\,\|\xi\|_2^2 + \|y_t\|_2^2 + \|F_t\|_2^2\,\middle|\,\mathcal{G}_t\right]\,\mathrm{d}A_t\right] \\
&\overset{\text{(II.23)}}{\leq} 3\mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\delta A_t}\|\xi\|_2^2\,\mathrm{d}A_t\right] + 3\mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\delta A_t}\|F_t\|_2^2\,\mathrm{d}A_t\right] \\
&\qquad + 6\mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\delta A_t}\mathbb{E}\left[\,\|\xi\|_2^2 + \|F_t\|_2^2\,\middle|\,\mathcal{G}_t\right]\,\mathrm{d}A_t\right] \\
&\overset{\text{Lemma I.23}}{=} 9\mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\delta A_t}\|\xi\|_2^2\,\mathrm{d}A_t\right] + 9\mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\delta A_t}\|F_t\|_2^2\,\mathrm{d}A_t\right] \\
&\underset{\text{(II.29)}}{\overset{\text{Corollary I.36}}{\leq}} \frac{9\mathrm{e}^{\delta\Phi}}{\delta}\|\xi\|_{\mathbb{L}^2_\delta}^2 + 9\Lambda^{\gamma,\delta,\Phi}\left\|\frac{f}{\alpha}\right\|_{\mathbb{H}^2_{\gamma\vee\delta}}^2.
\end{aligned}$$

We now need an estimate for $\mathbb{E}\left[\int_{(0,T]} \mathrm{dTr}[\langle\eta\rangle_s]\right]$, *i.e.* the second summand of (II.34), which is given by

$$\begin{aligned}
\mathbb{E}\left[\mathrm{Tr}[\langle\eta\rangle_T]\right] &= \mathbb{E}\left[\|\xi - y_0 + F_0\|^2\right] \leq 3\mathbb{E}\left[\|\xi\|_2^2 + \|y_0\|^2 + |F_0|^2\right] \\
&\overset{\text{(II.23)}}{\leq} 9\mathbb{E}\left[\|\xi\|_2^2\right] + 9\mathbb{E}\left[|F_0|^2\right] \overset{\text{(II.25)}}{\leq} 9\|\xi\|_{\mathbb{L}^2_\delta}^2 + \frac{9}{\gamma\vee\delta}\left\|\frac{f}{\alpha}\right\|_{\mathbb{H}^2_{\gamma\vee\delta}}^2,
\end{aligned}$$

where we used the fact that $\mathbb{E}\left[\|y_0\|^2\right] \leq 2\mathbb{E}\left[\|\xi\|_2^2 + |F_0|^2\right]$.

Then (II.34) yields

$$\|\eta\|_{\mathcal{H}^2_\delta}^2 \leq 9\left(1 + \mathrm{e}^{\delta\Phi}\right)\|\xi\|_{\mathbb{L}^2_\delta}^2 + 9\left(\frac{1}{\gamma\vee\delta} + \delta\Lambda^{\gamma,\delta,\Phi}\right)\left\|\frac{f}{\alpha}\right\|_{\mathbb{H}^2_{\gamma\vee\delta}}^2. \tag{II.35}$$

□

Remark II.17. An alternative framework can be provided if we define the norms in Section II.3 using another positive and increasing function h instead of the exponential function. In order to obtain the required *a priori* estimates, we need to assume that h is sub–multiplicative[7] and that it shares some common properties with the exponential function. However, we need to assume that the process A defined in **(F4)** is $\mathbb{P}-$a.s. bounded by a positive constant. We provide the detailed calculation for the case $h(x) = (1+x)^\zeta$, for $x \in \mathbb{R}_+$ and $\zeta \geq 1$ in Appendix A.1.1.

Now we are going to state in a convenient way for later use the pathwise estimates we have obtained in Lemma II.16. These estimates will allow us to prove in Chapter IV that specific sequences are uniformly integrable. We will not need all of them in Chapter IV, however we provide all the available information we have.

Lemma II.18. *Let ξ, T, C, α, A and Φ as in Lemma II.16. Assume that the $n-$dimensional $\mathbb{G}-$semimartingales y_t^1 and y_t^2 can be decomposed as follows*

$$y_t^i = \xi + \int_{(t,T]} f_s^i\,\mathrm{d}C_s - \int_t^T \mathrm{d}\eta_s^i \text{ for } i = 1,2 \tag{II.36}$$

where f^1, f^2 are $n-$dimensional $\mathbb{G}-$optional processes such that

$$\mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\beta A_t}\frac{\|f_t^i\|_2^2}{\alpha_t^2}\,\mathrm{d}C_t\right] < \infty,$$

[7] In the proof of Sato [64, Proposition 25.4] we can find a convenient tool for constructing sub–multiplicative functions.

for $i = 1, 2$ and for some $\beta \in \mathbb{R}_+$ and η^1, $\eta^2 \in \mathcal{H}^2(\mathbb{G}; \mathbb{R}^n)$. Then, for $\gamma, \delta \in (0, \beta]$, with $\gamma \neq \delta$,

$$\begin{aligned}\int_{(0,T]} \mathrm{e}^{\beta A_t} \|y_t^i\|_2^2 \,\mathrm{d}A_t \leq \frac{2}{\beta} \mathrm{e}^{\beta\Phi} \mathrm{e}^{\beta A_T} \sup_{t\in[0,T]} \mathbb{E}\Big[\|\xi^i\|_2^2 \Big| \mathcal{G}_t\Big] \\ + \frac{2}{\gamma(\beta-\gamma)} \mathrm{e}^{(\beta-\gamma)\Phi} \mathrm{e}^{(\beta-\gamma)A_T} \sup_{t\in[0,T]} \mathbb{E}\Big[\int_{(0,T]} \mathrm{e}^{\beta A_t} \frac{\|f_s^i\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s \Big| \mathcal{G}_t\Big]\end{aligned} \tag{II.37}$$

and

$$\int_{(0,T]} \mathrm{e}^{\beta A_t} \|y_t^1 - y_t^2\|_2^2 \,\mathrm{d}A_t \leq \frac{\mathrm{e}^{(\beta-\gamma)\Phi}}{\gamma(\beta-\gamma)} \mathrm{e}^{(\beta-\gamma)A_T} \sup_{t\in[0,T]} \mathbb{E}\Big[\int_{(0,T]} \mathrm{e}^{\beta A_t} \frac{\|f_s^1 - f_s^2\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s \Big| \mathcal{G}_t\Big] \tag{II.38}$$

Moreover, for the martingale parts $\eta^1, \eta^2 \in \mathcal{H}^2(\mathbb{G}; \mathbb{R}^n)$ of the aforementioned decompositions, we have

$$\sup_{t\in[0,T]} \|\eta_t^1 - \eta_0^1\|_2^2 \leq 6 \sup_{t\in[0,T]} \mathbb{E}\Big[\|\xi^1\|_2^2 + \frac{1}{\beta} \int_{(0,T]} \mathrm{e}^{\beta A_s} \frac{\|f_s^1\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s \Big| \mathcal{G}_t\Big] + 3 \int_{(0,T]} \mathrm{e}^{\beta A_t} \frac{\|f_s^1\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s \tag{II.39}$$

and

$$\begin{aligned}\sup_{t\in[0,T]} \|(\eta_t^1 - \eta_t^2) - (\eta_0^1 - \eta_0^2)\|_2 \\ \leq \frac{6}{\beta} \int_{(0,T]} \mathrm{e}^{\beta A_t} \frac{\|f_s^1 - f_s^2\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s + \frac{3}{\beta} \sup_{t\in[0,T]} \mathbb{E}\Big[\int_{(0,T]} \mathrm{e}^{\beta A_s} \frac{\|f_s^1 - f_s^2\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s \Big| \mathcal{G}_t\Big].\end{aligned} \tag{II.40}$$

Proof. For the following assume $\gamma, \delta \in (0, \beta]$ with $\gamma \neq \delta$.

• We will prove Inequality (II.37) for $i = 1$ by following analogous to Lemma II.16 calculations. The sole difference will be that we are going to apply the conditional form of the Cauchy–Schwartz Inequality. Moreover, by Identity (II.36), we have

$$\|y_t^1\|_2^2 = \Big\|\mathbb{E}\Big[\xi^1 + \int_{(t,T]} f_s^1 \,\mathrm{d}C_s \Big| \mathcal{G}_t\Big]\Big\|_2^2. \tag{II.41}$$

In view of these comments, we have

$$\begin{aligned}
&\int_{(0,T]} \mathrm{e}^{\beta A_t} \|y_t^1\|_2^2 \,\mathrm{d}A_t \int_{(0,T]} \mathrm{e}^{\beta A_t} \Big\|\mathbb{E}\Big[\xi^1 + \int_{(t,T]} f_s^1 \,\mathrm{d}C_s \Big| \mathcal{G}_t\Big]\Big\|_2^2 \,\mathrm{d}A_t \\
&\quad\leq\quad 2\int_{(0,T]} \mathrm{e}^{\beta A_t} \|\mathbb{E}[\xi^1|\mathcal{G}_t]\|_2^2 \,\mathrm{d}A_t + 2\int_{(0,T]} \mathrm{e}^{\beta A_t} \Big\|\mathbb{E}\Big[\int_{(t,T]} f_s^1 \,\mathrm{d}C_s \Big| \mathcal{G}_t\Big]\Big\|_2^2 \,\mathrm{d}A_t \\
&\quad\overset{\text{C-S Ineq.}}{\leq}\quad 2\int_{(0,T]} \mathrm{e}^{\beta A_t} \|\mathbb{E}[\xi^1|\mathcal{G}_t]\|_2^2 \,\mathrm{d}A_t + 2\int_{(0,T]} \mathrm{e}^{\beta A_t} \mathbb{E}\Big[\frac{1}{\gamma}\mathrm{e}^{-\gamma A_t} \Big| \mathcal{G}_t\Big] \mathbb{E}\Big[\int_{(t,T]} \mathrm{e}^{\gamma A_t} \frac{\|f_s^1\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s \Big| \mathcal{G}_t\Big] \,\mathrm{d}A_t \\
&\quad\leq\quad 2\int_{(0,T]} \mathrm{e}^{\beta A_t} \mathbb{E}\Big[\|\xi^1\|_2^2 \Big| \mathcal{G}_t\Big] \,\mathrm{d}A_t + \frac{2}{\gamma}\int_{(0,T]} \mathrm{e}^{(\beta-\gamma)A_t} \mathbb{E}\Big[\int_{(t,T]} \mathrm{e}^{\gamma A_t} \frac{\|f_s^1\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s \Big| \mathcal{G}_t\Big] \,\mathrm{d}A_t \\
&\quad\leq\quad 2\sup_{t\in[0,T]} \mathbb{E}\Big[\|\xi^1\|_2^2 \Big| \mathcal{G}_t\Big] \int_{(0,T]} \mathrm{e}^{\beta A_t} \,\mathrm{d}A_t + \frac{2}{\gamma}\int_{(0,T]} \mathrm{e}^{(\beta-\gamma)A_t} \mathbb{E}\Big[\int_{(0,T]} \mathrm{e}^{\beta A_t} \frac{\|f_s^1\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s \Big| \mathcal{G}_t\Big] \,\mathrm{d}A_t \\
&\quad\overset{\text{Cor. I.36}}{\leq}\quad 2\sup_{t\in[0,T]} \mathbb{E}\Big[\|\xi^1\|_2^2 \Big| \mathcal{G}_t\Big] \int_{(0,T]} \mathrm{e}^{\beta A_t} \,\mathrm{d}A_t \\
&\qquad\qquad + \frac{2}{\gamma} \sup_{t\in[0,T]} \mathbb{E}\Big[\int_{(0,T]} \mathrm{e}^{\beta A_t} \frac{\|f_s^1\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s \Big| \mathcal{G}_t\Big] \int_{(0,T]} \mathrm{e}^{(\beta-\gamma)A_t} \,\mathrm{d}A_t \\
&\quad\overset{\text{Cor. I.36}}{\leq}\quad \frac{2}{\beta}\mathrm{e}^{\beta\Phi}\mathrm{e}^{\beta A_T} \sup_{t\in[0,T]} \mathbb{E}\Big[\|\xi^1\|_2^2 \Big| \mathcal{G}_t\Big] \\
&\qquad\qquad + \frac{2}{\gamma(\beta-\gamma)} \mathrm{e}^{(\beta-\gamma)\Phi} \mathrm{e}^{(\beta-\gamma)A_T} \sup_{t\in[0,T]} \mathbb{E}\Big[\int_{(0,T]} \mathrm{e}^{\beta A_t} \frac{\|f_s^1\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s \Big| \mathcal{G}_t\Big]
\end{aligned}$$

• We will prove Inequality (II.38). We will follow analogous arguments as in the previous case, but we are going to use instead of (II.41) the identity

$$\|y_t^1 - y_t^2\|_2^2 = \Big\|\mathbb{E}\Big[\int_{(t,T]} f_s^1 - f_s^2 \,\mathrm{d}C_s \Big| \mathcal{G}_t\Big]\Big\|_2^2. \tag{II.42}$$

Now we have

$$\begin{aligned}
\int_{(0,T]} \mathrm{e}^{\beta A_t} \|y_t^1 - y_t^2\|_2^2 \,\mathrm{d}A_t &\overset{\text{(II.42)}}{=} \int_{(0,T]} \mathrm{e}^{\beta A_t} \left\| \mathbb{E}\Big[\int_{(t,T]} f_s^1 - f_s^2 \,\mathrm{d}C_s \Big| \mathcal{G}_t \Big] \right\|_2^2 \mathrm{d}A_t \\
&\overset{\text{C-S Ineq.}}{\leq} \int_{(0,T]} \mathrm{e}^{\beta A_t} \mathbb{E}\Big[\frac{1}{\gamma} \mathrm{e}^{-\gamma A_t} \Big| \mathcal{G}_t \Big] \mathbb{E}\Big[\int_{(t,T]} \mathrm{e}^{\gamma A_t} \frac{\|f_s^1 - f_s^2\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s \Big| \mathcal{G}_t \Big] \,\mathrm{d}A_t \\
&\leq \frac{1}{\gamma} \int_{(0,T]} \mathrm{e}^{(\beta-\gamma)A_t} \mathbb{E}\Big[\int_{(0,T]} \mathrm{e}^{\beta A_t} \frac{\|f_s^1 - f_s^2\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s \Big| \mathcal{G}_t \Big] \,\mathrm{d}A_t \\
&\overset{\text{Cor. I.36}}{\leq} \frac{\mathrm{e}^{(\beta-\gamma)\Phi}}{\gamma(\beta-\gamma)} \mathrm{e}^{(\beta-\gamma)A_T} \sup_{t\in[0,T]} \mathbb{E}\Big[\int_{(0,T]} \mathrm{e}^{\beta A_t} \frac{\|f_s^1 - f_s^2\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s \Big| \mathcal{G}_t \Big]
\end{aligned}$$

• Now we are going to prove (II.39) for $i = 1$. We use initially the analogous to Inequality (II.25) in order to obtain

$$\left\| \int_{(0,T]} f_s^1 \,\mathrm{d}C_s \right\|_2^2 \leq \frac{1}{\beta} \int_{(0,T]} \mathrm{e}^{\beta A_t} \frac{\|f_s^1\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s. \tag{II.43}$$

Moreover, by Identity (II.41) we obtain

$$\begin{aligned}
\|y_t^1\|_2^2 &\overset{\text{(II.41)}}{\leq} \left\| \mathbb{E}\Big[\xi^1 + \int_{(t,T]} f_s^1 \,\mathrm{d}C_s \Big| \mathcal{G}_t \Big] \right\|_2^2 \leq 2\mathbb{E}\Big[\|\xi^1\|_2^2 + \left\| \int_{(t,T]} f_s^1 \,\mathrm{d}C_s \right\|_2^2 \Big| \mathcal{G}_t \Big] \\
&\overset{\text{(II.43)}}{\leq} 2\mathbb{E}\Big[\|\xi^1\|_2^2 + \frac{1}{\beta} \int_{(0,T]} \mathrm{e}^{\beta A_s} \frac{\|f_s^1\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s \Big| \mathcal{G}_t \Big]
\end{aligned}$$

and consequently

$$\sup_{t\in[0,T]} \|y_t^1\|_2^2 \leq 2 \sup_{t\in[0,T]} \mathbb{E}\Big[\|\xi^1\|_2^2 + \frac{1}{\beta} \int_{(0,T]} \mathrm{e}^{\beta A_s} \frac{\|f_s^1\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s \Big| \mathcal{G}_t \Big]. \tag{II.44}$$

Now, by Identity (II.36) we have that

$$\begin{aligned}
\sup_{t\in[0,T]} \|\eta_t^1 - \eta_0^1\|_2^2 &= \sup_{t\in[0,T]} \left\| y_t^1 - y_0^1 + \int_{(0,T]} f_s^1 \,\mathrm{d}C_s \right\|_2^2 \leq 6 \sup_{t\in[0,T]} \|y_t^1\|_2^2 + 3 \left\| \int_{(0,T]} f_s^1 \,\mathrm{d}C_s \right\|_2^2 \\
&\underset{\text{(II.44)}}{\overset{\text{(II.43)}}{\leq}} 6 \sup_{t\in[0,T]} \mathbb{E}\Big[\|\xi^1\|_2^2 + \frac{1}{\beta} \int_{(0,T]} \mathrm{e}^{\beta A_s} \frac{\|f_s^1\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s \Big| \mathcal{G}_t \Big] + 3 \int_{(0,T]} \mathrm{e}^{\beta A_t} \frac{\|f_s^1\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s.
\end{aligned}$$

• We are going to prove, now, the Inequality (II.40). We use initially the analogous to Inequality (II.25) in order to obtain

$$\left\| \int_{(0,T]} (f_s^1 - f_s^2) \,\mathrm{d}C_s \right\|_2^2 \leq \frac{1}{\beta} \int_{(0,T]} \mathrm{e}^{\beta A_t} \frac{\|f_s^1 - f_s^2\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s. \tag{II.45}$$

Moreover, by Identity (II.42) we have by Conditional Cauchy-Schwartz Inequality (analogously to the second case)

$$\sup_{t\in[0,T]} \|y_t^1 - y_t^2\|_2^2 \leq \frac{1}{\beta} \sup_{t\in[0,T]} \mathbb{E}\Big[\int_{(0,T]} \mathrm{e}^{\beta A_s} \frac{\|f_s^1 - f_s^2\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s \Big| \mathcal{G}_t \Big]. \tag{II.46}$$

By Identity (II.36), we have

$$(\eta_t^1 - \eta_t^2) - (\eta_0^1 - \eta_0^2) = (y_t^1 - y_t^2) - (y_0^1 - y_0^2) + \int_{(0,t]} (f_s^1 - f_s^2) \,\mathrm{d}C_s.$$

Finally, we have

$$\begin{aligned}
\sup_{t\in[0,T]} \|(\eta_t^1 - \eta_t^2) - (\eta_0^1 - \eta_0^2)\|_2 &\leq \sup_{t\in[0,T]} \left\| (y_t^1 - y_t^2) - (y_0^1 - y_0^2) + \int_{(0,t]} (f_s^1 - f_s^2) \,\mathrm{d}C_s \right\|_2^2 \\
&\leq 6 \sup_{t\in[0,T]} \|y_t^1 - y_t^2\|_2^2 + \frac{3}{\beta} \int_{(0,T]} \mathrm{e}^{\beta A_s} \frac{\|f_s^1 - f_s^2\|_2^2}{\alpha_s^2} \,\mathrm{d}C_s
\end{aligned}$$

$$\underset{\text{(II.46)}}{\overset{\text{(II.45)}}{\leq}} \frac{6}{\beta}\int_{(0,T]} \mathrm{e}^{\beta A_t}\frac{\left\|f_s^1-f_s^2\right\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s + \frac{3}{\beta}\sup_{t\in[0,T]}\mathbb{E}\Big[\int_{(0,T]} \mathrm{e}^{\beta A_s}\frac{\left\|f_s^1-f_s^2\right\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s\Big|\mathcal{G}_t\Big].$$

□

Remark II.19. Viewing (II.20) as a BSDE whose generator does not depend on y and η, then this BSDE has a solution, which can be uniquely determined by the pair (y,η). Indeed, consider the data $(\mathbb{G},T,\xi,f,C)$ and the processes α and A, which all satisfy the respective assumptions of Lemma II.16 for some $\hat{\beta}>0$. Then the semimartingale

$$y_t = \mathbb{E}\Big[\xi+\int_{(t,T]} f_s\,\mathrm{d}C_s\Big|\mathcal{G}_t\Big] = \mathbb{E}\Big[\xi+\int_{(0,T]} f_s\,\mathrm{d}C_s\Big|\mathcal{G}_t\Big] - \int_{(0,t]} f_s\,\mathrm{d}C_s,\ t\in\mathbb{R}_+$$

satisfies $y_T=\xi$ and for $\eta_\cdot := \mathbb{E}\big[\xi+\int_{(0,T]} f_s\,\mathrm{d}C_s\big|\mathcal{G}_\cdot\big]$

$$\begin{aligned} y_t - y_T &= \mathbb{E}\Big[\xi+\int_{(t,T]} f_s\,\mathrm{d}C_s\Big|\mathcal{G}_t\Big] - \xi = \mathbb{E}\Big[\xi+\int_{(0,T]} f_s\,\mathrm{d}C_s\Big|\mathcal{G}_t\Big] - \int_{(0,t]} f_s\,\mathrm{d}C_s - \xi \\ &= \mathbb{E}\Big[\xi+\int_{(0,T]} f_s\,\mathrm{d}C_s\Big|\mathcal{G}_t\Big] + \int_{(t,T]} f_s\,\mathrm{d}C_s - \int_{(0,T]} f_s\,\mathrm{d}C_s - \xi = \eta_t + \int_{(t,T]} f_s\,\mathrm{d}C_s - \eta_T. \end{aligned}$$

Now, one candidate for a process $\overline{X}$ that we can choose such that $(\mathbb{G},T,\overline{X},\xi,C,f)$ become standard data for any arbitrarily chosen integrator C, is the natural pair of the zero martingale (under $\mathbb{G}$), which we will denote by 0. Hence, given the standard data $(\mathbb{G},T,0,\xi,C,f)$ the quadruple (y,Z,U,η) satisfies the BSDE

$$y_t = \xi + \int_{(t,T]} f_s\,\mathrm{d}C_s - \int_t^T \mathrm{d}\eta_s \text{ for } t\in[\![0,T]\!],$$

for any pair (Z,U). Assume now that there exists a quadruple $(\widetilde{y},\widetilde{Z},\widetilde{U},\widetilde{\eta})$ which satisfies

$$\widetilde{y}_t = \xi + \int_{(t,T]} f_s\,\mathrm{d}C_s - \int_t^T \mathrm{d}\widetilde{\eta}_s \text{ for } t\in[\![0,T]\!].$$

Then, the pair $(y-\widetilde{y},\eta-\widetilde{\eta})$ satisfies

$$y-\widetilde{y}_t = -\int_t^T \mathrm{d}\big(\eta-\widetilde{\eta}\big)_s \text{ for } t\in[\![0,T]\!]$$

and by Lemma II.16, for $\xi=0$ and $f=0$, we conclude that $\|y-\widetilde{y}\|_{\mathcal{S}^2} = \|\eta-\widetilde{\eta}\|_{\mathcal{H}^2} = 0$. Therefore y and $\widetilde{y}$, resp. η and $\widetilde{\eta}$, are indistinguishable, which implies our initial statement that every solution can be uniquely determined by the pair (y,η).

In order to obtain the *a priori* estimates for the BSDE (II.1), we will have to consider solutions (Y^i,Z^i,U^i,N^i), $i=1,2$, associated with the data $(\mathbb{G},T,\overline{X},\xi^i,C,f^i)$, $i=1,2$ under $\hat{\beta}$, where we also assume that f^1,f^2 have common r,ϑ bounds. Denote the difference between the two solutions by $(\delta Y,\delta Z,\delta U,\delta N)$, as well as $\delta\xi := \xi^1-\xi^2$ and

$$\delta_2 f_t := (f^1-f^2)(t,Y_t^2,Z_t^2,U^2(t,\cdot)),\ \psi_t := f^1(t,Y_t^1,Z_t^1,U^1(t,\cdot)) - f^2(t,Y_t^2,Z_t^2,U^2(t,\cdot)).$$

We have the identity

$$\delta Y_t = \delta\xi + \int_{(t,T]}\psi_s\,\mathrm{d}C_s - \int_t^T \delta Z_s\,\mathrm{d}X_s^\circ - \int_t^T\int_{\mathbb{R}^n}\delta U(s,\cdot)\widetilde{\mu}(\mathrm{d}s,\mathrm{d}x) - \int_t^T \mathrm{d}\delta N_s. \tag{II.47}$$

For the wellposedness of this last BSDE we need the following lemma.

Lemma II.20. *The processes*

$$\int_0^\cdot \delta Z_s\,\mathrm{d}X_s^\circ \quad \textit{and} \quad \int_0^\cdot\int_{\mathbb{R}^n}\delta U(s,\cdot)\widetilde{\mu}\,(\mathrm{d}t,\mathrm{d}x)$$

are square-integrable martingales with finite associated $\|\cdot\|_{\hat{\beta}}$*−norms.*

Proof. The square-integrability is obvious. The inequalities

$$\mathbb{E}\left[\mathrm{Tr}[\langle\delta Z\cdot X^\circ\rangle]\right] \leq 2\mathbb{E}\left[\mathrm{Tr}[\langle Z^1\cdot X^\circ\rangle]\right] + 2\mathbb{E}\left[\mathrm{Tr}[\langle Z^2\cdot X^\circ\rangle]\right],$$
$$\mathbb{E}\left[\mathrm{Tr}[\langle\delta U\star\widetilde{\mu}\rangle]\right] \leq 2\mathbb{E}\left[\mathrm{Tr}[\langle U^1\star\widetilde{\mu}\rangle]\right] + 2\mathbb{E}\left[\mathrm{Tr}[\langle U^2\star\widetilde{\mu}\rangle]\right],$$

together with Lemma II.5 guarantee that

$$\mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\hat{\beta}A_t}\|c_t\delta Z_t\|_2^2\,\mathrm{d}C_t\right] + \mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\hat{\beta}A_t}\,|||\delta U_t(\cdot)|||_t^2\,\mathrm{d}C_t\right] < \infty. \qquad \square$$

Therefore, by defining

$$H_t := \int_0^t \delta Z_s\,\mathrm{d}X_s^\circ + \int_0^t\int_{\mathbb{R}^n} \delta U(t,\cdot)\widetilde{\mu}(\mathrm{d}t,\mathrm{d}x) + \int_0^t \mathrm{d}\delta N_s, \tag{II.48}$$

we can treat the BSDE (II.47) exactly as the BSDE (II.20), where the martingale H will play the role of the martingale η.

Proposition II.21 (*A priori estimates for the BSDE* (II.1))**.** *Let* $(\mathbb{G}, T, \overline{X}, \xi^i, C, f^i)$, *be standard data under* $\hat{\beta}$ *for* $i = 1, 2$. *Then* $\psi/\alpha \in \mathbb{H}^2_{\hat{\beta}}$ *and, if* $M^\Phi(\hat{\beta}) < 1/2$, *the following estimates hold*

$$\|(\alpha\delta Y, \delta Z, \delta U, \delta N)\|^2_{\hat{\beta}} \le \widetilde{\Sigma}^\Phi(\hat{\beta})\|\delta\xi\|^2_{\mathbb{L}^2_{\hat{\beta}}} + \Sigma^\Phi(\hat{\beta})\left\|\frac{\delta_2 f}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat{\beta}}},$$

$$\|(\delta Y, \delta Z, \delta U, \delta N)\|^2_{\star,\hat{\beta}} \le \widetilde{\Sigma}^\Phi_\star(\hat{\beta})\|\delta\xi\|^2_{\mathbb{L}^2_{\hat{\beta}}} + \Sigma^\Phi_\star(\hat{\beta})\left\|\frac{\delta_2 f}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat{\beta}}},$$

where

$$\widetilde{\Sigma}^\Phi(\hat{\beta}) := \frac{\widetilde{\Pi}^{\hat{\beta},\Phi}}{1-2M^\Phi(\hat{\beta})},\ \widetilde{\Sigma}^\Phi_\star(\hat{\beta}) := \min\left\{\widetilde{\Pi}^{\hat{\beta},\Phi}_\star + 2M^\Phi_\star(\hat{\beta}), \widetilde{\Sigma}^\Phi(\hat{\beta}),\ 8 + \frac{16}{\hat{\beta}}\widetilde{\Sigma}^\Phi(\hat{\beta})\right\},$$

$$\Sigma^\Phi(\hat{\beta}) := \frac{2M^\Phi(\hat{\beta})}{1-2M^\Phi(\hat{\beta})},\ \Sigma^\Phi_\star(\hat{\beta}) := \min\left\{2M^\Phi_\star(\hat{\beta})(1+\Sigma^\Phi(\hat{\beta})),\ \frac{16}{\hat{\beta}}(1+\Sigma^\Phi(\hat{\beta}))\right\}.$$

Proof. For the integrability of ψ, using the Lipschitz property **(F3)** of f^1, f^2, we get

$$\|\psi_t\|_2^2 \le 2r_t\|\delta Y_t\|_2^2 + 2\theta^c_t\|c_t\delta Z_t\|_2^2 + 2\theta^d_t\,|||\delta U_t(\cdot)|||_t^2 + 2\|\delta_2 f_t\|_2^2.$$

Hence by the definition of α, which implies that

$$\frac{r}{\alpha^2} \le \alpha^2 \quad \text{and} \quad \frac{\theta^c}{\alpha^2}, \frac{\theta^d}{\alpha^2} \le 1, \tag{II.49}$$

we get

$$\begin{aligned}
\frac{\|\psi_t\|_2^2}{\alpha_t^2} &\le 2\left(\alpha_t^2\|\delta Y_t\|_2^2 + \|c_t\delta Z_t\|_2^2 + |||\delta U_t(\cdot)|||_t^2 + \frac{|\delta_2 f|^2}{\alpha^2}\right) \\
&\le 2\alpha_t^2\|\delta Y_t\|_2^2 + 2\|c_t\delta Z_t\|_2^2 + 2\,|||\delta U_t(\cdot)|||_t^2 \\
&\quad + \frac{4}{\alpha^2}\left(\|f^1(s,0,0,\mathbf{0})\|_2^2 + r_t\|Y_t^2\|_2^2 + \theta^c_t\|c_tZ_t^2\|_2^2 + \theta^d_t\,|||\delta U_t(\cdot)|||_t^2\right) \\
&\quad + \frac{4}{\alpha^2}\left(\|f^2(s,0,0,\mathbf{0})\|_2^2 + r_t\|\delta Y_t\|_2^2 + \theta^c_t\|c_t\delta Z_t^2\|_2^2 + \theta^d_t\,|||\delta U_t(\cdot)|||_t^2\right) \\
&\le 6\big(\alpha_t^2\|\delta Y_t\|_2^2 + \|c_t\delta Z_t\|_2^2 + |||\delta U_t(\cdot)|||_t^2\big) + \frac{4}{\alpha^2}\left(\|f^1(s,0,0,\mathbf{0})\|_2^2 + \|f^2(s,0,0,\mathbf{0})\|_2^2\right),
\end{aligned} \tag{II.50}$$

where, having used once more that (II.49) it follows that $\frac{\psi}{\alpha} \in \mathbb{H}^2_{\hat{\beta}}$. Next, for the $\|\cdot\|_{\hat{\beta}}$−norm, we have

$$\begin{aligned}
\|(\delta Y, \delta Z, \delta U, \delta N)\|^2_{\hat{\beta}} &= \|\alpha\delta Y\|^2_{\mathbb{H}^2_{\hat{\beta}}} + \|\delta Z\|^2_{\mathbb{H}^{2,\circ}_{\hat{\beta}}} + \|\delta U\|^2_{\mathbb{H}^{2,\natural}_{\hat{\beta}}} + \|\delta N\|^2_{\mathcal{H}^{2,\perp}_{\hat{\beta}}} \\
&\overset{\text{(II.48)}}{=} \|\alpha\delta Y\|^2_{\mathbb{H}^2_{\hat{\beta}}} + \|H\|^2_{\mathcal{H}^2_{\hat{\beta}}} \overset{\text{(II.21)}}{\le} \widetilde{\Pi}^{\hat{\beta},\Phi}\|\delta\xi\|^2_{\mathbb{L}^2_{\hat{\beta}}} + M^\Phi(\hat{\beta})\|\frac{\psi}{\alpha}\|^2_{\mathbb{H}^2_{\hat{\beta}}} \\
&\le \widetilde{\Pi}^{\hat{\beta},\Phi}\|\delta\xi\|^2_{\mathbb{L}^2_{\hat{\beta}}} + 2M^\Phi(\hat{\beta})\left(\|\alpha\delta Y\|^2_{\mathbb{H}^2_{\hat{\beta}}} + \|\delta Z\|^2_{\mathbb{H}^{2,\circ}_{\hat{\beta}}} + \|\delta U\|^2_{\mathbb{H}^{2,\natural}_{\hat{\beta}}}\right) + 2M^\Phi(\hat{\beta})\left\|\frac{\delta_2 f}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat{\beta}}} \\
&\le \widetilde{\Pi}^{\hat{\beta},\Phi}\|\delta\xi\|^2_{\mathbb{L}^2_{\hat{\beta}}} + 2M^\Phi(\hat{\beta})\left(\|\alpha\delta Y\|^2_{\mathbb{H}^2_{\hat{\beta}}} + \|H\|^2_{\mathcal{H}^2_{\hat{\beta}}}\right) + 2M^\Phi(\hat{\beta})\left\|\frac{\delta_2 f}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat{\beta}}}.
\end{aligned}$$

Therefore, this implies

$$\|(\alpha\delta Y,\delta Z,\delta U,\delta N)\|^2_{\hat\beta}\le\widetilde\Sigma^\Phi(\hat\beta)\|\delta\xi\|^2_{\mathbb{L}^2_{\hat\beta}}+\Sigma^\Phi(\hat\beta)\left\|\frac{\delta_2 f}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat\beta}}. \tag{II.51}$$

We can obtain *a priori* estimates for the $\|\cdot\|_{\star,\hat\beta}-$norm by arguing in two different ways:

- The identity (II.47) gives

$$\begin{aligned}
\|(\delta Y,\delta Z,\delta U,\delta N)\|^2_{\star,\hat\beta}&=\|\delta Y\|^2_{\mathcal{S}^2_{\hat\beta}}+\|\delta Z\|^2_{\mathbb{H}^{2,\circ}_{\hat\beta}}+\|\delta U\|^2_{\mathbb{H}^{2,\natural}_{\hat\beta}}+\|\delta N\|^2_{\mathcal{H}^{2,\perp}_{\hat\beta}}\\
&\overset{(\text{II.48})}{=}\|\delta Y\|^2_{\mathcal{S}^2_{\hat\beta}}+\|H\|^2_{\mathcal{H}^2_{\hat\beta}}\overset{(\text{II.22})}{\le}\widetilde\Pi^{\hat\beta,\Phi}_\star\|\delta\xi\|^2_{\mathbb{L}^2_{\hat\beta}}+M^\Phi_\star(\hat\beta)\|\frac{\psi}{\alpha}\|^2_{\mathbb{H}^2_{\hat\beta}}\\
&\overset{(\text{II.50})}{\le}\widetilde\Pi^{\hat\beta,\Phi}_\star\|\delta\xi\|^2_{\mathbb{L}^2_{\hat\beta}}+2M^\Phi_\star(\hat\beta)\|\alpha\delta Y\|^2_{\mathbb{H}^2_{\hat\beta}}+2M^\Phi_\star(\hat\beta)\|H\|^2_{\mathcal{H}^2_{\hat\beta}}+2M^\Phi_\star(\hat\beta)\left\|\frac{\delta_2 f}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat\beta}}\\
&\overset{(\text{II.51})}{\le}\widetilde\Pi^{\hat\beta,\Phi}_\star\|\delta\xi\|^2_{\mathbb{L}^2_{\hat\beta}}+2M^\Phi_\star(\hat\beta)\left\|\frac{\delta_2 f}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat\beta}}+2M^\Phi_\star(\hat\beta)\left(\widetilde\Sigma^\Phi(\hat\beta)\|\delta\xi\|^2_{\mathbb{L}^2_{\hat\beta}}+\Sigma^\Phi(\hat\beta)\left\|\frac{\delta_2 f}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat\beta}}\right)\\
&=\left(\widetilde\Pi^{\hat\beta,\Phi}_\star+2M^\Phi_\star(\hat\beta)\widetilde\Sigma^\Phi(\hat\beta)\right)\|\delta\xi\|^2_{\mathbb{L}^2_{\hat\beta}}+2M^\Phi_\star(\hat\beta)\left(1+\Sigma^\Phi(\hat\beta)\right)\left\|\frac{\delta_2 f}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat\beta}}.
\end{aligned}$$

- The identity (II.23) gives

$$\begin{aligned}
\|\delta Y\|^2_{\mathcal{S}^2_{\hat\beta}}&=\mathbb{E}\left[\sup_{0\le t\le T}\left(\mathrm{e}^{\frac{\hat\beta}{2}A_t}\|\delta Y_t\|_2\right)^2\right]\overset{(\text{II.23})}{\le}\mathbb{E}\left[\sup_{0\le t\le T}\mathbb{E}\left[\mathrm{e}^{\frac{\hat\beta}{2}A_t}\|\delta\xi\|_2+\mathrm{e}^{\frac{\hat\beta}{2}A_t}\left\|\int_{(t,T]}\psi_s\,\mathrm{d}C_s\right\|_2\Big|\mathcal{G}_t\right]^2\right]\\
&\overset{(\text{II.25})}{\le}2\mathbb{E}\left[\sup_{0\le t\le T}\mathbb{E}\left[\sqrt{\mathrm{e}^{\hat\beta A_t}\|\delta\xi\|^2_2+\frac{1}{\hat\beta}\int_{(t,T]}\mathrm{e}^{\hat\beta A_s}\frac{\|\psi_s\|^2_2}{\alpha^2_s}\,\mathrm{d}C_s}\Big|\mathcal{G}_t\right]^2\right]\\
&\le 2\mathbb{E}\left[\sup_{0\le t\le T}\mathbb{E}\left[\sqrt{\mathrm{e}^{\hat\beta A_T}\|\delta\xi\|^2_2+\frac{1}{\hat\beta}\int_{(0,T]}\mathrm{e}^{\hat\beta A_s}\frac{\|\psi_s\|^2_2}{\alpha^2_s}\,\mathrm{d}C_s}\Big|\mathcal{G}_t\right]^2\right]\\
&\le 8\mathbb{E}\left[\mathrm{e}^{\hat\beta A_T}\|\delta\xi\|^2_2+\frac{1}{\hat\beta}\int_{(0,T]}\mathrm{e}^{\hat\beta A_s}\frac{\|\psi_s\|^2_2}{\alpha^2_s}\,\mathrm{d}C_s\right]\\
&\overset{(\text{II.50})}{\le}8\|\delta\xi\|^2_{\mathbb{L}^2_{\hat\beta}}+\frac{16}{\hat\beta}\left(\left\|\frac{\delta_2 f}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat\beta}}+\|\alpha\delta Y\|^2_{\mathbb{H}^2_{\hat\beta}}+\|\delta Z\|^2_{\mathbb{H}^{2,\circ}_{\hat\beta}}+\|\delta U\|^2_{\mathbb{H}^{2,\natural}_{\hat\beta}}\right),
\end{aligned} \tag{II.52}$$

where, in the second and fifth inequality we used the inequality $a+b\le\sqrt{2(a^2+b^2)}$ and Doob's inequality respectively. Then we can derive the required estimate

$$\begin{aligned}
\|(\delta Y,\delta Z,\delta U,\delta N)\|^2_{\star,\hat\beta}&=\|\delta Y\|^2_{\mathcal{S}^2_{\hat\beta}}+\|\delta Z\|^2_{\mathbb{H}^{2,\circ}_{\hat\beta}}+\|\delta U\|^2_{\mathbb{H}^{2,\natural}_{\hat\beta}}+\|\delta N\|^2_{\mathcal{H}^{2,\perp}_{\hat\beta}}\\
&\overset{(\text{II.48})}{=}\|\delta Y\|^2_{\mathcal{S}^2_{\hat\beta}}+\|H\|^2_{\mathcal{H}^2_{\hat\beta}}\underset{(\text{II.52})}{\overset{(\text{II.48})}{\le}}8\|\delta\xi\|^2_{\mathbb{L}^2_{\hat\beta}}+\frac{16}{\hat\beta}\left\|\frac{\delta_2 f}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat\beta}}+\frac{16}{\hat\beta}\|\alpha\delta Y\|^2_{\mathbb{H}^2_{\hat\beta}}+\frac{16}{\hat\beta}\|H\|^2_{\mathcal{H}^2_{\hat\beta}}\\
&\overset{(\text{II.51})}{\le}\left(8+\frac{16}{\hat\beta}\widetilde\Sigma^\Phi(\hat\beta)\right)\|\delta\xi\|^2_{\mathbb{L}^2_{\hat\beta}}+\frac{16}{\hat\beta}\left(1+\Sigma^\Phi(\hat\beta)\right)\left\|\frac{\delta_2 f}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat\beta}}.
\end{aligned}$$

□

II.6. Proof of the theorem

We will use now the previous estimates to obtain the existence of a unique solution using a fixed point argument. The reader should keep in mind that the fixed point argument boils the existence and uniqueness of the solution of the BSDE (II.1) down to a martingale representation problem.

Proof of Theorem II.14. Let (y,z,u,w) be such that $(\alpha y,z,u,w)\in\mathbb{H}^2_{\hat\beta}\times\mathbb{H}^{2,\circ}_{\hat\beta}\times\mathbb{H}^{2,\natural}_{\hat\beta}\times\mathcal{H}^{2,\perp}$. Then the process M defined by

$$M_\cdot := \mathbb{E}\left[\xi+\int_{(0,T]} f(s,y_s,z_s,u(s,\cdot))\,\mathrm{d}C_s\middle|\mathcal{G}_\cdot\right]+w_\cdot\in\mathcal{H}^2(\mathbb{G};\mathbb{R}^n),$$

and by Proposition I.165 it has a unique, up to indistinguishability, orthogonal decomposition

$$M_\cdot = M_0+\int_0^\cdot Z_s\mathrm{d}X^\circ_s+\int_0^\cdot\int_{\mathbb{R}^\ell}U(s,x)\,\widetilde{\mu}(\mathrm{d}s,\mathrm{d}x)+L_\cdot,$$

where $(Z,U,L)\in\mathbb{H}^{2,\circ}\times\mathbb{H}^{2,\natural}\times\mathcal{H}^{2,\perp}$. In view of the identity

$$M_T-M_t=\int_t^T Z_s\mathrm{d}X^\circ_s+\int_t^T\int_{\mathbb{R}^\ell}U(s,x)\widetilde{\mu}(\mathrm{d}s,\mathrm{d}x)+\int_t^T\mathrm{d}L_s,\ 0\le t\le T,$$

we obtain

$$\begin{aligned}&\mathbb{E}\left[\xi+\int_{(t,T]} f(s,y_s,z_s,u(s,\cdot))\,\mathrm{d}C_s\middle|\mathcal{G}_t\right]\\&\qquad=\xi+\int_{(t,T]} f(s,y_s,z_s,u(s,\cdot))\,\mathrm{d}C_s-\int_t^T Z_s\mathrm{d}X^\circ_s-\int_t^T\int_{\mathbb{R}^\ell}U(s,x)\widetilde{\mu}(\mathrm{d}s,\mathrm{d}x)-\int_t^T\mathrm{d}N_s,\end{aligned}$$

where $N:=L-w$. Define now

$$Y_t:=\mathbb{E}\left[\xi+\int_{(t,T]} f(s,y_s,z_s,u(s,\cdot))\,\mathrm{d}C_s\middle|\mathcal{G}_t\right].$$

In order to construct a contraction using Lemma II.16, we need to choose $\delta>\gamma$. Then by Lemma II.13 we can choose $\gamma^\star\in(0,\hat\beta]$ such that $\inf_{(\gamma,\delta)\in\mathcal{C}_{\hat\beta}}\Pi^\Phi(\gamma,\delta)=\Pi^\Phi(\gamma^\star(\hat\beta),\hat\beta)$. Now we get that $(\alpha Y, Z\cdot X^\circ+U\star\widetilde{\mu}+N)\in\mathbb{H}^2_{\hat\beta}\times\mathcal{H}^2_{\hat\beta}$, and due to the orthogonality of the martingales, recall Lemma I.164, we conclude that $(\alpha Y,Z,U,N)\in\mathbb{H}^2_{\hat\beta}\times\mathbb{H}^{2,\circ}_{\hat\beta}\times\mathbb{H}^{2,\natural}_{\hat\beta}\times\mathcal{H}^{2,\perp}_{\hat\beta}$. Hence, the operator

$$S:\left(\mathbb{H}^2_{\hat\beta}\times\mathbb{H}^{2,\circ}_{\hat\beta}\times\mathbb{H}^{2,\natural}_{\hat\beta}\times\mathcal{H}^{2,\perp}_{\hat\beta},\|\cdot\|_{\hat\beta}\right)\longrightarrow\left(\mathbb{H}^2_{\hat\beta}\times\mathbb{H}^{2,\circ}_{\hat\beta}\times\mathbb{H}^{2,\natural}_{\hat\beta}\times\mathcal{H}^{2,\perp}_{\hat\beta},\|\cdot\|_{\hat\beta}\right)$$

that maps the processes $(\alpha y,z,u,w)$ to the processes $(\alpha Y,Z,U,N)$ defined above, is indeed well-defined.

Let $(\alpha y^i,z^i,u^i,w^i)\in\mathbb{H}^2_{\hat\beta}\times\mathbb{H}^{2,\circ}_{\hat\beta}\times\mathbb{H}^{2,\natural}_{\hat\beta}\times\mathcal{H}^{2,\perp}_{\hat\beta}$ for $i=1,2$, with

$$S\left(\alpha y^i,z^i,u^i,w^i\right)=(\alpha Y^i,Z^i,U^i,N^i),\quad\text{for}\quad i=1,2.$$

Denote, as usual, $\delta y,\delta z,\delta u,\delta w$ the difference of the processes and

$$\psi_t:=f\left(t,y^1_t,z^1_t,u^1(t,\cdot)\right)-f\left(t,y^2_t,z^2_t,u^2_t(\cdot)\right).$$

It is immediate that $\frac{\psi}{\alpha}\in\mathbb{H}^2_{\hat\beta}$ and that

$$\begin{aligned}\|S\left(\alpha y^1,z^1,u^1,w^1\right)-S\left(\alpha y^2,z^2,u^2,w^2\right)\|^2_{\hat\beta}&=\|\alpha\delta Y\|^2_{\mathbb{H}^2_{\hat\beta}}+\|\delta Z\|^2_{\mathbb{H}^{2,\circ}_{\hat\beta}}+\|\delta U\|^2_{\mathbb{H}^{2,\natural}_{\hat\beta}}+\|\delta N\|^2_{\mathcal{H}^{2,\perp}_{\hat\beta}}\\&\underset{\text{Lemma II.16}}{\overset{\delta\xi=0}{\le}}M^\Phi(\hat\beta)\left\|\frac{\psi}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat\beta}}\overset{\text{(II.50)}}{\le}2M^\Phi(\hat\beta)\left(\|\alpha\delta y\|^2_{\mathbb{H}^2_{\hat\beta}}+\|\delta z\|^2_{\mathbb{H}^{2,\circ}_{\hat\beta}}+\|\delta u\|^2_{\mathbb{H}^{2,\natural}_{\hat\beta}}\right)\\&\overset{\text{Lemma II.16}}{\le}2M^\Phi(\hat\beta)\left\|\left(\alpha y^1,z^1,u^1,w^1\right)-\left(\alpha y^2,z^2,u^2,w^2\right)\right\|^2_{\hat\beta}.\end{aligned}$$

Hence, for $M^\Phi(\hat\beta)<1/2$, we can apply Banach's fixed point theorem to obtain the existence of a unique fixed point $(\widetilde{Y},Z,U,N)$. To obtain a solution in the desirable spaces, we substitute $\widetilde{Y}$ in the triplet with Y, the corresponding càdlàg version; indeed, $\mathbb{G}$ satisfies the usual conditions and $\widetilde{Y}$ is a semimartingale.

The exact same reasoning using the $\|\cdot\|_{\mathcal{S}^2_{\hat\beta}}$−norm for Y leads to a contraction when $M^\Phi_\star(\hat\beta)<1/2$. □

Corollary II.22 (Picard approximation)**.** *Assume that $M^\Phi(\hat\beta)<1/2$ (resp. $M^\Phi_\star(\hat\beta)<1/2$) and define a sequence $(\Upsilon^{(p)})_{p\in\mathbb{N}}$ on $\mathbb{H}^2_{\hat\beta}\times\mathbb{H}^{2,\circ}_{\hat\beta}\times\mathbb{H}^{2,\natural}_{\hat\beta}\times\mathcal{H}^{2,\perp}_{\hat\beta}$ (resp. on $\mathcal{S}^2_{\hat\beta}\times\mathbb{H}^{2,\circ}_{\hat\beta}\times\mathbb{H}^{2,\natural}_{\hat\beta}\times\mathcal{H}^{2,\perp}_{\hat\beta}$) such that $\Upsilon^{(0)}$ is the zero element of the product space and $\Upsilon^{(p+1)}$ is the solution of*

$$Y_t^{(p+1)} = \xi + \int_t^T f(s, Y_s^{(p)}, Z_s^{(p)}, U^{(p)}(s,\cdot))\,\mathrm{d}C_s - \int_t^T Z_s^{(p+1)} dX_s^\circ - \int_t^T \int_{\mathbb{R}^\ell} U^{(p+1)}(s,x)\widetilde{\mu}(\mathrm{d}s,\mathrm{d}x) - \int_t^T dN_s^{(p+1)}$$

Then

(i) *The sequence* $(\Upsilon^{(p)})_{p\in\mathbb{N}}$ *converges in* $\|\cdot\|_{\hat{\beta}}$ *(resp. in* $\|\cdot\|_{\star,\hat{\beta}}$*), to the solution of the BSDE* (II.1).

(ii) *The following convergence holds*

$$\left(Z^{(p)}, U_s^{(p)}, N^{(p)}\right) \xrightarrow[p\to\infty]{} (Z, U, N), \text{ in } \mathbb{H}_{\hat{\beta}}(X^\circ) \times \mathbb{H}_{\hat{\beta}}(X^d) \times \mathbb{H}_{\hat{\beta}}(X^\perp).$$

(iii) *There exists a subsequence* $(\Upsilon^{(p_j)})_{j\in\mathbb{N}}$ *which converges* $\mathrm{d}\mathbb{P}\otimes \mathrm{e}^{\hat{\beta}A}\mathrm{d}C - a.e.$

Proof. As in the proof of Theorem II.14, we obtain, for $p \geq 1$,

$$\begin{aligned} \|\Upsilon^{(p+1)} - \Upsilon^{(p)}\|_{\hat{\beta}}^2 &\leq \left(2M^\Phi(\hat{\beta})\right)^p \|\Upsilon^{(1)}\|_{\hat{\beta}}^2 \\ \left(\text{resp. } \|\Upsilon^{(p+1)} - \Upsilon^{(p)}\|_{\star,\hat{\beta}}^2 \leq \left(2M_\star^\Phi(\hat{\beta})\right)^p \|\Upsilon^{(1)}\|_{\star,\hat{\beta}}^2\right), \end{aligned} \tag{II.53}$$

and consequently, since $\sum_{p\in\mathbb{N}} \|\Upsilon^{(p+1)} - \Upsilon^{(p)}\|_{\hat{\beta}}^2 < \infty$ (resp. $\sum_{p\in\mathbb{N}} \|\Upsilon^{(p+1)} - \Upsilon^{(p)}\|_{\star,\hat{\beta}}^2 < \infty$), the sequence $(\Upsilon^{(p)})_{p\in\mathbb{N}}$ is Cauchy under $\|\cdot\|_{\hat{\beta}}$ (resp. under $\|\cdot\|_{\star,\hat{\beta}}$). Denote by Υ the unique limit on the product space. Then, it coincides with the unique fixed point for the contraction S (see the proof of Theorem II.14 above) due to the construction of $(\Upsilon^{(p)})_{p\in\mathbb{N}}$, which proves (i).

For (ii), the result is immediate by the Cauchy property of the sequence $(\Upsilon^{(p)})_{p\in\mathbb{N}}$, Itô's isometry, the stability[8] of the respective closed linear space generated by X° and $X^\natural$ in conjunction with their orthogonality (recall Lemma I.164), which makes $\mathcal{H}^{2,\perp}$ to be a closed subspace; see [35, Theorem 6.16].

Finally, for (iii), by the $\|\cdot\|_{\hat{\beta}}$−convergence, we can extract a subsequence $(p_j)_{j\in\mathbb{N}}$ such that

$$\|\Upsilon^{(p_{j+1})} - \Upsilon^{(p_j)}\|_{\hat{\beta}} \leq 2^{-2j}, \text{ for every } j \in \mathbb{N}. \tag{II.54}$$

Define, for any $\varepsilon \geq 0$, $N^{p,\varepsilon} := \left\{ (\omega,t) \in \Omega \times [\![0,T]\!],\ |Y_t^{(p)}(\omega) - Y_t(\omega)| > \varepsilon \right\}$. Then we have

$$\begin{aligned} \mathrm{d}\mathbb{P}\otimes \mathrm{e}^{\hat{\beta}A}\mathrm{d}C\left(\limsup_{j\to\infty} N^{p_j,\varepsilon}\right) &= \lim_{j\to\infty} \mathrm{d}\mathbb{P}\otimes \mathrm{e}^{\hat{\beta}A}\mathrm{d}C\left(\bigcup_{i=j}^\infty \left[|Y^{(p_i)} - Y| > \varepsilon\right]\right) \\ &\leq \lim_{j\to\infty} \frac{1}{\varepsilon^2} \sum_{i=j}^\infty \mathbb{E}\left[\int_0^T \mathrm{e}^{\hat{\beta}A_t} |Y_t^{(p_i)} - Y_t|^2 \mathrm{d}C_t\right] \\ &\leq \lim_{j\to\infty} \frac{1}{\varepsilon^2} \sum_{i=j}^\infty \|Y^{(p_i)} - Y\|_{\hat{\beta}}^2 \\ &\leq \lim_{j\to\infty} \frac{1}{\varepsilon^2} \sum_{i=j}^\infty \left(\sum_{m=1}^\infty 2^m \left\|Y^{(p_{i+m+1})} - Y^{(p_{i+m})}\right\|_{\mathbb{H}_{\hat{\beta}}^2}^2\right) \\ &\leq \lim_{j\to\infty} \frac{1}{\varepsilon^2} \sum_{m=i}^\infty \left(\sum_{m=1}^\infty 2^m \left\|\Upsilon^{(p_{i+m+1})} - \Upsilon^{(p_{i+m})}\right\|_{\hat{\beta}}^2\right) \\ &\overset{\text{(II.54)}}{\leq} \lim_{j\to\infty} \frac{1}{\varepsilon^2} \sum_{i=j}^\infty \left(\sum_{m=1}^\infty 2^m 2^{-2(i+m)}\right) = 0, \text{ for any } \varepsilon > 0. \end{aligned}$$

Hence

$$\mathrm{d}\mathbb{P}\otimes \mathrm{e}^{\hat{\beta}A}\mathrm{d}C\left(\limsup_{j\to\infty} N^{p_j,0}\right) \leq \sum_{n\in\mathbb{N}} \mathrm{d}\mathbb{P}\otimes \mathrm{e}^{\hat{\beta}A}\mathrm{d}C\left(\limsup_{j\to\infty} N^{p_j,1/n}\right) = 0.$$

Following the same arguments, we have the almost sure convergence of $Z^{p_j}, U^{p_j}, N^{p_j}$ to the corresponding processes of the $\|\cdot\|_\beta$−solution of the BSDE (II.1). Moreover, using the same steps, we can obtain the analogous result for the $\|\cdot\|_{\star,\hat{\beta}}$−norm. □

[8] At this point the stability is understood as "closed under stopping the processes". For the precise statement see He et al. [35, Definition 6.16].

II.7. Related literature

Let us now compare our work with the papers by Bandini [6] and Cohen and Elliott [19] who also consider BSDEs in stochastically discontinuous filtrations. The setting in [19] is rather different from ours. Indeed, in our case a driving martingale X is given right from the start, and as a consequence the process C with respect to which the generator f is integrated is linked to the predictable bracket of X. However, the authors of [19] do not choose any X from the start, but consider instead a general martingale representation theorem involving countably many orthogonal martingales, which only requires the space of square integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ to be separable to hold. Furthermore, their process C can, unlike our case, be chosen arbitrary (in the sense that it does not have to be related to the driving martingales), but with the restriction that it has to be deterministic. Moreover, it has to assign positive measure on every interval, see Definition 5.1 therein, hence C cannot be piecewise constant; the latter would naturally arise from a discrete-time martingale with independent increments, which is exactly the situation one encounters when devising numerical schemes for BSDEs. Therefore, their setting cannot be embedded into our framework, and *vice versa*.

On the other hand, in [6], the author considers a BSDE driven by a pure–jump martingale without an orthogonal component, which is a special case of (II.1). The martingale in this setting should actually have jumps of finite activity, hence many of the interesting models for applications in mathematical finance, such as the generalized hyperbolic, CGMY, and Meixner processes, are excluded. Such a restriction is not present at all in our framework. Otherwise, the assumptions and the conclusion in [6] are analogous to the present work. A direct comparison is however not possible, *i.e.* we cannot deduce the existence and uniqueness results in her work from our setting, since the assumptions are not exactly comparable. In particular, the integrability condition (iii) on page 3 in [6] is not compatible with **(F5)**.

Let us also compare our result with the literature on BSDEs with random terminal time. Royer [62], for instance, considers a BSDE driven by Brownian motion, where the terminal time is a $[0, \infty)-$valued stopping time. Hence, her setting can be embedded in ours, by assuming the absence of jumps and of the orthogonal component, and further requiring that C is a continuous process. She shows existence and uniqueness of a solution under the assumptions that the generator is uniformly Lipschitz in z and continuous in y, and the terminal condition is bounded. Moreover, she requires that either the generator is strictly monotone in y and $f(t, 0, 0)$ is bounded (for all t) or that the generator is monotone in y and $f(t, 0, 0) = 0$ (for all t). These conditions are not directly covered by our Assumptions **(F1)**–**(F5)**, however if we consider her conditions and assume in addition that the generator is Lipschitz in y, then we can recover the existence and uniqueness result from our main theorem. Let us point out that BSDEs with constant terminal time are related to semi–linear parabolic PDEs, while BSDEs with random terminal time are associated to semi–linear elliptic PDEs.

We would also like to comment briefly on the choice of the norms we consider here. They are mostly inspired from the ones defined in the seminal work of El Karoui and Huang [29], and are equivalent to the usual norms found in the literature when the process A and time T are both bounded. Bandini [6] uses different spaces, where the norm is defined using the Doléans–Dade stochastic exponential instead of the natural exponential. In our setting where A is allowed to be unbounded, we can only say that our norm dominates hers. This means that we require stronger integrability conditions, but as a result we will also obtain a solution of the BSDE with stronger integrability properties. In any case, our method could be adapted to this choice of the norm, albeit with modified computations in our estimates. We remind the reader Remark II.17 for a short discussion about the definition of the norms.

Let us conclude this section by commenting on the condition (II.19). We start with the observation that the analysis of the counterexample of Confortola *et al.* [22] made in Section II.2 does not allow for a general statement of wellposedness of the BSDE when $\Phi \geq 1$. In this light, the result of Cohen and Elliott [19, Theorem 6.1], which implies that the condition $\Phi < 1$ ensures the wellposedness of the BSDE, lies in the optimal range for Φ. Analogously in the case of Bandini [6], once her results are translated using the Lipschitz assumption in **(F3)**, $\Phi < \frac{1}{\sqrt{2}}$ also ensures the wellposedness of the BSDE. On the contrary, condition (II.19) which reads as $\Phi < 1/(18\mathrm{e})$, may seem much more restrictive. The first immediate remark we can make is that the stochasticity of the integrator C considerably deteriorates the condition on Φ. In [19] the integrator is deterministic, while in [6] and in our case the integrator is stochastic. However, we would like to remind the reader, that, as explained above, the level of generality we are working with is substantially higher than in these two references. We also want to emphasise the fact that our condition is clearly not the sharpest one possible, but we believe it is the sharpest that can be obtained using our method of proof. The main possibilities for improvement are, in our view, twofold:

- First of all, in specific situations (e.g. T bounded, f Lipschitz, less general driving processes, ...) one could most probably improve the *a priori* estimates of Lemma II.16 by refining several of the inequalities.
- Second, as highlighted in Remark II.17, we actually have a degree of freedom in choosing the norms we are interested in. In this paper, we used exponentials, while Bandini [6] used stochastic exponentials, but other choices, leading to potentially better estimates, could also be considered.

We leave this interesting problem of finding the optimal Φ open for future research.

CHAPTER III

Stability of Martingale Representations

This chapter deals with the stability of martingale representations. In Section I.7 we have discussed about martingale representations, therefore let us describe now roughly how do we understand their stability property. For simplicity we assume that we are working on a probability space endowed with a single filtration $\mathbb{F}$. Given, now, two sequences $(X^k)_{k\in\overline{\mathbb{N}}}$ and $(Y^k)_{k\in\overline{\mathbb{N}}}$ consisting both of square-integrable martingales, we obtain for every $k \in \overline{\mathbb{N}}$ the orthogonal decompositions of Y^k with respect to $(\mathbb{F}, X^{k,c}, \mu^{X^{k,d}})$, say

$$Y^k = Z^k \cdot X^{k,c} + U^k \star \widetilde{\mu}^{(X^{k,d},\mathbb{F})} + N^k \text{ for } k \in \overline{\mathbb{N}}. \tag{III.1}$$

Assuming that the sequences $(X^k)_{k\in\overline{\mathbb{N}}}$ and $(Y^k)_{k\in\overline{\mathbb{N}}}$ are $\delta_{\mathrm{J}_1}-$convergent, the martingale representations (III.1) are stable if the sequence consisting of the orthogonal martingales $(N^k)_{k\in\overline{\mathbb{N}}}$ is convergent as well as the sum of the stochastic integrals with respect to the natural pair of X^k converges to the sum of the stochastic integrals with respect to natural pair of X^∞. Schematically, given that the convergence indicated with the solid arrows hold, then also the convergence indicated with the dashed arrows hold.

$$\begin{array}{c|ccccccc}
X^k & Y^k & = & Z^k \cdot X^{k,c} & + & U^k \star \widetilde{\mu}^{(X^k,\mathbb{F})} & + & N^k \\
\downarrow & \downarrow & & & & \downarrow & & \downarrow \\
X^\infty & Y^\infty & = & Z^\infty \cdot X^{\infty,c} & + & U^\infty \star \widetilde{\mu}^{(X^\infty,\mathbb{F})} & + & N^\infty
\end{array}$$

Theorem III.3 demonstrates that under the suitable framework, which is presented in Section III.1, not only the martingale representations are stable, but also their associated optional as well as predictable quadratic variation processes. The former stability is a well-known result in the literature and it is a consequence of the fact that $(X^k)_{k\in\overline{\mathbb{N}}}$, $(Y^k)_{k\in\overline{\mathbb{N}}}$ are P-UT. However, the latter stability consists a new result. We need to underline of course that it was Mémin's Theorems, see Theorems I.159 and I.160 or [53, Theorem 11, Corollary 12], that provided us with the stability of the canonical decompositions of special semimartingales.[1]

This result, although interesting on its own, is also crucial for obtaining the stability of BSDEs. To be more precise, it will be the enhanced version of Theorem III.3, namely Theorem III.50, which we will deploy in order to obtain finally the stability of BSDEs. Commenting briefly on the connection between the martingale-representation-stability and of the BSDE-stability, the reader may recall that in the proof of Theorem II.14 we have translated the existence and uniqueness of the solution of a BSDE into a martingale representation problem. In Chapter IV we will translate the one stability problem to the other *mutatis mutandis*.

The structure of the chapter follows. In Section III.1 we set the framework and we present the main theorem. Then, we provide in Section III.2 the outline of the proof and in the subsequent sections we will develop the necessary machinery for the final step of the proof, which will be presented in Section III.7. We have devoted Section III.9 for the comparison with the existing literature.

III.1. Framework and statement of the main theorem

The probability space $(\Omega, \mathcal{G}, \mathbb{P})$ will be fixed throughout the chapter. The framework that we are going to set will be also valid for the whole chapter except for Section III.8. Let us, now, fix an arbitrary sequence of càdlàg $\mathbb{R}^\ell-$valued processes $(X^k)_{k\in\overline{\mathbb{N}}}$ for which we assume

$$\sup_{k\in\overline{\mathbb{N}}} \mathbb{E}\left[\int_{(0,\infty)\times\mathbb{R}^\ell} \|x\|_2^2 \, \mu^{X^k}(\mathrm{d}s, \mathrm{d}x)\right] < \infty. \tag{III.2}$$

[1] Mémin mentions in [53] that this result is not completely new, since Aldous [3] has also presented an analogous one by means of the so-called *prediction process*. However, the author could not access any copy of [3] in order to provide further details on this matter.

Then we fix an arbitrary sequence of filtrations $(\mathbb{G}^k)_{k\in\overline{\mathbb{N}}}$ with $\mathbb{G}^k := (\mathcal{G}^k_t)_{t\geq 0}$, for every $k \in \overline{\mathbb{N}}$, on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and an arbitrary sequence of real-valued random variables $(\xi^k)_{k\in\overline{\mathbb{N}}}$.

We make the following additional assumptions:

(M1) The filtration $\mathbb{G}^\infty$ is quasi-left-continuous and the process X^∞ is $\mathbb{G}^\infty-$quasi-left-continuous.

(M2) The process $X^k \in \mathcal{H}^2(\mathbb{G}^k;\mathbb{R}^\ell)$ for every $k \in \overline{\mathbb{N}}$. Moreover, $X^k_\infty \xrightarrow{\mathbb{L}^2(\mathcal{G};\mathbb{R}^\ell)} X^\infty_\infty$.

(M3) The martingale X^∞ possesses the $\mathbb{G}^\infty-$predictable representation property.

(M4) The filtrations converge weakly, *i.e.* $\mathbb{G}^k \xrightarrow{\text{w}} \mathbb{G}^\infty$.

(M5) The random variable $\xi^k \in \mathbb{L}^2(\mathcal{G}^k_\infty;\mathbb{R})$, for every $k \in \overline{\mathbb{N}}$, and $\xi^k \xrightarrow{\mathbb{L}^2(\mathcal{G}_\infty;\mathbb{R})} \xi^\infty$.

Remark III.1. In view of Proposition I.158, conditions **(M2)** and **(M4)** imply that

$$(X^k, \mathbb{G}^k) \xrightarrow{\text{ext}} (X^\infty, \mathbb{G}^\infty).$$

Remark III.2. In **(M5)**, we have imposed an additional measurability assumption for the sequence of random variables $(\xi^k)_{k\in\overline{\mathbb{N}}}$, since we require that ξ^k is $\mathcal{G}^k_\infty-$measurable for any $k \in \overline{\mathbb{N}}$, instead of just being $\mathcal{G}-$measurable. We could spare that additional assumption at the cost of a stronger hypothesis in **(M4)**, namely that the weak convergence of the $\sigma-$algebrae

$$\mathcal{G}^k_\infty \xrightarrow{\text{w}} \mathcal{G}^\infty_\infty,$$

holds in addition. To sum up, in the statement of Theorem III.3 the pair **(M4)** and **(M5)** can be substituted by the following

(M3′) The filtrations converge weakly as well as the final $\sigma-$algebrae, that is

$$\mathbb{G}^k \xrightarrow{\text{w}} \mathbb{G}^\infty \text{ and } \mathcal{G}^k_\infty \xrightarrow{\text{w}} \mathcal{G}^\infty_\infty.$$

(M4′) The sequence $(\xi^k)_{k\in\overline{\mathbb{N}}} \subset \mathbb{L}^2(\mathcal{G}_\infty;\mathbb{R})$ and satisfies $\xi^k \xrightarrow{\mathbb{L}^2(\mathcal{G}_\infty;\mathbb{R})} \xi^\infty$.

Before we state the main theorem of the chapter, it would be convenient for the reader to recall Notation I.8.

Theorem III.3. *Let conditions* **(M1)** – **(M5)** *hold and define the $\mathbb{G}^k-$martingales $Y^k := \Pi^{\mathbb{G}^k}_o(\xi^k) = \mathbb{E}[\xi^k|\mathcal{G}^k_\cdot]$, for $k \in \overline{\mathbb{N}}$. Assume that the orthogonal decomposition of Y^k with respect to $\big(\mathbb{G}^k, X^{k,c}, \mu^{X^{k,d}}\big)$*[2] *is given by*

$$Y^k = Y^k_0 + Z^k \cdot X^{k,c} + U^k \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)} + N^k \text{ for } k \in \mathbb{N}$$

and

$$Y^\infty = Y^\infty_0 + Z^\infty \cdot X^{\infty,c} + U^\infty \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)} \text{ for } k = \infty.$$

Let $Q^k := Z^k \cdot X^{k,c} + U^k \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}$ for $k \in \overline{\mathbb{N}}$, then the following convergence hold

$$\big(Y^k, Q^k, N^k\big) \xrightarrow{(\mathrm{J}_1(\mathbb{R}^3),\mathbb{L}^2)} \big(Y^\infty, Q^\infty, N^\infty\big), \tag{III.3}$$

$$\big([Y^k], [Y^k, Q^k], [Y^k, N^k], [Y^k, X^k], [N^k]\big) \xrightarrow{(\mathrm{J}_1(\mathbb{R}^3\times\mathbb{R}^\ell\times\mathbb{R}),\mathbb{L}^1)} \big([Y^\infty], [Y^\infty, Q^\infty], [Y^\infty, N^\infty], [Y^\infty, X^\infty], [N^\infty]\big) \tag{III.4}$$

and

$$\big(\langle Y^k\rangle, \langle Y^k, X^k\rangle, \langle Q^k\rangle, \langle N^k\rangle\big) \xrightarrow{(\mathrm{J}_1(\mathbb{R}\times\mathbb{R}^\ell\times\mathbb{R}^2),\mathbb{L}^1)} \big(\langle Y^\infty\rangle, \langle Y^\infty, X^\infty\rangle, \langle Q^\infty\rangle, \langle N^\infty\rangle\big). \tag{III.5}$$

Moreover, the three above convergence can be assumed to hold jointly in $\mathbb{D}(\mathbb{R}^{9+2\ell})$. Additionally, the analogous convergence for the terminal values of the processes appear above hold.

[2]Observe that we regard the random measure associated to the jumps of the purely discontinuous part $X^{k,d}$ of the $\mathbb{G}^k-$martingale X^k, for every $k \in \overline{\mathbb{N}}$.

III.2. Outline of the proof of the main theorem

In this subsection we intend to ease the understanding of the technical subsections that follow. To this end, we present some helpful results, the strategy we are going to follow and an overview of the main arguments we are going to use in order to prove Theorem III.3. We will reuse the introductory scheme, but now updated so that it carries the information of Conditions **(M1)** - **(M5)**. However, as we have already mentioned, the discussion we intend to do requires some preliminary results, which are going to be also frequent referenced in the next sections. So, let us collect them in the next lemmata, which are essentially Mémin's Theorem, see Theorem I.160.

Lemma III.4. *Under the framework of Theorem III.3*:

(i) *The convergence* $(Y^k, \mathbb{G}^k) \xrightarrow{\text{ext}} (Y^\infty, \mathbb{G}^\infty)$ *holds.*

(ii) *The convergence* $Y_0^k \xrightarrow{(\delta_{|\cdot|},\mathbb{P})} Y_0^\infty$ *holds*

(iii) *The process* $\langle Y^\infty \rangle$ *is continuous.*

(iv) *The convergence*

$$(Y^k, [Y^k], \langle Y^k \rangle) \xrightarrow{\left(\mathrm{J}_1(\mathbb{R}\times\mathbb{R}^2),\mathbb{L}^2\times\mathbb{L}^1\right)} (Y^\infty, [Y^\infty], \langle Y^\infty \rangle)$$

holds. In particular,

$$\mathcal{Y}^k \xrightarrow{\left(\mathrm{J}_1(\mathbb{R}\times\mathbb{R}^3),\mathbb{L}^2\times\mathbb{L}^1\right)} \mathcal{Y}^\infty,$$

where $\mathcal{Y}^k := (Y^k, [Y^k], \langle Y^k \rangle, [Y^k] - \langle Y^k \rangle)$, *for every* $k \in \overline{\mathbb{N}}$.

(v) *The sequences* $\left([Y^k]_\infty\right)_{k\in\overline{\mathbb{N}}}$, $\left(\langle Y^k \rangle_\infty\right)_{k\in\overline{\mathbb{N}}}$ *and* $\left(\sup_{s>0} |Y^k|^2\right)_{k\in\overline{\mathbb{N}}}$ *are uniformly integrable.*

(vi) *The sequence* $(Y^k)_{k\in\mathbb{N}}$ *is* P-UT.

Proof. (i) See Proposition I.158.

(ii) It is immediate by the definition of the metric $\delta_{\mathrm{J}_1}(\mathbb{R})$.

(iii) Since Y^∞ is uniformly integrable, we can conclude by Theorem I.16 that it is $\mathbb{G}^\infty-$quasi-left-continuous. Then, we can conclude the continuity of $\langle Y^\infty \rangle$ by Theorem I.62.

(iv) For the first convergence see Theorem I.160.(i) and combine Conditions **(M1)**, **(M4)** and **(M5)** with (i). We can conclude the second convergence by the continuity of $\langle Y^\infty \rangle$ and Corollary I.122.

(v) For the sequences $\left([Y^k]_\infty\right)_{k\in\overline{\mathbb{N}}}$, $\left(\langle Y^k \rangle_\infty\right)_{k\in\overline{\mathbb{N}}}$ see Theorem I.160.(ii). The uniform integrability of the third sequence is derived by Burkholder–Davis–Gundy Inequality, see Theorem I.101, and the de La Vallée Poussin–Meyer Criterion, see Corollary I.48.

(vi) We will apply Proposition I.153. By construction, Y^k is a $\mathbb{G}^k-$martingale for every $k \in \overline{\mathbb{N}}$ and the convergence $Y^k \xrightarrow{(\mathrm{J}_1(\mathbb{R}),\mathbb{P})} Y^\infty$ is true by (iv). It is only left to prove that $\sup_{k\in\mathbb{N}} \mathbb{E}\left[\sup_{0\le s\le t} |Y_s^k|^2\right] < \infty$, for every $t \in \mathbb{R}_+$. However, we have for every $t \in \mathbb{R}_+$

$$\sup_{k\in\mathbb{N}} \mathbb{E}\Big[\sup_{0\le s\le t} |Y_s^k|^2\Big] \le \sup_{k\in\mathbb{N}} \mathbb{E}\Big[\sup_{s\ge 0} |Y_s^k|^2\Big] \overset{\text{Theorem I.57.(ii)}}{=} \sup_{k\in\mathbb{N}} \mathbb{E}\big[|Y_\infty^k|^2\big] = \sup_{k\in\mathbb{N}} \mathbb{E}\big[[Y^k]_\infty\big] \overset{\text{(v)}}{<} \infty.$$

□

We provide the analogous results for $(X^k)_{k\in\overline{\mathbb{N}}}$. We will not describe the proof as we did in the previous lemma since the arguments are completely analogous; the only difference will be that some equalities may be inequalities whose constants depend on ℓ.

Lemma III.5. *Under the framework of Theorem III.3*:

(i) *The convergence* $(X^k, \mathbb{G}^k) \xrightarrow{\text{ext}} (X^\infty, \mathbb{G}^\infty)$ *holds.*

(ii) *The convergence*

$$(X^k, [X^k], \langle X^k \rangle) \xrightarrow{\left(\mathrm{J}_1(\mathbb{R}^\ell\times\mathbb{R}^{\ell\times\ell}\times\mathbb{R}^{\ell\times\ell}),\mathbb{L}^2\times\mathbb{L}^1\times\mathbb{L}^1\right)} (X^\infty, [X^\infty], \langle X^\infty \rangle)$$

holds. In particular,

$$\mathcal{X}^k \xrightarrow{\left(\mathrm{J}_1(\mathbb{R}^\ell\times\mathbb{R}^{\ell\times 3\ell}),\mathbb{L}^2\times\mathbb{L}^1\right)} \mathcal{X}^\infty,$$

where $\mathcal{X}^k := (X^k, [X^k], \langle X^k \rangle, [X^k] - \langle X^k \rangle)$ *for evey* $k \in \overline{\mathbb{N}}$.

(iii) *The process* $\langle X^\infty \rangle$ *is continuous.*

(iv) *The sequences* $\left(\mathrm{Tr}\big[[X^k]_\infty\big]\right)_{k\in\overline{\mathbb{N}}}$, $\left(\mathrm{Tr}\big[\langle X^k \rangle_\infty\big]\right)_{k\in\overline{\mathbb{N}}}$ *and* $\left(\sup_{s>0} \|X^k\|_2^2\right)_{k\in\overline{\mathbb{N}}}$ *are uniformly integrable.*

(v) *The sequence* $(X^k)_{k\in\mathbb{N}}$ *is* P-UT.

Lemma III.6. *Under the framework of Theorem III.3 the convergence*

$$\big(X^k, Y^k, [Y^k, X^k], \langle Y^k, X^k\rangle\big) \xrightarrow{\left(\mathrm{J}_1(\mathbb{R}^\ell\times\mathbb{R}\times\mathbb{R}^\ell\times\mathbb{R}^\ell),\mathbb{P}\right)} \big(X^\infty, Y^\infty, [Y^\infty, X^\infty], \langle Y^\infty, X^\infty\rangle\big) \tag{III.6}$$

holds.

Proof. We will apply Theorem I.160 for the sequences $(X^{k,i}+Y^k)_{k\in\overline{\mathbb{N}}}$ and $(X^{k,i}-Y^k)_{k\in\overline{\mathbb{N}}}$, for every $i=1,\dots,\ell$. Let us start with the following notational abuse:

$$X^k+Y^k := (X^{k,1}+Y^k,\dots,X^{k,\ell}+Y^k) \text{ and } X^k-Y^k := (X^{k,1}-Y^k,\dots,X^{k,\ell}-Y^k) \text{ for } k\in\overline{\mathbb{N}}.$$

Observe now that $(X^k+Y^k)_\infty, (X^k-Y^k)_\infty \in \mathbb{L}^2(\mathcal{G};\mathbb{R}^\ell)$ for every $k\in\overline{\mathbb{N}}$. Therefore $X^k+Y^k, X^k-Y^k \in \mathcal{H}^2(\mathbb{G}^k;\mathbb{R}^\ell)$ and for every $k\in\overline{\mathbb{N}}$. By Conditions **(M2)** and **(M5)** we have

$$(X^k+Y^k)_\infty \xrightarrow{\mathbb{L}^2(\mathcal{G};\mathbb{R}^\ell)} (X^\infty+Y^\infty)_\infty \text{ as well as } (X^k-Y^k)_\infty \xrightarrow{\mathbb{L}^2(\mathcal{G};\mathbb{R}^\ell)} (X^\infty-Y^\infty)_\infty$$

Moreover, by Lemma III.4.(iii) we have that Y^∞ is $\mathbb{G}^\infty-$quasi-left-continuous, which in conjunction with **(M1)** implies that $X^\infty+Y^\infty$ and $X^\infty-Y^\infty$ are $\mathbb{G}^\infty-$quasi-left-continuous. Now, by Proposition I.158 we obtain

$$\big(X^k+Y^k,\mathbb{G}^k\big) \xrightarrow{\text{ext}} \big(X^\infty+Y^\infty,\mathbb{G}^\infty\big) \text{ as well as } \big(X^k-Y^k,\mathbb{G}^k\big) \xrightarrow{\text{ext}} \big(X^\infty-Y^\infty,\mathbb{G}^\infty\big).$$

Now, we can apply Theorem I.160 in order to obtain

$$\big(X^k+Y^k,[X^k+Y^k],\langle X^k+Y^k\rangle,\mathbb{G}^k\big) \xrightarrow{\text{ext}} \big(X^\infty+Y^\infty,[X^\infty+Y^\infty],\langle X^\infty+Y^\infty\rangle,\mathbb{G}^\infty\big)$$

as well as

$$\big(X^k-Y^k,[X^k-Y^k],\langle X^k-Y^k\rangle,\mathbb{G}^k\big) \xrightarrow{\text{ext}} \big(X^\infty-Y^\infty,[X^\infty-Y^\infty],\langle X^\infty-Y^\infty\rangle,\mathbb{G}^\infty\big).$$

Finally, by using Corollary I.125 and the polarisation identity we derive the required convergence. □

Now we can proceed to outline the proof of Theorem III.3. The first part of the statement amounts to showing the following convergence

$$\begin{array}{ccccccc} Y^k & = & Y_0^k & + & Z^k\cdot X^{k,c} + U^k\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)} & + & N^k \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y^\infty & = & Y_0^\infty & + & Z^\infty\cdot X^{\infty,c} + U^\infty\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)} & + & N^\infty \end{array} \tag{III.7}$$

However, Lemma III.4 yields directly the two left convergence. Thus, the sum on the right-hand of (III.7) converges. We will be able conclude if for every weak-limit of $(N^k)_{k\in\mathbb{N}}$, name $\overline{N}$, we can determine a filtration $\mathbb{F}:=(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, which may depend on $\overline{N}$, such that the following hold:

(A) For every $t\in\mathbb{R}_+$ holds $\mathcal{G}_t^\infty\subset\mathcal{F}_t$. Alternatively written, following Notation I.11, $\mathbb{G}^\infty\subset\mathbb{F}$.
(B) $X^\infty\in\mathcal{H}^2(\mathbb{F};\mathbb{R}^\ell)$, $Y^\infty\in\mathcal{H}^2(\mathbb{F},\mathbb{R})$.
(Γ) $[X^\infty]-\langle X^\infty\rangle^{\mathbb{G}^\infty}\in\mathcal{M}(\mathbb{F};\mathbb{R})$.
(Δ) $\overline{N}\in\mathcal{M}(\mathbb{F};\mathbb{R})$ and it is sufficiently integrable.
(E) $M_{\mu^{X^\infty,d}}\big[\Delta\overline{N}\big|\widetilde{\mathcal{P}}^{\mathbb{F}}\big]=0$ and $\langle X^{\infty,c,i},\overline{N}^c\rangle^{\mathbb{F}}=0$ for every $i=1,\dots,\ell$.
(ΣT) $M_{\mu^{X^\infty,d}}[\Delta Y^\infty\big|\widetilde{\mathcal{P}}^{\mathbb{F}}]=M_{\mu^{X^\infty,d}}[\Delta Y^\infty\big|\widetilde{\mathcal{P}}^{\mathbb{G}^\infty}]$ and $\langle X^{\infty,c,i},Y^\infty\rangle^{\mathbb{F}}=\langle X^{\infty,c,i},Y^\infty\rangle^{\mathbb{G}^\infty}$ for every $i=1,\dots,\ell$.

Let us comment briefly the implications of the above conditions. By (Γ) we obtain that X^∞ is also $\mathbb{F}-$quasi-left-continuous. We will see in Subsection III.6.2 that this property will allow us to conclude that

$$\langle X^{\infty,c}\rangle^{\mathbb{F}}=\langle X^{\infty,c}\rangle^{\mathbb{G}^\infty} \text{ and } \nu^{(X^{\infty,d},\mathbb{F})}\Big|_{\widetilde{\mathcal{P}}^{\mathbb{G}^\infty}}=\nu^{(X^{\infty,d},\mathbb{G}^\infty)}. \tag{III.8}$$

Now we will explain the outcome of specific combinations between (A)-(ΣT).

- The combination of (A) and (B) will be useful to prove (ΣT), which in turn will enable us to prove that the orthogonal decomposition of Y^∞ with respect to $\big(\mathbb{F},X^{\infty,c},\mu^{X^{\infty,d}}\big)$ is indistinguishable from the orthogonal decomposition of Y^∞ with respect to $\big(\mathbb{G}^\infty,X^{\infty,c},\mu^{X^{\infty,d}}\big)$. Observe that by **(M2)** and (B) we have $X^\infty\in\mathcal{H}^2(\mathbb{G}^\infty;\mathbb{R}^\ell)\cap\mathcal{H}^2(\mathbb{F};\mathbb{R}^\ell)$. Analogously, by the construction of Y^∞ and (B) we will also obtain that $Y^\infty\in\mathcal{H}^2(\mathbb{G}^\infty;\mathbb{R})\cap\mathcal{H}^2(\mathbb{F};\mathbb{R})$. Therefore, both of the aforementioned orthogonal decompositions are indeed well-defined. This result will be presented in Subsection III.6.2.

• The combination of (B) and (Δ) will be useful to prove (E). This, in turn, will enable us to prove that $\overline{N}$ is orthogonal to the space generated by the natural pair of the $\mathbb{F}-$martingale X^∞. This result will be provided in Subsection III.6.1. Observe that we did not claim that $\overline{N} \in \mathcal{H}^{2,\perp}(\mathbb{F}, X^{\infty,c}, \mu^{X^{\infty,d}})$, because $\overline{N}$ is not *a priori* square-integrable.

Exploiting these properties we will be able to identify $\overline{N}$, *i.e.* the arbitrary weak-limit of $(N^k)_{k\in\mathbb{N}}$, with the zero process by proving (finally) that $[Y^\infty, \overline{N}] \in \mathcal{M}(\mathbb{F};\mathbb{R})$. For this conclusion, we will make use of

- The fact that the representation of Y^∞ is not affected by the expansion of the filtration.
- Condition **(M3)**. The latter can be stated alternatively as $\mathcal{H}^{2,\perp}(\mathbb{G}^\infty, X^c, \mu^{X^{\infty,d}};\mathbb{R}) = \{0\}$.
- $\overline{N}$ is orthogonal to sufficiently many stochastic integrals with respect to the natural pair of X^∞.

Intuitively, this is what we need. The orthogonal martingales should converge to an element which is orthogonal to the space generated by the natural pair of X^∞ and the "information that is collected from Y^∞ through stochastic integrals with respect to the natural pair of X^∞ does not increase". This step will be the final one of the proof and is going to be presented in Section III.7.

Let us now provide some more information on the roadway we will follow in order to obtain (A)-(ΣT). Proving the properties (A)-(ΣT) under the framework **(M1)** - **(M5)** requires a series of preparatory results. We start be providing Proposition III.7 in Section III.3. This proposition translates the problem of determining the predictable objects $\langle X^{\infty,c,i}, L\rangle^{\mathbb{F}}$, for $i = 1,\dots,\ell$, and $M_{X^d}\left[\Delta L \middle| \widetilde{\mathcal{P}}^{\mathbb{F}}\right]$ into a problem of determining a rich enough family of $\mathbb{F}-$martingales. In the previously used notation, L is a uniformly integrable $\mathbb{F}-$martingale such that the objects are well-defined. With the aforementioned proposition we will prove in Subsection III.6.2 that (ΣT) is true, while with a simplification stated as Corollary III.8 we will prove in Subsection III.6.1 that (E) is valid.

Having a closer look on the problem-translation mentioned above, this amounts to ensuring the existence of a (suitable) deterministic function[3] $h : \left(\mathbb{R}^\ell, \mathcal{B}(\mathbb{R}^\ell)\right) \to \left(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\right)$ and a (suitable) family $\mathcal{J}$ of subsets of $\mathbb{R}^\ell$ such that for every $I \in \mathcal{J}$ holds

$$\text{for (E):} \qquad \left[(h\mathbb{1}_I) \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, \overline{N}\right], [X^{\infty,i}, \overline{N}] \in \mathcal{M}(\mathbb{F};\mathbb{R}), \text{ for every } i = 1,\dots,\ell, \tag{III.9}$$

$$\text{and for } (\Sigma\text{T}): \qquad \begin{aligned} &\left[(h\mathbb{1}_I) \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, Y^\infty\right] - \left\langle (h\mathbb{1}_I) \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, Y^\infty\right\rangle^{\mathbb{G}^\infty} \in \mathcal{M}(\mathbb{F};\mathbb{R}) \\ &[X^{\infty,i}, Y^\infty] - \langle X^{\infty,i}, Y^\infty\rangle^{\mathbb{G}^\infty} \in \mathcal{M}(\mathbb{F};\mathbb{R}), \text{ for every } i = 1,\dots,\ell. \end{aligned} \tag{III.10}$$

In order to prove (III.9) and (III.10) we will make use of Proposition I.147. Therefore, we need to weakly approximate $X^\infty, Y^\infty, \overline{N}$ and $(h\mathbb{1}_I) \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}$ for every $I \in \mathcal{J}$. While we can weakly approximate X^∞, Y^∞ and $\overline{N}$ by known sequences and some subsequence, this is not the case for the martingales $(h\mathbb{1}_I) \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}$. This is the purpose of Section III.4. At the beginning of Section III.4 we will discuss how to choose the function h and the family $\mathcal{J}$ and then we will present the results which provide the required family of convergent martingale-sequences.

Let us now briefly comment on (A), *i.e.* how to construct the filtration $\mathbb{F}$. As we have already mentioned, we are going to apply Proposition I.147. In the next two lines we follow the notation of this proposition. Having in our mind **(M4)**, a simple observation reveals that we can enrich the natural filtration of the limit process H^∞ provided we can improve the convergence $\Theta^k \xrightarrow{\mathcal{L}} \Theta^\infty$ to

$$(\Theta^k, \mathbb{E}[\mathbb{1}_{G^\infty}|\mathcal{G}^k]) \xrightarrow{\mathcal{L}} (\Theta^\infty, \mathbb{E}[\mathbb{1}_{G^\infty}|\mathcal{G}^\infty]), \text{ for every } G^\infty \in \mathcal{G}^\infty_\infty.$$

In other words, we will obtain that $(\Theta^\infty, \mathbb{E}[\mathbb{1}_{G^\infty}|\mathcal{G}^\infty])$ is an $\mathbb{F}^{\mathbb{E}[\mathbb{1}_{G^\infty}|\mathcal{G}^\infty]} \vee \mathbb{F}^{H^\infty}$-martingale, for every $G^\infty \in \mathcal{G}^\infty_\infty$. Then, it will be almost straightforward to obtain that Θ^∞ is martingale with respect to the filtration $\mathbb{F} := \mathbb{G}^\infty \vee \mathbb{F}^{H^\infty}$. This issue will be discussed also in Section III.6. Finally, once we have all the above results, we will present in Section III.7 the proof of Theorem III.3.

III.3. To be (orthogonal), or not to be, here is the answer

This section constitutes the first part of the preparatory results we are going to use in Section III.7. In order to avoid as many technicalities as possible amid the flow of the preparatory results, we provide in this section Proposition III.7 and Corollary III.8. It is also a good chance to provide some comments around these results. Generally speaking, the former result provides the martingale conditions, by which we can conclude if the projections (under two different filtrations) of a martingale Y on the spaces

[3]Recall Remark I.73 for the usage of deterministic functions as predictable functions.

generated (under the two different filtration) by the natural pair of a martingale X remain unaffected. The latter provides the martingale conditions, by which we can conclude if a martingale N is orthogonal to the space generated by the natural pair of a martingale X. The probability space is $(\Omega, \mathcal{G}, \mathbb{P})$, as it has been already mentioned, but for this section we make the following assumption:

■ $$\mathbb{H}^1, \mathbb{H}^2 \text{ are fixed filtrations such that } \mathbb{H}^1 \subset \mathbb{H}^2.$$

Let us now provide some more helpful comments. Assume initially that $X, Y \in \mathcal{M}(\mathbb{H}^1, \mathbb{R}) \cap \mathcal{M}(\mathbb{H}^2, \mathbb{R})$, that is to say, that they are both uniformly integrable martingales with respect to both given filtrations, and they are such that $[X, Y] \in \mathcal{A}(\mathbb{H}^1; \mathbb{R}) \cap \mathcal{A}(\mathbb{H}^2; \mathbb{R})$. By Theorem I.64 we know that there exists an $\mathbb{H}^1-$predictable process $[X, Y]^{(p,\mathbb{H}^1)}$ of finite variation as well as an $\mathbb{H}^2-$predictable process $[X, Y]^{(p,\mathbb{H}^2)}$ of finite variation such that

$$[X, Y] - [X, Y]^{(p,\mathbb{H}^1)} \in \mathcal{M}(\mathbb{H}^1, \mathbb{R}) \text{ and } [X, Y] - [X, Y]^{(p,\mathbb{H}^2)} \in \mathcal{M}(\mathbb{H}^2, \mathbb{R}).$$

If we assume further that $[X, Y]-[X, Y]^{(p,\mathbb{H}^1)} \in \mathcal{M}(\mathbb{H}^2; \mathbb{R})$, then it is a direct consequence of Corollary I.65 that $[X, Y]^{(p,\mathbb{H}^1)} = [X, Y]^{(p,\mathbb{H}^2)}$. The Proposition III.7 provides a criterion, by which we can refine the above. In other words, it provides the conditions under which

$$\langle X^c, Y^c\rangle^{\mathbb{H}^1} = \langle X^c, Y^c\rangle^{\mathbb{H}^2} \text{ and } M_{\mu^{X^d}}\big[\Delta Y \big| \widetilde{\mathcal{P}}^{\mathbb{H}^1}\big] = M_{\mu^{X^d}}\big[\Delta Y \big| \widetilde{\mathcal{P}}^{\mathbb{H}^2}\big]. \tag{III.11}$$

For this, however, we need to improve the integrability of the martingales and assume that X, Y lie in $\mathcal{H}^2(\mathbb{H}^1, \mathbb{R})$ as well as in $\mathcal{H}^2(\mathbb{H}^2, \mathbb{R})$. The integrability of X, Y and the Kunita–Watanabe Inequality, see Theorem I.100, imply that $\mathrm{Var}\big([X, Y]\big) \in \mathcal{A}^+(\mathbb{H}^1; \mathbb{R}) \subset \mathcal{A}^+(\mathbb{H}^2; \mathbb{R})$. which verifies that $[X, Y] \in \mathcal{A}^+(\mathbb{H}^1; \mathbb{R}) \subset \mathcal{A}^+(\mathbb{H}^2; \mathbb{R})$ as well. Roughly speaking, one could now translate (III.11) as "the coefficients[4] of the orthogonal decompositions of Y with respect to $\big(\mathbb{H}^1, X^c, \mu^{X^d}\big)$ and the coefficients of the orthogonal decompositions of Y with respect to $\big(\mathbb{H}^2, X^c, \mu^{X^d}\big)$ are the same". This, however, does not imply that the two orthogonal decompositions are indistinguishable. It may be the case that either $\langle X^c\rangle^{\mathbb{H}^1} \neq \langle X^c\rangle^{\mathbb{H}^2}$ or $\nu^{(X^d,\mathbb{H}^2)}\Big|_{\widetilde{\mathcal{P}}^{\mathbb{H}^1}} \neq \nu^{(X^d,\mathbb{H}^1)}$, which would lead in general in different stochastic integrals. Now, Condition (Γ) will verify that we have also (III.8).

As we have already commented, we are going to apply the next proposition for the pair X^∞, Y^∞, while the role of the two filtration $\mathbb{H}^1, \mathbb{H}^2$ will be played by $\mathbb{G}^\infty, \mathbb{F}$. Moreover, we anticipate once more in order to inform the reader that Lemma III.45 ensure that $\langle X^{\infty,c}\rangle^{\mathbb{G}^\infty} = \langle X^{\infty,c}\rangle^{\mathbb{F}}$ and $\nu^{(X^{\infty,d},\mathbb{F})}\Big|_{\widetilde{\mathcal{P}}^{\mathbb{G}^\infty}} = \nu^{(X^{\infty,d},\mathbb{G}^\infty)}$. So, we will also assume that the latter holds in the next proposition.

Proposition III.7. *Assume that the following hold:*

(i) $\mathbb{H}^1 \subset \mathbb{H}^2$, i.e. $\mathcal{H}^1_t \subset \mathcal{H}^2_t$ *for every* $t \in \mathbb{R}_+$.

(ii) $X \in \mathcal{M}(\mathbb{H}^1; \mathbb{R}^\ell) \cap \mathcal{M}(\mathbb{H}^2; \mathbb{R}^\ell)$, $Y \in \mathcal{M}(\mathbb{H}^1; \mathbb{R}) \cap \mathcal{M}(\mathbb{H}^2; \mathbb{R})$ *and* X *is* $\mathbb{H}^1-$*quasi–left–continuous.*

(iii) $h : \big(\mathbb{R}^\ell, \mathcal{B}(\mathbb{R}^\ell)\big) \longrightarrow \big(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\big)$ *is such that* $h(\Delta X^d_t) > 0$ *whenever* $\Delta X^d_t \neq 0$.

(iv) $\nu^{(X^d,\mathbb{H}^2)}\Big|_{\widetilde{\mathcal{P}}^{\mathbb{H}^1}} = \nu^{(X^d,\mathbb{H}^1)}$

(v) $[X, Y] - \langle X, Y\rangle^{\mathbb{H}^1} \in \mathcal{M}(\mathbb{H}^2, \mathbb{R}^\ell)$.

(vi) $\mathcal{I}$ *is a family of subsets of* $\mathbb{R}^\ell$ *such that* $\sigma(\mathcal{I}) = \mathcal{B}(\mathbb{R}^\ell)$ *and the martingale* $(h\mathbb{1}_A) \star \widetilde{\mu}^{(X^d,\mathbb{H}^1)}$[5] *is well–defined and such that* $(h\mathbb{1}_A) \star \widetilde{\mu}^{(X^d,\mathbb{H}^1)} \in \mathcal{H}^2(\mathbb{H}^1; \mathbb{R})$ *for every* $A \in \mathcal{I}$. *Moreover,*

$$\big[(h\mathbb{1}_A) \star \widetilde{\mu}^{(X^d,\mathbb{H}^1)}, Y\big] - \big\langle (h\mathbb{1}_A) \star \widetilde{\mu}^{(X^d,\mathbb{H}^1)}, Y\big\rangle^{\mathbb{H}^1} \in \mathcal{M}(\mathbb{H}^2; \mathbb{R}), \text{ for every } A \in \mathcal{I}.$$

(vii) $|\Delta Y|\mu^X \in \widetilde{\mathcal{A}}_\sigma(\mathbb{H}^1)$,[6]

Then, we have that

$$\langle X^{c,i}, Y^c\rangle^{\mathbb{H}^1} = \langle X^{c,i}, Y^c\rangle^{\mathbb{H}^2} \quad \text{for every } i = 1, \dots \ell \tag{III.12}$$

and

$$M_{\mu^{X^d}}[\Delta Y|\widetilde{\mathcal{P}}^{\mathbb{H}^1}] = M_{\mu^{X^d}}[\Delta Y|\widetilde{\mathcal{P}}^{\mathbb{H}^2}] \; M_{\mu^{X^d}} - a.e.. \tag{III.13}$$

[4]Although this terminology sounds naive, it is not far from the reality. One should observe the way the orthogonal decompositions are obtained and would realise that the above quantities are determining the integrands of the stochastic integrals taken with respect to the natural pair of a square integrable martingale.

[5]Recall Remark I.73 for the usage of deterministic functions as predictable functions.

[6]For the notation, recall Definition I.75. Moreover, observe that (i) and (vii) imply that also $|\Delta Y|\mu^X \in \widetilde{\mathcal{A}}_\sigma(\mathbb{H}^2)$.

Proof. We will prove initially (III.13). To this end, we are going to translate condition (vi) into

$$\mathbb{E}\Big[\big(Wh\Delta Y\big)*\mu^{X^d}_\infty\Big]=\mathbb{E}\Big[\big(WhM_{\mu^{X^d}}[\Delta Y|\widetilde{\mathcal{P}}^{\mathbb{H}^1}]\big)*\mu^{X^d}_\infty\Big] \text{ for every } \mathbb{H}^2-\text{measurable function } W. \tag{III.14}$$

Before we proceed, recall that we have assumed X to be $\mathbb{H}^1-$quasi-left-continuous. Thus, by Proposition I.87 we obtain that

$$\Delta\big((h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)}\big)_s=h\big(\Delta X^d_s\big)\mathbb{1}_A(\Delta X^d_s) \tag{III.15}$$

By the $\mathbb{H}^2-$martingale property of $\big[(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y\big]-\langle(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y\rangle^{\mathbb{H}^1}$, we obtain for every $0\le t<u<\infty$ and every $C\in\mathcal{H}^2_t$, that

$$\begin{aligned}\mathbb{E}\Big[\mathbb{1}_C\,\mathbb{E}\Big[\,[(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y]_u-\langle(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y\rangle^{\mathbb{H}^1}_u\Big|\mathcal{H}^2_t\Big]\Big]\\ =\mathbb{E}\Big[\,\mathbb{1}_C\,[(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y]_t-\mathbb{1}_C\,[(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y]_t\Big],\end{aligned}$$

equivalently

$$\begin{aligned}\mathbb{E}\Big[\mathbb{1}_C\,\mathbb{E}\big[\,[(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y]_u\big|\mathcal{H}^2_t\big]-\mathbb{1}_C\,[(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y]_t\Big]\\ =\mathbb{E}\Big[\mathbb{1}_C\,\mathbb{E}\big[\,\langle(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y\rangle^{\mathbb{H}^1}_u\big|\mathcal{H}^2_t\big]-\mathbb{1}_C\,\langle(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y\rangle^{\mathbb{H}^1}_t\Big],\end{aligned}$$

equivalently

$$\begin{aligned}\mathbb{E}\Big[\mathbb{1}_C\,[(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y]_u-\mathbb{1}_C\,[(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y]_t\Big]\\ =\mathbb{E}\Big[\mathbb{1}_C\,\langle(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y\rangle^{\mathbb{H}^1}_u-\mathbb{1}_C\,[(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y]_t\Big].\end{aligned} \tag{III.16}$$

By Definition I.96, since $(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)}\in\mathcal{H}^{2,d}(\mathbb{H}^1;\mathbb{R})$, we have that

$$[(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y]_u-[(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y]_t=\sum_{t<s\le u}\Big(\Delta\big((h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)}\big)_s\Delta Y_s\Big).$$

Moreover, in view of by Theorem I.93 and (vii), which ensures the existence of $M_{\mu^{X^d}}\big[\Delta Y\big|\widetilde{\mathcal{P}}^{\mathbb{H}^1}\big]$, we can write (III.16) equivalently

$$\begin{aligned}\mathbb{E}\Big[\mathbb{1}_C\sum_{t<s\le u}\Big(\Delta\big((h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)}\big)_s\Delta Y_s\Big)\Big]\\ =\mathbb{E}\Big[\mathbb{1}_C\int_{(t,u]\times\mathbb{R}^\ell}\Big(h(x)\mathbb{1}_A(x)M_{\mu^{X^d}}\big[\Delta Y\big|\widetilde{\mathcal{P}}^{\mathbb{H}^1}\big](s,x)\Big)\nu^{(X^d,\mathbb{H}^1)}(\mathrm{d}s,\mathrm{d}x)\Big],\end{aligned}$$

equivalently by (III.15),

$$\begin{aligned}\mathbb{E}\Big[\int_{(t,u]\times\mathbb{R}^\ell}\big(\mathbb{1}_Ch(x)\mathbb{1}_A(x)\Delta Y_s\big)\mu^{X^d}(\mathrm{d}s,\mathrm{d}x)\Big]\\ =\mathbb{E}\Big[\int_{(t,u]\times\mathbb{R}^\ell}\Big(\mathbb{1}_Ch(x)\mathbb{1}_A(x)M_{\mu^{X^d}}\big[\Delta Y\big|\widetilde{\mathcal{P}}^{\mathbb{H}^1}\big](s,x)\Big)\nu^{(X^d,\mathbb{H}^1)}(\mathrm{d}s,\mathrm{d}x)\Big],\end{aligned}$$

equivalently

$$\begin{aligned}\mathbb{E}\Big[\int_{(0,\infty)\times\mathbb{R}^\ell}\big[\mathbb{1}_C\mathbb{1}_{(t,u]}(s)\mathbb{1}_A(x)\big]\big(h(x)\Delta Y_s\big)\mu^{X^d}(\mathrm{d}s,\mathrm{d}x)\Big]\\ =\mathbb{E}\Big[\int_{(0,\infty)\times\mathbb{R}^\ell}\big[\mathbb{1}_C\mathbb{1}_{(t,u]}(s)\mathbb{1}_A(x)\big]\Big(h(x)M_{\mu^{X^d}}\big[\Delta Y\big|\widetilde{\mathcal{P}}^{\mathbb{H}^1}\big](s,x)\Big)\nu^{(X^d,\mathbb{H}^1)}(\mathrm{d}s,\mathrm{d}x)\Big].\end{aligned} \tag{III.17}$$

Now, by monotone class theorem we can conclude that condition (III.14) holds, because

$$\begin{aligned}\widetilde{\mathcal{P}}^{\mathbb{H}^2}&=\mathcal{P}^{\mathbb{H}^2}\otimes\mathcal{B}(\mathbb{R}^\ell)=\sigma\big(P\times A,\text{ where } P\in\mathcal{P}^{\mathbb{H}^2}\text{ and } A\in\mathcal{B}(\mathbb{R}^\ell)\big)\\ &\overset{\text{Th. I.14.(ii)}}{=}\sigma\big(C\times(t,u]\times A,\text{ where } 0\le t<u, C\in\mathcal{H}^2_t\text{ and } A\in\mathcal{I}\big),\end{aligned}$$

where we have from (vi) that $\sigma(\mathcal{I})=\mathcal{B}(\mathbb{R}^\ell)$.

The next observation is that the function $\widetilde{\Omega}\ni(\omega,s,x)\longmapsto h(x)\in\mathbb{R}$ is $\mathbb{H}^2-$predictable, since it is deterministic and continuous. Moreover, the function $h(x)$ is positive $M_{\mu^{X^d}}-a.e.$; recall that μ^{X^d} has been defined using the random set $[\Delta X^d\neq 0]$. Therefore, we obtain

$$\mathbb{E}\Big[(U\Delta Y)*\mu^{X^d}_\infty\Big]=\mathbb{E}\Big[\big(UM_{\mu^{X^d}}\big[\Delta Y\big|\widetilde{\mathcal{P}}^{\mathbb{H}^1}\big]\big)*\mu^{X^d}_\infty\Big],\text{ for every } \mathbb{H}^2-\text{measurable function } U, \tag{III.18}$$

by substituting in (III.14) the $\mathbb{H}^2-$predictable function $W(\omega,s,x)$ with the $\mathbb{H}^2-$predictable function $\frac{U(\omega,s,x)}{h(x)}$, where U is an arbitrary $\mathbb{H}^2-$predictable function.

On the other hand, condition (vii) ensures the well-posedness of $M_{\mu^{X^d}}\left[\Delta Y\middle|\widetilde{\mathcal{P}}^{\mathbb{H}^2}\right]$. Indeed, there exists a positive $\mathbb{H}^1-$predictable function V such that $\mathbb{E}\left[(V\Delta Y)*\mu^{X^d}_\infty\right]<\infty$. But V, by (i), is also $\widetilde{\mathcal{P}}^{\mathbb{H}^2}-$measurable, which verifies our claim. By definition now of $M_{\mu^{X^d}}\left[\Delta Y\middle|\widetilde{\mathcal{P}}^{\mathbb{H}^2}\right]$ holds

$$\mathbb{E}\Big[(U\Delta Y)*\mu^{X^d}_\infty\Big]=\mathbb{E}\Big[\big(UM_{\mu^{X^d}}\left[\Delta Y\middle|\widetilde{\mathcal{P}}^{\mathbb{H}^2}\right]\big)*\mu^{X^d}_\infty\Big],\ \text{for every } \mathbb{H}^2-\text{measurable function } U. \tag{III.19}$$

Now, recalling that $\mathbb{H}^1\subset\mathbb{H}^2$, which implies that $\mathcal{P}^{\mathbb{H}^1}\subset\mathcal{P}^{\mathbb{H}^2}$, it is only left to observe that $M_{\mu^{X^d}}\left[\Delta Y\middle|\widetilde{\mathcal{P}}^{\mathbb{H}^1}\right]$, $M_{\mu^{X^d}}\left[\Delta Y\middle|\widetilde{\mathcal{P}}^{\mathbb{H}r}\right]$ are both $\mathbb{H}^2-$predictable functions with

$$\mathbb{E}\Big[\big(UM_{\mu^{X^d}}\left[\Delta Y\middle|\widetilde{\mathcal{P}}^{\mathbb{H}^1}\right]\big)*\mu^{X^d}_\infty\Big]\overset{\text{(III.18)}}{\underset{\text{(III.19)}}{=}}\mathbb{E}\Big[\big(UM_{\mu^{X^d}}\left[\Delta Y\middle|\widetilde{\mathcal{P}}^{\mathbb{H}^2}\right]\big)*\mu^{X^d}_\infty\Big]\ \text{for every } \mathbb{H}^2-\text{measurable function } U.$$

This implies that

$$M_{\mu^{X^d}}\left[\Delta Y\middle|\widetilde{\mathcal{P}}^{\mathbb{H}^1}\right]=M_{\mu^{X^d}}\left[\Delta Y\middle|\widetilde{\mathcal{P}}^{\mathbb{H}^2}\right]\ M_{\mu^{X^d}}-\text{almost everywhere}. \tag{III.20}$$

We proceed now to show the validity of (III.12). By condition (ii) we have that $[X,Y]^i\in\mathcal{A}(\mathbb{H}^2;\mathbb{R})$ for every $i=1,\dots,\ell$, hence the process $\langle X,Y\rangle^{\mathbb{H}^2}$ is well-defined and such that $[X,Y]-\langle X,Y\rangle^{\mathbb{H}^2}\in\mathcal{M}(\mathbb{H}^2,\mathbb{R}^\ell)$. But then

$$\langle X,Y\rangle^{\mathbb{H}^2}-\langle X,Y\rangle^{\mathbb{H}^1}=[X,Y]-\langle X,Y\rangle^{\mathbb{H}^1}-\big([X,Y]-\langle X,Y\rangle^{\mathbb{H}^2}\big)\in\mathcal{M}(\mathbb{H}^2,\mathbb{R}^\ell)\cap\mathcal{V}_{\text{pred}}(\mathbb{H}^2;\mathbb{R}^\ell).$$

By Corollary I.65 it has to be indistinguishable from the zero process, *i.e.*

$$\langle X,Y\rangle^{\mathbb{H}^2}=\langle X,Y\rangle^{\mathbb{H}^1} \tag{III.21}$$

By Theorem I.93, we obtain for every $i=1,\dots,\ell$

$$\begin{aligned}\langle X^{d,i},Y\rangle^{\mathbb{H}^1}&\overset{\text{III.20}}{=}\langle\pi^i\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y\rangle^{\mathbb{H}^1}=(\pi^iM_{\mu^{X^d}}[\Delta Y|\widetilde{\mathcal{P}}^{\mathbb{H}^1}])*\nu^{(X^d,\mathbb{H}^1)}\\&\overset{\text{(III.20)}}{\underset{\text{(iv)}}{=}}(\pi^iM_{\mu^{X^d}}[\Delta Y|\widetilde{\mathcal{P}}^{\mathbb{H}^2}])*\nu^{(X^d,\mathbb{H}^2)}=\langle\pi^i\star\widetilde{\mu}^{(X^d,\mathbb{H}^2)},Y\rangle^{\mathbb{H}^2}=\langle X^{d,i},Y\rangle^{\mathbb{H}^2}.\end{aligned} \tag{III.22}$$

The combination of (III.21) and (III.22) yields

$$\langle X^{c,i},Y^c\rangle^{\mathbb{H}^1}=\langle X^i,Y\rangle^{\mathbb{H}^1}-\langle X^{d,i},Y^d\rangle^{\mathbb{H}^1}\overset{\text{(III.21)}}{\underset{\text{(III.22)}}{=}}\langle X^i,Y\rangle^{\mathbb{H}^2}-\langle X^{d,i},Y^d\rangle^{\mathbb{H}^2}=\langle X^{c,i},Y^c\rangle^{\mathbb{H}^1},$$

for every $i=1,\dots,\ell$. □

Now we can provide the following corollary, which can be proven by using completely analogous (but simpler) arguments to the previous proposition.

Corollary III.8. *Assume the framework of Proposition III.7 where* (ii), (v) *and* (vi) *are substituted by the following*

(ii′) $X\in\mathcal{H}^2(\mathbb{H}^1;\mathbb{R}^\ell)\cap\mathcal{H}^2(\mathbb{H}^2;\mathcal{H}^2)$ *and* $Y\in\mathcal{M}(\mathbb{F};\mathbb{R})$.

(v′) $[X,Y]^i\in\mathcal{A}(\mathbb{H}^1;\mathbb{R})$ *for every* $i=1,\dots,\ell$ *and* $[X,Y]\in\mathcal{M}(\mathbb{H}^2,\mathbb{R}^\ell)$.

(vi′) $\mathcal{I}$ *is a family of subsets of* $\mathbb{R}^\ell$ *such that* $\sigma(\mathcal{I})=\mathcal{B}(\mathbb{R}^\ell)$, *the martingale* $(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)}$ *is well–defined and such that* $(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)}\in\mathcal{H}^2(\mathbb{H}^1;\mathbb{R})$ *for every* $A\in\mathcal{I}$. *Moreover,*

$$\left[(h\mathbb{1}_A)\star\widetilde{\mu}^{(X^d,\mathbb{H}^1)},Y\right]\in\mathcal{M}(\mathbb{H}^2;\mathbb{R}),\ \textit{for every } A\in\mathcal{I}.$$

Then, we have that

$$\langle X^{c,i},Y^c\rangle=0\ \textit{for every } i=1,\dots\ell\ \textit{and}\ M_{\mu^{X^d}}[\Delta Y|\widetilde{\mathcal{P}}^{\mathbb{H}^1}]=0\ M_{\mu^{X^d}}-a.e..$$

Proof. The result in immediate by analogous arguments to Proposition III.7.

□

III.4. Constructing a sufficiently rich family of convergent martingale-sequences

This section is devoted to constructing, for a given function $h : \big(\mathbb{R}^\ell, \mathcal{B}(\mathbb{R}^\ell)\big) \longrightarrow (\mathbb{R}_+, \mathcal{B}(\mathbb{R}))$, a family of sequences of purely discontinuous martingales, namely

$$\big(h_I^k \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big)_{k\in\mathbb{N}}$$

for $I \in \mathcal{J}$ with $\sigma(\mathcal{J}) = \mathcal{B}(\mathbb{R}^\ell)$. Additionally, we will prove that for $I \in \mathcal{J}$

$$\big(X^k, h_I^k \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big) \xrightarrow{\left(\mathrm{J}_1(\mathbb{R}^{\ell+1}),\mathbb{L}^2\right)} \big(X^\infty, (h\mathbb{1}_I) \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}\big) \tag{III.23}$$

and

$$\begin{aligned}\Big(X^k, \big[Y^k, h_I^k \star \widetilde{\mu}^{(X^{k,d})}\big] - \big\langle Y^k, h_I^k \star \widetilde{\mu}^{(X^{k,d},}\big\rangle, \mathbb{G}^k\Big) &\xrightarrow{\left(\mathrm{J}_1(\mathbb{R}^\ell\times\mathbb{R}),\mathbb{L}^2\times\mathbb{L}^1\right)} \\ &\Big(X^\infty, \big[Y^k, (h\mathbb{1}_I) \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big] - \big\langle Y^k, (h\mathbb{1}_I) \star \widetilde{\mu}^{(X^{k,d},}\big\rangle\Big).\end{aligned} \tag{III.24}$$

Recall the discussion in Section III.2 for the reason we need such a family. In particular, (III.24) is connected with Proposition III.7.(vi). We divide this section into two subsections. In Subsection III.4.1 we provide Proposition III.17, which provides the validity of (III.23), associated with its preparatory results. The respective results of Convergence (III.24) are presented in Subsection III.4.2, where the main result of the subsection is Proposition III.25.

Let us now comment on the choice of the function h (we need only one such function in order to apply later Proposition III.7 and Corollary III.8), of the set $\mathcal{J}$ and of the integrands h_I^k, for $k \in \mathbb{N}$, $I \in \mathcal{J}$. Let us start with the definition of $M_{\mu^{X^{\infty,d}}}[\Delta L|\widetilde{\mathcal{P}}^{\mathbb{F}}]$, for $\mathbb{F}$ a filtration such that $\mathbb{G}^\infty \subset \mathbb{F}$, and $L \in \mathcal{M}(\mathbb{F};\mathbb{R})$ such that $\Delta L\mu^{X^{\infty,d}} \in \widetilde{\mathcal{A}}_\sigma(\mathbb{F})$. The $\mathbb{F}-$predictable function $M_{\mu^{X^{\infty,d}}}[\Delta L|\widetilde{\mathcal{P}}^{\mathbb{F}}]$ is defined as the $\mathbb{F}-$predictable function such that it holds

$$\mathbb{E}\big[(W\Delta L) * \mu_\infty^{X^{\infty,d}}\big] = \mathbb{E}\big[(WM_{\mu^{X^{\infty,d}}}[\Delta L|\widetilde{\mathcal{P}}^{\mathbb{F}}]) * \mu_\infty^{X^{\infty,d}}\big] \text{ for every } \mathbb{F}-\text{predictable function } W.$$

In the results of the previous section a positive function h has been intervened in the above products of integrands with the property that $h(\Delta X_t^{\infty,d}) > 0$ whenever $\Delta X_t^{\infty,d} \neq 0$; we have adapted the notation of the current section. The reason is quite obvious: we have to retain the information whenever a jump happens at a given $(\omega, t) \in [\Delta X^{\infty,d} \neq 0]$. However, we have the flexibility to alter the jump values $\Delta X_t^{\infty,d}(\omega)$ in a $\mathbb{G}^\infty-$predictable way. For example we could choose $h(x) = \|x\|_1$, which is positive whenever a jump occurs for $\Delta X^{\infty,d}$ and, moreover, does not allow for any cancellations between the jumps $\Delta X^{\infty,d,i}$, for $i = 1, \ldots, \ell$. But, this is not a satisfactory choice for our purpose since we will need better integrability properties. The next choice would be to distinguish between "big" and "small" jumps, so we are going to examine the function

$$\mathbb{R}^\ell \ni x \overset{\mathrm{R}_p}{\longmapsto} \sum_{i=1}^{\ell} (|x^i| \wedge 1)^p \in \mathbb{R}, \text{ for } p \in [1, \infty). \tag{III.25}$$

where $a \wedge b = \min\{a, b\}$ for $a, b \in \mathbb{R}$. Now, we have that

$$\sum_{0<s\leq\cdot} \mathrm{R}_p(\Delta X_s^{\infty,d}) \leq \sum_{0<s\leq\cdot} \|\Delta X_s^{\infty,d}\|_2 \in \mathbb{L}^1(\mathcal{G};\mathbb{R}), \text{ for every } p \in [1, \infty).$$

This property will allow us to construct sequences of martingales with (more than) sufficient integrability. We will constraint ourselves in the case $p = 2$.

Now that we have chosen the function which will play the role of h, we need to approximate the $\mathbb{G}^\infty-$submartingale $(\mathrm{R}_p\mathbb{1}_I) * \mu^{X^{\infty,d}}$. We comment briefly how we obtain the approximation and then why we resolve in this way. In view of Lemma III.5.(i) we can obtain the last mentioned approximation by means of Proposition I.134, which lead us also to the proper choice of $\mathcal{J}$ as well as of h_I^k, for $k \in \mathbb{N}$, $I \in \mathcal{J}$. If we translate Proposition I.134 to the current framework, we are allowed to choose the sets I from the set $\mathcal{J}(X^{\infty,d})$, see its definition at the beginning of the next section, and the integrands h_I^k have to be equal with $\mathrm{R}_2\mathbb{1}_I$. The reader may have realised that we have obtained convergent sequences of submartingales instead of martingales as we have promised. This is true. But we have done so because these are easier to be constructed. Indeed, using Proposition I.134 we create for every $k \in \overline{\mathbb{N}}$ the process $\mathrm{R}_2\mathbb{1}_I * \mu^{X^{\infty,d}}$ which is $\mathbb{G}^k-$adapted, with piecewise-constant paths and with positive jumps, *i.e.* its path

are non-decreasing $\mathbb{P}-a.s.$. Now, it is the results of Memin, more specifically Theorem I.159, which will allow us to obtain the initially required convergence

$$(\mathrm{R}_2\mathbb{1}_I)\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\xrightarrow{(\mathrm{J}_1,\mathbb{P})}(\mathrm{R}_2\mathbb{1}_I)\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}. \tag{III.26}$$

The $\mathbb{L}^2-$convergence will be easily obtained as a result of the choice of R_2.

III.4.1. The convergence (III.23) **is true.** Throughout this chapter, we will consider X^k to be a $\mathbb{G}^k-$martingale, for every $k\in\overline{\mathbb{N}}$. Before we proceed, let us introduce some notation that will be used throughout the rest of Chapter III.

Notation III.9. For a fixed $i\in\{1,\dots,\ell\}$, following the notation used in Subsection I.6.1, we introduce the sets

- $W(X^{\infty,i}(\omega)):=\{u\in\mathbb{R}\setminus\{0\},\exists t>0 \text{ with } \Delta X_t^{\infty,d,i}(\omega)=u\}$.
- $V(X^{\infty,i}):=\{u\in\mathbb{R}\setminus\{0\},\mathbb{P}\big([\Delta X_t^{\infty,d,i}=u,\ \text{for some } t>0]\big)>0\}$.
- $\mathcal{I}(X^{\infty,i}):=\{(v,w)\subset\mathbb{R}\setminus\{0\},vw>0 \text{ and } v,w\notin V(X^{\infty,i})\}$.
- $\mathcal{J}(X^\infty):=\Big\{\prod_{i=1}^\ell I_i,\ \text{where } I_i\in\mathcal{I}(X^{\infty,i})\cup\big\{\mathbb{R}\big\} \text{ for every } i=1,\dots,\ell\Big\}\setminus\big\{\mathbb{R}^\ell\big\}$.
- For every $I:=I_1\times\cdots\times I_\ell\in\mathcal{J}(X^\infty)$, we set

$$J_I:=\big\{i\in\{1,\dots,\ell\},I_i\neq\mathbb{R}\big\}\neq\emptyset. \tag{III.27}$$

- Let $k\in\overline{\mathbb{N}}$, $I:=I_1\times\cdots\times I_\ell\in\mathcal{J}(X^\infty)$ and $g:\Omega\times\mathbb{R}^\ell\longrightarrow\mathbb{R}$. Then we define the $\mathbb{R}^{\ell+1}-$valued process

$$\widehat{X}^k[g,I]:=\big(X^k,X^{k,g,I}\big),\ \text{with } X^{k,g,I}:=(g\mathbb{1}_I)*\mu^{X^{k,d}}.$$

By Lemma I.138 we have that the set $V(X^{\infty,i})$ is at most countable, for every $i=1,\dots,\ell$. Moreover, observe that an alternative representation of $X^{k,g,I}$ is given by

$$X^{k,g,I}_\cdot(\omega)=\sum_{0<t\leq\cdot}g(\omega,\Delta X_t^{k,d}(\omega))\mathbb{1}_I(\Delta X_t^{k,d}(\omega))=\int_{(0,\cdot]\times\mathbb{R}^\ell}g(\omega,x)\mathbb{1}_I(x)\mu^{X^{k,d}}(\omega;\mathrm{d}s,\mathrm{d}x).$$

Recall, moreover, that, due to **(M2)** and Theorem I.57.(ii), the random variable X^k_∞ exists $\mathbb{P}-a.s.$. Consequently, the process $X^{k,g,I}$, as well as the process $\widehat{X}^k[g,I]$, are $\mathbb{P}-a.s.$ well–defined.

Proposition III.10. *Let condition* **(M2)** *hold and fix an $I\in\mathcal{J}(X^\infty)$ and a function $g:(\Omega\times\mathbb{R}^\ell,\mathcal{G}\otimes\mathcal{B}(\mathbb{R}^\ell))\longrightarrow(\mathbb{R},\mathcal{B}(\mathbb{R}))$. For the function g we assume that there exists $\Omega^C\subset\Omega$ with $\mathbb{P}(\Omega^C)=1$, and*

$$\mathbb{R}^\ell\ni x\longmapsto g(\omega,x)\in\mathbb{R}\ \ \textit{is continuous on}\ \ C(\omega)\ \ \textit{for every}\ \ \omega\in\Omega^C,$$

where

$$C(\omega):=\prod_{i=1}^\ell A_i(\omega),\ \textit{with}\ A_i(\omega):=\begin{cases}W(X^{\infty,i}(\omega)),\ \textit{if } i\in J_I,\\ W(X^{\infty,i}(\omega))\cup\{0\},\ \textit{if } i\in\{1,\dots,\ell\}\setminus J_I,\end{cases}$$

Then, it holds

$$\widehat{X}^k[g,I]\xrightarrow[k\to\infty]{(\mathrm{J}_1(\mathbb{R}^{\ell+1}),\mathbb{P})}\widehat{X}^\infty[g,I].$$

Proof. Let us fix an $I:=I_1\times\cdots\times I_\ell\in\mathcal{J}(X^\infty)$ and recall that the space $(\mathbb{D}(\mathbb{R}^{\ell+1}),d_{\mathrm{J}_1(\mathbb{R}^{\ell+1})})$ is Polish. Therefore, it is sufficient, by Theorem I.141, to prove that for every subsequence $(\widehat{X}^{k_l}[g,I])_{l\in\mathbb{N}}$, there exists a further subsequence $(\widehat{X}^{k_{l_m}}[g,I])_{m\in\mathbb{N}}$ for which

$$\widehat{X}^{k_{l_m}}[g,I]\xrightarrow[m\to\infty]{\mathrm{J}_1(\mathbb{R}^{\ell+1})}\widehat{X}^\infty[g,I],\ \mathbb{P}-a.s.. \tag{III.28}$$

Let $(\widehat{X}^{k_l}[g,I])_{l\in\mathbb{N}}$ be fixed hereinafter. For every $i\in J_I$, since $I_i\in\mathcal{I}(X^{\infty,i})$, there exists $\Omega^{I_i}\subset\Omega$ such that

(i) $\mathbb{P}\big(\Omega^{I_i}\big)=1$,
(ii) $\Delta X_t^{\infty,d,i}(\omega)\notin\partial I_i$, for every $t\in\mathbb{R}_+$ and $\omega\in\Omega^{I_i}$,

where ∂A denotes the $|\cdot|-$boundary of the set A.

Condition **(M2)** implies the convergence $X^k \xrightarrow{(\mathrm{J}_1(\mathbb{R}^\ell),\mathbb{P})} X^\infty$. Hence, for the subsequence $(X^{k_l})_{l\in\mathbb{N}}$, there is a further subsequence $(X^{k_{l_m}})_{m\in\mathbb{N}}$ for which holds

$$X^{k_{l_m}} \xrightarrow{\mathrm{J}_1(\mathbb{R}^\ell)} X^\infty,\ \mathbb{P}-a.s.. \tag{III.29}$$

Let $\Omega_{\mathrm{sub}} \subset \Omega$ be such that $\mathbb{P}(\Omega_{\mathrm{sub}}) = 1$ and such that the convergence (III.29) holds for every $\omega \in \Omega_{\mathrm{sub}}$. Define $\Omega_{\mathrm{sub}}^{J_I} := \Omega_{\mathrm{sub}} \cap \big(\cap_{i\in J_I} \Omega^{I_i}\big)$. Then

(i′) $\mathbb{P}(\Omega_{\mathrm{sub}}^{J_I}) = 1$,

(ii′) $\Delta X_t^{\infty,d,i}(\omega) \notin \partial I_i$, for every $t \in \mathbb{R}_+$, for every $i \in J_I$ and every $\omega \in \Omega_{\mathrm{sub}}^{J_I}$.

By the last property we can conclude that $\partial I_i \cap W(X^{\infty,i}(\omega)) = \emptyset$, for every $\omega \in \Omega_{\mathrm{sub}}^{J_I}, i \in J_I$. Therefore, the function $\mathbb{R}^\ell \ni x \xmapsto{g_I(\omega)} g(\omega,x)\mathbb{1}_I(x)$ is continuous on the set $C(\omega)$ for $\mathbb{P}-$almost every $\omega \in \Omega_{\mathrm{sub}}^{J_I}$.

We can now conclude once we apply, for each $\omega \in \Omega_{\mathrm{sub}}^{J_I}$, Proposition I.134 for the sequence $(X^{k_{l_m}}(\omega))_{m\in\mathbb{N}}$ and for the function $\mathbb{R}^\ell \ni x \longmapsto g(\omega,x) \in \mathbb{R}$, which verifies (III.28). □

For the following results, recall the continuous function R_p introduced in (III.25).

Corollary III.11. *Let condition* **(M2)** *hold. Then, for every $I \in \mathcal{J}(X^\infty)$ it holds*

$$\widehat{X}^k[\mathrm{R}_p, I] \xrightarrow{(\mathrm{J}_1(\mathbb{R}^{\ell+1}),\mathbb{P})} \widehat{X}^\infty[\mathrm{R}_p, I], \quad \textit{for every } p \in \mathbb{R}_+.$$

Proof. Let $p \in \mathbb{R}_+$. It suffices to apply the above proposition to the function R_p, which is continuous. □

Let us now provide more details on the strategy of the proof for this step. Proposition III.17 below is the most important result of this sub–section, since it provides us with a rich enough family of converging martingale sequences. It is the family for which we have already commented and satisfies convergence (III.26). To this end, we are going to apply Theorem I.159 to the sequence $(X^{k,\mathrm{R}_2,I})_{k\in\overline{\mathbb{N}}}$, where $I \in \mathcal{J}(X^\infty)$. However, all of this requires to make sure that this sequence indeed verifies the requirements of the aforementioned theorem, for every $I \in \mathcal{J}(X^\infty)$. This is the subject for the remainder of this subsection. Before we proceed we provide a helpful for the calculations remark which is direct by Assumption (III.2). Then we will set some convenient notation.

Remark III.12. Let us fix initially a $k \in \overline{\mathbb{N}}$. In view of Assumption (III.2) we obtain that

$$\mathbb{E}\Big[\mathrm{Tr[q]} * \mu_\infty^{X^k}\Big] = \mathbb{E}\Big[\mathrm{Tr[q]} * \nu_\infty^{(X^k,\mathbb{G}^k)}\Big]. \tag{III.30}$$

This implies that we can associate to the process X^k a square-integrable $\mathbb{G}^k-$martingale, which in **(M2)** we have denoted with the same symbol X^k. [7] For the calculations, this means that $\sum_{s>0} \|\Delta X^{k,d}\|_2^2 \in \mathcal{A}^+(\mathbb{G}^k;\mathbb{R})$. Now, letting k run through $\overline{\mathbb{N}}$, we have that Assumption (III.2) and the Equality (III.30) yield that

$$\sup_{k\in\overline{\mathbb{N}}} \mathbb{E}\Big[\mathrm{Tr[q]} * \mu_\infty^{X^k}\Big] = \sup_{k\in\overline{\mathbb{N}}} \mathbb{E}\Big[\mathrm{Tr[q]} * \nu_\infty^{(X^k,\mathbb{G}^k)}\Big] < \infty,$$

which in turn, in view of Proposition I.85, implies

$$\begin{aligned}\sup_{k\in\overline{\mathbb{N}}} \mathbb{E}\Big[\mathrm{Tr}\big[\big[X^{k,d}\big]_\infty\big]\Big] &= \sup_{k\in\overline{\mathbb{N}}} \mathbb{E}\Big[\sum_{s>0}\big\|\Delta X_s^{k,d}\big\|_2^2\Big] = \sup_{k\in\overline{\mathbb{N}}} \mathbb{E}\Big[\mathrm{Tr}[q] * \mu_\infty^{X^k,d}\Big] = \sup_{k\in\overline{\mathbb{N}}} \mathbb{E}\Big[\sum_{s>0} \Delta\Big(\mathrm{Tr}[q] * \widetilde{\mu}^{(X^k,\mathbb{G}^k)}\Big)_s\Big] \\ &\le \sup_{k\in\overline{\mathbb{N}}} \mathbb{E}\Big[\mathrm{Tr}[q] * \mu_\infty^{X^k}\Big] + \sup_{k\in\overline{\mathbb{N}}} \mathbb{E}\Big[\mathrm{Tr}[q] * \nu_\infty^{(X^k,\mathbb{G}^k)}\Big] < \infty.\end{aligned} \tag{III.31}$$

We end this remark with following comment. The reader may recall that from Lemma III.5.(iv) we have better integrability properties for the sequence $\big(\mathrm{Tr}\big[\big[X^{k,d}\big]_\infty\big]\big)_{k\in\overline{\mathbb{N}}}$, however the above bound is immediate by Assumption (III.2) and we would prefer to refer to the above bounds whenever these are sufficient for our needs.

Let us, now, fix $I := \prod_{i=1}^\ell I_i \in \mathcal{J}(X^\infty)$, hence the set J_I is a fixed non–empty subset of $\{1,\dots,\ell\}$. Moreover, we define $\mathbb{R}^\ell \ni x \xmapsto{\mathrm{R}_{2,A}} \mathrm{R}_2(x)\mathbb{1}_A(x)$, for every $A \subset \mathbb{R}^\ell$.

[7] We underline again at this point that, according to our notation, the integer-valued measure μ^{X^k} associates to the jumps of the process X^k, while the integer valued measure $\mu^{X^{k,d}}$ associates to the jumps of the $\mathbb{G}^k-$martingale X^k, whose purely discontinuous part is $X^{k,d}$.

Lemma III.13. *The process $X^{k,\mathrm{R}_2,I}$ is a $\mathbb{G}^k-$special semimartingale, for every $k \in \overline{\mathbb{N}}$. In particular, its $\mathbb{G}^k$–canonical decomposition is given by*

$$X^{k,\mathrm{R}_2,I}_{\cdot} = \int_0^{\cdot}\int_{\mathbb{R}^\ell} \mathrm{R}_{2,I}(x)\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}(\mathrm{d}s,\mathrm{d}x) + \int_{(0,\cdot]\times I} \mathrm{R}_2(x)\nu^{(X^{k,d},\mathbb{G}^k)}(\mathrm{d}s,\mathrm{d}x)$$

or equivalently

$$\mathrm{R}_{2,I} * \mu^{X^{k,d}} = \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)} + \mathrm{R}_{2,I} * \nu^{(X^{k,d},\mathbb{G}^k)}.$$

Proof. Let $k \in \overline{\mathbb{N}}$. Observe that by construction the process $X^{k,\mathrm{R}_2,I}$ is $\mathbb{G}^k-$adapted and càdlàg. The function $\mathrm{R}_{2,I}$ is positive, hence the process $X^{k,\mathrm{R}_2,I}$ is a $\mathbb{G}^k-$submartingale of finite variation, as its paths are $\mathbb{P}-a.s.$ non–decreasing. Before we proceed, we need to show the integrability of $\mathrm{R}_{2,I} * \mu^{X^{k,d}}_\infty$ in order to make use of Proposition I.85. But, we have

$$\begin{aligned}\mathbb{E}\Big[\int_{(0,\infty)\times\mathbb{R}^\ell} \mathrm{R}_{2,I}(x)\,\mu^{X^{k,d}}(\mathrm{d}s,\mathrm{d}x)\Big] &\le \mathbb{E}\Big[\int_{(0,\infty)\times\mathbb{R}^\ell} \|x\|^2\mu^{X^{k,d}}(\mathrm{d}s,\mathrm{d}x)\Big]\\ &= \mathbb{E}\Big[\mathrm{T}\big[\big[X^{k,d}\big]\big]\Big] \overset{\text{(III.31)}}{\le} \infty, \end{aligned} \tag{III.32}$$

where the last equality holds because of Theorem I.79. Moreover, (III.32) yields also that $X^{k,\mathrm{R}_2,I} \in \mathcal{S}_{\mathrm{sp}}(\mathbb{G}^k;\mathbb{R})$, by Proposition I.98.(ii). We concluded the last property because the process in increasing. Moreover, by Theorem I.79 and Condition (III.32) (which implicitly makes use of (III.12)) we obtain

$$\mathbb{E}\Big[\int_{(0,\infty)\times\mathbb{R}^\ell} \mathrm{R}_{2,I}(x)\,\nu^{(X^{k,d},\mathbb{G}^k)}(\mathrm{d}s,\mathrm{d}x)\Big] < \infty, \text{ for every } k \in \overline{\mathbb{N}}. \tag{III.33}$$

Therefore, we have

$$\begin{aligned} X^{k,\mathrm{R}_2,I}_{\cdot} &= (\mathrm{R}_2\mathbb{1}_I) * \mu^{X^{k,d}} = \Big[(\mathrm{R}_2\mathbb{1}_I) * \mu^{X^{k,d}} - (\mathrm{R}_2\mathbb{1}_I) * \nu^{(X^{k,d},\mathbb{G}^k)}\Big] + (\mathrm{R}_2\mathbb{1}_I) * \nu^{(X^{k,d},\mathbb{G}^k)}\\ &= (\mathrm{R}_2\mathbb{1}_I) \star \mu^{(X^{k,d},\mathbb{G}^k)} + (\mathrm{R}_2\mathbb{1}_I) * \nu^{(X^{k,d},\mathbb{G}^k)}\\ &= \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)} + \mathrm{R}_{2,I} * \nu^{(X^{k,d},\mathbb{G}^k)}, \end{aligned}$$

where in the third equality we have used Proposition I.85. The finite variation part $\mathrm{R}_{2,I} * \nu^{(X^{k,d},\mathbb{G}^k)}$ is predictable, since $\mathrm{R}_{2,I}$ is deterministic and the random measure $\nu^{(X^{k,d},\mathbb{G}^k)}$ is predictable (see Definition I.10 and Theorem I.79). Hence, we can conclude also via this route that $X^{k,\mathrm{R}_2,I}$ is a special semimartingale, since it admits a representation as the sum of a martingale and a predictable part of finite variation. □

Lemma III.14. (i) *The sequence $\big(X^{k,\mathrm{R}_2,I}_\infty\big)_{k\in\overline{\mathbb{N}}}$ is uniformly integrable.*

(ii) *The sequence $\big([X^{k,\mathrm{R}_2,I}]_\infty\big)_{k\in\overline{\mathbb{N}}}$ is uniformly integrable.*

Proof. (i) Using the definitions of R_2 and $X^{k,\mathrm{R}_2,I}$, we get

$$X^{k,\mathrm{R}_2,I}_\infty = \int_{(0,\infty)\times\mathbb{R}^\ell} \mathrm{R}_{2,I}(x)\,\mu^{X^{k,d}}(\mathrm{d}s,\mathrm{d}x) \le \int_{(0,\infty)\times\mathbb{R}^\ell} |x|^2\,\mu^{X^{k,d}}(\mathrm{d}s,\mathrm{d}x) = \mathrm{Tr}\big[[X^{k,d}]_\infty\big] \le \mathrm{Tr}\big[[X^k]_\infty\big].$$

Hence, from Lemma III.5.(iv) and Corollary I.29.(ii) we can conclude.

(ii) By Lemma III.13, the process $X^{k,\mathrm{R}_2,I}$ is a $\mathbb{G}^k-$special semimartingale for every $k \in \overline{\mathbb{N}}$, whose martingale part is purely discontinuous. Therefore, we have by Definition I.96 that

$$\begin{aligned}[X^{k,\mathrm{R}_2,I}]_\infty &= \sum_{s>0}\big|\mathrm{R}_2\big(\Delta X^{k,d}_s\big)\big|^2\mathbb{1}_I(\Delta X^{k,d}_s) \le \sum_{s>0}\big|\mathrm{R}_2\big(\Delta X^{k,d}_s\big)\big|^2 = \sum_{s>0}\Big(\sum_{i=1}^\ell\big(\big|\Delta X^{k,d,i}_s\big|\wedge 1\big)^2\Big)^2\\ &= \sum_{s>0}\Big[\sum_{i=1}^\ell\Big(\big|\Delta X^{k,d,i}_s\big|^2\mathbb{1}_{(0,1)}\big(\big|\Delta X^{k,d,i}_s\big|\big) + \mathbb{1}_{[1,\infty)}\big(\big|\Delta X^{k,d,i}_s\big|\big)\Big)\Big]^2\\ &\le 2\ell\sum_{s>0}\Big[\sum_{i=1}^\ell\Big(\big|\Delta X^{k,d,i}_s\big|^4\mathbb{1}_{(0,1)}\big(\big|\Delta X^{k,d,i}_s\big|\big) + \mathbb{1}_{[1,\infty)}\big(\big|\Delta X^{k,d,i}_s\big|\big)\Big)\Big]\\ &\le 2\ell\sum_{s>0}\Big[\sum_{i=1}^\ell\big(\Delta X^{k,d,i}_s\big)^2\Big] = 2\ell\mathrm{Tr}\big[[X^{k,d}]_\infty\big] \le 2\ell\mathrm{Tr}\big[[X^k]_\infty\big]. \end{aligned}$$

Thus, using Lemma III.5.(iv) and Corollary I.29.(ii) again, we have the required result. □

Lemma III.15. (i) *The sequence* $\big(\mathrm{Var}(\mathrm{R}_{2,I} * \nu^{(X^{k,d},\mathbb{G}^k)})_\infty\big)_{k\in\overline{\mathbb{N}}}$ *is tight in* $(\mathbb{R}, |\cdot|)$.

(ii) *The sequence* $\big(\sum_{s>0}\big(\Delta(\mathrm{R}_{2,I} * \nu^{(X^{k,d},\mathbb{G}^k)})_s\big)^2\big)_{k\in\overline{\mathbb{N}}}$ *is uniformly integrable.*

Proof. (i) We have already observed that $X^{k,\mathrm{R}_{2,I}}$ is a $\mathbb{G}^k-$submartingale for every $k \in \overline{\mathbb{N}}$ and consequently $\mathrm{R}_{2,I} * \nu^{(X^{k,d},\mathbb{G}^k)}$ is non–decreasing for every $k \in \overline{\mathbb{N}}$; a property which is also immediate since $\mathrm{R}_{2,I}$ is a positive function and $\nu^{(X^{k,d},\mathbb{G}^k)}$ is a (positive) measure. Therefore, it holds

$$\mathrm{Var}\big(\mathrm{R}_{2,I} * \nu^{(X^{k,d},\mathbb{G}^k)}\big) = \mathrm{R}_{2,I} * \nu^{(X^{k,d},\mathbb{G}^k)}, \quad \text{for every } k \in \overline{\mathbb{N}}.$$

In view of the above and due to Markov's inequality, it suffices to prove that $\sup_{k\in\overline{\mathbb{N}}} \mathbb{E}[\mathrm{R}_{2,I} * \nu_\infty^{(X^{k,d},\mathbb{G}^k)}] < \infty$. Indeed, for every $\varepsilon > 0$ it holds for $K := \frac{1}{\varepsilon}\sup_{k\in\overline{\mathbb{N}}} \mathbb{E}[\mathrm{R}_{2,I} * \nu_\infty^{(X^{k,d},\mathbb{G}^k)}] > 0$ that

$$\sup_{k\in\overline{\mathbb{N}}} \mathbb{P}\big[\mathrm{R}_{2,I} * \nu_\infty^{(X^{k,d},\mathbb{G}^k)} > K\big] \leq \frac{1}{K}\sup_{k\in\overline{\mathbb{N}}} \mathbb{E}[\mathrm{R}_{2,I} * \nu_\infty^{(X^{k,d},\mathbb{G}^k)}] < \varepsilon,$$

which yields the required tightness. Now, observe that we have

$$\mathbb{E}[\mathrm{R}_{2,I} * \nu_\infty^{(X^{k,d},\mathbb{G}^k)}] \overset{\text{Proposition I.85}}{=} \mathbb{E}[\mathrm{R}_{2,I} * \mu_\infty^{X^{k,d}}] \overset{\text{(III.32)}}{<} \infty. \tag{III.34}$$

We have concluded using inequality (III.32), which in turn makes use of Assumption (III.2). Therefore (III.34) yields that $\sup_{k\in\overline{\mathbb{N}}} \mathbb{E}[\mathrm{R}_{2,I} * \nu_\infty^{(X^{k,d},\mathbb{G}^k)}] < \infty$. (ii) Before we proceed, the reader may recall Remark I.80, *i.e.* that we use a version of $\nu^{(X^{k,d},\mathbb{G}^k)}$ for which $\nu^{(X^{k,d},\mathbb{G}^k)}(\{s\} \times \mathbb{R}^\ell \leq 1$ identically for every $k \in \overline{\mathbb{N}}$. In view of this property, we can apply Cauchy–Schwartz Inequality in order to obtain the following upper bound

$$\begin{aligned}\Big(\int_{\mathbb{R}^\ell} \mathrm{R}_{2,I}(x)\,\nu^{(X^{k,d},\mathbb{G}^k)}(\{s\} \times \mathrm{d}x)\Big)^2 &\leq \int_{\mathbb{R}^\ell} \mathrm{R}_{2,I}^2(x)\,\nu^{(X^{k,d},\mathbb{G}^k)}(\{s\} \times \mathrm{d}x) \int_{\mathbb{R}^\ell} \nu^{(X^{k,d},\mathbb{G}^k)}(\{s\} \times \mathrm{d}x)\\ &\leq \int_{\mathbb{R}^\ell} \mathrm{R}_{2,I}^2(x)\,\nu^{(X^{k,d},\mathbb{G}^k)}(\{s\} \times \mathrm{d}x). \end{aligned}\tag{III.35}$$

Now, we have, for every $k \in \overline{\mathbb{N}}$, that the following holds:

$$\begin{aligned}\sum_{s>0}\big(\Delta(\mathrm{R}_{2,I} * \nu^{(X^{k,d},\mathbb{G}^k)})_s\big)^2 &= \sum_{s>0}\Big(\int_{\mathbb{R}^\ell} \mathrm{R}_{2,I}(x)\,\nu^{(X^{k,d},\mathbb{G}^k)}(\{s\} \times \mathrm{d}x)\Big)^2\\ &\overset{\text{(III.35)}}{\leq} \sum_{s>0}\int_{\mathbb{R}^\ell} \mathrm{R}_{2,I}^2(x)\,\nu^{(X^{k,d},\mathbb{G}^k)}(\{s\} \times \mathrm{d}x) \leq \int_{(0,\infty)\times\mathbb{R}^\ell} \mathrm{R}_{2,I}^2(x)\,\nu^{(X^{k,d},\mathbb{G}^k)}(\mathrm{d}s,\mathrm{d}x)\\ &= \int_{(0,\infty)\times I}\Big[\sum_{i=1}^{\ell}(|x^i| \wedge 1)^2\Big]^2 \nu^{(X^{k,d},\mathbb{G}^k)}(\mathrm{d}s,\mathrm{d}x) \leq 2\ell\int_{(0,\infty)\times I}\sum_{i=1}^{\ell}(|x^i|^2 \wedge 1)\,\nu^{(X^{k,d},\mathbb{G}^k)}(\mathrm{d}s,\mathrm{d}x)\\ &= 2\ell\,\mathrm{R}_{2,I} * \nu_\infty^{(X^{k,d},\mathbb{G}^k)}. \end{aligned}\tag{III.36}$$

Using Lemma III.14.(i) and the de La Vallée Poussin–Meyer Criterion, see Corollary I.48, there exists a moderate Young function Φ such that

$$\sup_{k\in\overline{\mathbb{N}}} \mathbb{E}\big[\Phi\big(X_\infty^{k,\mathrm{R}_{2,I}}\big)\big] < \infty.$$

Then, using that $X^{k,\mathrm{R}_{2,I}}$ is an increasing process, hence it is equal to its supremum process, the decomposition of Lemma III.13 and applying Theorem I.55 we can conclude that

$$\sup_{k\in\overline{\mathbb{N}}} \mathbb{E}\big[\Phi\big(\mathrm{R}_{2,I} * \nu_\infty^{(X^{k,d},\mathbb{G}^k)}\big)\big] < \infty.$$

By de La Vallée Poussin–Meyer Criterion, the latter condition is equivalent to the uniform integrability of the sequence $\big(\mathrm{R}_{2,I} * \nu_\infty^{(X^{k,d},\mathbb{G}^k)}\big)_{k\in\overline{\mathbb{N}}}$. Then, by (III.36) and Corollary I.29.(ii) we can conclude the uniform integrability of the required sequence. □

Corollary III.16. (i) *The sequence* $\big(\big[\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big]_\infty\big)_{k\in\overline{\mathbb{N}}}$ *is uniformly integrable.*

(ii) *Consequently, it holds that* $\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)} \in \mathcal{H}^{2,d}(\mathbb{G}^k;\mathbb{R})$, *for every* $k \in \overline{\mathbb{N}}$.

(iii) *The sequence* $\big(\langle \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\rangle_\infty\big)_{k\in\overline{\mathbb{N}}}$ *is uniformly integrable.*

Proof. (i) Using Definition I.96 and that $\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}$ is a martingale of finite varation, we have

$$\begin{aligned}\left[\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\right]_\infty &= \sum_{s>0}\left(\Delta(\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)})_s\right)^2 = \sum_{s>0}\left(\Delta(X^{k,\mathrm{R}_2,I})_s - \Delta(\mathrm{R}_{2,I} * \nu^{(X^{k,d},\mathbb{G}^k)})_s\right)^2 \\ &\leq 2\sum_{s>0}\left(\Delta(X^{k,\mathrm{R}_2,I})_s\right)^2 + 2\sum_{s>0}\left(\Delta(\mathrm{R}_{2,I} * \nu^{(X^{k,d},\mathbb{G}^k)})_s\right)^2 \\ &= 2\left[X^{k,\mathrm{R}_{2,I}}\right]_\infty + 2\sum_{s>0}\left(\Delta(\mathrm{R}_{2,I} * \nu^{(X^{k,d},\mathbb{G}^k)})_s\right)^2,\end{aligned}$$

where in the last equality we have used that $X^{k,\mathrm{R}_{2,I}}$ is a semimartingale whose paths have finite variation and Definition I.96. In view now of the above inequality, Lemma III.14.(ii), Lemma III.15.(ii) and (ii),(iv) of Corollary I.29 we can conclude the required property. This shows (i).

(ii) In addition, (i) implies the integrability of $\left[\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\right]$, hence from [41, Proposition I.4.50.c)] we get that $\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)} \in \mathcal{H}^{2,d}(\mathbb{G}^k, \infty; \mathbb{R})$.

(iii) It is immediate by (i) and Theorem I.55.

□

Proposition III.17. *Let conditions* **(M1)**, **(M2)** *and* **(M4)** *hold. Then the following convergence holds:*

$$\left(X^k, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\right) \xrightarrow{(\mathrm{J}_1(\mathbb{R}^{\ell+1}),\mathbb{L}^2)} \left(X^\infty, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}\right). \tag{III.37}$$

Proof. As we have already pointed out on page 83, we are going to apply Theorem I.159 to the sequence $(X^{k,\mathrm{R}_2,I})_{k\in\overline{\mathbb{N}}}$. By Lemma III.13, this is a sequence of $\mathbb{G}^k-$special semimartingales, for every $k \in \overline{\mathbb{N}}$.

In view of **(M1)**, which states that X^∞ is $\mathbb{G}^\infty-$quasi–left–continuous, and using Proposition I.87, we get that the compensator $\nu^{(X^{\infty,d},\mathbb{G}^\infty)}$ associated to $\mu^{X^{\infty,d}}$ is an atomless random measure. Therefore, the finite variation part of the $\mathbb{G}^\infty-$canonical decomposition of $X^{\infty,\mathrm{R}_2,I}$ is a continuous process. Moreover, by Theorem I.16 and **(M1)**, which states that the filtration $\mathbb{G}^\infty$ is quasi–left–continuous, it suffices to show that the martingale part of $X^{\infty,\mathrm{R}_2,I}$ is uniformly integrable. The latter holds by Corollary III.16.(ii).

Lemma III.14.(ii) yields that condition (i) of Theorem I.159 holds. Lemma III.15.(i) yields that condition (ii) of the aforementioned theorem also holds. Moreover, from Corollary III.11 for $p = 2$, we obtain the convergence

$$\left(X^k, X^{k,\mathrm{R}_2,I}\right) \xrightarrow{(\mathrm{J}_1(\mathbb{R}^{\ell+1}),\mathbb{P})} \left(X^\infty, X^{\infty,\mathrm{R}_2,I}\right). \tag{III.38}$$

The last convergence in conjunction with conditions **(M2)** and **(M4)**, Remark III.1 and Corollary I.125, is equivalent to the convergence $(X^{k,\mathrm{R}_2,I}, \mathbb{G}^k) \xrightarrow{\text{ext}} (X^{\infty,\mathrm{R}_2,I}, \mathbb{G}^\infty)$. Therefore, condition (iii) of Theorem I.159 is also satisfied.

Applying now Theorem I.159 to the sequence $(X^{k,\mathrm{R}_2,I})_{k\in\overline{\mathbb{N}}}$, and keeping in mind the decomposition from Lemma III.13, we obtain the convergence

$$\left(X^{k,\mathrm{R}_2,I}, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\right) \xrightarrow{(\mathrm{J}_1(\mathbb{R}^2),\mathbb{P})} \left(X^{\infty,\mathrm{R}_2,I}, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}\right). \tag{III.39}$$

Using Corollary I.125, we can combine the convergence in (III.38) and (III.39) to obtain

$$\left(X^k, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\right) \xrightarrow{(\mathrm{J}_1(\mathbb{R}^\ell\times\mathbb{R}),\mathbb{P})} \left(X^\infty, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}\right).$$

Observe that, in view of the last three convergence and Corollary I.125, we can actually conclude the convergence

$$\left(X^k, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}, \mathbb{G}^k\right) \xrightarrow{\text{ext}} \left(X^\infty, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, \mathbb{G}^\infty\right), \tag{III.40}$$

which will be useful for later reference.

The last result can be further strengthened to an $\mathbb{L}^2-$convergence in view of the following arguments: Let $\alpha^k = \left(X^k, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\right)$, then by Vitali's Convergence Theorem, see Theorem I.32, the latter is equivalent to showing that $d^2_{\mathrm{J}_1}(\alpha^k, \alpha^\infty)$ is uniformly integrable. Moreover, by the inequality

$$d^2_{\mathrm{J}_1}(\alpha^k, \alpha^\infty) \leq \left(d_{\mathrm{J}_1}(\alpha^k, 0) + d_{\mathrm{J}_1}(0, \alpha^\infty)\right)^2 \leq 2d^2_{\mathrm{J}_1}(\alpha^k, 0) + 2d^2_{\mathrm{J}_1}(0, \alpha^\infty) \leq 2\|\alpha^k\|^2_\infty + 2\|\alpha^\infty\|^2_\infty,$$

and Corollary I.29.(ii),(iv), it suffices to show that $(\|\alpha^k\|_\infty^2)_{n\in\overline{\mathbb{N}}}$ is uniformly integrable.

By Corollary III.16.(i) we know that the sequence $\big([\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^k,\mathbb{G}^k)}]_\infty\big)_{k\in\overline{\mathbb{N}}}$ is uniformly integrable. Therefore, using de La Vallée Poussin–Meyer Criterion, there exists a moderate Young function ϕ such that

$$\sup_{k\in\overline{\mathbb{N}}}\mathbb{E}\Big[\phi\Big(\big[\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^k,\mathbb{G}^k)}\big]_\infty\Big)\Big]<\infty. \tag{III.41}$$

Proposition I.51 yields that the map $\mathbb{R}_+\ni x\overset{\psi}{\longmapsto}\psi(x):=\phi(\frac{1}{2}x^2)$ is again moderate and Young. We can apply now the Burkholder–Davis–Gundy (BDG) inequality Theorem I.101 (in conjunction with Proposition I.53.(v)) to the sequence of martingales $\big(\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^k,\mathbb{G}^k)}\big)_{k\in\overline{\mathbb{N}}}$ using the function ψ, and we obtain that

$$\begin{aligned}\sup_{k\in\overline{\mathbb{N}}}\mathbb{E}\Big[\phi\Big(\frac{1}{2}\sup_{s>0}\big|(\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^k,\mathbb{G}^k)})_s\big|^2\Big)\Big]&=\sup_{k\in\overline{\mathbb{N}}}\mathbb{E}\Big[\psi\Big(\sup_{s>0}\big|(\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^k,\mathbb{G}^k)})_s\big|\Big)\Big]\\ \overset{\text{BDG}}{\leq}C_\psi\sup_{k\in\overline{\mathbb{N}}}\mathbb{E}\Big[\psi\big(\big[\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^k,\mathbb{G}^k)}\big]_\infty^{\frac{1}{2}}\big)\Big]&=C_\psi\sup_{k\in\overline{\mathbb{N}}}\mathbb{E}\Big[\phi\Big(\frac{1}{2}\big[\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^k,\mathbb{G}^k)}\big]_\infty\Big)\Big]\overset{\text{(III.41)}}{<}\infty.\end{aligned} \tag{III.42}$$

Hence the sequence $\big(\sup_{s>0}|\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^k,\mathbb{G}^k)}|^2\big)_{k\in\overline{\mathbb{N}}}$ is uniformly integrable, again from de La Vallée Poussin–Meyer Criterion. Moreover, $\big(\mathrm{Tr}\big[[X^k]_\infty\big]\big)_{k\in\overline{\mathbb{N}}}$ is a uniformly integrable sequence; it was proven in Lemma III.5.(iv). Using analogous arguments and recalling to the above inequality, we can conclude that the sequence $\big(\sup_{s>0}\|X_s^k\|_2^2\big)_{k\in\overline{\mathbb{N}}}$ is also uniformly integrable. Hence, the family

$$\Big(\sup_{s>0}|X_s^k|^2+\sup_{s>0}\big|\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^k,\mathbb{G}^k)}\big|^2\Big)_{k\in\overline{\mathbb{N}}}$$

is uniformly integrable, which allows us to conclude. □

Lemma III.18. *The sequence* $(\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)})_{k\in\overline{\mathbb{N}}}$ *possesses the* P-UT *property, consequently*

$$\big(\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)},[\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}]\big)\xrightarrow{(\mathrm{J}_1(\mathbb{R}^2),\mathbb{P})}\big(\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)},[\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}]\big) \tag{III.43}$$

Proof. We will apply Proposition I.153. To this end, let us verify that the requirements are fulfilled. Firstly recall that $\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}$ is $\mathbb{G}-$martingale, for every $k\in\overline{\mathbb{N}}$. Then, (III.42) yields that the sequence $(\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)})_{k\in\overline{\mathbb{N}}}$ is $\mathbb{L}^2-$bounded. Finally, (III.39) verifies the convergence

$$\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\xrightarrow{(\mathrm{J}_1(\mathbb{R}),\mathbb{P})}\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}.$$

□

III.4.2. The convergence (III.24) **is true.** Now that we made our choice for h, the convergence (III.24) reads

$$\begin{aligned}\big(X^k,\big[Y^k,\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big]-\big\langle Y^k,\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big\rangle\big)&\xrightarrow{(\mathrm{J}_1(\mathbb{R}^\ell\times\mathbb{R}),\mathbb{L}^2\times\mathbb{L}^1)}\\ &\big(X^\infty,\big[Y^k,\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big]-\big\langle Y^k,\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big\rangle\big).\end{aligned} \tag{III.44}$$

We comment once more that (III.44) will be used later to apply Proposition III.7. As one would expect, the strategy is very similar to the previous subsection. In other words, we will apply Theorem I.159 for the sequences $\big(\big[Y^k+\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big]\big)_{k\in\overline{\mathbb{N}}}$ and $\big(\big[Y^k-\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big]\big)_{k\in\overline{\mathbb{N}}}$ and then we will use the polarisation identity to conclude.

Notation III.19. The following notation will be valid only for this subsection.

- $\mathrm{R}^{k,2,I}:=\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}$, for every $k\in\overline{\mathbb{N}}$.
- $S^{k,+}:=\big[Y^k+\mathrm{R}^{k,2,I}\big]\in\mathcal{A}^+(\mathbb{G}^k;\mathbb{R})$, for every $k\in\overline{\mathbb{N}}$.
- $S^{k,-}:=\big[Y^k-\mathrm{R}^{k,2,I}\big]\in\mathcal{A}^+(\mathbb{G}^k;\mathbb{R})$, for every $k\in\overline{\mathbb{N}}$.

The following sequence of lemmata amounts to verifying that the requirements of Theorem I.159 are fulfilled for the sequences $(S^{k,+})_{k\in\overline{\mathbb{N}}}$ and $(S^{k,-})_{k\in\overline{\mathbb{N}}}$. Obviously, we will provide the proofs only for the former sequence, since the arguments will be exactly the same for the latter.

Lemma III.20. *For every $k \in \overline{\mathbb{N}}$ holds $S^{k,+}, S^{k,-} \in \mathcal{S}_{\mathrm{sp}}(\mathbb{G}^k;\mathbb{R})$ with canonical decomposition*

$$S^{k,+} = M^{k,+} + \langle Y^k + \mathrm{R}^{k,2,I}\rangle \text{ and } S^{k,-} = M^{k,-} + \langle Y^k - \mathrm{R}^{k,2,I}\rangle, \tag{III.45}$$

where $M^{k,+}, M^{k,-} \in \mathcal{M}(\mathbb{G}^k;\mathbb{R})$, $\langle Y^k + \mathrm{R}^{k,2,I}\rangle, \langle Y^k - \mathrm{R}^{k,2,I}\rangle \in \mathcal{A}^+_{\mathrm{pred}}(\mathbb{G}^k;\mathbb{R})$ for every $k \in \overline{\mathbb{N}}$.

Proof. By Lemma III.4.(v) and Lemma III.14.(i) we have that $Y^k, \mathrm{R}^{k,2,I} \in \mathcal{H}^2(\mathbb{G}^k;\mathbb{R})$ for every $k \in \overline{\mathbb{N}}$. Therefore, $Y^k + \mathrm{R}^{k,2,I} \in \mathcal{H}^2(\mathbb{G}^k;\mathbb{R})$ as well, for every $k \in \overline{\mathbb{N}}$. By Proposition I.99 we have that $S^{k,+} \in \mathcal{A}^+(\mathbb{G}^k;\mathbb{R})$, which in virtue of Proposition I.98 implies that $S^{k,+} \in \mathcal{S}_{\mathrm{sp}}(\mathbb{G}^k;\mathbb{R})$, for every $k \in \overline{\mathbb{N}}$.

Now, again by Proposition I.99 we obtain the $\mathbb{G}^k-$canonical decomposition

$$S^{k,+} = \left[Y^k + \mathrm{R}^{k,2,I}\right] = M^{k,+} + \langle Y^k + \mathrm{R}^{k,2,I}\rangle,$$

for some uniformly integrable $\mathbb{G}^k-$martingale, where the required properties hold for every $k \in \overline{\mathbb{N}}$. □

Lemma III.21. *The processes $S^{\infty,+}, S^{\infty,-}$ are $\mathbb{G}^\infty-$quasi-left-continuous.*

Proof. Since $Y^\infty, \mathrm{R}^{\infty,2,I} \in \mathcal{H}^2(\mathbb{G}^\infty;\mathbb{R}) \subset \mathcal{M}(\mathbb{G}^\infty;\mathbb{R})$, we obtain by Theorem I.16 that the $\mathbb{R}^2-$valued martingale $(Y^\infty, \mathrm{R}^{\infty,2,I})$ is $\mathbb{G}^\infty-$quasi-left-continuous. Therefore, by Proposition I.19 there exists a sequence $(\tau^m)_{m\in\mathbb{N}}$ of $\mathbb{G}^\infty-$totally inaccessible stopping times which exhausts the jumps of $(Y^\infty, \mathrm{R}^{\infty,2,I})$. On the other hand, for every $\mathbb{G}^\infty-$stopping time τ holds

$$\Delta S^\infty_\tau = \Delta[Y^\infty + \mathrm{R}^{\infty,2,I}]_\tau = \left(\Delta(Y^\infty + \mathrm{R}^{\infty,2,I})_\tau\right)^2 = \left(\Delta Y^\infty_\tau + \Delta \mathrm{R}^{\infty,2,I}_\tau\right)^2.$$

Therefore, the sequence $(\tau^m)_{m\in\mathbb{N}}$ exhausts also the jumps of S^∞ and now we can conclude by using Proposition I.19 again. □

Lemma III.22. *The sequences $([S^{k,+}]^{\frac{1}{2}}_\infty)_{k\in\overline{\mathbb{N}}}$, $([S^{k,-}]^{\frac{1}{2}}_\infty)_{k\in\overline{\mathbb{N}}}$ are uniformly integrable.*

Proof. We will dominate the sequence of positive random variables $([S^{k,+}]_\infty)_{k\in\overline{\mathbb{N}}}$ by a uniformly integrable sequence; recall Corollary I.29.(ii). Observe that, since $S^{k,+} \in \mathcal{A}^+(\mathbb{G}^k;\mathbb{R})$, then by Definition I.96 holds $[S^{k,+}]_\infty = \sum_{s>0}\left(\Delta S^{k,+}_s\right)^2$. Therefore,

$$\begin{aligned}
[S^{k,+}]^{\frac{1}{2}}_\infty &= \Big(\sum_{s>0}\left(\Delta S^{k,+}_s\right)^2\Big)^{\frac{1}{2}} = \Big(\sum_{s>0}\left(\Delta(Y^k + \mathrm{R}^{k,2,I})^2_s\right)^2\Big)^{\frac{1}{2}} = \Big(\sum_{s>0}\left(\Delta Y^k_s + \Delta \mathrm{R}^{k,2,I}_s\right)^4\Big)^{\frac{1}{2}}\text{[8]}\\
&\le \sum_{s>0}\left(\Delta Y^k_s + \Delta \mathrm{R}^{k,2,I}_s\right)^2 \le 2\sum_{s>0}\left(\Delta Y^k_s\right)^2 + 2\sum_{s>0}\left(\Delta \mathrm{R}^{k,2,I}_s\right)^2 \le 2[Y^k]_\infty + 2[\mathrm{R}^{k,2,I}_s].
\end{aligned}$$

By Lemma III.4.(v) we have that $([Y^k]_\infty)_{k\in\overline{\mathbb{N}}}$ is uniformly integrable and by Lemma III.14.(i) we recall the uniform integrability of $([\mathrm{R}^{k,2,I}]_\infty)_{k\in\overline{\mathbb{N}}}$. Now, we use (i),(iv) and (ii) of Corollary I.29 to conclude the desired uniform integrability. □

Lemma III.23. *The sequences $(S^{k,+}_\infty)_{k\in\overline{\mathbb{N}}}$, $(S^{k,-}_\infty)_{k\in\overline{\mathbb{N}}}$ are uniformly integrable. As a particular consequence, the sequences $\left(\langle Y^k + \mathrm{R}^{k,2,I}\rangle_\infty\right)_{k\in\overline{\mathbb{N}}}$, $\left(\langle Y^k - \mathrm{R}^{k,2,I}\rangle_\infty\right)_{k\in\overline{\mathbb{N}}}$ are uniformly integrable and tight in $\mathbb{R}$.*

Proof. We start by proving that the sequence $(S^{k,+}_\infty)_{k\in\overline{\mathbb{N}}}$ is uniformly integrable. To this end we use that $S^{k,+}$ is increasing in order to dominate the terminal $S^{k,+}_\infty$ by elements consisting a uniformly integrable family. We have

$$\begin{aligned}
0 \le S^{k,+}_\infty &= [Y^k + \mathrm{R}^{k,2,I}]_\infty = [Y^k]_\infty + 2[Y^k, \mathrm{R}^{k,2,I}]_\infty + [\mathrm{R}^{k,2,I}]_\infty\\
&\le [Y^k]_\infty + 2\mathrm{Var}\big([Y^k, \mathrm{R}^{k,2,I}]\big)_\infty + [\mathrm{R}^{k,2,I}]_\infty\\
&\le [Y^k]_\infty + 2[Y^k]^{\frac{1}{2}}_\infty[\mathrm{R}^{k,2,I}]^{\frac{1}{2}}_\infty + [\mathrm{R}^{k,2,I}]_\infty\\
&\le [Y^k]_\infty + [Y^k]_\infty + [\mathrm{R}^{k,2,I}]_\infty + [\mathrm{R}^{k,2,I}]_\infty\\
&= 2[Y^k]_\infty + 2[\mathrm{R}^{k,2,I}]_\infty.
\end{aligned}$$

In the second inequality we used the Kunita–Watanabe Inequality, see Theorem I.100. In the third inequality we used Young Inequality, see Lemma I.44.(i), applied for the pair of Young functions $(\mathrm{quad}, \mathrm{quad}^\star) =$

[8] We use $\omega-$wise the property $\|\cdot\|_{\vartheta_2} \le \|\cdot\|_{\vartheta_1}$ for $\vartheta_1 \le \vartheta_2$, where the norms are those associated to the classical spaces $\ell^\vartheta(\mathbb{N}) := \{x := (x^k)_{k\in\mathbb{N}} \subset \mathbb{R}, \|x\|_\vartheta := \left(\sum_{k\in\mathbb{N}}|x^k|^\vartheta\right)^{\frac{1}{\vartheta}} < \infty\}$.

(quad, quad), see Example I.46. Now, by Lemma III.4.(v), Lemma III.14.(ii) and Corollary I.29 we conclude.

Before we proceed to prove the uniform integrability of the sequence $\big(\langle Y^k+\mathrm{R}^{k,2,I}\rangle_\infty\big)_{k\in\overline{\mathbb{N}}}$, we denote by Φ the moderate Young function associated to the uniformly integrable family $(S^{k,+}_\infty)_{k\in\overline{\mathbb{N}}}$ by the de La Vallée Poussin–Meyer Criterion, see Corollary I.48. Then, it holds $\sup_{k\in\overline{\mathbb{N}}}\mathbb{E}\big[\Phi(S^{k,+}_\infty)\big]<\infty$. Now, by the canonical decompositions (III.45) and Theorem I.55 we have that

$$\sup_{k\in\overline{\mathbb{N}}}\mathbb{E}\big[\Phi(\langle Y^k+\mathrm{R}^{k,2,I}\rangle_\infty)\big]\leq(2\overline{c}_\Phi)^{\overline{c}_\Phi}\sup_{k\in\overline{\mathbb{N}}}\mathbb{E}\big[\Phi(S^{k,+}_\infty)\big]<\infty.$$

In other words, we have that Φ and $\big(\langle Y^k+\mathrm{R}^{k,2,I}\rangle_\infty\big)_{k\in\overline{\mathbb{N}}}$ satisfy the de La Vallée Poussin–Meyer Criterion. Therefore, $\big(\langle Y^k+\mathrm{R}^{k,2,I}\rangle_\infty\big)_{k\in\overline{\mathbb{N}}}$ is also uniformly integrable. Finally, the sequence $\big(\langle Y^k+\mathrm{R}^{k,2,I}\rangle_\infty\big)_{k\in\overline{\mathbb{N}}}$ is tight in $\mathbb{R}$ as a direct consequence of Markov's Inequality and the $\mathbb{L}^1-$boundedness of $\big(\langle Y^k+\mathrm{R}^{k,2,I}\rangle_\infty\big)_{k\in\overline{\mathbb{N}}}$; by Theorem I.25. □

Lemma III.24. *The following convergence are valid*

$$(X^k,S^{k,+},\mathbb{G}^k)\xrightarrow{\text{ext}}(X^\infty,S^{\infty,+},\mathbb{G}^\infty)\ \textit{and}\ (X^k,S^{k,-},\mathbb{G}^k)\xrightarrow{\text{ext}}(X^\infty,S^{\infty,-},\mathbb{G}^\infty)$$

Proof. We need to prove initially the joint convergence of $(Y^k,\mathrm{R}^{k,2,I})_{k\in\overline{\mathbb{N}}}$. In view of the already proven convergence (we provide their associated numbering)

$$\big(X^k,Y^k,[Y^k,X^k],\langle Y^k,X^k\rangle\big)\xrightarrow{\big(\mathrm{J}_1(\mathbb{R}^\ell\times\mathbb{R}\times\mathbb{R}^\ell\times\mathbb{R}^\ell),\mathbb{P}\big)}\big(X^\infty,Y^\infty,[Y^\infty,X^\infty],\langle Y^\infty,X^\infty\rangle\big)\tag{III.6}$$

and

$$\big(X^k,\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)},\mathbb{G}^k\big)\xrightarrow{\text{ext}}\big(X^\infty,\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)},\mathbb{G}^\infty\big),\tag{III.40}$$

we can conclude the convergence

$$\big(X^k,Y^k,\mathrm{R}^{k,2,I},\mathbb{G}^k\big)\xrightarrow{\text{ext}}\big(X^\infty,Y^\infty,\mathrm{R}^{\infty,2,I},\mathbb{G}^\infty\big).$$

The conclusion was drawn by Corollary I.125, because they share the sequence $(X^k)_{k\in\mathbb{N}}$. The sequences $(Y^k)_{k\in\mathbb{N}}$, $(\mathrm{R}^{k,2,I})_{k\in\mathbb{N}}$ are P-UT by Lemma III.4.(vi), Lemma III.18. Then, by Remark I.150 we have that $(Y^k+\mathrm{R}^{k,2,I})_{k\in\mathbb{N}}$ is also P-UT. On the other hand, by Corollary I.122, Theorem I.152 and the last remark we can conclude that

$$\big(X^k,Y^k,\mathrm{R}^{k,2,I},Y^k+\mathrm{R}^{k,2,I},S^{k,+},\mathbb{G}^k\big)\xrightarrow{\text{ext}}\big(X^\infty,Y^\infty,\mathrm{R}^{\infty,2,I},Y^\infty+\mathrm{R}^{\infty,2,I},S^\infty,\mathbb{G}^\infty\big),\tag{III.46}$$

which implies the required convergence. □

Proposition III.25. *The following convergence is valid*

$$\begin{aligned}\big(X^k,\big[Y^k,\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big]-\big\langle Y^k,\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big\rangle\big)&\xrightarrow{\big(\mathrm{J}_1(\mathbb{R}^\ell\times\mathbb{R}),\mathbb{L}^2\times\mathbb{L}^1\big)}\\&\big(X^\infty,\big[Y^k,\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big]-\big\langle Y^k,\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big\rangle\big).\end{aligned}\tag{III.44}$$

Proof. In view of Lemma III.20 - Lemma III.24 we apply Theorem I.159 for the sequences $(S^{k,+})_{k\in\overline{\mathbb{N}}}$ and $(S^{k,-})_{k\in\overline{\mathbb{N}}}$. Therefore

$$\big(X^k,[Y^k+\mathrm{R}^{k,2,I}],\langle Y^k+\mathrm{R}^{k,2,I}\rangle,\mathbb{G}^k\big)\xrightarrow{\text{ext}}\big(X^\infty,[Y^\infty+\mathrm{R}^{\infty,2,I}],\langle Y^\infty+\mathrm{R}^{\infty,2,I}\rangle,\mathbb{G}^\infty\big)$$

and

$$\big(X^k,[Y^k-\mathrm{R}^{k,2,I}],\langle Y^k-\mathrm{R}^{k,2,I}\rangle,\mathbb{G}^k\big)\xrightarrow{\text{ext}}\big(X^\infty,[Y^\infty-\mathrm{R}^{\infty,2,I}],\langle Y^\infty-\mathrm{R}^{\infty,2,I}\rangle,\mathbb{G}^\infty\big),$$

We are allowed to conclude the joint convergence with $(X^k)_{k\in\overline{\mathbb{N}}}$ because of Lemma III.24. Now, because of the common convergent sequence $(X^k)_{k\in\overline{\mathbb{N}}}$ we can obtain the joint convergence of the two last convergence, *i.e.*

$$\begin{aligned}\big(X^k,[Y^k+\mathrm{R}^{k,2,I}],\langle Y^k+\mathrm{R}^{k,2,I}\rangle,[Y^k-\mathrm{R}^{k,2,I}],\langle Y^k-\mathrm{R}^{k,2,I}\rangle,\mathbb{G}^k\big)&\xrightarrow{\text{ext}}\\&\big(X^\infty,[Y^\infty+\mathrm{R}^{\infty,2,I}],\langle Y^\infty+\mathrm{R}^{\infty,2,I}\rangle,[Y^\infty-\mathrm{R}^{\infty,2,I}],\langle Y^\infty-\mathrm{R}^{\infty,2,I}\rangle\mathbb{G}^\infty\big).\end{aligned}$$

Finally, we conclude the convergence III.44 by Corollary I.122, the polarisation identity and in view of the integrability properties provided by Lemma III.23. □

III.5. The orthogonal martingale-sequence

Now that we have obtained the families of converging martingales by Proposition III.17 and Proposition III.25, we proceed to collect some properties of the sequence $(N^k)_{k\in\mathbb{N}}$, where N^k has been obtained by the orthogonal decomposition of Y^k with respect to $(\mathbb{G}^k, X^{k,c}, \mu^{X^{k,d}})$; see Theorem III.3. It may seem naive that we devote a section only for a pair of lemmata, however, we will do so, in order to categorise the preparatory results in an accessible way. The Subsection III.6.1 will provide some additional information for the arbitrary weak limit of $(N^k)_{k\in\mathbb{N}}$.

Lemma III.26. *Assume the setting of Theorem III.3. Then, for the sequence $(N^k)_{k\in\mathbb{N}}$ the following are true:*

(i) *The sequence $(\langle N^k\rangle)_{k\in\mathbb{N}^k}$ is $C-$tight in $\mathbb{D}(\mathbb{R})$.*
(ii) *The sequence $(N^k)_{k\in\mathbb{N}}$ is tight in $\mathbb{D}(\mathbb{R})$.*
(iii) *The sequence $\big((N^k, \langle N^k\rangle)\big)_{k\in\mathbb{N}}$ is tight in $\mathbb{D}(\mathbb{R}^2)$.*
(iv) *The sequence $(N^k)_{k\in\mathbb{N}}$ is $\mathbb{L}^2-$bounded,* i.e. $\sup_{k\in\mathbb{N}}\mathbb{E}\Big[\sup_{t\in[0,\infty]}|N^k_t|^2\Big] < \infty$.
(v) *The sequence $(\langle N^k\rangle_\infty)_{k\in\mathbb{N}^k}$ is uniformly integrable.*
(vi) *The sequence $([N^k]_\infty)_{k\in\mathbb{N}^k}$ is $\mathbb{L}^1-$bounded,* i.e. $\sup_{k\in\mathbb{N}}\mathbb{E}\big[[N^k]_\infty\big] < \infty$.

Proof. (i) The orthogonal decompositions of Y^k with respect to $(\mathbb{G}^k, X^{k,c}, \mu^{X^{k,d}})$ and Proposition I.165 yield

$$\langle Z^k \cdot X^{k,c} + U^k \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}, N^k\rangle = 0, \text{ for every } k \in \mathbb{N}. \tag{III.47}$$

By (iii)-(iv) of Lemma III.4 we get that the sequence $(\langle Y^k\rangle)_{k\in\mathbb{N}}$ is C$-$tight. Moreover, (III.47) implies that

$$\langle Y^k\rangle = \langle Z^k \cdot X^{k,c} + U^k \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\rangle + \langle N^k\rangle, \text{ for every } k \in \mathbb{N}, \tag{III.48}$$

which in turn yields that the process $\langle Y^k\rangle$ strongly majorizes both $\langle Z^k \cdot X^{k,c} + U^k \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\rangle$ and $\langle N^k\rangle$, for every $k \in \mathbb{N}$; see Definition I.144. We can conclude thus the C$-$tightness of $(\langle N^k\rangle)_{k\in\mathbb{N}}$ by Proposition I.145.

(ii) Using Theorem I.146 to obtain the tightness of $(N^k)_{k\in\mathbb{N}}$, it suffices to show that $(\langle N^k\rangle)_{k\in\mathbb{N}}$ is C$-$tight and $(N^k_0)_{k\in\mathbb{N}}$ is tight. The first statement follows from (i), while for the second one we get that $N^k_0 = 0$, by the definition of the orthogonal decomposition of Y^k with respect to $(\mathbb{G}^k, X^{k,c}, \mu^{X^{k,d}})$, for every $k \in \mathbb{N}$.

(iii) This is immediate in view of Lemma I.143 and (i)-(ii).

(iv) We have by Doob's $\mathbb{L}^2-$inequality that

$$\mathbb{E}\Big[\sup_{t\in[0,\infty]}\big|N^k_t\big|^2\Big] \le 4\mathbb{E}\Big[\big|N^k_\infty\big|^2\Big] = 4\mathbb{E}\Big[\langle N^k\rangle_\infty\Big] \le 4\mathbb{E}\Big[\langle Y^k\rangle_\infty\Big], \tag{III.49}$$

where we used Identity (III.48). By Lemma III.4.(v) we obtain that the sequence $(\langle Y^k\rangle_\infty)_{k\in\mathbb{N}}$ is uniformly integrable and in particular $\mathbb{L}^1-$bounded.

(v) Recall (III.48), which implies that $(\langle Y^k\rangle_\infty)_{k\in\mathbb{N}}$ strongly majorizes $(\langle N^k\rangle_\infty)_{k\in\mathbb{N}}$. Moreover, the former sequence is uniformly integrable, see the comment in the proof of (iv) or Lemma III.4.(v), and by Corollary I.29.(ii) we can conclude.

(vi) The identity $\mathbb{E}\big[[N^k]_\infty\big] = \mathbb{E}\big[\langle N^k\rangle_\infty\big]$ for every $k \in \mathbb{N}$ and the uniform integrability of the sequence $(\langle N^k\rangle_\infty)_{k\in\mathbb{N}}$, which implies its $\mathbb{L}^1-$boundedness, allow us to conclude.

□

Lemma III.27. *Let $(N^{k_l})_{l\in\mathbb{N}}$ be a subsequence of $(N^k)_{k\in\mathbb{N}}$ and assume that there exists a real-valued càdlàg process $\overline{N}$ such that $N^{k_l} \xrightarrow{\mathcal{L}} \overline{N}$. Then, $(N^{k_l})_{l\in\mathbb{N}}$ is* P-UT *and, moreover, $[N^{k_l}] \xrightarrow{\mathcal{L}} [\overline{N}]$.*

Proof. This is immediate by Proposition I.153 and Lemma III.26.(iv). □

III.6. $\overline{\Theta}^{\infty,I}$ is an $\mathbb{F}$–martingale, for $I \in \mathcal{J}(X^\infty)$.

Let us start this section by returning to the discussion made in Section III.2. We remind the reader (the reader may recall the point after the two bullet points), that our ultimate aim is to identify the arbitrary weak limit of $(N^k)_{k\in\mathbb{N}}$, named $\overline{N}$, with N^∞, which is the element of $\mathcal{H}^{2,\perp}(\mathbb{G}^\infty, X^{\infty,c}, \mu^{X^{\infty,d}})$ given by the orthogonal decomposition of Y^∞ with respect to $(\mathbb{G}^\infty, X^{\infty,c}, \mu^{X^{\infty,d}})$.

Let us deviate for a while from the discussion we started, in order to provide some notation which will visualise easier our comments. In view of Lemma III.26.(iii) which verifies the tightness of $(N^k, \langle N^k\rangle)_{k\in\mathbb{N}}$ in $\mathbb{D}(\mathbb{R}^2)$, we will assume throughout the rest of the chapter that the sequence $(N^{k_l}, \langle N^{k_l}\rangle)_{l\in\mathbb{N}}$ is a weakly convergent subsequence extracted by an arbitrarily chosen subsequence of $(N^k, \langle N^k\rangle)_{k\in\mathbb{N}}$. Let us denote by $(\overline{N}, \overline{\Xi})$ the weak-limit of $\big((N^{k_l}, \langle N^{k_l}\rangle)\big)_{l\in\mathbb{N}}$, *i.e.*

$$(N^{k_l}, \langle N^{k_l}\rangle) \xrightarrow[l\to\infty]{\mathcal{L}} (\overline{N}, \overline{\Xi}), \tag{III.50}$$

where by Lemma III.26.(i) we know that $\overline{\Xi}$ is a continuous and increasing process. Having a closer examine of Convergence (III.50), we will realise that there is some connection between $\overline{N}$ and $\overline{\Xi}$. Indeed, observe that by the orthogonal decomposition of Y^k with respect to $(\mathbb{G}^k, X^{k,c}, \mu^{X^{k,d}})$ we have

$$\langle Y^k, N^k\rangle = \langle N^k\rangle \text{ for every } k \in \mathbb{N}. \tag{III.51}$$

Thus, *if* the convergence

$$(Y^{k_l}, N^{k_l}) \xrightarrow{\mathcal{L}} (Y^\infty, \overline{N}) \text{ in } \mathbb{D}(\mathbb{R}^2) \tag{III.52}$$

is true, then the convergence (III.50) translates to

$$[Y^{k_l}, N^{k_l}] - \langle N^{k_l}\rangle \xrightarrow[l\to\infty]{\mathcal{L}} \big[Y^\infty, \overline{N}\,\big] - \overline{\Xi}. \tag{III.53}$$

Indeed, in view of Lemma III.4.(vi) and Lemma III.27, we need only to prove that $(Y^{k_l}, N^{k_l}) \xrightarrow{\mathcal{L}} (Y^\infty, \overline{N})$ in $\mathbb{D}(\mathbb{R}^2)$ so that convergence (III.53) is valid. Having in (III.53) sufficient integrability for the approximating martingale sequence, we will finally obtain one more convergent martingale-sequence. It will be this convergence with which we will identify $\overline{\Xi}$ with $\langle N^\infty\rangle$. But let us postpone this discussion until Subsection III.6.2.

We return now to the discussion we deviated from. We draw the attention of the reader at the point where we know how to translate the sufficient requirements (A)-(ΣT) into the martingale conditions (III.9)-(III.10), which we state again below for the their convenience:

$$\big[\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, \overline{N}\,\big], [X^{\infty,i}, \overline{N}] \in \mathcal{M}(\mathbb{F};\mathbb{R}), \text{ for every } i = 1,\dots,\ell \tag{III.9}$$

and

$$\begin{aligned} &\big[\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, Y^\infty\,\big] - \big\langle \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, Y^\infty\big\rangle^{\mathbb{G}^\infty} \in \mathcal{M}(\mathbb{F};\mathbb{R}) \\ &[X^{\infty,i}, Y^\infty] - \langle X^{\infty,i}, Y^\infty\rangle^{\mathbb{G}^\infty} \in \mathcal{M}(\mathbb{F};\mathbb{R}), \text{ for every } i = 1,\dots,\ell. \end{aligned} \tag{III.10}$$

Pay attention to the fact that we have substituted the integrand $h\mathbb{1}_I$ with $\mathrm{R}_{2,I}$, where $I \in \mathcal{J}(X^\infty)$, *i.e.* the choice we have done in the previous sections, and that $\overline{N}$ now denotes the weak limit of $(N^k)_{k\in\mathbb{N}}$ determined by Convergence (III.50). Observe, now, that we have already constructed an approximating martingale-sequence for the processes appearing in (III.10), see Section III.4, but this is not the case for (III.9). But, *if* the convergence

$$(X^{k_l}, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k_l,d},\mathbb{G}^{k_l})}, N^{k_l}) \xrightarrow{\mathcal{L}} (X^\infty, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, \overline{N}) \text{ in } \mathbb{D}(\mathbb{R}^\ell \times \mathbb{R} \times \mathbb{R}) \tag{III.54}$$

holds, we can obtain (III.9), because all the martingale-sequences are P-UT; see Lemma III.5.(v), Lemma III.18, Lemma III.27 and Theorem I.152. Therefore, we know how to approximate every element in (III.9) - (III.10). The only object that has not been specified is the filtration $\mathbb{F}$. A sufficiently clear idea on the way we will construct $\mathbb{F}$ has been already provided in Section III.2 and we advise the reader to refresh their memory; see comments for (A). For this construction we will use Proposition I.147, so now (following the notation of the aforementioned proposition) we have only to clarify how the martingale-sequence $(\Theta^{k_l})_{l\in\overline{\mathbb{N}}}$ should be chosen. In reality, it is sufficient to collect in Θ^k all the $k-$elements of the convergent $\mathbb{G}^k-$martingale-sequences we will make use of. The following notation describes precisely what we tried to explain with words.

Notation III.28. In the following $I \in \mathcal{J}(X^\infty)$ and $k \in \mathbb{N}$.

- $H^{k,I} := \big(X^k, \langle X^k\rangle^{\mathbb{G}^k}, Y^k, \langle Y^k\rangle^{\mathbb{G}^k}, \langle X^k, Y^k\rangle^{\mathbb{G}^k}, N^k, \langle N^k\rangle,$
$\langle \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\rangle^{\mathbb{G}^k}, \langle \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}, X^k\rangle^{\mathbb{G}^k}, \langle \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}, Y^k\rangle^{\mathbb{G}^k}\big);$
- $\overline{H}^{\infty,I} := \big(X^\infty, \langle X^\infty\rangle^{\mathbb{G}^\infty}, Y^\infty, \langle Y^\infty\rangle^{\mathbb{G}^\infty}, \langle X^\infty, Y^\infty\rangle^{\mathbb{G}^\infty}, \overline{N}, \overline{\Xi},$
$\langle \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}\rangle^{\mathbb{G}^\infty}, \langle \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^k)}, X^\infty\rangle^{\mathbb{G}^\infty}, \langle \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, Y^\infty\rangle^{\mathbb{G}^k}\big);$
- $H^{k,I,G} := (H^{k,I}, \mathbb{E}[\mathbb{1}_G|\mathcal{G}^k]),$ for $G \in \mathcal{G}^\infty_\infty$;
- $\overline{H}^{\infty,I,G} := (H^{\infty,I}, \mathbb{E}[\mathbb{1}_G|\mathcal{G}^\infty]),$ for $G \in \mathcal{G}^\infty_\infty$;
- $\Theta^{k,I} := (X^k, [X^k] - \langle X^k\rangle^{\mathbb{G}^k}, N^k, [X^k, N^k], [\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}, N^k], Y^k, [Y^k, N^k] - \langle N^k\rangle$
$[X^k, Y^k] - \langle X^k, Y^k\rangle^{\mathbb{G}^k}, [\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}, Y^k] - \langle \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}, Y^k\rangle^{\mathbb{G}^k});$
- $\Theta^{k,I,G} := (\Theta^{k,I}, \mathbb{E}[\mathbb{1}_G|\mathcal{G}^k]),$ for $G \in \mathcal{G}^\infty_\infty$;
- $\overline{\Theta}^{\infty,I} := (X^\infty, [X^\infty] - \langle X^\infty\rangle^{\mathbb{G}^\infty}, \overline{N}, [X^\infty, \overline{N}], [\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, \overline{N}], Y^\infty,$
$[X^\infty, Y^\infty] - \langle X^\infty, Y^\infty\rangle^{\mathbb{G}^\infty}, [\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, Y^\infty] - \langle \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, Y^\infty\rangle^{\mathbb{G}^\infty});$
- $\overline{\Theta}^{\infty,I,G} := (\overline{\Theta}^{\infty,I}, \mathbb{E}[\mathbb{1}_G|\mathcal{G}^\infty]),$ for $G \in \mathcal{G}^\infty_\infty$;
- $E_1 := \mathbb{R}^\ell \times \mathbb{R}^{\ell\times\ell} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^\ell \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^\ell \times \mathbb{R};$
- $E_2 := \mathbb{R}^\ell \times \mathbb{R}^{\ell\times\ell} \times \mathbb{R} \times \mathbb{R}^\ell \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^\ell \times \mathbb{R}$

After this discussion, let us summarize what we need to prove so that we can apply Proposition I.147 for the subsequence $(\Theta^{k_l,I,G})_{l\in\mathbb{N}}$:

□ The family $\big\{\|\Theta^{k_l,I,G}_t\|_1\big\}$ is uniformly integrable for every $I \in \mathcal{J}(X^\infty)$ and for every $G \in \mathcal{G}^\infty_\infty$, which will be proven in Lemma III.30.

□ The Convergence (III.52) and (III.54) are valid for every $I \in \mathcal{J}(X^\infty)$. The two aforementioned convergence can be combined to

$$(X^{k_l}, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k_l,d},\mathbb{G}^{k_l})}, Y^{k_l}, N^{k_l}) \xrightarrow{\mathcal{L}} (X^\infty, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^\infty,\mathbb{G}^\infty)}, Y^\infty, \overline{N}) \text{ in } \mathbb{D}(\mathbb{R}^\ell \times \mathbb{R}^3), \tag{III.55}$$

for every $I \in \mathcal{J}(X^\infty)$. This is the purpose of Lemma III.31, which provides the tightness of $(H^{k,I,G}, \Theta^{k,I,G})_{k\in\mathbb{N}}$[9] in $\mathbb{D}(E \times \mathbb{R})$, for every $I \in \mathcal{J}(X^\infty)$ and for every $G \in \mathcal{G}^\infty_\infty$. Observe that on the validity of (III.50), the tightness of $(\Theta^{k,I,G})_{k\in\mathbb{N}}$ implies the validity of (III.55).

Definition III.29. For the pair $(\overline{N}, \overline{\Xi})$ as determined by Convergence (III.50), define $\mathbb{F} := \mathbb{G}^\infty \vee \mathbb{F}^{(\overline{N},\overline{\Xi})}$, *i.e.*

$$\mathcal{F}_t := \sigma\big(\mathcal{G}_t \cup \big\{\overline{N}_s, s \le t\big\} \cup \big\{\overline{\Xi}_s, s \le t\big\}\big).$$

Observe that the filtration $\mathbb{F}$ depends on $\overline{N}$ and $\overline{\Xi}$, but we notationally suppressed this dependence. Moreover, it does not necessarily hold that $\mathbb{F}$ is right–continuous. However, in view of [27, Theorem VI.3, p. 69] and the càdlàg property of Θ^∞, we can conclude also that Θ^∞ is an $\mathbb{F}_+$−martingale, where $\mathbb{F}_+ := (\mathcal{F}_{t+})_{t\in\mathbb{R}_+}$ and $\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$.

Lemma III.30. *The family of random variables* $\big(\big\|\Theta^{k,I,G}_t\big\|_1\big)_{k\in\mathbb{N},t\in\mathbb{R}_+}$ *is uniformly integrable for every* $I \in \mathcal{J}(X^\infty)$ *and* $G \in \mathcal{G}^\infty_\infty$.

Proof. Let $I \in \mathcal{J}(X^\infty)$ and $G \in \mathcal{G}^\infty_\infty$ be fixed. Recall that

$$\begin{aligned}\big\|\Theta^{k,I,G}_t\big\|_1 := {} & \big\|X^k_t\big\|_1 + \big\|[X^k]_t - \langle X^k\rangle_t\big\|_1 + \big|N^k_t\big| + \big\|[X^k, N^k]_t\|_1 \\ & + \big\|[N^k, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^\infty)}]_t\big\| + \big|Y^k_t\big| + \big|[Y^k, N^k]_t - \langle N^k\rangle_t\big| \\ & + \big\|[X^k, Y^k]_t - \langle X^k, Y^k\rangle_t\big\|_1 \\ & + \big|[\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}, Y^k]_t - \langle \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}, Y^k\rangle^{\mathbb{G}^k}_t| + \big|\mathbb{E}[\mathbb{1}_G|\mathcal{G}^k_t]\big|\end{aligned} \tag{III.56}$$

We are going to prove that for each summand of $\|M^{k,I,G}_t\|_1$ the associated family indexed by $\{k \in \mathbb{N}, t \in [0,\infty]\}$ is uniformly integrable. Then, we can conclude also for their sum $(\|M^{k,I,G}_t\|_1)_{k\in\mathbb{N},t\in[0,\infty]}$ by Corollary I.29.(iv).

Let us first recall that the following families are uniformly integrable

[9] With the exact same arguments we can prove that $(H^{k,I,G})_{k\in\overline{\mathbb{N}}}$ is tight in $\mathbb{D}(E\times\mathbb{R})$. For our convenience, we preferred to state Lemma III.31 for the martingale sequence.

- $\big(\mathrm{Tr}\big[[X^k]_\infty\big]\big)_{k\in\overline{\mathbb{N}}}$, $\big(\mathrm{Tr}\big[\langle X^k\rangle_\infty\big]\big)_{k\in\overline{\mathbb{N}}}$
- $\big([Y^k]_\infty\big)_{k\in\overline{\mathbb{N}}}$, $\big(\langle Y^k\rangle_\infty\big)_{k\in\overline{\mathbb{N}}}$,
- $\big(\big[\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big]_\infty\big)_{k\in\overline{\mathbb{N}}}$, $\big(\big\langle\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big\rangle_\infty\big)_{k\in\overline{\mathbb{N}}}$ and
- $(\langle N^k\rangle_\infty)_{k\in\mathbb{N}}$,

see Lemma III.5.(iv), Lemma III.4.(v), Corollary III.16 and Lemma III.26.(v), while by Lemma III.26.(iv) we obtain that

- $([N^k]_\infty)_{k\in\mathbb{N}}$ is $\mathbb{L}^1-$bounded.

Now, in view of the following relationships

$$\sup_{k\in\mathbb{N},t\in\mathbb{R}_+}\mathbb{E}\big[|Y^k_t|^2\big]\leq\sup_{k\in\mathbb{N},t\in\mathbb{R}_+}\mathbb{E}\big[[Y^k]_\infty\big]<\infty,\quad\sup_{k\in\mathbb{N},t\in\mathbb{R}_+}\mathbb{E}\big[|N^k_t|^2\big]\leq\sup_{k\in\mathbb{N},t\in\mathbb{R}_+}\mathbb{E}\big[[N^k]_\infty\big]<\infty$$

and

$$\sup_{k\in\mathbb{N},t\in\mathbb{R}_+}\mathbb{E}\big[|X^{k,i}_t|^2\big]\leq\sup_{k\in\mathbb{N},t\in\mathbb{R}_+}\mathbb{E}\big[[X^k]^{ii}_\infty\big]\leq\sup_{k\in\mathbb{N},t\in\mathbb{R}_+}\mathbb{E}\Big[\mathrm{Tr}\big[[X^k]_\infty\big]\Big]<\infty\text{ for every }i=1,\dots,\ell,$$

we conclude by the de La Vallée Poussin Theorem, see Theorem I.27, the uniform integrability of the families $\big\{\|X^k_t\|_1, l\in\mathbb{N}\text{ and }t\in\mathbb{R}_+\big\}$, $\big\{|Y^k_t|, l\in\mathbb{N}\text{ and }t\in\mathbb{R}_+\big\}$ and $\big\{|Y^k_t|, l\in\mathbb{N}\text{ and }t\in\mathbb{R}_+\big\}$.

The family $\{\mathbb{E}[\mathbb{1}_G|\mathcal{G}^k], k\in\mathbb{N}\text{ and }t\in\mathbb{R}_+\}$ is uniformly integrable by Corollary I.29.(iii). Finally, for the rest of the summands we apply either Lemma A.2 or Lemma A.3 of Appendix A.2, in order to obtain the uniform integrability of the families

- $\big(\|\mathrm{Var}\big([X^k,N^k]\big)_\infty\|_1\big)_{k\in\mathbb{N}}$, $\big(\|\mathrm{Var}\big([X^k,Y^k]\big)_\infty\|_1\big)_{k\in\mathbb{N}}$, $\big(\|\mathrm{Var}\big(\langle X^k,Y^k\rangle\big)_\infty\|_1\big)_{k\in\mathbb{N}}$,
- $\big(\mathrm{Var}\big([\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)},N^k]\big)_\infty\big)_{k\in\mathbb{N}}$, $\big(\mathrm{Var}\big(\langle\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)},N^k\rangle\big)_\infty\big)_{k\in\mathbb{N}}$ and
- $\big(\mathrm{Var}\big([Y^k,N^k]\big)_\infty\big)_{k\in\mathbb{N}}$,

This proves our claim that each summand is uniformly integrable. □

Lemma III.31. *The sequence* $\big(H^{k,I,G},\Theta^{k,I,G}\big)_{k\in\mathbb{N}}$ *is tight in* $\mathbb{D}(E_1\times\mathbb{R}\times E_2\times\mathbb{R})$ *for every* $I\in\mathcal{J}(X^\infty)$ *and for every* $G\in\mathcal{G}^\infty_\infty$.

Proof. Let $I\in\mathcal{J}(X^\infty)$ and $G\in\mathcal{G}^\infty_\infty$ be fixed. We claim that the sequence $\Phi^k:=(X^k,Y^k,N^k)$ is tight in $\mathbb{D}(\mathbb{R}^\ell\times\mathbb{R}^2)$, and that the tightness of $(\Phi^k)_{k\in\mathbb{N}}$ is sufficient to show the tightness of $(\Theta^{k,I,G})_{k\in\mathbb{N}}$. For the following we are going to prove for simplicity that $(\Theta^{k,I,G})_{k\in\mathbb{N}}$ is tight in $\mathbb{D}(E_2\times\mathbb{R})$, since (simpler or) analogous arguments prove the tightness of $(H^{k,I,G})_{k\in\mathbb{N}}$ in $\mathbb{D}(E_1\times\mathbb{R})$.

Let us argue initially for our second claim. The space $\mathbb{D}(E\times\mathbb{R})$ is Polish. Hence in this case tightness is equivalent to sequential compactness. Therefore, it suffices to provide a weakly convergent subsequence for every subsequence of $(\Theta^{k,I,G})_{l\in\mathbb{N}}$. To this end, let us consider a subsequence $(\Theta^{k_m,I,G})_{m\in\mathbb{N}}$. Assuming the tightness of $(\Phi^k)_{k\in\mathbb{N}}$, there exists a weakly convergent in $\mathbb{D}(\mathbb{R}^{\ell+2})$ subsequence $(\Phi^{k_{m_n}})_{n\in\mathbb{N}}$, converging to, say, $(X^\infty,Y^\infty,\widetilde{N})$. Moreover, we remind the reader that the following sequences are P-UT

- $(X^k)_{k\in\mathbb{N}}$; by Lemma III.5.(v). Hence also the subsequence $(X^{k_{m_n}})_{n\in\mathbb{N}}$.
- $(Y^k)_{k\in\mathbb{N}}$; by Lemma III.4.(vi). Hence also the subsequence $(Y^{k_{m_n}})_{n\in\mathbb{N}}$.
- $(N^{k_{m_n}})_{n\in\mathbb{N}}$; in view of the $\mathbb{L}^2-$boundedness of the martingale-sequence $(N^k)_{k\in\mathbb{N}}$ and the convergence $N^{k_{m_n}}\xrightarrow{\mathcal{L}}\widetilde{N}$ apply Lemma III.27.
- $(\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)})_{k\in\mathbb{N}}$; see Lemma III.18. Hence also the subsequence $(\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k_{m_n},d},\mathbb{G}^{k_{m_n}})})_{m\in\mathbb{N}}$.

Then, by Theorem I.152, Proposition III.17 and the above

$$\begin{array}{c}(X^{k_{m_n}},Y^{k_{m_n}},N^{k_{m_n}},[X^{k_{m_n}},N^{k_{m_n}}],[Y^{k_{m_n}},N^{k_{m_n}}],[\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^{k_{m_n},d},\mathbb{G}^{k_{m_n}})},N^{k_{m_n}}])\\ \Big\downarrow\mathcal{L}\\ (X^\infty,Y^\infty,N^\infty,[X^\infty,N^\infty],[Y^\infty,N^\infty],[\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^\infty,d},\mathbb{G}^\infty),N^\infty])\end{array}$$

Moreover, recall the following convergence:

$$(X^k,[X^k],\langle X^k\rangle)\xrightarrow{(\mathrm{J}_1(\mathbb{R}^\ell\times\mathbb{R}^{\ell\times\ell}\times\mathbb{R}^{\ell\times\ell}),\mathbb{P})}(X^\infty,[X^\infty],\langle X^\infty\rangle^{\mathbb{G}^\infty}),$$

by Lemma III.5, and

$$\big(X^k,Y^k,[Y^k,X^k],\langle Y^k,X^k\rangle\big)\xrightarrow{\left(\mathrm{J}_1(\mathbb{R}^\ell\times\mathbb{R}\times\mathbb{R}^\ell\times\mathbb{R}^\ell),\mathbb{P}\right)}\big(X^\infty,Y^\infty,[Y^\infty,X^\infty],\langle Y^\infty,X^\infty\rangle\big);\tag{III.6}$$

$$\big(X^k, \big[Y^k, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big] - \big\langle Y^k, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big\rangle\big) \xrightarrow{(\mathrm{J}_1(\mathbb{R}^\ell\times\mathbb{R}),\mathbb{L}^2\times\mathbb{L}^1)} \big(X^\infty, \big[Y^k, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big] - \big\langle Y^k, \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)}\big\rangle\big). \tag{III.44}$$

Hence, by Theorem I.152 and Corollary I.125, (this is the reason we actually needed for all the convergence the joint convergence with $(X^k)_{k\in\mathbb{N}}$) we also get the convergence in $\mathbb{D}(E)$

$$\begin{aligned}&\big(X^{k_{m_n}}, [X^{k_{m_n}}] - \langle X^{k_{m_n}}\rangle, N^{k_{m_n}}, [X^{k_{m_n}}, N^{k_{m_n}}], [\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k_{l_m},d},\mathbb{G}^{k_{m_n}})}, N^{k_{m_n}}], Y^{k_{m_n}},\\ &\qquad [Y^{k_{m_n}}, N^{k_{m_n}}], [X^{k_{m_n}}, Y^{k_{m_n}}], [\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k_{l_m},d},\mathbb{G}^{k_{m_n}})}, Y^{k_{m_n}}] - \langle \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{k_{l_m},d},\mathbb{G}^{k_{m_n}})}, Y^{k_{m_n}}\rangle\big)\\ &\qquad\qquad \Big\downarrow \mathcal{L}\\ &\big(X^\infty, [X^\infty] - \langle X^\infty\rangle, \widetilde{N}, [X^\infty, \widetilde{N}], [\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, \widetilde{N}], Y^\infty,\\ &\qquad [Y^\infty, \widetilde{N}], [X^\infty, Y^\infty], [\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, Y^\infty] - \langle \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, Y^\infty\rangle\big).\end{aligned}$$

In view of the $C-$tightness of $(\langle N^k\rangle)_{k\in\mathbb{N}}$, see Lemma III.26, we can pass to a further subsequence -which we will denote again by $(k_{m_n})_{n\in\mathbb{N}}$- so that

$$\langle N^{k_{m_n}}\rangle \xrightarrow[n\to\infty]{\mathcal{L}} \widetilde{\Xi}.$$

Hence, we can finally obtain a jointly weakly convergent in $\mathbb{D}(E\times\mathbb{R})$

$$\begin{aligned}&\Theta^{k_{m_n},I,G}\\ &\Big\downarrow \mathcal{L}\\ &\big(X^\infty, [X^\infty] - \langle X^\infty\rangle, \widetilde{N}, [X^\infty, \widetilde{N}], [\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, \widetilde{N}], Y^\infty, [Y^\infty, \widetilde{N}] - \widetilde{\Xi},\\ &\qquad [X^\infty, Y^\infty], [\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, Y^\infty] - \langle \mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}, Y^\infty\rangle, \mathbb{E}[\mathbb{1}_G|\mathcal{G}^\infty_\cdot]\big),\end{aligned}$$

where we have used that $(X^k, \mathbb{E}[\mathbb{1}_G|\mathcal{G}^k_\cdot]) \xrightarrow[k\to\infty]{(\mathrm{J}_1(\mathbb{R}^{\ell+1}),\mathbb{P})} (X^\infty, \mathbb{E}[\mathbb{1}_G|\mathcal{G}^\infty_\cdot])$ to conclude.

In order to prove our initial claim that $(\Phi^k)_{k\in\mathbb{N}}$ is tight, we will apply Theorem I.146 to the $\mathbb{L}^2-$bounded sequences $(X^k)_{k\in\mathbb{N}}, (Y^k)_{k\in\mathbb{N}}$, and $(N^k)_{k\in\mathbb{N}}$. The sequences $(X^k_0)_{k\in\mathbb{N}}, (Y^k_0)_{k\in\mathbb{N}}$, and $(N^k_0)_{k\in\mathbb{N}}$ are clearly tight in $\mathbb{R}^\ell$, $\mathbb{R}$ and $\mathbb{R}$ respectively, as $\mathbb{L}^2-$bounded. The sequences $\big(\mathrm{Tr}\big[\langle X^k\rangle\big]\big)_{k\in\mathbb{N}}$, $(\langle Y^k\rangle)_{k\in\mathbb{N}}$ and $(\langle N^k\rangle)_{k\in\mathbb{N}}$ are $C-$tight; see respectively Lemma III.5, Lemma III.4 and Lemma III.26.(i). Finally, the sequence $(\Psi^k)_{k\in\mathbb{N}}$, where

$$\Psi^k := \mathrm{Tr}\big[\langle X^k\rangle\big] + \langle Y^k\rangle + \langle N^k\rangle,$$

is $C-$tight as the sum of $C-$tight sequences; see Lemma I.143. This concludes the proof. □

Remark III.32. In the proof of the previous lemma, in order to prove the tightness of the sequence Φ^k we have used Theorem I.146, which in turn makes use of *Aldous' criterion for tightness*; see [41, Section VI.4a]. This allows us to conclude that every weak–limit point of $(N^k)_{k\in\mathbb{N}}$ is quasi–left–continuous for its natural filtration. Recall that by Condition **(M1)** we have in particular that X^∞ and Y^∞ are quasi–left–continuous for their natural filtrations. To sum up, for the arbitrary weak–limit point $\overline{N}$ of $(N^k)_{k\in\mathbb{N}}$, it holds

$$\mathbb{P}(\Delta X^{\infty,i}_t = 0) = \mathbb{P}(\Delta Y^\infty_t = 0) = \mathbb{P}(\Delta \overline{N}_t = 0) = 1, \text{ for every } t\in\mathbb{R}_+, i = 1,\dots,\ell.$$

Remark III.33. In the rest of the section we will use the pair $(\overline{N}, \overline{\Xi})$ obtained by (III.50). This can be done without loss of generality, since the pair $(\overline{N}, \overline{\Xi})$ was a weak-limit of an arbitrarily chosen subsequence of $\big((N^k, \langle N^k\rangle\big)_{k\in\mathbb{N}}$. At this point we introduce the following convention for the rest of the chapter: whenever the subsequence $(k_l)_{\in\mathbb{N}}$ is used, it will be understood as the sequence of $\mathbb{N}$ which is determined by the convergence (III.50).

Corollary III.34. *For every $I\in\mathcal{J}(X^\infty)$, $G\in\mathcal{G}^\infty_\infty$, we have that $\Theta^{k_l,I,G} \xrightarrow[l\to\infty]{\mathcal{L}} \overline{\Theta}^{\infty,I,G}$ and that the process $\overline{\Theta}^{\infty,I,G}$ is a uniformly integrable $\mathbb{F}^{\overline{H}^{\infty,I,G}}-$martingale.*

Proof. Let us fix an $I\in\mathcal{J}(X^\infty)$ and a $G\in\mathcal{G}^\infty_\infty$. By Lemma III.31, *i.e.* by the tightness in $\mathbb{D}(E\times\mathbb{R})$ of the sequence $(\Theta^{k,I,G})_{k\in\mathbb{N}}$, we obtain that the sequence $(\Theta^{k_l,I,G})_{l\in\mathbb{N}}$ is tight in $\mathbb{D}(E\times\mathbb{R})$. Therefore, it is sufficient to prove the convergence in law of each element of the subsequence $(\Theta^{k_l,I,G})_{l\in\mathbb{N}}$. We will abstain from providing all of them in detail, since we will use exactly the same arguments as in Lemma III.31, but for the weakly convergent subsequence $(\Theta^{k_l})_{l\in\mathbb{N}}$ with weak limit $\overline{\Theta}^{\infty,I,G}$.

In order to show the second statement, we have only to recall Lemma III.30 and that $\Theta^{k_l,I,G}$ is $\mathbb{E}\times\mathbb{R}-$valued $\mathbb{G}^{k_l}-$martingale for every $l \in \mathbb{N}$. Now we can conclude that $\overline{\Theta}^{\infty,I,G}$ is a uniformly integrable martingale with respect to its natural filtration by Proposition I.147. Finally, in order to obtain the martingale property with respect to $\mathbb{F}^{\overline{H}^{\infty,I,G}}$, *i.e.* the usual augmentation of the natural filtration of $\overline{\Theta}^{\infty,I,G}$, we apply [27, Theorem VI.3, p. 69]. □

At this point we remind the reader some properties which we will make use in the coming proposition.

Remark III.35. (i) For every $t \in [0,\infty]$ it holds $\mathcal{F}_t = \mathcal{G}^\infty_t \vee \mathcal{F}^{(\overline{N},\overline{\Xi})}_t$.
(ii) The inclusion $\mathbb{F}^{\overline{H}^{\infty,I,G}} \subset \mathbb{F}$ holds for every fixed $G \in \mathcal{G}^\infty_\infty$. In particular, for every fixed $G \in \mathcal{G}^\infty_\infty$, every $\mathbb{F}^{\overline{H}^{\infty,I,G}}-$predictable process is also $\mathbb{F}-$predictable.

Proposition III.36. *The process $\overline{\Theta}^{\infty,I}$ is a uniformly integrable $\mathbb{F}-$martingale, for every $I \in \mathcal{J}(X^\infty)$.*

Proof. Let us fix an $I \in \mathcal{J}(X^\infty)$. By applying Corollary III.34 for every $G \in \mathcal{G}^\infty_\infty$ we have that $\overline{\Theta}^{\infty,I}$ is a uniformly integrable $\mathbb{F}^{\overline{H}^{\infty,I,G}}-$martingale. Therefore we have only to prove that actually $\overline{\Theta}^{\infty,I}$ is also an $\mathbb{F}-$martingale. By Lemma A.7, it is sufficient to prove that for every $0 \le t < u \le \infty$ the following condition holds

$$\int_\Lambda \mathbb{E}\big[\,\overline{\Theta}^{\infty,I}_u \big| \mathcal{F}_t\big]\,\mathrm{d}\mathbb{P} = \int_\Lambda \overline{\Theta}^{\infty,I}_t\,\mathrm{d}\mathbb{P}, \tag{III.57}$$

for every $\Lambda \in \mathcal{A}_t := \big\{\Gamma\cap\Delta \,|\, \Gamma \in \mathcal{G}^\infty_t, \Delta \in \mathcal{F}^{(\overline{N},\overline{\Xi})}_t\big\}$. The class $\mathcal{A}_t$ is indeed a $\pi-$system[10] with $\sigma(\mathcal{A}_t) = \mathcal{F}_t$.

Let us, therefore, fix $0 \le t < u \le \infty$ and $\Lambda \in \mathcal{A}_t$, where $\Lambda = \Gamma\cap\Delta$, for some $\Gamma \in \mathcal{G}^\infty_t$, $\Delta \in \mathcal{F}^{(\overline{N},\overline{\Xi})}_t$. Observe that in particular $\Lambda \in \mathcal{F}^{\overline{H}^{\infty,I,\Gamma}}_t$. Now we obtain

$$\int_\Lambda \mathbb{E}\big[\,\overline{\Theta}^{\infty,I}_u \big| \mathcal{F}_t\big]\,\mathrm{d}\mathbb{P} = \int_\Lambda \mathbb{E}\Big[\mathbb{E}\big[\,\overline{\Theta}^{\infty,I}_u \big| \mathcal{F}_t\big]\Big|\mathcal{F}^{\overline{H}^{\infty,I,\Gamma}}_t\Big]\,\mathrm{d}\mathbb{P} = \int_\Lambda \mathbb{E}\Big[\overline{\Theta}^{\infty,I}_u \Big|\mathcal{F}^{\overline{H}^{\infty,I,\Gamma}}_t\Big]\,\mathrm{d}\mathbb{P} = \int_\Lambda \overline{\Theta}^{\infty,I}_t\,\mathrm{d}\mathbb{P}, \tag{III.58}$$

where the first equality holds because $\Lambda \in \mathcal{F}^{\overline{H}^{\infty,I,\Gamma}}_t$, therefore we use the definition of the conditional expectation with respect to the $\sigma-$algebra $\mathcal{F}^{\overline{H}^{\infty,I,\Gamma}}_t$. In the second equality we have used the tower property and Remark III.35.(ii), while for the third one we have used that $\overline{\Theta}^{\infty,I}$ is a $\mathbb{F}^{\overline{H}^{\infty,I,\Gamma}}-$martingale; we used Corollary III.34 for $\Gamma \in \mathcal{G}^\infty_t \subset \mathcal{G}^\infty_\infty$. Therefore, we can conclude that $\overline{\Theta}^{\infty,I}$ is a uniformly integrable $\mathbb{F}-$martingale. □

III.6.1. $\overline{N}$ is sufficiently integrable. The uniform integrability of the $\mathbb{F}-$martingale $\overline{N}$ implies neither the integrability of $[\,\overline{N}\,]^{\frac12}_\infty$ nor of $\sup_{t\in[0,\infty)}|\overline{N}_t|$. In Lemma III.39 we will prove that there exists a suitable $\Psi \in \mathcal{YF}_{\mathrm{mod}}$ whose Young conjugate $\Psi^\star$ is also moderate and such that $\Psi(\sup_{t\in[0,\infty)]}|\overline{N}_t|)$ and $\Psi\big([\,\overline{N}\,]^{\frac12}_\infty\big)$ are integrable.[11] This result is crucial in showing that $M_{\mu^{X^\infty}}\big[\Delta\overline{N}\,\big|\widetilde{\mathcal{P}}^{\mathbb{F}}\big]$ is well–defined, *i.e.* $|\Delta\overline{N}|\mu^{X^{\infty,d}} \in \widetilde{\mathcal{A}}_\sigma(\mathbb{F})$; see Corollary III.40 and Proposition III.41.

Since one may wonder about the necessity of such a function Ψ, let us clarify the situation. Let us first recall the notation of Example I.46, where we have defined $\Upsilon_\vartheta(x) := \frac{1}{\vartheta}x^\vartheta$, for every $\vartheta \in [1,\infty]$. Now, in view of the $\mathbb{L}^2-$boundedness of $(N^k)_{k\in\mathbb{N}}$, recall Lemma III.26, we can prove Lemma III.39 for Υ_ϑ, for every $\vartheta \in [1,2)$. If, moreover, $\vartheta \in (1,2)$, then the Young conjugate of Υ_ϑ is $\Upsilon_{\vartheta^\star}$, with $\vartheta \in (2,\infty)$ which is also moderate. However, if we were to prove Corollary III.40, we should have assumed the uniform integrability of the sequence $\big(\mathrm{Tr}\big[[X^\infty]^{\frac{\vartheta^\star}{2}}_\infty\big]\big)_{k\in\mathbb{N}}$. But this condition is equivalent to the convergence

$$X^k_\infty \xrightarrow{\|\cdot\|_{\Upsilon_{\vartheta^\star}}} X^\infty_\infty \text{ for some } \vartheta^\star \in (2,\infty),^{12}$$

which is stronger than the one assumed to **(M2)**. Recall that in **(M2)** we have assumed the convergence of the terminal random variables to hold under $\|\cdot\|_{\mathbb{L}^2(\mathcal{G};\mathbb{R}^\ell)}$.

The choice of the function Ψ which we are going to utilise is given in Definition III.37. However, before we reach at that point, let us comment on the way we chose this function. We have shown in Lemma III.14 that the sequence $\big(\mathrm{Tr}\big[[X^k]_\infty\big]\big)_{k\in\overline{\mathbb{N}}}$ is uniformly integrable. The orthogonal decomposition of Y^∞ with

[10] A $\pi-$system is a non–empty family of sets which is closed under finite intersections.
[11] It is this specific point that we had to obtain Proposition I.51 in Subsection I.2.2.
[12] See Notation I.52. Actually, this is the convergence under the norm of the classical $\mathbb{L}^\vartheta-$space.

respect to $(\mathbb{G}^\infty, X^{\infty,c}, \mu^{X^{\infty,d}})$ implies then that the (finite) family $\{\langle Z^\infty \cdot X^{\infty,c}\rangle_\infty, [U^\infty \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}]_\infty\}$ is uniformly integrable. Therefore, the family

$$\mathcal{U} := \big\{\mathrm{Tr}\big[[X^k]_\infty\big], k \in \overline{\mathbb{N}}\big\} \cup \big\{\langle Z^\infty \cdot X^{\infty,c}\rangle_\infty, [U^\infty \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}]_\infty\big\} \tag{III.59}$$

is uniformly integrable as a finite union of uniformly integrable families of random variables; see Corollary I.29.(iv). By the de La Vallée Poussin–Meyer Criterion, see Corollary I.48, there exists a moderate Young function with associated moderate sequence $\mathcal{A}(\mathcal{U})$ such that

$$\sup_{\Gamma\in\mathcal{U}} \mathbb{E}[\Phi_{\mathcal{A}(\mathcal{U})}(|\Gamma|)] < \infty. \tag{III.60}$$

Since the moderate sequence $\mathcal{A}(\mathcal{U})$ for which (III.60) is satisfied is not unique, we will hereinafter fix one and we will denote it $\mathcal{A}(\mathcal{U})$ as well.

Notation III.37. For the rest of the chapter $\Psi_{\mathcal{U}} := (\Phi_{\mathcal{A}(\mathcal{U})} \circ \mathrm{quad})^\star$, where $\Phi_{\mathcal{A}(\mathcal{U})}$ is the moderate Young function for which (III.60) holds. In other words, $\Psi_{\mathcal{U}}$ is the Young conjugate of $\Phi_{\mathcal{A}} \circ \mathrm{quad}$.

Lemma III.38. *It holds $\Phi_{\mathcal{A}(\mathcal{U})}, \Psi_{\mathcal{U}} \in \mathcal{YF}_{\mathrm{mod}}$, which is equivalent to*

$$(\overline{c}_{\Psi_{\mathcal{U}}})^\star = \underline{c}_{\Phi_{\mathcal{A}(\mathcal{U})}\circ\mathrm{quad}}, (\underline{c}_{\Psi_{\mathcal{U}}})^\star = \overline{c}_{\Phi_{\mathcal{A}(\mathcal{U})}\circ\mathrm{quad}}$$

and all the constants are finite. Moreover, there exists $\Upsilon_{\mathcal{U}} \in \mathcal{YF}$ such that $\Upsilon_{\mathcal{U}} \circ \Psi_{\mathcal{U}} = \mathrm{quad}$.

Proof. See Proposition I.51 and Corollary I.45. □

Lemma III.39. *The weak–limit $\overline{N}$ given by convergence* (III.50) *and the Young function $\Psi_{\mathcal{U}}$ from Notation III.37 satisfy*

$$\mathbb{E}\Big[\Psi_{\mathcal{U}}\Big(\sup_{t\in[0,\infty)} |\overline{N}_t|\Big)\Big] < \infty \quad \textit{and} \quad \mathbb{E}\Big[\Psi_{\mathcal{U}}\Big([\overline{N}]_\infty^{\frac{1}{2}}\Big)\Big] < \infty.$$

Proof. By Convergence (III.50), Corollary III.34, Remark III.32 and Proposition I.140 we have that

$$N_t^{k_l} \xrightarrow[l\to\infty]{\mathcal{L}} \overline{N}_t, \text{ for every } t \in \mathbb{R}_+.$$

The function $\Psi_{\mathcal{U}}$ is continuous and convex, since it admits a representation via a Lebesgue integral with positive and non-decreasing integrand. By the continuity of $\Psi_{\mathcal{U}}$ and the above convergence we can also conclude that

$$\Psi_{\mathcal{U}}(|N_t^{k_l}|) \xrightarrow[l\to\infty]{\mathcal{L}} \Psi_{\mathcal{U}}(|\overline{N}_t|), \text{ for every } t \in \mathbb{R}_+. \tag{III.61}$$

By Lemma III.38, there exists a Young function $\Upsilon_{\mathcal{U}}$ such that

$$\sup_{l\in\mathbb{N}} \mathbb{E}\Big[\Upsilon_{\mathcal{U}}\Big(\Psi_{\mathcal{U}}\big(\sup_{t\in[0,\infty)} |N_t^{k_l}|\big)\Big)\Big] = \sup_{l\in\mathbb{N}} \mathbb{E}\Big[\sup_{t\in[0,\infty)} |N_t^{k_l}|^2\Big] \overset{\text{Lem. III.26.(iv)}}{<} \infty. \tag{III.62}$$

By the above equality and de La Vallée Poussin Theorem, see Theorem I.27, we obtain the uniform integrability of the family $\big(\Psi_{\mathcal{U}}(|N_t^{k_l}|)\big)_{l\in\mathbb{N},t\in[0,\infty)}$. On the other hand, convergence (III.61) and the Dunford–Pettis Compactness Criterion, see Theorem I.26, yield that the set

$$\mathcal{Q} := \big\{\Psi_{\mathcal{U}}(|N_t^{k_l}|), t \in [0,\infty), l \in \mathbb{N}\big\} \cup \big\{\Psi_{\mathcal{U}}(|\overline{N}_t|), t \in [0,\infty)\big\}$$

is uniformly integrable, since we augment the relatively weakly compact set $\big(\Psi_{\mathcal{U}}(|N_t^{k_l}|)\big)_{t\in[0,\infty),l\in\mathbb{N}}$ merely by aggregating it with the weak–limits $\Psi_{\mathcal{U}}(|\overline{N}_t|)$, for $t \in [0,\infty)$. In particular, the subset $\overline{\mathcal{N}}_{\Psi_{\mathcal{U}}} := \big(\Psi_{\mathcal{U}}(|\overline{N}_t|)\big)_{t\in[0,\infty)}$ is uniformly integrable and in particular $\mathbb{L}^1-$bounded; see Theorem I.25. Now, the $\mathbb{L}^1-$boundedness of $\overline{\mathcal{N}}_{\Psi_{\mathcal{U}}}$ and the fact that $\overline{N}$ is a uniformly integrable $\mathbb{F}-$martingale, see Proposition III.36, imply that the random variable $\overline{N}_\infty := \lim_{t\uparrow\infty} \overline{N}_t$ exists $\mathbb{P}-$almost surely, see Theorem I.57. Using the uniform integrability of $\overline{\mathcal{N}}_{\Psi_{\mathcal{U}}}$ and the continuity of $\Psi_{\mathcal{U}}$ once again, we have that

$$\Psi_{\mathcal{U}}(|\overline{N}_t|) \xrightarrow[t\uparrow\infty]{\mathbb{L}^1(\mathcal{G};\mathbb{R})} \Psi_{\mathcal{U}}(|\overline{N}_\infty|), \tag{III.63}$$

i.e. $\Psi_{\mathcal{U}}(|\overline{N}_\infty|) \in \mathbb{L}^1(\mathcal{G};\mathbb{R})$; see Vitali's Convergence Theorem which is stated as Theorem I.32. Recall now that the function $\Psi_{\mathcal{U}}$ is moderate and convex, see Lemma III.38 and recall Definition I.37. By the integrability of $\Psi_{\mathcal{U}}(|\overline{N}_\infty|)$ we have that $\|\overline{N}_\infty\|_{\Psi_{\mathcal{U}}} < \infty$, where $\|\Theta\|_{\Psi_{\mathcal{U}}} := \inf\big\{\lambda > 0, \mathbb{E}\big[\Psi_{\mathcal{U}}\big(\frac{|\Theta|}{\lambda}\big)\big] \leq 1\big\}$ is the norm of the Orlicz space associated to the Young function $\Psi_{\mathcal{U}}$, see Proposition I.53.

Now we are ready to apply Doob's Inequality in the form of Theorem I.104 since the Young conjugate of $\Psi_{\mathcal{U}}$ is $\Phi_{\mathcal{A}(\mathcal{U})} \circ \mathrm{quad} \in \mathcal{YF}_{\mathrm{mod}}$ with associated constant $(\overline{c}_{\Psi_{\mathcal{U}}})^\star < \infty$; see Lemma III.38. The above yield

$$\Big\| \sup_{t\in\mathbb{R}_+} |\overline{N}_t| \Big\|_{\Psi_{\mathcal{U}}} \leq (\underline{c}_{\Psi_{\mathcal{U}}})^\star \big\|\overline{N}_\infty\big\|_{\Psi_{\mathcal{U}}} < \infty. \tag{III.64}$$

Inequality (III.64) yields therefore the finiteness of $\| \sup_{t\in[0,\infty)} |\overline{N}_t| \|_{\Psi_{\mathcal{U}}}$, which in conjunction with the fact that $\Psi_{\mathcal{U}}$ is moderate, *i.e.* $\overline{c}_{\Psi_{\mathcal{U}}} < \infty$, delivers also the finiteness of $\mathbb{E}[\Psi_{\mathcal{U}}(\sup_{t\in[0,\infty)} |\overline{N}_t|)]$; see Proposition I.53.(v).

Finally, we make use of the finiteness of $\mathbb{E}\big[\Psi_{\mathcal{U}}(\sup_{t\in[0,\infty)} |\overline{N}_t|)\big]$, of the Burkholder–Davis–Gundy Inequality, Theorem I.101, and of the fact that $\Psi_{\mathcal{U}} \in \mathcal{YF}_{\mathrm{mod}}$ to conclude that

$$\mathbb{E}\big[\Psi_{\mathcal{U}}\big([\,\overline{N}\,]_\infty^{\frac12}\big)\big] < \infty. \qquad \square$$

At this point, in order not to disrupt the flow of results regarding $\overline{N}$, we have to anticipate for a while in the next subsection. In particular we will convey the main message of Corollary III.44 and Lemma III.45, which state that the $\mathbb{F}-$martingale X^∞ is $\mathbb{F}-$quasi-left-continuous and such that $\nu^{(X^{\infty,d},\mathbb{F})}|_{\mathcal{P}^{\mathbb{G}^\infty}} = \nu^{(X^{\infty,d},\mathbb{G}^\infty)}$. Now, we can proceed to the next

Corollary III.40. *The weak–limit $\overline{N}$ in convergence* (III.50) *satisfies*

$$\mathbb{E}\Big[\int_{(0,\infty)\times\mathbb{R}^\ell} |\Delta\overline{N}_s|\, \|x\|_1\ \mu^{X^{\infty,d}}(\mathrm{d}s,\mathrm{d}x)\Big] < \infty.$$

In other words, $|\Delta\overline{N}|\mu^{X^{\infty,d}} \in \widetilde{\mathcal{A}}_\sigma(\mathbb{F})$; recall for this Definition I.75. Moreover, the processes $[Y^\infty,\overline{N}]$, $\langle Z^\infty\cdot X^{\infty,c},\overline{N}^c\rangle$, $[U^\infty\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)},\overline{N}^d]$ and $[X^\infty,\overline{N}]$ are of class (D) *for the filtration $\mathbb{F}$.*

Proof. We prove initially that the required expectation is indeed finite.

$$\begin{aligned}
\mathbb{E}\Big[\int_{(0,\infty)\times\mathbb{R}^\ell} |\Delta\overline{N}_s|\, \|x\|_1\, \mu^{X^{\infty,d}}(\mathrm{d}s,\mathrm{d}x)\Big] &= \sum_{i=1}^\ell \mathbb{E}\Big[\int_{(0,\infty)\times\mathbb{R}^\ell} |\Delta\overline{N}_s|\, |x^i| \mu^{X^{\infty,d}}(\mathrm{d}s,\mathrm{d}x)\Big] \\
&= \sum_{i=1}^\ell \mathbb{E}\Big[\sum_{s>0} |\Delta\overline{N}_s|\, |\Delta X_s^{\infty,d,i}|\Big] \leq \sum_{i=1}^\ell \mathbb{E}\Big[\big(\sum_{s>0}(\Delta\overline{N}_s)^2\big)^{\frac12}\ \big(\sum_{s>0}(\Delta X_s^{\infty,d,i})^2\big)^{\frac12}\Big] \\
&\leq \sum_{i=1}^\ell \mathbb{E}\Big[\,[\,\overline{N}\,]_\infty^{\frac12}\,[X^{\infty,i}]_\infty^{\frac12}\Big] \underset{\text{Not. III.37}}{\overset{\text{Lem. I.44.(i)}}{\leq}} \sum_{i=1}^\ell \mathbb{E}\Big[\Psi_{\mathcal{U}}\big([\,\overline{N}\,]_\infty^{\frac12}\big) + \Phi_{\mathcal{A}(\mathcal{U})}\circ\mathrm{quad}\big([X^{\infty,i}]_\infty^{\frac12}\big)\Big] \\
&\leq \mathbb{E}\Big[\Psi_{\mathcal{U}}\big([\,\overline{N}\,]_\infty^{\frac12}\big) + \Phi_{\mathcal{A}(\mathcal{U})}\big(\mathrm{Tr}\big[[X^\infty]_\infty\big]\big)\Big] \underset{\text{Lemma III.39}}{\overset{\text{(III.60)}}{<}} \infty,
\end{aligned}$$

where in the last inequality we used also the convexity of $\Phi_{\mathcal{A}(\mathcal{U})}$ in order to take out the coefficient $\frac12$, which appears due to the definition of the function quad.

We are going to prove the second statement only for $[Y^\infty,\overline{N}]$. It is sufficient to prove for the associated total variation process that $\mathrm{Var}\big([Y^\infty,\overline{N}\,]\big)_\infty \in \mathbb{L}^1(\Omega,\mathcal{G},\mathbb{P})$. For the following calculation we are going to use Property (III.60) and Lemma III.39, exactly as we did above.

$$\begin{aligned}
\mathrm{Var}\big([Y^\infty,\overline{N}\,]\big)_\infty &= \mathrm{Var}\big([Z^\infty\cdot X^{\infty,c} + U^\infty\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)},\overline{N}\,]\big)_\infty \\
&\leq \mathrm{Var}\big([Z^\infty\cdot X^{\infty,c},\overline{N}\,]\big)_\infty + \mathrm{Var}\big([U^\infty\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)},\overline{N}\,]\big)_\infty \\
&= \mathrm{Var}\big(\langle Z^\infty\cdot X^{\infty,c},\overline{N}^c\rangle\big)_\infty + \mathrm{Var}\big([U^\infty\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)},\overline{N}^d]\big)_\infty \\
&\overset{\text{KW Ineq.}}{\leq} \langle Z^\infty\cdot X^{\infty,c}\rangle_\infty^{\frac12}\langle\overline{N}^c\rangle_\infty^{\frac12} + [U^\infty\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}]_\infty^{\frac12}\big[\overline{N}^d\,\big]_\infty^{\frac12} \\
&\leq \langle Z^\infty\cdot X^{\infty,c}\rangle_\infty^{\frac12}\big[\overline{N}\,\big]_\infty^{\frac12} + [U^\infty\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}]_\infty^{\frac12}\big[\overline{N}\,\big]_\infty^{\frac12} \\
&\overset{\text{Lem. I.44.(i)}}{\leq} \big[\Phi_{\mathcal{A}}\big(\langle Z^\infty\cdot X^{\infty,c}\rangle_\infty\big) + \Phi_{\mathcal{A}}\big(\big[U^\infty\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}\big]_\infty\big)\big]\,\Psi_{\mathcal{U}}\big(\big[\overline{N}\,\big]_\infty^{\frac12}\big) \underset{\text{Lem. III.39}}{\overset{\text{(III.60)}}{<}} \infty.
\end{aligned}$$

$\square$

We conclude this sub–sub–section with the following result.

Proposition III.41. *The weak–limit* $\overline{N}$ *of convergence* (III.50) *satisfies*

$$\langle X^{\infty,c,i}, \overline{N}^c\rangle^{\mathbb{F}} = 0 \quad \textit{for every } i = 1, \dots \ell \quad \textit{and} \quad M_{\mu^{X^{\infty,d}}}[\Delta\overline{N}|\widetilde{\mathcal{P}}^{\mathbb{F}}] = 0. \tag{III.65}$$

Proof. We will apply Corollary III.8 to the pair of $\mathbb{F}-$martingales $(X^\infty, \overline{N})$. The following verify that the the requirements of the aforementioned corollary are indeed satisfied.

(i) By Definition III.29 holds $\mathbb{G}^\infty \subset \mathbb{F}$.

(ii′) By **(M2)** holds $X^\infty \in \mathcal{H}^2(\mathbb{G}^\infty; \mathbb{R}^\ell)$ and by Lemma III.39 we have that $\overline{N} \in \mathcal{M}(\mathbb{F}; \mathbb{R})$. By Proposition III.36 we have that X^∞ is also an $\mathbb{F}-$martingale.

(iii) The function $\mathrm{R}_{2,I}$ satisfies the required properties.

(iv) This is Lemma III.45.

(v′) This is Corollary III.40. Moreover, by Proposition III.36 we have that $[X^{\infty,i}, \overline{N}]$ is an $\mathbb{F}-$martingale for every $i = 1, \dots, \ell$.

(vi′) By Lemma A.6 we have that $\sigma\big(\mathcal{J}(X^\infty)\big) = \mathcal{B}(\mathbb{R}^\ell)$. Moreover, Proposition III.36 verifies that

$$\big[\mathrm{R}_{2,I} \star \widetilde{\mu}^{(X^d,\mathbb{G}^\infty)}, Y^\infty\big] \in \mathcal{M}(\mathbb{F}; \mathbb{R}) \text{ for every } I \in \mathcal{J}(X^\infty).$$

(vii) This is Corollary III.40.

The above verify our claim. □

III.6.2. The orthogonal decomposition of Y^∞ is not affected. In this sub–sub–section we use the notation and framework of Section III.6. Recall that the filtration $\mathbb{F}$ has been defined in (III.29), and that the subsequence $(k_l)_{l\in\mathbb{N}}$ is fixed and such that convergence (III.50) holds.

Lemma III.42. *Let* $X_0 + X^{\infty,c,\mathbb{F}} + \mathrm{Id}_\ell \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{F})}$ *be the canonical representation of* X^∞ *as* $\mathbb{F}-$*martingale and* $X_0 + X^{\infty,c} + \mathrm{Id}_\ell \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}$ *be the canonical representation of* X^∞ *as* $\mathbb{G}^\infty-$*martingale. Then the respective parts are indistinguishable,* i.e.

$$X^{\infty,c} = X^{\infty,c,\mathbb{F}} \quad \textit{and} \quad \mathrm{Id}_\ell \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)} = \mathrm{Id}_\ell \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{F})} \ \textit{up to indistinguishability.}$$

Therefore, we will simply denote the continuous part of the $\mathbb{F}-$martingale X^∞ by $X^{\infty,c}$ and we will use indifferently $\mathrm{Id}_\ell \star \widetilde{\mu}^{(\mathbb{G}^\infty, X^{\infty,c}, \mu^{X^{\infty,d}})}$ and $\mathrm{Id}_\ell \star \widetilde{\mu}^{(\mathbb{F}, X^{\infty,c}, \mu^{X^{\infty,d}})}$ to denote the discontinuous part of the $\mathbb{F}-$martingale X^∞.

Proof. By Proposition III.36 the process X^∞ is an $\mathbb{F}-$martingale whose canonical representation is given by

$$X^\infty = X_0 + X^{\infty,c,\mathbb{F}} + \mathrm{Id}_\ell \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{F})},$$

see [41, Corollary II.2.38]. However X^∞ is $\mathbb{G}^\infty-$adapted, which in conjunction with [34, Theorem 2.2] implies that the process $X^{\infty,c,\mathbb{F}} + \mathrm{Id}_\ell \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{F})}$ is a $\mathbb{G}^\infty-$martingale. On the other hand, the canonical representation of the process X^∞ as a $\mathbb{G}^\infty-$martingale is

$$X^\infty = X_0 + X^{\infty,c} + \mathrm{Id}_\ell \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}.$$

Hence, by Theorem I.6 we can conclude the indistinguishability of the respective parts due to the uniqueness of the decomposition. □

Lemma III.43. *The following holds* $\langle X^\infty\rangle^{\mathbb{G}^\infty} = \langle X^\infty\rangle^{\mathbb{F}}$.

Therefore, we will denote the dual predictable projection of X^∞ simply by $\langle X^\infty\rangle$.

Proof. In Proposition III.36 we showed that the process $[X^\infty] - \langle X^\infty\rangle^{\mathbb{G}^\infty}$ is an $\mathbb{F}-$martingale. On the other hand, $\big(\mathrm{Tr}\big[[X^\infty]_t\big]\big)_{t\in[0,\infty]}$ is uniformly integrable and in particular of class (D) for the filtration $\mathbb{F}$ since the process is increasing. Consequently, by Theorem I.64, there exists a unique $\mathbb{F}-$predictable process, say $\langle X^\infty\rangle^{\mathbb{F}}$, such that $[X^\infty] - \langle X^\infty\rangle^{\mathbb{F}}$ is a uniformly integrable $\mathbb{F}-$martingale. Recall that by definition $\mathbb{G}^\infty \subset \mathbb{F}$, therefore $\mathcal{P}^{\mathbb{G}^\infty} \subset \mathcal{P}^{\mathbb{F}}$, which allows us to conclude that $\langle X^\infty\rangle^{\mathbb{G}^\infty} - \langle X^\infty\rangle^{\mathbb{F}}$ is a uniformly integrable $\mathbb{F}-$predictable $\mathbb{F}-$martingale of finite variation. Therefore, by Corollary I.65 we obtain that $\langle X^\infty\rangle^{\mathbb{G}^\infty} - \langle X^\infty\rangle^{\mathbb{F}} = 0$ up to an evanescent set. □

Corollary III.44. *The process* X^∞ *is* $\mathbb{F}-$*quasi–left–continuous. Therefore,*

$$\nu^{(X^{\infty,d},\mathbb{F})}(\omega; \{t\} \times \mathbb{R}^\ell) = 0 \quad \textit{for every } (\omega, t) \in \Omega \times \mathbb{R}_+. \tag{III.66}$$

PROOF. The $\mathbb{F}-$quasi–left–continuity of X^∞ is immediate by the continuity of $\langle X^\infty\rangle$ and Theorem I.62. By Proposition I.87 we conclude also (III.66). □

Now, we can obtain some useful properties about the predictable quadratic covariation of the continuous and the purely discontinuous martingale part of X^∞.

Lemma III.45. *The following hold* $\langle X^{\infty,c}\rangle^{\mathbb{F}} = \langle X^{\infty,c}\rangle^{\mathbb{G}^\infty}$ *and* $\nu^{(X^{\infty,d},\mathbb{F})}|_{\mathcal{P}^{\mathbb{G}^\infty}} = \nu^{(X^{\infty,d},\mathbb{G}^\infty)}$.

Therefore, we will denote the dual predictable projection of $X^{\infty,c}$ simply by $\langle X^{\infty,c}\rangle$.

PROOF. For the reader's convenience we separate the proof in two parts.

(i) Firstly we prove that $\nu^{(X^{\infty,d},\mathbb{F})}|_{\mathcal{P}^{\mathbb{G}^\infty}} = \nu^{(X^{\infty,d},\mathbb{G}^\infty)}$. Indeed, X^∞ is both $\mathbb{G}^\infty-$ and $\mathbb{F}-$quasi–left–continuous, it holds $\mathbb{P}-$almost surely for every $t \in \mathbb{R}_+$

$$\int_{\mathbb{R}^\ell} x\,\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}(\{t\}\times \mathrm{d}x) \overset{\textbf{(M1)}}{=} \int_{\mathbb{R}^\ell} x\,\mu^{X^{\infty,d}}(\{t\}\times \mathrm{d}x) \overset{\text{(III.66)}}{=} \int_{\mathbb{R}^\ell} x\,\widetilde{\mu}^{(X^{\infty,d},\mathbb{F})}(\{t\}\times \mathrm{d}x). \tag{III.67}$$

Consequently, for every non–negative, $\mathbb{G}^\infty$–predictable function θ, using Theorem I.79, it holds

$$\mathbb{E}\big[\theta * \nu_\infty^{(X^{\infty,d},\mathbb{F})}\big] \overset{\text{(III.67)}}{=} \mathbb{E}\Big[\sum_{s>0}\theta(s,\Delta X^{\infty,d})\mathbb{1}_{[\Delta X^{\infty,d}\neq 0]}\Big] = \mathbb{E}\big[\theta * \nu_\infty^{(X^{\infty,d},\mathbb{G}^\infty)}\big].$$

Therefore we can conclude that $\nu^{(X^{\infty,d},\mathbb{F})}\big|_{\mathcal{P}^{\mathbb{G}^\infty}} = \nu^{(X^{\infty,d},\mathbb{G}^\infty)}$.

(ii) Now we prove that $\langle X^{\infty,c}\rangle^{\mathbb{G}^\infty} = \langle X^{\infty,c}\rangle^{\mathbb{F}}$. We will combine the previous part with Lemma III.43. The map Id_ℓ is both $\mathbb{G}^\infty-$ and $\mathbb{F}-$predictable function as deterministic and continuous. Then

$$\begin{aligned}\langle X^{\infty,c}\rangle^{\mathbb{F}} &= \langle X^\infty\rangle - \langle \mathrm{Id}_\ell \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{F})}\rangle^{\mathbb{F}} \underset{\text{Cor. III.44}}{\overset{\text{Lem. III.42}}{=}} \langle X^\infty\rangle - |\mathrm{Id}_\ell|^2 * \nu^{(X^{\infty,d},\mathbb{F})}\\ &\overset{\text{(i)}}{=} \langle X^\infty\rangle - |\mathrm{Id}_\ell|^2 * \nu^{(X^{\infty,d},\mathbb{G}^\infty)} \overset{\textbf{(M1)}}{=} \langle X^\infty\rangle - \langle \mathrm{Id}_\ell \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}\rangle^{\mathbb{G}^\infty} = \langle X^{\infty,c}\rangle^{\mathbb{G}^\infty}.\end{aligned}$$ □

In view of the previous lemmata, we are able to prove that every $\mathbb{G}^\infty-$stochastic integral with respect to X^∞ is also an $\mathbb{F}-$martingale. The exact statement is provided in the next results.

Lemma III.46. *For every* $Z \in \mathbb{H}^2(\mathbb{G}^\infty, X^{\infty,c};\mathbb{R}^\ell)$ *holds* $Z\cdot X^{\infty,c} \in \mathcal{H}^{2,c}(\mathbb{F};\mathbb{R})$

PROOF. By Lemma III.45 we can easily conclude that $\mathbb{H}^2(\mathbb{G}^\infty, X^{\infty,c};\mathbb{R}^\ell) \subset \mathbb{H}^2(\mathbb{F}, X^{\infty,c};\mathbb{R}^\ell)$. We are going to prove the required property initially for simple integrands. Assume that ρ,σ are $\mathbb{G}^\infty-$stopping times such that $\rho \le \sigma$ $\mathbb{P}-a.s.$ and that ψ is an $\mathbb{R}^\ell-$valued, bounded and $\mathcal{G}_\rho^\infty-$measurable random variable. Then, see Theorem I.67.(i), the $\mathbb{G}^\infty-$stochastic integral $(\psi\mathbb{1}_{]\!]\rho,\sigma]\!]})\cdot X^{\infty,c}$ is defined as

$$\psi\mathbb{1}_{]\!]\rho,\sigma]\!]}\cdot X^{\infty,c} = \sum_{i=1}^{\ell}\int_0^{\cdot}\psi^i\mathbb{1}_{]\!]\rho,\sigma]\!]}\mathrm{d}X^{\infty,c,i}. \tag{III.68}$$

Treating now ψ as an $\mathcal{F}_\rho-$measurable variable, since $\mathbb{G}^\infty \subset \mathbb{F}$, and using that $X^{\infty,c}$ is an $\mathbb{F}-$martingale, Proposition III.36, yields that the representation in (III.68) is also an $\mathbb{F}-$martingale.

Let now $Z \in \mathbb{H}^2(\mathbb{G}^\infty, X^{\infty,c};\mathbb{R}^\ell)$ such that Z is bounded. In this case $Z\cdot X^{\infty,c} = \sum_{i=1}^\ell Z^i\cdot X^{\infty,c,i}$. But, using classical arguments, we can approximate Z^i, $i=1,\dots,\ell$, by a sequence of simple $\mathbb{G}^\infty-$predictable processes, name $(Z^{k,i})_{k\in\mathbb{N}}$, $i=1,\dots,\ell$. On the other hand, using the previous step we obtain that $Z^{k,i}\cdot X^{\infty,c,i} \in \mathcal{H}^{2,c}(\mathbb{F};\mathbb{R})$, as linear combination of elements of $\mathcal{H}^{2,c}(\mathbb{F};\mathbb{R})$. Therefore, $Z^i\cdot X^{\infty,c,i} \in \mathcal{H}^{2,c}(\mathbb{F};\mathbb{R})$, for $i=1,\dots,\ell$, as limit of square-integrable $\mathbb{F}-$martingales. Thus, also $Z\cdot X^{\infty,c} \in \mathcal{H}^{2,c}(\mathbb{F};\mathbb{R})$.

Finally, by Theorem I.67 we can conclude for an arbitrary $Z \in \mathbb{H}^2(\mathbb{G}^\infty, X^{\infty,c};\mathbb{R})$, since the process $Z\cdot X^{\infty,c}$ can be approximated by $\big((Z\mathbb{1}_{[\|Z\|_2\le k]})\cdot X^{\infty,c}\big)_{k\in\mathbb{N}} \subset \mathcal{H}^{2,c}(\mathbb{F};\mathbb{R})$. □

Lemma III.47. *For every* $U \in \mathbb{H}^2(\mathbb{G}^\infty, \mu^{X^{\infty,d}};\mathbb{R})$ *holds* $U\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)} \in \mathcal{H}^{2,d}(\mathbb{F};\mathbb{R})$. *Moreover, the processes* $U\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}$, $U\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{F})}$ *are indistinguishable.*

PROOF. Let $U \in \mathbb{H}^2(\mathbb{G}^\infty, \mu^{X^{\infty,d}};\mathbb{R})$. The inclusion $\mathcal{P}^{\mathbb{G}^\infty} \subset \mathcal{P}^{\mathbb{F}}$ and the equalities

$$\begin{aligned}\mathbb{E}\Big[\sum_{s>0}\Big|\int_{\mathbb{R}^\ell} U(s,x)\widetilde{\mu}^{(\mathbb{F},X^{\infty,c},\mu^{X^{\infty,d}})}(\{s\}\times\mathrm{d}x)\Big|^2\Big] &\overset{\text{Lemma III.42}}{=} \mathbb{E}\Big[\sum_{s>0}\Big|\int_{\mathbb{R}^\ell} U(s,x)\mu^{X^\infty}(\{s\}\times\mathrm{d}x)\Big|^2\Big]\\ &\overset{\textbf{(M1)}}{=} \mathbb{E}\Big[\sum_{s>0}\Big|\int_{\mathbb{R}^\ell} U(s,x)\widetilde{\mu}^{(\mathbb{G}^\infty,X^{\infty,c},\mu^{X^{\infty,d}})}(\{s\}\times\mathrm{d}x)\Big|^2\Big] < \infty\end{aligned}$$

yield that $U \in \mathbb{H}^2(\mu^{X^{\infty,d}},\mathbb{F};\mathbb{R})$, hence the $\mathbb{F}-$martingale $U\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{F})}$ is well-defined.

The crucial property for the following comes from Lemma III.45. More precisely, we exploit the fact that

$$\nu^{(X^{\infty,d},\mathbb{F})}|_{\mathcal{P}^{\mathbb{G}^\infty}} = \nu^{(X^{\infty,d},\mathbb{G}^\infty)}$$

in order to obtain from Theorem I.90 that for $A \in \widetilde{\mathcal{P}}^{\mathbb{G}^\infty}$ the (local) $\mathbb{G}^\infty-$martingales $\mathbb{1}_A \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}$, $\mathbb{1}_A \star \widetilde{\mu}^{X^{(\infty,d},\mathbb{F})}$ are indistinguishable. Therefore, in view of the (local) $\mathbb{F}-$martingale property of the latter process, we can conclude also that $\mathbb{1}_A \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}$ is a (local) $\mathbb{F}-$martingale.

Now, the arguments we are going to use are similar to Lemma III.46. Let us initially fix an $i \in \{1,\dots,\ell\}$ and assume that a U^i is a function of the form

$$U^i(\omega,s,x) := \xi^i(\omega)\mathbb{1}_{(\rho_1^i(\omega),\rho_2^i(\omega)]}(s)\mathbb{1}_{I^i}(x)\pi^i(x), \tag{III.69}$$

where $I^i \in \mathcal{B}(\mathbb{R}^\ell)$, ρ_1^i, ρ_2^i are $\mathbb{G}^\infty-$predictable times and ξ^i is a $\mathcal{G}^\infty_{\rho_1^i}-$measurable and bounded random variable. Recall that π^i denotes the $i-$canonical projection $\mathbb{R}^\ell \ni x \mapsto x^i \in \mathbb{R}$. It is immediate that U^i is a $\mathbb{G}^\infty-$predictable function. Observe that for every $\mathbb{G}^\infty-$stopping time τ

$$\Delta\big(U^i \star \widetilde{\mu}^{X^{(\infty,d},\mathbb{G}^\infty)}\big)_\tau = U(\tau,\Delta X^{\infty,d}_\tau) \overset{\text{(III.69)}}{=} \xi^i\mathbb{1}_{I^i}(\Delta X^{\infty,d}_\tau)\Delta X^{\infty,d,i}_\tau,$$

i.e. we can easily conclude that $U^i \in \mathbb{H}^2(\mathbb{G}^\infty,\mu^{X^{\infty,d}};\mathbb{R})$. Then, we have that $U^i \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}$ and $U^i \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{F})}$ are well-defined and the former lies in $\mathcal{H}^{2,d}(\mathbb{G}^\infty;\mathbb{R})$ while the latter lies in $\mathcal{H}^{2,d}(\mathbb{F};\mathbb{R})$. We will prove now that they are indistinguishable. In view of the above discussion and Proposition I.86, we have that

$$\begin{aligned} U^i \star \widetilde{\mu}^{X^{(\infty,d},\mathbb{G}^\infty)} &= (\xi^i\mathbb{1}_{(\rho_1^i,\rho_i^2]}\pi^i)\cdot\big(\mathbb{1}_{I^i} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}\big) \\ &= (\xi^i\mathbb{1}_{(\rho_1^i,\rho_i^2]}\pi^i)\cdot\big(\mathbb{1}_{I^i} \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{F})}\big) = U^i \star \widetilde{\mu}^{X^{(\infty,d},\mathbb{F})}. \end{aligned} \tag{III.70}$$

The Identity (III.70) allows us to conclude that $U^i \star \widetilde{\mu}^{X^{(\infty,d},\mathbb{G}^\infty)} \in \mathcal{H}^{2,d}(\mathbb{F};\mathbb{R})$.

Now we let i run over $\{1,\dots,\ell\}$ and define

$$U(\omega,s,x) := \sum_{i=1}^{\ell} \xi^i(\omega)\mathbb{1}_{(\rho^i(\omega),\rho^{i+1}(\omega)]}(s)\mathbb{1}_{I^i}(x)\pi^i(x), \tag{III.71}$$

where $I^i \in \mathcal{B}(\mathbb{R}^\ell)$ for every $i = 1,\dots,\ell$, $(\rho^i)_{i=1,\dots,\ell}$ is a finite family of $\mathbb{G}^\infty-$predictable times and ξ^i is a $\mathcal{G}^\infty_{\rho^i}-$measurable and bounded random variable for every $i \in \{1,\dots,\ell\}$, we can conclude by linearity that $U \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)} \in \mathcal{H}^{2,d}(\mathbb{F};\mathbb{R})$.

The case $U \in \mathbb{H}^2(\mathbb{G}^\infty,\mu^{X^{\infty,d}};\mathbb{R})$ and U is bounded can be concluded by utilising the usual approximation arguments. For the general case we have the following. Let $U \in \mathbb{H}^2(\mathbb{G}^\infty,\mu^{X^{\infty,d}};\mathbb{R})$ and observe that the sequence $(U^k)_{k\in\mathbb{N}}$, where $U^k := U\mathbb{1}_{[\frac{1}{k}\le|U|\le k]}$ for $k \in \mathbb{N}$, is such that $U^k \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)} \in \mathcal{H}^{2,d}(\mathbb{F};\mathbb{R})$. Moreover,

$$W^k := [U \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)} - U^k \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}] = \sum_{s>0}\Big(U(s,\Delta X^{\infty,d}_s)\mathbb{1}_{[\frac{1}{k}\le|U|\le k]}\Big)^2 \xrightarrow[k\to\infty]{} 0 \ \mathbb{P}-a.s..$$

Using the fact that $0 \le W^k \le [U \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}]_\infty$, where $[U \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}]_\infty \in \mathbb{L}^1(\mathcal{G};\mathbb{R})$, we have that the sequence $(W^k) - k \in \mathbb{N}$ is uniformly integrable. Thus, we apply Vitali's Theorem to conclude that the above convergence holds in $\mathbb{L}^1$, which in turn implies the $\mathbb{L}^2-$convergence of $\big(U^k \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}_\infty\big)_{k\in\mathbb{N}}$ to $U \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}_\infty$. This allows to conclude that the limit $U \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)} \in \mathcal{H}^{2,d}(\mathbb{F};\mathbb{R})$ as limit of square-integrable $\mathbb{F}-$martingales. □

Proposition III.48. *For the $\mathbb{F}-$martingale Y^∞ holds*

$$\langle X^{\infty,c,i}, Y^{\infty,c}\rangle^{\mathbb{F}} = \langle X^{\infty,c,i}, Y^{\infty,c}\rangle^{\mathbb{G}^\infty} \text{ for every } i = 1,\dots\ell$$

and

$$M_{\mu^{X^{\infty,d}}}[\Delta Y^\infty|\widetilde{\mathcal{P}}^{\mathbb{F}}] = M_{\mu^{X^{\infty,d}}}[\Delta Y^\infty|\widetilde{\mathcal{P}}^{\mathbb{G}^\infty}].$$

Proof. We will apply Proposition III.7 to the pair (X^∞, Y^∞) and the pair of filtrations $(\mathbb{G}^\infty,\mathbb{F})$. We verify that the aforementioned pair indeed satisfies the requirements of Proposition III.7:

(i) By Definition III.29 holds $\mathbb{G}^\infty \subset \mathbb{F}$.

(ii) By **(M2)** holds $X^\infty \in \mathcal{H}^2(\mathbb{G}^\infty;\mathbb{R}^\ell)$ and by construction of Y^∞ we have that $Y^\infty \in \mathcal{H}^2(\mathbb{G}^\infty;\mathbb{R})$. By Proposition III.36 we have that they are also $\mathbb{F}-$martingales.

(iii) The function $\mathrm{R}_{2,I}$ satisfies the required properties.

(iv) This is Lemma III.45.

(v) Since both X^∞, Y^∞ are square-integrable martingales, we apply Kunita–Watanabe Inequality I.24 in order to obtain that $\mathrm{Var}\big([X^{\infty,i},Y^\infty]\big)_\infty \in \mathbb{L}^1(\mathcal{G};\mathbb{R})$ for every $i=1,\dots,\ell$. Therefore, we can conclude that $[X^{\infty,i},Y^\infty]$ is of class (D) for the filtration $\mathbb{F}$, for every $i=1,\dots,\ell$. Moreover, by Proposition III.36 we have that $[X^{\infty,i},Y^\infty]-\langle X^{\infty,i},Y^\infty\rangle^{\mathbb{G}^\infty}$ is an $\mathbb{F}-$martingale for every $i=1,\dots,\ell$.

(vi) By Lemma A.6 we have that $\sigma\big(\mathcal{J}(X^\infty)\big)=\mathcal{B}(\mathbb{R}^\ell)$. Moreover, Proposition III.36 verifies that

$$\big[\mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^d,\mathbb{G}^\infty)},Y^\infty\big]-\big\langle \mathrm{R}_{2,I}\star\widetilde{\mu}^{(X^d,G^\infty)},Y^\infty\big\rangle^{\mathbb{G}^\infty}\in\mathcal{M}(\mathbb{F};\mathbb{R}),\ \text{for every } I\in\mathcal{J}(X^\infty).$$

(vii) Since X^∞, Y^∞ are both square-integrable, we have that $|\Delta Y^\infty|\mu^{X^\infty,d}\in\widetilde{\mathcal{A}}_\sigma(\mathbb{F})$.

In view of the above, the claim of the proposition is true. □

The following proposition justifies the title of the current subsection.

Corollary III.49. *The orthogonal decomposition of Y^∞ with respect to $(\mathbb{F},X^{\infty,c},\mu^{X^{\infty,d}})$ is given by*

$$Y^\infty = Y_0 + Z^\infty\cdot X^{\infty,c}+U^\infty\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{F})}, \tag{III.72}$$

where $Z^\infty\in\mathbb{H}^2(\mathbb{G}^\infty,X^{\infty,c};\mathbb{R})$ and $U^\infty\in\mathbb{H}^2(\mathbb{G}^\infty,\mu^{X^{\infty,d}};\mathbb{R})$ are determined by the orthogonal decomposition of Y^∞ with respect to $(\mathbb{G}^\infty,X^{\infty,c},\mu^{X^{\infty,d}})$; see Theorem III.3. In other words, the orthogonal decomposition of Y^∞ with respect to $(\mathbb{F},X^{\infty,c},\mu^{X^{\infty,d}})$ is indistinguishable from the orthogonal decomposition of Y^∞ with respect to $(\mathbb{G}^\infty,X^{\infty,c},\mu^{X^{\infty,d}})$.

Proof. It is immediate by Lemma III.46, Lemma III.47 and Proposition III.48.

□

III.7. Step 1 is valid

This subsection is devoted to the proof of the main theorem. In view of the preparatory results obtained in the previous sections as well as of the outline of the proof presented in subsection III.2, the following proof basically amounts to proving that $N^\infty=\overline{N}$ up to indistinguishability and $\langle N^\infty\rangle=\overline{\overline{\Xi}}$. We will divide the proof in two main parts. Initially we will prove that the Convergence (III.3) is true, where we will have obtained already the convergence $\langle N^k\rangle\xrightarrow{\mathcal{L}}\langle N^\infty\rangle$. Then we will prove that the Convergence (III.4) and (III.5) are also true.

Proof of Theorem III.3. By Lemma III.26.(iii) we have that the sequence $\big((N^k,\langle N^k\rangle)\big)_{k\in\mathbb{N}}$ is tight in $\mathbb{D}(\mathbb{R}^2)$. Therefore, an arbitrary subsequence $\big((N^{k_l},\langle N^{k_l}\rangle)\big)_{l\in\mathbb{N}}$ has a further subsequence indexed by $(k_{l_m})_{m\in\mathbb{N}}$ which converges in law, say to $(\overline{N},\overline{\Xi})$, *i.e.*

$$(N^{k_{l_m}},\langle N^{k_{l_m}}\rangle)\xrightarrow[m\to\infty]{\mathcal{L}}(\overline{N},\overline{\Xi}), \tag{III.73}$$

where $\overline{N}$ is a càdlàg process and $\overline{\Xi}$ is a continuous and increasing process. The continuity of $\overline{\Xi}$ follows from Lemma III.26.(i). Therefore, we can use the results of subsection III.6 for the subsequence $\big((N^{k_{l_m}},\langle N^{k_{l_m}}\rangle)\big)_{m\in\mathbb{N}}$ and the pair $(\overline{N},\overline{\Xi})$.

By Proposition III.41 and Corollary III.49 we conclude

$$\begin{aligned}\langle Y^\infty,\overline{N}\rangle &= \langle Z^\infty\cdot X^{\infty,c},\overline{N}^c\rangle+\langle U^\infty\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)},\overline{N}^d\rangle\\ &= \langle Z^\infty\cdot X^{\infty,c},\overline{N}^c\rangle+\langle U^\infty\star\widetilde{\mu}^{(X^{\infty,d},\mathbb{F})},\overline{N}^d\rangle\\ &= Z^\infty\cdot\langle X^{\infty,c},\overline{N}^c\rangle+\big(U^\infty M_{\mu^{X^\infty,d}}[\Delta\overline{N}|\widetilde{\mathcal{P}}^{\mathbb{F}}]\big)*\nu^{(X^{\infty,d},\mathbb{F})}\overset{\text{(III.65)}}{=}0,\end{aligned} \tag{III.74}$$

i.e. $[Y^\infty,\overline{N}]$ is an $\mathbb{F}-$martingale. On the other hand, we have proven in Proposition III.36 that $[Y^\infty,\overline{N}]-\overline{\Xi}$ is an $\mathbb{F}-$martingale as well. By subtracting the two martingales we obtain that $\overline{\Xi}$ is also an $\mathbb{F}-$martingale. However, $\overline{\Xi}$ is an $\mathbb{F}-$predictable process of finite variation and a martingale, therefore it has to be constant; see Corollary I.65. Now, we have that

$$\langle N^{k_{l_m}}\rangle\xrightarrow[m\to\infty]{\mathcal{L}}\overline{\Xi}\ \text{ implies that }\ \langle N^{k_{l_m}}\rangle_0\xrightarrow[m\to\infty]{\mathcal{L}}\overline{\Xi}_0.$$

Recall that by definition $\langle N^k\rangle_0 = 0$ for every $k \in \mathbb{N}$, hence $\overline{\Xi}_0 = 0$. Therefore $\overline{\Xi} = 0$ and, since the limit is a deterministic process, the convergence above is equivalent to the following

$$\langle N^{k_{l_m}}\rangle \xrightarrow[m\to\infty]{(\mathrm{J}_1(\mathbb{R}),\mathbb{P})} 0.$$

Since the limit above is common for every subsequence and $(\mathbb{D}, \mathrm{J}_1(\mathbb{R}))$ is Polish, we can conclude from [28, Theorem 9.2.1] that

$$\langle N^k\rangle \xrightarrow{(\mathrm{J}_1(\mathbb{R}),\mathbb{P})} 0.$$

Using Lemma III.26.(v) and [35, Theorem 1.11] we can strengthen the above convergence to

$$\langle N^k\rangle \xrightarrow{(\mathrm{J}_1(\mathbb{R}),\mathbb{L}^1)} 0. \tag{III.75}$$

Then, we can also conclude that $N^k \xrightarrow{(\mathrm{J}_1(\mathbb{R}),\mathbb{L}^2)} 0$. Indeed, for every $R > 0$ and by Doob's $\mathbb{L}^2-$inequality we obtain

$$\mathbb{E}\Big[\sup_{t\in[0,R]} |N^k_t|^2\Big] \leq 4\mathbb{E}[|N^k_R|^2] = 4\mathbb{E}[\langle N^k\rangle_R] \xrightarrow[k\to\infty]{} 0,$$

which implies the convergence $\mathbb{E}[d^2_{\mathrm{lu}}(N^k, 0)] \longrightarrow 0$.

Using the convergence of $Y^k \xrightarrow{(\mathrm{J}_1(\mathbb{R}),\mathbb{L}^2)} Y^\infty$ and the convergence of $(N^k)_{k\in\mathbb{N}}$ to the zero process, which is trivially continuous, we can obtain the joint convergence

$$(Y^k, N^k) \xrightarrow{(\mathrm{J}_1(\mathbb{R}^2),\mathbb{L}^2)} (Y^\infty, 0).$$

Moreover, using the orthogonal decompositions of Y^k and Y^∞ and the previous results, we obtain

$$Z^k \cdot X^{k,c} + U^k \star \widetilde{\mu}^{(X^{k,d},\mathbb{G}^k)} = Y^k - N^k - Y^k_0 \xrightarrow{(\mathrm{J}_1(\mathbb{R}),\mathbb{L}^2)} Y^\infty - Y^\infty_0 = Z^\infty \cdot X^{\infty,c} + U^\infty \star \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)},$$

which yields then (III.3).

Thus, it is only left to prove convergence (III.5). Since the sequences $(X^k)_{k\in\overline{\mathbb{N}}}$ and $(Y^k)_{k\in\overline{\mathbb{N}}}$ satisfy the conditions of Theorem I.160 we obtain in particular that the sequences $(Y^k + X^k)_{k\in\overline{\mathbb{N}}}$ and $(Y^k - X^k)_{k\in\overline{\mathbb{N}}}$ also satisfy the conditions of this theorem. Therefore we can conclude that

$$\langle Y^k + X^k\rangle \xrightarrow{(\mathrm{J}_1(\mathbb{R}),\mathbb{P})} \langle Y^\infty + X^\infty\rangle \quad\text{and}\quad \langle Y^k - X^k\rangle \xrightarrow{(\mathrm{J}_1(\mathbb{R}),\mathbb{P})} \langle Y^\infty - X^\infty\rangle.$$

By the continuity of the limiting processes, recall **(M1)** and [41, Theorem 4.2], using the identity

$$\langle Y^k, X^k\rangle = \frac{1}{4}\big(\langle Y^k + X^k\rangle - \langle Y^k - X^k\rangle\big), \text{ for every } k \in \overline{\mathbb{N}},$$

and convergence (III.75), we can conclude that

$$(\langle Y^k\rangle, \langle Y^k, X^k\rangle, \langle N^k\rangle) \xrightarrow{\left(\mathrm{J}_1(\mathbb{R}\times\mathbb{R}^\ell\times\mathbb{R}),\mathbb{L}^1\right)} (\langle Y^\infty\rangle, \langle Y^\infty, X^\infty\rangle, 0).$$

In order to strengthen the last convergence to an $\mathbb{L}^1-$convergence, we only need to recall that by Theorem I.160 the sequences $\big(\mathrm{Tr}\big[\langle X^k\rangle_\infty\big]\big)_{k\in\overline{\mathbb{N}}}$ and $\big(\langle Y^k\rangle_\infty\big)_{k\in\overline{\mathbb{N}}}$ are uniformly integrable. Now Lemma A.2 provides the uniform integrability of $\big(\big\|\mathrm{Var}(\langle Y^k, X^k\rangle)_\infty\big\|_1\big)_{k\in\overline{\mathbb{N}}}$, which allows us to conclude.

In view of the convergence of $(N^k)_{k\in\mathbb{N}}$ to the zero process, we can also obtain the associated convergence for the terminal random variables. □

III.8. Corollaries of the main theorem

In this section we provide two corollaries of Theorem III.3. The first one, which generalises Madan, Pistorius, and Stadje [50, Corollary 2.6] will be the useful for obtaining the stability property of BSDEs, which will be stated as Theorem IV.10. The second result deals with the special case that the sequence of filtrations are generated by processes with independent increments. It is this property that provides us the flexibility to work on possibly different probability spaces and, thus, obtain weak approximations.

Theorem III.50. *Assume the framework of Theorem III.3, where condition* **(M2)** *is substituted by the following:*

(M2′) *For every $k \in \mathbb{N}$,*[13] *we define $\overline{X}^k : (X^{k,\circ}, X^{k\natural}) \in \mathcal{H}^2(\mathbb{G}^k;\mathbb{R}^\ell) \times \mathcal{H}^{2,d}(\mathbb{G}^k;\mathbb{R}^\ell)$ such that*

[13] Observe that X^∞ is assumed as in Theorem III.3.

(i) $M_{\mu^{X^{k,\natural}}}\left[\Delta X^{k,\circ}\middle|\widetilde{\mathcal{P}}^{\mathbb{G}^k}\right] = 0$ *and*

(ii) $(X^{k,\circ}_\infty, X^{k,\natural}_\infty) \xrightarrow[k\to\infty]{\mathbb{L}^2(\mathcal{G};\mathbb{R}^\ell)} (X^{\infty,c}_\infty, X^{k,\natural}_\infty)$.

The convergence of Theorem III.3 hold and additionally

$$\big(\langle Y^k, X^{k,\circ}\rangle, \langle Y^k, X^{k,\natural}\rangle\big) \xrightarrow{(\mathrm{J}_1(\mathbb{R}^2\times\mathbb{R}^\ell\times\mathbb{R}^\ell),\mathbb{L}^1)} \big(\langle Y^\infty, X^{\infty,c}\rangle, \langle Y^\infty, X^{\infty,d}\rangle\big).$$

Remark III.51. The Condition **(M2′)** is clearly stronger than **(M2)**. So, as one may expect, we have some better properties comparing to the framework of Theorem III.3. Let us comment briefly on two immediate effects.

(i) Unlike Condition **(M2)**, under **(M2′)** we can approximate the continuous part of $X^{\infty,c}$ with a convergent sequence of martingales which may have purely discontinuous parts. This property turns out to be crucial for numerical schemes, where one needs to approximate the continuous part with discrete-time martingales, *i.e.* purely discontinuous martingales.

(ii) The separate convergence of the sequence of terminal random variables $(X^{k,\circ}_\infty, X^{k,\natural}_\infty)$ allows to apply Theorem III.3 for each sequence separately. This property allows in particular for the following convergence

$$\big(\langle Y^k, X^{k,\circ}\rangle, \langle Y^k, X^{k,\natural}\rangle\big) \xrightarrow{(\mathrm{J}_1(\mathbb{R}^2\times\mathbb{R}^\ell\times\mathbb{R}^\ell),\mathbb{L}^1)} \big(\langle Y^\infty, X^{\infty,c}\rangle, \langle Y^\infty, X^{\infty,d}\rangle\big),$$

which will be crucial for obtaining the stability property of BSDEs.

Proof of Theorem III.50. For the following recall Lemma I.164, which verifies that $\langle X^{k,\circ}, X^{k\natural}\rangle = 0$ for every $k \in \mathbb{N}$. Now, we can apply initially the Theorem I.160 for the sequence $(X^{k,\circ})_{k\in\mathbb{N}}$ in order to prove the convergence

$$\langle X^{k,\circ}\rangle \xrightarrow{(\mathrm{J}_1(\mathbb{R}^\ell),\mathbb{P})} \langle X^{\infty,c}\rangle \tag{III.76}$$

and analogously for

$$\langle X^{k,\natural}\rangle \xrightarrow{(\mathrm{J}_1(\mathbb{R}^\ell),\mathbb{P})} \langle X^{\infty,d}\rangle. \tag{III.77}$$

Then, using the aforementioned orthogonality, we refine the convergence of to the space of Itô generated by $X^{k,\circ}$ □

III.9. Comparison with the literature

In this section we are going to compare Theorem III.3 with the related literature, which mainly amounts to Jacod, Méléard, and Protter [42, Theorem 3.3] and Briand, Delyon, and Mémin [15, Theorem 5]. In the former, Jacod et al. study the stability of (locally) square-integrable martingale representations in the Galtchouk–Kunita–Watanabe sense. This means, that the sequence $(Y^k)_{k\in\overline{\mathbb{N}}}$ (we use the notation we have introduced for Theorem III.3) may consist of locally square-integrable martingales, while $(X^k)_{k\in\overline{\mathbb{N}}}$ is a sequence of square-integrable martingales, which will serve as the Itô integrator. Using the analogous to our introduction scheme we could sketch [42, Theorem 3.3] as follows

$$\begin{array}{ccccccc} Y^k & = & Y^k_0 & + & Z^k\cdot X^k & + & N^k \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y^\infty & = & Y^\infty_0 & + & Z^\infty\cdot X^\infty & + & N^\infty \end{array}$$

Comparing to our framework, in [42] there is no need for an analogous to **(M3)** condition. On the other hand, in [42] the sequences are adapted to a single filtration. This, in particular, allows for significant simplifications comparing to working in different stochastic bases. For example, one is allowed to calculate the predictable quadratic variation $\langle Y^k, X^m\rangle$ for $k \neq m$, which in general is not well-defined when the filtration depends on the index of the sequence.

Let us discuss now about the work of Briand et al. [15]. The framework here can be regarded as a special case of our framework. Indeed, we retrieve the framework of [15] once we restrict X^∞ to be a Brownian motion, $(X^k)_{k\in\mathbb{N}}$ to be a purely discontinuous martingale that approximates Brownian motion and, finally, assume that $\mathbb{G}^k = \mathbb{F}^{X^k}$.

CHAPTER IV

Stability of Backward Stochastic Differential Equations with Jumps

This is the last chapter of the present dissertation and it deals with the stability property of BSDEs. The term stability will be used in an analogous sense as in Chapter III, *i.e.* if for a given convergent sequence of "inputs" we get a convergent sequence of "outputs", then we will say that the stability holds. In Chapter II has been made clear what the "input" for a BSDE should be, namely a sextuple $\mathcal{SD} := (\mathbb{G}, T, \overline{X}, \xi, C, f)$ that satisfies Conditions **(F1)** - **(F5)** under (some) $\hat{\beta}$. In other words, $\mathcal{SD}$ are standard data under some $\hat{\beta}$. Thus, we will say that "Backward Stochastic Differential Equations are Stable", if for an arbitrary sequence of standard data $\left(\mathcal{SD}^k\right)_{k\in\mathbb{N}}$ which converges to the standard data $\mathcal{SD}^\infty$, with Υ^k being the associated to $\mathcal{SD}^k$ solution of Equation (II.1) for every $k \in \overline{\mathbb{N}}$, the sequence of solutions $(\Upsilon^k)_{k\in\mathbb{N}}$ converges to the solution Υ^∞. Of course, we will have to impose additional structural and integrability conditions so that we can finally obtain the desired property, but we will be more precise later.

The first result that dealt with the stability of BSDEs with Lipschitz generator f is [39]. In this work, the authors perturb the terminal value ξ and the integrator f, while the rest of the data are assumed fixed (in particular the driving martingale is Brownian motion), in order to obtain the required property. In other words, in [39] the approximating sequence consists of continuous semimartingales. This is not satisfactory from the numerical point of view, where we need the approximating sequence to consist of discrete-time semimartingales. To this direction, and restricting our interest in the case of a Lipschitz generator, Bally [5], Briand, Delyon, and Mémin [14], Chevance [18], Henry-Labordère, Tan, and Touzi [36], Ma, Protter, San Martin, and Torres [49], Zhang [65] among others propose numerical schemes for approximating BSDEs driven by Brownian motion. In the case the driving martingale is either a purely discontinuous martingale or a jump-diffusion, the literature is less abundant. We mention the work of Aazizi [1], of Lejay and Torres [46], which deals with the jump-diffusion whose purely discontinuous martingale is a compensated Poisson case, of Bouchard and Elie [11], Bouchard, Elie, and Touzi [12], and of Madan et al. [50], which can be regarded as the most general among the aforementioned works. In our framework we do not distinguish between continuous-time and discrete-time approximations of the driving martingale, which can be a general square-integrable martingale, and, moreover, we allow for filtration perturbation. However, we postpone a more detailed comparison with the related literature for Section IV.6.

The structure of the chapter follows. In the first section we set part of the framework by determining the sequence of standard data as well as the notation we are going to use in the chapter. In Section IV.1 we provide the conditions the sequence of standard data should satisfy and we state the main theorem of the chapter. Then, in Section IV.3 we provide the main arguments for the proof of the main theorem. It will be evident there that the stability of martingale representation plays an important role in obtaining the stability property of BSDEs, as we have already commented. This is not surprising, as the reader may recall that in Theorem II.14 we obtained the existence and uniqueness of the solution of a BSDE by translating the problem into a martingale representation one. The technical lemmata which verify our claims in the body of the proof are presented in Sections IV.4 and IV.5.

IV.1. Notation

The probability space $(\Omega, \mathcal{G}, \mathbb{P})$ will be fixed throughout the chapter. Moreover, as we have already informed the reader in the introduction of Chapter I, for this chapter the dimension ℓ of the driving martingales will be equal to 1. The solution of a BSDE will be also real-valued. Moreover, we are going to borrow the notation as was set in Section II.3. However, before we proceed, let us fix

- a sequence of filtrations $(\mathbb{G}^k)_{k\in\overline{\mathbb{N}}}$,

- a sequence of stopping times $(T^k)_{k\in\overline{\mathbb{N}}}$,
- a sequence $(\overline{X}^k)_{k\in\mathbb{N}}$, where $\overline{X}^k := (X^{k,\circ}, X^{k,\natural})$ such that $(\overline{X}^k)^{T^k} \in \mathcal{H}^2(\mathbb{G}^k;\mathbb{R}) \times \mathcal{H}^{2,d}(\mathbb{G}^k;\mathbb{R})$ for every $k \in \mathbb{N}$ and
- X^∞ such that $(X^\infty)^{T^\infty} \in \mathcal{H}^2(\mathbb{G}^\infty;\mathbb{R})$. We denote by $\overline{X}^\infty$ the natural pair of X^∞.

Now, we fix a sequence $(\mathcal{SD}^k)_{k\in\overline{\mathbb{N}}}$ where $\mathcal{SD}^k := (\mathbb{G}^k, T^k, \overline{X}^k, \xi^k, C^{\overline{X}^k}, f^k)$. The following assumption will be true for the rest of the chapter.

■ $(\mathcal{SD}^k)_{k\in\overline{\mathbb{N}}}$ is a sequence of standard data *under the same* $\hat{\beta}$.[1]

In other words, for every $k \in \overline{\mathbb{N}}$, the sextuple $\mathcal{SD}^k$, satisfies Conditions **(F1)** - **(F5)** of Section II.4 under $\hat{\beta}$. The reader should keep in mind that $\hat{\beta}$ is universal.

Notation IV.1. For every $k \in \overline{\mathbb{N}}$ the $\mathbb{G}^k-$stopping time T^k is fixed. Therefore, in order to ease notation, we will denote the stopped processes $(\overline{X}^k)^{T^k}$, $(X^{k,\circ})^{T^k}$, $(X^{k,\natural})^{T^k}$, $(C^{\overline{X}^k})^{T^k}$ simply as $\overline{X}^k$, $X^{k,\circ}$, $X^{k,\natural}$, $C^{\overline{X}^k}$, for every $k \in \overline{\mathbb{N}}$. We will do so also for $f^k\mathbb{1}_{[\![0,T^k]\!]}$, *i.e.* we will denote it simply by f^k, for every $k \in \overline{\mathbb{N}}$.

Notation IV.2. We provide here the notation which is associated to the standard data $\mathcal{SD}^k$, for every $k \in \overline{\mathbb{N}}$, through the conditions **(F1)** - **(F5)**.

- We adopt Notation II.2 as well as Notation II.6, since the pair $(\overline{X}^k, C^{\overline{X}^k})$ satisfies Assumption $(\mathbb{C})$ for every $k \in \overline{\mathbb{N}}$.
- For every $k \in \overline{\mathbb{N}}$, the stochastic bounds associated to the generator f^k by **(F3)** are denoted by r^k and $\vartheta^k := (\theta^{k,c}, \theta^{k,d})$.
- For every $k \in \overline{\mathbb{N}}$, the associated processes by **(F4)** are denoted by $\alpha^k := (\max\{\sqrt{r^k}, \theta^{k,c}, \theta^{k,d}\})^{\frac{1}{2}}$ and $A^k := \int_{(0,\cdot]}(\alpha^k)^2\,\mathrm{d}C^{\overline{X}^k}$. Moreover, the bound associated to A^k by (II.18) will be denoted by Φ^k, for every $k \in \overline{\mathbb{N}}$.

In view of the above, we can adopt also Notation II.4, which provides the associated normed spaces, whose notation we immediately simplify. We provide initially the simplifications for the standard data $\mathcal{SD}^k$ indexed by the positive integers.

Notation IV.3. For any $\beta \geq 0$, $(\omega, t) \in \Omega \times \mathbb{R}_+$ and $k \in \mathbb{N}$ we define:

- $\mathcal{P}^k := \mathcal{P}^{\mathbb{G}^k}$, $\widetilde{\mathcal{P}}^k := \widetilde{\mathcal{P}}^{\mathbb{G}^k}$.
- $\mathbb{L}^2_{\beta,k} := \mathbb{L}^2_{A^k,\beta}(\mathcal{G}^k_{T^k};\mathbb{R})$.
- $\mathbb{H}^2_{\beta,k} := \mathbb{H}^2_{A^k,\beta}(\mathbb{G}^k, \overline{X}^k;\mathbb{R})$, $\mathbb{H}^{2,\circ}_{\beta,k} := \mathbb{H}^2_\beta(\mathbb{G}^k, X^{k,\circ};\mathbb{R})$, $\mathbb{H}^{2,\natural}_{\beta,k} := \mathbb{H}^2_\beta(\mathbb{G}^k, \mu^{X^{k,\natural}};\mathbb{R})$.
- $\mathcal{S}^2_{\beta,k} := \mathcal{S}^2_{A^k,\beta}(\mathbb{G}^k, \overline{X}^k;\mathbb{R})$, $\mathcal{H}^{2,\perp}_{\beta,k} := \mathcal{H}^{2,\perp}_{A^k,\beta}(\mathbb{G}^k, X^{k,\circ}, \mu^{X^{k,\natural}};\mathbb{R})$, $\mathcal{H}^2_{\beta,k} := \mathcal{H}^2_{A^k,\beta}(\mathbb{G}^k;\mathbb{R})$.
- $|||\cdot|||_{t,k} := |||\cdot|||^{\overline{X}^k}_t$, $\mathfrak{H}_{\omega,t,k} := \mathfrak{H}^{\overline{X}^k}_{\omega,t}$, $\mathfrak{H}_k := \mathfrak{H}^{\overline{X}^k}$.
- $\|\cdot\|_{\beta,k} := \|\cdot\|_{A^k,\beta,\overline{X}^k}$, $\|\cdot\|_{\star,\beta,k} := \|\cdot\|_{\star,A^k,\beta,\overline{X}^k}$.
- $C^k := C^{\overline{X}^k}, c^k := c^{\overline{X}^k}$, $\mu^k := \mu^{X^{k,\natural}}$, $\nu^k := \nu^{(X^{k,\natural},\mathbb{G}^k)}$, $\widetilde{\mu}^k := \widetilde{\mu}^{(X^{k,\natural},\mathbb{G}^k)}$.

The following is the analogous notation for the standard data $\mathcal{SD}^\infty$.

Notation IV.4. For any $\beta \geq 0$, $(\omega, t) \in \Omega \times \mathbb{R}_+$

- $\mathcal{P}^\infty := \mathcal{P}^{\mathbb{G}^\infty}$, $\widetilde{\mathcal{P}}^\infty := \widetilde{\mathcal{P}}^{\mathbb{G}^\infty}$.
- $\mathbb{L}^2_{\beta,\infty} := \mathbb{L}^2_{A^\infty,\beta}(\mathcal{G}^\infty_{T^\infty};\mathbb{R})$.
- $\mathbb{H}^2_{\beta,\infty} := \mathbb{H}^2_{A^\infty,\beta}(\mathbb{G}^\infty, \overline{X}^\infty;\mathbb{R})$, $\mathbb{H}^{2,c}_{\beta,\infty} := \mathbb{H}^2_\beta(\mathbb{G}^\infty, X^{\infty,c};\mathbb{R})$, $\mathbb{H}^{2,d}_{\beta,\infty} := \mathbb{H}^2_\beta(\mathbb{G}^\infty, \mu^{X^{\infty,d}};\mathbb{R})$.
- $\mathcal{S}^2_{\beta,\infty} := \mathcal{S}^2_{A^\infty,\beta}(\mathbb{G}^\infty, \overline{X}^\infty;\mathbb{R})$, $\mathcal{H}^{2,\perp}_{\beta,\infty} := \mathcal{H}^{2,\perp}_{A^\infty,\beta}(\mathbb{G}^\infty, X^{\infty,c}, \mu^{X^{\infty,d}};\mathbb{R})$, $\mathcal{H}^2_{\beta,\infty} := \mathcal{H}^2_{A^\infty,\beta}(\mathbb{G}^\infty;\mathbb{R})$.
- $|||\cdot|||_{t,\infty} := |||\cdot|||^{\overline{X}^\infty}_t$, $\mathfrak{H}_{\omega,t,\infty} := \mathfrak{H}^{\overline{X}^\infty}_{\omega,t}$, $\mathfrak{H}_\infty := \mathfrak{H}^{\overline{X}^\infty}$.
- $\|\cdot\|_{\beta,\infty} := \|\cdot\|_{A^\infty,\beta,\overline{X}^\infty}$, $\|\cdot\|_{\star,\beta,\infty} := \|\cdot\|_{\star,A^\infty,\beta,\overline{X}^\infty}$.
- $C^\infty := C^{\overline{X}^\infty}, c^\infty := c^{\overline{X}^\infty}$, $\mu^\infty := \mu^{X^{\infty,d}}$, $\nu^\infty := \nu^{(X^{\infty,d},\mathbb{G}^\infty)}$, $\widetilde{\mu}^\infty := \widetilde{\mu}^{(X^{\infty,d},\mathbb{G}^\infty)}$.

Notation IV.5. We introduce some convenient notation for two subsets of $\mathbb{R}_+ \times \mathbb{R}$ as well as the notation for sets of continuous functions with compact support. D is an arbitrary subset of $\mathbb{R}$.

- $\widetilde{E}_0 := (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R})$ and $\widetilde{E} := (\mathbb{R}_+ \times \mathbb{R}) \setminus \widetilde{E}_0$.

[1] We can assume that $\hat{\beta}$ is sufficiently large. In other words, we can assume that $\hat{\beta}$ is such that the values $M^\Phi(\hat{\beta})$, $M^\Phi_\star(\hat{\beta})$ determined in Lemma II.13 are arbitrarily close to $18\mathrm{e}\Phi$.

- Let $f : (D, |\cdot|) \to (\mathbb{R}, |\cdot|)$, its support is the set $\mathrm{supp}(f) := \overline{\{s \in \mathbb{R}_+, f(s) \neq 0\}}^{|\cdot|} \cap D$.
- Let $f : (\mathbb{R}_+ \times \mathbb{R}, \|\cdot\|_2) \to (\mathbb{R}, |\cdot|)$, its support is the set $\mathrm{supp}(f) := \overline{\{s \in \mathbb{R}_+ \times \mathbb{R}, f(s) \neq 0\}}^{\|\cdot\|_2}$.
- $\mathrm{C}_c(\mathbb{R}_+; \mathbb{R}) := \Big\{ Z : \big(\mathbb{R}_+, |\cdot|\big) \to (\mathbb{R}, |\cdot|), Z$ is continuous with compact support$\Big\}$.
- $\mathrm{C}_{c,0}(\mathbb{R}; \mathbb{R}) := \Big\{ U : \big(\mathbb{R}, |\cdot|\big) \to (\mathbb{R}, |\cdot|), U$ is continuous with compact support $\mathrm{supp}(U)$ and such that $\inf\{\delta_{\mathbb{R}}(x, 0), x \in \mathrm{supp}(U)\} > 0\Big\}$.
- $\mathrm{C}_{c,0}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R}) := \Big\{ U : \big(\mathbb{R}_+ \times \mathbb{R}, \|\cdot\|_2\big) \to (\mathbb{R}, |\cdot|), U$ is continuous with compact support $\mathrm{supp}(U)$ and such that $\inf\big\{\|x - y\|_2, x \in \mathrm{supp}(U), y \in \widetilde{E}_0\big\} > 0\Big\}$.

Remark IV.6. The set $\mathrm{C}_{c,0}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$ is non-empty. For example, let $U : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ whose support $\mathrm{supp}(U)$ is a compact subset of the open set $\widetilde{E}$. Then $\widetilde{U}$ lies in $\mathrm{C}_{c,0}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$, since $\widetilde{E}_0 = (\mathbb{R}_+ \times \mathbb{R}) \setminus \widetilde{E}$ is a closed subset of $\mathbb{R}_+ \times \mathbb{R}$ and $\mathrm{supp}(U) \cap \widetilde{E}_0 = \emptyset$. In particular, $U(t, \cdot) \in \mathrm{C}_c(\mathbb{R}; \mathbb{R})$ for every $t \in \mathbb{R}_+$.

Remark IV.7. Let μ^X be the integer-valued random measure associated to the càdlàg process X. The way that the stochastic integral with respect to μ^X has been defined is such that $\mu^X(\widetilde{E}_0) = 0$. In other words, $\mu^X(\mathbb{R}_+ \times \mathbb{R}) = \mu^X(\widetilde{E})$. Therefore, for a filtration $\mathbb{G}$ such that $\nu^{(X,\mathbb{G})}$ is well-defined, it holds $\nu^{(X,\mathbb{G})}(\widetilde{E}_0) = 0$; this is a consequence of Theorem I.79. Moreover, this property will be used in Subsection IV.5.1 in order to attain the Lusin approximation we will make use of.

We have introduced the Notation IV.1 - IV.5 in order to ease notation. However, we will use interchangeably the complete notation when we want to underline a property. For example we will do so when we set the framework of the chapter, which is described by the conditions **(B1)** - **(B9)**. The reader will realise that the conditions **(B1)** - **(B5)** are nothing more than the framework of Theorem III.50, which is not surprising in view of the comments presented at the beginning of the chapter. The remaining conditions are such that we can obtain a convergent sequence of Lebesgue–Stieltjes integrals. We need only to comment beforehand on Condition **(B6)**, which states that the sequence $(A^k)_{k \in \overline{\mathbb{N}}}$ has to be bounded. This condition implies the equivalence of the norms in the weighted spaces indexed by β with the respective whose norm does not depend on A, *i.e.* $\beta = 0$. However, by Condition **(B6)** it is not necessarily implied that the sequence $(C^k)_{k \in \overline{\mathbb{N}}}$ is bounded. We will assume, however, that C^k is $\mathbb{P}-$almost surely finite, for every $k \in \overline{\mathbb{N}}$, which is clearly weaker from the above statement. We provide more comments on the nature of the Conditions **(B1)** - **(B9)** in Subsection IV.5.6.

IV.2. Framework and statement of the main theorem

The sequence of standard data $(\mathcal{SD}^k)_{k \in \overline{\mathbb{N}}}$ satisfies the following conditions:

(B1) The filtration $\mathbb{G}^\infty$ is quasi-left-continuous and the process $\overline{X}^\infty$ is $\mathbb{G}^\infty-$quasi-left-continuous..

(B2) The process $\overline{X}^k = (X^{k,\circ}, X^{k\natural}) \in \mathcal{H}^2(\mathbb{G}^k; \mathbb{R}) \times \mathcal{H}^{2,d}(\mathbb{G}^k; \mathbb{R})$, for every $k \in \overline{\mathbb{N}}$. Moreover,

$$(X^{k,\circ}_\infty, X^{k,\natural}_\infty) \xrightarrow{\mathbb{L}^2(\mathcal{G};\mathbb{R})} (X^{\infty,\circ}_\infty, X^{\infty,\natural}_\infty).$$

(B3) The martingale X^∞ possesses the $\mathbb{G}^\infty$-Predictable Representation Property.

(B4) The filtrations converge weakly, *i.e.* $\mathbb{G}^k \xrightarrow{\mathrm{w}} \mathbb{G}^\infty$.

(B5) The random variable $\xi^k \in \mathbb{L}^2(\mathcal{G}^k_{T^k}; \mathbb{R})$ for every $k \in \overline{\mathbb{N}}$. Moreover, $\xi^k \xrightarrow{\mathbb{L}^2(\mathcal{G};\mathbb{R})} \xi^\infty$.

(B6) The sequence $(A^k_\infty)_{k \in \overline{\mathbb{N}}}$ is bounded.

(B7) The generators possess additionally the following properties:

(i) For every $(y, z, U) \in \mathbb{R} \times \mathbb{R} \times \mathrm{C}_{c,0}(\mathbb{R})$ and for every $k \in \overline{\mathbb{N}}$ the process $\big(f^k(t, y, z, U(t, \cdot)\big)_{t \in \mathbb{R}_+}$ is càdlàg $\mathbb{P}$-almost surely.

(ii) For K_1, resp. K_2, compact subset of $\mathbb{R}$, resp. $\mathbb{R}$, and $U \in \mathrm{C}_{c,0}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$, if $(\lambda^k)_{k \in \mathbb{N}} \subset \Lambda$[2] such that

$$\sup_{t \in \mathbb{R}_+} |\lambda^k(t) - t| \xrightarrow[k \to \infty]{} 0$$

[2]Recall Definition I.108.

then $\mathbb{P}-$almost surely

$$\sup_{t\in[0,N]}\Big\{\big|f^{\infty}\big(\lambda^k(t),y,z,U(t,\cdot)\big)-f^{\infty}\big(t,y,z,U(t,\cdot)\big)\big|,z\in K_2\Big\}\xrightarrow[k\to\infty]{}0 \text{ for every } N\in\mathbb{N}.$$

(iii) For K_1, resp. K_2, compact subset of $\mathbb{R}$, resp. $\mathbb{R}$, and $U\in \mathrm{C}_{c,0}(\mathbb{R}_+\times\mathbb{R};\mathbb{R})$ it holds

$$\sup_{t\in\mathbb{R}_+}\Big\{\big|f^k(t,y,z,U(t,\cdot))-f(t,y,z,U(t,\cdot))\big|,(y,z)\in K_1\times K_2\Big\}\xrightarrow[k\to\infty]{}0\ \mathbb{P}-\text{almost surely.}$$

(iv) The sequences of stochastic bounds $(r^k)_{k\in\overline{\mathbb{N}}}$, $(\vartheta^k)_{k\in\overline{\mathbb{N}}}$ are $\mathbb{P}$-almost surely bounded on $\mathbb{R}_+$. Additionally,

$$r^k\xrightarrow{\|\cdot\|_\infty}r^\infty,\ \theta^{k,c}\xrightarrow{\|\cdot\|_\infty}\theta^{\infty,c},\ \theta^{k,d}\xrightarrow{\|\cdot\|_\infty}\theta^{\infty,d}\ \mathbb{P}\text{-almost surely}$$

and $\alpha^\infty(\omega)$ is $\mathrm{d}C^\infty(\omega)$-almost everywhere continuous for $\mathbb{P}-$almost every ω. Moreover, $\mathrm{d}\langle X^{\infty,c}\rangle_s/\mathrm{d}C^\infty_s$ has locally bounded paths $\mathbb{P}-$almost surely.

(B8) The sequence $\left(\int_{(0,\infty)}\mathrm{e}^{\hat\beta A^k_t}\frac{\left|f^k(t,0,0,\mathbf{0})\right|^2}{(\alpha^k_t)^2}\,\mathrm{d}C^k_t\right)_{k\in\overline{\mathbb{N}}}$ is uniformly integrable.

(B9) The sequences $(C^k)_{k\in\overline{\mathbb{N}}}$ and $(\Phi^k)_{k\in\overline{\mathbb{N}}}$ satisfy

(i) $C^k\xrightarrow{(\mathrm{J}_1(\mathbb{R}),\mathbb{P})}C^\infty$,

(ii) $C^k_\infty:=\lim_{t\to\infty}C^k_t\in\mathbb{R}_+\ \mathbb{P}-a.s.$ with $C^k_\infty\xrightarrow{\mathbb{P}}C^\infty_\infty$ and

(iii) $\Phi^k\xrightarrow{|\cdot|}\Phi^\infty:=0$.

Remark IV.8. The convergence in Condition **(B5)** implies that $\sup_{k\in\overline{\mathbb{N}}}\left\|\xi^k\right\|_{\mathbb{L}^2(\mathcal{G};\mathbb{R})}<\infty$. In view of the Condition **(B6)** the above bound is equivalent to $\sup_{k\in\overline{\mathbb{N}}}\left\|\xi^k\right\|_{\mathbb{L}^2_{A^k,\hat\beta}(\mathcal{G}^k_{T^k};\mathbb{R})}<\infty$. Moreover, in view of the Condition **(B6)** again, the Condition **(B8)** is equivalent to

$$\left(\int_{(0,\infty)}\frac{\left|f^k(t,0,0,\mathbf{0})\right|^2}{(\alpha^k_t)^2}\,\mathrm{d}C^k_t\right)_{k\in\overline{\mathbb{N}}}\text{is uniformly integrable.}$$

In the proofs of the lemmata we will need in the next sections we will always indicate the process A^k so that it is clear its role. Moreover, this inconvenience may turn out to be fruitful in the case it helps the reader to provide any suggestions for the improvement of the Condition **(B6)**. These suggestions are more than welcome by the author.

Remark IV.9. Recall Remark II.15 and that $\mathcal{SD}^k$ is standard data under $\hat\beta$, for every $k\in\overline{\mathbb{N}}$. In view of Condition **(B9)**.**(iii)** we have that the equation

$$Y_t=\xi^k+\int_{(t,T^k]}f^k(s,Y_s,Z_s,U(s,\cdot))\,\mathrm{d}C^k_s-\int_t^{T^k}Z_s\,\mathrm{d}X^{k,\circ}_s-\int_t^{T^k}\int_{\mathbb{R}}U(s,x)\,\widetilde{\mu}^k(\mathrm{d}s,\mathrm{d}x)-\int_t^{T^k}\mathrm{d}N^k_s\quad\text{(IV.1)}$$

has a unique solution (Y^k,Z^k,U^k,N^k) finally for $k\in\mathbb{N}$. Without loss of generality, possibly by restricting ourselves in the case $\sup_{k\in\mathbb{N}}\Phi^k<(18\mathrm{e})^{-1}$ and for $\hat\beta$ large enough (recall for this Foonote 1), we will assume that the BSDE (IV.1) admits a unique solution for every $k\in\mathbb{N}$. Moreover, observe that for the standard data $\mathcal{SD}^\infty$ we have assumed $\Phi^\infty=0$, therefore there exists a unique solution of the BSDE

$$Y_t=\xi^\infty+\int_{(t,T^\infty]}f^\infty(s,Y_s,Z_s,U(s,\cdot))\,\mathrm{d}C^\infty_s-\int_t^{T^\infty}Z_s\,\mathrm{d}X^{\infty,c}_s-\int_t^{T^\infty}\int_{\mathbb{R}}U(s,x)\,\widetilde{\mu}^\infty(\mathrm{d}s,\mathrm{d}x).\quad\text{(IV.2)}$$

Observe that because of Condition **(B3)** the orthogonal martingale N^∞ vanishes.

We proceed now to the statement of the theorem of the chapter. In the body of its proof we will present the main arguments, which are based on the Moore–Osgood Theorem, see Theorem A.8 and the references therein, while the remaining technical parts will be presented in a series of sections.

Theorem IV.10. *Let the Conditions* **(B1)** - **(B9)** *hold for the sequence* $(\mathcal{SD}^k)_{k\in\overline{\mathbb{N}}}$ *of standard data under the same* $\hat\beta$. *For every* $k\in\mathbb{N}$ *denote by* $\Upsilon^k:=(Y^k,Z^k,U^k,N^k)$ *the unique solution of the BSDE* (IV.1) *associated to the standard data* $\mathcal{SD}^k$ *and by* $\Upsilon^\infty:=(Y^\infty,Z^\infty,U^\infty,0)$ *the associated to* $\mathcal{SD}^\infty$ *solution. Then,*

$$(Y^k,Z^k\cdot X^{k,\circ}+U^k\star\widetilde{\mu}^k,N^k)\xrightarrow{\left(\mathrm{J}_1(\mathbb{R}^3),\mathbb{L}^2\right)}\left(Y^\infty,Z^\infty\cdot X^{\infty,c}+U^\infty\star\widetilde{\mu}^\infty,0\right)\quad\text{(IV.3)}$$

and

$$(\langle Y^k\rangle, \langle Z^k \cdot X^{k,\circ}\rangle, \langle U^k \star \widetilde{\mu}^k\rangle, \langle Y^k, X^{k,\circ}\rangle, \langle Y^k, X^{k,\natural}\rangle) \xrightarrow{(\mathrm{J}_1(\mathbb{R}^5),\mathbb{L}^1)} (\langle Y^\infty\rangle, \langle Z^\infty \cdot X^{\infty,c}\rangle, \langle U^\infty \star \widetilde{\mu}^\infty\rangle, \langle Y^\infty, X^c\rangle, \langle Y^\infty, X^d\rangle). \tag{IV.5}$$

IV.3. Proof of Theorem IV.7

For the rest of the chapter Conditions **(B1)** - **(B9)** will be assumed to be in force for the sequence of $(\mathcal{SD}^k)_{k\in\overline{\mathbb{N}}}$. Before we proceed to the proof of Theorem IV.10 we present the next result, which gives uniform *a priori* estimates of the tail of the Picard approximations, see Corollary II.22. Proposition IV.11 generalizes Briand et al. [15, Corollary 10].[3] In the following, for $k \in \overline{\mathbb{N}}$, we will keep the meaning of $(\Upsilon^k)_{k\in\overline{\mathbb{N}}}$ as defined in Theorem IV.10.

Proposition IV.11. *There exist $k_0, k_{\star,0} \in \mathbb{N}$ such that*

$$\lim_{p\to\infty} \sup_{k\geq k_0} \left\|\Upsilon^k - \Upsilon^{k,(p)}\right\|^2_{\hat{\hat{\beta}},k} = 0 \text{ and } \lim_{p\to\infty} \sup_{k\geq k_{\star,0}} \left\|\Upsilon^k - \Upsilon^{k,(p)}\right\|^2_{\star,\hat{\hat{\beta}},k} = 0,$$

where $\Upsilon^{k,(p)}$ is the p-element of the Picard approximation of the solution Υ^k associated to the standard data $\mathcal{SD}^k$.

Proof. Let us initially restate for the convenience of the reader the constants $\Pi^\Phi(\gamma,\delta)$, $\Pi^\Phi_\star(\gamma,\delta)$ as well as the values $M^\Phi(\beta)$, $M^\Phi_\star(\beta)$ of Lemma II.16, where $\beta > 0$ and $\gamma, \delta \in (0,\beta]$ such that $\gamma < \delta$. So, from Lemma II.16, we have

$$\Pi^\Phi(\gamma,\delta) := \frac{9}{\delta} + (2+9\delta)\frac{\mathrm{e}^{(\delta-\gamma)\Phi}}{\gamma(\delta-\gamma)}$$

with

$$M^\Phi(\beta) := \inf_{(\gamma,\delta)\in\mathcal{C}_\beta} \Pi^\Phi(\gamma,\delta) = \frac{9}{\beta} + \frac{\Phi^2(2+9\beta)}{\sqrt{\beta^2\phi^2+4}-2} \exp\Big(\frac{\beta\Phi + 2 - \sqrt{\beta^2\Phi^2+4}}{2}\Big)$$

and

$$\Pi^\Phi_\star(\gamma,\delta) := \frac{8}{\gamma} + \frac{9}{\delta} + 9\delta\frac{\mathrm{e}^{(\delta-\gamma)\Phi}}{\gamma(\delta-\gamma)} \text{with } M^\Phi_\star(\beta) := \inf_{(\gamma,\delta)\in\mathcal{C}_\beta} \Pi^\Phi_\star(\gamma,\delta) = \Pi^\Phi_\star(\overline{\gamma}^\Phi_\star(\beta),\beta).$$

We observe that, when we fix $\gamma,\delta \in (0,\hat{\hat{\beta}}]^2$, then the functions

$$\mathbb{R}_+ \ni x \mapsto \Pi^x(\gamma,\delta) \in \mathbb{R}_+ \text{ and } \mathbb{R}_+ \ni x \mapsto \Pi^x_\star(\gamma,\delta) \in \mathbb{R}_+ \tag{IV.6}$$

are decreasing and continuous under the usual topology of $\mathbb{R}$. Hence, by the assumption **(B9).(iii)** and by Footnote 1 on p. 106, we can assume that there exist Φ^{k_0} and $\Phi^{k_{\star,0}}$ such that

- $M^{\Phi^{k_0}}(\hat{\beta}) = \Pi^{\Phi^{k_0}}(\overline{\gamma}^{\Phi^{k_0}},\hat{\beta}) < \frac{1}{2}$, for some $\overline{\gamma}^{\Phi^{k_0}} \in (0,\hat{\hat{\beta}}]^2$ and
- $M^{\Phi^{k_{\star,0}}}_\star(\hat{\beta}) = \Pi^{\Phi^{k_{\star,0}}}_\star(\overline{\gamma}^{\Phi^{k_{\star,0}}}_\star,\hat{\beta}) < \frac{1}{2}$, for some $\overline{\gamma}^{\Phi^{k_{\star,0}}} \in (0,\hat{\hat{\beta}}]^2$.

Consequently, by Lemma II.13 and the fact that the functions in (IV.6) are decreasing, we obtain

$$\Pi^{\Phi^k}(\overline{\gamma}^{\Phi^k},\hat{\beta}) = M^{\Phi^k}(\hat{\beta}) \leq \Pi^{\Phi^k}(\overline{\gamma}^{\Phi^{k_0}},\hat{\beta}) \leq M^{\Phi^{k_0}}(\hat{\beta}) < \frac{1}{2} \text{ for every } k \geq k_0,$$

as well as

$$\Pi^{\Phi^k}_\star(\overline{\gamma}^{\Phi^k}_\star,\hat{\beta}) = M^{\Phi^k}_\star(\hat{\beta}) \leq \Pi^{\Phi^k}_\star(\overline{\gamma}^{\Phi^{k_{\star,0}}}_\star,\hat{\beta}) \leq M^{\Phi^{k_{\star,0}}}_\star(\hat{\beta}) < \frac{1}{2} \text{ for } k \geq k_{\star,0}. \tag{IV.7}$$

Let us, now, for the rest of the proof to deal only with the second convergence, since we can readily adapt the same arguments for the first one. By Corollary II.22, more precisely by (II.53), we derive

$$\begin{aligned}\|\Upsilon^k - \Upsilon^{k,(p)}\|^2_{\star,\hat{\hat{\beta}},k} &\leq \sum_{n=0}^\infty 2^{n+1}\|\Upsilon^{k,(p+n)} - \Upsilon^{k,(p+n+1)}\|^2_{\star,\hat{\hat{\beta}},k} \\ &\leq \sum_{n=0}^\infty 2^{n+1}2^{-(p+n)}\|\Upsilon^{k,(1)}\|^2_{\star,\hat{\hat{\beta}},k} = 4^{1-p}\|\Upsilon^{k,(1)}\|^2_{\star,\hat{\hat{\beta}},k}.\end{aligned} \tag{IV.8}$$

[3] For more details, the interested reader should consult Section IV.6, where we will provide the comparison with the related literature.

Choosing in the Picard approximation our starting point to be the zero element for every $k \in \mathbb{N}$, we obtain by Lemma II.16

$$\left\|\Upsilon^{k,(1)}\right\|^2_{\star,\hat{\beta},k} \leq \widetilde{\Pi}^{\hat{\beta},\Phi^k}_{\star} \left\|\xi^k\right\|^2_{\mathbb{L}^2_{\hat{\beta},k}} + \Pi^{\Phi^k}_{\star}(\overline{\gamma}^{\Phi^k}_{\star},\hat{\beta}) \left\| \frac{f^k\big(\cdot,0,0,\mathbf{0}\big)}{\alpha^k} \right\|^2_{\mathbb{H}^2_{\hat{\beta},k}}.$$

In view of Condition **(B5)**, Condition **(B8)**, Remark IV.8 and the fact that the function $\mathbb{R}_+ \ni x \mapsto \widetilde{\Pi}^{\hat{\beta},x}_{\star} \in \mathbb{R}_+$ (resp. $\mathbb{R}_+ \ni x \mapsto \widetilde{\Pi}^{\hat{\beta},x} \in \mathbb{R}_+$) is decreasing, we can obtain with the help of (IV.7)

$$\sup_{k\geq k^\star_0} \|\Upsilon^{k,(1)}\|^2_{\star,\hat{\beta},k} \leq \widetilde{\Pi}^{\hat{\beta},\Phi^{k_\star,0}} \sup_{k\geq k^\star_0} \|\xi^k\|^2_{\mathbb{L}^2_{\hat{\beta}}(\mathcal{G}^k_{T^k})} + \Pi^{\Phi^k}_{\star}(\overline{\gamma}^{\Phi^{k_\star,0}}_{\star},\hat{\beta}) \sup_{k\geq k^\star_0} \left\| \frac{f^k\big(\cdot,0,0,\mathbf{0}\big)}{\alpha^k} \right\|^2_{\mathbb{H}^2_{\star,\hat{\beta},k}} < \infty, \quad \text{(IV.9)}$$

which implies the desired result in conjunction with Inequality (IV.8). □

IV.3.1. Main part of the proof of Theorem IV.10. As we have already commented, we will provide below the main arguments for proving Theorem IV.10. The technical results which verify our argumentation will be provided in the coming sections.

Proof of Theorem IV.10. We will keep the notation of Proposition IV.11 and we will denote by $(\Upsilon^{\infty,(p)})_{p\in\mathbb{N}}$ the Picard scheme associated to $\mathcal{SD}^\infty$. The starting point of the Picard scheme associated to the standard data $\mathcal{SD}^k$ will be the zero element, for every $k \in \overline{\mathbb{N}}$. In other words, we define $\Upsilon^{k,(0)} := (0,0,\mathbf{0},0)$, for every $k \in \overline{\mathbb{N}}$. Recall that for every $k \in \overline{\mathbb{N}}$ we have stopped the processes indexed by k at time T^k. Therefore, we can substitute for every $k \in \overline{\mathbb{N}}$ the terminal time T^k by ∞ and we will do so for our convenience. Then, by the construction of Picard scheme (recall Corollary II.22) holds for every $p \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} Y^{\infty,(p+1)}_t &= \xi^\infty + \int_{(t,T^\infty]} f^\infty\big(s, Y^{\infty,(p)}_s, Z^{\infty,(p)}_s, U^{\infty,(p)}(s,\cdot)\big)\,\mathrm{d}C^\infty_s \\ &\qquad - \int_t^{T^\infty} Z^{\infty,(p+1)}_s\,\mathrm{d}X^{\infty,c}_s - \int_t^{T^\infty}\int_{\mathbb{R}} U^{\infty,(p+1)}(s,x)\widetilde{\mu}^\infty(\mathrm{d}s,\mathrm{d}x) \\ &= \xi^\infty + \int_{(t,\infty)} f^\infty\big(s, Y^{\infty,(p)}_s, Z^{\infty,(p)}_s, U^{\infty,(p)}(s,\cdot)\big)\,\mathrm{d}C^\infty_s \\ &\qquad - \int_t^{\infty} Z^{\infty,(p+1)}_s\,\mathrm{d}X^{\infty,c}_s - \int_t^{\infty}\int_{\mathbb{R}} U^{\infty,(p+1)}(s,x)\widetilde{\mu}^\infty(\mathrm{d}s,\mathrm{d}x) \end{aligned} \quad \text{(IV.11)}$$

Analogously, for every $k \in \mathbb{N}$ and for every $p \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} Y^{k,(p+1)}_t &= \xi^k + \int_{(t,T^k]} f^k\big(s, Y^{k,(p)}_s, Z^{k,(p)}_s, U^{k,(p)}(s,\cdot)\big)\,\mathrm{d}C^k_s \\ &\qquad - \int_t^{T^k} Z^{k,(p+1)}_s\,\mathrm{d}X^{k,\circ}_s - \int_t^{T^k}\int_{\mathbb{R}} U^{k,(p+1)}(s,x)\,\widetilde{\mu}^k(\mathrm{d}s,\mathrm{d}x) - \int_t^{T^k} dN^{k,(p+1)}_s \\ &= \xi^k + \int_{(t,\infty)} f^k\big(s, Y^{k,(p)}_s, Z^{k,(p)}_s, U^{k,(p)}(s,\cdot)\big)\,\mathrm{d}C^k_s \\ &\qquad - \int_t^{\infty} Z^{k,(p+1)}_s\,\mathrm{d}X^{k,\circ}_s - \int_t^{\infty}\int_{\mathbb{R}} U^{k,(p+1)}(s,x)\,\widetilde{\mu}^k(\mathrm{d}s,\mathrm{d}x) - \int_t^{\infty} dN^{k,(p+1)}_s \end{aligned} \quad \text{(IV.12)}$$

By Proposition IV.11 we get that $\Upsilon^{k,(p)} \xrightarrow[p\to\infty]{\|\cdot\|_{\hat{\beta},k}} \Upsilon^k$ as well as $\Upsilon^{k,(p)} \xrightarrow[p\to\infty]{\|\cdot\|_{\star,\hat{\beta},k}} \Upsilon^k$, both finally uniformly for $k \in \mathbb{N}$. In particular, the latter convergence implies $Y^{k,(p)} \xrightarrow[p\to\infty]{\mathcal{S}^2_{\hat{\beta},k}} Y^k$ finally uniformly in k, which in turn implies $Y^{k,(p)} \xrightarrow[p\to\infty]{(\mathrm{J}_1(\mathbb{R}),\mathbb{L}^2)} Y^k$ finally uniformly in k. Hence, by orthogonality of the respective parts, Itô's isometry and Doob's inequality (see also the proof of Theorem II.14 on p. 68) we obtain

$$(Y^{k,(p)}, Z^{k,(p)}\cdot X^{k,\circ}, U^{k,(p)}\star\widetilde{\mu}^k, N^{k,(p)}) \xrightarrow[p\to\infty]{(\|\cdot\|_\infty,\mathbb{L}^2)} (Y^k, Z^k\cdot X^{k,\circ}, U^k\star\widetilde{\mu}^k, N^k)$$

finally uniformly in k. At this point we will combine two facts in order to rewrite the above convergence under the J_1-topology. The first one is that the convergence under $\|\cdot\|_\infty-$norm allows us to conclude

the convergence of the sum of two convergent sequences. The second is that the a $\|\cdot\|_\infty-$convergent sequence is also J_1-convergent. Therefore,

$$(Y^{k,(p)}, Z^{k,(p)}\cdot X^{k,\circ} + U^{k,(p)}\star\widetilde{\mu}^k, N^{k,(p)}) \xrightarrow[p\to\infty]{(\mathrm{J}_1(\mathbb{R}^3),\mathbb{L}^2)} (Y^k, Z^k\cdot X^{k,\circ} + U^k\star\widetilde{\mu}^k, N^k) \tag{IV.13}$$

finally uniformly in k. Consequently, in order to apply the Moore-Osgood Theorem, see Theorem A.8, it is sufficient to prove that for every $p\in\mathbb{N}\cup\{0\}$ holds

$$\begin{aligned}(Y^{k,(p)}, Z^{k,(p)}\cdot X^{k,\circ} + U^{k,(p)}\star\widetilde{\mu}^k, N^{k,(p)}) &\xrightarrow[k\to\infty]{(\mathrm{J}_1(\mathbb{R}^3),\mathbb{L}^2)} \\ &(Y^{\infty,(p)}, Z^{\infty,(p)}\cdot X^{\infty,c} + U^{\infty,(p)}\star\widetilde{\mu}^\infty, N^{\infty,(p)}).\end{aligned} \tag{IV.14}$$

To this end, let us transform the BSDE (IV.11) and BSDE (IV.12) into martingales.[4] The aforementioned transformation will allow us to utilise the stability of martingale representations. For fixed $p\in\mathbb{N}\cup\{0\}$ we have

$$\begin{aligned}
M_t^{\infty,(p)} &:= Y_t^{\infty,(p+1)} + \int_{(0,t]} f^\infty\big(s, Y_s^{\infty,(p)}, Z_s^{\infty,(p)}, U^{\infty,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^\infty && \text{(IV.15)}\\
&\overset{\text{(IV.11)}}{=} \xi^\infty + \int_{(0,\infty)} f^\infty\big(s, Y_s^{\infty,(p)}, Z_s^{\infty,(p)}, U^{\infty,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^\infty \\
&\qquad - \int_t^\infty Z_s^{\infty,(p+1)}\,\mathrm{d}X_s^{\infty,c} - \int_t^\infty\int_{\mathbb{R}} U^{\infty,(p+1)}(s,x)\widetilde{\mu}^\infty(\mathrm{d}s,\mathrm{d}x)\\
&\overset{\text{(IV.11)}}{=} Y_0^{\infty,(p+1)} + \int_0^\infty Z_s^{\infty,(p+1)}\,\mathrm{d}X_s^{\infty,c} + \int_0^\infty\int_{\mathbb{R}} U^{\infty,(p+1)}(s,x)\widetilde{\mu}^\infty(\mathrm{d}s,\mathrm{d}x)\\
&\qquad - \int_t^\infty Z_s^{\infty,(p+1)}\,\mathrm{d}X_s^{\infty,c} - \int_t^\infty\int_{\mathbb{R}} U^{\infty,(p+1)}(s,x)\widetilde{\mu}^\infty(\mathrm{d}s,\mathrm{d}x)\\
&= Y_0^{\infty,(p+1)} + \int_0^t Z_s^{\infty,(p+1)}\,\mathrm{d}X_s^{\infty,c} + \int_0^t\int_{\mathbb{R}} U^{\infty,(p+1)}(s,x)\widetilde{\mu}^\infty(\mathrm{d}s,\mathrm{d}x) && \text{(IV.16)}
\end{aligned}$$

is a square integrable $\mathbb{G}^\infty$-martingale on $\mathbb{R}_+$. Analogously, for $k\in\mathbb{N}$ and $p\in\mathbb{N}\cup\{0\}$ we have

$$\begin{aligned}
M_t^{k,(p)} &:= Y_t^{k,(p+1)} + \int_{(0,t]} f^k\big(s, Y_s^{k,(p)}, Z_s^{k,(p)}, U^{k,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^k && \text{(IV.17)}\\
&\overset{\text{(IV.12)}}{=} \xi^k + \int_{(0,\infty)} f^k\big(s, Y_s^{k,(p)}, Z_s^{k,(p)}, U^{k,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^k \\
&\qquad - \int_t^\infty Z_s^{k,(p+1)}\,\mathrm{d}X_s^{k,\circ} - \int_t^\infty\int_{\mathbb{R}} U^{k,(p+1)}(s,x)\,\widetilde{\mu}^k(\mathrm{d}s,\mathrm{d}x) - \int_t^\infty dN_s^{k,(p+1)}\\
&\overset{\text{(IV.12)}}{=} Y_0^{k,(p+1)} + \int_0^\infty Z_s^{k,(p+1)}\,\mathrm{d}X_s^{k,\circ} + \int_0^\infty\int_{\mathbb{R}} U^{k,(p+1)}(s,x)\,\widetilde{\mu}^k(\mathrm{d}s,\mathrm{d}x) + \int_0^\infty dN_s^{k,(p+1)}\\
&\qquad - \int_t^\infty Z_s^{k,(p+1)}\,\mathrm{d}X_s^{k,\circ} - \int_t^\infty\int_{\mathbb{R}} U^{k,(p+1)}(s,x)\,\widetilde{\mu}^k(\mathrm{d}s,\mathrm{d}x) - \int_t^\infty dN_s^{k,(p+1)}\\
&= Y_0^{k,(p+1)} + \int_0^t Z_s^{k,(p+1)}\,\mathrm{d}X_s^{k,\circ} + \int_0^t\int_{\mathbb{R}} U^{k,(p+1)}(s,x)\,\widetilde{\mu}^k(\mathrm{d}s,\mathrm{d}x) + \int_0^t dN_s^{k,(p+1)} && \text{(IV.18)}
\end{aligned}$$

is a square integrable $\mathbb{G}^k$-martingale on $\mathbb{R}_+$. At this point, we can obtain the convergence

$$(Z^{k,(p+1)}\cdot X^{k,\circ} + U^{k,(p+1)}\star\widetilde{\mu}^k, N^{k,(p+1)}) \xrightarrow[k\to\infty]{(\mathrm{J}_1(\mathbb{R}^3),\mathbb{L}^2)} (Z^{\infty,(p+1)}\cdot X^{\infty,c} + U^{\infty,(p+1)}\star\widetilde{\mu}^\infty, 0) \tag{IV.19}$$

if we apply Theorem III.50 to the sequence $(M^{k,(p)})_{k\in\overline{\mathbb{N}}}$. In view of the Conditions **(B1) - (B4)**, we need only to prove the convergence

$$M_\infty^{k,(p)} \xrightarrow[k\to\infty]{\mathbb{L}^2(\mathcal{G};\mathbb{R})} M_\infty^{\infty,(p)} \tag{IV.20}$$

for every $p\in\mathbb{N}\cup\{0\}$. However, in view of (IV.15) and (IV.17) and recalling that for the Picard schemes holds $Y_\infty^{k,(p+1)} = \xi^k$, for every $k\in\overline{\mathbb{N}}$ and $p\in\mathbb{N}\cup\{0\}$, we obtain immediately a pair of sufficient conditions for the Convergence (IV.20) which read

[4]Recall that the same technique was used in the proof of Theorem II.14 in order to utilise finally the orthogonal decomposition of square-integrable martingales.

□ $\xi^k \xrightarrow{\mathbb{L}^2(\mathcal{G};\mathbb{R})} \xi^\infty$; but this is Condition **(B5)**.

□ $\int_{(0,\infty)} f^k\big(s, Y^{k,(p)}_s, Z^{k,(p)}_s, U^{k,(p)}(s,\cdot)\big)\,\mathrm{d}C^k_s \xrightarrow[k\to\infty]{\mathbb{L}^2(\mathcal{G};\mathbb{R})} \int_{(0,\infty)} f^\infty\big(s, Y^{\infty,(p)}_s, Z^{\infty,(p)}_s, U^{\infty,(p)}(s,\cdot)\big)\,\mathrm{d}C^\infty_s$ for every $p \in \mathbb{N}\cup\{0\}$. We will name for later reference the one which corresponds to the Lebesgue-Stieltjes integrals by

$$\int_{(0,\infty)} f^k\big(s, Y^{k,(p)}_s, Z^{k,(p)}_s, U^{k,(p)}(s,\cdot)\big)\,\mathrm{d}C^k_s \xrightarrow[k\to\infty]{\mathbb{L}^2(\mathcal{G};\mathbb{R})} \int_{(0,\infty)} f^\infty\big(s, Y^{\infty,(p)}_s, Z^{\infty,(p)}_s, U^{\infty,(p)}(s,\cdot)\big)\,\mathrm{d}C^\infty_s. \tag{LS$^{(p)}_\infty$}$$

Assume for the following that (LS$^{(p)}_\infty$) is valid for every $p \in \mathbb{N}\cup\{0\}$, *i.e.* we can apply Theorem III.50 for the martingale sequence $(M^{k,(p)})_{k\in\overline{\mathbb{N}}}$. Therefore, we obtain the convergence for every $p \in \mathbb{N}\cup\{0\}$

$$(M^{k,(p)}, Z^{k,(p+1)}\cdot X^{k,\circ} + U^{k,(p+1)}\star\widetilde{\mu}^k, N^{k,(p+1)}) \xrightarrow{(\mathrm{J}_1(\mathbb{R}^3),\mathbb{L}^2)} (M^{\infty,(p)}, Z^{\infty,(p+1)}\cdot X^{\infty,c} + U^{\infty,(p+1)}\star\widetilde{\mu}^\infty, 0) \tag{IV.21}$$

Comparing now the above convergence with Convergence (IV.14), we realise that they differ only on the first element. Recall once again the definition of $M^{k,(p)}$, for $k \in \overline{\mathbb{N}}$; see (IV.15) and (IV.17). Now, we can obtain the convergence

$$(M^{k,(p)}, Y^{k,(p+1)}) \xrightarrow{(\mathrm{J}_1(\mathbb{R}^2),\mathbb{L}^2)} (M^{\infty,(p)}, Y^{\infty,(p+1)}) \text{ for every } p \in \mathbb{N}\cup\{0\},$$

if the convergence

$$\int_{(0,\cdot]} f^k\big(s, Y^{k,(p)}_s, Z^{k,(p)}_s, U^{k,(p)}(s,\cdot)\big)\,\mathrm{d}C^k_s \xrightarrow{(\mathrm{J}_1(\mathbb{R}),\mathbb{L}^2)} \int_{(0,\cdot]} f^\infty\big(s, Y^{\infty,(p)}_s, Z^{\infty,(p)}_s, U^{\infty,(p)}(s,\cdot)\big)\,\mathrm{d}C^\infty_s. \tag{LS$^{(p)}$}$$

holds for every $p \in \mathbb{N}\cup\{0\}$. The last argument is indeed valid in view of the continuity of the limit process (recall **(B9).(iii)** and in particular that $\Phi^\infty = 0$), Proposition I.120 and Corollary I.122.

To sum up, in order to obtain the convergence (IV.14) for every $p \in \mathbb{N}\cup\{0\}$ we need to prove the Convergence (LS$^{(p)}$) as well as the Convergence (LS$^{(p)}_\infty$).

Let us comment now on the way that we will obtain the Convergence (IV.5). Initially recall the form of the sequence $(Y^{k,(p)})_{k\in\overline{\mathbb{N}}}$ for every $p \in \mathbb{N}\cup\{0\}$. By (IV.15) and (IV.17) we have for $p \in \mathbb{N}\cup\{0\}$

$$Y^{k,(p+1)} = M^{k,(p)} - \int_{(0,\cdot]} f^k\big(s, Y^{k,(p)}_s, Z^{k,(p)}_s, U^{k,(p)}(s,\cdot)\big)\,\mathrm{d}C^\infty_s$$

where $Y^{k,(0)}$ is the zero process $k \in \overline{\mathbb{N}}$ and $M^{k,(0)}$ is determined by the triplet $\big(Z^{k,(0)}, U^{k,(0)}, N^{k,(0)}\big)$ for every $k \in \overline{\mathbb{N}}$. Therefore, for $p \in \mathbb{N}$, $Y^{k,(p)}$ is a $\mathbb{G}^k-$semimartingale, for every $k \in \overline{\mathbb{N}}$, but not necessarily in $\mathcal{S}_{\mathrm{sp}}(\mathbb{G}^k;\mathbb{R})$ because of the presence of the càdlàg and $\mathbb{G}^k-$adapted process Lebesgue–Stieltjes integral. In particular, for $(M^{k,(p)})_{k\in\overline{\mathbb{N}}}$ we conclude that it is P-UT by Lemma I.143 as $\mathrm{J}_1(\mathbb{R})-$convergent martingale sequence and $\mathbb{L}^2-$bounded; we have assumed the validity of (IV.20). For the sequence (for fixed $p \in \mathbb{N}\cup\{0\}$) of the Lebesgue–Stieltjes integrals we can conclude it is also P-UT by Proposition I.151 and the convergence (LS$^{(p)}$). Therefore, $(Y^{k,(p)})_{k\in\overline{\mathbb{N}}}$ is P-UT, for every $p \in \mathbb{N}$, as sum of P-UT sequences. Consequently, by Theorem I.152 we obtain the convergence

$$\big[Y^{k,(p)}\big] \xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R}),\mathbb{P}} \big[Y^{\infty,(p)}\big] \text{ for every } p \in \mathbb{N}.$$

Not surprisingly, we will apply Theorem I.159 for the sequence $([Y^{k,(p)}])_{k\in\overline{\mathbb{N}}}$, for every $p \in \mathbb{N}$. It is immediate by Proposition I.98 that for every $k \in \overline{\mathbb{N}}$ $[Y^{k,(p)}]$ is a special $\mathbb{G}^k$-martingale with canonical decomposition $[Y^{k,(p)}] = ([Y^{k,(p)}] - \langle Y^{k,(p)}\rangle) + \langle Y^{k,(p)}\rangle$, for every $p \in \mathbb{N}$. Then, using standard arguments and the polarisation identity we are able to obtain Convergence (IV.5). □

It is clear by the arguments provided in the proof of Theorem IV.10 that we are going to prove by induction on $p \in \mathbb{N}\cup\{0\}$ the Convergence (LS$^{(p)}$) and (LS$^{(p)}_\infty$), which we restate here for the convenience

of the reader:

$$\int_{(0,\cdot]} f^k\big(s, Y_s^{k,(p)}, Z_s^{k,(p)}, U^{k,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^k \xrightarrow[k\to\infty]{(\mathrm{J}_1(\mathbb{R}),\mathbb{L}^2)} \int_{(0,\cdot]} f^\infty\big(s, Y_s^{\infty,(p)}, Z_s^{\infty,(p)}, U^{\infty,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^\infty \tag{LS$^{(p)}$}$$

and

$$\int_{(0,\infty)} f^k\big(s, Y_s^{k,(p)}, Z_s^{k,(p)}, U^{k,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^k \xrightarrow[k\to\infty]{\mathbb{L}^2(\mathcal{G};\mathbb{R})} \int_{(0,\infty)} f^\infty\big(s, Y_s^{\infty,(p)}, Z_s^{\infty,(p)}, U^{\infty,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^\infty. \tag{LS$_\infty^{(p)}$}$$

Notation IV.12. In the next sections we will additionally use the following notation.

- $L_t^{k,(p)} := \int_{(0,t]} f^k\big(s, Y_s^{k,(p)}, Z_s^{k,(p)}, U^{k,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^k$, for $t\in[0,\infty)$, $k\in\overline{\mathbb{N}}$ and $p\in\mathbb{N}\cup\{0\}$.
- $L_\infty^{k,(p)} := \int_{(0,\infty)} f^k\big(s, Y_s^{k,(p)}, Z_s^{k,(p)}, U^{k,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^k$, for $k\in\overline{\mathbb{N}}$ and $p\in\mathbb{N}\cup\{0\}$.
- $\Gamma^{k,(p)} := \int_{(0,\infty)} \mathrm{e}^{\hat\beta A_s^k} \frac{\left|f^k\big(s, Y_s^{k,(p)}, Z_s^{k,(p)}, U^{k,(p)}(s,\cdot)\big)\right|^2}{(\alpha_s^k)^2}\,\mathrm{d}C_s^k$, for $k\in\overline{\mathbb{N}}$ and $p\in\mathbb{N}\cup\{0\}$.
- $\Delta^{k,(p)} := \mathrm{Tr}\Big[\langle Z^{k,(p)}\cdot X^{k,\circ} + U^{k,(p)}\star\widetilde{\mu}^k\rangle_\infty\Big]$, for $k\in\mathbb{N}$ and $p\in\mathbb{N}\cup\{0\}$.
- $\Delta^{\infty,(p)} := \mathrm{Tr}\Big[\langle Z^{\infty,(p)}\cdot X^{\infty,c} + U^{k,(p)}\star\widetilde{\mu}^\infty\rangle_\infty\Big]$, $p\in\mathbb{N}\cup\{0\}$.
- $\mu_{C^k(\omega)}$ denotes the measure on $\big(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+)\big)$ with associated distribution function $C^k(\omega)$ $\mathbb{P}-$almost surely.

IV.4. The first step of the induction is valid

Recall that in the proof of Proposition IV.11 we have set $\big(Y^{k,(0)}, Z^{k,(0)}, U^{k,(0)}\big) := (0,0,\mathbf{0})$ for every $k\in\overline{\mathbb{N}}$. Now we provide some useful lemmata that we will utilise for proving the first step of the induction in Proposition IV.16.

Lemma IV.13. *The sequence* $(C^k)_{k\in\overline{\mathbb{N}}}$ *is* P-UT.

Proof. Firstly recall the fact that $(C^k)_{k\in\overline{\mathbb{N}}}$ is a sequence of increasing processes. Therefore, $\mathrm{Var}(C^k) = C^k$, for every $k\in\overline{\mathbb{N}}$. On the other hand, Condition **(B9).(i)** implies that $(C^k)_{k\in\overline{\mathbb{N}}}$ is tight in $\mathbb{D}(\mathbb{R})$, which in turn implies that $\mathrm{Var}(C^k)_t$ is tight in $\mathbb{R}$ for every $t\in\mathbb{R}_+$. Now, we can conclude by Proposition I.151. □

Lemma IV.14. *For any subsequence* $(C^{k_l})_{l\in\mathbb{N}}$ *there exists a further subsequence* $(C^{k_{l_m}})_{m\in\mathbb{N}}$ *such that*

$$C^{k_{l_m}} \xrightarrow[m\to\infty]{\mathrm{J}_1(\mathbb{R})} C^\infty \ \mathbb{P}-\textit{almost surely as well as } C_\infty^{k_{l_m}} \xrightarrow[m\to\infty]{\mathrm{J}_1(\mathbb{R})} C_\infty^\infty \ \mathbb{P}-\textit{almost surely.}$$

Moreover, $\mu_{C^{k_{l_m}}} \xrightarrow{\mathrm{w}} \mu_{C^\infty}$ $\mathbb{P}-$*almost surely.*

Proof. The first statement is direct by Theorem I.141. Indeed, the fact that $(\mathbb{D}(\mathbb{R}),\delta_{\mathrm{J}_1})$ and $(\mathbb{R},\delta_{|\cdot|})$ are both Polish and Conditions **(i)**, **(ii)** of **(B9)** allow us to verify the statement. Passing possibly to a further subsequence we can assume without loss of generality that both convergent sequences are indexed by $(k_{l_m})_{k\in\mathbb{N}}$. The second statement is also true in view of Condition **(B9).(iii)**, where the condition $\Phi^\infty = 0$ implies that C^∞ is continuous, Lemma A.13 and Proposition A.12. □

Lemma IV.15. *The sequence* $\big\{\sup_{t\in\mathbb{R}_+}\big|L_t^{k,(0)}\big|^2, k\in\overline{\mathbb{N}}\big\}$ *is uniformly integrable.*

Proof. Using Cauchhy–Schwarz Inequality as in (II.25) we can obtain for every $t\in[0,\infty]$ and $k\in\overline{\mathbb{N}}$

$$|L_t^{k,(0)}|^2 \le \frac{1}{\hat\beta}\int_{(0,t]} \mathrm{e}^{\hat\beta A_s^k}\frac{\left|f^k(s,0,0,\mathbf{0})\right|^2}{(\alpha_s^k)^2}\,\mathrm{d}C_s^k \le \frac{1}{\hat\beta}\int_{(0,\infty)} \mathrm{e}^{\hat\beta A_s^k}\frac{\left|f^k(s,0,0,\mathbf{0})\right|^2}{(\alpha_s^k)^2}\,\mathrm{d}C_s^k.$$

Since, by **(B8)**, the right-hand side is independent of t and uniformly integrable, we obtain the required result by Corollary I.29.(iv).

□

Proposition IV.16. *The first step of the induction is valid, i.e.*

$$\int_{(0,\cdot]} f^k\big(s, Y_s^{k,(0)}, Z_s^{k,(0)}, U^{k,(0)}(s,\cdot)\big)\,\mathrm{d}C_s^k \xrightarrow[k\to\infty]{(\mathrm{J}_1(\mathbb{R}),\mathbb{L}^2)} \int_{(0,\cdot]} f^\infty\big(s, Y_s^{\infty,(0)}, Z_s^{\infty,(0)}, U^{\infty,(0)}(s,\cdot)\big)\,\mathrm{d}C_s^\infty. \tag{LS$^{(0)}$}$$

and

$$\int_{(0,\infty)} f^k\big(s, Y_s^{k,(0)}, Z_s^{k,(0)}, U^{k,(0)}(s,\cdot)\big)\,\mathrm{d}C_s^k \xrightarrow[k\to\infty]{\mathbb{L}^2(\mathcal{G};\mathbb{R}^n)} \int_{(0,\infty)} f^\infty\big(s, Y_s^{\infty,(0)}, Z_s^{\infty,(0)}, U^{\infty,(0)}(s,\cdot)\big)\,\mathrm{d}C_s^\infty. \tag{LS$^{(0)}_\infty$}$$

Proof. By definition $\big(Y^{k,(0)}, Z^{k,(0)}, U^{k,(0)}\big) := (0,0,\mathbf{0})$ for every $k \in \overline{\mathbb{N}}$. We are going to apply Vitali's theorem (see Theorem I.32), *i.e.* we will prove initially the Convergence (LS$^{(0)}$) and (LS$^{(0)}_\infty$) in probability. This will be proven by means of Theorem I.141. Let a subsequence $\big(L_\infty^{k_l,(0)}\big)_{l\in\mathbb{N}}$. By Lemma IV.14 there exists a further subsequence $(k_{l_m})_{m\in\mathbb{N}}$ such that $\mu_{C^{k_{l_m}}} \xrightarrow{\mathrm{w}} \mu_{C^\infty}$ $\mathbb{P}-$almost surely. Consequently, we have also that $\sup_{m\in\overline{\mathbb{N}}} \mu_{C^{k_{l_m}}}(\mathbb{R}_+) < \infty$ $\mathbb{P}-$almost surely.

Therefore,

$$\begin{aligned}
&\left|\int_{(0,\infty)} f^{k_{l_m}}\big(s,0,0,\mathbf{0}\big)\,\mathrm{d}C_s^{k_{l_m}} - \int_{(0,\infty)} f^\infty\big(s,0,0,\mathbf{0}\big)\,\mathrm{d}C_s^\infty\right| \\
&\quad\le \left|\int_{(0,\infty)} f^{k_{l_m}}\big(s,0,0,\mathbf{0}\big)\,\mathrm{d}C_s^{k_{l_m}} - \int_{(0,\infty)} f^\infty\big(s,0,0,\mathbf{0}\big)\,\mathrm{d}C_s^{k_{l_m}}\right| \\
&\qquad + \left|\int_{(0,\infty)} f^\infty\big(s,0,0,\mathbf{0}\big)\,\mathrm{d}C_s^{k_{l_m}} - \int_{(0,\infty)} f^\infty\big(s,0,0,\mathbf{0}\big)\,\mathrm{d}C_s^\infty\right| \\
&\quad\le \int_{(0,\infty)} \left|f^{k_{l_m}}\big(s,0,0,\mathbf{0}\big) - f^\infty\big(s,0,0,\mathbf{0}\big)\right|_\infty \mathrm{d}C_\infty^{k_{l_m}} \\
&\qquad + 2\left|\int_{(0,\infty)} f^\infty\big(s,0,0,\mathbf{0}\big)\,\mathrm{d}C_s^{k_{l_m}} - \int_{(0,\infty)} f^\infty\big(s,0,0,\mathbf{0}\big)\,\mathrm{d}C_s^\infty\right| \\
&\quad\le \left|f^{k_{l_m}}\big(s,0,0,\mathbf{0}\big) - f^\infty\big(s,0,0,\mathbf{0}\big)\right|_\infty \sup_{k\in\overline{\mathbb{N}}} \mu_{C^{k_{l_m}}}(\mathbb{R}_+) \\
&\qquad + \left|\int_{(0,\infty)} f^\infty\big(s,0,0,\mathbf{0}\big)\,\mathrm{d}C_s^{k_{l_m}} - \int_{(0,\infty)} f^\infty\big(s,0,0,\mathbf{0}\big)\,\mathrm{d}C_s^\infty\right| \xrightarrow[m\to\infty]{} 0 \ \mathbb{P}-a.s.,
\end{aligned} \tag{IV.22}$$

The first summand converges to zero in view of Condition **(B7).(iii)** and the finiteness of $\sup_{m\in\overline{\mathbb{N}}} \mu_{C^{k_{l_m}}}(\mathbb{R}_+)$. The second summand converges to zero in view of the càdlàg property of the paths of $f^\infty(\cdot,0,0,\mathbf{0})$ and $\mu_{C^{k_{l_m}}} \xrightarrow{\mathrm{w}} \mu_{C^\infty}$; use either Theorem A.14 or Mazzone [51, Theorem 1]. Therefore, the subsequence $(L_\infty^{k_{l_m},(0)})_{m\in\mathbb{N}}$ converges $\mathbb{P}-a.s.$ to $L_\infty^{\infty,(0)}$. By Theorem I.141, we obtain the convergence $L_\infty^{k,(0)} \xrightarrow{\mathbb{P}} L_\infty^{\infty,(0)}$, which can be strengthened to (LS$^{(0)}_\infty$) in view of Lemma IV.15 and Vitali's theorem.

We will prove now the Convergence (LS$^{(0)}$). Initially we will obtain the convergence in probability. To this end, recall that by Lemma IV.13 the sequence $(C^k)_{k\in\overline{\mathbb{N}}}$ is P-UT. Now, by Proposition I.151, it is sufficient, since C^∞ is continuous, to prove the convergence of the integrands under the J_1-topology in probability. But this is immediate in view of **(B7).(iii)**. To conclude indeed the Convergence (LS$^{(0)}$) we have only to recall Lemma IV.15 and the inequality for $\alpha,\beta \in \mathbb{D}(\mathbb{R}^n)$

$$\delta_{\mathrm{J}_1(\mathbb{R}^n)}(\alpha,\beta) \le \|\alpha\|_\infty + \|\beta\|_\infty.$$

□

Although we have accomplished the aim this section, we will need to obtain some further results. The reason is that we need to complement our induction hypothesis with the assumption that the uniform integrability of the sequence[5]

$$\left(\int_{(0,\infty)} \mathrm{e}^{\hat\beta A_s^k} \frac{\big\|f^k\big(s, Y^{k,(p)}, Z_s^{k,(p)}, U^{k,(p)}(s,\cdot)\big)\big\|_2^2}{(\alpha_s^k)^2}\,\mathrm{d}C_s^k\right)_{k\in\overline{\mathbb{N}}}$$

[5]We could have abstained from writing the processes A^k, for $k \in \overline{\mathbb{N}}$, since they are bounded.

is inherited by the uniform integrability of the sequence

$$\left(\int_{(0,\infty)} \mathrm{e}^{\hat{\beta}A^k_s} \frac{\left\|f^k\big(s, Y^{k,(p-1)}, Z^{k,(p-1)}_s, U^{k,(p-1)}(s,\cdot)\big)\right\|_2^2}{(\alpha^k_s)^2}\,\mathrm{d}C^k_s\right)_{k\in\overline{\mathbb{N}}}$$

and that of the sequences

$$\left(\left|\xi^k\right|^2\right)_{k\in\overline{\mathbb{N}}} \text{ and } \left(\int_{(0,\infty)} \mathrm{e}^{\hat{\beta}A^k_s} \frac{\left\|f^k\big(s,0,0,\mathbf{0}\big)\right\|_2^2}{(\alpha^k_s)^2}\,\mathrm{d}C^k_s\right)_{k\in\overline{\mathbb{N}}}.$$

Corollary IV.17. *The convergence*

$$\big(Y^{k,(1)}, Z^{k,(1)}\cdot X^{k,\circ} + U^{k,(1)}\star\widetilde{\mu}^k, N^{k,(1)}\big) \xrightarrow[k\to\infty]{(\mathrm{J}_1(\mathbb{R}^3),\mathbb{L}^2)} \big(Y^{\infty,(1)}, Z^{\infty,(1)}\cdot X^{\infty,c} + U^{\infty,(1)}\star\widetilde{\mu}^\infty, 0\big)$$

and

$$\big(\langle Z^{k,(1)}\cdot X^{k,\circ}\rangle, \langle U^{k,(1)}\star\widetilde{\mu}^k\rangle, \langle N^{k,(1)}\rangle\big) \xrightarrow[k\to\infty]{(\mathrm{J}_1(\mathbb{R}^3),\mathbb{L}^2)} \big(\langle Z^{\infty,(1)}\cdot X^{\infty,c}\rangle, \langle U^{\infty,(1)}\star\widetilde{\mu}^\infty\rangle, 0\big)$$

are valid.

Proof. In view of the discussion in the body of the main theorem, the validity of $\mathrm{LS}^{(0)}$ and $\mathrm{LS}^{(0)}_\infty$ allows us to apply Theorem III.50 and obtain (IV.21) for $p=0$ and the second convergence of the statement above. However, the subsequent to (IV.21) comments allow us to obtain also the first convergence of the theorem. □

Remark IV.18. Observe that $Y^{k,(1)}_\infty, Y^{(1)}_\infty$ are well-defined, for every $k\in\mathbb{N}$, $\mathbb{P}$-almost surely. Indeed, by (IV.15) and (IV.17)

$$Y^{\infty,(1)}_t = M^{\infty,(0)}_t - \int_0^t f^\infty\big(s, Y^{\infty,(0)}_s, Z^{\infty,(0)}_s, U^{\infty,(0)}(s,\cdot)\big)\,\mathrm{d}C^\infty_s$$

and

$$Y^{k,(1)}_t = M^{k,(0)}_t - \int_{(0,t]} f^k\big(s, Y^{k,(0)}_s, Z^{k,(0)}_s, U^{k,(0)}(s,\cdot)\big)\,\mathrm{d}C^k_s \text{ for every } k\in\mathbb{N}.$$

The limit at ∞ of the martingale part exists due to their square integrability and the limit of the Lebesgue-Stieltjes integrals exists $\mathbb{P}$-almost surely.

The following results will allow us to conclude the uniform integrability of the families $\big\{\big|L^{k,(0)}_t\big|^2, t\in[0,\infty)$ and $k\in\overline{\mathbb{N}}\big\}$ and $\big(\mathrm{Tr}\big[\langle M^{k,(0)}\rangle_\infty\big]\big)_{k\in\overline{\mathbb{N}}}$. The reader may recall Notation IV.12 on p. 113.

Lemma IV.19. *The sequence* $\big(\sup_{t\in\mathbb{R}_+}\big|M^{k,(0)}_t - M^{k,(0)}_0\big|^2\big)_{k\in\overline{\mathbb{N}}}$ *is uniformly integrable. In particular, the sequence* $(\Delta^{k,(1)})_{k\in\overline{\mathbb{N}}}$ *is uniformly integrable.*

Proof. Let us start with an observation. The martingale $M^{k,(0)}$, for $k\in\overline{\mathbb{N}}$, was defined in (IV.15) for $k=\infty$ and in (IV.17) for $k\in\mathbb{N}$. However, we will use the equivalent forms (IV.16) and (IV.18). Now, we proceed to prove the claim of the lemma. We will make use of the pathwise estimates in Lemma II.18 and in particular (II.39). In view of **(B6)** and of Corollary I.29.(i) we can substitute the constants that appear by one independent of k, say B. We have, now, for every $k\in\overline{\mathbb{N}}$ that

$$\begin{aligned}&\sup_{t\in\mathbb{R}_+}\big|M^{k,(0)}_t - M^{k,(0)}_0\big|^2\\ &\qquad\le C\sup_{t\in\mathbb{R}_+}\mathbb{E}\left[\left|\xi^k\right|^2 + \int_{(0,\infty)} \mathrm{e}^{\hat{\beta}A^k_s}\frac{\left|f^k\big(s,0,0,\mathbf{0}\big)\right|^2}{(\alpha^k_s)^2}\,\mathrm{d}C^k_s\middle|\mathcal{G}_t\right] + C\int_{(0,\infty)} \mathrm{e}^{\hat{\beta}A^k_s}\frac{\left|f^k\big(s,0,0,\mathbf{0}\big)\right|^2}{(\alpha^k_s)^2}\,\mathrm{d}C^k_s.\end{aligned}$$

At this point, we observe that the right hand side consists of elements of uniformly integrable sequence. Let us argue a bit more on this. By **(B5)** and Vitali's Theorem we have that the sequence $\left(\left|\xi^k\right|^2\right)_{k\in\overline{\mathbb{N}}}$ is uniformly integrable and by **(B8)** we have that the sequence

$$\left(\int_{(0,\infty)} \mathrm{e}^{\hat{\beta}A^k_s}\frac{\left|f^k\big(s,0,0,\mathbf{0}\big)\right|^2}{(\alpha^k_s)^2}\,\mathrm{d}C^k_s\right)_{k\in\overline{\mathbb{N}}}.$$

Observe that the elements of the latter sequence appear in the above bound twice. By Corollary I.29.(iv), it is therefore sufficient for the validity of our claim to prove that the sequence

$$\Big(\sup_{t\in\mathbb{R}_+}\mathbb{E}\Big[\big|\xi^k\big|^2+\int_{(0,\infty)}\mathrm{e}^{\hat\beta A^k_s}\frac{\big|f^k\big(s,0,0,\mathbf{0}\big)\big|^2}{(\alpha^k_s)^2}\,\mathrm{d}C^k_s\Big|\mathcal{G}^k_t\Big]\Big)_{k\in\overline{\mathbb{N}}} \tag{IV.23}$$

is uniformly integrable. But this is Lemma I.106 in conjunction with Corollary I.29.(iv).

We proceed now to prove that the second sequence is uniformly integrable. We will prove initially that the sequence $\big(\langle M^{k,(0)}-M^{k,(0)}_0\rangle_\infty\big)_{k\in\overline{\mathbb{N}}}$ is uniformly integrable. By the previous step and by de La Vallée Poussin–Meyer Criterion there exists $\Upsilon\in\mathcal{YF}_{\mathrm{mod}}$ such that

$$\sup_{k\in\overline{\mathbb{N}}}\mathbb{E}\Big[\Upsilon\Big(\sup_{t\in\mathbb{R}_+}\big|M^{k,(0)}_t-M^{k,(0)}_0\big|^2\Big)\Big]<\infty. \tag{IV.24}$$

We apply now Theorem I.55 for every $k\in\overline{\mathbb{N}}$ for the $\mathbb{G}^k-$submartingale $\big|M^{k,(0)}_\cdot-M^{k,(0)}_0\big|^2$ and we obtain

$$\mathbb{E}\Big[\Upsilon\big(\langle M^{k,(0)}-M^{k,(0)}_0\rangle_\infty\big)\Big]\le(2\bar c_\Upsilon)^{\bar c_\Upsilon}\,\mathbb{E}\Big[\Upsilon\Big(\sup_{t\in\mathbb{R}_+}\big|M^{k,(0)}_t-M^{k,(0)}_0\big|^2\Big)\Big],$$

which, in view of (IV.24) and the de La Vallée Poussin–Meyer Criterion, yields the uniform integrability of the sequence $\mathrm{Tr}\Big[\big\langle M^{k,(0)}-M^{k,(0)}_0\big\rangle_\infty\Big]$.

Now, in order to prove the uniform integrability of $(\Delta^{k,(1)})$ we have initially to use the fact that $N^{k,(1)}\in\mathcal{H}^{2,\perp}(\mathbb{G}^k,X^{k,\circ},\mu^{X^{k,\natural}};\mathbb{R}^n)$ for every $k\in\mathbb{N}$ and $N^{\infty,(1)}\in\mathcal{H}^{2,\perp}(\mathbb{G}^\infty,X^{\infty,c},\mu^{X^{\infty,d}};\mathbb{R}^n)$. Then the domination

$$0\le\Delta^{k,(1)}\le\Delta^{k,(1)}+\langle N^{k,(1)}\rangle_\infty=\langle M^{k,(0)}\rangle_\infty$$

and the uniform integrability of the sequence consisting of the dominant elements of the above inequality imply the required result. □

Lemma IV.20. *The sequence* $(\Gamma^{k,(1)})_{k\in\overline{\mathbb{N}}}$ *is uniformly integrable.*

Proof. We will prove the pathwise estimates we have obtained in Lemma II.18 and more precisely the Estimate (II.37). As we did in the previous lemma, in view of **(B6)** and of Corollary I.29.(i), we will substitute the coefficients that appear on the right-hand side of the Estimate (II.37) by a constant independent of k, say C. Let, now, $k\in\mathbb{N}$. By definition of $\Gamma^{k,(1)}$ we have

$$\begin{aligned}
\Gamma^{k,(1)} &= \int_{(0,\infty)}\mathrm{e}^{\hat\beta A^k_s}\frac{\big|f^k\big(s,Y^{k,(1)}_s,Z^{k,(1)}_s,U^{k,(1)}(s,\cdot)\big)\big|^2}{(\alpha^k_s)^2}\,\mathrm{d}C^k_s\\
&\underset{\text{(II.49)}}{\overset{\text{(II.16)}}{\le}}\int_{(0,\infty)}\mathrm{e}^{\hat\beta A^k_s}\Big[(\alpha^k_s)^2\big|Y^{k,(1)}_s\big|^2+\big|c^k_sZ^{k,(1)}_s\big|^2+\big|\big|\big|U^{k,(1)}_s(\cdot)\big|\big|\big|\Big]\,\mathrm{d}C^k_s+\int_{(0,\infty)}\mathrm{e}^{\hat\beta A^k_s}\frac{\big|f^k\big(s,0,0,\mathbf{0}\big)\big|^2}{(\alpha^k_s)^2}\,\mathrm{d}C^k_s\\
&\underset{\text{Lem. II.5}}{\overset{\text{(II.17)}}{=}}\int_{(0,\infty)}\mathrm{e}^{\hat\beta A^k_s}\big|Y^{k,(1)}_s\big|^2\,\mathrm{d}A^k_s+\int_{(0,\infty)}\mathrm{e}^{\hat\beta A^k_s}\,\mathrm{d}\langle Z^{k,(1)}\cdot X^{k,\circ}+U^{k,(1)}\star\widetilde\mu^k\rangle_s\\
&\qquad+\int_{(0,\infty)}\mathrm{e}^{\hat\beta A^k_s}\frac{\big|f^k\big(s,0,0,\mathbf{0}\big)\big|^2}{(\alpha^k_s)^2}\,\mathrm{d}C^k_s\\
&\underset{\text{Lem. II.18}}{\overset{\text{(II.17)}}{\le}}C\sup_{t\in\mathbb{R}_+}\mathbb{E}\Big[\big|\xi\big|^2\Big|\mathcal{G}^k_t\Big]+C\sup_{t\in\mathbb{R}_+}\mathbb{E}\Big[\int_{(0,\infty)}\mathrm{e}^{\hat\beta A^k_s}\frac{\big|f^k\big(s,0,0,\mathbf{0}\big)\big|^2}{(\alpha^k_s)^2}\,\mathrm{d}C^k_s\Big|\mathcal{G}^k_t\Big]\\
&\qquad+\mathrm{e}^{\hat\beta A^k_\infty}\langle Z^{k,(1)}\cdot X^{k,\circ}+U^{k,(1)}\star\widetilde\mu^k\rangle_\infty+\int_{(0,\infty)}\mathrm{e}^{\hat\beta A^k_s}\frac{\big|f^k\big(s,0,0,\mathbf{0}\big)\big|^2}{(\alpha^k_s)^2}\,\mathrm{d}C^k_s
\end{aligned}$$

The analogous bound can be proven to hold also for $k=\infty$. We observe again at this point that the elements of the right hand side forms a uniformly integrable sequence. For the sequence associated to the first two summands we use analogous arguments to (IV.23). For the sequence associated to the third summand we use Condition **(B6)** and Lemma IV.19. Finally, the sequence associated to the last summand is uniformly integrable by **(B8)**. Then, Corollary I.29.(iv) allows us to conclude. □

Proposition IV.21. *The sequence* $\big\{\sup_{t\in\mathbb{R}_+}\big|L^{k,(1)}_t\big|^2,k\in\overline{\mathbb{N}}\big\}$ *is uniformly integrable.*

Proof. Using the Cauchy–Schwarz Inequality as in (II.25) we can obtain for every $t \in [0,\infty]$ and $k \in \overline{\mathbb{N}}$

$$\begin{aligned}|L_t^{k,(1)}|^2 &\leq \frac{1}{\hat{\beta}} \int_{(0,t]} \mathrm{e}^{\hat{\beta} A_s^k} \frac{\left|f^k\big(s, Y^{k,(1)}, Z^{k,(1)}, U^{k,(1)}(t,\cdot)\big)\right|^2}{(\alpha_s^k)^2} \, \mathrm{d}C_s^k \\ &\leq \frac{1}{\hat{\beta}} \int_{(0,\infty)} \mathrm{e}^{\hat{\beta} A_s^k} \frac{\left|f^k\big(s, Y^{k,(1)}, Z^{k,(1)}, U^{k,(1)}(t,\cdot)\big)\right|^2}{(\alpha_s^k)^2} \, \mathrm{d}C_s^k.\end{aligned}$$

Since, the right-hand side is independent of t and uniformly integrable by Lemma IV.20, we obtain the required result by Corollary I.29.(iv). □

Remark IV.22. The validity of Lemma IV.20 allows us indeed to complement our induction step with the statement

$$\text{the sequence } \big(\Gamma^{k,(p)}\big)_{k\in\overline{\mathbb{N}}} \text{ is uniformly integrable} \tag{$\Gamma^{(p)}-$UI}$$

which will allow us to prove that the sequence $\big\{\sup_{t\in\mathbb{R}_+}\big|L_t^{k,(p)}\big|^2, k \in \overline{\mathbb{N}}\big\}$ is uniformly integrable.

IV.5. The $p-$step of the induction is valid

In this section we assume that the Convergence

$$L^{k,(p-1)} \xrightarrow[k\to\infty]{\left(\mathrm{J}_1(\mathbb{R}),\mathbb{L}^2\right)} L^{\infty,(p-1)} \tag{LS$^{(p-1)}$}$$

and

$$L_\infty^{k,(p-1)} \xrightarrow[k\to\infty]{\mathbb{L}^2(\mathcal{G};\mathbb{R})} L_\infty^{\infty,(p-1)} \tag{LS$_\infty^{(p-1)}$}$$

as well as the statement

$$\text{the sequences } \big(\Gamma^{k,(p-1)}\big)_{k\in\overline{\mathbb{N}}} \text{ and } \big(\Delta^{k,(p-1)}\big)_{k\in\overline{\mathbb{N}}} \text{ are uniformly integrable} \tag{Step $(p-1)$ UI}$$

are true for some arbitrary but fixed $p \in \mathbb{N}$. We will prove the Convergence (LS$^{(p)}$) and (LS$_\infty^{(p)}$) as well as the statement

$$\text{the sequences } \big(\Gamma^{k,(p)}\big)_{k\in\overline{\mathbb{N}}} \text{ and } \big(\Delta^{k,(p)}\big)_{k\in\overline{\mathbb{N}}} \text{ are uniformly integrable} \tag{Step (p) UI}$$

are true.

Comparing to the first step of the induction the p-step is more involved, so let us articulate in the way we are going to reduce the complexity. In view of Vitali's Theorem, it is sufficient to prove initially that the Convergence (LS$^{(p)}$) and (LS$_\infty^{(p)}$) holds in probability and then we have to prove that the sequences are sufficiently uniformly integrable. Now, in order to obtain the aforementioned convergence in probability, we are going to use that $\big(\mathbb{D}(\mathbb{R}), \delta_{\mathrm{J}_1(\mathbb{R})}\big)$ and $\big(\mathbb{R}, |\cdot|\big)$ are Polish spaces. Therefore, in view of Theorem I.141, it is sufficient to prove that from every subsequence $\big(L^{k_l,(p)}\big)_{l\in\mathbb{N}}$, resp. $\big(L_\infty^{k_l,(p)}\big)_{l\in\mathbb{N}}$, we can extract a further subsequence $\big(L^{k_{l_m},(p)}\big)_{m\in\mathbb{N}}$, resp. $\big(L_\infty^{k_{l_m},(p)}\big)_{m\in\mathbb{N}}$, such that

$$L^{k_{l_m},(p)} \xrightarrow[m\to\infty]{\mathrm{J}_1(\mathbb{R})} L^{\infty,(p)} \ \mathbb{P}-a.s. \tag{IV.25}$$

resp.

$$L_\infty^{k_{l_m},(p)} \xrightarrow[m\to\infty]{|\cdot|} L_\infty^{\infty,(p)} \ \mathbb{P}-a.s.. \tag{IV.26}$$

To this end let us fix an $\varepsilon > 0$ and assume that for $\mathbb{P}-$almost every ω there exist suitable $\big(\widetilde{Z}^{\varepsilon,(p)}(\omega,s)\big)_{s\in\mathbb{R}_+}$ and $\big(\widetilde{U}^{\varepsilon,(p)}(\omega,s,x)\big)_{(s,x)\in\mathbb{R}_+\times\mathbb{R}^\ell}$ such that

$$\left|\int_{(0,\infty)} f^\infty\big(,s,Y_s^{\infty,(p)},\widetilde{Z}_s^{\varepsilon,(p)},\widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^\infty - L_\infty^{\infty,(p)}\right| < \frac{\varepsilon}{4} \ \mathbb{P}-a.s.. \tag{IV.27}$$

Then, using the set inclusions

$$\Big[\Big|\sum_{i=1}^4 X^k\Big| > \varepsilon\Big] \subseteq \Big[\sum_{i=1}^4 |X^k| > \frac{\varepsilon}{4}\Big] \subseteq \Big[\bigcup_{i=1}^4 \Big[|X^k| > \frac{\varepsilon}{4}\Big]\Big] \subseteq \bigcup_{i=1}^4 \Big[|X^k| > \frac{\varepsilon}{4}\Big]$$

we obtain Convergence (IV.26) if[6] for all but finitely many $m \in \mathbb{N}$ holds

$$\begin{aligned}
&\Big[\big|L_\infty^{m,(p)} - L^{\infty,(p)}\big|\Big] \subseteq \\
&\quad \Big[\Big|L_\infty^{m,(p)} - \int_{(0,\infty)} f^m\big(s, Y_s^{m,(p)}, \widetilde{Z}_s^{\varepsilon,(p)}, \widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^m\Big| > \frac{\varepsilon}{4}\Big] \\
&\bigcup\Big[\Big|\int_{(0,\infty)} (f^m - f^\infty)\big(s, Y_s^{m,(p)}, \widetilde{Z}_s^{\varepsilon,(p)}, \widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^m\Big| > \frac{\varepsilon}{4}\Big] \qquad \text{(IV.28)}\\
&\bigcup\Big[\Big|\int_{(0,\infty)} f^\infty\big(s, Y_s^{m,(p)}, \widetilde{Z}_s^{\varepsilon,(p)}, \widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^m - \int_{(0,\infty)} f^\infty\big(s, Y_s^{\infty,(p)}, \widetilde{Z}_s^{\varepsilon,(p)}, \widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^\infty\Big| > \frac{\varepsilon}{4}\Big] \\
&\bigcup\Big[\Big|\int_{(0,\infty)} f^\infty\big(s, Y_s^{\infty,(p)}, \widetilde{Z}_s^{\varepsilon,(p)}, \widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^\infty - L_\infty^{\infty,(p)}\Big| > \frac{\varepsilon}{4}\Big] = \emptyset.
\end{aligned}$$

Therefore, we obtain the Convergence of the the sequence in $(\mathrm{LS}_\infty^{(p)})$ in probability, if we prove that the first three sets of the right-hand side of (IV.28) are empty for all but finitely many $m \in \mathbb{N}$. The fourth one is the empty set if (IV.27) is true.

The analogous decomposition can be done for (IV.25), where the $|\cdot|$ has to be substituted by the metric $\delta_{\mathrm{J}_1(\mathbb{R})}$. Returning now to the uniform integrability that the sequences should satisfy, we will need to prove that the family $\big(\sup_{t\in\mathbb{R}_+} |L_t^{k,(p)}|^2\big)_{k\in\overline{\mathbb{N}}}$ is uniformly integrable, which is a sufficient condition for concluding both the Convergence $(\mathrm{LS}^{(p)})$ and $(\mathrm{LS}_\infty^{(p)})$.

■ From now on we fix an arbitrary subsequence $\big(L^{k_l,(p)}\big)_{l\in\mathbb{N}}$, resp. $\big(L_\infty^{k_l,(p)}\big)_{l\in\mathbb{N}}$.

Let us end this part by collecting all the information we have available for the next subsections. We will state them as a remark so that it is easily referred. Moreover, for notational convenience we can assume that the sequence for which the forthcoming convergence are obtained $\mathbb{P}-$almost surely is indexed by $(k_{l_m})_{m\in\mathbb{N}}$. This can be done without loss of generality, since we pass to a further subsequence finitely many times.

Remark IV.23. (i) By Lemma IV.14 we have that there exist a $\delta_{\mathrm{J}_1(\mathbb{R})}-$convergent subsequence $\big(C^{k_{l_m}}\big)_{m\in\overline{\mathbb{N}}}$ as well as a $\delta_{|\cdot|}-$convergent subsequence $\big(C_\infty^{k_{l_m}}\big)_{m\in\overline{\mathbb{N}}}$.

(ii) By Condition **(B7).(iii)** we obtain in particular that

$$f^{k_{l_m}}\big(\cdot,0,0,\mathbf{0}\big) \xrightarrow{\mathrm{J}_1(\mathbb{R})} f^\infty\big(\cdot,0,0,\mathbf{0}\big)\ \mathbb{P}-a.s..$$

(iii) The Convergence $(\mathrm{LS}^{(p-1)})$, $(\mathrm{LS}_\infty^{(p-1)})$, which are assumed true as the induction assumption, allows us to obtain that

$$L^{k_{l_m},(p-1)} \xrightarrow[m\to\infty]{\mathrm{J}_1(\mathbb{R})} L^{\infty,(p-1)} \text{ as well as } L_\infty^{k_{l_m},(p-1)} \xrightarrow[m\to\infty]{|\cdot|} L_\infty^{\infty,(p-1)}\ \mathbb{P}-a.s..$$

(iv) In view of the discussion made in the main body of the proof of Theorem IV.10, see in particular on p. 111, the validity of the Convergence $(\mathrm{LS}^{(p-1)})$ and $(\mathrm{LS}_\infty^{(p-1)})$ allows us to apply Theorem III.50 for the martingale sequence $(M^{k,(p-1)})_{k\in\overline{\mathbb{N}}}$. Therefore, we can apply Lemma I.106 because of the uniform integrability of the sequence $\big(\big|M_\infty^{k,(p-1)}\big|^2\big)_{k\in\overline{\mathbb{N}}}$ in order to obtain that

$$\Big(\sup_{t\in\mathbb{R}_+} \big|M_t^{k,(p-1)}\big|^2\Big)_{k\in\overline{\mathbb{N}}} \text{ is uniformly integrable.} \qquad \text{(IV.29)}$$

Moreover, we obtain that the following convergence are true

$$(M^{k,(p-1)}, \langle Z^{k,(p)}\cdot X^{k,\circ}\rangle, \langle U^{k,(p)}\star\widetilde{\mu}^k\rangle) \xrightarrow[k\to\infty]{\big(\mathrm{J}_1(\mathbb{R}^3),\mathbb{L}^2\big)} (M^{\infty,(p-1)}, \langle Z^{\infty,(p)}\cdot X^{\infty,c}\rangle, \langle U^{\infty,(p)}\star\widetilde{\mu}^\infty\rangle). \qquad \text{(IV.30)}$$

and

$$(\langle Z^{k,(p)}\cdot X^{k,\circ}, X^{k,\circ}\rangle, \langle U^{k,(p)}\star\widetilde{\mu}^k, X^{k,\natural}\rangle) \xrightarrow[k\to\infty]{\big(\mathrm{J}_1(\mathbb{R}^2),\mathbb{L}^2\big)} (\langle Z^{\infty,(p)}\cdot X^{\infty,c}, X^{\infty,c}\rangle, \langle U^{\infty,(p)}\star\widetilde{\mu}^\infty, X^{\infty,d}\rangle). \qquad \text{(IV.31)}$$

[6]For notational convenience we index the $k_{l_m}-$element in the next expression simply by m.

(v) In view of (iv) we can assume that

$$(M^{k_{l_m},(p-1)}, \langle Z^{k_{l_m},(p)} \cdot X^{k_{l_m},\circ}\rangle, \langle U^{k_{l_m},(p)} \star \widetilde{\mu}^{k_{l_m}}\rangle) \xrightarrow[m\to\infty]{\mathrm{J}_1(\mathbb{R}^3)} (M^{\infty,(p-1)}, \langle Z^{\infty,(p)} \cdot X^{\infty,c}\rangle, \langle U^{\infty,(p)} \star \widetilde{\mu}^{\infty}\rangle)\ \mathbb{P}-a.s. \tag{IV.32}$$

and

$$(\langle Z^{k_{l_m}(p)} \cdot X^{k_{l_m}\circ}, X^{k_{l_m}\circ}\rangle, \langle U^{k_{l_m}(p)} \star \widetilde{\mu}^{k}, X^{k_{l_m}\natural}\rangle) \xrightarrow[m\to\infty]{\mathrm{J}_1(\mathbb{R}^2)} (\langle Z^{\infty,(p)} \cdot X^{\infty,c}, X^{\infty,c}\rangle, \langle U^{\infty,(p)} \star \widetilde{\mu}^{\infty}, X^{\infty,d}\rangle)\ \mathbb{P}-a.s.. \tag{IV.33}$$

(vi) Recall (IV.15) and (IV.17), *i.e.* $Y^{k,(p)} = M^{k,(p-1)} - L^{k,(p-1)}$ for every $k \in \overline{\mathbb{N}}$. we claim that

$$\Big(\sup_{t\in\mathbb{R}_+} \big|Y_t^{k,(p)}\big|^2\Big)_{k\in\overline{\mathbb{N}}} \text{ is uniformly integrable.} \tag{IV.34}$$

This can be concluded as follows. We have

$$\sup_{t\in\mathbb{R}_+} \big|Y_t^{k,(p)}\big|^2 \le 2\sup_{t\in\mathbb{R}_+} \big|M_t^{k,(p-1)}\big|^2 + 2\sup_{t\in\mathbb{R}_+} \big|L_t^{k,(p-1)}\big|^2 \text{ for every } k \in \overline{\mathbb{N}}.$$

For the sequence $\Big(\sup_{t\in\mathbb{R}_+} \big|L_t^{k,(p-1)}\big|^2\Big)_{k\in\overline{\mathbb{N}}}$ we can conclude its uniform integrability by arguing analogously to Proposition IV.21 since $(\Gamma^{k,(p-1)})_{k\in\overline{\mathbb{N}}}$ has been assumed uniformly integrable; see (Step $(p-1)$ UI). Now, we can conclude our initial statement by means of Corollary I.29.(iv) and because of (IV.29).

(vii) In view of Condition **(B9).(iii)**, which implies that $L^{\infty,(p-1)}$ is a continuous process, (iii), (v), Proposition I.121 and Corollary I.122, we can conclude that

$$Y^{k_{l_m},(p)} \xrightarrow[m\to\infty]{\mathrm{J}_1(\mathbb{R}^n)} Y^{\infty,(p)}\ \mathbb{P}-a.s..$$

Remark IV.24. The purpose of the above remarks was not only to collect all the available information, but also to provide us with an $\Omega_{\text{sub}} \subset \Omega$ with $\mathbb{P}(\Omega_{\text{sub}}) = 1$ such that the above properties hold for every $\omega \in \Omega_{\text{sub}}$. In the next subsections whenever we say a property holds $\mathbb{P}-$almost surely for a subsequence indexed by $(k_{l_m})_{m\in\overline{\mathbb{N}}}$, the reader understands for every $\omega \in \Omega_{\text{sub}}$.

Now, we are ready to proceed to prove our claim under this convenient framework.

IV.5.1. Lusin approximation. In this section we provide a pathwise approximation for the fourth summand of (IV.28). The reader may revise Notation II.2, Notation II.6 and Notation IV.5. We start with a lemma which justifies the validity of the approximation we will obtain.

Lemma IV.25. *Let $\widetilde{Y}, \widetilde{Z}$ and $\widetilde{U}$ with the following properties*

- $\widetilde{Y} : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ *is a process with càdlàg and bounded $\omega-$paths for $\omega \in \Omega_{\text{sub}}$.*
- $\widetilde{Z} : \Omega_{\text{sub}} \times \mathbb{R}_+ \to \mathbb{R}$ *is such that $\widetilde{Z}(\omega,\cdot) \in \mathrm{C}_c(\mathbb{R}_+;\mathbb{R})$ for every ω.*
- $\widetilde{U} : \Omega_{\text{sub}} \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *is such that $\widetilde{U}(\omega,\cdot) \in \mathrm{C}_{c,0}(\mathbb{R}_+ \times \mathbb{R};\mathbb{R})$ for every ω.*

Then,

(i) *For every $\omega \in \Omega_{\text{sub}}$*

$$\int_{(0,\infty)} \big|\widetilde{Z}(\omega,s)\big|^2 \,\mathrm{d}\langle X^{k_{l_m},\circ}\rangle_s(\omega) < \infty \text{ for every } m \in \mathbb{N} \text{ and } \int_{(0,\infty)} \big|\widetilde{Z}(\omega,s)\big|^2 \,\mathrm{d}\langle X^{\infty,c}\rangle_s(\omega) < \infty.$$

(ii) *For every $s \in \mathbb{R}_+$, for every $(\omega,t) \in \Omega_{\text{sub}} \times \mathbb{R}_+$ and for every $m \in \overline{\mathbb{N}}$ the quantity $\big|\big|\big|\widetilde{U}_s(\cdot)\big|\big|\big|_{t,k_{l_m}}(\omega)$ is well-defined.*

(iii) *For every $(\omega,s),(\omega,t) \in \Omega_{\text{sub}} \times \mathbb{R}_+$ and for every $m \in \overline{\mathbb{N}}$*

$$f^{k_{l_m}}\big(\omega,t,\widetilde{Y}_t(\omega),\widetilde{Z}(\omega,t),\widetilde{U}(\omega,t,\cdot)\big) \text{ is well-defined and real number}$$

as well as

$$f^{k_{l_m}}\big(\omega,t,\widetilde{Y}_t(\omega),\widetilde{Z}(\omega,s),\widetilde{U}(\omega,s,\cdot)\big) \text{ is well-defined and real number.}$$

Moreover,

$$\int_{(0,\infty)} \big|f^{k_{l_m}}\big(\omega,t,\widetilde{Y}_t(\omega),\widetilde{Z}(\omega,t),\widetilde{U}(\omega,t,\cdot)\big)\big| \,\mathrm{d}C_t^k(\omega) < \infty$$

as well as

$$\int_{(0,\infty)} \left| f^{k_{l_m}}\left(\omega, t, \widetilde{Y}_t(\omega), \widetilde{Z}(\omega, s), \widetilde{U}(\omega, s, \cdot)\right)\right| \mathrm{d}C_t^k(\omega) < \infty.$$

Proof. For each $\omega \in \Omega_{\text{sub}}$ $\widetilde{Z}(\omega.\cdot)$ and $\widetilde{U}(\omega, \cdot)$ are Borel measurable as continuous functions. Therefore, no measurability issues arise.

(i) It is clear.

(ii) For the well-posedness of $\left|\left|\left|\widetilde{U}_s(\cdot)\right|\right|\right|_{t,k_{l_m}}(\omega)$ it is sufficient to prove (recall its definition by (II.9)) that

$$\widehat{K}_t^{\overline{X}^{k_{l_m}}}\left(|\widetilde{U}_s(\cdot)|^2\right) < \infty \text{ and } \left|\widehat{K}_t^{\overline{X}^{k_{l_m}}}\left(\widetilde{U}_s(\cdot)\right)\right|^2 < \infty.$$

By Assumption ($\mathbb{C}$), for every $m \in \overline{\mathbb{N}}$ the transition kernel $\widehat{K}^{\overline{X}^{k_{l_m}}}$ is such that it satisfies the properties (I.71), *i.e.* for every $m \in \overline{\mathbb{N}}$

$$K_t^{\overline{X}^{k_{l_m}}}(\omega; \{0\}) = 0, \int_{\mathbb{R}} (|x|^2 \wedge 1)\, K_t^{\overline{X}^{k_{l_m}}}(\omega; \mathrm{d}x) \leq 1 \text{ and } \Delta C^{\overline{X}^{k_{l_m}}}(\omega) K_t^{\overline{X}^{k_{l_m}}}(\omega; \mathbb{R}) \leq 1.$$

In the following, we will use the second property in conjunction with the fact that $\text{supp}(\widetilde{U}(\omega))$ is compact with

$$\inf\left\{\|z - w\|_2, z \in \text{supp}(\widetilde{U}(\omega)) \text{ and } w \in \widetilde{E}_0\right\} > 0.$$

Let us define

$$\Pi := \left\{x \in \mathbb{R}, \exists s \in \mathbb{R}_+ \text{ such that } (s, x) \in \text{supp}\big(\widetilde{U}(\omega, \cdot)\big)\right\}.$$

Now, we have

$$\begin{aligned}
0 &< \inf\left\{\left(|t-s|^2 + |x-y|^2\right)^{\frac{1}{2}}, (t,x) \in \text{supp}\big(\widetilde{U}(\omega,\cdot)\big) \text{ and } (s,y) \in \widetilde{E}_0\right\} \\
&\leq \inf\left\{\left(|t-s|^2 + |x|^2\right)^{\frac{1}{2}}, (t,x) \in \text{supp}\big(\widetilde{U}(\omega,\cdot)\big) \text{ and } (s,y) \in \mathbb{R}_+ \times \{0\}\right\} \\
&= \inf\{|x|, \exists t \in \mathbb{R}_+ \text{ such that } (t,x) \in \text{supp}\big(\widetilde{U}(\omega,\cdot)\big)\} \\
&= \inf\{|x|, x \in \Pi\}
\end{aligned}$$

The last property, in turn, implies that the function $\mathbb{R} \ni x \longmapsto (|x|^2 \wedge 1)^{-1} \mathbb{1}_\Pi(x) \in \mathbb{R}_+$ is bounded from below by 1 and from above by the real number

$$\sup\left\{(|x|^2 \wedge 1)^{-1}, x \in \Pi\right\} = \inf\left\{|x|^2 \wedge 1, x \in \Pi\right\}^{-1} = \left(\inf\{|x|, x \in \Pi\} \wedge 1\right)^{-1} < \infty. \tag{IV.35}$$

On the other hand, we have in particular,

$$\int_{\mathbb{R}} \left[(|x|^2 \wedge 1)\mathbb{1}_\Pi(x)\right] K_s^{\overline{X}^\infty}(\omega; \mathrm{d}x) \leq 1, \text{ for every } s \in \mathbb{R}_+. \tag{IV.36}$$

Combining (IV.35) and (IV.36) we can conclude that

$$\widehat{K}_t^{\overline{X}^{k_{l_m}}}\left(\widetilde{U}_s(\cdot)\right)$$

is a real-number. We can argue analogously for $\widehat{K}_t^{\overline{X}^{k_{l_m}}}\left(\left|\widetilde{U}_s(\cdot)\right|^2\right)$.

(iii) The Lipschitz property of $f^{k_{l_m}}$ for every $m \in \overline{\mathbb{N}}$ allows us to conclude.

□

Proposition IV.26. *For every $\varepsilon > 0$, there exist $\widetilde{Z}^{\varepsilon,(p)} : \Omega_{\text{sub}} \times \mathbb{R}_+ \to \mathbb{R}$ and $\widetilde{U}^{\varepsilon,(p)} : \Omega_{\text{sub}} \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ (which are defined ω-by-ω) with the following properties:*

(i) $\widetilde{Z}^{\varepsilon,(p)}(\omega, \cdot) \in \mathrm{C}_c(\mathbb{R}_+; \mathbb{R})$ *for every* ω.

(ii) $\widetilde{U}^{\varepsilon,(p)}(\omega, \cdot)(\omega, \cdot) \in \mathrm{C}_{c,0}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$ *for every* ω.

(iii) *The pair* $\left(\widetilde{Z}^{\varepsilon,(p)}, \widetilde{U}^{\varepsilon,(p)}\right)$[7] *satisfies*

$$\left|\int_{(0,\infty)} \left[f^\infty\left(s, Y_s^{\infty,(p)}, \widetilde{Z}_s^{\varepsilon,(p)}, \widetilde{U}^{\varepsilon,(p)}(s,\cdot)\right) - f^\infty\left(s, Y_s^{\infty,(p)}, Z_s^{\infty,(p)}, U^{\infty,(p)}(s,\cdot)\right)\right] \mathrm{d}C_s^\infty\right| < \frac{\varepsilon}{4}\ \mathbb{P} - a.s..$$

[7] We will omit the ω in order to simplify notation. Moreover, we will denote $Z^{\varepsilon,(p)}(\omega, s)$ by $Z_s^{\varepsilon,(p)}$ when we do not write the ω.

PROOF. Let us fix an $\varepsilon > 0$.

□ We deal initially with the $\widetilde{Z}^{\varepsilon,(p)}-$approximation. We will simplify also the notation for $X^{\infty,c}$ by simply referring to it as X^c, since no confusion can arise at this point. We claim that we can determine a $\widetilde{Z}$ such that $\widetilde{Z}(\omega,\cdot)$ is continuous and of compact support which additionally satisfies $\mathbb{P}-a.s.$

$$\int_{(0,\infty)} \mathrm{e}^{\hat{\beta}A_s^\infty} \left|Z_s - \widetilde{Z}\right|^2 \mathrm{d}\langle X^c\rangle_s < \frac{\varepsilon^2\hat{\beta}}{32}. \tag{IV.37}$$

We remind the reader that the process $\langle X^c\rangle$ is increasing, continuous (recall **(B1)**), $\mathbb{G}^\infty-$predictable whose limit $\lim_{t\to\infty}\langle X^c\rangle_\infty$ is a real-number $\mathbb{P}-$almost surely, due to the fact that $X^c \in \mathcal{H}^2(\mathbb{G}^\infty;\mathbb{R})$. In particular, $\langle X^c\rangle(\omega)$ is $\mathcal{B}(\mathbb{R}_+)-$measurable for every $\omega \in \Omega_{\text{sub}}$. Therefore, we can associate to $\langle X^c\rangle(\omega)$ a finite measure on $\big(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\big)$ for every $\omega \in \Omega_{\text{sub}}$. We denote the associated measure by $\langle X^c\rangle$ in order to keep notation simple. Since $(\mathbb{R}_+, \delta_{|\cdot|})$ is Polish we have that the measure $\langle X^c\rangle$ is a Radon measure; see Bogachev [10, 7.1.1. Definition, 7.1.7. Theorem]. Moreover, $(\mathbb{R}_+, \delta_{|\cdot|})$ is locally compact, which allows us to apply Lusin's Theorem, see [10, 7.1.13. Theorem]. Therefore, by using standard arguments we obtain finally that the space $\mathrm{C}_c(\mathbb{R}_+)$ is dense[8] on

$$\mathbb{L}^2_{\hat{\beta},A^\infty}\big(\langle X^c\rangle(\omega)\big) := \Big\{f : \big(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\big) \to \big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big), \int_{(0,\infty)} \mathrm{e}^{\hat{\beta}A_s^\infty}|f_s|^2 \mathrm{d}\langle X^c\rangle(\omega)\Big\}$$

for every $\omega \in \Omega_{\text{sub}}$.

□ We deal now with the $\widetilde{U}^{\varepsilon,(p)}-$approximation. The arguments are more or less analogous to the previous case. We ease notation by assigning also the symbol U to the $\mathbb{G}^\infty-$predictable random function $U^{\infty,(p)}$. We start with the fact that ν^∞ as well as $\mathrm{q} * \nu^\infty_\infty$ are well-defined for every $\omega \in \Omega_{\text{sub}}$. In view of Remark IV.7, we will assume that $\nu^\infty(\omega)$ is a measure on $\big(\widetilde{E}, \mathcal{B}(\widetilde{E})\big)$. The space $\widetilde{E}$ is metric, but not Polish, since it is not complete. Therefore, we have that the measure $\nu^\infty(\omega; \mathrm{d}, s, \mathrm{d}x)$ is regular; see Bogachev [10, 7.1.1 Definition, 7.1.7. Theorem]. However, we can use that it is locally compact and that ν^∞ is tight, as a $\sigma-$compact measure. Now, Remark IV.7 allows us to use indifferently the measurable spaces $\big(\widetilde{E}, \mathcal{B}(\widetilde{E})\big)$ and $\big(\mathbb{R}_+ \times \mathbb{R}, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})\big)$, since the measure $\nu^\infty(\omega)$ is for every $\omega \in \Omega_{\text{sub}}$ well-defined on both with zero mass on $\widetilde{E}_0$. After this discussion, we can apply Lusin's Theorem, see [10, 7.1.13. Theorem], in order to obtain (using standard arguments as described before) that the space $\mathrm{C}_{c,0}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$ is dense in

$$\mathbb{L}^2_{\hat{\beta}A^\infty}\big(\nu^\infty(\omega)\big) := \Big\{f : \big(\mathbb{R}_+ \times \mathbb{R}, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})\big) \to \big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big), (\mathrm{e}^{\hat{\beta}A^\infty}|f|^2) * \nu^\infty_\infty(\omega)\Big\} \text{ for every } \omega \in \Omega_{\text{sub}}.$$

Therefore, we can find for every $\omega \in \Omega_{\text{sub}}$ a function $\widetilde{U}(\omega,\cdot) \in \mathrm{C}_{c,0}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$ such that

$$\int_{(0,\infty)} \mathrm{e}^{\hat{\beta}A_s^\infty(\omega)} \Big(\big|\big|\big|U_s(\cdot) - \widetilde{U}_s(\cdot)\big|\big|\big|_{s,\infty}(\omega)\Big)^2 \mathrm{d}C_s^\infty(\omega) < \frac{\varepsilon^2\hat{\beta}}{32}. \tag{IV.38}$$

□ Now we are ready to provide the required estimation in view of the following set inclusions

$$\begin{aligned}
&\Big[\Big|\int_{(0,\infty)} f^\infty\big(s, Y_s^{\infty,(p)}, \widetilde{Z}_s^{\varepsilon,(p)}, \widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^\infty - L_\infty^{\infty,(p)}\Big| \geq \frac{\varepsilon}{4}\Big] \\
&= \Big[\Big|\int_{(0,\infty)} f^\infty\big(s, Y_s^{\infty,(p)}, \widetilde{Z}_s^{\varepsilon,(p)}, \widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^\infty - L_\infty^{\infty,(p)}\Big|^2 \geq \frac{\varepsilon^2}{16}\Big] \\
&\subseteq \Big[\int_{(0,\infty)} \Big|f^\infty\big(s, Y_s^{\infty,(p)}, \widetilde{Z}_s^{\varepsilon,(p)}, \widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big) - f^\infty\big(s, Y_s^{\infty,(p)}, Z_s^{\infty,(p)}, U^{\infty,(p)}(s,\cdot)\big)\Big|^2 \mathrm{d}C_s^\infty \geq \frac{\varepsilon^2}{16}\Big] \\
&\underset{(\text{II.25})}{\overset{(\text{II.16})}{\subseteq}} \Big[\frac{1}{\hat{\beta}}\int_{(0,\infty)} \mathrm{e}^{\hat{\beta}A_s^\infty}\Big[\big|c_s^\infty(\widetilde{Z}_s^{\varepsilon,(p)} - Z_s)\big|^2 + \big|\big|\big|\widetilde{U}_s^{\varepsilon,(p)}(\cdot) - U_s(\cdot)\big|\big|\big|_{s,\infty}\Big]\,\mathrm{d}C_s^\infty \geq \frac{\varepsilon^2}{16}\Big]^{9} \\
&\subseteq \Big[\frac{1}{\hat{\beta}}\int_{(0,\infty)} \mathrm{e}^{\hat{\beta}A_s^\infty}\big|c_s^\infty(\widetilde{Z}_s^{\varepsilon,(p)} - Z_s)\big|^2 \mathrm{d}C_s^\infty \geq \frac{\varepsilon^2}{32}\Big] \\
&\qquad + \Big[\frac{1}{\hat{\beta}}\int_{(0,\infty)} \mathrm{e}^{\hat{\beta}A_s^\infty}\big|\big|\big|\widetilde{U}_s^{\varepsilon,(p)}(\cdot) - U_s(\cdot)\big|\big|\big|^2_{s,\infty}\,\mathrm{d}C_s^\infty \geq \frac{\varepsilon^2}{32}\Big] \underset{(\text{IV.38})}{\overset{(\text{IV.37})}{\subseteq}} \emptyset.
\end{aligned}$$

[8] We prove initially that $\mathrm{C}_c(\mathbb{R}_+)$ is dense in the space of simple functions, which is dense in the space $\mathbb{L}^2\big(\langle X^c\rangle(\omega)\big)$.

□

IV.5.2. Weak convergence of the integrators. In this section we are going to use partially the information provided in Remark IV.23 in order to prove that $\mathbb{P}-$almost surely holds

$$\int_{(0,\cdot]} f^{\infty}\big(s, Y_s^{k_{l_m},(p)}, \widetilde{Z}_s^{\varepsilon,(p)}, \widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^{k_{l_m}} \xrightarrow[m\to\infty]{\mathrm{J}_1} \int_{(0,\cdot]} f^{\infty}\big(s, Y_s^{\infty,(p)}, \widetilde{Z}_s^{\varepsilon,(p)}, \widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^{\infty}$$

as well as

$$\int_{(0,\infty)} f^{\infty}\big(s, Y_s^{k_{l_m},(p)}, \widetilde{Z}_s^{\varepsilon,(p)}, \widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^{k_{l_m}} \xrightarrow[m\to\infty]{|\cdot|} \int_{(0,\infty)} f^{\infty}\big(s, Y_s^{\infty,(p)}, \widetilde{Z}_s^{\varepsilon,(p)}, \widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^{\infty}.$$

In other words, we will prove that the third set of the union of (IV.28) is finally empty. These results are presented in Proposition IV.32, which is actually an application of Theorem A.14. More precisely, we will apply Proposition A.16 and Corollary A.18. The aforementioned theorem provides a characterisation of the weak convergence of finite measures on the positive real-line based on relatively compact subsets of $\mathbb{D}$. This characterisation, which is new to the best of author's knowledge, generalises the classical characterisation based on relatively compact sets of the space of continuous functions. One could comment that this is not surprising, since the relatively compact sets of $\mathbb{D}$ under the $\delta_{\mathrm{J}_1}-$metric admit an Arzela–Ascoli-type characterisation; see Theorem I.111.

Therefore, in order to proceed we have to provide the preparatory results for applying Theorem A.14. Recall that in Remark IV.24 we have fixed an Ω_{sub} with $\mathbb{P}(\Omega_{\mathrm{sub}}) = 1$.

Remark IV.27. For notational convenience, we will omit the superscripts ε and (p) from $\widetilde{Z}^{\varepsilon,(p)}$, $\widetilde{U}^{\varepsilon,(p)}$. Hence, in the rest of the Section IV.5, we assume that a fixed $0 < \varepsilon < \frac{1}{\widehat{\beta}}$ is given and $\widetilde{Z}$, $\widetilde{U}$ satisfy the properties of Proposition IV.26. Moreover, the $k_{l_m}-$element of a sequence indexed on $k \in \mathbb{N}$ will be simply denoted as indexed by m.

Lemma IV.28. *For every $m \in \overline{\mathbb{N}}$, we define*

$$f_t^{\infty}[Y^{m,(p)}](\omega) := f^{\infty}\big(\omega, t, Y_t^{m,(p)}(\omega), \widetilde{Z}(\omega,t), \widetilde{U}(\omega,t,\cdot)\big)$$

for $\omega \in \Omega_{\mathrm{sub}}$ and for $t \in \mathbb{R}_+$. Then, for every $t \in \mathbb{R}_+$

$$\lim_{s\downarrow t} f^{\infty}\big(\omega, s, Y_s^{m,(p)}(\omega), \widetilde{Z}(\omega,s), \widetilde{U}(\omega,s,\cdot)\big) = f_t^{\infty}[Y^{m,(p)}](\omega)$$

and

$$\begin{aligned} f_{t-}^{\infty}[Y^{m,(p)}](\omega) &= f^{\infty}\big(\omega, t-, Y_{t-}^{m,(p)}(\omega), \widetilde{Z}(\omega,t-), \widetilde{U}(\omega,t-,\cdot)\big) \\ &= f^{\infty}\big(\omega, t-, Y_{t-}^{m,(p)}(\omega), \widetilde{Z}(\omega,t), \widetilde{U}(\omega,t,\cdot)\big). \end{aligned}$$

In other words, $f_{\cdot}^{\infty}[Y^{m,(p)}](\omega) \in \mathbb{D}(\mathbb{R})$ for every $\omega \in \Omega_{\mathrm{sub}}$ and every $m \in \mathbb{N}$..

Proof. Let us fix an $m \in \overline{\mathbb{N}}$ and $(\omega, t) \in \Omega_{\mathrm{sub}} \times \mathbb{R}_+$. Firstly, the reader may recall Lemma IV.25 for the well-posedness of $f_t^{\infty}[Y^{m,(p)}](\omega)$. We distinguish now two cases.

□ Let us assume that $s \downarrow t$. We will prove that $f_s^{\infty}[Y^{m,(p)}] \xrightarrow[s\downarrow t]{|\cdot|} f_t^{\infty}[Y^{m,(p)}]$. To this end, we calculate an upper bound for $|f_s^{\infty}[Y^{m,(p)}] - f_t^{\infty}[Y^{m,(p)}]|^2$.

$$\begin{aligned} &|f_s^{\infty}[Y^{m,(p)}] - f_t^{\infty}[Y^{m,(p)}]|^2 \\ &\quad\leq\ 2|f_s^{\infty}[Y^{m,(p)}] - f^{\infty}\big(s, Y_t^{m,(p)}, \widetilde{Z}_t, \widetilde{U}(t,\cdot)\big)|^2 + 2|f^{\infty}\big(s, Y_t^{m,(p)}, \widetilde{Z}_t, \widetilde{U}(t,\cdot)\big) - f_t^{\infty}[Y^{m,(p)}]|^2 \\ &\quad\overset{(\mathrm{II.16})}{\leq}\ 2r_s^{\infty}|Y_s^{m,(p)} - Y_t^{m,(p)}|^2 + 2\theta_s^{\infty,c}|c_s^{\infty}(\widetilde{Z}_s - \widetilde{Z}_t)|^2 + 2\theta_s^{\infty,d}\big\|\widetilde{U}_s(\cdot) - \widetilde{U}_t(\cdot)\big\|_{s,\infty}^2 \\ &\qquad\qquad + 2|f^{\infty}\big(s, Y_t^{m,(p)}, \widetilde{Z}_t, \widetilde{U}(t,\cdot)\big) - f_t^{\infty}[Y^{m,(p)}]|^2. \end{aligned} \tag{IV.39}$$

First of all recall that by Condition **(B7).(iv)** the bounds $r^{\infty}, \theta^{\infty,c}$ and $\theta^{\infty,d}$ are $\mathbb{P}-$almost surely bounded, while the process $\big|\mathrm{d}\langle X^{\infty,c}\rangle_s / \mathrm{d}C_s^{\infty}\big|^2$ has locally bounded paths $\mathbb{P}-$almost surely. Let us associate to the $\mathbb{P}-$almost surely bounded processes the stochastic bounds $C(r^{\infty})$, $C(\theta^{\infty,c})$ and $C(\theta^{\infty,d})$.

[9] We have used the fact that $\theta^{\infty,c}, \theta^{\infty,d} \leq \alpha^{\infty}$; see (II.49).

Since the process $\mathrm{d}\langle X^{\infty,c}\rangle_s/\mathrm{d}C^\infty_s$ has locally bounded paths, for the point (ω,t) there exists a $\delta=\delta(\omega,t)$ such that

$$C\big(\mathrm{RN};B(t,\delta)\big)^{10} := \sup_{s\in(t-\delta,t+\delta)}\Big|\frac{\mathrm{d}\langle X^{\infty,c}\rangle_s}{\mathrm{d}C^\infty_s}\Big|^2<\infty. \tag{IV.40}$$

The first summand convergences to 0 because $Y^{m,(p)}$ is càdlàg $\mathbb{P}-$almost surely and $0\le r^\infty_s\le C(r^\infty)<\infty$ $\mathbb{P}-$almost surely.

The second summand can be written for s such that $|s-t|<\delta$ as

$$2\theta^{\infty,c}_s|c^\infty_s(\widetilde{Z}_s-\widetilde{Z}_t)|^2\le 2C(\theta^{\infty,c})C\big(\mathrm{RN};B(t,\delta)\big)\big|\widetilde{Z}_s-\widetilde{Z}_t\big|^2\xrightarrow[s\downarrow t]{}0, \tag{IV.41}$$

since $\widetilde{Z}$ is (uniformly) continuous.

For the third summand we collect initially some facts which will help us to conclude. Recall that the pair $(\overline{X}^\infty,C^\infty)$ satisfies Assumption $(\mathbb{C})$; see on p. 49. Therefore, we have in particular that

$$\int_{\mathbb{R}}(|x|^2\wedge 1)\,K^{\overline{X}^\infty}_s(\omega;\mathrm{d}x)\le 1,\ \text{for every } s\in\mathbb{R}_+,$$

which in turn implies for

$$\Pi:=\big\{x\in\mathbb{R},\exists s\in\mathbb{R}_+ \text{ such that } (s,x)\in\mathrm{supp}(\widetilde{U})\big\}$$

that

$$\int_{\mathbb{R}}\big[(|x|^2\wedge 1)\mathbb{1}_\Pi(x)\big]\,K^{\overline{X}^\infty}_s(\omega;\mathrm{d}x)\le 1,\ \text{for every } s\in\mathbb{R}_+. \tag{IV.42}$$

On the other hand, $\mathrm{supp}(\widetilde{U})$ is a compact subset of the open $\widetilde{E}=(\mathbb{R}_+\times\mathbb{R})\setminus\widetilde{E}_0$; recall Proposition IV.26 and Notation IV.5. This implies that

$$\inf\big\{|z-w|,z\in\mathrm{supp}(\widetilde{U}) \text{ and } w\in\widetilde{E}_0\big\}>0, \tag{IV.43}$$

as the intersection of the compact set $\mathrm{supp}(\widetilde{U})$ with the closed set $\widetilde{E}_0$ is empty. Using (IV.43) and treating for the first two forthcoming lines the sets $\mathrm{supp}(\widetilde{U})$ and $\widetilde{E}_0$ as subsets of the product $\mathbb{R}_+\times\mathbb{R}$ we have

$$\begin{aligned}
0&<\inf\big\{\big(|t-s|^2+|x-y|^2\big)^{\frac12},(t,x)\in\mathrm{supp}(\widetilde{U}) \text{ and } (s,y)\in\widetilde{E}_0\big\}\\
&\le\inf\big\{\big(|t-s|^2+|x|^2\big)^{\frac12},(t,x)\in\mathrm{supp}(\widetilde{U}) \text{ and } (s,y)\in\mathbb{R}_+\times\{0\}\big\}\\
&=\inf\{|x|,\exists t\in\mathbb{R}_+ \text{ such that } (t,x)\in\mathrm{supp}(\widetilde{U})\}\\
&=\inf\{|x|,x\in\Pi\}
\end{aligned}$$

The last property, in turn, implies that the function $\mathbb{R}\ni x\longmapsto(|x|^2\wedge 1)^{-1}\mathbb{1}_\Pi(x)\in\mathbb{R}_+$ is bounded from below by 1 and from above by the real number

$$\sup\big\{(|x|^2\wedge 1)^{-1},x\in\Pi\big\}=\inf\big\{|x|^2\wedge 1,x\in\Pi\big\}^{-1}=\big(\inf\{|x|,x\in\Pi\}\wedge 1\big)^{-1}<\infty. \tag{IV.44}$$

Using these observations and recalling Notation II.2 as well as that C^∞ is continuous by **(B9).(iii)**, we have for the third summand

$$\begin{aligned}
0\ &\le\ 2\theta^{\infty,d}_s\big|\big|\big|\widetilde{U}_s(\cdot)-\widetilde{U}_t(\cdot)\big|\big|\big|^2_{s,\infty}\le 2C(\theta^{\infty,d})\ \widehat{K}^{\overline{X}^\infty}_s\big(|\widetilde{U}_s(\cdot)-\widetilde{U}_t(\cdot)|^2\big)\\
&=\ 2C(\theta^{\infty,d})\int_{\mathbb{R}}\big(|\widetilde{U}(s,x)-\widetilde{U}(t,x)|^2\big)\,K^{\overline{X}^\infty}_s(\mathrm{d}x)^{11}\\
&\overset{\text{(IV.42)}}{\le}\ 2C(\theta^{\infty,d})\sup_{x\in\Pi}\Big[(|x|^2\wedge 1)^{-1}\,|\widetilde{U}(s,x)-\widetilde{U}(t,x)|^2\Big]\\
&\le\ 2C(\theta^{\infty,d})\sup_{x\in\Pi}(|x|^2\wedge 1)^{-1}\sup_{x\in\Pi}|\widetilde{U}(s,x)-\widetilde{U}(t,x)|^2\xrightarrow[s\downarrow t]{}0.
\end{aligned} \tag{IV.45}$$

For the conclusion, we have used the fact that $\widetilde{U}$ is uniformly continuous (as continuous with compact metrisable support) and that $|(t,x)-(s,x)|=|t-s|$ for every $s,t\in\mathbb{R}_+$ and $x\in\mathbb{R}$.

Finally, for the fourth summand we have only to make use of the right-continuity of the path

$$\Big(f^\infty\big(\omega,s,Y^{m,(p)}_t(\omega),\widetilde{Z}(\omega,t),\widetilde{U}(\omega,t,\cdot)\big)\Big)_{s\in\mathbb{R}_+};$$

[10] We abstain from writing the dependence on ω, which is clear in view of the preceding comments.

[11] We have again abused notation by omitting the ω.

recall Condition **(B7).(i)** and the fact that $\widetilde{U}(t,\cdot)\in \mathrm{C}_c(\mathbb{R})$.

□ Let us assume now that $s\uparrow t$. Then, completely analogously to the previous case we obtain

$$\begin{aligned}&|f^\infty_s[Y^{m,(p)}]-f^\infty\big(t-,Y^{m,(p)}_{t-},\widetilde{Z}_t,\widetilde{U}(t,\cdot)\big)|^2\\ &\overset{(\mathrm{II}.16)}{\leq} 2r^\infty_s|Y^{m,(p)}_s-Y^{m,(p)}_{t-}|^2+2\theta^{\infty,c}_s|c^\infty_s(\widetilde{Z}_s-\widetilde{Z}_t)|^2+2\theta^{\infty,d}_s\big|\big|\big|\widetilde{U}_s(\cdot)-\widetilde{U}_t(\cdot)\big|\big|\big|^2_{s,\infty}\\ &\quad+2|f^\infty\big(s,Y^{m,(p)}_{t-},\widetilde{Z}_t,\widetilde{U}(t,\cdot)\big)-f^\infty\big(t-,Y^{m,(p)}_{t-},\widetilde{Z}_t,\widetilde{U}(t,\cdot)\big)|^2.\end{aligned}$$

For the first three summands we can argue as in the previous case. For the fourth summand, we use the left-continuity of the path

$$\Big(f^\infty\big(\omega,s,Y^{m,(p)}_{t-}(\omega),\widetilde{Z}(\omega,t),\widetilde{U}(\omega,t,\cdot)\big)\Big)_{s\in\mathbb{R}_+},$$

which enables us to obtain the equality

$$f^\infty\big(\omega,t-,Y^{m,(p)}_{t-}(\omega),\widetilde{Z}(\omega,t),\widetilde{U}(\omega,t,\cdot)\big)=\lim_{s\uparrow t}f^\infty\big(\omega,s,Y^{m,(p)}_{t-}(\omega),\widetilde{Z}(\omega,t),\widetilde{U}(\omega,t,\cdot)\big).$$

In view of the last comment, we can conclude. □

Remark IV.29. Using the same arguments we can prove also for every $m\in\mathbb{N}$ that $\big(f^m_t[Y^{m,(p)}](\omega)\big)_{t\in\mathbb{R}_+}\in\mathbb{D}(\mathbb{R})$, where for every $t\in\mathbb{R}_+$

$$f^m_t[Y^{m,(p)}](\omega):=f^m\big(\omega,t,Y^{m,(p)}_t(\omega),\widetilde{Z}(\omega,t),\widetilde{U}(\omega,t,\cdot)\big). \tag{IV.46}$$

We inform the reader that we will adopt the notation of Lemma IV.28 for the rest of the subsection.

Lemma IV.30. *It holds* $f^\infty[Y^{m,(p)}](\omega)\xrightarrow[m\to\infty]{\mathrm{J}_1(\mathbb{R})}f^\infty[Y^{\infty,(p)}](\omega)$ *for every* $\omega\in\Omega_{\mathrm{sub}}$.

Proof. By Remark IV.23.(vii) we have that $Y^{m,(p)}\xrightarrow[m\to\infty]{\mathrm{J}_1}Y^{\infty,(p)}$ $\mathbb{P}-$almost surely. Therefore, by Theorem I.109 there exists for almost every ω a sequence $(\lambda^m_\omega)_{m\in\mathbb{N}}\subset\Lambda$ (see Definition I.108) such that

$$\sup_{t\in\mathbb{R}_+}|\lambda^m_\omega(t)-t|\xrightarrow[m\to\infty]{}0 \text{ and } \sup_{t\in[0,N]}\big|Y^{m,(p)}_{\lambda^m_\omega(t)}(\omega)-Y^{\infty,(p)}_t(\omega)\big|\xrightarrow[m\to\infty]{}0 \text{ for every } N\in\mathbb{N}. \tag{IV.47}$$

Using again Theorem I.109, it is sufficient to prove that

$$\sup_{t\in[0,N]}\big|f^\infty_{\lambda^m_\omega(t)}[Y^{m,(p)}](\omega)-f^\infty_t[Y^{\infty,(p)}](\omega)\big|\xrightarrow[m\to\infty]{}0 \text{ for every } N\in\mathbb{N}. \tag{IV.48}$$

To this end, using analogous to (IV.39) arguments we obtain (where we omit the dependence in ω to ease notation)

$$\begin{aligned}&\big|f^\infty_{\lambda^m(t)}[Y^{m,(p)}]-f^\infty_t[Y^{\infty,(p)}]\big|^2\\ &\leq 2\big|f^\infty_{\lambda^m(t)}[Y^{m,(p)}]-f^\infty\big(\lambda^m(t),Y^{\infty,(p)}_t,\widetilde{Z}_t,\widetilde{U}(t,\cdot)\big)\big|^2+2\big|f^\infty\big(\lambda^m(t),Y^{\infty,(p)}_t,\widetilde{Z}_t,\widetilde{U}(t,\cdot)-f^\infty_t[Y^{\infty,(p)}]\big|^2\\ &\overset{(\mathrm{II}.16)}{\leq} 2r^\infty_{\lambda^m(t)}\big|Y^{m,(p)}_{\lambda^m(t)}-Y^{\infty,(p)}_t\big|^2+2\theta^{\infty,c}_{\lambda^m(t)}\big|c^\infty_{\lambda^m(t)}(\widetilde{Z}_{\lambda^m(t)}-\widetilde{Z}_t)\big|^2\\ &\quad+2\theta^{\infty,d}_{\lambda^m(t)}\big|\big|\big|\widetilde{U}_{\lambda^m(t)}(\cdot)-\widetilde{U}_t(\cdot)\big|\big|\big|^2_{\lambda^m(t),\infty}+2\big|f^\infty\big(\lambda^m(t),Y^{\infty,(p)}_t,\widetilde{Z}_t,\widetilde{U}(t,\cdot)\big)-f^\infty_t[Y^{m,(p)}]\big|^2.\end{aligned}$$

Let us, now, fix an $N\in\mathbb{N}$. Using the above bound, we obtain (IV.48) if we can prove that the following hold

- $\sup_{t\in[0,N]}\big|Y^{m,(p)}_{\lambda^m_\omega(t)}(\omega)-Y^{\infty,(p)}_t(\omega)\big|^2\xrightarrow[m\to\infty]{}0$, since $r^\infty_\cdot(\omega)$ is bounded. But this is true by Convergence (IV.47).
- $\sup_{t\in[0,N]}\big|c^\infty_{\lambda^m(t)}(\widetilde{Z}_{\lambda^m(t)}-\widetilde{Z}_t)\big|^2\xrightarrow[m\to\infty]{}0$, in view of the $\mathbb{P}-$almost surely boundedness of $\theta^{\infty,c}_\cdot$. Name $C(\theta^{\infty,c}_\cdot)$ the bound of $\theta^{\infty,c}_\cdot$. Arguing as in (IV.40) and using that the set $[0,N+1]$ is compact, we can determine[12] a finite bound $C(\mathrm{RN};[0,N+1])$ of $\mathrm{d}\langle X^{\infty,c}\rangle_s/\mathrm{d}C^\infty_s$ on $[0,N+1]$, *i.e.*

$$C(\mathrm{RN};[0,N+1]):=\sup_{t\in[0,N+1]}\frac{\mathrm{d}\langle X^{\infty,c}\rangle_s}{\mathrm{d}C^\infty_s}.$$

[12]Since $\mathrm{d}\langle X^{\infty,c}\rangle_s/\mathrm{d}C^\infty_s$ is locally bounded, using the notation of (IV.40) we have that $\bigcup_{t\in(0,N+1]}(t-\delta(\omega,t)\wedge 0,t+\delta(\omega,t))\cup\big[0,\delta(0,\omega)\big)$ is an open cover of $[0,N+1]$ in $\mathbb{R}_+$. The compactness of $[0,N+1]$ implies that there is a finite cover. Therefore, we can use the maximum of the bounds which associate to the finite cover.

Therefore, analogously to (IV.41) we obtain (we omit ω)

$$\sup_{t\in[0,N]}\Big(2\theta^{\infty,c}_{\lambda^m(t)}|c^{\infty}_{\lambda^m(t)}(\widetilde{Z}_s-\widetilde{Z}_t)|^2\Big)\leq 2C(\theta^{\infty,c})C\big(\mathrm{RN};[0,N+1]\big)\sup_{t\in[0,N]}\big|\widetilde{Z}_{\lambda^m(t)}-\widetilde{Z}_t\big|^2, \tag{IV.49}$$

when $\lambda^m(t)\in[0,N+1]$. Now, $\widetilde{U}$ is uniformly continuous, as continuous with compact metrisable support. In other words, for the given $\varepsilon>0$ (recall Remark IV.27) there exists a $\delta>0$ such that $\big|\widetilde{Z}_u-\widetilde{Z}_w\big|<\varepsilon$ whenever $|u-w|<\delta$. Without loss of generality, we can assume that $\delta\leq 1$. On the other hand, in view of (IV.47), for every given $\delta>0$ there exists an $M\in\mathbb{N}$ such that $\sup_{t\in[0,N]}|\lambda^m(t)-t|<\delta$ for every $m\geq M$. Combining the last two properties, we can finally obtain for every $t\in[0,N]$ that $\lambda^m(t)\in[0,N+1]$ for all but finitely many m and, moreover,

$$\sup_{t\in[0,N]}\big|\widetilde{Z}_{\lambda^m(t)}-\widetilde{Z}_t\big|^2\xrightarrow[m\to\infty]{}0,$$

which implies the required one, in view of (IV.49).

- $\big|\big|\big|\widetilde{U}_{\lambda^m(t)}(\cdot)-\widetilde{U}_t(\cdot)\big|\big|\big|^2_{\lambda^m(t),\infty}\xrightarrow[m\to\infty]{}0$, in view of the $\mathbb{P}-$almost surely boundedness of $\theta^{\infty,d}$. Name $C(\theta^{\infty,d})$ the bound of $\theta^{\infty,d}$. Arguing as in (IV.45) we can obtain (we omit the ω)

$$\begin{aligned}
0 &\leq \sup_{t\in[0,N]}2\theta^{\infty,d}_{\lambda^m(t)}\big|\big|\big|\widetilde{U}_{\lambda^m(t)}(\cdot)-\widetilde{U}_t(\cdot)\big|\big|\big|^2_{\lambda^m(t),\infty}\leq 2C(\theta^{\infty,d})\sup_{t\in[0,N]}\widehat{K}^{\overline{X}^{\infty}}_{\lambda^m(t)}\big(|\widetilde{U}_{\lambda^m(t)}(\cdot)-\widetilde{U}_t(\cdot)|^2\big)\\
&= 2C(\theta^{\infty,d})\sup_{t\in[0,N]}\int_{\mathbb{R}}\big(|\widetilde{U}(\lambda^m(t),x)-\widetilde{U}(t,x)|^2\big)\,K^{\overline{X}^{\infty}}_{\lambda^m(t)}(\mathrm{d}x)\\
&\overset{\text{(IV.42)}}{\leq} 2C(\theta^{\infty,d})\sup_{t\in[0,N]}\sup_{x\in\Pi}\Big[(|x|^2\wedge 1)^{-1}\,|\widetilde{U}(\lambda^m(t),x)-\widetilde{U}(t,x)|^2\Big]\\
&\overset{\text{(IV.44)}}{\leq} 2C(\theta^{\infty,d})\sup_{x\in\Pi}(|x|^2\wedge 1)^{-1}\sup_{t\in[0,N]}\sup_{x\in\Pi}|\widetilde{U}(\lambda^m(t),x)-\widetilde{U}(t,x)|^2\xrightarrow[m\to\infty]{}0.
\end{aligned}\tag{IV.50}$$

We have used again the uniform continuity of $\widetilde{U}$ as well as the fact that $\lim_{m\to\infty}\sup_{t\in\mathbb{R}_+}|\lambda^m(t)-t|=0$.

- $\sup_{t\in[0,N]}\big|f^{\infty}\big(\lambda^m(t),Y^{\infty,(p)}_t,\widetilde{Z}_t,\widetilde{U}(t,\cdot)\big)-f^{\infty}_t[Y^{m,(p)}]\big|^2\xrightarrow[m\to\infty]{}0$. But, this is immediate in view of Condition **(B7).(ii)** and the boundedness of $\big(\big|Y^{\infty,(p)}_t\big|\big)_{t\in\mathbb{R}_+}$ and $(\big|\widetilde{Z}_s\big|)_{s\in\mathbb{R}_+}$. For the former set recall Remark IV.23.(vi), while for the latter set it is immediate by the continuity of $\widetilde{Z}$ and the compactness of its support. Therefore, there exist compact sets $K_1\subset\mathbb{R}$ and $K_2\subset\mathbb{R}$ such that $(Y^{\infty,(p)}_t)_{t\in\mathbb{R}_+}\subset K_1$ and $(\widetilde{Z}_t)_{t\in\mathbb{R}_+}\subset K_2$ and we can indeed call **(B7).(ii)**.

□

Lemma IV.31. *For compact $K_1,K_2\subset\mathbb{R}$, and $U\in\mathrm{C}_{c,0}(\mathbb{R}_+\times\mathbb{R};\mathbb{R})$ it holds*

$$\sup_{t\in\mathbb{R}_+}\Big\{\big|f^{\infty}\big(t,y,z,U(t,\cdot)\big)\big|,(y,z)\in K_1\times K_2\Big\}<\infty\ \mathbb{P}-a.s..$$

Proof. Let $K_1,K_2\subset\mathbb{R}$ be compact and $U\in\mathrm{C}_{c,0}(\mathbb{R}_+\times\mathbb{R};\mathbb{R})$. The claim is immediate by the Lipschitz property of f^{∞}, Condition **(B7).(iv)**, the boundedness of K_1,K_2 as well as of the boundedness of the image of U. □

Proposition IV.32. *It holds*

$$\int_{(0,\cdot]}f^{\infty}\big(s,Y^{k_{l_m},(p)}_s,\widetilde{Z}^{\varepsilon,(p)}_s,\widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C^{k_{l_m}}_s\xrightarrow[m\to\infty]{\mathrm{J}_1}\int_{(0,\cdot]}f^{\infty}\big(s,Y^{\infty,(p)}_s,\widetilde{Z}^{\varepsilon,(p)}_s,\widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C^{\infty}_s$$

as well as

$$\int_{(0,\infty)}f^{\infty}\big(s,Y^{k_{l_m},(p)}_s,\widetilde{Z}^{\varepsilon,(p)}_s,\widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C^{k_{l_m}}_s\xrightarrow[m\to\infty]{|\cdot|}\int_{(0,\infty)}f^{\infty}\big(s,Y^{\infty,(p)}_s,\widetilde{Z}^{\varepsilon,(p)}_s,\widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C^{\infty}_s.$$

Proof. We will apply Proposition A.16 and Corollary A.18. In view of Remark IV.23.(i), Lemma A.13 and Proposition A.12, we have that the associated to $(C^m)_{m\in\overline{\mathbb{N}}}$ measures converge weakly. In the previous lemma we have proven that we have the the sequence of the integrands is $\delta_{\mathrm{J}_1(\mathbb{R})}-$convergent. We have only to prove that the family $\big\{f^{\infty}_t[Y^{m,(p)}](\omega),t\in\mathbb{R}_+\text{ and }m\in\overline{\mathbb{N}}\big\}$ is bounded for every $\omega\in\Omega_{\mathrm{sub}}$. In view of Lemma IV.31 we need only to prove that $\big\{Y^{m,(p)}_t(\omega),t\in\mathbb{R}_+\text{ and }m\in\overline{\mathbb{N}}\big\}$ is bounded for every $\omega\in\Omega_{\mathrm{sub}}$.

To this end we are going to apply Lemma A.19. Let us verify its requirements. The first one is satisfied by Remark IV.23.(vi), while (ii) is satisfied because $(Y^{m,(p)})_{m\in\overline{\mathbb{N}}}$ is a compact set of $\mathbb{D}(\mathbb{R})$ as $\mathrm{J}_1(\mathbb{R})-$convergent sequence Remark IV.23.(vii); see also Theorem I.111. For the third requirement, we have by Remark IV.23.(vi) that

$$\begin{aligned}\mathbb{E}\Big[\limsup_{(m,t)\to(\infty,\infty)}\big|Y_t^{m,(p)}\big|\Big] &= \mathbb{E}\Big[\lim_{n\to\infty}\big|Y_{t_n}^{m_n,(p)}\big|\Big]^{13} \leq \mathbb{E}\Big[\lim_{n\to\infty}\sup_{t\in\mathbb{R}_+}\big|Y_t^{m_n,(p)}\big|\Big]\\ &\leq \liminf_{n\to\infty}\mathbb{E}\Big[\sup_{t\in\mathbb{R}_+}\big|Y_t^{m_n,(p)}\big|\Big] \leq \sup_{k\in\overline{\mathbb{N}}}\mathbb{E}\Big[\sup_{t\in\mathbb{R}_+}\big|Y_t^{k,(p)}\big|\Big] < \infty.\end{aligned}$$

Therefore, $\mathbb{P}\big(\big[\limsup_{(m,t)\to(\infty,\infty)}\big|Y_t^{m,(p)}\big| = \infty\big]\big) = 0$ which allows us to conclude that

$$\mathbb{P}\big(\big[\sup_{m\in\mathbb{N}}|Y^{m,(p)}|_\infty < \infty\big]\big) = 1. \tag{IV.51}$$

Using that $\mathbb{P}\big(\big[|Y^{\infty,(p)}|_\infty < \infty\big]\big) = 1$, we conclude also that $\sup_{m\in\overline{\mathbb{N}}}\big|Y^{m,(p)}(\omega)\big|_\infty < \infty$ for every $\omega\in\Omega_{\mathrm{sub}}$, without loss of generality. □

IV.5.3. The drivers converge uniformly on compacts. In this subsection, which actually consists of a single result, we will justify that for all but finally many $m\in\mathbb{N}$ holds

$$\left|\int_{(0,\infty)}(f^m - f^\infty)\big(s, Y_s^{m,(p)}, \widetilde{Z}_s^{\varepsilon,(p)}, \widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^m\right| < \frac{\varepsilon}{4}. \tag{IV.52}$$

Lemma IV.33. *The Inequality* (IV.52) *is true for all but finitely many* $m\in\mathbb{N}$.

Proof. In view of Condition **(B7).(iii)** and the compactness of the support of $\widetilde{Z}(\omega,\cdot)$, it is sufficient to prove that the set $\big\{\big|Y_t^{m,(p)}\big|, t\in\mathbb{R}_+ \text{ and } m\in\overline{\mathbb{N}}\big\}$ is bounded. But, this is proven in (IV.51). Then, since $\sup_{m\in\overline{\mathbb{N}}}(C_\infty^m) < \infty$ we can conclude. □

IV.5.4. The predictable quadratic variations converge. In this subsection we are going to prove that for all but finitely many $m\in\mathbb{N}$ holds

$$\left|L_\infty^{m,(p)} - \int_{(0,\infty)} f^m\big(s, Y_s^{m,(p)}, \widetilde{Z}_s^{\varepsilon,(p)}, \widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C_s^m\right| < \frac{\varepsilon}{4},$$

in other words we are going to prove that the second set of the right hand side of (IV.28) is finally empty. At this part the Condition **(B7).(iii)** in conjunction with Convergence (IV.32) will provide us the required result. In the whole subsection the measures are treated as Borel measures.

We start with a lemma for the convergence of the sequence $(A^m)_{m\in\overline{\mathbb{N}}}$.

Lemma IV.34. *The sequence* $(A^m)_{m\in\overline{\mathbb{N}}}$ *is* $\|\cdot\|_\infty-$*convergent. Consequently,*

$$\mathrm{e}^{\hat{\beta}A^m} \xrightarrow{\|\cdot\|_\infty} \mathrm{e}^{\hat{\beta}A^\infty}.$$

Proof. Recall that by definition $A_\cdot^m := \int_{(0,\cdot]}(\alpha_s^m)^2\,\mathrm{d}C_s^m \in \mathcal{V}^+(\mathbb{G}^m;\mathbb{R})$ for every $m\in\overline{\mathbb{N}}$. Therefore, we obtain the locally uniform convergence once we prove the pointwise convergence for $t\in\mathbb{R}_+$; see Jacod and Shiryaev [41, Proposition VI.2.15 c)]. We can then conclude the uniform convergence by Lemma A.13. In total, we need to prove that $A_t^m \longrightarrow A_t^\infty$ for every $t\in[0,\infty]$.

By Condition **(B7).(iv)** and by the definition of $(\alpha_\cdot^m)^2 := \max\big\{\sqrt{r_\cdot^m}, \theta_\cdot^c, \theta_\cdot^d\big\}$ (see **(F4)** on p. 58) for every $k\in\overline{\mathbb{N}}$ we have $\alpha_\cdot^m \xrightarrow{\|\cdot\|_\infty} \alpha_\cdot^\infty$. Now, recall that α^∞ is continuous $C^\infty-$almost everywhere (see **(B7).(iii)**) and the weak convergence of the measures associated to the sequence $(C^m)_{m\in\overline{\mathbb{N}}}$ in order to apply the characterisation of Mazzone [51, Theorem 1]. In view of the above, we obtain for $t\in\mathbb{R}_+$

$$\begin{aligned}\big|A_t^m - A_t^\infty\big| &= \left|\int_{(0,\cdot]}(\alpha_s^m)\,\mathrm{d}C_s^m - \int_{(0,\cdot]}(\alpha_s^\infty)\,\mathrm{d}C_s^m\right| + \left|\int_{(0,\cdot]}(\alpha_s^\infty)\,\mathrm{d}C_s^m - \int_{(0,\cdot]}(\alpha_s^\infty)\,\mathrm{d}C_s^\infty\right|\\ &\leq \int_{(0,\cdot]}\big|(\alpha_s^m) - (\alpha_s^\infty)\big|\,\mathrm{d}C_s^m + \left|\int_{(0,\cdot]}(\alpha_s^\infty)\,\mathrm{d}C_s^m - \int_{(0,\cdot]}(\alpha_s^\infty)\,\mathrm{d}C_s^\infty\right|\\ &\leq \big\|\alpha^m - \alpha^\infty\big\|_\infty \sup_{m\in\overline{\mathbb{N}}} C_\infty^m + \left|\int_{(0,\cdot]}(\alpha_s^\infty)\,\mathrm{d}C_s^m - \int_{(0,\cdot]}(\alpha_s^\infty)\,\mathrm{d}C_s^\infty\right| \xrightarrow[m\to\infty]{} 0.\end{aligned}$$

[13] The equality holds for some subsequence $\big((m_n,t_n)\big)_{n\in\mathbb{N}}$, which may depend on the path, such that $(m_n,t_n)\to(\infty,\infty)$.

Analogously we can prove the convergence for $t = \infty$. □

The previous lemma in conjunction with Convergence (IV.32) - (IV.33) provides the following results.

Lemma IV.35. *The following convergence are true*

$$\int_{(0,\cdot]} \mathrm{e}^{\hat{\beta} A_s^m} (Z_s^{m,(p)})^2 \mathrm{d}\langle X^{m,\circ}\rangle_s \xrightarrow[m\to\infty]{\mathrm{J}_1(\mathbb{R})} \int_{(0,\cdot]} \mathrm{e}^{\hat{\beta} A_s^\infty} (Z_s^{\infty,(p)})^2 \mathrm{d}\langle X^{\infty,c}\rangle_s \tag{IV.53}$$

and

$$\int_{(0,\cdot]} \mathrm{e}^{\hat{\beta} A_s^m} Z_s^{m,(p)} \mathrm{d}\langle X^{m,\circ}\rangle_s \xrightarrow[m\to\infty]{\mathrm{J}_1(\mathbb{R})} \int_{(0,\cdot]} \mathrm{e}^{\hat{\beta} A_s^\infty} Z_s^{\infty,(p)} \mathrm{d}\langle X^{\infty,c}\rangle_s. \tag{IV.54}$$

The same is true when the domain of integration is the whole positive real line and the metric is substituted by $|\cdot|$. Moreover, for every $R : \mathbb{R}_+ \to \mathbb{R}$ bounded function, which can be approximated uniformly by piecewise-constant functions, the following convergence is true

$$\int_{(0,\cdot]} \mathrm{e}^{\hat{\beta} A_s^m} R_s Z_s^{m,(p)} \, \mathrm{d}\langle X^{m,\circ}\rangle_s \xrightarrow[m\to\infty]{\mathrm{J}_1(\mathbb{R})} \int_{(0,\cdot]} \mathrm{e}^{\hat{\beta} A_s^\infty} R_s Z_s^{m,(p)} \, \mathrm{d}\langle X^{\infty,c}\rangle_s. \tag{IV.55}$$

The same is true when the domain of integration is the whole positive real line and the metric is substituted by $|\cdot|$. In particular, R can be chosen to be continuous with compact support.

Proof. Observe that in each case, the limit-process is continuous. Therefore, the convergence under Skorokhod topology coincides with the locally uniform topology, and the same is true for (IV.32) - (IV.33), which are special case of (IV.53) and (IV.54) respectively. So, we will use the fact that the aforementioned convergence hold under the locally uniform topology. Now, we can prove the claim using the same argument as in the proof of Lemma IV.34. More precisely, in view of Convergence (IV.32) - (IV.33), we use the associativity of the Lebesgue–Stieltjes integral in order to prove initially that the required convergence are true when their integrands are multiplied by $\mathrm{e}^{\hat{\beta} A^\infty}$. Now, in view of the boundedness of $(A^m)_{m\in\overline{\mathbb{N}}}$ by **(B6)**, we can conclude the uniform convergence of $(\mathrm{e}^{\hat{\beta} A^m})_{m\in\mathbb{N}}$ to $\mathrm{e}^{\hat{\beta} A^\infty}$. Observe that the sequence $(\mathrm{e}^{\hat{\beta} A^m})_{m\in\overline{\mathbb{N}}}$ remains uniformly bounded. Then, using these last convergence, we use the fact that all measures are finite in order to conclude (IV.53) and (IV.54).

For the Convergence (IV.55), in view of (IV.33), we can repeat the previous arguments for the case $(\mathrm{e}^{\hat{\beta} A^m} S)_{m\in\overline{\mathbb{N}}}$, where S is piecewise constant. Now, we can conclude again the general case, for the class described in the statement since the convergence of the integrands will be under the uniform topology. □

Lemma IV.36. *The following convergence are true*

$$\begin{aligned}\int_{(0,\cdot]\times\mathbb{R}} \mathrm{e}^{\hat{\beta} A_s^m} \left(U^{m,(p)}(s,x)\right)^2 \nu^m(\mathrm{d}s,\mathrm{d}x) - \sum_{s\leq\cdot} \mathrm{e}^{\hat{\beta} A_s^m} \left(\int_{\mathbb{R}} U(s,x)^{m,(p)} \, \nu^m(\{s\}\times\mathrm{d}x)\right)^2 \\ \xrightarrow[m\to\infty]{\mathrm{J}_1(\mathbb{R})} \int_{(0,\cdot]\times\mathbb{R}} \mathrm{e}^{\hat{\beta} A_s^\infty} \left(U^{\infty,(p)}(s,x)\right)^2 \nu^\infty(\mathrm{d}s,\mathrm{d}x)\end{aligned} \tag{IV.56}$$

and

$$\begin{aligned}\int_{(0,\cdot]\times\mathbb{R}} \mathrm{e}^{\hat{\beta} A_s^m} U^{m,(p)}(s,x) x \, \nu^m(\mathrm{d}s,\mathrm{d}x) \\ - \sum_{s\leq\cdot} \mathrm{e}^{\hat{\beta} A_s^m} \int_{\mathbb{R}} U^{m,(p)}(s,x) \, \nu^m(\{s\}\times\mathrm{d}x) \int_{\mathbb{R}} x \, \nu^m(\{s\}\times\mathrm{d}x) \\ \xrightarrow[m\to\infty]{\mathrm{J}_1(\mathbb{R})} \int_{(0,\cdot]\times\mathbb{R}} \mathrm{e}^{\hat{\beta} A_s^\infty} U^{\infty,(p)}(s,x) \nu^\infty(\mathrm{d}s,\mathrm{d}x)\end{aligned} \tag{IV.57}$$

The same is true when the domain of integration is the whole positive real line and the metric is substituted by $|\cdot|$. Moreover, for every $S : \widetilde{E} \to \mathbb{R}$ bounded function which vanishes outside of a compact set of $\widetilde{E}$

and which is the uniform limit of piecewise constant functions,[14] *the following convergence is true*

$$\begin{aligned}\int_{(0,\cdot]} &\mathrm{e}^{\hat{\beta}A^m_s} U^{m,(p)}(s,x)S(s,x)\nu^m(\mathrm{d}s,\mathrm{d}x)\\ &-\sum_{s\leq\cdot}\mathrm{e}^{\hat{\beta}A^m_s}\int_{\mathbb{R}} U(s,x)^{m,(p)}\,\nu^m(\{s\}\times\mathrm{d}x)\int_{\mathbb{R}} S(s,x)\,\nu^m(\{s\}\times\mathrm{d}x) \qquad \text{(IV.58)}\\ &\xrightarrow[m\to\infty]{\mathrm{J}_1(\mathbb{R})}\int_{(0,\cdot]}\mathrm{e}^{\hat{\beta}A^\infty_s}U^{\infty,(p)}(s,x)S(s,x)\,\nu^\infty(\mathrm{d}s,\mathrm{d}x)\end{aligned}$$

The same is true when the domain of integration is the whole positive real line and the metric is substituted by $|\cdot|$. *In particular, the function* S *can be chosen to lie in* $\mathrm{C}_{c,0}(\mathbb{R}_+\times\mathbb{R};\mathbb{R})$.

Proof. The convergence (IV.56) and (IV.57) are simply the translation of (IV.32) - (IV.33). The (IV.58) needs to be treated in a more careful way. □

Lemma IV.37. *The following inequality is true for all but finitely many* $m\in\mathbb{N}$

$$\left|L^{m,(p)}_\infty-\int_{(0,\infty)} f^m\big(s,Y^{m,(p)}_s,\widetilde{Z}_s,\widetilde{U}(s,\cdot)\big)\,\mathrm{d}C^m_s\right|<\frac{\varepsilon}{4}.$$

Proof. Using the Cauchy–Schwartz Inequality as in (II.25) we obtain

$$\begin{aligned}&\left|L^{m,(p)}_\infty-\int_{(0,\infty)} f^m\big(s,Y^{m,(p)}_s,\widetilde{Z}^{\varepsilon,(p)}_s,\widetilde{U}^{\varepsilon,(p)}(s,\cdot)\big)\,\mathrm{d}C^m_s\right|^2\\ &\quad=\left|\int_{(0,\infty)}\Big[f^m\big(s,Y^{m,(p)}_s,Z^{m,(p)}_s,U^{m,(p)}(s,\cdot)\big)-f^m\big(s,Y^{m,(p)}_s,\widetilde{Z}_s,\widetilde{U}(s,\cdot)\big)\Big]\,\mathrm{d}C^m_s\right|^2\\ &\quad=\int_{(0,\infty)}\mathrm{e}^{\hat{\beta}A^m_s}(\alpha^m_s)^{-2}\big|f^m\big(s,Y^{m,(p)}_s,Z^{m,(p)}_s,U^{m,(p)}(s,\cdot)\big)-f^m\big(s,Y^{m,(p)}_s,\widetilde{Z}_s,\widetilde{U}(s,\cdot)\big)\big|^2\,\mathrm{d}C^m_s\\ &\quad\underset{\text{(II.16)}}{\overset{\text{(II.49)}}{\leq}}\int_{(0,\infty)}\mathrm{e}^{\hat{\beta}A^m_s}\Big[\big|c^m_s(Z^{m,(p)}_s-\widetilde{Z}_s)\big|^2+\big|\big|\big|U^{m,(p)}_s(\cdot)-\widetilde{U}_s(\cdot)\big|\big|\big|_{s,k}\Big]\,\mathrm{d}C^m_s\\ &\quad=\int_{(0,\infty)}\mathrm{e}^{\hat{\beta}A^m_s}(Z^{m,(p)}_s-\widetilde{Z}_s)^2\,\mathrm{d}\langle X^{m,\mathrm{o}}\rangle_s\\ &\qquad\qquad+\int_{(0,\infty)}\mathrm{e}^{\hat{\beta}A^m_s}\Big\{(U^{m,(p)}(s,x))^2-2U^{m,(p)}(s,x)\widetilde{U}(s,x)+(\widetilde{U}(s,x))^2\Big\}\,\mathrm{d}\nu^m(\mathrm{d}s,\mathrm{d}x)\\ &\qquad-\sum_{s>0}\mathrm{e}^{\hat{\beta}A^m_s}\Bigg\{\bigg(\int_{\mathbb{R}}U(s,x)^{m,(p)}\,\nu^m(\{s\}\times\mathrm{d}x)\bigg)^2\\ &\qquad\qquad+\int_{\mathbb{R}}\widetilde{U}^{m,(p)}(s,x)\,\nu^m(\{s\}\times\mathrm{d}x)\int_{\mathbb{R}}U(s,x)^{m,(p)}\,\nu^m(\{s\}\times\mathrm{d}x)\\ &\qquad\qquad\qquad+\bigg(\int_{\mathbb{R}}\widetilde{U}^{m,(p)}(s,x)\,\nu^m(\{s\}\times\mathrm{d}x)\bigg)^2\Bigg\}\end{aligned}$$

By Lemma IV.35 and Lemma IV.36 we have that the right hand side of the above inequality converges to the value

$$\int_{(0,\infty)}\mathrm{e}^{\hat{\beta}A^\infty_s}(Z^{\infty,(p)}_s-\widetilde{Z}_s)^2\,\mathrm{d}\langle X^{\infty,c}\rangle_s+\int_{(0,\cdot]}\mathrm{e}^{\hat{\beta}A^\infty_s}\big(U^{\infty,(p)}(s,x)-\widetilde{U}(s,x)\big)^2\,\nu^\infty(\mathrm{d}s,\mathrm{d}x)\underset{\text{(IV.38)}}{\overset{\text{(IV.37)}}{<}}\frac{\varepsilon\hat{\beta}^2}{64}.$$

Now recall that we have assumed (without loss of generality) in Remark IV.27 that $\varepsilon<\frac{1}{\hat{\beta}}$. Therefore, we can conclude the claim of the lemma.

□

[14] In this case a piecewise constant function f if of the form $f=\sum_{i=1}^p f_i\mathbb{1}_{\Pi_i}$, where $(\Pi_i)_{i=1,\dots,p}$ is a pairwise disjoint family of rectangular subsets of $\widetilde{E}$.

IV.5.5. Uniform integrability. For the following the reader may recall Notation IV.12. In this subsection we will prove that the statement (Step (p) UI) is true *i.e.* $\big(\Gamma^{k,(p)}\big)_{k\in\overline{\mathbb{N}}}$ and $\big(\Delta^{k,(p)}\big)_{k\in\overline{\mathbb{N}}}$ are uniformly integrable. This in turn will ensure that the sequence $\big(\sup_{t\in\mathbb{R}_+}\big|L_t^{k,(p)}\big|^2\big)_{k\in\overline{\mathbb{N}}}$ is uniformly integrable and they will allow to proceed with the induction. Recall that we need the uniform integrability of $\big(\sup_{t\in\mathbb{R}_+}\big|L_t^{k,(p)}\big|^2\big)_{k\in\overline{\mathbb{N}}}$ in order to apply Vitali's Theorem and finally prove that the Convergence $(\mathrm{LS}^{(p)})$ and $(\mathrm{LS}_\infty^{(p)})$ hold in $\mathbb{L}^2-$mean and not only in probability.

For the following lemmata we will make use of the pathwise estimates obtained in Lemma II.18. As we did in Lemmata IV.19 and IV.20, in view of Condition **(B6)** we will substitute the coefficients of the aforementioned inequalities with a universal constant, say C. After these comments, we can proceed.

Lemma IV.38. *The sequence* $\Big(\sup_{t\in\mathbb{R}_+}\big|M_t^{k,(p)}-M_0^{k,(p)}\big|^2\Big)_{k\in\overline{\mathbb{N}}}$ *is uniformly integrable. In particular, the sequence* $\big(\Delta^{k,(p)}\big)_{k\in\overline{\mathbb{N}}}$ *is uniformly integrable.*

Proof. The arguments are very similar to the one used to prove Lemma IV.19. However, we provide all the details for the sake of completeness.

The martingale $M^{k,(p)}$, for $k\in\overline{\mathbb{N}}$, was defined in (IV.15) for $k=\infty$ and in (IV.17) for $k\in\mathbb{N}$. However, we will use the equivalent forms (IV.16) and (IV.18). Now, we proceed to prove the first claim of the lemma. We will make use of the pathwise estimates in Lemma II.18 and in particular (II.39). We have, now, for every $k\in\overline{\mathbb{N}}$ that

$$\begin{aligned}
&\sup_{t\in\mathbb{R}_+}\big|M_t^{k,(p)}-M_0^{k,(p)}\big|^2\\
&\quad\le C\sup_{t\in\mathbb{R}_+}\mathbb{E}\Bigg[\big|\xi^k\big|^2+\int_{(0,\infty)}\mathrm{e}^{\hat\beta A_s^k}\frac{\big|f^k\big(s,Y_s^{k,(p-1)},Z_s^{k,(p-1)},U^{k,(p-1)}(s,\cdot)\big)\big|^2}{(\alpha_s^k)^2}\,\mathrm{d}C_s^k\Bigg|\mathcal{G}_t\Bigg]\\
&\qquad\qquad+C\int_{(0,\infty)}\mathrm{e}^{\hat\beta A_s^k}\frac{\big|f^k\big(s,Y_s^{k,(p-1)},Z_s^{k,(p-1)},U^{k,(p-1)}(s,\cdot)\big)\big|^2}{(\alpha_s^k)^2}\,\mathrm{d}C_s^k\\
&\quad\le C\sup_{t\in\mathbb{R}_+}\mathbb{E}\Big[\big|\xi^k\big|^2+\Gamma^{k,(p-1)}\Big|\mathcal{G}_t\Big].+C\Gamma^{k,(p-1)}
\end{aligned}$$

The right hand side is uniformly integrable; this is true in view of Lemma I.106 and the uniform integrability of the sequence $\big(\big|\xi^k\big|^2\big)_{k\in\overline{\mathbb{N}}}$ and of $\big(\Gamma^{k,(p)}\big)_{k\in\overline{\mathbb{N}}}$.

In view of the uniform integrability of the sequence $\Big(\sup_{t\in\mathbb{R}_+}\big|M_t^{k,(p)}-M_0^{k,(p)}\big|^2\Big)_{k\in\overline{\mathbb{N}}}$ we can apply Theorem I.55 in order to obtain the uniform integrability of $\big(\langle M^{k,(p)}-M_0^{k,(p)}\rangle_\infty\big)_{k\in\overline{\mathbb{N}}}$. Then, we have the required property for the sequence $\big(\Delta^{k,(p)}\big)_{k\in\overline{\mathbb{N}}}$. □

Lemma IV.39. *The sequence* $\big(\Gamma^{k,(p)}\big)_{k\in\overline{\mathbb{N}}}$ *is uniformly integrable.*

Proof. We will make use of Lemma II.18 and in particular the Let $k\in\mathbb{N}$. Then, we have

$$\begin{aligned}
\Gamma^{k,(p)}&=\int_{(0,\infty)}\mathrm{e}^{\hat\beta A_s^k}\frac{\big|f^k\big(s,Y_s^{k,(p)},Z_s^{k,(p)},U^{k,(p)}(s,\cdot)\big)\big|^2}{(\alpha_s^k)^2}\,\mathrm{d}C_s^k\\
&\underset{\text{(II.49)}}{\overset{\text{(II.16)}}{\le}}\int_{(0,\infty)}\mathrm{e}^{\hat\beta A_s^k}(\alpha_s^k)^2\big|Y_s^{k,(p)}\big|^2\,\mathrm{d}C_s^k+\int_{(0,\infty)}\mathrm{e}^{\hat\beta A_s^k}\big|c_s^k(Z_s^{k,(p)})\big|^2+\big|\big|\big|U_s^{k,(p)}(\cdot)\big|\big|\big|_{s,k}\Big]\,\mathrm{d}C_s^k\\
&\underset{\text{(II.49)}}{\overset{\text{(II.37)}}{\le}}C\sup_{t\in\mathbb{R}_+}\mathbb{E}\Big[\big|\xi^k\big|^2+\Gamma^{k,(p-1)}\Big|\mathcal{G}_t^k\Big]+C\big\langle(Z_s^{k,(p)}-Z_s^{k,(p-1)})\cdot X^{k,\circ}+\big(U^{k,(p)}-U^{k,(p-1)}\big)\star\widetilde{\mu}^k\big\rangle_\infty.
\end{aligned}$$

The analogous bound can be obtained for $k=\infty$. Now we can conclude. □

Lemma IV.40. *The sequence* $\big(\sup_{t\in\mathbb{R}_+}\big|L_t^{k,(p)}\big|^2\big)_{k\in\overline{\mathbb{N}}}$ *is uniformly integrable.*

Proof. Apply Cauchy-Schwartz and use the previous lemma. □

IV.5.6. Some comments on the framework of the chapter. In this section we would like to comment on the nature of the conditions we have imposed to settle the framework of Chapter IV. Let us start with the first four Conditions **(B1)** - **(B5)**. We have repeatedly commented that they are inherited

by Theorem III.50. One may question the necessity of the convergence imposed in **(B2)**, which is stronger comparing to the analogous of Theorem III.3, *i.e.*

$$X^{k,\circ}_\infty + X^{k,\natural}_\infty \xrightarrow[k\to\infty]{\mathbb{L}^2(\mathcal{G};\mathbb{R}^n)} X^{\infty,c}_\infty + X^{\infty,d}_\infty$$

instead of

$$\overline{X}^k_\infty \xrightarrow[k\to\infty]{\mathbb{L}^2(\mathcal{G};\mathbb{R}^n)} \overline{X}^\infty_\infty.$$

The main reason is that in Subsection IV.5.4 we need to utilize the convergence

$$\big(\langle X^{m,\circ}\rangle, \langle X^{m,\natural}\rangle\big) \xrightarrow[m\to\infty]{\mathrm{J}_1(\mathbb{R}^{\ell\times\ell}\times\mathbb{R}^{\ell\times\ell})} \big(\langle X^{\infty,c}\rangle, \langle X^{\infty,d}\rangle\big)\ \mathbb{P}-a.s.,$$

a convergence which is not guaranteed by Theorem III.3. On the other hand, Condition **(B2)** is the one that provides us the flexibility to approximate the continuous part of X^∞ with purely discontinuous martingales, a property which is essential when the discussion comes to numerical schemes.

For the Condition **(B6)** we have provided some comments at the end if Section IV.1. Therefore, we proceed to Condition **(B7)**.

(B7).(ii) This condition is usually met when we need to guarantee the convergence of compositions under the J_1-topology, *e.g.* see Kurtz and Protter [44, Lemma 2.1]. In our case we do not need to be that abstract since we can exploit the Lipschitz property of the generators. The latter allows us to obtain for K_2 compact subset of $\mathbb{R}$ and $U \in \mathrm{C}_{c,0}(\mathbb{R}_+ \times \mathbb{R})$ and $N \in \mathbb{N}$ that, if $(y^k)_{k\in\overline{\mathbb{N}}}$ such that

$$y^k \xrightarrow{\mathrm{J}_1} y^\infty$$

then

$$\sup_{t\in[0,N]} \Big\{ \big| f^\infty\big(\omega, \lambda^k(t), y^k_{\lambda^k(t)}, z, U(\lambda^k(t),\cdot)\big) - f^\infty\big(\omega, t, y^\infty, z, U(t,\cdot)\big)\big|, z \in K_2 \Big\} \xrightarrow[k\to\infty]{} 0,$$

since U is uniformly continuous.

(B7).(iii) This condition is a generalisation of the assumption that the generators are globally Lipschitz which share the same Lipschitz constants. This global property implies in particular that the sequence of generators forms for every point (y, z, u) a uniformly equicontinuous family of functions. It is then relatively straightforward to prove that the convergence for every point (y, z, u) implies the convergence **(B7).(iii)**. For example, when there is no dependence on u, see in the proof of Briand et al. [15, Proposition 11].

(B7).(iv) In view of the predictability of the bounds, it is natural to assume the convergence under the uniform topology. We will comment the assumption of the local boundedness of the Radon–Nikodým derivative when we discuss about Condition **(B9)**.

We proceed now to the discussion about Condition **(B9)**.

(B9).(i) Let us fix a $k \in \mathbb{N}$. The analogous arguments hold also for the case $k = \infty$. Recall that the pair $(\overline{X}^k, C^k)$ satisfies Assumption Section I.7. The existence of an integrator C^k is ensured by Lemma I.166, where we have assumed specific positive functions, say f^1, f^2, of $\langle X^{k,\circ}\rangle$ and $\mathrm{Id}_\ell * \nu^{(X^\natural,\mathbb{G}^k)}$ with the property that . In other words, we have set $C^k := f^1\big(\langle X^{k,\circ}\rangle\big) + f^2\big(\mathrm{Id}_\ell * \nu^{(X^\natural,\mathbb{G}^k)}\big)$.

On the other hand, in view of the Conditions **(B1)** - **(B4)** and Theorem I.160, we know that the sequence $(\langle X^{k,\circ}\rangle, \langle X^{k,\natural}\rangle)$ converges under the J_1-topology to $(\langle X^{\infty,c}\rangle, \langle X^{\infty,d}\rangle)$. Since we are interested at this point for absolute continuity of measures, we can choose for every $k \in \overline{\mathbb{N}}$ the same positive and continuous function f, such that

$$C^k := f^1\big(\langle X^{k,\circ}\rangle\big) + f^2\big(\langle X^{k,\natural}\rangle\big) \text{ for every } k \in \mathbb{N} \text{ and } C^\infty := f^1\big(\langle X^{\infty,c}\rangle\big) + f^2\big(\langle X^{\infty,d}\rangle\big).$$

We can, possibly, alternatively choose a sequence $(f^k)_{k\in\mathbb{N}}$, which may allow dependence on the time, as soon as it respects the J_1-topology.

(B9).(ii) This is completely analogous to the previous case once we recall that Theorem I.160 verifies also the convergence

$$(\langle X^{k,\circ}\rangle_\infty, \langle X^{k,\natural}\rangle_\infty) \xrightarrow{\mathbb{L}^2(\mathbb{G};\mathbb{R})} (\langle X^{\infty,c}\rangle_\infty, \langle X^{\infty,d}_\infty\rangle).$$

(B9).(iii) Since we have assumed that the bound determined by (II.18) is deterministic and in view of **(B3)**, this assumption is completely natural.

In view of the comment on the way we propose to choose C^∞, the (local) boundedness of the the Radon–Nikodým derivative $\mathrm{d}\langle X^{\infty,c}\rangle/\mathrm{d}C^\infty$ is clear.

IV.6. Comparison with the related literature

In this section we are going to compare our result with the existing literature. More precisely, the comparison will be made with Madan et al. [50] which is the most general result for discrete-time approximations of BSDEs. However, we need to underline that the work of Briand et al. [15] has greatly inspired the author. In the following we will translate the notation to the one we have used.

We start with the Condition **(B1)**, *i.e.* the limit-filtration has to be quasi-left-continuous, which is common between the two works. Actually, we may recall that Mémin [53, Counter-example 3] which justifies the necessity of **(B1)**. However, in our case we are not restricted in (square-integrable) Lévy martingale, but it can be an arbitrary (square-integrable) one.

We proceed to the conditions imposed on the sequence of driving martingales $(\overline{X}^{k,\circ}, X^{k\natural})_{k\in\overline{\mathbb{N}}}$. In [50] the integrators can be multidimensional, while we have presented the real-valued case. However, in view of the results in Chapter III, we can readily adapt the dimension of the integrators to a higher one. The same holds true for the dimension of the solution of the BSDE, in view of the main theorem of Chapter II. In Chapter IV we preferred to present the result in an as simple as possible manner. In both cases, they are square-integrable martingales whose terminal values converge in mean to the respective terminal values of the natural pair of X^∞. Now we proceed to describe a few improvements that take place under our framework.

- The time horizon in our case can be infinite, while in [50] it is assumed finite.
- The terminal values in [50] need to converge in a stronger sense than in $\mathbb{L}^2-$mean, see [50, Conditions (2.2) and (2.5)], while in our case the convergence is in $\mathbb{L}^2-$mean. Actually, we have explained at the beginning of Subsection III.6.1 the role of Lemma III.39 in order to obtain this sharp convergence.
- In our framework we need $M_{\mu^{X^{k,\natural}}}\big[\Delta X^{k,\circ}\big|\widetilde{\mathcal{P}}^{\mathbb{G}^k}\big] = 0$ for every $k \in \mathbb{N}$, which is a weaker condition compared to independence assumption imposed in [50]. Let us argue about it. To this end, let us fix a finite time horizon T, a $k \in \mathbb{N}$ and a partition $(t^m)_{m=0,\dots,M}$ of $[0,T]$. We will assume that $X^{k,\circ}, X^{k,\natural}$ have independent increments and are constant on the interval $[t^m, t^{m+1})$ for $m = 0,\dots,M-1$. In this setting we have for every bounded $\mathbb{G}^k-$predictable function U

$$M_{\mu^{X^{k,\natural}}}\Big[UM_{\mu^{X^{k,\natural}}}\big[\Delta X^{k,\circ}\big|\widetilde{\mathcal{P}}^{\mathbb{G}^k}\big]\Big] = M_{\mu^{X^{k,\natural}}}\Big[U\Delta X^{k,\circ}\Big] = \mathbb{E}\Big[\sum_{m=0}^{M-1} U(t^m, \Delta X^{k,\natural}_{t^m})\Delta X^{k,\circ}_{t^m}\Big] = 0.$$

Finally, regarding the properties the generator should possess, we will restrict ourselves on mentioning that Madan et al. [50, Assumption 1] demands the generators to be uniformly Lipschitz. In Subsection IV.5.6 we have commented that the uniform Lipschitz condition translates the pointwise convergence into seemingly stronger kind of convergence. For this reason, we could say that we have used finally the usual convergence assumptions as they translate in a weaker framework.

APPENDIX A

Auxiliary results

A.1. Auxiliary results of Chapter II

Proof of Lemma II.13. Let $(\gamma,\delta)\in\mathcal{C}_\beta$. We shall begin by obtaining the critical points of the map Π^Φ. We have

$$\frac{\partial}{\partial\gamma}\Pi^\Phi(\gamma,\delta)=(2+9\delta)\,\mathrm{e}^{(\delta-\gamma)\Phi}\,\frac{\Phi\gamma^2+(2-\delta\Phi)\gamma-\delta}{\gamma^2(\delta-\gamma)^2},$$
$$\frac{\partial}{\partial\delta}\Pi^\Phi(\gamma,\delta)=-\frac{9}{\delta^2}+\mathrm{e}^{(\delta-\gamma)\Phi}\left\{\frac{\big[9+(2+9\delta)\Phi\big](\delta-\gamma)}{\gamma(\delta-\gamma)^2}-\frac{2+9\delta}{\gamma(\delta-\gamma)^2}\right\}.$$

The only possible critical points for Π^Φ are therefore such that $\delta=-2/9$ or $\gamma=\frac{\delta\Phi-2\pm\sqrt{4+\delta^2\Phi^2}}{2\Phi}$. However, the values $\delta=-2/9$ and $\gamma=\frac{\delta\Phi-2-\sqrt{4+\delta^2\Phi^2}}{2\Phi}$ are ruled out as negative. For $0<\delta\le\beta$ we have

$$\Big(\frac{\delta\Phi-2+\sqrt{4+\delta^2\Phi^2}}{2\Phi},\delta\Big)\in\mathcal{C}_\beta.$$

Let us define $\overline{\gamma}^\Phi(\delta):=\frac{\delta\Phi-2+\sqrt{4+\delta^2\Phi^2}}{2\Phi}$, for $0<\delta\le\beta$. It is easy to verify that $\overline{\gamma}^\Phi(\delta)\in(0,\delta)$. Then, some tedious calculations yield that

$$\frac{\partial\Pi^\Phi}{\partial\delta}\big(\bar\gamma^\Phi(\delta),\delta\big)=-\frac{9}{\delta^2}-\frac{\exp[\big(\delta-\bar\gamma^\Phi(\delta)\big)\Phi]}{\bar\gamma^\Phi(\delta)\big(\delta-\bar\gamma^\Phi(\delta)\big)^2}\cdot\frac{2\bar\gamma^\Phi(\delta)\Phi+9\bar\gamma^\Phi(\delta)+2}{(\bar\gamma^\Phi(\delta)\Phi+1)^2}<0$$

therefore Π^Φ does not admit any critical point on $\mathcal{C}_\beta$, for which $0<\gamma<\delta<\beta$. Hence, the infimum on this set is necessarily attained on its boundary. The cases where at least one among δ and γ goes to 0, or where their difference goes to 0, lead to the value ∞. The only remaining case is therefore $0<\gamma<\delta=\beta$, where β is fixed. Then we get

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\Pi^\Phi\big(\gamma,\beta\big)=(2+9\beta)\,\mathrm{e}^{(\beta-\gamma)\Phi}\,\frac{\Phi\gamma^2+(2-\beta\Phi)\gamma-\beta}{\gamma^2(\beta-\gamma)^2},$$

and $\Pi^\Phi(\gamma,\beta)$ viewed as a function of γ attains its minimum at its critical point given by $\overline{\gamma}^\Phi(\beta)$, since $\frac{\mathrm{d}\Pi^\Phi}{\mathrm{d}\gamma}\big(\gamma,\beta\big)<0$ on $(0,\overline{\gamma}^\Phi(\beta))$ and $\frac{\mathrm{d}\Pi^\Phi}{\mathrm{d}\gamma}\big(\gamma,\beta\big)>0$ on $(\overline{\gamma}^\Phi(\beta),\beta)$.

Now, we proceed to the second case, and start by determining the critical points of $\Pi^\Phi_\star$. It holds

$$\frac{\partial}{\partial\gamma}\Pi^\Phi_\star(\gamma,\delta)=-\frac{8}{\gamma^2}+9\delta\mathrm{e}^{(\delta-\gamma)\Phi}\frac{\Phi\gamma^2-(\delta\Phi-2)\gamma-\delta}{\gamma^2(\delta-\gamma)^2},$$
$$\frac{\partial}{\partial\delta}\Pi^\Phi_\star(\gamma,\delta)=-\frac{9}{\delta^2}+9\mathrm{e}^{(\delta-\gamma)\Phi}\frac{(1+\delta\Phi)(\delta-\gamma)-\delta}{\gamma(\delta-\gamma)^2}.$$

Following analogous computations as above we can prove that, for $(\gamma,\delta)\in\mathcal{C}_\beta$, the equation

$$\frac{\partial}{\partial\gamma}\Pi^\Phi_\star(\gamma,\delta)=0\Leftrightarrow P_\delta(\gamma):=8(\delta-\gamma)^2-9\delta\mathrm{e}^{(\delta-\gamma)\Phi}\big(\Phi\gamma^2-(\delta\Phi-2)\gamma-\delta\big)=0$$

has a unique root, say $\overline{\gamma}^\Phi_\star(\delta)$, which moreover satisfies $\overline{\gamma}^\Phi_\star(\delta)\in(\overline{\gamma}^\Phi(\delta),\delta)$. This can be proved because the function $P_\delta:(0,\delta)\to\mathbb{R}$ is decreasing, for each fixed $\delta\in(0,\beta)$, with $P_\delta\big(\overline{\gamma}^\Phi(\delta)\big)>0$ and $P_\delta(\delta)<0$. Now observe that for $\gamma>\frac{\delta^2\Phi}{1+\delta\Phi}$ it holds $\frac{\partial}{\partial\delta}\Pi^\Phi_\star(\gamma,\delta)<0$ and that $P_\delta\big(\frac{\delta^2\Phi}{1+\delta\Phi}\big)>0$. Using the monotonicity of P_δ we have that $\overline{\gamma}^\Phi_\star(\delta)>\frac{\delta^2\Phi}{1+\delta\Phi}$ and therefore also $\frac{\partial}{\partial\delta}\Pi^\Phi_\star(\overline{\gamma}^\Phi_\star(\delta),\delta)<0$. Arguing as above we can conclude that the infimum is attained for $\delta=\beta$ at the point $\big(\overline{\gamma}^\Phi_\star(\beta),\beta\big)$.

Finally, the limiting statements follow by straightforward but tedious computations. □

A.1.1. The non-exponential submultiplicative case. In this subsection we provide the *a priori* estimates of the semi–martingale decomposition (II.20) in the case $h_\zeta : \mathbb{R} \to [1, \infty)$, where $h_\zeta(x) = 1$ for $x < 0$ and $h_\zeta(x) := (1 + x)^\zeta$, for $x > 0$ where ζ is a fixed number greater than 1, with the additional assumption that the process A defined in **(F4)** is $\mathbb{P}-$a.s. bounded by Ψ. However, using completely analogous arguments the *a priori* estimates can be obtained for a submultiplicative function h. In [64, Proposition 25.4] (see the proof of (iii)) we can find an easy criterion for an increasing function $h : \mathbb{R} \to \mathbb{R}_+$ to be submultiplicative. Namely, it suffices for the function $h : \mathbb{R} \to (0, \infty)$ to be flat on $(-\infty, b]$ and the function $\log(h(\cdot))$ to be concave on (b, ∞), for some $b \in \mathbb{R}_+$. It is immediate that the function h_ζ as defined above is submultiplicative. Since ζ is fixed, we will simply denote h_ζ as h in the rest of the section.

Before we proceed to the proof we provide the properties of the submultiplicative function we will need, presented however for the function h_ζ, as well as some notation. For We start with properties.

(i) $h : \mathbb{R}_+ \to [1, \infty)$ and h is non-decreasing.

(ii) h is submultiplicative, *i.e.* there exists $c_h > 0$ such that

$$h(x + y) \le c_h h(x)h(y), \text{ for all } x, y \in \mathbb{R}_+.$$

(iii) It holds $x \le h(x)$ for every $x \in \mathbb{R}_+$.

(iv) The indefinite integral $\int h(s)^{-\gamma}\,\mathrm{d}s$ can be explicitly calculated. We define for $\gamma > \frac{1}{\zeta}$

$$\begin{aligned} H_\gamma(x) &:= \int_{(0,x]} h^{-\gamma}(s)\,\mathrm{d}s = \frac{1}{1-\gamma\zeta}\big[(1+x)^{1-\gamma\zeta} - 1\big] \\ &= \frac{1}{-\gamma\zeta+1}\big[(1+x)^{-\gamma\zeta+1} - 1\big] = \frac{1}{-\gamma\zeta+1}\big[\big(h(x)\big)^{\frac{-\gamma\zeta+1}{\zeta}} - 1\big] \\ &= \frac{1}{-\gamma\zeta+1}\big[\big(h(x)\big)^{-\gamma+\frac{1}{\zeta}} - 1\big] = \frac{1}{-\gamma\zeta+1}\big[h^{\vartheta(\gamma)}(x) - 1\big], \end{aligned} \tag{A.1}$$

where $\vartheta(\gamma) := -\gamma + \frac{1}{\zeta} < 0$. Moreover,

$$\begin{aligned} \int_{(t,u]} h^{-\gamma}(s)\mathrm{d}s &= H_\gamma(u) - H_\gamma(t) = |H_\gamma(t)| - |H_\gamma(u)| \\ &= \frac{1}{|-\gamma\zeta+1|}[h^{\vartheta(\gamma)}(t) - h^{\vartheta(\gamma)}(u)] = \frac{1}{\gamma\zeta-1}[h^{\vartheta(\gamma)}(t) - h^{\vartheta(\gamma)}(u)]. \end{aligned} \tag{A.2}$$

(v) The set $\Gamma_H := \big\{\gamma \in \mathbb{R}_+, H_\gamma : \mathbb{R}_+ \to (-\infty, 0] \text{ is increasing}\big\}$ is non–empty. In view of the restriction in (iv), the set Γ_H is indeed non-empty, *i.e.*

$$\Gamma_H = (\frac{1}{\zeta}, \infty). \tag{A.3}$$

(vi) There exists $\beta \in \mathbb{R}_+$ such that

$$\mathbb{E}\Big[\int_{(0,T]} h^\beta(A_s)\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s\Big] < \infty.$$

By [64, Proposition 25.4] we can calculate the submultiplicative constant

$$c_h = \exp\big[\ln\big(h(2\cdot 0)\big) - \ln\big(h(0)\big)\big] = 1.$$

Lemma A.1. *Assume the framework of Lemma II.16 except for the definition of the $\beta-$norms, where we substitute the exponential function with the function h, for $\zeta \ge 1$;* e.g *see* (vi)*. Moreover, assume that A is bounded by the positive constant Ψ. Then, it holds for $\frac{1}{\zeta} < \gamma < \delta < \hat{\beta}$*

$$\begin{aligned} \|\alpha y\|_{\mathbb{H}_\delta^2}^2 + \|\eta\|_{\mathbb{H}_\delta^2}^2 \le &\left[2h(\Psi)h^\delta(\Phi) + 9 + \frac{9[h(\Psi)]^{1-\frac{1}{\zeta}}[h(\Phi)]^{\delta-\frac{1}{\zeta}}}{\delta - \frac{1}{\zeta} + 1}\right]\|\xi\|_{\mathbb{L}_\delta^2}^2 \\ &+ \left[\frac{2[h(\Psi)]^{1+\frac{1}{\zeta}}[h(\Phi)]^{\delta-\gamma+\frac{1}{\zeta}}}{\delta-\gamma+\frac{1}{\zeta}+1} + \frac{9h(\Psi)[h(\Phi)]^{\delta-\gamma}}{\delta-\gamma+1} + \frac{9}{\delta\zeta-1}\right]\left\|\frac{f}{\alpha}\right\|_{\mathbb{H}_\delta^2}, \end{aligned}$$

and

$$\|y\|_{\mathcal{S}_\delta^2}^2 + \|\eta\|_{\mathbb{H}_\delta^2}^2 \le \big[8 + 2h(\Psi)h^\delta(\Phi)\big]\,\|\xi\|_{\mathbb{H}_\delta^2}^2$$

$$+\left[\frac{[h(\Psi)]^{\frac{1}{\zeta}}}{\gamma\zeta-1}+\frac{2[h(\Psi)]^{1+\frac{1}{\zeta}}[h(\Phi)]^{\delta-\gamma+\frac{1}{\zeta}}}{\delta-\gamma+\frac{1}{\zeta}+1}\right]\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_\delta}.$$

Proof. Recall the identity

$$y_t=\xi+\int_{(t,T]}f_s\,\mathrm{d}C_s-\int_{(t,T]}\mathrm{d}\eta_s=\mathbb{E}\left[\xi+\int_{(t,T]}f_s\,\mathrm{d}C_s\middle|\mathcal{G}_t\right],\tag{A.4}$$

and introduce the anticipating function

$$F_t=\int_{(t,T]}f_s\,\mathrm{d}C_s.$$

For $\gamma\in\Gamma_H$, we have by the Cauchy-Schwarz inequality,

$$\begin{aligned}\|F_t\|_2^2 &\le \int_{(t,T]}h^{-\gamma}(A_s)\,\mathrm{d}A_s\int_{(t,T]}h^{\gamma}(A_s)\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s\\ &= \int_{(A_t,A_T]}h^{-\gamma}(A_{L_s})\,\mathrm{d}s\int_{(t,T]}h^{\gamma}(A_s)\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s\\ &\overset{\text{Lemma I.34}}{\underset{\text{(i)}}{\le}} \int_{(A_t,A_T]}h^{-\gamma}(s)\,\mathrm{d}s\int_{(t,T]}h^{\gamma}(A_s)\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s\\ &\overset{\text{(A.2)}}{=} \frac{1}{\gamma\zeta-1}[h^{\vartheta(\gamma)}(A_t)-h^{\vartheta(\gamma)}(A_T)]\int_{(t,T]}h^{\gamma}(A_s)\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s\\ &\le \frac{1}{\gamma\zeta-1}h^{\vartheta(\gamma)}(A_t)\int_{(t,T]}h^{\gamma}(A_s)\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s.\end{aligned}\tag{A.5}$$

For $\delta\in\mathbb{R}_+$ and by integrating (A.5) w.r.t. $h^\delta(A_t)\,\mathrm{d}A_t$ it follows

$$\begin{aligned}\int_{(0,T]}h^{\delta}(A_t)\|F_t\|_2^2\,\mathrm{d}A_t &\overset{\text{(A.5)}}{\le} \frac{1}{\gamma\zeta-1}\int_{(0,T]}h^{\delta+\vartheta(\gamma)}(A_t)\int_{(t,T]}h^{\gamma}(A_s)\frac{\|f_s\|_2^2}{\alpha_s^2}\mathrm{d}C_s\,\mathrm{d}A_t\\ &= \frac{1}{\gamma\zeta-1}\int_{(0,T]}h^{\gamma}(A_s)\frac{\|f_s\|_2^2}{\alpha_s^2}\int_{(0,s)}h^{\delta+\vartheta(\gamma)}(A_t)\,\mathrm{d}A_t\,\mathrm{d}C_s\\ &\le \frac{1}{\gamma\zeta-1}\int_{(0,T]}h^{\gamma}(A_s)\frac{\|f_s\|_2^2}{\alpha_s^2}\int_{(0,s]}h^{\delta+\vartheta(\gamma)}(A_t)\,\mathrm{d}A_t\,\mathrm{d}C_s\\ &\le \frac{1}{\gamma\zeta-1}\int_{(0,T]}h^{\gamma}(A_s)\frac{\|f_s\|_2^2}{\alpha_s^2}\int_{(A_0,A_s]}h^{\delta+\vartheta(\gamma)}(A_{L_t})\,\mathrm{d}t\,\mathrm{d}C_s,\end{aligned}\tag{A.6}$$

where in the equality we used Tonelli's Theorem. By [64, Proposition 25.4] we have that $h^{\delta+\vartheta(\gamma)}$ remains submultiplicative if $\delta+\vartheta(\gamma)>0$, which is equivalent to $\delta>|\vartheta(\gamma)|$. Therefore, we can apply Lemma I.34.(vii) for the function $g(x)=h^{\delta+\vartheta(\gamma)}(x)$. Then, Inequality (A.6) becomes

$$\begin{aligned}\int_{(0,T]}h^{\delta}(A_t)\|F_t\|_2^2\,\mathrm{d}A_t &\overset{\text{Lemma I.34}}{\le} c_h^{\delta+\vartheta(\gamma)}h^{\delta+\vartheta(\gamma)}(\Phi)\int_{(0,T]}h^{\gamma}(A_s)\frac{\|f_s\|_2^2}{\alpha_s^2}\int_{(A_0,A_s]}h^{\delta+\vartheta(\gamma)}(t)\,\mathrm{d}t\,\mathrm{d}C_s\\ &\overset{c_h=1}{=} \frac{h^{\delta+\vartheta(\gamma)}(\Phi)}{\delta+\vartheta(\gamma)+1}\int_{(0,T]}h^{\gamma}(A_s)\frac{\|f_s\|_2^2}{\alpha_s^2}\big(h^{\delta+\vartheta(\gamma)+1}(A_s)-h^{\delta+\vartheta(\gamma)+1}(A_0)\big)\mathrm{d}C_s\\ &\underset{h(A_0)=1}{\overset{A_0=0}{\le}} \frac{h^{\delta+\vartheta(\gamma)}(\Phi)}{\delta+\vartheta(\gamma)+1}\int_{(0,T]}\big[h(A_s)\big]^{\delta+\frac{1}{\zeta}+1}\frac{\|f_s\|_2^2}{\alpha_s^2}\mathrm{d}C_s,\end{aligned}\tag{A.7}$$

which is integrable if $\delta+\frac{1}{\zeta}+1\le\beta$; see (vi). Define

$$\varrho:=\delta+\frac{1}{\zeta}+1 \text{ and } \Lambda^{\gamma,\delta,\Phi}:=\frac{h^{\delta+\vartheta(\gamma)}(\Phi)}{\delta+\vartheta(\gamma)+1}.$$

To sum up, for

$$\gamma\in\Gamma_H,\delta>\gamma-\frac{1}{\zeta} \text{ and } 0<\varrho\le\beta,\tag{A.8}$$

we have

$$\mathbb{E}\Big[\int_{(0,T]} h^\delta(A_t)\|F_t\|_2^2 \mathrm{d}A_t\Big] \le \Lambda^{\gamma,\delta,\Phi}\Big\|\frac{f}{\alpha}\Big\|^2_{\mathbb{H}^2_\varrho}. \tag{A.9}$$

For the estimate of $\|\alpha y\|_{\mathbb{H}^2_\delta}$ we first use the fact that

$$\begin{aligned}
\|\alpha y\|^2_{\mathbb{H}^2_\delta} &= \mathbb{E}\Big[\int_{(0,T]} h^\delta(A_t)\|y_t\|_2^2 \,\mathrm{d}A_t\Big] \\
&\le 2\mathbb{E}\Big[\int_{(0,T]} \mathbb{E}\left[h^\delta(A_t)\|\xi\|_2^2 + h^\delta(A_t)\|F_t\|_2^2 \middle| \mathcal{G}_t\right] \mathrm{d}A_t\Big] \\
&= 2\mathbb{E}\Big[\int_{(0,\infty)} \mathbb{E}\left[h^\delta(A_t)\|\xi\|_2^2 + h^\delta(A_t)\|F_t\|_2^2 \middle| \mathcal{G}_t\right] \mathrm{d}A_t^T\Big] \\
&= 2\mathbb{E}\Big[\int_{(0,\infty)} h^\delta(A_t)\|\xi\|_2^2 + h^\delta(A_t)\|F_t\|_2^2 \,\mathrm{d}A_t^T\Big] \\
&= 2\mathbb{E}\Big[\int_{(0,T]} h^\delta(A_t)\|\xi\|_2^2 \,\mathrm{d}A_t\Big] + 2\mathbb{E}\Big[\int_{(0,T]} h^\delta(A_t)\|F_t\|_2^2 \,\mathrm{d}A_t\Big] \\
&= 2\mathbb{E}\Big[\int_{(A_0,A_T]} h^\delta(A_{L_s})\|\xi\|_2^2 \,\mathrm{d}s\Big] + 2\mathbb{E}\Big[\int_{(0,T]} h^\delta(A_t)\|F_t\|_2^2 \,\mathrm{d}A_t\Big] \\
&= 2c_h^\delta h^\delta(\Phi)\mathbb{E}\Big[\|\xi\|_2^2 \int_{(A_0,A_T]} h^\delta(s)\,\mathrm{d}s\Big] + 2\mathbb{E}\Big[\int_{(0,T]} h^\delta(A_t)\|F_t\|_2^2 \,\mathrm{d}A_t\Big] \\
&\underset{c_h=1}{=} \begin{cases} 2h^\delta(\Phi)\mathbb{E}\Big[\|\xi\|_2^2 h^\delta(\lambda)(A_T - A_0)\Big] + 2\mathbb{E}\Big[\int_{(0,T]} h^\delta(A_t)\|F_t\|_2^2 \,\mathrm{d}A_t\Big], \text{ for } \lambda \in [A_0, A_T] \text{ or} \\ \frac{2}{\delta\zeta+1} h^\delta(\Phi)\mathbb{E}\Big[\|\xi\|_2^2\big(h^{\delta+\frac{1}{\zeta}}(A_T) - h^{\delta+\frac{1}{\zeta}}(A_0)\big)\Big] + 2\mathbb{E}\Big[\int_{(0,T]} h^\delta(A_t)\|F_t\|_2^2 \,\mathrm{d}A_t\Big] \end{cases}
\end{aligned} \tag{A.10}$$

$$\underset{A_0=0,(\text{A.9})}{\overset{(\text{i}),(\text{iii})}{\le}} \begin{cases} 2h^\delta(\Phi)\|\xi\|^2_{\mathbb{L}^2_{\delta+1}} + 2\Lambda^{\gamma,\delta,\Phi,c_h}\Big\|\frac{f}{\alpha}\Big\|^2_{\mathbb{H}^2_\varrho}. \\ \frac{2}{\delta\zeta+1} h^\delta(\Phi)\|\xi\|^2_{\mathbb{L}^2_{\delta+\frac{1}{\zeta}}} + 2\Lambda^{\gamma,\delta,\Phi,c_h}\Big\|\frac{f}{\alpha}\Big\|^2_{\mathbb{H}^2_\varrho}. \end{cases} \tag{A.11}$$

In the second equality we have used that the processes $\|\xi\|_2^2 \mathbb{1}_{\Omega\times[0,\infty]}(\cdot)$ and $|F(\cdot)|^2$ are uniformly integrable, hence their optional projections are well defined. Indeed, using (A.5) and recalling that $\mathbb{E}[\|F_0\|_2^2] < \infty$, we can conclude the uniform integrability of $|F(\cdot)|^2$. Then, it holds that

$$\begin{aligned}
\Pi_o^{\mathbb{G}}\left(h^\delta(A.)\|\xi\|_2^2 + h^\delta(A.)|F(\cdot)|^2\right)_t &= h^\delta(A_t)\Pi_o^{\mathbb{G}}\left(\|\xi\|_2^2 \mathbb{1}_{\Omega\times[0,\infty]}(\cdot)\right)_t + h^\delta(A_t)\Pi_o^{\mathbb{G}}\left(|F(\cdot)|^2\right)_t \\
&= h^\delta(A_t)\mathbb{E}\left[\|\xi\|_2^2 \middle| \mathcal{G}_t\right] + h^\delta(A_t)\mathbb{E}\left[\|F_t\|_2^2 \middle| \mathcal{G}_t\right] \\
&= \mathbb{E}\left[h^\delta(A_t)\|\xi\|_2^2 + h^\delta(A_t)\|F_t\|_2^2 \middle| \mathcal{G}_t\right],
\end{aligned}$$

which justifies the use of Corollary I.23.

For the estimate of $\|y\|_{\mathcal{S}^2_\delta}$ we have

$$\begin{aligned}
\|y\|_{\mathcal{S}^2_\delta} &= \mathbb{E}\Big[\sup_{0\le t\le T}\left(h^{\frac{\delta}{2}}(A_t)\|y_t\|_2\right)^2\Big] \le \mathbb{E}\Big[\sup_{0\le t\le T}\left(h^{\frac{\delta}{2}}(A_t)\mathbb{E}\big[\|\xi\|_2 + \|F_t\|_2 \big| \mathcal{G}_t\big]\right)^2\Big] \\
&= \mathbb{E}\Big[\sup_{0\le t\le T}\mathbb{E}\big[h^{\frac{\delta}{2}}(A_t)\|\xi\|_2 + h^{\frac{\delta}{2}}(A_t)\|F_t\|_2 \big| \mathcal{G}_t\big]^2\Big] \le 2\mathbb{E}\left[\sup_{0\le t\le T}\mathbb{E}\left[\sqrt{h^\delta(A_t)\|\xi\|_2^2 + h^\delta(A_t)\|F_t\|_2^2}\,\middle| \mathcal{G}_t\right]^2\right] \\
&\overset{(\text{A.5})}{\le} 2\mathbb{E}\left[\sup_{0\le t\le T}\mathbb{E}\left[\sqrt{h^\delta(A_T)\|\xi\|_2^2 + \frac{1}{\gamma\zeta-1}h^{\delta+\vartheta(\gamma)}(A_t)\int_{(t,T]} h^\gamma(A_s)\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s}\,\middle| \mathcal{G}_t\right]^2\right] \\
&\le 2\mathbb{E}\left[\sup_{0\le t\le T}\mathbb{E}\left[\sqrt{h^\delta(A_T)\|\xi\|_2^2 + \frac{1}{\gamma\zeta-1}\int_{(t,T]} [h(A_s)]^{\gamma+\delta+\vartheta(\gamma)}\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s}\,\middle| \mathcal{G}_t\right]^2\right] \\
&\le 2\mathbb{E}\left[\sup_{0\le t\le T}\mathbb{E}\left[\sqrt{h^\delta(A_T)\|\xi\|_2^2 + \frac{1}{\gamma\zeta-1}\int_{(0,T]} [h(A_s)]^{\delta+\frac{1}{\zeta}}\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s}\,\middle| \mathcal{G}_t\right]^2\right]
\end{aligned}$$

$$\begin{aligned}
&\leq 8\mathbb{E}\left[h^{\delta}(A_T)\|\xi\|_2^2 + \frac{1}{\gamma\zeta-1}\int_{(0,T]}[h(A_s)]^{\delta+\frac{1}{\zeta}}\frac{\|f_s\|_2^2}{\alpha_s^2}\,\mathrm{d}C_s\right] \\
&= 8\|\xi\|_{\mathbb{H}_\delta^2}^2 + \frac{1}{\gamma\zeta-1}\left\|\frac{f}{\alpha}\right\|_{\mathbb{H}^2_{\delta+\frac{1}{\zeta}}}^2 . \qquad \text{(A.12)}
\end{aligned}$$

where in the second and third inequalities we used the inequality $a+b \leq \sqrt{2(a^2+b^2)}$ and (A.5) respectively.

What remains is to control $\|\eta\|_{\mathcal{H}_\delta^2}$. We remind once more the reader that

$$\int_t^T \mathrm{d}\eta_s = \xi - y_t + F_t,$$

hence

$$\mathbb{E}\left[\|\xi - y_t - F_t\|_2^2 \middle| \mathcal{G}_t\right] = \mathbb{E}\left[\int_{(t,T]}\mathrm{dTr}[\langle\eta\rangle_s]\middle|\mathcal{G}_t\right]. \qquad \text{(A.13)}$$

In addition, we have

$$h^{\delta}(A_s) = (1+A_s)^{\delta\zeta} = \delta\zeta\int_{(A_0,A_s]}(1+x)^{\delta\zeta-1}\mathrm{d}x + 1 = \delta\zeta\int_{(A_0,A_s]}h^{\delta-\frac{1}{\zeta}}(x)\mathrm{d}x + 1 \qquad \text{(A.14)}$$

$$\begin{aligned}
\int_{(0,T]}h^{\delta}(A_s)\,\mathrm{dTr}[\langle\eta\rangle_s] &\overset{\text{(A.14)}}{\leq} \delta\zeta\int_{(0,T]}\int_{(A_0,A_s]}h^{\delta-\frac{1}{\zeta}}(t)\,\mathrm{d}t\;\mathrm{dTr}[\langle\eta\rangle_s] + \mathrm{Tr}[\langle\eta\rangle_T] \\
&\underset{\text{(i)}}{\overset{\text{Lemma I.34}}{\leq}} \delta\zeta\int_{(0,T]}\int_{(A_0,A_s]}h^{\delta-\frac{1}{\zeta}}(A_{L_t})\,\mathrm{d}t\;\mathrm{dTr}[\langle\eta\rangle_s] + \mathrm{Tr}[\langle\eta\rangle_T] \\
&= \delta\zeta\int_{(0,T]}\int_{(0,s]}h^{\delta-\frac{1}{\zeta}}(A_t)\,\mathrm{d}A_t\mathrm{dTr}[\langle\eta\rangle_s] + \mathrm{Tr}[\langle\eta\rangle_T] \\
&\leq \delta\zeta\int_{(0,T]}h^{\delta-\frac{1}{\zeta}}(A_t)\int_{(t,T]}\mathrm{dTr}[\langle\eta\rangle_s]\mathrm{d}A_t + \mathrm{Tr}[\langle\eta\rangle_T],
\end{aligned}$$

so that

$$\|\eta\|_{\mathcal{H}_\delta^2} \leq \delta\zeta\mathbb{E}\left[\int_{(0,T]}h^{\delta-\frac{1}{\zeta}}(A_t)\int_{(t,T]}\mathrm{dTr}[\langle\eta\rangle_s]\,\mathrm{d}A_t\right] + \mathbb{E}\left[\mathrm{Tr}[\langle\eta\rangle_T]\right]. \qquad \text{(A.15)}$$

We now need an estimate for $\mathbb{E}\left[\int_{(0,T]}\mathrm{dTr}[\langle\eta\rangle_s]\right]$, which is given by

$$\begin{aligned}
\mathbb{E}\left[\mathrm{Tr}[\langle\eta\rangle_T]\right] &= \mathbb{E}\left[|\xi - y_0 + F(0)|^2\right] \leq 3\mathbb{E}\left[\|\xi\|_2^2 + \|y_0\|_2^2 + \|F_0\|_2^2\right] \\
&\overset{\text{(A.4)}}{\leq} 9\mathbb{E}\left[\|\xi\|_2^2\right] + 9\mathbb{E}\left[\|F_0\|_2^2\right] \underset{\text{(i)}}{\overset{\text{(A.5)}}{\leq}} 9\|\xi\|_{\mathbb{L}_\delta^2} + 9\frac{1}{\delta\zeta-1}\left\|\frac{f}{\alpha}\right\|_{\mathbb{H}_\delta^2} \quad \text{for } \delta\in\Gamma_H \\
&\overset{\text{(A.5)}}{\leq} 9\|\xi\|_{\mathbb{L}_\delta^2} + \frac{9}{\delta\zeta-1}\left\|\frac{f}{\alpha}\right\|_{\mathbb{H}_\delta^2}, \qquad \text{(A.16)}
\end{aligned}$$

where we used the fact that $\mathbb{E}\left[\|y_0\|_2^2\right] \leq 2\mathbb{E}\left[\|\xi\|_2^2 + \|F_0\|_2^2\right]$.

For the first summand on the right-hand-side of (A.15), we have

$$\begin{aligned}
\mathbb{E}\left[\int_{(0,T]}h^{\delta-\frac{1}{\zeta}}(A_t)\int_{(t,T]}\mathrm{dTr}[\langle\eta\rangle_s]\,\mathrm{d}A_t\right] &\overset{\text{Lemma I.23}}{=} \mathbb{E}\left[\int_{(0,T]}h^{\delta-\frac{1}{\zeta}}(A_t)\mathbb{E}\left[\int_{(t,T]}\mathrm{dTr}[\langle\eta\rangle_s]\middle|\mathcal{G}_t\right]\mathrm{d}A_t\right] \\
&\overset{\text{(A.13)}}{=} \mathbb{E}\left[\int_{(0,T]}h^{\delta-\frac{1}{\zeta}}(A_t)\mathbb{E}\left[|\xi - y_t + F_t|^2\middle|\mathcal{G}_t\right]\mathrm{d}A_t\right] \\
&\leq 3\mathbb{E}\left[\int_{(0,T]}h^{\delta-\frac{1}{\zeta}}(A_t)\mathbb{E}\left[\|\xi\|_2^2 + \|y_t\|_2^2 + \|F_t\|_2^2\middle|\mathcal{G}_t\right]\mathrm{d}A_t\right] \\
&\overset{\text{(A.4)}}{\leq} 3\mathbb{E}\left[\int_{(0,T]}h^{\delta-\frac{1}{\zeta}}(A_t)\|\xi\|_2^2\,\mathrm{d}A_t\right] + 3\mathbb{E}\left[\int_{(0,T]}h^{\delta-\frac{1}{\zeta}}(A_t)\|F_t\|_2^2\,\mathrm{d}A_t\right]
\end{aligned}$$

$$
\begin{aligned}
&+ 6\mathbb{E}\left[\int_{(0,T]} h^{\delta-\frac{1}{\zeta}}(A_t)\mathbb{E}\left[\,\|\xi\|_2^2 + \|F_t\|_2^2 \middle| \mathcal{G}_t\right] \mathrm{d}A_t\right] \\
&\overset{\text{Lem. I.23}}{=} 9\mathbb{E}\left[\int_{(0,T]} h^{\delta-\frac{1}{\zeta}}(A_t)\|\xi\|_2^2 \,\mathrm{d}A_t\right] + 9\mathbb{E}\left[\int_{(0,T]} h^{\delta-\frac{1}{\zeta}}(A_t)\|F_t\|_2^2 \,\mathrm{d}A_t\right] \\
&\overset{(\gamma\in\Gamma_H)}{=} 9\mathbb{E}\left[\int_{(A_0,A_T]} h^{\delta-\frac{1}{\zeta}}(A_{L_t})\|\xi\|_2^2 \,\mathrm{d}t\right] + 9\mathbb{E}\left[\int_{(0,T]} h^{\delta-\frac{1}{\zeta}}(A_t)\|F_t\|_2^2 \,\mathrm{d}A_t\right] \\
&\underset{(A.9)}{\overset{\text{Cor. I.36}}{\leq}} 9h^{\delta-\frac{1}{\zeta}}(\Phi)\mathbb{E}\left[\|\xi\|_2^2 \int_{(A_0,A_T]} h^{\delta-\frac{1}{\zeta}}(t)\,\mathrm{d}t\right] + 9\Lambda^{\gamma,\delta-\frac{1}{\zeta},\Phi}\left\|\frac{f}{\alpha}\right\|_{\mathbb{H}^2_{\delta+1}} \qquad \text{for } \delta+1\leq\beta \\
&\leq \frac{9h^{\delta-\frac{1}{\zeta}}(\Phi)}{\delta-\frac{1}{\zeta}+1}\mathbb{E}\left[\|\xi\|_2^2\left[h^{\delta-\frac{1}{\zeta}+1}(A_T)-1\right]\mathrm{d}t\right] + 9\Lambda^{\gamma,\delta-\frac{1}{\zeta},\Phi}\left\|\frac{f}{\alpha}\right\|_{\mathbb{H}^2_{\delta+1}} \\
&\leq \frac{9h^{\delta-\frac{1}{\zeta}}(\Phi)}{\delta-\frac{1}{\zeta}+1}\|\xi\|_{\delta-\frac{1}{\zeta}+1} + 9\Lambda^{\gamma,\delta-\frac{1}{\zeta},\Phi}\left\|\frac{f}{\alpha}\right\|_{\mathbb{H}^2_{\delta+1}}. \qquad (A.17)
\end{aligned}
$$

In total, Inequality (A.15) can be written

$$
\begin{aligned}
\|\eta\|_{\mathcal{H}^2_\delta} &\leq \delta\zeta\mathbb{E}\left[\int_0^T h^{\delta-\frac{1}{\zeta}}(A_t)\int_{(t,T]} \mathrm{dTr}[\,\langle\eta\rangle_s]\,\mathrm{d}A_t\right] + \mathbb{E}\left[\mathrm{Tr}[\,\langle\eta\rangle_T]\right] \\
&\underset{(A.17)}{\overset{(A.16)}{\leq}} \frac{9h^{\delta-\frac{1}{\zeta}}(\Phi)}{\delta-\frac{1}{\zeta}+1}\|\xi\|_{\delta-\frac{1}{\zeta}+1} + 9\Lambda^{\gamma,\delta-\frac{1}{\zeta},\Phi}\left\|\frac{f}{\alpha}\right\|_{\mathbb{H}^2_{\delta+1}} + 9\|\xi\|_{\mathbb{L}^2_\delta} + \frac{9}{\delta\zeta-1}\left\|\frac{f}{\alpha}\right\|_{\mathbb{H}^2_\delta} \qquad (A.18)
\end{aligned}
$$

In order to be able to construct a contraction, we need to have on the right hand side of (A.11), resp. (A.12), and (A.18) only $\|\frac{f}{\alpha}\|_{\mathbb{H}^2_\delta}$. One way to overcome this problem is to assume that A is $\mathbb{P}-a.s.$ bounded, say by Ψ. Then, the Inequality (A.11) can be written as

$$
\begin{aligned}
\|\alpha y\|^2_{\mathbb{H}^2_\delta} &\leq \begin{cases} 2h^\delta(\Phi)\|\xi\|^2_{\mathbb{L}^2_{\delta+1}} + 2\Lambda^{\gamma,\delta,\Phi,c_h}\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_\varrho} \\ \frac{2}{\delta\zeta+1}h^\delta(\Phi)\|\xi\|^2_{\mathbb{L}^2_{\delta+\frac{1}{\zeta}}} + 2\Lambda^{\gamma,\delta,\Phi,c_h}\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_\varrho} \end{cases} \\
&\leq \begin{cases} 2h(\Psi)h^\delta(\Phi)\|\xi\|^2_{\mathbb{L}^2_\delta} + \frac{2[h(\Psi)]^{1+\frac{1}{\zeta}}[h(\Phi)]^{\delta+\vartheta(\gamma)}}{\delta+\vartheta(\gamma)+1}\left\|\frac{f}{\alpha}\right\|_{\mathbb{H}^2_\delta}, \\ \frac{2}{\delta\zeta+1}h^{\frac{1}{\zeta}}(\Psi)h^\delta(\Phi)\|\xi\|^2_{\mathbb{L}^2_\delta} + \frac{2[h(\Psi)]^{1+\frac{1}{\zeta}}[h(\Phi)]^{\delta+\vartheta(\gamma)}}{\delta+\vartheta(\gamma)+1}\left\|\frac{f}{\alpha}\right\|_{\mathbb{H}^2_\delta}. \end{cases} \qquad (A.19)
\end{aligned}
$$

the Inequality (A.12) can be written as

$$
\|y\|_{\mathcal{S}^2_\delta} \leq 8\|\xi\|^2_{\mathbb{H}^2_\delta} + \frac{1}{\gamma\zeta-1}\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\delta+\frac{1}{\zeta}}} \leq 8\|\xi\|^2_{\mathbb{H}^2_\delta} + \frac{[h(\Psi)]^{\frac{1}{\zeta}}}{\gamma\zeta-1}\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_\delta} \qquad (A.20)
$$

and the Inequality (A.18) can be written as

$$
\begin{aligned}
\|\eta\|_{\mathcal{H}^2_\delta} &\leq \left(9 + \frac{9[h(\Psi)]^{1-\frac{1}{\zeta}}[h(\Phi)]^{\delta-\frac{1}{\zeta}}}{\delta-\frac{1}{\zeta}+1}\right)\|\xi\|_\delta + \left(9h(\Psi)\Lambda^{\gamma,\delta-\frac{1}{\zeta},\Phi} + \frac{9}{\delta\zeta-1}\right)\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_\delta} \\
&= \left(9 + \frac{9[h(\Psi)]^{1-\frac{1}{\zeta}}[h(\Phi)]^{\delta-\frac{1}{\zeta}}}{\delta-\frac{1}{\zeta}+1}\right)\|\xi\|_\delta + \left(\frac{9h(\Psi)[h(\Phi)]^{\delta-\gamma}}{\delta-\gamma+1} + \frac{9}{\delta\zeta-1}\right)\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_\delta}. \qquad (A.21)
\end{aligned}
$$

□

A.2. Auxiliary results of Chapter III

Lemma A.2. *Let $(L^k)_{k\in\mathbb{N}}$ be a sequence of $\mathbb{R}^p-$valued processes and $(N^k)_{k\in\mathbb{N}}$ be a sequence of $\mathbb{R}^q-$valued processes such that*

(i) $\left(\mathrm{Tr}\left[[L^k]_\infty\right]\right)_{k\in\mathbb{N}}$ *is uniformly integrable and*

(ii) $\left(\mathrm{Tr}\left[[N^k]_\infty\right]\right)_{k\in\mathbb{N}}$ *is bounded in* $\mathbb{L}^1(\mathcal{G};\mathbb{R})$, i.e. $\sup_{k\in\mathbb{N}}\mathbb{E}\left[\mathrm{Tr}\left[[N^k]_\infty\right]\right] < \infty$.

Then $\big(\big\|\mathrm{Var}\big([L^k,N^k]\big)_\infty\big\|_1\big)_{k\in\mathbb{N}}$ *is uniformly integrable.*

Proof. By Corollary I.29.(iv), it is sufficient to prove that the sequence $\big(\mathrm{Var}\big([L^{k,i},N^{k,j}]\big)_\infty\big)_{k\in\mathbb{N}}$ is uniformly integrable, for every $i=1,\dots,p$ and $j=1,\dots,q$. We will use Theorem I.25 in order to prove the required property. Let i,j be arbitrary but fixed. The $\mathbb{L}^1$–boundedness of the sequence $(\mathrm{Var}([L^{k,i},N^{k,j}])_\infty)_{k\in\mathbb{N}}$ is obtained by means of the Kunita–Watanabe inequality in the form I.26 and the $\mathbb{L}^1-$boundedness of the sequences $\big([L^{k,i}]_\infty\big)_{k\in\mathbb{N}}$ and $\big([N^{k,j}]_\infty\big)_{k\in\mathbb{N}}$; the former is $\mathbb{L}^1-$bounded as uniformly integrable. By the Kunita–Watanabe inequality again, but now in the form (I.24), and the Cauchy–Schwarz inequality, we obtain for any $A\in\mathcal{G}$

$$\begin{aligned}\int_A \mathrm{Var}\big([L^{k,i},N^{k,j}]\big)_\infty\,\mathrm{d}\mathbb{P} &\overset{\text{K-W ineq.}}{\leq} \int_A [L^{k,i}]_\infty^{\frac12}[N^{k,j}]_\infty^{\frac12}\mathrm{d}\mathbb{P} \overset{\text{C-S ineq.}}{\leq} \Big(\int_A [L^{k,i}]_\infty\,\mathrm{d}\mathbb{P}\Big)^{\frac12}\Big(\int_A [N^{k,j}]_\infty\,\mathrm{d}\mathbb{P}\Big)^{\frac12}\\ &\leq \Big(\int_A [L^{k,i}]_\infty\,\mathrm{d}\mathbb{P}\Big)^{\frac12}\Big(\sup_{k\in\mathbb{N}}\int_\Omega [N^{k,j}]_\infty\,\mathrm{d}\mathbb{P}\Big)^{\frac12}\\ &\leq C\Big(\int_A [L^{k,i}]_\infty\,\mathrm{d}\mathbb{P}\Big)^{\frac12},\end{aligned}\tag{A.22}$$

where $C^2:=\sup_{k\in\mathbb{N}}\mathbb{E}\Big[\mathrm{Tr}\big[[N^k]_\infty\big]\Big]$. Note that for the third inequality we used that $[N^{k,j}]_\infty\geq 0$ for all $k\in\mathbb{N}$.

Now we use [35, Theorem 1.9] for the uniformly integrable sequence $\big([L^{k,i}]_\infty\big)_{k\in\mathbb{N}}$. For every $\varepsilon>0$, there exists $\delta>0$ such that, whenever $\mathbb{P}(A)<\delta$, it holds

$$\sup_{k\in\mathbb{N}}\int_A [L^{k,i}]_\infty\,\mathrm{d}\mathbb{P}<\frac{\varepsilon^2}{C^2}\Longleftrightarrow \sup_{k\in\mathbb{N}}\Big(\int_A [L^{k,i}]_\infty\,\mathrm{d}\mathbb{P}\Big)^{1/2}<\frac{\varepsilon}{C}.$$

This further implies $\sup_{k\in\mathbb{N}}\mathbb{E}\big[\mathbb{1}_A\mathrm{Var}\big([L^{k,i},N^{k,j}]_\infty\big)\big]\overset{\text{(A.22)}}{<}\varepsilon$, which is the required condition. □

The proof for the predictable quadratic variation is completely analogous. We state the result, however, for our convenience.

Lemma A.3. *Let* $(L^k)_{k\in\mathbb{N}}$ *be a sequence of* $\mathbb{R}^p-$*valued processes and* $(N^k)_{k\in\mathbb{N}}$ *be a sequence of* $\mathbb{R}^q-$*valued processes such that*

(i) $\langle L^k\rangle$ *and* $\langle N^k\rangle$ *are well-defined for every* $k\in\mathbb{N}$,
(ii) $\big(\mathrm{Tr}\big[\langle L^k\rangle_\infty\big]\big)_{k\in\mathbb{N}}$ *is uniformly integrable and*
(iii) $\big(\mathrm{Tr}\big[\langle N^k\rangle_\infty\big]\big)_{k\in\mathbb{N}}$ *is bounded in* $\mathbb{L}^1(\mathcal{G};\mathbb{R})$, i.e. $\sup_{k\in\mathbb{N}}\mathbb{E}\big[\mathrm{Tr}\big[[N^k]_\infty\big]\big]<\infty$.

Then $\big(\big\|\mathrm{Var}\big(\langle L^k,N^k\rangle\big)_\infty\big\|_1\big)_{k\in\mathbb{N}}$ *is uniformly integrable.*

In the following we provide some results which can be seen as simple exercises in measure theory.

Lemma A.4. *Let* D *be a countable and dense subset of* $\mathbb{R}$. *Then every open* $U\subset\mathbb{R}$ *is the countable union of intervals with endpoints in* D, *i.e. there exists a sequence of intervals* $\big((a_k,b_k)\big)_{k\in\mathbb{N}}$ *with* $a_k,b_k\in D$, *for every* $k\in\mathbb{N}$, *such that* $U=\cup_{k\in\mathbb{N}}(a_k,b_k)$.

Proof. Let U be an open subset of $\mathbb{R}$. For every $x\in U$ there exists an $r_x>0$ such that $B(x,r_x)\subset U$, where $B(y,r)=\{z\in\mathbb{R},|y-z|<r\}$. Due to the density of D we can choose $a_x,b_x\in D$ such that $a_x<b_x$ and $(a_x,b_x)\subset B(x,r_x)\subset U$. Then clearly $\cup_{x\in U}(a_x,b_x)\subset U$. On the other hand, D is countable and therefore $D\times D$ is also countable. Moreover, we can naturally associate to each ordered pair $\{\{a\},\{a,b\}\}$ the interval (a,b) if $a<b$ or the empty set if $a\geq b$, which proves that there are at most countably many intervals with endpoints in D. □

Corollary A.5. *Let* $U\subset\mathbb{R}\setminus\{0\}$ *be open. Then for every* $i=1,\dots,\ell$ *there exists a countable subfamily of* $\mathcal{I}(X^{\infty,i})$, *which has been introduced in Section III.4, name* $(U^{i,k})_{k\in\mathbb{N}}$, *such that* $U=\cup_{k\in\mathbb{N}}U^{i,k}$.

Proof. Let U be an open subset of $\mathbb{R}\setminus\{0\}$ and $i\in\{1,\dots,\ell\}$. Then, there exist open sets U^+,U^- such that $U^+\subset(0,\infty)$, $U^-\subset(-\infty,0)$ and $U=U^+\cup U^-$. On the other hand, by [41, Lemma VI.3.12] we can conclude that the set $\mathcal{I}(X^{\infty,i})$ is at most countable. Therefore, the complement of $\mathcal{I}(X^{\infty,i})$, denote it by $\mathcal{I}(X^{\infty,i})^c$, is dense in $\mathbb{R}$. Indeed, if $\mathcal{I}(X^{\infty,i})^c$ was not dense, there would exist an open subset V of $\mathbb{R}$ such that $\mathcal{I}(X^{\infty,i})^c\cap V=\emptyset$ or equivalently $V\subset\mathcal{I}(X^{\infty,i})$. However this is a contradiction, since V is uncountable and $\mathcal{I}(X^{\infty,i})$ is countable.

The above allow us to apply Lemma A.4 for $D = \mathcal{I}(X^{\infty,i})$ in order to find two countable families $(U^{+,i,k})_{k\in\mathbb{N}}$ and $(U^{-,i,k})_{k\in\mathbb{N}}$ such that

$$U^+ = \bigcup_{k\in\mathbb{N}} U^{+,i,k} \text{ and } U^- = \bigcup_{k\in\mathbb{N}} U^{-,i,k}.$$

The required countable family is $(U^{+,i,k})_{k\in\mathbb{N}} \cup (U^{-,i,k})_{k\in\mathbb{N}}$. □

Lemma A.6. *It holds $\sigma\big(\mathcal{J}(X^\infty)\big) = \mathcal{B}(\mathbb{R}^\ell)$, where $\mathcal{J}(X^\infty)$ has been introduced in Subsection III.4 and $\sigma\big(\mathcal{J}(X^\infty)\big)$ denotes the $\sigma-$algebra in $\mathbb{R}^\ell$ generated by the family $\mathcal{J}(X^\infty)$.*

Proof. The space $\mathbb{R}^\ell$ is a finite product of the second countable metric space $\mathbb{R}$. Therefore, it holds

$$\mathcal{B}(\mathbb{R}^\ell) = \underbrace{\mathcal{B}(\mathbb{R}) \otimes \cdots \otimes \mathcal{B}(\mathbb{R})}_{\ell-\text{times}}$$

and to this end, it is sufficient to prove that $\sigma(\mathcal{I}(X^{\infty,i})) = \mathcal{B}(\mathbb{R})$ for every $i = 1, \ldots, \ell$.

By Corollary A.5 we have

$$\begin{aligned}\sigma(\mathcal{I}(X^{\infty,i})) &= \sigma(\{U \subset \mathbb{R}\setminus\{0\}, U \text{ is open}\}) = \sigma(\{U \subset \mathbb{R}\setminus\{0\}, U \text{ is open}\} \cup \{\mathbb{R}\} \cup \{0\}) \\ &= \sigma(\{U \subset \mathbb{R}, U \text{ is open}\}) = \mathcal{B}(\mathbb{R}),\end{aligned}$$

which allows us to conclude. □

Lemma A.7. *Let $(\Sigma, \mathcal{S})$ be a measurable space and ϱ_1, ϱ_2 be two finite signed measures on this space. Let $\mathcal{A}$ be a family of sets with the following properties:*

(i) *$\mathcal{A}$ is a $\pi-$system,* i.e. *if $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.*
(ii) *It holds $\sigma(\mathcal{A}) = \mathcal{S}$*
(iii) *$\varrho_1(A) = \varrho_2(A)$ for every $A \in \mathcal{A}$.*

Then it holds $\varrho_1 = \varrho_2$.

Proof. Let ϱ_i^+, ϱ_i^- be the (positive) measures obtained by the *Jordan decomposition* of ϱ_i, for $i = 1, 2$. By (iii) we obtain

$$\varrho_1^+(A) - \varrho_1^-(A) = \varrho_2^+(A) - \varrho_2^-(A), \text{ for every } A \in \mathcal{A}.$$

However, the above equality can be translated into equality of measures, *i.e.*

$$\varrho_1^+(A) + \varrho_2^-(A) = \varrho_2^+(A) + \varrho_1^-(A), \text{ for every } A \in \mathcal{A}.$$

The class

$$\mathcal{C} = \{A \in \mathcal{S}, \varrho_1^+(A) + \varrho_2^-(A) = \varrho_2^+(A) + \varrho_1^-(A)\}$$

can be proven to be a $\lambda-$system.[1] Now we can conclude that the (positive) measures $\varrho_1^+ + \varrho_2^-$ and $\varrho_2^+ + \varrho_1^-$ coincide on $\mathcal{D}(\mathcal{A}) \subset \mathcal{C}$, where we have denoted by $\mathcal{D}(\mathcal{A})$ the $\lambda-$system generated by $\mathcal{A}$. Using now the property (i) and the $\pi - \lambda$ Lemma,[2] see [4, Lemma 4.11], we can conclude that the (positive) measures $\varrho_1^+ + \varrho_2^-$ and $\varrho_2^+ + \varrho_1^-$ coincide on $\sigma(\mathcal{A}) = \mathcal{S}$.

Let, now, P_1, N_1, resp. P_2, N_2, the *Hahn decomposition* of the signed measure ϱ_1, resp. ϱ_2, where with P_i we denote the set of positive mass of ϱ_i and with N_i we denote the set of negative mass of ϱ_i, for $i = 1, 2$. In view of the $\varrho_i^+ \perp \varrho_i^-$, for $i = 1, 2$ and of the following equalities

$$\begin{aligned}\Omega &= (P_1 \cap P_2) \cup (P_1 \cap N_2) \cup (N_1 \cap P_2) \cup (N_1 \cap N_2) \\ \varrho_1^+(P_1 \cap P_2) &= \varrho_1^+(P_1 \cap P_2) + \varrho_2^-(P_1 \cap P_2) = \varrho_2^+(P_1 \cap P_2) + \varrho_1^-(P_1 \cap P_2) = \varrho_2^+(P_1 \cap P_2) \\ \varrho_1^+(P_1 \cap N_2) &+ \varrho_2^-(P_1 \cap N_2) = \varrho_2^+(P_1 \cap N_2) + \varrho_1^-(P_1 \cap N_2) = 0 \\ 0 &= \varrho_1^+(N_1 \cap P_2) + \varrho_2^-(N_1 \cap P_2) = \varrho_2^+(N_1 \cap P_2) + \varrho_1^-(N_1 \cap P_2) \\ \varrho_2^-(N_1 \cap N_2) &= \varrho_1^+(N_1 \cap N_2) + \varrho_2^-(N_1 \cap N_2) = \varrho_2^+(N_1 \cap N_2) + \varrho_1^-(N_1 \cap N_2) = \varrho_1^-(N_1 \cap N_2)\end{aligned}$$

we can conclude that $P_1 \cap P_2$, $N_1 \cap N_2$ is a common Hahn decomposition of the signed measures ϱ_1 and ϱ_2. Therefore

[1] A $\lambda-$system is also referred to as *Dynkin system* or *$d-$system* or *Sierpiński class*. In [4, Foonote 4, p. 135] the interested reader can find references for this ambiguity in terminology. In [10] is used the term *$\sigma-$additive class*.

[2] Also known as *Dynkin's Lemma*.

• For every $A \in \{B \subset (P_1 \cap P_2), B \in \mathcal{S}\}$ we obtain $\varrho_1^+(A) = \varrho_2^+(A)$, hence we can conclude that $\varrho_1^+ = \varrho_2^+$.

• For every $A \in \{B \subset (N_1 \cap N_2), B \in \mathcal{S}\}$ we obtain $\varrho_1^-(A) = \varrho_2^-(A)$, hence we can conclude that $\varrho_1^- = \varrho_2^-$.

□

A.3. Auxiliary results of Chapter IV

A.3.1. Moore–Osgood Theorem. The Moore–Osgood Theorem, whose well-known form is Theorem A.8, provides sufficient conditions for the existence of the limit of a doubly-indexed sequence and it can be seen as a special case of Rudin [63, Theorem 7.11]. Here we provide a second form, since the second time[3] we need to apply the aforementioned theorem we need to relax the existence of the pointwise limits; see Condition (ii) of Theorem A.8. For more comments on the existence of the iterated limits and of the joint limit of a doubly-indexed sequence the interested reader should consult Hobson [37, Chapter VI, Sections 336-338]. Specifically for the validity of the Theorem A.9 see [37, Chapter VI, Section 337, p. 466] and the reference therein.

Theorem A.8. *Let (Γ, d_Γ) be a metric space and $(\gamma_{k,p})_{k,p\in\overline{\mathbb{N}}}$ be a sequence. If*

(i) $\lim\limits_{p\to\infty} \sup\limits_{k\in\mathbb{N}} d_\Gamma(\gamma_{k,p}, \gamma_{k,\infty}) = 0$ *and*

(ii) $\lim\limits_{k\to\infty} d_\Gamma(\gamma_{k,p}, \gamma_{\infty,p}) = 0$ *for all* $p \in \mathbb{N}$,

then the joint limit $\lim\limits_{k,p\to\infty} \gamma_{n,k}$ *exists. In particular holds* $\lim\limits_{k,p\to\infty} \gamma_{k,p} = \lim\limits_{p\to\infty} \gamma_{\infty,p} = \lim\limits_{k\to\infty} \gamma_{k,\infty}$.

Theorem A.9. *Let $(\mathbb{R}, |\cdot|)$ and $(\vartheta_{k,p})_{k,p\in\overline{\mathbb{N}}}$ be a sequence. If*

(i) $\lim\limits_{p\to\infty} \sup\limits_{k\in\mathbb{N}} |\vartheta_{k,p} - \vartheta_{k,\infty}| = 0$ *and*

(ii) $\lim\limits_{p\to\infty} \big(\limsup\limits_{k\to\infty} \vartheta_{k,p} - \liminf\limits_{k\to\infty} \vartheta_{k,p}\big) = 0$,

then the joint limit $\lim\limits_{k,p\to\infty} \vartheta_{k,p}$ *exists.*

A.3.2. Weak convergence of measures on the positive real line.

Definition A.10. Let $(\mu_k)_{k\in\overline{\mathbb{N}}}$ be a countable family of measures on $\big(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\big)$. We will say that the sequence $(\mu_k)_{k\in\mathbb{N}}$ converges weakly to the measure μ_∞ if for every $f : \big(\mathbb{R}_+, \|\cdot\|_\infty\big) \to \big(\mathbb{R}, |\cdot|\big)$ continuous and bounded holds

$$\lim_{k\to\infty} \left| \int_{[0,\infty)} f(x)\, \mu_k(\mathrm{d}x) - \int_{[0,\infty)} f(x)\, \mu_\infty(\mathrm{d}x) \right| = 0.$$

We denote the weak convergence of $(\mu_k)_{k\in\mathbb{N}}$ to μ_∞ by $\mu_k \xrightarrow{\mathrm{w}} \mu_\infty$.

In this section we will use the set

$$\mathcal{W}_{0,\infty}^+ := \big\{C \in \mathbb{D}(\mathbb{R}) \;\big|\; C \text{ is increasing, with } C_0 = 0 \text{ and } \lim_{t\to\infty} C_t \in \mathbb{R}\big\}.$$

Remark A.11. We can extend every element of $\mathcal{W}_{0,\infty}^+$, name C its arbitrary element, on $[0,\infty]$ such that it is left-continuous at the symbol ∞ by defining $C_\infty := \lim_{t\to\infty} C_t$.

We provide in the following proposition some convenient equivalence for the weak convergence of finite measures. The statement is tailor-made to our later needs, but the interested reader may consult Bogachev [10, Section 8.1 - Section 8.3]. Then we provide in Theorem A.14 a new, to the best knowledge of the author, characterisation of weak convergence of finite measures to an atomless measure defined on the positive real line, which uses relatively compact sets of the Skorokhod space instead of relatively compact sets of the space of continuous functions defined on $\mathbb{R}_+$ endowed with the $\|\cdot\|_\infty$−topology.

Proposition A.12. *Let $(\mu_k)_{k\in\overline{\mathbb{N}}}$ be sequence of finite measures on $\big(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\big)$, and denote the associated distribution functions by C^k, for $k \in \overline{\mathbb{N}}$, for which the finiteness of the measure translates to $C_\infty^k := \lim_{t\to\infty} C_t^k \in \mathbb{R}_+$ for every $k \in \overline{\mathbb{N}}$. We assume the following*

(i) *The sequence $(\mu_k)_{k\in\mathbb{N}}$ is bounded,* i.e. $\sup_{k\in\mathbb{N}} \mu_k(\mathbb{R}_+) < \infty$. *Equivalently,* $\sup_{k\in\mathbb{N}} C_\infty^k < \infty$.

[3] The first one is in the proof of Theorem IV.10, while the second is in the proof of Proposition A.16

(ii) *The sequence* $(\mu_k)_{k\in\mathbb{N}}$ *is tight. In other words, for every* $\varepsilon > 0$ *there exists a compact set* I *such that* $\sup_{k\in\mathbb{N}} \mu_k(\mathbb{R}_+ \setminus I) < \varepsilon$.[4] *Equivalently,* $\sup_{k\in\mathbb{N}} |C^k_\infty - C^k_N| < \varepsilon$ *for some* $N \in \mathbb{R}_+$.

(iii) $\mu_k(\{0\}) = 0$ *for every* $k \in \overline{\mathbb{N}}$. *Equivalently,* $C^k \in \mathcal{W}^+_{0,\infty}$ *for every* $k \in \overline{\mathbb{N}}$.

(iv) μ_∞ *is* atomless, i.e. $\mu_\infty(\{t\}) = 0$ *for every* $t \in \mathbb{R}_+$. *Equivalently,* C^∞ *is continuous.*

The following are equivalent:

(I) $\mu_k \xrightarrow{\mathrm{w}} \mu$.

(II) $C^k_t \xrightarrow[k\to\infty]{} C^\infty_t$, *for every* $t \in \mathbb{R}_+$.

(III) $C^k \xrightarrow{\mathrm{J}_1(\mathbb{R})} C^\infty$.

(IV) $C^k \xrightarrow{\mathrm{lu}} C^\infty$.

(V) $\sup_{I\in\mathcal{I}_N} |\mu_k(I) - \mu_\infty(I)| \xrightarrow[k\to\infty]{} 0$ *for every* $N \in \mathbb{N}$, *where* $\mathcal{I}_N := \big\{ I \subset [0,N] \ \mid \ I \text{ is interval} \big\}$.

Proof. The equivalence between (I) and (II) is a classical result, *e.g.* see Bogachev [10, Proposition 8.1.8]. The equivalences between (II),(III) and (IV) are provided by Jacod and Shiryaev [41, Theorem VI.2.15.c.(i), Proposition VI.1.17 b)]. We obtain the equivalence between (IV) and (V) in view of the validity of the following inequalities for every $N \in \mathbb{N}$:

$$\begin{aligned}
\sup_{t\in[0,N]} |C^k_t - C^\infty_t| &\le \sup_{I\in\mathcal{I}_N} |\mu_k(I) - \mu_\infty(I)| \quad (\text{because } C^k_0 = C^\infty_0 = 0)\\
&\le \sup_{s,t\in[0,N]} |C^k_{t-} - C^k_s - C^\infty_t + C^\infty_s| + \sup_{s,t\in[0,N]} |C^k_t - C^k_s - C^\infty_t + C^\infty_s|\\
&\quad + \sup_{s,t\in[0,N]} |C^k_{t-} - C^k_{s-} - C^\infty_{t-} + C^\infty_{s-}|\\
&\le 3 \sup_{t\in[0,N]} |C^k_t - C^\infty_t| + 2 \sup_{t\in[0,N]} |C^k_{t-} - C^\infty_{t-}| + \sup_{t\in[0,N]} |C^k_{t-} - C^\infty_t|\\
&\le 5 \sup_{t\in[0,N]} |C^k_t - C^\infty_t| + \sup_{t\in[0,N]} |C^k_{t-} - C^k_t| + \sup_{t\in[0,N]} |C^k_t - C^\infty_t| \quad (C^k \text{is càdlàg for every } k \in \overline{\mathbb{N}})\\
&\le 6 \sup_{t\in[0,N]} |C^k_t - C^\infty_t| + \sup_{t\in[0,N]} |C^k_{t-} - C^k_t|.
\end{aligned}$$

Then Jacod and Shiryaev [41, Lemma VI.2.5] implies that

$$\begin{aligned}
\limsup_{k\to\infty} \sup_{I\in\mathcal{I}_N} |\mu_k(I) - \mu_\infty(I)| &\le 6 \lim_{k\to\infty} \sup_{t\in[0,N]} |C^k_t - C^\infty_t| + \limsup_{k\to\infty} \sup_{t\in[0,N]} |C^k_{t-} - C^k_t|\\
&\le \sup_{t\in[0,N]} |C^\infty_{t-} - C^\infty_t| = 0.
\end{aligned}$$

□

The following lemma provides a useful criterion to conclude the equivalence between $\delta_{\|\cdot\|_\infty}$-convergence and $\delta_{\mathrm{J}_1}(\mathbb{R}_+)$-convergence in $\mathbb{D}\big([0,\infty]\big)$.

Lemma A.13. *Let* $(C^k)_{k\in\overline{\mathbb{N}}} \subset \mathcal{W}^+_{0,\infty}$ *which satisfies the following properties:*

(i) $C^k \xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R})} C^\infty$.

(ii) $C^k_\infty \xrightarrow[k\to\infty]{|\cdot|} C^\infty_\infty$.

(iii) *The limit function* C^∞ *is continuous.*

Then it holds $C^k \xrightarrow[k\to\infty]{\|\cdot\|_\infty} C$.

Proof. Let us fix an arbitrary $\varepsilon > 0$. We start by exploiting the fact that $C^\infty \in \mathcal{W}^+_{0,\infty}$. Then, there exists $K > 0$ such that

$$\sup_{t\ge K} (C^\infty_\infty - C^\infty_t) < \frac{\varepsilon}{4}. \tag{A.23}$$

By (ii), there exists $k_1 = k_1(\varepsilon) \in \mathbb{N}$ such that

$$\sup_{k\ge k_1} |C^k_\infty - C^\infty_\infty| < \frac{\varepsilon}{4}. \tag{A.24}$$

[4] Without loss of generality we can assume that I is of the form $[0,N]$, for some $N > 0$.

By (iii) there exists $\bar{s} > K$ such that $C_{\bar{s}} = \frac{1}{2}(C_K + C_\infty^\infty)$. By (i), (iii) and Proposition I.113, there exists $k_2 = k_2(\varepsilon, \bar{s})$ such that

$$\sup_{k \geq k_2} \sup_{0 \leq t \leq \bar{s}} |C_t^k - C_t^\infty| < \frac{1}{4}(C_\infty^\infty - C_K) < \frac{\varepsilon}{16}, \tag{A.25}$$

where the last inequality is valid in view of (A.23), since $\bar{s} > K$. Using the fact that the functions are increasing we derive

$$\begin{aligned} \sup_{k \geq k_1 \vee k_2} \Big(\sup_{\bar{s} \leq t < \infty} C_\infty^k - C_t^k \Big) &\leq \sup_{k \geq k_1 \vee k_2} \big(C_\infty^k - C_{\bar{s}}^k \big) \\ &\leq \sup_{k \geq k_1 \vee k_2} \big| C_\infty^k - C_\infty^\infty \big| + \sup_{k \geq k_1 \vee k_2} \big| C_\infty^\infty - C_{\bar{s}}^\infty \big| + \sup_{k \geq k_1 \vee k_2} \big| C_{\bar{s}}^\infty - C_{\bar{s}}^k \big| \\ &\underset{(A.25)}{\overset{(A.24)}{<}} \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{16} < \frac{\varepsilon}{2}. \end{aligned} \tag{A.26}$$

Therefore, by combining the above, we obtain for every $k \geq k_1 \vee k_2$

$$\sup_{t \in \mathbb{R}_+} |C_t^k - C_t^\infty| \leq \sup_{0 \leq t \leq \bar{s}} |C_t^k - C_t^\infty| + \sup_{\bar{s} \leq t \leq \infty} |C_t^k - C_t^\infty| \underset{(A.26)}{\overset{(A.25)}{<}} < \varepsilon.$$

□

Recall that the Kolmogorov metric $\delta_{\mathrm{Kolm}}(C^1, C^2) := \|C^1 - C^2\|_\infty$, for $C^1, C^2 \in \mathcal{W}_{0,\infty}^+$, dominates the Lévy metric, which metrises the weak topology on the space of finite measures. Hence the previous lemma can be seen as a criterion for the weak convergence $\mu_{C^k} \xrightarrow{\mathrm{w}} \mu_{C^\infty}$, where μ_{C^k}, for $k \in \overline{\mathbb{N}}$ is the finite measures associated to the increasing function $C^k \in \mathcal{W}_{0,\infty}^+$.

The next step is to provide a characterisation of the aforementioned weak convergence of measures using relatively compact sets of $\mathbb{D}$. Since, however, a relatively compact set of $\mathbb{D}$ is, in general, only locally uniformly bounded, see Theorem I.111, we have to restrict ourselves to those relatively compact subsets which are uniformly bounded, so that the integrals make sense. The following theorem can be regarded as an extension of Parthasarathy [56, Theorem 6.8] in the special case that the limiting measure is an atomless measure on $\mathbb{R}_+$. For $\alpha \in \mathbb{D}(\mathbb{R}^p)$ the modulus w'_N has been defined in Definition I.108.

Theorem A.14. *Let the measurable space $\big(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\big)$, $(\mu_k)_{k \in \overline{\mathbb{N}}}$ be a sequence of finite measures such that μ_∞ is an atomless finite measure and $\mathcal{A} \subset \mathbb{D}(\mathbb{R})$. Assume that the following conditions are true*

(i) $\mu_k \xrightarrow{\mathrm{w}} \mu_\infty$.
(ii) *$\mathcal{A}$ is uniformly bounded,* i.e. $\sup_{\alpha \in \mathcal{A}} \|\alpha\|_\infty < \infty$ *and*
(iii) *$\mathcal{A}$ is relatively compact. In other words,* $\lim_{\zeta \downarrow 0} \sup_{\alpha \in \mathcal{A}} w'_N(\alpha, \zeta) = 0$ *for every* $N \in \mathbb{N}$.

Then,

$$\lim_{k \to \infty} \sup_{\alpha \in \mathcal{A}} \left| \int_{[0,\infty)} \alpha(x)\, \mu_k(\mathrm{d}x) - \int_{[0,\infty)} \alpha(x)\, \mu_\infty(\mathrm{d}x) \right| = 0. \tag{A.27}$$

Proof. We will decompose initially the quantity whose convergence we intent to prove as follows

$$\begin{aligned} &\sup_{\alpha \in \mathcal{A}} \left| \int_{[0,\infty)} \alpha(x)\, \mu_k(\mathrm{d}x) - \int_{[0,\infty)} \alpha(x)\, \mu_\infty(\mathrm{d}x) \right| \\ &\quad \leq \sup_{\alpha \in \mathcal{A}} \left| \int_{[0,N]} \alpha(x)\, \mu_k(\mathrm{d}x) - \int_{[0,N]} \alpha(x)\, \mu_\infty(\mathrm{d}x) \right| \\ &\qquad\qquad + \sup_{\alpha \in \mathcal{A}} \left| \int_{[N,\infty)} \alpha(x)\, \mu_k(\mathrm{d}x) \right| + \sup_{\alpha \in \mathcal{A}} \left| \int_{[N,\infty)} \alpha(x)\, \mu_\infty(\mathrm{d}x) \right| \end{aligned} \tag{A.29}$$

for some $N > 0$ to be determined and then the first summand of the right-hand side will be decomposed as follows

$$\sup_{\alpha \in \mathcal{A}} \left| \int_{[0,N]} \alpha(x)\, \mu_k(\mathrm{d}x) - \int_{[0,N]} \alpha(x)\, \mu_\infty(\mathrm{d}x) \right|$$

$$\begin{aligned}
&\leq \sup_{\alpha\in\mathcal{A}}\left|\int_{[0,N]}\alpha(x)\,\mu_k(\mathrm{d}x)-\int_{[0,N]}\alpha(x)\,\widetilde{\mu}_k^{\alpha}(\mathrm{d}x)\right| \\
&\qquad + \sup_{\alpha\in\mathcal{A}}\left|\int_{[0,N]}\alpha(x)\,\widetilde{\mu}_k^{\alpha}(\mathrm{d}x)-\int_{[0,N]}\alpha(x)\,\widetilde{\mu}_\infty^{\alpha}(\mathrm{d}x)\right| \\
&\qquad\qquad + \sup_{\alpha\in\mathcal{A}}\left|\int_{[0,N]}\alpha(x)\,\widetilde{\mu}_\infty^{\alpha}(\mathrm{d}x)-\int_{[0,N]}\alpha(x)\,\mu_\infty(\mathrm{d}x)\right|
\end{aligned} \tag{A.31}$$

where the measures $\widetilde{\mu}_k^{\alpha}$, for $k\in\overline{\mathbb{N}}$ and $\alpha\in\mathcal{A}$, will be constructed given the measure μ_k and the function α. For the second and third summand of (A.29) we will prove that they become arbitrarily small for large $N>0$. Then, we will conclude once we obtain the convergence to 0 as $k\to\infty$ of the summands of (A.31). To this end, let us fix an $\varepsilon>0$.

□ Condition (i) implies that $\sup_{k\in\overline{\mathbb{N}}}\mu_k(\mathbb{R}_+)<\infty$as well as the tightness of the sequence by the convergence $\mu_k(\mathbb{R}_+)\xrightarrow[k\to\infty]{}\mu_\infty(\mathbb{R}_+)$. Hence, for $M:=\sup_{\alpha\in\mathcal{A}}\|\alpha\|_\infty<\infty$, which is Condition (ii), there exists $N=N(\varepsilon,M)>0$ such that

$$\sup_{k\in\overline{\mathbb{N}}}\mu_k\big([N,\infty)\big)<\frac{\varepsilon}{2M},\ \text{which implies}\ \sup_{k\in\overline{\mathbb{N}},\alpha\in\mathcal{A}}\left|\int_{[N,\infty)}\alpha(x)\,\mu_k(\mathrm{d}x)\right|<\frac{\varepsilon}{2}.$$

In other words, we have proven that the second and third summand of the right-hand side of (A.29) become arbitrarily small for large N. In the following N is assumed fixed, but large enough so that the above hold.

□ We proceed now to construct for every $k\in\overline{\mathbb{N}}$ and $\alpha\in\mathcal{A}$ a suitable measure $\widetilde{\mu}_k^{\alpha}$ on $\big(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+)\big)$ such that we can obtain the convergence of the summands in (A.31). Define $K_N:=\sup_{k\in\overline{\mathbb{N}}}\,\mu_k\left([0,N]\right)<\infty$. By (iii) there exists $\zeta=\zeta(\varepsilon,K_N)>0$ such that for every $\zeta'<\zeta$ it holds $\sup_{\alpha\in\mathcal{A}}w'_N(\alpha,\zeta')<\frac{\varepsilon}{4K_N}$. By the definition of w' (see Definition I.108) for every $\alpha\in\mathcal{A}$ and every $\zeta'<\zeta$, there exists a ζ'-sparse set $P_\alpha^{N,\zeta'}$, *i.e.*

$$\begin{aligned}
P_\alpha^{N,\zeta'}:=\Big\{0=t(0;P_\alpha^{N,\zeta'})<t(1;P_\alpha^{N,\zeta'})<\ldots<t(\kappa^\alpha;P_\alpha^{N,\zeta'})=N:& \\
\min_{i=1,\ldots,\kappa^\alpha-1}\big(t(i;P_\alpha^{N,\zeta'})-t(i-1;P_\alpha^{N,\zeta'})\big)>\zeta'\Big\},&
\end{aligned}$$

such that

$$\begin{aligned}
\max_{i=1,\ldots,\kappa^\alpha}\sup\Big\{|\alpha(s)-\alpha(u)|:s,u\in\big[t(i-1;P_\alpha^{N,\zeta'}),t(i;P_\alpha^{N,\zeta'})\big)\Big\}&<w'_N(\alpha,\zeta')+\frac{\varepsilon}{4K_N} \\
&<\sup_{\alpha\in\mathcal{A}}w'_N(\alpha,\zeta')+\frac{\varepsilon}{4K_N}<\frac{\varepsilon}{2K_N}.
\end{aligned} \tag{A.32}$$

To sum up, for every $(\alpha,\zeta')\in\mathcal{A}\times(0,\zeta)$ we can find a partition of $[0,N]$, $P_\alpha^{N,\zeta'}$, satisfying (A.32). For the following, given a finite measure on $\big(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+)\big)$, name λ, and a sparse set $P_\alpha^{N,\zeta'}$

- ▹ we will denote $A(i;P_\alpha^{N,\zeta'}):=[t(i-1;P_\alpha^{N,\zeta'}),t(i;P_\alpha^{N,\zeta'}))$ for $i=1,\ldots,\kappa^\alpha$ and
- ▹ we define the finite measure $\widetilde{\lambda}^\alpha:\big(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+)\big)\to[0,\infty)$ by

$$\widetilde{\lambda}^\alpha(\cdot):=\sum_{i=1}^{\kappa^\alpha}\lambda\big(A(i;P_\alpha^{N,\zeta'})\big)\delta_{t(i-1;P_\alpha^{N,\zeta'})}(\cdot),$$

where δ_x is the Dirac measure sitting at the point x. With the above notation we have

$$\begin{aligned}
\left|\int_{[0,N]}\alpha(x)\,\lambda(\mathrm{d}x)-\int_{[0,N]}\alpha(x)\,\widetilde{\lambda}^\alpha(\mathrm{d}x)\right| &= \left|\sum_{i=1}^{\kappa^\alpha}\int_{A(i;P_\alpha^{N,\zeta'})}\alpha(x)\,\lambda(\mathrm{d}x)-\sum_{i=1}^{\kappa^\alpha}\int_{A(i;P_\alpha^{N,\zeta'})}\alpha(x)\,\widetilde{\lambda}^\alpha(\mathrm{d}x)\right| \\
&\leq \sum_{i=1}^{\kappa^\alpha}\left|\int_{A(i;P_\alpha^{N,\zeta'})}\alpha(x)\,\lambda(\mathrm{d}x)-\int_{A(i;P_\alpha^{N,\zeta'})}\alpha(x)\,\widetilde{\lambda}^\alpha(\mathrm{d}x)\right| \\
&= \sum_{i=1}^{\kappa^\alpha}\left|\int_{A(i;P_\alpha^{N,\zeta'})}\alpha(x)\,\lambda(\mathrm{d}x)-\alpha\big(t(i-1;P_\alpha^{N,\zeta'})\lambda\left([t(i-1;P_\alpha^{N,\zeta'}),t(i;P_\alpha^{N,\zeta'}))\right)\right| \\
&= \sum_{i=1}^{\kappa^\alpha}\left|\int_{A(i;P_\alpha^{N,\zeta'})}\alpha(x)-\alpha\big(t(i-1;P_\alpha^{N,\zeta'})\lambda(\mathrm{d}x)\right| \leq \sum_{i=1}^{\kappa^\alpha}\int_{A(i;P_\alpha^{N,\zeta'})}\left|\alpha(x)-\alpha\big(t(i-1;P_\alpha^{N,\zeta'})\right|\lambda(\mathrm{d}x)
\end{aligned}$$

$$\overset{(A.32)}{\leq} \frac{\varepsilon}{2K_N} \sum_{i=1}^{\kappa^\alpha} \int_{A(i;P_\alpha^{N,\zeta'})} \lambda(\mathrm{d}x) = \frac{\varepsilon}{2K_N} \lambda\left([0,N]\right). \tag{A.33}$$

For every $k \in \overline{\mathbb{N}}$ and $\alpha \in \mathcal{A}$ we define $\widetilde{\mu}_k^\alpha := \widetilde{(\mu_k)}^\alpha$. Now, using the approximation (A.33) for μ_k, for $k \in \overline{\mathbb{N}}$, we obtain

$$\sup_{k\in\overline{\mathbb{N}},\alpha\in\mathcal{A}} \left| \int_{[0,N]} \alpha(x)\, \mu_k(\mathrm{d}x) - \int_{[0,N]} \alpha(x)\, \widetilde{\mu}_k^\alpha(\mathrm{d}x) \right| < \frac{\varepsilon}{2}. \tag{A.34}$$

To sum up, we have constructed the measures $\widetilde{\mu}_k^\alpha$ such that the first and third summand of (A.31) become arbitrarily small uniformly on $k \in \overline{\mathbb{N}}$ and on $\alpha \in \mathcal{A}$.

□ We can conclude (A.27) once we obtain the validity of

$$\lim_{k\to\infty} \sup_{\alpha\in\mathcal{A}} \left| \int_{[0,N]} \alpha(x)\, \widetilde{\mu}_k^\alpha(\mathrm{d}x) - \int_{[0,N]} \alpha(x)\, \widetilde{\mu}_\infty^\alpha(\mathrm{d}x) \right| = 0.$$

Indeed, for fixed $\zeta' \in (0,\zeta)$,

$$\begin{aligned}
&\left| \int_{[0,N]} \alpha(x)\, \widetilde{\mu}_k^\alpha(\mathrm{d}x) - \int_{[0,N]} \alpha(x)\, \widetilde{\mu}_\infty^\alpha(\mathrm{d}x) \right| = \left| \sum_{i=1}^{\kappa^\alpha} \int_{A(i;P_\alpha^{N,\zeta'})} \alpha(x)\, \widetilde{\mu}_k^\alpha(\mathrm{d}x) - \sum_{i=1}^{\kappa^\alpha} \int_{A(i;P_\alpha^{N,\zeta'})} \alpha(x)\, \widetilde{\mu}_\infty^\alpha(\mathrm{d}x) \right| \\
&= \left| \sum_{i=1}^{\kappa^\alpha} \Big(\int_{A(i;P_\alpha^{N,\zeta'})} \alpha(x)\, \widetilde{\mu}_k^\alpha(\mathrm{d}x) - \int_{A(i;P_\alpha^{N,\zeta'})} \alpha(x)\, \widetilde{\mu}_\infty^\alpha(\mathrm{d}x) \Big) \right| \\
&= \left| \sum_{i=1}^{\kappa^\alpha} \alpha\big(t(i;P_\alpha^{N,\zeta'})\big) \Big[\mu_k\big(A(i;P_\alpha^{N,\zeta'})\big) - \mu\big(A(i;P_\alpha^{N,\zeta'})\big) \Big] \right| \\
&\overset{(ii)}{\leq} \|\alpha\|_\infty \sum_{i=1}^{\kappa^\alpha} \Big| \mu_k\big(A(i;P_\alpha^{N,\zeta'})\big) - \mu_\infty\big(A(i;P_\alpha^{N,\zeta'})\big) \Big| \\
&\leq \sup_{\alpha\in\mathcal{A}} \|\alpha\|_\infty \cdot \kappa^\alpha \cdot \sup_{i=1,\dots,\kappa^\alpha} \Big| \mu_k\big(A(i;P_\alpha^{N,\zeta'})\big) - \mu_\infty\big(A(i;P_\alpha^{N,\zeta'})\big) \Big| \\
&\leq \sup_{\alpha\in\mathcal{A}} \|\alpha\|_\infty \cdot \kappa^\alpha \cdot \sup_{I\in\mathcal{I}_N} \big| \mu_k(I) - \mu_\infty(I) \big|.
\end{aligned} \tag{A.35}$$

Let us now denote $m(P_\alpha^{N,\zeta'}) := \min\big\{ t(i;P_\alpha^{N,\zeta'}) - t(i-1;P_\alpha^{N,\zeta'}), i = 1,\dots,\kappa^\alpha \big\} > \zeta'$ by the definition of $P_\alpha^{N,\zeta'}$. Therefore,

$$\sup_{\alpha\in\mathcal{A}} \kappa^\alpha \leq \sup_{\alpha\in\mathcal{A}} \left\lceil \frac{N}{m(P_\alpha^{N,\zeta'})} \right\rceil^{5} \leq \left\lceil \frac{N}{\zeta'} \right\rceil < \infty.$$

Hence, for every fixed sparse set $P_\alpha^{N,\zeta'}$ which satisfies (A.32) we obtain

$$\sup_{\alpha\in\mathcal{A}} \left| \int_{[0,N]} \alpha(x)\, \widetilde{\mu}_k^\alpha(\mathrm{d}x) - \int_{[0,N]} \alpha(x)\, \widetilde{\mu}_\infty^\alpha(\mathrm{d}x) \right| \leq \sup_{\alpha\in\mathcal{A}} \|\alpha\|_\infty \cdot \sup_{\alpha\in\mathcal{A}} \kappa^\alpha \cdot \sup_{I\in\mathcal{I}_N} \left| \mu_k(I) - \mu(I) \right| \xrightarrow[k\to\infty]{(i)} 0,$$

where we have used Proposition A.12.(V) which is equivalent to (i). □

Remark A.15. We have presented the previous theorem for $\mathcal{A} \subset \mathbb{D}(\mathbb{R})$. However, the result can be readily adapted for $\mathcal{A} \subset \mathbb{D}(\mathbb{R}^p)$.

Proposition A.16. *Let the measurable space $\big(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\big)$ and $(\mu_k)_{k\in\overline{\mathbb{N}}}$ be a sequence of finite measures such that $\mu_k(\{0\}) = 0$ for every $k \in \overline{\mathbb{N}}$ and μ_∞ is atomless. Additionally, let $\mathcal{A} := (\alpha^k)_{k\in\overline{\mathbb{N}}} \subset \mathbb{D}(\mathbb{R}^p)$. Assume the following to be true:*

(i) $\mu_k \xrightarrow{\mathrm{w}} \mu_\infty$.

(ii) *$\mathcal{A}$ is uniformly bounded,* i.e. $\sup_{\alpha\in\mathcal{A}} \|\alpha\|_\infty < \infty$.

(iii) *$\mathcal{A}$ is $\delta_{\mathrm{J}_1(\mathbb{R}^p)}-$convergent. In other words, $\alpha^k \xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R}^p)} \alpha^\infty$.*

Then,

$$\lim_{k\to\infty} \left\| \int_{[0,\infty)} \alpha^k(x)\, \mu_k(\mathrm{d}x) - \int_{[0,\infty)} \alpha^\infty(x)\, \mu_\infty(\mathrm{d}x) \right\|_2 = 0.$$

[5] We denote by $\lceil x \rceil$ the least integer greater than or equal to x.

Proof. Let us denote

$$\gamma^{k,m} := \left\| \int_{[0,\infty)} \alpha^k(x)\, \mu_m(\mathrm{d}x) - \int_{[0,\infty)} \alpha^k(x)\, \mu_\infty(\mathrm{d}x) \right\|_2 \text{ for } k \in \overline{\mathbb{N}} \text{ and } m \in \mathbb{N}.$$

We are going to apply Theorem A.9 in order to obtain the required result. By Theorem A.14 we have that

$$\lim_{m\to\infty} \sup_{k\in\overline{\mathbb{N}}} \left|\gamma^{k,m}\right| = 0.$$

In other words, Condition (i) of Theorem A.9 is satisfied. Let us prove, now, that Condition (ii) of the aforementioned theorem is also satisfied, *i.e.* we need to prove that

$$\lim_{m\to\infty} \big(\limsup_{k\to\infty} \gamma^{k,m} - \liminf_{k\to\infty} \gamma^{k,m}\big) = 0.$$

However, it is sufficient to prove that $\lim_{m\to\infty} \limsup_{k\to\infty} \gamma^{k,m} = 0$, since the elements of the doubly-indexed sequence are positive. To this end, we have

$$\begin{aligned}
&\limsup_{k\to\infty} \gamma^{k,m} \\
&\quad \le \limsup_{k\to\infty} \left\| \int_{[0,\infty)} \alpha^k(x)\, \mu_m(\mathrm{d}x) - \int_{[0,\infty)} \alpha^\infty(x)\, \mu_m(\mathrm{d}x) \right\|_2 \\
&\qquad\qquad + \limsup_{k\to\infty} \left\| \int_{[0,\infty)} \alpha^\infty(x)\, \mu_m(\mathrm{d}x) - \int_{[0,\infty)} \alpha^\infty(x)\, \mu_\infty(\mathrm{d}x) \right\|_2 \\
&\qquad\qquad\qquad + \limsup_{k\to\infty} \left\| \int_{[0,\infty)} \alpha^\infty(x)\, \mu_\infty(\mathrm{d}x) - \int_{[0,\infty)} \alpha^k(x)\, \mu_\infty(\mathrm{d}x) \right\|_2 .
\end{aligned}$$

We are going to prove now that the two last summands of the right-hand side of the above inequality are equal to zero. We start with the second and we realise that we have only to use that $\mu_k \xrightarrow{\mathrm{w}} \mu_\infty$ and Theorem A.14 for the special case of a singleton, in order to verify our claim. The third summand is also equal to zero as an outcome of the bounded convergence theorem. Indeed, we have that the the sequence $\mathcal{A}$ is uniformly bounded and by Proposition I.121 we have the pointwise convergence $\alpha^k(x) \longrightarrow \alpha^\infty(x)$ for $k \to \infty$, at every point x which is point of continuity of α^∞. Since the set $\{x \in \mathbb{R}_+, \Delta\alpha^\infty(x) \neq 0\}$ is at most countable, we can conclude that it is a $\mu_\infty-$null set. Therefore, since every element of $\mathcal{A}$ is Borel measurable, we can conclude.

Finally, we deal with the first summand. Let us provide initially some helpful results. Using again Proposition I.121 we have that

$$\limsup_{k\to\infty} \left\|\alpha^k(x) - \alpha^\infty(x)\right\|_2 \le \left\|\Delta\alpha^\infty(x)\right\|_2 \text{ for every } x \in \mathbb{R}_+, \tag{A.36}$$

because the only possible accumulation points of the sequence $\big(\alpha^{k,i}(x)\big)_{k\in\mathbb{N}}$ are $\alpha^{\infty,i}(x)$ and $\alpha^{\infty,i}(x-)$, for every $i = 1, \dots, p$ and every $x \in \mathbb{R}_+$; recall Remark I.119. Now observe that the function $\mathbb{R}_+ \ni x \longmapsto \left\|\Delta\alpha^\infty(x)\right\|_2 \in \mathbb{R}_+$ is Borel measurable and $\mu_\infty-$almost everywhere equal to the zero function. This observation allows us to apply Mazzone [51, Theorem 1] in order to conclude that

$$\int_{[0,\infty)} \left\|\Delta\alpha^\infty(x)\right\|_2 \mu_m(\mathrm{d}x) \xrightarrow[m\to\infty]{} \int_{[0,\infty)} \left\|\Delta\alpha^\infty(x)\right\|_2 \mu_\infty(\mathrm{d}x) = 0. \tag{A.37}$$

In view of the above and the boundedness of the sequence $\mathcal{A}$, we apply Fatou's lemma and we obtain for every $m \in \mathbb{N}$

$$\begin{aligned}
&\limsup_{k\to\infty} \left\| \int_{[0,\infty)} \alpha^k(x)\, \mu_m(\mathrm{d}x) - \int_{[0,\infty)} \alpha^\infty(x)\, \mu_m(\mathrm{d}x) \right\|_2 \\
&\qquad \le \int_{[0,\infty)} \limsup_{k\to\infty} \left\|\alpha^k(x) - \alpha^\infty(x)\right\|_2 \mu_m(\mathrm{d}x) \overset{(A.36)}{\le} \int_{[0,\infty)} \left\|\Delta\alpha^\infty(x)\right\|_2 \mu_m(\mathrm{d}x).
\end{aligned}$$

Now, we can conclude that the Condition (ii) of Theorem A.9 is indeed satisfied by combining the above bound with Convergence (A.37). □

Remark A.17. Let us adopt the notation of Theorem A.14 and Proposition A.16 and assume that the measurable space is $\big([0,T], \mathcal{B}([0,T])\big)$, for some $T \in \mathbb{R}_+$. We claim that we can adapt the aforementioned results without loss of generality, which can be justified as follows. The limit measure μ_∞ is atomless, so the generality is not harmed if in Theorem A.14 the integrands α^k, for some $k \in \overline{\mathbb{N}}$, have a jump at point T.

Indeed, by the weak convergence $\mu_k \Rightarrow \mu_\infty$ and Proposition A.12 we have that the sequence of distribution functions $(C^k)_{k\in\overline{\mathbb{N}}}$ which is associated to the sequence $(\mu^k)_{k\in\overline{\mathbb{N}}}$ converges uniformly. Therefore, we can easily conclude that

$$\limsup_{k\to\infty} \left\|\Delta\alpha^k(T)\right\|_2 \mu_k(\{T\}) = \limsup_{k\to\infty} \left\|\Delta\alpha^k(T)\right\|_2 \Delta C_T^k \leq \left\|\Delta\alpha^\infty(T)\right\|_2 \Delta C_T^\infty = 0.$$

In other words, we can simply assume that the distribution functions are constant after time T in order to reduce the general case to the compact-interval case.

The following corollary is almost evident due to the fact that the limit measure is atomless. However we state it in the form we will need it in Subsection IV.5.2 and we provide its complete (rather trivial) proof.

Corollary A.18. *Let the measurable space $\big(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\big)$, $(\mu_k)_{k\in\overline{\mathbb{N}}}$ be a sequence of finite measures, where $\mu^k(\{0\}) = 0$ for every $k \in \overline{\mathbb{N}}$ and μ_∞ is atomless. Let, moreover, $\mathcal{A} := (\alpha^k)_{k\in\overline{\mathbb{N}}} \subset \mathbb{D}(\mathbb{R}^p)$ and the following to be true:*

(i) $\mu_k \xrightarrow{\text{w}} \mu_\infty$.
(ii) *$\mathcal{A}$ is uniformly bounded,* i.e. $\sup_{\alpha\in\mathcal{A}} \|\alpha\|_\infty < \infty$ *and*
(iii) *$\mathcal{A}$ is $\delta_{\mathrm{J}_1(\mathbb{R}^p)}-$convergent.*

Then,

$$\int_{[0,\cdot]} \alpha^k(x)\,\mu_k(\mathrm{d}x) \xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R}^p)} \int_{[0,\cdot]} \alpha^\infty(x)\,\mu_\infty(\mathrm{d}x). \tag{A.38}$$

Proof. Let C^k denote the distribution function associated to the measure μ_k for every $k \in \overline{\mathbb{N}}$ and

$$\gamma^k(t) := \int_{[0,t]} \alpha^k(x)\,\mu_k(\mathrm{d}x) = \int_{[0,t]} \alpha^k(s)\,\mathrm{d}C_s^k \in \mathbb{D}(\mathbb{R}^p) \text{ for } k \in \overline{\mathbb{N}}.$$

We are going to apply Proposition I.118 in order to prove the required convergence. To this end, let us fix a $t^\infty \in \mathbb{R}_+$ and a sequence $(t^k)_{k\in\mathbb{N}}$ such that $t^k \longrightarrow t^\infty$ as $k \to \infty$. We need to prove that the requirements (i) - (iii) of the aforementioned theorem are satisfied.

(i) $\lim_{k\to\infty}\Big(\left\|\gamma^k(t^k)-\gamma^\infty(t^\infty)\right\|_2 \wedge \left\|\gamma^k(t^k)-\gamma^\infty(t^\infty-)\right\|_2\Big) = 0$. However, γ^∞ is continuous. Therefore, it is sufficient to prove

$$\lim_{k\to\infty} \left\|\gamma^k(t^k) - \gamma^\infty(t^\infty)\right\|_2 = 0.$$

To this end, we have

$$\gamma^k(t^k) = \int_{[0,t^k]} \alpha^k(s)\,\mathrm{d}C_s^k = \int_{[0,t^k\wedge t^\infty]} \alpha^k(s)\,\mathrm{d}C_s^k + \int_{[t^k\wedge t^\infty,t^\infty]} \alpha^k(s)\,\mathrm{d}C_s^k + \int_{[t^\infty,t^k\wedge t^\infty]} \alpha^k(s)\,\mathrm{d}C_s^k.$$

The first summand converges to $\gamma^\infty(t^\infty)$, because the sequence $(C^k\mathbb{1}_{[0,t^k]})_{k\in\overline{\mathbb{N}}}$ is $\|\cdot\|_\infty-$convergent and $(\alpha^k)_{k\in\overline{\mathbb{N}}}$ is uniformly bounded $\delta_{\mathrm{J}_1(\mathbb{R}^p)}-$convergent. In other words, the conditions of Proposition A.16 are satisfied. For the second summand we have

$$\begin{aligned}
\left|\int_{[t^k\wedge t^\infty,t^\infty]} \alpha^k(s)\,\mathrm{d}C_s^k\right| &\leq \mathrm{Var}\Big(\int_{[0,\cdot]} \alpha^k(s)\,\mathrm{d}C_s^k\Big)_{t^\infty} - \mathrm{Var}\Big(\int_{[0,\cdot]} \alpha^k(s)\,\mathrm{d}C_s^k\Big)_{t^k\wedge t^\infty} \\
&= \int_{[t^k\wedge t^\infty,t^\infty]} \big(\alpha^k(s)\vee 0\big)\,\mathrm{d}C_s^k + \int_{[t^k\wedge t^\infty,t^\infty]} \big(-\alpha^k(s)\vee 0\big)\,\mathrm{d}C_s^k \\
&\leq \sup_{k\in\overline{\mathbb{N}}} \left\|\alpha^k\right\|_\infty \mu_k\big([t^k\wedge t^\infty, t^\infty]\big) \\
&= \sup_{k\in\overline{\mathbb{N}}} \left\|\alpha^k\right\|_\infty \big(C_{t^\infty}^k - C_{t^k\wedge t^\infty}^k\big) \xrightarrow[k\to\infty]{} 0,
\end{aligned}$$

where we used Proposition A.12 in order to conclude the convergence. Analogously we can prove that the third summand converges also to 0.

(ii) If $\lim_{k\to\infty} \left\|\gamma^k(t^k)-\gamma^\infty(t^\infty)\right\|_2 = 0$ and $(s^k)_{k\in\mathbb{N}}$ such that $t^k \geq s^k$ for every $k \in \mathbb{N}$ and $s^k \xrightarrow[k\to\infty]{} t^\infty$, then

$$\lim_{k\to\infty} \left\|\gamma^k(s^k) - \gamma^\infty(t^\infty)\right\|_2 = 0.$$

We can repeat the exact same arguments as in (i) in order to prove the convergence for every $(s^k)_{k\in\mathbb{N}}$ such that $s^k \xrightarrow[k\to\infty]{} t^\infty$.

(iii) If $\lim_{k\to\infty}\left\|\gamma^k(t^k)-\gamma^\infty(t^\infty-)\right\|_2=0$ and $(s^k)_{k\in\mathbb{N}}$ such that $s^k\geq t^k$ for every $k\in\mathbb{N}$ and $s^k\xrightarrow[k\to\infty]{}t^\infty$, then

$$\lim_{k\to\infty}\left\|\gamma^k(s^k)-\gamma^\infty(t^\infty-)\right\|_2=0.$$

due to the continuity of γ^∞, we can apply the same argument as in the previous cases.

□

The following lemma will be helpful in proving that a relatively compact set of $\mathbb{D}(\mathbb{R}^p)$ is uniformly bounded. Observe that every relatively compact subset of $\mathbb{D}(\mathbb{R}^p)$ satisfies (ii) of the following lemma; see Theorem I.111.

Lemma A.19. *Let $(\alpha^k)_{k\in\mathbb{N}}\subset\mathbb{D}\big(\mathbb{R}^p\big)$ such that*

(i) $\alpha^k(\infty):=\lim_{t\to\infty}\alpha^k(t)$ *exists and* $\left\|\alpha^k(\infty)\right\|_2<\infty$,

(ii) $\sup_{k\in\mathbb{N}}\sup_{t\in[0,N]}\left\|\alpha^k(t)\right\|_2<\infty$ *for every* $N\in\mathbb{N}$, *and*

(iii) $\limsup_{(k,t)\to(\infty,\infty)}\left\|\alpha^k(t)\right\|_2<\infty$.

Then the sequence $(\alpha^k)_{k\in\mathbb{N}}$ is uniformly bounded, i.e. $\sup_{k\in\mathbb{N}}\|\alpha\|_\infty<\infty$.

Proof. Assume that the sequence is unbounded, *i.e.* for every $M\in\mathbb{N}$ there exists $k(M)\in\mathbb{N}$ such that $\|\alpha^{k(M)}\|_\infty>M$. We can extract, then, a subsequence of $\big(k(M)\big)_{M\in\mathbb{N}}$, name $\big(k(M_n)\big)_{n\in\mathbb{N}}$, such that $k(M_l)<k(M_m)$ whenever $l<m$ and $M_n\to\infty$ as $n\to\infty$. This can be done as follows. For $n=1$ we define $k(M_1):=k(1)$ and for $i\in\mathbb{N}\setminus\{1\}$ we set

$$k(M_{i+1}):=\min\Big\{k(M)\in\mathbb{N},\|\alpha^{k(M)}\|_\infty>\Big\lceil\|\alpha^{k(M_i)}\|_\infty\Big\rceil\Big\},$$

where $\lceil x\rceil$ denotes the least integer greater than or equal to x. Since we have assumed that for every $k\in\mathbb{N}$ the value $\left\|\alpha^k(\infty)\right\|_2$ is finite, we have in particular (using that $\alpha^k\in\mathbb{D}(\mathbb{R}^p)$) that $\|\alpha^k\|_\infty<\infty$. Therefore,

$$\Big\{k(M)\in\mathbb{N},\|\alpha^{k(M)}\|_\infty>\Big\lceil\|\alpha^{k(M_i)}\|_\infty\Big\rceil\Big\}\neq\emptyset$$

and its minimum exists as $\mathbb{N}$ is a well-ordered set under the usual order. In view of the these comments, the subsequence $\big(k(M_n)\big)_{n\in\mathbb{N}}$ is well-defined and, according to our initial assumption, it has to be unbounded.

By definition of $\alpha^{k(M)}$, there exists $t_{k(M)}\in\mathbb{R}_+$ such that $\|\alpha^{k(M)}(t_{k(M)})\|_\infty>M$, a property which holds in particular for the subsequence indexed by $(k(M_n))_{n\in\mathbb{N}}$. Let us distinguish, now, two cases for the sequence $(t_{k(M_n)})_{n\in\mathbb{N}}$.

□ If $(t_{k(M_n)})_{n\in\mathbb{N}}$ is bounded, then (ii) leads to a contradiction, for $N:=\sup\{t_{k(M_n)},n\in\mathbb{N}\}$.

□ If $(t_{k(M_n)})_{n\in\mathbb{N}}$ is unbounded, then there exists a subsequence $(t_{k(M_{n_l})})_{l\in\mathbb{N}}$ such that $t_{k(M_{n_l})}\longrightarrow\infty$ as $l\to\infty$. Therefore, we have also

$$\lim_{l\to\infty}\left\|\alpha^{k(M_{n_l})}(t_{k(M_{n_l})})\right\|_\infty\geq\lim_{l\to\infty}M_{n_l}=\infty.$$

But, then (iii) leads to a contradiction.

Now, we have only to verify that the set $\{\|\alpha^k\|_\infty,k=1,\dots,k(1)\}$ is bounded in order to conclude that the sequence $(\alpha^k)_{k\in\mathbb{N}}$ is $\|\cdot\|_\infty$-bounded. But the above is clear since it is a finite set of finite numbers; use that $\|\alpha^k(\infty)\|_2<\infty$ and that $\alpha^k\in\mathbb{D}(\mathbb{R}^p)$.

□

Bibliography

[1] S. Aazizi. Discrete-time approximation of decoupled forward-backward stochastic differential equations driven by pure jump lévy processes. *Adv. in Appl. Probab.*, 45(3):791–821, 09 2013.

[2] D. Aldous. Stopping times and tightness. *Ann. Probab.*, 6(2):335–340, 04 1978.

[3] D. Aldous. Weak convergence and the general theory of processes. Incomplete draft, University of California, Berkeley, 1981.

[4] C. D. Aliprantis and K. Border. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer, 3rd edition, 2007.

[5] V. Bally. Approximation scheme for solutions of bsde. In N. El Karoui and L. Mazliak, editors, *Backward stochastic differential equations*, volume 364 of *Chapman & Hall/CRC Research Notes in Mathematics Series*, pages 177 – 191. Longman, 1997.

[6] E. Bandini. Existence and uniqueness for backward stochastic differential equations driven by a random measure, possibly non quasi-left continuous. *Electronic Communications in Probability*, 20, 2015.

[7] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[8] J.-M. Bismut. *Analyse Convexe et Probabilités*. PhD thesis, Faculté des Sciences de Paris, 1973.

[9] J.-M. Bismut. Conjugate convex functions in optimal stochastic control. *Journal of Mathematical Analysis and Applications*, 44(2):384–404, 1973.

[10] V. Bogachev. *Measure Theory*. Analysis. Springer-Verlag, Berlin Heidelberg, 2007.

[11] B. Bouchard and R. Elie. Discrete-time approximation of decoupled Forward–Backward SDE with jumps. 118:53 – 75, 01 2008.

[12] B. Bouchard, R. Elie, and N. Touzi. Discrete-time approximation of BSDEs and probabilistic schemes for fully non-linear pdes. In H. Albrecher, W. Runggaldier, and W. Schachermayer, editors, *Advanced Financial Modelling*, Radon Series on Computational and Applied Mathematics, pages 91 – 124. De Gruyter, 2009.

[13] B. Bouchard, D. Possamaï, X. Tan, and C. Zhou. A unified approach to *a priori* estimates for supersolutions of BSDEs in general filtrations. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* (*B*), to appear, 2015.

[14] P. Briand, B. Delyon, and J. Mémin. Donsker-type theorem for BSDEs. *Electron. Commun. Probab.*, 6:1–14, 2001. doi: 10.1214/ECP.v6-1030. URL http://dx.doi.org/10.1214/ECP.v6-1030.

[15] P. Briand, B. Delyon, and J. Mémin. On the robustness of backward stochastic differential equations. *Stochastic Processes and their Applications*, 97:229–253, 2002.

[16] R. Buckdahn. Backward stochastic differential equations driven by a martingale. *preprint*, 1993.

[17] R. Carbone, B. Ferrario, and M. Santacroce. Backward stochastic differential equations driven by càdlàg martingales. *Theory of Probability & its Applications*, 52(2):304–314, 2008.

[18] D. Chevance. *Résolution numériques des équations differentielles stochastiques rétrogrades,*. PhD thesis, Université de Provence – Aix – Marseille I, 1997.

[19] S. Cohen and R. Elliott. Existence, uniqueness and comparisons for BSDEs in general spaces. *The Annals of Probability*, 40(5):2264–2297, 2012.

[20] S. Cohen and R. Elliott. *Stochastic Calculus and Applications*. Springer, 2nd edition, 2015.

[21] S. Cohen, R. Elliott, and C. Pearce. A general comparison theorem for backward stochastic differential equations. *Advances in Applied Probability*, 42(3):878–898, 2010.

[22] F. Confortola, M. Fuhrman, and J. Jacod. Backward stochastic differential equation driven by a marked point process: An elementary approach with an application to optimal control. *Ann. Appl. Probab.*, 26(3):1743–1773, 06 2016.

[23] F. Coquet, J. Mémin, and L. Słomiński. On weak convergence of filtrations. *Séminaire de Probabilités*, XXXV:306–328, 2001.

[24] M. Davis and P. Varaiya. Dynamic programming conditions for partially observable stochastic systems. *SIAM Journal on Control and Optimization*, 11(2):226–261, 1973.

[25] M. Davis and P. Varaiya. The multiplicity of an increasing family of $\sigma-$fields. *The Annals of Probability*, 2(5):958–963, 1974.

[26] C. Dellacherie and P.-A. Meyer. *Probabilities and Potential.* North-Holland, 1978.

[27] C. Dellacherie and P.-A. Meyer. *Probabilities and Potential, B: Theory of Martingales.* North-Holland, 1982.

[28] R. Dudley. *Real Analysis and Probability.* Cambridge University Press, 2nd edition, 2002.

[29] N. El Karoui and S.-J. Huang. A general result of existence and uniqueness of backward stochastic differential equations. In N. El Karoui and L. Mazliak, editors, *Backward stochastic differential equations*, volume 364 of *Chapman & Hall/CRC Research Notes in Mathematics Series*, pages 27–36. Longman, 1997.

[30] N. El Karoui and L. Mazliak. *Backward Stochastic Differential Equations.* Chapman & Hall/CRC Research Notes in Mathematics Series. Taylor & Francis, 1997.

[31] N. El Karoui, S. Peng, and M.-C. Quenez. Backward stochastic differential equations in finance. *Mathematical Finance*, 7(1):1–71, 1997.

[32] P. Embrechts and M. Hofert. A note on generalized inverses. *Mathematical Methods of Operations Research*, 77(3):423–432, 2013.

[33] S. Ethier and T. Kurtz. *Markov Processes: Characterization and Convergence.* J. Wiley & Sons, 1986.

[34] H. Föllmer and P. Protter. Local martingales and filtration shrinkage. *ESAIM Probab. Stat.*, 15(In honor of Marc Yor, suppl.):S25–S38, 2011. ISSN 1292-8100.

[35] S. He, J. Wang, and J. A. Yan. *Semimartingale Theory and Stochastic Calculus.* CRC Press, 1992.

[36] P. Henry-Labordère, X. Tan, and N. Touzi. A numerical algorithm for a class of BSDEs via the branching process. *Stochastic Processes and their Applications*, 124(2):1112 – 1140, 2014. ISSN 0304-4149.

[37] E. Hobson. *The theory of functions of a real variable and the theory of Fourier's series.* Cambridge: University Press, 1907.

[38] D. Hoover. Convergence in distribution and Skorokhod convergence for the general theory of processes. *Probability Theory and Related Fields*, 89(3):239–259, 1991.

[39] Y. Hu and S. Peng. A stability theorem of backward stochastic differential equations and its application. *Comptes Rendus de l'Académie des Sciences, Série I: Mathématique*, 324(9):1059–1064, 1997.

[40] J. Jacod. *Calcul Stochastique et Problèmes de Martingales.* Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2006.

[41] J. Jacod and A. Shiryaev. *Limit Theorems for Stochastic Processes.* Springer, 2nd edition, 2013.

[42] J. Jacod, S. Méléard, and P. Protter. Explicit form and robustness of martingale representations. *The Annals of Probability*, 28:1747–1780, 2000.

[43] M. Krasnosel'skii and Y. Rutickii. *Convex Functions and Orlicz Spaces.* P. Noordfoff Ltd., Groningen, 1961.

[44] T. Kurtz and P. Protter. Weak limit theorems for stochastic integrals and stochastic differential equations. *The Annals of Probability*, 19(3):1035–1070, 07 1991.

[45] R. Leadbetter, S. Cambanis, and V. Pipiras. *A Basic Course in Measure and Probability: Theory for Applications.* Cambridge University Press, 2014.

[46] A. Lejay and M. S. Torres. Numerical approximation of backward stochastic differential equations with jumps, 2013.

[47] E. Lenglart, D. Lépingle, and M. Pratelli. Présentation unifiée de certaines inégalités de la théorie des martingales. *Séminaire de Probabilités*, XIV:26–52, 1980.

[48] R. Long. *Martingale Spaces and Inequalities.* Vieweg+Teubner Verlag, 1993.

[49] J. Ma, P. Protter, J. San Martin, and S. Torres. Numberical method for backward stochastic differential equations. *The Annals of Applied Probability*, 12(1):302–316, 2002.

[50] D. Madan, M. Pistorius, and M. Stadje. Convergence of BSΔEs driven by random walks to BSDEs: the case of (in)finite activity jumps with general driver. *Stochastic Processes and their Applications*, 126:1553–1584, 2016.

[51] F. Mazzone. A characterization of almost everywhere continuous functions. *Real Analysis Exchange*, 21(1):317–319, 1995.
[52] P.-A. Meyer. Sur le lemme de la Vallée Poussin et un théorème de Bismut. *Séminaire de Probabilités*, XII:770–774, 1978.
[53] J. Mémin. Stability of Doob–Meyer decomposition under extended convergence. *Acta Mathematicae Applicatae Sinica*, 19:177–190, 2003.
[54] A. Papapantoleon, D. Possamaï, and A. Saplaouras. Existence and uniqueness results for BSDEs with jumps: the whole nine yards. *Preprint,* `arXiv:1607.04214`, 2016.
[55] E. Pardoux and S. Peng. Adapted solution of a backward stochastic differential equation. *System and Control Letters*, 14(1):55–61, 1990.
[56] K. Parthasarathy. *Probability Measures on Metric Spaces.* Academic Press, 1972.
[57] Y. V. Prokhorov. Convergence of random processes and limit theorems in probability theory. *Theory of Probability and Applications*, I(2):157 – 214, 1956.
[58] M. Rao and Z. Ren. *Theory of Orlicz Spaces.* Marcel Dekker, 1991.
[59] R. Rebolledo. La méthode des martingales appliquée à l'étude de la convergence en loi de processus. *Mémoires de la Société Mathématique de France*, 62:I1–V125, 1979.
[60] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion.* Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2004.
[61] R. Rockafellar. *Convex Analysis.* Princeton landmarks in mathematics and physics. Princeton University Press, 1970.
[62] M. Royer. BSDEs with a random terminal time driven by a monotone generator and their links with PDEs. *Stochastics and Stochastic Reports*, 76(4):281–307, 2004.
[63] W. Rudin. *Principles of Mathematical Analysis.* WCB/McGraw-Hill, third edition, 1976.
[64] K. Sato. *Lévy Processes and Infinitely Divisible Distributions.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999. ISBN 9780521553025.
[65] J. Zhang. A numerical scheme for bsdes. *Ann. Appl. Probab.*, 14(1):459–488, 02 2004.