Matroidal subdivisions, Dressians and tropical Grassmannians

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Zusammenfassung

In dieser Arbeit untersuchen wir verschiedene Aspekte von tropischen linearen Räumen und deren Modulräumen, den tropischen Grassmannschen und Dressschen. Tropische lineare Räume sind dual zu Matroidunterteilungen. Motiviert durch das Konzept der Splits, dem einfachsten Fall einer polytopalen Unterteilung, wird eine neue Klasse von Matroiden eingeführt, die mit Techniken der polyedrischen Geometrie untersucht werden kann. Diese Klasse ist sehr groß, da sie alle Paving-Matroide und weitere Matroide enthält. Die strukturellen Eigenschaften von Split-Matroiden können genutzt werden, um neue Ergebnisse in der tropischen Geometrie zu erzielen. Vor allem verwenden wir diese, um Strahlen der tropischen Grassmannschen zu konstruieren und die Dimension der Dressschen zu bestimmen. Dazu wird die Beziehung zwischen der Realisierbarkeit von Matroiden und der von tropischen linearen Räumen weiter entwickelt.

Die Strahlen einer Dressschen entsprechen den Facetten des Sekundärpolytops eines Hypersimplexes. Eine besondere Klasse von Facetten bildet die Verallgemeinerung von Splits, die wir Multi-Splits nennen und die Herrmann ursprünglich als k-Splits bezeichnet hat. Wir geben eine explizite kombinatorische Beschreibung aller Multi-Splits eines Hypersimplexes. Diese korrespondieren mit Nested-Matroiden. Über die tropische Stiefelabbildung erhalten wir eine Beschreibung aller Multi-Splits für Produkte von Simplexen. Außerdem präsentieren wir Berechnungen für explizite untere Schranken der Anzahl der Facetten einiger Sekundärpolytope von Hypersimplexen.

Berechnungen und Algorithmen spielen auch im Weiteren eine wichtige Rolle. Wir führen eine neue Methode zum Berechnen von tropischen linearen Räumen und sogar allgemeiner von dualen Komplexen von polyedrischen Unterteilungen ein. Diese Methode basiert auf einem Algorithmus von Ganter (1984) für endliche Hüllensysteme. Außerdem beschreiben wir die Implementierung eines algebraischen Teilkörpers der formalen Puiseux-Reihen. Dieser kann eingesetzt werden zum Lösen von linearen Programmen und konvexen Hüllenproblemen, die jeweils von einem reelen Parameter abhängen. Darüber hinaus ist dieses Werkzeug sowohl für tropische Konvexgeometrie als auch tropische algebraische Geometrie wertvoll.

Tropische Varietäten, wie zum Beispiel tropische lineare Räume oder tropische Grassmannsche, sind der gemeinsame Schnitt von endlich vielen tropischen Hyperflächen. Die Menge der zu den Hyperflächen gehörenden Polynome bildet eine tropische Basis. Für den allgemeinen Fall geben wir eine explizite obere Schranke für den Grad an, den die Polynome in einer tropischen Basis benötigen. Als Anwendung berechnen wir f-Vektoren von tropischen Varietäten und veranschaulichen die Unterschiede zwischen Gröbnerbasen und tropischen Basen.

Abstract

In this thesis we study various aspects of tropical linear spaces and their moduli spaces, the tropical Grassmannians and Dressians. Tropical linear spaces are dual to matroid subdivisions. Motivated by the concept of splits, the simplest case of a subdivision, a new class of matroids is introduced, which can be studied via techniques from polyhedral geometry. This class is very large as it strictly contains all paving matroids. The structural properties of these split matroids can be exploited to obtain new results in tropical geometry, especially on the rays of the tropical Grassmannians and the dimension of the Dressian. In particular, a relation between matroid realizability and certain tropical linear spaces is elaborated.

The rays of a Dressian correspond to facets of the secondary polytope of a hypersimplex. A special class of facets is obtained by a generalization of splits, called multi-splits or originally, in Herrmann's work, k-splits. We give an explicit combinatorial description of all multi-splits of the hypersimplex. These are in correspondence to nested matroids and, via the tropical Stiefel map, also to multi-splits of products of simplices. Hence, we derive a description for all multi-splits of a product of simplices. Moreover, a computational result leads to explicit lower bounds on the total number of facets of secondary polytopes of hypersimplices.

Other computational aspects are also part of our research: A new method for computing tropical linear spaces and more general duals of polyhedral subdivisions is developed and implemented in the software `polymake`. This is based on Ganter's algorithm (1984) for finite closure systems. Additionally, we describe the implementation of a subfield of the field of formal Puiseux series. This is employed for solving linear programs and computing convex hulls depending on a real parameter. Moreover, this approach is useful for computations in convex and algebraic tropical geometry.

Tropical varieties, as for example tropical linear spaces or tropical Grassmannians, are intersections of finitely many tropical hypersurfaces. The set of corresponding polynomials is a tropical basis. We give an explicit upper bound for the degree of a general tropical basis of a homogeneous polynomial ideal. As an application f-vectors of tropical varieties are discussed. Various examples illustrate differences between Gröbner bases and tropical bases.

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Contents

1 Introduction to tropical linear spaces

1.1 Matroids and subdivisions

Linear spaces are one of the most basic and important objects in mathematics. They describe the set of solutions of a linear system of equations. Such systems can be solved efficiently by Gaussian elimination and they are relevant in all areas of mathematics and in practice. A key property of a linear space is the existence of a basis, i.e., a fixed set of vectors such that every vector is a unique finite linear combination of those vectors. The Steinitz exchange lemma states that any set of linearly independent vectors can be extended to a basis by a subset of any given basis. This key property defines an abstract combinatorial object – a matroid, that unifies both matrices and graphs. A *matroid* is a finite collection of elements, such that any subset of an independent set is an independent set itself, and each independent set can be extended to a maximal independent set, which is called a *basis*. The cardinality of any basis is the same, which is the *rank* of the matroid. There are many other cryptomorphic definitions of matroids and their equivalence is not obvious. Matroids have been developed independently by Whitney [Whi35] and Nakasawa; see [NK09]. The books of Oxley and White [Oxl11, Whi86] provide an introduction and overview about matroids. Key examples of matroids are *realizable matroids*, i.e., a collection of finitely many vectors, and *graphical matroids*, i.e., a collection of edges of a graph. A set of edges is independent if they are circuit free.

In the first chapters of this thesis we study matroids with polyhedral methods. A *polytope* is the convex hull of finitely many points; see the books of Ziegler [Zie95, Zie00]. The convex hull of the characteristic vectors of the bases of a matroid form the *matroid polytope*. Many properties of the matroid can be directly read off this polytope. For example the basis exchange property is encoded in the edge directions of the matroid polytope. Gel′fand, Goresky, MacPherson and Serganova [GGMS87] showed that the matroid polytopes are exactly those 0/1-polytopes whose edge directions are roots of Coxeter type A, i.e., differences $e_i - e_j$ of standard unit vectors.

Studying matroids in polyhedral terms goes back to Edmonds [Edm70]. All matroid polytopes are subpolytopes of a slice through vertices of the unit cube. The intersection of the n-dimensional cube with the hyperplane $x_1 + \ldots + x_n = d$ is called the *hypersimplex*

$\Delta(d,n)$. The hypersimplex itself is a matroid polytope of the uniform matroid. Hypersimplices appear in algebraic combinatorics, graph theory, optimization, phylogenetics, matroid theory, tropical geometry, analysis and number theory; see [DLRS10, Subsection 6.3.6]. Hypersimplices are a central object in the next three chapters.

A *subdivision* of a polytope is a covering of the polytope with subpolytopes called *cells*, such that any two cells do not overlap and intersect each other in a common face. A precise definition and a broad overview can be found in [DLRS10]. A subdivision of the hypersimplex into matroid polytopes is a *matroid subdivision*. The simplest example of a subdivision is a *split*, i.e., a subdivision into two polytopes. Splits have been studied by Herrmann and Joswig in [HJ08, HJ10]. Additionally they considered subdivisions composed of compatible splits. The matroid polytopes that occur in a subdivision of compatible splits give rise to a new class of matroids. We introduce this new class of *split matroids* in Chapter 2. The class of split matroids contains all paving matroids. Paving matroids are a well studied class in matroid theory and it is conjectured that almost all matroids are paving; [Oxl11, Conjecture 15.5.10]. This conjecture would imply that almost all matroids are split matroids. Moreover, we show that split matroids are closed under dualization and taking minors. Cameron and Mayhew [CM17] recently proved that the class of split matroids has exactly five excluded minors. They are listed in Section 2.5.

Splits in subdivisions of a polytope arise in various areas of mathematics, as any subdivision is a unique composition of compatible splits and a split-prime subdivision. This and similar decompositions have been studied in [BD92, Hir06, HJ08, Koi14]. The subdivisions of a hypersimplex that consist of split matroid polytopes are exactly those that totally decompose, i.e., they have a trivial split-prime remainder. Every rank two matroid is split and hence the matroid subdivisions of the second hypersimplex $\Delta(2,n)$ are exactly those that are split decomposable. Split decomposable subdivisions are dual to metric trees, which play a key role in phylogenetic analysis; see for example [BHV01, SS04].

Splits, that we mentioned before, are examples of coarsest (non-trivial) subdivisions. Herrmann and Joswig [HJ08] were the first who systematically studied coarsest subdivisions, while finest subdivisions, also known as *triangulations*, occur in almost all fields of mathematics; see [DLRS10]. Herrmann [Her11] introduced a class of coarsest subdivisions generalizing splits. We call such a subdivision a *multi-split*. This is a subdivision where all maximal cells meet in a common $(k-1)$-codimensional cell, when k is the number of maximal cells. Another relevant class of subdivisions are *regular subdivisions*. They are induced by a height function. It is a fundamental result of Gel′fand, Kapranov and Zelevinsky [GKZ08] that the regular subdivisions are in bijection to the boundary complex of a polytope – the *secondary polytope*. The secondary polytope is the convex hull of the GKZ-vectors, which correspond to triangulations and can be expressed in terms of the volumes of the occurring simplices. Multi-splits are necessarily regular subdivisions and hence correspond to facets of a secondary polytope. The purpose of Chapter 3 is to give an explicit combinatorial description of all multi-splits of a hypersimplex. It turns out that all of them are matroid subdivisions consisting of nested matroid polytopes. These subdivisions are related via the tropical Stiefel map, investigated in [HJS12, FR15], to subdivisions of products of simplices. Thus, tropical convexity and tropical point configurations play a major role in Chapter 3; for the connections see also [DS04, HJS12].

1.2 The Dressian

A polyhedral fan is a collection of polyhedral cones such that two cones do not intersect in their relative interior and that they meet in a common face. All the lifting vectors that induce the same subdivision form a relative open cone. The closure of these cones form a fan, which is called a *secondary fan*. This is the normal fan of a secondary polytope. We call a lifting function that induces a matroidal subdivision a *tropical Plücker vector*. The name tropical Plücker vector is due to the following classical concept. As mentioned before a realizable matroid is obtained by the columns of a matrix. These are finitely many points in a vector space. The bases of this matroid correspond to the non-vanishing maximal minors of the matrix. The vector of these minors is the classical *Plücker vector*. There are variants of matroids that take partial information of the minors into account. The most prominent example for these are oriented matroids; see [BLV78]. A unifying concept to this approach has been introduced by Baker and Bowler [BB17]. In this thesis we are interested in the case of valuated minors. A *valuation* on a field $\mathbb{K}$ is a map $\mathrm{val} : \mathbb{K} \to \mathbb{R} \cup \{-\infty\}$ such that

$$\mathrm{val}(x \cdot y) = \mathrm{val}(x) + \mathrm{val}(y) \;\text{ and }\; \mathrm{val}(x+y) \leq \max(\mathrm{val}(x), \mathrm{val}(y)) \ .$$

A tropical Plücker vector is the abstract version of a classical Plücker vector where each basis of the underlying matroid is equipped with the valuation of its minor. Dress and Wenzel [DW92] developed them as valuated matroids. The coordinate-wise valuation is one of many possibilities of tropicalization of a linear space. Hence, they and their abstract genearlization are called tropical Plücker vectors.

The set of all tropical Plücker vectors forms a subfan in the secondary fan. This fan is called the *Dressian*; see [HJJS09]. We study this space in Chapter 2 and Chapter 3. In particular, we take a closer look at the dimension and rays of the Dressian. The moduli spaces of labeled metric trees are precisely the Dressians of the second hypersimplex $\Delta(2, n)$. The Dressian in general is the moduli space of tropical linear spaces. A *tropical linear space* is the dual of a regular matroid subdivision of the hypersimplex $\Delta(d, n)$ or a matroid polytope:

$$\left\{ x \in \mathbb{R}^n \;\middle|\; \mathrm{argmin}_I \left\{ \pi_I - \sum_{i \in I} x_i \right\} \text{ is the collection of bases of a loop-free matroid} \right\} \ ,$$

where the lifting π is the corresponding tropical Plücker vector. A tropical linear space is naturally equipped with the structure of a polyhedral complex, inherited by the subdivision. In Chapter 4 we discuss a common algorithm that efficiently computes tropical linear spaces and tight spans, i.e., duals of not necessarily regular subdivisions. Important examples of tropical linear spaces are derived from corank vectors. These are tropical Plücker vectors that consist of coranks that are given by a fixed matroid. A tropical linear space is called the *Bergman fan* of a matroid in the case that the tropical Plücker vector is zero for the bases and ∞ at the non bases of this matroid. They arise as linear spaces with trivial valuation. A Bergman fan is a "local" version of tropical linear space and a geometric embedding of the lattice of flats of a matroid; see [FS05, Theorem 4.1]. Bergman fans have been introduced as "logarithmic limit sets" of algebraic varieties in [Ber71], and have been further studied for example in [Stu02, FS05, AK06]. Rincón provides an algorithm in [Rin13] which computes Bergman fans with a finer polyhedral structure, while our algorithm always computes the tropical linear space with the coarsest structure that is derived from the subdivision. This is beside the higher generality another advantage of our procedure, as coarse means more efficient for a computer.

Tropical linear spaces have been studied by Speyer [Spe05, Spe08, Spe09]. His conjecture about the f-vector of a tropical linear space is an open question. The ith entry in the *f-vector* of a polyhedral complex is the number of its i-dimensional faces. Speyer conjectured that the maximal number of i-dimensional cells in a matroid subdivision is given by a subdivision into series-parallel matroids. He proves the statement for realizable matroids in characteristic 0; see [Spe09]. Our implementation in `polymake` [GJ00] allows us to construct explicit examples of tropical linear spaces. They can be investigated to study Speyer's conjecture or other properties. The f-vectors of some tropical linear spaces and tropical varieties are computed in Section 4.5 and Section 5.5. Kastner, Shaw and Winz used our implementation to study the homology of tropical linear spaces [KSW17].

1.3 The tropical Grassmannian

Realizable tropical linear spaces arise as tropicalizations, for example the point-wise valuation, of classical linear spaces. The Grassmannian is the moduli space that parameterizes all d-dimensional subspaces in $\mathbb{K}^n$. This is the simplest case of a moduli space. The *tropical Grassmannian* is the tropicalization of the classical Grassmannian and parameterizes the tropicalization of linear spaces over $\mathbb{K}^n$. These objects and their relations have been studied in the case $\mathbb{K} = \mathbb{C}\{\{t\}\}$ by Speyer and Sturmfels; see [SS04] and [Spe05]. For other fields take a look at [MS15], where the main focus lies on lines in the n-dimensional projective space. These tropicalizations of linear spaces correspond to labeled metric trees.

An important tool to study both algebraic and tropical varieties, such as the Grassmannians, are Gröbner bases. For example, they are used to solve systems of polynomial equations, which is much more complicated than solving systems of linear equations. Before we introduce Gröbner bases formally we impose some terminology. To simplify the situation we restrict ourselves to the case of trivially valuated coefficients. The more general case is discussed in Chapter 5.

Let $f = \sum_\alpha c_\alpha\, x^\alpha \in \mathbb{K}[x_1, x_2, \ldots, x_n]$ be a polynomial. The *initial form* of f with respect to the vector $w \in \mathbb{R}^n$ is the polynomial

$$\mathrm{in}_w(f) \;=\; \sum_{\substack{\alpha\cdot w \\ \text{is maximal}}} c_\alpha\, x^\alpha \;\in\; \mathbb{K}[x_1, x_2, \ldots, x_n] \;.$$

Let $I \subset \mathbb{K}[x_1, x_2, \ldots, x_n]$ be an ideal. The *initial ideal* of I with respect to $w \in \mathbb{R}^n$ is the ideal $\mathrm{in}_w(I) = \{\mathrm{in}_w(f) \,|\, f \in I\}$. A *Gröbner basis* with respect to $w \in \mathbb{R}^n$ of an ideal I is a finite set of polynomials, such that they generate the ideal I and their initial forms generate the initial ideal. There is a link between Gröbner bases and polyhedral geometry. An ideal I introduces an equivalence relation on the vector space $\mathbb{R}^n$. Two vectors v and w are considered as equivalent if their initial ideals coincide, i.e., $\mathrm{in}_v(I) = \mathrm{in}_w(I)$. This relation divides $\mathbb{R}^n$ into relative open cones. The set of closures of these cones forms a polyhedral fan, which is the normal fan of the state polytope; see [Stu96]. The first who studied the Gröbner fan were Mora and Robbiano [MR88].

The collection of all closed cones whose corresponding initial ideals are monomial-free forms a subfan of the Gröbner fan. This fan is a *tropical hypersurface* if the ideal $I = \langle f \rangle$ is a principal ideal. Note that the tropical hypersurface of f is equal to the $(n-1)$-skeleton of the normal fan to the Newton polytope of f. The just described subfan of the Gröbner fan is related to tropical varieties via the fundamental theorem of tropical algebraic geometry; see [SS04, Theorem 2.1] or [MS15, Theorem 3.2.3]. This theorem states that for an ideal I in a polynomial ring in n variables over an algebraically closed field with non-trivial valuation the following three subsets coincide:

(i) The closure of the set $\{\mathrm{val}(v) \,|\, v \in (\mathbb{K} \setminus \{0\})^n$ and $f(v) = 0$ for all $f \in I\}$,

(ii) the set of all vectors $w \in \mathbb{R}^n$ such that $\mathrm{in}_w(I)$ contains no monomial, and

(iii) the intersection $\bigcap_{f\in I} \mathcal{T}(f)$, where $\mathcal{T}(f)$ is the tropical hypersurface of f.

Note that alternatively the condition to take the topological closure in (i) can be replaced by the request that the valuation is surjective. For this purpose Markwig [Mar10] introduced the field of generalized Puiseux series. In any case, the condition of being algebraically closed is required. The key feature of generalized Puiseux series is that they have real exponents and that their valuation groups, i.e., the image of the valuation map, are the real numbers.

The classical Grassmannian is the vanishing locus of the Plücker ideal in the variables indexed by all d subsets of $[n]$. This ideal is generated by the famous 3-term Plücker relations

$$p_{Iab} \cdot p_{Icd} - p_{Iac} \cdot p_{Ibd} + p_{Iad} \cdot p_{Ibc} \;\;,$$

where $I \in \binom{[n]}{d-2}$ and $a, b, c, d \in [n] \setminus I$. See [Stu08, Chapter 3] for details and an explicit specification of a quadratic Gröbner basis. The integrality of the coefficients of the 3-term

Plücker relations are a witness for the fact that tropical Grassmannians only depend on the characteristic of the underlying field.

In contrast to classical algebraic geometry, the intersection of tropical hypersurfaces of generators of an ideal is not the tropical variety itself. A finite set of generators whose tropical hypersurfaces cut out the tropical variety is called a *tropical basis*. Clearly, tropical bases play an important role in tropical algebraic geometry. The relation between tropical bases and Gröbner bases has not been conclusively determined. Tropical bases, their degree and relation to Gröbner bases is the core of Chapter 5.

A set which is the intersection of tropical hypersurfaces of generators is called a *tropical prevariety*. The Dressians that we defined combinatorially are the tropical prevarieties of the Grassmannians, which are the intersection of all tropical hypersurfaces of the 3-term Plücker relations. A Dressian itself is independent of the characteristic of the underlying field and it contains all the tropical Grasmannians as sets.

A concrete and rounding-error free implementation of tropicalization needs an appropriate field. The field of generalized real Puiseux series is applicable to tropicalize algebraic varieties. Furthermore, this field is totally ordered. This allows to consider polyhedra and their tropicalizations, i.e., the solution set of linear systems of inequalities [Jos17]. This leads to tropical geometry, tropical convexity in combination with classical linear optimization. The most remarkable result in that area is the counterexample of the "continuous analogue of the Hirsch conjecture" by Allamigeon, Benchimol, Gaubert and Joswig [ABGJ14]. Chapter 6 follows the setting of [ABGJ15, ABGJ14] and investigates tropical polytopes, polytopes over Puiseux series and classical polytopes that occur as evaluation of those over fields of Puiseux series. A construction and implementation of an appropriate field – the field of *Puiseux fractions* – is the central aspect of Chapter 6.

1.4 About this thesis

In this thesis we discuss various aspects of tropical linear spaces and related mathematical objects. In particular, we study them via matroids. The chapters of this thesis can be read independently and in any order, even though they are all related to each other.

In Chapter 2, we combine the concept of polytopal splits with matroid theory. This leads to the new class of matroids that we call *split matroids*. In this chapter we describe the relation of matroid polytopes and classical matroid properties. Moreover, we characterize split matroids without any direct reference to polytopal properties. However, split matroids can be studied via techniques from polyhedral geometry. We show that the structural properties of the split matroids can be exploited to obtain new results in tropical geometry, especially on the rays of the tropical Grassmannians and Dressians. This chapter is a collaboration with Michael Joswig and closely follows the publication "Matroids from hypersimplex splits" in *Journal of Combinatorial Theory, Series A* volume 151 pages 254–284 [JS17].

This publication is available via https://dx.doi.org/10.1016/j.jcta.2017.05.001. The exposé is enriched by comments about the explicit construction of realizable and non-realizable matroids and some clarifications in the text and proofs. Newest results of Cameron and Mayhew concerning split matroids are mentioned in Section 2.5.

Chapter 3 treats a generalization of splits. We present an explicit combinatorial description of a special class of facets of the secondary polytopes of hypersimplices. These facets correspond to polytopal subdivisions called multi-splits. We show a relation between the cells in a multi-split of the hypersimplex and nested matroids. Moreover, we derive a description of all multi-splits of a product of simplices. Additionally, we present a computational result to derive explicit lower bounds on the number of facets of secondary polytopes of hypersimplices. This chapter is available as "Multi-splits and tropical linear spaces from nested matroids" [Sch17] at arXiv:1707.02814.

In Chapter 4 we focus on computational aspects of tropical linear spaces and their bounded parts and more general duals of polyhedral subdivisions. We introduce an abstract framework that deals with the boundary cells of such a subdivision. Our method is based on Ganter's algorithm (1984) for finite closure systems and implemented in polymake. This chapter is based on a joint work with Simon Hampe and Michael Joswig. It has been presented at "International conference on effective methods in algebraic geometry 2017" and closely follows the preprint "Algorithms for tight spans and tropical linear spaces" [HJS17] at arXiv:1612.03592. A computational result about the Grassmannians of different characteristics is added as well as an example of tropical convexity.

In Chapter 5 we deal with arbitrary ideals. We give an explicit upper bound for the degree of a tropical basis of a homogeneous polynomial ideal. As an application f-vectors of tropical varieties are discussed. Various examples illustrate differences between Gröbner and tropical bases. This chapter is a joint work with Michael Joswig and published as "The degree of a tropical basis" [JS18]. This article is available via https://dx.doi.org/10.1090/proc/13787. First published in Proc. Amer. Math. Soc. 146 (March 2018), published by the American Mathematical Society. ©2017 American Mathematical Society.

Chapter 6 is again about computational aspects. We describe the implementation of a subfield of the field of formal Puiseux series in polymake. We demonstrate how that can be used for solving linear programs and computing convex hulls depending on a real parameter. Moreover, this approach is also a tool for computations of tropical polytopes and tropicalization of linear spaces. This chapter is a joint work with Michael Joswig, Georg Loho and Benjamin Lorenz. It is published as "Linear programs and convex hulls over fields of Puiseux fractions" in *Mathematical aspects of computer and information sciences: 6th International Conference, MACIS 2015* pages 429–445 [JLLS16]. The final publication is available at Springer via https://dx.doi.org/10.1007/978-3-319-32859-1_37. The code examples in this thesis are updated to the current polymake version.

2 Matroids from hypersimplex splits

2.1 Introduction

The purpose of this chapter is to introduce, to characterize and to exploit a new class of matroids which we call *split matroids.* We will argue that there are good reasons to study these matroids for the sake of matroid theory itself. Additionally, however, they also give rise to a large and interesting class of tropical linear spaces. In this way we can use split matroids to answer some questions which previously arose in the investigation of tropical Grassmannians [SS04] and Dressians [HJJS09, HJS12].

The split matroids are motivated via polyhedral geometry in the following way. For a given matroid M the convex hull of the characteristic vectors of the bases of M is the *matroid polytope* $P(M)$. The hypersimplices $\Delta(d, n)$ are the matroid polytopes corresponding to the uniform matroids $U_{d,n}$. If M has rank d and n elements, the matroid polytope $P(M)$ is a subpolytope of $\Delta(d, n)$. Studying matroids in polyhedral terms goes back to Edmonds [Edm70].

A *split* of a polytope is a subdivision with precisely two maximal cells. These subdivisions are necessarily regular, and the cells are matroid polytopes. The hyperplane spanned by the intersection of the two maximal cells is the corresponding *split hyperplane.* Clearly this hyperplane determines the split, and it yields a facet of both maximal cells. As our first contribution we show the following converse. Each facet of a matroid polytope $P(M)$ corresponds to either a hypersimplex facet or a hypersimplex split (Proposition 2.7). We call the latter the *split facets* of $P(M)$. The hypersimplex facets correspond to matroid deletions and contractions, and the hypersimplex splits have been classified in [HJ08]. Now the matroid M is a *split matroid* if the split facets of $P(M)$ satisfy a compatibility condition. We believe that these matroids are interesting since they form a large class but feature stronger combinatorial properties than general matroids. “Large” means that they comprise the paving matroids and their duals as special cases (Theorem 2.19). It is conjectured that asymptotically almost all matroids are paving matroids [MNWW11] and [Oxl11, 15.5.8]. In particular, this would imply that almost all matroids are split. In Section 2.6 we present statistical data that compare paving and split matroids based on a census of small matroids which has been obtained by Matsumoto, Moriyama, Imai and Bremner [MMIB12].

We characterize the split matroids in terms of deletions and contractions, i.e., in pure matroid language (Theorem 2.11 and Proposition 2.15). In this way it becomes apparent that the basic concepts of matroid splits and split matroids make several appearances in the matroid literature. For instance, a known characterization of paving matroids implicitly makes use of this technique; see [Oxl11, Proposition 2.1.24]. Splits also occur in a recent matroid realizability result by Chatelain and Ramírez Alfonsín [CRA14]. Yet, to the best of our knowledge, so far split matroids have not been recognized as an interesting class of matroids in their own right.

One motivation to study matroid polytopes comes from tropical geometry; see Maclagan and Sturmfels [MS15]. Tropical geometry is related to the study of an algebraic variety defined over some field with a discrete valuation. And a *tropical variety* is the image of such a variety under the valuation map. In particular, a *tropical linear space* corresponds to a polytopal subdivision of the hypersimplices where each cell is a matroid polytope; see De Loera, Rambau and Santos [DLRS10] for general background on subdivisions of polytopes. The *Dressian* $\mathrm{Dr}(d,n)$ is the polyhedral fan of lifting functions for the (regular) matroid subdivisions of $\Delta(d,n)$. By definition, this is a subfan of the secondary fan. In general, $\mathrm{Dr}(d,n)$ has maximal cones of various dimensions, i.e., it is not pure. In the work of Dress and Wenzel [DW92] these lifting functions occur as "valuated matroids". Using split matroids we provide exact asymptotic bounds for $\dim \mathrm{Dr}(d,n)$ (Theorem 2.31).

A tropical linear space is *realizable* if it arises as the tropicalization of a classical linear space. It is known from work of Speyer [Spe05, Spe09] that the realizability of tropical linear spaces is related with the realizability of matroids. Here we give a first characterization of matroid realizability in terms of certain tropical linear spaces (Theorem 2.35). The subset of $\mathrm{Dr}(d,n)$ which corresponds to the realizable tropical linear spaces is the *tropical Grassmannian*. The latter is also equipped with a fan structure which is inherited from the Gröbner fan of the (d,n)-Plücker ideal. Yet it is still quite unclear how these two fan structures are related. Here we obtain a new structural result by showing that, via split matroids, one can construct very many non-realizable tropical linear spaces which correspond to rays of the Dressian (Theorem 2.45). It was previously unknown if *any* such ray exists. The Dressian rays correspond to those tropical linear spaces which are most degenerate. Once they are known it is "only" necessary to determine the common refinements among them to describe the entire Dressians. In this way the rays yield a condensed form of encoding. It is worth noting that the Dressians have far fewer rays than maximal cones. For instance, $\mathrm{Dr}(3,8)$ has 4748 maximal cones but only twelve rays, up to symmetry [HJS12, Theorem 31].

2.2 Matroid polytopes and their facets

Throughout this chapter let M be a matroid of rank d with ground set $[n] = \{1, 2, \ldots, n\}$. Frequently, we use the term (d, n)-*matroid* in this situation. We quickly browse through the basic definitions; further details about matroid theory can be found in the books of Oxley [Oxl11] and White [Whi86]. We use the notation of Oxley [Oxl11] for specific matroids and operations. The matroid M is defined by its *bases*. They are d-element subsets of $[n]$ which satisfy an abstract version of the basis exchange condition from linear algebra. Subsets of bases are called *independent*, and a dependent set which is minimal with respect to inclusion is a *circuit*. An element $e \in [n]$ is a *loop* if it is not contained in any basis, and it is a *coloop* if it is contained in all the bases. Let S be a subset of $[n]$. Its *rank*, denoted by $\mathrm{rk}(S)$, is the maximal size of an independent set contained in S. The set S is a *flat* if for all $e \in [n] - S$ we have $\mathrm{rk}(S + e) = \mathrm{rk}(S) + 1$. The entire ground set and, in the case of loop-freeness, also the empty set are flats; the other flats are called *proper flats*. The set of flats of M, partially ordered by inclusion, forms a geometric lattice, the *lattice of flats*. The matroid M is *connected* if there is no *separator set* $S \subsetneq [n]$ with $\mathrm{rk}(S) + \mathrm{rk}([n] - S) = d$. A connected matroid with at least two elements does not have any loops or coloops. A disconnected (d, n)-matroid decomposes in a *direct sum* of an (r, m)-matroid M' and a rank $d - r$ matroid M'' on $\{m + 1, \ldots, n\}$, i.e., a basis is the union of a basis of M and a basis of N. We write $M' \oplus M''$ for the direct sum.

For a flat F of rank r we define the *restriction* $M|F$ of F with respect to M as the matroid on the ground set F whose bases are the sets in the collection

$$\{\sigma \cap F \mid \sigma \text{ basis of } M \text{ and } \#(\sigma \cap F) = r\} \quad .$$

Dually, the *contraction* M/F of F with respect to M is the matroid on the ground set $[n] - F$ whose bases are given by

$$\{\sigma - F \mid \sigma \text{ basis of } M \text{ and } \#(\sigma \cap F) = r\} \quad .$$

The restriction $M|F$ is a matroid of rank r, while the contraction M/F is a matroid on the complement of rank $d - r$.

Via its characteristic function on the elements, a basis of M can be read as a 0/1-vector of length n with exactly d ones. The joint convex hull of all such points in $\mathbb{R}^n$ is the *matroid polytope* $P(M)$ of M. A basic reference to polytope theory is Ziegler's book [Zie00]. It is immediate that the matroid polytope of any (d, n)-matroid is contained in the $(n-1)$-dimensional simplex

$$\Delta \;=\; \left\{ x \in \mathbb{R}^n \;\middle|\; x_1 \geq 0,\, x_2 \geq 0,\, \ldots,\, x_n \geq 0,\, \sum_{i=1}^{n} x_i = d \right\} \quad .$$

Combinatorial properties of M directly translate into geometric properties of $P(M)$ and vice versa. For instance, Edmonds [Edm70, (8) and (9)] gave the exterior description

$$P(M) \;=\; \left\{ x \in \Delta \;\middle|\; \sum_{i \in F} x_i \leq \mathrm{rk}(F), \text{ where } F \text{ ranges over all flats} \right\} \tag{2.1}$$

of the matroid polytope $P(M)$ in terms of the flats. The set

$$P_M(F) \ := \ \left\{ x \in P(M) \ \middle| \ \sum_{i \in F} x_i = \mathrm{rk}(F) \right\}$$

is the face of $P(M)$ defined by the flat F. Clearly, some flats lead to redundant inequalities. A *flacet* of M is a flat which defines a facet of $P(M)$ and which is minimal with respect to inclusion among all flats that define the same facet.

They have been characterized in purely combinatorial terms by Fujishige [Fuj84, Theorems 3.2 and 3.4] and, independently, by Feichtner and Sturmfels [FS05, Propositions 2.4 and 2.6] as follows.

Proposition 2.1.

(i) The dimension of $P(M)$ equals n minus the number of connected components of M.

(ii) A proper flat F whose restriction $M|F$ and contraction M/F both are connected is a flacet of M.

(iii) For each proper flat F we have

$$P_M(F) = P(M|F) \times P(M/F) = P(M|F \oplus M/F) \ .$$

Remark 2.2. Proposition 2.1(ii) characterizes the flacets of a connected matroid. For a disconnected matroid the notion of a flacet is somewhat subtle. First, in the disconnected case there are proper hyperplanes which contain the entire matroid polytope. Such a hyperplane is not facet defining, and the corresponding flat is not a flacet. Second, for any given facet the defining inequality is never unique. In our definition we choose a specific representative by picking the inclusion minimal flat. If a flat is a direct sum $F \oplus G$, then $P_M(F \oplus G)$ is the intersection of the two faces $P_M(F)$ and $P_M(G)$. In particular, the restriction to a flacet is always connected, while the contraction is not.

The *hypersimplex* $\Delta(d, n)$ is the matroid polytope of the uniform matroid $U_{d,n}$ of rank d on n elements. Its vertices are all the 0/1-vectors of length n with exactly d ones. As $\Delta(d, n)$ is the intersection of the unit cube $[0, 1]^n$ with the hyperplane $\sum x_i = d$, the $2n$ facets of $[0, 1]^n$ give rise to a facet description for $\Delta(d, n)$. In this case the flacets are the n flats with one element. The matroid polytope of any (d, n)-matroid is a subpolytope of $\Delta(d, n)$. The following converse, obtained by Gel′fand, Goresky, MacPherson and Serganova, is a fundamental characterization. The vertex-edge graph of the (d, n)-hypersimplex is called the *Johnson graph* $J(d, n)$. This is a $[d \cdot (n - d)]$-regular undirected graph with $\binom{n}{d}$ nodes; each of its edges corresponds to the exchange of two bits.

Proposition 2.3 ([GGMS87, Theorem 4.1])**.** *A subpolytope P of $\Delta(d, n)$ is a matroid polytope if and only if the vertex-edge graph of P is a subgraph of the Johnson graph $J(d, n)$.*

In the subsequent sections we will be interested in polytopal subdivisions of hypersimplices and, more generally, arbitrary matroid polytopes. The following concept is at the heart of our deliberations. A *split* of a polytope P is a polytopal subdivision Σ of P with exactly two maximal cells. The two maximal cells share a common codimension-1 cell, and its affine span is the *split hyperplane* of Σ.

Proposition 2.4 ([HJ08, Lemma 5.1])**.** *For any proper non-empty subset $S \subsetneq [n]$ and any positive integer $\mu < d$ with $d - \#S < \mu < n - \#S$ the (S, μ)-*hyperplane *equation*

$$\mu \sum_{i \in S} x_i \;=\; (d-\mu) \sum_{j \notin S} x_j \tag{2.2}$$

defines a split of $\Delta(d,n)$. Conversely, each split of $\Delta(d,n)$ arises in this way.

The split equation above is given in its homogeneous form. Since the hypersimplices are not full-dimensional this can be rewritten in many ways. For instance, taking $\sum_i x_i = d$ into account yields the inhomogeneous equation

$$\sum_{i \in S} x_i \;=\; d - \mu \;\;, \tag{2.3}$$

which is equivalent to (2.2). Note that (2.3) has a similar shape as the inequalities in the exterior description (2.1) of the matroid polytopes. A direct computation shows that the intersection of $\Delta(d,n)$ with the (S, μ)-hyperplane is the product of hypersimplices

$$\Delta(d-\mu, S) \times \Delta(\mu, [n]-S) \;\;, \tag{2.4}$$

where we use a complementary pair of subsets of $[n]$ (instead of cardinalities) in the second arguments of the hypersimplex notation to fix the embedding into $\Delta(d,n)$ as a subpolytope.

Remark 2.5. By [HJ08, Observation 3.1] a hyperplane H which separates an arbitrary polytope P defines a split of P if and only if H does not intersect any edge of P in its relative interior: Clearly, if H separates any edge of P, it does not define a subdivision of P without new vertices. Conversely, if no edge of P gets separated, then H induces a split with the two maximal cells $P \cap H^+$ and $P \cap H^-$, where H^+ and H^- are the two affine halfspaces defined by H. In view of Proposition 2.3 we conclude that the (maximal) cells of any split of a hypersimplex form matroid polytopes. See also [HJJS09, Proposition 3.4].

We want to express Proposition 2.4 in terms of matroids and their flats.

Lemma 2.6. *Let F be a proper flat such that $0 < \operatorname{rk}(F) < \#F$. If there is an element e in $[n] - F$ which is not a coloop, then the $(F, d - \operatorname{rk}(F))$-hyperplane defines a split of $\Delta(d,n)$. In this case the intersection of $\Delta(d,n)$ with that split hyperplane equals*

$$\Delta(\operatorname{rk}(F), F) \times \Delta(d - \operatorname{rk}(F), [n]-F) \;\;,$$

and, in particular, the face $P_M(F) = P(M|F) \times P(M/F)$ is the intersection of $P(M)$ with the split hyperplane.

Proof. Pick an element $e \in [n]$ in the complement of F which is not a coloop. This yields $\mathrm{rk}([n]-e) = d$, whence the submodularity of the rank function implies

$$\begin{aligned} \#F - \mathrm{rk}(F) &\leq \#F - \mathrm{rk}(F) + \#([n]-(F+e)) - \mathrm{rk}([n]-(F+e)) \\ &\leq \#([n]-e) - \mathrm{rk}([n]-e) \\ &= n-1-d \ . \end{aligned}$$

With our assumption $0 < \mathrm{rk}(F) < \#F$ we obtain

$$d - \#F \ < \ d - \mathrm{rk}(F) \ \leq \ n - \#F - 1 \ ,$$

which is precisely the condition in Proposition 2.4 for $S = F$ and $\mu = d - \mathrm{rk}(F)$. This means that the $(F, d-\mathrm{rk}(F))$-hyperplane defines a split of $\Delta(d,n)$. The intersection with $\Delta(d,n)$ can be read off from (2.4). □

The value $d - \mathrm{rk}(F)$ is determined by the flat F, whence we will shorten the notation of $(F, d-\mathrm{rk}(F))$-hyperplane to *F-hyperplane*. Throughout the rest of this chapter we will assume that $n \geq 2$, i.e., M has at least two elements. If M is additionally connected, this forces that M does not have any loops or coloops. The relevance of the previous lemma for the investigation of matroid polytopes stems from the following observation.

Proposition 2.7. *Suppose that M is connected. Each facet of $P(M)$ is defined by the F-hyperplane for some flat F with $0 < \mathrm{rk}(F) < \#F$, or it is induced by one of the hypersimplex facets. In particular, the facets of $P(M)$ are either induced by hypersimplex splits or hypersimplex facets.*

Proof. Consider an arbitrary facet Φ of the polytope $P(M)$. From (2.1) we know that Φ is either induced by an inequality of the form $\sum_{i\in F} x_i \leq \mathrm{rk}(F)$ for some flat F of M, or Φ corresponds to one of the non-negativity constraints. The latter yield hypersimplex facets, and the same also holds for the singleton flats. We are left with the case where F has at least two elements.

The connectivity implies that M has no coloops, as we assumed that M has at least two elements. Suppose that $\mathrm{rk}(F) = \#F$. Then the restriction $M|F$ to the flat consists of coloops and thus is disconnected. Since M is connected, this implies that the hyperplane $\sum_{i\in F} x_i = \mathrm{rk}(F)$ cuts out a face of codimension higher than one. A similar argument works if $\mathrm{rk}(F) = 0$ as in this case the contraction M/F is disconnected. We conclude that $0 < \mathrm{rk}(F) < \#F$. Now the claim follows from Lemma 2.6. □

We call a flacet F a *split flacet* if the *F-hyperplane* is a split of $\Delta(d,n)$. Notice that Lemma 2.6 explains this notion in matroid terms.

Example 2.8. Let $\mathcal{S}$ be the matroid on $n = 6$ elements and rank $d = 2$, with the three non-bases 12, 34 and 56; i.e., $\mathcal{S}$ has exactly twelve bases. We call this matroid the *snowflake matroid* for its relationship with the snowflake tree discussed in Example 2.29 below. The pairs 12, 34 and 56 form flats of rank one. The matroid polytope $P(\mathcal{S})$ has nine facets: the six non-negativity constraints $x_i \geq 0$, together with $x_1 + x_2 \leq 1$, $x_3 + x_4 \leq 1$ and $x_5 + x_6 \leq 1$. These are split flacets, written as in (2.3).

Two splits of a polytope P are *compatible* if their split hyperplanes do not meet in a relatively interior point of P.

Consider a (d, n)-matroid M, with k connected components $C_1, \ldots, C_k$. The affine hull of the matroid polytope $P(M)$ is a $(n-k)$-dimensional affine space, that intersects the unit cube $[0, 1]^n$ in the direct product of hypersimplices

$$\Delta(\mathrm{rk}(C_1), C_1) \times \cdots \times \Delta(\mathrm{rk}(C_k), C_k) \ .$$

This is a matroid polytope which plays a major role in the next definition as well as in Section 3.3. Note that for a connected (d, n)-matroid the above product is the hypersimplex $\Delta(d, n)$.

Definition 2.9. The (d, n)-matroid M is a *split matroid* if its split flacets form a compatible system of splits of the affine hull of $P(M)$ intersected with the unit cube $[0, 1]^n$.

The matroid polytopes of the (d, n)-split matroids are exactly those whose faces of codimension at least two are contained in the boundary of the (d, n)-hypersimplex. The notion of a split matroid is a bit subtle in the disconnected case which we will look into next. See also Proposition 2.15 (which characterizes the connected components of a split matroid) and Example 2.17 below.

Lemma 2.10. *Let M be a split matroid which is disconnected. Then each connected component of M is a split matroid, too.*

Proof. Let C be some connected component of the (d, n)-matroid M. Assume that $M|C$ has $n' = \#C$ elements and rank d'. Let F and G be two distinct split flacets of the connected matroid $M|C$. Notice that this can only happen if $M|C$ is not uniform. Now F is a flat of M, and Lemma 2.6 gives us the F-hyperplane H_F which yields a split of $\Delta(d, n)$ and a valid inequality of $P(M)$. Let us denote by $H(M)$ the intersection of the affine hull of $P(M)$ with the unit cube and for other matroids respectively. Notice that we may assume that $D = [n] - C$ contains an element which is not a coloop. We have $H(M) = H(M|C) \times H(M|D) = \Delta(d', C) \times H(M|D) \subseteq \Delta(d, n)$ and hence

$$\begin{aligned} H_F \cap H(M) &= \Delta(\mathrm{rk}(F), F) \times \Delta(d - \mathrm{rk}(F), [n] - F) \cap \Delta(d', C) \times H(M|D) \\ &= \Delta(\mathrm{rk}(F), F) \times \Delta(d' - \mathrm{rk}(F), C - F) \times H(M|D) \ . \end{aligned} \tag{2.5}$$

That intersection contains an interior point of $H(M)$, which is why this defines a facet of $P(M)$. By construction this defines a split flacet of M. The same applies to G, yielding another split hyperplane H_G, which also yields a split flacet of M. Since M is a split matroid these two split flacets of M are compatible. The explicit description in (2.5) shows that the split flacets F and G of $M|C$ are compatible, too. We conclude that $M|C$ is a split matroid. □

We conclude that it suffices to analyze those split matroids which are connected. The following characterization of split matroids does not require any reference to splits of a polytope, since a flacet of a connected matroid is non-split if and only if it is of size one.

Theorem 2.11. *Let M be a connected matroid. The matroid M is a split matroid if and only if for each split flacet F the restriction $M|F$ and the contraction M/F both are uniform.*

Proof. Assume that M is a split matroid and F is a split flacet. Let r be the rank of F. As F does not correspond to a hypersimplex facet we know that $r < d$. Hence F is not the entire ground set $[n]$. In particular, all conditions for Lemma 2.6 are satisfied. Moreover, the intersection of any two facets of the matroid polytope $P(M)$ is contained in the boundary of the hypersimplex $\Delta(d, n)$. This implies that the intersection of the split hyperplane of F with $P(M)$ coincides with the intersection of that hyperplane with $\Delta(d, n)$. By Lemma 2.6 we have that $M|F$ is the uniform matroid of rank r on the set F, and M/F is the uniform matroid of rank $d - r$ on the set $[n] - F$.

To prove the converse, let F and G be two distinct split flacets of M with uniform restrictions and contractions. We need to show that the hypersimplex splits corresponding to F and G are compatible. By Proposition 2.1(iii) and Lemma 2.6 we have

$$P_M(F) \;=\; P(M|F) \times P(M/F) \;=\; \Delta(\mathrm{rk}(F), F) \times \Delta(d - \mathrm{rk}(F), [n] - F)\ . \tag{2.6}$$

This implies that $P_M(F)$ is exactly the intersection of the F-hyperplane with $\Delta(d, n)$. In particular, since the G-hyperplane is a valid inequality for $P_M(F)$, the F- and G-hyperplanes do not share any points in the relative interior of $\Delta(d, n)$. This means that the corresponding hypersimplex splits are compatible. □

Remark 2.12. Equation (2.6) says that the face $P_M(F)$ corresponding to a flacet F of a split matroid is the matroid polytope of a partition matroid, i.e., a direct sum of uniform matroids.

A flat is called *cyclic* if it is a union of circuits. This notion gives rise to yet another cryptomorphic way of defining matroids; see [BdM08, Theorem 3.2]. A matroid whose cyclic flats form a chain with respect to inclusion is called *nested*. Such matroids will play a role in Section 2.4 below.

Proposition 2.13. *Each flacet F of a matroid M with at least two elements is a cyclic flat. This property holds even if M is not connected.*

Proof. Let F be a flacet of M. The restriction $M|F$ is connected, even if M itself is not connected; see also Remark 2.2. Thus for each $e \in F$ there exists a circuit $e \in C \subseteq F$ in $M|F$ that connects e with another element of F. This circuit of $M|F$ is a minimal dependent set in M. Hence F is a cyclic flat. □

The compatibility relation among the hypersimplex splits was completely described in [HJ08, Proposition 5.4]. The following is a direct consequence. Notice that this characterization of split compatibility is a tightening of the submodularity property of the rank function.

Proposition 2.14. *Assume that M is connected. Let F and G be two distinct split flacets. The splits obtained from the F- and the G-hyperplane are compatible if and only if*

$$\#(F \cap G) + d \ \leq \ \mathrm{rk}(F) + \mathrm{rk}(G) \ .$$

For instance, this condition is satisfied if $F \cap G$ is an independent set and $F + G$ contains a basis.

Proof. The F- and the G-hyperplane both define splits. [HJ08, Proposition 5.4] states that two splits are compatible if and only if one of the following four inequalities holds.

$$\begin{aligned} \#(F \cap G) \ &\leq \ \mathrm{rk}(F) + \mathrm{rk}(G) - d \\ \#(F - G) \ &\leq \ \mathrm{rk}(F) - \mathrm{rk}(G) \\ \#(G - F) \ &\leq \ \mathrm{rk}(G) - \mathrm{rk}(F) \\ \#([n] - F - G) \ &\leq \ d - \mathrm{rk}(F) - \mathrm{rk}(G) \end{aligned}$$

We will show that the last three conditions never hold for a connected matroid.

We denote by $H \subseteq F \cap G$ the inclusion maximal cyclic flat that is contained in $F \cap G$. Then $c := \#(F \cap G) - H$ is the number of coloops in $M|(F \cap G)$. By Proposition 2.13 the flacet F is a cyclic flat, too. Now [BdM08, Theorem 3.2] implies that

$$\#(F-G) \ = \ \#(F-H)-c \ > \ \mathrm{rk}(F)-\mathrm{rk}(H)-c \ = \ \mathrm{rk}(F)-\mathrm{rk}(G \cap F) \ \geq \ \mathrm{rk}(F)-\mathrm{rk}(G) \ .$$

Similarly we get $\#(G - F) > \mathrm{rk}(G) - \mathrm{rk}(F)$. The submodularity of the rank function yields

$$\begin{aligned} \#([n] - (F + G)) + \mathrm{rk}(F) + \mathrm{rk}(G) &\geq \mathrm{rk}([n] - (F + G)) + \mathrm{rk}(F + G) + \mathrm{rk}(F \cap G) \\ &\geq \mathrm{rk}([n]) + \mathrm{rk}(F \cap G) \\ &\geq d \ . \end{aligned}$$

The above equality holds if and only if the matroid is the direct sum $F \oplus G \oplus ([n] - (F + G))$ and the set $[n] - (F + G)$ consists of coloops.

If $F \cap G$ is independent and $F + G$ has full rank d, we have

$$\#(F \cap G) \;=\; \mathrm{rk}(F \cap G) + \underbrace{\mathrm{rk}(F + G) - d}_{=\,0} \;\leq\; \mathrm{rk}(F) + \mathrm{rk}(G) - d \;. \tag{2.7}$$

□

Proposition 2.15. *A matroid M is a split matroid if and only if at most one connected component is a non-uniform split matroid and all other connected components are uniform.*

Proof. We only need to discuss the case that M is disconnected. First assume that M is a direct sum of uniform matroids and at most one non-uniform split matroid $M|C$. Let F and G be a split flacets of M. By assumption the F-hyperplane does not separate the matroid polytope of any of the uniform matroids. Hence F is a flacet of $M|C$. Similarly is G a flacet of the split matroid $M|C$. In particular, the intersection of the F-hyperplane with the G-hyperplane restricted to $P(M|C)$ contains no interior point of $P(M|C)$. This implies that the intersection of the F-hyperplane with the G-hyperplane contains no interior point of $P(M) = P(M|C) \times P(M/C)$.

Now assume that M is a disconnected split (d, n)-matroid. From Lemma 2.10 we know that each connected component is a split matroid. Let C_1, C_2 be two connected components of M, and let F, G be split flacets of C_1 and C_2, respectively. These split flacets exist if and only if neither $M|C_1$ nor $M|C_2$ is uniform. Let $x_F \in P(M|C_1)$ be a point on the relative interior of the facet defined by $\sum_{i \in F} x_i = \mathrm{rk}(F)$. Similarly, let $x_G \in P(M|C_2)$ be a point on the relative interior of the facet defined by G. Finally, let x_H be a point in the relative interior of $P(M/(C_1 + C_2))$.

We have seen in Lemma 2.10 that the F-hyperplane is a facet of $P(M)$. Hence F is a flacet of M, and G is similar. By construction the point $(x_F, x_G, x_H) \in P(M|C_1) \times P(M|C_2) \times P(M/(C_1 + C_2))$ lies in the interior of $P(M)$ as well as on the F- and G-hyperplanes. We conclude that the flacets F and G are incompatible. Since this cannot happen in a split matroid, we may conclude that either $M|C_1$ or $M|C_2$ are uniform. □

Example 2.16. For instance, the direct sum of the $(2, 4)$-matroid with five bases, which is a split matroid, with an isomorphic copy is not a split matroid.

Example 2.17. The 12-, 34- and the 56-hyperplanes, corresponding to the split flacets of the snowflake matroid $\mathcal{S}$ from Example 2.8 are pairwise compatible. For instance, we have $\#(\{1, 2\} \cap \{3, 4\}) = 0 \leq 1 + 1 - 2$. This shows that the snowflake matroid is a split matroid; see also Figure 2.1a below. Note that the direct sum of the snowflake matroid with a coloop $U_{1,1}$ is a split matroid, too. In particular, the 12- and 34-hyperplanes do not intersect in the interior of $\Delta(2, 6) \times \Delta(1, 1)$. However, they do intersect in the interior of $\Delta(3, 7)$, as $\#(\{1, 2\} \cap \{3, 4\}) = 0 > 1 + 1 - 3$ shows.

Example 2.18. For a different kind of example consider the $(3,6)$-matroid with the eight non-bases 134, 234, 345, 346, 156, 256, 356 and 456. This matroid has exactly the two flacets 34 and 56. The 34- and the 56-hyperplanes are not compatible. Hence this is not a split matroid.

A rank-d matroid whose circuits have either d or $d+1$ elements is a *paving* matroid. It is conjectured that asymptotically almost all matroids are paving; see [Oxl11, Conjecture 15.5.10] and [MNWW11, Conjecture 1.6]. A paving matroid whose dual is also paving is called *sparse paving*. It is known that a matroid is paving if and only if there is no minor isomorphic to the direct sum of the uniform matroid $U_{2,2}$ and $U_{0,1}$; see [Oxl11, page 126]. The following is a geometric characterization of the paving matroids.

Theorem 2.19. *Suppose that the (d,n)-matroid M is connected. Then M is paving if and only if it is a split matroid such that each split flacet has rank $d-1$.*

Proof. Let M be paving, and let F be a split flacet. Then F is a corank-1 flat of M, i.e., F is a proper flat of maximal rank $d-1$. Since there are no circuits with fewer than d elements, the restriction $M|F$ is a uniform matroid of rank $d-1$. The contraction M/F is a loop-free matroid of rank 1, and thus uniform. By Theorem 2.11 we find that M is a split matroid, and each split flacet of M has rank $d-1$.

Conversely, let M be a matroid such that the split flacets correspond to a compatible system of splits of $\Delta(d,n)$ such that, moreover, each split flacet is of rank $d-1$. Let F be such a split flacet. Then, by Lemma 2.6 we have $P_M(F) = \Delta(d-1,F) \times \Delta(1,[n]-F)$. It follows that the restriction $M|F$ does not have a circuit with fewer than d elements.

Now consider a set C of size $d-1$ or less which is contained in no split flacet, and let $D \subseteq [n]-C$ be some set of size $d-\#C$ in the complement of C. Let $\bar{x} = e_{C\cup D} = \sum_{i\in C\cup D} e_i$. Then, for any flacet F, we have

$$\sum_{i\in F} \bar{x}_i \;=\; \sum_{i\in F\cap C} \bar{x}_i + \sum_{i\in F\cap D} \bar{x}_i \;<\; \#C + d - \#C \;=\; d \ ,$$

as C is not contained in F. This shows that $\bar{x}$ satisfies the flacet inequality $\sum_{i\in F} x_i \leq d-1$. Further, the inequalities imposed by the hypersimplex facets also hold, and so $\bar{x}$ is contained in $P(M)$. Since $\bar{x} = e_{C+D}$ is a vertex of $\Delta(d,n)$ it follows that it must also be a vertex of the subpolytope $P(M)$. Therefore, $C+D$ is a basis of M, whence C is an independent set. We conclude that M does not have any circuit with fewer than d elements. Any circuit of a rank-d matroid with more than d elements has exactly $d+1$ elements. This is why M is a paving matroid. □

Remark 2.20. Each split flacet of a paving matroid M corresponds to a partition matroid, or more precisely the facets that are supported by a split-hyperplane are matroid polytopes of partition matroids. Moreover, the split flacets are exactly the corank-1 flats of M that contain a circuit; see also Remark 2.12. In this way, the split flacets of a paving matroid implicitly occur in the matroid literature, e.g., in the proof of [Oxl11, Proposition 2.1.24].

We want to look into a construction which yields very many split matroids. Let σ be some d-element subset of $[n]$. That is, σ is a basis of the uniform matroid $U_{d,n}$, and $e_\sigma = \sum_{i\in\sigma} e_i$ is a vertex of $\Delta(d,n)$. Its neighbors in the Johnson graph $J(d,n)$ lie on the $(\sigma, d-1)$-hyperplane in $\Delta(d,n)$. More precisely, from (2.4) we can see that the convex hull of the neighbors of e_σ equals

$$\Delta(d-1,\sigma) \times \Delta(1,[n]-\sigma) \ ,$$

which is the product of a $(d-1)$-simplex and an $(n-d-1)$-simplex. The resulting split is called the *vertex split* with respect to σ or e_σ. Two vertex splits are compatible if and only if the two vertices do not span an edge. In this way the compatible systems of vertex splits of $\Delta(d,n)$ bijectively correspond to the stable sets in the Johnson graph $J(d,n)$. The following observation is similar to [BPvdP15, Lemma 8].

Corollary 2.21. *Again let M be a (d,n)-matroid which is connected. Then M is sparse paving if and only if the conclusion of Theorem 2.19 holds and additionally the splits are vertex splits.*

Proof. For each rank $d-1$ split flacet F of the split matroid M we have $M/F = U_{1,[n]-F}$ and $M|F = U_{d-1,F}$. The dual of M is a matroid of rank $n-d$ on n elements. The matroid polytope $P(M^*)$ is the image of $P(M)$ under coordinate-wise transformation $x_i \mapsto 1-x_i$. It follows that the split flacets of M^* are the complements of the split flacets of M. Thus, for the split flacet $[n]-F$ in M^* we obtain

$$\begin{aligned} M^*|([n]-F) \ &= \ (M/F)^* \ = \ U^*_{1,[n]-F} \ = \ U_{n-\#F-1,[n]-F} \quad \text{and} \\ M^*/([n]-F) \ &= \ (M|F)^* \ = \ U^*_{d-1,F} \ = \ U_{\#F-d+1,F} \ . \end{aligned}$$

This implies that M^* is paving if and only if each split flacet F has cardinality d. □

The following two examples illustrate the differences between paving and split matroids. The class of split matroids is strictly larger. In contrast to the class of paving matroids the class of split matroids is closed under dualization.

Example 2.22. The $(\{1,2,3,4\},2)$-hyperplane yields a split of the hypersimplex $\Delta(4,8)$. The two maximal cells correspond to split matroids which are not paving nor are their duals.

Yet there are still plenty of matroids which are not split.

Example 2.23. Up to symmetry there are 15 connected matroids of rank three on six elements. Among these there are exactly four which are non-split. One such example is the nested matroid given by the columns of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & \lambda \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \ ,$$

where $\lambda \neq 0,1$. This matroid is realizable over any field with more than two elements.

Knuth gave the following construction for stable sets in Johnson graphs [Knu74]. Due to Corollary 2.21 this is the same as a compatible set of vertex splits, which arise from the split flacets of a sparse paving matroid.

Example 2.24. The function

$$(x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} i \cdot x_i \mod n$$

defines a proper coloring of the nodes of $J(d, n)$ with n colors. Each color class forms a stable set, and there must be at least one stable set of size at least $\frac{1}{n}\binom{n}{d}$.

For special choices of d and n larger stable sets in $J(d, n)$ are known.

Example 2.25. Identifying a natural number between 0 and $2^k - 1$ with its binary representation yields a 0/1-vector of length k. All quadruples of such vectors that sum up to 0 modulo 2 form a stable set S in $J(4, 2^k)$ of size $n(n-1)(n-2)/24$, where $n = 2^k$. Fixing one vector and restricting to those quadruples in S which contain that vector gives a stable set of size $(n-1)(n-2)/6$ in $J(3, 2^k - 1)$. The latter construction also occurs in [Duk04, Theorem 3.1] and [HJJS09, Theorem 3.6].

In a way, the sparse paving matroids are those split matroids which are the easiest to get at. We sum up our discussion in the following characterization.

Theorem 2.26. *The following collections of sets are in bijection with one another:*

(i) The split flacets of sparse paving connected matroids of rank d on n elements,

(ii) the cyclic flats of sparse paving connected matroids of rank d on n elements,

(iii) the sets of compatible vertex splits of $\Delta(d, n)$,

(iv) the stable sets of the graph $J(d, n)$,

(v) the sets of binary vectors of length n with constant weight d and Hamming distance at least 4.

Proof. Each split flacet of a sparse paving matroid M is a cyclic flat by Proposition 2.13. The proof of Theorem 2.19 shows that cyclic flats of rank $d-1$ are split flacets of M. Further, the cyclic flats of a connected paving matroid are those of rank $d-1$, the empty set and the entire ground set $[n]$. This establishes that (i) and (ii) are equivalent.

Corollary 2.21 is exactly the equivalence of (i) and (iii).

By Proposition 2.14 two vertex splits of $\Delta(d, n)$ are compatible if and only if the two vertices do not span an edge. The compatible systems of vertex splits of $\Delta(d, n)$ bijectively correspond to the stable sets in the vertex-edge graph of $\Delta(d, n)$, which is the Johnson graph $J(d, n)$. This means that (iii) is equivalent to (iv).

The vertices of the hypersimplex $\Delta(d, n)$ are all binary vectors of length n with constant weight d. The Hamming distance of two such vectors v and w is the number of coordinates where $v_i \neq w_i$. This number is twice the distance of the vertices in the Johnson graph $J(d, n)$. Note that odd numbers do not occur as Hamming distances. Hamming distance at least 4 means that the vertices are not adjacent in $J(d, n)$. This yields the equivalence of (iv) and (v). $\square$

A table with lower bounds on the maximal size of such a set for $n \leq 28$ is given in [BSSS90, Table I-A]. Notice that this data also gives lower bounds on the total number of (d, n)-matroids; see, e.g., [BPvdP15].

2.3 Matroid subdivisions and tropical linear spaces

In this section we want to exploit the structural information that we gathered about split matroids to derive new results about tropical linear spaces, the tropical Grassmannians and the related Dressians [SS04, HJJS09]. We begin with some basics on general polyhedral subdivisions; see [DLRS10] for further details.

Let P be some polytope. A polytopal subdivision of P is *regular* if it is induced by a lifting function on the vertices of P. Examples are given by the Delaunay subdivisions where the lifting function is the Euclidean norm squared. The lifting functions on P which induce the same polytopal subdivision, Σ, form a relatively open polyhedral cone, the *secondary cone* of Σ. The *secondary fan* of P comprises all secondary cones. The inclusion relation on the closures of the secondary cones of P imposes a partial ordering. This is dual to the set of regular polytopal subdivisions of P partially ordered by refinement. The secondary fan has a non-trivial lineality space which accounts for the various choices of affine bases. Usually we will ignore these linealities. In particular, whenever we talk about dimensions we refer to the dimension of a secondary fan modulo its linealities.

A *tropical Plücker vector* $\pi \in \mathbb{R}^{\binom{n}{d}}$ is a lifting function on the vertices of the hypersimplex $\Delta(d, n)$ such that the regular subdivision induced by π is a *matroid subdivision*, i.e., each of its cells is a matroid polytope. The cells of the dual of a matroid subdivision that correspond to loop-free matroid polytopes form a subcomplex. This subcomplex of that matroid subdivision is the *tropical linear space* defined by π. The *Dressian* $\mathrm{Dr}(d, n)$ is the subfan of the secondary fan of the hypersimplex $\Delta(d, n)$ comprising the tropical Plücker vectors. According to Remark 2.5 each split of a hypersimplex is a regular matroid subdivision and hence it defines a ray of the corresponding Dressian.

Let M be a (d, n)-matroid. The matroid polytope $P(M)$ is a subpolytope of $\Delta(d, n)$. Restricting the tropical Plücker vectors to vertices of $P(M)$ and looking at regular subdivisions of $P(M)$ into matroid polytopes gives rise to the *Dressian* $\mathrm{Dr}(M)$ of the matroid M; see [HJJS09, Section 6]. The rank of any subset S of $[n]$ coincides with the rank of the flat spanned by S. Restricting the rank function of M to all subsets of $[n]$ of a fixed cardinality k yields the *k-rank vector* of M. The *dual-rank function* of M is the rank function of M^*, the dual matroid of M, and the *corank function* is the difference between d and the rank function. The *k-corank vector* of M is the map

$$\rho_k(M) : \binom{[n]}{k} \to \mathbb{N}, \ S \mapsto d - \mathrm{rk}_M(S) \ .$$

The regular subdivision of $\Delta(k,n)$ with lifting function $\rho_k(M)$ is the *k-corank subdivision* induced by the matroid M. Usually we will omit the size k in those definitions if k equals d. The following known result says that the k-corank subdivision is a matroid subdivision.

Lemma 2.27. *The k-corank vector $\rho_k(M)$ of the (d,n)-matroid M is a (k,n)-tropical Plücker vector. Moreover, the matroid polytope $P(M)$ occurs as a cell in the k-corank subdivision induced by M. That cell is maximal if and only if M is connected.*

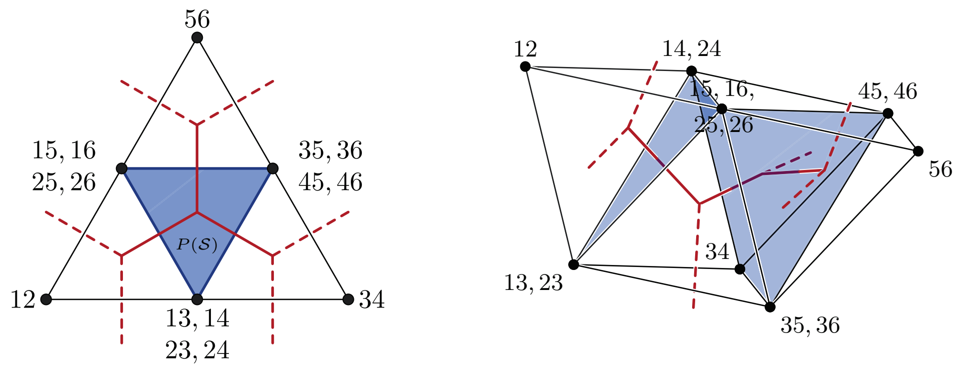
Proof. Speyer showed that $\rho_k(M)$ is a tropical Plücker vector such that the matroid polytope $P(M)$ occurs as a cell [Spe05, Proposition 4.5.5]. The dimension of that cell can be read off from Proposition 2.1. □

Example 2.28. With $d = 2$ and $n = 4$ let M be the matroid with the five bases 12, 13, 14, 23 and 24. We pick $k = d = 2$. The rank of the unique non-basis 34 equals 1, whence $\rho_2(M) = (0,0,0,0,0,1)$. The matroid subdivision induced by $\rho_2(M)$ splits the hypersimplex $\Delta(2,4)$ into two Egyptian pyramids. Every subset of $\{1,2,3,4\}$ with cardinality $k = 3$ contains a basis, and thus $\rho_3(M) = (0,0,0,0)$. There are no loops in M, whence for $k = 1$ the corank vector $\rho_1(M)$ equals $(1,1,1,1)$. Here and below the ordering of the k-subsets of $[n]$ in the corank vectors is lexicographic.

Example 2.29. The corank subdivision of the matroid $\mathcal{S}$ in Example 2.8 is a matroid subdivision of $\Delta(2,6)$ whose tropical linear space is the snowflake tree, hence the name snowflake matroid for $\mathcal{S}$. See Figure 2.1a, on the next page, for a visualization.

By Proposition 2.7 the facets of any matroid polytope are either hypersimplex facets or induced by hypersimplex splits. In the following we will be interested in the set of hypersimplex splits arising from the split flacets of a given matroid. The next result explains what happens if that matroid is a split matroid.

Proposition 2.30. *Let M be a split (d,n)-matroid which is connected. Then the corank vector $\rho(M)$ is contained in the relative interior of a simplicial cone of* $\mathrm{Dr}(d,n)$*, and the dimension of that cone is given by the number of split flacets of M. In particular, $\rho(M)$ is a ray if and only if it induces a split of $\Delta(d,n)$. This is the case if and only if M is a nested matroid with exactly three cyclic flats.*



a) Subdivision induced by snowflake matroid $\mathcal{S}$. b) Subdivision dual to caterpillar tree.

Figure 2.1: Two subdivisions of $\Delta(2, 6)$ and their tropical linear spaces.

Proof. Let H be the set of hypersimplex splits corresponding to the split flacets of M. By definition the splits in H are compatible. Since each subset of a compatible set of splits is again compatible it follows that the secondary cone spanned by H is a simplicial cone.

Recall that M is nested if the cyclic flats form a chain. The empty set and $[n]$ are two cyclic flats in any connected matroid. Assume that the matroid M is nested with precisely three cyclic flats. Then the third cyclic flat F induces the only split, since the restriction $M|F$ and the contraction $M\backslash F$ are uniform matroids.

Conversely, if the matroid M is split with a unique split flacet F, then obviously $\emptyset \subsetneq F \subsetneq [n]$. Each circuit C of M with fewer than $d + 1$ elements leads to valid inequality of the polytope $P(M)$. This inequality separates $P(M)$ from those vertices of the hypersimplex with $x_i = 1$ for $i \in C$. Hence, the only split flacet F contains the circuit C. The restriction $M|F$ is a uniform matroid and thus $\mathrm{rk}(C) = \mathrm{rk}(F)$. We get that F is the closure of C. Hence we may conclude that M is nested. $\square$

Our next result generalizes [HJJS09, Theorem 3.6], which settled the case $d = 3$.

Theorem 2.31. *For the dimension of the Dressian we have*

$$\frac{1}{n}\binom{n}{d} - 1 \;\leq\; \dim \mathrm{Dr}(d, n) \;\leq\; \binom{n-2}{d-1} - 1 \;.$$

Proof. Speyer showed that the *spread* of any matroid subdivision of the hypersimplex $\Delta(d, n)$, i.e., its number of maximal cells, does not exceed $\binom{n-2}{d-1}$ [Spe05, Theorem 3.1]. The dimension of a secondary cone of a subdivision Σ is the size of a maximal linearly independent family of coarsest subdivisions which are refined by Σ. As each (coarsest) subdivision has at least two maximal cells, the dimension of the secondary cone is at most the spread minus one. This follows from the fact that at least k (linearly independent) rays are necessary in order to generate a cone of dimension k. It follows that $\dim \mathrm{Dr}(d,n) \leq \binom{n-2}{d-1} - 1$. The lower bound is given by Knuth's construction of stable sets in $J(d,n)$; see Example 2.24. $\square$

This gives the following asymptotic estimates.

Corollary 2.32. *For fixed d the dimension of the Dressian $\mathrm{Dr}(d,n)$ is of order $\Theta(n^{d-1})$. Further, the asymptotic dimension of the Dressian $\mathrm{Dr}(d,2d)$ is bounded from below by $\Omega(4^d d^{-3/2})$ and bounded from above by $O(4^d d^{-1/2})$.*

Proof. For fixed d the lower and the upper bound in Theorem 2.31 both grow as fast as n^{d-1} asymptotically. Stirling's formula yields that the binomial coefficient $\binom{2d}{d}$ grows like $2^{2d}/\sqrt{\pi d}$. Specializing the bounds in Theorem 2.31 to $n = 2d$ thus yields

$$\Omega\left(\frac{2^{2d-1}}{d\sqrt{\pi d}}\right) \;\leq\; \dim \mathrm{Dr}(d,2d) \;\leq\; O\left(\frac{2^{2d-2}}{\sqrt{\pi(d-1)}}\right) \quad .$$

Now the lower and the upper bound differ by a multiplicative factor of

$$\frac{d\sqrt{d}}{2\sqrt{d-1}} \ ,$$

which tends to $d/2$ when d goes to infinity. $\square$

The following example shows that not all matroid subdivisions are induced by a corank function.

Example 2.33. The matroid subdivision Σ of the hypersimplex $\Delta(2,6)$ induced by the lifting vector $(3,2,1,0,0,2,1,0,0,2,1,1,2,2,3)$ is not a corank subdivision. Lemma 3.38 in Chapter 3 provides a criterion that shows this fact. We give a hint how this claim can be verified without the criterion. This subdivision Σ has exactly 4 maximal cells which come as two pairs of isomorphic cells. One can check that Σ does not agree with the corank subdivision induced by any of these maximal cells or any of the three non maximal cells in the interior of the hypersimplex. The lifting-vector is obtained from a metric caterpillar tree with six leaves and unit edge lengths; see Figure 2.1b. Notice that the subdivision Σ is realizable by a tropical point configuration, while the corank subdivision induced by the snowflake matroid $\mathcal{S}$ is not; see [HJS12].

Tropical geometry studies the images under the valuation map of algebraic varieties over fields with a discrete valuation; see, e.g., [MS15, Chapter 3]. Let $\mathbb{K}\{\!\{t\}\!\}$ be the field of formal Puiseux series over an algebraically closed field $\mathbb{K}$. The valuation map $\mathrm{val} : \mathbb{K}\{\!\{t\}\!\} \to \mathbb{R} \cup \{\infty\}$ sends a Puiseux series to the exponent of the term of lowest order. Each d-dimensional subspace in the vector space $\mathbb{K}\{\!\{t\}\!\}^n$ can be written as the column span of a $d{\times}n$-matrix A. The maximal minors of A encode that subspace as a *Plücker vector* which is a point on the *Grassmannian* $\mathrm{Gr}_{\mathbb{K}\{\{t\}\}}(d,n)$, an algebraic variety over $\mathbb{K}\{\{t\}\}$. Tropicalizing the Plücker vector of A yields a tropical Plücker vector, i.e., a point on the Dressian $\mathrm{Dr}(d,n)$. In fact, the set of all tropical Plücker vectors which arise in this way is the *tropical Grassmannian* $\mathrm{TGr}_{\mathrm{char}\,\mathbb{K}}(d,n)$. The latter is the tropical variety which comes about as the tropicalization of $\mathrm{Gr}_{\mathbb{K}\{\{t\}\}}(d,n)$, and this is a $d(n-d)$-dimensional polyhedral fan which is a proper subset of $\mathrm{Dr}(d,n)$ unless $d=2$ or $(d,n)=(3,6)$; see [SS04] and [Spe05]. The precise relationship between the fan structures of $\mathrm{TGr}_{\mathrm{char}\,\mathbb{K}}(d,n)$ and $\mathrm{Dr}(d,n)$ is a topic of ongoing research. Since the Plücker ideal which defines $\mathrm{Gr}_{\mathbb{K}\{\{t\}\}}(d,n)$, is generated by polynomials with integer coefficients, the tropical variety $\mathrm{TGr}_{\mathrm{char}\,\mathbb{K}}(d,n)$ only depends on the characteristic of the field $\mathbb{K}\{\{t\}\}$ which agrees with the characteristic of $\mathbb{K}$. The tropical Plücker vectors that lie in the tropical Grassmannian are called *realizable*. We also say that such a tropical Plücker vector can be *lifted* to an ordinary Plücker vector. The following was stated in [Spe05, Example 4.5.4]. We indicate a short proof for the sake of completeness.

Proposition 2.34. *Let π be a (d,n)-tropical Plücker vector which can be lifted to an ordinary Plücker vector over $\mathbb{K}\{\{t\}\}$. Then the cells in the subdivision of $\Delta(d,n)$ induced by π necessarily correspond to matroids which are realizable over $\mathbb{K}$.*

Proof. By our assumption there exists an ordinary Plücker vector p which valuates to π. We can pick a matrix $A\in\mathbb{K}\{\{t\}\}^{d\times n}$ such that for each d-set I of columns we have $\det A_I=p_I$. It follows that $\mathrm{val}(\det A_I)=\pi_I$. Note that the matrix A is not unique.

Let M be the matroid corresponding to a cell. Up to a linear transformation we may assume that π is non-negative, and we have $\pi_I=0$ if and only if I is a basis of M. We will show that A can be chosen such that the valuation of each entry is non-negative.

We apply Gaussian elimination to the $n\geq d$ columns of A. This way the classical Plücker vector associated with A is multiplied with a non-zero scalar. Thus the tropical Plücker vector π is modified by adding a multiple of the all-ones vector. In each step, among the possible pivots, pick one whose valuation is minimal. Let γ be the product of all pivot elements, and let $c\,t^g$ for $c\neq 0$ be the term of lowest order. By construction $g=\mathrm{val}(c\,t^g)=\mathrm{val}(\gamma)$ is a lower bound for the valuations of the minors of A, which is actually attained. Since π is non-negative and since $\pi_I=0$ if I is a basis we conclude that $g=0$.

Including possibly trivial pivots with 1 we obtain exactly d pivots, one for each row of A. Multiplying each row with the inverse of the lowest order term of the corresponding pivot does not change π. The resulting matrix A' is a realization with entries whose valuations are non-negative. Hence we can evaluate the matrix $A'\in\mathbb{K}\{\{t\}\}^{d\times n}$ at $t=0$. This gives us the matrix $B\in\mathbb{K}^{d\times n}$ with the constant terms of A'. The matrix B realizes M since $\det B_I=0$ if and only if the lowest order term of $\det A'_I$ is constant in t. $\square$

Our next goal is to prove a characterization of realizability of Plücker vectors in the Dressian in terms of matroid realizability. In the proof we will use a standard construction from matroid theory which will also reappear further below. The *free extension* of the (d,n)-matroid M by an element $f = n+1$ is the $(d,n{+}1)$-matroid which arises from M by adding f to the ground set such that it is independent from each $(d{-}1)$-element subset of $[n]$.

Theorem 2.35. *Let M be a (d,n)-matroid. The corank vector $\rho(M)$ can be lifted to an ordinary Plücker vector over $\mathbb{K}\{\!\{t\}\!\}$ if and only if M is realizable over $\mathbb{K}$.*

Proof. Let $\rho(M)$ be realizable. Since $P(M)$ occurs as a cell in the matroid subdivision induced by $\rho(M)$ the matroid M is realizable due to Proposition 2.34.

Conversely, let us assume that the matroid M is realizable and the matrix $B \in \mathbb{K}^{d\times n}$ is a full rank realization. The matrix B has only finitely many entries, and these generate some extension field $\mathbb{L}$ of the prime field of $\mathbb{K}$. The field $\mathbb{L}$ may or may not be transcendental, but it is certainly not algebraically closed. Hence there exists an element $\alpha \in \mathbb{K} - \mathbb{L}$ which is algebraic over $\mathbb{L}$ of degree at least d. The vector $(1, \alpha, \ldots, \alpha^{d-1})$ is $\mathbb{L}$-linearly independent of any $d-1$ columns of B, as these minors are degree $d-1$ polynomials in α. We infer that even the free extension of M is realizable over $\mathbb{L}(\alpha) \subsetneq \mathbb{K}$. After altogether n free extensions, and a tower of n algebraic extensions over $\mathbb{L}$, we obtain a matrix $C \in \mathbb{K}^{d\times n}$ such that the block column matrix $[B|C]$ is a realization of the n-fold free extension of M. We define $A := B + t\cdot C$ which is a $d{\times}n$-matrix with coefficients in $\mathbb{K}\{\!\{t\}\!\}$.

For any d-subset I of $[n]$ and for any subset $S \subseteq I$ we denote by $D(S) \in \mathbb{K}\{\!\{t\}\!\}^{d\times n}$ the matrix whose k-th column is the k-th column of B if $k \in S$ and t times the k-th column of C otherwise. Then

$$\det A_I \;=\; \det(B_I + t\cdot C_I) \;=\; \sum_{S\subseteq I} \det D_I(S) \;\; .$$

Further, by choice of C, we have $\det D_I(S) = 0$ if and only if S is a dependent set in M, and $\mathrm{val}(\det D_I(S)) = d - \# S$ if S is independent. For a fixed set $S \subseteq I$ the Puiseux series $\det D_I(S)$ has a term $c(S)t^{g(S)}$ of lowest order, and we have $g(S) = \mathrm{val}(\det D_I(S)) = d - \# S$. The field $\mathbb{K}$ is an $\mathbb{L}$-vector space, and the set

$$\{\, c(S) \mid S \text{ independent subset of } I \}$$

of leading coefficients is linearly independent over $\mathbb{L}$. This is why we obtain $\mathrm{val}\det A_I = d - \mathrm{rk}(I)$, i.e., cancellation does not occur. That is, the ordinary Plücker vector of the matrix A tropicalizes to $\rho(M)$. □

On the end of this section let us remark some consequences of Theorem 2.35.

Remark 2.36. Via the von Staudt constructions for addition and multiplication [vS57] it is possible to encode the solvability of a polynomial equation into the realizability of a matroid. Brylawski and Kelly constructed in [BK78] a $(3, 2p+5)$-matroid which encodes that the p-th cyclotomic polynomial $\sum_{k=0}^{p-1} x^k$ has a solution $x \neq 1$. This polynomial factorizes to $(x-1)^{p-1}$ over fields of characteristic p. Hence, this matroid is realizable over an algebraically closed field $\mathbb{K}$ if and only if $p \neq \mathrm{char}(\mathbb{K})$.

Corollary 2.37. *Given a positive integer N, there are tropical Plücker vectors that are on the tropical Grassmannian $\mathrm{TGr}_{\mathrm{char}\,\mathbb{K}}(d,n)$ if and only if d, n are large enough and* $\mathrm{char}\,\mathbb{K}$ *does not divide N.*

An example for $N = 2$ is the corank-vector of the non-Fano matroid.

The following remark describes the opposite situation.

Remark 2.38. Let $N \in \mathbb{Z}$ be a positive integer. Kahn explicitly constructed a matroid M in [Kah82] with the property, that M is realizable over an algebraically closed field $\mathbb{K}$ if and only if the characteristic of $\mathbb{K}$ is non zero and divides N. This construction can be seen as the encoding of the algebraic equations $\sum_{k=0}^{p-1} x^k = 0$ for each p that divides N together with $x = 1$. Clearly, this matroid M is automatically not realizable over a field of characteristic 0.

Corollary 2.39. *It follows from Theorem 2.35 that there are tropical Plücker vectors that are on the tropical Grassmannian $\mathrm{TGr}_{\mathrm{char}\,\mathbb{K}}(d,n)$ if and only if d, n are large enough and* $\mathrm{char}\,\mathbb{K}$ *divides N.*

An example for $N = 2$ is the corank-vector of the Fano matroid.

2.4 Rays of the Dressian

The purpose of this section is to describe a large class of tropical linear spaces, which are tropically rigid, i.e., they correspond to rays of the corresponding Dressian. Before we can define a special construction for matroids we first browse through a few standard concepts.

Let M be a connected matroid of rank d with $[n]$ as its set of elements. The *parallel extension* of M at an element $e \in [n]$ by $s \notin [n]$ is the $(d, n+1)$-matroid whose flats are either flats of M which do not contain e or sets of the form $F + s$, where F is a flat containing e. Among all connected extensions the parallel extension is the one in which the shortest length of a circuit that contains the added element is minimal. In fact, that length equals two. Similarly, the free extension is characterized by the following property: Any circuit that contains the added element has length $d + 1$, and this is the maximal length of such a circuit.

In general a single-element *extension* N of M is a $(d, n + 1)$-matroid whose deletion $N \setminus (n + 1)$ is M; see [Oxl11, Section 7.2] for more details. Similar a *coextension* of M is a matroid whose contraction is M, or in other words the dual of an extension applied to the dual matroid M^*. That is, a coextension of a (d,n)-matroid is a $(d{+}1,n{+}1)$-matroid. Finally, a *series-extension* is a parallel coextension.

Definition 2.40. The *series-free lift* of M, denoted as $\operatorname{sf} M$, is the matroid of rank $d+1$ with $n+2$ elements obtained as the series-extension of M' at f by s, where M' is the free extension of M by f.

Note that $\operatorname{sf} M$ is connected as M is connected. In the sequel we want to show that the corank subdivision of $\operatorname{sf} M$ yields a ray of the Dressian $\operatorname{Dr}(d+1,n+2)$, whenever M is a (d,n)-split matroid. Let us first determine the rank function and the bases of $\operatorname{sf} M$. We write fs as shorthand for the two-element set $f+s=\{f,s\}$.

Lemma 2.41. *The set B of size $d{+}1$ is a basis in $\operatorname{sf} M$ if and only if one of the following conditions hold:*

(i) $fs \subseteq B$ *and* $\operatorname{rk}_M(B-fs) = d-1$, *or*

(ii) $f \in B$ *and* $s \notin B$ *and* $\operatorname{rk}_M(B-f) = d$, *or*

(iii) $f \notin B$ *and* $s \in B$ *and* $\operatorname{rk}_M(B-s) = d$.

Further, the rank of $S \subseteq [n]+fs$ is given by

$$\operatorname{rk}_{\operatorname{sf} M}(S) \;=\; \min\bigl(\operatorname{rk}_M(S-fs) + \#(fs\cap S),\; d+1\bigr)\;. \tag{2.8}$$

The split flacets of $\operatorname{sf} M$ *are those of M and additionally $[n]$, the ground set of M.*

Proof. Clearly each basis in $\operatorname{sf} M$ contains at least f or s. Conversely, any basis B of M extends to a basis of $\operatorname{sf} M$ with either f or s. A circuit of the free extension M' of M by f that contains f has size $d+1$. Hence each circuit of $\operatorname{sf} M$ that contains f and s has length $d+2$. In particular, this implies that each independent set B in M of size $d-1$ together with fs forms a basis of $\operatorname{sf} M$. Any set which is dependent over M is also dependent over $\operatorname{sf} M$.

The formula for the rank function is a direct consequence of the description of the bases. We see that there is no circuit of length at most $d+1$ that contains f, s or both. Proposition 2.13 says that there is no flacet that contains f or s. Contracting the set $[n]$ in $\operatorname{sf} M$ yields the uniform matroid of rank 1 on the two-element set fs, and this is connected. For S a subset of $[n]+fs$ and any set $F \neq [n]$ that does not contain fs we have

$$\begin{aligned}
\operatorname{rk}_{(\operatorname{sf} M)/F}(S) &= \operatorname{rk}_{\operatorname{sf} M}(S+F) - \operatorname{rk}_{\operatorname{sf} M}(F)\\
&= \min\{\operatorname{rk}_M(S+F-fs) + \#(fs\cap S),\, d+1\} - \operatorname{rk}_M(F)\\
&= \min\{\operatorname{rk}_{M/F}(S-fs) + \#(fs\cap S),\, d-\operatorname{rk}_M(F)+1\}\\
&= \operatorname{rk}_{\operatorname{sf}(M/F)}(S)
\end{aligned}$$

The matroid $\mathrm{sf}(M/F) = (\mathrm{sf}\, M)/F$ is connected if and only if M/F is connected. The restriction $\mathrm{sf}(M|F)$ coincides with $M|F$. Both, the restriction and contraction on F are connected in M if and only if they are connected in $\mathrm{sf}\, M$. We conclude that the split flacets of $\mathrm{sf}\, M$ are precisely the ones in our claim. □

Our next goal is to describe the maximal cells of the corank subdivision induced by $\mathrm{sf}\, M$. To this end we first define the matroid $\mathrm{sf}^*\, M$ as the free coextension of M by f, followed by the parallel extension at f by s. We call $\mathrm{sf}^*\, M$ the *parallel-cofree lift* of M. This new construction is related to the series-free lift by the equality

$$\mathrm{sf}^*\, M \;=\; \left(\mathrm{sf}(M^*)\right)^* \;.$$

A direct computation shows that the rank function is given by

$$\begin{aligned} \mathrm{rk}_{\mathrm{sf}^*\, M}(S) \;&=\; \min\left(\mathrm{rk}_{\mathrm{sf}\, M}(S) + \#(fs - S) - 1,\; \# S\right) \\ &=\; \min\left(\mathrm{rk}_M(S - fs) + 1,\; \# S\right) \;. \end{aligned} \tag{2.9}$$

One maximal cell of the corank subdivision induced by $\mathrm{sf}\, M$ is obvious, namely the matroid polytope $P(\mathrm{sf}\, M)$. This is the case as M, and thus also $\mathrm{sf}\, M$, is connected. Here is another one.

Lemma 2.42. *The corank subdivision of* $\mathrm{sf}^*\, M$ *coincides with the corank subdivision of* $\mathrm{sf}\, M$*. Hence the matroid polytope* $P(\mathrm{sf}^*\, M)$ *is a maximal cell of the corank subdivision of* $\Delta(d+1, n+2)$ *induced by* $\mathrm{sf}\, M$*. Further, the cells* $P(\mathrm{sf}\, M)$ *and* $P(\mathrm{sf}^*\, M)$ *intersect in a common cell of codimension one.*

Proof. Let S be a subset of $[n] + fs$ of size $d+1$. We have $\mathrm{rk}_M(S - fs) \leq d$. From (2.9) we deduce that $\mathrm{rk}_{\mathrm{sf}^*\, M}(S) = \mathrm{rk}_M(S - fs) + 1 \leq d + 1 = \# S$, while Lemma 2.41 gives $\mathrm{rk}_{\mathrm{sf}\, M}(S) = \mathrm{rk}_M(S - fs) + \#(fs \cap S) \leq \#(S - fs) + \#(fs \cap S) = d + 1$. Combining these two, let us arrive at the equation $\mathrm{rk}_{\mathrm{sf}^*\, M}(S) = \mathrm{rk}_{\mathrm{sf}\, M}(S) - \#(fs \cap S) + 1$. This implies

$$\rho(\mathrm{sf}^*\, M) + 1 \;=\; \rho(\mathrm{sf}\, M) + x_f + x_s \;.$$

As a consequence the corank subdivision of $\mathrm{sf}^*\, M$ coincides with the corank subdivision of $\mathrm{sf}\, M$. The common bases of the matroids $\mathrm{sf}\, M$ and $\mathrm{sf}^*\, M$ are the bases of the direct sum $M \oplus U_{1,fg}$. The corresponding matroid polytope yields the desired cell of codimension one. □

For each split flacet F of M we let N_F be the connected $(d+1, n+2)$-matroid with elements $[n] + fs$ which has the following list of cyclic flats: $\emptyset$, $[n] - F$ of rank $d - \mathrm{rk}(F)$, $[n] - F + fs$ of rank $d + 1 - \mathrm{rk}(F)$ and $[n] + fs$ of rank $d + 1$.

Note that these sets form a chain. This chain has a rank 0 element. The ranks are strictly increasing, and for each set the rank is less than the size. Hence these sets form the cyclic flats of a matroid. Its rank function is given by $\mathrm{rk}(S) = \min\{\mathrm{rk}(G) + \#(S - G)\}$ where G ranges over all cyclic flats; see [BdM08]. Hence, the rank function of N_F satisfies

$$\begin{aligned}\mathrm{rk}_{N_F}(S) = \min\{d+1,\ \#(S \cap F) + d + 1 - \mathrm{rk}_M(F), \\ \# S,\ \#(S \cap (F + fs)) + d - \mathrm{rk}_M(F)\}\ .\end{aligned} \tag{2.10}$$

This is a nested matroid with exactly two split flacets, namely $[n] - F$ and $[n] - F + fs$. The corresponding hypersimplex splits are not compatible, i.e., N_F is not a split matroid. The following result compares the corank in $\mathrm{sf}\, M$ with the corank in N_F.

Proposition 2.43. *For each split flacet F of M and any set $S \subseteq [n]+fs$ with $\#(S) = d+1$ we have*

$$d + 1 - \mathrm{rk}_{\mathrm{sf}\, M}(S) + \mathrm{rk}_M(F) - \#(S \cap F) \ \geq\ d + 1 - \mathrm{rk}_{N_F}(S)\ . \tag{2.11}$$

Proof. Since the size of S equals $d+1$ the equation (2.10) simplifies to

$$d{+}1{-}\mathrm{rk}_{N_F}(S) \ = \ \max\{0, \mathrm{rk}_M(F){-}\#(S{\cap}F), \mathrm{rk}_M(F){+}1{-}\#(S{\cap}F){-}\#(S{\cap}fs)\} \tag{2.12}$$

if we subtract both sides from $d+1$. That expression is the corank of S in the nested matroid N_F. This corank function gives the $(d+1, n+2)$-tropical Plücker vector $\rho(N_F)$. In the sequel we will make frequent use of the inequality

$$\mathrm{rk}_M(S - fs) \ \leq\ \mathrm{rk}_M(F) + \#(S - F - fs) \ = \ \#(S - fs) - \#(S \cap F) + \mathrm{rk}_M(F)\ , \tag{2.13}$$

which is a consequence of the submodularity and monotonicity of the rank function together with $\mathrm{rk}_M(S - F - fs) \leq \#(S - F - fs)$.

To prove (2.11) we distinguish three cases. First, if neither f nor s are in S the inequality (2.11) is equivalent to

$$d + 1 - \mathrm{rk}_M(S) + \mathrm{rk}_M(F) \ \geq\ \max\{\#(S \cap F),\, \mathrm{rk}_M(F) + 1\} \tag{2.14}$$

as one add $\#(S{\cap}F)$ on both sides of (2.12). The inequality (2.14) follows from $\mathrm{rk}_M(S) \leq d$ and (2.13) with $\#(S - fs) = d + 1$.

Second, if $\#(fs \cap S) = 1$, by applying (2.8) the inequality (2.11) is equivalent to

$$d - \mathrm{rk}_M(S - fs) + \mathrm{rk}_M(F) \ \geq\ \max\{\#(S \cap F),\, \mathrm{rk}_M(F)\}\ ,$$

which holds due to the same arguments as in the first case where we combine (2.13) with $\#(S - fs) = d$.

Third, in the remaining case we have $s, f \in S$, which yields $\mathrm{rk}_M(S - fs) \leq \#(S - fs) = d - 1$. This implies that the inequality (2.11) is equivalent to

$$d - 1 - \mathrm{rk}_M(S - fs) + \mathrm{rk}_M(F) - \#(S \cap F) \ \geq\ \max\{0,\, \mathrm{rk}_M(F) - \#(S \cap F)\} \tag{2.15}$$

again by applying (2.8) and $\mathrm{rk}_M(S - fs) \leq d-1$ on the left hand side, and (2.12) on the rigfht hand side. If the maximum is attained at $\mathrm{rk}_M(F) - \#(S \cap F)$ that inequality holds trivially. We are left with the situation where the maximum on the right is attained solely by zero. This means that $\mathrm{rk}_M(F) < \#(S \cap F)$, which yields

$$d - \#(S \cap F) + \mathrm{rk}_M(F) \;\geq\; d-1 \;\geq\; \mathrm{rk}_M(S - fs) \;. \tag{2.16}$$

Where the second inequality follows from the fact that we are in the case $\#(S-fs) = d-1$. If $\mathrm{rk}_M(S - fs) < d-1$ then (2.15) is immediate. So we may assume that $\mathrm{rk}_M(S - fs) = d-1$. From Lemma 2.41 we deduce that S is a basis of $\mathrm{sf}\, M$. Since F is also a flacet of $\mathrm{sf}\, M$ we get $\mathrm{rk}_M(F) \geq \#(S \cap F)$. However, this contradicts $\mathrm{rk}_M(F) < \#(S \cap F)$, and we conclude that the maximum to the right of (2.15) cannot be attained at zero only. This final contradiction completes our proof. □

Lemma 2.44. *Let M be a (d,n)-split matroid. Then for each split flacet F of M the matroid polytope $P(N_F)$ is a maximal cell of the corank subdivision of $\Delta(d+1,n+2)$ induced by* $\mathrm{sf}\, M$*. Further, the cell $P(N_F)$ shares a split flacet with $P(\mathrm{sf}\, M)$ and another one with $P(\mathrm{sf}^*\, M)$.*

Proof. We want to show that equality holds in (2.11) if S is a basis of N_F. In other words the corank lifting of N_F agrees with the corank lift of $\mathrm{sf}\, M$ on $P(N_F)$, up to an affine transformation. Moreover, the bases of N_F are lifted to height zero, while the lifting function is strictly positive on all other bases; see inequality (2.11). This implies that $P(N_F)$ is a maximal cell in the corank subdivision of $\mathrm{sf}\, M$.

The matroid M is split, hence the contraction M/F on the flacet F is a uniform matroid of rank $d - \mathrm{rk}(F)$. Therefore, the rank function satisfies

$$\mathrm{rk}_M(S + F - fs) - \mathrm{rk}_M(F) \;=\; \min\{\#(S - F - fs),\, d - \mathrm{rk}_M(F)\} \;.$$

Together with (2.8) from Lemma 2.41 we get

$$\begin{aligned} \mathrm{rk}_{\mathrm{sf}\, M}(S) \;&\leq\; \mathrm{rk}_{\mathrm{sf}\, M}(S + F) \\ &=\; \min\{\mathrm{rk}_M(S + F - fs) + \#(S \cap fs),\, d+1\} \\ &=\; \min\{\#(S - F) + \mathrm{rk}_M(F),\, d + \#(S \cap fs),\, d+1\} \;. \end{aligned} \tag{2.17}$$

Similarly as before, the restriction $M|F$ is uniform and hence holds

$$\mathrm{rk}_M(S \cap F) = \;\min\{\#(S \cap F),\, \mathrm{rk}_M(F)\} \;\leq\; d \;. \tag{2.18}$$

The same formula holds for the rank in $\mathrm{sf}\, M$, as again (2.8) shows.

From (2.10) we get

$$d+1 \;=\; \mathrm{rk}_{N_F}(S) \;\leq\; d + 1 - \mathrm{rk}_M(F) + \#(S \cap F)$$

for any basis S of N_F, and hence $\mathrm{rk}_M(F) \leq \#(S \cap F)$. Combined with (2.18) we have $\mathrm{rk}_M(S \cap F) = \mathrm{rk}_M(F)$, due to (2.8) this is equivalent to $\mathrm{rk}_{\mathrm{sf}\,M}(S \cap F) = \mathrm{rk}_{\mathrm{sf}\,M}(F)$ for any bases S of N_F. The submodularity of the rank function implies

$$\mathrm{rk}_{\mathrm{sf}\,M}(S) + \mathrm{rk}_{\mathrm{sf}\,M}(F) \;\geq\; \mathrm{rk}_{\mathrm{sf}\,M}(S+F) + \mathrm{rk}_{\mathrm{sf}\,M}(S \cap F) \;=\; \mathrm{rk}_{\mathrm{sf}\,M}(S+F) + \mathrm{rk}_{\mathrm{sf}\,M}(F) \ ,$$

and therefore also equality in (2.17) for bases of N_F. Hence holds

$$d + 1 - \mathrm{rk}_{\mathrm{sf}\,M}(S) - \#(S \cap F) + \mathrm{rk}_M(F) \;\leq\; 0 \ , \tag{2.19}$$

where we have used $\mathrm{rk}_M(F) \leq \#(S \cap F)$ that we have obtained from (2.10) before, and $\mathrm{rk}_M(F) + 1 \leq \#(S \cap F) + \#(S \cap fs)$ which follows similar also from (2.10).

The inequality (2.19) shows that equality holds in (2.11) whenever S is a basis of N_F. As a consequence $P(N_F)$ is a maximal cell of the corank subdivision of $\mathrm{sf}\,M$. Moreover, the equality in (2.11) for $S = F$ shows that $P(N_F)$ intersects $P(\mathrm{sf}\,M)$ in the facet

$$P_{\mathrm{sf}\,M}(F) \;=\; P_{N_F}([n] - F + fs) \ , \tag{2.20}$$

as F is a flacet of $\mathrm{sf}\,M$. Note that the equality of the two facets in (2.20) follows from (2.6). By Lemma 2.42 the same kind of argument holds for $\mathrm{sf}^*\,M$. That is, $P(N_F)$ intersects $P(\mathrm{sf}\,M)$ in the facet $P_{\mathrm{sf}^*\,M}(F + fs) = P_{N_F}([n] - F)$. □

From the above we know that, for a split matroid M, the matroid polytopes of $\mathrm{sf}\,M$, $\mathrm{sf}^*\,M$ and the nested matroid N_F for each flacet of M form maximal cells of the corank subdivision induced by $\mathrm{sf}\,M$. It might be possible to obtain a similar result from the description of the corank subdivision of the cube given by Fink and Moci; see [FM17, Proposition 5.10]. However, the following result describes the corresponding Plücker vector in the Dressian and the relation to the tropical Grassmannian.

Theorem 2.45. *Let M be a connected (d,n)-split matroid. Then the corank vector $\rho(\mathrm{sf}\,M)$ is a ray in the Dressian $\mathrm{Dr}(d+1, n+2)$. Moreover, it can be lifted to an ordinary Plücker vector over $\mathbb{K}\{\!\{t\}\!\}$ if and only if M is realizable over $\mathbb{K}$.*

Proof. Let Σ be the matroid subdivision of $\Delta(d+1, n+2)$ induced by $\rho(\mathrm{sf}\,M)$. By Lemma 2.27, Lemma 2.42 and Lemma 2.44 the matroid polytopes $P(\mathrm{sf}\,M)$, $P(\mathrm{sf}^*\,M)$ and $P(N_F)$, for each flacet of M, form maximal cells of Σ. Further, those results show that for each flacet of these three kinds of matroids there are precisely two maximal cells in that list which contain that flacet. Since the dual graph of Σ is connected this shows that these are all the maximal cells of Σ.

Moreover, for each flacet F of M, the three maximal cells $P(\mathrm{sf}\,M)$, $P(\mathrm{sf}^*\,M)$ and $P(\mathrm{sf}\,N_F)$ form a triangle in the tropical linear space. It follows from [HJS12, Proposition 28] that Σ does not admit a non-trivial coarsening, i.e., $\rho(\mathrm{sf}\,M)$ is a ray of the secondary fan and thus of the Dressian.

Finally, by Theorem 2.35, the tropical Plücker vector $\rho(\text{sf } M)$ can be lifted to an ordinary Plücker vector over $\mathbb{K}\{\{t\}\}$ if and only if sf M is realizable over $\mathbb{K}$. As $\mathbb{K}$ is algebraically closed a matroid is realizable over $\mathbb{K}$ if and only if any free extension or any series extension is realizable. $\square$

Another general construction for producing tropical Plücker vectors and thus tropical linear spaces arises from point configurations in tropical projective tori. This has been investigated in [HJS12], [Rin13] and [FR15]. In the latter reference the resulting tropical linear spaces are called *Stiefel tropical linear spaces*. These two constructions are not mutually exclusive; there are Stiefel type rays which also arise via Theorem 2.45. Complete descriptions of the Dressians $\mathrm{Dr}(3, n)$ are known for $n \leq 8$. All their rays are of Stiefel type or they arise from connected matroids of rank two via Theorem 2.45.

Via our method non-realizable matroids of rank three lead to interesting phenomena in rank four. In particular, the following consequence of the above answers [HJS12, Question 36].

Corollary 2.46. *The Dressian* $\mathrm{Dr}(d, n)$ *contains rays which do not admit a realization in any characteristic for* $d = 4$ *and* $n \geq 11$ *as well as for* $d \geq 5$ *and* $n \geq 10$. *There are rays of the Dressian* $\mathrm{Dr}(4, 9)$ *that are not realizable in characteristic* 2 *and others that are not realizable in any other characteristic.*

Proof. The non-Pappus $(3, 9)$-matroid and the Vamos $(4, 8)$-matroid are not realizable in any characteristic. Both are connected and paving and hence split. The construction in Theorem 2.45 leads to non-realizable rays in $\mathrm{Dr}(4, 11)$ and $\mathrm{Dr}(5, 10)$. Each free extension or coextension of such a matroid is again connected and split. Thus we obtain non-realizable rays in all higher Dressians.

Applying Theorem 2.45 to the Fano and the non-Fano $(3, 7)$-matroids we obtain two rays in $\mathrm{Dr}(4, 9)$. The first one is realizable solely in characteristic 2, whereas the other one is realizable in all other characteristics. $\square$

Example 2.47. Once again consider the snowflake matroid $\mathcal{S}$ from Examples 2.8 and 2.29. The corank vector of the series-free lift sf $\mathcal{S}$ is a ray in $\mathrm{Dr}(3, 8)$. Since $\mathcal{S}$ has three split flacets the corank subdivision has $3 + 2 = 5$ maximal cells. This is the, up to symmetry, unique ray of $\mathrm{Dr}(3, 8)$ which does not arise from point configuration in the tropical projective 2-torus; see [HJS12, Fig. 7]. A projection of this subdivision to three dimensions is shown in Figure 2.2.

2.5 Remarks and open questions

It is interesting to characterize the class of split matroids in terms of their minors. To this end we have the following contribution.

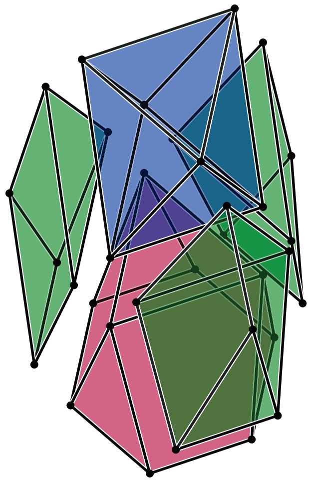


Figure 2.2: Projection of the corank subdivision of $\Delta(3, 8)$ induced by $\mathrm{sf}\,\mathcal{S}$ or, equivalently, induced by $\mathrm{sf}^*\,\mathcal{S}$. There are *five* maximal cells, three of them correspond to nested matroids.

Proposition 2.48. *The class of split matroids is closed under duality as well as under taking minors.*

Proof. The matroid polytope $P(M^*)$ of the dual M^* of a (d, n)-matroid M is the image of $P(M) \subset \mathbb{R}^n$ under the coordinate-wise transformation $x_i \mapsto 1 - x_i$. In particular, $P(M^*)$ is affinely isomorphic with $P(M)$. In view of Proposition 2.15 we may assume that M is connected. In this case any flacet F of M is mapped to the flacet $[n] - F$ of M^*. The compatibility relation among the splits is preserved under affine transformations. It follows that M^* is split if and only if M is.

Assume that M is a split matroid. Next we will show that the deletion $M|([n] - e)$ of an element $e \in [n]$ is again split. Since we already know that the class of split matroids is closed under duality it will follow that the class of split matroids is minor closed.

Let F be a split flacet of $M|([n] - e)$. The F-hyperplane separates at least one vertex of $\Delta(d, [n] - e)$ from $P(M|([n] - e))$. This implies that the closure of F in M is a split flacet of M. For that closure there are two possibilities. So either F or $F + e$ is a split flacet of M.

Let us suppose that F and G are two split flacets of $M|([n] - e)$ which are incompatible. That is, there is some point x in the relative interior of $\Delta(d, [n] - e)$ which lies on the F- and G-hyperplanes. We aim at finding a contradiction by distinguishing four cases which arise from the two possibilities for the closures of the two flacets F and G.

First, suppose that F and $G + e$ are split flacets of M. Then there exists some element $h \in G - F$, for otherwise e would be in the closure of F in M. For each $\varepsilon > 0$ we define the vector $\hat{x} \in \mathbb{R}^n$ with

$$\hat{x}_e = \varepsilon , \quad \hat{x}_h = x_h - \varepsilon \quad \text{and} \quad \hat{x}_i = x_i \text{ for all other elements } i \ . \tag{2.21}$$

If $\varepsilon > 0$ is sufficiently small then the vector $\hat{x}$ is contained in the relative interior of $\Delta(d, n)$. By construction $\hat{x}$ lies on the F- and $(G + e)$-hyperplanes, so that the corresponding splits are not compatible. This contradicts that M is a split matroid.

The second case where $F + e$ and G are split flacets of M is symmetric to the previous.

Thirdly suppose that F and G are split flacets of M. Assume that $M|([n]-e)$ is connected. Then we have $\#(F \cap G) + d > \mathrm{rk}(F) + \mathrm{rk}(G)$ from Proposition 2.14, and the same result implies that F and G are incompatible split flacets of M. Again this is a contradiction to M being split. So we assume that $M|([n] - e)$ is disconnected. Then there exists an element $h \in [n] - F - G - e$, and we may construct a relatively interior point $\hat{x} \in \Delta(d, n)$ as in (2.21). As before this leads to a contradiction to the assumption that M is a split matroid.

In the fourth and final case $F + e$ and $G + e$ are split flacets of M. As in the third case the desired contradiction arises from Proposition 2.14, provided that $M|([n] - e)$ is connected. It remains to consider the situation where $M|([n] - e)$ is disconnected. Note that Proposition 2.14 would imply that F and G are incompatible in their connected component if $F \subsetneq G$ or $F \supsetneq G$, which again contradicts that $M|([n]-e)$ is split. Therefore we find elements $f \in F - G$, $g \in G - F$ and $h \in [n] - F - G - e$. As a minor variation to (2.21) we let

$$\hat{x}_e = \varepsilon , \quad \hat{x}_f = x_f - \varepsilon , \quad \hat{x}_g = x_g - \varepsilon , \quad \hat{x}_h = x_h + \varepsilon \quad \text{and else} \quad \hat{x}_i = x_i \ .$$

The vector $\hat{x}$ lies on the $(F + e)$- and $(G + e)$-hyperplanes, as well as in the relative interior of $\Delta(d, n)$. This entails that the flacets $F + e$ and $G + e$ are incompatible, and this concludes the proof. □

So it is natural to ask for the excluded minors. We have seen several of those in this chapter. The only disconnected minimal excluded minor is the $(4, 8)$-matroid in Example 2.16. One can show that the rank of a connected excluded minor must be at least 3. The class of split matroids is also closed under dualization. Hence the number of elements is at least 6. There are precisely four excluded minors of rank 3 on 6 elements, up to symmetry. One of them is the matroid in Example 2.18, and a second one is its dual. The third example is the nested matroid $\mathrm{sf}(\mathrm{sf}\, U_{1,2})$; see Example 2.23. Finally, the fourth case has an extra split and is represented by the vectors: $(1, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, $(0, 0, 1)$, $(0, 1, 1)$.

Recently Cameron and Mayhew showed the following.

Theorem 2.49 (Theorem 1.4 [CM17])**.** *The excluded minors for the class of split matroids are the five minors listed above.*

The four connected minors are exactly those $(3,6)$-matroids with chains of cyclic flats of length four. This implies that chains of cyclic flats of connected split matroids are at most of lenght three. The certificate for non-splitting of Cameron and Mayhew relies on this fact.

Here is another class of matroids of recent interest; see, e.g., Fife and Oxley [FO17]. A *laminar* family $\mathcal{L}$ of subsets of $[n]$ satisfies for all sets $A, B \in \mathcal{L}$ either $A \cap B = \emptyset$, $A \subseteq B$ or $B \subseteq A$. Furthermore, let c be any real valued function on $\mathcal{L}$, and which is called a *capacity function*. A set I is an independent set of the *laminar matroid* $L = L([n], \mathcal{L}, c)$ if $\#(I \cap A) \leq c(A)$ for all $A \in \mathcal{L}$. Here the triplet $([n], \mathcal{L}, c)$ is called a *presentation* of L. By [FO17, Theorem 2.7] each loop-free laminar matroid has a unique canonical presentation where the laminar family is the set of closures of the circuits, and the capacity function assigns to each set in the laminar family its rank. The class of split matroids and the class of laminar matroids are not contained in one another: The Fano matroid is a split matroid, but it is not laminar as it has closed circuits of size three which share exactly one element. On the other hand the nested matroid from Example 2.23 is not split. However, each nested matroid is laminar [FO17, Proposition 4.4].

It may be of general interest to look at tropical linear spaces where the matroidal cells correspond to matroids from a restricted class. For instance, Speyer [Spe09] looks at series-parallel matroids, and he conjectures that the tropical linear spaces arising from them maximize the f-vector. The tight span of a tropical linear space, i.e., the subcomplex of the bounded cells, whose maximal cells are matroid polytopes of split matroids are necessarily one-dimensional, i.e., they are trees.

Example 2.50. Each connected matroid of rank 2 is a split matroid, which follows directly from Proposition 2.14 and by duality every connected $(n-2,n)$-matroid; see Proposition 2.48. Hence, every tight span is one-dimensional if $d = 2$ or $d = n-2$; see also [SS04] for the relation of pyhlogenetic trees and tropical linear spaces.

Conceptually, it would be desirable to be able to write down all rays of all the Dressians and the tropical Grassmannians. Due to the intricate nature of matroid combinatorics, however, it seems somehow unlikely that this can ever be done in an explicit way. The next best thing is to come up with as many ray classes as possible. In [HJS12] tropical point configurations are used as data, whereas here we look at split matroids and their corank subdivisions. A third class of rays comes from the nested matroids. Their analysis is the purpose of Chapter 3. It can be shown that the corank subdivision of a connected matroid M is a "k-split" in the sense of Herrmann [Her11] if and only if M is a nested matroid with $k+1$ cyclic flats. Again, see Chapter 3.

A *polymatroid* is a polytope associated with a submodular function. This generalizes matroids given by their rank functions. Since splits are defined for arbitrary polytopes there is an obvious notion of a "split polymatroid". It seems promising to investigate them.

Another generalization of classical matroids are *delta-matroids* introduced in [Bou87]. Maximal independent sets in delta-matroids do not have same cardinality. Valuated delta-matroids has been studied in tropical context in [Rin12]. The investigation of the corresponding Dressian and splits seems to be likely.

All three polymatroids, delta-matroids and tropical Plücker vectors are closely related to "integral discrete functions" which occur in discrete convex analysis; see, e.g., Murota [Mur03]. In that language a tropical Plücker vector is the same as an "M-concave function" on the vertices of the underling matroid polytope. It would be interesting to investigate the notation of splits and realizability in terms of M-convexity. Hirai took a first step in this direction in [Hir06], where he studies splits of "polyhedral convex functions".

2.6 Some matroid statistics

Matsumoto, Moriyama, Imai and Bremner classified matroids of small rank with few elements [MMIB12]. A summary is given in the appendix Table A.2. Based on the census of [MMIB12] we determined the percentages of paving and split matroids. The results are given in Table 2.1 and Table 2.2. That computation employed `polymake` [GJ00], and the results are accessible via the new database at db.polymake.org. In all tables we marked entries with $-$ that have not been computed due to time and memory constraints.

Filtering all 190214 matroids of rank 4 on 9 elements for paving, sparse paving and split matroids took about 2000 sec with `polymake` version 3.1 (AMD Phenom II X6 1090T with 3.6 GHz single-threaded, running openSUSE 42.1). We expect that the computation for all $(4, 10)$-matroids, which is the next open case, would take much more than 600 CPU days.

Example 2.51. All matroids of rank d on $d + 2$ elements are split matroids. Table 2.1 shows that most of these are not paving.

Table 2.1: The percentage of paving among the isomorphism classes of all matroids of rank d on n elements.

$d\backslash n$	3	4	5	6	7	8	9	10	11	12
2	67	57	46	43	38	36	33	32	30	29
3	100	50	31	24	21	21	30	52	78	91
4		100	40	22	17	34	77	–	–	–
5			100	33	14	12	63	–	–	–
6				100	29	10	14	–	–	–
7					100	25	7	17	–	–
8						100	22	5	19	–
9							100	20	4	16
10								100	18	3
11									100	17

Table 2.2: The percentage of split matroids among the isomorphism classes of all matroids of rank d on n elements.

$d\backslash n$	3	4	5	6	7	8	9	10	11	12
2	100	100	100	100	100	100	100	100	100	100
3	100	100	100	89	75	60	52	61	80	91
4		100	100	100	75	60	82	–	–	–
5			100	100	100	60	82	–	–	–
6				100	100	100	52	–	–	–
7					100	100	100	61	–	–
8						100	100	100	80	–
9							100	100	100	91
10								100	100	100
11									100	100

3 Multi-splits of hypersimplices

3.1 Introduction

It is a natural idea to decompose a difficult problem into smaller pieces. There are many natural situations in which one has fixed a finite set of points, i.e, a *point configuration*. For example these points could be the exponent vectors of the monomials in a polynom or structured points in convex position. In general, all convex combinations of a finite point configuration form a polytope.

In a typical situation one asked for a specific subdivision or even all subdivisions of a polytope into smaller polytopes whose vertices are points of a given point configuration. For example we are using a subdivision into simplices when we compute the volume of a polytope, and these subdivisions are in correspondence with Gröbner bases in toric ideals. Other famous examples for subdivisions are placing, minimum weight, and Delaunay triangulations. For an overview of these and further applications see the monograph [DLRS10] by De Loera, Rambau, and Santos.

All subdivisions form a finite lattice with respect to coarsening and refinement. Gel′fand, Kapranov and Zelevinsky showed that the sublattice of regular subdivisions is the face-lattice of a polytope; see [GKZ08]. This polytope is called the *secondary polytope* of the subdivision. The vertices of the secondary polytope correspond to finest subdivisions, i.e., triangulations. Concrete coordinates for these vertices can be read off from the volumes of the appearing simplices. In this coordinatization they are called GKZ-vectors, due to the names of the three mentioned authors. An important example in combinatorics is the *associahedron*, which is the secondary polytope of a convex n-gon; see [CSZ15]. It is remarkable that the number of triangulations of an n-gon is the Catalan number $\frac{1}{n-1}\binom{2n-4}{2n-2}$ and the number of diagonals is $\frac{n(n-3)}{2}$, a triangular number minus one. A subdivision into two maximal cells is a coarsest subdivision and called *split*. The coarsest subdivisions of the n-gon are the splits along the diagonals. This example shows that the associahedron has $\frac{1}{n-1}\binom{2n-4}{2n-2}$ vertices and only $\frac{n(n-3)}{2}$ facets. It is expectable that in general the number of facets of the secondary polytope is much smaller than the number of vertices.

Herrmann and Joswig were the first who systematically studied splits and hence facets of the secondary polytope. Herrmann introduced a generalization of splits in [Her11]. A multi-split is a coarsest subdivision, such that all maximal cells meet in a common cell.

The purpose of this chapter is to further explore the facet structure of the secondary polytope for two important classes of polytopes – products of simplices and hypersimplices. In particular, we investigate their multi-splits. Triangulations of products of simplices have been studied in algebraic geometry, optimization and game theory; see [DLRS10, Section 6.2]. An additional motivation to study splits of products of simplices is their relation to tropical convexity [DS04], tropical geometry and matroid theory.

The focus of our interest is on hypersimplices, that we have seen in the previous chapter. Recall, the *hypersimplex* $\Delta(d,n)$ is the intersection of the n-dimensional 0/1-cube with the hyperplane $x_1 + \ldots + x_n = d$. Hypersimplices appear frequently in mathematics. For example, they appear in algebraic combinatorics, graph theory, optimization, analysis and number theory (see [DLRS10, Subsection 6.3.6]), as well as in phylogenetics, matroid theory and tropical geometry. Chapter 2 already gave us plenty examples of the relations between splits of a hypersimplex and tropical geometry by using methods from matroid theory. Both matroid theory and tropical geometry are closely related to phylogenetics. Bandelt and Dress [BD92] were the first who studied the split decomposition of a finite metric in phylogenetic analysis. Later Hirai [Hir06], Herrmann, Joswig [HJ08] and Koichi [Koi14] developed split decompositions of polyhedral subdivisions. In particular they discussed subdivisions of hypersimplices. The special case of a subdivision of a hypersimplex $\Delta(2,n)$ corresponds to a class of finite pseudo-metrics. The matroid subdivisions of $\Delta(2,n)$ are totally split-decomposable and correspond to phylogenetic trees with n labeled leaves; see [HJ08] and [SS04].

A product of simplices appears as vertex figures of any vertex of a hypersimplex. Moreover, a subdivision of a product of simplices extends to a subdivision of a hypersimplex via the tropical Stiefel map. This lift has been studied in [HJS12], [Rin13] and [FR15].

This chapter comprises three main results, that combine polyhedral and matroid theory as well as tropical geometry. In Section 3.2 we show that any multi-split of a hypersimplex is the image of a multi-split of a product of simplices under the tropical Stiefel map (Theorem 3.15). To reach this goal we introduce the concept of "negligible" points in a point configuration. With this tool we are able to show that the point configuration consisting of the vertices of a product of simplices suffice to describe a given multi-split of the hypersimplex. This already implies that all multi-splits of a hypersimplex are subdivisions into matroid polytopes.

In Section 3.3 we define a relation depending on matroid properties of the occurring cells. We use this relation to enumerate all multi-splits of hypersimplices (Proposition 3.33) and show that all maximal cells in a multi-split of a hypersimplex correspond to matroid polytopes of nested matroids (Theorem 3.31). This generalizes the last statement of Proposition 2.30, which treats 2-splits, i.e. multi-splits with exactly two maximal cells. As a consequence of the enumeration of all multi-splits of a hypersimplex we get the

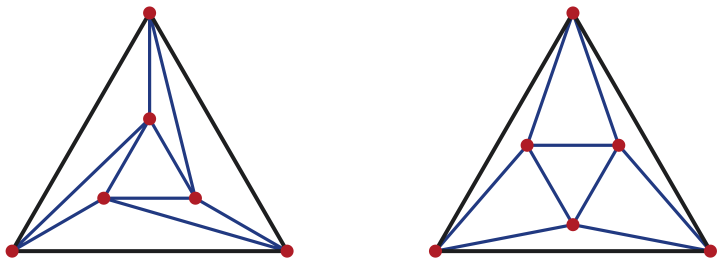


Figure 3.1: A non-regular and a combinatorially isomorphic regular subdivision, whose lifted polytope is an octahedron with a single facet, whose inner normal is negative on the last coordinate.

enumeration of all multi-splits of a product of simplices (Theorem 3.35). Nested matroids are a well studied class in matroid theory. Hampe recently introduced the "intersection ring of matroids" in [Ham17] and showed that every matroid is a linear combination of nested matroids in this ring. Moreover, matroid polytopes of nested matroids describe the intersection of linear hyperplanes in a matroid subdivision locally. Hence they occur frequently in those subdivisions; see Chapter 2.

In the last Section 3.4 we take a closer look at coarsest matroid subdivisions of the hypersimplex in general. Regular matroid subdivisions are important in tropical geometry as they are dual to tropical linear spaces. They give rise to a "valuated matroid" introduced by Dress and Wenzel [DW92]. Coarsest matroid subdivisions have been studied in [HJS12]. We compare two constructions of matroid subdivisions, those that are in the image of the tropical Stiefel map and those that appear as a corank vector of a matroid. We present our computational results on the number of coarsest matroid subdivisions of the hypersimplex $\Delta(d, n)$ for small parameters d and n (Proposition 3.40), which illustrate how fast the number of combinatorial types of matroid subdivisions grows.

3.2 Multi-splits of the hypersimplex

In this section we will study a natural class of coarsest subdivisions, called "multi-splits". Our goal is to show that any "multi-split" of the hypersimplex can be derived from a "multi-split" of a product of simplices. We assume that the reader has a basic background on subdivisions and secondary fans. The basics could be found in [DLRS10]. We will shortly introduce our notation and definitions.

We consider a finite set of points in $\mathbb{R}^n$ as a *point configuration* $\mathcal{P}$, i.e., each point occurs once in $\mathcal{P}$. A subdivision Σ of $\mathcal{P}$ is a collection of subsets of $\mathcal{P}$, such that faces of the convex hulls of these sets, that we call *cells*, intersect in common faces, the sets that correspond to a face are in the collection, and all cells cover the convex hull of all the points. The *lower convex hull* of a polytope $Q \subset \mathbb{R}^{n+1}$ is the collection of all faces with an inner facet normal with a strictly positive $(n+1)$-coordinate. A subdivision on the

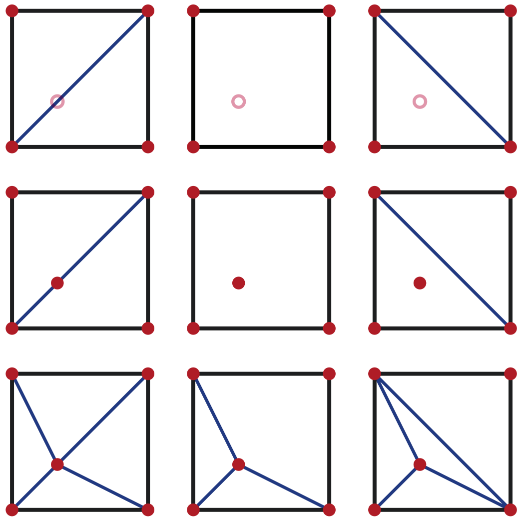


Figure 3.2: Nine (regular) subdivisions of the five points of Example 3.1. The inner point is negligible in all subdivisions in the middle row. This point is lifted above the lower convex hull in the regular subdivisions in the top row. The subdivision in the middle of the top row is a 1-split, the left and right in the second row are 2-splits and in the middle of the bottom row is a 3-split.

point configuration $\mathcal{P} \subset \mathbb{R}^n$ is *regular* when it is combinatorially isomorphic to the lower convex hull of a polytope $Q \subset \mathbb{R}^{n+1}$ and the vertices of the polytope Q project to the point configuration $\mathcal{P}$ by omitting the last coordinate; see Figure 3.1. The polytope is called the *lifted polytope*. The $(n+1)$-coordinate is called the *height*. The heights of the points in $\mathcal{P}$ form the *lifting vector*.

The set of all lifting vectors whose projection of the lower convex hull coincides form an open cone. The closure of such a cone is called a *secondary cone*. The collection of all secondary cones is the *secondary fan* of the point configuration $\mathcal{P}$. We call a point $q \in \mathcal{P}$ *negligible in the subdivision* Σ if there is a cell containing the point q and q does not occur as a vertex of any 1-dimensional cell. In particular, a negligible point q lies in a cell C if and only if $q \in \operatorname{conv}(C \setminus \{q\})$. For a regular subdivision this means that q is lifted to a redundant point and to the lower convex hull of the lifted polytope.

Example 3.1. Consider the point configuration of the following five points $(0,0)$, $(3,0)$, $(0,3)$, $(3,3)$, $(1,1)$. All nine possible subdivisions of that point configuration are regular and $(1,1)$ is negligible in three of them; see Figure 3.2.

A negligible point $q \in \mathcal{P}$ can be omitted in the subdivision Σ. More precisely we have the following relation between the subdivisions of $\mathcal{P}$ and those subdivisions of $\mathcal{P} \setminus \{q\}$.

Proposition 3.2. *Let $q \in \mathcal{P}$, such that $q \in \operatorname{conv}(C \setminus \{q\})$. Consider the following map on the set of all subdivisions of $\mathcal{P}$ where q is negligible.*

$$\Sigma \mapsto \{C \setminus \{q\} \mid C \in \Sigma\}$$

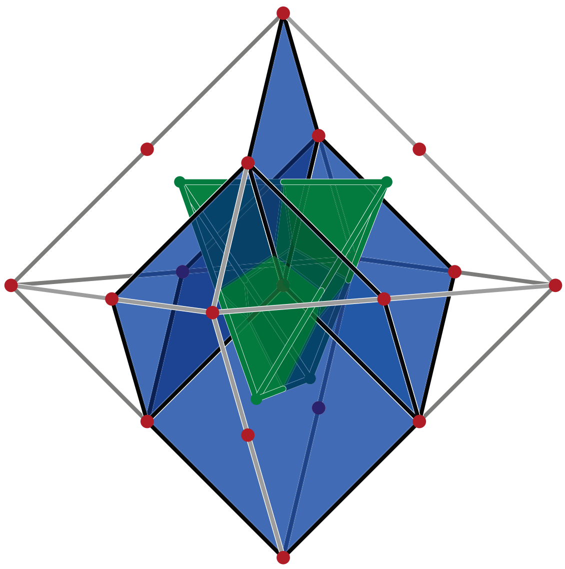


Figure 3.3: A 4-split in the hypersimplex $\Delta(4, 8)$ projected to a dilated octahedron scaled by the factor of two, and the corresponding tight span, which is a 3-simplex.

This map is a bijection onto all subdivisions of $\mathcal{P} \setminus \{q\}$.

A *k-split* of a point configuration $\mathcal{P}$ is a coarsest subdivision Σ of the convex hull P of $\mathcal{P}$ with k maximal faces and a common $(k-1)$-codimensional face. We call this face the *common cell* and denote this polytope by H^{Σ}. We shorten the notation if the point configuration $\mathcal{P}$ is the vertex set of a polytope P and write this as k-split of P. If we do not specify the number of maximal cells we will call such a coarsest subdivision a *multi-split*.

Example 3.3. The point configuration of the points in Example 3.1 has four coarsest subdivisions. These are a 1-split, two 2-splits and a 3-split; see Figure 3.2.

Example 3.4. Figure 3.3 illustrates a 4-split in the hypersimplex $\Delta(4, 8)$ projected to a dilated octahedron scaled by the factor of two and its tight span, i.e., the bounded cells in the dual complex which is a 3-simplex. The interior faces look like a tropical hyperplane.

Example 3.5. In general not all coarsest subdivisions are multi-splits. An extremal example is a 4-dimensional cross polytope with perturbed vertices, such that five points do not lie in a common hyperplane. The secondary polytope of this polytope has 29 facets, non of which is a multi-split.

Example 3.6. Another example for a coarsest subdivision that is not a multi-split is illustrated in Figure 3.4 on the next page.

A 2-split is also known as *split*. In Chapter 2 we derived a new class of matroids from those. Splits have been studied by several people in phylogenetic analysis, metric spaces

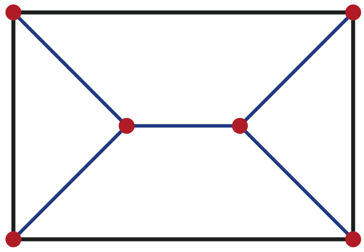


Figure 3.4: A coarsest regular subdivision, which is not a multi-split.

and polyhedral geometry. For example by Bandelt and Dress [BD92], Hirai [Hir06], Herrmann and Joswig [HJ08] and Koichi [Koi14]. The more general multi-splits have been introduced by Herrmann in [Her11] under the term k-split. The main result there is the following.

Proposition 3.7 ([Her11, Theorem 4.9])**.** *Each k-split is a regular subdivision. The dual complex of the lower cells, i.e., the subcomplex in the polar of the lifted polytope, is a k-simplex modulo its lineality space.*

Proposition 3.7 implies that the subdivision of a multi-split corresponds to a ray in the secondary fan, i.e., this is a coarsest regular and non-trivial subdivision. Furthermore, the number of cells of a fixed codimension of a k-split that meets the interior is the same as the number of faces of the same codimension of a k-simplex. In particular, the number of maximal and those of non-trivial inclusionwise minimal cells equals k. Note that the latter are the $(n-k+1)$-dimensional cells.

We recall the main construction of Proposition 3.7, which proves the regularity. The subdivision Σ is induced by a complete fan $\mathcal{F}^\Sigma$ with k maximal cones, a lineality space $\operatorname{aff} H^\Sigma$ and an apex at $a \in \mathbb{R}^n$. Here "induced" means a cell of Σ is the intersection of a cone of $\mathcal{F}^\Sigma$ with P. The apex a is not unique, it can be any point in H^Σ. Later we will take specific choices for it. A lifting function that induces the multi-split is given by the following. All points in $\mathcal{P} \cap \operatorname{aff} H^\Sigma$ are lifted to height zero. The height of a point $p \in \mathcal{P}$ that is contained in a ray of $\mathcal{F}^\Sigma$ is the shortest distance to the affine space $\operatorname{aff} H^\Sigma$. Each other point in the point configuration $\mathcal{P}$ is a non-negative linear combination of those rays. The height of a point is given by the linear combination with the same coefficients multiplied with the heights of points in the rays of $\mathcal{F}^\Sigma$.

The following Lemma summarizes important properties of the common cell H^Σ.

Lemma 3.8. *The common cell H^Σ is the intersection of the affine space* $\operatorname{aff} H^\Sigma$ *with P and* $\operatorname{aff} H^\Sigma$ *intersects P in its relative interior. Hence, the relative interior of the common cell H^Σ is contained in the relative interior of the polytope P.*

Proof. Let us assume that $\mathcal{F}^\Sigma$ is the complete fan of the k-split Σ. The intersection of all maximal cones in $\mathcal{F}^\Sigma$ is an affine space which shows $H^\Sigma = \operatorname{aff} H^\Sigma \cap P$. The dual cell of H^Σ is a k-simplex by Proposition 3.7, and therefore a bounded polytope. Cells in the boundary of the polytope P are dual to unbounded polyhedra. Hence, this implies that H^Σ is not contained in any proper face of P. □

Note that in the case that the point configuration $\mathcal{P}$ is the vertex set of a polytope P there is at least one vertex that is contained in the common cell H^Σ. Even more if P is not a point then H^Σ is at least 1-dimensional, otherwise it would be a face of P.

Let $N(v)$ be the set of vertices that are neighbors of v in the vertex-edge graph of P and $\varepsilon = \min_{u\in N(v)} \sum_{w\in N(v)} \langle w - v,\, u - v\rangle$. The intersection of the polytope P with a hyperplane that (weakly) separates the vertex v from all other vertices and does not pass through v is the *vertex figure* of v

$$\mathrm{VF}(v) = \left\{ x \in P \;\middle|\; \sum_{w\in N(v)} \langle w - v,\, x - v\rangle = \varepsilon \right\}$$

Our goal is to relate a k-split of a polytope to a k-split in a vertex figure.

We will focus on a particular class of convex polytopes, the hypersimplices. We define for $d, n \in \mathbb{Z}$, $I \subseteq [n]$ and $0 \le d \le \#(I)$ the polytope

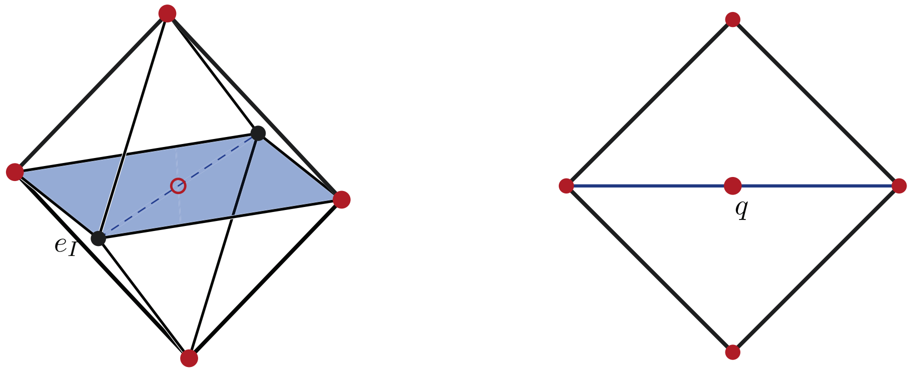
$$\Delta(d, I) \;=\; \left\{ x \in [0,1]^n \;\middle|\; \sum_{i\in I} x_i = d \text{ and } \sum_{i\notin I} x_i = 0 \right\} .$$

The (d, n)-*hypersimplex* is the polytope $\Delta(d, [n])$, that we denote also by $\Delta(d, n)$. Clearly, the polytope $\Delta(d, I)$ is a fixed embedding of the hypersimplex $\Delta(d, \#(I))$ into n-dimensional space. We define the $(n-1)$-simplex Δ_{n-1} as the hypersimplex $\Delta(1, n)$ which is isomorphic to $\Delta(n-1, n)$.

Example 3.9. The vertex figure $\mathrm{VF}(e_I)$ of $e_I = \sum_{i\in I} e_i$ in the hypersimplex $\Delta(d, n)$ is

$$\begin{aligned}
\mathrm{VF}(e_I) &= \left\{ x \in \Delta(d,n) \;\middle|\; \sum_{i\in I} \sum_{j\in[n]-I} \langle e_j - e_i,\, x - e_I\rangle = n \right\} \\
&= \left\{ x \in \Delta(d,n) \;\middle|\; \langle d e_{[n]-I} - (n-d) e_I,\, x - e_I\rangle = n \right\} \\
&= \Delta(d-1, I) \times \Delta(1, [n] - I)
\end{aligned}$$

We say that a subdivision Σ' on $\mathcal{P}' \subsetneq \mathbb{R}^n$ is *induced* by another subdivision Σ on $\mathcal{P} \subsetneq \mathbb{R}^n$ if for all $\sigma \in \Sigma$ with $\dim(\operatorname{conv}\sigma \cap \operatorname{conv}\mathcal{P}') = 0$ we have $\operatorname{conv}\sigma \cap \operatorname{conv}(\mathcal{P}') \subseteq \mathcal{P}'$ and $\Sigma' = \{\operatorname{conv}\sigma \cap \mathcal{P}' \mid \sigma \in \Sigma\}$. Note that this is not the same concept as a subdivision that is "induced" by a fan.



a) A 2-split of the octahedron. b) The induced 2-split with the interior point q.

Figure 3.5: A 2-split Σ in the octahedron $\Delta(2,4)$, with the common cell H^Σ and the induced 2-split in the vertex figure $\mathrm{VF}(e_I)$.

Example 3.10. A subdivision of the octahedron into two egyptian pyramids is a 2-split. The common cell is a square. Figure 3.5 illustrates this subdivision as well as the induced subdivision of the vertex figure. The induced subdivision is a 2-split of a square on a point configuration with five points, the four vertices and an interior point q. The point q is the intersection of the vertex figure $\mathrm{VF}(e_I)$ and the convex hull of the two vertices that are not in the vertex figure. The interior point q is negligible.

The situation of Example 3.10 generalizes to k-splits of arbitrary polytopes.

Proposition 3.11. *Let Σ be a k-split of the polytope P and $v \in H^\Sigma$ be a vertex of P. Then each cone of $\mathcal{F}^\Sigma$ intersects the vertex figure $\mathrm{VF}(v)$ of v. In particular, the subdivision Σ induces a k-split on a point configuration that is contained in $\mathrm{VF}(v)$.*

Proof. Let us assume without loss of generality that the vertex v is the apex of $\mathcal{F}^\Sigma$, this implies $v \in H^\Sigma$. Each ray of $\mathcal{F}^\Sigma$ is a cone of the form $\{\lambda(w-v) \mid \lambda \geq 0\} + \operatorname{aff} H^\Sigma$ for another vertex $w \in P$. Hence, each ray intersects the vertex figure $\mathrm{VF}(v)$ of $v \in H^\Sigma$. This implies that the intersection of a ℓ-dimensional cone with $\mathrm{VF}(v)$ is $\ell-1$ dimensional. We conclude that the induced subdivision is again a k-split. $\square$

Our main goal is to classify all multi-splits of the hypersimplices. Recall from Example 3.9 that for the hypersimplex the vertex figure of $e_I = \sum_{i\in I} e_i$ with $\# I = d$ is the product of simplices

$$\begin{aligned}\mathrm{VF}(e_I) &= \left\{ x \in \Delta(d,n) \;\middle|\; \sum_{i\in I} x_i = d-1 \right\} \\ &= \Delta(d-1, I) \times \Delta(1, [n]-I) \simeq \Delta_{d-1} \times \Delta_{n-d-1} \quad .\end{aligned}$$

The intersection of the vertex figure of e_I and the line spanned by the two vertices e_I and e_J with $J \in \binom{[n]}{d}$ is a point q with coordinates

$$q_i = \begin{cases} 1 & \text{if } i \in I \cap J \\ \frac{\#(I-J)-1}{\#(I-J)} & \text{if } i \in I - J \\ \frac{1}{\#(I-J)} & \text{if } i \in J - I \\ 0 & \text{if } i \notin I \cup J \end{cases}$$

We denote by $\mathcal{Q}_I$ the set of all these intersection points. They include the vertices of the vertex figure of e_I. For those we have $\#(I-J) = 1$. A lifting function λ of $\Delta(d,n)$ induces a lifting on each point $q \in \mathcal{Q}_I$ by taking

$$\lambda(q) = \lambda\left(\frac{\#(I-J)-1}{\#(I-J)} e_I + \frac{1}{\#(I-J)} e_J\right) = \frac{\#(I-J)-1}{\#(I-J)} \lambda(e_I) + \frac{1}{\#(I-J)} \lambda(e_J) \ .$$

From Proposition 3.11 follows that for each k-split of $\Delta(d,n)$ there exists a d-set I and a vertex e_I such that the k-split on $\Delta(d,n)$ induces a k-split on the point configuration $\mathcal{Q}_I$. Our goal is to show that all interior points of $\operatorname{conv} \mathcal{Q}_I$ are negligible.

Before we discuss this in general let us take a closer look on a key example where $d = n-d$. In the example, the point configuration consists only of the vertices and exactly one additional point. This example will be central in the rest of the argumentation.

Consider for a moment the point configuration $\mathcal{P}_j$ with the vertices of $\Delta_{j-1} \times \Delta_{j-1}$ and exactly one additional point q which is $\sum_{i=1}^{2j} \frac{1}{j} e_i$ the barycenter of $\Delta_{j-1} \times \Delta_{j-1}$. Note that for $j > 2$ is $\# \mathcal{P}_j$ smaller than $\# \mathcal{Q}_I$, even if $j = \# I$ and $n = 2j$.

Lemma 3.12. *There is no $(2j-1)$-split of $\mathcal{P}_j$.*

Proof. Let us assume we have given a $(2j-1)$-split Σ of $\mathcal{P}_j$. The dimension of $\Delta_{j-1} \times \Delta_{j-1}$ is $2j-2$, hence the common cell H^Σ is $(2j-2)-(k-1)$ dimensional. In our situation the dimension is 0. The only 0 dimensional cell in the interior is $\{q\} = H^\Sigma$. Let $\mathcal{F}^\Sigma$ be the complete fan that induces Σ. The apex of $\mathcal{F}^\Sigma$ has to be q. Proposition 3.7 shows that this fan has $k = 2j-1$ rays. Each of these $2j-1$ rays intersects $\Delta_{j-1} \times \Delta_{j-1}$ in a point on the boundary. An intersection point has to be an element of the point configuration and hence it is a vertex of $\Delta_{j-1} \times \Delta_{j-1}$. The convex hull Q of all $2j-1$ vertices that we obtain as an intersection of the boundary with a ray is a $2j-2$-dimensional simplex in $\mathbb{R}^{2j}$. This simplex Q contains q in its interior, since $\mathcal{F}^\Sigma$ is complete. By Lemma 3.8 we have that q is in the relative interior of $\operatorname{conv} \mathcal{P}_j$. Hence, no coordinate of q is integral, while the vertices are 0/1-vectors. This implies that for each of the $2j$ coordinates of q there is a vertex of the simplex that is 1 in this coordinate. A vertex of $\Delta_{j-1} \times \Delta_{j-1}$ has only two non zero entries, hence there is at least one coordinate $\ell \in [2j]$ such that only one vertex $w \in Q$ fulfills $w_\ell = 1$. We deduce that the coefficient of w in the convex combination of the vertices that sums up to q is $\frac{1}{j}$.

The simplex Q is of dimension $2j-2$, which is the dimension of $\Delta_{j-1} \times \Delta_{j-1}$. Hence another vertex v exists in Q, such that the support of v and the support w intersect non-trivially. The coefficient of w is $\frac{1}{j}$, hence the coefficient of v has to be 0. This contradicts the fact that q is in the interior of the simplex. $\square$

Remark 3.13. The proof of Lemma 3.12 shows that the barycenter of $\Delta_{j-1} \times \Delta_{j-1}$ is on the boundary of the constructed simplex Q. In fact, the arguments of the proof apply to any $(j+1)$-dimensional subpolytope of $\Delta_{j-1} \times \Delta_{j-1}$, instead of the subpolytope Q. Hence, in any triangulation the barycenter is contained in a j-dimensional simplex.

Our next step is to reduce the general case to the case where $2d = n$, which is equivalent to $\# I = d = n - d$, and the point configuration is $\mathcal{Q}_I$. This is close to the situation in Lemma 3.12, but still not the same.

For any non-vertex $p \in \mathcal{Q}_I$ we define

$$F_p = \left\{ x \in \Delta(d,n) \;\middle|\; \sum_{i \in I} x_i = d-1 \text{ and } x_j = p_j \text{ for all } p_j \in \{0,1\} \right\} \; .$$

By definition the set $\mathcal{Q}_I \cap \operatorname{relint} F_p$ consists of the single point p and there is a unique d-set J such that $p \in \operatorname{conv}(e_I, e_J)$. Clearly $\#(I-J) = \# I - \#(I \cap J) = \# J - \#(I \cap J) = \#(J-I)$ and

$$p_j \text{ is non-integral if and only if } j \in I - J \text{ or } j \in J - I \; .$$

The coordinatewise affine transformation $x_j \mapsto 1 - x_j$ if $j \in I - J$ and $x_j \mapsto x_j$ if $j \in J - I$ is an isomorphism between the face F_q of the vertex figure $\operatorname{VF}(e_I)$ and the product of simplices $\Delta_{j-1} \times \Delta_{j-1}$ for $j = d - \# I \cap J$. The point p is mapped to the barycenter.

The intersection of F_p with the common cell H^Σ is either $\{p\}$ or F_p. Hence, the only possibilities for a multi-split of the point configuration $\mathcal{Q}_I \cap F_q$ are $2j-1$ or 1 maximal cell. The multi-split is induced by the polytope $\Delta(d,n) \cap \operatorname{aff}\{e_I, F_q\}$. A 1-split cannot be induced by a polytope. Therefore it has to be a $2j-1$-split. All together we get the following result for arbitrary multi-splits.

Lemma 3.14. *Let Σ be a multi-split of the point configuration $\mathcal{Q}_I$. All points of $\mathcal{Q}_I \setminus \{0,1\}^n$ are negligible in Σ.*

Proof. To each $q \in \mathcal{Q}_I$ we assign the set $\{i \in [n] \mid q_i \notin \mathbb{Z}\}$ of non-integral support. A point $q \in \mathcal{Q}_I$ is a 0/1-vector if and only if its non-integral support is empty. Consider a ray R in the fan $\mathcal{F}^\Sigma$, i.e., the dimension of R is $\dim(H^\Sigma) + 1$. Let $V_R \subseteq \mathcal{Q}_I$ be the set of all points of the intersection $R \cap \mathcal{Q}_I$. Fix a point

$$p \in \{q \in V_R \mid \text{the non-integral support of } q \text{ is non-empty}\}$$

whose non-integral support is inclusionwise minimal in the above set. Our goal is to show that such a p does not exist and hence the above set is empty. This implies that any point $q \in V_R$ is integral.

From [Her11, Proposition 4.8] it follows that the face F_p is either trivially subdivided or a multi-split. In a trivial subdivision the interior point p is not a vertex of $R \cap \operatorname{conv} \mathcal{Q}_I$. By construction all the non-integral points in F_p except for p are negligible, otherwise p would not be a vertex of V_R. Moreover, p is the only interior point and $k = 2j - 1$, where j is the size of the non-integral support. This contradicts Lemma 3.12. We conclude that the above constructed set is empty. Hence all non-integral points in $\mathcal{Q}_I$ are negligible. □

Proposition 3.11 and Lemma 3.14 show that the induced subdivision is a subdivision of the vertex figure $\mathrm{VF}(e_I)$, which is a product of simplices. This reverses a construction that lifts regular subdivisions of the product of simplices $\Delta_{d-1} \times \Delta_{n-d-1}$ to the hypersimplex $\Delta(d, n)$. This lift has been studied in the context of tropical convexity in [HJS12], [Rin13] and [FR15]. We define the tropical Stiefel map of a regular subdivision on the product of simplices $\Delta_{d-1} \times \Delta_{n-d-1}$. We denote by $\lambda(i, j) \in \mathbb{R}$ the height of the vertex $(e_i, e_j) \in \Delta_{d-1} \times \Delta_{n-d-1}$. The *tropical Stiefel map* π is defined on sets $A \subseteq \{1, \ldots, d\}, B \subseteq \{d+1, \ldots, n\}$ with $\# A = \# B$

$$\pi : (A, B) \mapsto \min_{\omega \in \mathrm{Sym}(B)} \sum_{i \in A} \lambda(i, \omega_i)$$

where $\mathrm{Sym}(B)$ is the symmetry group on the set B. Note that $\pi(\{i\}, \{j\}) = \lambda(i, j)$.

Let $e_I \in \Delta(d, n)$ be a vertex and λ be a lifting on $\Delta_{d-1} \times \Delta_{n-d-1}$. Then the tropical Stiefel map defines a lifting on a vertex $e_J \in \Delta(d, n)$ by taking the height $\pi(I - J, J - I)$. The polytope $\Delta_{d-1} \times \Delta_{n-d-1}$ is isomorphic to $\mathrm{VF}(e_I) = \Delta_{d-1,I} \times \Delta_{1,[n]-I} \subsetneq \Delta(d, n)$. The tropical Stiefel map extends a lifting of the vertex figure $\mathrm{VF}(e_I)$ to the entire hypersimplex $\Delta(d, n)$. The dual complex of the extended subdivision of $\Delta(d, n)$ is isomorphic to the dual complex of the subdivision of $\Delta_{d-1} \times \Delta_{n-d-1}$; see [HJS12, Theorem 7]. In particular, the Stiefel map extends a k-split of $\Delta_{d-1} \times \Delta_{n-d-1}$ to a k-split of $\Delta(d, n)$.

From Lemma 3.14 we deduce.

Theorem 3.15. *Any k-split of the hypersimplex $\Delta(d, n)$ is the image of a k-split of a product of simplices $\Delta_{d-1} \times \Delta_{n-d-1}$ under the Stiefel map. In particular, the k-split Σ is an extension of a k-split of $\Delta(d, I) \times \Delta(n - d, [n] - I)$ if and only if $e_I \in H^\Sigma$.*

Proof. For any k-split Σ of the hypersimplex $\Delta(d, n)$ and any vertex $e_I \in H^\Sigma$ the k-split Σ induces a k-split on the point configuration $\mathcal{Q}_I$. By Proposition 3.2 and Lemma 3.14 this is a subdivision on the vertex figure $\mathrm{VF}(e_I)$, which is a product of simplices. The Stiefel map extends this k-split to a k-split on $\Delta(d, n)$ by coning over the cells. This k-split coincides with Σ on $\mathrm{VF}(e_I)$ and hence do both k-splits on the hypersimplex $\Delta(d, n)$. □

Remark 3.16. Note that the construction of subdivisions of the hypersimplex $\Delta(d,n)$ from those of $\Delta_{d-1} \times \Delta_{n-d-1}$ in [HJS12] is more general than the definition we gave, as they take non-regular subdivisions into account.

Moreover, our definition of the tropical Stiefel map is the special case of "pointed support sets" in [FR15]. However, if one restricts the image of the Stiefel map that lifts subdivisions of $\Delta_{d-1} \times \Delta(n-1)$ to those of $\Delta(d, d+n)$ on the face supported by the equation $\sum_{i=1}^{d} x_i = 0$, then one gets all the liftings that occur as a Stiefel lifting on $\Delta(d,n)$ of Fink and Rincón. The restriction corresponds to a deletion of the first d elements in the underlying matroids.

The image of the tropical Stiefel map that we take as definition commutes with the dualization map $e_I \mapsto e_{[n]-I}$, that maps the vertices of $\Delta(d,n)$ to those of $\Delta(n-d,d)$, while in the general case this is false as the following example shows.

The following example illustrates properties of the Stiefel map of Fink and Rincón.

Example 3.17. Consider the Stiefel lifting on $\Delta_3 \times \Delta_5$ given by the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

This induces a subdivision on $\Delta(4,10)$, where the face $\Delta(4,10) \cap \{(0,0,0,0)\} \times \Delta(4,6)$ is subdivided into four maximal cells. Ignoring the first four coordinates leads to a subdivision of $\Delta(4,6)$, which contains the matroid polytope of the matroid with non-bases 3456, 1256 and 3456 as a maximal cell.

This is the dual of the snowflake matroid from Example 2.8. The matroid polytope of the snowflake matroid does not appear in the Stiefel map as it is a non-transversal matroid; see [FR15, Corollary 5.6] and [Oxl11, Example 1.6.3].

3.3 Matroid subdivisions and multi-splits

In this section we will further analyze multi-splits of the hypersimplex. Our goal is to describe the polytopes that occur as maximal cells. We will see that these polytopes correspond to a particular class of matroids.

A subpolytope P of the hypersimplex $\Delta(d,n)$ is called a *matroid polytope* if the vertex-edge graph of P is a subgraph of the vertex-edge graph of $\Delta(d,n)$. Note that the vertices of a matroid polytope are 0/1-vectors and a subset of those of the hypersimplex.

The vertices of a matroid polytope P are the characteristic vectors of the bases of a matroid $\mathrm{M}(P)$. The convex hull of the characteristic vectors of the bases of a matroid

M is the matroid polytope $\mathrm{P}(M)$. See [Oxl11] and [Whi86] for the basic background of matroid theory and [Edm70] for a polytopal description, that we used as definition.

We will give three examples of classes of matroids that are important for this section.

Example 3.18. Clearly the hypersimplex $\Delta(d,n)$ itself is a matroid polytope. The matroid $\mathrm{M}(\Delta(d,n))$ is called the *uniform matroid* of rank d on $[n]$ elements. The d-subsets of $[n]$ are exactly the bases of $\mathrm{M}(\Delta(d,n))$. The uniform matroid has the maximal number of bases among all (d,n)-matroids.

Example 3.19. Let $C_1,\ldots,C_k$ be a partition of the set $[n]$ and $d_i \leq \#(C_i)$ non-negative integers. The matroid $\mathrm{M}(\Delta(d_1,C_1)\times\cdots\times(\Delta(d_k,C_k))$ is called a *partition matroid* of rank $d_1+\ldots+d_k$ on $[n]$. A d-subset S of $[n]$ is a basis of this matroid if $\#(S\cap C_i)=d_i$.

Example 3.20. Let $\emptyset \subsetneq F_1 \subsetneq \ldots \subsetneq F_k \subseteq [n]$ be an ascending chain of sets and $0 \leq r_1 < r_2 < \ldots < r_k$ be integers with $r_\ell < \#(F_\ell)$ for all $\ell \leq k$. The polytope

$$P \;=\; \left\{ x \in \Delta(d,n) \;\middle|\; \sum_{i\in F_\ell} x_i \leq r_\ell \right\}$$

is a matroid polytope. This follows from the analysis of all 3-dimensional octahedral faces of the hypersimplex. None of those is separated by more than one of the additional inequalities and hence the polytope is a matroid polytope. The matroid $\mathrm{M}(P)$ is called a *nested matroid* of rank $r_k+\#([n]-F_k)$ on $[n]$. The sets $F_1,\ldots,F_k$ are the *cyclic flats* of the nested matroid $\mathrm{M}(P)$ if $r_1=0$. If $r_1\neq 0$, then the above and $\emptyset$ are the cyclic flats.

Remark 3.21. There are many cryptomorphic definitions for matroids. Bonin and de Mier introduced in [BdM08] the definition via cyclic flats and their ranks, i.e., unions of minimal dependent sets. In this chapter we only need the very special case of nested matroids, where the lattice of cyclic flats is a chain, see also Chapter 2 where cyclic flats play an important role.

A *matroid subdivision* of $\Delta(d,n)$ is a subdivision into matroid polytopes, i.e., one in which all the (maximal) cells in the subdivision are matroid polytopes. The lifting function of a regular subdivision of a matroid polytope is called a *tropical Plücker vector*, since it arises as the valuation of a classical Plücker vector. Note that the tropical Plücker vectors form a subfan in the secondary fan of the hypersimplex $\Delta(d,n)$. This fan is called the *Dressian* $\mathrm{Dr}(d,n)$.

Each multi-split of the hypersimplex $\Delta(d,n)$ is a matroid subdivision as Theorem 3.15 in combination with the following proposition shows.

Proposition 3.22 ([Rin13],[HJS12])**.** *The image of any lifting function on $\Delta_{d-1}\times\Delta_{n-d-1}$ under the Stiefel map is a tropical Plücker vector.*

From now on let Σ be a k-split of the hypersimplex $\Delta(d,n)$. We investigate which matroid polytopes appear in the subdivision Σ.

Let us briefly introduce some matroid terms. A set S is *independent* in the matroid M if it is a subset of a basis of M. The rank $\operatorname{rk}(S)$ of a set S is the maximal size of an independent set in S. An important operation on a matroid M is the *restriction* $M|F$ to a subset F of the ground set. The set F is the ground set of $M|F$. A set $S \subseteq F$ is independent in $M|F$ if and only if S is independent in M. It is also common to denote the restriction $M|([n]-F)$ by $M \setminus F$ which is called the *deletion* of F. A matroid M is called *connected* if there is no set $\emptyset \subsetneq S \subsetneq [n]$ with $\mathrm{P}(M) = \mathrm{P}(M|S) \times \mathrm{P}(M|([n]-S))$. For each matroid M there is a unique partition $C_1,\dots,C_k$ of $[n]$, such that $\mathrm{P}(M) = \mathrm{P}(M|C_1) \times \cdots \times \mathrm{P}(M|C_k)$. The sets $C_1,\dots,C_k$ are called *connected components* of M. The element $e \in [n]$ is called a *loop* if $\{e\}$ is a connected component and $\mathrm{P}(M|\{e\}) = \Delta(0,\{e\})$. If instead $\mathrm{P}(M|\{e\}) = \Delta(1,\{e\})$, then e is called a *coloop*. The dual operation of the restriction is the *contraction* M/F. The ground set of the matroid M/F is $[n]-F$. A set S is independent in M/F if $\operatorname{rk}_M(S+F) = \#(S) + \operatorname{rk}_M(F)$.

The following describes a relation of the connected components of a matroid and its matroid polytope.

Lemma 3.23 ([Fuj84, Theorem 3.2] and [FS05, Propositions 2.4])**.** *The number of connected components of a matroid M on the ground set $[n]$ equals the difference $n - \dim \mathrm{P}(M)$.*

Example 3.24. An element e is a loop in a partition matroid $\mathrm{M}(\Delta(d_1,C_1) \times \cdots \times (\Delta(d_k,C_k))$ if and only if $e \in C_\ell$ and $\operatorname{rk}(C_\ell) = d_\ell = 0$. The element is a coloop if instead $\operatorname{rk}(C_\ell) = d_\ell = \#(C_\ell)$. The other connected components are those sets C_ℓ with $0 < d_\ell < \#(C_\ell)$.

A nested matroid is loop-free if $d_1 > 0$ and coloop-free if $F_k = [n]$. A loop- and coloop-free nested matroid is connected.

At first we consider the common cell H^Σ in a k-split Σ of $\Delta(d,n)$.

Proposition 3.25. *The common cell H^Σ is a matroid polytope of a loop and coloop-free partition matroid with k connected components.*

Proof. The common cell H^Σ is a cell in a matroid subdivision and hence a matroid polytope. The dimension of this polytope is $n-k$. From Lemma 3.23 follows that the corresponding matroid $M = \mathrm{M}(H^\Sigma)$ has k connected components. Let $C_1,\dots,C_k$ be the connected components of M and $d_\ell = \operatorname{rk}_M(C_\ell)$. Clearly, this is a partition of the ground set $[n]$ and the sum $d_1 + \ldots + d_k$ equals d. The polytope $H^\Sigma = \mathrm{P}(M)$ is the intersection of $\Delta(d,n)$ with an affine space. Hence, there are no further restrictions to the polytope and each matroid polytope $\mathrm{P}(M|C_\ell)$ is equal to $\Delta(d_\ell, C_\ell)$. The common cell H^Σ intersects $\Delta(d,n)$ in the interior, hence $0 < d_\ell < \#(C_\ell)$ and the matroid M is loop and coloop-free. $\square$

We define the relation $\preceq_P$ on the connected components $C_1, \dots, C_k$ of $\mathrm{M}(H^\Sigma)$ depending on a cell $P \in \Sigma$ by

$$C_a \preceq_P C_b \text{ if and only if for each } v \in H^\Sigma \text{ and for each } i \in C_a \text{ and } j \in C_b \text{ with } v_i = 1 \text{ and } v_j = 0 \text{ we have } v + e_j - e_i \in P \ . \tag{3.1}$$

Lemma 3.26. *Let $C_1, \dots, C_k$ be the connected components of the matroid $\mathrm{M}(H^\Sigma)$. The matroid polytope P of a cell in Σ defines a partial order on the connected components $C_1, \dots, C_k$ via $C_a \preceq_P C_b$.*

Proof. Let $i, j \in [n]$ and $v \in H^\Sigma$ be a vertex with $v_i = 1$, $v_j = 0$. Then $v - e_i + e_j \in H^\Sigma$ if and only if there is a circuit in $\mathrm{M}(H^\Sigma)$ containing both i and j. The vector v is the characteristic vector of a basis in $\mathrm{M}(H^\Sigma)$ and adding $e_j - e_i$ corresponds to a basis exchange. This implies that i and j are in the same connected component, i.e., $\preceq_P$ is reflexive.

Let $C_1 \preceq_P C_2 \preceq_P C_1$ and $i \in C_1$, $j \in C_2$. Take $v, w \in H^\Sigma$ with $v_i = w_j = 1$ and $v_j = w_i = 0$. By assumption we have $v - e_i + e_j, w + e_i - e_j \in P$ and since H^Σ is convex

$$\frac{1}{2}(v - e_i + e_j) + \frac{1}{2}(w + e_i - e_j) = \frac{1}{2}v + \frac{1}{2}w \in H^\Sigma \ .$$

A convex combination of points in P lies in H^Σ if and only if all the points are in H^Σ. Hence, we got $v - e_i + e_j \in H^\Sigma$ and therefore $C_1 = C_2$.

Let $C_1 \preceq_P C_2 \preceq_P C_3$, $i \in C_1$, $j \in C_3$ and $v \in H^\Sigma$ with $v_i = 1$ and $v_j = 0$. Consider the cone $Q = \{\lambda x + y \,|\, y \in H^\Sigma, x + y \in P \text{ and } \lambda \geq 0\}$. This is the cone in the fan $\mathcal{F}^\Sigma$ that contains P with the same dimension as P. Let $k, \ell \in C_2$ be indices with $v_k = 1$ and $v_\ell = 0$ and $w = v - e_k + e_\ell$. Then $v, w \in H^\Sigma$ and $v - e_k + e_j$, $w + e_k - e_\ell$, $v - e_i + e_\ell \in P$. That implies

$$v - e_i + e_j \ = \ \frac{1}{3}\left(v + 3(e_j - e_k) + w + 4(e_k - e_\ell) + v + 3(e_\ell - e_i)\right) \in Q \ .$$

Clearly $v - e_i + e_j \in \Delta(d, n)$ and hence $v - e_i + e_j \in P$. This shows that $\preceq_P$ is transitive. □

Before we further investigate the relation $\preceq_P$ we take a look at rays of $\mathcal{F}^\Sigma$. The next Lemma describes the $(n - k + 1)$-dimensional cells in Σ. The k-split Σ has exactly k of these cells and each maximal cell contains $k - 1$ of those; see Proposition 3.7.

Lemma 3.27. *For each $(n-k+1)$-dimensional cell of Σ there are $a, b \in [n]$ such that the cell equals*

$$R_{a,b} \ = \ \left(H^{\Sigma} + \{\,\mu(e_a - e_b) \ | \ \mu \geq 0\}\right) \cap \Delta(d,n) \ .$$

Proof. Let R be a $(n-k+1)$-dimensional cell in Σ. This cell R is a matroid polytope. Hence, all the edges of R are of the form $e_i - e_j$ for some i and j. The common cell H^{Σ} is $(n-k)$-dimensional. Therefore, there is an edge in R that connects a vertex of the common cell H^{Σ} with a vertex v that is not in H^{Σ}. The direction of this edge is unique as $\dim R = 1 + \dim H^{\Sigma}$. These arguments show that R is of the desired form. □

Now we are able to further investigate $\preceq_P$ and hence the cells in the k-split Σ.

Lemma 3.28. *For a connected matroid* $\mathrm{M}(P)$ *the relation* $\preceq_P$ *is a total ordering on the connected components of* $\mathrm{M}(H^{\Sigma})$.

Proof. Let us assume that C_1 and C_2 are two incomparable connected components of $M(H^{\Sigma})$. We define

$$F = \bigcup_{C \preceq_P C_1} C \text{ and } G = \bigcup_{C \preceq_P C_2} C \ .$$

Pick $i \in C_1$ and $j \in C_2$. The matroid $M = \mathrm{M}(P)$ is connected hence there is a circuit A containing both i and j. The set $A \cap F \cap G$ is independent in M, as $i \notin G$. Let $S \supseteq A \cap F \cap G$ be a maximal independent set in $F \cap G$. Let N be the connected component of i in the minor $(M/S) \setminus (F \cap G - S)$. Note that the sets F and G are closed in the matroid $\mathrm{M}(H^{\Sigma})$, and that the elements of $F \cap G - S$ are exactly the loops of the contraction $\mathrm{M}(H^{\Sigma})/S$ and hence also of the matroid M/S. Moreover, $A - S$ is a circuit in M/S, and hence is $j \in N$. We conclude that $C_1, C_2 \subset N$.

The equation $\sum_{\ell \in N} x_\ell = \mathrm{rk}(N)$ defines a face of $\mathrm{P}(M)$. This face is contained in $\mathrm{P}(N) \times \Delta(d - \mathrm{rk}(N), [n] - N) \subsetneq \Delta(\mathrm{rk}(N), N) \times \Delta(d - \mathrm{rk}(N), [n] - N)$.

[Her11, Proposition 4.8] states that the induced subdivision on a face of a k-split is either trivial or a multi-split with fewer than k maximal cells. We are in the latter case, as the induced subdivision on $\Delta(\mathrm{rk}(N), N)$ is non-trivial. This comes from the fact that the incomparable components C_1 and C_2 are contained in N, therefore the hypersimplex $\Delta(\mathrm{rk}(C_1 \cup C_2), C_1 \cup C_2)$ is non-trivially subdivided, and hence the corresponding face of $\Delta(\mathrm{rk}(N), N)$.

Hence, we can assume without loss of generality that $F \cap G = \emptyset$. Clearly, the following two inequalities are valid for $\mathrm{P}(M)$ and the face that they define includes H^{Σ}

$$\sum_{i \in F} x_i \leq \mathrm{rk}(F) \text{ and } \sum_{i \in G} x_i \leq \mathrm{rk}(G) \ .$$

Let R be the unique ray in Σ that is not contained in $\mathrm{P}(M)$. There is a vertex $v \notin H^\Sigma$ of $\Delta(d,n)$ that is contained in both R and in $H^\Sigma - e_a + e_b$ for some $a, b \in [n]$. The rays in Σ positively span the complete space. Hence, we get the bounds

$$\mathrm{rk}(F) + 1 \geq \sum_{i \in F} v_i > \mathrm{rk}(F) \text{ and } \mathrm{rk}(G) + 1 \geq \sum_{i \in G} v_i > \mathrm{rk}(G) \ .$$

This implies that $b \in F \cap G$. We conclude that either $F \preceq_P G$ or $G \preceq_P F$. □

Example 3.29. Consider the octahedron $\Delta(2,4)$. The hyperplane $x_1 + x_2 = x_3 + x_4$ through the four vertices $e_1 + e_3$, $e_2 + e_3$, $e_1 + e_4$ and $e_2 + e_4$ strictly separates the vertices $e_1 + e_2$ and $e_3 + e_4$. Moreover the hyperplane splits $\Delta(2,4)$ into two maximal cells, the corresponding subdivision Σ is a 2-split. The partition matroid $\mathrm{M}(H^\Sigma)$ has four bases and two connected components $C_1 = \{1,2\}$ and $C_2 = \{3,4\}$.

Let M be the $(2,4)$-matroid with the following five bases $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$. The polytope $\mathrm{P}(M)$ is an egyptian pyramid and a maximal cell in Σ. The inequality $x_1 + x_2 \leq 1$ is valid for $\mathrm{P}(M)$ and hence $C_2 \not\preceq_P C_1$. It is easy to verify that $C_1 \preceq_P C_2$ as $e_3 + e_4 \in \mathrm{P}(M)$.

We derive the following description for the maximal cells of a k-split, which we already saw in Example 3.29.

Lemma 3.30. *Let P be a maximal cell of the k-split Σ of $\Delta(d,n)$. Furthermore, let $C_1 \preceq_P \ldots \preceq_P C_k$ be the order of the connected components of $M(H^\Sigma)$. Then $x \in P \subsetneq \mathbb{R}^n$ if and only if $x \in \Delta(d,n)$ with*

$$\sum_{\ell=1}^{h} \sum_{i \in C_\ell} x_i \leq \sum_{\ell=1}^{h} \mathrm{rk}_M(C_\ell) \text{ for } h \leq k \ . \tag{3.2}$$

Proof. First, we will show that each $x \in P$ fulfills the inequalities (3.2). The following equation holds for each $v \in H^\Sigma \subsetneq P$

$$\sum_{i \in C_\ell} v_i \ = \ \mathrm{rk}_M(C_\ell) \ .$$

Lemma 3.27 shows that a ray of $\mathcal{F}^\Sigma$ is of the form $H^\Sigma + \mathrm{pos}(e_j - e_i)$ for some $i, j \in [n]$. Clearly, for each pair (i,j) of such elements and every point $v \in H^\Sigma$ with coordinates $v_j = 0$ and $v_i = 1$ we get $v + e_j - e_i \in \Delta(d,n) - H^\Sigma$.

Hence, $v + e_j - e_i \in P$ implies that $C_a \preceq_P C_b$ for $i \in C_a$ and $j \in C_b$. This is $a \leq b$. This proves (3.2) for all points that are in a ray and in P. Each point $x \in P$ is a positive combination of vectors in rays of the fan $\mathcal{F}^\Sigma$, hence the inequalities (3.2) are valid for all vectors in P.

Conversely, we will show that each point in $\Delta(d,n)$, that is valid for (3.2), is already in P. The left hand side of (3.2) is a totally unimodular system, i.e., all square minors are either -1, 0 or 1. Hence all the vertices of the polytope are integral, even if we add the constraints $0 \leq x_i \leq 1$. This is precisely a statement of [Sch86, Theorem 19.3].

Take a vertex v of $\Delta(d,n)$ that is valid. Either $v \in H^\Sigma$ and hence $v \in P$ or at least an inequality of (3.2) is strict. In this case let $a = \min\{\ell \in [n] \mid \sum_{i \in C_\ell} v_i < \mathrm{rk}_M(C_\ell)\}$ and $b = \min\{\ell \in [n] \mid \sum_{i \in C_\ell} v_i > \mathrm{rk}_M(C_\ell)\}$. Note that both sides of the inequality (3.2) for $h = k$ sum up to d. Hence, both of the minima exist and $a < b$, otherwise the inequality (3.2) with $h = a$ would be invalid. Pick $i \in C_b$ with $v_i = 0$ and $j \in C_a$ with $v_j = 1$. The vector $w = v - e_j + e_i$ is another vertex of $\Delta(d,n)$, that is valid for (3.2). Moreover, $w \in P$ implies that $v \in P$ since $C_a \preceq_P C_b$. We conclude that P has the desired exterior description. □

Now we are able to state our second main result, which allows us to construct all k-splits of the hypersimplex explicitly and relate them to nested matroids.

Theorem 3.31. *A maximal cell in any k-split Σ of $\Delta(d,n)$ is the matroid polytope $P(M)$ of a connected nested matroid M.*

More precisely, the cyclic flats of M are the $k+1$ sets $\emptyset \subsetneq C_1 \subsetneq C_1 \cup C_2 \subsetneq \ldots \subsetneq \bigcup_{i=1}^k C_i = [n]$, where $C_1 \preceq_P \ldots \preceq_P C_k$ are the connected components of $M(H^\Sigma)$.

Moreover, the other k maximal cells are given by a cyclic permutation of the sets $C_1, C_2 \ldots, C_k$. In particular, each maximal cell in a multi-split of $\Delta(d,n)$ determines all the cells.

Proof. Fix a maximal cell P in Σ and let $C_1 \preceq_P \ldots \preceq_P C_k$ be the connected components of the partition matroid $N = \mathrm{M}(H^\Sigma)$. We define $F_\ell = \bigcup_{i=1}^{\ell} C_i$ for all $1 \leq \ell \leq k$. We have

$$0 < \mathrm{rk}_N(F_1) < \ldots < \mathrm{rk}_N(F_{\ell-1}) < \mathrm{rk}_N(F_{\ell-1}) + \mathrm{rk}_N(C_\ell) = \mathrm{rk}_N(F_\ell) < \ldots < \mathrm{rk}_N(F_k) = d.$$

The sets F_ℓ and $\emptyset$ are the cyclic flats of nested matroid M with ranks given by $\mathrm{rk}_N(F_\ell)$ respectively 0; see Example 3.20. The matroid polytope $P(M)$ of M is exactly described by Lemma 3.30. This implies that the maximal cell P is the matroid polytope $P(M)$ with the desired $k+1$ cyclic flats.

The intersection of all maximal cells of the k-split Σ excluding the cell $P(M)$ is a ray of $\mathcal{F}^\Sigma$. This ray $R_{a,b}$ contains a vertex $w \in \Delta(d,n)$ of the form $v + e_a - e_b$, where $v \in H^\Sigma$. We can choose this vertex w, such that $w \notin P(M)$. We deduce from (3.2) the following strict inequalities for w:

$$\sum_{\ell=1}^{h} \sum_{i \in C_\ell} w_i \;>\; \sum_{\ell=1}^{h} \mathrm{rk}_M(C_\ell) \text{ for all } h < k \;.$$

As $w = v + e_a - e_b$ and $\sum_{i \in C_\ell} v_i = \mathrm{rk}(C_\ell)$, we get for $h = 1$ that $a \in C_1$ and from $h = k-1$ that $b \in C_k$. This implies that for every maximal cell $Q \neq \mathrm{P}(M)$ of Σ we have $C_k \preceq_Q C_1$.

Moreover, each maximal cell $Q \neq \mathrm{P}(M)$ shares a facet with $\mathrm{P}(M)$. Let $\sum_{\ell=1}^{m} \sum_{i \in C_\ell} x_i = \sum_{\ell=1}^{m} \mathrm{rk}_M(C_\ell)$ be the facet defining equation. This facet implies $C_m \not\preceq_Q C_{m+1}$. All the other inequalities of (3.2) are valid for Q. We conclude that $C_{m+1} \preceq_Q \ldots \preceq_Q C_k \preceq_Q C_1 \preceq_Q \ldots \preceq_Q C_m$. $\square$

Note that there is a finer matroid subdivision for any k-split of the hypersimplex $\Delta(d,n)$, except for the case $k = d = 2$ and $n = 4$. Moreover, each matroid polytope of a connected nested matroid with at least four cyclic flats on at least $k + d + 1$ elements occurs in a coarsest matroid subdivision, which is not a k-split.

In contrast we have that for each connected nested (d,n)-matroid M with $k + 1$ cyclic flats there is a unique k-split of the hypersimplex $\Delta(d,n)$ that contains $\mathrm{P}(M)$ as a maximal cell. Conversely, a k-split of the hypersimplex $\Delta(d,n)$ determines k such nested matroids. Furthermore, each k-split Σ determines a unique loop- and coloop-free partition matroid $\mathrm{M}(H^\Sigma)$, while each ordering of the connected components of $\mathrm{M}(H^\Sigma)$ leads to a unique connected nested (d,n)-matroid with $k + 1$ cyclic flats. We conclude the following enumerative relations.

Corollary 3.32. *The following three sets are pairwise in bijection:*

1. *The loop- and coloop-free partition (d,n)-matroids with k connected components,*
2. *the collections of all connected nested (d,n)-matroids with $k+1$ cyclic flats, whose pairwise set differences of all of those cyclic flats coincide,*
3. *the collections of k-splits of $\Delta(d,n)$ with the same interior cell.*

Moreover, the collections in (2) have all the same size $k!$ and those in (3) are of size $(k-1)!$.

Now we are able to count k-times all k-splits of the hypersimplex $\Delta(d,n)$ by simply counting nested matroids, i.e., ascending chains of subsets. The following is a natural generalization of the formulae that count 2-splits in [HJ08, Theorem 5.3] and 3-splits in [Her11, Corollary 6.4].

Proposition 3.33. *The total number of k-splits in the hypersimplex $\Delta(d,n)$ equals*

$$\frac{1}{k} \sum_{\alpha_1=2}^{\beta_1-2(k-1)} \cdots \sum_{\alpha_{k-1}=2}^{\beta_{k-1}-2} \mu_k^{d,n}(\alpha_1,\ldots,\alpha_{k-1}) \prod_{j=1}^{k-1} \binom{\beta_j}{\alpha_j}$$

where $\beta_i = n - \sum_{\ell=1}^{i-1} \alpha_\ell$ and

$$\mu_k^{d,n}(\alpha_1,\ldots,\alpha_{k-1}) \;=\; \#\left(\left\{ x \in \mathbb{Z}^k \;\middle|\; \sum_{i=1}^{k} x_i = d \text{ and } 0 < x_j < \alpha_j \text{ for } j \leq k \right\}\right)$$

with $\alpha_k = \beta_k$.

Proof. Fix non-negative numbers $\alpha_1, \ldots, \alpha_k$ that sum up to n. The number of connected nested (d,n)-matroids with $k+1$ cyclic flats $\emptyset = F_0 \subsetneq F_1, \ldots, F_k = [n]$ that satisfy $\#(F_j - F_{j-1}) = \alpha_j$ is determined by the following product of binomial coefficients weighted by the number $\mu_k^{d,n}$ of possibilities for ranks on the cyclic flats

$$\mu_k^{d,n}(\alpha_1, \ldots, \alpha_{k-1}) \prod_{j=1}^{k} \binom{n - \alpha_1 - \ldots - \alpha_{j-1}}{\alpha_j} .$$

Clearly, the rank function satisfies $0 < \operatorname{rk}(F_j) - \operatorname{rk}(F_{j-1}) < \#(F_j - F_{j-1}) = \alpha_j$, hence $\alpha_j \geq 2$. Moreover, the last binomial coefficient is equal to one. The number α_k is determined by $\alpha_k = n - \sum_j^{k-1} \alpha_j$. We get that the number of connected nested (d,n)-matroids with $k+1$ cyclic flats is given by

$$\sum_{a_1=2}^{\beta_1 - 2(k-1)} \cdots \sum_{\alpha_{k-1}=2}^{\beta_{k-1}-2} \mu_k^{d,n}(\alpha_1, \ldots, \alpha_{k-1}) \prod_{j=1}^{k-1} \binom{\beta_j}{\alpha_j} .$$

We derive the number of k-splits by division by k. This completes the proof. □

See Table A.3 for concrete values of the formula. Table A.4 lists the number of k-splits up to symmetry, i.e., coordinate permutation.

Example 3.34. Consider the case that $n = d + k = 2k$. The number of loop- and coloop-free partition (k,n)-matroids equals $(2k-1)!! = (2k-1)(2k-3)\cdots 1$, as $\alpha_j = 2$ for all $j \leq k$. The number of k-splits in $\Delta(k, 2k)$ equals $(k-1)!(2k-1)!!$ and those of connected nested matroids with $k+1$ cyclic flats $k!(2k-1)!!$. Note that in this case all these k-splits, partitions and nested matroids are equivalent under reordering of the $[n]$ elements.

We will now use the combination of Theorem 3.15 and Theorem 3.31 to obtain an enumeration of all k-splits of the product of simplices $\Delta_{d-1} \times \Delta_{\ell-1}$. Note that the sizes of the common cells of multi-splits of a hypersimplex vary, and in general there are more multi-splits in a hypersimplex than in one of its vertex figures. Hence, we cannot simply count the k-splits of a hypersimplex by a double counting argument.

Suppose we fix a k-split with the connected components $C_1, \ldots C_k$ of $\mathrm{M}(H^\Sigma)$. Any d-set I of $[n]$ determines a vertex figure of the hypersimplex $\Delta(d,n)$. The vertex e_I is in the common cell if and only if for all i we have $\#(C_i \cap I) = \operatorname{rk}(C_i)$. Clearly, the set I splits C_i into the two sets $C_i \cap I$ and $C_i - I$. This argumentation leads to the following theorem.

Theorem 3.35. *The k-splits of $\Delta_{d-1} \times \Delta_{\ell-1}$ are in bijection with collections of k pairs $(A_1, B_1), \ldots, (A_k, B_k)$, such that $A_1, \ldots, A_k$ is a partition of $[d]$ and $B_1, \ldots, B_k$ is a partition of $[\ell]$. In particular, the number of k-splits of $\Delta_{d-1} \times \Delta_{\ell-1}$ equals*

$$\frac{1}{k} \left(\sum_{\alpha_1=1}^{\beta_1-(k-1)} \cdots \sum_{\alpha_{k-1}=1}^{\beta_{k-1}-1} \prod_{j=1}^{k-1} \binom{\beta_j}{\alpha_j} \right) \cdot \left(\sum_{\gamma_1=1}^{\delta_1-(k-1)} \cdots \sum_{\gamma_{k-1}=1}^{\delta_{k-1}-1} \prod_{j=1}^{k-1} \binom{\delta_j}{\gamma_j} \right) ,$$

where $\beta_i = d - \sum_{j=1}^{i-1} \alpha_j$ and $\delta_i = \ell - \sum_{j=1}^{i-1} \gamma_j$.

3.4 Coarsest matroid subdivisions

We have enumerated specific coarsest matroid subdivisions. In this section we will compare two constructions for coarsest matroid subdivisions. We have seen already the first of these constructions for matroid subdivisions. The Stiefel map lifts rays of $\Delta_{d-1} \times \Delta_{n-d-1}$ to rays of the Dressian $\mathrm{Dr}(d, n)$. This construction for rays has been studied in [HJS12] under the name of "tropically rigid point configurations". Other (coarsest) matroid subdivisions can be constructed via matroids. Let M be a (d, n)-matroid. We recall, that the *corank vector* of M is the map

$$\rho_M : \binom{[n]}{d} \to \mathbb{N}, \quad S \mapsto d - \mathrm{rk}_M(S) \ .$$

The corank vector is a tropical Plücker vector. Moreover, the induced subdivision contains the matroid polytope $\mathrm{P}(M)$ as a cell; see [Spe05, Example 4.5.4] and Proposition 2.34.

There are coarsest matroid subdivisions, obtained from corank vectors, that are not in the image of the Stiefel map; see [HJS12, Figure 7] and Theorem 2.45.

There are matroid subdivisions that are both induced by the Stiefel map and corank subdivisions.

Example 3.36. We have seen in Theorem 3.15, that every multi-split of the hypersimplex is induced by the Stiefel map. Moreover, each multi-split is a corank subdivision. The maximal cells are nested matroids. This follows from Theorem 3.31 combined with the methods of Section 2.4.

A subdivision that is induced by a corank vector satisfies the following criteria. With these we are able to certify that a matroid subdivision is not induced by a corank vector.

Lemma 3.37. *Let M be a (d, n)-matroid and Σ the corank subdivision of $\mathrm{P}(M)$. For each vertex v of the hypersimplex $\Delta(d, n)$ a (maximal) cell $\sigma \in \Sigma$ exists, such that $v \in \sigma$ and $\sigma \cap \mathrm{P}(M) \neq \emptyset$. In other words, the cell $P(M)$ together with the neighboring cells cover all vertices of $\Delta(d, n)$.*

Proof. Let M be a (d, n)-matroid and Σ the corresponding corank subdivision of the hypersimplex $\Delta(d, n)$. Furthermore, let v be a vertex of the hypersimplex $\Delta(d, n)$. Then $v = e_S$ for a set $S \in \binom{[n]}{d}$. Given a maximal independent subset I of S, the set I can be enlarged to a basis B with $d - \mathrm{rk}(S)$ elements of an arbitrary basis of M. Hence, there is a sequence $S_{d-\mathrm{rk}(S)}, \ldots, S_0 \in \binom{[n]}{d}$, such that $S_{d-\mathrm{rk}(S)} = S$, $\#(S_j \cap S_{j+1}) = d - 1$, $S_0 = B$ and $d - \mathrm{rk}(S_j) = j$ for all $0 \leq j < d - \mathrm{rk}(S)$. Thus, the corresponding $d - \mathrm{rk}(S) + 1$ vertices of the lifted polytope of $\Delta(d, n)$ lie on the hyperplane $\sum_{i \in B} x_i = x_{n+1}$, where x_{n+1} is the height coordinate, i.e., the corank. This hyperplane determines a face of the lifted polytope and hence a cell $\sigma \in \Sigma$. Both vertices v and $e_B \in \mathrm{P}(M)$ are contained in this cell σ. □

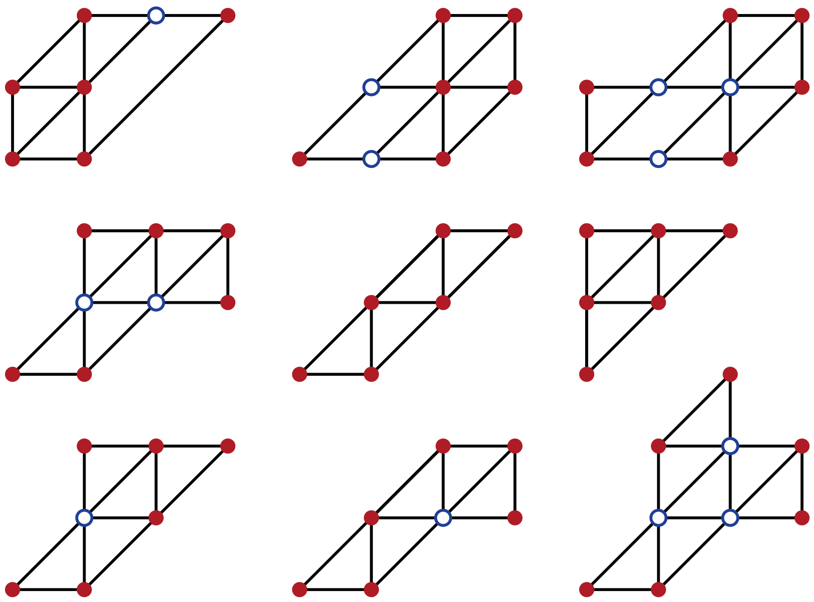


Figure 3.6: The nine rigid tropical point configurations of Example 3.39, each of which is a tropical convex hull of six points.

Lemma 3.38. *Let Σ be a subdivision of the hypersimplex $\Delta(d, n)$, such that the subdivision is the corank subdivision of a connected matroid and induced by a regular subdivision of the product of simplices via the Stiefel map. The subdivision on the product of simplices $\Delta_{d-1} \times \Delta(n-d-1)$ is realizable with a 0/1-vector as lifting function.*

Proof. Clearly, the corank subdivision Σ of the matroid M is regular. Moreover, if Σ is induced by the Stiefel map, then there is a vertex v that is contained in each maximal cell. The matroid polytope $\mathrm{P}(M)$ is a maximal cell as M is connected. Hence, the vertex v is a vertex of $\mathrm{P}(M)$ and the characteristic vector of a basis of M. This implies that the neighbors of v are of corank 0 and 1. This shows that the restriction of the corank lifting to the neighbors of v has the required form. □

We will apply Lemma 3.38 to tropical point configurations. These are vectors in the tropical torus $\mathbb{R}^d/(1,\dots,1)\mathbb{R}$. The line segment in the tropical torus between the two points v and w is the set $\{u \in \mathbb{R}^d/(1,\dots,1)\mathbb{R} \mid \lambda, \mu \in \mathbb{R} \text{ and } u_i = \min(v_i + \lambda,\, w_i + \mu)\}$. Note that such a line segment consists of several ordinary line segments, with additional (pseudo-)vertices. The *tropical convex hull* of a set of points is the smallest set such that all line segments between points are in this set. Such a tropical convex hull of finitely many points decomposes naturally in a polyhedral complex. The cells in the tropical convex hull of a tropical point configuration of $(n-d)$ points in $\mathbb{R}^d/(1,\dots,1)\mathbb{R}$ are in bijection with the cells of a regular subdivision of the product $\Delta_{d-1} \times \Delta_{n-d-1}$, where the height of $e_i + e_j$ is the j-th coordinate of the i-th point in the tropical point configuration;

see [DS04, Lemma 22]. A tropical point configuration is *tropically rigid* if it induces a coarsest (non-trivial) subdivision on the product of simplices $\Delta_{d-1} \times \Delta_{n-d-1}$.

A tropical point configuration corresponds to a corank subdivision if the points are realizable by 0/1 coordinates in $\mathbb{R}^d$ or equivalently by -1, 0 and 1 in the tropical torus. In particular, there is a point that has lattice distance at most one to each other point. This criteria certify that the next examples are not corank subdivisions.

The following illustrates examples of coarsest non-corank subdivisions.

Example 3.39. Figure 3.6 shows nine rigid tropical point configurations out of 36 symmetry classes. They correspond to nine coarsest subdivisions of $\Delta_2 \times \Delta_5$. The Stiefel map of those induces coarsest matroid subdivisions of the hypersimplex $\Delta(3,9)$. None of those is a corank subdivision. Proposition 3.40 shows that these are all rigid tropical point configurations that do not lift to a corank subdivision of the hypersimplex $\Delta(3,9)$.

We lifted those to rays of the hypersimplex $\Delta(3,9)$ and checked whether they are equivalent to corank liftings. For this computation we used both the software `polymake` [GJ00] and `mptopcom` [JJK17]. Before we state our computational result, note that there is a natural symmetry action of the symmetric group on n elements on the hypersimplex $\Delta(d,n)$. This group acts on the hypersimplex, by permutation of the coordinate directions. From our computations we got the following result.

Proposition 3.40. *The nine liftings illustrated as tropical point configuration in Figure 3.6 lead to coarsest regular subdivisions of $\Delta(3,9)$. These are, up to symmetry, all coarsest regular subdivisions of $\Delta(3,9)$ that are induced by the Stiefel map and not by a corank lift.*

We will close with two enumerative results about the number of coarsest regular matroid subdivisions of the hypersimplex $\Delta(d,n)$ for small parameters d and n. With the previously mentioned methods we have computed all coarsest regular subdivisions of $\Delta_{d-1} \times \Delta_{n-d-1}$ for small parameters of d and n and lifted them to the hypersimplex. Note that this is a massive computation, as there are 7402421 symmetry classes of triangulations for the product $\Delta_3 \times \Delta_4$ and the acting symmetric group has 9! elements. Another example is $\Delta_2 \times \Delta_6$ where the number of symmetry classes of triangulations in the regular flip component is 533242 and the group has $10! = 3628800$ elements. For each symmetry class a convex hull computation is necessary and after that another reduction that checks for symmetry.

The number of all these subdivisions up to symmetry is listed in Table 3.1a on the last page. Note that we do not count the number of coarsest regular subdivisions of $\Delta_{d-1} \times \Delta_{n-d-1}$.

For our second result we computed all corank subdivisions for all matroids in the `polymake` database available at db.polymake.org. This database is based on a classification of matroids of small rank with few elements of Matsumoto, Moriyama, Imai

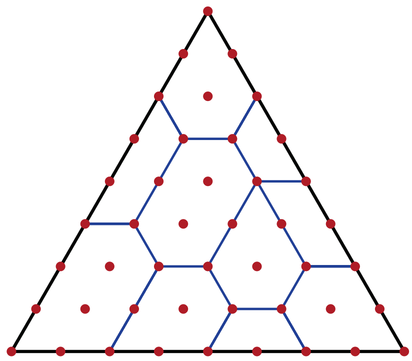


Figure 3.7: A coarsest non-regular subdivision of $8 \cdot \Delta_2$, that transfers to the product of simplices $\Delta_2 \times \Delta_7$ via the Cayley trick, and to the hypersimplex $\Delta(3, 11)$ via the tropical Stiefel map.

and Bremner [MMIB12]. We got the coarsest regular subdivisions by computing the secondary cones. The number of all of these subdivisions is given in Table 3.1b. Note that the product $\Delta_2 \times \Delta_7$ and hence the hypersimplex $\Delta(3, 11)$ has a coarsest non-regular subdivision; see Figure 3.7 and [DLRS10, Exercise 9.5].

Combining both techniques we got the following result.

Proposition 3.41. *The number of coarsest matroid subdivisions of $\Delta(d, n)$ for $d \leq 4$ and $n \leq 10$, excluding $d = 4$, $n = 10$, is bounded from below by the numbers listed in Table 3.2.*

Table 3.1: Numbers of symmetry classes of coarsest matroid subdivisions in the hypersimplex $\Delta(d, n)$.

a) The number in the Stiefel image.

$d \backslash n$	4	5	6	7	8	9	10
2	1	1	2	2	3	3	4
3		1	3	5	11	36	207
4			2	5	39	2949	–

b) The number of corank subdivisions.

$d \backslash n$	4	5	6	7	8	9	10
2	1	1	2	2	3	3	4
3		1	3	5	12	38	139
4			2	5	33	356	–

Table 3.2: The number of coarsest matroid subdivisions in $\Delta(d,n)$ that are either corank subdivisions or in the image of the Stiefel map.

a) The number without any identifications.

$d\backslash n$	4	5	6	7	8	9	10
2	3	10	25	56	119	246	501
3		10	65	616	15470	1220822	167763972
4			25	616	217945	561983523	–

b) The number of symmetry classes.

$d\backslash n$	4	5	6	7	8	9	10
2	1	1	2	2	3	3	4
3		1	3	5	12	47	287
4			2	5	43	3147	–

4 Algorithms for tropical linear spaces

4.1 Introduction

Tropical linear spaces are among the most basic objects in tropical geometry [MS15, Chapter 4]. In polyhedral geometry language they form polyhedral complexes which are dual to regular matroid subdivisions of hypersimplices. We have studied special classes of those in the previous two chapters. We already have seen that hypersimplices correspond to matroid polytopes of uniform matroids. Research on matroid subdivisions and related objects goes back to Dress and Wenzel [DW92] and to Kapranov [Kap93]. Speyer instigated a systematic study in the context of tropical geometry [Spe08], while suitable algorithms have been developed and implemented by Rincón [Rin13].

Here we present a new algorithm for computing tropical linear spaces, which is implemented in the software system `polymake` [GJ00]. Moreover, we report on computational experiments. Our approach has two key ingredients. First, our method is completely polyhedral – in contrast to Rincón's algorithm [Rin13], which primarily rests on exploiting matroid data. Employing the polyhedral structure has the advantage that this procedure naturally lends itself to interesting generalizations and variations. In particular, this includes tropical linear spaces corresponding to non-trivially valuated matroids. Second, our method fundamentally relies on an algorithm of Ganter [Gan87, GR91] for enumerating the closed sets in a finite closure system. This procedure is a variant of breadth-first-search in the Hasse diagram of the poset of closed sets. As a consequence the computational costs grow only linearly with the number of edges in the Hasse diagram, i.e., the number of covering pairs among the closed sets. So this complexity is asymptotically optimal in the size of the output, and this is what makes our algorithm highly competitive in practice. The challenge is to implement the closure operator and to intertwine it with the search in such a way that it does not impair the output-sensitivity.

Kaibel and Pfetsch employed Ganter's algorithm for enumerating face lattices of convex polytopes [KP02], and this was later extended to bounded subcomplexes of unbounded polyhedra [HJP13]. Here this is generalized further to arbitrary regular subdivisions and their duals. Such a dual has been called *tight span* in [HJS12] as it generalizes the tights spans of finite metric spaces studied by Isbell [Isb64] and Dress [Dre84]. The tight span of an arbitrary polytopal complex may be seen as a special case of the dual block complex of a cell complex; e.g., see [Mun84, §64]. From a topological point of view subdivisions of point configurations are cell decompositions of balls, which, in turn, are special cases of manifolds with boundary. The duality of manifolds with boundary is classically known as Lefschetz duality (e.g., see [Mun84, §70]), and this generalizes Poincaré duality as well as cone polarity. We introduce a new object, called the *extended tight span*, which contains the tight span, but which additionally takes duals of certain boundary cells into account. This works for an arbitrary polytopal subdivision, Σ, in which case the extended tight span of Σ with respect to a set $\Gamma \subset \partial\Sigma$ is just a partially ordered set. If, however, Σ is regular, then the tight span can be equipped with a natural polyhedral structure. We give an explicit coordinatization. In this way tropical linear spaces arise as the extended tight spans of matroid subdivisions with respect to those boundary cells which correspond to loop-free matroids. While a tropical linear space can be given several polyhedral structures, the structure as an extended tight span is the coarsest. Algorithmically, this has the advantage of being the sparsest, i.e., being the one which takes the least amount of memory. In this sense, this is the canonical polyhedral structure of a tropical linear space.

This chapter is organized as follows. We start out with recalling basic facts about general closure systems with a special focus on Ganter's algorithm [Gan87, GO16]. Next we introduce the extended tight spans, and this is subsequently specialized to tropical linear spaces. We compare the performance of Rincón's algorithm [Rin13] with our new method. To exhibit one application the chapter closes with a case study on the f-vectors of tropical linear spaces.

4.2 Closure systems, lower sets and matroids

While we are mainly interested in applications to tropical geometry, it turns out that it is useful to start out with some fundamental combinatorics. This is the natural language for Ganter's procedure, which we list as Algorithm 4.1 below.

Definition 4.1. A *closure operator* on a set S is a function $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$ on the power set of S, which fulfills the following axioms for all subsets $A, B \subseteq S$:

(i) $A \subseteq \mathrm{cl}(A)$ (Extensiveness).

(ii) If $A \subseteq B$ then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ (Monotonicity).

(iii) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ (Idempotency).

A subset A of S is called *closed*, if $\mathrm{cl}(A) = A$. The set of all closed sets of S with respect to some closure operator is called a *closure system*.

Classical examples include the following. If the set S carries a topology then the function which sends any subset A to the smallest closed set (defined as the complement of an open set) containing A is a closure operator, called the *topological closure*. If the set S is equipped with a group structure then the function which sends any subset A to the smallest subgroup containing A is a closure operator. Throughout the following we are particularly interested in the case where the set $S = [n]$ is finite.

The closed sets of a closure system (S, cl) are partially ordered by inclusion. The resulting poset is the *closure poset* induced by (S, cl). The *Hasse diagram* of (S, cl) is the directed graph whose nodes are the closed sets and whose arcs correspond to the covering relations of the closure poset. We assume that all arcs are directed upward, i.e., toward the larger set. Ganter's Algorithm 4.1 computes the Hasse diagram of a finite closure system; see [Gan87, GO16] and [GR91], where you will also find a different version that enumerates all closed sets and not their inclusions. As its key property each closed set is pushed to the queue precisely once, and this entails that the running time is linear in the number of edges of the Hasse diagram, i.e., the algorithm is *output-sensitive*.

Algorithm 4.1: Produces the Hasse diagram of a finite closure system.

Input: A set S and a closure operator cl on S
Output: The Hasse diagram of (S, cl)
$H \leftarrow$ empty graph
Queue $\leftarrow [\mathrm{cl}(\emptyset)]$
add node for closed set $\mathrm{cl}(\emptyset)$ to H
while *Queue is not empty* **do**
  $N \leftarrow$ first element of Queue, remove N from Queue
  forall *minimal* $N_i := \mathrm{cl}(N \cup \{i\})$, *where* $i \in S \backslash N$ **do**
    if N_i *does not occur as a node in* H *yet* **then**
      add new node for closed set N_i to H
      add N_i to Queue
    add arc from N to N_i to H
return H

Example 4.2. Based on Algorithm 4.1, Kaibel and Pfetsch [KP02] proposed a method to compute the face lattice of a convex polytope P. This can be done in two different ways: A face of a polytope can either be identified by its set of vertices or by the set of facets it is contained in.

In the first case, the set S is the set of vertices and the closure of a set is the smallest face containing this set.

In the second case, the set S is the set of facets. Let $F \subseteq S$. The intersection of the facets in F is a face Q_F of P. The closure of F is defined as the set of all facets which contain Q_F. Note that, with this approach, Algorithm 4.1 actually computes the face lattice with inverted relations.

In both cases, the closed sets are exactly the faces of P and the closure operator is given in terms of the vertex–facet incidences.

Example 4.3. Instead of polytopes, one can also compute the face lattice of a polyhedral fan in much the same manner. The crucial problem is to define the closure of a set of rays which is not contained in any cone. The solution to this is to extend the set S to contain not only all rays, but also an additional artificial element, say ∞. Now the closure of a set $F \subseteq S$ is either the smallest cone containing it, if it exists, or the full set S. In particular, this ensures that the length of a maximal chain in the face lattice of a k-dimensional fan is always $k+1$.

The following class of closure systems is ubiquitous in combinatorics and tropical geometry. The monographs by White [Whi86] and Oxley [Oxl11] provide introductions to the subject.

Definition 4.4. Let S be a finite set equipped with a closure operator $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$. The pair (S, cl) is a *matroid* if the following holds in addition to the closure axioms:

(iv) If $A \subseteq S$ and $x \in S$, and $y \in \mathrm{cl}(A \cup \{x\}) \setminus \mathrm{cl}(A)$, then $x \in \mathrm{cl}(A \cup \{y\})$ (MacLane–Steinitz Exchange).

This is one of many ways to define a matroid; see [Oxl11, Lemma 1.4.3] for explicit cryptomorphisms. The closed sets of a matroid are called *flats*.

Remark 4.5. For matroids it is not necessary to check for the minimality of the closed sets N_i in Algorithm 4.1. In view of Axiom (iv) this is always satisfied. This application of the algorithm also demonstrates that, while the empty set is typically closed, this does not always need to be the case. In fact, the closure of the empty set in a matroid is the set of all *loops*.

For our applications it will be relevant to consider special closure systems which are derived from other closure systems in the following way. A *lower set* Λ of the closure system (S, cl) is a subset of the closed sets such that for all pairs of closed sets with $A \subseteq B$ we have that $B \in \Lambda$ implies $A \in \Lambda$.

Proposition 4.6. *Let* (S, cl) *be a closure system with lower set* Λ*. Then the function* cl_Λ *which is defined by*

$$\mathrm{cl}_\Lambda(A) = \begin{cases} \mathrm{cl}(A) & \text{if } \mathrm{cl}(A) \in \Lambda \\ S & \text{otherwise} \end{cases}, \tag{4.1}$$

is a closure operator on S.

Proof. Extensiveness and idempotency are obvious. We need to show that monotonicity holds. To this end consider two closed sets $A \subseteq B \subseteq S$. Suppose first that A lies in the lower set Λ. Then $\mathrm{cl}_\Lambda(A) = \mathrm{cl}(A) \subseteq \mathrm{cl}(B) \subseteq \mathrm{cl}_\Lambda(B)$. If, however, $A \notin \Lambda$, then $B \notin \Lambda$ as Λ is a lower set. In this case we have $\mathrm{cl}_\Lambda(A) = S = \mathrm{cl}_\Lambda(B)$. □

Example 4.7. An unbounded convex polyhedron is *pointed* if it does not contain any affine line. In that case the polyhedron is projectively equivalent to a convex polytope, with a marked face, the *face at infinity*; see, e.g., [JT13, Theorem 3.36]. So we arrive at the situation where we have a convex polytope P with a marked face F. Now the set of faces of P which intersects F trivially forms a lower set Λ in the closure system of faces of P. In this way combining Example 4.2 with Proposition 4.6 and Algorithm 4.1 gives a method to enumerate the bounded faces of an unbounded polyhedron. Ignoring the entire set S, which is closed with respect to cl_Λ but not bounded, recovers the main result from [HJP13].

Example 4.8. For a d-polytope P and $k \le d$ the faces of dimension at most k form a lower set. This is the *k-skeleton* of the polytope P.

The closure operators from all examples in this section are implemented in `polymake` [GJ00].

4.3 Extended tight spans

It is the goal of this section to describe duals of polytopal complexes in terms of closure systems. Via Algorithm 4.1 this gives means to deal with them effectively. For details on polyhedral subdivisions we refer to the monograph [DLRS10].

Let $\mathcal{P} \subset \mathbb{R}^d$ be a finite point configuration, and let Σ be a polytopal subdivision of $\mathcal{P}$. That is, Σ is a polytopal complex whose vertices lie in $\mathcal{P}$ and which covers the convex hull $P = \operatorname{conv} \mathcal{P}$. This definition coincides with those given in Chapter 3, where the definition includes a description of the encoding of a polyhedral complex as a collection of finite sets. As in this chapter we call the elements of Σ *cells*; the set of maximal cells is denoted by $\Sigma^{\max}$ and the maximal boundary facets (meaning the maximal cells of Σ contained in the facets of P) by Δ_Σ. Now we obtain a closure operator on the set of sets $S_\Sigma := \Sigma^{\max} \cup \Delta_\Sigma$ by letting

$$\mathrm{cl}^\Sigma(F) \ := \ \begin{cases} \emptyset & \text{if } F = \emptyset \ , \\ \{g \in S_\Sigma \mid \bigcap_{\sigma \in F} \sigma \subseteq g\} & \text{otherwise} \ . \end{cases} \tag{4.2}$$

for any $F \subseteq S_\Sigma$. Note that cl^Σ is basically the same as the dual operator in Example 4.2. In fact, the closed, non-empty sets in S_Σ correspond to the cells of Σ, while the poset relation is the inverse containment relation.

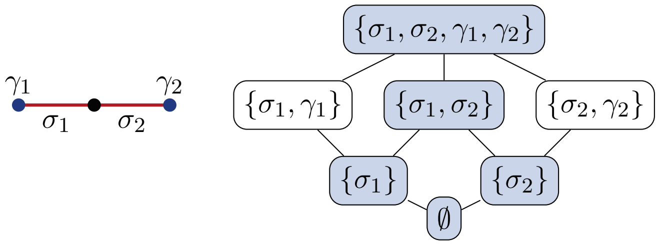


Figure 4.1: A regular subdivision and its extended tight span for $\Gamma = \emptyset$ and $\Gamma = \Delta(\Sigma) = \{\gamma_1, \gamma_2\}$, respectively. The latter is marked in blue.

Now let Γ be a collection of boundary faces of Σ. This defines a lower set Λ_Γ for the closure system $(S_\Sigma, \mathrm{cl}^\Sigma)$, which consists of all sets F, such that $\bigcap_{\sigma \in F} \sigma \nsubseteq \tau$ for any $\tau \in \Gamma$. We will denote the corresponding closure operator by $\mathrm{cl}^\Sigma_\Gamma := \mathrm{cl}^\Sigma_{\Lambda_\Gamma}$ and we call the resulting closure system $(S_\Sigma, \mathrm{cl}^\Sigma_\Gamma)$ the *extended tight span* of Σ with respect to Γ.

If $\Gamma = \Delta_\Sigma$, the closed sets are all cells of Σ which are not contained in the boundary. This is exactly the *tight span* of a polytopal subdivision defined in [HJJS09], which is dual to the interior cells. Note that this can also be obtained as the closure system of $(\Sigma^{\max}, \mathrm{cl}^\Sigma)$.

Example 4.9. Let $\mathcal{P}$ be $\{-1, 0, 1\}$ with the convex hull $[-1, 1]$ and Σ be its subdivision into intervals $\sigma_1 = [-1, 0]$ and $\sigma_2 = [0, 1]$, that is $\Sigma^{\max} = \{\sigma_1, \sigma_2\}$. The subdivision and its corresponding extended tight spans for $\Gamma = \emptyset$ and $\Gamma = \Delta_\Sigma = \{\{-1\}, \{1\}\} = \{\gamma_1, \gamma_2\}$ can be seen in Figure 4.1. This example also demonstrates that we need to declare the closure of the empty set to be itself to ensure monotonicity.

If the subdivision is regular, i.e., induced by a height function $h : \mathcal{P} \to \mathbb{R}$, we can actually coordinatize the extended tight span. Any regular subdivision with fixed height function is dual to a *dual complex* N_Σ, which is a complete polyhedral complex in $\mathbb{R}^d$. More precisely, for every point $x \in \mathbb{R}^d$ there is a cell of Σ, consisting of all points $p \in \mathcal{P}$ which minimize $h(p) - p \cdot x$. Points which induce the same cell form an open polyhedral cell and the topological closures of these cells form the dual complex. It is a well-known fact that there is a bijective, inclusion-reversing relation between the cells of Σ and the cells of N_Σ.

In particular, each maximal cell of $\Sigma^{\max}$ is dual to a vertex and every boundary facet in $\Delta(\Sigma)$ is dual to a ray of N_Σ. Hence, every closed set F of $(S_\Sigma, \mathrm{cl}^\Sigma_\Gamma)$ corresponds to a polyhedral cell ρ_F and together these cells form a subcomplex of N_Σ. More precisely, we denote by

$$T_{\Sigma,\Gamma} := \left\{ \rho_F \;\middle|\; F \subsetneq S_\Sigma \text{ closed w.r.t. } \mathrm{cl}^\Sigma_\Gamma \right\} \tag{4.3}$$

the *coordinatized extended tight span* of Σ with respect to Γ. Its face lattice is by definition the poset of closed sets of $(S_\Sigma, \mathrm{cl}^\Sigma_\Gamma)$.

4.4 Tropical linear spaces

In this section we will finally investigate the objects that we are most interested in: valuated matroids and tropical linear spaces. We prefer to see the latter as special cases of extended tight spans. Valuated matroids were first studied by Dress and Wenzel [DW92]; see [MS15, Chapter 4] for their role in tropical geometry.

Let us introduce some notation. For a subset B of $[n]$ of size r, let $e_B := \sum_{i \in B} e_i$. For a collection $M \subseteq \binom{[n]}{r}$ of such subsets we let

$$P_M \ := \ \operatorname{conv}\{e_B \mid B \in M\} \tag{4.4}$$

be the subpolytope of the hypersimplex $\Delta(r, n)$ which is spanned by those vertices which correspond to elements in M. In this language matroids were characterized by Gel′fand, Goresky, MacPherson and Serganova [GGMS87] as follows.

Proposition 4.10. *The set M comprises the bases of a matroid if and only if the vertex–edge graph of the polytope P_M is a subgraph of the vertex–edge graph of $\Delta(r, n)$ or, equivalently, if every edge of P_M is parallel to $e_i - e_j$ for some i and j.*

Throughout the following, let M be (the set of bases of) a matroid on n elements. In that case P_M is the *matroid polytope* of M. The matroid M is said to be *loop-free* if $\bigcup_{B \in M} B = [n]$. The *rank* of M is r, the size of any basis. If P_M is the full hypersimplex, then $M = U_{r,n}$ is a *uniform matroid.* The above description fits well with our geometric approach. Any function $v : M \to \mathbb{R}$ gives rise to a regular subdivision on P_M, which we denote by $\Sigma_{M,v}$. The pair (M, v) is a *valuated matroid* if every cell of $\Sigma_{M,v}$ is again a matroid polytope. Then $\Sigma_{M,v}$ is called a *matroid subdivision.*

Example 4.11. The set M of subsets of $\{1, 2, 3, 4\}$ with exactly two elements has cardinality six. Their characteristic vectors are the vertices of a regular octahedron embedded in 4-space. If we let v be the map which sends five vertices to 0 and the sixth one to 1, then (M, v) is a valuated matroid.

We will define tropical linear spaces as duals of valuated matroids. To this end let (M, v) be a valuated matroid of rank r on n elements. For a vector $x \in \mathbb{R}^n$, we define the set

$$M_x \ := \ \{B \in M \mid v(B) - e_B \cdot x \text{ is minimal}\} \ . \tag{4.5}$$

From the definition of the dual complex in Section 4.3 we see that the elements of M_x correspond to a cell of $\Sigma_{M,v}$ and thus define a matroid. Note that for any $\lambda \in \mathbb{R}$ we clearly have $M_x = M_{x+\lambda\mathbf{1}}$.

Definition 4.12. The *tropical linear space* associated with the valuated matroid (M, v) is the set

$$B(M, v) \ := \ \{x \in \mathbb{R}^n \ \mid \ M_x \text{ is loop-free}\} / \mathbb{R}\mathbf{1} \ . \tag{4.6}$$

Note that our definition of a valuated matroid, as well as that of a tropical linear space are with respect to minimum as tropical addition. The following is our main result. While it is easy to prove, it is relevant since it entails a new effective procedure for enumerating the cells of a tropical linear space via Algorithm 4.1.

Theorem 4.13. *Let Γ be the set of boundary faces of $\Sigma := \Sigma_{M,v}$ which correspond to matroids with loops. Then*

$$B(M,v) \ = \ T_{\Sigma,\Gamma}/\mathbb{R}\mathbf{1} \ , \tag{4.7}$$

where $T_{\Sigma,\Gamma}$ is the coordinatized extended tight span defined in Section 4.3.

Proof. Let N_Σ denote the dual complex of Σ. From our definition it is immediately clear that $B(M,v)$ is a subcomplex of $N_\Sigma/\mathbb{R}\mathbf{1}$. It consists of all cells whose dual cell in Σ is the polytope of a loop-free matroid. Since any cell in Σ corresponds to a loop-free matroid, if and only if it is not contained in a boundary facet of a matroid with loops, the claim follows. □

We call the resulting polyhedral structure of $B(M,v)$ *canonical*.

Remark 4.14. Note that one can naturally replace Γ by the subset of *maximal* boundary faces corresponding to matroids with loops. These faces are defined by the equations $x_i = 0$ for $i \in [n]$.

Example 4.15. If the valuation is constant then the matroid subdivision is trivial. It follows that the dual complex coincides with the normal fan of the matroid polytope P_M. In this case $B(M,v)$ is the *Bergman fan* of M, in its coarsest possible subdivision; see [Ham14a] for a proof.

The polyhedral complex $B(M,v)$ reflects quite a lot of the combinatorics of the matroid M, for instance, the rank of M equals $\dim(B(M,v)) + 1$. If L is the lineality space of $B(M,v)$, then the number of connected components of M is $\dim(L) + 1$; see [FS05].

Furthermore, it is remarkable that the bounded cells in a tropical linear space do not have to be a tropical convex set. As this tight span is the intersection of a tropical linear space and the negative of its dual, which is a tropical space with respect to max instead of taking the minimum; see [Spe05, Proposition 2.12]. The following example illustrates this fact.

Example 4.16. Consider the matroid of rank 3 on 8 elements from Example 2.47 that occures also in [HJS12, Section 5]. This matroid has the the following 30 bases:

$$\begin{gathered} 123\ 124\ 126\ 127\ 128\ 134\ 136\ 137\ 138\ 234\ 235\ 236\ 237\ 238\ 245 \\ 247\ 248\ 256\ 257\ 258\ 267\ 268\ 345\ 347\ 348\ 356\ 357\ 358\ 367\ 368\,. \end{gathered}$$

Its tight span is non-planar with f-vector $(5,7,3)$; see Figure 2.2 or [HJS12, Figure 7]. The five tropical points are the origin and the following points:

$$(0,1,1,1,0,1,1,1),\ (1,1,1,0,1,0,1,1),\ (1,1,1,1,1,1,0,0),\ (0,1,1,0,0,0,0,0)\,.$$

The vector $(0,1,1,0,0,0,1,1)$ is in the min-tropical convex hull of these points but not in the max-tropical convex hull. Hence this tight span is not tropically convex.

4.4.1 Performance comparison

As mentioned in Section 4.1, there is an algorithm by Rincón [Rin13] for computing Bergman fans, i.e., tropical linear spaces with trivial valuation. An extension which can also deal with trivially valued arbitrary matroids which may not be realizable has been implemented in `polymake`'s bundled extension `a-tint` [Ham14b]. It is this implementation we refer to in the following discussion. The original software `TropLi` by Rincón only takes realizable matroids as input.

Table 4.1: Comparing running times for computing Bergman fans.

(n,r)	# matroids	Rincón	Hasse	CH	ETS
(6,2)	23	0.0	0.2	0.8	0.0
(6,3)	38	0.0	0.4	1.6	0.0
(7,2)	37	0.0	0.3	1.6	0.0
(7,3)	108	0.0	1.5	5.8	0.2
(8,2)	58	0.0	0.4	1.9	0.0
(8,3)	325	0.3	6.0	21.4	0.8
(8,4)	940	1.8	48.7	86.5	9.2

Rincón's and our algorithm are very difficult to compare for two reasons. First of all, a matroid can be encoded in numerous ways. For instance, in terms of closures, as in Definition 4.4, or in terms of bases, as in Proposition 4.10. Many further variants exist, and the conversion between these representations is often a non-trivial computational task. Below we will assume that all matroids are given in terms of their bases. The second problem is that the two algorithms essentially compute very different things. Our algorithm computes the full face lattice of the canonical polyhedral structure of a tropical linear space. On the other hand, Rincón's algorithm only computes the rays and the maximal cones of the Bergman fan, albeit in a finer subdivision. In this setup it is therefore to be expected that our approach is significantly slower. In particular, to compute the regular subdivision and its boundary cells (including the loopfree ones), we need to apply a convex hull algorithm to the matroid polytope before we can make use of our algorithm. Still, the discussion has merit when separating the timings for the different steps; see Table 4.1. We compute Bergman fans of all (isomorphism classes of) matroids of a given rank r on a given ground set $[n]$ as provided at http://www-imai.

is.s.u-tokyo.ac.jp/~ymatsu/matroid); see also [MMIB12]. Each matroid is given only in terms of its bases. We first apply Rincón's algorithm and then compute the Hasse diagram of the face lattice of the fan as described in Example 4.3. For our approach we split the computations into two steps: First we compute the convex hull of the matroid polytope, displayed under "CH" and then measure the running time of our closure algorithm "ETS" (extended tight span) separately. Times were measured on an AMD Phenom II X6 1090T with 3.6 GHz using a single thread and `polymake` version 3.1. We employed the double description method implemented in the `Parma Polyhedral Library` (via `polymake`'s interface) for computing the convex hulls [PPL12].

The results show that almost all of the time in our algorithm is spent computing the facets of the matroid polytope. On the other hand, if one aims to obtaining the same amount of information, i.e., the full face lattice, for Rincón's algorithm, this increases the computation time dramatically. This demonstrates that the finer subdivision produced by this algorithm is significantly worse in terms of complexity than the canonical subdivision.

We also like to point out that for non-trivial valuations our algorithm is, to the best of our knowledge, currently the only feasible method for computing tropical linear spaces.

4.5 A case study on f-vectors of tropical linear spaces

Throughout the rest of this chapter we will restrict ourselves to valuations of uniform matroids. Equivalently, we study matroid subdivisions of hypersimplices (and their lifting functions). Speyer was the first to conduct a thorough study of the combinatorics of tropical linear spaces [Spe08]. He conjectured the following.

Conjecture 4.17 (Speyer's f-vector conjecture)**.** Let $n \geq 1$ and $1 \leq r \leq n$. Let v be any valuation on $U_{r,n}$. Then the number of $(i-1)$-dimensional bounded faces of $B(U_{r,n}, v)$ is at most $\binom{n-2i}{r-i}\binom{n-i-1}{i-1}$.

To study this problem, one is naturally interested in some form of moduli space of all possible valuations on $U := U_{r,n}$. This role is played by the *Dressian* $\mathrm{Dr}(r,n)$ of Chapter 2; see [HJJS09, HJS12]. It is a subfan of the secondary fan of $P_U = \Delta(r,n)$, consisting of all cones which correspond to matroidal subdivisions. As a set it contains the *tropical Grassmannian* $\mathrm{TGr}_p(r,n)$ for any characteristic $p \geq 0$ introduced by Speyer and Sturmfels [SS04] in the case $p = 0$. This is the tropicalization of the ordinary Grassmannian over an algebraically closed field of characteristic p, and it consists of all cones of the secondary fan which correspond to *realizable* valuations on U, i.e., those which can be realized as valuated vector matroids in characteristic p; see [MS15, Chapter 4]. However, this inclusion is generally strict. In fact, the Dressian is not even pure in general.

Remark 4.18. For $r = 2$, the Grassmannians agree for any characteristic and the Dressian $\mathrm{Dr}(r, n)$ is equal to this tropical Grassmannian. Combinatorially, this is the *space of phylogenetic trees*; e.g., see [Kap93, §1.3] and [MS15, §4.3]. For $r = 3$ and $3 \leq n \leq 6$, the equality $\mathrm{Dr}(r, n) = \mathrm{TGr}_p(r, n)$ still holds on the level of sets for each p. This is trivial for $n = 3, 4$, as there are no non-trivial subdivisions of P_U in that case. For $n = 5$ it follows from duality and the statement for $\mathrm{Dr}(2, 5)$. The Dressian $\mathrm{Dr}(3, 6)$ was computed in [SS04]. Note that, while the Dressian and the Grassmannian may agree as sets, they can have different polyhedral structures. Understanding the precise relation between these structures is still an open problem for general parameters. The cases $(3, 7)$ and $(3, 8)$ are the first where the Dressian differs from the Grassmannian. The Dressian $\mathrm{Dr}(3, 7)$ was computed in [HJJS09]. In particular, the possible combinatorial types of the corresponding tropical planes (and thus, their possible f-vectors) were listed. The polyhedral structure of $\mathrm{Dr}(3, 8)$ was computed in [HJS12]. In Corollary 2.46 we presented differences in the rays of Dressians and Grassmannians.

4.5.1 The Dressian $\mathrm{Dr}(3, 8)$

We wish to compute f-vectors of uniform tropical planes in $\mathbb{R}^8/\mathbb{R}\mathbf{1}$, i.e., tropical linear spaces corresponding to valuations on $U_{3,8}$. To this end, we make use of the data obtained in [HJS12], which is available at `http://svenherrmann.net/DR38/dr38.html`. There is a natural S_8-symmetry on the Dressian and the web-page provides representatives for each cone orbit.

We computed tropical linear spaces for each cone by choosing an interior point as valuation. For the sake of legibility, we only include results for the maximal cones of the Dressian. There are 14 maximal cones of dimension nine and 4734 maximal cones of dimension eight. The full data can be obtained at `http://page.math.tu-berlin.de/~hampe/dressian38.php`.

Convention 4.19. The f-vector of a tropical linear space L is the f-vector of its canonical polyhedral structure. The *bounded f-vector* of L is the f-vector of the bounded part of this structure. All counts are given modulo the S_8-symmetry on the Dressian.

There is only one bounded f-vector $(n-2, n-3)$ for a tropical linear space that corresponds to a maximal cone in the Dressian $\mathrm{Dr}(2, n)$, since this linear space has the combinatorics of a binary tree with n labeled leaves. The generic tropical linear spaces in the Dressian $\mathrm{Dr}(3, 6)$ have a bounded f-vector which is either $(5, 4, 0)$ or $(6, 6, 1)$; see [HJJS09]. In the case of $(3, 7)$ the (generic) bounded f-vectors read $(7, 6, 0)$, $(9, 10, 2)$ and $(10, 12, 3)$. This could be derived from the data of the maximal cones provided in [HJJS09] using our algorithm.

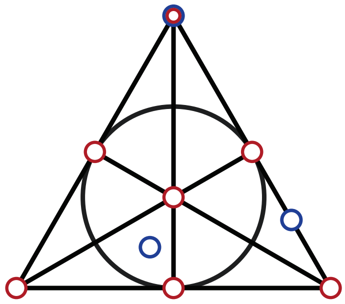


Figure 4.2: The three loop-free extensions of the Fano matroid.

Theorem 4.20. *Every generic tropical plane in* $\mathbb{R}^8/\mathbb{R}\mathbf{1}$ *has one of four possible* f*-vectors:*

- ▷ *If it corresponds to a nine-dimensional cone in the Dressian, its* f*-vector is* $(13, 55, 63)$ *and its bounded* f*-vector is* $(13, 15, 3)$*. There are nine different combinatorial types of such planes; see Figure 4.3.*
- ▷ *If it corresponds to one of the 4734 eight-dimensional maximal cones in the Dressian, there are three possibilities:*
  - ∘ *There are 51 planes with* f*-vector* $(13, 56, 64)$ *and bounded* f*-vector* $(13, 16, 4)$*.*
  - ∘ *There are 1079 planes with* f*-vector* $(14, 58, 65)$ *and bounded* f*-vector* $(14, 18, 5)$*.*
  - ∘ *There are 3604 planes with* f*-vector* $(15, 60, 66)$ *and bounded* f*-vector* $(15, 20, 6)$*.*

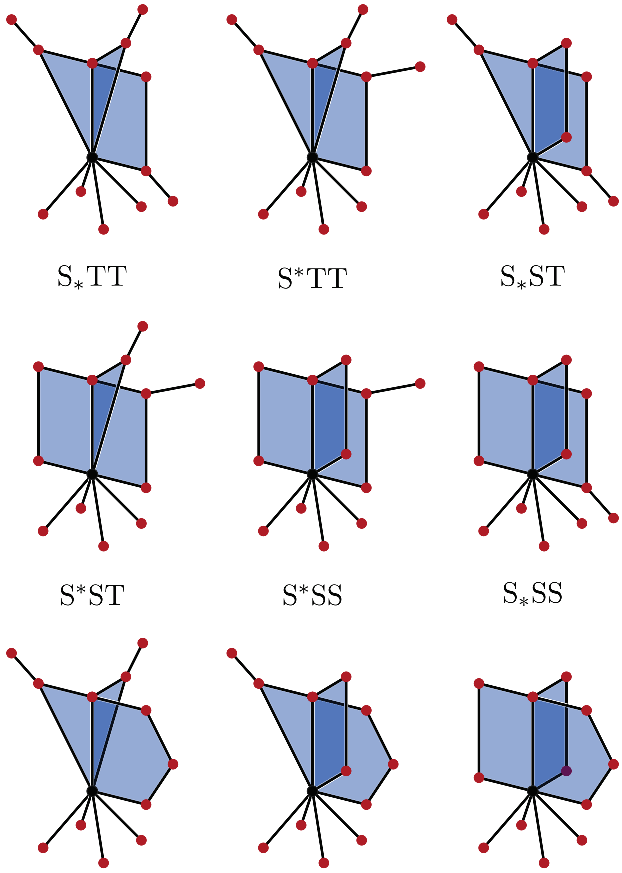
  There are 3013 different combinatorial types of such planes.

The maximal bounded f-vector $(15, 20, 6)$ agrees with the upper bound in Conjecture 4.17.

Remark 4.21. Each of the nine different combinatorial types that correspond to a nine-dimensional cone contains a vertex (marked in black in Figure 4.3), which in turn corresponds to the matroid polytope of a parallel extension of the *Fano matroid*. This is a certificate that these tropical linear spaces are not realizable over any field of characteristic greater than two; see [Oxl11, Chapter 6 and Appendix]. Figure 4.2 illustrates the connected extensions of the Fano matroid, these are those that are loop-free.

Further computer experiments reveal the following details, that involves corank vectors we have introduced in Chapter 2 which we will meet again in the next subsection.

Proposition 4.22. *Let* p *be* 0*,* 3*,* 5 *or* 7*. The intersection of the relative interior of a cone* C *in the Dressian* $\mathrm{Dr}(3, 8)$ *and the tropical Grassmannian* $\mathrm{TGr}_p(3, 8)$ *is trivial if and only if the corank vector of a Fano matroid extension is contained in the boundary of* C*.*



S_*TT S^*TT S_*ST

S^*ST S^*SS S_*SS

PTT PST PSS

Figure 4.3: The various combinatorial types of bounded parts of tropical linear spaces corresponding to nine-dimensional cones in the Dressian. Note that all of them share the same f-vector $(13, 15, 3)$. The naming convention is P = pentagon, S = square, T = triangle. The star $*$ indicates where the square has an additional edge attached.

4.6 Outlook

4.6.1 Higher Dressians

We have given an algorithm which computes tropical linear spaces for arbitrary valuations in reasonable time; computing all tropical linear spaces for $\mathrm{Dr}(3,8)$ above only took a few hours on a standard personal computer. This indicates that it is feasible to apply this algorithm more ambitiously, e.g., to Dressians with larger parameters. However, in these cases not much data is currently available. Computing higher Dressians is a challenging task in itself. The next step would be to look at $\mathrm{Dr}(4,8)$. While computing the full Dressian is, at the moment, beyond our means, we can consider the following construction by Speyer [Spe08]. Let M be a matroid of rank r on n elements. We define an associated valuation on $U_{r,n}$ by

$$v_M(B) := r - \mathrm{rk}_M(B) \ , \tag{4.8}$$

where $B \in \binom{[n]}{r}$ is a basis of the uniform matroid and

$$\mathrm{rk}_M(B) = \max_{B' \in M} \{|B \cap B'|\} \tag{4.9}$$

is the rank of B in M. Speyer showed that the corank indeed defines a valuation and that the matroid polytope P_M appears as a cell in the induced regular subdivision, see Lemma 2.27. This subdivision is the *corank subdivision* of M that we have seen in Chapter 2.

There are 940 isomorphism classes of matroids of rank four on eight elements [MMIB12]; our computation is based on the data from http://www-imai.is.s.u-tokyo.ac.jp/~ymatsu/matroid. For computing the tropical linear spaces given by the valuations defined above we employed the enriched version available at db.polymake.org. This is certainly not enough to provide a global view on the Dressian $\mathrm{Dr}(4,8)$, but it gives us a first glimpse of relevant combinatorial features. There are 62 different bounded f-vectors of such tropical linear spaces, so we cannot list them all. Also, up to combinatorial isomorphism, there are 465 different subdivisions of the hypersimplex induced by these matroids. As an example, consider the matroid $M := U_{1,2}^{\oplus 4}$; see [Oxl11, Chapter 4.2] for more on direct sums of matroids. The bounded f-vector of the tropical linear space induced by v_M is $(14,24,12,1)$. In particular, the last two entries already achieve the respective maxima conjectured by Speyer, which are $(20,30,12,1)$. Experiments suggests that this is generally true, i.e., if $M = U_{1,2}^{\oplus d}$, then the valuation on $U_{d,2d}$ gives a linear space whose bounded f-vector maximizes the last two entries. Among valuations of the form v_M on $U_{4,8}$, the maximal number of edges is in fact also 24. However, the maximal number of vertices is 15. This is achieved by the unique matroid with 56 bases and 14 rank three flats. For experts: This is a *sparse paving* matroid, which has the maximal number 16 of *cyclic flats* among all matroid of rank four on eight elements.

4.6.2 Further optimization

Many of the objects considered here, such as polytopes, fans and matroids, exhibit symmetries which are also visible in the corresponding closure systems. It seems desirable, therefore, to exploit this during the computation. For every orbit of a closed set, only one representative would be computed. In a first approach, this could be achieved by considering equivalent sets to be the same in Algorithm 4.1: Once, when collecting all minimal closures $\mathrm{cl}(N \cup \{i\})$ and again when checking if N_i is already in the graph. One could then easily recover the list of all closed sets in the end, though reconstructing the full poset structure (i.e. without symmetry) would require significant computational work.

As mentioned in Subsection 4.4.1, the most expensive part in our computations is a convex hull algorithm for computing the subdivision and the facets of the matroid polytope. It is known that the facets can be described in terms of the combinatorics of the matroid [FS05]. It is unclear if such a description can be given for the regular subdivision.

The degree of a tropical basis

5

5.1 Introduction

Computations with ideals in polynomial rings require an explicit representation in terms of a finite set of polynomials which generate that ideal. The size, i.e., the amount of memory required to store this data, depends on four parameters: the number of variables, the number of generators, their degrees and the sizes of their coefficients. For purposes of computational complexity it is of major interest to obtain explicit bounds for these parameters. An early step in this direction is Hermann's degree bound [Her26] on solutions of linear equations over $\mathbb{Q}[x_1, \ldots, x_n]$. In practice, however, not all generating sets are equally useful, and so it is important to seek complexity results for generating sets which have additional desirable properties. A landmark result here is the worst case space complexity estimate for Gröbner bases by Mayr and Meyer [MM82].

Tropical geometry associates with an algebraic variety a piecewise linear object in the following way. Let $\mathbb{K}$ be a field with a real-valued valuation, which we denote as val. We consider an ideal I in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ and its vanishing locus $\mathrm{V}(I)$, which is an affine variety. The *tropical variety* $\mathcal{T}(I)$ is defined as the topological closure of the set

$$\mathrm{val}(\mathrm{V}(I)) \;=\; \Big\{ \big(\mathrm{val}(z_1), \ldots, \mathrm{val}(z_n)\big) \;\Big|\; z \in \mathrm{V}(I) \cap (\mathbb{K} \setminus \{0\})^n \Big\} \quad \subset \; \mathbb{R}^n \; . \tag{5.1}$$

In general, $\mathcal{T}(I)$ is a polyhedral complex whose dimension agrees with the Krull dimension of I; see Bieri and Groves [BG84]. If, however, the ideal I has a generating system of polynomials whose coefficients are mapped to zero by val, that polyhedral complex is a fan. This is the *constant coefficient case*. A major technical challenge in tropical geometry is the fact that, in general, intersections of tropical varieties do not need to be tropical varieties. Therefore, the following concept is crucial for an approach via computational commutative algebra. A finite generating subset T of I is a *tropical basis* if the tropical variety $\mathcal{T}(I)$ coincides with the intersection of the finitely many tropical hypersurfaces $\mathcal{T}(f)$ for $f \in T$; see [MS15, §2.6] for the details.

Our main result states that each such ideal has a tropical basis whose degree does not exceed a certain bound which is given explicitly. While the bound which we are currently able to achieve is horrendous, to the best of our knowledge this is the first result of this kind. The main result comes in two versions: Theorem 5.5 covers the case of constant coefficients, while Theorem 5.10 deals with the general case. Moreover, we present examples of tropical bases which exhibit several interesting features. We close this chapter with an application to f-vectors of tropical varieties and two open questions.

5.2 Degree bounds

In this section we will assume that the valuation on the field $\mathbb{K}$ is trivial, i.e., we are in the constant coefficient case. Throughout the following let I be a homogeneous ideal in the polynomial ring $R := \mathbb{K}[x_1, \ldots, x_n]$. Bogart et al. were the first to describe an algorithm for computing a tropical basis [BJS+07, Theorem 11]. This algorithm is implemented in `Gfan`, a software package for computing Gröbner fans and tropical varieties [Jen]. Since our proof rests on the method of Bogart et al. we need to give a few more details. Every weight vector $w \in \mathbb{R}^n$ gives rise to a generalized term order on R. The generalization lies in the fact that this order may only be partial, which is why the initial form $\mathrm{in}_w(f)$ of a polynomial f does not need to be a monomial. Now the tropical variety of I can be described as the set

$$\mathcal{T}(I) \;=\; \{w \in \mathbb{R}^n \mid \mathrm{in}_w(I) \text{ does not contain any monomial}\} \quad ,$$

where the *initial ideal* $\mathrm{in}_w(I)$ is generated from all initial forms of polynomials in I. Declaring two weight vectors equivalent whenever their initial ideals agree yields a stratification of $\mathbb{R}^n$ into relatively open polyhedral cones; this is the *Gröbner fan* of I. Each maximal cone of the Gröbner fan corresponds to a proper term order or, equivalently, to a monomial initial ideal and a reduced Gröbner basis. A Gröbner basis is *universal* if it is a Gröbner basis for each term order. By construction $\mathcal{T}(I)$ is a subfan of the Gröbner fan. A polynomial $f \in I$ is a *witness* for a weight vector $w \in \mathbb{R}^n$ if its initial form $\mathrm{in}_w(f)$ is a monomial. Such a polynomial f certifies that the Gröbner cone containing w is not contained in $\mathcal{T}(I)$. The algorithm in [BJS+07] now checks each Gröbner cone and adds witnesses to a universal Gröbner basis to obtain a tropical basis.

An ideal I contains the monomial $x^m = x_1^{m_1} \cdots x_n^{m_n}$ if and only if the quotient

$$I : x^m \;=\; \{f \in R \mid x^m f \in I\}$$

contains a unit. The ideal

$$(I : x^m)^\infty \;=\; \bigcup_{k \in \mathbb{N}} (I : x^{km})$$

is called the *saturation* of I with respect to x^m. Since the ring R is Noetherian there exists a smallest number k such that $I : x^{km} = (I : x)^\infty$. That number k is the *saturation exponent*. Hence the total degree of any witness does not exceed αn, where α is the maximal saturation exponent of all initial ideals of I with respect to $x_1 \cdots x_n$. We need to get a grip on that parameter α. The *degree* of a finite set of polynomials is the maximal total degree which occurs.

Proposition 5.1. *Let I be a homogeneous ideal. The saturation exponent α of I with respect to $x_1 \cdots x_n$ is bounded by*

$$\alpha \;\leq\; \deg U \;\;,$$

where U is a universal Gröbner basis for I.

Proof. Since U is universal it contains a Gröbner basis $\{f_1, \ldots, f_s\}$ for the reverse lexicographic order. By [Eis95, Proposition 15.12] the set

$$\left\{ \frac{f_1}{\gcd(x_n, f_1)}, \frac{f_2}{\gcd(x_n, f_2)}, \ldots, \frac{f_s}{\gcd(x_n, f_s)} \right\}$$

is a Gröbner basis for $I : x_n$. Thus the saturation exponent of I with respect to x_n is bounded by the degree $\deg_{x_n}(U)$ of U in the variable x_n. Permuting the variables implies a similar statement for x_i. It follows that $\alpha = \max_{1 \leq i \leq n} \deg_{x_i} U \;\leq\; \deg U$. □

Notice that the tropical variety of a homogeneous ideal I coincides with the tropical variety of the saturated ideal $I : (x_1 \cdots x_n)^\infty$. For the next step we need to determine the degree of a universal Gröbner basis. The key ingredient is a result of Mayr and Ritscher [MR10]. Here and below d is the *degree* of I, i.e., the minimum of the degrees of all generating sets, and r is the Krull dimension.

Proposition 5.2 (Mayr and Ritscher)**.** *Assume that $r \geq 1$. Each reduced Gröbner basis G of the ideal I satisfies*

$$\deg G \;\leq\; 2 \left(\frac{d^{n-r} + d}{2} \right)^{2^{r-1}} \quad . \tag{5.2}$$

Lakshman and Lazard [LL91] give an asymptotic bound of the degree on zero-dimensional ideals, that is, for $r = 0$. For Gröbner bases one could argue that the degree is more interesting than the number of polynomials. This is due to the following simple observation.

Remark 5.3. A reduced Gröbner basis of degree e (of any ideal in R) can contain at most $\binom{e+n-1}{e} = \binom{e+n-1}{n-1}$ polynomials. The reason is that no two leading monomials can divide one another.

We are ready to bound the degree of a universal Gröbner basis. In view of the previous remark this also entails a bound on the number of polynomials. Since we will use Proposition 5.2, throughout this section we will assume that $r \geq 1$.

Corollary 5.4. *There is a universal Gröbner basis for I whose degree is bounded by* (5.2)*.*

Proof. The union of the reduced Gröbner bases for all term orders is universal. The claim follows since the bound in Proposition 5.2 is uniform. □

For our main result we apply the bounds which we just obtained to the output of the algorithm in [BJS+07].

Theorem 5.5. *Suppose that the valuation* val *on the coefficients is trivial. There is a universal Gröbner basis U and a tropical basis T of the homogenous ideal I with*

$$\deg T \;\leq\; \max\{\deg U, \alpha n\} \;\leq\; n \deg U \;\leq\; 2n \left(\frac{d^{n-r}+d}{2}\right)^{2^{r-1}} . \tag{5.3}$$

Proof. The number αn bounds the degree of a witness, and so the first inequality follows from the correctness of the algorithm [BJS+07, Theorem 11]. For a weight vector w we abbreviate $J := \mathrm{in}_w(I)$. From U we can obtain a universal Gröbner basis H for J, and this satisfies $\deg H \leq \deg U$. The initial ideal of J coincides with the initial ideal of I with respect to a perturbation of the term order that yields J in direction w. From Proposition 5.1 we thus get the second inequality. Finally, the third inequality follows from (5.2) and Corollary 5.4. □

Replacing (5.2) by other estimates gives variations of the last inequality in (5.3). For example, the bound

$$\deg G \;\leq\; 2\left(\frac{d^2}{2}+d\right)^{2^{n-1}} \tag{5.4}$$

of Dubé [Dub90] does not rely on the dimension r. Multiplying that bound by n also yields an upper bound on the degree of a tropical basis. Note that the results of this section hold for arbitrary characteristic of $\mathbb{K}$.

5.3 Examples

Throughout this section, we will be looking at the case $\mathbb{K} = \mathbb{C}$, and val sends each non-zero complex number to zero. In particular, as above, we are considering constant coefficients.

It is known that, in general, a universal Gröbner basis does not need to be a tropical basis; see [BJS+07, Ex. 10] or [MS15, Ex. 2.6.7]. That is, it cannot be avoided to compute witness polynomials. In fact, the following example, which is a simple modification of [BJS+07, Ex. 10], shows that adding witnesses may even increase the degree.

Example 5.6. Let $I \subset \mathbb{C}[x, y, z]$ be the ideal generated by the six degree 3 polynomials

$$\begin{array}{ccc} x^2y + xy^2\,, & x^2z + xz^2\,, & y^2z + yz^2\,, \\ x^3 + x^2y + x^2z\,, & xy^2 + y^3 + y^2z\,, & xz^2 + yz^2 + z^3 \end{array}$$

These six generators together with the ten polynomials of degree 3 below form a universal Gröbner basis for I.

$$\begin{array}{ccc} x^3 - xy^2 - xz^2\,, & x^2y - y^3 + yz^2\,, & x^2z + y^2z - z^3\,, \\ x^3 - xy^2 + x^2z\,, & xy^2 + y^3 - yz^2\,, & xz^2 - y^2z + z^3\,, \\ x^3 + x^2y - xz^2\,, & x^2y - y^3 - y^2z\,, & x^2z - yz^2 - z^3\,, \\ & x^3 + y^3 + z^3 & \end{array}$$

The monomial x^2yz of degree 4 is contained in I. This is a witness to the fact that the tropical variety $\mathcal{T}(I)$ is empty. Since, however, there is no monomial of degree 3 contained in I, any tropical basis must have degree at least 4. One such tropical basis, T, is given by the six generators and the monomial x^2yz. This also shows that a tropical basis does not need to contain a universal Gröbner basis.

A tropical basis does not even need to be any Gröbner basis, as the next example shows.

Example 5.7. Consider the three polynomials

$$x^5\,, \quad x^4 + x^2y^2 + y^4\,, \quad y^5$$

in $\mathbb{C}[x, y]$. They form a tropical basis for the ideal they generate. However each Gröbner basis has to include at least one of the S-polynomials $x^3y^2 + xy^4$ or $x^4y + x^2y^3$.

For conciseness Example 5.6 and Example 5.7 address tropical varieties which are empty. One can modify the above to obtain ideals and systems of generators with similar properties for tropical varieties of arbitrarily high dimension. We leave the details to the reader.

It is obvious that the final upper bound in (5.3) is an extremely coarse estimate. However, better bounds on the degree of the universal Gröbner basis can clearly be exploited. The following example may serve as an illustration.

Example 5.8. Let $I = \langle xy - zw + uv \rangle \subset \mathbb{C}[x, y, z, u, v, w]$. In this case we have $d = 2$, $n = 6$ and $r = 5$. Since I is a principal ideal the single generator forms a Gröbner basis, which is even universal and also a tropical basis. The degree of that universal Gröbner basis is $d = 2$, which needs to be compared with the upper bound of 2^{17} from (5.2). For the saturation exponent we have $\alpha = 1 \leq 2$, and the degree of the tropical basis equals $d = 2$. This is rather close to the bound $\alpha n = 6$, whereas the final upper bound in (5.3) is as much as $3 \cdot 2^{18}$.

Our final example generalizes the previous. In fact, Example 5.8 re-appears below for $D = 2$ and $N = 4$.

Example 5.9. The Plücker ideal $I_{D,N}$ captures the algebraic relations among the $D\times D$-minors of a generic $D\times N$-matrix with coefficients in the field $\mathbb{K}$. This is a homogeneous prime ideal in the polynomial ring over $\mathbb{K}$ with $n = \binom{N}{D}$ variables. The variety $\mathrm{V}(I_{D,N})$ is the *Grassmannian* of D-planes in $\mathbb{K}^N$. Its tropicalization $\mathcal{T}(I_{D,N})$ is the *tropical Grassmannian* of Speyer and Sturmfels [SS04]; see also [MS15, §4.3].

The Plücker ideal is generated by quadratic relations; see [Stu08, Theorem 3.1.7]. Its dimension equals $r = (N - D)D + 1$; see [SS04, Cor 3.1]. From this data we derive that there is a tropical basis $T_{D,N}$ of degree

$$\deg T_{D,N} \;\leq\; 2 \cdot \binom{N}{D} \cdot \left(2^{\binom{N}{D}-ND+D^2-2} + 1\right)^{2^{ND-D^2}} \quad .$$

To the best of our knowledge explicit tropical bases for $I_{D,N}$ are known only for $D = 2$ and $(D, N) \in \{(3, 6),\ (3, 7)\}$; see [SS04] and [HJJS09]. Note that for $D = 2$ the degree of a universal Gröbner basis grows with n while the quadratic 3-term Plücker relations form a tropical basis.

5.4 Non-constant coefficients

Recently, Markwig and Ren [MR16] presented a new algorithm which extends [BJS+07] to the case of non-constant coefficients. We will use their method to generalize Theorem 5.5 accordingly. To this end we will browse through our exposition in Section 5.2 and indicate the necessary changes to the arguments.

Let $\mathbb{K}$ be a field equipped with a non-trivial discrete valuation val. The *valuation ring* $\mathfrak{r} := \{\, a \in \mathbb{K} \mid \mathrm{val}(a) \geq 0 \}$ has a unique maximal ideal $\mathfrak{m} := \{\, a \in \mathbb{K} \mid \mathrm{val}(a) > 0 \}$. The ideal $\mathfrak{m}$ is generated by a prime element, which we denote as t. The *residue field* of $\mathbb{K}$ is the quotient $\Bbbk := \mathfrak{r}/\mathfrak{m}$. The *initial form* of a homogeneous polynomial $f = \sum_{u\in\mathbb{N}^n} c_u x^u$ in $\mathbb{K}[x_1, \ldots, x_n]$ with respect to a weight vector $w \in \mathbb{R}^n$ is

$$\mathrm{in}_w(f) \;=\; \sum_{\substack{w\cdot u-\mathrm{val}(c_u)\\ \text{maximal}}} \overline{t^{-\mathrm{val}(c_u)} c_u}\, x^u \;\in\; \Bbbk[x_1, \ldots, x_n] \;,$$

where $\overline{\cdot}$ describes the canonical projection from $\mathfrak{r}$ to $\Bbbk$. The initial ideal $J = \mathrm{in}_w(I)$, the tropical variety $\mathcal{T}(I)$ of an ideal I, witnesses and tropical bases are defined as in the constant coefficient case. The key difference to the classical case is that the stratification of $\mathbb{R}^n$ by initial ideals yields a polyhedral complex, the *Gröbner complex* $\Gamma(I)$, which does not need to be a fan; see [MS15, Section 2.5]. The tropical variety $\mathcal{T}(I)$ is a subcomplex of $\Gamma(I)$.

Let $w \in \mathbb{R}^n$ be a generic vector, i.e., an interior point of some maximal cell of $\Gamma(I)$. Like in the classical case, a *Gröbner basis* of I with respect to w is a set of generators such that their initial forms with respect to w generate the entire initial ideal $\mathrm{in}_w(I)$. Further, a Gröbner basis is *universal* if it works for all weight vectors. Again, a universal Gröbner basis enhanced with a witness for each cell in $\Gamma(I) \setminus \mathcal{T}(I)$ forms a tropical basis. As before, the degree of a witness with respect to w is bounded by the saturation exponent of $(\mathrm{in}_w(I) : x)^\infty$.

We are ready to state and prove the following generalization of Theorem 5.5.

Theorem 5.10. *Suppose that* val *is a non-constant discrete valuation on* $\mathbb{K}$. *There is a universal Gröbner basis* U *and a tropical basis* T *of the homogeneous ideal* I *with*

$$\deg T \;\leq\; \max\{\deg U, \alpha n\} \;\leq\; n \deg U \;\leq\; 2n \left(\frac{d^2}{2} + d\right)^{2^{n-1}} . \tag{5.5}$$

Proof. Our proof is based on the algorithm of Markwig and Ren [MR16], which is a direct generalization of [BJS+07]. Let U be a universal Gröbner basis of the ideal I. For $w \in \mathbb{R}^n$ the set $\{\mathrm{in}_w(f) \mid f \in U\}$ is a Gröbner basis of J. By Proposition 5.1, the saturation exponent of the saturation $(J : x)^\infty$ is bounded by the degree $\deg U$. This establishes the first two inequalities in (5.5). The final inequality follows from Dubé's bound (5.4). That result was extended to non-constant coefficients by Chan and MacLagan; see [CM13, Theorem 3.1]. □

Remark 5.11. The canonical valuation on the field of Puiseux series $\mathbb{C}\{\{t\}\}$ is not discrete, and the valuation ring is not Noetherian; see [MS15, Remark 2.4.13]. However, the computation of a tropical basis for any finitely generated ideal can be restricted to a polynomial ring over an appropriate discretely valuated subfield. The degree bound in Theorem 5.10 does not depend on the choice of that subfield. Thus Theorem 5.10 also holds for $\mathbb{K} = \mathbb{C}\{\{t\}\}$, provided that I is finitely generated.

5.5 The f-vector of a tropical variety

The *f-vector* of a polyhedral complex, which counts the number of cells by dimension, is a fundamental combinatorial complexity measure. In this section we will give an explicit bound on the f-vector of a tropical variety $\mathcal{T}(I)$, with arbitrary valuation on the field $\mathbb{K}$, in terms of the number s of polynomials in a tropical basis T and the degree d of a tropical basis, T. Notice that in the previous sections 'd' was the degree of I.

First we discuss the case of a tropical hypersurface, that is, $s = 1$, as in Example 5.8. Let $g \in R$ be an arbitrary homogeneous polynomial of degree d. As in Section 5.4, here we are admitting non-constant coefficients. A tropical hypersurface $\mathcal{T}(g)$ is dual to the regular subdivision of the Newton polytope $N(g)$ of g, which is gotten from lifting the lattice points in $N(g)$, which correspond to the monomials in g, to the valuation of their coefficients [MS15, Proposition 3.1.6]. See the monograph [DLRS10] for details on polytopal subdivisions of finite point sets. The polynomial g has at most $\binom{d+n-1}{n-1}$ monomials, which correspond to the lattice points in the dth dilation of the $(n-1)$-dimensional simplex $d \cdot \Delta_{n-1}$. The *standard simplex* Δ_{n-1} is the $(n-1)$-dimensional convex hull of the n standard basis vectors $e_1, \ldots, e_n$. The maximal f-vector of a polytopal subdivision of $d \cdot \Delta_{n-1}$ by lattice points is (simultaneously for all dimensions) attained for a unimodular triangulation [BM85, Theorem 2]. If Δ is such a unimodular triangulation, then its vertices use all lattice points in $d \cdot \Delta_{n-1}$. The converse does not hold if $n \geq 4$. The f-vector of Δ equals

$$f_j^{\Delta} \;=\; \sum_{i=0}^{j} (-1)^{i+j} \binom{j}{i} \binom{di+d+n-1}{n-1} \;; \tag{5.6}$$

see [DLRS10, Theorem 9.3.25]. By duality the bound in (5.6) translates into a bound on the f-vector for the tropical hypersurface $\mathcal{T}(g)$:

$$f_j^{\mathcal{T}(g)} \;\leq\; f_{n-j-1}^{\Delta} \;\leq\; \sum_{i=1}^{n-j} (-1)^{n+i-j} \binom{n-j-1}{i-1} \binom{di+n-1}{n-1} \;. \tag{5.7}$$

From the above computation we can derive the following general result.

Proposition 5.12. *Let I be a homogeneous ideal in R. Then the f-vector of the tropical variety $\mathcal{T}(I)$ with a tropical basis T, consisting of s polynomials of degree at most d, is bounded by*

$$f_j \;\leq\; \sum_{i=1}^{n-j} (-1)^{n+i-j} \binom{n-j-1}{i-1} \binom{sdi+n-1}{n-1} \;.$$

Proof. Let g denote the product $h_1 \cdots h_s$ of all polynomials in the tropical basis T. The tropical hypersurface of g is the support of the $(n-1)$-skeleton of the polyhedral complex dual to a regular subdivision of the Newton polytope $N(g)$; see [MS15, Proposition 3.1.6]. This polytope is the Minkowski sum of all Newton polytopes $N(h)$ for $h \in T$. Moreover, the polyhedral subdivision of $N(g)$ dual to $\mathcal{T}(g)$ is the common refinement of the subdivisions of the Newton polytopes for the polynomials in T. The tropical variety $\mathcal{T}(I)$ is a subcomplex of this refinement since, by the definition of T, we have

$$\mathcal{T}(I) \;=\; \bigcap_{f \in T} \mathcal{T}(f) \;.$$

The polynomial g is of degree at most sd. From the inequality (5.7) we get the claim. □

Let us now discuss the special case of a tropical hypersurface $\mathcal{T}(g)$ with constant coefficients. That is, we assume that the valuation map applied to each coefficient of the homogeneous polynomial g yields zero. In this case the lifting is trivial and thus $\mathcal{T}(g)$ is

Table 5.1: The vectors $\lambda(d,n)$ for small values of d and n. A star indicates only a lower bound, which is due to the fact that we could not complete our ad hoc computation with the given resources.

$n\backslash d$	2	3	4	5
2	(2)	(2)	(2)	(2)
3	$(4,4)$	$(6,6)$	$(6,6)$	$(8,8)$
4	$(7,12,8)$	$(12,18,10)$	$(15,24,16)$	$(20,36,22)$
5	$(11,30,30,10)$	$(20,48,50,20)$	$(28,83,86,33)^*$	$(33,96,101,36)^*$

dual to a lattice polytope contained in the simplex $d\cdot\Delta_{n-1}$; see [MS15, Proposition 3.1.10]. We introduce the parameter

$$\lambda_j(d,n) \;=\; \max\left\{ f_j^P \;\middle|\; P \text{ is a lattice polytope in } d\cdot\Delta_{n-1}\right\} \quad ,$$

which measures how combinatorially complex tropical hypersurfaces (with constant coefficients) can be. We arrive at the following conclusion.

Corollary 5.13. *Let I be a homogeneous ideal in R which is generated by polynomials with constant coefficients. Then the f-vector of the tropical variety $\mathcal{T}(I)$ a tropical basis T, consisting of s polynomials of degree at most d, is bounded by*

$$f_j \;\le\; \lambda_{n-j-1}(sd,n) \quad . \tag{5.8}$$

Notice that the $(n-1)$-simplex has $\lambda_0(1,n)=n$ vertices and an interval has $\lambda_0(d,2)=2$ vertices. The number of vertices $\lambda_0(d,n)$ does not exceed the sum of the number of vertices in $(d-1)\cdot\Delta_{n-1}$ and $d\cdot\Delta_{n-2}$. Hence, e.g., the number of $(n-1)$-cells of $\mathcal{T}(I)$ in (5.8) is bounded by

$$f_{n-1} \;\le\; \lambda_0(sd,n) \;\le\; \sum_{i=0}^{sd-2} 2\binom{i+n-3}{i} + \sum_{i=0}^{n-3}(n-i)\binom{i+sd-2}{i} \quad .$$

We calculated the numbers $\lambda_j(d,n)$ for small values of d and n with `polymake` [GJ00]. The result is summarized in Table 5.1. Note that, e.g., for $d=2$ and $n=4$ there is no polytope that maximizes f_j simultaneously for all j. We expect that it is difficult to explicitly determine the values for $\lambda_j(d,n)$. The somewhat related question of determining the (maximal) f-vectors of 0/1-polytopes is a challenging open problem; see [Zie00].

5.6 Open questions

For constant coefficients, Hept and Theobald [HT09] developed an algorithm for computing tropical bases, which is based on projections.

Question A. Can their approach by used to obtain better degree bounds?

Our current techniques employ the Gröbner complex of an ideal, i.e., a universal Gröbner basis. Yet, as Example 5.7 shows tropical bases and Gröbner bases are not related in a straightforward way.

Question B. Is it possible to directly obtain a tropical basis from the generators of an ideal, i.e., without the need to compute any Gröbner basis?

Notice that the algorithm of Hept and Theobald [HT09] uses elimination (Gröbner bases). However, one may ask if techniques from polyhedral geometry can further be exploited to obtain yet another method for computing tropical bases and tropical varieties.

Polyhedral computations over Puiseux fractions 6

6.1 Introduction

It is well known and not difficult to see that the standard concepts from linear programming (LP), e.g., the Farkas Lemma and LP duality, carry over to an arbitrary ordered field; e.g., see [CK70, Section II] or [Jer73, §2.1]. Traces of this can already be found in Dantzig's monograph [Dan63, Chapter 22]. This entails that any algorithm whose correctness rests on these LP corner stones is valid over any ordered field. In particular, this holds for the simplex method and usual convex hull algorithms. A classical construction, due to Hilbert, turns a field of rational functions, e.g., with real coefficients, into an ordered field; see [vdW93, §147]. In [Jer73] Jeroslow discussed these fields in the context of linear programming in order to provide a rigorous foundation of the so-called "big M method". The purpose of this note is to describe the implementation of the simplex method and of a convex hull algorithm over fields of this kind in the open source software system `polymake` [GJ00].

Hilbert's ordered field of rational functions is a subfield of the field of formal Puiseux series $\mathbb{R}\{\{t\}\}$ with real coefficients. The latter field is real-closed by the Artin–Schreier Theorem [SGHL07, Theorem 12.10]; by Tarski's Principle (cf. [Tar48]) this implies that $\mathbb{R}\{\{t\}\}$ has the same first order properties as the reals. The study of polyhedra over $\mathbb{R}\{\{t\}\}$ is motivated by tropical geometry [DY07], especially tropical linear programming [ABGJ15]. The connection of the latter with classical linear programming has recently lead to a counter-example [ABGJ14] to a "continuous analogue of the Hirsch conjecture" by Deza, Terlaky and Zinchenko [DTZ09]. In terms of parameterized linear optimization (and similarly for the convex hull computations) our approach amounts to computing with sufficiently large (or, dually, sufficiently small) positive real numbers. Here we do *not* consider the more general algorithmic problem of stratifying the parameter space to describe all optimal solutions of a linear program for *all* choices of parameters; see, e.g., [JKM08] for work into that direction.

This chapter is organized as follows. We start out with summarizing known facts on ordered fields. Then we describe a specific field, $\mathbb{Q}\{t\}$, which is the field of rational functions with rational coefficients and rational exponents. This is a subfield of the formal

Puiseux series $\mathbb{Q}\{\{t\}\}$, which we call the field of *Puiseux fractions*. It is our opinion that this is a subfield which is particularly well suited for exact computations with (some) Puiseux series; see [MC13] for an entirely different approach. In the context of tropical geometry Markwig [Mar10] constructed a much larger field, which contains the classical Puiseux series as a proper subfield. For our applications it is relevant to study the evaluation of Puiseux fractions at sufficiently large rational numbers. In Section 6.3 we develop what this yields for comparing convex polyhedra over $\mathbb{R}\{\{t\}\}$ with ordinary convex polyhedra over the reals. The tropical geometry point of view enters the picture in Section 6.4. We give an algorithm for solving the dual tropical convex hull problem, i.e., the computation of generators of a tropical cone from an exterior description. Allamigeon, Gaubert and Goubault gave a combinatorial algorithm for this in [AGG13], while we use a classical (dual) convex hull algorithm and apply the valuation map. The benefit of our approach is more geometric than in terms of computational complexity: in this way we will be able to study the fibers of the tropicalization map for classical versus tropical cones for specific examples. Section 6.5 sketches the `polymake` implementation of the Puiseux fraction arithmetic and the LP and convex hull algorithms. The LP solver is a dual simplex algorithm with steepest edge pivoting, and the convex hull algorithm is the classical beneath-and-beyond method [Ede87] [Jos03]. An overview with computational results is given in Section 6.6.

6.2 Ordered fields and rational functions

A field $\mathbb{F}$ is *ordered* if there is a total ordering $\leq$ on the set $\mathbb{F}$ such that for all $a, b, c \in \mathbb{F}$ the following conditions hold:

(i) if $a \leq b$ then $a + c \leq b + c$,

(ii) if $0 \leq a$ and $0 \leq b$ then $0 \leq a \cdot b$.

Any ordered field necessarily has characteristic zero. Examples include the rational numbers $\mathbb{Q}$, the reals $\mathbb{R}$ and any subfield in between.

Given an ordered field $\mathbb{F}$ we can look at the ring of univariate polynomials $\mathbb{F}[t]$ and its quotient field $\mathbb{F}(t)$, the field of rational functions in the indeterminate t with coefficients in $\mathbb{F}$. On the ring $\mathbb{F}[t]$ we obtain a total ordering by declaring $p < q$ whenever the leading coefficient of $q - p$ is a positive element in $\mathbb{F}$. Extending this ordering to the quotient field by letting

$$\frac{u}{v} < \frac{p}{q} \quad :\Longleftrightarrow \quad uq < vp \quad ,$$

where the denominators v and q are assumed positive, turns $\mathbb{F}(t)$ into an ordered field; see, e.g., [vdW93, §147]. This ordered field is called the "Hilbert field" by Jeroslow [Jer73].

By definition, the exponents of the polynomials in $\mathbb{F}[t]$ are natural numbers. However, conceptually, there is no harm in also taking negative integers or even arbitrary rational numbers as exponents into account, as this can be reduced to the former by clearing denominators and subsequent substitution. For example,

$$\frac{2t^{3/2} - t^{-1}}{1 + 3t^{-1/3}} \;=\; \frac{2t^{5/2} - 1}{t + 3t^{2/3}} \;=\; \frac{2s^{15} - 1}{s^6 + 3s^4} \ , \tag{6.1}$$

where $s = t^{1/6}$. In this way that fraction is written as an element in the field $\mathbb{Q}(t^{1/6})$ of rational functions in the indeterminate $s = t^{1/6}$ with rational coefficients. Further, if $p \in \mathbb{F}(t^{1/\alpha})$ and $q \in \mathbb{F}(t^{1/\beta})$, for natural numbers α and β, then the sum $p + q$ and the product $p \cdot q$ are contained in $\mathbb{F}(t^{1/\gcd(\alpha,\beta)})$. This shows that the union

$$\mathbb{F}\{t\} \;=\; \bigcup_{\nu \geq 1} \mathbb{F}(t^{1/\nu}) \tag{6.2}$$

is again an ordered field. We call its elements *Puiseux fractions*. The field $\mathbb{F}\{t\}$ is a subfield of the field $\mathbb{F}\{\!\{t\}\!\}$ of *formal Puiseux series*, i.e., the formal power series with rational exponents of common denominator. For an algorithmic approach to general Puiseux series see [MC13].

The map val which sends the rational function p/q, where $p, q \in \mathbb{F}[t^{1/\nu}]$, to the number $\deg_t p - \deg_t q$ defines a non-Archimedean valuation on $\mathbb{F}(t)$. Here we let $\mathrm{val}(0) = -\infty$. As usual the *degree* of a non-zero polynomial is the largest occurring exponent. The valuation map extends to Puiseux series. More precisely, for $f, g \in \mathbb{F}\{t\}$ we have the following:

(i) $\mathrm{val}(f \cdot g) = \mathrm{val}(f) + \mathrm{val}(g)$,

(ii) $\mathrm{val}(f + g) \leq \max(\mathrm{val}(f), \mathrm{val}(g))$.

If $\mathbb{F} = \mathbb{R}$ is the field of real numbers we can evaluate a Puiseux fraction $f \in \mathbb{R}\{t\}$ at a real number τ to obtain the real number $f(\tau)$. This map is defined for all $\tau > 0$ except for the finitely many poles, i.e., zeros of the denominator. Restricting the evaluation to positive numbers is necessary since we are allowing rational exponents. The valuation map satisfies the equation

$$\lim_{\tau \to \infty} \log_\tau |f(\tau)| \;=\; \mathrm{val}(f) \ . \tag{6.3}$$

That is, seen on a logarithmic scale, taking the valuation of f corresponds to interpreting t like an infinitesimally large number. Reading the valuation map in terms of the limit (6.3) is known as *Maslov dequantization*, see [Mas86].

Occasionally, it is also useful to be able to interpret t as a *small* infinitesimal. To this end, one can define the *dual degree* $\deg^*$, which is the smallest occurring exponent. This gives rise to the *dual valuation* map $\mathrm{val}^*(p/q) = \deg^*_t p - \deg^*_t q$ which yields

$$\mathrm{val}^*(f+g) \;\geq\; \min(\mathrm{val}^*(f), \mathrm{val}^*(g)) \quad \text{and} \quad \lim_{\tau \to 0} \log_\tau |f(\tau)| \;=\; \mathrm{val}^*(f) \ .$$

Changing from the primal to the dual valuation is tantamount to substituting t by t^{-1}.

Remark 6.1. The valuation theory literature often employs the dual definition of a valuation. The equation (6.3) is the reason why we usually prefer to work with the primal.

Up to isomorphism of valuated fields the valuation on the field $\mathbb{F}(t)$ of rational functions is unique, e.g., see [vdW93, §147]. As a consequence the valuation on the slightly larger field of Puiseux fractions is unique, too.

To close this section let us look at the algorithmically most relevant case $\mathbb{F} = \mathbb{Q}$. Then, in general, the evaluation map sends positive rationals to not necessarily rational numbers, again due to fractional exponents. By clearing denominators in the exponents one can see that evaluating at $\sigma > 0$ ends up in the totally real number field $\mathbb{Q}(\sqrt[\nu]{\sigma})$ for some positive integer ν. For instance, evaluating the Puiseux fraction from Example (6.1) would give an element of $\mathbb{Q}(\sqrt[6]{\sigma})$.

6.3 Parameterized polyhedra

Consider a matrix $A \in \mathbb{F}\{t\}^{m\times(d+1)}$. Then the set

$$C \ := \ \left\{x \in \mathbb{F}\{t\}^{d+1} \;\middle|\; A \cdot x \geq 0\right\}$$

is a polyhedral cone in the vector space $\mathbb{F}\{t\}^{d+1}$. Equivalently, C is the set of feasible solutions of a linear program with $d+1$ variables over the ordered field $\mathbb{F}\{t\}$ with m homogeneous constraints, the rows of A. The Farkas–Minkowski–Weyl Theorem establishes that each polyhedral cone is finitely generated. A proof for this result on polyhedral cones over the reals can be found in [Zie95, §1.3 and §1.4] under the name "Main theorem for cones". It is immediate to verify that the arguments given hold over any ordered field. Therefore, there is a matrix $B \in \mathbb{F}\{t\}^{(d+1)\times n}$, for some $n \in \mathbb{N}$, such that

$$C \ = \ \{ B \cdot a \mid a \in \mathbb{F}\{t\}^n,\, a \geq 0 \} \ . \tag{6.4}$$

The columns of B are points and the cone C is the non-negative linear span of those.

Let L be the *lineality space* of C, i.e., L is the unique maximal linear subspace of $\mathbb{F}\{t\}^{d+1}$ which is contained in C. If $\dim L = 0$, then the cone C is *pointed*. Otherwise, the set C/L is a pointed polyhedral cone in the quotient space $\mathbb{F}\{t\}^{d+1}/L$. A *face* of C is the intersection of C with a supporting hyperplane. The faces are partially ordered by inclusion. Each face contains the lineality space. Adding the entire cone C as an additional top element we obtain a lattice, the *face lattice* of C. The maximal proper faces are the *facets* which form the co-atoms in the face lattice. The *combinatorial type* of C is the isomorphism class of the face lattice (e.g., as a partially ordered set). Notice that our definition says that each cone is combinatorially equivalent to its quotient modulo its lineality space.

Picking a positive element τ yields matrices $A(\tau) \in \mathbb{F}^{m\times(d+1)}$ and $B(\tau) \in \mathbb{F}^{(d+1)\times n}$ as well as a polyhedral cone $C(\tau) = \{x \in \mathbb{F}^{d+1} \mid A(\tau) \cdot x \geq 0\}$ by evaluating the Puiseux fractions at the parameter τ. Here and below we will assume that τ avoids the at most finitely many poles of the $(m+n)\cdot(d+1)$ coefficients of A and B.

Theorem 6.2. *There is a positive element $\tau_0 \in \mathbb{F}$ so that for every $\tau > \tau_0$ we have*

$$C(\tau) \;=\; \{\, B(\tau)\cdot\alpha \mid \alpha \in \mathbb{F}^n,\, \alpha \geq 0\} \quad ,$$

and evaluating at τ maps the lineality space of C to the lineality space of $C(\tau)$. Moreover, the polyhedral cones C and $C(\tau)$ over $\mathbb{F}\{t\}$ and $\mathbb{F}$, respectively, share the same combinatorial type.

Proof. First we show that an orthogonal basis of the lineality space L evaluates to an orthogonal basis of the lineality space of $C(\tau)$. For this, consider two vectors $x, y \in \mathbb{F}\{t\}^{d+1}$ and pick τ large enough to avoid their poles and zeros. Then, the scalar product of x and y vanishes if and only if the scalar product of $x(\tau)$ and $y(\tau)$ does. Hence, the claim follows.

Now we can assume that the polyhedral cone C is pointed, i.e., it does not contain any linear subspace of positive dimension. If this is not the case the subsequent argument applies to the quotient C/L.

Employing orthogonal bases, as for the lineality spaces above, shows that the evaluation maps the linear hull of C to the linear hull of $C(\tau)$, preserving the dimension. So we may assume that C is full-dimensional, as otherwise the arguments below hold in the linear hull of C.

Let $\ell \leq \binom{m}{d}$ be the number of d-element sets of linearly independent rows of the matrix A. For each such set of rows the set of solutions to the corresponding homogeneous system of linear equations is a one-dimensional subspace of $\mathbb{F}\{t\}^{(d+1)}$. For each such system of homogeneous linear equations pick two non-zero solutions, which are negatives of each other. We arrive at 2ℓ vectors in $\mathbb{F}\{t\}^{(d+1)}$ which we use to form the columns of the matrix $Z \in \mathbb{F}\{t\}^{(d+1)\times 2\ell}$.

By the Farkas–Minkowski–Weyl theorem, we may assume that the columns of B from (6.4) only consist of the rays of C and that the rays of C form a subset of the columns of Z. In particular, the columns of B occur in Z. Since the cone C is pointed, the matrix B contains at most one vector from each opposite pair of the columns of Z. This entails that B has at most ℓ columns.

Further, the real matrix $Z(\tau)$ contains all rays of $C(\tau)$ for each τ that avoids the poles of A and Z. In the following, we want to show that those columns of $Z(\tau)$ which form the rays of $C(\tau)$ are exactly the columns of $B(\tau)$.

We define $s(j,k) \in \mathbb{F}\{t\}$ to be the scalar product of the jth row of A and the kth column of Z. The $m \cdot 2\ell$ signs of the scalar products $s(j,k)$, for $j \in [m]$ and $k \in [2\ell]$, form the *chirotope* of the linear hyperplane arrangement defined by the rows of A (in fact, due to taking two solutions for each homogenous system of linear equations, we duplicate the information of the chirotope). For almost all $\tau \in \mathbb{F}$ evaluating the Puiseux fractions $s(j,k)$ at τ yields an element of $\mathbb{F}$. For sufficiently large τ the sign of $s(j,k)$ agrees with its evaluation. This follows from the definition of the ordering on $\mathbb{F}\{t\}$, cf. [Jer73, Proposition, §1.3].

Let $\tau_0 \in \mathbb{F}$ be larger than all the at most finitely many poles of A and Z. Further, let τ_0 be large enough such that the chirotope of $A(\tau)$ agrees with the chirotope of A for all $\tau > \tau_0$.

By construction the rays of C correspond to the non-negative columns of the chirotope whose support, given by the 0 entries, is maximal with respect to inclusion; these are exactly the columns of B. The corresponding columns of the chirotope of $A(\tau)$, for $\tau > \tau_0$, yield the rays of $C(\tau)$, which, hence, are the columns of $B(\tau)$.

The same holds for the facets of C and $C(\tau)$. The facets of C correspond to the non-negative rows of the chirotope whose support, given by the 0 entries, is maximal with respect to inclusion.

Now the claim follows since the face lattice of a polyhedral cone is determined by the incidences between the facets and the rays. □

A statement related to Theorem 6.2 occurs in Benchimol's PhD thesis [Ben14]. The Proposition 5.12 in [Ben14] discusses the combinatorial structure of tropical polyhedra (arising as the feasible regions of tropical linear programs). Yet here we consider the relationship between the combinatorial structure of Puiseux polyhedra and their evaluations over the reals. As in the proof of [Ben14, Proposition 5.12] we could derive an explicit upper bound on the optimal τ_0. To this end one can estimate the coefficients of the Puiseux fractions in Z, which are given by determinantal expressions arising from submatrices of A. Their poles and zeros are bounded by Cauchy bounds (e.g., see [RS02, Theorem 8.1.3]) depending on those coefficients. We leave the details to the reader.

A *convex polyhedron* is the intersection of finitely many linear inequalities. It is called a *polytope* if it is bounded. Restricting to cones allows a simple description in terms of homogeneous linear inequalities. Yet this encompasses arbitrary polytopes and polyhedra, as they can equivalently be studied through their homogenizations. In fact, all implementations in `polymake` are based on this principle. For further reading we refer to [Zie95, §1.5]. We visualize Theorem 6.2 with a very simple example.

Example 6.3. Consider the polytope P in $\mathbb{R}\{t\}^2$ for large t defined by the four inequalities

$$x_1, x_2 \geq 0, \qquad x_1 + x_2 \leq 3, \qquad x_1 - x_2 \leq t \ .$$

The evaluations at $\tau \in \{0,1,3\}$ are depicted in Figure 6.1. For $\tau = 0$ we obtain a triangle, for $\tau = 1$ a quadrangle and for $\tau \geq 3$ a triangle again. The latter is the combinatorial type of the polytope P over the field of Puiseux fractions with real coefficients.

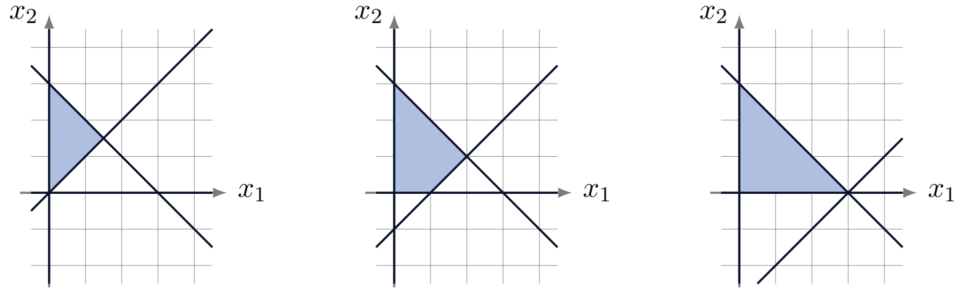


Figure 6.1: Polygon depending on a real parameter as defined in Example 6.3

Corollary 6.4. *The set of combinatorial types of polyhedral cones which can be realized over $\mathbb{F}\{t\}$ is the same as over $\mathbb{F}$.*

Proof. One inclusion is trivial since $\mathbb{F}$ is a subfield of $\mathbb{F}\{t\}$. The other inclusion follows from the preceding result. ☐

For $A \in \mathbb{F}\{t\}^{m\times d}$, $b \in \mathbb{F}\{t\}^m$ and $c \in \mathbb{F}\{t\}^d$ we consider the linear program $\mathrm{LP}(A, b, c)$ over $\mathbb{F}\{t\}$ which reads as

$$\begin{array}{ll} \text{maximize} & c^\top \cdot x \\ \text{subject to} & A \cdot x = b\,,\ \ x \geq 0\ \ . \end{array} \tag{6.5}$$

For each positive $\tau \in \mathbb{F}$ (which avoids the poles of the Puiseux fractions which arise as coefficients) we obtain a linear program $\mathrm{LP}(A(\tau), b(\tau), c(\tau))$ over $\mathbb{F}$. Theorem 6.2 now has the following consequence for parametric linear programming.

Corollary 6.5. *Let $x^* \in \mathbb{F}\{t\}^d$ be an optimal solution to the LP (6.5) with optimal value $v \in \mathbb{F}\{t\}$. Then there is a positive element $\tau_0 \in \mathbb{F}$ so that for every $\tau > \tau_0$ the vector $x^*(\tau)$ is an optimal solution for $\mathrm{LP}(A(\tau), b(\tau), c(\tau))$ with optimal value $v(\tau)$.*

The above corollary was proved by Jeroslow [Jer73, §2.3]. His argument, based on controlling signs of determinants, is essentially a local version of our Theorem 6.2. Moreover, determining all the rays of a polyhedral cone can be reduced to solving sufficiently many LPs. This could also be exploited to derive another proof of Theorem 6.2 from Corollary 6.5.

Remark 6.6. It is worth to mention the special case of a linear program over the field $\mathbb{F}\{t\}$, where the coordinates of the linear constraints, in fact, are elements of the field $\mathbb{F}$ of coefficients, but the coordinates of the linear objective function are arbitrary elements in $\mathbb{F}\{t\}$. That is, the feasible domain is a polyhedron P over $\mathbb{F}$. Evaluating the objective function at some $\tau \in \mathbb{F}$ makes one of the vertices of P optimal. Solving for all values of τ, in general, amounts to computing the entire normal fan of the polyhedron P. This is equivalent to solving the dual convex hull problem over $\mathbb{F}$ for the given inequality

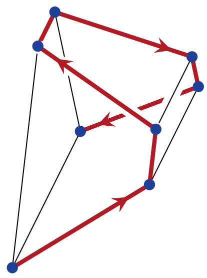


Figure 6.2: The 3-dimensional Goldfarb–Sit cube.

description of P; see also [JKM08]. Here we restrict our attention to solving parametric linear programs via Corollary 6.5.

The next example is a slight variation of a construction of Goldfarb and Sit [GS79]. This is a class of linear optimization problems on which certain versions of the simplex method perform poorly.

Example 6.7. We fix $d > 1$ and pick a positive $\delta \leq \frac{1}{2}$ as well as a positive $\varepsilon < \frac{\delta}{2}$. Consider the linear program

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{d} \delta^{d-i} x_i \\ \text{subject to} & \quad 0 \leq x_1 \leq \varepsilon^{d-1} \\ & x_{j-1} \leq \delta x_j \leq \varepsilon^{d-j}\delta - x_{j-1} \qquad \text{for } 2 \leq j \leq d \ . \end{array}$$

The feasible region is combinatorially equivalent to the d-dimensional cube. Applying the simplex method with the "steepest edge" pivoting strategy to this linear program with the origin as the start vertex visits all the 2^d vertices. Moreover, the vertex-edge graph with the orientation induced by the objective function is isomorphic to (the oriented vertex-edge graph of) the Klee–Minty cube [KM72]. See Figure 6.2 for a visualization of the 3-dimensional case.

We may interpret this linear program over the reals or over $(\mathbb{R}\{\delta\})\{\varepsilon\}$, the field of Puiseux fractions in the indeterminate ε with coefficients in the field $\mathbb{R}\{\delta\}$. This depends on whether we want to view δ and ε as indeterminates or as real numbers. Here we consider the ordering induced by the dual valuation val*, i.e., δ and ε are *small* infinitesimals, where $\varepsilon \ll \delta$. Two more choices arise from considering ε a constant in $\mathbb{R}\{\delta\}$ or, conversely, δ a constant in $\mathbb{R}\{\varepsilon\}$. Note that our constraints on δ and ε are feasible in all four cases.

Our third and last example is a class of linear programs occurring in [ABGJ14]. For these the central path of the interior point method with a logarithmic barrier function has a total curvature which is exponential as a function of the dimension.

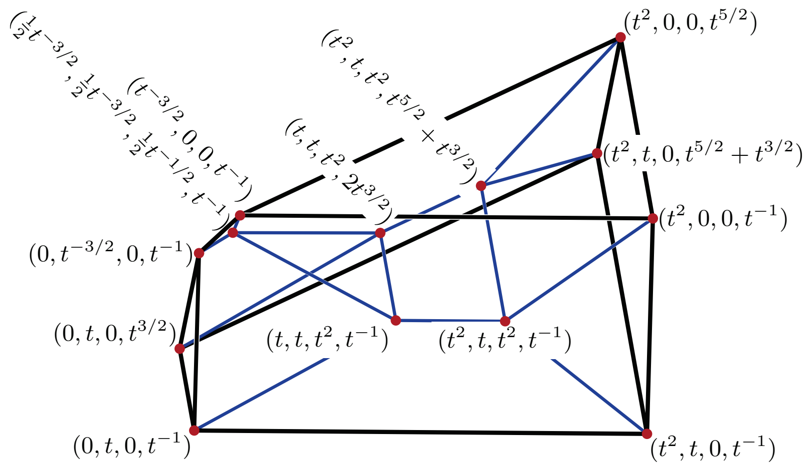


Figure 6.3: The Schlegel diagram of the feasible polytope in Example 6.8 for $r = 1$ and $t > 1$

Example 6.8. Given a positive integer r, we define a linear program over the field $\mathbb{Q}\{t\}$ (with the primal valuation) in the $2r + 2$ variables $u_0, v_0, u_1, v_1, \ldots, u_r, v_r$ as follows:

$$\begin{aligned} &\text{minimize} && v_0 \\ &\text{subject to} && u_0 \leq t\,,\ \ v_0 \leq t^2 \\ &&& \left.\begin{aligned} &u_i \leq t u_{i-1}\,,\ \ u_i \leq t v_{i-1} \\ &v_i \leq t^{1-\frac{1}{2^i}}(u_{i-1} + v_{i-1}) \end{aligned}\right\} \qquad \text{for } 1 \leq i \leq r \\ &&& u_r \geq 0\,,\ \ v_r \geq 0\,. \end{aligned}$$

Here it would be interesting to know the exact value for the optimal τ_0 in Theorem 6.2, as a function of r. Experimentally, based on the method described below, we found $\tau_0 = 1$ for $r = 1$ and $\tau_0 = 2^{2^{r-1}}$ for r at most 5. We conjecture the latter to be the true bound in general. Figure 6.3 shows the Schlegel projection of the feasibility polytope on the facet $u_1 = 0$ for $r = 1$ and $t > 1$.

To find the optimal bound for a given constraint matrix A we can use the following method. One can solve the dual convex hull problem for the cone C, which is the feasible region in homogenized form, to obtain a matrix B whose columns are the rays of C. This also yields a submatrix of A corresponding to the rows which define facets of C. Without loss of generality we may assume that that submatrix is A itself. Let τ_0 be the largest zero or pole of any (Puiseux fraction) entry of the matrix $A \cdot B$. Then for every value $\tau > \tau_0$ the sign patterns of $(A \cdot B)(\tau)$ and $A \cdot B$ coincide, and so do the combinatorial types of C and $C(\tau)$. Determining the zeros and poles of a Puiseux fraction amounts to factorizing univariate polynomials.

6.4 Tropical dual convex hulls

Tropical geometry is the study of the piecewise linear images of algebraic varieties, defined over a field with a non-Archimedean valuation, under the valuation map; see [MS15] for an overview. The motivation for research in this area comes from at least two different directions. First, tropical varieties still retain a lot of interesting information about their classical counterparts. Therefore, passing to the tropical limit opens up a path for combinatorial algorithms to be applied to topics in algebraic geometry. Second, the algebraic geometry perspective offers opportunities for optimization and computational geometry. Here we will discuss how classical convex hull algorithms over fields of Puiseux fractions can be applied to compute tropical convex hulls; see [Jos09] for a survey on the subject; a standard algorithm is the tropical double description method of [AGG10].

The *tropical semiring* $\mathbb{T}$ consists of the set $\mathbb{R} \cup \{-\infty\}$ together with $u \oplus v = \max(u, v)$ as the addition and $u \odot v = u + v$ as the multiplication. Extending these operations to vectors turns $\mathbb{T}^{d+1}$ into a semimodule. A *tropical cone* is the sub-semimodule

$$\operatorname{tcone}(G) = \{\, \lambda_1 \odot g_1 \oplus \cdots \oplus \lambda_n \odot g_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{T}\}$$

generated from the columns $g_1, \ldots, g_n$ of the matrix $G \in \mathbb{T}^{(d+1)\times n}$. Similar to classical cones, tropical cones admit an exterior description [GK11]. It is known that every tropical cone is the image of a classical cone under the valuation map $\operatorname{val}\colon \mathbb{R}\{\!\{t\}\!\} \to \mathbb{T}$; see [DY07]. Based on this idea, we present an algorithm for computing generators of a tropical cone from a description in terms of tropical linear inequalities; see Algorithm 6.1 below.

Before we can start to describe that algorithm we first need to discuss matters of general position in the tropical setting. The *tropical determinant* of a square matrix $U \in \mathbb{T}^{\ell\times\ell}$ is given by

$$\operatorname{tdet}(U) \;=\; \bigoplus_{\sigma\in S_\ell} u_{1\pi(1)} \odot \cdots \odot u_{\ell\pi(\ell)} \quad . \tag{6.6}$$

Here S_ℓ is the symmetric group of degree ℓ; computing the tropical determinant is the same as solving a linear assignment optimization problem. Consider a pair of matrices $H^+, H^- \in \mathbb{T}^{m\times(d+1)}$ which serve as an exterior description of the tropical cone

$$Q \;=\; \left\{ z \in \mathbb{T}^{(d+1)} \;\middle|\; H^+ \odot z \geq H^- \odot z \right\} \quad . \tag{6.7}$$

In contrast to the classical situation we have to take two matrices into account. This is due to the lack of an additive inverse operation. We will assume that $\mu(i,j) := \min(H^+_{ij}, H^-_{ij}) = -\infty$ for any pair $(i,j) \in [m]\times[d+1]$, i.e., for each coordinate position at most one of the corresponding entries in the two matrices is finite. Then we can define

$$\chi(i,j) \;:=\; \begin{cases} 1 & \text{if } \mu(i,j) = H^+_{ij} \neq -\infty \\ -1 & \text{if } \mu(i,j) = H^-_{ij} \neq -\infty \\ 0 & \text{otherwise} \;. \end{cases}$$

For each term $u_{1\pi(1)} \odot \cdots \odot u_{\ell\pi(\ell)}$ in (6.6) we define its *sign* as

$$\operatorname{sign}(\pi) \cdot \chi(1,\pi(1)) \cdots \chi(\ell,\pi(\ell)) \ ,$$

where $\operatorname{sign}(\pi)$ is the sign of the permutation π. Now the exterior description (6.7) of the tropical cone Q is *tropically sign-generic* if for each square submatrix U of $H^+ \oplus H^-$ we have $\operatorname{tdet}(U) \neq -\infty$ and, moreover, the signs of all terms $u_{1\pi(1)} \odot \cdots \odot u_{\ell\pi(\ell)}$ which attain the maximum in (6.6) agree. By looking at $1{\times}1$-submatrices U we see that in this case all coefficients of the matrix $H^+ \oplus H^-$ are finite and thus $\chi(i,j)$ is never 0.

Algorithm 6.1: A dual tropical convex hull algorithm

Input: pair of matrices $H^+, H^- \in \mathbb{T}^{m\times(d+1)}$ which provide a tropically sign-generic exterior description of the tropical cone Q from (6.7)
Output: generators for Q
pick two matrices $A^+, A^- \in \mathbb{R}\{\{t\}\}^{m\times(d+1)}$ with strictly positive entries such that $\operatorname{val}(A^+) = H^+$ and $\operatorname{val}(A^-) = H^-$
apply a classical dual convex hull algorithm to determine a matrix $B \in \mathbb{R}\{\{t\}\}^{(d+1)\times n}$ such that
$\{\, B\cdot a \mid a \in \mathbb{R}\{\{t\}\}^n,\ a \geq 0\} = \left\{x \in \mathbb{R}\{\{t\}\}^{(d+1)} \;\middle|\; (A^+ - A^-)\cdot x \geq 0,\ x \geq 0\right\}$
return $\operatorname{val}(B)$

Correctness of Algorithm 6.1. The main lemma of tropical linear programming [ABGJ15, Theorem 16] says the following. In the tropically sign-generic case, an exterior description of a tropical cone can be obtained from an exterior description of a classical cone over Puiseux series by applying the valuation map to the constraint matrix coefficient-wise. This statement assumes that the classical cone is contained in the non-negative orthant. We infer that

$$\begin{aligned} Q &= \left\{ z \in \mathbb{T}^{m\times(d+1)} \;\middle|\; H^+ \odot z \geq H^- \odot z \right\} \\ &= \operatorname{val}\left(\left\{ x \in \mathbb{R}\{\{t\}\}^{m\times(d+1)} \;\middle|\; A^+\cdot x \geq A^-\cdot x, x \geq 0 \right\} \right) \\ &= \operatorname{val}\left(\{\, B\cdot a \mid a \in \mathbb{R}\{\{t\}\}^n, x \geq 0\} \right) \ . \end{aligned}$$

Now [DY07, Proposition 2.1] yields

$$Q \ = \ \operatorname{val}(\{\, B\cdot a \mid a \in \mathbb{R}\{\{t\}\}^n, x \geq 0\}) \ = \ \operatorname{tcone}(\operatorname{val}(B)) \ .$$

This ends the proof. □

The correctness of our algorithm is not guaranteed if the genericity condition is not satisfied. The crucial properties of the lifted matrices A^+, A^- are not necessarily fulfilled. It is an open question of how an exterior description over $\mathbb{T}$ is related to an exterior description over $\mathbb{R}\{\{t\}\}$ in the general setting. We are even lacking a convincing concept for the "facets" of a general tropical cone.

6.5 Implementation

As a key feature the polymake system is designed as a Perl/C++ hybrid, that is, both programming languages are used in the implementation and also both programming languages can be employed by the user to write further code. One main advantage of Perl is the fact that it is interpreted; this makes it suitable as the main front end for the user. Further, Perl has its strengths in the manipulation of strings and file processing. C++ on the other hand is a compiled language with a powerful template mechanism which allows to write very abstract code which, nonetheless, is executed very fast. Our implementation, in C++, makes extensive use of these features. The implementation of the dual steepest edge simplex method, contributed by Thomas Opfer [Opf11], and the beneath-beyond method for computing convex hulls (see [Ede87] and [Jos03]) are templated. Therefore polymake can handle both computations for arbitrary number field types which encode elements in an ordered field.

Based on this mechanism we implemented the type RationalFunction which depends on two generic template types for coefficients and exponents. Note that the field of coefficients here does not have to be ordered. Our proof-of-concept implementation employs the classical Euclidean GCD algorithm for normalization. Currently the numerator and the denominator are chosen coprime such that the denominator is normalized with leading coefficient one. For the most interesting case $\mathbb{F} = \mathbb{Q}$ it is known that the coefficients of the intermediate polynomials can grow quite badly, e.g., see [vzGG03, Example 1]. Therefore, as expected, this is the bottleneck of our implementation. In a number field or in a field with a non-Archimedean valuation the most natural choice for a normalization is to pick the elements of the ring of integers as coefficients. The reason for our choice is that this more generic design does not make any assumption on the field of coefficients. This makes it very versatile, and it fits the overall programming style in polymake. A fast specialization to the rational coefficient case could be based on [vzGG03, Algorithm 11.4]. This is left for a future version.

The polymake implementation of Puiseux fractions $\mathbb{F}\{t\}$ closely follows the construction described in Section 6.2. The new number type is derived from RationalFunction with overloaded comparison operators and new features such as evaluating and converting into TropicalNumber. An extra template parameter MinMax allows to choose whether the indeterminate t is a small or a large infinitesimal.

There are other implementations of Puiseux series arithmetic, e.g., in Magma [BCP97] or MATLAB [MAT14]. However, they seem to work with finite truncations of Puiseux series and floating-point coefficients. This does not allow for exact computations of the kind we are interested in.

6.6 Computations

We briefly show how our polymake implementation can be used. Further, we report on timings for our LP solver, tested on the Goldfarb–Sit cubes from Example 6.7, and for our (dual) convex hull code, tested on the polytopes with a "long and winding" central path from Example 6.8.

6.6.1 Using **polymake**

The following code defines a 3-dimensional Goldfarb–Sit cube over the field $\mathbb{Q}\{t\}$, see Example 6.7. We use the parameters $\varepsilon = t$ and $\delta = \frac{1}{2}$. The template parameter Min indicates that the ordering is induced by the dual valuation val*, and hence the indeterminate t plays the role of a small infinitesimal.

```
polytope > set_var_names<UniPolynomial<Rational,Rational>>('t');
polytope > $monomial = new UniPolynomial<Rational,Rational>('t');
polytope > $t = new PuiseuxFraction<Min>($monomial);
polytope > $p = goldfarb_sit(3,2*$t,1/2);
```

The polytope object, stored in the variable $p, is generated with a facet description from which further properties will be derived below. It is already equipped with a LinearProgram subobject encoding the objective function from Example 6.7. The following lines show the maximal value and corresponding vertex of this linear program as well as the vertices derived from the outer description. Below, we present timings for such calculations.

```
polytope > print $p->LP->MAXIMAL_VALUE;
(1)
polytope > print $p->LP->MAXIMAL_VERTEX;
(1) (0) (0) (1)
polytope > print $p->VERTICES;
(1) (0) (0) (0)
(1) (t^2) (2*t^2) (4*t^2)
(1) (0) (t) (2*t)
(1) (t^2) (t -2*t^2) (2*t -4*t^2)
(1) (0) (0) (1)
(1) (t^2) (2*t^2) (1 -4*t^2)
(1) (0) (t) (1 -2*t)
(1) (t^2) (t -2*t^2) (1 -2*t + 4*t^2)
```

As an additional benefit of our implementation we get numerous other properties for free. For instance, we can compute the parameterized volume, which is a polynomial in t.

```
polytope > print $p->VOLUME;
(t^3 -4*t^4 + 4*t^5)
```

That polynomial, as an element of the field of Puiseux fractions, has a valuation, and we can evaluate it at the rational number $\frac{1}{12}$.

Table 6.1: Timings (in seconds) for the Goldfarb–Sit cubes of dimension d with $\delta = \frac{1}{2}$. For ε we tried a small infinitesimal as well as two rational numbers, one with a short binary encoding and another one whose encoding is fairly large. For comparison we also tried both parameters as indeterminates.

d	m	n	$\mathbb{Q}\{\varepsilon\}$ ε	$\mathbb{Q}$ $\varepsilon = \frac{1}{6}$	$\mathbb{Q}$ $\varepsilon = \frac{2}{17^{4500}}$	$(\mathbb{Q}\{\delta\})\{\varepsilon\}$ $\varepsilon \ll \delta$
3	6	8	0.010	0.003	0.005	0.101
4	8	16	0.026	0.001	0.017	0.353
5	10	32	0.064	0.002	0.065	1.034
6	12	64	0.157	0.007	0.253	2.877
7	14	128	0.368	0.006	0.829	7.588
8	16	256	0.843	0.016	2.643	19.226
9	18	512	1.906	0.039	7.703	47.806
10	20	1024	4.258	0.090	21.908	118.106
11	22	2048	9.383	0.191	59.981	287.249
12	24	4096	20.583	0.418	160.894	687.052

```
polytope > print $p->VOLUME->val;
3
polytope > print $p->VOLUME->evaluate(1/12);
25/62208
```

6.6.2 Linear programs

We have tested our implementation by computing the linear program of Example 6.7 with polyhedra defined over Puiseux fractions.

The simplex method in `polymake` is an implementation of a (dual) simplex with a (dual) steepest edge pricing. We set up the experiment to make sure our Goldfarb–Sit cube LPs behave as badly as possible. That is, we force our implementation to visit all $n = 2^d$ vertices, when d is the dimension of the input. Table 6.1 illustrates the expected exponential growth of the execution time of the linear program. In three of our four experiments we choose δ as $\frac{1}{2}$. The computation over $\mathbb{Q}\{\varepsilon\}$ costs a factor of about 80 in time, compared with the rational cubes for a modest $\varepsilon = \frac{1}{6}$. However, taking a small ε whose binary encoding takes more than 18,000 bits is substantially more expensive than the computations over the field $\mathbb{Q}\{\varepsilon\}$ of Puiseux fractions. Taking δ as a second small infinitesimal is possible but prohibitively expensive for dimensions larger than twelve.

6.6.3 Convex hulls

We have also tested our implementation by computing the vertices of the polytope from Example 6.8. For this we used the client `long_and_winding` which creates the

Table 6.2: Timings (in seconds) for convex hull computation of the feasibility set from Example 6.8. All timings represent an average over ten iterations. If any test exceeded a one hour time limit this and all larger instances of the experiment were skipped and marked −.

r	d	m	n	$\mathbb{Q}\{t\}$	$\mathbb{Q}$
1	4	7	11	0.018	0.000
2	6	10	28	0.111	0.000
3	8	13	71	0.754	0.010
4	10	16	182	15.445	0.036
5	12	19	471	1603.051	0.150
6	14	22	1226	−	0.737
7	16	25	3201	−	4.001
8	18	28	8370	−	25.093
9	20	31	21901	−	223.240
10	22	34	57324	−	1891.133

$d = (2r + 2)$-dimensional polytope given by $m = 3r + 4$ facet-defining inequalities. Over the rationals we evaluated the inequalities at 2^{2^r} which probably gives the correct combinatorics; see the discussion at the end of Example 6.8. This very choice forces the coordinates of the defining inequalities to be integral, such that the polytope is rational. The number of vertices n is derived from that rational polytope. The running times grow quite dramatically for the parametric input. This overhead could be reduced via a better implementation of the Puiseux fraction arithmetic.

6.6.4 Experimental setup

Everything was calculated on the same Linux machine with `polymake` perpetual beta version 2.15-beta3 which includes the new number type, the templated simplex algorithm and the templated beneath-and-beyond convex hull algorithm. The lines of code that we present here are updatet to `polymake` version 3.1, this effects only the initialization of the variable `$monomial`. All timings were measured in CPU seconds and averaged over ten iterations. The simplex algorithm was set to use only one thread.

All tests were done on `openSUSE` 13.1 (x86_64), with Linux kernel 3.11.10-25, `clang` 3.3 and `perl` 5.18.1. The rational numbers use a `C++`-wrapper around the `GMP` library version 5.1.2. As memory allocator `polymake` uses the `pool_allocator` from `libstdc++`, which was version 4.8.1 for the experiments.

The hardware for all tests was:

Intel(R) Core(TM) i7-3930K CPU @ 3.20GHz
bogomips: 6400.21
MemTotal: 32928276 kB

Conclusion

We have investigated tropical linear spaces and their moduli spaces. Classical linear spaces are a topic in a basic course in mathematics, while their tropical analogous is combinatorially rich. Also for a better understanding of the tropicalization of another variety it is helpful to study tropical linear spaces. We have seen that they are related to various areas in mathematics. We enlightened the relations to polyhedral, algebraic and computational geometry, as well as matroid theory. All of these areas can benefit from this thesis with its theorems, algorithms, examples, data and implementations. Beside these visible results in this thesis, there are also many features and enhancements added in the software `polymake` [GJ00]. In particular, the online database of "small matroids" developed by Matsumoto, Moriyama, Imai and Bremner [MMIB12, Table 1] has been made accessible at db.polymake.org. Furthermore, an interactive model of the Dressian $\mathrm{Dr}(2,6)$ is online available at gallery.discretization.de; see also [JMS+16].

We have shown how useful the concept of corank vectors and corank subdivisions are for investigating the Dressian. Moreover, in Theorem 2.35 they are the key to relate realizability of matroids and tropical linear spaces. Fink and Moci further developed this line of research and improved some of our results. They showed that the maximal cells of the corank subdivision of a matroid on the 0/1-cube are in bijection to the cyclic flats of the matroid, see [FM17, Proposition 5.10]. We introduced split matroids, which have many characterizations and applications as they carry geometric features. We used the combination of split matroids or sparse paving matroids and corank vectors to approximate the dimension of the Dressian (Theorem 2.31 and Corollary 2.32), which confirms a huge difference between a Grassmannian and a Dressian. Additionally taking series-lifts into account we were able to construct many new rays of the Dressian, among them some that are non-realizable. In particular, this construction answered a question of Herrmann, Joswig and Speyer in [HJS12, Question 36]. Proving this result included a refined connection between the realizability of a matroid and a tropical linear space. Furthermore, we showed that all multi-splits of the hypersimplex can be constructed by iterating the process of series-lifts. Our analysis provides a full description of all multi-splits of hypersimplices (Theorem 3.31) and products of simplices (Theorem 3.35). Theorem 3.31 and Proposition 3.33 tightened and generalized several results of Herrmann about 3-splits [Her11, Proposition 6.3, Corollary 6.4 and Theorem 6.5]. They are important as they can help to get new insights into the structure of secondary polytopes.

We are the first who gave an explicit bound on the degree of a tropical basis (Theorem 5.5 and Theorem 5.10) and hopefully our examples in Section 5.3 of ideals are useful to clarify the relation of tropical bases and Gröbner bases. Clearly, the description and implementation of the field of Puiseux fractions in Chapter 6 is a feasible tool not only for tropical geometry, but also for many parametrized polyhedral problems.

We developed some new concepts, such as split matroids, negligible points and extended tight spans. All of those have been created in the context of investigating the interaction of matroids and polyhedral geometry in the light of tropical geometry. There are many questions left open in this interplay. We will close with some of them concerning the above mentioned terms. Analogous to split matroids, a matroid could be called *multi-split* if two facets of the matroid polytope intersect at most in a fixed codimension in the relative interior of the hypersimplex. How do these multi-split matroids fit into the overall asymptotic behaviour of all matroids? Is there a weak decompostion of a matroid subdivision into multi-split subdivisions, which is compatible with the intersection ring of matroids introduced by Hampe [Ham17]? What is the connection between negligible points and fine subdivisions? Can one apply Algorithm 4.1 with an appropriate closure operator to the computation of triangulations or coarsest subdivisions?

Appendix

In this appendix we present enumeration results of various kinds. We list the number of triangulations of products of simplices in Table A.1, which is a byproduct of the computation of rays of their secondary fan.

We saw some tables that are based on the census of small matroids which has been obtained by Matsumoto, Moriyama, Imai and Bremner [MMIB12]. All the computations for these tables were done with `polymake` [GJ00].

Moreover, we list the number of multi-splits in Table A.3 of the formula in Proposition 3.33 and the number of multi-splits up to symmetry in Table A.4.

Table A.1: The number of triangulations of the product $\Delta_{d-1} \times \Delta_{n-d-1}$; compare also with [San05] for $d = 3$.

$d\backslash n$	7	8	9	10
3	4488	376200	58652640	16119956160
4	4488	4533408	–	–

Table A.2: The number of isomorphism classes of all matroids of rank d on n elements, see [MMIB12, Table 1]

$d\backslash n$	4	5	6	7	8	9	10	11	12
2	7	13	23	37	58	87	128	183	259
3	4	13	38	108	325	1275	10037	298491	31899134
4	1	5	23	108	940	190214	4886380924	–	–
5		1	6	37	325	190214	–	–	–
6			1	7	58	1275	4886380924	–	–
7				1	8	87	10037	–	–
8					1	9	128	298491	–
9						1	10	183	31899134
10							1	11	259
11								1	12

Table A.3: The total number of k-splits in the hypersimplex $\Delta(d, n)$. This is the evaluation of the formula in Proposition 3.33.

d	k \ n	6	7	8	9	10	11	12	13	14
2	2	25	56	119	246	501	1012	2035	4082	8177
3	2	35	91	210	456	957	1969	4004	8086	16263
	3	30	210	980	3836	13650	45870	148632	470184	1463462
4	2	25	91	245	582	1293	2761	5753	11804	23985
	3		210	1540	7476	30240	110550	379764	1252680	4020016
	4			630	7560	56700	341880	1817970	8923200	41489448
5	2		56	210	582	1419	3223	7007	14807	30706
	3			980	7476	37590	156750	588126	2065206	6938932
	4				7560	94500	734580	4569180	24959220	125381256
	5					22680	415800	4573800	39279240	290930640

Table A.4: The number of k-splits in the hypersimplex $\Delta(d, n)$ with respect to symmetry on the coordinates.

d	k \ n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	2	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9
3	2		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	3			1	1	2	3	4	5	7	8	10	12	14	16	19	21	23
4	2			2	3	5	6	8	9	11	12	14	15	17	18	20	21	23
	3				1	2	4	6	9	12	16	20	25	30	36	42	49	56
	4					1	1	2	3	5	6	9	11	15	18	23	27	34
5	2				2	4	6	8	10	12	14	16	18	20	22	24	26	28
	3					2	4	8	12	18	24	32	40	50	60	72	84	98
	4						1	2	4	7	11	16	23	31	41	53	67	83
	5							1	1	2	3	5	7	10	13	18	23	30

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