Recalculation of the derivation of the image source model for a rectangular room with rigid boundaries

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Remarks

This document is meant to be a helpful resource to comprehend the derivation of the image source model for a rectangular room according to [AB79, Appendix A] in more detail. If you find any mistakes, please contact us! We are happy to improve this document.

The following conventions for the Fourier transform are used in this document: The Fourier transform of the sound pressure $p(x,t)$ with respect to time t reads

$$P(x,\omega) = \mathcal{F}_t\left\{p(x,t)\right\} = \int\limits_{-\infty}^{\infty} p(x,t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t \tag{1}$$

and the Fourier transform with respect to space x reads

$$P(k_x,t) = \mathcal{F}_x\left\{p(x,t)\right\} = \int\limits_{-\infty}^{\infty} p(x,t)\mathrm{e}^{\mathrm{j}k_x x}\mathrm{d}x, \tag{2}$$

where ω denotes the angular frequency and k_x denotes the wavenumber in x-direction. A monochromatic plane wave in 3D space is then written as

$$p(\mathbf{x},t) = \mathrm{e}^{-\mathrm{j}\mathbf{kx}}\mathrm{e}^{\mathrm{j}\omega t} \tag{3}$$

with field point $\mathbf{x} = (x,y,z)^{\mathrm{T}}$, and propagates in direction of the wavenumber vector $\mathbf{k} = (k_x,k_y,k_z)^{\mathrm{T}}$, $|\mathbf{k}| = k$. Please note that these conventions differ from the ones used in the original paper [AB79] where for both spatial and temporal Fourier transform the opposite sign in the exponential function is used.

Derivation

The inhomogeneous Helmholtz equation with an exciting source in $\mathbf{x_0} = (x_0, y_0, z_0)^{\mathrm{T}}$ is

$$\nabla^2 P(\mathbf{x}, \mathbf{x_0}, \omega) + \left(\frac{\omega}{c}\right)^2 P(\mathbf{x}, \mathbf{x_0}, \omega) = -\delta(\mathbf{x} - \mathbf{x_0}), \tag{4}$$

where c denotes the speed of sound, exhibits for a rectangular room with rigid walls (Neumann boundary conditions) and the volume $V = L_x \cdot L_y \cdot L_z$ the solution

$$\boxed{P(\mathbf{x}, \mathbf{x_0}, \omega) = \frac{1}{V} \sum_{\mathbf{m}=-\infty}^{\infty} \frac{\psi_{\mathbf{m}}(\mathbf{x}) \psi_{\mathbf{m}}(\mathbf{x_0})}{k_{\mathbf{m}}^2 - k^2}} \tag{5}$$

with

$$\psi_{\mathbf{m}}(\mathbf{x}) = \cos\left(\frac{m_x \pi}{L_x} x\right) \cos\left(\frac{m_y \pi}{L_y} y\right) \cos\left(\frac{m_z \pi}{L_z} z\right), \tag{6}$$

$$\mathbf{m} = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} \tag{7}$$

and the modal wave vector

$$\mathbf{k_m} = \begin{pmatrix} \frac{m_x \pi}{L_x} \\ \frac{m_y \pi}{L_y} \\ \frac{m_z \pi}{L_z} \end{pmatrix} = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} \quad , \quad |\mathbf{k_m}|^2 = k_{\mathbf{m}}^2. \tag{8}$$

Please note that source and receiver point can be interchanged in eq. (4). The inverse formulation can be found in [AB79, eq. (A1)] but the formulation here is preferred for clarity.
Eq. (5) shall now be converted to an expression containing exponential functions to arrive at [AB79, eq. (A5)], which is done with the exponential expansion of the cosine

$$\cos(x) = \frac{1}{2} \left(\mathrm{e}^{\mathrm{j}x} + \mathrm{e}^{-\mathrm{j}x}\right). \tag{9}$$

Expanding the numerator product of eq. (5) and reordering gives

$$
\begin{aligned}
\psi_{\mathbf{m}}(\mathbf{x})\psi_{\mathbf{m}}(\mathbf{x_0}) =& \cos\left(\frac{m_x\pi}{L_x}x\right)\cos\left(\frac{m_y\pi}{L_y}y\right)\cos\left(\frac{m_z\pi}{L_z}z\right)...\\
&\cdot\cos\left(\frac{m_x\pi}{L_x}x_0\right)\cos\left(\frac{m_y\pi}{L_y}y_0\right)\cos\left(\frac{m_z\pi}{L_z}z_0\right)
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
=&\frac{1}{2}\left(\mathrm{e}^{\mathrm{j}k_x x}+\mathrm{e}^{-\mathrm{j}k_x x}\right)\frac{1}{2}\left(\mathrm{e}^{\mathrm{j}k_x x_0}+\mathrm{e}^{-\mathrm{j}k_x x_0}\right)...\\
&\cdot\frac{1}{2}\left(\mathrm{e}^{\mathrm{j}k_y y}+\mathrm{e}^{-\mathrm{j}k_y y}\right)\frac{1}{2}\left(\mathrm{e}^{\mathrm{j}k_y y_0}+\mathrm{e}^{-\mathrm{j}k_y y_0}\right)...\\
&\cdot\frac{1}{2}\left(\mathrm{e}^{\mathrm{j}k_z z}+\mathrm{e}^{-\mathrm{j}k_z z}\right)\frac{1}{2}\left(\mathrm{e}^{\mathrm{j}k_z z_0}+\mathrm{e}^{-\mathrm{j}k_z z_0}\right)
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
=&\frac{1}{4}\left(\mathrm{e}^{\mathrm{j}k_x(x+x_0)}+\mathrm{e}^{\mathrm{j}k_x(x-x_0)}+\mathrm{e}^{-\mathrm{j}k_x(x-x_0)}+\mathrm{e}^{-\mathrm{j}k_x(x+x_0)}\right)...\\
&\cdot\frac{1}{4}\left(\mathrm{e}^{\mathrm{j}k_y(y+y_0)}+\mathrm{e}^{\mathrm{j}k_y(y-y_0)}+\mathrm{e}^{-\mathrm{j}k_y(y-y_0)}+\mathrm{e}^{-\mathrm{j}k_y(y+y_0)}\right)...\\
&\cdot\frac{1}{4}\left(\mathrm{e}^{\mathrm{j}k_z(z+z_0)}+\mathrm{e}^{\mathrm{j}k_z(z-z_0)}+\mathrm{e}^{-\mathrm{j}k_z(z-z_0)}+\mathrm{e}^{-\mathrm{j}k_z(z+z_0)}\right)
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
=&\frac{1}{4^3}(2\cdot\cos(k_x(x+x_0))+2\cdot\cos(k_x(x-x_0)))...\\
&\cdot(2\cdot\cos(k_y(y+y_0))+2\cdot\cos(k_y(y-y_0)))...\\
&\cdot(2\cdot\cos(k_z(z+z_0))+2\cdot\cos(k_z(z-z_0)))
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
=&\frac{2^3}{4^3}(\cos(k_x(x+x_0))+\cos(k_x(x-x_0)))...\\
&\cdot(\cos(k_y(y+y_0))\cos(k_z(z+z_0))+\cos(k_y(y+y_0))\cos(k_z(z-z_0))...\\
&+\cos(k_y(y-y_0))\cos(k_z(z+z_0))+\cos(k_y(y-y_0))\cos(k_z(z-z_0)))
\end{aligned}
\tag{14}
$$

$$
=\frac{1}{8}(\cos(k_x(x+x_0))\cos(k_y(y+y_0))\cos(k_z(z+z_0))+...). \tag{15}
$$

Looking only at the first out of eight completely expanded triple cosine terms yields

$$\cos(k_x(x+x_0))\cos(k_y(y+y_0))\cos(k_z(z+z_0)) \tag{16}$$

$$=\frac{1}{2}\left(\mathrm{e}^{\mathrm{j}k_x(x+x_0)}+\mathrm{e}^{-\mathrm{j}k_x(x+x_0)}\right)\frac{1}{2}\left(\mathrm{e}^{\mathrm{j}k_y(y+y_0)}+\mathrm{e}^{-\mathrm{j}k_y(y+y_0)}\right)\frac{1}{2}\left(\mathrm{e}^{\mathrm{j}k_z(z+z_0)}+\mathrm{e}^{-\mathrm{j}k_z(z+z_0)}\right) \tag{17}$$

$$\begin{aligned}=\frac{1}{8}\Big(&\mathrm{e}^{\mathrm{j}k_x(x+x_0)}\mathrm{e}^{\mathrm{j}k_y(y+y_0)}\mathrm{e}^{\mathrm{j}k_z(z+z_0)}+\mathrm{e}^{\mathrm{j}k_x(x+x_0)}\mathrm{e}^{\mathrm{j}k_y(y+y_0)}\mathrm{e}^{-\mathrm{j}k_z(z+z_0)}\ldots\\
&+\mathrm{e}^{\mathrm{j}k_x(x+x_0)}\mathrm{e}^{-\mathrm{j}k_y(y+y_0)}\mathrm{e}^{\mathrm{j}k_z(z+z_0)}+\mathrm{e}^{\mathrm{j}k_x(x+x_0)}\mathrm{e}^{-\mathrm{j}k_y(y+y_0)}\mathrm{e}^{-\mathrm{j}k_z(z+z_0)}\ldots\\
&+\mathrm{e}^{-\mathrm{j}k_x(x+x_0)}\mathrm{e}^{\mathrm{j}k_y(y+y_0)}\mathrm{e}^{\mathrm{j}k_z(z+z_0)}+\mathrm{e}^{-\mathrm{j}k_x(x+x_0)}\mathrm{e}^{\mathrm{j}k_y(y+y_0)}\mathrm{e}^{-\mathrm{j}k_z(z+z_0)}\ldots\\
&+\mathrm{e}^{-\mathrm{j}k_x(x+x_0)}\mathrm{e}^{-\mathrm{j}k_y(y+y_0)}\mathrm{e}^{\mathrm{j}k_z(z+z_0)}+\mathrm{e}^{-\mathrm{j}k_x(x+x_0)}\mathrm{e}^{-\mathrm{j}k_y(y+y_0)}\mathrm{e}^{-\mathrm{j}k_z(z+z_0)}\Big).\end{aligned} \tag{18}$$

Each product of three exponential functions can be written as one exponential function with a longer exponent. This exponent then contains a scalar product of a certain vector $\mathbf{k_m}$ (with all three vector components running from $-\infty$ to ∞, cf. eq. (5) together with (8)) and a vector for field and source points $\mathbf{R}_p = (x \pm x_0, y \pm y_0, z \pm z_0)$, e.g.

$$\mathrm{e}^{\mathrm{j}(k_x(x+x_0)+k_y(y+y_0)+k_z(z+z_0))} = \mathrm{e}^{\mathbf{k_m}\cdot\mathbf{R}_p}. \tag{19}$$

There obviously exist 8 different vectors $\mathbf{R}_p$ if considering all $\pm$-combinations and the 8 combinations of exponential functions from eq. (18) arise for each $\mathbf{R}_p$.
Considering now all terms arising from the 8 triple cosines in eq. (15), there are 8 vectors $\mathbf{k_m}$ resulting from all possible combinations of the components and their opposite numbers (because the indices in eq. (5) run from $-\infty$ to ∞), and from each vector $\mathbf{k_m}$, the same (!) exponential functions as in eq. (18) result. Thus, the sum in eq. (5) contains each exponential function 8 times, which cancels the $\frac{1}{8}$ from eq. (15) (the factor $\frac{1}{8}$ from eq. (18) remains, however) and this results in [AB79, eq. (A5)]

$$P(\mathbf{x},\mathbf{x_0},\omega) = \frac{1}{8V}\sum_{\mathbf{m}=-\infty}^{\infty}\sum_{p=1}^{8}\frac{\mathrm{e}^{\mathrm{j}\mathbf{k_m}\cdot\mathbf{R}_p}}{k_{\mathbf{m}}^2 - k^2}. \tag{20}$$

The exponent of the exponential function in eq. (20) might as well be negative, as this only changes the order in which the terms are summed up:

$$P(\mathbf{x},\mathbf{x_0},\omega) = \frac{1}{8V}\sum_{\mathbf{m}=-\infty}^{\infty}\sum_{p=1}^{8}\frac{\mathrm{e}^{-\mathrm{j}\mathbf{k_m}\cdot\mathbf{R}_p}}{k_{\mathbf{m}}^2 - k^2}. \tag{21}$$

This was done to be able to directly apply the Fourier conventions used throughout this derivation, cf. eq. (1) and (2).
Because of the sifting property of the Dirac function for time instant t_0 [GRS01]

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\mathrm{d}t = f(t_0), \tag{22}$$

eq. (20) can be rewritten as [AB79, eq. (A8)]

$$P(\mathbf{x},\mathbf{x_0},\omega) = \frac{1}{8V}\sum_{p=1}^{8}\iiint\limits_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{j}\boldsymbol{\xi}\cdot\mathbf{R}_p}}{|\boldsymbol{\xi}|^2-k^2}\sum_{\mathbf{m}=-\infty}^{\infty}\delta(\boldsymbol{\xi}-\mathbf{k_m})\mathrm{d}^3\boldsymbol{\xi} \tag{23}$$

with $\boldsymbol{\xi} = (\xi_x, \xi_y, \xi_z)^{\mathrm{T}}$. Due to the Fourier series of the Dirac comb with sampling period T [Wik18a]

$$\sum_{n=-\infty}^{\infty}\delta(t-nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{j}n\frac{2\pi}{T}t}, \tag{24}$$

the following expression can be used for each component of $\mathbf{m}$ in eq. (23), here exemplarily shown for m_x:

$$\sum_{m_x=-\infty}^{\infty}\delta(\xi_x-\frac{m_x\pi}{L_x}) = \frac{L_x}{\pi}\sum_{m_x=-\infty}^{\infty}\mathrm{e}^{\mathrm{j}m_x 2L_x\xi_x}. \tag{25}$$

When considering all 3 dimensions of $\mathbf{m}$, this leads to

$$\sum_{\mathbf{m}=-\infty}^{\infty}\delta(\boldsymbol{\xi}-\mathbf{k_m}) = \frac{L_xL_yL_z}{\pi^3}\sum_{\mathbf{m}=-\infty}^{\infty}\mathrm{e}^{\mathrm{j}m_x2L_x\xi_x}\mathrm{e}^{\mathrm{j}m_y2L_y\xi_y}\mathrm{e}^{\mathrm{j}m_z2L_z\xi_z} \tag{26}$$

$$= \frac{V}{\pi^3}\sum_{\mathbf{m}=-\infty}^{\infty}\mathrm{e}^{\mathrm{j}2(m_xL_x\xi_x+m_yL_y\xi_y+m_zL_z\xi_z)} \tag{27}$$

$$= \frac{V}{\pi^3}\sum_{\mathbf{m}=-\infty}^{\infty}\mathrm{e}^{\mathrm{j}\mathbf{R_m}\cdot\boldsymbol{\xi}}, \tag{28}$$

where the exponent holds the scalar product of the vector $\boldsymbol{\xi}$ with

$$\mathbf{R_m} = 2\cdot\begin{pmatrix} m_xL_x \\ m_yL_y \\ m_zL_z \end{pmatrix}. \tag{29}$$

Substituting eq. (28) in (23) cancels the room volume V and leads to

$$P(\mathbf{x},\mathbf{x_0},\omega) = \frac{1}{8\pi^3}\sum_{p=1}^{8}\sum_{\mathbf{m}=-\infty}^{\infty}\iiint\limits_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{j}\boldsymbol{\xi}(\mathbf{R}_p-\mathbf{R_m})}}{|\boldsymbol{\xi}|^2-k^2}\mathrm{d}^3\boldsymbol{\xi}. \tag{30}$$

The sign in the exponent can be inverted again, because adding instead of subtracting $\mathbf{R_m}$ inside the brackets only changes the order of summation (cf. [AB79, eq. (A10)])

$$P(\mathbf{x},\mathbf{x_0},\omega) = \frac{1}{8\pi^3}\sum_{p=1}^{8}\sum_{\mathbf{m}=-\infty}^{\infty}\iiint\limits_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{j}\boldsymbol{\xi}(\mathbf{R}_p+\mathbf{R_m})}}{|\boldsymbol{\xi}|^2-k^2}\mathrm{d}^3\boldsymbol{\xi}. \tag{31}$$

(Again, this step was done to better agree with the Fourier convention used here.)
It can be shown, that the integral in eq. (31) is a plane wave decomposition (PWD) of a point source at the point $\mathbf{R}_p+\mathbf{R_m}$.

Plane wave decomposition of a point source

Cf. e.g. [Wik18b], here rewritten according to the Fourier conventions used here (see eq. (1) and (2)), [Wik18b] uses the same convention as [AB79].

The solution of the three-dimensional inhomogeneous Helmholtz equation for the field $\varphi(\mathbf{x})$

$$\nabla^2\varphi(\mathbf{x}) + k_0^2\varphi(\mathbf{x}) = -\delta(\mathbf{x}) \tag{32}$$

is given by a point source

$$\varphi(r) = \frac{\mathrm{e}^{-\mathrm{j}k_0 r}}{4\pi r} \tag{33}$$

with wavenumber k_0 and distance to the origin r. Assuming that the Fourier transform of $\varphi(\mathbf{x})$, denoted as $\hat{\varphi}(\mathbf{k})$, exists, the inverse transform is given by

$$\varphi(\mathbf{x}) = \frac{1}{8\pi^3}\iiint_{-\infty}^{\infty} \hat{\varphi}(\mathbf{k})\mathrm{e}^{-\mathrm{j}\mathbf{k}\mathbf{x}}\mathrm{d}\mathbf{k}. \tag{34}$$

When substituting the complete inverse Fourier transform in eq. (32), the prefactor $-\mathrm{j}k_{x,y,z}$ appears for each derivation. Note further that the Dirac impulse is the inverse Fourier transform of 1:

$$\delta(\mathbf{x}) = \frac{1}{8\pi^3}\iiint_{-\infty}^{\infty} 1\cdot\mathrm{e}^{-\mathrm{j}\mathbf{k}\mathbf{x}}\mathrm{d}\mathbf{k}. \tag{35}$$

Thus, eq. (32) can be rewritten as

$$\frac{1}{8\pi^3}\iiint_{-\infty}^{\infty}(-k_x^2 - k_y^2 - k_z^2 + k_0^2)\hat{\varphi}(\mathbf{k})\mathrm{e}^{-\mathrm{j}\mathbf{k}\mathbf{x}}\mathrm{d}\mathbf{k} = -\frac{1}{8\pi^3}\iiint_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{j}\mathbf{k}\mathbf{x}}\mathrm{d}\mathbf{k}. \tag{36}$$

The integrands can be considered in isolation (or the Fourier transforms are performed), which yields

$$(\underbrace{-k_x^2 - k_y^2 - k_z^2}_{=-|\mathbf{k}|^2} + k_0^2)\hat{\varphi}(\mathbf{k}) = -1 \tag{37}$$

$$\Leftrightarrow \quad \hat{\varphi}(\mathbf{k}) = \frac{1}{|\mathbf{k}|^2 - k_0^2}. \tag{38}$$

Substituting eq. (38) in (34) gives

$$\varphi(\mathbf{x}) = \frac{1}{8\pi^3}\iiint_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{j}\mathbf{k}\mathbf{x}}}{|\mathbf{k}|^2 - k_0^2}\mathrm{d}\mathbf{k} = \frac{\mathrm{e}^{-\mathrm{j}k_0 r}}{4\pi r}. \tag{39}$$

Eq. (39) expressed with the nomenclature used so far reads

$$\frac{\mathrm{e}^{-\mathrm{j}k|\mathbf{R}|}}{4\pi|\mathbf{R}|} = \frac{1}{8\pi^3}\iiint\limits_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{j}\boldsymbol{\xi}\mathbf{R}}}{|\boldsymbol{\xi}|^2 - k^2}\mathrm{d}^2\boldsymbol{\xi} \tag{40}$$

where $|\mathbf{R}|$ denotes the distance between source and field point. Consequently, the integrals in equation eq. (31) disappear and it can be rewritten as [AB79, eq. (A13)]

$$\boxed{P(\mathbf{x}, \mathbf{x_0}, \omega) = \sum_{p=1}^{8}\sum_{\mathbf{m}=-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{j}k|\mathbf{R}_p+\mathbf{R_m}|}}{4\pi|\mathbf{R}_p + \mathbf{R_m}|}}. \tag{41}$$

To obtain the desired time domain solution, the inverse Fourier transform with respect to time is applied

$$p(\mathbf{x}, \mathbf{x_0}, t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\sum_{p=1}^{8}\sum_{\mathbf{m}=-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{j}k|\mathbf{R}_p+\mathbf{R_m}|}}{4\pi|\mathbf{R}_p + \mathbf{R_m}|}\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega \tag{42}$$

$$= \sum_{p=1}^{8}\sum_{\mathbf{m}=-\infty}^{\infty} \frac{1}{4\pi|\mathbf{R}_p + \mathbf{R_m}|}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}\omega\frac{|\mathbf{R}_p+\mathbf{R_m}|}{c}}\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega. \tag{43}$$

With the Fourier transform of a Dirac shifted by time period τ [GRS01]

$$\int\limits_{-\infty}^{\infty} \delta(t-\tau)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \mathrm{e}^{-\mathrm{j}\omega\tau}, \tag{44}$$

the result (now not depending anymore on the choice of Fourier transform conventions) is [AB79, eq. (A14)]

$$\boxed{p(\mathbf{x}, \mathbf{x_0}, t) = \sum_{p=1}^{8}\sum_{\mathbf{m}=-\infty}^{\infty} \frac{\delta(t - \frac{|\mathbf{R}_p+\mathbf{R_m}|}{c})}{4\pi|\mathbf{R}_p + \mathbf{R_m}|}}. \tag{45}$$

Obviously, the sound field inside a rectangular room with rigid walls is a superposition of delayed and weighted Dirac pulses. The delay and weighting factors for each Dirac depend on the speed of sound and the distance between the (image) sources and the field point, which is given by $|\mathbf{R}_p + \mathbf{R_m}|$.

The index p can be reformulated to directly account for the 8 possible $\mathbf{R}_p$ versions if summing over the vector $\mathbf{u} = (u, v, w)^T$ with every component being 0 or 1 and using $\mathbf{R_u}$ instead of $\mathbf{R}_p$

$$\mathbf{R_u} = \begin{pmatrix} x - x_0 + 2ux_0 \\ y - y_0 + 2vy_0 \\ z - z_0 + 2wz_0 \end{pmatrix}. \tag{46}$$

Now, the result is [LJ08, eq. (5)]

$$\boxed{p(\mathbf{x}, \mathbf{x_0}, t) = \sum_{\mathbf{u}=0}^{1} \sum_{\mathbf{m}=-\infty}^{\infty} \frac{\delta(t - \frac{|\mathbf{R_u}+\mathbf{R_m}|}{c})}{4\pi|\mathbf{R_u} + \mathbf{R_m}|}}, \tag{47}$$

where the sum over the vector $\mathbf{u}$ equals a triple summation:

$$\sum_{\mathbf{u}=0}^{1} = \sum_{u=0}^{1} \sum_{v=0}^{1} \sum_{w=0}^{1}. \tag{48}$$

References

[AB79] Jont B. Allen and David A. Berkley. Image method for efficiently simulating small-room acoustics. *J. Acoust. Soc. Am.*, 65(4):943–950, 1979.

[GRS01] Bernd Girod, Rudolf Rabenstein, and Alexander Stenger. *Signals and Systems*. Wiley, Chichester, 2001.

[LJ08] Eric A Lehmann and Anders M Johansson. Prediction of energy decay in room impulse responses simulated with an image-source model. *J. Acoust. Soc. Am.*, 124(1):269–277, 2008.

[Wik18a] Wikipedia.org. Dirac comb, https://en.wikipedia.org/wiki/Dirac_comb, read 24.01.2018.

[Wik18b] Wikiversity.org. Waves in composites and metamaterials/point sources and EM vector potentials, http://en.wikiversity.org/wiki/Waves_in_composites_and_metamaterials/Point_sources_and_EM_vector_potentials, read 23.01.2018.