Message routeing and percolation in interference limited multihop networks

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Abstract

English:

Message routeing and percolation in interference limited multihop networks

This thesis consists of two main parts. In the first part, we investigate a probabilistic model for routeing of messages in relay-augmented multihop ad-hoc networks, where each transmitter sends one message to the origin. Given the (random) transmitter locations, we weight the family of random, uniformly distributed message trajectories by an exponential probability weight, favouring trajectories with low interference (measured in terms of signal-to-interference ratio) and trajectory families with little congestion (measured in terms of the number of pairs of hops using the same relay). Under the resulting Gibbs distribution, the system targets the best compromise between entropy, interference, and congestion for a common welfare, instead of an optimization of the individual trajectories. We also discuss a game-theoretic relation of our Gibbsian model with a joint optimization of message trajectories opposite to a selfish optimization.

In the limit of high spatial density of users, we describe the totality of all the message trajectories in terms of empirical measures. Employing large deviations arguments, we derive a characteristic variational formula for the limiting free energy and analyse the minimizer of the formula, which describe the most likely shape of the trajectory flow. The empirical measures of the message trajectories well describe the interference, but not the congestion; the latter requires introducing an additional empirical measure. Our results remain valid under replacing the two penalization terms with more general functionals of these two empirical measures.

In the special case where congestion is not penalized, we derive qualitative properties of this minimizer. We analytically identify the emerging typical scenarios in three extreme regimes. We analyse the typical number of hops and the typical length of a hop, and the deviation of the trajectory from the straight line, (1) in the limit of a large communication area and large distances, and (2) in the limit of a strong interference weight. In both regimes, the typical trajectory approaches a straight line quickly, in regime (1) with equal hop lengths. Interestingly, in regime (1), the typical length of a hop diverges logarithmically in the distance of the transmitter to the origin. We further analyse (3) local and global repulsive effects of a densely populated subarea on the trajectories. Our findings are illustrated by numerical examples.

In the second part of the thesis, we study signal-to-interference plus noise ratio (SINR) percolation for Cox point processes, i.e., Poisson point processes with a random intensity measure. SINR percolation was first studied by Dousse et al. in the case of a two-dimensional Poisson point process. It is a version of continuum percolation where the connection between two points depends on the locations of all points of the point process. Continuum percolation for Cox point processes was recently studied by Hirsch, Jahnel, and Cali.

We study the SINR graph model for a stationary Cox point process in two or higher dimensions. We show that under suitable moment or boundedness conditions on the path-loss function and the intensity measure, this graph has an infinite connected component if the spatial density of points is large enough and the interferences are sufficiently reduced (without vanishing). This holds in all dimensions larger than 1 if the intensity measure is asymptotically essentially connected, and also if the intensity measure is only stabilizing but the connection radius is large.

A prominent example of the intensity measure is the two-dimensional Poisson–Voronoi tessellation. We show that its total edge length in a given square has all exponential moments. We conclude that its SINR graph has an infinite cluster if the path-loss function is bounded and has a power-law decay of exponent at least 3.

Both models investigated in the thesis describe multihop networks where the signal-to-interference (plus noise) ratio is decisive for determining the quality of service. Based on these properties, we conclude the thesis with establishing relations between the two models and the recent work by Hirsch, Jahnel, Keeler, and Patterson about probabilities of frustration events in highly dense interference limited relay-augmented ad-hoc networks. We also investigate how the choice of path-loss function influences the results of our thesis and preliminary work, and we discuss the most relevant open questions related to our two main subjects.

Deutsch:

Nachrichtenvermittlung und Perkolation in interferenzbegrenzten Netzwerken mit mehreren Hops

Diese Dissertation besteht aus zwei Hauptteilen. Im ersten Teil analysieren wir ein probabilistisches Modell für Nachrichtenvermittlung in zufälligen dichten Netzwerken, in denen jeder Benutzer genau eine Nachricht zum Ursprung schickt, und Nachrichten von den anderen Benutzern (die ebenso als Relais funktionieren) weitergeleitet werden können. Gegeben die (zufälligen) Orte der Sender, gewichten wir die Familie der zufälligen, gleichverteilten Nachrichtentrajektorien mit einem exponentiellen Wahrscheinlichkeitsgewicht, das Trajektorien mit niedriger Interferenz (gemessen im Sinne der *signal-to-interference ratio (SIR)*) und Familien von Trajektorien mit kleiner Nachrichtenstauung (gemessen durch die Anzahl der Paare von Hops, die das gleiche Relais benutzen) bevorzugt. Unter der resultierenden Gibbs-Verteilung erzielt das System genau den besten Kompromiss zwischen Entropie, Interferenz und Nachrichtenstauung für das öffentliche Wohl, statt einer Optimierung der individuellen Trajektorien. Wir diskutieren auch eine spieltheoretische Beziehung unseres Modells mit einer gemeinsamen Optimierung der Nachrichtentrajektorien gegenüber einer eigennützigen Optimierung.

Im Grenzwert der hohen räumlichen Dichte der Benutzer beschreiben wir die Gesamtheit von allen Nachrichtentrajektorien im Sinne von empirischen Maßen. Unter der Verwendung der Methode der großen Abweichungen verzweigen wir eine charakterische Variationsformel für die begrenzende freie Energie. Zudem analysieren wir den Minimierer dieser Formel, der die wahrscheinlichste Gestalt des Abflusses der Trajektorien beschreibt. Die empirischen Maße der Nachrichtentrajektorien beschreiben die Interferenz gut, aber nicht die Nachrichtenstauung; letztere benötigt ein zusätzliches empirisches Maß. Unsere Ergebnisse bleiben mit allgemeineren Funktionalen dieser zwei empirischen Maße gültig.

Im Spezialfall, in dem Nachrichtenstauung nicht bestraft wird, beschreiben wir qualitative Eigenschaften dieses Minimierers. Wir bestimmen die entstehenden Szenarien in drei Extremfällen analytisch. Wir analysieren die typische Anzahl der Hops und die typische Länge eines Hops, und die Abweichung der Trajektorie von der Gerade, (1) im Grenzwert eines großen Kommunikationsgebietes und großer Abstände, (2) im Grenzwert eines starken Interferenzgewichtes. In beiden Fällen erreicht die typische Trajektorie die Gerade schnell, im Fall (1) mit gleichen Hoplängen. Interessanterweise divergiert im Fall (1) die typische Länge eines Hops logarithmisch im Abstand vom Ursprung. Wir analysieren weiterhin (3) lokale und globale abstoßende Effekte eines dicht bevölkerten Teilgebietes auf die Trajektorien. Unsere Erkenntnisse werden mit numerischen Beispielen illustriert.

Im zweiten Teil der Dissertation untersuchen wir die *signal-to-interference plus noise ratio (SINR)*-Perkolation für Cox-Punktprozesse, d. h. Poisson-Punktprozesse mit zufälligem Intensitätsmaß. Die SINR-Perkolation wurde zuerst von Dousse et al. studiert, für den Fall des zweidimensionalen Poisson-Punktprozesses. Sie ist eine Version der kontinuerlichen Perkolation, wobei die Verbindung zwischen zwei Punkten von den Orten aller Punkte des Punktprozesses abhängt. Die kontinuerliche Perkolation für Cox-Punktprozesse wurde vor Kurzem von Hirsch, Jahnel und Cali studiert.

Wir studieren den SINR-Graphen eines stationären Cox-Punktprozesses in zwei oder mehr Dimensionen. Wir zeigen, dass dieser Graph unter bestimmten Annahmen an die Pfadverlust-Funktion bzw. das Intensitätsmaß, wie Beschränktheit oder Existenz von exponentiellen Momenten, eine unbeschränkte zusammenhängende Komponente hat, wenn die räumliche Dichte der Punkte groß genug ist und die Interferenzen genügend reduziert sind (ohne zu verschwinden). Dies gilt in allen Dimensionen größer als eins, wenn das Intensitätsmaß asymptotisch wesentlich zusammenhängend (*asymptotically essentially connected*) ist oder auch wenn das Intensitätsmaß nur stabilisierend ist, aber der Verbindungsradius groß ist.

Ein prominentes Beispiel des Intensitätsmaßes ist die zweidimensionelle Poisson-Voronoi-Parkettierung. Wir zeigen, dass ihre totale Kantenlänge in einem gegebenen Quadrat alle exponentiellen Momente hat. Wir folgern, dass ihr SINR-Graph ein unendliches Cluster hat, wenn die Pfadverlust-Funktion beschränkt ist und ihr Verfall nach einem Potenzgesetz mit Exponent nicht kleiner als 3 erfolgt.

Die beiden Modelle, die wir in dieser Dissertation betrachten, beschreiben Netzwerke mit mehreren Hops, wobei die SI(N)R für die Empfangsqualität entscheidend ist. Auf der Grundlage von diesen Eigenschaften schließen wir die Dissertation mit der Etablierung von Beziehungen zwischen den zwei Modellen und der kürzlich erschienenen Arbeit von Hirsch, Jahnel, Keeler und Patterson über die Wahrscheinlichkeiten von Frustrationsereignissen in dichten interferenzbegrenzten zufälligen Netzwerken mit Relais. Wir untersuchen auch, wie die Auswahl der Pfadverlust-Funktion die Ergebnisse dieser Dissertation und einiger vorläufiger Arbeiten beeinflusst, und wir diskutieren die wichtigsten offenen Fragen im Zusammenhang zu unseren zwei Hauptthemen.

Contents

1. Introduction

1.1. Main goals and findings

The quality of service in large ad-hoc telecommunication networks has received particular interest in recent years. In such models, users of the network are usually situated according to a point process in Euclidean space. In many works investigating this subject, the network under consideration is *interference limited*. In other words the quality of service of a transmission depends on the ratio between the signal of the transmission and the interference coming from all other transmitters in the network plus some external noise, measured at the receiver. Also, in many recent models the network is *multihop*, i.e., messages travel from transmitter to receiver in multiple hops using other users or some additional equipment (e.g., base stations or relay stations) as relays. See e.g. [FM07, BB09, BB09b, H10] for overviews of the mathematical research on interference limited multihop networks and Section 5.1 of the present thesis for historical remarks on this subject.

In such models, in order to be able to derive a clear picture one has to make a certain approximation in limiting settings. Two mathematical settings are frequently used: the *high-density limit* (sometimes called a hydrodynamic limit or a mean-field limit), where the number of users in a compact fixed area diverges, and the *thermodynamic limit*, where the area diverges but the number of users per space unit remains fixed. The former models a situation like at concerts, demonstrations or sports events, while the latter one models large area networks with moderate user density.

A number of papers on interference limited or multihop networks (e.g. [HJKP18, HJP18, HJ17] in the high-density limit, [HJKP16] in the thermodynamic limit, and [GT08, TL14] in the limit of a high interference given the distribution of the point process of users) use large deviations methods. This has several advantages. Indeed, the corresponding large deviation principles often come with a law of large numbers for certain empirical measures and with exponential bounds on the probabilities of deviations from the limit. This suggests that the qualitative behaviour of the network is already close to the limit for relatively moderate values of the diverging parameter.

The present thesis investigates two phenomena for interference limited multihop networks: *message routeing* in the high-density limit and *percolation* in the thermodynamic limit. The first part originates from the author's joint work [KT18, KT19] with W. König and deals with a Gibbsian model for message routeing in highly dense multihop networks. The second part is an extended version of the single-author paper [T18] and deals with signal-to-interference ratio percolation for Cox point processes. Though both parts use large deviations methods, they will feature more prominently when we address the Gibbsian model. Further, the last chapter of the body of the thesis, Chapter 5, establishes connections between these two subjects and relates them to other models about interference limited multihop networks. This material was not part of the three aforementioned papers. The organization of the thesis will be sketched in Section 1.2. In Section 1.3 we will discuss the relations of the thesis to the three aforementioned papers and we will motivate the thesis for those readers who are familiar with some of the papers. See Section A.4 in the Appendix for bibliographic details of these papers.

Let us now introduce the two models and give a heuristic summary of our main results.

1.1.1. Message routeing: a Gibbsian approach

1.1.1.1. Our model

One of the most prominent problems in random networks is the question of how to conduct a messages through the system in an optimal way. Optimality is often measured in terms of determining the shortest path from the transmitter to the recipient, or if interference is considered, determining the path that yields the least interference. If many messages are considered at the same time, an additional aspect of optimality may be to achieve a minimal amount of congestion.

Many investigations concern the question for just one single transmitter/recipient pair, which is a question that every single participant faces. However, a strategy found in such a setting may lead to a selfish routeing, and it is quite likely that the totality of all these routeings for all the individuals is far from optimal for the community of all the users [CCS16, Section 1]. Instead, the entire system may work even better if an optimal *compromise* is realized, by which we mean a joint strategy that leads to an optimum for the entire system, though possibly not for every participant.

In this thesis we present a probabilistic ansatz for describing a jointly favourable routeing for an unbounded number of transmitter/recipient pairs, which takes into account the following three crucial properties of the family of message trajectories: entropy (i.e., counting complexity of the number of trajectory families satisfying certain properties), interference, and congestion. That is, we consider a situation in which all the messages are directed through the system in a random way, such that each hop prefers a low interference, and such that the total amount of congestion is preferred to be low. Parameters control the strengths of influence of the three effects.

Let us describe our model in words. Let the users be located randomly as the sites of a Poisson point process on a compact subset of $\mathbb{R}^d$, $d \geq 1$, which we fix. Each user sends out precisely one message, which arrives at a unique base station located at the origin. We consider the entire collection of possible trajectories of the messages through the system. We employ an ad-hoc relaying system with multiple hops such that all the users act as relays for the handoffs of the messages. The maximal number of hops is $k_{\max} \in \mathbb{N}$ for each message. Each k-hop message trajectory (where $k \in \{1, \dots, k_{\max}\}$ is random) is random and *a priori* uniformly distributed. The family of all trajectories is *a priori* independent.

Now, the probability distribution of this family is given in terms of a Gibbs ansatz by introducing two exponential weight terms. That is, we define a quenched measure on trajectories given the locations of the users. The first one weights the total amount of interference measured in terms of the signal-to-interference ratio (SIR) for each hop. The second term weights the total congestion, i.e., the number of times that any two trajectories use the same relay. Under the arising measure there is a competition between all the three decisive effects of the trajectory family: entropy, interference, and congestion. Furthermore, the users form a random environment for the family, which not only determines the starting sites of all the trajectories, but also has a decisive effect on interference and congestion. While the latter has a smoothing effect on the fine details of the spatial distribution of all the trajectories, the effect of the former is not so clear to predict because the superposition of signals have a very non-local influence.

The interesting feature is that the total penalty is given to the entire trajectory collection in terms of a probability weight in the spirit of a common welfare. In Section 2.6 we give a game-theoretic discussion of the two weight terms in the exponent in the light of traffic theory; more precisely we ask under what circumstances the optimization of the sum of these two terms can be called selfish or non-selfish. In Section 2.5.3 we also make a connection between this optimization and our model from the viewpoint of stochastic algorithms. According to the results of [M18], realizing our Gibbsian system numerically using Monte Carlo Markov chain methods is on average much more effective than finding the actual optimum. This gives another motivation for our model. We further compare the Gibbsian approach with the optimization of the sum of the two terms in Section 2.5.4, using the observations of [CLSK13] about general combinatorial optimization problems for networks. In Section 2.7.2 we will recall further works [BC12, CBK17] that use Gibbs sampling and other Monte Carlo Markov chain methods in the context of telecommunication networks.

1.1.1.2. The behaviour of the model in the high-density limit

Our main interest is in understanding the spatial distribution of the totality of all the message trajectories under the Gibbs distribution. The measure under consideration is a highly complex object, as it depends on all the user locations and on many detailed properties and quantities. However, we make a substantial step towards a thorough understanding by deriving an asymptotic formula for the logarithmic behaviour of the normalization constant in the limit of a high spatial density of the users. The limiting situation is then described in terms of a large deviations rate function and a variational formula, whose minimizers describe the optimal joint choices of the trajectories. This formula is deterministic and depends only on general spatial considerations, not on the individual users. It has the form of entropy (arising as an exponential rate of counting complexity) plus energy (a limiting expression arising from the sum of the penalty terms for interference and congestion).

The main objects in terms of which we achieve this description are the empirical measures of the trajectories of the messages sent out by the users, disintegrated with respect to the lengths and rescaled to finite asymptotic size. These measures turn out to converge in the weak topology in the high-density limit. The counting complexity of the statistics of the message trajectories can be written in terms of multinomial expressions and afterwards, in the limit of finer and finer decompositions of the space, approximated in terms of relative entropies, using Stirling's formula. The interference term can also be handled in a standard way [HJKP18], since it is a continuous function of the collection of empirical measures of message trajectories.

However, a key finding of our work is that the congestion term is a highly discontinuous function of these empirical measures. Indeed, one cannot express its limiting behaviour in terms of these measures. Instead, one needs to substantially enlarge the probability space of trajectories and to introduce another collection of empirical measures, the ones of the locations of users (relays) who receive given numbers of incoming messages (counted with multiplicity if a message trajectory hits the same relay multiple times). The congestion expression then turns out to be a lower semicontinuous function of these empirical measures, and hence the limiting congestion term is still expressible in terms of the weak limits of these measures. Again, using explicit combinatorics and Stirling's formula, we arrive at explicit entropic terms describing the statistics of these measures.

These two families of empirical measures together enable us to describe all the properties of the message trajectories that we are interested in. We establish a full large deviation principle for the tuple of all these measures with an explicit rate function and obtain in particular that they converge towards the minimizer of a characteristic variational formula. We also derive that all minimizers satisfy certain positivity properties, which makes it possible to characterize the minimizers in terms of Euler-Lagrange equations. We also verify that the minimizer is unique, which was not yet shown in [KT18]. However, unfortunately, due to the complexity of the congestion term, the description of the minimizer is rather implicit. Nevertheless, in the special case when congestion is not penalized, the minimizer, which is also unique, turns out to be given by an explicit expression that is amenable to further investigation.

Apart from the potential value for understanding a new type of message routeing models in telecommunication, our investigations provide also some interesting mathematical research on topological fine properties of random paths in random environment in a high-density setting. This subject has received considerable interest for various types of such processes over the last decades, see the references in [KT18, Section 1.1.2]. We formulate our model in a more general, slightly abstract, way in order to bring its mathematical essence to the surface. That is, we consider a random complete geometric graph in a compact subset of $\mathbb{R}^d$ (where the vertices are the users and the edges are the straight line segments between any two users) and a distribution of trajectories that has an interaction (the interference) with all the locations of the nodes and suppresses local clumping (the congestion). We present our results in terms of two general penalty terms that have similar mathematical properties as the interference term respectively the congestion term.

In Section 2.7.1 we will briefly recall the subject of the recent works [HJKP18, HJP18, HJ17] on large deviations in highly dense relay-augmented networks. In particular, the setting of [HJKP18] is related to our Gibbsian model in multiple ways and has also some connections to signal-to-interference ratio percolation, which will be explained in Section 5.2.

1.1.1.3. Routeing properties in the highly dense system

Having carried out our large deviations analysis in a more abstract setting, we shall focus on features of the highly dense network that are relevant for applications in telecommunications. Thus, we consider the case of the true interference penalty term, with congestion not being penalized. We analyse the unique minimizer of the variational formula, which is the almost sure coordinatewise weak limit of the crucial empirical measures. Our goal is to understand the global effects that are induced in the Gibbsian system exclusively by entropy and energy (i.e., interference penalization) into geometric properties of the trajectory collection. As our model depends on various parameters (size and form of the communication area, density of users, choice of the interference term, strength of interference and congestion weighting, etc.), this can be done rigorously only in certain *limiting* regimes, namely:

(R1) large communication area and long distances (and large hop numbers),

(R2) strong interference penalization, and

(R3) high local density of users on a subset of the communication area.

We are interested in geometric properties such as typical hop lengths, the average number of hops, and the typical shape of the trajectory. In regimes (1) and (2), we expect that the typical trajectories approach straight lines, and in (1) there is an additional question about the typical length of a hop and the number of hops. Here, we would like to understand how the quality of service becomes bad in a large telecommunication area served by a single base station, and how many and how long hops the messages would like to take if the constraint $k_{\max}$ on the maximum number of hops is dropped.

However, the regime (3) and our questions here are of a different nature. We would like to determine if the presence of a subarea with a particularly high population density has a significant (positive or negative) impact on the effective use of the relaying system: on the one hand, the trajectories have more available relays in such an area, but on the other hand, the interference achieves high values there. This is a trade-off between entropy and energy that we want to understand.

In regimes (1) – (2), we will see that the typical trajectory follows a straight line with exponential decay of probabilities of macroscopic deviations from this shape. Moreover, in regime (1) we will also find simple formulas for the asymptotic number of hops and the average length of a hop, which turns out to be the same for each hop of the trajectory. However, one of our most striking findings is that, in regime (1), the typical hop length diverges as a power of the logarithm of the distance between the transmitter and the base station, and hence the typical number of hops is sublinear in the distance. This effect seems to come from the facts that the total mass of the intensity measure of the communication area diverges and that, *a priori*, i.e., before switching on the interference weight, every message trajectory of a given length has the same weight, even very unreasonable ones that have long spatial detours, e.g., many loops.

However, in regime (3), we encounter different effects. First, we identify the following global effect on the total number of relaying hops in the entire system: if the communication area is small (in the sense that all the interferences in the system do not vary much), then the total number of relaying hops vanishes exponentially fast in the diverging parameter of the dense population, regardless of the choice of the densely populated subset, as long as it has positive Lebesgue measure. In some cases, we also detect a local effect on the relaying hops if the densely populated subset is very small: we demonstrate that a certain neighbourhood of that subset is definitely unfavourable for relaying hops for practically all the other users. This is a very clear effect coming from the high interference of the densely populated area, which expels the trajectories away.

Some of our results about routeing properties in the network are easy to guess, and the main value of our work is the explicit characterization of the quantities and the derivation of exponential bounds for deviations. We formulated our results in quite simple settings, by putting the communication area equal to a ball and the user density equal to the Lebesgue measure, but it is clear that they can be extended into various directions with respect to more complex shapes and/or user distributions. We expect that the behaviour of the system in the limiting regimes is similar if congestion is also penalized, as the effect of congestion is *a priori* not spatial, but combinatorial; however, the implicitness of the minimizer in this case hinders us from verifying such results.

Based on our explicit formulas, we also provide simulations in Section 3.5. They illustrate that most of the effects that we derived analytically in limiting settings, i.e., for large values of the parameters, already appear in a very pronounced way for quite moderate values of the parameters.

Let us note that the kinds of questions that we investigate in this thesis in the high-density limit are also of relevance in the thermodynamic limit, both mathematically and for applications in telecommunications. However, the experience of related work is that large deviations methods lead in the high-density setting to much more handy formulas (see e.g. [HJKP18, HJP18, HJ17]) than in the thermodynamic limit (see e.g. [HJKP16]). This is why we decided to analyse our Gibbsian model in the high-density limit.

1.1.2. Percolation: signal-to-interference ratio percolation for Cox point processes

1.1.2.1. Prior work

Continuum percolation was introduced by Gilbert [G61]. In his random graph model, two points of a homogeneous Poisson point process X^λ in $\mathbb{R}^2$ with intensity $\lambda > 0$ are connected by an edge if their distance is less than a fixed connection radius $r > 0$. He showed that this model undergoes a phase transition: there is a critical intensity $\lambda_c(r) \in (0,\infty)$ such that almost surely, for $\lambda < \lambda_c(r)$, the graph consists of finite connected components, while for $\lambda > \lambda_c(r)$, it *percolates*, i.e., it has an infinite connected component. The motivation to study this setting was to model a telecommunication network, in which the points of X^λ are the users, and transmissions between users are only possible along the edges of the graph. In this view, long-distance communication is only possible if the graph percolates.

The model of [G61] has been widely studied and generalized in the Poisson case, see e.g. [MR96, FM07, BB09] for overviews. The question of percolation is related to connectivity properties of the graph restricted to large compact sets [MR96, Sections 3,4], and this way it arises naturally in a thermodynamic limit. After 2010, the theory has also been extended to various other kinds of point processes, for example sub-Poisson [BY10, BY13], Ginibre and Gaussian zero [GKP16], and Gibbsian [J16, S13]. The case of Gibbsian point processes was also studied earlier, see the references in [J16].

[HJC17] considered Gilbert's graph model for a Cox point process, that is, a Poisson point process in a random environment. More precisely, let $\lambda > 0$ and a stationary random measure Λ on $\mathbb{R}^d$, $d \geq 2$ be given. A Cox point process X^λ with intensity measure $\lambda\Lambda$ is characterized by the property that conditional on Λ, X^λ is a Poisson point process with intensity $\lambda\Lambda$. In [HJC17], it was shown that under certain stabilization and connectedness conditions on Λ, $0 < \lambda_c(r) < \infty$ holds. More precisely, $\lambda_c(r) > 0$ if Λ is stabilizing, and $\lambda_c(r) < \infty$ under the stronger assumption that Λ is asymptotically essentially connected. These assertions are to be understood in the annealed sense, i.e., under a probability measure that governs Λ and X^λ jointly.

According to [HJC17], the most important examples of Λ for telecommunication are given by a stationary tessellation process, e.g., a Poisson–Voronoi, Poisson–Delaunay or Poisson line tessellation. The edge set of such a tessellation process can be used for modelling a telecommunication network on a street system, where the points of the Cox point process are the users, situated on the streets. The randomness of the tessellation process can be interpreted as the statistical variability of street systems in different areas. While Poisson–Delaunay tessellations fit well for modelling rural areas, Poisson–Voronoi tessellations are good approximations for various kinds of urban environments [CGHJNP18].[1] The definitions of these tessellation processes can be found in Section A.2 in the Appendix.

Another variant of Gilbert's graph model motivated by telecommunication is the signal-to-interference plus noise ratio (SINR) graph, which was considered in [DBT05, DFMMT06, FM07] in the case of a homogeneous Poisson point process with intensity $\lambda > 0$ in $\mathbb{R}^2$.[2] Here, two points are connected if the SINR between them is larger than a given threshold $\tau > 0$ in both directions. The SINR of a transmission from $x \in \mathbb{R}^d$ to $y \in \mathbb{R}^d$ has the form [GK00, DBT05]

$$\mathrm{SINR}(x,\, y,\, X^\lambda) = \frac{P\ell(|x-y|)}{N_0 + \gamma P I(x,y)}. \tag{1.1}$$

Here ℓ is a *path-loss function*, assumed to be monotone increasing, describing propagation of signal strength over distance, $P > 0$ is the transmitted power, $N_0 \geq 0$ is the external noise, and $I(x,y)$, the interference for the transmission from x to y, is the sum of $\ell(|X_i - y|)$ over all Poisson points $X_i \in X^\lambda \setminus \{x, y\}$, and $\gamma \geq 0$ is a factor expressing how strongly interference is cancelled compared to the signal. The motivation for the SINR model is that in real telecommunication networks, even if the transmitter is close to the receiver, the transmission may be unsuccessful due to too many other transmitters standing near the receiver, see [FM07, Section 1.2.5].[3]

[1] According to [CGHJNP18], Poisson line tessellations fit also well for certain urban environments, however, they are not stabilizing, and thus the results of [HJC17] do not hold for them. In the present thesis we will not investigate the case of tessellations, apart from formulating the most relevant open questions regarding their percolation properties in Section 5.3.2.1.

[2] The term *interference limited network* was used in the monograph [FM07] as a synonym for SINR graph; nevertheless, we find it rightful to use this expression for any network in which SI(N)R plays an important role, hence the title of this thesis.

[3] We note that in the Gibbsian model introduced in Section 1.1.1 the notion of SIR is similar to the one of SINR in (1.1), however, with γ being proportional to the reciprocal of the density parameter of the Poisson point process of users. This choice of γ

For $\lambda > 0$, let us write $\gamma^*(\lambda)$ for the supremum of all $\gamma > 0$ for which the SINR graph percolates. If $\gamma = 0$, then the SINR graph equals Gilbert's graph with radius $r_{\mathrm{B}} = \ell^{-1}(\tau N_0/P)$ and contains all SINR graphs with positive γ. Thus, for $\lambda < \lambda_{\mathrm{c}}(r_{\mathrm{B}})$, we have $\gamma^*(\lambda) = 0$. [DFMMT06] showed that under suitable integrability and boundedness assumptions on ℓ, for any $\lambda > \lambda_{\mathrm{c}}(r_{\mathrm{B}})$, one has $\gamma^*(\lambda) > 0$. Further, the following assertions were derived in [DBT05, FM07] about $\lambda \mapsto \gamma^*(\lambda)$. SINR graphs with $\gamma > 0$ have degrees bounded by $1 + 1/(\tau\gamma)$, which yields that $\gamma^*(\lambda) \leq 1/\tau$ for all $\lambda > 0$. Further, $\gamma^*(\lambda) = \mathcal{O}(1/\lambda)$ holds as $\lambda \to \infty$, and also $\gamma^*(\lambda) = \Omega(1/\lambda)$ if ℓ has bounded support.[4] In [BY13], a more general notion of SINR graphs was considered, and the results of [DFMMT06] were extended to the case of sub-Poisson point processes in this context.

1.1.2.2. Our goals and findings

In the present thesis we investigate SINR percolation for Cox point processes, combining the benefits of modelling both user locations and connections between the users more realistically than in Gilbert's original model. To the best of our knowledge, there has been no work preceding ours that considered SINR percolation also in $d \geq 3$ dimensions, despite the fact that some results of previous work about $d = 2$ extend to $d \geq 3$ without additional effort. We formulate our results for $d \geq 2$ whenever possible, and we point out which assertions of prior work extend to the higher dimensions.

Let us summarize our most important results regarding SINR percolation for Cox point processes. First, we give general sufficient criteria for the existence of an infinite connected component in this model. We consider the above defined SINR graph for a Cox point process X^λ with intensity measure $\lambda\Lambda$ on $\mathbb{R}^d$, $d \geq 2$. We show that if Λ is asymptotically essentially connected, then for λ sufficiently large, we have $\gamma^*(\lambda) > 0$ if any of the following additional assumptions is satisfied: (a) ℓ has bounded support, (b) $\Lambda(A)$ is almost surely bounded for any compact set $A \subset \mathbb{R}^d$, (c) for any compact set $A \subset \mathbb{R}^d$, there exists $\alpha > 0$ such that $\mathbb{E}[\exp(\alpha\Lambda(A))] < \infty$, and ℓ satisfies a certain stronger decay property.

Second, we verify that these results apply to one of the most realistic models. We show that the random intensity measure Λ given by a two-dimensional Poisson–Voronoi tessellation satisfies the exponential moment assumption in (c), and thus the SINR graph of the Cox point process with this intensity percolates for large λ and small $\gamma > 0$ for path-loss functions of the form $\ell(r) = \min\{1, r^{-\alpha}\}$ or $\ell(r) = (1 + r)^{-\alpha}$ for $\alpha \geq 3$, which were considered relevant for modelling signal propagation in [DBT05]. In fact, in [T18] it was only shown that the total edge length of Λ satisfies $\mathbb{E}[\exp(\alpha\Lambda(A))] < \infty$ for compact sets $A \subset \mathbb{R}^2$ and small $\alpha > 0$ (depending on A). The author thanks B. Jahnel for his ideas and hints that were used in order to extend the result to all $\alpha > 0$. B. Jahnel and the author recently showed [JT19] that the same assertion holds for the Poisson–Delaunay tessellation in two dimensions. In [JT19], other types of planar tessellations were also considered, see Section 5.3.2.1 for further details. Moreover, for the particular case of the homogeneous Poisson point process, our results imply that $\gamma^*(\lambda) > 0$ holds for λ sufficiently large. This yields a generalization of [DFMMT06, Theorem 1] to $d \geq 3$ dimensions, while the question whether $\gamma^*(\lambda) > 0$ holds for each $\lambda > \lambda_c(r_B)$ remains open.

Further, in the case when Λ is only stabilizing, we show that if the connection radius r_{B} is sufficiently large, then $\gamma^*(\lambda) > 0$ holds for sufficiently large λ in case (b) or (c) above. We verify this statement for all dimensions $d \geq 2$, whereas [T18] contained the result only for $d = 2$.

We also provide estimates on $\gamma^*(\lambda)$. First, we conclude that the degree bound $1 + 1/(\tau\gamma)$ and the estimate that $\gamma^*(\lambda) \leq \frac{1}{\tau}$ for all $\lambda > 0$ also hold in the Cox case. Second, we observe that if the number of Cox points who can successfully submit to a given point is bounded by some $k \in \mathbb{N}$ for all points, then every point can only receive messages from its k nearest neighbours. This together with a high-confidence result of [BB08] leads us to the conjecture that in the two-dimensional Poisson case, $\gamma^*(\lambda) \leq \frac{1}{4\tau}$ holds. Further, for $d = 2$, for b-dependent Cox point processes with intensity measures that are locally bounded away from 0, we show that $\gamma^*(\lambda) = \mathcal{O}(1/\lambda)$ holds as $\lambda \to \infty$, and also $\gamma^*(\lambda) = \Omega(1/\lambda)$ if also the support of ℓ is compact. These generalize [DBT05, Theorem 4].

makes it possible to simplify the notion of SIR, modulo error terms that become negligible in the high-density limit. Namely, we choose $N_0 = 0$, $P = 1$, and we include also $\ell(|x - y|)$ in the interference sum. See Section 2.1.4 for the definition of SIR in the Gibbsian setting and Section 2.3.1.1 for further details.

[4] The $\mathcal{O}(\cdot)$ and $\Omega(\cdot)$ notations will be recalled at the beginning of Section 2.1.

Most of our results require stabilization of Λ, i.e., decay of its spatial correlations with distance. We complement these assertions with infinite-range dependent examples of Λ, both ones to which our results extend and ones for which they do not hold. E.g., [BY13] presented an example of a Cox point process satisfying the degenerate property $\lambda_{\mathrm{c}}(r) = 0$ for all $r > 0$. There, Gilbert's graph model percolates with probability 1 for all $r, \lambda > 0$. We observe that there is a rich class of Cox point processes for which there exists $r > 0$ such that Gilbert's graph with connection radius r percolates with positive probability for all $\lambda > 0$, and with probability in (0,1) for some choice of λ. Further, for a large subclass of these processes, for any $r_B > 0$ and $\lambda > 0$, the SINR graph also percolates with positive probability for some $\gamma > 0$.

1.1.3. Conclusions, connections between the two models

After stating, proving, and discussing our main results regarding message routeing and percolation, we conclude the thesis by relating our two main subjects to each other as well as to other work. First, we put our settings into context of the mathematical research on interference limited (i.e., SI(N)R-constrained) networks. We provide historical remarks about related prior work, in particular we recall some of the main results of [GK00, FDTT07] about the information-theoretic capacity of random wireless networks. Based on these results, we analyse the effect of boundedness or unboundedness of the path-loss function in a high-density limit. This explains certain features of the Gibbsian model, in which the path-loss function is assumed to be bounded near zero, and we also point out that in case the path-loss function is unbounded, the high-density limit behaves similarly to a rescaled version of a thermodynamic limit. Further, we discuss the effects of other properties of the path-loss function (e.g., integrable tail, strict decay or convexity of its reciprocal) on the results of the respective parts of the thesis, including also SINR percolation.

Second, we summarize some results of the paper [HJKP18] about probabilities of rare frustration events, i.e., events of bad quality of service, in the high-density limit. This setting turns out to have connections to both subjects of this thesis. Indeed, the degree bounds of SINR graphs derived in context of SINR percolation help give simple *a priori* bounds on frustration probabilities in the setting of [HJKP18]. On the other hand, the model of [HJKP18] is also related to a variant of our Gibbsian model with a "hard" penalization of interference, i.e., with trajectories having some hops with SIR value below a given threshold being not only penalized but forbidden. Using results of [HJKP18] and the author's master's thesis [T16], we provide sufficient conditions for the well-definedness of this variant. We also comment on the discretization procedure in the high-density setting, which originates from [HJKP18] but appears in Chapter 2 in an altered and somewhat generalized form.

Third, we list the most relevant open questions related to the subject of this thesis in Section 5.3. We discuss what answers we expect to them (if applicable) and what we find the main difficulties regarding them.

We have finished the introduction of the main subjects of the present thesis in words. Nevertheless, as partially already indicated, the further chapters will also contain various discussions about interpretations of the most important quantities in the two models, relations to prior work about random spatial telecommunication networks as well as to other fields of mathematics, extensions of our results, and conjectures and open questions. These discussions involve the formal definition of the models under consideration, and sometimes they also rely on arguments of some of our proofs. Therefore, they are not included in the present introductory chapter; they will appear in the later chapters at due times.

1.2. Organization of this thesis

In the following, after explaining the relations of this thesis to its main ingredients [KT18], [KT19], and [T18] in Section 1.3, we will move on to the mathematical setting. The rest of the thesis is organized as follows.

Chapters 2 and 3 are devoted to the analysis of our Gibbsian model sketched in Section 1.1.1. In Chapter 2 we provide the formal definition of this model and investigate its behaviour in the high-density limit. We derive the limiting free energy, identify the minimizer of the corresponding variational formula, and we carry out the proof of the underlying large deviation principle. In this chapter we also discuss modelling questions regarding our Gibbsian approach, possible extensions of our model, and relations of our setting to other models for wireless networks, Markov chain Monte Carlo, and traffic theory. Afterwards, in Chapter 3 we analyse properties of the minimizer in case congestion is not penalized, in the limiting regimes (1), (2), (3). This is accompanied by the related discussions and simulation results. Chapter 4 includes our results about SINR percolation for Cox point processes introduced in Section 1.1.2, together with their proofs and corresponding discussions. Finally, Chapter 5 contains the concluding discussions and open questions outlined in Section 1.1.3. Each of these chapters contains a description about its internal organization at its beginning.

In the Appendix, Sections A.1–A.3 provide auxiliary material for the body of the thesis; cross-references pointing towards them will appear at due times. Section A.4 contains bibliographic details of the papers the thesis is based on, Section A.5 includes a statement about the author's own contribution in the co-authored parts of the thesis, and an index of notations can be found in Section A.6, followed by an index, the references, and the Curriculum Vitae of the author.

1.3. Relation of this thesis to the papers [KT18, KT19, T18]

In this section we point out the novelties of the present thesis with respect to content and structure, compared to the union of the three papers [KT18, KT19, T18] it is based on. We also explain what principles we followed during turning the extended material of the two papers [KT18, KT19] about the Gibbsian model into Chapters 2 and 3 of this thesis, and we comment on differences between the target groups of [KT18], [KT19], and the corresponding parts of this dissertation. While these details can also be helpful for readers who are not familiar with any of these papers, the main goal of this section is to provide motivation for the present thesis to readers who have already encountered some of the papers. Reading this section or any part of the papers [KT18, KT19, T18] is not a prerequisite for understanding the remaining sections in the body of the thesis.

[KT18] introduced the Gibbsian model and derived its properties in the high-density limit, using large deviations methods. It identified the limiting free energy (i.e., the logarithmic rate of the normalizing constant of the Gibbs distribution) in terms of a large deviations rate function and a variational formula and expressed the minimizer(s) of that formula. Although it included various modelling discussions, its focus was mainly mathematical, and its methods were probabilistic. It formulated its results for a version of the Gibbsian model with two general penalty terms, including the ones penalizing interference and congestion as a special case.

In contrast, the follow-up paper [KT19] focused on applications in telecommunications and thus considered only this special case. In the case when congestion is not penalized, it analysed qualitative properties of the highly dense network described in terms of the unique minimizer of the variational formula, such as the typical number of hops, the typical length of a hop, and the typical shape of the trajectory. It also included numerical results about this minimizer. Further, it made connections between the Gibbsian model and other fields such as traffic theory (more precisely, congestion games) and Markov chain Monte Carlo, in case congestion is also penalized. In order to understand the assertions of [KT19] it suffices to know the model of [KT18] in this special case, and the results of that paper without reference to the congestion term and without proofs.

Thus, for readers interested only in the mathematical properties of the Gibbsian model, it suffices to consult [KT18], and those who want to focus only on the applications can read [KT19] without checking [KT18]. The authors of these papers intended to optimize the content and presentation in favour of the respective target group. It is nevertheless desirable to present the entire material within a monograph, following the logical order of the subject, in favour of the readers who are interested both in theoretical aspects and in applications of the Gibbsian model. This is the programme of Chapters 2 and 3 of the present thesis. Here, the material is not split into two separate works, and thus overlaps and cross-references can be reduced radically.

In Chapter 2, the main results come exclusively from [KT18] but the related discussions partially originate from [KT19]. These discussions follow the statements of the main results and precede their proofs.[5] On a formal mathematical level, they are not preliminaries of the proofs, nevertheless, some of them provide interpretations or informal ideas that may be useful in order to understand the proofs (see e.g. Section 2.3.2). Chapter 3 tells about routeing properties in the Gibbsian system, originating exclusively from [KT19].

The part of the thesis about SINR percolation for Cox point processes, Chapter 4, follows the organization of the single paper [T18] it is based on. For the reader's convenience, we spell out more details of some auxiliary

[5] There are a small number of counterexamples: discussions strongly relying on proofs follow the proofs at the end of the chapter.

results that were omitted from [T18] for brevity. See Theorem 4.1.9 (rectangle crossings of Poisson–Boolean models), Lemma 4.2.1 (b-dependent discrete percolation processes), Section A.2 in the Appendix (Poissonian tessellations), and Section A.3 also in the Appendix (degree bounds on general SINR graphs). Also in the other chapters, we use the opportunities given by the form of a monograph: we make our text more reader-friendly by using footnotes, an index of notations, and an index (cf. Section 1.2).

As already explained, the thesis improves the main results of the three papers in the following ways.

(i) We verify the uniqueness of the minimizer of the variational formula corresponding to the Gibbsian model in case both interference and congestion are penalized. The corresponding results of Chapter 2 (e.g., the convergence of the crucial empirical measures) are reformulated according to this.

(ii) We prove the existence of *all* exponential moments of the total edge length of the two-dimensional Poisson–Voronoi tessellation in compact sets (instead of only the existence of some exponential moments, depending on the set).

(iii) We extend the result about presence of percolation in SINR graphs of stabilizing Cox point processes for large connection radii to the case of $d \geq 3$ dimensions.

The following additional discussions in the main chapters are new compared to the three papers.

(i) In Section 2.3.2, where we interpret the limiting entropy term of the Gibbsian model, we also discuss the case when the numbers of incoming hops are not separated.

(ii) In Section 2.4.4 we provide more details of a simple time dependent variant of the Gibbsian model. We explain that the limiting free energy can be derived similarly to the original model, but the description of the minimizer of the variational formula is highly involved, although it has a clear interpretation.

(iii) Section 2.5 establishes connections between our Gibbs distribution and the optimization of the interference term plus the congestion term. Here, in the new Section 2.5.4, we make this connection more rigorous using general observations of [CLSK13] about combinatorial optimization.

(iv) In Section 2.11.2 we explain in detail how the description of the minimizer of the variational formula corresponding to the Gibbsian model extends to the case when congestion is not penalized.

(v) In Section 2.12.1 we describe the behaviour of the highly dense system in case only congestion is penalized and interference is not. We complement this by analysing the *a priori* case (where the congestion term is also dropped) in Section 2.12.2.

(vi) In Section 4.2.3.6 we comment on *information theoretically secure SINR graphs* in the Cox case. Percolation properties of such SINR graphs were first studied in [VI14] in the two-dimensional Poisson case.

Finally, our new concluding chapter, Chapter 5, the content of which we outlined in Section 1.1.3, relates our results about message routeing and the ones about percolation to each other as well as to other models on interference limited multihop networks. This puts our work into a more general context and provides a more thorough discussion of its subjects. Apart from a few of the open questions listed in Section 5.3, the content of this chapter was not included in [KT18, KT19, T18]. On the other hand, this chapter uses some results of the author's master's thesis [T16].

2. A Gibbsian model for message routeing in highly dense multihop networks

In this chapter we introduce the mathematical setting of the Gibbsian model explained in Section 1.1.1. We present, discuss, and prove our main results regarding its behaviour in the high-density limit. We perform this in the setting of two general exponential penalty terms, including the ones penalizing interference and congestion as a special case. We also discuss the mathematical essence of our model and its relevance for telecommunications, possible extensions, and relations to other fields.

We introduce the model and necessary notation in Section 2.1. We present our main results in Sections 2.2.1 (the limiting free energy of the model), 2.2.2 (the description of the minimizer), 2.2.3 (the large deviation principle and the convergence of the empirical measures), and 2.2.4 (results in case congestion is not penalized), which together form Section 2.2.

We continue with various kinds of discussions. In Section 2.3 we interpret the ingredients of our model and the relevant objects that arise in the high-density limit. In Section 2.4 we discuss possible extensions of the model. Afterwards, in Section 2.5 we provide motivations for a Gibbsian ansatz for routeing of messages in ad-hoc telecommunication networks. In particular, we relate our model to the optimization of the sum of the interference term and the congestion term, and we make a connection between this optimization and our model from the viewpoint of Markov chain Monte Carlo. In Section 2.6 we discuss this optimization problem in terms of traffic theory. Lastly, in Section 2.7 we make some remarks on the related literature about highly dense relay-augmented ad-hoc networks and about using Gibbs sampling in the context of telecommunication networks.

Finally, we prove the main results of the chapter: in Section 2.8 we prepare for the proofs by introducing our methods and deriving formulas for the probability terms, in Section 2.9 we put all this together to a proof of the limiting free energy, the large deviation principle and the convergence of the empirical measures, in Section 2.10 we analyse the minimizer of the characteristic variational formula, and in Section 2.11 we extend the proofs to the case when congestion is not penalized. Finally, in Section 2.12, based on elements of the preceding proofs, we comment on the case when interference is not penalized while congestion possibly is.

2.1. The Gibbsian model

Now we turn to the mathematical setting. We will use the following notations throughout the thesis. We write $\mathbb{N} = \{1,2,\dots\}$ and $\mathbb{N}_0 = \{0,1,2,\dots\}$. For $n \in \mathbb{N}_0$, we put $[n] = \{1,\dots,n\}$; in particular, $[0] = \varnothing$. For a countable set A, we denote by $\#A$ the cardinality of A, writing $\#A = \infty$ if A is countably infinite. For any topological space V and for any $A \subseteq V$, we denote the interior of A by A^o, the closure of A by $\overline{A}$, the boundary of A by ∂A, and the complement of A by A^{c}. Further, $\mathcal{M}(V)$ denotes the set of all finite nonnegative Borel measures on the topological space V, which we equip with the weak topology. Finally, in any metric space (X,d_0), for any $x \in X$ and $r > 0$, we let $B_r(x) = \{y \in X : d_0(x,y) < r\}$ be the open ball of radius r around x in X. For $d \in \mathbb{N}$ and $x \in \mathbb{R}^d$, we write $|x|$ for the Euclidean norm of x, and for $p \in [1,\infty]$, we write $\|x\|_p$ for the ℓ^p-norm of x. We denote by $\mathrm{dist}_p(\varphi,\psi) = \inf\{\|x-y\|_p : x \in \varphi, y \in \psi\}$ the ℓ^p-distance between two sets $\varphi,\psi \subset \mathbb{R}^d$ for $p \in [1,\infty]$, and we also write dist instead of dist_2. For any two functions $f,g \colon [0,\infty) \to [0,\infty)$, we write $f(x) = \mathcal{O}(g(x))$ (as $x \to \infty$) if there exists $c > 0$ and $K > 0$ such that for all $x > K$ we have $f(x) \leq cg(x)$, $f(x) = \Omega(g(x))$ if $g(x) = \mathcal{O}(f(x))$, and $f(x) = \Theta(g(x))$ if $f(x) = \mathcal{O}(g(x))$ and $f(x) = \Omega(g(x))$ both hold. Further, in any limit, we write $f(x) = o(g(x))$ if $f(x)/g(x)$ tends to zero, $f(x) \sim g(x)$ if $f(x)/g(x)$ tends to one, and $f(x) \asymp g(x)$ if the quotient of the two sides stays bounded and bounded away from zero. Lastly, for $a,b \in \mathbb{R}$, we write $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$.

The rest of the notation introduced in Chapters 2 and 3 is valid in these chapters, and in Chapters 4 and 5 only in case we indicate it. We introduce now the model of the present chapter. We are working in $\mathbb{R}^d$ with some fixed $d \in \mathbb{N}$. Our model is defined as follows. Let $W \subset \mathbb{R}^d$ be compact, the territory of our telecommunication system, containing the origin o of $\mathbb{R}^d$.

2.1.1. Users

Let $\mu \in \mathcal{M}(W)$ be an absolutely continuous measure on W with $\mu(W) > 0$. Note that we do not require that $\mathrm{supp}(\mu) = W$. For $\lambda > 0$, we denote by X^λ a Poisson point process in W with intensity measure $\lambda\mu$. The points $X_i \in X^\lambda$ are interpreted as the *locations of the users* in the system, while the origin o of $\mathbb{R}^d$ is the single *base station*. We assume that $X^\lambda = \{X_i : i \in I^\lambda\}$ with $I^\lambda = \{1,\dots,N(\lambda)\}$ and $(N(\lambda))_{\lambda>0}$ a homogeneous Poisson process on $\mathbb{N}_0$ with intensity $\mathbb{E}[N(1)] = \mu(W)$, and $(X_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence of W-distributed random variables with distribution $\mu(\cdot)/\mu(W)$, defined on one probability space $(\Omega,\mathcal{F},\mathbb{P})$. Since μ has a density, all points X_i are mutually different with probability one. Further, X^λ is increasing in λ, and its *empirical measure*, normalized by $1/\lambda$,

$$L_\lambda = \frac{1}{\lambda}\sum_{i\in I^\lambda} \delta_{X_i}, \tag{2.1}$$

converges towards μ almost surely as $\lambda \to \infty$.

These assumptions on the users can be relaxed, see Section 2.4.1.

2.1.2. Message trajectories

We now introduce the collection of trajectories sent out from the users to o, i.e., for uplink communication. For any $i \in I^\lambda$, we call a vector of the form

$$S^i = (S^i_{-1} = K_i, S^i_0 = X_i, S^i_1 \in X^\lambda, \dots, S^i_{K_i-1} \in X^\lambda, S^i_{K_i} = o) \in \bigcup_{k\in\mathbb{N}} \Big(\{k\} \times \{X_i\} \times W^{k-1} \times \{o\}\Big), \tag{2.2}$$

a *message trajectory* from X_i to o with K_i hops. That is, S^i starts from X_i and ends in o after K_i hops from user to user in X^λ. Hence, each user sends exactly one message to o, and each user has the function of a relay. We fix a number $k_{\max} \in \mathbb{N}$ and write $\mathcal{S}^i_{k_{\max}}(X^\lambda)$ for the set of all possible realizations of the random variable S^i with $K_i \le k_{\max}$, i.e., with no more than $k_{\max}$ hops. Hence, elements $s^i = (s^i_{-1}, s^i_0, s^i_1, \dots, s^i_{s^i_{-1}-1}, s^i_{s^i_{-1}})$ of $\mathcal{S}^i_{k_{\max}}(X^\lambda)$ satisfy $s^i_{-1} \in \{1,\dots,k_{\max}\}$, $s^i_0 = X_i$, and $s^i_{s^i_{-1}} = o$. We write $\mathcal{S}_{k_{\max}}(X^\lambda) = \times_{i\in I^\lambda} \mathcal{S}^i_{k_{\max}}(X^\lambda)$ for the set of all possible realizations of the families $S = (S^i)_{i\in I^\lambda}$. The assumption that we choose a finite upper bound $k_{\max}$ on the number of hops will be discussed in Section 2.4.3.

Given $i \in I^\lambda$, we consider each trajectory S^i in (2.2) as an $\mathcal{S}^i_{k_{\max}}(X^\lambda)$-valued random variable. Its *a priori* measure is defined by the formula

$$s^i \mapsto \frac{1}{N(\lambda)^{s^i_{-1}-1}}, \qquad s^i \in \mathcal{S}^i_{k_{\max}}(X^\lambda). \tag{2.3}$$

That is, its restriction to $\{s^i \in \mathcal{S}^i_{k_{\max}}(X^\lambda) : s^i_{-1} = k\}$ is the uniform distribution on the set of all k-hop trajectories from X_i to o for any $k \in [k_{\max}]$, and its total mass is equal to $k_{\max}$. Recall that it fixes the starting point X_i and the terminal point o.

Under our joint *a priori* measure, all the trajectories are independent; indeed, it gives the value

$$s = (s^i)_{i\in I^\lambda} \mapsto \prod_{i\in I^\lambda} \frac{1}{N(\lambda)^{s^i_{-1}-1}} \tag{2.4}$$

to the configuration $s \in \mathcal{S}_{k_{\max}}(X^\lambda)$. Thus, it gives a total mass of $k_{\max}^{N(\lambda)}$ to $\mathcal{S}_{k_{\max}}(X^\lambda)$.

2.1.3. Gibbsian trajectory distribution

In this section we define the central object of this study: a Gibbs distribution on the set $\mathcal{S}_{k_{\max}}(X^\lambda)$ of collections of trajectories, which is the joint *a priori* measure (2.4) weighted by two exponential penalty terms. After providing the abstract definitions, in Section 2.1.4 we sketch the key example that is relevant for application in telecommunication: the case of penalizing interference and congestion. The general conditions on the ingredients of the Gibbs distribution in this section arise naturally from the properties of this example.

We introduce the following general notation. For two sets A,B, let A^B denote the set of functions with domain B mapping to A. For $k \in \mathbb{N}$, elements of the product space $W^k = W^{\{0,1,\dots,k-1\}}$ are denoted as $(x_0,\dots,x_{k-1})$. For $l = 0,\dots,k-1$, the l-th marginal of a measure $\nu_k \in \mathcal{M}(W^k)$ is denoted by $\pi_l \nu_k \in \mathcal{M}(W)$, i.e., $\pi_l\nu_k(A) = \nu_k(W^{\{0,\dots,l-1\}} \times A \times W^{\{l+1,\dots,k-1\}})$ for any Borel set $A \subseteq W$.

For fixed $k \in [k_{\max}]$ and for a collection of trajectories $s \in \mathcal{S}_{k_{\max}}(X^\lambda)$, we define

$$R_{\lambda,k}(s) = \frac{1}{\lambda} \sum_{i \in I^\lambda \colon s^i_{-1}=k} \delta_{(s^i_0,\dots,s^i_{k-1})}, \tag{2.5}$$

the empirical measure of all the k-hop trajectories, which is an element of $\mathcal{M}(W^k)$. By the assumption that each user sends out exactly one message, we have

$$\sum_{k=1}^{k_{\max}} \pi_0 R_{\lambda,k}(s) = L_\lambda. \tag{2.6}$$

For $k \in [k_{\max}]$, we choose a continuous function $f_k \colon \mathcal{M}(W) \times W^k \to \mathbb{R}$ that is bounded from below (here we assume that $\mathcal{M}(W)$ is equipped with the weak topology). Using (2.6), we put

$$\mathfrak{S}(s) = \lambda \sum_{k=1}^{k_{\max}} \Big\langle R_{\lambda,k}(s)(\cdot), f_k(L_\lambda,\cdot)\Big\rangle = \sum_{k=1}^{k_{\max}} \sum_{i \in I^\lambda \colon s^i_{-1}=k} f_k(L_\lambda, s^i_0,\dots,s^i_{k-1}), \tag{2.7}$$

where we write $\langle \nu, f\rangle$ for the integral of the function f against the measure ν. Moreover, we define

$$m_i(s) = \sum_{j \in I^\lambda} \sum_{l=1}^{s^j_{-1}-1} \mathbb{1}\{s^j_l = s^i_0\}, \quad i \in I^\lambda, \tag{2.8}$$

as the number of incoming hops into the user (relay) $s^i_0 = X_i$ of any of the trajectories.

We pick a function $\eta \colon \mathbb{N}_0 \to \mathbb{R}$, bounded from below such that $\lim_{m\to\infty} \eta(m)/m = \infty$. Then we put

$$\mathfrak{M}(s) = \sum_{i \in I^\lambda} \eta(m_i(s)). \tag{2.9}$$

Now, we define

$$\mathrm{P}^{\gamma,\beta}_{\lambda,X^\lambda}(s) := \frac{1}{Z^{\gamma,\beta}_\lambda(X^\lambda)} \Big(\prod_{i \in I^\lambda} \frac{1}{N(\lambda)^{s^i_{-1}-1}} \Big) \exp\Big\{ -\gamma\mathfrak{S}(s) - \beta\mathfrak{M}(s)\Big\}, \tag{2.10}$$

where $\gamma > 0$ and $\beta > 0$ are parameters. This is the Gibbs distribution with independent reference measure given in (2.4), subject to two exponential weights with the terms (2.7) and (2.9). Here

$$Z^{\gamma,\beta}_\lambda(X^\lambda) = \sum_{r \in \mathcal{S}_{k_{\max}}(X^\lambda)} \Big(\prod_{i \in I^\lambda} \frac{1}{N(\lambda)^{r^i_{-1}-1}} \Big) \exp\Big\{ -\gamma\mathfrak{S}(r) - \beta\mathfrak{M}(r)\Big\} \tag{2.11}$$

is the normalizing constant, which we will refer to as *partition function*. Note that $\mathrm{P}^{\gamma,\beta}_{\lambda,X^\lambda}(\cdot)$ is random conditional on X^λ and it is a probability measure on $\mathcal{S}_{k_{\max}}(X^\lambda)$. In the jargon of statistical mechanics, it is a *quenched* measure, which we will consider almost surely with respect to the process $(X^\lambda)_{\lambda>0}$. In the *annealed* setting, one would average out over $(X^\lambda)_{\lambda>0}$, see Section 2.4.5.

2.1.4. The key example: penalization of interference and congestion

In this section we sketch the most important example for the exponents $\mathfrak{S}$ and $\mathfrak{M}$ in (2.10), where $\mathfrak{S}$ registers interference and $\mathfrak{M}$ expresses congestion in a telecommunication network. In the modelling discussions of Sections 2.3, 2.4, 2.5, and 2.6, we will most often consider this special setting. Analysing the qualitative properties of the network with this choice of $\mathfrak{S}$ in the special case $\beta = 0$ (i.e., no penalization of congestion) is the main topic of Chapter 3.

Now we introduce interference. We choose a *path-loss function*, which describes the propagation of signal strength over distance. This is a monotone decreasing, continuous function $\ell\colon [0,\infty) \to (0,\infty)$. Examples used in practice include $\ell(r) = \min\{1, r^{-\alpha}\}$, for some $\alpha > 0$, which corresponds to ideal Hertzian propagation, see e.g. [GT08, Section II.], and $\ell(r) = (1+r)^{-\alpha}$, which is similar to the previous one but strictly monotone decreasing[6]. For further examples, see [BB09, Section 2.3.1]. The *signal-to-interference ratio (SIR)* of a transmission from $X_i \in X^\lambda$ to $x \in W$ in the presence of the users in X^λ is given as

$$\mathrm{SIR}(X_i,\, x,\, L_\lambda) = \frac{\ell(|X_i - x|)}{\frac{1}{\lambda}\sum_{j\in I^\lambda} \ell(|X_j - x|)}. \tag{2.12}$$

We call the denominator of the r.h.s of (2.12) the *interference* at x. The definition (2.12) comes from [HJKP18], cf. Section 5.2.1. Note that it is almost the same as the definition (1.1) of SINR with $N_0 = 0$, $P > 0$ arbitrary, and $\gamma = 1/\lambda$, the only difference is that here the sum in the denominator also contains the term $\ell(|X_i - x|)$.

Now, given a trajectory configuration $s = (s^i)_{i\in I^\lambda} \in \mathcal{S}_{k_{\max}}(X^\lambda)$, we put

$$\mathfrak{S}(s) = \sum_{i\in I^\lambda} \sum_{l=1}^{s^i_{-1}} \mathrm{SIR}(s^i_{l-1}, s^i_l, L_\lambda)^{-1} = \lambda \sum_{k=1}^{k_{\max}} \int_{W^k} R_{\lambda,k}(s)(\mathrm{d}x_0, \dots, \mathrm{d}x_{k-1}) \sum_{l=1}^{k} \frac{\int_W \ell(|y - x_l|) L_\lambda(\mathrm{d}y)}{\ell(|x_{l-1} - x_l|)}, \tag{2.13}$$

where for $k \in [k_{\max}]$, we write $x_k = o$. Then (2.13) is a special case of (2.7) with

$$f_k(\nu, x_0, \dots, x_{k-1}) = \sum_{l=1}^{k} \frac{\int_W \ell(|y - x_l|)\nu(\mathrm{d}y)}{\ell(|x_{l-1} - x_l|)}, \quad x_k = o, k \in [k_{\max}]. \tag{2.14}$$

Regarding this interference penalty term, three natural questions arise. The first one is why we choose (2.12) as the definition of SIR (which should also be compared to the conventional definition (1.1) of SINR). The second one is what is the interpretation of the interference penalty term (2.13). These two questions are related to the high-density setting, and we will answer them in Section 2.3.1. The third one is what is the goal and the effect of choosing the message trajectories according to a uniform *a priori* distribution weighted exponentially by a factor depending on the SIR values of all transmissions in the system. This is a question about the motivation of our Gibbsian approach, and we will discuss it in Section 2.5.1.

Next, we introduce congestion. We define $\eta(m) = m(m-1)$ and (as in (2.9)) we put

$$\mathfrak{M}(s) = \sum_{i\in I^\lambda} \eta(m_i(s)) = \sum_{i\in I^\lambda} m_i(s)(m_i(s) - 1), \quad s \in \mathcal{S}_{k_{\max}}(X^\lambda). \tag{2.15}$$

Note that $\eta(m_i(s)) = m_i(s)(m_i(s) - 1)$ is the number of ordered pairs of hops arriving at the relay $X_i = s^i_0$. We will explain and motivate this choice in Section 2.5.2.

In the downlink scenario, instead of users sending messages to the base station, the base station sends exactly one message to each of the users, using the same relaying rules. One can define a Gibbsian model analogously, now for trajectories from o to X_i instead of the other way around. For this, the interference term and the congestion term have to be redefined in an obvious way. We are certain that analogues of all our results are true and can be proved in the same way, hence we abstained from spelling them out.

For possible extensions of this model involving time dependence or users transmitting multiple messages, see Section 2.4.

[6] These assumptions on the path-loss function ℓ are valid until the end of Chapter 3. In Chapter 4, in the context of SINR percolation, we will make new assumptions on it, keeping the usual notation ℓ.

2.2. Behaviour of the Gibbsian model in the high-density limit

2.2.1. The limiting free energy

The main goal of the present chapter is the description of this model in the high-density limit $\lambda \to \infty$ in the quenched setting. Our first result describes the limiting *free energy*, i.e., the exponential behaviour of the partition function $Z_\lambda^{\gamma,\beta}(X^\lambda)$. One expects that this is entirely governed by the large deviations behaviour of the empirical measures $((R_{\lambda,k})(S)_{k\in[k_{\max}]})_{\lambda>0}$. This expectation is supported by the fact that, for $i \in I^\lambda$ and $s \in \mathcal{S}_{k_{\max}}(X^\lambda)$, we can express $m_i(s)$ defined in (2.8) in terms of $(R_{\lambda,k}(s))_{k\in[k_{\max}]}$ as follows

$$m_i(s) = \lambda \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k_{\max}} \pi_l R_{\lambda,k}(s)(\{s_0^i\}). \tag{2.16}$$

Surprisingly, it turns out that the limiting free energy cannot be described entirely in terms of these measures. The reason is that the function in (2.16) that maps them onto $m_i(s)$ is highly discontinuous in the limit $\lambda \to \infty$; even a proper formulation of such continuity would be awkward since both i and s depend on λ.

One therefore needs to substantially extend the probability space and to choose an additional family of empirical measures such that the congestion term $\mathfrak{M}(s)$ can be written as a (lower semi-)continuous function of these measures in the limit $\lambda \to \infty$. A natural choice of such a family is the one of the measures

$$P_{\lambda,m}(s) = \frac{1}{\lambda} \sum_{i\in I^\lambda :\, m_i(s)=m} \delta_{s_0^i}, \quad m \in \mathbb{N}_0. \tag{2.17}$$

Then for $m \in \mathbb{N}_0$, $P_{\lambda,m}(s) \in \mathcal{M}(W)$ is the empirical measure of the users s_0^i whose number of incoming hops equals m. For any $s \in \mathcal{S}_{k_{\max}}(X^\lambda)$ the following hold

$$(i) \quad \sum_{k=1}^{k_{\max}} \pi_0 R_{\lambda,k}(s) = L_\lambda, \qquad (ii) \quad \sum_{m=0}^{\infty} P_{\lambda,m}(s) = L_\lambda, \qquad (iii) \quad \sum_{m=0}^{\infty} m P_{\lambda,m}(s) = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l R_{\lambda,k}(s). \tag{2.18}$$

Condition (i) expresses our assumption that each user transmits precisely one message, (ii) says that each user serves as a relay for precisely m message trajectories for precisely one $m \in \mathbb{N}_0$, and (iii) says that the relays can be calculated in two ways: according to the number of incoming hops and according to the index of the hop of a trajectory that uses it. Moreover, we can write (2.9) in terms of $(P_{\lambda,m}(s))_{m\in\mathbb{N}_0}$ as follows

$$\mathfrak{M}(s) = \sum_{i\in I^\lambda} \eta(m_i(s)) = \lambda \sum_{m=0}^{\infty} \eta(m) P_{\lambda,m}(s)(W).$$

We note that the function $\mathcal{M}(W)^{\mathbb{N}_0} \to \mathbb{R} \cup \{\infty\}$, $(\xi_m)_{m\in\mathbb{N}_0} \mapsto \sum_{m=0}^{\infty} \eta(m)\xi_m(W)$ is lower semicontinuous, and even continuous on $\{(\xi_m)_{m\in\mathbb{N}_0} \colon \sum_{m=0}^{\infty} \eta(m)\xi_m(W) \le \alpha\}$ for any $\alpha \in \mathbb{R}$.

The limiting free energy will be described in terms of the following kind of families of measures, and it will turn out that all subsequential limits of the families $((R_{\lambda,k}(S))_{k=1}^{k_{\max}}, (P_{\lambda,m}(S))_{m=0}^{\infty})$ in the quenched limit $\lambda \to \infty$ are of this kind.

Definition 2.2.1. *An* admissible trajectory setting *is a collection of measures* $\Psi = ((\nu_k)_{k=1}^{k_{\max}}, (\mu_m)_{m=0}^{\infty})$ *with* $\nu_k \in \mathcal{M}(W^k)$ *for all* k *and* $\mu_m \in \mathcal{M}(W)$ *for all* m*, satisfying the following properties.*

$$(i) \quad \sum_{k=1}^{k_{\max}} \pi_0 \nu_k = \mu, \qquad (ii) \quad \sum_{m=0}^{\infty} \mu_m = \mu, \qquad (iii) \quad M := \sum_{m=0}^{\infty} m\mu_m = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l \nu_k. \tag{2.19}$$

The measure ν_k is the measure of the k-hop trajectories and μ_m the measure of the users that receive precisely m incoming hops; note that there is no reason that they be normalized (like for μ). Observe that in (2.18), L_λ, $R_{\lambda,k}(s)$, and $P_{\lambda,m}(s)$ play the role of μ, ν_k, and μ_m, respectively. In particular, after replacing μ with L_λ, $((R_{\lambda,k}(s))_{k\in[k_{\max}]}, (P_{\lambda,m}(s))_{m\in\mathbb{N}_0})$ satisfies the definition of an admissible trajectory setting. This way, the measure M defined in (iii) can be interpreted as the measure of all relaying hops, i.e., the one of all incoming hops arriving at the users (relays) in the system. See Section 2.4.2 for a modified version of our model where the assumption (i) is relaxed. By

$$\mathcal{H}_V(\nu \mid \widetilde{\nu}) = \begin{cases} \int_V \mathrm{d}\nu \, \log \frac{\mathrm{d}\nu}{\mathrm{d}\widetilde{\nu}} - \nu(V) + \widetilde{\nu}(V), & \text{if the density } \frac{\mathrm{d}\nu}{\mathrm{d}\widetilde{\nu}} \text{ exists,} \\ +\infty & \text{otherwise,} \end{cases} \tag{2.20}$$

we denote the relative entropy [GZ93, Section 2.3] of a Borel measure ν with respect to another Borel measure $\widetilde{\nu}$ on a measurable subset V of $\mathbb{R}^n$, $n \in \mathbb{N}$.

For an admissible trajectory setting $\Psi = ((\nu_k)_{k=1}^{k_{\max}}, (\mu_m)_{m=0}^{\infty})$ we define

$$\mathrm{S}(\Psi) = \sum_{k=1}^{k_{\max}} \int_{W^k} \mathrm{d}\nu_k \, \widetilde{f}_k, \qquad \text{where } \widetilde{f}_k(x_0,\dots,x_{k-1}) = f_k(\mu,x_0,\dots,x_{k-1}), \tag{2.21}$$

and

$$\mathrm{M}(\Psi) = \sum_{m=0}^{\infty} \eta(m)\, \mu_m(W) \tag{2.22}$$

and

$$\mathrm{I}(\Psi) = \sum_{k=1}^{k_{\max}} \mathcal{H}_{W^k}\big(\nu_k \mid \mu \otimes M^{\otimes(k-1)}\big) + \sum_{m=0}^{\infty} \mathcal{H}_W(\mu_m \mid \mu c_m) + \mu(W)\Big(1 - \sum_{k=1}^{k_{\max}} M(W)^{k-1}\Big) - \frac{1}{\mathrm{e}}, \tag{2.23}$$

where we recall $M = \sum_{m\in\mathbb{N}_0} m\mu_m$ from Definition 2.2.1(iii) and η defined before (2.9), and $c_m = \exp(-1/(\mathrm{e}\mu(W))(\mathrm{e}\mu(W))^{-m}/m!$ are the weights of the Poisson distribution with parameter $1/(\mathrm{e}\mu(W))$. In Section 2.3.2 we argue that $\mathrm{I}(\Psi)$ is well-defined as an element of $(-\infty,\infty]$ and $\Psi \mapsto \mathrm{I}(\Psi)$ is a lower semicontinuous function that is bounded from below, and we provide an interpretation for $\mathrm{I}(\cdot)$. A tedious but elementary calculation shows that I is convex. Given this, it is in fact not difficult to derive that I is strictly convex on its level sets, thanks to the fact that the term $\sum_{m=0}^{\infty} \mathcal{H}_W(\mu_m \mid \mu c_m)$ is strictly convex on its level sets, see Section A.1 in the Appendix. In Section 2.2.3, I will turn out to govern the large deviations of the trajectory configuration. The terms $\mathrm{S}(\cdot)$ and $\mathrm{M}(\cdot)$ are analogues of the penalty terms $\mathfrak{S}(\cdot)$ and $\mathfrak{M}(\cdot)$ in the high-density setting, respectively.

We fix all the parameters $W, \mu, k_{\max}, f_k, \eta, \gamma$, and β of the model. Our first main result is the following.

Theorem 2.2.2 (Quenched exponential rate of the partition function)**.** *For $\mathbb{P}$-almost all $\omega \in \Omega$,*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log Z_\lambda^{\gamma,\beta}(X^\lambda(\omega)) = - \inf_{\Psi \text{ admissible trajectory setting}} \big(\mathrm{I}(\Psi) + \gamma\mathrm{S}(\Psi) + \beta\mathrm{M}(\Psi)\big). \tag{2.24}$$

See Section 2.3 for a discussion and Section 2.9.4 for the proof.

2.2.2. Description of the minimizer

From the variational formula in (2.24), descriptive information about the typical behaviour of the network can be deduced, especially in the case of the specific choice of $\mathfrak{M}$ and $\mathfrak{S}$ from Section 2.1.4, see Sections 2.2.3 and 2.3.2. Hence, it is important to derive the Euler–Lagrange equations and to characterize the minimizer in most explicit terms. Our main results in this respect are the following.

Proposition 2.2.3 (Characterization of the minimizer)**.** *Let $k_{\max} > 1$. The variational formula in* (2.24) *has a unique minimizer $\Psi = ((\nu_k)_{k=1}^{k_{\max}},(\mu_m)_{m=0}^{\infty})$, which is the unique solution to the following equations.*

$$\nu_k(\mathrm{d}x_0,\dots,\mathrm{d}x_{k-1}) = \mu(\mathrm{d}x_0)\, A(x_0) \prod_{l=1}^{k-1} \big(C(x_l)M(\mathrm{d}x_l)\big)\mathrm{e}^{-\gamma \widetilde{f}_k(x_0,\dots,x_{k-1})}, \qquad k \in [k_{\max}], \tag{2.25}$$

$$\mu_m(\mathrm{d}x) = \mu(\mathrm{d}x)\, B(x) \frac{(C(x)\mu(W))^{-m}}{m!} \mathrm{e}^{-\beta\eta(m)}, \qquad m \in \mathbb{N}_0. \tag{2.26}$$

where $A,B,C\colon W \to [0,\infty)$ are functions such that the conditions in (2.19) *are satisfied.*

The case $k_{\max} = 1$ is trivial. Indeed, in this case the only admissible trajectory setting is $\Psi = (\nu_1,(\mu_m)_{m\in\mathbb{N}_0})$ with $\mu_0 = \nu_1 = \mu$ and $\mu_m = 0$ otherwise, therefore this Ψ minimizes (2.24).

The proof of Proposition 2.2.3 is in Section 2.10.

While explicit formulas for the functions A and B can, given the function C, easily be derived from (i) and (ii) in (2.19) (see (2.99)), the condition for C coming from (iii) is deeply involved and cannot be easily solved intrinsically; see (2.100) – (2.102). We have no argument for its existence to offer other than via proving the existence of a minimizer Ψ and deriving the Euler–Lagrange equations. By convexity of I, S, and M, every solution Ψ to these equations is a minimizer, and by strict convexity of I on its level sets, the minimizer is unique. We will interpret the equations (2.25) – (2.26) in Section 2.3.3. The equations (2.25) – (2.26) become more explicit in case $\beta = 0$, where uniqueness of the minimizer still holds, see Section 2.2.4.

2.2.3. Large deviations for the empirical trajectory measure

Actually, the minimizer of the variational formula in (2.24) receives a rigorous interpretation in terms of important objects that describe the network. As we have already mentioned, the family of empirical measures

$$\Psi_\lambda(s) = \big((R_{\lambda,k}(s))_{k\in[k_{\max}]},(P_{\lambda,m}(s))_{m\in\mathbb{N}_0}\big) \tag{2.27}$$

satisfies the definition of an admissible trajectory setting, apart from the fact that in Definition 2.2.1, μ has to replaced by L_λ everywhere, where we recall that L_λ converges to μ almost surely as $\lambda\to\infty$. According to our remarks after Definition 2.2.1, $R_{\lambda,k}(s)$ and $P_{\lambda,m}(s)$ play the roles of ν_k and μ_m, respectively, in an admissible trajectory setting, which explains this term. Furthermore, for $s\in\mathcal{S}_{k_{\max}}(X^\lambda)$, we can express the term $\mathfrak{M}$ as

$$\mathfrak{M}(s) = \lambda \mathrm{M}(\Psi_\lambda(s)).$$

Moreover, for the continuous penalization term we have

$$\mathfrak{S}(s) \approx \lambda \mathrm{S}(\Psi_\lambda(s)), \tag{2.28}$$

where we typically do not have an identity, because $\mathfrak{S}(s) = \lambda\sum_{k=1}^{k_{\max}}\int_{W^k}\mathrm{d}R_{\lambda,k}(s) f_k(L_\lambda,\cdot)$, which is usually not equal to $\lambda\mathrm{S}(\Psi_\lambda(s)) = \lambda\sum_{k=1}^{k_{\max}}\int_{W^k}\mathrm{d}R_{\lambda,k}(s) f_k(\mu,\cdot)$. However, since $L_\lambda \Longrightarrow \mu$ almost surely (where we write $\Longrightarrow$ for weak convergence of finite measures), this difference vanishes in the limit, see Proposition 2.9.2.

We consider now the distribution of $\Psi_\lambda(S)$ with S distributed under the product reference measure introduced in (2.4), normalized to a probability measure, $\mathrm{P}^{0,0}_{\lambda,X^\lambda}$; note that the normalization $Z^{0,0}_\lambda(X^\lambda)$ is equal to $k_{\max}^{N(\lambda)}$. Our next main result, Theorem 2.2.4, is a large deviation principle (LDP; see (2.30) – (2.31)) and the convergence towards the minimizer of the variational formula. See Section 2.12.2 for a discussion about the case $\beta = \gamma = 0$.

Theorem 2.2.4 (LDP and convergence for the empirical measures)**.** *The following statements hold almost surely with respect to* $(X^\lambda)_{\lambda>0}$.

(i) The distribution of $\Psi_\lambda(S)$ *under* $\mathrm{P}^{0,0}_{\lambda,X^\lambda}$ *satisfies an LDP as* $\lambda\to\infty$ *with scale* λ *on the set*

$$\mathcal{A} = \Big(\prod_{k=1}^{k_{\max}}\mathcal{M}(W^k)\Big)\times\mathcal{M}(W)^{\mathbb{N}_0} \tag{2.29}$$

with rate function given by $\mathcal{A}\ni\Psi\mapsto \mathrm{I}(\Psi)+\mu(W)\log k_{\max}$*, which we define as* ∞ *if* Ψ *is not an admissible trajectory setting.*

(ii) For any $\gamma,\beta\in(0,\infty)$*, the distribution of* $\Psi_\lambda(S)$ *under* $\mathrm{P}^{\gamma,\beta}_{\lambda,X^\lambda}$ *converges towards the unique minimizer of the variational formula in* (2.24)*.*

For the reader's convenience, we recall that the LDP states that the rate function $\mathrm{I}+\mu(W)\log k_{\max}$ is lower semicontinuous and

$$\limsup_{\lambda\to\infty}\frac{1}{\lambda}\log \mathrm{P}^{0,0}_{\lambda,X^\lambda}(\Psi_\lambda(S)\in F) \quad\le\quad -\inf_F\big(\mathrm{I}+\mu(W)\log k_{\max}\big), \tag{2.30}$$

$$\liminf_{\lambda\to\infty}\frac{1}{\lambda}\log \mathrm{P}^{0,0}_{\lambda,X^\lambda}(\Psi_\lambda(S)\in G) \quad\ge\quad -\inf_G\big(\mathrm{I}+\mu(W)\log k_{\max}\big), \tag{2.31}$$

for any closed set F and any open set G in $\mathcal{A}$. See [DZ98] for more on large deviations theory. On $\mathcal{A}$, we consider the product topology that is induced by weak convergence in each factor; this is equal to coordinatewise weak convergence, see Section 2.9.3 for more details.

The proof of Theorem 2.2.4(i) is carried out in Section 2.9.5. Assertion (ii) is a simple consequence of (i), since the functionals S and M are bounded and continuous on the set $B_C = \{\Psi \in \mathcal{A} \colon \mathrm{M}(\Psi) \le C\}$ for any C, and B_C is compact in $\mathcal{A}$ (see Lemma 2.10.1). Denoting the level sets of the rate function $\mathrm{I} + \mu(W) \log k_{\max}$ by $\Phi_\alpha = \{\Psi \in \mathcal{A} \colon \mathrm{I}(\Psi) + \mu(W) \log k_{\max} \le \alpha\}$ for $\alpha \in \mathbb{R}$, we see that $\Phi_\alpha \cap B_C$ is compact for any α and C. Thus, Varadhan's lemma [DZ98, Lemmas 4.3.4, 4.3.6] can be applied to prove the assertion (ii).

2.2.4. Dropping the congestion term

Proposition 2.2.3 yields a rather implicit description of the minimizer of (2.24) in the case $\beta,\gamma > 0$ that we have considered so far, although we have showed that the minimizer is unique. However, in the special case $\beta = 0$, where the congestion functional $\mathfrak{M}$ (2.9) is absent, the situation is much better. Indeed, it turns out that the minimizer is explicitly given in terms of the parameters of the model and it is also unique. For the specific choice of Section 2.1.4 where $\mathfrak{S}$ penalizes interference (2.13), on base of this knowledge, we are able in Chapter 3 to derive a number of relevant qualitative properties of the trajectories.

In what follows, we call $\Sigma = (\nu_k)_{k\in[k_{\max}]}$ with $\nu_k \in \mathcal{M}(W^k)$ for all $k \in [k_{\max}]$ an *asymptotic routeing strategy* if we have $\sum_{k=1}^{k_{\max}} \pi_0 \nu_k = \mu$. In (2.19) we see that the first coordinate, Σ, of an admissible trajectory setting Ψ is an asymptotic routeing strategy, and in (2.21) we see that $\mathrm{S}(\Psi)$ depends only on Σ. We will therefore write $\mathrm{S}(\Sigma)$ for $\mathrm{S}(\Psi)$. Further, we write $M = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l \nu_k$, in accordance with (2.19) but with no regard to the measures $(\mu_m)_{m\in\mathbb{N}_0}$. We define an entropic term J for asymptotic routeing strategies as follows.

$$\mathrm{J}(\Sigma) = \sum_{k=1}^{k_{\max}} \mathcal{H}_{W^k}(\nu_k \mid \mu^{\otimes k}) + \mu(W)\Big(1 - \sum_{k=1}^{k_{\max}} M(W)^{k-1}\Big) + M(W) \log \mu(W). \tag{2.32}$$

Similarly to I in (2.23), J describes counting complexity in the high-density limit, but without reference to the measures $(\mu_m)_m$, see Section 2.11.1. We will interpret J in Section 2.3.2.

The following proposition summarizes the analogues of Theorem 2.2.2, Proposition 2.2.3, and Theorem 2.2.4 in case $\beta = 0$, after dropping all the measures μ_m.

Proposition 2.2.5. *The following statements hold almost surely with respect to* $(X^\lambda)_{\lambda>0}$.

(1) The distribution of

$$\Sigma_\lambda(S) = (R_{\lambda,k}(S))_{k\in[k_{\max}]} \tag{2.33}$$

under $\mathrm{P}^{0,0}_{\lambda,X^\lambda}$ *satisfies an LDP as* $\lambda \to \infty$ *with scale* λ *on the set* $\mathcal{A}_0 = \prod_{k=1}^{k_{\max}} \mathcal{M}(W^k)$ *with rate function given by* $\mathcal{A}_0 \ni \Psi \mapsto \mathrm{J}(\Sigma) + \mu(W) \log k_{\max}$*, which we define as* ∞ *if* Σ *is not an asymptotic routeing strategy. Further, the rate function is* good*, i.e., it has compact level sets.*

(2) For any $\gamma \in (0,\infty)$,

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log Z^{\gamma,0}_\lambda(X^\lambda) = - \inf_{\Sigma \text{ asymptotic routeing strategy}} \big(\mathrm{J}(\Sigma) + \gamma \mathrm{S}(\Sigma)\big). \tag{2.34}$$

(3) Let $\gamma > 0$ *and* $k_{\max} > 1$*. The variational formula on the r.h.s. of* (2.34) *exhibits a unique minimizer* $\Sigma = (\nu_k)_{k\in[k_{\max}]}$ *given as*

$$\nu_k(\mathrm{d}x_0, \dots, \mathrm{d}x_{k-1}) = \mu(\mathrm{d}x_0) A(x_0) \prod_{l=1}^{k-1} \frac{\mu(\mathrm{d}x_l)}{\mu(W)} \mathrm{e}^{-\gamma \widetilde{f}_k(x_0,\dots,x_{k-1})}, \quad k \in [k_{\max}], \tag{2.35}$$

where

$$\frac{1}{A(x_0)} = \sum_{k=1}^{k_{\max}} \int_{W^{k-1}} \prod_{l=1}^{k-1} \frac{\mu(\mathrm{d}x_l)}{\mu(W)} \mathrm{e}^{-\gamma \widetilde{f}_k(x_0,\dots,x_{k-1})}, \qquad x_0 \in W. \tag{2.36}$$

(4) For any $\gamma \in (0,\infty)$, the distribution of $\Sigma_\lambda(S)$ under $\mathrm{P}^{\gamma,0}_{\lambda,X^\lambda}$ converges to the minimizer of the variational formula in (2.34).

Knowing the uniqueness of the minimizer, one can interpret the convergence in (4) as a law of large numbers for the empirical trajectory measure family $\Sigma_\lambda(S)$, which follows from the LDP in (1); and similarly for the case of $\Psi_\lambda(s)$ in Theorem 2.2.4 (ii).

Proposition 2.2.5 is proved in Section 2.11.1. An interpretation of the equations (2.35) – (2.36) can be found in Section 2.3.3. In the special setting of Section 2.1.4, we derive simple geometric properties of the unique minimizer (2.35) in Section 2.3.4.

Let us explain in what way Proposition 2.2.5 is the special case of the aforementioned results for $\beta = 0$ and in what way it differs. It is true that the LDP in the assertion (1) directly follows from the LDP in Theorem 2.2.4(i) via the contraction principle [DZ98, Theorem 4.2.1] for the projection map $(\Sigma,(\mu_m)_m) \mapsto \Sigma$, however, with rate function given by

$$\mathrm{J}(\Sigma) = \inf_{(\mu_m)_{m\in\mathbb{N}_0}\colon \Psi=(\Sigma,(\mu_m)_{m\in\mathbb{N}_0}) \text{ admissible trajectory setting}} \mathrm{I}(\Psi). \tag{2.37}$$

It is an elementary but tedious task to identify this as in (2.32) by identifying

$$\mu_m(\mathrm{d}x) = \mu(\mathrm{d}x)\frac{\Big(\frac{M(\mathrm{d}x)}{\mu(\mathrm{d}x)}\Big)^m}{m!}\mathrm{e}^{-M(\mathrm{d}x)/\mu(\mathrm{d}x)}, \qquad m \in \mathbb{N}_0, \tag{2.38}$$

as the unique minimizer on the right-hand side of (2.37), given $M = \sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\nu_k$. However, we chose an alternative route for proving the LDP with explicit identification of J, which is a variant of the proof of Theorem 2.2.4(i) that involves only asymptotic routeing strategies. From (2.37) it is clear that the variational formula in (2.34) is indeed the special case of (2.24) for $\beta = 0$, i.e.,

$$\inf_{\Sigma \text{ asymptotic routeing strategy}} \big(\mathrm{J}(\Sigma) + \gamma\mathrm{S}(\Sigma)\big) = \inf_{\Psi \text{ admissible trajectory setting}} \big(\mathrm{I}(\Psi) + \gamma\mathrm{S}(\Psi)\big). \tag{2.39}$$

Note also that the minimizer Ψ is unique. In fact, Ψ is of the same form as the minimizer (2.25) – (2.26) of the variational formula (2.24) for $\gamma,\beta > 0$, see Section 2.11.2. This raises the additional question whether or not the measures $(P_{\lambda,m}(S))_{m\in\mathbb{N}_0}$ converge to the minimizer $(\mu_m)_{m\in\mathbb{N}_0}$ in (2.38) under $\mathrm{P}^{\gamma,0}_{\lambda,X^\lambda}$ for M corresponding to the minimal ν_k's of (2.35). Since the congestion term, which gave rise to a strong compactness argument, is now absent, this question cannot immediately be decided using Varadhan's lemma (equivalently, we have no information about the exponential tightness of $((P_{\lambda,m}(S))_{m\in\mathbb{N}_0})_{\lambda>0}$). We nevertheless believe that the answer is positive because the candidate Ψ for the limit is the unique minimizer of $\Psi' \mapsto \mathrm{I}(\Psi') + \gamma\mathrm{S}(\Psi')$. Moreover, this compactness property was also used in the proof of Theorem 2.2.2, which is another reason that we had to redo the proofs of Proposition 2.2.5(2) and (3), given our proof of the assertion (1).

Aiming for a complete description of the Gibbsian model also in case $\min\{\gamma,\beta\} = 0$, we will comment on the case $\beta > \gamma = 0$ of penalizing only congestion in Section 2.12.1 and on the *a priori* case $\beta = \gamma = 0$ in Section 2.12.2, using elements of the preceding proofs. Already the case $\beta > \gamma = 0$ in Section 2.1.4 is rather uninteresting for applications in telecommunications because in this case the network is not interference limited, and the penalization has no spatial effect. In case $\beta = \gamma = 0$, there is no penalization at all, and message trajectories are uniformly distributed. It is nevertheless instructive to spend a few words on these cases in order to see how they fit into the picture corresponding to the case $\gamma,\beta > 0$. For example, we find it interesting that while for $\gamma > \beta = 0$, the limiting behaviour of the system can be described in terms of $(R_{\lambda,k}(\cdot))_{k\in[k_{\max}]}$, for $\beta > \gamma = 0$, the measures $(P_{\lambda,m}(\cdot))_{m\in\mathbb{N}_0}$ alone are not sufficient for analysing the high-density limit.

2.3. Discussion: modelling questions and the high-density limit

2.3.1. The notion of SIR and the interference term

This section includes the following modelling discussions. In Section 2.3.1.1 we explain our definition (2.12) of SIR used for the Gibbsian model and its relation to the conventional definition (1.1) of SI(N)R, and we discuss about the continuity of the path-loss function at 0. In Section 2.3.1.2 we comment on the relevance of our choice of the interference penalization term $\mathfrak{S}(s)$ in (2.7).[7]

2.3.1.1. The notion of SIR and its adaptation to the high-density setting

Note that the conventional definition of interference of a transmission from X_i to x is $\sum_{j \in I^\lambda \setminus \{i\}} \ell(|X_j - x|)$, more precisely $\sum_{X_j \in X^\lambda \setminus \{X_i, x\}} \ell(|X_j - x|)$, cf. (1.1), in contrast to our definition in (2.12), where we added a factor of $\frac{1}{\lambda}$, following [HJKP18, Section 1]. According to this convention, we should say "total received power" instead of "interference" and thus "STIR" (signal-to-total received power ratio) instead of "SIR", cf. [KB14, Section II.]. As we are interested in the limit $\lambda \to \infty$, where it makes no difference whether or not we add the terms $\frac{1}{\lambda}\ell(|X_i - x|)$ and $\frac{1}{\lambda}\ell(|x-x|) = \frac{1}{\lambda}\ell(0)$ to the denominator, we stick to our notions "SIR" and "interference". For the same reason, our model does not include noise. However, note also our additional factor of $1/\lambda$, which we think is appropriate, at least mathematically, to our setting, in which we consider the high-density limit $\lambda \to \infty$. We actually scale the "usual" SIR by the density parameter. Indeed, in order to cope with an enormous number of messages in a system with one base station and a fixed bandwidth[8], one can either distribute the messages over a longer time stretch or decompose the messages into many smaller ones. The factor of $1/\lambda$ is a crude approximation of a combination of these two strategies.

The assumption that the path-loss function ℓ is continuous at 0 comes from [DBT05, DFMMT06, GT08, HJKP18] and differs from the works [GK00, KB14], which make mathematical use of the perfect scaling of the path-loss function $\ell(r) = r^{-\alpha}$, which is for this reason one of the standard choices. However, for small r, this is an unrealistic choice, cf. [GK00, Section I.A], [GT08, Section I.]. See Section 5.1.1 for further historical remarks on the notion of SI(N)R. In Section 5.1.2 we will discuss the effect of the choice of ℓ in both models investigated in this thesis and we will relate these to other models on interference limited networks. In particular, we will point out the main differences between our high-density limit and other high-density settings with an unbounded path-loss function.

2.3.1.2. The interference term

The interference term $\mathfrak{S}(s)$ in (2.7) quantifies the quality of the transmission of the messages in case they use the trajectories s^i from X_i to o. The choice of the *reciprocals* of the SIRs comes from the fact that the *bandwidth* used for a transmission is defined [SPW07] as

$$\frac{\varrho}{\log_2(1 + \mathrm{SIR}(\cdot))}, \tag{2.40}$$

where ϱ is the data transmission rate, and SIR is defined as in (2.12) without the factor of $1/\lambda$ in the denominator of (2.12). This quantity is of order $1/\lambda$ for λ large, under the assumption that $L_\lambda \Longrightarrow \mu$. In the high-density setting $\lambda \to \infty$ that we study, (2.40) can be approached well by (a constant times) the reciprocals of the SIR, since $\log(1 + x) \sim x$ as $x \to 0$. [SPW07, Section 3] suggests that in case of multihop communication, the used bandwidth equals the sum of the used bandwidth values corresponding to the individual hops, which explains our choice of the sum over l in (2.7). We note that the idea of using a sum of reciprocals of SIR values as a cost function to be minimized appeared also in [BC12], see Section 2.7.2 for further details.

2.3.2. The entropy term

Let us now explain some important properties of the entropy term

$$\mathrm{I}(\Psi) = \sum_{k=1}^{k_{\max}} \mathcal{H}_{W^k}\big(\nu_k \mid \mu \otimes M^{\otimes(k-1)}\big) + \sum_{m=0}^{\infty} \mathcal{H}_W(\mu_m \mid \mu c_m) + \mu(W)\Big(1 - \sum_{k=1}^{k_{\max}} M(W)^{k-1}\Big) - \frac{1}{\mathrm{e}} \tag{2.41}$$

[7] The subject of Section 2.5.1 is also related to the penalization of interference. There, we will motivate the exponential form of our Gibbs distribution and relate it to a variant of the model where trajectories with bad SIR values are forbidden.

[8] See Section 2.3.1.2 for a discussion about the bandwidth.

defined in (2.23).

According to (i) and (iii) in (2.19), we have that $M(W) \leq (k_{\max} - 1)\mu(W)$, further, the first term on the right-hand side of (2.23) is bounded from below. Moreover, since by (ii) in (2.19), we have

$$\sum_{m\in\mathbb{N}_0} c_m = \sum_{m\in\mathbb{N}_0} \frac{\mu_m(W)}{\mu(W)} = 1, \tag{2.42}$$

it follows that $\sum_{m=0}^{\infty} \mathcal{H}_W(\mu_m \mid \mu c_m)$ is nonnegative. These together with elementary properties of the relative entropy [DZ98, Section 6.2] imply that $\mathrm{I}(\Psi)$ is well-defined as an element of $(-\infty,\infty]$ and $\Psi \mapsto \mathrm{I}(\Psi)$ is a lower semicontinuous function that is bounded from below. More precisely, the LDP in Theorem 2.2.4 implies that the infimum of $\mathrm{I}(\Psi)$ over admissible trajectory settings equals $-\mu(W)\log k_{\max}$, which equals the almost sure limit of $1/\lambda$ times the logarithm of the total mass $k_{\max}^{N(\lambda)}$ of the joint *a priori* measure (2.4).

Let us now provide an interpretation of $\mathrm{I}(\cdot)$. Let $\lambda > 0$ and $s \in \mathcal{S}_{k_{\max}}(X^\lambda)$. Recall the empirical measure family $\Psi_\lambda(s) = \big((R_{\lambda,k}(s))_{k\in[k_{\max}]}, (P_{\lambda,m}(s))_{m\in\mathbb{N}_0}\big)$ from (2.27) and the constraints (2.18), which are similar to the ones (2.19) but with μ replaced by the rescaled empirical measure L_λ everywhere.

Informally speaking, for $\lambda > 0$ large, $\mathrm{I}(\Psi)$ asymptotically describes the following crucial counting term:

$$\mathrm{I}(\Psi) \approx -\frac{1}{\lambda} \log \frac{\#\Big\{s \in \mathcal{S}_{k_{\max}}(X^\lambda) \colon R_{\lambda,k}(s) \approx \nu_k,\ \forall k \in [k_{\max}] \text{ and } P_{\lambda,m}(s) \approx \mu_m,\ \forall m \in \mathbb{N}_0\Big\}}{N(\lambda)^{\sum_{i\in I^\lambda}(\widetilde{s}^i_{-1}-1)}}, \tag{2.43}$$

where $\widetilde{s}$ in the denominator is an arbitrarily chosen element of the set in the numerator. In this way, it fully describes the distribution of $\Psi_\lambda(s)$ on an exponential level.

A major part of the proof of Theorem 2.2.2 consists in making (2.43) rigorous and verifying the corresponding formal statement. In the beginning of Section 2.8, we will argue why taking "=" signs instead of "≈" in the numerator of the right-hand side of (2.43) is not applicable. Instead, first, in Section 2.8.1 we will introduce a spatial discretization procedure and formulate a rigorous discrete version of these "≈" relations for fixed λ and fixed fineness parameter of the discretization. In Section 2.8.2 we will derive explicit combinatorial formulas for the cardinality of the trajectories in this setting. Next, in Section 2.9.1 we will conclude that $1/\lambda$ times the logarithm of the quotient of the obtained counting complexity and the term $N(\lambda)^{\sum_{i\in I^\lambda}(\widetilde{s}^i_{-1}-1)}$ tends to $\mathrm{I}(\Psi)$ in the limit $\lambda \to \infty$ followed by the fineness parameter of the discretization tending to zero.

The characterization of $\mathrm{I}(\Psi)$ that arises directly from this argument is not exactly (2.41) but another, however identical expression (2.63). The reason why we chose (2.41) as the definition of $\mathrm{I}(\Psi)$ is that it is given in terms of objects that are commonly used in large deviations theory: relative entropies, multiples of total masses of the corresponding measures plus an additive constant. In particular, we find it natural in (2.41) that each μ_m is compared to the intensity measure μ multiplied by the weight of a Poisson distribution at m. Indeed, in case $\beta = 0$, for the minimizer Ψ of the variational formula (2.35), $(\frac{\mathrm{d}\mu_m}{\mathrm{d}\mu}(x))_{m\in\mathbb{N}_0}$ is the Poisson distribution with parameter $\frac{\mathrm{d}M}{\mathrm{d}\mu}(x)$ for each $x \in W$. Roughly speaking, the relative entropies plus the linear term $\mu(W)(1 - \sum_{k=1}^{k_{\max}} M(W)^{k-1})$ arise from $1/\lambda$ times the logarithmic rates of certain multinomial expressions in the above mentioned discretized counting procedure, after carrying out the limit $\lambda \to \infty$ followed by the fineness parameter tending to zero.

On the same informal level as for $\mathrm{I}(\Psi)$ in case of an admissible trajectory setting Ψ, for an asymptotic routeing strategy Σ, the entropic term

$$\mathrm{J}(\Sigma) = \sum_{k=1}^{k_{\max}} \mathcal{H}_{W^k}(\nu_k \mid \mu^{\otimes k}) + \mu(W)\Big(1 - \sum_{k=1}^{k_{\max}} M(W)^{k-1}\Big) + M(W)\log\mu(W) \tag{2.44}$$

asymptotically describes the counting term

$$\mathrm{J}(\Sigma) \approx -\frac{1}{\lambda} \log \frac{\#\Big\{s \in \mathcal{S}_{k_{\max}}(X^\lambda) \colon R_{\lambda,k}(s) \approx \nu_k,\ \forall k \in [k_{\max}]\}}{N(\lambda)^{\sum_{i\in I^\lambda}(\widetilde{s}^i_{-1}-1)}}, \tag{2.45}$$

where $\widetilde{s}$ in the denominator is an arbitrarily chosen element of the set in the numerator. This is very similar to (2.41) but with no reference to the measures $P_{\lambda,m}(s)$. The first two summands in (2.44) also appear in (2.41).

Roughly speaking, the term $M(W)\log\mu(W)$ is $1/\lambda$ times the logarithmic rate of the counting complexity of the number of possible choices of the relays of all trajectories from the set of users (with repetitions allowed), normalized by the joint *a priori* measure; see Lemmas 2.11.2 and 2.11.3 for a more precise description.

2.3.3. Interpretation of the minimizer

In case $\beta,\gamma > 0$, Proposition 2.2.3 tells us quite some information about the limiting trajectory distribution and the limiting spatial distribution of users with a given number of incoming hops under the measure $\mathrm{P}^{\gamma,\beta}_{\lambda,X^\lambda}$. Indeed, both have densities that are $\mu^{\otimes k}$-almost everywhere positive. It is remarkable that the k-hop trajectories follow a distribution that comes from choosing independently all the k sites with measures that do not depend on k (the starting point according to $A(x)\,\mu(\mathrm{d}x)$ and all the other $k-1$ sites each according to $C(x)\,M(\mathrm{d}x)$), exponentially weighted with the term $\gamma\widetilde{f}_k$. Furthermore, all the measures of the users receiving m incoming hops superpose each other on the full set $\mathrm{supp}(\mu)$, and at each space point x, this number m is distributed according to some Poisson distribution, exponentially weighted with the term $\beta\eta(m)$. Finally, the minimizer is unique.

As for the case $\beta = 0,\gamma > 0$, the minimizer also exhibits all the properties enumerated for $\beta,\gamma > 0$. In the k-hop trajectories, the starting point is chosen according to $A(x)\,\mu(\mathrm{d}x)$ and all the other $k-1$ sites according to the measure $\mu(\mathrm{d}x)/\mu(W)$, weighted with $\gamma\widetilde{f}_k$. Moreover, the number of incoming hops at a given relay at the site $x \in W$ is Poisson distributed with parameter equal to $M(\mathrm{d}x)/\mu(\mathrm{d}x)$.

Actually, the characterization (2.25) – (2.26) of the minimizer in case $\gamma,\beta > 0$ is also valid for $\gamma > \beta = 0$. We will explain this in more detail in Section 2.11.2, following all the proofs that this discussion refers to.

2.3.4. Rotation symmetry

Let us assume that $\beta = 0$ (cf. Section 2.2.4) and let us consider the special setting of Section 2.1.4 where $\mathfrak{S}$ penalizes interference. If $W = \overline{B_r(o)} \subset \mathbb{R}^d$ is a closed ball and μ is invariant under rotations, then the measures $(\nu_k)_k$ in (2.35) are also invariant under rotations of the entire trajectory, i.e., for any orthogonal $d \times d$-matrix O, we have that $\nu_k(\mathrm{d}x_0,\dots,\mathrm{d}x_{k-1}) = \nu_k^O(\mathrm{d}x_0,\dots,\mathrm{d}x_{k-1}) \equiv \nu_k(\mathrm{d}(Ox_0),\dots,\mathrm{d}(Ox_{k-1}))$ for any $k \in [k_{\max}]$. This is easily seen by an inspection of the formulas for the entropy term J in (2.32) and for the interference term S in (2.21), as the function $(x,y) \mapsto \int_W \ell(|z-y|)\,\mu(\mathrm{d}z)/\ell(|x-y|)$ is invariant under multiplication of both arguments with the same orthogonal matrix.

2.4. Extensions and variants of the model

In this section we discuss possible extensions and variants of the Gibbsian model. The proofs of Sections 2.8 and 2.9 immediately generalize to some extensions, in some other cases we have a clear picture about what results we expect but additional technical difficulties need to be handled, and there are variants where also serious modelling questions need to be addressed in order to formulate correct and relevant assertions. We present the extensions and variants in an increasing order of expected involvedness. While some of them are interesting for primarily mathematical reasons (e.g., allowing an unbounded number of hops or considering the annealed version of the Gibbsian model), other ones aim for making the modelling of a telecommunication network more realistic (e.g., considering users whose distribution is not Poissonian, allowing users to send no message or multiple messages, or introducing time dependence in the model).

2.4.1. Non-Poissonian users

In fact, the main results of this chapter hold for any collection of (random or non-random) point processes $((X_i)_{i=1,\dots,N(\lambda)})_{\lambda>0}$ on W for which $L_\lambda = \frac{1}{\lambda}\sum_{i=1}^{N(\lambda)}\delta_{X_i}$ converges weakly (almost surely, if random) to μ as $\lambda \to \infty$. Neither the independence or monotonicity in λ, nor the Poissonity of $(N(\lambda))_{\lambda>0}$ is used for the proofs. For example, our results remain also true for the deterministic set $X^\lambda = W \cap (\frac{1}{\lambda}\mathbb{Z}^d)$ and μ the Lebesgue measure on W.

2.4.2. Sending no message or multiple messages

One easily sees from the proofs in Sections 2.8 and 2.9 that Theorems 2.2.2 and 2.2.4 as well as Proposition 2.2.5 can be extended to cases where users send out no message or multiple messages. This models the standard situation in which large messages are cut into many smaller ones, who independently find their ways through the system. For this, the trajectory probability space has to be enlarged: to each user $X_i \in X^\lambda$, we attach the number $P_i \in \mathbb{N}_0$ of transmitted messages, and for each $j \in \{1, \dots, P_i\}$, there is an independent trajectory $X_i \to o$. The empirical trajectory measure $R_{\lambda,k}(\cdot)$ must be augmented by these trajectories. The main additional assumption then is that $\sum_{k=1}^{k_{\max}} \pi_0 R_{\lambda,k}(S)$ converges to some measure $\mu_0 \in \mathcal{M}(W)$ with $0 \neq \mu_0 \ll \mu$. (Also the case that μ_0 is not absolutely continuous with respect to μ is interesting, but will need additional work.) The interference term $\mathfrak{S}(\cdot)$ introduced in (2.13) also has to be changed. According to [BB09, Sections 2.3.1, 5.1], the SIR of the transmission of one of the P_i messages from X_i to $x \in W$ should be defined as

$$\frac{\ell(|X_i - x|)}{\frac{1}{\lambda}\sum_{j \in I^\lambda} \ell(|X_j - x|) P_j}.$$

One could also incorporate (possibly random) sizes of the messages, which would require an additional enlargement of the trajectory space.

2.4.3. Allowing an unbounded number of hops

If the upper bound $k_{\max}$ for the length of the trajectories is dropped, then the *a priori* measure defined in (2.4) has infinite total mass, and therefore the entropy function I is not bounded from below. However, since the function f_k in (2.13) is positive and bounded away from zero, for any $\gamma > 0$, $\beta \geq 0$, the total probability mass of all the k-hop trajectories under the Gibbs distribution is upper bounded by some geometrically decaying term in k. Hence, the definition of the model of Section 2.1.4 is no problem for $k_{\max} = \infty$ and $\gamma > 0$, $\beta \geq 0$, and the same holds for the general model of Section 2.1.3 as long as f_k is positive and bounded away from 0. We believe that all our results in this chapter about its limiting behaviour as $\lambda \to \infty$ remain essentially true (apart from the LDP under the *a priori* measure, since I and J cannot be turned into rate functions by adding a constant to them). However, proofs will require an additional cutting argument, which might become rather nasty. Further, from a modelling point of view, we find the necessity of taking an arbitrary number of hops in a fixed compact communication area with a bounded path-loss function questionable. Therefore, we decided to state and prove the results of the present chapter only for finite $k_{\max}$. Our belief that the results remain unchanged is supported by the fact that also the minimizing object $\Psi = ((\nu_k)_{k=1}^{k_{\max}}, (\mu_m)_{m=0}^{\infty})$ defined in Proposition 2.2.3 enjoys a geometric upper bound for $\nu_k(W^k)$ in k. Thus, the collections of measures defined by (2.25) – (2.26) are also well-defined and form the set of minimizers of the variational formula (2.24) in case $k_{\max} = \infty$, where uniqueness of the minimizer can be verified using the same strict-convexity argument (cf. Section A.1 in the Appendix) as before. Even the results about the qualitative properties of the minimizer in the limiting regime (1) of large communication areas and large transmitter–receiver distances extend to the case $k_{\max} = \infty$, see Section 3.2.

2.4.4. Time dependent versions of the Gibbsian model

The model of Section 2.1.4 can also be made time dependent. If one, e.g., introduces $k_{\max}$ discrete time slots indexed by $[k_{\max}]$ and assumes the l-th hop of any message trajectory to happen at time l for any $l \in [k_{\max}]$, then the interference of a transmission at time l is obtained from the starting points of all hops that happen at the same time [GK00, Section I.A]. Further, the congestion term can also be adapted to the time dependent situation via counting numbers of incoming hops at each time step separately; indeed, if two message trajectories use the same relay but at different times, this does not have to be penalized, cf. Section 2.5.2.

It is instructive to provide some details of this variant of the model for $\gamma > 0$ and $\beta = 0$.[9] For $l \in [k_{\max}]$,

[9] Given our results in Section 2.2 and their proofs, the case $\beta > 0$ is not much more difficult, but the disintegration of the numbers of incoming hops has to be performed according to the numbers of incoming hops at each time slot $l \in [k_{\max} - 1]$ at which

$x,y \in W$ and a trajectory configuration $s \in \mathcal{S}_{k_{\max}}(X^\lambda)$, we define the time dependent interference[10] at receiver $y \in W$ at time slot l as follows

$$I_l(y,\, s) = \sum_{j \in I^\lambda \colon s^j_{-1} \geq l} \ell(|s^j_{l-1} - y|)$$

and a time dependent SIR for the transmission from $x \in W$ to $y \in W$ at the time slot l as follows

$$\mathrm{SIR}_l(x,\, y,\, s,\, \lambda) = \frac{\ell(|x-y|)}{\frac{1}{\lambda} I_l(y,\, s)}.$$

Then we define a new interference penalty term as

$$\widetilde{\mathfrak{S}}(s) = \sum_{i \in I^\lambda} \sum_{l=1}^{k-1} \mathrm{SIR}_l^{-1}(s^i_{l-1},\, s^i_l,\, s,\, \lambda)^{-1}.$$

Then for $\gamma > 0$ (and $\beta = 0$), one can define a Gibbs distribution on $\mathcal{S}_{k_{\max}}(X^\lambda)$ analogously to (2.10) but with $\mathfrak{S}(\cdot)$ replaced by $\widetilde{\mathfrak{S}}(\cdot)$. Now, we observe that in this variant of the Gibbsian model, the configuration space is the same as in the original one, further, the interference term is still a continuous (although not linear) function of the empirical measure family $(R_{\lambda,k}(s))_{k \in [k_{\max}]}$. In particular, the combinatorics of the system is unchanged, and its logarithmic behaviour can be described in the limit $\lambda \to \infty$ in terms of the entropic term $\mathrm{J}(\Sigma)$ for asymptotic routeing strategies Σ. Moreover, the limiting interference penalty term has the form

$$\widetilde{\mathrm{S}}(\Sigma) = \sum_{k=1}^{k_{\max}} \int_{W^k} \nu_k(\mathrm{d}x_0, \dots, \mathrm{d}x_{k-1}) \sum_{l=1}^{k_{\max}} \frac{\sum_{k'=l}^{k_{\max}} \int_W \pi_{l-1}\nu_{k'}(\mathrm{d}y)\ell(|y-x_l|)}{\ell(|x_{l-1}-x_l|)}, \qquad x_k = o.$$

Thus, we expect that the limiting free energy can be expressed as the negative infimum of the variational formula $\mathrm{J}(\Sigma) + \widetilde{\mathrm{S}}(\Sigma)$, analogously to Proposition 2.2.5. However, the time dependent interference term induces an interaction between message trajectories, similarly to the congestion term (see Section 2.5.2 for more details of the latter one). This makes the description of the minimizers of the variational formula much more involved than in the original model for $\beta = 0$, due to the fact that while $\Sigma \mapsto \mathrm{S}(\Sigma)$ is linear, $\Sigma \mapsto \widetilde{\mathrm{S}}(\Sigma)$ is a quadratic function of the measures ν_k and their marginals. Deriving the Euler–Lagrange equations similarly to Section 2.10.2, we arrive at the implicit characterization of minimizer(s) $\widetilde{\Sigma} = (\widetilde{\nu}_k)_{k \in [k_{\max}]}$

$$\begin{aligned} \widetilde{\nu}_k(\mathrm{d}x_0, \dots, \mathrm{d}x_{k-1}) &= \mu(\mathrm{d}x_0) \widetilde{A}(x_0) \prod_{l=1}^{k-1} \frac{\mu(\mathrm{d}x_l)}{\mu(W)} \\ &\times \exp\Big[-\gamma \sum_{l=1}^{k} \Big(\frac{\sum_{k'=l}^{k_{\max}} \int_W \pi_{l-1}\widetilde{\nu}_{k'}(\mathrm{d}y)\ell(|y-x_l|)}{\ell(|x_{l-1}-x_l|)} + \sum_{k'=l}^{k_{\max}} \int_{W^k} \widetilde{\nu}_{k'}(\mathrm{d}z_0, \dots, \mathrm{d}z_{k-1}) \frac{\ell(|x_{l-1}-z_l|)}{\ell(|z_{l-1}-z_l|)} \Big) \Big], \end{aligned} \tag{2.46}$$

for $k \in [k_{\max}]$ and $x_k = o$, where $\widetilde{A} \colon W \to (0,\infty)$ is a function that makes $\widetilde{\Sigma}$ satisfy $\sum_{k \in [k_{\max}]} \pi_0 \widetilde{\nu}_k = \mu$. Since the entropy term J is the same as in the non-time dependent case, and it is strictly convex on its level sets according to the equation (2.37) and the strict convexity of I on its level sets, uniqueness of the minimizer follows similarly to the case of the original Gibbsian model. Nevertheless, the form of minimizer (2.46) is rather implicit; it can be interpreted as follows. All terms apart from the exponential arise the same way as the analogous terms in (2.35), which we explained in Section 2.3.3. Further, in the exponential there are two summands for fixed $l \in [k_{\max}]$ arising from inverse SIR in the high-density limit. Using the terminology of [BC12, Section II.A], the first one is the "selfish" part of the interference penalization, which is small whenever the SIR of the transmission from x_{l-1} to x_l is large, whereas the other one is the "altruistic" part, which is small whenever $\ell(|x_{l-1} - z_l|)$ is small compared to the signal strength $\ell(|z_{l-1} - z_l|)$ of the l-th hop of the trajectories of length at least l on average.

relaying may occur. This makes the notation of the congestion term and the entropy term more involved, but the effect of these terms is very similar to the original model. In particular, for $k_{\max} = 2$, both of these terms are unchanged, unlike the interference term. Since here we want to focus on the impact of the time dependent interference penalization, we stick to the case $\beta = 0$.

[10] Again, this is conventionally not called an interference but a total received power, cf. Section 2.3.1.1.

More realistic and mathematically much more demanding time dependent versions of our model can be set up in various ways; for example, one could allow for a much longer time horizon (e.g., of order λ, and then dropping the factor of λ in the interference term), which must come with the possibility of messages standing still for many separate time units. Furthermore, one could allow users to transmit multiple messages over time. One could also introduce mobility of users, similarly to [HJKP18] or even more generally. The new notion of SIR comes with significant changes in the behaviour of the system in the high-density limit, and we decided to defer such investigations to a later work.

For historical remarks on time dependent models for interference limited multihop networks in the context of capacity, see Section 5.1.

2.4.5. The annealed setting

Of mathematical interest might also be the annealed setting, where we average also over the locations of the users. In order to get an interesting result we have to assume that L_λ satisfies a large deviation principle on the set $\mathcal{M}(W)$ with some good rate function H. (In the case of a Poisson point process with intensity measure $\lambda\mu$, H would be [HJP18, Proposition 3.6] the relative entropy with respect to μ, see (2.20).) Then the large-λ exponential rate of the annealed partition function should be equal to the negative infimum over $\mu_0 \in \mathcal{M}(W)$ of $\mathrm{H}(\mu_0)$ plus the quenched rate function terms from the right-hand side of (2.24) with μ replaced by μ_0 everywhere. Also our other results on the LDP and the form of the minimizer(s) should have some analogue, which we do not spell out.

2.5. Why a Gibbsian ansatz?

Let us give some motivation for our choice of the probability of the form $\exp\{-(\gamma\mathfrak{S}(s) + \beta\mathfrak{M}(s))\}$ (modulo a counting factor) for a trajectory collection s in (2.10).

2.5.1. Penalization of interference

The interference term $\mathfrak{S}(s)$ in (2.13) quantifies the joint quality of the transmissions of the messages in terms of a sum of an individual interference term over all the $N(\lambda)$ trajectories and over all of their hops.

In a mathematical description of a telecommunication system, one typically requires [GK00, DBT05, FM07, HJKP16, HJKP18] that the SIR be larger than a given threshold $\tau > 0$, in order that the signal can be successfully transmitted. However, our model is designed to investigate the global throughput, where we do not want to consider any single message, but the total quality of transmission in the entire system, similarly to the setting of [FDTT07], see Section 5.1.2. This quantity $\mathfrak{S}(s)$ is the sum of all the reciprocal values of the SIRs of all the (hops of the) messages, which we explained in Section 2.3.1.2. It is exponentially weighted with a negative factor, which "softly" keeps all the SIRs at high values on average.

One can also modify our Gibbs distribution in such a way that trajectory families exhibiting hops with $\mathrm{SIR}(s^i_{l-1}, s^i_l, X^\lambda)$ less than or equal to τ have probability zero, simply by changing the interference penalization value (2.7) to ∞ for such families. This means a change from the penalization function $x \mapsto \gamma/x$ (applied to $\mathrm{SIR}(s^i_{l-1}, s^i_l, X^\lambda)$) into the function $x \mapsto \infty \times \mathbb{1}_{[0,\tau]}(x)$ (using the convention $\infty \times 0 = 0$). Combining the hard and the soft penalization is also possible, using the function $x \mapsto \infty \times \mathbb{1}_{[0,\tau]}(x) + \gamma/x$, similarly to [BC12, Section III.A]; see Section 2.7.2 for further details of the setting of [BC12]. For $\tau > 0$ small enough, almost surely, the modified model (with hard or combined hard-soft penalization) eventually becomes well-posed as λ tends to infinity, at least along an increasing, diverging sequence $(\lambda_n)_{n\in\mathbb{N}}$ of intensities. Further, taking a subsequence is not necessary if τ is sufficiently small. Indeed, in Section 5.2.3 we will argue that these follow from the results of [HJKP18] on probabilities of frustration events, combined with Borel–Cantelli type arguments. We expect that analogues of the results of this chapter are valid for these modified settings, but additional topological problems have to be addressed, see also Section 5.2.3.

One of our motivations is to explore the physical effect of the penalization of the joint probability of the random paths, which are *a priori* randomly picked with equal probability: Does the (soft) requirement of a good transmission quality force the trajectories already to choose geometrically the shortest route? What hop lengths do they choose? We would like to understand the interplay between entropy and interference-energy (and possibly congestion), and the result coming out of this by optimizing their relation.

2.5.2. Penalization of congestion

The congestion term $\mathfrak{M}(s)$ in (2.15) counts the ordered pairs of incoming hops arriving at the relays in the system. This is certainly an important characteristics of the quality of service, as too high an accumulation of many messages at relays results in a delay. Hence, it is natural to suppress the occurrence of such events, in order to increase the value of the model for realistic modelling.

An important property of this term is that it introduces dependence between the trajectories of different messages, unlike the interference term. Indeed, while $\mathfrak{S}(s)$ can be decomposed into a sum of terms depending on the respective trajectories, each summand in $\mathfrak{M}(s)$ involves many different trajectories. This is not only true in the special case of penalizing interference and congestion, but in general in our setting introduced in Section 2.1.3.

2.5.3. Relation to an optimization problem via Monte Carlo Markov chains

In the light of the above, it is certainly interesting to minimize the cost function $s \mapsto \gamma\mathfrak{S}(s) + \beta\mathfrak{M}(s)$ for fixed $\beta,\gamma \in (0,\infty)$. Computationally, this is in general a hard problem for high densities λ because the cardinality of $\mathcal{S}_{k_{\max}}(X^\lambda)$ increases superexponentially in $N(\lambda)$, and $N(\lambda)$ is of linear order in λ. Thus, computing all values of $s \mapsto \gamma\mathfrak{S}(s) + \beta\mathfrak{M}(s)$ and then extracting the maximum is only feasible for small λ.

Now, our Gibbsian trajectory distribution opens the possibility to optimize this cost function via the well-known approach of *simulated annealing*. Furthermore, for λ large, it is substantially less complex to realize the Gibbs distribution using Monte Carlo Markov chains than to directly minimize the cost function.

Indeed, our Gibbs distribution favours trajectory collections with small values of the cost function. Now, let us investigate the computational complexity of the numerical realization of the Gibbs distribution $\mathrm{P}^{\gamma,\beta}_{\lambda,X^\lambda}$, using Monte Carlo Markov chains (see e.g. [H02]). The recent master's thesis of Morgenstern [M18] investigates this question. Given the intensity λ and the realization of the point process of users X^λ, the author finds irreducible and aperiodic Markov chains on the state space $\mathcal{S}_{k_{\max}}(X^\lambda)$, both of Gibbs sampler and Metropolis types, having the Gibbs distribution as their stationary distribution, with respect to which they are reversible. These chains therefore converge towards $\mathrm{P}^{\gamma,\beta}_{\lambda,X^\lambda}$ as the number of Markovian steps tends to infinity.

The following results have been verified in [M18] in the special setting of Section 2.1.4:

- The chains can be constructed in such a way that computing their transition matrices takes only a polynomial number of operations in $N(\lambda)$.
- For both types of chains, in the limit $N(\lambda) \asymp \lambda \to \infty$, the mixing time is at most exponential in λ. This, together with the previous observation, provides an at most exponential upper bound on the number of operations needed in order to simulate the Gibbs distribution up to a given error $\varepsilon > 0$ in total variation distance. This is certainly much more efficient than evaluating all the trajectory collections.
- In a variant of the Gibbsian model where any user (relay) can receive at most a fixed number $m_{\max} \in \mathbb{N}$ of incoming hops, the mixing time is even polynomial in λ. This is a realistic modelling since in most applications in multihop networks, each relay has a bounded capacity for receiving incoming hops (see e.g. [HJP18, HJ17]). Note that this variant is always well-defined if $m_{\max} \geq k_{\max} - 1$. The proofs of the results of Chapter 2 indicate that for large $m_{\max}$, this variant behaves similarly to the original model in the high-density limit.
- These Monte Carlo Markov chains can also be used in order to find the optimum of the cost function $s \mapsto \gamma\mathfrak{S}(s) + \beta\mathfrak{M}(s)$ for a fixed λ and a fixed realization of X^λ, using simulated annealing. Here, one lets the transition probability of the t-th step of the chain depend on t via replacing (γ,β) with (γ_t,β_t) such that $\gamma_t,\beta_t \to \infty$ sufficiently slowly as $t \to \infty$. [M18, Theorem 7.1] shows that if one chooses $\beta_t = \frac{\beta}{\gamma}\gamma_t \leq c_0 \log t$ for a suitably chosen $c_0 = c_0(\lambda) \asymp \lambda/N(\lambda)^2$, then the Markov chain converges to the uniform distribution on the set of minimizers of the cost function.

2.5.4. Interpretation of the Gibbsian model in terms of combinatorial optimization

The observations of [CLSK13] on general combinatorial network optimization problems provide a more explicit connection between the Gibbs distribution (2.10) and the optimization of the function $\gamma\mathfrak{S}+\beta\mathfrak{M}$ over $\mathcal{S}_{k_{\max}}(X^\lambda)$ (in the general context of Section 2.1.3). Indeed, according to [CLSK13, Section 2], for $\beta,\gamma \geq 0$ with $\max\{\beta,\gamma\} > 0$, the optimum value

$$-\min_{s\in\mathcal{S}_{k_{\max}}(X^\lambda)} \Big(\gamma\mathfrak{S}(s)+\beta\mathfrak{M}(s)\Big)$$

equals λ times the optimum value

$$-\min\Big\{\sum_{s\in\mathcal{S}_{k_{\max}}(X^\lambda)} p_s(\frac{\gamma}{\lambda}\mathfrak{S}(s)+\frac{\beta}{\lambda}\mathfrak{M}(s))\colon p_s\geq 0,\ \forall s\in\mathcal{S}_{k_{\max}}(X^\lambda),\ \sum_{s\in\mathcal{S}_{k_{\max}}(X^\lambda)} p_s=1\Big\}. \tag{2.47}$$

On the other hand, if we consider a Gibbs distribution with weights

$$w_\lambda^{\gamma,\beta}(s)=\frac{\exp(-\gamma\mathfrak{S}(s)-\beta\mathfrak{M}(s))}{\sum_{r\in\mathcal{S}_{k_{\max}}(X^\lambda)}\exp(-\gamma\mathfrak{S}(r)-\beta\mathfrak{M}(r))},\qquad s\in\mathcal{S}_{k_{\max}}(X^\lambda), \tag{2.48}$$

then its (non-limiting) free energy

$$\frac{1}{\lambda}\log\sum_{s\in\mathcal{S}_{k_{\max}}(X^\lambda)} \mathrm{e}^{-\gamma\mathfrak{S}(s)+\beta\mathfrak{M}(s))}=\frac{1}{\lambda}\log\sum_{s\in\mathcal{S}_{k_{\max}}(X^\lambda)}\exp^{-\lambda(\frac{\gamma}{\lambda}\mathfrak{S}(s)+\frac{\beta}{\lambda}\mathfrak{M}(s))}$$

equals the optimum value

$$-\min\Big\{\sum_{s\in\mathcal{S}_{k_{\max}}(X^\lambda)} p_s(\frac{\gamma}{\lambda}\mathfrak{S}(s)+\frac{\beta}{\lambda}\mathfrak{M}(s))+\frac{1}{\lambda}\sum_{s\in\mathcal{S}_{k_{\max}}(X^\lambda)} p_s\log p_s\colon p_s\geq 0,\ \forall s\in\mathcal{S}_{k_{\max}}(X^\lambda),\ \sum_{s\in\mathcal{S}_{k_{\max}}(X^\lambda)} p_s=1\Big\}. \tag{2.49}$$

That is, the free energy solves the minimization problem in (2.47), off by an *entropy term* $\frac{1}{\lambda}\log\sum_s p_s\log p_s$. The approximation gap is upper bounded by $\frac{1}{\lambda}\log\#\mathcal{S}_{k_{\max}}(X^\lambda)$. For almost all realizations of $(X^\lambda)_{\lambda>0}$, this gap tends to infinity logarithmically as $\lambda\to\infty$, cf. Section 2.9.1. Further, the minimizing $(p_s)_{s\in\mathcal{S}_{k_{\max}}(X^\lambda)}$ in (2.49) equals the collection of Gibbsian weights $(w_\lambda^{\gamma,\beta}(s))_{s\in\mathcal{S}_{k_{\max}}(X^\lambda)}$.

The difference of the Gibbs distribution in (2.48) and the one in (2.10) (with the special choices of $\mathfrak{S}$ and $\mathfrak{M}$ according to Section 2.1.4) that we use throughout this chapter is that the latter one normalizes each s by its *a priori* weight $N(\lambda)^{-\sum_{i\in I^\lambda}(s^i_{-1}-1)}$. Mathematically speaking, this is necessary in order to keep the logarithmic rate of combinatorial terms on the same scale λ as the interference term and the congestion term (instead of the scale $\lambda\log\lambda$, cf. Section 2.9.1). The interpretation of this normalization is that *a priori*, each user chooses its number of hops uniformly in $[k_{\max}]$, and given this number, the user chooses its trajectory uniformly from the set of all k-hop trajectories, independently of the other users' choices. Without this normalization, larger numbers of hops would be strongly preferred by the Gibbs distribution.

Note that for trajectory configurations s where there exists $k\in[k_{\max}]$ such that each user chooses a k-hop trajectory, the *a priori* normalization of s equals $N(\lambda)^{-(k-1)N(\lambda)}=\exp(-(k-1)N(\lambda)\log N(\lambda))$. Thus, if we considered only trajectory configurations where each user takes the same number k of hops, for some $k\in\mathbb{N}$ that is also fixed, then the resulting Gibbs distribution would be of the form (2.48) but with an additional combinatorial term $-(k-1)N(\lambda)\log N(\lambda)$ in the exponents and with the partition function (normalizing constant of the Gibbs distribution) accordingly defined. Thus, the above observations about the Gibbs distribution without the *a priori* normalization would also apply in this setting, modulo this additional term at all corresponding places. (The arguments of [CLSK13] would also provide an optimization gap of at most $\frac{1}{\lambda}\log\#\mathcal{S}_{k_{\max}}(X^\lambda)$ in this case.)

2.6. Game-theoretic interpretation of the optimization problem

In Section 2.5.3 we explained how our model that we introduced in Section 2.1 can be employed for obtaining a stochastic simulation algorithm for finding minimizer(s) s of the cost function $\gamma\mathfrak{S}(s)+\beta\mathfrak{M}(s)$ in the special case of Section 2.1.4. In this section we give a more thorough discussion of this optimization problem from a game-theoretic point of view. In particular, we explain in which sense the optimum in our model is selfish or non-selfish and give a number of explicit examples for illustration. Note that in the term $\mathfrak{S}(s)$ there is no interaction between the trajectories (only with the users), but in the term $\mathfrak{M}(s)$. We therefore keep both $\beta > 0$ and $\gamma > 0$ fixed.

Let $X^\lambda = \{X_1,\dots,X_n\}$ be fixed, where $n \in \mathbb{N}$. For the rest of this section, we simplify the notation as follows. We write $\mathcal{S} = \mathcal{S}_{k_{\max}}(X^\lambda)$ and for $i \in [n]$, $\mathcal{S}^i = \mathcal{S}^i_{k_{\max}}(X^\lambda)$. Let now $s = (s^i)_{i=1}^n \in \mathcal{S}$ be a collection of message trajectories. We recall that for $i \in [n]$, s^i_{-1} is the number of hops taken by the i-th trajectory s^i sent out from X_i to o. Then, in terms of interference and congestion, the *individual cost* $C_i(s)$ of s^i with respect to the entire collection s is the individual interference penalization of s^i, together with the congestion penalization at all the relays that s^i uses:

$$C_i(s) = \gamma \sum_{l=1}^{s^i_{-1}} \mathrm{SIR}(s^i_{l-1},s^i_l,X^\lambda)^{-1} + \beta \sum_{l=1}^{s^i_{-1}-1} \sum_{j=1}^{n} (m_j(s)-1)\mathbb{1}\{s^i_l = X_j\}. \tag{2.50}$$

The *total cost* of the trajectory collection s is defined as

$$C(s) = \sum_{i=1}^n C_i(s) = \gamma\mathfrak{S}(s) + \beta\mathfrak{M}(s) = \gamma \sum_{i=1}^n \sum_{l=1}^{s^i_{-1}} \mathrm{SIR}(s^i_{l-1},s^i_l,X^\lambda)^{-1} + \beta \sum_{j=1}^n m_j(s)(m_j(s)-1).$$

We say that s is *system-optimal* if $C(s) \leq C(s')$ for all $s' \in \mathcal{S}$.

For a collection $s = (s^i)_{i=1}^n$ of trajectories we write $s = s_i(s^i,s^{-i})$, where $s^{-i} = (s^j)_{j\in[n],j\neq i}$. Now, given i and $s^{-i} = (s^j)_{j\neq i}$ with $s^j \in \mathcal{S}^j$ for all $j \neq i$, a *best response* of the i-th user for s^{-i} is $u^i \in \mathcal{S}^i$ such that $C_i(s_i(u^i,s^{-i})) \leq C_i(s_i(s^i,s^{-i})) = C_i(s)$ for all $s^i \in \mathcal{S}^i$. We say [NRTV07, Section 1.3.3] that $s = (s^i)_{i=1}^n$ is a *pure Nash equilibrium* if s^i is a best response for $s^{-i} = (s^j)_{j\neq i}$ for all $i \in [n]$.

Claim 2.6.1. *For $\beta,\gamma,\lambda > 0$, given X^λ (with $n > 0$), a pure Nash equilibrium always exists.*

Proof. The claim follows from the well-known result [NRTV07, Theorem 18.12] that unweighted atomic congestion games always have a pure Nash equilibrium. Indeed, the cost functions C_i, $i \in [n]$, and C are the individual respectively total costs in an unweighted atomic congestion game (atomic instance) [NRTV07, Section 18], which is defined as follows. For each $i \in [n]$, the set of all possible paths $s^i \in \mathcal{S}^i$ of length at most $k_{\max}$ from X_i to o via users in $X_j \in X^\lambda$ without visiting the same X_j twice can be seen as the set of the strategies of the i-th user (player) X_i. Indeed, for the sake of optimization of individual and total costs, we can neglect trajectories with loops since removing any loop from the trajectory of the i-th user strictly decreases C_i and does not increase C_j for $j \neq i$, neither C. Since almost surely the points of X^λ are pairwise distinct, the sets of possible strategies of different users are disjoint, in other words, the game is unweighted. Further, each user has a finite number of strategies.

The cost function in this game is defined as follows. Each hop from X_i to X_j has a constant cost equal to $\gamma\mathrm{SIR}(X_i,X_j,X^\lambda)^{-1}$, and each used relay X_j has a linear cost equal to $\beta(m_j(s)-1)$, depending on the trajectory collection s. This way, by (2.50), the cost of the strategy of X_i corresponding to $s \in \mathcal{S}$ equals $C_i(s)$. Thus, the claim follows. □

Now, if there exists a system-optimal $s \in \mathcal{S}$ such that $C(s) < C(s')$ for all Nash equilibria s', then we call s a *non-selfish optimum*, since there exists $i \in [n]$ such that s^i is not the best response of the i-th user for the remaining coordinates of the trajectory collection. Example 2.6.2 shows a two-dimensional example that has a non-selfish optimum, Remark 2.6.3 discusses the question of uniqueness of the Nash equilibrium, and Remark 2.6.4 tells more about the relation of the individual and the total costs.

Example 2.6.2. Let $d = 2$, $\lambda = 1$ and $k_{\max} = 2$, and let $X^\lambda = X^1 = \{X_1, X_2, X_3\}$, ℓ and $\gamma > 0$ be chosen in the following way. X_1, X_2, X_3 and o, X_2, X_3 are vertices of two equilateral triangles with X_1 being in the interior of the latter triangle, so that $|X_1 - X_2| = |X_1 - X_3| = |X_1|$ and $|X_2| = |X_3|$, so that $\gamma\mathrm{SIR}(X_1,o,X^1)^{-1} = \gamma\mathrm{SIR}(X_i,X_1,X^1)^{-1} = 1$ and $\gamma\mathrm{SIR}(X_i,o,X^1)^{-1} = 1 + q$ for all $i \in \{2,3\}$ for some $q > 0$ (see Figure 1).

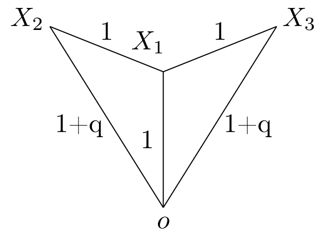


Figure 1. Interference weights per hop in Example 2.6.2. In the relevant cases, the congestion at X_1 is $\beta y(y-1)$, where y is the number of elements of $\{X_2,X_3\}$ relaying through X_1.

Number of hops of s^2	Number of hops of s^3	$C_2(s)$	$C_3(s)$	$C(s)$
1	1	$2+q$	$2+q$	$5+2q$
2	1	2	$2+q$	$5+q$
2	2	$2+\beta$	$2+\beta$	$5+2\beta$

Table 2.1. Individual and total costs in standard representatives of the relevant cases in Example 2.6.2.

The boundedness of $\ell(|\cdot-\cdot|)$ away from 0 on $W\times W$ implies that for any $\beta>0$ and $i\in\{2,3\}$, any $s^i\in\mathcal{S}^i$ that uses some X_j with $j\in\{2,3\}$ as a relay is suboptimal both with respect to total and individual costs. Indeed, leaving out this relay and moving on to the next hop of the same trajectory instead decreases $C_i(s)$ without increasing any $C_m(s)$, $m\neq i$. Using analogous arguments, one easily concludes that in any system-optimal trajectory and also in any Nash equilibrium, X_1 submits directly to o, and the two users X_2,X_3 use either the direct link to o or the two-hop path via X_1 to o. Table 2.1 shows the individual costs and the total cost in some standard representatives of these cases.

The positive parameters q and β can be chosen such that the following holds. Given that X_2 uses its two-hop path $X_2\to X_1\to o$, the best response of X_3 is to also use its two-hop path $X_3\to X_1\to o$ and vice versa, so that both users using their two-hop paths forms the unique Nash equilibrium, but the system optima are the trajectory collections in which only one of them relays via X_1 and the other one submits directly to o. According to Table 2.1, this holds if $q>0$ and $\beta\in(q/2,q)$. Thus, in such cases, the optimum is non-selfish.

Similar effects occur in all dimensions $d\geq 2$, with $d+1$ users $X_1,X_2,\dots,X_{d+1}$ situated so that $|X_j-X_1|=|X_i-X_1|$ and $|X_1|<|X_i|=|X_j|$ for all $i,j\geq 2$. In such cases, one can choose the parameters in such a way that for all $j\geq 2$, knowing that X_1 transmits directly to o and each X_i, $j\neq i\geq 2$ relays through X_1, the best response of X_j is to use also the relayed link via X_1, but with respect to total costs it would be better if X_j transmitted directly to o. Note that if this holds, it may still happen that neither of these two joint strategies is system-optimal. ◇

Remark 2.6.3. In the setting of our Gibbsian model, Nash equilibria are not necessarily unique. Consider Example 2.6.2 in the boundary case $\beta=q$. Then one easily checks that the system exhibits three different Nash equilibria, namely: the one in the second line of Table 2.1, the same with the roles of s^2 and s^3 interchanged, and the one in the last line of the table. Also for $\beta>q$, there are two Nash equilibria, namely the ones where exactly one of s^2,s^3 uses the direct hop to o and the other one relays via X_1, by the symmetry between X_2 and X_3.

Remark 2.6.4. Now we show that in general, if plugging in an additional relay to a trajectory decreases the total cost, then it also decreases the individual cost of the transmitter of that trajectory. Thus, in particular, a situation opposite to Example 2.6.2 is not possible, that is, for any choice of ℓ,β,γ, it cannot happen that $C(s)$ is a non-selfish system optimum while $C_2(s)$ or $C_3(s)$ is a Nash equilibrium.

Indeed, consider Figure 2 with $\lambda>0$, $X_i,X_h\in X^\lambda$, and $x\in X^\lambda\cup\{o\}$, where the direct hop from X_i to x has interference penalization $p_0>0$, while the two-hop path via X_h has interference penalization p_1+p_2 with $p_1,p_2>0$. Now, if $s^{-i}=(s^j)_{j\neq i}$ is given and the number of incoming hops at X_h coming from all trajectories but the one of X_i equals $m\geq 0$, then the direct link from X_i to x has individual cost p_0+K and the $X_i\to X_h\to o$

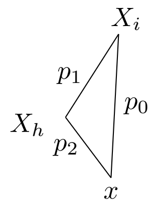


Figure 2. A situation opposite to Example 2.6.2 is not possible.

relayed link has individual cost $m + p_1 + p_2 + K$, for some $K \geq 0$. On the other hand, the total cost of the collection with the $X_i \to x$ direct link is $2m + p_1 + p_2 + K'$ and the one with the $X_i \to X_h \to x$ relayed link is $p_0 + K'$, for some $K' \geq 0$. Thus, if plugging in the relay X_h increases the individual cost C_i, then it also increases the total cost C. This implies the claim.

2.7. Related work on telecommunication networks

For historical remarks on the notion of SI(N)R and the mathematical research on interference limited networks, see Section 5.1.1.

2.7.1. Large deviation principles for highly dense relay-augmented networks

Let us remind the reader on some recent work on the quality of service in highly dense relay-augmented ad-hoc networks. A multihop network with users distributed according to a Poisson point process with diverging intensity was investigated in [HJKP18]. Using large deviations methods, that paper derived the asymptotic behaviour of rare frustration events such as many users having an unlikely bad quality of service for an unusually long period of time. The author's master's thesis [T16] extended the results of [HJKP18] to the case when users have random fadings (interpreted as loudness values of the users). See Section 5.2.1 for further details of these two works. [HJP18] also describes frustration probabilities in a network, where relays have a bounded capacity and users become frustrated when their connection to a relay is refused because it is already occupied; see also [HJ17].

One difference between these works and the Gibbsian model of the present thesis is that the latter one uses a notion of quality of service for the entire system rather than for single transmissions. In particular, trajectories with bad SIR are *a priori* not excluded. There is a random mechanism for choosing the message trajectories of all users, given the user locations, and our results hold almost surely with respect to the point process of user locations in the high-density limit. For these results, users need not form a Poisson point process and can even be located deterministically, see Section 2.4.1. This is also a difference from [HJKP18, HJP18, HJ17], where user locations are not fixed and their randomness is (at least partially) responsible for unlikely frustration events.

2.7.2. Markov chain Monte Carlo for telecommunication networks

Gibbs sampling was used for various aspects of modelling telecommunication networks, e.g., in [CBK17] for optimal placement of contents in a cellular network, and in [BC12] for power control and associating users to base stations. These Monte Carlo Markov chain methods are used to decrease some kind of cost in the system via a random mechanism, with no easily implementable deterministic methods being available. Our Gibbsian model also has this property if both interference and congestion are penalized. As we explained in Section 2.5.3, the recent master's thesis of Morgenstern [M18] investigated the use of a Gibbs sampler or a Metropolis algorithm for an experimental realization of our Gibbsian system. Further details of using Gibbs distributions and Monte Carlo Markov chains for general combinatorial optimization problems in networks can be found in [CLSK13].

2.8. The distribution of the empirical measures

Having seen in Section 2.2.3 that the Gibbsian model can be entirely described in terms of the family $\Psi_\lambda(s)$, i.e., of the crucial empirical measures $R_{\lambda,k}(s)$ and $P_{\lambda,m}(s)$ defined in (2.5) respectively (2.17), we now consider the question how to describe their distributions. We have to quantify the number of message trajectory families s that give the same family of empirical measures. The plain and short (but wrong) answer is

$$\sum_{s\in\mathcal{S}_{k_{\max}}(X^\lambda)\colon R_{\lambda,k}(s)=\nu_k\,\forall k,\ P_{\lambda,m}(s)=\mu_m\,\forall m}\ \prod_{i\in I^\lambda}\frac{1}{N(\lambda)^{s^i_{-1}-1}}\approx \mathrm{e}^{-\lambda \mathrm{I}(\Psi)},\tag{2.51}$$

where we recall $\mathrm{I}(\Psi)$ from (2.23) and recall that $\Psi=((\nu_k)_{k\in[k_{\max}]},(\mu_m)_{m\in\mathbb{N}_0})$. From such an assertion, it is indeed not far to conclude Theorem 2.2.2, but the problem is that this statement is not true like this. Actually, there are very many Ψ's such that the left-hand side is equal to zero, for example if any of the ν_k's or μ_m's has values outside $\frac{1}{\lambda}\mathbb{N}_0$. However, if we do not consider single Ψ's, but open sets of Ψ's, then the idea behind (2.51) is sustainable. Therefore, we proceed in a standard way by decomposing the area W into finitely many subsets and count the message trajectories only according to the discretization sets that they visit. In Section 2.8.1 we introduce necessary notation for carrying out this strategy and we comment on the relevance of the discretization procedure. Next, in Section 2.8.2 we derive explicit formulas for the distribution of the empirical measures in this discretization.

For the purpose of the present chapter, where we consider the high-density limit $\lambda\to\infty$, we later need to take this limit and afterwards the limit as the fineness parameter δ of the decomposition of W goes to zero. The outcome of these parts of the procedure is formulated in Proposition 2.9.1. In Proposition 2.9.2 the consequences for the interference term and for the congestion term are formulated.

2.8.1. Our discretization procedure

Let us now head towards the formulation of the discretization procedure. We proceed by triadic spatial discretization of the Poisson point process $(X^\lambda)_{\lambda>0}$, similarly to the approach of [HJKP18]. See Section 5.2.4 for a discussion about the essence of this discretization and the main differences between its version in [HJKP18] and the one in the present chapter. To be more precise, we perform the following discretization argument. Note that we may assume that our communication area W is taken as $W=[-r,r]^d$, by accordingly extending μ trivially. We write $\mathbb{B}=\{3^{-n}\colon n\in\mathbb{N}_0\}$. For $\delta\in\mathbb{B}$, we define the set

$$W_\delta=\{[x-r\delta,x+r\delta]^d\colon x\in(2r\delta\mathbb{Z})^d\cap W\}$$

of congruent subcubes of W of side length $2r\delta$ and centres in $(2r\delta\mathbb{Z})^d$. Note that W_δ is a finite set, o is a centre of an element of W_δ, and any intersection of two distinct elements of W_δ has zero Lebesgue measure. Elements of W_δ will be called *δ-subcubes.* We will assume that for all $\delta\in\mathbb{B}$, the δ-subcubes are canonically numbered as $W^\delta_1,\dots,W^\delta_{\delta^{-d}}$, which can be done e.g. according to the increasing lexicographic order of the midpoints of the subcubes. Now, for Lebesgue-almost every $x\in W$, for all $\delta\in\mathbb{B}$ there exists a unique W^δ_j that contains x, let $W_{\mathbb{B}}$ be the set of all $x\in W$ satisfying this property.

Now, if $\nu\in\mathcal{M}(W)$, then for any $\delta\in\mathbb{B}$, we define $\nu^\delta(\cdot)=\nu(\cdot\mid\mathcal{F}_\delta)\in\mathcal{M}(W)$ as the conditional version of ν given $\mathcal{F}_\delta=\sigma(W_\delta)$, that is, the measure on W that has in each box W^δ_i a constant Lebesgue density and mass equal to $\nu(W^\delta_i)$. Since $\mathcal{F}_\delta\subset\mathcal{F}_{\delta'}$ for $\delta,\delta'\in\mathbb{B}$ with $\delta'<\delta$, we see that $(\nu^\delta)^{\delta'}=(\nu^{\delta'})^\delta=\nu^\delta$ by the tower property. We also write $L^\delta_\lambda:=(L_\lambda)^\delta$ for $\lambda>0$ and $\delta\in\mathbb{B}$, where the empirical measure L_λ was defined in (2.1). We proceed analogously for W^k, $k\in[k_{\max}]$ instead of W. Note that $\nu^\delta\Longrightarrow\nu$ as $\delta\downarrow 0$, which can be shown by a martingale convergence argument, since the union of all the $\mathcal{F}_\delta$ generates the Borel-σ-field on W.

Now we are able to define what a *standard setting* is, the interpretation of which will be given right after the definition. Roughly speaking, the measures ν_k and μ_m appearing in its definition will later play the role of the measures appearing in (2.51), their δ-approximations are defined as above, and their (δ,λ)-versions approach them in the limit $\lambda\to\infty$, followed by $\delta\downarrow 0$. The latter ones satisfy the constraints of (2.18) restricted to $\mathcal{F}_\delta^{\otimes k}$ respectively $\mathcal{F}_\delta$, hence, the collection of the measures ν_k and μ_m will later turn out to be eligible for the variational problem in (2.24) (under some mild additional assumption, see Lemma 2.8.5).

Definition 2.8.1. *A* standard setting *is a collection of measures*

$$\underline{\Psi} = \Big((\nu_k)_{k=1}^{k_{\max}}, ((\nu_k^\delta)_{k=1}^{k_{\max}})_{\delta\in\mathbb{B}}, ((\nu_k^{\delta,\lambda})_{k=1}^{k_{\max}})_{\delta\in\mathbb{B},\lambda>0}, \\ (\mu_m)_{m=0}^\infty, ((\mu_m^\delta)_{m=0}^\infty)_{\delta\in\mathbb{B}}, ((\mu_m^{\delta,\lambda})_{m=0}^\infty)_{\delta\in\mathbb{B},\lambda>0}, (\mu^{\delta,\lambda})_{\delta\in\mathbb{B},\lambda>0} \Big) \tag{2.52}$$

with the following properties: for any $\delta,\delta' \in \mathbb{B}$, $\lambda > 0$, $k \in [k_{\max}]$, $m \in \mathbb{N}_0$, *and* $i,i_0,\dots,i_{k-1} = 1,\dots,\delta^{-d}$, *respectively,* $\nu_k \in \mathcal{M}(W^k)$ *and* $\mu_m \in \mathcal{M}(W)$, *and*

1. $\mu^{\delta,\lambda} = L_\lambda^\delta$.
2. $\nu_k^{\delta,\lambda} \in \mathcal{M}(W^k)$. *Further,* $\sum_{k=1}^{k_{\max}} \pi_0 \nu_k^{\delta,\lambda} = \mu^{\delta,\lambda}$, *moreover,* $\lambda\nu_k^{\delta,\lambda}(W_{i_0}^\delta \times \dots \times W_{i_{k-1}}^\delta) \in \mathbb{N}_0$.
3. *If* $\delta' \le \delta$, *then* $\nu_k^{\delta',\lambda}(\cdot \mid \mathcal{F}_\delta^{\otimes k}) = \nu_k^{\delta,\lambda}(\cdot)$.
4. $\nu_k^{\delta,\lambda} \overset{\lambda\to\infty}{\Longrightarrow} \nu_k^\delta$.
5. $\mu_m^{\delta,\lambda} \in \mathcal{M}(W)$ *with the property that* $\sum_{m=0}^\infty \mu_m^{\delta,\lambda} = \mu^{\delta,\lambda}$, *moreover,* $\lambda\mu_m^{\delta,\lambda}(W_i^\delta) \in \mathbb{N}_0$.
6. $\sum_{m=0}^\infty m\mu_m^{\delta,\lambda} = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l \nu_k^{\delta,\lambda}$.
7. *If* $\delta' \le \delta$, *then* $\mu_m^{\delta',\lambda}(\cdot \mid \mathcal{F}_\delta) = \mu_m^{\delta,\lambda}(\cdot)$.
8. $\mu_m^{\delta,\lambda} \overset{\lambda\to\infty}{\Longrightarrow} \mu_m^\delta$.

Remark 2.8.2. Immediate properties of a standard setting Ψ are the following.

(A) If $\delta' \le \delta$, then $\mu^{\delta',\lambda}(\cdot \mid \mathcal{F}_\delta) = \mu^{\delta,\lambda}(\cdot)$.

(B) $\mu^{\delta,\lambda} \overset{\lambda\to\infty}{\Longrightarrow} \mu^\delta$ since $L_\lambda \Longrightarrow \mu$ as $\lambda \to \infty$.

(C) $\mu^\delta(\cdot) = \mu(\cdot \mid \mathcal{F}_\delta)$. In particular, $\mu^\delta \overset{\delta\downarrow 0}{\Longrightarrow} \mu$.

(D) $\nu_k^\delta(\cdot) = \nu_k(\cdot \mid \mathcal{F}_\delta^{\otimes k})$. In particular, $\nu_k^\delta \overset{\delta\downarrow 0}{\Longrightarrow} \nu_k$.

(E) $\mu_m^\delta(\cdot) = \mu_m(\cdot \mid \mathcal{F}_\delta)$. In particular, $\mu_m^\delta \overset{\delta\downarrow 0}{\Longrightarrow} \mu_m$.

Remark 2.8.3 (Standard settings and message trajectories)**.** The properties of Remark 2.8.2 explain the meaning of the δ-indexed coordinates of a standard setting. Let us now interpret how one can obtain the (δ,λ)-dependent coordinates of a standard setting starting from a fixed trajectory collection $s \in \mathcal{S}_{k_{\max}}(X^\lambda)$. Let

$$P_\lambda(s) = \frac{1}{\lambda} \sum_{i\in I^\lambda} \delta_{s_0^i} \tag{2.53}$$

denote the empirical measure of the starting sites of the trajectories, then $P_\lambda(s) = L_\lambda$, by our assumption that each user is picked precisely once in such a configuration. Hence, its δ-discretized version $P_\lambda^\delta(s)$ equals $\mu^{\delta,\lambda}$.[11] Let us now choose $\nu_k^{\delta,\lambda} = R_{\lambda,k}^\delta(s)$ (recall (2.5)). Then the requirement $\sum_{k=1}^{k_{\max}} \pi_0\nu_k^{\delta,\lambda} = \mu^{\delta,\lambda} = L_\lambda^\delta$ in (2) holds by (2.18). Further, let us choose $\mu_m^{\delta,\lambda} = P_{\lambda,m}^\delta(s)$ (recall (2.17)). Then the constraints $\sum_{m=0}^\infty \mu_m^{\delta,\lambda} = \mu^{\delta,\lambda}$ in (5) and $\sum_{m=0}^\infty m\mu_m^{\delta,\lambda} = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l\nu_k^{\delta,\lambda}$ in (6) also hold by (2.18). Note that all the other requirements of Definition 2.8.1 are satisfied because of the tower property respectively because of the convergence of L_λ towards μ.

In the proof of Theorem 2.2.2, it will be essential to verify that, for certain standard settings, $\sum_{m=0}^\infty m\mu_m^\delta(W)$ converges to $\sum_{m=0}^\infty m\mu_m(W)$ as $\delta \downarrow 0$, which is not implied by Definition 2.8.1. However, similarly to the de la Vallée Poussin theorem about uniform integrability, the superlinear increase of $m \mapsto \eta(m)$ yields a handy criterion for this, which will imply that $\Psi = ((\nu_k)_{k=1}^{k_{\max}}, (\mu_m)_{m=0}^\infty)$ is an admissible trajectory setting.

Definition 2.8.4. *A* controlled standard setting *is a standard setting* $\underline{\Psi}$ *as in* (2.52) *with the following extra property:*

$$\lim_{\lambda\to\infty} \sum_{m=0}^\infty \eta(m)\mu_m^{\delta,\lambda}(W) = \sum_{m=0}^\infty \eta(m)\mu_m^\delta(W) < \infty, \quad \textit{for all } \delta \in \mathbb{B}. \tag{2.54}$$

[11] With a slight abuse of notation, we write $P_\lambda^\delta(s)$ instead of $P_\lambda(s)^\delta$, and similarly we write $R_{\lambda,k}^\delta(s)$ and $P_{\lambda,m}^\delta(s)$.

Note that by part (D) of Remark 2.8.2, we have $\sum_{k=1}^{k_{\max}} k\nu_k^\delta(W^k) = \sum_{k=1}^{k_{\max}} k\nu_k(W^k)$ for any standard setting. Using this, we verify the following lemma.

Lemma 2.8.5. *Let $\underline{\Psi}$ be a controlled standard setting as in (2.52). Then $\Psi = ((\nu_k)_{k=1}^{k_{\max}}, (\mu_m)_{m=0}^\infty)$ is an admissible trajectory setting.*

Proof. Part (2) of Definition 2.8.1 claims that for all $\delta \in \mathbb{B}$ and $\lambda > 0$ we have $\sum_{k=1}^{k_{\max}} \pi_0 \nu_k^{\delta,\lambda} = \mu^{\delta,\lambda}$. By parts (B) and (C) of Remark 2.8.2, we have $\lim_{\delta\downarrow 0}\lim_{\lambda\to\infty} \nu_k^{\delta,\lambda} = \nu_k$ in the weak topology of $\mathcal{M}(W^k)$, for any fixed $k \in [k_{\max}]$. Similarly, by part (4) of Definition 2.8.1 and part (D) of Remark 2.8.2, we have $\lim_{\delta\downarrow 0}\lim_{\lambda\to\infty} \mu^{\delta,\lambda} = \mu$ in the weak topology of $\mathcal{M}(W)$. Moreover, since taking marginals is a continuous operation, also $\lim_{\delta\downarrow 0}\lim_{\lambda\to\infty} \pi_0\nu_k^{\delta,\lambda} = \pi_0\nu_k$ for all k in the weak topology of $\mathcal{M}(W)$. Thus, we have (i) in (2.19) for $(\nu_k)_{k=1}^{k_{\max}}$. In order to see that (ii) holds for $(\mu_m)_{m=0}^\infty$, one can additionally use part (5) of Definition 2.8.1, together with (2.54) and dominated convergence. Finally, by part (6) of Definition 2.8.1, (2.54), the fact that $\lim_{m\to\infty} \eta(m)/m = \infty$, and dominated convergence, we see that for any controlled setting $\underline{\Psi}$, we also have

$$\sum_{m=0}^\infty m\mu_m = \lim_{\delta\downarrow 0}\sum_{m=0}^\infty m\mu_m^\delta = \lim_{\delta\downarrow 0}\lim_{\lambda\to\infty}\sum_{m=0}^\infty m\mu_m^{\delta,\lambda} = \lim_{\delta\downarrow 0}\lim_{\lambda\to\infty}\sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\nu_k^{\delta,\lambda} = \sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\nu_k \tag{2.55}$$

in the weak topology of $\mathcal{M}(W)$. This implies (iii) in (2.19) for Ψ. Hence, Ψ is an admissible trajectory setting. $\square$

A converse of Lemma 2.8.5 also holds, in the following sense: for any admissible trajectory setting $\Psi = ((\nu_k)_{k=1}^{k_{\max}}, (\mu_m)_{m=0}^\infty)$, there exists a standard setting containing it, which can be chosen controlled if $\sum_{m\in\mathbb{N}_0} \eta(m)\mu_m(W) < \infty$. This will be the content of Proposition 2.9.3, a preliminary result for Theorems 2.2.2 and 2.2.4(i).

2.8.2. The distribution of the empirical measures

In this section we describe the combinatorics of the system. For a standard setting $\underline{\Psi}$ as in Definition 2.8.1, let us introduce the configuration set

$$J^{\delta,\lambda}(\underline{\Psi}) = \Big\{ s \in \mathcal{S}_{k_{\max}}(X^\lambda) \;\Big|\; R^\delta_{\lambda,k}(s) = \nu_k^{\delta,\lambda}\ \forall k,\quad P^\delta_{\lambda,m}(s) = \mu_m^{\delta,\lambda}\ \forall m \Big\} \tag{2.56}$$

for fixed $\delta \in \mathbb{B}$ and $\lambda > 0$. In words, $J^{\delta,\lambda}(\underline{\Psi})$ is the set of families of trajectories such that the δ-coarsenings of the empirical measures of the trajectories and the hop numbers are given by the respective measures in the standard setting $\underline{\Psi}$. Note that $J^{\delta,\lambda}(\underline{\Psi})$ depends only on the δ-λ depending measures in the collection $\underline{\Psi}$.

In case $\mu^{\delta,\lambda}(W) > 0$, we will refer to the entity s_0^i, $i = 1,\dots,\lambda\mu^{\delta,\lambda}(W)$, as the i-th user or i-th transmitter, the entity s^i as the trajectory of the i-th user, s^i_{-1} as the length (number of hops) of s^i, s^i_l as the l-th relay of s^i (for $l = 1,\dots,s^i_{-1}-1$), finally $m_i(s)$ as the number of incoming hops at the relay s_0^i.

The combinatorics of computing $\#J^{\delta,\lambda}(\underline{\Psi})$ is given as follows.

Lemma 2.8.6 (Cardinality of $J^{\delta,\lambda}(\underline{\Psi})$). *For any $\delta \in \mathbb{B}$ and $\lambda > 0$, and for any standard setting $\underline{\Psi}$,*

$$\#J^{\delta,\lambda}(\underline{\Psi}) = N^1_{\delta,\lambda}(\underline{\Psi}) \times N^2_{\delta,\lambda}(\underline{\Psi}) \times N^3_{\delta,\lambda}(\underline{\Psi}), \tag{2.57}$$

where

$$N^1_{\delta,\lambda}(\underline{\Psi}) = \prod_{i=1}^{\delta^{-d}} \binom{\lambda\mu^{\delta,\lambda}(W_i^\delta)}{((\lambda\nu_k^{\delta,\lambda}(W_i^\delta \times W_{i_1}^\delta \times \ldots \times W_{i_{k-1}}^\delta))_{i_1,\ldots,i_{k-1}=1}^{\delta^{-d}})_{k=1}^{k_{\max}}}, \tag{2.58}$$

$$N^2_{\delta,\lambda}(\underline{\Psi}) = \prod_{i=1}^{\delta^{-d}} \binom{\lambda\mu^{\delta,\lambda}(W_i^\delta)}{(\lambda\mu_m^{\delta,\lambda}(W_i^\delta))_{m\in\mathbb{N}_0}}, \tag{2.59}$$

$$N^3_{\delta,\lambda}(\underline{\Psi}) = \prod_{i=1}^{\delta^{-d}} \frac{\Big(\lambda\sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\nu_k^{\delta,\lambda}(W_i^\delta)\Big)!}{\prod_{m=0}^\infty m!^{\lambda\mu_m^{\delta,\lambda}(W_i^\delta)}} = \prod_{i=1}^{\delta^{-d}} \frac{(\lambda\sum_{m=0}^\infty m\mu_m^{\delta,\lambda}(W_i^\delta))!}{\prod_{m=0}^\infty m!^{\lambda\mu_m^{\delta,\lambda}(W_i^\delta)}}. \tag{2.60}$$

Proof. We proceed in three steps by counting first the trajectories, registering only the partition sets W_i^δ that they travel through, second, for each $m \in \mathbb{N}_0$, the sets of relays in each partition set that receive precisely m ingoing hops, and finally the choices of the relays for each hop in each partition set. Since every choice in the three steps can be freely combined with the other ones, the product of the three cardinalities is equal to the number of all trajectory configurations with the requested coarsened empirical measures.

(A) *Number of the transmitters of trajectories passing through given sequences of δ-subcubes.* For each configuration $s \in J^{\delta,\lambda}(\underline{\Psi})$ defined in (2.56), in each δ-subcube W_i^δ, $i = 1,\dots,\delta^{-d}$, there are $\lambda\mu^{\delta,\lambda}(W_i^\delta)$ users. Out of them exactly $\lambda\nu_k^{\delta,\lambda}(W_i^\delta \times W_{i_1}^\delta \times \dots W_{i_{k-1}}^\delta)$ take exactly k hops, having their first relay in $W_{i_1}^\delta$, their second relay in $W_{i_2}^\delta$ etc. and their $(k-1)$st relay in $W_{i_{k-1}}^\delta$, for any $k \in [k_{\max}]$ and $i_1,\dots,i_{k-1} = 1,\dots,\delta^{-d}$. Such choices in different subcubes W_i^δ corresponding to the transmitters are independent. Thus, the total number of such choices equals the number $N^1_{\delta,\lambda}(\underline{\Psi})$ defined in (2.58). Note that for $i = 1,\dots,\delta^{-d}$,

$$\sum_{k=1}^{k_{\max}} \sum_{i_1,\dots,i_{k-1}=1}^{\delta^{-d}} \nu_k^{\delta,\lambda}(W_i^\delta \times W_{i_1}^\delta \times \dots \times W_{i_{k-1}}^\delta) = \sum_{k=1}^{k_{\max}} \pi_0 \nu_k^{\delta,\lambda}(W_i^\delta) = \mu^{\delta,\lambda}(W_i^\delta),$$

where we used part (2) of Definition 2.8.1; hence the multinomial expressions in (2.58) are well-defined.

(B) *Number of incoming hops.* In this step, for any δ-subcube W_i^δ, we count all the possible ways to distribute the incoming hops among the relays (= users) $X_j \in W_i^\delta$, under the two constraints that in W_i^δ there are $\lambda\mu^{\delta,\lambda}(W_i^\delta)$ potential relays, and for any $m \in \mathbb{N}_0$, exactly $\lambda\mu_m^{\delta,\lambda}(W_i^\delta)$ receive precisely m incoming hops. Such choices are clearly independent of each other for different δ-subcubes. Hence, the total number of such choices equals the number $N^2_{\delta,\lambda}(\underline{\Psi})$ defined in (2.59). Again, the constraint (5) from Definition 2.8.1 implies that the multinomial expression (2.59) is well-defined. Clearly, all choices in this part are independent of the choices in part (A).

(C) *Number of assignments of the hops to the relays.* Assume that we have made one possible choice in part (A) and one possible choice in part (B). We now derive the number of possible ways of distributing, for any i, all the incoming hops in W_i^δ among the relays in W_i^δ. Let us call this number M_i, then we know from part (A) that $M_i = \lambda \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l \nu_k^{\delta,\lambda}(W_i^\delta)$, since each such hop is the l-th of some of the trajectories for some l. The cardinality of the set of relays in W_i^δ is equal to $\lambda\mu^{\delta,\lambda}(W_i^\delta) = \lambda\sum_{m=0}^\infty \mu_m^{\delta,\lambda}(W_i^\delta)$, and in part (B) we decomposed it into sets, indexed by m, in which each relay receives precisely m ingoing hops. Let us call such a relay an m-relay. Think of each such relay as being replaced by precisely m copies (in particular those with $m = 0$ are discarded), then we have $\lambda\sum_{m=0}^\infty m\mu_m^{\delta,\lambda}(W_i^\delta)$ virtual relays in W_i^δ. (Note that this is equal to M_i by (6).) Now, if all these m copies of the m-relays were distinguishable, then the number of ways to distribute the M_i ingoing hops to the relays would be simply equal to $M_i!$. However, since these m copies are identical, we overcount by a factor of $m!$ for any m-relay. This means that the number of hops into W_i^δ is equal to $M_i!/\prod_{m=0}^\infty (m!)^{\lambda\mu_m^{\delta,\lambda}(W_i^\delta)}$. Since all these cardinalities can freely be combined with each other, we have deduced that the number of possible choices is equal to the number $N^3_{\delta,\lambda}(\underline{\Psi})$ defined in (2.60).

We also see that all the choices in the three parts are independent of each other, i.e., can be freely combined with each other and yield different combinations. Hence, we arrived at the assertion. □

2.9. The limiting free energy and the LDP: proof of Theorems 2.2.2 and 2.2.4

In this section we prove Theorems 2.2.2 and 2.2.4(i), that is, we derive the variational formula in (2.24) for the high-density (i.e., $\lambda \to \infty$) exponential rate of the partition function, and we verify the LDP for the empirical measures. Our first step is to derive the large-λ exponential rate of the combinatorial formulas for the empirical measures of Lemma 2.8.6 in Section 2.9.1. Furthermore, in Section 2.9.2 we formulate and prove how the interference term and the congestion term behave in the limits $\lambda \to \infty$, followed by $\delta \downarrow 0$. In Section 2.9.3, given an admissible trajectory setting, we construct a standard setting containing it. Using all these, in Section 2.9.4 we prove Theorem 2.2.2 and in Section 2.9.5 we complete the proof of Theorem 2.2.4(i).

For the rest of this section, we fix the set $\Omega_1 \subset \Omega$ of full $\mathbb{P}$-measure on which we do our quenched investigations:

$$\Omega_1 = \Big\{\omega \in \Omega \colon X_i(\omega) \in W_{\mathbb{B}}\ \forall i \in \mathbb{N}, \quad \lim_{\lambda\to\infty} \frac{\#\{i \in I^\lambda(\omega) \colon X_i(\omega) \in W_j^\delta\}}{\lambda} = \mu(W_j^\delta),\ \forall j = 1,\dots,\delta^{-d},\ \forall \delta \in \mathbb{B}\Big\}. \tag{2.61}$$

That $\mathbb{P}(\Omega_1) = 1$ holds follows immediately from the Restriction Theorem [K93, Section 2.2] combined with the Poisson Law of Large Numbers [K93, Section 4.2] and the fact that μ is absolutely continuous.

2.9.1. The asymptotics of the combinatorics

Fix any $\omega \in \Omega_1$, and let the quantities I^λ and X^λ refer to this ω. Denote

$$N^0_{\delta,\lambda}(\underline{\Psi}) = \prod_{i=1}^{\delta^{-d}} \prod_{k=1}^{k_{\max}} \prod_{l=1}^{k-1} N(\lambda)^{\lambda \pi_l \nu_k^{\delta,\lambda}(W_i^\delta)}. \tag{2.62}$$

For a measurable subset V of $\mathbb{R}^d$ and $\nu,\widetilde{\nu} \in \mathcal{M}(V)$, let us write $H_V(\nu|\widetilde{\nu}) = \int_V \mathrm{d}\nu \log \frac{\mathrm{d}\nu}{\mathrm{d}\widetilde{\nu}}$ if the density $\frac{\mathrm{d}\nu}{\mathrm{d}\widetilde{\nu}}$ exists and $H_V(\nu|\widetilde{\nu}) = \infty$ otherwise. (The difference between $H_V(\nu|\widetilde{\nu})$ and the relative entropy $\mathcal{H}_V(\nu|\widetilde{\nu})$ defined in (2.20) is the additive term $\widetilde{\nu}(V) - \nu(V)$.) Let us recall $M = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l \nu_k = \sum_{m=0}^\infty m\mu_m$ from (2.19) and $c_m = \exp(-1/(\mathrm{e}\mu(W))(\mathrm{e}\mu(W))^{-m}/m!$ from (2.23). Note that the rate function I defined in (2.23) has also the representation

$$\mathrm{I}(\Psi) = \sum_{k=1}^{k_{\max}} H_{W^k}(\nu_k \mid \mu^{\otimes k}) - H_W(M|\mu) + \sum_{m=0}^\infty H_W(\mu_m \mid \mu c_m) - \frac{1}{\mathrm{e}}, \tag{2.63}$$

which we are going to use here. The equivalence between (2.63) and (2.23) will be verified in Section A.1 in the Appendix. Recall (2.42), which implies that the third term in (2.63) is invariant under replacing H with $\mathcal{H}$. We now identify the large-λ exponential rate of the cardinality of $J^{\delta,\lambda}(\underline{\Psi})$ both on the scale $\lambda\log\lambda$ and λ:

Proposition 2.9.1 (Exponential rates of counting terms)**.** *Let $\underline{\Psi}$ be a controlled standard setting. Let us write $\Psi = ((\nu_k)_{k=1}^{k_{\max}},(\mu_m)_{m=0}^\infty)$. We have*

$$\lim_{\delta\downarrow 0} \lim_{\lambda\to\infty} \frac{1}{\lambda} \log \frac{\# J^{\delta,\lambda}(\underline{\Psi})}{N^0_{\delta,\lambda}(\underline{\Psi})} = -\mathrm{I}(\Psi),$$

almost surely, as an identity in $[-\infty,\infty)$. Moreover, if $\mathrm{I}(\Psi) < \infty$, then

$$\lim_{\delta\downarrow 0} \lim_{\lambda\to\infty} \frac{1}{\lambda\log\lambda} \log \# J^{\delta,\lambda}(\underline{\Psi}) = M(W) < \infty,$$

almost surely.

Proof. Recall that Ψ is an admissible trajectory setting, according to Lemma 2.8.5. In particular, $\mathrm{I}(\Psi) \in (-\infty,\infty]$ is well-defined.

We use Stirling's formula $\lambda! = (\lambda/\mathrm{e})^\lambda \mathrm{e}^{o(\lambda)}$ in the limit $\mathbb{N} \ni \lambda \to \infty$, which leads to

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \binom{a^{(\lambda)}}{a_1^{(\lambda)},\dots,a_n^{(\lambda)}} = -\sum_{i=1}^n a_i \log \frac{a_i}{a}, \tag{2.64}$$

for any integers $a_1^{(\lambda)},\dots,a_n^{(\lambda)}$ that sum up to $a^{(\lambda)}$ and satisfy $\frac{1}{\lambda} a_i^{(\lambda)} \stackrel{\lambda\to\infty}{\to} a_i$ for all $i = 1,\dots,n$ with positive numbers $a_1,\dots,a_n$ satisfying $\sum_{i=1}^n a_i = a$.

From (2.58) we obtain that

$$\begin{aligned} I_\delta^1(\underline{\Psi}) &= -\lim_{\lambda\to\infty} \frac{1}{\lambda} \log N^1_{\delta,\lambda}(\underline{\Psi}) \\ &= \sum_{i=1}^{\delta^{-d}} \sum_{k=1}^{k_{\max}} \sum_{i_1,\dots,i_{k-1}=1}^{\delta^{-d}} \nu_k^\delta(W_i^\delta \times W_{i_1}^\delta \times \dots \times W_{i_{k-1}}^\delta) \log \frac{\nu_k^\delta(W_i^\delta \times W_{i_1}^\delta \times \dots \times W_{i_{k-1}}^\delta)}{\mu^\delta(W_i^\delta)}, \end{aligned}$$

where we also used that all the measures $\nu_k^{\delta,\lambda}$ and $\mu^{\delta,\lambda}$ converge as $\lambda\to\infty$ to ν_k^{δ} and μ^{δ}, respectively.

Now we add the term $\prod_{l=1}^{k-1}\mu^\delta(W_{i_l}^\delta)$ both in the numerator and the denominator under the logarithm and separate these two terms. In the former, we write its logarithm as $\sum_{l=1}^{k-1}\log\mu^\delta(W_{i_l}^\delta)$, interchange this sum on l with all the other sums on the $i_0,\dots,i_{k-1}$ and write the sums over $i_0,\dots,i_{l-1},i_{l+1},\dots,i_{k-1}$ in terms of the l-th marginal measure of ν_k^δ. This gives

$$\begin{aligned} I_\delta^1(\underline{\Psi}) = &\sum_{i=1}^{\delta^{-d}}\sum_{k=1}^{k_{\max}}\sum_{i_1,\dots,i_{k-1}=1}^{\delta^{-d}}\nu_k^\delta(W_i^\delta\times W_{i_1}^\delta\times\dots\times W_{i_{k-1}}^\delta)\log\frac{\nu_k^\delta(W_i^\delta\times W_{i_1}^\delta\times\dots\times W_{i_{k-1}}^\delta)}{\mu^\delta(W_i^\delta)\prod_{l=1}^{k-1}\mu^\delta(W_{i_l}^\delta)}\\ &+\sum_{i=1}^{\delta^{-d}}\sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\nu_k^\delta(W_i^\delta)\log\mu^\delta(W_i^\delta). \end{aligned}\tag{2.65}$$

In the same way as for I_1^δ, we obtain

$$I_\delta^2(\underline{\Psi}) = -\lim_{\lambda\to\infty}\frac{1}{\lambda}\log N_{\delta,\lambda}^2(\underline{\Psi}) = \sum_{i=1}^{\delta^{-d}}\sum_{m=0}^{\infty}\mu_m^\delta(W_i^\delta)\log\frac{\mu_m^\delta(W_i^\delta)}{\mu^\delta(W_i^\delta)}.\tag{2.66}$$

Using (2.61), on Ω_1 we have that the asymptotic behaviour of (2.62) is the following

$$N_{\delta,\lambda}^0(\underline{\Psi}) = N(\lambda)^{\lambda\sum_{i=1}^{\delta^{-d}}\sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\nu_k^{\delta,\lambda}(W_i^\delta)} = (\lambda\mu(W))^{\lambda(1+o(1))\sum_{i=1}^{\delta^{-d}}\sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\nu_k^{\delta,\lambda}(W_i^\delta)}.$$

On the other hand, also by Stirling's formula, we can identify the large-λ rate of the quotient of the counting terms in (2.60) and (2.62) as follows:

$$\begin{aligned} I_\delta^{3,0}(\underline{\Psi}) &= -\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\frac{N_{\delta,\lambda}^3(\underline{\Psi})}{N_{\delta,\lambda}^0(\underline{\Psi})}\\ &= -\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\prod_{i=1}^{\delta^{-d}}\frac{\big(\frac{1}{\mathrm{e}\mu(W)}\sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\nu_k^{\delta,\lambda}(W_i^\delta)\big)^{\lambda\sum_{k'=1}^{k_{\max}}\sum_{l'=1}^{k'-1}\pi_{l'}\nu_{k'}^{\delta,\lambda}(W_i^\delta)}}{\prod_{m=0}^{\infty}m!^{\lambda\mu_m(W_i^\delta)}}\\ &= -\sum_{i=1}^{\delta^{-d}}\sum_{k'=1}^{k_{\max}}\sum_{l'=1}^{k'-1}\pi_{l'}\nu_{k'}^\delta(W_i^\delta)\left(\log\sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\nu_k^\delta(W_i^\delta)-(1+\log\mu(W))\right)+\sum_{i=1}^{\delta^{-d}}\sum_{m=0}^{\infty}\mu_m^\delta(W_i^\delta)\,\log(m!), \end{aligned}\tag{2.67}$$

where for the last term we used the fact that $\underline{\Psi}$ is controlled (see also Lemma 2.8.5), together with dominated convergence. We can summarize the sum of the terms in (2.65), (2.66), and (2.67) as

$$\begin{aligned} -\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\frac{\#J^{\delta,\lambda}(\underline{\Psi})}{N_{\delta,\lambda}^0(\underline{\Psi})} &= I_\delta^1(\underline{\Psi})+I_\delta^2(\underline{\Psi})+I_\delta^{3,0}(\underline{\Psi})\\ &= \sum_{k=1}^{k_{\max}}\sum_{i_0,\dots,i_{k-1}=1}^{\delta^{-d}}\nu_k^\delta(W_{i_0}^\delta\times\dots\times W_{i_{k-1}}^\delta)\log\frac{\nu_k^\delta(W_{i_0}^\delta\times\dots\times W_{i_{k-1}}^\delta)}{\prod_{l=0}^{k-1}\mu^\delta(W_{i_l}^\delta)}\\ &\quad+\sum_{i=1}^{\delta^{-d}}\sum_{m=0}^{\infty}\mu_m^\delta(W_i^\delta)\Big(\log\frac{\mu_m^\delta(W_i^\delta)}{\mu^\delta(W_i^\delta)}+m(1+\log\mu(W))+\log(m!)\Big)\\ &\quad-\sum_{i=1}^{\delta^{-d}}\left(\sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\nu_k^\delta(W_i^\delta)\right)\log\frac{\sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\nu_k^\delta(W_i^\delta)}{\mu^\delta(W_i^\delta)}, \end{aligned}\tag{2.68}$$

where in the first line on the right-hand side we changed the summing index i into i_0. Since we have

$$\sum_{m=0}^{\infty}\mu_m^\delta(W) = \sum_{m=0}^{\infty}\mu_m(W) = \mu(W),$$

and thus

$$\sum_{i=1}^{\delta^{-d}}\sum_{m=0}^{\infty}\mu_m^\delta(W_i^\delta)\Big(\log\frac{\mu_m^\delta(W_i^\delta)}{\mu^\delta(W_i^\delta)}+m(1+\log\mu(W))+\log(m!)\Big)=\sum_{m=0}^{\infty}\mu_m^\delta(W_i^\delta)\log\frac{\mu_m^\delta(W_i^\delta)}{c_m\mu^\delta(W_i^\delta)}-\frac{1}{\mathrm{e}},$$

we obviously arrived at the discrete version of the entropy terms in (2.63), more precisely, the entropy of the measures in (2.63) with respect to the σ-field $\mathcal{F}_\delta$, respectively $\mathcal{F}_\delta^{\otimes k}$. Now, according to [G11, Proposition (15.6)], the limit of these entropies as $\delta\downarrow 0$ is equal to their corresponding continuous version, i.e., the right-hand side of (2.68) converges to $\mathrm{I}(\Psi)$. The first part of Proposition 2.9.1 follows.

Moreover, if $\mathrm{I}(\Psi)<\infty$, then we have by continuity

$$\begin{aligned}\lim_{\delta\downarrow 0}\lim_{\lambda\to\infty}\frac{1}{\lambda\log\lambda}\log\#J^{\delta,\lambda}(\underline{\Psi})&=\lim_{\delta\downarrow 0}\lim_{\lambda\to\infty}\frac{1}{\lambda\log\lambda}\log N^0_{\delta,\lambda}(\underline{\Psi})\\&=\lim_{\delta\downarrow 0}\lim_{\lambda\to\infty}\sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\sum_{i=1}^{\delta^{-d}}\pi_l\nu_k^{\delta,\lambda}(W_i^\delta)=\sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\nu_k(W)=\sum_{k=1}^{k_{\max}}(k-1)\nu_k(W^k)\in[0,\infty),\end{aligned}$$

where in the last identity we used that by Fubini's theorem, $\pi_0\nu_k(W)=\nu_k(W^k)$ holds for all k. Hence, using that Ψ is an admissible trajectory setting, we conclude the second part of Proposition 2.9.1. □

2.9.2. Approximations for the penalization terms

The limiting relations between the penalization terms depending on the numbers of incoming hops in (2.9) and (2.22), and between the continuous penalization terms in (2.7) and (2.21) are given as follows.

Proposition 2.9.2. *Let $\underline{\Psi}$ be a controlled standard setting. Let us write $\Psi=((\nu_k)_{k=1}^{k_{\max}},(\mu_m)_{m=0}^{\infty})$ for the admissible trajectory setting contained in $\underline{\Psi}$. Then, almost surely,*

$$\lim_{\delta\downarrow 0}\lim_{\lambda\to\infty}\sup_{s\in J^{\delta,\lambda}(\underline{\Psi})}\Big|\frac{1}{\lambda}\mathfrak{M}(s)-\mathrm{M}(\Psi)\Big|=0,\tag{2.69}$$

and

$$\lim_{\delta\downarrow 0}\lim_{\lambda\to\infty}\sup_{s\in J^{\delta,\lambda}(\underline{\Psi})}\Big|\frac{1}{\lambda}\mathfrak{S}(s)-\mathrm{S}(\Psi)\Big|=0.\tag{2.70}$$

Proof. Throughout the proof, we perform our analysis on Ω_1. First, we show (2.69). Consider some $s\in J^{\delta,\lambda}(\underline{\Psi})$ for $\lambda>0$ and $\delta\in\mathbb{B}$. Additionally assume that $s_l^i\in W_{\mathbb{B}}$ for all $i\in I^\lambda$ and $l=0,\dots,k$ (which is always the case for $s=S=(S^i)_{i\in I^\lambda}$ on Ω_1).

Then $P_\lambda^\delta(s)=\mu^{\delta,\lambda}$ and $P_{\lambda,m}^\delta(s)=\mu_m^{\delta,\lambda}$ for all $m\in\mathbb{N}_0$, see (2.52) and (2.53), and the definition (2.56) of $J^{\delta,\lambda}(\underline{\Psi})$. Recall that $m_i(s)$ is the number of incoming hops at relay X_i for the trajectory configuration s. Hence we have

$$\begin{aligned}\mathfrak{M}(s)=\sum_{i\in I^\lambda}\eta(m_i(s))&=\sum_{m=0}^{\infty}\eta(m)\#\{i\in I^\lambda\colon m_i(s)=m\}=\lambda\sum_{m=0}^{\infty}\eta(m)P_{\lambda,m}(s)(W)\\&=\lambda\sum_{m=0}^{\infty}\eta(m)P_{\lambda,m}^\delta(s)(W)=\lambda\sum_{m=0}^{\infty}\eta(m)\mu_m^{\delta,\lambda}(W),\end{aligned}$$

for all such s. Note that by part (8) of Definition 2.8.1 and part (E) of Remark 2.8.2, we obtain that $\mu_m^{\delta,\lambda}$ tends to μ_m as $\lambda\to\infty$ followed by $\delta\downarrow 0$. Now, (2.54) in Definition 2.8.4, together with the fact that the total mass of μ_m^δ equals the one of μ_m for any m, implies the assertion in (2.69).

We continue with verifying (2.70). Let us fix an arbitrary controlled standard setting $\underline{\Psi}$. Our goal is to prove that (2.70) holds for this $\underline{\Psi}$. Using that, for an admissible trajectory setting $\Psi=((\nu_k)_{k=1}^{k_{\max}},(\mu_m)_{m=0}^{\infty})$, $\mathrm{S}(\Psi)$ depends only on $(\nu_k)_{k=1}^{k_{\max}}$, we have for any $\lambda>0$, $\delta\in\mathbb{B}$, $s\in J^{\delta,\lambda}(\underline{\Psi})$ and $k\in[k_{\max}]$ that

$$\frac{1}{\lambda}\mathfrak{S}(s)-\mathrm{S}(\Psi)=\langle R_{\lambda,k}(s),f_k(L_\lambda,\cdot)\rangle-\langle\nu_k,f_k(\mu,\cdot)\rangle.$$

In the rest of Section 2.9, we will often have to verify convergence of certain (sequences of) measures in the (coordinatewise) weak topology. In order to keep our arguments clear and short, for $k \in \mathbb{N}$, we fix a metric $d_k(\cdot,\cdot)$ on $\mathcal{M}(W^k)$ that generates the weak topology on this space. It turns out to be convenient to choose d_k to be the *Lipschitz bounded metric* [DZ98, Section D.2] on $\mathcal{M}(W^k)$, that is,

$$d_k(\nu_k^1,\nu_k^2) = \sup\{|\langle \nu_k^1, f\rangle - \langle \nu_k^2, f\rangle| \colon f \in \mathrm{Lip}_1(W^k)\} \tag{2.71}$$

for all k, where $\mathrm{Lip}_1(W^k)$ is the set of Lipschitz continuous functions taking W^k to $\mathbb{R}$ with Lipschitz parameter less than or equal to 1 and with uniform bound 1. For $k \in [k_{\max}]$ we have

$$\begin{aligned}\Big|\frac{1}{\lambda}\mathfrak{S}(s) - \mathrm{S}(\Psi)\Big| &= \Big|\langle R_{\lambda,k}(s), f_k(L_\lambda,\cdot)\rangle - \langle \nu_k, f_k(\mu,\cdot)\rangle\Big| \\ &\leq \Big|\langle R_{\lambda,k}(s), f_k(L_\lambda,\cdot)\rangle - \langle \nu_k^{\delta,\lambda}, f_k(L_\lambda,\cdot)\rangle\Big| + \Big|\langle \nu_k^{\delta,\lambda}, f_k(L_\lambda,\cdot)\rangle - \langle \nu_k^{\delta,\lambda}, f_k(L_\lambda^\delta,\cdot)\rangle\Big| \\ &\quad + \Big|\langle \nu_k^{\delta,\lambda}, f_k(L_\lambda^\delta,\cdot)\rangle - \langle \nu_k^{\delta,\lambda}, f_k(\mu,\cdot)\rangle\Big| + \Big|\langle \nu_k^{\delta,\lambda}, f_k(\mu,\cdot)\rangle - \langle \nu_k, f_k(\mu,\cdot)\rangle\Big|.\end{aligned} \tag{2.72}$$

Now, we claim that all the four terms on the right-hand side tend to 0 in the limit $\lambda \to \infty$ followed by $\delta \downarrow 0$. Indeed, for the first term, let $g \in \mathrm{Lip}^1(W^k)$. Then we have

$$\begin{aligned}\Big|\langle R_{\lambda,k}(s),g\rangle - \langle \nu_k^{\delta,\lambda}, g\rangle &= \Big|\int_{W^k} g(y) R_{\lambda,k}(s)(\mathrm{d}y) - \int_{W^k} g(y) R_{\lambda,k}^\delta(s)(\mathrm{d}y)\Big| \\ &\leq \sum_{i_0,\dots,i_{k-1}=1}^{\delta^{-d}} \sup_{y,z \in W_{i_0}^\delta \times \dots \times W_{i_{k-1}}^\delta} |g(y)-g(z)| R_{\lambda,k}(s)(W_{i_0}^\delta \times \ldots \times W_{i_{k-1}}^\delta) \\ &\leq 2\delta\sqrt{dk} R_{\lambda,k}(s)(W) \leq 2\delta\sqrt{dk} L_\lambda(W),\end{aligned}$$

which tends to 0 as $\lambda \to \infty$ followed by $\delta \downarrow 0$. It follows that $R_{\lambda,k}(s) - \nu_k^{\delta,\lambda}$ tends weakly to 0 as $\lambda \to \infty$ followed by $\delta \downarrow 0$. We note that for any $\alpha > 0$, the restriction of f_k to $\mathcal{M}_{\leq\alpha}(W) \times W^k$ is bounded, where we wrote $\mathcal{M}_{\leq\alpha}(V)$ for the set of measures on the space V with total mass $\leq \alpha$. Indeed, since W is compact, $\mathcal{M}_{\leq\alpha}(W)$ with the weak topology is also a compact, metrizable space by Prohorov's theorem. Thus, the continuous function $f_k \colon \mathcal{M}_{\leq\alpha}(W) \times W^k \to \mathbb{R}$ is uniformly continuous and therefore bounded. Now, since eventually $L_\lambda \in \mathcal{M}_{\leq 2\mu(W)}(W)$, the first term on the right-hand side of (2.72) tends to 0.

As for the second term, note that for any $\delta \in \mathbb{B}$, $L_\lambda - L_\lambda^\delta$ tends to $\mu - \mu^\delta$ as $\lambda \to \infty$, which tends to 0 as $\delta \downarrow 0$. Thus, by the fact that f_k is continuous and bounded on $\mathcal{M}_{\leq 2\mu(W)}(W) \times W^k$ and eventually $L_\lambda, L_\lambda^\delta, \nu_k^{\delta,\lambda} \in \mathcal{M}_{\leq 2\mu(W)}(W)$ for all $\delta \in \mathbb{B}$, the second term also tends to 0 as first $\lambda \to \infty$ and afterwards $\delta \downarrow 0$. An analogous argument applies for the third term, using that L_λ^δ converges to μ as first $\lambda \to \infty$ and then $\delta \downarrow 0$ by part (B) of Remark 2.8.2 and the definition of μ^δ, $\delta \in \mathbb{B}$. The fourth term tends to zero since it easily follows from part (4) of Definition 2.8.1 and part (D) of Remark 2.8.2 that $\nu_k^{\delta,\lambda}$ converges weakly to ν_k, and f_k is continuous and bounded on $\mathcal{M}_{\leq 2\mu(W)}(W) \times W^k$. We conclude (2.70) and Proposition 2.9.2. □

2.9.3. Existence of standard settings

Recall that we equip $\mathcal{A}$ defined in (2.29) with the product topology of the weak topologies of the factors $\mathcal{M}(W^k)$, $\mathcal{M}(W)$, and that this is the topology of coordinatewise weak convergence. For $k \in \mathbb{N}$, let $d_k(\cdot,\cdot)$ be the Lipschitz bounded metric (2.71) on $\mathcal{M}(W^k)$, which generates the weak topology on this space. Then,

$$d_0(\Psi^1,\Psi^2) = \sum_{k=1}^{k_{\max}} d_k(\nu_k^1,\nu_k^2) + \sum_{m=0}^{\infty} 2^{-m} d_1(\mu_m^1,\mu_m^2), \quad \Psi^1,\Psi^2 \in \mathcal{A}, \tag{2.73}$$

is a metric on $\mathcal{A}$ that generates the product topology. Complying with our general notation, in the rest of this section, for $\varrho > 0$ and $\Psi \in \mathcal{A}$, $B_\varrho(\Psi) = \{\Psi' \in \mathcal{A} \colon d_0(\Psi',\Psi) < \varrho\}$ will denote the open ϱ-ball around Ψ in $(\mathcal{A},d_0)$. We have the following.

Proposition 2.9.3. *On Ω_1, for any admissible trajectory setting (see Definition 2.2.1), $\Psi = ((\nu_k)_k,(\mu_m)_m)$, there exists a standard setting $\underline{\Psi}$ containing it. If $\sum_m \eta(m)\mu_m(W) < \infty$, then $\underline{\Psi}$ can be chosen to be a controlled standard setting.*

Proof. We fix an admissible trajectory setting Ψ and construct $\underline{\Psi}$ as follows. As is required in Definition 2.8.1, the measures ν_k^δ for $k \in [k_{\max}]$ and μ_m^δ for $m \in \mathbb{N}_0$ are the δ-coarsenings of the measures ν_k and μ_m, respectively, and $\mu^{\delta,\lambda} = L_\lambda^\delta$. Now, for $\delta \in \mathbb{B}$ and $\lambda > 0$, pick some measures $\nu_k^{\delta,\lambda}$ and $\mu_m^{\delta,\lambda}$ with values in $\frac{1}{\lambda}\mathbb{N}_0$ such that the requirements (2) $\sum_{k=1}^{k_{\max}} \pi_0 \nu_k^{\delta,\lambda} = \mu^{\delta,\lambda}$, (5) $\sum_{m=0}^\infty \mu_m^{\delta,\lambda} = \mu^{\delta,\lambda}$ and (6) $\sum_{m=0}^\infty m\mu_m^{\delta,\lambda} = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l \nu_k^{\delta,\lambda}$ of Definition 2.8.1 are met, such that $\nu_k^{\delta,\lambda} \Longrightarrow \nu_k^\delta$ and $\mu_m^{\delta,\lambda} \Longrightarrow \mu_m^\delta$ as $\lambda \to \infty$, and such that the collection $\underline{\Psi}$ of all these measures is a standard setting containing Ψ, which is controlled if $\sum_m \eta(m)\mu_m(W) < \infty$.

We claim that this can be done by taking suitable up- and downroundings of the numbers

$$\nu_k'^{\delta,\lambda}(W_{i_0}^\delta \times \ldots \times W_{i_{k-1}}^\delta) = \nu_k^\delta(W_{i_0}^\delta \times \ldots \times W_{i_{k-1}}^\delta)\frac{L_\lambda^\delta(W_{i_0}^\delta)}{\mu^\delta(W_{i_0}^\delta)}\mathbb{1}\{\mu^\delta(W_{i_0}^\delta) > 0\}, \quad k \in [k_{\max}], \tag{2.74}$$

for all $i_0,\ldots,i_{k-1} = 1,\ldots,\delta^{-d}$, and dividing by λ, analogously for the μ_m's. Now, using the d-metric defined in (2.73), we prove that the convergences required in Definition 2.8.1 hold for such $\underline{\Psi}$.

First, we prove the convergence of the δ-coarsenings $\Psi^\delta = ((\nu_k^\delta)_k,(\mu_m^\delta)_m)$ to Ψ in the d_0-metric. We claim that for any $\varrho > 0$, there exists $\delta_0 \in \mathbb{B}$ such that $\Psi^\delta \in B_\varrho(\Psi)$ for all $\mathbb{B} \ni \delta \le \delta_0$. Indeed, for $k \in [k_{\max}]$, $\nu_k \in \mathcal{M}(W^k)$ and $\delta \in \mathbb{B}$ we see that the distance between ν_k and its δ-coarsening is of order δ with respect to the Lipschitz bounded metric:

$$\begin{aligned} d_k(\nu_k,\nu_k^\delta) &= \sup_{f\in \mathrm{Lip}_1(W^k)} \Big| \sum_{i_0,\ldots,i_{k-1}=1}^{\delta^{-d}} \Big(\int_{W_{i_0}^\delta \times\ldots\times W_{i_{k-1}}^\delta} f(x)\nu_k(\mathrm{d}x) - \int_{W_{i_0}^\delta \times\ldots\times W_{i_{k-1}}^\delta} f(x)\nu_k^\delta(\mathrm{d}x)\Big)\Big| \\ &\le \sup_{f\in \mathrm{Lip}_1(W^k)} \sum_{i_0,\ldots,i_{k-1}=1}^{\delta^{-d}} \sup_{x,y\in W_{i_0}^\delta \times\ldots\times W_{i_{k-1}}^\delta} |f(x)-f(y)|\nu_k(W_{i_0}^\delta \times \ldots \times W_{i_{k-1}}^\delta) \le 2\delta\nu_k(W^k)\sqrt{kd}, \end{aligned}$$

where we wrote $x = (x_0,\ldots,x_{k-1})$ and $y = (y_0,\ldots,y_{k-1})$; and analogously for μ_m. Thus, we have

$$d_0(\Psi,\Psi^\delta) \le 2\delta\sqrt{d}\Big[\sum_{k=1}^{k_{\max}} \nu_k(W^k)\sqrt{k} + \sum_{m=0}^\infty \mu_m(W)2^{-m}\Big].$$

Since $\sum_{m=0}^\infty \mu_m(W) < \infty$ by (ii) in (2.19), there exists a constant C, depending only on Ψ, such that $\Psi^\delta \in B_\varrho(\Psi)$ for any $\delta \le C\varrho$.

Second, we ignore the up- or downroundings in the construction of Ψ and prove the following. For $\delta \in \mathbb{B}$ and $\lambda > 0$, let $\Psi'^{\delta,\lambda}$ be the collection of the measures introduced in (2.74). We claim that on Ω_1, for $\delta \in \mathbb{B}$ fixed, we have

$$\limsup_{\lambda\to\infty} d_0(\Psi^\delta,\Psi'^{\delta,\lambda}) = 0.$$

Indeed, for any $k \in [k_{\max}]$ and $i_0,\ldots,i_{k-1} = 1,\ldots,\delta^{-d}$, $d_k(\nu_k^\delta,\nu_k'^{\delta,\lambda})$ is bounded from above by

$$\sup_{f\in \mathrm{Lip}_1(W^k)} \sum_{i_0,\ldots,i_{k-1}=1}^{\delta^{-d}} \nu_k^\delta(W_{i_0}^\delta \times \ldots \times W_{i_{k-1}}^\delta)\Big|\frac{L_\lambda^\delta(W_{i_0}^\delta)}{\mu^\delta(W_{i_0}^\delta)} - 1\Big|\|f\|_\infty \le \nu_k^\delta(W^k) \max_{i_0\in[\delta^{-d}]}\Big|\frac{L_\lambda^\delta(W_{i_0}^\delta)}{\mu^\delta(W_{i_0}^\delta)} - 1\Big|. \tag{2.75}$$

Using a similar estimate on $d_1(\mu_m^\delta,\mu_m'^{\delta,\lambda})$ for $m \in \mathbb{N}_0$, we obtain

$$d_0(\Psi^\delta,\Psi'^{\delta,\lambda}) \le \Big(\sum_{k=1}^{k_{\max}} \nu_k^\delta(W^k) + \sum_{m=0}^\infty 2^{-m}\mu_m^\delta(W)\Big) \max_{i_0\in[\delta^{-d}]}\Big|\frac{L_\lambda^\delta(W_{i_0}^\delta)}{\mu^\delta(W_{i_0}^\delta)} - 1\Big|,$$

which tends to 0 on Ω_1 as $\lambda \to \infty$, according to (2.61).

Now, if we add the suitable up- and downroundings, we only change distances in the d-metric by an error term of order $1/\lambda$, which vanishes as $\lambda \to \infty$. This implies that $\underline{\Psi}$ is a standard setting. It also follows easily that if $\sum_m \eta(m)\mu_m(W) < \infty$, then $\underline{\Psi}$ is controlled. $\square$

2.9.4. Proof of Theorem 2.2.2

Abbreviate

$$\mathfrak{Y}(r) = \Big(\prod_{i\in I^\lambda} N(\lambda)^{-(r^i_{-1}-1)}\Big)\exp\Big\{-\gamma\mathfrak{S}(r)-\beta\mathfrak{M}(r)\Big\}, \qquad \lambda>0, r\in\mathcal{S}_{k_{\max}}(X^\lambda),$$

and note that the partition function is given as

$$Z_\lambda^{\gamma,\beta}(X^\lambda) = \sum_{r\in\mathcal{S}_{k_{\max}}(X^\lambda)} \mathfrak{Y}(r). \tag{2.76}$$

Then Theorem 2.2.2 says that its large-λ negative exponential rate is given as the infimum of $\mathrm{I}(\Psi)+\gamma\mathrm{S}(\Psi)+\beta\mathrm{M}(\Psi)$, taken over all admissible trajectory settings Ψ. Throughout the proof, we assume that the configuration $X^\lambda = X^\lambda(\omega)$ comes from some $\omega\in\Omega_1$ defined in (2.61).

Having proved Propositions 2.9.1, 2.9.2, and 2.9.3, our strategy to prove Theorem 2.2.2 is the following. First, Proposition 2.9.3 gives a standard way of constructing from an admissible trajectory setting Ψ satisfying $\mathrm{I}(\Psi)+\gamma\mathrm{S}(\Psi)+\beta\mathrm{M}(\Psi)<\infty$ a controlled standard setting $\underline{\Psi}$ that contains Ψ. Then the lower bound for the partition function is easily given in terms of the objects that are contained in any such $\underline{\Psi}$ and using the logarithmic asymptotics for their combinatorics from Propositions 2.9.1 and 2.9.2, and finally taking the infimum over all such Ψ, respectively $\underline{\Psi}$. The upper bound needs more care, since the entire sum over r has to be handled. First of all, we show that the sum can be restricted for all $\lambda>0$, modulo some error term that is negligible on the exponential scale, to the sum of those configurations whose congestion exponent is at most $C\lambda$ for some appropriate large constant $C>0$. This sum can be decomposed, for any $\delta\in\mathbb{B}$, to sums on configurations coming from a particular choice of empirical measures on the δ-partitions of W. The number of these empirical measures and the sum on the partitions is negligible in the limit $\lambda\to\infty$, and the asymptotics of the sums on r in these partitions can be evaluated with the help of our spatial discretization procedure, using arguments of the proofs of Propositions 2.9.1 and 2.9.2 in the limit $\lambda\to\infty$, followed by $\delta\downarrow 0$. Using these, we arrive at the formula (2.24).

Let us give the details. We start with the proof of the lower bound. For any admissible trajectory setting Ψ such that $\mathrm{I}(\Psi)+\gamma\mathrm{S}(\Psi)+\beta\mathrm{M}(\Psi)<\infty$, we pick a controlled standard setting $\underline{\Psi}$ as in Proposition 2.9.3 and recall the configuration class $J^{\delta,\lambda}(\underline{\Psi})$ from (2.56). Then, for any $\lambda>0$ and $\delta\in\mathbb{B}$,

$$Z_\lambda^{\gamma,\beta}(X^\lambda) \ge \sum_{r\in J^{\delta,\lambda}(\underline{\Psi})}\mathfrak{Y}(r) \ge \frac{\# J^{\delta,\lambda}(\underline{\Psi})}{\sup_{r\in J^{\delta,\lambda}(\underline{\Psi})}\prod_{i\in I^\lambda} N(\lambda)^{-(r^i_{-1}-1)}}\exp\Big\{-\sup_{r\in J^{\delta,\lambda}(\underline{\Psi})}\big(\gamma\mathfrak{S}(r)+\beta\mathfrak{M}(r)\big)\Big\}. \tag{2.77}$$

Hence,

$$\begin{aligned}\liminf_{\lambda\to\infty}\frac{1}{\lambda}\log Z_\lambda^{\gamma,\beta}(X^\lambda) &\ge \liminf_{\delta\downarrow 0}\liminf_{\lambda\to\infty}\frac{1}{\lambda}\log\frac{\# J^{\delta,\lambda}(\underline{\Psi})}{\sup_{r\in J^{\delta,\lambda}(\underline{\Psi})}\prod_{i\in I^\lambda} N(\lambda)^{-(r^i_{-1}-1)}}\\ &\quad-\gamma\limsup_{\delta\downarrow 0}\limsup_{\lambda\to\infty}\sup_{r\in J^{\delta,\lambda}(\underline{\Psi})}\frac{1}{\lambda}\mathfrak{S}(r)-\beta\limsup_{\delta\downarrow 0}\limsup_{\lambda\to\infty}\sup_{r\in J^{\delta,\lambda}(\underline{\Psi})}\frac{1}{\lambda}\mathfrak{M}(r)\\ &= -\mathrm{I}(\Psi)-\gamma\mathrm{S}(\Psi)-\beta\mathrm{M}(\Psi).\end{aligned} \tag{2.78}$$

In the last step we also used Propositions 2.9.1 and 2.9.2 together with the fact that $\underline{\Psi}$ is controlled. Now take the supremum over all such Ψ on the r.h.s. of (2.78) to conclude that the lower bound in (2.24) holds.

The upper bound of Theorem 2.2.2 requires some additional work. We start from (2.76). For $C>0$ we have

$$Z_\lambda^{\gamma,\beta}(X^\lambda) = \sum_{r\in\mathcal{S}_{k_{\max}}(X^\lambda)\colon\mathfrak{M}(r)\le\lambda C}\mathfrak{Y}(r) + \sum_{r\in\mathcal{S}_{k_{\max}}(X^\lambda)\colon\mathfrak{M}(r)>\lambda C}\mathfrak{Y}(r). \tag{2.79}$$

Since the total mass of our *a priori* measure has a bounded large-λ exponential rate (see Section 2.1.2), and $\mathfrak{S}$, $\mathfrak{M}$ are bounded from below, we see that

$$\limsup_{C\to\infty}\limsup_{\lambda\to\infty}\frac{1}{\lambda}\log\sum_{r\in\mathcal{S}_{k_{\max}}(X^\lambda)\colon\mathfrak{M}(r)>\lambda C}\mathfrak{Y}(r) = -\infty.$$

Thus, for C sufficiently large, the exponential rate of $Z_\lambda^{\gamma,\beta}(X^\lambda)$ is equal to the one of the first term on the right-hand side of (2.79). We additionally require C so large that

$$\inf_{\Psi \text{ adm. traj. setting}, \ \mathrm{M}(\Psi)\le C}(\mathrm{I}(\Psi)+\gamma\mathrm{S}(\Psi)+\beta\mathrm{M}(\Psi)) = \inf_{\Psi \text{ adm. traj. setting}}(\mathrm{I}(\Psi)+\gamma\mathrm{S}(\Psi)+\beta\mathrm{M}(\Psi)). \tag{2.80}$$

Let us write $\mathcal{S}_{k_{\max},C}(X^\lambda)=\{r\in\mathcal{S}_{k_{\max}}(X^\lambda)\colon \mathfrak{M}(r)\le\lambda C\}$ and $Z_\lambda^{\gamma,\beta,C}(X^\lambda)=\sum_{r\in\mathcal{S}_{k_{\max},C}(X^\lambda)}\mathfrak{Y}(r)$. The upper bound of Theorem 2.2.2 follows as soon as we show that

$$\limsup_{\lambda\to\infty}\frac{1}{\lambda}\log Z_\lambda^{\gamma,\beta,C}(X^\lambda)\le -\inf_{\Psi \text{ admissible trajectory setting}, \ \mathrm{M}(\Psi)\le C}(\mathrm{I}(\Psi)+\gamma\mathrm{S}(\Psi)+\beta\mathrm{M}(\Psi)). \tag{2.81}$$

For fixed $\lambda>0$ and $\delta\in\mathbb{B}$, let us say that a collection of measures $\Psi^{\delta,\lambda}=((\nu_k^{\delta,\lambda})_{k=1}^{k_{\max}},(\mu_m^{\delta,\lambda})_{m=0}^\infty)$ lies in $G(\delta,\lambda)=G(\delta,\lambda)(X^\lambda)$ if all these measures take values in $\frac{1}{\lambda}\mathbb{N}_0$ only and satisfy the constraints $\sum_{k=1}^{k_{\max}}\pi_0\nu_k^{\delta,\lambda}=L_\lambda^\delta$, $\sum_{m=0}^\infty\mu_m^{\delta,\lambda}=L_\lambda^\delta$, and $\sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\nu_k^{\delta,\lambda}=\sum_{m=0}^\infty m\mu_m^{\delta,\lambda}$. We will write $J^{\delta,\lambda}(\Psi^{\delta,\lambda})$ for the set $J^{\delta,\lambda}(\underline{\Psi})$ defined in (2.56). Then the union of $J^{\delta,\lambda}(\Psi^{\delta,\lambda})$ over all $\Psi^{\delta,\lambda}$ with $\sum_{m=0}^\infty\eta(m)\mu_m^{\delta,\lambda}(W)\le C$ is equal to

$$\big\{((R^\delta_{\lambda,k}(r))_{k\in[k_{\max}]},(P^\delta_{\lambda,m}(r))_{m\in\mathbb{N}_0})\colon r\in\mathcal{S}_{k_{\max},C}(X^\lambda)\big\},$$

since these three equations characterize the tuple of the measures $(R^\delta_{\lambda,k}(S))_{k=1}^{k_{\max}}$ and $(P^\delta_{\lambda,m}(S))_{m=0}^\infty$ if $(S^i)_{i\in I^\lambda}\in\mathcal{S}_{k_{\max},C}(X^\lambda)$.

Using this, we can estimate, for any $\delta\in\mathbb{B}$,

$$Z_\lambda^{\gamma,\beta,C}(X^\lambda)=\sum_{\Psi^{\delta,\lambda}\in G(\delta,\lambda)\colon \mathrm{M}(\Psi^{\delta,\lambda})\le C}\ \sum_{r\in J^{\delta,\lambda}(\Psi^{\delta,\lambda})}\mathfrak{Y}(r)\le \#G(\delta,\lambda)\sup_{\Psi^{\delta,\lambda}\in G(\delta,\lambda)\colon \mathrm{M}(\Psi^{\delta,\lambda})\le C}\ \sum_{r\in J^{\delta,\lambda}(\Psi^{\delta,\lambda})}\mathfrak{Y}(r). \tag{2.82}$$

Hence,

$$\begin{aligned}
&\limsup_{\lambda\to\infty}\frac{1}{\lambda}\log Z_\lambda^{\gamma,\beta,C}(X^\lambda)\\
&\quad\le\limsup_{\delta\downarrow0}\limsup_{\lambda\to\infty}\frac{1}{\lambda}\log\#G(\delta,\lambda)\\
&\quad\quad+\limsup_{\delta\downarrow0}\limsup_{\lambda\to\infty}\frac{1}{\lambda}\log\sup_{\Psi^{\delta,\lambda}\in G(\delta,\lambda)\colon \mathrm{M}(\Psi^{\delta,\lambda})\le C}\Big[\frac{\#J^{\delta,\lambda}(\Psi^{\delta,\lambda})}{\inf_{r\in J^{\delta,\lambda}(\Psi^{\delta,\lambda})}\prod_{i\in I^\lambda}N(\lambda)^{-(r^i_{-1}-1)}}\\
&\quad\quad-\gamma\liminf_{\delta\downarrow0}\liminf_{\lambda\to\infty}\inf_{r\in J^{\delta,\lambda}(\Psi^{\delta,\lambda})}\frac{1}{\lambda}\mathfrak{S}(r)-\beta\liminf_{\delta\downarrow0}\liminf_{\lambda\to\infty}\inf_{r\in J^{\delta,\lambda}(\Psi^{\delta,\lambda})}\frac{1}{\lambda}\mathfrak{M}(r)\Big].
\end{aligned}\tag{2.83}$$

According to Lemma 2.9.4 below, the first term on the right-hand side is equal to zero. Now pick a sequence $(\delta_n)_n$ and for each n a sequence $(\lambda_{n,j})_j$ along which the superior limits as $n\to\infty$, respectively $j\to\infty$, are realized. Now choose, for any n and j, a maximizer $\widetilde{\Psi}^{\delta_n,\lambda_{n,j}}$. Pick λ_0 so large that $N(\lambda)\le 2\mu(W)\lambda$ for all $\lambda\ge\lambda_0$. Hence,

$$\bigcup_{\lambda>\lambda_0,\delta\in\mathbb{B}}G(\delta,\lambda)\subseteq\Big(\prod_{k=1}^{k_{\max}}\mathcal{M}_{\le2\mu(W)}(W^k)\Big)\times\mathcal{M}_{\le2\mu(W)}(W)^{\mathbb{N}_0}, \tag{2.84}$$

where we recall that $\mathcal{M}_{\le\alpha}(V)$ is the set of measures on a space V with total mass $\le\alpha$. Note that $\mathcal{M}_{\le2\mu(W)}(W^k)$ is compact in the weak topology of $\mathcal{M}(W^k)$ for any k, according to Prohorov's theorem.

Without loss of generality (using two diagonal sequence arguments), we can assume that for all $n\in\mathbb{N}$, $\widetilde{\Psi}^{\delta_n,\lambda_{n,j}}$ converges coordinatewise weakly to a collection of measures $\widetilde{\Psi}^{\delta_n}=((\widetilde{\nu}_k^{\delta_n})_{k=1}^{k_{\max}},(\widetilde{\mu}_m^{\delta_n})_{m=0}^\infty)$ as $j\to\infty$, and $\widetilde{\Psi}^{\delta_n}$ converges coordinatewise weakly to a collection of measures $\widetilde{\Psi}$ as $n\to\infty$. Then, it is clear that $\widetilde{\Psi}$ satisfies (i) from (2.19), and also that

$$\lim_{n\to\infty}\lim_{j\to\infty}\sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\widetilde{\nu}_k^{\delta_n,\lambda_{n,j}}=\sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\widetilde{\nu}_k.$$

In order to see that (iii) holds for $\widetilde{\Psi}$, it remains to show that $\lim_{n\to\infty}\lim_{j\to\infty}\sum_{m=0}^{\infty} m\widetilde{\mu}_m^{\delta_n,\lambda_{n,j}} = \sum_{m=0}^{\infty} m\widetilde{\mu}_m$. For $N\in\mathbb{N}$ and for any continuous function $f\colon W\to\mathbb{R}$, we estimate

$$\Big|\Big\langle \sum_{m=0}^{\infty} m(\widetilde{\mu}_m^{\delta_n,\lambda_{n,j}} - \widetilde{\mu}_m), f\Big\rangle\Big| \leq \sum_{m=0}^{N} m\left|\langle\widetilde{\mu}_m^{\delta_n,\lambda_{n,j}} - \widetilde{\mu}_m, f\rangle\right| + \sum_{m=N+1}^{\infty} \|f\|_\infty m\left|\widetilde{\mu}_m^{\delta_n,\lambda_{n,j}}(W) - \widetilde{\mu}_m(W)\right|.$$

The first term on the r.h.s. clearly tends to 0 as $j\to\infty$, followed by $n\to\infty$, for any fixed N. The second term can further be estimated from above as follows

$$\|f\|_\infty \sum_{m>N} \eta(m)\Big(\sup_{\widetilde{m}>N}\frac{\widetilde{m}}{\widetilde{\eta}(m)}\Big)(\widetilde{\mu}_m^{\delta_n,\lambda_{n,j}}(W) + \widetilde{\mu}_m(W)) \leq 2\|f\|_\infty \Big(\sup_{\widetilde{m}>N}\frac{\widetilde{m}}{\widetilde{\eta}(m)}\Big)C.$$

By the assumption that $(\eta(N)/N)\to\infty$ as $N\to\infty$, the right-hand side tends to 0. One can analogously show that $\sum_{m=0}^{\infty}\widetilde{\mu}_m^{\delta_n,\lambda_{n,j}}$ tends to $\sum_{m=0}^{\infty}\widetilde{\mu}_m$ as $j\to\infty$ followed by $n\to\infty$, and hence condition (ii) from (2.19) holds. Also we have $\sum_{m=0}^{\infty}\eta(m)\widetilde{\mu}_m(W)\leq C$. Altogether, $\widetilde{\Psi}$ is an admissible trajectory setting.

Now, using the arguments of the proofs of Propositions 2.9.1 and 2.9.2 (which also involve the coarsened limits $\widetilde{\Psi}^{\delta_n}$ for fixed $n\in\mathbb{N}$) for the subsequential limits $j\to\infty$ followed by $n\to\infty$, we conclude that

$$\lim_{n\to\infty}\lim_{j\to\infty}\frac{\# J^{\delta_n,\lambda_{n,j}}(\widetilde{\Psi}^{\delta_n,\lambda_{n,j}})}{\inf_{r\in J^{\delta_n,\lambda_{n,j}}(\widetilde{\Psi}^{\delta_n,\lambda_{n,j}})}\prod_{i\in I^{\lambda_{n,j}}} N(\lambda_{n,j})^{-(r_{-1}^i-1)}} = \mathrm{I}(\widetilde{\Psi})$$

and, using the boundedness and continuity of each f_k on $\mathcal{M}_{\leq 2\mu(W)}(W)\times W^k$,

$$\lim_{n\to\infty}\lim_{j\to\infty}\inf_{r\in J^{\delta_n,\lambda_{n,j}}(\widetilde{\Psi}^{\delta_n,\lambda_{n,j}})}\frac{1}{\lambda_{n,j}}\mathfrak{S}(r) = \mathrm{S}(\widetilde{\Psi}).$$

Furthermore, the lower semicontinuity of $\mathcal{M}(W)^{\mathbb{N}_0}\to(-\infty,\infty]$, $(\nu_m)_{m\in\mathbb{N}_0}\mapsto\sum_{m\in\mathbb{N}_0}\eta(m)\nu_m(W)$, together with Fatou's lemma implies that

$$-\beta\liminf_{n\to\infty}\liminf_{j\to\infty}\inf_{r\in J^{\delta_n,\lambda_{n,j}}(\widetilde{\Psi}^{\delta_n,\lambda_{n,j}})}\frac{1}{\lambda_{n,j}}\mathfrak{M}(r)\leq -\beta\mathrm{M}(\widetilde{\Psi}). \tag{2.85}$$

Thus, we conclude that (2.81) (and therefore the upper bound in Theorem 2.2.2) holds, as soon as Lemma 2.9.4 is formulated and verified. This we do now. Recall that we are working with fixed $\omega\in\Omega_1$ and that the notion of $G(\delta,\lambda)$ depends on ω via $G(\delta,\lambda) = G(\delta,\lambda)(X^\lambda(\omega))$.

Lemma 2.9.4. *For any $\delta\in\mathbb{B}$ and $\omega\in\Omega_1$, we have*

$$\limsup_{\lambda\to\infty}\frac{1}{\lambda}\log\#G(\delta,\lambda) = 0.$$

Proof. For $\lambda>0$, let $G_1(\delta,\lambda)$ denote the set of $(\nu_k^{\delta,\lambda})_{k=1}^{k_{\max}}$ satisfying part (2) of Definition 2.8.1. It is easily seen that its cardinality increases only polynomially in λ. Now, given $(\nu_k^{\delta,\lambda})_{k=1}^{k_{\max}}\in G_1(\delta,\lambda)$, we will give an upper bound for the number of $(\mu_m^{\delta,\lambda})_{m=0}^{\infty}$ such that the pair of these tuples is in $G(\delta,\lambda)$. This is much more demanding, since there is *a priori* no upper bound for m. We will provide a λ-dependent one.

For any $\lambda>0$, $\Psi^{\delta,\lambda}\in G(\delta,\lambda)$ and $j=1,\dots,\delta^{-d}$ we have that

$$\lambda\sum_{m=0}^{\infty} m\mu_m^{\delta,\lambda}(W_j^\delta) = \lambda\sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\nu_k^{\delta,\lambda}(W_j^\delta)\leq(k_{\max}-1)N(\lambda),$$

and the numbers $\mu_0^{\delta,\lambda}(W_j^\delta),\dots,\mu_{(k_{\max}-1)N(\lambda)}^{\delta,\lambda}(W_j^\delta)$, are $\frac{1}{\lambda}$ times nonnegative integers. In particular, $\mu_m^{\delta,\lambda}(W_j^\delta) = 0$ for $m>(k_{\max}-1)N(\lambda)$.

Let $\varepsilon>0$ be fixed. We claim that for all sufficiently large $\lambda>0$, there are not more than $\varepsilon N(\lambda)\sim\varepsilon\lambda\mu(W)$ nonzero ones out of these quantities. Indeed, if there were at least $\lceil\varepsilon N(\lambda)\rceil$ nonzero ones, denoted

$\mu^{\delta,\lambda}_{m_0}(W^\delta_j),\dots,\mu^{\delta,\lambda}_{m_{\lceil\varepsilon N(\lambda)\rceil-1}}(W^\delta_j)$ with $0\leq m_0<m_1<\ldots<m_{\lceil\varepsilon N(\lambda)\rceil-1}\leq(k_{\max}-1)N(\lambda)$, then we could estimate

$$
\begin{aligned}
(k_{\max}-1)N(\lambda)&\geq\sum_{m=0}^{(k_{\max}-1)N(\lambda)}\lambda m\mu^{\delta,\lambda}_m(W^\delta_j)\geq\sum_{i=0}^{\lceil\varepsilon N(\lambda)\rceil-1}\lambda m_i\mu^{\delta,\lambda}_{m_i}(W^\delta_j)\mathbb{1}\{\mu^{\delta,\lambda}_{m_i}(W^\delta_j)>0\}\\
&=\sum_{i=0}^{\lceil\varepsilon N(\lambda)\rceil-1}\lambda m_i\mu^{\delta,\lambda}_{m_i}(W^\delta_j)\mathbb{1}\{\mu^{\delta,\lambda}_{m_i}(W^\delta_j)\geq\frac{1}{\lambda}\}\geq\sum_{i=0}^{\lceil\varepsilon N(\lambda)\rceil-1}m_i\geq\sum_{m=0}^{\lceil\varepsilon N(\lambda)\rceil-1}m\sim\frac12(\varepsilon N(\lambda))(\varepsilon N(\lambda)-1),
\end{aligned}
$$

which would be a contradiction for all $\lambda>0$ sufficiently large.

Now, $\#G(\delta,\lambda)$ can be estimated as follows. Let us first fix $(\nu^{\delta,\lambda}_k)_{k=1}^{k_{\max}}\in G_1(\delta,\lambda)$, i.e., satisfying part (2) from Definition 2.8.1, and let us count the number of $(\mu^{\delta,\lambda}_m)_{m=0}^{(k_{\max}-1)N(\lambda)}$ such that $((\nu^{\delta,\lambda}_k)_{k=1}^{k_{\max}},(\mu^{\delta,\lambda}_m)_{m=0}^{(k_{\max}-1)N(\lambda)})$ lies in $G(\delta,\lambda)$. Out of the $k_{\max}\delta^{-d}N(\lambda)$ quantities $\mu^{\delta,\lambda}_0(W^\delta_j),\dots,\mu^{\delta,\lambda}_{(k_{\max}-1)N(\lambda)}(W^\delta_j)$, $j=1,\dots,\delta^{-d}$, at most $\lceil\varepsilon N(\lambda)\rceil\delta^{-d}$ are nonzero. The number of ways to choose them equals $\binom{k_{\max}N(\lambda)\delta^{-d}}{\lceil\varepsilon N(\lambda)\rceil\delta^{-d}}$. Having chosen $\lceil\varepsilon N(\lambda)\rceil\delta^{-d}$ potentially nonzero ones so that the remaining $k_{\max}\delta^{-d}N(\lambda)-\lceil\varepsilon N(\lambda)\rceil\delta^{-d}$ ones are equal to zero, the potentially nonzero ones sum up to $N(\lambda)$ according to part (5) of Definition 2.8.1, and each one has a value in $\frac{1}{\lambda}\mathbb{N}_0$. For this, there are at most $\binom{N(\lambda)+\lceil\varepsilon N(\lambda)\rceil\delta^{-d}-1}{\lceil\varepsilon N(\lambda)\rceil\delta^{-d}-1}$ combinations, for any choice of the set of the potentially nonzero ones. Since $\omega\in\Omega_1$, we have $N(\lambda)=N(\lambda)(\omega)=\lambda(\mu(W)+o(1))$ as $\lambda\to\infty$ (where the $o(1)$ term depends on ω). Therefore, using Stirling's formula as in (2.64), we have the following estimate in the limit $\lambda\to\infty$

$$
\begin{aligned}
\#G(\delta,\lambda)&\leq\#G_1(\delta,\lambda)\binom{k_{\max}N(\lambda)\delta^{-d}}{\lceil\varepsilon N(\lambda)\rceil\delta^{-d}}\binom{N(\lambda)+\lceil\varepsilon N(\lambda)\rceil\delta^{-d}-1}{\lceil\varepsilon N(\lambda)\rceil\delta^{-d}-1}\\
&=\mathrm{e}^{o(\lambda)}\exp\Big(-\lambda\mu(W)\Big((k_{\max}-\varepsilon)\delta^{-d}\log\frac{(k_{\max}-\varepsilon)\delta^{-d}}{k_{\max}\delta^{-d}}+\varepsilon\delta^{-d}\log\frac{\varepsilon\delta^{-d}}{k_{\max}\delta^{-d}}\Big)\Big)\\
&\quad\times\exp\Big(-\lambda\mu(W)\Big(\varepsilon\delta^{-d}\log\frac{\varepsilon\delta^{-d}}{1+\varepsilon\delta^{-d}}+\log\frac{1}{1+\varepsilon\delta^{-d}}\Big)\Big).
\end{aligned}
$$

Letting $\varepsilon\downarrow0$, we conclude that $\limsup_{\lambda\to\infty}\frac{1}{\lambda}\log\#G(\delta,\lambda)=0$. □

2.9.5. The large deviation principle: proof of Theorem 2.2.4(i)

In this section we prove Theorem 2.2.4(i). The combinatorial essence of this theorem has already been proved in Proposition 2.9.1, including the relations with δ-coarsenings. What remains to be done is to relate this to the coordinatewise weak convergence on $\mathcal{A}$. We will be able to use some of the arguments of Section 2.9.4.

The lower semicontinuity of $\mathrm{I}+\mu(W)\log k_{\max}$ was already discussed in Section 2.2.1, the nonnegativity in Section 2.2.3. These together mean that $\mathrm{I}+\mu(W)\log k_{\max}$ is a rate function.

We proceed with the proof of the lower bound. Let $G\subseteq\mathcal{A}$ be open. If $\inf_G\mathrm{I}=\infty$, then there is nothing to show, therefore let us assume that there exists $\Psi\in G$ with $\mathrm{I}(\Psi)<\infty$. Let us choose a standard setting $\underline{\Psi}$ containing Ψ according to the construction in the proof of Proposition 2.9.3. Since G is open, there exists $\varrho>0$ such that $B_\varrho(\Psi)\subseteq G$. Let us choose $\delta_0\in\mathbb{B}$ and, for any $\mathbb{B}\ni\delta\leq\delta_0$, some $\lambda_0=\lambda_0(\delta)>0$ such that $\Psi^\delta,\Psi^{\delta,\lambda}\in B_\varrho(\Psi)$ for any $\lambda>\lambda_0$. Now we can estimate, for these δ and λ,

$$
\begin{aligned}
\mathrm{P}^{0,0}_{\lambda,X^\lambda}(\Psi_\lambda(S)\in G)&\geq\mathrm{P}^{0,0}_{\lambda,X^\lambda}(\Psi_\lambda(S)\in B_\varrho(\Psi))\geq\mathrm{P}^{0,0}_{\lambda,X^\lambda}\big((\Psi_\lambda(S))^\delta=\Psi^{\delta,\lambda}\big)\\
&=\frac{1}{Z^{0,0}_\lambda(X^\lambda)}\sum_{r\in J^{\delta,\lambda}(\Psi^{\delta,\lambda})}\frac{1}{\prod_{i\in I^\lambda}N(\lambda)^{r^i_{-1}-1}}\geq\frac{\#J^{\delta,\lambda}(\Psi^{\delta,\lambda})}{k_{\max}^{N(\lambda)}\sup_{r\in J^{\delta,\lambda}(\Psi^{\delta,\lambda})}\prod_{i\in I^\lambda}N(\lambda)^{r^i_{-1}-1}}.
\end{aligned}
$$

Now, using Proposition 2.9.1 and the fact that $N(\lambda)/\lambda\to\mu(W)$, we obtain

$$
\liminf_{\lambda\to\infty}\frac{1}{\lambda}\log\mathrm{P}^{0,0}_{\lambda,X^\lambda}(\Psi_\lambda(S)\in G)\geq-\mu(W)\log k_{\max}-\mathrm{I}(\Psi).
$$

Note that $\underline{\Psi}$ is not necessarily controlled because $\mathrm{M}(\Psi) < \infty$ is not guaranteed. However, since for all $\delta \in \mathbb{B}$, $s = 1, \dots, \delta^{-d}$ and $\lambda > 0$, $\mu_m^{\delta,\lambda}(W_s^\delta)/\mu_m^\delta(W_s^\delta)$ does not depend on m (apart from some error terms the sum of which over m is of order $1/\lambda$, cf. (2.74)), we easily see that Proposition 2.9.1 holds for this $\underline{\Psi}$ as well. Now, take the supremum over $\Psi \in G \cap \{\mathrm{I} < \infty\}$ to conclude that the lower bound holds.

We continue with the upper bound. Let $F \subseteq \mathcal{A}$ be closed. Let us choose an increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive numbers along which the limes superior in (2.30) is realized. For $\lambda > 0$, let us put

$$O(\lambda) = \big\{\Psi \in \mathcal{A} \colon \mathrm{P}^{0,0}_{\lambda,X^\lambda}(\Psi_\lambda(S) = \Psi) > 0\big\}.$$

If for all but finitely many $n \in \mathbb{N}$ we have $F \cap O(\lambda_n) = \varnothing$, then

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log \mathrm{P}^{0,0}_{\lambda,X^\lambda}(\Psi_\lambda(S) \in F) = -\infty. \tag{2.86}$$

Therefore, without loss of generality, we can assume that $O(\lambda_n) \cap F$ is non-empty for all $n \in \mathbb{N}$. For $\delta \in \mathbb{B}$ and $A \subset \mathcal{A}$, let us write $A^\delta = \{\Psi^\delta \colon \Psi \in A\}$, where Ψ^δ is the coordinatewise δ-coarsened version of Ψ. Then we have

$$\begin{aligned} \mathrm{P}^{0,0}_{\lambda_n,X^{\lambda_n}}\big(\Psi_{\lambda_n}(S) \in F\big) &= \mathrm{P}^{0,0}_{\lambda_n,X^{\lambda_n}}\big(\Psi_{\lambda_n}(S) \in F \cap O(\lambda_n)\big) = \mathrm{P}^{0,0}_{\lambda_n,X^{\lambda_n}}\big((\Psi_{\lambda_n}(S))^\delta \in (F \cap O(\lambda_n))^\delta\big) \\ &\leq \#(F \cap O(\lambda_n))^\delta \sup_{\Psi \in F \cap O(\lambda_n)} \frac{\#J^{\delta,\lambda_n}(\Psi^\delta)}{k_{\max}^{N(\lambda_n)} \inf_{r \in J^{\delta,\lambda_n}(\Psi^\delta)} \prod_{i \in I^{\lambda_n}} N(\lambda_n)^{r^i_{-1}-1}}. \end{aligned} \tag{2.87}$$

It is clear that $(F \cap O(\lambda_n))^\delta \subseteq G(\delta,\lambda_n) = (O(\lambda_n))^\delta$ for all $n \in \mathbb{N}$ and $\delta \in \mathbb{B}$, where $G(\delta,\lambda_n)$ was defined in Section 2.9.4. Hence, by Lemma 2.9.4,

$$\limsup_{\delta\downarrow 0} \limsup_{n\to\infty} \frac{1}{\lambda_n} \log \#(F \cap O(\lambda_n))^\delta = 0.$$

It remains to show that

$$\limsup_{\delta\downarrow 0} \limsup_{n\to\infty} \frac{1}{\lambda_n} \log \Big[\sup_{\Psi \in F \cap O(\lambda_n)} \frac{\#J^{\delta,\lambda_n}(\Psi^\delta)}{\inf_{r \in J^{\delta,\lambda_n}(\Psi^\delta)} \prod_{i \in I^{\lambda_n}} N(\lambda_n)^{r^i_{-1}-1}} \Big] \leq -\inf_{\Psi \in F} \mathrm{I}(\Psi). \tag{2.88}$$

One can do this analogously to the proof of the upper bound of Theorem 2.2.2 starting from (2.83). Indeed, using Prohorov's theorem together with a diagonal sequence argument, we find $\Psi^* \in \mathcal{A}$ that the maximizer in (2.87) converges to along a subsequence of δ's and λ_n's. The limit lies in F because F is closed. Using the lower semicontinuity of I together with Fatou's lemma, we conclude that the left-hand side of (2.88) is not larger than $-\mathrm{I}(\Psi^*)$, which itself is not larger than $-\inf_F \mathrm{I}$. This finishes the proof of the upper bound in Theorem 2.2.4(i).

2.10. Analysis of the minimizer

This section is devoted to the proof of Proposition 2.2.3. In particular, in Section 2.10.1 we show that the infimum in (2.24) is uniquely attained, and for the minimizer $\Psi = ((\nu_k)_{k=1}^{k_{\max}}, (\mu_m)_{m=0}^\infty)$, for any $k \in [k_{\max}]$, $\mu^{\otimes k}$ is absolutely continuous with respect to ν_k and μ is absolutely continuous with respect to each μ_m. Further, for all $k \in [k_{\max}]$ and $m \in \mathbb{N}_0$, there exist constants $c_k > 0$, $k \in [k_{\max}]$, and $c'_m > 0$, $m \in \mathbb{N}_0$ such that $\nu_k(A) \geq c_k \mu^{\otimes k}(A)$ for all $A \subseteq W^k$ measurable and $\mu_m(A') \geq c'_m(A')$ for all $A' \subseteq W$ measurable. This is a prerequisite for perturbing the minimizer in many admissible directions. In Section 2.10.2 we finish the proof of Proposition 2.2.3 by deriving the Euler–Lagrange equations. For the rest of the section, we fix all parameters W, μ, γ, β, and $k_{\max}$. Moreover, we use the following representation of I from (2.23).

$$\mathrm{I}(\Psi) = \sum_{k=1}^{k_{\max}} H_{W^k}(\mu \otimes M^{\otimes(k-1)}) + \sum_{m=0}^{\infty} \Big[H_W(\mu_m \mid \mu) - \mu_m(W) \log \frac{(\mathrm{e}\mu(W))^{-m}}{m!} \Big]. \tag{2.89}$$

2.10.1. Existence, uniqueness, and positivity of the minimizer

We start with the following lemma, which follows almost immediately from the arguments of the proof of the upper bound of Theorem 2.2.2 in Section 2.9.4, given the strict convexity of I on its level sets.

Lemma 2.10.1. *The variational formula in* (2.24) *has precisely one minimizer.*

Proof. Recall that the three functionals I, S, M are lower semicontinuous and convex. Furthermore, it is clear that we can restrict the infimum in (2.24) to those Ψ that satisfy also $\mathrm{M}(\Psi) \le C$ for any sufficiently large C. But, as we have seen in Section 2.9.4, this set of Ψ's is compact. This implies that the set of minimizers is non-empty, compact, and convex (where we also used that the set of admissible trajectory settings is closed under convex combinations).

Further, I is strictly convex on its level sets (cf. Section A.1 in the Appendix), and therefore so is $\mathrm{I}+\gamma\mathrm{S}+\beta\mathrm{M}$ (which is not identically $+\infty$ either). This implies uniqueness of the minimizer; indeed, assume that $\Psi_1 \neq \Psi_2$ are minimizers, then by convexity of the set of minimizers, $\Psi = (\Psi_1+\Psi_2)/2$ is also a minimizer, and by strict convexity of $\mathrm{I}+\gamma\mathrm{S}+\beta\mathrm{M}$ on its level sets, it follows that $\mathrm{I}(\Psi)+\gamma\mathrm{S}(\Psi)+\beta\mathrm{M}(\Psi) < \mathrm{I}(\Psi_1)+\gamma\mathrm{S}(\Psi_1)+\beta\mathrm{M}(\Psi_1) < \infty$, which contradicts with the assumption that Ψ_1 is a minimizer. $\square$

Now we prove that, for each minimizer Ψ, $\mu^{\otimes k}$ is absolutely continuous with respect to ν_k and μ is absolutely continuous with respect to each μ_m, and the corresponding Radon–Nikodym derivatives are even bounded away from 0. (Note that the opposite absolute continuities are true by finiteness of the relative entropies in (2.23).) We need to show this only for $k_{\max} > 1$, as we explained after Proposition 2.2.3. Let us start with verifying the absolute continuities.

Lemma 2.10.2. *If $k_{\max} > 1$ and $\Psi = ((\nu_k)_{k=1}^{k_{\max}}, (\mu_m)_{m=0}^{\infty})$ is the unique minimizer of* (2.24)*, then $\mu^{\otimes k} \ll \nu_k$ for any $k \in [k_{\max}]$ and $\mu \ll \mu_m$ for any $m \in \mathbb{N}_0$.*

Proof. The essence of the proof is the following. The functionals $\mathrm{M}(\cdot)$ and $\mathrm{S}(\cdot)$ are linear in each μ_m respectively ν_k, as well as the third term in $\mathrm{I}(\cdot)$ in (2.23) in each μ_m. On the other hand, the function $x \mapsto x\log x$ has the slope $-\infty$ at $x \downarrow 0$. We show the following assertions about the minimizer Ψ step by step as follows. Recall that $M = \sum_{m\in\mathbb{N}_0} m\mu_m = \sum_{k\in[k_{\max}]}\sum_{l=1}^{k-1}\pi_l\nu_k$. We write $\ge$ and $>$, respectively, between measures in $\mathcal{M}(W^k)$ if their difference lies in $\mathcal{M}(W^k)$, respectively in $\mathcal{M}(W^k)\setminus\{0\}$.

Fix a measurable set $A \subset W$ such that $\mu(A) > 0$. Then we have:

1. $M(A) > 0$.
2. for any $m_1 < m_0 < m_2$ such that $\mu_{m_1}(A) > 0$ and $\mu_{m_2}(A) > 0$, also $\mu_{m_0}(A) > 0$.
3. $\mu_0(A) > 0$.
4. $\mu_m(A) > 0$ for any $m \ge k_{\max}$.
5. $\nu_k(A^k) > 0$ for any $k \in [k_{\max}]$.

Indeed, these steps are verified respectively as follows. In each of the steps, for $\varepsilon \in (0,1)$, we construct an admissible trajectory setting $\Psi^\varepsilon = ((\nu_k^\varepsilon)_{k=1}^{k_{\max}}, (\mu_m^\varepsilon)_{m=0}^{\infty})$ such that $\mathrm{I}(\Psi^\varepsilon)+\gamma\mathrm{S}(\Psi^\varepsilon)+\beta\mathrm{M}(\Psi^\varepsilon) < \mathrm{I}(\Psi)+\gamma\mathrm{S}(\Psi)+\beta\mathrm{M}(\Psi)$ for sufficiently small $\varepsilon > 0$, and therefore Ψ is not a minimizer of (2.24).

1. If $M(A) = 0$, then in particular $\mu_0(A) = \nu_1(A) = \mu(A)$ and $\mu_m(A) = 0$ for all $m > 0$. Also, $\pi_1\nu_2(A) = \nu_2(W\times A) = 0$, according to the definition of M.

   Let us define Ψ^ε as follows: $\nu_2^\varepsilon = (1-\varepsilon)\nu_2 + \varepsilon(\mu^{\otimes 2})/\mu(W)$, $\nu_k^\varepsilon = (1-\varepsilon)\nu_k$ for $k \neq 2$, $\mu_1^\varepsilon = (1-\varepsilon)\mu_1 + \varepsilon\mu$ and $\mu_m^\varepsilon = (1-\varepsilon)\mu_m$ for $m \neq 1$. Then we compute and estimate the three terms of the entropy $\mathrm{I}(\Psi)$ as follows.

$$\begin{aligned}
&\sum_{k=1}^{k_{\max}} H_{W^k}\big(\nu_k^\varepsilon \mid \mu \otimes (M^\varepsilon)^{\otimes(k-1)}\big) \\
&\le \sum_{k=1}^{k_{\max}} H_{W\times(W\setminus A)^{k-1}}((1-\varepsilon)\nu_k \mid \mu \otimes (M^\varepsilon)^{\otimes(k-1)}) + H_{W\times A}\Big(\frac{\varepsilon\mu^{\otimes 2}}{\mu(W)} \mid \varepsilon\mu^{\otimes 2}\Big) + \mathcal{O}(\varepsilon) \\
&\le \sum_{k=1}^{k_{\max}} H_{W^k}(\mu \otimes M^{\otimes(k-1)}) + \mathcal{O}(\varepsilon),
\end{aligned}$$

furthermore,

$$\begin{aligned}
&\sum_{m=0}^{\infty} H_W(\mu_m^\varepsilon \mid \mu) - \mu_m^\varepsilon(W) \log \frac{(\mathrm{e}\mu(W))^{-m}}{m!} \\
&\le H_W((1-\varepsilon)\mu_m \mid \mu) - \mu_m(W) \log \frac{(\mathrm{e}\mu(W))^{-m}}{m!} + \mu(A)\varepsilon \log \varepsilon + \mathcal{O}(\varepsilon).
\end{aligned} \tag{2.90}$$

For the second term we used the convexity of the relative entropy in the form

$$H_W((1-\varepsilon)\mu_1 + \varepsilon\mu \mid \mu) \le (1-\varepsilon) H_W(\mu_1 \mid \mu) \le H_W(\mu_1 \mid \mu) + \mathcal{O}(\varepsilon). \tag{2.91}$$

This in turn follows from [HJKP18, Lemmas 3.10, 3.11], which implies that, for any $k \in \mathbb{N}$, $\xi,\eta \in \mathcal{M}(W^k)$ with $\eta \neq 0$ and $\xi \ll \eta$,

$$\Big| H_{W^k}(\xi \mid \eta) - H_{W^k}((1-\varepsilon)\xi \mid \eta) \Big| \underset{\varepsilon\downarrow 0}{\asymp} \varepsilon.$$

It follows that, as $\varepsilon \downarrow 0$,

$$\mathrm{I}(\Psi^\varepsilon) + \gamma \mathrm{S}(\Psi^\varepsilon) + \beta \mathrm{M}(\Psi^\varepsilon) - \big[\mathrm{I}(\Psi) + \gamma \mathrm{S}(\Psi) + \beta \mathrm{M}(\Psi)\big] \le \mathcal{O}(\varepsilon) + \mu(A)\varepsilon \log \varepsilon, \tag{2.92}$$

which is negative for all sufficiently small $\varepsilon > 0$. Thus, Ψ is not a minimizer.

2. If $M(A) > 0$ but $\mu_{m_1}(A) > 0$, $\mu_{m_2}(A) > 0$, and $\mu_{m_0}(A) = 0$ for some $m_1 < m_0 < m_2$, then let $\nu_k^\varepsilon = \nu_k$ for all $k \in [k_{\max}]$, and let $\mu_{m_0}^\varepsilon = (1-\varepsilon)\mu_{m_0} + \varepsilon(\alpha_1\mu_{m_1} + \alpha_2\mu_{m_2})$, $\mu_{m_1}^\varepsilon = (1-\alpha_1\varepsilon)\mu_{m_1}$, $\mu_{m_2}^\varepsilon = (1-\varepsilon\alpha_2)\mu_{m_2}$, where $\alpha_1,\alpha_2 \in (0,1)$ are such that $\alpha_1 + \alpha_2 = 1$ and $m_1\alpha_1 + m_2\alpha_2 = m_0$. Then, Ψ^ε is an admissible trajectory setting with $M^\varepsilon = M$. It follows similarly to the previous computation that $\mathrm{I}(\Psi^\varepsilon) + \gamma \mathrm{S}(\Psi^\varepsilon) + \beta \mathrm{M}(\Psi^\varepsilon) < \mathrm{I}(\Psi) + \gamma \mathrm{S}(\Psi) + \beta \mathrm{M}(\Psi)$ for all sufficiently small $\varepsilon > 0$. However, instead of (2.90), we have

$$\begin{aligned}
&\sum_{m=0}^{\infty} H_W(\mu_m^\varepsilon \mid \mu) - \mu_m^\varepsilon(W) \log \frac{(\mathrm{e}\mu(W))^{-m}}{m!} \\
&\le \sum_{m=0}^{\infty} H_W(\mu_m \mid \mu) - \mu_m(W) \log \frac{(\mathrm{e}\mu(W))^{-m}}{m!} + (\alpha_1\mu_{m_1}(A) + \alpha_2\mu_{m_2}(A))\varepsilon \log \varepsilon + \mathcal{O}(\varepsilon),
\end{aligned}$$

as $\varepsilon \downarrow 0$.

3. If $M(A) > 0$ but $\mu_0(A) = 0$, let $\nu_k^\varepsilon = (1-\varepsilon)\nu_k$ for all $1 < k \le k_{\max}$, $\mu_m^\varepsilon = (1-\varepsilon)\mu_m$ for all $m > 0$, $\mu_0^\varepsilon = \varepsilon\mu + (1-\varepsilon)\mu_0$ and $\nu_1^\varepsilon = (1-\varepsilon)\nu_1 + \varepsilon\mu$. It is again sufficient to consider the entropy terms in I. The summands on $k > 1$ can be estimated as follows.

$$\begin{aligned}
\sum_{k=2}^{k_{\max}} H_{W^k}(\nu_k^\varepsilon \mid \mu \otimes (M^\varepsilon)^{(k-1)}) &= \sum_{k=2}^{k_{\max}} H_{W^k}((1-\varepsilon)\nu_k \mid (1-\varepsilon)^{k-1}\mu \otimes M^{k-1}) \\
&\le \sum_{k=2}^{k_{\max}} H_{W^k}(\nu_k \mid \mu \otimes M^{(k-1)}) + \mathcal{O}(\varepsilon).
\end{aligned}$$

The summand for $k = 1$ can be estimated with the help of (2.91). For the summand for $m = 0$, we have

$$\begin{aligned}
H_W(\mu_0^\varepsilon \mid \mu) &= H_{W\setminus A}((1-\varepsilon)\mu_0 + \varepsilon\mu \mid \mu) + \mu(A)\varepsilon \log \varepsilon \\
&\le H_{W\setminus A}((1-\varepsilon)\mu_0 \mid \mu) + \mu(A)\varepsilon \log \varepsilon + \mathcal{O}(\varepsilon) = H_W(\mu_0 \mid \mu) + \mu(A)\varepsilon \log \varepsilon + \mathcal{O}(\varepsilon).
\end{aligned}$$

while the remaining sum is handled as follows.

$$\begin{aligned}&\sum_{m=1}^{\infty} H_W(\mu_m^\varepsilon \mid \mu) - \mu_m^\varepsilon(W)\log\frac{(\mathrm{e}\mu(W))^{-m}}{m!}\\&= \sum_{m=1}^{\infty} H_W((1-\varepsilon)\mu_m \mid \mu) - \mu_m(W)\log\frac{(\mathrm{e}\mu(W))^{-m}}{m!} + \mathcal{O}(\varepsilon).\end{aligned}$$

Thus, (2.92) holds also here, which implies the claim.

4. If $M(A) > 0$ but $\mu_{m_0}(A) = 0$ for some $m_0 \geq k_{\max}$, let $\mu_{m_0}^\varepsilon = (1-\varepsilon)\mu_{m_0} + \varepsilon M/m_0$, $\mu_m^\varepsilon = (1-\varepsilon)\mu_m$ for $m \notin \{0,m_0\}$, and $\nu_k^\varepsilon = \nu_k$ for all $k \in [k_{\max}]$. Then,

$$\sum_{m=1}^{\infty} m\mu_m^\varepsilon = (1-\varepsilon)\sum_{m=1}^{\infty} m\mu_m + \frac{\varepsilon m_0}{m_0}\sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\nu_k = \sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\nu_k.$$

On the other hand, we have

$$\mu - \sum_{m=1}^{\infty}\mu_m^\varepsilon \geq \mu - (1-\varepsilon)\sum_{m=1}^{\infty}\mu_m - \frac{\varepsilon(k_{\max}-1)}{m_0}\mu \geq (1-\varepsilon)\mu - (1-\varepsilon)\sum_{m=1}^{\infty}\mu_m = (1-\varepsilon)\mu_0.$$

Therefore, if we put $\mu_0^\varepsilon = \mu - \sum_{m=1}^{\infty}\mu_m^\varepsilon$, then $\mu_0^\varepsilon \geq (1-\varepsilon)\mu_0 \geq 0$, in particular Ψ^ε is an admissible trajectory setting. Now we can proceed analogously to (3) to conclude that $\mathrm{I}(\Psi^\varepsilon) + \gamma\mathrm{S}(\Psi^\varepsilon) + \beta\mathrm{M}(\Psi^\varepsilon) < \mathrm{I}(\Psi) + \gamma\mathrm{S}(\Psi) + \beta\mathrm{M}(\Psi)$ for sufficiently small $\varepsilon > 0$.

The proof of (5) is very similar to the ones of (2), (3), and (4), thus we leave it to the reader. □

The proof of the following lemma is similar to the one of Lemma 2.10.2, therefore we omit its proof.

Lemma 2.10.3. *If $k_{\max} > 1$ and $\Psi = ((\nu_k)_{k=1}^{k_{\max}},(\mu_m)_{m=0}^{\infty})$ is the unique minimizer of (2.24), then for each $k \in [k_{\max}]$, there exists $c_k > 0$ such that for all $A \subseteq W^k$ measurable, $\nu_k(A) \geq c_k\mu^{\otimes k}(A)$ holds. Similarly, for each $m \in \mathbb{N}_0$, there exists $c_m' > 0$ such that for all $A' \subseteq W$ measurable, $\mu_m(A') \geq c_m'\mu(A')$ holds.*

2.10.2. Deriving the Euler–Lagrange equations

In this section we finish the proof of Proposition 2.2.3. Let us assume that $k_{\max} > 1$. According to the results of Section 2.10.1, now we see that (2.24) exhibits a unique minimizer, which has almost everywhere positive Lebesgue density on the corresponding powers of supp μ. Knowing this, we now carry out the perturbation analysis for the minimizer of the optimization problem in (2.24) and derive the shape of the minimizer in most explicit terms.

We use the method of Lagrange multipliers in the framework of a perturbation argument. Let $\Psi = ((\nu_k)_{k=1}^{k_{\max}},(\mu_m)_{m=0}^{\infty})$ minimize (2.24). Fix any collection of signed measures $\Phi = ((\tau_k)_{k=1}^{k_{\max}},(\sigma_m)_{m=0}^{\infty})$ such that only finitely many σ_m's are different from zero, the Radon–Nikodym derivative $\frac{\mathrm{d}\tau_k}{\mathrm{d}\mu^{\otimes k}}$ is a simple function for each k, also $\frac{\mathrm{d}\sigma_m}{\mathrm{d}\mu}$ is a simple function for each m, further, they satisfy the following constraints:

$$(i)\quad \sum_{k=1}^{k_{\max}}\pi_0\tau_k = 0, \qquad (ii)\quad \sum_{m=0}^{\infty}\sigma_m = 0, \qquad (iii)\quad M_\Phi := \sum_{m=0}^{\infty} m\sigma_m = \sum_{k=1}^{k_{\max}}\sum_{l=1}^{k-1}\pi_l\tau_k. \tag{2.93}$$

Then it follows from Lemma 2.10.3 that, for any $\varepsilon \in \mathbb{R}$ with sufficiently small $|\varepsilon|$, $\Psi + \varepsilon\Phi = ((\nu_k + \varepsilon\tau_k)_{k=1}^{k_{\max}},(\mu_m + \varepsilon\sigma_m)_{m=0}^{\infty})$ is a collection of (non-negative!) measures that satisfies (2.19) and is therefore admissible in the variational formula in (2.24). That (2.19) is satisfied follows easily from (2.93). Furthermore, using the notation of Section 2.10.1, the non-negativity follows from the fact that each τ_k respectively each σ_m is a finite linear combination of measures of the form $\mathbb{1}_A\,\mathrm{d}\mu^{\otimes k}$ with $A \subset W^k$ respectively of the form $\mathbb{1}_B\,\mathrm{d}\mu$ with $B \subset W$, and we have $\mathbb{1}_A\,\mathrm{d}\mu^{\otimes k} \leq c_k^{-1}\mathbb{1}_A\,\nu_k$ respectively $\mathbb{1}_B\,\mathrm{d}\mu \leq c_m'^{-1}\mathbb{1}_B\,\mathrm{d}\mu_m$. Since only finitely many such summands are involved, there is a constant $C > 0$ such that $|\tau_k| \leq C\nu_k$ and $|\sigma_m| \leq C\mu_m$ for any $k \in [k_{\max}]$ and $m \in \mathbb{N}_0$, and hence it suffices to take $|\varepsilon| < 1/C$.

From minimality, we deduce that

$$0=\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\Big(\mathrm{I}(\Psi+\varepsilon\Phi)+\gamma\mathrm{S}(\Psi+\varepsilon\Phi)+\beta\mathrm{M}(\Psi+\varepsilon\Phi)\Big). \tag{2.94}$$

We calculate the latter two terms as

$$\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\Big(\gamma\mathrm{S}(\Psi+\varepsilon\Phi)+\beta\mathrm{M}(\Psi+\varepsilon\Phi)\Big)=\gamma\sum_{k\in[k_{\max}]}\langle\tau_k,\widetilde{f}_k\rangle+\beta\sum_{m\in\mathbb{N}_0}\eta(m)\sigma_m(W),$$

where, as before, we used the notation $\langle\nu,f\rangle$ for the integral of a function f with respect to a measure ν. Abbreviating $M=\sum_{k\in[k_{\max}]}\sum_{l=1}^{k-1}\pi_l\nu_k$ and using the representation (2.63) of $\mathrm{I}(\cdot)$, we see that

$$\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\mathrm{I}(\Psi+\varepsilon\Phi)=\sum_{k\in[k_{\max}]}\Big\langle\tau_k,1+\log\frac{\mathrm{d}\nu_k}{\mathrm{d}\mu^{\otimes k}}\Big\rangle-\Big\langle M_\Phi,1+\log\frac{\mathrm{d}M}{\mathrm{d}\mu}\Big\rangle+\sum_{m\in\mathbb{N}_0}\Big\langle\sigma_m,1+\log\frac{\mathrm{d}\mu_m}{\mathrm{d}(c_m\mu)}\Big\rangle, \tag{2.95}$$

where we recall that $c_m=\frac{\mathrm{e}\mu(W))^{-m}}{m!}\mathrm{e}^{-1/(\mathrm{e}\mu(W))}$. Summarizing, we obtain from (2.94) that

$$0=\Big\langle\Phi,\big((h_k)_{k\in[k_{\max}]},(g_m)_{m\in\mathbb{N}_0}\big)\Big\rangle, \tag{2.96}$$

where

$$h_k=\gamma\widetilde{f}_k+2-k+\log\frac{\mathrm{d}\nu_k}{\mathrm{d}(\mu\otimes M^{\otimes(k-1)})}\qquad\text{and}\qquad g_m=\beta\eta(m)+1+\log\frac{\mathrm{d}\mu_m}{\mathrm{d}\mu}-\log\frac{(\mathrm{e}\mu(W))^{-m}}{m!}.$$

We conceive Φ as an element of the vector space

$$\mathcal{A}_\pm=\prod_{k\in[k_{\max}]}\mathcal{M}_\pm(W^k)\times\mathcal{M}_\pm(W)^{\mathbb{N}_0}$$

where $\mathcal{M}_\pm$ is the set of finite signed measures equipped with the weak topology, and $((h_k)_{k\in[k_{\max}]},(g_m)_{m\in\mathbb{N}_0})$ as a function on $\prod_{k\in[k_{\max}]}W^k\times W^{\mathbb{N}_0}$. The condition in (2.93) means that Φ is perpendicular to any function in

$$\begin{aligned}\mathcal{F}=\Big\{&((\varphi_k)_{k\in[k_{\max}]},(\psi_m)_{m\in\mathbb{N}_0})\colon\varphi_k\colon W^k\to\mathbb{R},\psi_m\colon W\to\mathbb{R}\text{ bounded and measurable for any }k,m,\\&\exists\widetilde{A},\widetilde{B},\widetilde{C}\colon W\to\mathbb{R}\colon\varphi_k(x_0,\dots,x_{k-1})=\widetilde{A}(x_0)+\sum_{l=1}^{k-1}\widetilde{C}(x_l),\\&\text{and }\psi_m(x)=\widetilde{B}(x)-m\widetilde{C}(x)\text{ for }x,x_0,\dots,x_{k-1}\in W\Big\}.\end{aligned}$$

We have shown that if Φ is perpendicular to any simple function in $\mathcal{F}$, then it is also perpendicular to $((h_k)_{k\in[k_{\max}]},(g_m)_{m\in\mathbb{N}_0})$. Since $\mathcal{F}$ is a closed linear subspace of $\mathcal{A}_\pm$, it follows that it contains this element. That is, there are three functions $\widetilde{A},\widetilde{B},\widetilde{C}$ on W such that, for any k respectively m,

$$h_k(x_0,\dots,x_{k-1})=\widetilde{A}(x_0)+\sum_{l=1}^{k-1}\widetilde{C}(x_l)\qquad\text{and}\qquad g_m(x)=\widetilde{B}(x)-m\widetilde{C}(x),\qquad x,x_0,\dots,x_{k-1}\in W.$$

Using an obvious substitution, this is equivalent to the existence of three positive functions A,B,C such that

$$\nu_k(\mathrm{d}x_0,\dots,\mathrm{d}x_{k-1})=\mu(\mathrm{d}x_0)\,A(x_0)\prod_{l=1}^{k-1}\big(C(x_l)M(\mathrm{d}x_l)\big)\mathrm{e}^{-\gamma\widetilde{f}_k(x_0,\dots,x_{k-1})},\qquad k\in[k_{\max}], \tag{2.97}$$

$$\mu_m(\mathrm{d}x)=\mu(\mathrm{d}x)\,B(x)\frac{(C(x)\mu(W))^{-m}}{m!}\mathrm{e}^{-\beta\eta(m)},\qquad m\in\mathbb{N}_0. \tag{2.98}$$

From (i) and (ii) in (2.19), we can identify A and B as

$$\begin{aligned}\frac{1}{A(x_0)}&=\sum_{k\in[k_{\max}]}\int_{W^{k-1}}\prod_{l=1}^{k-1}\big(C(x_l)M(\mathrm{d}x_l)\big)\mathrm{e}^{-\gamma\widetilde{f}_k(x_0,\dots,x_{k-1})},\\\frac{1}{B(x)}&=\sum_{m\in\mathbb{N}_0}\frac{(C(x)\mu(W))^{-m}}{m!}\mathrm{e}^{-\beta\eta(m)}.\end{aligned} \tag{2.99}$$

Furthermore, condition (iii) says that

$$\frac{1}{C(x)} = \frac{1}{C(x)} \frac{\mu(\mathrm{d}x)}{M(\mathrm{d}x)} \varphi\Big(\frac{1}{C(x)\mu(W)}\Big) = \Gamma(C\,\mathrm{d}M,x), \qquad x \in W, \tag{2.100}$$

where $\varphi(\alpha) = \sum_{m\in\mathbb{N}_0} m\frac{\alpha^m}{m!}\mathrm{e}^{-\beta\eta(m)} / \sum_{m\in\mathbb{N}_0} \frac{\alpha^m}{m!}\mathrm{e}^{-\beta\eta(m)}$ for $\alpha \in [0,\infty)$ and

$$\Gamma(\mathrm{d}\widetilde{M},x) = \int_W \mu(\mathrm{d}x_0) \frac{\sum_{k\in[k_{\max}]} \int_{W^{k-2}} \prod_{l=1}^{k-2} \widetilde{M}(\mathrm{d}x_l)\, F_k(x_0,x_1,\dots,x_{k-2},x)}{\sum_{k\in[k_{\max}]} \int_{W^{k-1}} \prod_{l=1}^{k-1} \widetilde{M}(\mathrm{d}x_l)\, \mathrm{e}^{-\gamma\widetilde{f}_k(x_0,\dots,x_{k-1})}}, \tag{2.101}$$

where

$$F_k(x_0,x_1,\dots,x_{k-2},x) = \sum_{l=1}^{k-1} \mathrm{e}^{-\gamma\widetilde{f}_k(x_0,y^l)}, \tag{2.102}$$

y^l is the vector of length $k-1$, consisting of $x_1,\dots,x_{k-2}$; augmented by x at the l-th place, and $\widetilde{M}(\mathrm{d}x) = C(x)M(\mathrm{d}x)$. This ends our derivation of the Euler–Lagrange equations for any minimizer Ψ of (2.24).

Since $\mathrm{I} + \gamma\mathrm{S} + \beta\mathrm{M}$ is convex, it follows that any admissible trajectory setting Ψ satisfying (2.97) – (2.102) is a minimizer of (2.24). Further, Lemma 2.10.1 guarantees that (2.24) has precisely one minimizer, which also satisfies these equations according to Lemma 2.10.3.

2.11. No penalization of congestion

This section consists of two parts. In Section 2.11.1 we carry out the proof of Proposition 2.2.5. Afterwards, in Section 2.11.2 we argue why the description of minimizer that we obtained in Section 2.10.2 is also valid in the case $\gamma > \beta = 0$.

2.11.1. Proof of Proposition 2.2.5

We proceed analogously to Sections 2.8 and 2.9, and thus we start with part (2), i.e., with verifying (2.34). We use the discretization argument from Section 2.8.1 again. We now provide the definition of a *transmission setting*, the analogue of Definition 2.8.1 of a standard setting with no reference to users receiving given numbers of incoming hops.

Definition 2.11.1. *A* transmission setting *is a collection of measures*

$$\underline{\Sigma} = \Big(\Sigma = (\nu_k)_{k=1}^{k_{\max}}, ((\nu_k^\delta)_{k=1}^{k_{\max}})_{\delta\in\mathbb{B}}, ((\nu_k^{\delta,\lambda})_{k=1}^{k_{\max}})_{\delta\in\mathbb{B},\lambda>0}, (\mu^{\delta,\lambda})_{\delta\in\mathbb{B},\lambda>0}\Big) \tag{2.103}$$

such that for any $\delta,\delta' \in \mathbb{B}$, $\lambda > 0$, $k \in [k_{\max}]$, and $s,s_0,\dots,s_{k-1} = 1,\dots,\delta^{-d}$, respectively, $\nu_k \in \mathcal{M}(W^k)$, and parts (1), (2), (3), *and* (4) *of Definition 2.8.1 hold.*

Recall that Definition 2.11.1 implies parts (A), (B), (C), and (D) of Remark 2.8.2. Further, it is easy to see that for any transmission setting $\underline{\Sigma}$, Σ is an asymptotic routeing strategy.

The following lemma describes the combinatorics of the choices of message trajectories in the system. We recall the empirical measures $(R_{\lambda,k}(s))_{k\in[k_{\max}]}$ from (2.5).

Lemma 2.11.2. *Let $\underline{\Sigma}$ be a transmission setting. For $\delta \in \mathbb{B}$ and $\lambda > 0$ let*

$$K^{\delta,\lambda}(\underline{\Sigma}) = \big\{s \in \mathcal{S}_{k_{\max}}(X^\lambda) : R^\delta_{\lambda,k}(s) = \nu_k^{\delta,\lambda}\ \forall k = 1,\dots,k_{\max}\big\}.$$

Then we have $\#K^{\delta,\lambda}(\underline{\Sigma}) = N^1_{\delta,\lambda}(\underline{\Sigma}) \times N^4_{\delta,\lambda}(\underline{\Sigma})$, where $N^1_{\delta,\lambda}(\underline{\Sigma})$ equals $N^1_{\delta,\lambda}(\underline{\Psi})$ from (2.58) *for any standard setting $\underline{\Psi}$ containing $\underline{\Sigma}$, and*

$$N^4_{\delta,\lambda}(\underline{\Sigma}) = \prod_{j=1}^{\delta^{-d}} (\lambda\mu^{\delta,\lambda}(W_j^\delta))^{\lambda \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l \nu_k^{\delta,\lambda}(W_j^\delta)}.$$

Proof. We proceed in two steps by counting first the trajectories, registering only the partition sets W_i^δ that they travel through, second, the choices of the relays for each hop in each partition set. Since every choice in the two steps can be freely combined with the other one, the product of the two cardinalities is equal to the number of all trajectory configurations with the prescribed coarsened empirical measures.

(A) *Number of the transmitters of trajectories passing through given sequences of δ-subcubes.* This is equal to the corresponding quantity in the proof of Lemma 2.8.6, hence it equals $N^1_{\delta,\lambda}(\underline{\Sigma})$.

(B) *Number of assignments of the hops to the relays.* For each $i = 1, \dots, \delta^{-d}$, there are $\lambda \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l \nu_k(W_i^\delta)$ incoming hops arriving to the relays in W_i^δ in total. Each incoming hop arriving at W_i^δ can choose any of the $\lambda\mu^{\delta,\lambda}(W_i^\delta)$ users as the corresponding relay. Such choices between different hops in W_i^δ are independent, moreover, all the choices in W_i^δ are independent from all the choices in W_j^δ for $j \neq i$. It follows that the number of assignments equals $N^4_{\delta,\lambda}(\underline{\Sigma})$.

We also see that all the choices in the two parts are independent of each other, i.e., they can be freely combined with each other and yield different combinations. Hence, we arrived at the assertion. □

Using the arguments of the proof of Proposition 2.9.1, the next lemma immediately follows.

Lemma 2.11.3. *Let $\underline{\Sigma}$ be a transmission setting. Then the following assertions hold almost surely.*

$$\lim_{\delta\downarrow 0} \lim_{\lambda\to\infty} \frac{1}{\lambda} \log \frac{\# K^{\delta,\lambda}(\underline{\Sigma})}{N^0_{\delta,\lambda}(\underline{\Sigma})} = -\mathrm{J}(\Sigma) \in [-\infty,\infty), \tag{2.104}$$

where $N^0_{\delta,\lambda}(\underline{\Sigma})$ equals $N^0_{\delta,\lambda}(\underline{\Psi})$ from (2.62) for any standard setting $\underline{\Psi}$ containing $\underline{\Sigma}$. Moreover, if the r.h.s. of (2.104) is finite, then

$$\lim_{\delta\downarrow 0} \lim_{\lambda\to\infty} \frac{1}{\lambda \log \lambda} \log \# K^{\delta,\lambda}(\underline{\Sigma}) = M(W) = \sum_{k=1}^{k_{\max}} (k-1)\nu_k(W).$$

Now, the identity in (2.34) follows from the proof of Theorem 2.2.2, using transmission settings instead of standard settings and replacing Proposition 2.9.1 with our Lemma 2.11.3. There is one more major change in the proof. Indeed, instead of the compactness of $\{\Psi\colon \Psi \text{ adm. trajectory setting}, \mathrm{M}(\Psi) \leq y\}$ for all $y \geq 0$ in Section 2.9.4 and the fact that any level set of $\mathrm{I} + \gamma\mathrm{S} + \beta\mathrm{M}$ is contained in a larger level set of M, one shall use the following argument. Using that S is continuous on the set of asymptotic routeing strategies, and that J is lower semicontinuous, bounded from below, and it has compact level sets [DZ98, Section 6.2], it follows that each level set of $\mathrm{J} + \gamma\mathrm{S}$ is included in a larger level set of S. Now, for all $y \in \mathbb{R}$, the set $\{\Sigma\colon \Sigma \text{ asymptotic routeing strategy}, \mathrm{S}(\Sigma) \leq y\}$ is compact, because it is closed and contained in the set $\{\Sigma\colon \Sigma \text{ asymptotic routeing strategy}, \sum_{k=1}^{k_{\max}} k\nu_k(W^k) \leq y'\}$ for all sufficiently large $y' \in \mathbb{R}$, and such sets are compact by Prohorov's theorem. These together allow us to conclude (2.34).

From this, parts (1) and (4) of Proposition 2.2.5 can be derived analogously to how Theorem 2.2.4 was derived from Theorem 2.2.2 in Section 2.9.5. The additional fact that the rate function $\mathrm{J} + \mu(W)\log k_{\max}$ has compact level sets holds because relative entropies with respect to fixed reference measures have compact level sets [DZ98, Section 6.2].

Lastly, we verify (3), i.e., we prove that (2.35) is the unique minimizer of (2.34). The fact that the set of minimizers of the variational formula on the right-hand side of (2.34) is non-empty, compact, and convex follows similarly to Lemma 2.10.1, again by Prohorov's theorem and the compactness of the sets $\{\Sigma\colon \Sigma \text{ asymptotic routeing strategy}, \mathrm{S}(\Psi) \leq y\}$, $y > 0$. Further, an argument analogous to Lemmas 2.10.2 and 2.10.3 shows that for all minimizers $\Sigma = (\nu_k)_{k\in[k_{\max}]}$, we have that $\nu_k \ll \mu^{\otimes k} \ll \nu_k$ and $\frac{\mathrm{d}\nu_k}{\mathrm{d}\mu^{\otimes k}}$ is bounded away from zero, for all $k \in [k_{\max}]$. Deriving the Euler–Lagrange equations similarly to Section 2.10.2 (but now for the variational formula (2.34)), it follows that (2.35) – (2.36) hold for any minimizer $\Sigma = (\nu_k)_{k\in[k_{\max}]}$ of (2.34). This also implies that the minimizer Σ is unique.[12] Thus, we conclude Proposition 2.2.5. □

[12] Alternatively, one could argue by (2.37) and the strict convexity of I on its level sets in order to conclude that J is strictly convex on its level sets and thus the minimizer must be unique, cf. Section 2.10.1.

2.11.2. The Euler–Lagrange equations in case of no penalization of congestion

Now, we argue that the equations (2.97) – (2.102) also hold for $\Psi = ((\nu_k)_{k\in[k_{\max}]}, (\mu_m)_{m\in\mathbb{N}_0})$ where $(\nu_k)_{k\in[k_{\max}]}$ is the unique minimizer (2.35) of the variational formula (2.34) corresponding to the case $\gamma > \beta = 0$ and $(\mu_m)_{m\in\mathbb{N}_0}$ is defined according to (2.38) with $M = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l \nu_k$. Indeed, it is easy to check that if we define the function $C\colon W \to [0,\infty)$ as $C(x) = \mu(\mathrm{d}x)/(M(\mathrm{d}x)\mu(W))$, then C satisfies (2.100) with $\beta = 0$. Thus, if we define the functions A and B according to (2.99) with this choice of C and with $\beta = 0$, then the measure ν_k defined according to (2.25) is the same as the one in (2.35), and the measure μ_m defined by (2.26) is the same as (2.38) (for the same value of $\gamma > 0$), for all $k \in [k_{\max}]$ and $m \in \mathbb{N}_0$. This verifies the claim. Here we used that for $\beta = 0$, φ in (2.100) is just the identity on $[0,\infty)$.

In fact, one easy way to conjecture that (2.38) is the unique minimizer of the variational formula (2.37) is to heuristically assume that all minimizers of (2.24) for $\beta = 0$ are of the form (2.25) – (2.26).[13] Then, (2.99) – (2.100) must hold for any of these minimizers because of the constraints in (2.19), and since φ is just the identity, it follows immediately that the equations (2.25) – (2.26) are equivalent to the ones (2.35) – (2.38), which also provides an alternative proof for the uniqueness of the minimizer of (2.24). Note that this is only a heuristic because the compactness argument in the proof of Lemma 2.10.1, which guaranteed existence of a minimizer of (2.24) for $\beta > 0$, is not valid in case $\beta = 0$. As already mentioned in Section 2.2.4, we decided to overcome this problem by deriving the behaviour of the highly dense system for $\beta = 0$ in terms of the measures $(R_{\lambda,k}(\cdot))_{k\in[k_{\max}]}$ only (cf. Proposition 2.2.5 and the proofs in Section 2.11), and to identify (2.38) as the unique minimizer of (2.37), omitting the details of this computation from the present thesis.

2.12. Discussion about no penalization of interference

2.12.1. Penalizing only congestion

Of mathematical interest can also be the case $\beta > \gamma = 0$, where congestion is penalized and interference is not. As already mentioned, the effect of the congestion term $\mathfrak{M}$ is purely combinatorial and not spatial. Since the compactness provided by this term (cf. Section 2.10.1) is present in this case, one can simply use the same methods as for $\gamma,\beta > 0$ in order to extend the results of Sections 2.2.1, 2.2.2 (including the uniqueness of minimizer), and 2.2.3 (including the convergence of the empirical measures $\Psi_\lambda(S)$ to the unique minimizer) to the case $\beta > \gamma = 0$.[14] Then, it is easy to inspect in (2.100) that the function C is constant and therefore so is the Radon–Nikodym derivative $\mathrm{d}M/\mathrm{d}\mu$ for the minimizer Ψ (cf. (2.19)). The uniqueness of the minimizer follows analogously to Lemma 2.10.1, but the characterization of the constants C and $\mathrm{d}M/\mathrm{d}\mu$ is still rather implicit.

Since *a priori*, the limiting entropy term I is also purely combinatorial, it is interesting to ask whether the combinatorics of the system can well be described in terms of non-spatial objects in the high-density limit, instead of applying our spatial discretization procedure by virtue of brute force. On the other hand, while for $\gamma > \beta = 0$, the empirical measures $(R_{\lambda,k}(\cdot))_{k\in[k_{\max}]}$ were sufficient to describe the combinatorics, the ones $(P_{\lambda,m}(\cdot))_{m\in\mathbb{N}_0}$ are unsatisfactory for the same purpose for $\beta > \gamma = 0$. Indeed, admissible trajectory configurations exhibit the property that each message takes at most $k_{\max}$ hops, and in general this property cannot be checked knowing only the numbers of incoming hops of the users.

2.12.2. The *a priori* case.

In Section 2.2.3 we mentioned that in the *a priori* case $\beta = \gamma = 0$, for $\lambda > 0$, the partition function $Z_\lambda^{0,0}(X^\lambda)$ equals $k_{\max}^{N(\lambda)}$. Taking the logarithm of this expression and dividing that by λ, we obtain $\frac{N(\lambda)}{\lambda}\log k_{\max}$, which tends to $\mu(W)\log k_{\max}$, almost surely. Now, the question arises whether this explicit expression for the limiting free energy can also be obtained in terms of a variational formula analogous to (2.24) for the case $\beta,\gamma > 0$ or

[13] Given this assumption, the minimizer of (2.24) must also be unique thans to the strict convexity of I on its level sets.

[14] However, note that the arguments of Section 2.4.3 about the well-behavedness of the model of Section 2.1.4 in case $k_{\max} = \infty$ do not apply if the interference term is dropped.

to (2.34) for $\gamma > \beta = 0$, where now we have merely an entropy term since neither interference nor congestion is penalized. For $\lambda > 0$, the distribution $\mathrm{P}^{0,0}_{\lambda,X^\lambda}$ is a product of the distributions of the individual message trajectories of the users, which are uniform in the sense that any number $k \in [k_{\max}]$ of hops has probability $1/k_{\max}$ and given the number of hops k, any k-hop trajectory from the user to o has the same probability. According to this, it is very plausible to believe that for the admissible trajectory setting

$$\Psi = (\Sigma = (\nu_k)_{k\in[k_{\max}]},(\mu_m)_{m\in\mathbb{N}_0}) = ((\tfrac{\mu^{\otimes k}}{k_{\max}\mu(W)^{k-1}})_{k\in[k_{\max}]},(d_m\mu)_{m\in\mathbb{N}_0}), \tag{2.105}$$

where d_m are the weights of the Poisson distribution with $(k_{\max}-1)/2$, which parameter equals $M(\mathrm{d}x)/\mu(\mathrm{d}x)$ for all $x \in W$, and we wrote $M = \sum_{k\in[k_{\max}]}\sum_{l=1}^{k-1}\pi_l\nu_k$, the following holds. Ψ is the unique minimizer of (2.24) for $\beta = \gamma = 0$ and Σ is the unique minimizer of (2.34) for $\gamma = 0$.

Indeed, an easy computation yields that $\mathrm{I}(\Psi) = \mathrm{J}(\Sigma) = -\mu(W)\log k_{\max}$, and we already know from Theorem 2.2.4 and Proposition 2.2.5(1) that $-\mu(W)\log k_{\max}$ equals $\inf\{\mathrm{I}(\Psi)\colon \Psi \text{ admissible trajectory setting}\} = \inf\{\mathrm{J}(\Sigma)\colon \Sigma \text{ asymptotic routeing strategy}\}$. The uniqueness of the minimizer of these variational formulas follows from the strict convexity of I respectively J on their level sets.

Thus, our main results also apply to the case $\beta = \gamma = 0$, although the convergence of the empirical measures $\Psi_\lambda(S)$ to the minimizer of the corresponding variational formula cannot be derived the same way as for $\beta,\gamma > 0$. Nevertheless, the weak convergence of L_λ to μ together with the form (2.105) of the minimizer implies that the statement also holds for $\beta = \gamma = 0$.

3. Routeing properties in the Gibbsian model

In this chapter we analyse routeing properties of the unique minimizer (2.35) of the variational formula (2.34) in the special case of Section 2.1.4 where $\mathfrak{S}(\cdot)$ penalizes interference (and congestion is not penalized, i.e., $\beta = 0$). We start with expressing the formulas for the minimizer in this special case and commenting on the notion of the typical trajectory in Section 3.1. Each of the following three sections is devoted to one of our three theoretical investigations, which form the core of this chapter, i.e., the analysis of the large-distance limit (1) in Section 3.2, the limit of strong interference penalization (2) in Section 3.3, and the limit of high local density of users (3) in Section 3.4. Each of these sections gives the question, the results, the proofs, and a discussion in the respective setting. Finally, Section 3.5 gives numerical plots and studies about qualitative properties of our model.

3.1. Interpretation of the limiting trajectory distribution

Since in the special case of case Section 2.1.4 with $\beta = 0$, f_k is defined according to (2.14), for $k_{\max} > 1$, (2.35) reads as

$$\nu_k(\mathrm{d}x_0,\dots,\mathrm{d}x_{k-1}) = \mu(\mathrm{d}x_0)A(x_0)\prod_{l=1}^{k-1}\frac{\mu(\mathrm{d}x_l)}{\mu(W)}\mathrm{e}^{-\gamma\sum_{l=1}^{k} g(x_{l-1},x_l)}, \quad x_k = o,\ k \in [k_{\max}], \tag{3.1}$$

and the normalizing function A is defined as

$$\frac{1}{A(x_0)} = \sum_{k=1}^{k_{\max}}\int_{W^{k-1}}\prod_{l=1}^{k-1}\frac{\mu(\mathrm{d}x_l)}{\mu(W)}\mathrm{e}^{-\gamma\sum_{l=1}^{k} g(x_{l-1},x_l)}, \qquad x_0 \in W, \tag{3.2}$$

where $g\colon W \times W \to \mathbb{R}$ is defined as follows

$$g(x,y) = \frac{\int_W \mu(\mathrm{d}z)\ell(|z-y|)}{\ell(|x-y|)}, \tag{3.3}$$

so that

$$\widetilde{f}_k(x_0,\dots,x_{k-1}) = \sum_{l=1}^{k} g(x_{l-1},x_l), \qquad \forall x_0,\dots,x_{k-1} \in W.$$

With this choice of the parameters, the results of Section 2.2.4 apply.

The purpose of the present chapter is to make further qualitative assertions about the "typical" trajectory from a given transmission site $x_0 \in W$ to the origin, after having taken the limit $\lambda \to \infty$, in the limiting regimes (1), (2), and (3) that were sketched in Section 1.1.1. A definition of the "typical" trajectory as a random variable is not immediate, due to the nature of this setting. In this chapter we will focus on the probability measure on $\bigcup_{k\in[k_{\max}]}(\{k\} \times W^{k-1})$ given by its Radon–Nikodym derivative

$$T_{x_0}(k,x_1,\dots,x_{k-1}) = \frac{\nu_k(\mathrm{d}x_0,\mathrm{d}x_1,\dots,\mathrm{d}x_{k-1})}{\big(\sum_{k=1}^{k_{\max}}\pi_0\nu_k(\mathrm{d}x_0)\big)\mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_{k-1})} = \frac{\nu_k(\mathrm{d}x_0,\mathrm{d}x_1,\dots,\mathrm{d}x_{k-1})}{\mu(\mathrm{d}x_0)\mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_{k-1})} \tag{3.4}$$

with respect to $\sum_{k\in[k_{\max}]}(\delta_k \otimes \mu^{\otimes(k-1)})$. This function is the main object of our study in the present chapter. We normalized T_{x_0} in such a way that $\sum_{k\in[k_{\max}]}\int_{W^{k-1}} T_{x_0}(k,x_1,\dots,x_{k-1})\mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_{k-1}) = 1$. According to Proposition 2.2.5,

$$T_{x_0}(k,x_1,\dots,x_{k-1}) = A(x_0)\mu(W)^{-(k-1)}\prod_{l=1}^{k-1}\mathrm{e}^{-\gamma\sum_{l=1}^{k} g(x_{l-1},x_l)}. \tag{3.5}$$

We will use the convention that the 0th coordinate of T_{x_0} is the one corresponding to the variable k and the l-th is the one corresponding to x_l, for $l \in [k-1]$. This way, the marginal $\pi_0 T_{x_0}$ is a measure on $[k_{\max}]$.

Let us recall that also the measure $M = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l \nu_k$ carries interesting information about the system. According to Sections 2.2.1 and 2.2.4, at a position $x \in W$, the typical number of incoming hops of a user at x is Poisson distributed with parameter $M(\mathrm{d}x)/\mu(\mathrm{d}x)$, and the total mass $M(W)$ is the amount of relaying hops in the entire system, in particular it is zero if every message hops directly into o without any relaying hop. Part of our analysis will also be devoted explicitly to M, see Section 3.4.

3.2. Large communication areas with large transmitter–receiver distances

This section is devoted to the analysis of the highly dense telecommunication system described in Section 3.1 (according to the results of Section 2.2.4) in regime (1), i.e., in the limit of a large communication area coupled with a large distance of the user from the base station. We present our main results in Section 3.2.1 and prove them in Section 3.2.2. Section 3.2.3 includes discussions related to this regime.

3.2.1. The typical number, length, and direction of hops in a large-distance limit

In this section the main object of interest is the typical shape of the trajectory from a certain site to the origin, in particular the typical length of any of the hops, the number of hops, and the spatial progress of the trajectory, in particular whether or not it runs along the straight line or how strongly it deviates from it. We will answer these questions for the special choice that W is a closed ball around the origin, μ is the Lebesgue measure on W, and the path-loss function ℓ corresponds to ideal Hertzian propagation so that $b = \int_{\mathbb{R}^d} \ell(|x|)\mathrm{d}x < \infty$, that is, $\ell(r) = \min\{1, r^{-\alpha}\}$ for some $\alpha > d$.

Furthermore, in order to obtain a transparent picture and to derive a neat result, we will have to assume that the starting site of our trajectory is far away from the origin. In such a setting, it is plausible to expect that, as the radius of the ball tends to infinity, a proportion of users that tends to one takes the same order of magnitude of number of hops. This also gives information about the typical length and direction of each hop in large but still compact communication areas.

We will see that this setting exhibits the interesting property that the typical number of hops diverges to infinity as the distance of the user x_0 from o tends to infinity, however, in a sublinear way, more precisely, like the distance divided by a power of its logarithm. Second, using the asymptotics of the value of this largest summand in (3.2), one can conclude about the typical length of the hops and about how much they deviate from the straight line between the transmitter and the receiver o. In our specific setting, we will be able to obtain precise and explicit asymptotics for all these quantities.

We denote the radius of the communication area $W = \overline{B_r(o)}$ by r and we recall that $k_{\max}$ is the maximal hop number. We consider the limit of large r and large $k_{\max}$. We consider one user placed at $x_0 \in W$ with a distance from the origin $|x_0| = r_0$ being large, such that $r > r_0$, but $r \asymp r_0$. Then one can say that for large r, x_0 is a "typical" location of a user in $\overline{B_r(o)}$, chosen uniformly at random.

In our first result, Theorem 3.2.1, we examine the "typical" number of hops of a trajectory from x_0 to o as a random variable under the marginal distribution $\pi_0 T_{x_0}$ on $\mathbb{N}$. According to (3.5), in the present setting, this is given by

$$\pi_0 T_{x_0}(k) = A(x_0) a_k(x_0) \qquad \text{where } a_k(x_0) = \int_{(B_r(o))^{k-1}} \prod_{l=1}^{k-1} \Big(\omega_d r^{-d} \, \mathrm{e}^{-\gamma g(x_{l-1}, x_l)} \, \mathrm{d}x_l \Big), \qquad x_k = o, \tag{3.6}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$, and we recall that $g(x_{l-1}, x_l) = \frac{\int_W \mathrm{d}y \, \ell(|y - x_l|)}{\ell(|x_{l-1} - x_l|)}$. It is *a priori* not clear what the relation between r and $k_{\max}$ in the limiting setting should be in order to obtain interesting assertions. Nevertheless, while A depends on $k_{\max}$ via the identity $1/A(x_0) = \sum_{k=1}^{k_{\max}} a_k(x_0)$, we observe that the terms $a_k(x_0)$ can also be defined for $k > k_{\max}$ analogously to (3.6) for $k \le k_{\max}$. Thus, it will be our first task to find the asymptotics of $a_k(x_0)$ without any reference to $k_{\max}$. We encounter a large deviation principle on a quite surprising scale.

Theorem 3.2.1 (Large deviations for the hop number)**.** *Fix $t \in (0,\infty)$. Then, in the limit $r_0 \to \infty$ with $r > r_0 = |x_0| \asymp r$, for any choice of $r_0 \mapsto k(r_0) \in \mathbb{N}$,*

$$\frac{1}{r_0 \log^{1-1/\alpha} r_0} \log a_{k(r_0)}(x_0) \begin{cases} = -(dt + b\gamma t^{1-\alpha}) + o(1) & \text{if } \frac{k(r_0)}{r_0 \log^{-1/\alpha} r_0} \to t, & (3.7)\\ \leq -b\gamma t^{1-\alpha} + o(1) & \text{if } \frac{k(r_0)}{r_0 \log^{-1/\alpha} r_0} \leq t + o(1) & (3.8)\\ \leq -dt + o(1) & \text{if } \frac{k(r_0)}{r_0 \log^{-1/\alpha} r_0} \geq t + o(1). & (3.9)\end{cases}$$

where we recall that $b = \int_{\mathbb{R}^d} \mathrm{d}y\, \ell(|y|)$.

The upper bound in (3.8) will follow from the convexity of $1/\ell(|\cdot|)$ and a comparison between the functionals $(x,y) \mapsto g(x,y)$ and $(x,y) \mapsto b/\ell(|x-y|)$.

Note that Theorem 3.2.1 identifies the growth of $\log a_{k(r_0)}(x_0)$ on the scale $r_0 \log^{1-1/\alpha} r_0$ for $k(r_0)$ on the scale $r_0 \log^{-1/\alpha} r_0$; indeed, the second and third line rule out small and large values of $k(r_0)$ on that scale, and the first line identifies the precise dependence on the prefactor. In more technical terms, $a_k(x_0)$ satisfies, with $k(r_0) \asymp r_0 \log^{-1/\alpha}(r_0)$, a large deviation principle on the scale $r_0 \log^{1-1/\alpha} r_0$ with rate function $t \mapsto dt + b\gamma t^{1-\alpha}$, cf. (2.30) – (2.31). It is easily seen that this rate function has a unique minimizer

$$t^* = \arg\min_{t\in(0,\infty)} \big(dt + b\gamma t^{1-\alpha}\big) = \Big(\frac{b\gamma(\alpha-1)}{d}\Big)^{1/\alpha} \tag{3.10}$$

with minimum value

$$dt^* + b\gamma (t^*)^{1-\alpha} = \frac{(b\gamma)^{1/\alpha}}{(\alpha-1)d}\Big[d + \big((\alpha-1)d\big)^{1/\alpha}\Big].$$

As a consequence, we have the following law of large numbers.

Corollary 3.2.2. *In the limit $r_0 \to \infty$ with $r > r_0 = |x_0| \asymp r$, any maximizer $k^*(r_0)$ of $\mathbb{N} \ni k \mapsto a_k(x_0)$ satisfies*

$$k^*(r_0) \sim t^* \frac{r_0}{\log^{1/\alpha} r_0}. \tag{3.11}$$

Further, if $k_{\max} \geq k^*(r_0)$ for at least one such maximizer for all sufficiently large $r_0 > 0$, then we have

$$\frac{1}{r_0 \log^{1-1/\alpha} r_0} \log \frac{1}{A(x_0)} \to -(dt^* + b\gamma (t^*)^{1-\alpha}). \tag{3.12}$$

If $k_{\max}$ is smaller than all the maximizers, then the asymptotics of $A(x_0)$ depend on those of $a_{k_{\max}}(x_0)$ rather than on $a_{k^*(r_0)}(r_0)$, and (3.12) has to be adapted accordingly. We note that (3.12) requires only a lower bound on $k_{\max}$, and in Corollary 3.2.2, $k_{\max}$ could be equal to $+\infty$ for each r_0, as it can be easily seen from the proof of the corollary[15]. (3.12) says that the asymptotic logarithmic behaviour of $1/A(x_0)$ on scale $r_0 \log^{1-1/\alpha} r_0$ coincides with the one of the single maximal summand $a_{k^*(r_0)}(r_0)$. Formulated in terms of the marginal distribution $\pi_0 T_{x_0}$ of T_{x_0} on the length k of the path from x_0 to o, since the behaviour of the Lebesgue measure restricted to $\overline{B_r(o)}$ is subexponential in r_0 in the large-distance limit that we are considering, we have that

$$\pi_0 T_{x_0}\Big([t^* - \varepsilon, t^* + \varepsilon]^{\mathrm{c}} \frac{r_0}{\log^{1/\alpha}(r_0)}\Big)$$

tends to zero exponentially fast on the scale $r_0 \log^{1-1/\alpha} r_0$ for all $\varepsilon > 0$. In Section 3.2.3.1 we give an explanation of how these scales arise.

In the proof of the lower bound of (3.7), considering a trajectory with equal hop lengths was sufficient, i.e., $t^* r_0/\log^{1/\alpha} r_0$ hops along the same straight line directed from x_0 to o with length $r_0/k(r_0) \sim \frac{1}{t^*}\log^{1/\alpha} r_0$ each. We now show, again in terms of a large deviations estimate on the scale $r_0 \log^{1-1/\alpha} r_0$, that macroscopic deviations from this optimal hop length on the scale $\log^{1/\alpha} r_0$ have extremely small probability.

[15] See Section 2.4.3 for a discussion about the behaviour of the Gibbsian model in case $k_{\max}$ is equal to infinity or tends to infinity in the limit $\lambda \to \infty$.

Proposition 3.2.3. *For $\varepsilon,\delta > 0$ and $k \in \mathbb{N}$, let*

$$D_{\varepsilon,\delta}(k,x_0) = \Big\{(x_1,\dots,x_{k-1}) \in B_r(o)^{k-1} \colon \exists I \subseteq [k-1] \colon \#I \geq \delta k, \\ \frac{1}{\#I}\sum_{l\in I} \frac{|x_{l-1}-x_l| - \big||x_{l-1}|-|x_l|\big|}{\log^{1/\alpha} r_0} > \varepsilon\Big\}, \qquad x_k = o. \tag{3.13}$$

Then, in the limit $r_0 \to \infty$ with $r > r_0 = |x_0| \asymp r$, for $k(r_0) \sim t^* r_0/\log^{1/\alpha} r_0$,

$$\limsup \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log T_{x_0}\big(k(r_0), D_{\varepsilon,\delta}(k(r_0),x_0)\big) < 0. \tag{3.14}$$

In words, the probability that there are $\asymp k(r_0)$ hops $x_l - x_{l-1}$, $l \in I$, for some index set I, in the trajectory of relays such that their average hop length $\frac{1}{\#I}\sum_{i=1}^{\#I} |x_{l_i} - x_{l_i-1}|$ deviates from the optimal hop length $\frac{1}{t^*}\log^{1/\alpha} r_0 \approx r_0/k^*(r_0)$ on that scale, decays exponentially fast to zero on the scale $r_0 \log^{1-1/\alpha} r_0$. (In the denominator of the summands in (3.13), we have removed the factor $\frac{1}{t^*}$ in order to simplify notation.)

We presented the results of this section for the path-loss functions ℓ of the form $\ell(r) = \min\{1, r^{-\alpha}\}$, $\alpha > d$, which makes the notation in the proofs less heavy. However, these assertions require only two properties of ℓ: the integrability of $\ell(|\cdot|)$ over $\mathbb{R}^d$ and the convexity of $1/\ell(|\cdot|)$, see Section 3.2.3.2.

The proofs of Theorem 3.2.1, Corollary 3.2.2, and Proposition 3.2.3 are carried out in Sections 3.2.2.1, 3.2.2.2, and 3.2.2.3, respectively. A discussion about these results and their proofs can be found in Section 3.2.3.1.

Certainly, the results of this section hold for much more general communication areas W, not only for balls. Essential for our approach is only that a – in every space dimension diverging – neighbourhood of the straight line between x_0 and o is contained in W in the limit considered. The parameter d appearing in the rate function goes back to our assumption that the volume of W grows like the d-th power of r; however, other powers than d in $[1,d]$ are also possible by putting other geometric assumptions on W.

3.2.2. Proof of the results of Section 3.2.1

All the three results of Section 3.2.1 tell about the limit $r_0 \to \infty$ with $r > r_0 \asymp r$, where $x_0 \in W = \overline{B_r(o)}$ has Euclidean norm $|x_0| = r_0$. Throughout this section, we will use the notation $\lim_{r,r_0}$ for this limit and refer to it as “our limit”.

3.2.2.1. Proof of Theorem 3.2.1

Our strategy for proving the three assertions (3.7), (3.8), and (3.9) is the following. First we verify the lower bound in (3.7). Then we prove (3.8) and afterwards (3.9), and we combine these two proofs in order to conclude the upper bound in (3.7).

Proof of (3.7), *lower bound.* Let us first consider $k(r_0)$ satisfying just $k(r_0) = o(r_0)$. We obtain a lower bound for $a_{k(r_0)}(x_0)$ defined in (3.6) by restricting the x_l-integral to the ball with radius one around $\frac{k(r_0)-l}{k(r_0)} x_0$ for $l = 1,\dots,k(r_0)-1$. Then, eventually, $1 \leq |x_{l-1} - x_l| \leq |x_0|/k(r_0) + 2$ for $l = 1,\dots,k(r_0)$. Note that $g(x_{l-1},x_l) \leq b/\ell(|x_{l-1}-x_l|) = b|x_{l-1}-x_l|^\alpha$, where we recall that $b = \int_{\mathbb{R}^d} \mathrm{d}y\, \ell(|y|)$. Hence, for any $\varepsilon \in (0,1)$, eventually,

$$g(x_{l-1},x_l) \leq b|x_{l-1}-x_l|^\alpha \leq (1+\varepsilon) b r_0^\alpha / k(r_0)^\alpha, \qquad l = 1,\dots,k(r_0),$$

where in the first step we used that $\frac{r_0}{k(r_0)} = \frac{|x_0|}{k(r_0)}$ tends to infinity in our limit. This gives

$$a_{k(r_0)}(x_0) \geq (\omega_d r^d)^{-k(r_0)+1} \mathrm{e}^{-\gamma b k(r_0)(1+\varepsilon)(r_0/k(r_0))^\alpha} \geq \mathrm{e}^{-(d+\varepsilon)k(r_0)\log r_0 - \gamma b k(r_0)(1+\varepsilon)(r_0/k(r_0))^\alpha},$$

where the second inequality holds eventually, since $r_0 \asymp r$. Now, an elementary optimization on $k(r_0)$ shows that $k(r_0) \asymp r_0 \log^{-1/\alpha} r_0$ is the relevant scale. Then, in the particular case that $k(r_0) \sim t r_0 \log^{-1/\alpha} r_0$ for some $t \in (0,\infty)$, carrying out the limit and making $\varepsilon \downarrow 0$ afterwards, we have

$$\liminf_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log a_{k(r_0)}(x_0) \geq -\Big(dt + \gamma b t^{1-\alpha}\Big),$$

which is the lower bound in (3.7). △

Proof of (3.8). This proof uses that $1/\ell$ is convex and that the numerator $\int_W \ell(|y-x_l|)\,\mathrm{d}y$ can be well approximated by b for sufficiently many l. These arguments lead to the following lemma.

Lemma 3.2.4. *Let $\varepsilon > 0$. If $k(r_0) \le \frac{1}{2} r_0$ for all r_0 sufficiently large, then eventually in our limit,*

$$\sum_{l=1}^{k(r_0)} g(x_{l-1},x_l) \ge (1-\varepsilon)^\alpha (b-\varepsilon) k(r_0)^{1-\alpha} r_0^\alpha \tag{3.15}$$

holds simultaneously for all $x_0,\dots,x_{k(r_0)-1} \in B_r(o)$ with $|x_0| = r_0$ and $x_{k(r_0)} = o$.

Proof. Let us define an auxiliary function $s\colon (0,\infty) \to (0,\infty)$ such that $r - s(r) \to \infty$ and $0 < r - s(r) = o(r)$ in our limit. Fix $\varepsilon \in (0,\frac{1}{4})$. The idea is to pick r sufficiently large such that

$$\int_{B_r(o)} \ell(|y-x|)\,\mathrm{d}y \ge b - \varepsilon, \qquad \forall x \in \overline{B_{s(r)}(o)}. \tag{3.16}$$

Let us assume that we are given a trajectory $(x_0,x_1,\dots,x_{k(r_0)-1},x_{k(r_0)})$ with $k(r_0) \le \frac{1}{2} r_0$, $|x_0| = r_0$, and $x_{k(r_0)} = o$. Let us define the index of the last hop outside $B_{s(r)}(o)$:

$$K(r_0,r) = \max\{l \in \{0,1,\dots,k(r_0)-1\} \colon |x_l| \ge s(r)\}, \tag{3.17}$$

which we want to understand as $K(r_0,r) = 0$ if there is no such hop. Let $r_0 > 0$ be sufficiently large so that $s(r) > (1-\varepsilon) r$ and (3.16) holds. Then we have

$$\sum_{l=1}^{k(r_0)} g(x_{l-1},x_l) \ge \sum_{l=K(r_0,r)+1}^{k(r_0)} g(x_{l-1},x_l) \ge \sum_{l=K(r_0,r)+1}^{k(r_0)} \frac{b-\varepsilon}{\ell(|x_{l-1}-x_l|)} \tag{3.18}$$

$$\ge \frac{(b-\varepsilon)(k(r_0)-K(r_0,r))}{\ell\Big(\frac{1}{k(r_0)-K(r_0,r)} \sum_{l=K(r_0,r)+1}^{k(r_0)} |x_{l-1}-x_l|\Big)} \tag{3.19}$$

$$\ge \frac{(b-\varepsilon)(k(r_0)-K(r_0,r))}{\ell\Big(\frac{1}{k(r_0)-K(r_0,r)} (1-\varepsilon) r_0\Big)} \tag{3.20}$$

$$\ge (1-\varepsilon)^\alpha (b-\varepsilon)(k(r_0)-K(r_0,r))^{1-\alpha} r_0^\alpha \ge (1-\varepsilon)^\alpha (b-\varepsilon)(k(r_0))^{1-\alpha} r_0^\alpha. \tag{3.21}$$

In (3.18) we used the fact that $x_{K(r_0,r)},\dots,x_{k(r_0)-1},x_{k(r_0)}$ lie in $B_{s(r)}(o)$ and therefore (3.16) can be applied for the numerator of each $g(x_{l-1},x_l)$ with $l > K(r_0,r)$. Next, (3.19) is an application of Jensen's inequality for $1/\ell(|\cdot|)$, and (3.20) uses the following fact. Either $K(r_0,r) = 0$, in which case

$$\sum_{l=K(r_0,r)+1}^{k(r_0)} |x_{l-1}-x_l| \ge \sum_{l=K(r_0,r)+1}^{k(r_0)} (|x_{l-1}|-|x_l|) \ge |x_0| = r_0 \ge 2k(r_0) \ge k(r_0) - K(r_0,r), \tag{3.22}$$

or $K(r_0,r) > 0$, and thus

$$\begin{aligned}\sum_{l=K(r_0,r)+1}^{k(r_0)} |x_{l-1}-x_l| &\ge \sum_{l=K(r_0,r)+1}^{k(r_0)} (|x_{l-1}|-|x_l|) \ge s(r) \\ &\ge (1-\varepsilon) r > (1-\varepsilon) r_0 \ge k(r_0) \ge k(r_0) - K(r_0,r).\end{aligned} \tag{3.23}$$

In both cases the argument in $\ell(\cdot)$ is ≥ 1, and we can write the term in terms of the α-norm, and the first step in (3.21) also follows. Hence, we have derived (3.15). □

By Lemma 3.2.4, for any $\varepsilon > 0$, we have eventually in our limit, under the assumptions of the lemma

$$\begin{aligned}&\int_{\overline{B_r(o)}^{k(r_0)-1}} \Big(\prod_{l=1}^{k(r_0)-1} \frac{\mathrm{d}x_l}{\mathrm{Leb}(\overline{B_r(o)})}\Big)\, \mathrm{e}^{-\gamma \sum_{l=1}^{k(r_0)} g(x_{l-1},x_l)}\\ &\leq \int_{\overline{B_r(o)}^{k(r_0)-1}} \Big(\prod_{l=1}^{k(r_0)-1} \frac{\mathrm{d}x_l}{\mathrm{Leb}(\overline{B_r(o)})}\Big)\, \mathrm{e}^{-\gamma(1-\varepsilon)^\alpha (b-\varepsilon) k(r_0)^{1-\alpha} r_0^\alpha} = \mathrm{e}^{-\gamma(1-\varepsilon)^\alpha (b-\varepsilon) k(r_0)^{1-\alpha} r_0^\alpha}.\end{aligned}$$

Now, let $t > 0$ and $r_0 \mapsto k(r_0)$ such that $k(r_0) \leq (t + o(1)) r_0 / \log^{1/\alpha} r_0$ (in particular $k(r_0) \leq \frac{1}{2} r_0$ eventually). Then,

$$\begin{aligned}\limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log a_{k(r_0)}(x_0) &\leq \limsup_{r,r_0} \frac{-(b-\varepsilon)(1-\varepsilon)^\alpha \gamma k(r_0)^{1-\alpha} r_0^\alpha}{r_0 \log^{1-1/\alpha} r_0}\\ &= \limsup_{r,r_0} -(b-\varepsilon)(1-\varepsilon)^\alpha \gamma \Big(\frac{k(r_0)\log^{1/\alpha}(r_0)}{r_0}\Big)^{1-\alpha} \leq -(b-\varepsilon)(1-\varepsilon)^\alpha \gamma t^{1-\alpha}\end{aligned} \tag{3.24}$$

for all $\varepsilon > 0$, and thus

$$\limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log a_{k(r_0)}(x_0) \leq -b\gamma t^{1-\alpha},$$

which is (3.8). △

Proof of (3.9). Note that for any $x \in \overline{B_r(o)}$, we have

$$\int_{B_r(o)} \ell(|y - re_1|)\,\mathrm{d}y \leq \int_{B_r(o)} \ell(|y-x|)\,\mathrm{d}y,$$

where $e_1 = (1,0,\dots,0)$ is the first unit vector of $\mathbb{R}^d$.

Let us introduce the quantity $b_0 = \lim_{r\to\infty} \int_{B_r(o)} \ell(|y-re_1|)\,\mathrm{d}y = \sup_{r\in(0,\infty)} \int_{B_r(o)} \ell(|y-re_1|)\,\mathrm{d}y \in (0,b)$. Now, for any $k\colon (0,\infty) \to \mathbb{N}$, in our limit,

$$\begin{aligned}\mathrm{Leb}(B_r(o))^{-(k(r_0)-1)} a_{k(r_0)}(x_0) &= \int_{(B_r(o))^{k(r_0)-1}} \Big(\prod_{l=1}^{k(r_0)-1} \mathrm{d}x_l\Big)\, \mathrm{e}^{-\gamma \sum_{l=1}^{k(r_0)} \frac{\int_{B_r(o)} \ell(|y-x_l|)\,\mathrm{d}y}{\ell(|x_{l-1}-x_l|)}}\\ &\leq \int_{(\mathbb{R}^d)^{k(r_0)-1}} \prod_{l=1}^{k(r_0)-1} \Big(\mathrm{d}x_l\, \mathrm{e}^{-\gamma \frac{b_0 - o(1)}{\ell(|x_{l-1}-x_l|)}}\Big)\\ &\leq \Big(\int_{\mathbb{R}^d} \mathrm{e}^{-\gamma \frac{b_0-o(1)}{\ell(|y|)}}\,\mathrm{d}y\Big)^{k(r_0)-1} = \mathcal{O}(1)^{k(r_0)} = \exp\big(o(k(r_0)\log r_0)\big),\end{aligned} \tag{3.25}$$

where the first step in the last line follows from an elementary substitution and a reversion of the order of integration. Now, recall that in our limit, we have $r \asymp r_0$. If $t > 0$ and $k(r_0) \geq (t + o(1)) r_0 / \log^{1/\alpha} r_0$, then

$$\mathrm{Leb}(B_r(o))^{-(k(r_0)-1)} = \exp(-(dt + o(1)) k(r_0) \log r_0)) \leq \exp\big(-(dt+o(1)) r_0 \log^{1-1/\alpha} r_0\big).$$

This implies (3.9). △

Proof of (3.7)*, upper bound.* We combine our arguments from the proofs of the upper bounds in (3.8) and (3.9) in order to obtain the upper bound in (3.7). Indeed, for $t > 0$, $k(r_0) \sim t r_0 / \log^{1/\alpha} r_0$, and $\varepsilon > 0$, let us write $g(x_{l-1},x_l) = \varepsilon g(x_{l-1},x_l) + (1-\varepsilon) g(x_{l-1},x_l)$, estimate the first term like in (3.25) and the second term with the help of (3.15). This gives eventually

$$\begin{aligned}a_{k(r_0)}(x_0) &\leq \int_{W^{k(r_0)-1}} \Big(\prod_{l=1}^{k(r_0)-1} \frac{\mathrm{d}x_l}{\mathrm{Leb}(B_r(o))}\Big)\, \mathrm{e}^{-\varepsilon\gamma \sum_{l=1}^{k(r_0)} \frac{b_0 - o(1)}{\ell(|x_{l-1}-x_l|)}}\, \mathrm{e}^{-(1-\varepsilon)(1-\varepsilon)^\alpha (b-\varepsilon)\gamma t^{1-\alpha} r_0 \log^{1-1/\alpha} r_0}\\ &\leq \exp\Big(-(dt-\varepsilon) r_0 \log^{1-1/\alpha} r_0 - (1-\varepsilon)^{\alpha+1} \gamma (b-\varepsilon) t^{1-\alpha} r_0 \log^{1-1/\alpha} r_0\Big).\end{aligned} \tag{3.26}$$

Carrying out our limit and letting $\varepsilon \downarrow 0$ implies the upper bound in (3.7). This finishes the proof of Theorem 3.2.1. □

3.2.2.2. Proof of Corollary 3.2.2

The assertion (3.11) follows immediately from Theorem 3.2.1. As for (3.12), let $k^*(r_0)$ be the smallest maximizer of $k \mapsto a_k(x_0)$, and let $r_0 \mapsto k_{\max}(r_0)$ satisfy the assumption of the corollary, i.e., $k_{\max}(r_0) \geq k^*(r_0)$. The lower bound easily follows from (3.7) by estimating $1/A(x_0)$ from below by the single summand $a_{k^*(r_0)}(x_0)$ and using (3.11). As for an upper bound, we first write

$$\begin{aligned}
&\limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log \frac{1}{A(x_0)} \leq \limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log \Big(\sum_{k=1}^{\lfloor \frac{1}{2} r_0 \rfloor} a_k(x_0) + \sum_{k=\lfloor \frac{1}{2} r_0 \rfloor +1}^{\infty} a_k(x_0) \Big) \\
&= \max\Big\{ \limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log \Big(\sum_{k=1}^{\lfloor \frac{1}{2} r_0 \rfloor} a_k(x_0) \Big), \limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log \Big(\sum_{k=\lfloor \frac{1}{2} r_0 \rfloor +1}^{\infty} a_k(x_0) \Big) \Big\}.
\end{aligned}$$

Then the proof of (3.9) implies that there exists a constant $D > 0$ such that we have

$$\sum_{k=\lfloor \frac{1}{2} r_0 \rfloor +1}^{\infty} a_k(x_0) \leq \sum_{k=\lfloor \frac{1}{2} r_0 \rfloor +1}^{\infty} (D r_0^d)^{-k} = \frac{(D r_0^d)^{-\lfloor \frac{1}{2} r_0 \rfloor +1}}{1 - \frac{1}{D r_0^d}} \leq \exp(-(\tfrac{1}{2} - o(1)) r_0 \log r_0),$$

therefore

$$\limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log \Big(\sum_{k=\lfloor \frac{1}{2} r_0 \rfloor +1}^{\infty} a_k(x_0) \Big) = -\infty.$$

Moreover, the assumption in Corollary 3.2.2 that $k_{\max}(r_0) \geq k^*(r_0)$ for all sufficiently large $r_0 > 0$ together with (3.7) yields

$$\begin{aligned}
&\limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log \Big(\sum_{k=1}^{\lfloor \frac{1}{2} r_0 \rfloor} a_k(x_0) \Big) \\
&\leq \limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log(\lfloor r_0/2 \rfloor) + \limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log a_{k^*}(x_0) = -(dt^* + \gamma b {t^*}^{1-\alpha}),
\end{aligned}$$

where we recall that $t^* = (b\gamma(\alpha-1)/d)^{1/\alpha}$ is the unique minimizer of $t \mapsto dt + t^{1-\alpha}$ on $(0,\infty)$, cf. (3.10). We conclude the upper bound in (3.12). □

3.2.2.3. Proof of Proposition 3.2.3

Let $\varepsilon,\delta > 0$ be fixed. First, let us note that by the definition of T_{x_0} and the fact that the behaviour of the Lebesgue measure restricted to $\overline{B_r(o)}$ is subexponential in our limit, (3.14) is equivalent to

$$\begin{aligned}
&\limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log \int_{D_{\varepsilon,\delta}(k(r_0),x_0)} \Big(\prod_{l=1}^{k(r_0)-1} \frac{\mathrm{d}x_l}{\mathrm{Leb}(B_r(o))} \Big) \mathrm{e}^{-\gamma \sum_{l=1}^{k(r_0)} g(x_{l-1},x_l)} \\
&< \limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log a_{k(r_0)}(x_0) = -\big(dt^* + b\gamma {t^*}^{1-\alpha}\big),
\end{aligned} \tag{3.27}$$

with $k(r_0) \sim t^* r_0 \log^{-1/\alpha} r_0$ and $x_{k(r_0)} = o$, where in the last step we used (3.7). For this, it suffices to show that there exists $\varepsilon_1 > 0$ such that for any choice of $x_0 \mapsto (x_1,\dots,x_{k(r_0)-1}) = (x_1(x_0),\dots,x_{k(r_0)-1}(x_0)) \in D_{\varepsilon,\delta}(k(r_0),x_0))$ writing $I = I(x_0,x_1,\dots,x_{k(r_0)-1})$ as in (3.13), we have

$$\liminf_{r,r_0} \frac{\sum_{l=1}^{k(r_0)} g(x_{l-1},x_l)}{k(r_0) \log r_0} = \liminf_{r,r_0} \frac{\sum_{l=1}^{k(r_0)} g(x_{l-1},x_l)}{t^* r_0 \log^{1-1/\alpha} r_0} \geq b {t^*}^{-\alpha} + \varepsilon_1. \tag{3.28}$$

Indeed, then one can argue analogously to (3.26) to conclude the first inequality in (3.27).

Now we prove (3.28). We will first replace the functional $(x,y)\mapsto g(x,y)$ with $(x,y)\mapsto \frac{b}{\ell(|x-y|)}$ everywhere and then argue for $g(x,y)$, estimating the numerator of $g(\cdot,\cdot)$ similarly to Lemma 3.2.4.

We have, first using Jensen's inequality for the convex function $|\cdot|^\alpha$, then by the definition of $D_{\varepsilon,\delta}(k(r_0),x_0)$ together with the fact that $\alpha>1$,

$$\begin{aligned}\frac{1}{\#I}\sum_{l\in I}|x_l-x_{l-1}|^\alpha &\geq \Big(\frac{1}{\#I}\sum_{l\in I}|x_l-x_{l-1}|\Big)^\alpha \geq \Big(\frac{1}{\#I}\sum_{l\in I}\big||x_l|-|x_{l-1}|\big|\big)+\varepsilon\log^{1/\alpha} r_0\Big)^\alpha\\ &\geq \Big(\frac{1}{\#I}\sum_{l\in I}\big||x_l|-|x_{l-1}|\big|\Big)^\alpha+(\varepsilon\log^{1/\alpha} r_0)^\alpha.\end{aligned}\tag{3.29}$$

Similarly, by Jensen's inequality and the triangle inequality,

$$\frac{\sum_{l\in[k(r_0)]\setminus I}|x_l-x_{l-1}|^\alpha}{k(r_0)-\#I}\geq\Big(\frac{1}{k(r_0)-\#I}\sum_{l\in[k(r_0)]\setminus I}|x_l-x_{l-1}|\Big)^\alpha\geq\sum_{l\in[k(r_0)]\setminus I}\Big(\frac{1}{k(r_0)-\#I}\big||x_l|-|x_{l-1}|\big|\Big)^\alpha.$$

Hence, more applications of Jensen's inequality yield

$$\begin{aligned}&\frac{1}{k(r_0)}\sum_{l\in[k(r_0)]}|x_l-x_{l-1}|^\alpha\\ &=\frac{\#I}{k(r_0)}\frac{1}{\#I}\sum_{l\in I}|x_l-x_{l-1}|^\alpha+\frac{k(r_0)-\#I}{k(r_0)}\frac{1}{k(r_0)-\#I}\sum_{l\in[k(r_0)]\setminus I}|x_l-x_{l-1}|^\alpha\\ &\geq\frac{\#I}{k(r_0)}\Big(\frac{1}{\#I}\sum_{l\in I}\big||x_l|-|x_{l-1}|\big|\Big)^\alpha+\frac{k(r_0)-\#I}{k(r_0)}\Big(\frac{\sum_{l\in[k(r_0)]\setminus I}\big||x_l|-|x_{l-1}|\big|}{k(r_0)-\#I}\Big)^\alpha+\frac{\#I(\varepsilon\log^{1/\alpha}r_0)^\alpha}{k(r_0)}\\ &\geq\Big(\frac{1}{k(r_0)}\sum_{l\in[k(r_0)]}\big||x_{l-1}|-|x_l|\big|\Big)^\alpha+\delta\varepsilon^\alpha\log r_0\geq\Big(\frac{r_0}{k(r_0)}\Big)^\alpha+\delta\varepsilon^\alpha\log r_0\\ &=(t^{*-\alpha}+\delta\varepsilon^\alpha)\log r_0,\end{aligned}\tag{3.30}$$

where in the first step in the penultimate line we used that $\#I\geq\delta k(r_0)$.

Now, we turn to $\ell(|\cdot|)$ instead of $|\cdot|^{-\alpha}$. Hence, we have to distinguish between $|\cdot|\leq 1$ and $|\cdot|>1$. Let us define $I'=I'(x_0,(x_1,\dots,x_{k(r_0)-1}))\subseteq[k(r_0)]$ as the set of $l\in[k(r_0)]$ such that $|x_l-x_{l-1}|\leq 1$. Without loss of generality, I' is not empty. Then, after passing to a subsequence, if needed, we have that $\#I'\sim\delta' k(r_0)$ for some $\delta'\in[0,1]$. Thus,

$$\frac{1}{\#I'}\sum_{l\in I'}\frac{|x_{l-1}-x_l|-\big||x_{l-1}|-|x_l|\big|}{\log^{1/\alpha}r_0}=\mathcal{O}(1/\log^{1/\alpha}r_0)=o(1).\tag{3.31}$$

Let us assume for a moment that $I\cap I'=\varnothing$ and $\delta'<1$. Splitting into I' and $[k(r_0)]\setminus I'$, we obtain

$$\begin{aligned}\frac{1}{k(r_0)}\sum_{l\in[k(r_0)]}\frac{1}{\ell(|x_l-x_{l-1}|)}&\geq\frac{1}{k(r_0)}\Big(\mathcal{O}(\#I')+\sum_{l\in[k(r_0)]\setminus I'}|x_l-x_{l-1}|^\alpha\Big)\\ &\geq\delta'-o(1)+\frac{1-\delta'-o(1)}{k(r_0)-\#I'}\sum_{l\in[k(r_0)]\setminus I'}|x_l-x_{l-1}|^\alpha.\end{aligned}\tag{3.32}$$

We want to apply to the last term a lower bound analogous to (3.30), i.e., for the sum over $[k(r_0)]\setminus I'$ instead of $[k(r_0)]$. For this, we need that the sum of the $||x_{l-1}|-|x_l||$ satisfies a lower bound against $\sim r_0$. Using that $I\cap I'=\varnothing$, we indeed see this as follows:

$$\sum_{l\in[k(r_0)]\setminus I'}\big||x_{l-1}|-|x_l|\big|\geq-(\delta'+o(1))k(r_0)+\sum_{l\in[k(r_0)]}\big||x_{l-1}|-|x_l|\big|\geq r_0(1-o(1)).$$

Now, making a computation analogous to (3.30) for the right-hand side of (3.32), we obtain in our limit

$$\frac{1}{k(r_0)}\sum_{l\in[k(r_0)]}\frac{1}{\ell(|x_l-x_{l-1}|)}\ge\delta'-o(1)+(1-\delta'-o(1))\Big[\Big(\frac{r_0}{\#([k(r_0)]\setminus I')}\Big)^\alpha+\frac{\delta}{1-\delta'}\varepsilon^\alpha\log r_0\Big]$$
$$\ge\Big((1-\delta')^{1-\alpha}t^{*-\alpha}+\delta\varepsilon^\alpha-o(1)\Big)\log r_0\ge(t^{*-\alpha}+\delta\varepsilon^\alpha-o(1))\log r_0. \qquad (3.33)$$

The case $I\cap I'\neq\varnothing$ can be handled analogously as long as $\delta''>0$, where $\delta''\in[0,\delta']$ is defined as $\delta''=\liminf_{r,r_0}\frac{\#(I\setminus I')}{k(r_0)}$. Indeed, in this case (3.31) implies that $\liminf_{r,r_0}\frac{1}{k(r_0)}\sum_{l\in I\setminus I'}(|x_{l-1}-x_l|-||x_{l-1}|-|x_l||)$ is also positive. Thus, in our limit, a lower estimate on $\frac{1}{k(r_0)}\sum_{l\in[k(r_0)]}(|x_{l-1}-x_l|-||x_{l-1}|-|x_l||)$ can still be obtained analogously to (3.32). Further, we observe that the corresponding lower bound on $\frac{1}{k(r_0)}\sum_{l\in[k(r_0)]}\frac{1}{\ell(|x_l-x_{l-1}|)}$ that is analogous to the first expression in the second line of (3.33) tends to infinity as $\delta''\downarrow 0$.

Hence, we have in any case that (3.33) holds with $\delta\varepsilon^\alpha$ replaced by some positive number. From this, (3.28) follows for $(x,y)\mapsto g(x,y)$ replaced by $(x,y)\mapsto\frac{b}{\ell(|x-y|)}$ for some $\varepsilon_1>0$.

In order to conclude (3.28), we now proceed similarly to the proof of (3.8), that is, we use uniform convergence of the interferences to b within $B_r(o)$ away from the boundary. Let us recall the auxiliary function s and the index $K(r_0,r)$ at (3.17). We essentially show that either a non-negligible part of the deviations from the straight line induced by the definition of $D_{\varepsilon,\delta}(k(r_0),x_0)$ takes place after the $K(r_0,r)$-th hop or the first $K(r_0,r)$ hops have a very high interference penalization value, and in both cases (3.28) holds.

For each x_0 with $|x_0|=r_0$, let us choose $(x_1(x_0),\dots,x_{k(r_0)-1}(x_0))\in D_{\varepsilon,\delta}(k(r_0),x_0)$. Let us use the notation $\tau(r_0)=\tau(x_0,x_1(x_0),\dots,x_{k(r_0)-1}(x_0))$ for $\tau(r_0)=\frac{K(r_0,r)}{k(r_0)}$. Let us further write $I(r_0)=I(x_0,x_1(x_0),\dots,x_{k(r_0)-1}(x_0))$ for a choice of a set I according to (3.13). According to (3.33), without loss of generality we can assume that $I'=I'(x_0,x_1(x_0),\dots,x_{k(r_0)-1}(x_0))=\varnothing$ for all x_0 considered.

In our limit, $\int_{B_r(o)}\ell(|z-y|)\mathrm{d}z=b-o(1)$ uniformly in $y\in\overline{B_{s(r)}(o)}$. Thus, in case $\tau(r_0)=0$, (3.33) implies that (3.28) holds with some $\varepsilon_1>0$. Hence, in order to conclude (3.28), we can assume that $\tau(r_0)\neq 0$ eventually in our limit. Further, by our assumptions on the function s, for any $\varepsilon_2>0$, eventually $s(r)>(1-\varepsilon_2)r_0$. Now, since $x_l(x_0)\in\overline{B_{s(r)}(o)}$ for all $l>K(r_0,r)$, similarly to (3.33), the convexity of $1/\ell(|\cdot|)$ implies the following

$$\frac{1}{k(r_0)}\sum_{l\in[k(r_0)]}g(x_{l-1},x_l)\ge\frac{1}{k(r_0)}\sum_{l=K(r_0,r)+1}^{k(r_0)}g(x_{l-1},x_l)\ge\frac{1}{k(r_0)}\sum_{l=K(r_0,r)+1}^{k(r_0)}\frac{b-o(1)}{\ell(|x_{l-1}-x_l|)}$$
$$\ge\kappa(\varepsilon_2)(1-\tau(r_0))^{1-\alpha}(b-o(1))t^{*-\alpha}\log r_0 \qquad (3.34)$$

for some function $\kappa\colon[0,1]\to\mathbb{R}$ with $\lim_{\varrho\downarrow0}\kappa(\varrho)=1$. Now, let $\varepsilon_3>0$. Taking first our limit and then $\varepsilon_2\downarrow0$, we see that if $\liminf_{r,r_0}\tau(r_0)$ is at least ε_3, then the proof of our goal (3.28) is finished; however, it is *a priori* not clear that ε_1 in (3.28) can be chosen uniformly bounded away from zero in the limit $\varepsilon_3\downarrow0$. Therefore, we will now assume that $\liminf_{r,r_0}\tau(r_0)=0$; this is a case that has to be handled separately, and the computations corresponding to this case will also allow for handling the limit $\varepsilon_3\downarrow0$ in the previous case. After passing to a subsequence, we can assume that $\lim_{r,r_0}\tau(r_0)=0$.

Let us first investigate the case that $\limsup_{r,r_0}\frac{1}{r_0}\sum_{l=K(r_0,r)+1}^{k(r_0)}(|x_{l-1}|-|x_l|)<1$. Then we have

$$\liminf_{r,r_0}\frac{1}{r_0}\sum_{l=1}^{K(r_0,r)}\big||x_{l-1}|-|x_l|\big|\ge\liminf_{r,r_0}\frac{1}{r_0}\sum_{l=1}^{K(r_0,r)}(|x_{l-1}|-|x_l|)>\varepsilon_4$$

for some $\varepsilon_4>0$ depending on $\limsup_{r,r_0}\frac{1}{r_0}\sum_{l=K(r_0,r)+1}^{k(r_0)}(|x_{l-1}|-|x_l|)$. Thus, using that $\int_{B_r(o)}\ell(|z-y|)\mathrm{d}z\ge b_0-o(1)$ uniformly for $y\in\overline{B_r(o)}$ in our limit (where b_0 was defined before (3.25)), a convexity argument similar to (3.30) yields

$$\liminf_{r,r_0}\frac{1}{k(r_0)\log r_0}\sum_{l\in[k(r_0)]}g(x_{l-1},x_l)\ge\liminf_{r,r_0}\frac{1}{t^{*\alpha}k(r_0)^{1-\alpha}r_0^\alpha}\sum_{l=1}^{K(r_0,r)}g(x_{l-1},x_l)$$
$$\ge\liminf_{r,r_0}\varepsilon_4^\alpha t^{*-\alpha}(b_0-o(1))\Big(\frac{K(r_0,r)}{k(r_0)}\Big)^{1-\alpha}=\infty. \qquad (3.35)$$

Next, we fix $\varepsilon_5 \in (0,\varepsilon)$ and we consider the case that $\liminf_{r,r_0} \frac{1}{r_0} \sum_{l=K(r_0,r)+1}^{k(r_0)} (|x_{l-1}| - |x_l|) \ge 1$ (observe that the total sum over all $l \in [k(r_0)]$ is telescoping and hence equal to r_0) and

$$\liminf_{r,r_0} \frac{1}{\#I(r_0)} \sum_{l\in I(r_0):\, l>K(r_0,r)} \frac{|x_{l-1}-x_l| - \big||x_{l-1}| - |x_l|\big|}{\log^{1/\alpha} r_0} > \varepsilon_5. \tag{3.36}$$

Then one can employ an estimate analogous to (3.30) in order to conclude (3.28).

Finally, we consider the case that $\liminf_{r,r_0} \frac{1}{r_0} \sum_{l=K(r_0,r)+1}^{k(r_0)} (|x_{l-1}| - |x_l|) \ge 1$ but (3.36) fails. After passing to a subsequence, we can assume that $\exists \lim_{r,r_0} w(r_0) \ge \varepsilon - \varepsilon_5$, where we put

$$w(r_0) = \frac{1}{\#I(r_0)} \sum_{l\in I(r_0)\cap[K(r_0,r)]} \frac{|x_{l-1}-x_l| - \big||x_{l-1}| - |x_l|\big|}{\log^{1/\alpha} r_0}.$$

Using also that $\#I(r_0) \ge \delta k(r_0) \sim \delta t^* r_0 \log^{1/\alpha} r_0$, we have

$$\varepsilon - \varepsilon_5 - o(1) \le \frac{1}{\#I(r_0)} \sum_{l\in I(r_0)\cap[K(r_0,r)]} \frac{|x_{l-1}-x_l|}{\log^{1/\alpha} r_0} \le \Big(\frac{1}{\delta t^*} + o(1)\Big) \sum_{l\in I(r_0)\cap[K(r_0,r)]} \frac{|x_{l-1}-x_l|}{r_0}.$$

Thus, a convexity argument similar to (3.30) implies

$$\begin{aligned}
&\liminf_{r,r_0} \frac{1}{k(r_0)\log r_0} \sum_{l\in[k(r_0)]} g(x_{l-1},x_l) \ge \liminf_{r,r_0} \frac{1}{k(r_0)\log r_0} \sum_{l\in I(r_0)\cap[K(r_0,r)]} g(x_{l-1},x_l) \\
&\ge \liminf_{r,r_0} \frac{1}{k(r_0)\log r_0} (b_0 - o(1)) \Big(\frac{r_0 \delta t^* (\varepsilon - \varepsilon_5)}{\#(I(r_0)\cap[K(r_0,r)])} \Big)^\alpha \#(I(r_0)\cap[K(r_0,r)]) \\
&\ge \liminf_{r,r_0} \tau(r_0)^{1-\alpha} b_0 (\delta(\varepsilon-\varepsilon_5))^\alpha = \infty.
\end{aligned}$$

Hence, in case $\liminf_{r,r_0} \tau(r_0) = 0$, (3.28) holds with a suitable choice of $\varepsilon_1 > 0$. Further, the computations corresponding to this case show that this ε_1 can be chosen in such a way that as $\liminf_{r,r_0} \tau(r_0)$ tends to zero, we have a lower bound on $\frac{1}{k(r_0)\log r_0} \sum_{l\in[k(r_0)]} g(x_{l-1},x_l)$ the $\liminf$ of which does not exceed ε_1. This allows for handling the limit $\varepsilon_3 \downarrow 0$ in the earlier case that $\liminf_{r,r_0} \tau(r_0) \ge \varepsilon_3$.

Taking infimum over the values of ε_1 in (3.28) corresponding to all the different cases, we see that this infimum is positive. We conclude that (3.28) holds with a suitable choice of $\varepsilon_1 > 0$. □

3.2.3. Discussion about the results of Section 3.2.1

This section discusses the relevance and some extensions of the results of Section 3.2.1. In Section 3.2.3.1 we interpret our large-distance limit and in Section 3.2.3.2 we explain how the choice of the path-loss function influences our results.

3.2.3.1. The large-distance limit

In Section 3.2.1 we consider the typical trajectory in a large homogeneous multihop communication system with one base station in the area W, after the high-density limit has been taken. According to the basic rules in this system, virtually every hop in the area W is homogeneously admitted (even those that do not bring the message any closer to the base station or even further away), but an exponential interference weight is given to the joint configuration of all the trajectories. It may appear somewhat irrelevant to consider a limit of large area, large distances, and many hops, since with an increasing number of hops the technical difficulties and annoying side-effects become larger, but our work is meant to reveal the basic effects emerging in such a setting, in particular the effect of the interference penalization, and our result in terms of a large deviation principle gives also bounds on deviations from the extreme regime.

Since the interference term in particular gives small weights to large hops, it may be expected that the typical trajectory turns out to follow a straight line with all the hops being of the same length, but it may also come as a surprise that the typical hop length diverges like a power of the logarithm of the distance. The reason for this is the fact that *a priori* all the hops (within the area) are admitted and that, in the distribution T_{x_0} of the typical trajectory, as W grows in size, a very small weight term $1/\mathrm{Leb}(W) = 1/\mu(W)$ for each hop appears. This favours a small number of hops. The best compromise between this effect and the interference effect turns out to be on a logarithmic scale.

One could think of a model in which the search for the next hop is done only in a neighbourhood of the current location, which would presumably lead to the removal of the small weight term $1/\mathrm{Leb}(W)$ per hop and finally to a number of hops that is linear in the distance from the origin, but this would make the decay of the path-loss function ℓ irrelevant and describe a fundamentally different organization of message routeing in the telecommunication system. Such an organization is found e.g. in the continuum percolation setting of [YCG11], where the optimal number of hops turns out to be asymptotically linear in the distance from the user to the origin in a large-distance limit. Further, [YCG11, Theorem 2.1] claims that the probability of having trajectories of a significantly unusual length decays exponentially fast, which is analogous to our Proposition 3.2.3.

3.2.3.2. The role of the choice of the path-loss function in the large-distance limit

We derived our large-distance statements for the path-loss function $\ell(r) = \min\{1, r^{-\alpha}\}$ for $\alpha > d$, since this ℓ describes the propagation of signal strength realistically, see e.g. [BB09, GT08, HJKP18]. However, following the proofs of the results of Section 3.2.1 presented in Section 3.2.2, we conclude that analogous results hold whenever the path-loss function ℓ has the following two properties: $\int_{\mathbb{R}^d} \ell(|x|)\mathrm{d}x < \infty$ and $1/\ell$ is convex. If ℓ satisfies these assumptions, then in our large-distance limit, in the optimal strategy (cf. Section 3.2.2.1, proof of (3.7), lower bound), the user takes $\asymp k(r_0)$ equal hops along $[x_0,o]$, where $r_0 \mapsto k(r_0)$ satisfies

$$\log(r_0) \sim \frac{1}{\ell\Big(\frac{r_0}{k(r_0)}\Big)}. \tag{3.37}$$

This shows that the optimal scale depends only on the tail behaviour of ℓ. Thus, for example, the results of Section 3.2.1 also hold for the path-loss function $\ell(r) = (K+r)^{-\alpha}$, $K > 0$, $\alpha > d$. In general, (3.37) shows that under the two above assumptions on ℓ, the optimal scale diverges to ∞ and is sublinear. The faster ℓ decays, the slower $r_0/k(r_0)$ grows. E.g., if $\ell(r) = \mathrm{e}^{-\alpha r}$ for some $\alpha > 0$, then the correct scale is $k(r_0) \asymp r_0/\log\log r_0$.

3.3. Strong penalization for the interference

This section is devoted to regime (2), i.e., the limit of strong penalization of interference. Our main result corresponding to this, Proposition 3.3.1, is stated in Section 3.3.1 and proved in Section 3.3.2.

3.3.1. Strong interference penalization makes message trajectories straight

Proposition 3.2.3 shows that in the large-distance limit, with μ being the Lebesgue measure in a large ball W, the typical message trajectory from the transmitter x_0 to $x_k = o$ under T_{x_0} does not deviate much from the straight line with high probability. In this proposition, $|x_0|$, $k = k(|x_0|)$, and the radius of W are assumed to tend to infinity in a certain coupled way. From an application point of view, it is also desirable to see a similar effect for a fixed compact communication area W, a fixed starting site x_0 and a fixed upper bound $k_{\max} \in \mathbb{N}$ on the hop number. One way to find such an effect is to consider the limit of a large interference penalization parameter γ. It is easily seen from (3.5) that this limiting behaviour should be entirely described by the minimizer of $W^{k-1} \ni (x_1, \dots, x_{k-1}) \mapsto \sum_{l=1}^{k} g(x_{l-1}, x_l)$. In this section, for $k \in [k_{\max}]$, we write ν_k^γ for the measure ν_k introduced in (3.1) and $T_{x_0}^\gamma$ for the measure T_{x_0} defined in (3.4) corresponding to the parameter γ. Our next result gives criteria under which this minimizer follows a straight line and we have exponential estimates for deviations of trajectories from that.

Let us consider the case where W is a closed ball $\overline{B_r(o)}$ with $r > 0$ and the path-loss function ℓ is strictly monotone decreasing (and satisfies the original condition that it is continuous and positive on $[0,\infty)$). A typical choice [BB09, Section 22.1.2] is $\ell(r) = (1+r)^{-\alpha}$. Further, let us assume that the intensity measure is rotationally invariant, i.e., $\mu \circ O^{-1} = \mu$ for any orthogonal $d \times d$ matrix O. Under these conditions, we conclude that any minimizer of $W^{k-1} \ni (x_1,\dots,x_{k-1}) \mapsto \sum_{l=1}^k g(x_{l-1},x_l)$ is of the form $x_l = c_l x_0$ for $l = 1,\dots,k-1$ with positive constants $1 > c_1 > \dots > c_{k-1} > 0$. Moreover, the total probability mass carried by trajectories deviating from the straight line through the transmitter and o in Euclidean distance at least by some fixed positive quantity decays exponentially fast as $\gamma \to \infty$.

More precisely, writing $[[x,y]] = \{\alpha x + (1-\alpha)y | \alpha \in \mathbb{R}\}$ for the line through $x,y \in \mathbb{R}^d$, we state the following.

Proposition 3.3.1. *Let $r > 0, W = \overline{B_r(o)}, k_{\max} \geq 2$, and let ℓ and μ be fixed. Let us assume that ℓ is strictly monotone decreasing and μ is rotationally invariant.*

(A) For $x_0 \in W$, let us write

$$\mathfrak{m}_{k_{\max}}(x_0) = \min_{k \in [k_{\max}]} \min_{x_1,\dots,x_{k-1} \in W} \sum_{l=1}^k g(x_{l-1},x_l), \qquad x_k = o.$$

Then, for any minimizer $k \in [k_{\max}]$ and $x_1,\dots,x_{k-1}$, there exist $1 > c_1 > \dots > c_{k-1} > 0$ such that $x_l = c_l x_0$ for all $l \in [k-1]$.

(B) For $k \in [k_{\max}]$ and $\varepsilon > 0$, let us define

$$D_k^\varepsilon(x_0) = \{(x_1,\dots,x_{k-1}) \in W^{k-1} \mid \exists l \in \{1,\dots,k-1\} \colon \mathrm{dist}(x_l,[[x_0,o]]) > \varepsilon\}. \tag{3.38}$$

Then, we have

$$\sup_{x_0 \in W} \sup_{k \in [k_{\max}]} \limsup_{\gamma \to \infty} \frac{1}{\gamma} \log \sup_{(x_1,\dots,x_{k-1}) \in D_k^\varepsilon(x_0)} T_{x_0}^\gamma(k,x_1,\dots,x_{k-1}) < 0. \tag{3.39}$$

The proof of the first part of this proposition is based on simple geometric arguments, while the proof of the second part additionally uses the Laplace method. Note that in the first part, a minimizer always exists because W is compact and g is continuous. The proof is carried out in Section 3.3.2.

We expect that Proposition 3.3.1(A) is not true in general if ℓ is not strictly monotone decreasing. Indeed, in this case modifying the position of a relay in a path that is optimal with respect to interference penalization may not change the penalization at all. This indicates that if $\mathfrak{m}_{k_{\max}}(x_0)$ is attained for some x_0 by a path along $[x_0,o]$, it may also be attained by a non-straight path.

3.3.2. Proof of Proposition 3.3.1

Throughout the proof, given any number of hops $k \in [k_{\max}]$, we will always assume that $x_k = o$.

We start with proving part (A). Let us fix $x_0 \in \overline{B_r(o)}$. The fact that $(x,y) \mapsto g(x,y)$ is bounded away from 0 implies that for $x_0 = o$, $\mathfrak{m}_{k_{\max}}(x_0)$ is uniquely attained at the 1-hop trajectory from x_0 to $x_1 = o$. Thus, we can assume that $x_0 \neq o$.

Let now $k \in [k_{\max}]$ and $(x_1,\dots,x_{k-1}) \in \overline{B_r(o)}^{k-1}$. Let us assume that $\sum_{l=1}^k g(x_{l-1},x_l) = \mathfrak{m}_{k_{\max}}(x_0)$. We show that there are $1 > c_1 > \dots > c_{k-1} > 0$ such that $x_j = c_j x_0$ for all $j \in [k-1]$, proceeding in the following steps.

(i) Let $\mathcal{H}$ denote the closed half-space of $\mathbb{R}^d$ that contains x_0 and whose boundary is orthogonal to the vector from x_0 to o and contains o. Then $(x_1,\dots,x_{k-1}) \in \mathcal{H}^{k-1}$.

(ii) $(x_1,\dots,x_{k-1}) \in (\mathcal{H} \cap [x_0,o])^{k-1}$, where we write $[x,y] = \{\alpha x + (1-\alpha)y \colon \alpha \in [0,1]\}$ for the closed segment between $x,y \in \mathbb{R}^d$.

(iii) $|x_0| > |x_1| > \dots > |x_{k-1}| > 0$.

We prove these claims respectively as follows.

(i) Assume that the assertion does not hold, then let us define another trajectory $(x_1',\dots,x_{k-1}') \in \mathcal{H}^{k-1}$ via $x_l' = x_l$ if $x_l \in \mathcal{H}$ and x_l' being the image of x_l under reflection across the boundary hyperplane of

$\mathcal{H}$ otherwise, for all $l \in [k-1]$. The rotation invariance of μ and W combined with the property that $|x_l| = |x_l'|$ implies that

$$\int_W \mu(\mathrm{d}y)\ell(|x_l - y|) = \int_W \mu(\mathrm{d}y)\ell(|x_l' - y|), \qquad \forall l \in [k_{\max}]. \tag{3.40}$$

But, since $|x_{l-1} - x_l| \geq |x_{l-1}' - x_l'|$ and ℓ is strictly decreasing,

$$\ell(|x_{l-1} - x_l|) \leq \ell(|x_{l-1}' - x_l'|), \tag{3.41}$$

where equality holds if and only if x_{l-1},x_l are both in $\mathcal{H}$ or both in $\mathbb{R}^d \setminus \mathcal{H}$. We conclude that $\sum_{l=1}^k g(x_{l-1},x_l) > \sum_{l=1}^k g(x_{l-1}',x_l')$, which contradicts $(x_1,\dots,x_{k-1})$ being the minimizer in (3.38).

(ii) The case $d=1$ is trivial. Let us consider the case $d \geq 2$. Assume $(x_1,\dots,x_{k-1}) \in \mathcal{H}^{k-1}$. Let us define another trajectory $(x_1',\dots,x_{k-1}') \in (\mathcal{H} \cap [x_0,o])^{k-1}$ such that for all $l \in [k-1]$, x_l' satisfies $|x_l'| = |x_l|$ and $x_l' \in [x_0,o]$. That is, $x_l' = x_0|x_l|/|x_0|$. Then, the radial symmetry of μ implies that (3.40) holds. Furthermore, the fact that ℓ is strictly decreasing but $|x_{l-1} - x_l| \geq |x_{l-1}' - x_l'|$ implies that also (3.41) is true in this case, where equality holds if and only if $x_l = x_l'$ for all $l \in [k-1]$, i.e., if $x_l \in [x_0,o]$ for all $l \in [k-1]$.

(iii) Let $(x_1,\dots,x_{k-1}) \in [x_0,o]^{k-1}$. In the following argument, we cancel in this trajectory all hops that increase the distance from o. This results in a smaller sum of the interference terms. Indeed, let us define $i_0 = 0$ and $i_j = \inf\{l \in [k] \colon |x_l| < |x_{i_{j-1}}|\}$, $j = 1,\dots,k$. Let m be the largest index j such that $i_j < \infty$, then it is clear that $1 \leq m \leq k$ since $x_0 \neq o$. Now, let us define an m-hop trajectory with relay sequence $(y_1,\dots,y_{m-1}) = (x_{i_1},\dots,x_{i_{m-1}})$, writing $y_0 = x_0$ and $y_m = o$. Let us further define $\varepsilon' = \min_{x,y \in \overline{B_r(o)}} g(x,y) > 0$. Then, since for any $j \in [m-1]$ we have that $|x_{i_j-1} - x_{i_j}| \geq |x_{i_{j-1}} - x_{i_j}|$, we conclude that

$$\sum_{j=1}^m g(y_{j-1},y_j) = \sum_{j=1}^m g(x_{i_{j-1}},x_{i_j}) \leq \sum_{j=1}^m g(x_{i_j-1},x_{i_j}) \leq \sum_{l=1}^k g(x_{l-1},x_l) - (k-m)\varepsilon'.$$

Thus, $(x_1,\dots,x_{k-1})$ can only minimize (3.38) if $k=m$, that is, if $|x_0| > |x_1| > \ldots > |x_{k-1}| > 0$.

This finishes the proof of part (A) of Proposition 3.3.1.

As for part (B), we note that the case $d=1$ is trivial since $D_k^\varepsilon(x_0) = \varnothing$ for all $x_0 \in \overline{B_r(o)}$. Throughout the rest of the proof, let $d \geq 2$. First, we fix $x_0 \in \overline{B_r(o)}$ and $k \in [k_{\max}]$, and we verify that

$$\limsup_{\gamma\to\infty} \frac{1}{\gamma} \log \sup_{(x_1,\dots,x_{k-1}) \in D_k^\varepsilon(x_0)} T_{x_0}^\gamma(k,x_1,\dots,x_{k-1}) < -\kappa \tag{3.42}$$

for some $\kappa > 0$ that neither depends on x_0 nor on k. This will imply (3.39).

Again, it is easy to see that if $x_0 = o$, then (3.42) holds for some $\kappa > 0$, let us therefore assume that $x_0 \neq o$. We first verify that there exists $\delta = \delta(\varepsilon) > 0$, independent of x_0 and k, such that

$$\mathfrak{m}_{k_{\max}}^\varepsilon(x_0) = \inf_{(x_1,\dots,x_{k-1}) \in D_k^\varepsilon(x_0)} \sum_{l=1}^k g(x_{l-1},x_l) \geq \mathfrak{m}_{k_{\max}} + \delta(\varepsilon). \tag{3.43}$$

In the construction of $(x_1,\dots,x_{k-1}) \mapsto (x_1',\dots,x_{k-1}')$ in the proof of (i) above, the fact that $\mathrm{dist}(x_l,[[x_0,o]]) = \mathrm{dist}(x_l',[[x_0,o]])$ for all $l \in [k-1]$ and $k \in [k_{\max}]$ implies that if $(x_1,\dots,x_{k-1}) \in D_k^\varepsilon(x_0)$, then $(x_1',\dots,x_{k-1}') \in D_k^\varepsilon(x_0) \cap \mathcal{H}^{k-1}$. It follows that the infimum in (3.43) can be realized along sequences of trajectories that have all their relays $x_1,\dots,x_{k-1}$ in $\mathcal{H}$.

Let now $(x_1,\dots,x_{k-1}) \in D_k^\varepsilon(x_0) \cap \mathcal{H}^{k-1}$ and consider the construction of $(x_1,\dots,x_{k-1}) \mapsto (x_1',\dots,x_{k-1}')$ in the proof of (ii) above. We observe the following. Since $x_0 \in [x_0,o]$ and $(x_1,\dots,x_{k-1}) \in D_k^\varepsilon(x_0)$, there exists $l_1 \in [k]$ such that

$$\mathrm{dist}(x_{l_1},[[x_0,o]]) > \mathrm{dist}(x_{l_1-1},[[x_0,o]]) + \frac{\varepsilon}{k} \geq \mathrm{dist}(x_{l_1-1},[[x_0,o]]) + \frac{\varepsilon}{k_{\max}},$$

where each $[[x_0,o]]$ can also be replaced by $[x_0,o]$. One easily sees that this bound holds uniformly in $x_0 \in W$ and $k \in [k_{\max}]$.

Now, the Pythagoras theorem together with the fact that ℓ is strictly monotone decreasing and bounded away from 0 yields that in this case there exists $\delta'(\varepsilon) > 0$ such that $\ell(|x_{l_1-1} - x_{l_1}|) < \ell(|x'_{l_1-1} - x'_{l_1}|) - \delta'(\varepsilon)$. Note that $\delta'(\varepsilon)$ depends only on ℓ, r, and ε but not on k, l_1 or x_0. On the other hand, by the rotational symmetry of μ, the identity (3.40) holds for all $l \in [k]$ for this choice of the relays x_l and x'_l. Therefore, we conclude that there exists a constant $\delta = \delta(\varepsilon) > 0$ such that for all $n \in \mathbb{N}$ we have

$$\sum_{l=1}^{k} g(x_{l-1},x_l) > \sum_{l=1}^{k} g(x'_{l-1},x'_l) + \delta(\varepsilon) \geq \mathfrak{m}_{k_{\max}} + \delta(\varepsilon).$$

This implies (3.43), and the construction shows that $\delta(\varepsilon) > 0$ can be chosen independently of x_0 and k.

We now finish the proof of part (B). Let us use the notation $A^\gamma(x_0) = A(x_0)$ for the normalization term in (3.2) corresponding to γ and recall the notation $T^\gamma_{x_0} = T_{x_0}$ from Proposition 3.3.1. It is clear from the Laplace method [DZ98, Section 4.3] that we have

$$A^\gamma(x_0) = \mathrm{e}^{\gamma \mathfrak{m}_{k_{\max}}(x_0) + o(\gamma)} \quad \text{as } \gamma \to \infty.$$

For any $(x_1,\dots,x_{k-1}) \in D^\varepsilon_k(x_0)$, using (3.1) and (3.43), we can estimate

$$T^\gamma_{x_0}(k,x_1,\dots,x_{k-1}) = \frac{\nu^\gamma_k(\mathrm{d}x_0,\dots,\mathrm{d}x_{k-1})}{\mu(\mathrm{d}x_0)\mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_{k-1})} \leq \mathrm{e}^{\gamma \mathfrak{m}_{k_{\max}}(x_0) - \gamma \mathfrak{m}^\varepsilon_{\max}(x_0) + o(\gamma)} \leq \mathrm{e}^{o(\gamma) - \gamma\delta(\varepsilon)}.$$

We conclude (3.42) (with $\kappa > 0$ being independent of $x_0 \in W$ and $k \in [k_{\max}]$). Thus, part (B) of Proposition 3.3.1 follows.

3.4. High local density of users

This section describes the behaviour of the system in regime (3), i.e., in the limit of a high local density of users in a subset of the communication area. We explain both global and local aspects of this limit, respectively in Section 3.4.1 and Section 3.4.2.

We consider the following question about the behaviour of our model given by (3.1), assuming always that $k_{\max} \geq 2$.

> Does the density of trajectories increase unboundedly in a densely populated subarea, or do the messages avoid such an area for the sake of having lower interference?[16]

In order to give substance to this question we replace our user density measure μ with

$$\mu^a = \mu + a\mathrm{Leb}|_\Delta \in \mathcal{M}(W), \qquad a \in (0,\infty), \tag{3.44}$$

where $\mathrm{Leb}|_\Delta$ is the Lebesgue measure concentrated on a compact set $\Delta \subseteq W$, seen as a measure on W, where we assume that $\mathrm{Leb}(\Delta) > 0$. We think of Δ as of a set of very high concentration of users and will consider the behaviour of the optimal path trajectory in the limit $a \to \infty$. We will from now on label all objects that depend on μ^a instead of μ with the index a. We will study the measure

$$M^a = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l \nu^a_k, \tag{3.45}$$

where ν^a_k is defined according to (3.1). We recall that it can be interpreted as the measure of all the incoming hops at a given location (see Sections 2.2.1 and 3.1). Note that the total mass $M^a(W)$ is zero if all messages

[16] This question appeared first in the introduction of the first version of [KT18] (*arXiv:1704.03499v1*), hence the quote environment.

go directly to the base station without any relaying hop; hence it is a measure for the total amount of relaying hops. Explicitly, according to (2.101) – (2.102), we have

$$M^a(\mathrm{d}x) = \mu^a(\mathrm{d}x) \int_W \mu^a(\mathrm{d}x_0) \frac{\sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \int_{W^{k-2}} \prod_{l' \in [k-1]\setminus\{l\}} \mu^a(\mathrm{d}x_{l'})\, \mathrm{e}^{-\gamma \sum_{l'=1}^{k} g^a(x_{l'-1},x_{l'})}\Big|_{x_l=x}}{\sum_{k=1}^{k_{\max}} \int_{W^{k-1}} \prod_{l=1}^{k-1} \mu^a(\mathrm{d}x_l) \mathrm{e}^{-\gamma \sum_{l=1}^{k-1} g^a(x_{l-1},x_l)}}. \tag{3.46}$$

Now we are interested in the behaviour of the measure M^a as $a \to \infty$. Since $(x,y) \mapsto \ell(|x-y|)$ is bounded away from 0 on $W \times W$, we first note that the large-a behaviour of the interference term is given by

$$\lim_{a\to\infty} \frac{1}{a} g^a(x,y) = \frac{\int_\Delta \mathrm{d}z\, \ell(|y-z|)}{\ell(|x-y|)} =: g_\Delta(x,y), \quad x,y \in W. \tag{3.47}$$

The limiting function g_Δ measures the interference only in relation with the interference coming from Δ. This ratio will turn out to be relevant and the effective interference term in the limit $a \to \infty$.

3.4.1. Global effects

Our first result is that, when the path-loss function $(x,y) \mapsto \ell(|x-y|)$ does not vary much on $W \times W$, the presence of the highly dense area Δ has a strongly repellent effect everywhere in the system and suppresses all the relaying hops; indeed, the total mass of the measure M^a tends to zero exponentially fast as $a \to \infty$, under our general assumptions on the path-loss function ℓ.

Proposition 3.4.1 (Criterion for exponential decay of the amount of relays)**.** *We have*

$$\sup_{x\in W} \limsup_{a\to\infty} \frac{1}{a} \log \frac{\mathrm{d}M^a}{\mathrm{dLeb}}(x) < 0 \tag{3.48}$$

if and only if

$$\min_{x_0\in W} \Big[\min_{x_1 \in W} \big(g_\Delta(x_0,x_1) + g_\Delta(x_1,o) \big) - g_\Delta(x_0,o) \Big] > 0. \tag{3.49}$$

Remark 3.4.2. (i) The inequality (3.48) implies an exponential decay of the total mass of M^a, i.e.,

$$\limsup_{a\to\infty} \frac{1}{a} \log M^a(W) < 0.$$

(ii) Since μ^a is clearly subexponential in $a \to \infty$, (3.48) is equivalent to a uniform exponential decay of the Radon–Nikodym derivative of M^a with respect to μ^a instead of $\mathrm{Leb}|_W$.

(iii) The condition in (3.49) says that the effective interference penalty for a two-hop trajectory is uniformly worse than the one of a direct hop to the origin. This condition involves only one- and two-hop trajectories and is valid even when $k_{\max}$ is much larger than 2.

(iv) Multiplying by two of the three denominators in (3.49) and using that the map $W \times W \ni (x,y) \mapsto \ell(|x-y|)$ is bounded and bounded away from zero, we easily see that (3.49) holds if and only if

$$\min_{x_0,x_1\in W} \Big[\ell(|x_1|) \int_\Delta \ell(|z-x_1|)\, \mathrm{d}z + \ell(|x_0-x_1|) \int_\Delta \ell(|z|)\, \mathrm{d}z - \frac{\ell(|x_1|)\ell(|x_0-x_1|)}{\ell(|x_0|)} \int_\Delta \ell(|z|)\mathrm{d}z \Big] > 0. \tag{3.50}$$

(v) A sufficient condition for (3.49) to hold is as follows. Let $p \in (0,1]$ be such that $p\ell_{\max} = \ell_{\min}$, where $\ell_{\max} = \max_{x,y\in W} \ell(|x-y|)$ and $\ell_{\min} = \min_{x,y\in W} \ell(|x-y|)$ are the maximal and the minimal path-loss values in the system, respectively. Then, a lower bound for the left-hand side of (3.50) is $\ell_{\max}^2 \mathrm{Leb}(\Delta)(2p^2 - \frac{1}{p})$. This is positive as long as p is larger than $2^{-1/3} \approx 0.794$.

Similarly, an upper bound on the left-hand side of (3.50) in terms of p is $\ell_{\max}^2 \mathrm{Leb}(\Delta)(2-p^3)$, but this is larger than zero for all $p \in (0,1]$, hence such a general estimate cannot be used for disproving (3.50) in any case.

(vi) In our numerical results in Examples 3.5.1 and 3.5.2 with $W = \Delta$, the condition (3.49) does not hold.

Proof of Proposition 3.4.1. Consider the quantity on the left-hand side of (3.48). Taking the limit $a \to \infty$, we obtain for fixed $x,x_0 \in W$ for the numerator of (3.46)

$$\begin{aligned}\lim_{a\to\infty} \frac{1}{a} \log\Big[\sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \int_{W^{k-2}} \prod_{l'\in[k-1]\setminus\{l\}} \mu^a(\mathrm{d}x_{l'}) \exp\Big(-\gamma \sum_{l'=1}^{k} g^a(x_{l'-1},x_{l'})\Big|_{x_l=x}\Big)\Big] \\ = -\gamma \min_{k\in[k_{\max}]\setminus\{1\}} \min_{l\in[k-1]} \min_{x_1,\dots,x_{l-1},x_{l+1},\dots,x_{k-1}\in W} \sum_{l'=1}^{k} g_\Delta(x_{l'-1},x_{l'})\Big|_{x_l=x}.\end{aligned} \tag{3.51}$$

On the other hand, for the denominator of (3.46) for x_0 fixed, we have

$$\begin{aligned}\lim_{a\to\infty} \frac{1}{a} \log\Big[\sum_{k=1}^{k_{\max}} \int_{W^{k-1}} \prod_{l=1}^{k-1} \mu^a(\mathrm{d}x_l) \exp\Big(-\gamma \sum_{l=1}^{k-1} g^a(x_{l-1},x_l)\Big)\Big] \\ = -\gamma \min_{k\in[k_{\max}]} \min_{x_1,\dots,x_{k-1}\in W} \sum_{l=1}^{k} g_\Delta(x_{l-1},x_l).\end{aligned} \tag{3.52}$$

These two assertions follow from the Laplace method [DZ98, Section 4.3] in a standard way, since the a-dependence of the integrating measure μ^a is clearly subexponential. Hence, we obtain that

$$\begin{aligned}\lim_{a\to\infty} \frac{1}{a} \log M^a(\mathrm{d}x) = -\gamma \min_{x_0\in W} \Big[\min_{k\in[k_{\max}]\setminus\{1\}} \min_{l\in[k-1]} \min_{x_1,\dots,x_{l-1},x_{l+1},\dots,x_{k-1}\in W} \sum_{l'=1}^{k} g_\Delta(x_{l'-1},x_{l'})\Big|_{x_l=x} \\ - \min_{k\in[k_{\max}]} \min_{x_1,\dots,x_{k-1}\in W} \sum_{l=1}^{k} g_\Delta(x_{l-1},x_l)\Big].\end{aligned} \tag{3.53}$$

Note that after taking supremum over $x \in W$ on the right-hand side of (3.53), we obtain a negative number if and only if

$$\min_{x_0\in W} \Big[\min_{k\in[k_{\max}]\setminus\{1\}} \min_{x_1,\dots,x_{k-1}\in W} \sum_{l=1}^{k} g_\Delta(x_{l-1},x_l) - g_\Delta(x_0,o)\Big] > 0. \tag{3.54}$$

Now, assume that the condition (3.49) does not hold. Then we may pick $x_0',x_1' \in W$ with $(g_\Delta(x_0',x_1')+g_\Delta(x_1',o)-g_\Delta(x_0',o)) \le 0$. But this implies that (3.54) is false, as is shown by taking $k = 2$, $x_0 = x_0'$, and $x = x_1'$. We conclude that (3.48) does not hold.

Conversely, let us assume that (3.48) is not satisfied and let us conclude that (3.49) also does not hold. Using (3.48) and (3.53), we can choose $x_0 \in W$, $k \in [k_{\max}] \setminus \{1\}$ and $x_1,\dots,x_{k-1} \in W$ such that

$$\sum_{l=1}^{k} g_\Delta(x_{l-1},x_l) \le g_\Delta(x_0,o), \qquad x_k = o. \tag{3.55}$$

Let k be minimal for x_0 with this property. We show that there exists $x_0',x_1' \in W$ such that $g_\Delta(x_0',x_1') + g_\Delta(x_1',o) \le g_\Delta(x_0',o)$, therefore (3.49) does not hold. Indeed, if this is not the case for $x_0' = x_{k-2}$ and $x_1' = x_{k-1}$, then we have

$$\sum_{l=1}^{k-2} g_\Delta(x_{l-1},x_l) + g_\Delta(x_{k-2},o) \le \sum_{l=1}^{k} g_\Delta(x_{l-1},x_l) \le g_\Delta(x_0,o) < \sum_{l=1}^{k-2} g_\Delta(x_{l-1},x_l) + g_\Delta(x_{k-2},o),$$

where in the last step we used the minimality of k for x_0. This is in contradiction with (3.49), and thus the proof is concluded. □

3.4.2. Local effects

The condition (3.49) can be applied to any $\Delta \subseteq W$ with $\mathrm{Leb}(\Delta) > 0$, in particular also to $\Delta = W$. In this sense, Proposition 3.4.1 is non-spatial. We now discuss among what conditions the spatial effect that the quality of service (interference penalization with interference coming only from Δ) is significantly worse for messages relaying through a neighbourhood of Δ than through an area sufficiently far away from Δ occurs in our model. For simplicity, we consider only the case $k_{\max} = 2$, a very small set Δ, and a special choice of the path-loss function. We will give arguments that suggest that, for any large a, it is strictly suboptimal to relay through a neighbourhood of Δ as opposed to circumventing Δ sufficiently far.

Analogously to (3.51) – (3.52), the large-a limit for the mass of all relaying hops from x_0 into a set $A \subset W$ (assumed being equal to the closure of its interior) and further to o is given by

$$-\lim_{a\to\infty}\frac{1}{a}\log T^a_{x_0}(2,A) = \gamma\Big[\Xi_{x_0}(A) - \min\big\{g_\Delta(x_0,o),\Xi_{x_0}(W)\big\}\Big], \tag{3.56}$$

where

$$\Xi_{x_0}(A) = \min_{x_1\in A}[g_\Delta(x_0,x_1) + g_\Delta(x_1,o)].$$

We want to discuss under what conditions $\Xi_{x_0}(A)$ is smaller for sets A that are bounded away from Δ than for A being a neighbourhood of Δ. For simplicity, let us do that for $W = \mathbb{R}^d$ and very small sets $\Delta = B_r(y_0)$ with $r \ll 1$ only, i.e., let us approximate

$$g_\Delta(x,y) \approx |\Delta|\frac{\ell(|y-y_0|)}{\ell(|y-x|)}, \qquad x,y\in\mathbb{R}^d. \tag{3.57}$$

Hence, we will put $\Delta = \{y_0\}$ and discuss the function

$$f_{x_0,y_0}(\varepsilon) = \min_{x_1\in W\colon |x_1-y_0|=\varepsilon}\Big[\frac{\ell(|x_1-y_0|)}{\ell(|x_0-x_1|)} + \frac{\ell(|y_0|)}{\ell(|x_1|)}\Big], \qquad \varepsilon\geq 0.$$

This is an approximation of $\Xi_{x_0}(\partial B_\varepsilon(y_0))$. We will see that, under quite general conditions, $f_{x_0,y_0}(\varepsilon) < f_{x_0,y_0}(0)$ for all $\varepsilon\in[0,\varepsilon_0]$ for some $\varepsilon_0 > 0$. This means that, for all sufficiently large a, the probability weight for trajectories $x_0 \to B_{\varepsilon_0-\delta}(y_0) \to o$ is exponentially smaller than the one for trajectories $x_0 \to B_{\varepsilon_0}(y_0)^{\mathrm{c}} \to o$ for any $\varepsilon_0 > \delta > 0$.

To do this, use the triangle inequality and the monotonicity of ℓ to see that

$$f_{x_0,y_0}(\varepsilon) \leq \widetilde{f}_{x_0,y_0}(\varepsilon) := \frac{\ell(\varepsilon)}{\ell(|x_0-y_0|+\varepsilon)} + \frac{\ell(|y_0|)}{\ell(|y_0|+\varepsilon)}.$$

Note that $\widetilde{f}_{x_0,y_0}(0) = f_{x_0,y_0}(0)$ and that

$$\widetilde{f}'_{x_0,y_0}(0) = \frac{\ell'(0)}{\ell(|x_0-y_0|)} - \frac{\ell(0)\ell'(|x_0-y_0|)}{\ell(|x_0-y_0|)^2} - \frac{\ell'(|y_0|)}{\ell(|y_0|)}.$$

Note that for the choice $\ell(r) = (1+r)^{-\alpha}$ for some $\alpha > 0$, this is negative as soon as $|x_0-y_0|(1+|x_0-y_0|)^{\alpha-1} > (1+|y_0|)^{-1}$, i.e., as soon as y_0 is sufficiently far away from o, given the distance of the transmission site x_0 from y_0. This proves the announced conclusion that a two-hop transmission from x_0 to the origin is strictly not optimal if the relaying hop uses a neighbourhood of y_0; here we used no information about the spatial relation of the three sites x_0, y_0, and o, but the fact that $\ell'(0) < 0$.[17] However, for the path-loss function $\ell(r) = \min\{1,r^{-\alpha}\}$, this argument does not work, since $\widetilde{f}'_{x_0,y_0}(0) > 0$ (because $\ell'(0) = 0$).

3.5. Numerical studies

In this section we give numerical illustrations of various properties of the minimizer $(\nu_k)_{k\in[k_{\max}]}$ of (3.1), which describes the limiting empirical trajectory measure according to Proposition 2.2.5 (adapted to the telecommunication setting of Section 2.1.4). We consider $k_{\max} = 2$, $d = 1,2$, ℓ satisfying $\ell(r) \sim r^{-4}$ as $r\to\infty$, W being a ball sufficiently large such that both direct communication and two-hop communication are non-negligible, and μ being the Lebesgue measure on W. We do not consider congestion, i.e., we put $\beta = 0$. We will look at the

[17] Note that $\ell'(0)$ can only be defined as a right derivative because ℓ is defined on the domain $[0,\infty)$.

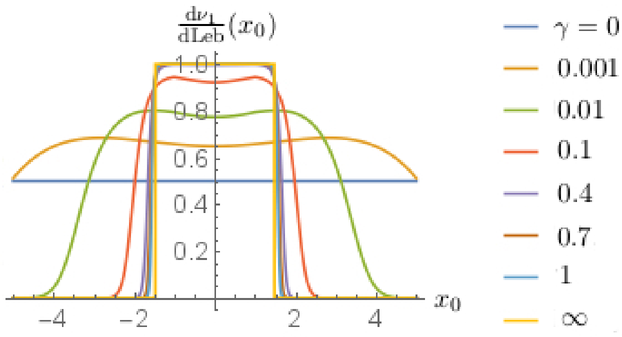


Figure 3. The graphs of $x_0 \mapsto \frac{d\nu_1}{d\mathrm{Leb}}(x_0)$ as in Example 3.5.1 for $\gamma = 0, 0.001, 0.01, 0.1, 0.4, 0.7, 1, \infty$.

areas where one-hop and two-hop communication dominate, respectively, and the approximation of a straight line of the latter trajectories for $d = 2$. We will see that the effects that we proved in Section 3.3 in the limit $\gamma \to \infty$ are already very pronounced for $\gamma = 1$.

First, let us choose $\ell(r) = \min\{1, r^{-4}\}$. Let $W = \overline{B_\varrho(o)}$ be a ball around the origin o. We will pick ϱ sufficiently large so that the effect of the path-loss function ℓ is strong enough in the sense that we can study areas in W from which a direct hop to o is preferred and areas from which a two-hop trajectory is preferred. We are interested in seeing how sharp the transition between these two areas is. By rotational invariance, we expect that the first area is a centred ball and the second the complement of a ball in W. Hence, we do not lose much when going to $d = 1$. Using the arguments of the first paragraph of Section 3.3.1, for large γ, we expect the transition close to the point where the interference term gives the transition from optimality of one-hop trajectories to two-hop trajectories, i.e., at the radius $|x_0|$, where the number

$$g(x_0, o) - \min_{x_1 \in W} \big(g(x_0, x_1) + g(x_1, o)\big) \tag{3.58}$$

switches the sign. Let r_0^* denote that point. Our main question is whether already for moderate values of γ, we see a pronounced transition in the measures $\nu_1(dx_0)$ and $\pi_0\nu_2(dx_0)$ of the form that $\nu_1(dx_0) \approx \mu(dx_0)$ for all x_0 with $|x_0| < r_0^*$ and $\pi_0\nu_2(dx_0) \approx \mu(dx_0)$ for all x_0 with $|x_0| > r_0^*$, with a fast change around r_0^*. Observe that both ν_1 and $\pi_0\nu_2$ have densities that are positive throughout the interior of the support of μ, according to Section 2.10.1.

In the following one-dimensional numerical example, the answer is yes, already for $\gamma = 1$. The plots presented here were created using Wolfram *Mathematica*.

Example 3.5.1. Let $k_{\max} = 2$, $d = 1$, $W = [-5,5] = \overline{B_5(o)} \subset \mathbb{R}$, $\mu = \mathrm{Leb}|_W$, and $\ell(r) = \min\{1, r^{-4}\}$. According to Proposition 2.2.5 (cf. also Section 3.1), the minimizing measures $\Sigma = (\nu_1, \nu_2)$ are given as follows. With

$$\frac{1}{A(x_0)} = \exp\Big(-\gamma \frac{\int_{-5}^{5} \ell(|y|)dy}{\ell(|x_0|)}\Big) + \frac{1}{10}\int_{-5}^{5} dx_1 \exp\Big(-\gamma\Big(\frac{\int_{-5}^{5} \ell(|y - x_1|)dy}{\ell(|x_0 - x_1|)} + \frac{\int_{-5}^{5} \ell(|y|)dy}{\ell(|x_1|)}\Big)\Big), \tag{3.59}$$

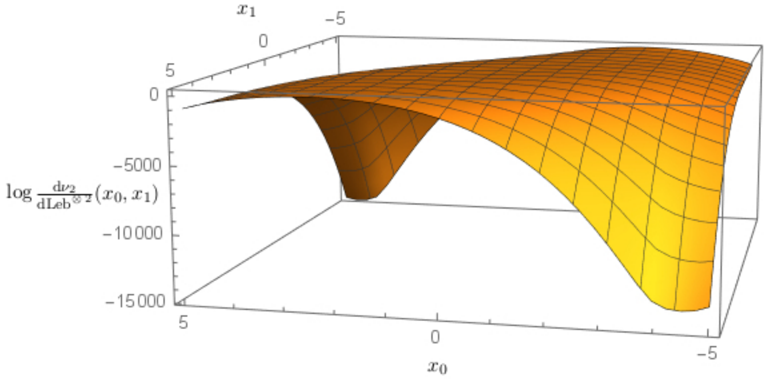
we have

$$\nu_1(dx_0) = dx_0 A(x_0) \exp\Big(-\gamma \frac{\int_{-5}^{5} \ell(|y|)dy}{\ell(|x_0|)}\Big) \tag{3.60}$$

and

$$\nu_2(dx_0, dx_1) = \frac{1}{10} dx_0 dx_1 A(x_0) \exp\Big(-\gamma\Big(\frac{\int_{-5}^{5} \ell(|y - x_1|)dy}{\ell(|x_0 - x_1|)} + \frac{\int_{-5}^{5} \ell(|y|)dy}{\ell(|x_1|)}\Big)\Big). \tag{3.61}$$

All integrals are numerically tractable for $\gamma \in [0,1]$. As seen in Figure 3, already for $\gamma = 1$, the density of ν_1 is very close to the step function with a jump at the point r_0^* where (3.58) switches its sign. Also the density of two-hops paths, $(x_0, x_1) \mapsto \frac{d\nu_2}{d\mathrm{Leb}^{\otimes 2}}(x_0, x_1)$, is extremely small for $|x_0 - x_1|$ large, already for $\gamma = 1$, so that we prefer to plot it on a logarithmic scale, see Figure 4.



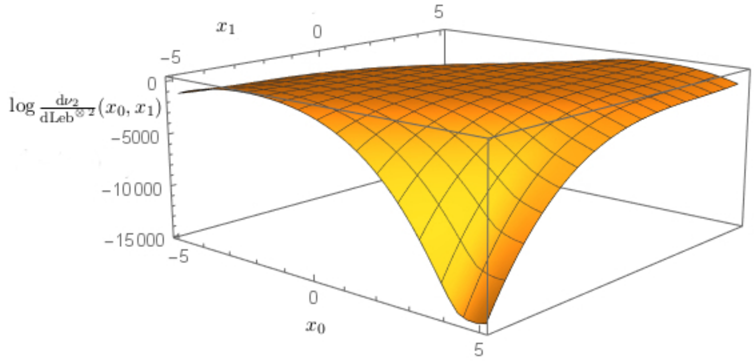


Figure 4. The graphs of $(x_0,x_1) \mapsto \log \frac{\mathrm{d}\nu_2}{\mathrm{dLeb}^{\otimes 2}}(x_0,x_1)$ as in Example 3.5.1 for $\gamma = 1$ from two different views.

It is not easy to read from Figure 4 what value(s) maximize(s) $\frac{\mathrm{d}\nu_2}{\mathrm{dLeb}^{\otimes 2}}(x_0,x_1)$ over $x_1 \in W$ for given $x_0 \in W$. Such a maximizer can be interpreted as an *optimal relay* of the transmitter x_0. Numerical simulations of the optimal relays turn out to be very noisy. Nevertheless, we see that this maximization problem has an interesting property. Since the normalized intensity measure $\mu/\mu(W)$ is the uniform distribution on W, for any $\gamma > 0$ and $x_0 \in W$ fixed, the set of optimal relays of x_0 equals the set of minimizers of

$$x_1 \mapsto g(x_0,x_1) + g(x_1,o) = \frac{\int_W \ell(|z-x_1|)\mathrm{d}z}{\ell(|x_0-x_1|)} + \frac{\int_W \ell(|z|)\mathrm{d}z}{\ell(|x_1|)}. \tag{3.62}$$

The interpretation of this phenomenon is that the optimal value with respect to interference penalization corresponds to the largest value of $x_1 \mapsto \frac{\mathrm{d}\nu_2}{\mathrm{dLeb}^{\otimes 2}}(x_0,x_1)$ for any $x_0 \in W$. Now, the second term in (3.62) is optimal if $|x_1| \le 1$. Further, since $x_1 \mapsto \int_W \ell(|z-x_1|)\mathrm{d}z$ is strictly monotone decreasing in $|x_1|$, one easily sees that if $|x_0| \le 1$, then $x_1 = \operatorname{sgn} x_0$ optimizes the first term over $\{|x_1| \le 1\}$. This implies that if $|x_0| < 1$, then $\frac{\mathrm{d}\nu_2}{\mathrm{dLeb}^{\otimes 2}}(x_0,x_1)$ is strictly larger for $x_1 = \operatorname{sgn} x_0$ than for any x_1 with $|x_1| < 1$. That is, all optimal relays of x_0 are further away from o than x_0 itself, and the first hop of an optimal message trajectory increases the distance from o! Nevertheless, we recall from Figure 3 that for $|x_0| \le 1$, one-hop communication dominates two-hop communication already for $\gamma = 1$, i.e., the density of ν_2 at an optimal two-hop path is still very low.

Since $x_1 \mapsto \int_W \ell(|z-x_1|)\mathrm{d}z$ varies much slower than $x_1 \mapsto \ell(|x_1|)$ for $|x_1| > 1$, we expect according to (3.62) that in general for $|x_0| \le 2$, all optimal relays of x_0 must be situated very close to $\operatorname{sgn} x_0$. In contrast, for $|x_0|$ significantly larger than 2, the optimal relays of x_0 are all about $x_0/2$. Indeed, in this regime, the optimization of $x_1 \mapsto \frac{\mathrm{d}\nu_2}{\mathrm{dLeb}^{\otimes 2}}(x_0,x_1)$ can be restricted to the set of x_1 such that both $|x_0 - x_1| = |x_0| - |x_1| > 1$ and $|x_1| > 1$ hold. On this set, the following three properties are satisfied: $x_1 \mapsto \int_W \ell(|z-x_1|)\mathrm{d}z$ does not vary much (at $x_1 = o$, its value is around 2.6613, while for $|x_1| = 4$, it is about 2.3328), the arguments of ℓ are $|x_1|$ and $|x_0 - x_1|$, which both belong to the set where ℓ is strictly monotone decreasing, and $1/\ell$ is convex. In the context of Example 3.5.2, where these properties hold everywhere (i.e., ℓ has no constant part), we will argue why they imply that the optimal relay of x_0 is close to $x_0/2$ for all $x_0 \in W$.

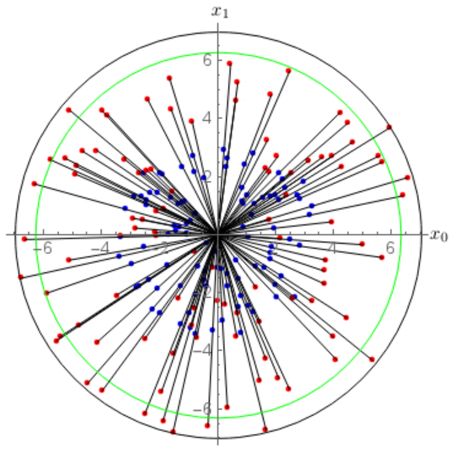


Figure 5. 100 uniformly distributed users x_0 in $W = \overline{B_7(o)}$ (in red), their approximate optimal relays $\approx x_0/2$ (blue) and the circle of radius ≈ 6.299 separating the regimes where one-hop and two-hop communication dominates, respectively (green). The outer black circle is the boundary of W.

Our numerical results suggest that (3.49) does not hold in this example for $\Delta = W$, since otherwise, by Proposition 3.4.1, $\pi_0\nu_2$ would be close to the zero measure on the entire communication area W for large γ. The same is true for Example 3.5.2. $\diamond$

In the next, two-dimensional, example, we analyse the concentration of the measure ν_2 (3.1) on the straight line between transmitter and receiver in the setting of Section 3.3, i.e., in case of a rotationally invariant intensity μ and a strictly monotone decreasing path-loss function ℓ.

Example 3.5.2. We choose $d = 2$, $k_{\max} = 2$, $W = \overline{B_7(o)} \subset \mathbb{R}^2$, $\mu = \mathrm{Leb}|_W$, and $\ell(r) = (1+r)^{-4}$. We note that ℓ is strictly monotone decreasing. Now, the one-hop trajectory measure ν_1 is a measure on $W \subset \mathbb{R}^2$ and the two-hop one ν_2 is a measure on $W^2 \subset \mathbb{R}^4$; they are defined as in (3.1), analogously to the concrete case (3.59) – (3.61), with a suitable adaptation to the new parameters.

Now, Proposition 3.3.1(A) implies that for any $x_0 \in W$, all maximizers of the function $x_1 \mapsto \frac{\mathrm{d}\nu_2}{\mathrm{dLeb}^{\otimes 2}}(x_0,x_1)$ are situated along the straight line segment $[x_0,o]$. The higher the value of γ is, the stronger the density $\frac{\mathrm{d}\nu_2}{\mathrm{dLeb}^{\otimes 2}}(x_0,\cdot)$ concentrates around the set of maximizers, according to part (B) of the same proposition. As in Example 3.5.1, since $\mu/\mu(W)$ is the uniform distribution on W, the maximum of $x_1 \mapsto \frac{\mathrm{d}\nu_2}{\mathrm{dLeb}^{\otimes 2}}(x_0,x_1)$ is taken at the minimizer(s) of $x_1 \mapsto g(x_0,x_1) + g(x_1,o)$ for any $\gamma > 0$.

Since these minimizers lie on $[x_0,o]$, we now numerically search for an approximate minimizer of the form $x_1 = \alpha x_0$ with $\alpha \in [0,1]$. It is highly expectable that the optimal α will be very close to $1/2$ for any $x_0 \in W$. Indeed, since $1/\ell$ is convex, for $c > 0$, $x_1 \mapsto c/\ell(|x_0 - \alpha x_0|) + c/\ell(|\alpha x_0 - o|)$ attains its minimum at $\alpha = 1/2$. On the other hand, for any $x_0 \in W$, the numerator of $g(x_0,\cdot)$, i.e., $x_1 \mapsto \int_W \ell(|z - x_1|)\mu(\mathrm{d}z)$, is close to a constant on $\frac{1}{2}W = \{x_0/2\colon x_0 \in W\}$. Its value at o is approximately 1.0022 and its minimal value in $\frac{1}{2}W$ at $|x_1| = 7/2$ is around 0.9798. Therefore, α much smaller than $1/2$ cannot be the optimal choice. Outside $\frac{1}{2}W$, the interference is not any more close to constant: its minimal value at $|x_1| = 7$ is about 0.4770, slightly less than the half of the value at o. Nevertheless, for α much larger than $1/2$ (close to 1), the additional penalty for a longer second hop is much larger than the gain due to the larger path-loss value and better interference of the first hop. Thus, we expect that such an α also cannot be optimal.

We simulate 100 points x_0 according to the uniform distribution $\mu/\mu(W)$ in W and we compute the minimizer of $\alpha \mapsto g(x_0,\alpha x_0) + g(\alpha x_0,x_0)$ for each of these points. We observe that in all these cases, the optimal α is very close to 0.5: it lies strictly between 0.49 and 0.51. We note that this holds both in the regime $\{|x_0| < r_0^*\}$ where one-hop and in the regime $\{|x_0| > r_0^*\}$ where two-hop communication dominates for large γ. In this example, the value of r_0^* lies between 6.2989 and 6.299, so that 18 points out of 100 in the sample belong to the regime $\{|x_0| > r_0^*\}$. These results are visualized in Figure 5.

4. Signal to interference plus noise ratio percolation for Cox point processes

In this section we turn to the mathematical setting of the SINR percolation model for Cox point processes, which we introduced in Section 1.1.2. The rest of this chapter is organized as follows. In Section 4.1 we define the model and present our main results. In particular, in Section 4.1.1 we summarize the results of [HJC17] about continuum percolation for Cox point processes and in Section 4.1.2 the ones of [DBT05, DFMMT06] about SINR percolation in the Poisson case. In Section 4.1.3 we present our main results about phase transitions in the Cox–SINR setting. Section 4.1.4 contains our assertions and conjectures about the critical interference cancellation factor. In Section 4.1.5 we discuss the applicability of the results of Sections 4.1.3 and 4.1.4 to the main examples of the intensity measure. Section 4.2 includes the proofs of the results of Section 4.1.3, which is followed by a discussion about our results and their proofs in Section 4.2.3. Finally, in Section 4.3 we verify the assertions of Section 4.1.4.

4.1. Model definition and main results

4.1.1. Continuum percolation for Cox point processes

In this section we recall the continuum percolation model defined in [HJC17, Section 2]. Let Λ be a random element in the space $\mathbb{M}$ of Borel measures on $\mathbb{R}^d$, equipped with the evaluation σ-field [LP17, Section 13.1], that is, the smallest σ-field that makes the mappings $B \mapsto \Lambda(B)$ measurable for all Borel sets $B \subseteq \mathbb{R}^d$. We always assume that $d \geq 2$. We define

$$Q_r(x) = x + [-r/2,r/2]^d$$

for $x \in \mathbb{R}^d$ and $r \geq 0$, further, we write $Q_r = Q_r(o)$, where o denotes the origin of $\mathbb{R}^d$. We assume that $\mathbb{E}[\Lambda(Q_1)] = 1$ and Λ is stationary, that is, $\Lambda(\cdot)$ equals $\Lambda(\cdot + x)$ in distribution for all $x \in \mathbb{R}^d$.

Then for $\lambda > 0$, we let X^λ be a Cox point process with intensity $\lambda\Lambda$. That is, conditional on Λ, X^λ is a Poisson point process with intensity $\lambda\Lambda$.[18] Note that the conditions on Λ imply that $\Lambda(\{x\}) = 0$ holds almost surely for all $x \in \mathbb{R}^d$ and thus X^λ is a *simple* point process. That is, one can write $X^\lambda = (X_i)_{i\in I}$ where the random index set I is such that almost surely, for all $i,j \in I$, $X_i \neq X_j$ unless $i = j$. Further, if $\Lambda \equiv \mathrm{Leb}$, then X^λ is a homogeneous Poisson point process with unit intensity. We will often simply say "Cox process" instead of "Cox point process". We denote by Λ_B the restriction of the random measure Λ to the set $B \subset \mathbb{R}^d$.

Let us give some examples of random intensity measures satisfying our assumptions. Any absolutely continuous example Λ has the form $\Lambda(\mathrm{d}x) = l_x \mathrm{d}x$ for a stationary non-negative random field $l = \{l_x\}_{x\in\mathbb{R}^d}$ with $\mathbb{E}[l_o] = 1$, see [HJC17, Example 2.1]. Examples include the modulated Poisson point process: $l_x = \lambda_1 \mathbb{1}\{x \in \Xi\} + \lambda_2 \mathbb{1}\{x \notin \Xi\}$ for a stationary random closed set Ξ and $\lambda_1,\lambda_2 \geq 0$, and intensities given by a shot-noise field: $l_x = \sum_{Y_i \in Y_\mathbf{S}} k(x - Y_i)$ for a non-negative integrable kernel $k\colon \mathbb{R}^d \to [0,\infty)$ with compact support and $Y_\mathbf{S}$ a Poisson point process with intensity $\lambda_\mathbf{S} > 0$. Relevant singular examples are the Poisson point processes on random street systems [HJC17, Example 2.2]. Here, $\Lambda(\mathrm{d}x) = \nu_1(S \cap \mathrm{d}x)$ for a stationary point process S with values in the space of line segments, e.g., a Poisson–Voronoi or Poisson–Delaunay tessellation, where ν_1 denotes one-dimensional Hausdorff measure. We will recall the definition of these tessellation processes in Section A.2 in the Appendix.

[18] Here we stick to the terminology of [HJC17], however, some other works on Cox point processes use different definitions. E.g., in [LP17], the *intensity measure* of a point process Φ is defined as the measure ν on $\mathbb{R}^d$ defined as $\nu(\cdot) = \mathbb{E}[\#(\Phi \cap \cdot)]$. According to this terminology, the intensity measure of X^λ defined above is λ times the Lebesgue measure because $\mathbb{E}[\#(X^\lambda \cap \cdot)]=\lambda\mathbb{E}[\Lambda(\cdot)] = \lambda\mathrm{Leb}(\cdot)$. Further, the random measure $\lambda\Lambda$ is called [LP17, Section 13.2] the *directing random measure* of X^λ.

For $r,\lambda > 0$, the *Gilbert graph* $g_r(X^\lambda)$ is defined as follows. Its vertex set is $X^\lambda = \{X_i : i \in I\}$, and $X_i, X_j \in X^\lambda$, $i \neq j$, are connected by an edge whenever their distance is less than the connection radius r. A *cluster* in a random graph is a maximal connected component, and we say that the graph *percolates* if it contains an infinite cluster. The critical intensity is defined as

$$\lambda_c(r) = \inf\{\lambda > 0 : \mathbb{P}(g_r(X^\lambda) \text{ percolates}) > 0\}.$$

Percolation of $g_r(X^\lambda)$ occurs if and only if the associated *Boolean model*, that is, $X^\lambda \oplus B_{r/2}(o) = \bigcup_{i\in I} B_{r/2}(X_i)$, has an unbounded connected component, see [HJC17, Section 7.1]. Here we wrote $B_R(x)$ for the open ℓ^2-ball of radius R around x for $x \in \mathbb{R}^d$ and $R > 0$. Note that for fixed $r > 0$, $\lambda \mapsto \mathbb{P}(g_r(X^\lambda) \text{ percolates})$ is monotone increasing in λ. Given $r > 0$, any intensity $\lambda \in (0,\lambda_c(r))$ is called *subcritical*, $\lambda = \lambda_c(r)$ *critical* and any $\lambda \in (\lambda_c(r),\infty)$ *supercritical.*

The next two definitions are crucial in [HJC17] for showing that a subcritical respectively supercritical phase exists. The first notion is stabilization, which means a certain decay of spatial correlations of the intensity measure with distance. We recall that $\mathrm{dist}_p(\varphi,\psi) = \inf\{\|x - y\|_p : x \in \varphi, y \in \psi\}$ denotes the ℓ^p-distance between two sets $\varphi,\psi \subset \mathbb{R}^d$ for $p \in [1,\infty]$.

Definition 4.1.1. *The random measure* Λ *is* stabilizing *if there exists a random field of* stabilization radii $R = \{R_x\}_{x\in\mathbb{R}^d}$ *defined on the same probability space as* Λ *such that, writing*

$$R(Q_n(x)) = \sup_{y\in Q_n(x)\cap\mathbb{Q}^d} R_y, \qquad n \geq 1,\ x \in \mathbb{R}^d,$$

the following hold.

1. (Λ,R) *is jointly stationary,*
2. $\lim_{n\to\infty} \mathbb{P}(R(Q_n) < n) = 1$,
3. *for all* $n \geq 1$, *for any bounded measurable function* $f : \mathbb{M} \to [0,\infty)$ *and finite* $\varphi \subseteq \mathbb{R}^d$ *with* $\mathrm{dist}_2(x,\varphi\setminus\{x\}) > 3n$ *for all* $x \in \varphi$, *the following random variables are independent:*

$$f(\Lambda_{Q_n(x)})\mathbb{1}\{R(Q_n(x)) < n\}, \qquad x \in \varphi.$$

A strong form of stabilization is *b-dependence*; for $b > 0$, Λ is called b-dependent if Λ_A and Λ_B are independent whenever $\mathrm{dist}_2(A,B) > b$. On the other hand, in this chapter, b-dependence of stochastic processes defined on discrete subsets of $\mathbb{R}^d$ with an explicitly given value of b will always be meant with dist_∞ instead of dist_2 on the discrete set.[19]

Let us write $\mathrm{supp}(\mu) = \{x \in \mathbb{R}^d : \mu(Q_\varepsilon(x)) > 0, \forall\varepsilon > 0\}$ for the support of a (possibly singular) measure μ. The second notion, asymptotic essential connectedness, indicates, in addition to stabilization, strong local connectivity of the intensity measure.

Definition 4.1.2. *The stabilizing random measure* Λ *with stabilization radii* R *is* asymptotically essentially connected *if for all* $n \geq 1$, *whenever* $R(Q_{2n}) < n/2$, *we have that* $\mathrm{supp}(\Lambda_{Q_n})$ *is non-empty and contained in a connected component of* $\mathrm{supp}(\Lambda_{Q_{2n}})$.

As for the main examples, it was shown in [HJC17, Section 3.1] that Poisson–Voronoi and Poisson–Delaunay tessellations are asymptotically essentially connected. Further, shot-noise fields are b-dependent but not asymptotically essentially connected in general. The modulated Poisson point process is also b-dependent if Ξ is a Poisson–Boolean model (that is, the Boolean model of a homogeneous Poisson point process), in this case it is also asymptotically essentially connected if $\lambda_1,\lambda_2 > 0$.

By [HJC17, Theorem 2.4, 2.5], the following holds about phase transitions of the Gilbert graph.

Theorem 4.1.3 ([HJC17])**.** *Let* $r > 0$. *If* Λ *is stabilizing, then* $\lambda_c(r) > 0$. *If* Λ *is asymptotically essentially connected, then* $\lambda_c(r) < \infty$.

[19] In fact, b-dependence is meant with respect to the ℓ^∞-distance also in the discrete models used in [HJC17], this is why we stick to this convention here. (C. Hirsch and B. Jahnel, personal communication, 2018.)

Let $r > 0$. Roughly speaking, the spatial decorrelation coming from stabilization of Λ makes it easy to verify, using discrete percolation techniques, that long-distance connections in $g_r(X^\lambda)$ do not exist for $\lambda > 0$ sufficiently small, see [HJC17, Section 5.1]. On the other hand, as $\lambda \to \infty$, X^λ fills the support of Λ with high probability. This fact together with the stabilization of Λ and the strong connectivity of the support of Λ can be used to verify percolation of $g_r(X^\lambda)$ for large λ if Λ is asymptotically connected, cf. [HJC17, Section 5.2].

4.1.2. Signal to interference plus noise ratio graph

In this section we follow [DFMMT06]. We choose a monotone decreasing *path-loss function* $\ell : [0,\infty) \to [0,\infty)$, which describes the propagation of signal strength over distance. Note that $\ell(|x-y|) \leq \ell(0)$ for all $x,y \in \mathbb{R}^d$. Further assumptions on ℓ will be made below using the following definitions. For two points X_i,X_j of the Cox point process X^λ, we define the *signal-to-interference-plus noise ratio (SINR)* of the transmission from X_i to X_j as follows

$$\mathrm{SINR}(X_i,\, X_j,\, X^\lambda) = \frac{P\ell(|X_i - X_j|)}{N_0 + \gamma \sum_{k \neq i,j} P\ell(|X_k - X_j|)}, \tag{4.1}$$

where $P > 0$ is the *transmitted power*, $N_0 \geq 0$ is the environmental *noise*, the sum in the denominator of (4.1) is called the *interference* (of the transmission from X_i to X_j), and $\gamma \geq 0$ is the *interference cancellation factor*. Then we fix $\tau > 0$ and say that the transmission from X_i to X_j is possible if and only if

$$\mathrm{SINR}(X_i,\, X_j,\, X^\lambda) > \tau. \tag{4.2}$$

We will tacitly exclude the degenerate case $\gamma = N_0 = 0$. Further, if $N_0 = 0$, we use the convention [BB09, Section 6.1] that the inequality (4.2) holds if $P\ell(|X_i - X_j|) > \tau\gamma \sum_{k \neq i,j} P\ell(|X_k - X_j|)$.[20]

We define the *directed SINR graph* $g^{\rightarrow}_{(\gamma,N_0,\tau,P)}(X^\lambda)$ on the vertex set X^λ via drawing a directed edge pointing from X_i towards X_j (denoted as $X_i \to X_j$) whenever $i \neq j$ and $\mathrm{SINR}(X_i,\, X_j,\, X^\lambda) > \tau$. Next, the (undirected) *SINR graph* $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ has vertex set X^λ, and $(X_i,X_j) \in X^\lambda \times X^\lambda$ is an edge in $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ if and only if both $X_i \to X_j$ and $X_j \to X_i$ are edges in $g^{\rightarrow}_{(\gamma,N_0,\tau,P)}(X^\lambda)$.

We note that [KY07] studied percolation in the directed SINR graph in the two-dimensional Poisson case. It obtained results that are very similar to the ones of [DBT05, DFMMT06, FM07] in the undirected case. In the present chapter we will focus on the undirected SINR graph, but we will also use some properties of the directed one in our arguments.

The notions of SINR and the (directed or undirected) SINR graph can be extended to general simple point processes, analogously to the Cox case. Some of the results of this chapter turn out to hold for a larger class of simple point processes, see Section 4.1.4.1. See Figure 6 for simulations of the SINR graph in the two-dimensional Poisson case. The SINR graph model has a number of generalizations and variants, including the case of random powers (cf. Section 4.2.3.4), the SINR graph with external interferers (cf. Section 4.2.3.5), and the information theoretically secure SINR graph (cf. Section 4.2.3.6).

As for $N_0 > 0$ and $\gamma = 0$, X_i,X_j are connected by an edge in $g_{(0,N_0,\tau,P)}(X^\lambda)$ if and only if the *signal-to-noise ratio (SNR)* between them is larger than τ, i.e.,

$$\mathrm{SNR}(X_i,\, X_j) = \mathrm{SNR}(X_j,\, X_i) = \frac{P\ell(|X_i - X_j|}{N_0} > \tau. \tag{4.3}$$

Whenever $\ell^{-1}(\tau N_0/P)$ is well-defined and positive (in particular, $\ell(0) > \tau N_0/P$), this is equivalent to $|X_i - X_j| \leq \ell^{-1}(\frac{\tau N_0}{P})$. In this case $g_{(0,N_0,\tau,P)}(X^\lambda)$ equals the Gilbert graph $g_{r_{\mathrm{B}}}(X^\lambda)$, where

$$r_{\mathrm{B}} = \ell^{-1}\Big(\frac{\tau N_0}{P}\Big). \tag{4.4}$$

For two graphs $G = (V,E),G' = (V,E')$ with the same vertex set V, we will write $G \preceq G'$ if $E \subseteq E'$, i.e., if all edges in G are also contained in G'. Now, for $\tau,P > 0$ and $N_0 > 0$, we have $g_{(\gamma,N_0,\tau,P)}(X^\lambda) \preceq g_{(\gamma',N_0,\tau,P)}(X^\lambda)$

[20] Using the strict inequality in (4.2) originates from [DFMMT06] and differs from [DBT05, HJKP18] where (4.2) was defined with "$\geq \tau$" instead of "$> \tau$". The reason why we prefer the strict inequality is that using the non-strict one, for $N_0 = 0$ and compactly supported ℓ, the transmission from X_i to X_j may be considered successful although $\ell(|X_i - X_j|) = 0$.

for all $0 \le \gamma' < \gamma$. Thus, $g_{(\gamma,N_0,\tau,P)}(X^\lambda) \preceq g_{r_{\mathrm{B}}}(X^\lambda)$, hence any edge $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ has length at most r_{B}. In contrast, if $N_0 = 0$ (and $\gamma > 0$), then the edge lengths of the SINR graph $g_{(\gamma,0,\tau,P)}(X^\lambda)$ are unbounded. On the other hand, while Gilbert graphs have no bound on the degrees of the vertices, we will show in Section 4.1.4 that all in-degrees in $g^{\rightarrow}_{(\gamma,N_0,\tau,P)}(X^\lambda)$ and thus also all degrees in $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ are bounded by $1 + 1/(\tau\gamma)$ for fixed $\gamma > 0$.

Now, we define

$$\gamma^*(\lambda) = \gamma^*(\lambda,N_0,\tau,P) := \sup\{\gamma > 0\colon \mathbb{P}\big(g_{(\gamma,N_0,\tau,P)}(X^\lambda) \text{ percolates}\big) > 0\} \tag{4.5}$$

for fixed $\lambda,\tau,P > 0$ and $N_0 \ge 0$. Further, we put

$$\lambda_{N_0,\tau,P} = \inf\{\lambda > 0\colon \gamma^*(\lambda') > 0, \forall \lambda' \ge \lambda\}. \tag{4.6}$$

Then for $\lambda < \lambda_{N_0,\tau,P}$, $\mathbb{P}(g_{(\gamma,N_0,\tau,P)}(X^\lambda) \text{ percolates}) = 0$ for all $\gamma > 0$. *A priori*, there is no reason for $\lambda_{N_0,\tau,P} = \inf\{\lambda > 0\colon \gamma^*(\lambda) > 0\}$ to hold, but this identity will turn out to be true in most of the cases we consider.

Now, let us fix $N_0 \ge 0$, $\tau,P > 0$, and let us make the following assumption on the path-loss function ℓ (which also implies that ℓ is monotone decreasing) for the rest of this chapter.

Assumption (ℓ).

(i) ℓ is continuous, constant on $[0,d_0]$ for some $d_0 \ge 0$, and on $[d_0,\infty) \cap \operatorname{supp} \ell$ it is strictly decreasing.

(ii) $1 \ge \ell(0) > \tau N_0/P$,

(iii) $\int_{\mathbb{R}^d} \ell(|x|)\mathrm{d}x < \infty$.

These constraints on ℓ are slightly stronger than the ones of [DFMMT06] because we allow d_0 to be positive, motivated by the facts that the proof of the main results of [DFMMT06] works also for $d_0 > 0$ and path-loss functions with $d_0 > 0$ are widely used in practice. E.g., the path-loss function $\ell(r) = \min\{1, r^{-\alpha}\}$, $\alpha > d$, corresponding to ideal Hertzian propagation satisfies Assumption (ℓ), cf. Section 2.1.4.

Let us recall [DFMMT06, Theorem 1] about the homogeneous Poisson case $\Lambda \equiv \mathrm{Leb}$ for $d = 2$.

Theorem 4.1.4 ([DFMMT06])**.** *If $\Lambda \equiv \mathrm{Leb}$, $d = 2$, and $N_0,\tau,P > 0$, then $\lambda_{N_0,\tau,P} = \lambda_{\mathrm{c}}(r_{\mathrm{B}}) \in (0,\infty)$.*

In words, for any intensity λ for which the SNR graph $g_{(0,N_0,\tau,P)}(X^\lambda) = g_{r_{\mathrm{B}}}(X^\lambda)$ is supercritical, there exists a small but positive γ such that $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ still percolates. The case $N_0 = 0$ will be discussed in Section 4.1.3.3 in the general Cox case.

4.1.3. Phase transitions

This section contains our main results about percolation properties of $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ depending on the parameters $N_0,\tau,P,\lambda,\gamma$. In Section 4.1.3.1 we present our main results for fixed $N_0,\tau,P > 0$. In this setting, the SNR radius r_{B} is fixed, and thus, according to Theorem 4.1.3, if Λ is asymptotically essentially connected, then the SNR graph percolates for large λ with positive probability (actually with probability 1 thanks to stabilization, see Section 4.2.3.1). We show that under additional assumptions on Λ and ℓ, we have $\gamma^*(\lambda) > 0$ for all sufficiently large λ. Under similar assumptions, in Section 4.1.3.2 we show that if Λ is only stabilizing, then one can choose the SNR radius r_{B} large enough such that $\gamma^*(\lambda) > 0$ occurs for sufficiently large $\lambda > 0$. In Section 4.1.3.3 we comment on the case $N_0 = 0$. For a discussion related to the results of the present section and their proofs, we refer the reader to Section 4.2.3.

4.1.3.1. The case of asymptotically essentially connected intensity

If Λ is asymptotically essentially connected, then the SINR graph percolates for large enough λ and accordingly chosen small enough $\gamma > 0$ under additional assumptions on ℓ and Λ.

Theorem 4.1.5. *Let $N_0,\tau,P > 0$.*

(1) $\lambda_{N_0,\tau,P} \geq \lambda_{\mathrm{c}}(r_{\mathrm{B}})$. *Further, if* Λ *is stabilizing, then* $\lambda_{N_0,\tau,P} > 0$.

(2) *If* Λ *is asymptotically essentially connected, then* $\lambda_{N_0,\tau,P} < \infty$ *holds if at least one of the following conditions is satisfied:*

  (a) ℓ *has compact support,*

  (b) $\Lambda(Q_1)$ *is almost surely bounded,*

  (c) $\mathbb{E}[\exp(\alpha\Lambda(Q_1))] < \infty$ *for some* $\alpha > 0$*, and* $\int_x^\infty r^{d-1}\ell(r)\,\mathrm{d}r = \mathcal{O}(1/x)$ *as* $x \to \infty$.

We already see that (1) is true. Indeed, it follows from Theorem 4.1.3 and the fact that for $N_0,\tau,P,\gamma > 0$, we have $g_{(\gamma,N_0,\tau,P)}(X^\lambda) \preceq g_{(0,N_0,\tau,P)}(X^\lambda)$. Note that (1) requires only that SNR graph be a well-defined Gilbert graph; for this, it suffices if $\ell : (0,\infty) \to [0,\infty)$ is monotone decreasing and the radius $r_B = \ell^{-1}(\tau N_0/P)$ is well-defined and positive. In particular, this allows that $\lim_{r\downarrow 0}\ell(r) = \infty$, which is excluded by Assumption (ℓ).

The proof of Theorem 4.1.5(2) is carried out in Section 4.2.1.1. In the rest of this chapter, we will write "Y is bounded (away from 0)" equivalently to "Y is almost surely bounded (away from 0)" for any nonnegative random variable Y.

Note that the condition (2c) includes the case that $\int_0^\infty r^d\ell(r)\,\mathrm{d}r < \infty$ and even the boundary case that $\ell(r) \asymp r^{-d-1}$ as $r \to \infty$. The latter one is important because in two dimensions, bounded path-loss functions ℓ satisfying $\ell(r) \asymp r^{-3}$ are used in modelling real-world networks, see [DBT05, Section II.C]. Regarding the question what kind of unbounded intensity measures Λ satisfy the exponential moment condition in (2c), our main example is the Poisson–Voronoi tessellation in two dimensions. See [GFSS06, CGHJNP18] for applications of this tessellation process in modelling statistical properties of real street systems.

Theorem 4.1.6. *Let* $\Lambda(\mathrm{d}x) = \nu_1(S \cap \mathrm{d}x)$*, where* S *is the Voronoi tessellation defined by a two-dimensional Poisson point process* $X_{\mathbf{S}}$ *with intensity* $\lambda_{\mathbf{S}} > 0$*. Then we have* $\mathbb{E}[\exp(\alpha\Lambda(Q_1))] < \infty$ *for all* $\alpha > 0$.

To the best of our knowledge, this result has not been proven before; we will verify it Section 4.2.1.2 and discuss its possible extensions in Section 4.2.3.7. B. Jahnel and the author recently verified the analogous statement for the Poisson–Delaunay tessellation in two dimensions [JT19], see Section 5.3.2.1 for further details. The existence of some exponential moments of $\Lambda(Q_1)$ remains open for both kinds of tessellations for $d \geq 3$. Since the Poisson–Voronoi tessellation is also asymptotically essentially connected, Theorems 4.1.5 and 4.1.6 immediately imply the following assertion, which also holds for the Poisson–Delaunay tessellation for $d = 2$ according to [JT19].

Corollary 4.1.7. *Let* $d = 2$*,* $\tau,P,N_0 > 0$*, and let* Λ *be given by a Poisson–Voronoi tessellation as in Theorem 4.1.6. Let* ℓ *(satisfying Assumption (*ℓ*)) be such that* $\ell(r) = \mathcal{O}(1/r^3)$ *as* $r \to \infty$*. Then* $\lambda_{N_0,\tau,P} < \infty$.

Further, in Section 4.1.5.1 we will discuss the applicability of all results of Sections 4.1.3 to each of the examples introduced in Section 4.1.1. We see that Theorem 4.1.5(2) holds in particular in the Poisson case and thus it generalizes Theorem 4.1.4 to $d \geq 3$ dimensions. However, it does not recover the identity that $\lambda_{N_0,\tau,P} = \lambda_{\mathrm{c}}(r_{\mathrm{B}})$ for all $N_0,\tau,P > 0$. This identity holds for $d = 2$ thanks to a Russo–Seymour–Welsh type result, Theorem 4.1.9, about the Poisson–Boolean model, which has no analogue in higher dimensions.

Let us outline the proof of Theorem 4.1.5(2). The proof is in the spirit of [DFMMT06, Section 3]: we aim to show that for large λ, the SNR graph has an infinite connected component within which the interferences at the vertices are uniformly bounded. Then, one can choose $\gamma > 0$ for which all connections in this infinite connected component of $g_{(0,N_0,\tau,P)}(X^\lambda)$ also exist in $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ for any small positive γ, and the same holds for $\lambda' > \lambda$ (possibly at the price of decreasing γ without vanishing). This will then imply that $\lambda_{N_0,\tau,P} < \infty$.

However, the proof is more challenging than in the Poisson case: the randomness of the intensity measure gives rise to potential increase of interferences and spatial correlations. While some proof techniques of [DFMMT06] are not available anymore, we can mostly replace them with other ones from [HJC17]. If Λ is asymptotically essentially connected, then percolation of the SNR graph for sufficiently large $\lambda > 0$ follows from a comparison with a certain site percolation process, introduced in [HJC17, Section 5.2]. Knowing this, it remains to show that the infinite cluster of open (good) sites in this site percolation process contains an infinite connected subset within which the interferences are uniformly bounded. Hence, we need to perform a Peierls argument (cf. [G99, Section 1.4]) for the process of open sites with small interferences. This turns out to be possible in all the three cases (2a), (2b), and (2c), and the most involved proof corresponds to the case (2c).

Actually, if Λ is absolutely continuous with bounded Lebesgue density, then $\lambda_{N_0,\tau,P} > 0$, and if its density is also bounded away from 0, then $\lambda_{N_0,\tau,P} < \infty$, without a stabilization assumption, cf. Section 4.2.3.2. In Section 4.2.3.3 we present a general construction of non-stabilizing examples for which $\lambda_{N_0,\tau,P} = 0$ holds. In some of these examples, there exists $\lambda > 0$ such that $\mathbb{P}(g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ percolates$) \in (0,1)$. We also present examples where $0 < \lambda_{N_0,\tau,P} < \infty$ but Λ is not stabilizing, however, in these examples $\Lambda(Q_1)$ is bounded, see Section 4.2.3.2 for the absolutely continuous case and Example 4.2.11 in Section 4.2.3.3 for the singular case.

4.1.3.2. The case of only stabilizing intensity

According to [HJC17, Section 2.1], stabilization of Λ does not imply that $\lambda_{\mathrm{c}}(r) < \infty$ for all $r > 0$, see Section 4.1.5.1 for more details. Now we show that if Λ is stabilizing with $\mathbb{E}[\Lambda(Q_1)] = 1$, then $\lambda_{\mathrm{c}}(r) < \infty$ holds for r large enough, and for the SINR graph, if r_{B} is large (in particular $N_0 > 0$) and (2b) or (2c) holds, then also $\lambda_{N_0,\tau,P} < \infty$.

The fact that $\lambda_{\mathrm{c}}(r) < \infty$ holds for r large for Λ stabilizing is actually a direct consequence of certain results of [HJC17], but since it was not stated explicitly in that paper, we present it as a corollary.

Corollary 4.1.8. *If Λ is stabilizing, then the following hold.*

1. *There exists $r_0 \geq 0$ such that $\lambda_{\mathrm{c}}(r) < \infty$ for all $r > r_0$.*
2. $\lim_{r\to\infty} \lambda_{\mathrm{c}}(r) = 0$.

The proof of Corollary 4.1.8 is carried out in Section 4.2.2.1. We will see that after recalling some elements of Palm calculus and the notion of percolation probability for Cox processes from [HJC17], the corollary follows immediately from [HJC17, Theorem 2.9].

Thus, SNR graphs of stabilizing Cox processes exhibit a supercritical phase if $r_{\mathrm{B}} = \ell^{-1}(\tau N_0/P)$ is large enough, and the critical intensity tends to zero as $r_{\mathrm{B}} \to \infty$. Hence, any intensity $\lambda > 0$ is SNR-supercritical for r_{B} sufficiently large. That is, percolation can be obtained via increasing P or reducing N_0 or τ. In practice, it depends on technological development and physical constraints whether such improvements are possible. We note that the paper [BY13] worked under the assumption that N_0 an τ are fixed and thus formulated its results for large r_{B} in terms of large P, cf. Section 4.2.3.5.

If $d = 2$, then for the Poisson point process, Theorem 4.1.4 guarantees that $\lambda_{N_0,\tau,P} = \lambda_{\mathrm{c}}(r_{\mathrm{B}}) < \infty$ for all $r_{\mathrm{B}} > 0$. This relies on the Russo–Seymour–Welsh type result [MR96, Corollary 4.1] that for $r > 0$ and $\lambda > \lambda_c(r)$, $3n \times n$ rectangles are crossed by a cluster of the Poisson–Boolean model $X^\lambda \oplus B_{r/2}(o)$ in the hard direction with probability tending to 1 as $n \to \infty$. This result is formulated more precisely and slightly more generally as follows (cf. [FM07, Theorem 2.7.1]). For $n,\alpha > 0$, let $R(\alpha,n)$ denote the rectangle $[0,\alpha n] \times [0,n] \subset \mathbb{R}^d$.

Theorem 4.1.9 (MR96, FM07)**.** *For $\lambda > 0$, let X^λ be a homogeneous Poisson point process on $\mathbb{R}^2$. Fix $r > 0$. For $n > 0$ and $\alpha > 1$, let $C(\alpha,n)$ denote the event that $R(\alpha,n)$ is horizontally crossed by the Poisson–Boolean model $X^\lambda \oplus B_{r/2}(o)$. That is, $C(\alpha,n)$ is defined as the event that there exists a connected component $\mathcal{C}$ of $X^\lambda \oplus B_{r/2}(o)$ such that for both vertical sides $\{0\} \times [0,n]$ and $\{\alpha n\} \times [0,n]$ of $R(\alpha,n)$, there exists a point in $\mathcal{C} \cap X^\lambda$ having distance less than $r/2$ from that side. If $\lambda > \lambda_{\mathrm{c}}(r)$, then we have for any $\alpha > 1$ that*

$$\lim_{n\to\infty} \mathbb{P}(C(\alpha,n)) = 1.$$

Now, in the coupled limit $\lambda \downarrow 0, r \uparrow \infty, \lambda r^d = \varrho > 0$, the rescaled Cox process $r^{-1}X^\lambda$ converges weakly to a Poisson point process with intensity ϱ [HJC17, Section 2.2.2]. Further, using arguments of [HJC17, Section 7.1], we will see that for fixed n, the probability that the Boolean model of the Cox process crosses a $3nr \times nr$ rectangle in a given direction converges to the probability that the limiting Poisson–Boolean model crosses a $3n \times n$ rectangle in the same direction. These together with the stabilization of Λ give us an opportunity to map the SINR graph to a renormalized percolation process, using the construction of [DFMMT06, Section 3] involving crossing probabilities. Moreover, if Λ is stabilizing, then interferences can be controlled similarly to the proof of Theorem 4.1.5 under the assumptions (2b) or (2c) on the stationary intensity measure Λ and the path-loss function ℓ, using this renormalized percolation process. These imply that $\lambda_{N_0,\tau,P} < \infty$ if r_{B} is large. Actually, it is even true that any $\lambda > 0$ exceeds $\lambda_{N_0,\tau,P}$ if r_{B} is sufficiently increased. Similarly, in higher dimensions $d \geq 3$, one can use discrete percolation arguments of [HJC17, Section 5.2] in order to verify that an analogous assertion holds.

Proposition 4.1.10. *Let $d \geq 2$ and $\lambda > 0$, and let Λ be stabilizing. If* $\operatorname{supp} \ell = [0,\infty)$ *and assumption* (2b) *or* (2c) *of Theorem 4.1.5 holds, then there exists $r_0 \geq 0$ such that if $r_{\mathrm{B}} \in [r_0,\infty)$, then $\lambda_{N_0,\tau,P} < \lambda$.*

We note that while $\lambda_{N_0,\tau,P} < \infty$ follows from the mere assumption that r_{B} is large (with no further reference to the separate values of τ,N_0,P), the function $\lambda \mapsto \gamma^*(\lambda)$ depends on finer details of the parameters λ,τ,N_0, and P. In particular, in Section 4.1.4 we will see that for all $\lambda > 0$, $\gamma^*(\lambda) \leq 1/\tau$ holds, regardless of the values of N_0 and P.

We will prove Proposition 4.1.10 in Section 4.2.2.2 and discuss its applicability to the main examples in Section 4.1.5.1. Although it is possible to provide one proof for all dimensions $d \geq 2$, we find it instructive to start with the case $d = 2$ and to verify the assertion using crossings of $3n \times n$ boxes in that case. Indeed, this discrete model lead to the assertion $\lambda_{N_0,\tau,P} = \lambda_{\mathrm{c}}(r_{\mathrm{B}})$ in the two-dimensional Poisson case, and thus using this model may be helpful for future investigations of the precise value of $\lambda_{N_0,\tau,P}$ in the stabilizing Cox case for $d = 2$ and large r_{B}. Afterwards, we will sketch the proof for $d \geq 3$.

Note that unlike Theorem 4.1.5, Proposition 4.1.10 does not tell about the case when ℓ has compact support. Indeed, in that case, r_{B} cannot be increased arbitrarily, and it may happen that $\lambda_{\mathrm{c}}(r_{\mathrm{B}}) = \infty$ for all r_{B} with $\ell(r_{\mathrm{B}}) > 0$. Then, SINR graphs also do not percolate for any possible $r_{\mathrm{B}} < \sup \operatorname{supp} \ell$ and $\lambda,\gamma \geq 0$.

Thus, although apart from the two-dimensional Poisson case we do not know whether $\lambda_{\mathrm{c}}(r_{\mathrm{B}}) = \lambda_{N_0,\tau,P}$ holds for given values of the parameters, at least we see that both critical intensities tend to zero as $r_{\mathrm{B}} \to \infty$. Proposition 4.1.10 relies on the fact for any $d \in \mathbb{N}$, if $\Lambda \equiv \mathrm{Leb}$, then if $\lambda r^d = \varrho$, $r^{-1}(X^\lambda \oplus B_{r/2}(o))$ has already the same distribution as the limiting Boolean model $X^\varrho \oplus B_{1/2}(o)$. This is the well-known *scale invariance* of the Poisson–Boolean model (cf. [DFMMT06, Section 3]). More generally, the scale invariance says that if $d \in \mathbb{N}$ and $\lambda,\lambda',r,r' > 0$ are such that $\lambda r^d = \lambda' r'^d$, then

$$\mathcal{B}(\lambda',r') \overset{\text{distr.}}{=} \sqrt[d]{\lambda/\lambda'}\mathcal{B}(\lambda,r) = \frac{r'}{r}\mathcal{B}(\lambda,r), \qquad g_{r'}(X^{\lambda'}) \overset{\text{distr.}}{=} \sqrt[d]{\lambda/\lambda'} g_r(X^\lambda) = \frac{r'}{r} g_r(X^\lambda), \tag{4.7}$$

where we write $\mathcal{B}(\widehat{\lambda},\widehat{r})$ for a Poisson–Boolean model $X^{\widehat{\lambda}} \oplus B_{\widehat{r}/2}(o) \subseteq \mathbb{R}^d$ with intensity $\widehat{\lambda}$ and connection radius $\widehat{r}$. For Poisson–Gilbert graphs, this has the consequence

$$\mathbb{P}(g_r(X^\lambda) \text{ percolates}) = \mathbb{P}\big(g_{r\sqrt[d]{\lambda/\lambda'}}(X^{\lambda'}) \text{ percolates}\big), \quad \forall r,\lambda,\lambda' > 0. \tag{4.8}$$

Equivalently, for Poisson–Boolean models,

$$\mathbb{P}(\mathcal{B}(\lambda,r) \text{ percolates}) = \mathbb{P}\big(\mathcal{B}(\lambda', r\sqrt[d]{\lambda/\lambda'}) \text{ percolates}\big). \tag{4.9}$$

4.1.3.3. The case of no environmental noise

We now consider the case $N_0 = 0$. Note that in this case P cancels in (4.1), and thus we will assume without loss of generality that $P = 1$. Further, we fix $\tau > 0$. Since for any $\tau,a > 0$ and $\gamma > 0$, one has $g_{(\gamma,a,\tau,1)}(X^\lambda) \preceq g_{(\gamma,0,\tau,1)}(X^\lambda)$, it follows that

$$\lambda_{0,\tau,1} \leq \inf_{a>0} \lambda_{a,\tau,1}. \tag{4.10}$$

In the Poisson case for $d = 2$, [DFMMT06, Section 3.4] claimed that $\lambda_{0,\tau,1} = 0$ and argued that this can be shown analogously to the statement of Theorem 4.1.4 that $\lambda_{N_0,\tau,1} < \infty$ for all $N_0 > 0$, and that the only difference is that there is no Boolean threshold. We now show that this claim is true if ℓ has unbounded support, but it fails in most of the relevant cases, in particular also in the two-dimensional Poisson case, if $\operatorname{supp} \ell$ is compact.

Let ℓ be such that $\operatorname{supp} \ell = [0,\infty)$. As for the case $d = 2$ and $\Lambda \equiv \mathrm{Leb}$, let us fix $\tau > 0$ (and $P = 1$), and let $\lambda > 0$ be arbitrary. By the scale invariance (4.8) and the fact that $\lambda_{\mathrm{c}}(1) \in (0,\infty)$, it follows that any $\lambda > 0$ satisfies $\lambda > \lambda_{\mathrm{c}}(r)$ for all sufficiently large $r > 0$. Choosing $r_{\mathrm{B}}(a) = \ell^{-1}(\tau a)$, we see that $r_{\mathrm{B}}(a)$ is well-defined for all sufficiently small noise powers $a > 0$, and $r_{\mathrm{B}}(a) \to \infty$ as $a \downarrow 0$. The proof of [DFMMT06, Theorem 1] implies that $\lambda_{a,\tau,1} = \lambda_c(\ell^{-1}(\tau a))$ whenever the right-hand side of this equation is well-defined. Thus, $g_{(\gamma,a,\tau,1)}(X^\lambda)$ percolates almost surely for all γ,a sufficiently small, and hence so does $g_{(\gamma,0,\tau,1)}(X^\lambda) \succeq g_{(\gamma,a,\tau,1)}(X^\lambda)$.

Now, for $d \geq 2$, in the general Cox case, if supp ℓ is unbounded, then letting $N_0 \downarrow 0$ is equivalent to letting $r_B \to \infty$. Since $g_{(\gamma,N_0,\tau,1)}(X^\lambda) \preceq g_{(\gamma,0,\tau,1)}(X^\lambda)$ for any $N_0 > 0$, Proposition 4.1.10 implies

$$\lambda_{0,\tau,1} \leq \inf_{N_0>0} \lambda_{N_0,\tau,1} = 0 \tag{4.11}$$

if Λ is stabilizing, supp ℓ is unbounded, and (2b) or (2c) holds. In contrast, if supp ℓ is bounded, then for any $d \geq 2$, $\lambda_{0,\tau,1} = 0$ is only true in the pathological case $\lambda_c(r_{\max}) = 0$, in particular it never occurs if Λ is stabilizing.

Corollary 4.1.11. *If $d \geq 2$ and $r_{\max} := \sup \operatorname{supp} \ell$ is finite, then $\lambda_{0,\tau,1} \geq \lambda_c(r_{\max})$.*

Proof. The statement is trivial if $\lambda_c(r_{\max}) = 0$. Else, note that for any $\lambda > 0$ and $X_i,X_j \in X^\lambda$, if $|X_i - X_j| \geq r_{\max}$, then $\mathrm{SINR}(X_i, X_j, X^\lambda) = 0$. Hence, $g_{(\gamma,0,\tau,1)}(X^\lambda) \preceq g_{r_{\max}}(X^\lambda)$ for any $\gamma > 0$. Choosing $0 < \lambda < \lambda_c(r_{\max})$, with probability 1, $g_{(\gamma,0,\tau,1)}(X^\lambda)$ does not percolate for any $\gamma > 0$. □

Thus, since $\Lambda \equiv$ Leb is stabilizing, an argument similar to the one in [BY13, Section 3.4.2] shows that [DFMMT06, Corollary 1] is false for all choices of ℓ with compact support.

4.1.4. Estimates on the critical interference cancellation factor

In the Poisson case $\Lambda \equiv$ Leb for $d = 2$, [DBT05, FM07] have derived the following bounds on the critical interference cancellation factor $\gamma^*(\lambda)$ defined in (4.5).

(A) $\forall \lambda > 0$, $\gamma^*(\lambda) \leq \frac{1}{\tau}$.

(B) $\gamma^*(\lambda) = \mathcal{O}(1/\lambda)$ as $\lambda \to \infty$.

(C) If ℓ has bounded support, then $\gamma^*(\lambda) = \Omega(1/\lambda)$ as $\lambda \to \infty$.

(A) implies that $\lambda \mapsto \gamma^*(\lambda)$ is bounded. In Section 4.1.4.1 we recover this bound for any simple point process and present conjectures regarding its possible improvements. For the Cox case, in Section 4.1.4.2 we provide sufficient conditions under which (B) holds or at least $\gamma^*(\lambda)$ tends to 0 as $\lambda \to \infty$, while in Section 4.1.4.3 we investigate generalizations of (C).

4.1.4.1. Intensity-independent bounds

In the Poisson case, (A) is a consequence of the fact [DBT05, Theorem 1] that SINR graphs with $\gamma > 0$ have bounded degrees. This assertion generalizes to any dimension and any simple point process. Recall the notion of the (directed or undirected) SINR graph of a general simple point process from Section 4.1.2.

Proposition 4.1.12. *Let P,$\tau > 0$, $N_0 > 0$. Then, for any simple point process $\Phi = \{X_i\}_{i \in I}$, almost surely,*

$$\forall \gamma > 0, \forall i \in I, \quad X_i \text{ has in-degree less than } 1 + \tfrac{1}{\tau\gamma} \text{ in } g^{\rightarrow}_{(\gamma,N_0,\tau,P)}(\Phi). \tag{4.12}$$

Thus, in the Cox case, for all $\lambda > 0$, $\gamma^*(\lambda) \leq \frac{1}{\tau}$, and $\mathbb{P}(g_{(\frac{1}{\tau},N_0,\tau,P)}(X^\lambda)$ percolates$) = 0$.

For $N_0 = 0$, the same proof implies the same assertion apart from the non-percolation for $\gamma = 1/\tau$. The proof of the bound (4.12) is analogous to the one of [DBT05, Theorem 1]. We note that it even holds in one dimension, and among the properties of ℓ it only uses that $\ell(|X_i - X_j|) > 0$ holds if there is an edge from X_i to X_j or from X_j to X_i in the directed SINR graph. The proof can be found in Section A.3 in the Appendix, in a more general context. Its arguments can also be used in order to derive stronger degree bounds if $N_0 > 0$ and to show that also the out-degrees in $g^{\rightarrow}_{(\gamma,N_0,\tau,P)}(\Phi)$ are bounded if ℓ has unbounded support; we refrain from presenting here the details.

By (4.12), if $\gamma \geq \frac{1}{\tau}$, degrees in $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ are at most 1, and thus all clusters of $g_{(N_0,\gamma,\tau,P)}(\Phi)$ are pairs or isolated points. This implies lack of percolation. We also expect that there is no infinite cluster if $\gamma \in [\frac{1}{2\tau}, \frac{1}{\tau})$,

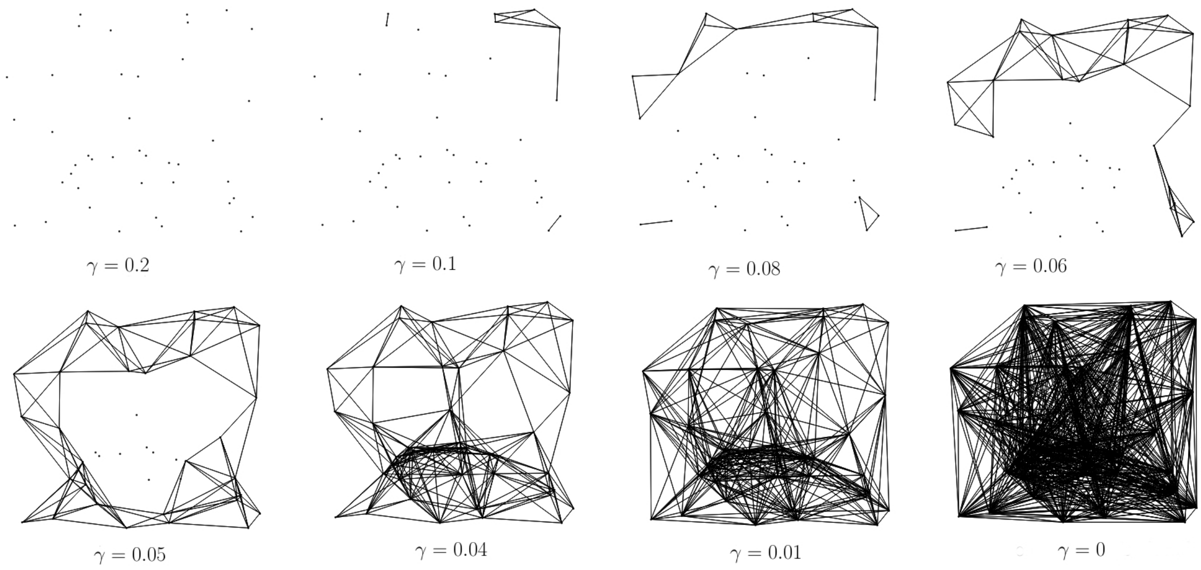


Figure 6. SINR graphs for $d = 2$ and $\Lambda \equiv$ Leb restricted to $[0,1]^2$ with different values of γ, where $N_0 = 2,\tau = 1,P = 1,\ell(r) = \min\{100,r^{-4}\}$, $\lambda = 40$. The realization for $\gamma = \frac{1}{5\tau}$ has no edges, and the one for $\gamma = \frac{1}{10\tau}$ is still highly disconnected. The ones for $\gamma \leq \frac{1}{25\tau}$ are connected, but the effect of bounded degrees is still prominent for $\gamma = \frac{1}{100\tau}$ in comparison with the almost complete graph corresponding to $\gamma = 0$.

where the degree bound is 2, for a large class of point processing including the stationary Poisson one. Indeed, in this regime, all clusters are isolated points, finite cycles or (possible in one or two directions infinite) paths. This reminds of one-dimensional percolation models, which are very often subcritical.

The degree constraints also relate SINR graphs to certain k-nearest neighbours graphs. For $k \in \mathbb{N}$ and a simple point process Φ, we write $g_{\mathbf{B}}(k,\Phi)$ for the undirected graph where $X_i,X_j \in \Phi$, $i \neq j$ are connected by an edge if and only if they are mutually among the k nearest neighbours of each other. This graph is almost surely well-defined under the additional assumption that Φ is *nonequidistant*, i.e., almost surely, for any $X_i,X_j,X_k,X_l \in X^\lambda$, $|X_i - X_j| = |X_k - X_l|$ implies that $\{i,j\} = \{k,l\}$ or that $i = j$ and $k = l$. Stationary Cox processes are clearly nonequidistant. We have the following.

Lemma 4.1.13. *Let Φ be a simple nonequidistant point process. If $\tau,P,\gamma > 0$ and $N_0 \geq 0$ are such that almost surely, all in-degrees in $g^{\to}_{(\gamma,N_0,\tau,P)}(\Phi)$ are at most $k \in \mathbb{N}$, then $g_{(\gamma,N_0,\tau,P)}(\Phi) \preceq g_{\mathbf{B}}(k,\Phi)$.*

The proof of Lemma 4.1.13 is carried out in Section 4.3.1. We use this lemma in order to derive a conjecture for the two-dimensional Poisson case. In this case Balister and Bollobás [BB08] studied the graph $g_{\mathbf{B}}(k,X^1)$, which has the same distribution as $\lambda^{1/2} g_{\mathbf{B}}(k,X^\lambda)$ for all $\lambda > 0$. In particular, $\mathbb{P}(g_{\mathbf{B}}(k,X^1)$ percolates$) = \mathbb{P}(g_{\mathbf{B}}(k,X^\lambda)$ percolates$)$ for all $\lambda > 0$. By [BB08, Section 3], with high confidence, $g_{\mathbf{B}}(k,X^1)$ percolates only if $k \geq 5$. That is, this assertion follows once one proves that certain high-dimensional integrals exceed certain deterministic values, but so far the integrals have only been evaluated using Monte Carlo methods. This is more than simulations but less than a proof. If the result holds, then by (4.12) and Lemma 4.1.13, it implies the following improvement of [DBT05, Theorem 1].

Conjecture 4.1.14. *Let $\Lambda \equiv$ Leb and $d = 2$. Then for any $N_0 \geq 0$ and $\lambda > 0$, $\gamma^*(\lambda) \leq \frac{1}{4\tau}$, and $\mathbb{P}(g_{(\frac{1}{4\tau},N_0,\tau,P)}(X^\lambda)$ percolates$) = 0$.*

Simulations suggest that the maximum of $\lambda \mapsto \gamma^*(\lambda)$ is even lower than $\frac{1}{4\tau}$, cf. Figure 6. Conversely, Theorem 4.1.5(2) implies that for $d \geq 2$, $g_{\mathbf{B}}(k,X^1)$ percolates for all k sufficiently large. This was proven in [BB08] for $d = 2$ and $k \geq 15$, and it is intuitively quite clear that this implies the same statement for any $d \geq 3$ for k sufficiently large, although this was not explicitly stated in [BB08].

4.1.4.2. Upper bounds for large intensities

For b-dependent Cox processes, under mild further assumptions on Λ and ℓ, we recover (B) in a weaker form. Further, for $d = 2$, under stronger additional assumptions, we verify that (B) holds in its original form. First, any $\gamma > 0$ becomes subcritical for large λ whenever the SINR graph has bounded edge length.

Proposition 4.1.15. *If Λ is b-dependent, $N_0 \geq 0$, τ,$P > 0$, further, $N_0 > 0$ or ℓ has bounded support, then*

$$\lim_{\lambda \to \infty} \gamma^*(\lambda) = 0. \tag{4.13}$$

We will prove Proposition 4.1.15 in Section 4.3.2.1. It is easy to inspect in this proof that if Λ is only stabilizing, we can always choose $d_0 = \max\{r \geq 0 \colon \ell(r) = \ell(0)\}$ so large that (4.13) holds. However, we are not aware of an application of a setting where d_0 is very large.

Second, for $d = 2$, (B) stays true for b-dependent Cox processes for which $\Lambda(Q_\delta)$ is bounded away from 0 for small enough $\delta > 0$.

Proposition 4.1.16. *If $d = 2$, N_0,τ,$P > 0$, Λ is b-dependent, and $\Lambda(Q_{\delta/2})$ is bounded away from 0 for some $\delta > 0$ such that $\ell(\delta) > \tau N_0/P$, then as $\lambda \to \infty$, $\gamma^*(\lambda) = O(1/\lambda)$.*

The proof of Proposition 4.1.16 will be carried out in Section 4.3.2.2. The applicability of the results of this section to the main examples will be discussed in Section 4.1.5.2.

4.1.4.3. Lower bounds for large intensities

In [DBT05, Section III-C], (C) was verified for the Poisson case for $d = 2$ and compactly supported ℓ. It can easily be generalized to a class of b-dependent Cox point processes as follows.

Corollary 4.1.17. *Let $d = 2$,* supp *ℓ bounded, Λ b-dependent such that $\Lambda(Q_{d_1})$ is bounded away from 0 for some $d_1 > 0$. Then we have $\gamma^*(\lambda) = \Omega(1/\lambda)$ as $\lambda \to \infty$.*

The proof of Corollary 4.1.17 will be sketched in Section 4.3.3. This follows the lines of the original proof of [DBT05, Theorem 2], using some additional observations. In Section 4.1.5.2 we will discuss the applicability of Corollary 4.1.17 to the main examples.

4.1.5. Applicability of the results to the main examples

4.1.5.1. Phase transitions

We now consider each of the relevant examples of Λ from [HJC17] recalled in Section 4.1.1 and discuss the applicability of Theorem 4.1.5 and Proposition 4.1.10 to them. For the sake of brevity, we will tacitly assume that $N_0 > 0$. The case $N_0 = 0$ can be handled according to Section 4.1.3.3.

Let us note that in the case of a modulated Poisson point process, Λ is b-dependent for some $b > 0$ as long as the stationary random closed set Ξ is *boundedly determined*, i.e., there exists $R > 0$, $b' > 0$, and a b'-dependent stationary point process $X_{\mathbf{S}}$ such that $\{o \in \Xi\}$ is measurable with respect to the σ-algebra generated by $X_{\mathbf{S}} \cap B_R$. This observation helps extend some results about Ξ being a Poisson–Boolean model (which is clearly boundedly determined) to all boundedly determined Ξ.

Now, all examples are stabilizing and therefore they exhibit a subcritical phase by Theorem 4.1.5(1), apart from general modulated Poisson point processes where Ξ is not boundedly determined.

For the Poisson point process modulated by a boundedly determined Ξ, Λ is b-dependent and $\Lambda(Q_1)$ is bounded. Further, if λ_1,$\lambda_2 > 0$, then Λ is also asymptotically essentially connected, and thus $\lambda_{N_0,\tau,P} < \infty$ holds for any $r_{\mathrm{B}} > 0$ under the general Assumption (ℓ) on ℓ. In particular, this covers the Poisson case $\Lambda \equiv$ Leb. Further, by stabilization, $\lambda_{N_0,\tau,P} < \infty$ holds for large r_{B} also if either λ_1 or λ_2 is zero, in case ℓ has unbounded support and satisfies Assumption (ℓ). It is easy to see that if $\lambda_1 > \lambda_2 = 0$ and Ξ is a Poisson–Boolean model, then there are cases where $\lambda_c(r_B) = \infty$ holds for all small enough $r_B > 0$. Indeed, if the Poisson–Boolean model is subcritical, then one can choose r_B so small that the Cox–Boolean model $X^\lambda \oplus B_{r_B/2}(o)$ is still contained in a subcritical Poisson–Boolean model for any $\lambda > 0$. Also for $\lambda_2 > \lambda_1 = 0$, a supercritical phase may be missing. Indeed, e.g. for $d = 2$ and $\lambda > 0$, for any supercritical Poisson–Boolean model $\mathcal{B}(\lambda_0,r_0)$ with intensity $\lambda_0 > 0$ and radius $r_0 > 0$, there exists $r_1 < r_0$ such that $\mathcal{B}(\lambda_0,r_1)$ has no unbounded vacant component [MR96, Section 4.6]. Then for $\Xi = \mathcal{B}(\lambda_0,r_0)$ and $\lambda > 0$, let the Cox process X^λ have intensity measure $\lambda\Lambda$, with $\Lambda = \lambda_2 \mathbb{1}_{\Xi^c}\mathrm{Leb}$ satisfying $\mathbb{E}[\Lambda(Q_1)] = 1$. Then for $r_B > 0$ small, for all $\lambda > 0$, the Cox–Boolean model $X^\lambda \oplus B_{r_B/2}(o)$ is included in $\mathcal{B}(\lambda_0,r_1)^c$ and thus has no unbounded cluster.

For a general, not boundedly determined Ξ, neither Theorem 4.1.5 nor Proposition 4.1.10 is applicable due to the possible lack of stabilization. However, $\Lambda(Q_1)$ is still bounded and Λ is absolutely continuous, and therefore a subcritical phase exists for $\lambda_1,\lambda_2 \geq 0$ thanks to a comparison to a Poisson–Gilbert graph. Further, if $\lambda_1,\lambda_2 > 0$, then a similar comparison yields that $\lambda_{N_0,\tau,P} < \infty$ holds for any r_B. We will verify these assertions in Section 4.2.3.2.

For the shot-noise field, it may again happen that $\lambda_c(r_B) = \infty$ for small $r_B > 0$. Indeed, if the underlying Poisson point process $X_{\mathbf{S}}$ is such that its Boolean model with connection radius $r/2 = \operatorname{diam}\operatorname{supp} k$ is subcritical and also r_B is small, then the Cox–Boolean model $X^\lambda \oplus B_{r_B/2}(o)$ is included in a subcritical Poisson–Boolean model for any $\lambda > 0$. Nevertheless, for any shot-noise field, Λ is b-dependent and, although $\Lambda(Q_1)$ is not bounded, it has all exponential moments thanks to Campbell's theorem [K93, Section 3.2]. Hence, for ℓ with unbounded support satisfying the decay condition in (2c), $\lambda_{N_0,\tau,P} < \infty$ holds for the shot-noise field if r_B is large.

Poisson–Voronoi and Poisson–Delaunay tessellations are asymptotically essentially connected, and thus by (2a), $\lambda_{N_0,\tau,P} < \infty$ holds for any $r_B > 0$ if ℓ has bounded support. However, these intensities do not satisfy the boundedness condition in (2b). By Theorem 4.1.6, in the Voronoi case for $d = 2$, $\Lambda(Q_1)$ has all exponential moments, thus the condition (2c) is also applicable. Thanks to the results of [JT19], the same holds for the two-dimensional Poisson–Delaunay tessellation. The same question remains open (and highly interesting) for both kinds of tessellations for $d \geq 3$.

4.1.5.2. Estimates on the critical interference cancellation factor

Let us now discuss the applicability of Propositions 4.1.15 and 4.1.16, and Corollary 4.1.17 to the main examples. Each of them requires b-dependence, and therefore they are only applicable to the Poisson point process modulated by a boundedly determined Ξ and to the shot-noise field. For these two examples, Proposition 4.1.15 immediately applies. For $d = 2$, Proposition 4.1.16 and Corollary 4.1.17 require also that $\Lambda(Q_\delta)$ be bounded away from 0 for some $\delta > 0$, which only applies for the modulated Poisson point process with a boundedly determined Ξ and with $\lambda_1,\lambda_2 > 0$ (for which it holds for all $\delta > 0$).

Poisson–Voronoi and Poisson–Delaunay tessellations, Poisson–Boolean models, and shot-noise fields can be regularized by augmenting their defining Poisson point process with a grid shifted by a uniform random vector in $[0,1]^d$ (see [BBM11]), resulting in a b-dependent intensity measure Λ' such that $\Lambda'(Q_{d_1})$ is bounded away from 0 for $d_1 > 0$ large enough. For such regularized intensity measures, Proposition 4.1.15 and Corollary 4.1.17 certainly apply. Further, if $d = 2$ and d_1 is small enough to satisfy the condition of Proposition 4.1.16, then also this result applies to the Cox point process with the regularized Poisson-based intensity.

4.2. Proof and discussion of phase transitions

This section includes the proofs of the results of Section 4.1.3. In particular, Section 4.2.1 is devoted to the proofs of the assertions of Section 4.1.3.1: in Section 4.2.1.1 we verify Theorem 4.1.5(2) and in Section 4.2.1.2 we prove Theorem 4.1.6. Further, Section 4.2.2 contains the results of Section 4.1.3.2: in Section 4.2.2.1 we show how Corollary 4.1.8 can be derived from the results of [HJC17], whereas in Section 4.2.2.2, using arguments of Section 4.2.1.1, we verify Proposition 4.1.10.

For the reader's convenience, we start with the precise statement and proof of the following assertion, which was used but not spelt out in [HJC17] and [T18] at multiple places.

Lemma 4.2.1. *Let Λ be stabilizing with stabilization radii R, let $n \geq 1$ and $p,r > 0$ with $p \geq r$. Let us define a site percolation process on $(\mathbb{Z}^d,\|\cdot\|_\infty)$ as follows: the site $z \in \mathbb{Z}^d$ is* open *if $R(Q_{pn}(nz)) < rn$ and closed otherwise. Then the process of open sites is b-dependent for all $b \geq \lfloor p+2r \rfloor$.*

Proof. For simplicity we write down the proof for $n = 1$, the case $n > 1$ is analogous. For any hypercube $Q \subset \mathbb{R}^d$, let us write $C(Q)$ for the centre of Q. For any $x \in \mathbb{R}^d$, the event $\{R(Q_p(x)) < r\}$ equals the event that $R(Q) := \sup_{w \in Q \cap \mathbb{Q}^d} R_w < r$ holds for any translate Q of Q_r contained entirely in $Q_p(x)$. Let now $z = (z_j)_{j \in [d]}$, $z' = (z'_j)_{j \in [d]}$ be elements of $\mathbb{Z}^d$. If for any two translates Q,Q' of Q_r contained entirely in $Q_p(z)$ respectively $Q_p(z')$ we have that $|C(Q)-C(Q')| > 3r$, then the events $\{R(Q_p(z)) < r\}$ and $\{R(Q_p(z')) < r\}$ are independent by the definition of stabilization. Now we claim that this condition is satisfied if $\|z-z'\|_\infty > p+2r$. Indeed, then we can fix a coordinate $i \in [d]$ such that $|z_i - z'_i| > p+2r$; let us assume without loss of generality that $z'_i > z_i$. Now, for any two translates Q,Q' of Q_r contained entirely in $Q_p(z)$ respectively $Q_p(z')$, let us write $C(Q) = (x_j)_{j \in [d]}$ and $C(Q') = (x'_j)_{j \in [d]}$, then we have $x_i \leq z_i + (p-r)/2$ and $x'_i \geq z'_i - (p-r)/2$. Hence, $x'_i - x_i \geq (z'_i - z_i) - (p-r) > p+2r-(p-r) = 3r$. Therefore, $|C(Q)-C(Q')| \geq \|C(Q)-C(Q')\|_\infty > 3r$, as required. This implies the lemma. Indeed, for $z,z' \in \mathbb{Z}^d$, we have that $\|z-z'\|_\infty \in \mathbb{N}_0$, and thus $\|z-z'\|_\infty > \lfloor p+2r \rfloor$ implies $\|z-z'\|_\infty > p+2r$. $\square$

It is easy to inspect from the proof that the lemma holds on $(\mathbb{Z}^d,\|\cdot\|_\infty)$ under a redefined notion of stabilization where we replace the Euclidean distance in Definition 4.1.1 with the distance with respect to any norm $\|\cdot\|$ on $\mathbb{R}^d$ that satisfies $\|x\| \leq \sqrt{d}\|x\|_\infty$ for all $x \in \mathbb{R}^d$. On the other hand, it does not always hold for $(\mathbb{Z}^d,\|\cdot\|_\infty)$ replaced by $(\mathbb{Z}^d,|\cdot|)$ for all $b \geq \lfloor p+2r \rfloor$, but since $|x| \leq \sqrt{d}\|x\|_\infty$ holds for all $x \in \mathbb{R}^d$, it clearly holds for $b \geq \sqrt{d}(p+2r)$ (where we have removed the downroundings because $|z-z'| \in \mathbb{N}_0$ does not necessarily hold for $z,z' \in \mathbb{Z}^d$). To see that $b \geq \lfloor p+2r \rfloor$ is not always sufficient for $(\mathbb{Z}^d,|\cdot|)$, take e.g. $d = 2$, $z = (0,0)$, $z' = (3,3)$, $p = 2$, and $r = 1$. Then $|z-z'| = 3\sqrt{2} > p+2r$, but for $x = (1/2,1/2)$ and $x' = (5/2,5/2)$, we have $Q_r(x) \subset Q_p(z)$ and $Q_r(x') \subset Q_p(z')$ while $|x-x'| = 2\sqrt{2} < 3r$, therefore the definition of stabilization does not guarantee that $\{R(Q_p(z)) < r\}$ and $\{R(Q_p(z')) < r\}$ are independent for $\Lambda \not\equiv \mathrm{Leb}$.

4.2.1. Proof of the results of Section 4.1.3.1

4.2.1.1. Proof of Theorem 4.1.5 (existence of supercritical phase)

For the proof we fix $N_0,\tau,P > 0$. Now, for $\gamma \geq 0$ and $\lambda > 0$, we use the simplified notation $g_{(\gamma)}(X^\lambda) = g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ (until the end of the present section). Further, we assume that Λ is asymptotically essentially connected. Thus, by Theorem 4.1.3, $\lambda_c(r) \in (0,\infty)$ holds for all $r > 0$. We recall that $g_{(0)}(X^\lambda) = g_{r_B}(X^\lambda)$, cf. (4.4).

The proof consists of four steps. First, we map our percolation problem to a discrete site percolation model. Second, we indicate why this discrete model has an unbounded cluster for large λ and accordingly chosen small $\gamma > 0$, conditional on the assumption that interferences can be sufficiently controlled. Third, we show that if the discrete model percolates, then so does $g_{(\gamma)}(X^\lambda)$. Fourth, we finish the proof of percolation in the discrete model by controlling the interferences. Thus, the structure of our proof is similar to the one of [DFMMT06, Theorem 1]. However, the discrete model we use originates from [HJC17] and not from [DFMMT06]. Further, the randomness of Λ makes the fourth step more involved than in the Poisson case. At this point, we use the assumptions (2a), (2b) respectively (2c).

STEP 1. *Mapping to a lattice percolation problem.*

Let $r \in (d_0,r_B)$, such r exists by (4.4) and (i) – (ii) in Assumption (ℓ). Following [HJC17, Section 5.2], for $n \geq 1$, we say that a site $z \in \mathbb{Z}^d$ is *n-good* if

1. $R(Q_n(nz)) < n/2$,
2. $X^\lambda \cap Q_n(nz) \neq \varnothing$,
3. every $X_i, X_j \in X^\lambda \cap Q_{3n}(nz)$ are connected by a path in $g_r(X^\lambda) \cap Q_{6n}(nz)$.

A site $z \in \mathbb{Z}^d$ is *n-bad* if z is not n-good.

Next, for $a \geq 0$, we define a "shifted" version ℓ_a of the path-loss function ℓ, similarly to [DFMMT06]. Note that any point of $Q_a(x)$ is at distance at most $\frac{a\sqrt{d}}{2}$ away from the centre x of $Q_a(x)$. We define $\ell_a\colon [0,\infty) \to [0,\infty)$ as follows

$$\ell_a(r) = \ell(0)\mathbb{1}\Big\{r < \frac{a\sqrt{d}}{2}\Big\} + \ell\Big(r - \frac{a\sqrt{d}}{2}\Big)\mathbb{1}\Big\{r \geq \frac{a\sqrt{d}}{2}\Big\}. \tag{4.14}$$

Note that $\ell_0 = \ell$. Now, we define the shot-noise processes

$$I_a(x) = \textstyle\sum_{X_i \in X^\lambda} \ell_a(|x - X_i|), \qquad I(x) = \textstyle\sum_{X_i \in X^\lambda} \ell(|x - X_i|), \qquad x \in \mathbb{R}^d.$$

Then $I_0(x) = I(x)$. By the triangle inequality, for $a \geq 0$, $I(x) \leq I_a(z)$ holds for any $z \in \mathbb{R}^d$ and $x \in Q_a(z)$. Further, we make the following observation, which was not included in [DFMMT06]. This will be a key argument in the proof of Theorem 4.1.5(2) in the case (2c).

Lemma 4.2.2. *Let $n \geq 1$, $a \geq n$, $k \in \mathbb{N}$, $z \in k\mathbb{Z}^d$. Then for any $y \in Q_k(z)^o \cap \mathbb{Z}^d$ and $S \subseteq \mathbb{R}^d$ measurable, we have almost surely*

$$\sum_{X_j \in X^\lambda \cap S} \ell_{ka}(|nz - X_j|) \geq \sum_{X_j \in X^\lambda \cap S} \ell_a(|ny - X_j|).$$

Proof. By the definition of ℓ_b, $b > 0$, it suffices to show that for all $x \in \mathbb{R}^d$ and $y \in Q_k(z)^o \cap \mathbb{Z}^d$, $|x - ny| \leq \frac{a\sqrt{d}}{2}$ implies $|x - nz| \leq \frac{ka\sqrt{d}}{2}$. But since $|nz - ny| \leq \frac{n(k-1)\sqrt{d}}{2}$, this follows by the triangle inequality:

$$|x - nz| \leq |x - ny| + |ny - nz| \leq \frac{a\sqrt{d}}{2} + \frac{(k-1)n\sqrt{d}}{2} \leq \frac{ka\sqrt{d}}{2}.$$

Thus, the proof is finished. $\square$

Now, for $z \in \mathbb{Z}^d$, $n \geq 1$, and $M > 0$, we define the following random variables

$$A_n(z) = \mathbb{1}\{z \text{ is } n\text{-good}\}, \qquad B_{n,M}(z) = \mathbb{1}\{I_{6n}(nz) \leq M\}, \qquad C_{n,M}(z) = A_n(z)B_{n,M}(z). \tag{4.15}$$

STEP 2. *Percolation in the lattice.*

If $\lambda > 0$ is sufficiently large, then for all n,M sufficiently large, the process of sites $z \in \mathbb{Z}^d$ with $C_{n,M}(z) = 1$ percolates with probability one (where $\mathbb{Z}^d$ is equipped with its nearest neighbour edges). This immediately follows by a Peierls argument once we have verified that the following holds.

Proposition 4.2.3. *Under the assumption* (2a), (2b) *or* (2c) *in Theorem 4.1.5, for all $\lambda > 0$ and for all sufficiently large $\lambda > 0$, $n \geq 1$, and $M > 0$, there exists a constant $q_C < 1$ such that for any $N \in \mathbb{N}$ and pairwise distinct sites $z_1, \dots, z_N \in \mathbb{Z}^d$, we have*

$$\mathbb{P}(C_{n,M}(z_1) = 0, \dots, C_{n,M}(z_N) = 0) \leq q_C^N. \tag{4.16}$$

Moreover, for any $\varepsilon > 0$, we can choose λ, n, and M large enough such that $q_C \leq \varepsilon$.

In order to verify this proposition we start with the results of [HJC17] about the n-good sites.

Lemma 4.2.4. *For all sufficiently large $n \geq 1$ and $\lambda > 0$, there exists $q_A < 1$ such that for any $N \in \mathbb{N}$ and pairwise distinct sites $z_1, \dots, z_N \in \mathbb{Z}^d$,*

$$\mathbb{P}(A_n(z_1) = 0, \dots, A_n(z_N) = 0) \leq q_A^N. \tag{4.17}$$

Moreover, for any $\varepsilon > 0$ and for sufficiently large λ, one can choose n so large that $q_A \leq \varepsilon$.

Proof. In [HJC17, Section 5.2] it was shown that for asymptotically essentially connected Λ, the process of n-good sites is 7-dependent, and percolation of n-good sites implies percolation of $g_r(X^\lambda)$ (the proof of the latter statement will be recalled in Step 3). Moreover, for $z \in \mathbb{Z}^d$, we have

$$\lim_{n\to\infty}\lim_{\lambda\to\infty} \mathbb{P}(A_n(z)=0)=0, \tag{4.18}$$

where the convergence is uniform in $z \in \mathbb{Z}^d$. Let now $N \in \mathbb{N}$ and $z_1,\dots,z_N \in \mathbb{Z}^d$ pairwise distinct. By 7-dependence, there exists $m \geq 1$ and a subset $\{k_j\}_{j=1}^m$ of $[N]$ such that $A_n(z_{k_1}),\dots,A_n(z_{k_m})$ are independent and $m \geq \frac{N}{8^d}$. Now, let $q_A' = \limsup_{n\to\infty} \mathbb{P}(A_n(o)=0)^{\frac{1}{8^d}}$. By (4.18), q_A' tends to zero as $\lambda \to \infty$. Further,

$$\mathbb{P}(A_n(z_1)=0,\dots,A_n(z_N)=0) \leq \mathbb{P}(A_n(z_{k_1})=0,\dots,A_n(z_{k_m})=0) \leq \mathbb{P}(A_n(o)=0)^{\frac{N}{8^d}} \leq q_A^N,$$

for λ,n sufficiently large and $q_A = 2q_A'$. This finishes the proof of the lemma. □

The main step of the proof of Proposition 4.2.3 is to prove the following assertion, in other words, to control the interferences.

Proposition 4.2.5. *Under the assumption* (2a), (2b) *or* (2c) *in Theorem 4.1.5, for all $\lambda > 0$ and for all sufficiently large $n \geq 1$ and $M > 0$, there exists a constant $q_B < 1$ such that for any $N \in \mathbb{N}$ and pairwise distinct sites $z_1,\dots,z_N \in \mathbb{Z}^d$, we have*

$$\mathbb{P}(B_{n,M}(z_1)=0,\dots,B_{n,M}(z_N)=0) \leq q_B^N. \tag{4.19}$$

Moreover, for any $\varepsilon > 0$, $\lambda > 0$, and large enough $n \geq 1$, we can choose M large enough such that $q_B \leq \varepsilon$.

Note the assumption that n has to be large enough; this was absent in the analogous result [DFMMT06, Proposition 2]. In fact, it is not necessary in case (2b) is satisfied. The proof of Proposition 4.2.5 is postponed until Step 4. There, it is easy to inspect that the proof also works if Λ is only stabilizing. Given Lemma 4.2.4 and Proposition 4.2.5, Proposition 4.2.3 can be concluded as follows.

Proof of Proposition 4.2.3. Let $N \in \mathbb{N}$ and let $z_1,\dots,z_N \in \mathbb{Z}^d$ be pairwise distinct. By the stationarity of Λ, $\{C_{n,M}(z_i)\}_{i=1}^N$ are identically distributed. Using Lemma 4.2.4 and Proposition 4.2.5, we obtain for sufficiently large n and M

$$\begin{aligned}
&\mathbb{P}(C_{n,M}(z_1)=0,\dots,C_{n,M}(z_N)=0) = \mathbb{P}(A_n(z_1)B_{n,M}(z_1)=\ldots=A_n(z_N)B_{n,M}(z_N)=0)\\
&\quad\leq \mathbb{P}\big(\big(\exists S\subseteq[N]\colon |S|\geq N/2,\ A_n(z_i)=0,\ \forall i\in S\big) \text{ or } \big(\exists S\subseteq[N]\colon |S|\geq N/2,\ B_{n,M}(z_i)=0,\ \forall i\in S\big)\big)\\
&\quad\leq 2\max\big\{\mathbb{P}\big(\exists S\subseteq[N]\colon |S|\geq N/2,\ A_n(z_i)=0,\ \forall i\in S\big),\mathbb{P}\big(\exists S\subseteq[N]\colon |S|\geq N/2,\ B_{n,M}(z_i)=0,\ \forall i\in S\big)\big\}\\
&\quad\leq 2\binom{N}{\lfloor N/2\rfloor}\max\{q_A^{N/2},q_B^{N/2}\} \leq 2\times 2^N\max\{\sqrt{q_A}^N,\sqrt{q_B}^N\}.
\end{aligned}$$

Putting $q_C = 4\max\{\sqrt{q_A},\sqrt{q_B}\}$ and choosing λ,n,M large enough, the proposition follows. □

STEP 3. *Percolation in the SINR graph.*

Now, let λ,n,M be such that the sites $z \in \mathbb{Z}^d$ with $C_{n,M}(z)=1$ percolate. If z is such that $C_{n,M}(z)=1$, then $I_{6n}(nz) \leq M$, and thus $I(x) \leq M$ for all $x \in Q_{6n}(nz)$. Now, as in [DFMMT06, Section 3.3], for z such that $C_{n,M}(z)=1$ and for $X_i,X_j \in X^\lambda \cap Q_{6n}(nz)$ with $|X_i - X_j| \leq r$, we have

$$\frac{P\ell(|X_i-X_j|)}{N_0+\gamma\sum_{k\neq i,j}P\ell(|X_k-X_j|)} \geq \frac{P\ell(r)}{N_0+\gamma PM}.$$

Choosing

$$\gamma' = \frac{N_0}{PM}\Big(\frac{\ell(r)}{\ell(r_{\mathrm{B}})}-1\Big) > 0, \tag{4.20}$$

(where the inequality holds because $d_0 < r < r_{\mathrm{B}}$), we have

$$\frac{P\ell(r)}{N_0+\gamma' PM} = \frac{P\ell(r_{\mathrm{B}})}{N_0} = \tau. \tag{4.21}$$

Thus, for $\gamma \in (0,\gamma')$, any two Cox points of distance less than r both lying within $Q_{6n}(nz)$ for z such that $C_{n,M}(z) = 1$ are connected in $g_{(\gamma)}(X^\lambda)$.

Finally, similarly to [HJC17, Section 5.2], we have the following. If there exists an infinite connected component $\mathcal{C}$ of n-good sites z with $I_{6n}(nz) \le M$, let $z,z' \in \mathcal{C}$ with $|z-z'| = 1$. Then by property (2) in the definition of n-goodness, there exist $X_i \in Q_n(nz), X_i' \in Q_n(nz')$. Note that X_i,X_i' are both contained in $Q_{3n}(nz)$. Thus by property (3), we find a path from X_i to X_i' in $g_r(X^\lambda) \cap Q_{6n}(nz)$. Since $I_{6n}(nz) \le M$, all the edges of this path also exist in $g_{(\gamma)}(X^\lambda)$. Hence, $g_{(\gamma)}(X^\lambda) \cap (\bigcup_{z\in\mathcal{C}} Q_{6n}(nz))$ contains an infinite path, which implies that $g_{(\gamma)}(X^\lambda)$ percolates.

Thus, Theorem 4.1.5 follows as soon as we have proven Proposition 4.2.5.

STEP 4. *Proof of Proposition 4.2.5.*

We start the proof with splitting the interference into two parts. For $x \in \mathbb{R}^d$ and $n \ge 1$, we put

$$I_{6n}^{\mathrm{in}}(x) = \sum_{X_i \in X^\lambda \cap Q_{12n\sqrt{d}}(x)} \ell_{6n}(|X_i - x|), \qquad I_{6n}^{\mathrm{out}}(x) = \sum_{X_i \in X^\lambda \setminus Q_{12n\sqrt{d}}(x)} \ell_{6n}(|X_i - x|).$$

Then, for $M > 0$, if $I_{6n}(x) > M$, then $I_{6n}^{\mathrm{in}}(x) > M/2$ or $I_{6n}^{\mathrm{out}}(x) > M/2$. Using a union bound, it suffices to verify Proposition 4.2.5 both with $B_{n,M}(z_i)$ replaced by $B_{n,M/2}^{\mathrm{in}}(z_i)$ and with $B_{n,M}(z_i)$ replaced by $B_{n,M/2}^{\mathrm{out}}(z_i)$ everywhere in (4.19) for all $i \in [N]$, where for $z \in \mathbb{Z}^d$, we write $B_{n,M}^{\mathrm{in}}(z) = \mathbb{1}\{I_{6n}^{\mathrm{in}}(nz) > M/2\}$ and $B_{n,M/2}^{\mathrm{out}}(z) = \mathbb{1}\{I_{6n}^{\mathrm{out}}(nz) > M/2\}$. Indeed, having these assertions, we can combine them analogously to the proof of Proposition 4.2.3. Clearly, it is enough to prove them without the factors 1/2 in the definitions of $B_{n,M}^{\mathrm{in}}(z_i)$ and $B_{n,M}^{\mathrm{out}}(z_i)$, $i \in [N]$. Now we construct a renormalized percolation process. A site $z \in \mathbb{Z}^d$ is *n-tame* if

1. $R(Q_{12n\sqrt{d}}(nz)) < n/2$,
2. $I_{6n}^{\mathrm{in}}(nz) \le M$.

A site $z \in \mathbb{Z}^d$ is *n-wild* if it is not n-tame. The process of n-tame sites is $\lceil 12n\sqrt{d} + 1\rceil$-dependent according to the definition of stabilization. Thus, using dependent percolation theory [LSS97, Theorem 0.0] (similarly to the proof of Lemma 4.2.4), in order to verify Proposition 4.2.5 with $B_{n,M}(\cdot)$ replaced by $B_{n,M}^{\mathrm{in}}(\cdot)$, it suffices to show that for all $\lambda > 0$, $\mathbb{P}(z \text{ is } n\text{-wild})$ can be made arbitrarily close to 0 uniformly in $z \in \mathbb{Z}^d$ by choosing first n sufficiently large and then M large enough accordingly. We have

$$\mathbb{P}(z \text{ is } n\text{-wild}) \le \mathbb{P}(R(Q_{12n\sqrt{d}}(nz)) \ge n/2) + \mathbb{P}(I_{6n}^{\mathrm{in}}(nz) > M).$$

The first term can be made arbitrarily small by choosing n large enough, according to the definition of stabilization. Further, by (4.14),

$$I_{6n}^{\mathrm{in}}(nz) = \sum_{X_i \in X^\lambda \cap Q_{12n\sqrt{d}}(nz)} \ell_{6n}(|X_i - nz|) \le \ell(0)\#\big(X^\lambda \cap Q_{12n\sqrt{d}}(nz)\big)$$

holds for all $z \in \mathbb{Z}^d$. In particular,

$$\mathbb{E}[I_{6n}^{\mathrm{in}}(nz)] \le \ell(0)\lambda\mathbb{E}[\Lambda(Q_{12n\sqrt{d}})] = (12n\sqrt{d})^d\ell(0)\lambda < \infty.$$

Thus, for any $n \ge 1$, $\mathbb{P}(I_{6n}^{\mathrm{in}}(nz) > M)$ can be made arbitrarily small uniformly in $z \in \mathbb{Z}^d$ by choosing M large enough. Thus, we conclude Proposition 4.2.5 with $B_{n,M}(\cdot)$ replaced by $B_{n,M}^{\mathrm{in}}(\cdot)$.

We can already conclude Proposition 4.2.5 in the case (2a) where ℓ has bounded support. Indeed, in this case, for sufficiently large n, the following holds for all $z \in \mathbb{Z}^d$

$$I_{6n}^{\mathrm{in}}(nz) \le I_{6n}(nz) = \sum_{X_i \in X^\lambda \cap Q_{6n+2\sup\operatorname{supp}\ell}(nz)} \ell_{6n}(|X_i - nz|) \le \sum_{X_i \in X^\lambda \cap Q_{12n\sqrt{d}}(nz)} \ell_{6n}(|X_i - nz|) = I_{6n}^{\mathrm{in}}(nz),$$

and thus the proposition follows.

It remains to verify Proposition 4.2.5 with $B_{n,M}(z_i)$ replaced by $B_{n,M}^{\mathrm{out}}(nz_i)$, in the cases (2b) and (2c). Here, without loss of generality, we will assume that supp $\ell = [0,\infty)$.

By Markov's inequality, for any $s > 0$,

$$\begin{aligned}\mathbb{P}(B^{\mathrm{out}}_{n,M}(z_1) = \ldots = B^{\mathrm{out}}_{n,M}(z_N) = 0) &= \mathbb{P}(I^{\mathrm{out}}_{6n}(nz_1) > M, \ldots, I^{\mathrm{out}}_{6n}(nz_N) > M)\\ &\le \mathbb{P}\Big(\sum_{i=1}^N I^{\mathrm{out}}_{6n}(nz_i) > NM\Big) \le \exp(-sNM)\mathbb{E}\Big[\exp\Big(s\sum_{i=1}^N \sum_{X_k\in X^\lambda\setminus Q_{12n\sqrt{d}}(nz_i)} \ell_{6n}(|nz_i - X_k|)\Big)\Big]. \end{aligned}\tag{4.22}$$

Applying the form of the Laplace functional of a Cox point process (cf. [K93, Sections 3.2, 6]) to the function $f(x) = s\sum_{i=1}^N \ell_{6n}(|x - z_i|)\mathbb{1}\{x \in \mathbb{R}^d \setminus Q_{12n\sqrt{d}}(nz_i)\}$, we obtain

$$\begin{aligned}&\mathbb{E}\Big[\exp\Big(s\sum_{i=1}^N \sum_{X_k\in X^\lambda\setminus Q_{12n\sqrt{d}}(nz_i)} \ell_{6n}(|nz_i - X_k|)\Big)\Big]\\ &\qquad = \mathbb{E}\Big[\exp\Big(\lambda\int_{\mathbb{R}^d}\Big(\exp\Big(s\sum_{i=1}^N \ell_{6n}(|nz_i - x|)\mathbb{1}\{x \in \mathbb{R}^d \setminus Q_{12n\sqrt{d}}(nz_i)\}\Big) - 1\Big)\Lambda(\mathrm{d}x)\Big)\Big]. \end{aligned}\tag{4.23}$$

Now, we provide an upper bound on $s\sum_{i=1}^N \ell_{6n}(|nz_i - x|)$. The sites $\{z_i\}_{i=1}^N$ are pairwise distinct, and therefore the sum $\sum_{i=1}^N \ell_{6n}(|x - nz_i|)$ can be upper bounded by $\sum_{z\in\mathbb{Z}^d} \ell_{6n}(|x - nz|)$. Further, the sites $\{z_i\}_{i=1}^N$ are contained in the hypercubic lattice $\mathbb{Z}^d$. Let us write Q_x for the cube of $\mathbb{Z}^d$ containing x; this is well-defined for Leb-a.e. $x \in \mathbb{R}^d$. Now, for such x, for $i \in \mathbb{N}_0$ and $z_0 \in \{z \in \mathbb{Z}^d : i \le \mathrm{dist}_\infty(z,Q_x) < (i+1)\}$, the contribution of $\ell(|nz_0 - x|)$ to the latter sum is at most $\ell(in)$. Thus, we have

$$\sum_{i=1}^N \ell_{6n}(|x - nz_i|) \le \sum_{z\in\mathbb{Z}^d} \ell_{6n}(|x - nz|) \le \sum_{i=1}^\infty \#\{z \in \mathbb{Z}^d : i \le \mathrm{dist}_\infty(z,Q_x) < (i+1)\}\ell_{6n}(in) =: K(n) \in [0,\infty].\tag{4.24}$$

Since ℓ is monotone decreasing, we have for any $n \ge 1$

$$K(n) \le 2^d + \sum_{i=0}^{\lceil 6\sqrt{d}/2\rceil} \ell(0)((2i+2)^d - (2i)^d) + \sum_{i=\lceil 6\sqrt{d}/2\rceil}^\infty ((2i+2)^d - (2i)^d)\ell(i - 6\sqrt{d}/2) =: K_0.$$

By (iii) in Assumption (ℓ), we have that $K_0 < \infty$. Thus, choosing $s \le 1/K_0$ in (4.23), we see that $s\sum_{i=1}^N \ell_{6n}(|nz_i - x|) \le 1$. Therefore, using that $\exp(y) - 1 \le 2y$ for all $y \le 1$, we have

$$\exp\Big(s\sum_{i=1}^N \ell_{6n}(|nz_i - x|)\mathbb{1}\{x \in \mathbb{R}^d \setminus Q_{12n\sqrt{d}}(nz_i)\}\Big) - 1 \le 2s\sum_{i=1}^N \ell_{6n}(|nz_i - x|)\mathbb{1}\{x \in \mathbb{R}^d \setminus Q_{12n\sqrt{d}}(nz_i)\}.$$

Plugging this into (4.23), we obtain

$$\mathbb{E}\Big[\exp\Big(s\sum_{i=1}^N \sum_{X_k\in X^\lambda\setminus Q_{12n\sqrt{d}}(nz_i)} \ell_{6n}(|nz_i - X_k|)\Big)\Big] \le \mathbb{E}\Big[\prod_{i=1}^N X_i\Big],$$

where we introduced the random variables

$$X_i = \exp\Big(2\lambda s\int_{\mathbb{R}^d\setminus Q_{12n\sqrt{d}}(nz_i)} \ell_{6n}(|x - nz_i|)\Lambda(\mathrm{d}x)\Big), \qquad i \in [N].$$

The random variables $\{X_i\}_{i=1}^N$ are identically distributed. Using that the following extended version of Hölder's inequality holds for any sequence $\{Y_i\}_{i\in\mathbb{N}}$ of identically distributed non-negative random variables

$$\mathbb{E}\Big(\prod_{i=1}^\infty Y_i^{p_i}\Big) \le \mathbb{E}(Y_1), \qquad p_i \ge 0\ \forall i,\ \sum_{i=1}^\infty p_i = 1,\tag{4.25}$$

we obtain

$$\mathbb{E}\Big(\exp\Big(2\lambda s\sum_{i=1}^N \int_{\mathbb{R}^d\setminus Q_{12n\sqrt{d}}(nz_i)} \ell_{6n}(|nz_i - x|)\Lambda(\mathrm{d}x)\Big)\Big) \le \mathbb{E}\Big(\exp\Big(2\lambda sN\int_{\mathbb{R}^d\setminus Q_{12n\sqrt{d}}} \ell_{6n}(|x|)\Lambda(\mathrm{d}x)\Big)\Big).\tag{4.26}$$

Now we shall consider the cases (2b) and (2c) separately. In the case (2b) where $\Lambda(Q_1)$ is bounded, we show that the right-hand side of (4.26) has a finite exponential growth rate on scale N as $N \to \infty$, even after extending the integration to $\mathbb{R}^d$. We claim that there exists $W \geq 0$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}\Big(\exp\Big(2\lambda s N \int_{\mathbb{R}^d} \ell_{6n}(|x|)\Lambda(\mathrm{d}x)\Big)\Big) \leq 2\lambda s W. \tag{4.27}$$

Indeed, let $\{Q^i\}_{i=1}^\infty$ be a subdivision of $\mathbb{R}^d$ into congruent copies of Q_1 (up to the boundaries). Then, for any $N \in \mathbb{N}$, by the extended version (4.25) of Hölder's inequality,

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(2\lambda s N \int_{\mathbb{R}^d} \ell_{6n}(|x|)\Lambda(\mathrm{d}x)\Big)\Big] &\leq \mathbb{E}\Big[\exp\Big(2\lambda s N \sum_{i=1}^\infty \max_{x\in Q^i} \ell_{6n}(|x|)\Lambda(Q^i)\Big)\Big] \\ &\leq \mathbb{E}\Big[\exp\Big(2\lambda s N\Big(\sum_{i=1}^\infty \max_{x\in Q^i}\ell_{6n}(|x|)\Big)\Lambda(Q_1)\Big)\Big],\end{aligned}$$

where $\sum_{i=1}^\infty \max_{x\in Q^i} \ell_{6n}(|x|)$ is finite by Assumption (ℓ) (iii) and the monotonicity of ℓ. Now, taking $\limsup_{N\to\infty} \frac{1}{N}\log$ on both sides implies the claim with $W = \sum_{i=1}^\infty \max_{x\in Q^i} \ell_{6n}(|x|)\mathrm{esssup}\Lambda(Q_1)$, which is finite since $\Lambda(Q_1)$ is bounded.

Using this, we continue the estimation (4.22) with $s \leq 1/K_0$ for n-good $z_1,\dots,z_N$ as follows

$$\begin{aligned}\mathbb{P}\Big(\sum_{i=1}^N I_{6n}^{\mathrm{out}}(nz_i) > nM\Big) &\leq \exp\Big(-NMs\Big)\mathbb{E}\Big(\exp\Big(2\lambda s N\int_{\mathbb{R}^d}\ell_{6n}(|x|)\Lambda(\mathrm{d}x)\Big)\Big) \\ &\leq \exp\Big(N\Big(-Ms + 2\lambda s(W+o(1))\Big)\Big) \leq q_B^N\end{aligned} \tag{4.28}$$

eventually, where W is chosen according to (4.27) and we have defined q_B as

$$q_B = \exp(-Ms + 3\lambda s W). \tag{4.29}$$

For any $\lambda > 0$, $q_B \leq 1$ for large enough $M > 0$ and $q_B \downarrow 0$ as $M \to \infty$, given that $s \leq 1/K_0$. Thus, Proposition 4.2.5 follows in the case (2b).

It remains to consider the case (2c). We continue estimating the right-hand side of (4.26). We extend the integration domain $\mathbb{R}^d \setminus Q_{12n\sqrt{d}}$ to $\mathbb{R}^d \setminus Q_{\lfloor 12n\sqrt{d}\rfloor}$ and we subdivide $\mathbb{R}^d \setminus Q_{\lfloor 12n\sqrt{d}\rfloor}$ into the union of concentric ℓ^∞-annuli $Q_{\lfloor 12n\sqrt{d}\rfloor+2} \setminus Q_{\lfloor 12n\sqrt{d}\rfloor}$,$Q_{\lfloor 12n\sqrt{d}\rfloor+4} \setminus Q_{\lfloor 12n\sqrt{d}\rfloor+2}$ etc. (up to the boundaries). Now for each $i \in \mathbb{N}_0$, let us write $A^i = Q_{\lfloor 12n\sqrt{d}\rfloor+2i+2} \setminus Q_{\lfloor 12n\sqrt{d}\rfloor+2i}$. Note that A^i is covered by the union of $(\lfloor 12n\sqrt{d}\rfloor + 2i + 2)^d - (\lfloor 12n\sqrt{d}\rfloor + 2i)^d$ congruent copies of $\Lambda(Q_1)$, and this number of copies equals $\mathrm{Leb}(A^i)$. Further, for $x \in A^i$, we have for all sufficiently large n (not depending on i)

$$\ell_{6n}(|x|) \leq \ell_{6n}\Big(\frac{\lfloor 12n\sqrt{d}\rfloor + 2i}{2}\Big) \leq \ell\Big(i - 1 + \frac{6n\sqrt{d}}{2}\Big) \leq \ell\Big(i + 2n\sqrt{d}\Big).$$

Hence, using also the extended version (4.25) of Hölder's inequality, we have

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(2\lambda s N\int_{\mathbb{R}^d\setminus Q_{12n\sqrt{d}}}\ell_{6n}(|x|)\Lambda(\mathrm{d}x)\Big)\Big] &\leq \mathbb{E}\Big[\exp\Big(2\lambda s N\int_{\mathbb{R}^d\setminus Q_{\lfloor 12n\sqrt{d}\rfloor}}\ell_{6n}(|x|)\Lambda(\mathrm{d}x)\Big)\Big] \\ \leq \mathbb{E}\Big[\exp\Big(2\lambda s N\sum_{i=0}^\infty \Lambda\big(A^i\big)\ell(i+2n\sqrt{d})\Big)\Big] &\leq \mathbb{E}\Big[\exp\Big(2\lambda s N\sum_{i=0}^\infty \ell(i+2n\sqrt{d})\mathrm{Leb}(A^i)\Lambda(Q_1)\Big)\Big].\end{aligned} \tag{4.30}$$

Now, since for $i \in \mathbb{N}_0$, $\mathrm{Leb}(A^i) \leq 2d(\lfloor 12n\sqrt{d}\rfloor + 2i + 2)^{d-1}$, for all sufficiently large n, the right-hand side of (4.30) is upper bounded by

$$\begin{aligned}&\mathbb{E}\Big[\exp\Big(4\lambda s N d\sum_{i=0}^\infty \big(2 + 2i + 7\times 2n\sqrt{d}\big)^{d-1}\ell\Big(i+2n\sqrt{d}\Big)\Lambda(Q_1)\Big)\Big] \\ &\leq \mathbb{E}\Big[\exp\Big(4\lambda s 7^{d-1} N d\sum_{i=0}^\infty \big(i + 2n\sqrt{d}\big)^{d-1}\ell\Big(i+2n\sqrt{d}\Big)\Lambda(Q_1)\Big)\Big].\end{aligned}$$

Thus, according to the constraint on ℓ in (2c), starting from (4.22), we have arrived at

$$\mathbb{P}(I^{\rm out}_{6n}(nz_1) > M, \dots, I^{\rm out}_{6n}(nz_N) > M) \leq \exp(-sNM)\mathbb{E}\Big[\exp\Big(Cs\frac{N}{n}\Lambda(Q_1)\Big)\Big] \in [0,\infty] \tag{4.31}$$

for a suitable $C > 0$, given that $s \leq \frac{1}{K_0}$.

Now, in order to be able to finish the proof of Proposition 4.2.5, we will first write $N = kn$, make use of Lemma 4.2.2 for $k \in \mathbb{N}$, conclude the proposition in the limit $\mathbb{N} \ni k \to \infty$, and then conclude about $kn < N < (k+1)n$. Lemma 4.2.2 implies that for any $n, k \in \mathbb{N}$ and for any pairwise distinct $z_1, \dots, z_{kn} \in \mathbb{Z}^d$, we have

$$\mathbb{P}(I^{\rm out}_{6n}(nz_1) > M, \dots, I^{\rm out}_{6n}(nz_{kn}) > M) \leq \mathbb{P}(I^{\rm out}_{6kn}(nz(z_1,k)) > M, \dots, I^{\rm out}_{6kn}(nz(z_{kn},k)) > M). \tag{4.32}$$

where for $z' \in \mathbb{Z}^d$ and $k \in \mathbb{N}$, we define $z(z',k)$ to be a point of $k\mathbb{Z}^d$ such that $z' \in Q_k(z)^o$. Now we argue that the right-hand side is bounded by $q'^N_B = q'^{nk}_B$, where $q'_B > 0$ can be made arbitrarily close to 0 by choosing M sufficiently large. Note that $z(z_1,k), \dots, z(z_N,k)$ are not necessarily distinct, and thus a direct application of (4.31) with N and n both replaced by kn is not possible. However, the following improvement of the estimate (4.24) holds in this case.

Lemma 4.2.6. *If ℓ satisfies Assumption (ℓ), then there exists $C_1 > 0$ such that for all $n \geq 1$, $x \in \mathbb{R}^d$, and (not necessarily distinct) $z_1, \dots, z_n \in \mathbb{Z}^d$ such that $nz_1, \dots, nz_n \in \mathbb{R}^d \setminus Q_{12n\sqrt{d}}(x)$, one has $\sum_{i=1}^n \ell_{6n}(|x - nz_i|) \leq C_1$.*

Proof. Under the assumptions of the lemma on all parameters, for all $i \in [n]$, we have $|x - nz_i| \geq 6n\sqrt{d}$, and thus $\ell_{6n}(|x - nz_i|) \leq \ell_{6n}(6n\sqrt{d}) = \ell(3n\sqrt{d})$. It follows that

$$\sum_{i=1}^n \ell_{6n}(|x - nz_i|) \leq n\ell(|3n\sqrt{d}|) = n^{2-d}n^{d-1}\ell(3\sqrt{d}n) \leq n^{2-d}\sum_{i=n}^\infty i^{d-1}\ell(3\sqrt{d}i) = n^{2-d}o(1) = o(1),$$

as $n \to \infty$, where in the penultimate step we used that $\sum_{i=n}^\infty i^{d-1}\ell(3\sqrt{d}i) = o(1)$ according to (i) and (iii) in Assumption (ℓ), and the last step holds because $d \geq 2$. In particular, $n \mapsto \sup\{\sum_{i=1}^n \ell_{6n}(|x - nz_i|) \colon z_i \in \mathbb{Z}^d \text{ and } nz_i \in \mathbb{R}^d \setminus Q_{12n\sqrt{d}}(x), \forall i \in [n]\}$ is bounded uniformly over $x \in \mathbb{R}^d$. We conclude the lemma. $\square$

Let now $C_1 > 0$ satisfy the assumption of Lemma 4.2.6 and let $K_1 := \max\{K_0, C_1\}$. Since the only place in the proof of the bound (4.31) where we used disjointness of $z_1, \dots, z_N$ was the estimate (4.24), we can replace this estimate with Lemma 4.2.6 in order to conclude (4.31) for $z_1, \dots, z_N$ not necessarily disjoint and $N = kn$. This gives that for $s < \min\{\frac{1}{K_1}, \alpha^*/(2C)\}$ and n sufficiently large, the right-hand side of (4.32) is upper bounded by $q'^N_B = q'^{nk}_B$ for $q'_B = \exp(-sM + 1)$, where C is chosen according to (4.31) and $\alpha^* = \sup\{\alpha > 0 \colon \mathbb{E}[\exp(\alpha\Lambda(Q_1))] < \infty\}$ is positive by the exponential moment condition in (2c). (Here, $\alpha^*/2C$ is defined as ∞ if $\alpha^* = \infty$.) Here, once we have chosen n sufficiently large, s is independent of n and k. (The parameter C depends also on λ.)

Thus, for any $kn \leq N < (k+1)n$ and pairwise distinct $z_1, \dots, z_N \in \mathbb{Z}^d$, we obtain

$$\mathbb{P}(I^{\rm out}_{6n}(nz_1) > M, \dots, I^{\rm out}_{6n}(nz_N) > M) \leq \mathbb{P}(I^{\rm out}_{6n}(nz_1) > M, \dots, I^{\rm out}_{6n}(nz_{kn}) > M) \leq q'^{kn}_B = q^{2kn}_B \leq q^N_B,$$

with $q_B = \sqrt{q'_B}$, where in the last step we used that $2k \geq k + 1 > N/n$ for $k \geq 1$. With this choice of q_B under the assumption (2c), we conclude the proof of Proposition 4.2.5. $\square$

4.2.1.2. Proof of Theorem 4.1.6

In the following, we write $D_a(x)$ for the closed disk (closed ℓ^2-ball) of radius $a > 0$ around $x \in \mathbb{R}^2$. By stationarity, it suffices to verify the theorem for D_1 instead of Q_1, which we do now.

We write $X_{\mathbf{S}} = (X_i)_{i \in I}$ and start the proof with two lemmas.

Lemma 4.2.7. *Let $b \geq a > 0$. Almost surely w.r.t. $X_{\mathbf{S}}$, in the event $A = \{X_{\mathbf{S}} \cap D_b \neq \varnothing\}$, we have*

$$\nu_1(S \cap D_a) \leq \sum_{X_j \in X_{\mathbf{S}} \cap D_{b+3a}} \nu_1(\partial \mathfrak{C}(X_j) \cap D_a), \tag{4.33}$$

where $\mathfrak{C}(X_j)$ is the Voronoi cell around X_j.

Proof. We know that almost surely, any cell of the Poisson–Voronoi tessellation is a convex polygon, therefore it is bounded.

In the event A, we can choose $X_i \in X_{\mathbf{S}} \cap D_b$. We claim that for any edge of S intersecting with D_a, the corresponding edge in the dual Delaunay graph connects two points in D_{b+3a}. Indeed, assume otherwise, then there exists $x \in D_a$, $X_j \in X_{\mathbf{S}} \cap D_{b+3a}^c$ such that $|x - X_j| = \min\{|x - X_l| \colon l \in I\}$. However, since $x \in D_a$ and $X_i \in D_b$, we have $|x - X_i| \leq \max_{y \in D_a, z \in D_b} |y - z| = 2a + (b - a) = b + a$, further, since $X_j \in D_{b+3a}^c$, $|x - X_j| \geq \mathrm{dist}_2(X_j, D_b) > (b + 3a) - a > b + a$. This contradicts with the assumption that $|x - X_j| = \min\{|x - X_l| \colon l \in I\}$, and hence the claim is proven.

Thus, for any Voronoi edge intersecting with $D_a \subseteq D_b$, the corresponding Delaunay edge has both endpoints in $X_{\mathbf{S}} \cap D_{b+3a}$, in particular, the Voronoi edge is in $\partial \mathfrak{C}(X_j)$ for some $X_j \in X_{\mathbf{S}} \cap D_{b+3a}$. The sum in (4.33) includes the length of the intersection of any such Voronoi edge with D_a among the summands at least once, hence the lemma follows. □

Lemma 4.2.8. *Let $x \in \mathbb{R}^2$, $a > 0$, and $X_j \in X^\lambda$. Then $\nu_1(\partial \mathfrak{C}(X_j) \cap D_a) \leq 2\pi a$.*

Proof. It suffices to show that for any polygon line $P \subset \mathbb{R}^2$ which is the boundary of a convex polygon P', $\nu_1(P \cap D_a)$ is bounded by $2\pi a$.

Now, if $P \subseteq D_a$, then $\nu_1(P \cap D_a)$ equals the perimeter of P', and it is elementary to show that this perimeter is at most $2\pi a$. Else, $P' \cap D_a$ is convex since both P' and D_a are convex, and $\nu_1(P \cap D_a)$ is bounded from above by the perimeter of $P' \cap D_a$, which is again at most $2\pi a$. □

Lemmas 4.2.7 and 4.2.8 have the following immediate corollary.

Corollary 4.2.9. *Let $b \geq a > 0$ and $x \in \mathbb{R}^2$. Then in the event $\{X_{\mathbf{S}} \cap D_b \neq \varnothing\}$, $\nu_1(S \cap D_a)$ is at most $2\pi a \#(X_{\mathbf{S}} \cap D_{b+3a})$.*

Proof of Theorem 4.1.6. Let us define

$$R := \inf\{r > 0 \colon D_r \cap X_{\mathbf{S}} \neq \varnothing\}.$$

Then we have that

$$\mathbb{P}(R \geq r) = \exp(-\lambda_{\mathbf{S}} \pi r^2), \qquad \forall r > 0; \tag{4.34}$$

this is a version of the statement that the Poisson–Voronoi tessellation is exponentially stabilizing [HJC17, Example 3.1]. Note that by the fact that almost surely, any two points $X_i \neq X_j$ of $X_{\mathbf{S}}$ have different Euclidean norms, it follows that $\mathbb{P}(\#(X_{\mathbf{S}} \cap D_R) = 1) = 1$. Further, conditional on R, for any $T > R$, $\#(X_{\mathbf{S}} \cap (D_T \setminus D_R))$ is Poisson distributed with parameter $\lambda_{\mathbf{S}} \pi (T^2 - R^2)$ (and certainly independent of $\#(X_{\mathbf{S}} \cap D_R)$). In particular, in the event $\{R \leq 1\}$, $D_1 \cap X_{\mathbf{S}} \neq \varnothing$, and therefore Corollary 4.2.9 applied for $a = b = 1$ implies that

$$\Lambda(D_1) \leq 2\pi \#(X_{\mathbf{S}} \cap D_4) = 2\pi \#(X_{\mathbf{S}} \cap (D_4 \setminus D_1)) + 2\pi \#(X_{\mathbf{S}} \cap D_1). \tag{4.35}$$

Now, given $\{R \leq 1\}$, the two terms on the right-hand side of (4.35) are independent, $\#(X_{\mathbf{S}} \cap (D_4 \setminus D_1))$ is a Poisson random variable with parameter $15 \lambda_{\mathbf{S}} \pi$, whereas $\#(X_{\mathbf{S}} \cap D_1)$ is a Poisson random variable with parameter $\lambda_{\mathbf{S}} \pi$ conditioned to be positive. Thus, the expression on the right-hand side of (4.35) has all exponential moments, and in order to finish the proof of the theorem it suffices to show that

$$\mathbb{E}[\exp(\alpha \Lambda(D_1)) \mathbb{1}\{R > 1\}] < \infty$$

holds for all $\alpha > 0$. Now, in the event $\{R > 1\}$, using that $D_R \cap X_{\mathbf{S}} \neq \varnothing$, we can apply Corollary 4.2.9 for $a = 1$, and $b = R$ to obtain that

$$\Lambda(D_1) \le 2\pi\#(X_{\mathbf{S}} \cap D_{R+3}) = 2\pi\#(X_{\mathbf{S}} \cap D_R) + 2\pi\#(X_{\mathbf{S}} \cap (D_{R+3} \setminus D_R)) = 2\pi(1 + \#(X_{\mathbf{S}} \cap (D_{R+3} \setminus D_R))),$$

almost surely, where given R, the right-hand side is stochastically dominated by $2\pi(1 + Y((6R + 9)\pi\lambda_{\mathbf{S}}))$, and for $\beta > 0$ we write $Y(\beta)$ for a Poisson random variable with parameter β. Thus, for $\alpha > 0$,

$$\begin{aligned}&\mathbb{E}[\exp(\alpha\Lambda(D_1))\mathbb{1}\{R > 1\}] \le \exp(2\pi\alpha)\mathbb{E}[\exp(2\pi\alpha Y((6R + 9)\pi\lambda_{\mathbf{S}}))\mathbb{1}\{R > 1\}]\\ &\le \mathbb{E}\big[\exp\big(2\pi\alpha + 2\pi(6R + 9)\lambda_{\mathbf{S}}\big(e^{2\pi\alpha} - 1\big)\big)\big].\end{aligned} \tag{4.36}$$

Now, by the estimate that for any non-negative integrable random variable Z,

$$\mathbb{E}[Z] \le \int_0^\infty \mathbb{P}(Z \ge t)\mathrm{d}t = \sum_{k=0}^\infty \int_k^{k+1} \mathbb{P}(Z \ge t)\mathrm{d}t \le \sum_{k=0}^\infty \mathbb{P}(Z \ge k),$$

we obtain for $c > 0$

$$\mathbb{E}[\exp(cR)] \le \exp(c) + \sum_{k=1}^\infty \mathbb{P}\big(\exp(cR) \ge k\big) = \exp(c) + \sum_{k=1}^\infty \mathbb{P}\Big(R \ge \frac{\log k}{c}\Big) = \exp(c) + \sum_{k=1}^\infty \mathrm{e}^{-\lambda_{\mathbf{S}}\pi\frac{(\log k)^2}{c^2}} < \infty.$$

Thus, the right-hand side of (4.36) is also finite for all $\alpha > 0$. This finishes the proof. □

4.2.2. Proof of the results of Section 4.1.3.2

4.2.2.1. Proof of Corollary 4.1.8

Before carrying out the proof, we recall Palm calculus for Cox processes from [HJC17, Section 2.2]. The *Palm version* $X^{\lambda,*}$ of a stationary point process X^λ of intensity $\lambda = \mathbb{E}[\#(X^\lambda \cap Q_1)] > 0$ is a point process whose distribution is defined via

$$\mathbb{E}\big[f(X^{\lambda,*})\big] = \frac{1}{\lambda}\mathbb{E}\Big[\sum_{X_i \in X^\lambda \cap Q_1} f(X^\lambda - X_i)\Big], \tag{4.37}$$

for any bounded measurable function $f\colon \mathbb{M}_{\mathrm{co}} \to [0,\infty)$, where $\mathbb{M}_{\mathrm{co}}$ is the set of σ-finite counting measures. In particular, $\mathbb{P}(o \in X^{\lambda,*}) = 1$.

For any infinite, locally finite graph $G = (V,E)$ and for a vertex $v \in V$, we say that $v \leftrightsquigarrow \infty$ in G if v is contained in an infinite connected component of G. Then, for $r > 0$,

$$\theta(\lambda,r) = \mathbb{P}\Big(o \leftrightsquigarrow \infty \text{ in } g_r(X^{\lambda,*})\Big) \tag{4.38}$$

denotes the *percolation probability* of the origin of the Cox–Gilbert graph $g_r(X^\lambda)$. Then $\lambda_{\mathrm{c}}(r) = \inf\{\lambda > 0\colon \theta(\lambda,r) > 0\}$, cf. [HJC17, Section 2.2].

Proof of Corollary 4.1.8. We first verify (1). Let $\overline{\theta}(\varrho)$ be the percolation probability of the Gilbert graph of a stationary Poisson point process with intensity $\varrho > 0$ and connection radius 1. By [HJC17, Theorem 2.9], for Λ stabilizing,

$$\lim_{r\uparrow\infty,\lambda\downarrow 0,\ \lambda r^d = \varrho} \theta(\lambda,r) = \overline{\theta}(\varrho).$$

Let $\varrho > 0$ satisfy $\overline{\theta}(\varrho) > 0$. If $\lambda r^d = \varrho$ for r large enough, then $\theta(\lambda,r) > 0$, thus $\lambda_{\mathrm{c}}(r) \le \frac{\varrho}{r^d} < \infty$. This verifies (1). Since $\frac{\varrho}{r^d} \to 0$ as $r \to \infty$, it follows also that $\lim_{r\to\infty} \lambda_{\mathrm{c}}(r) = 0$. But this is (2). □

4.2.2.2. Proof of Proposition 4.1.10

We start with the case $d = 2$. Let us write $\mathcal{B}(\widehat{\lambda},\widehat{r})$ for the Poisson–Boolean model with intensity $\widehat{\lambda} > 0$ and connection radius $\widehat{r} > 0$. Further, for $r > 0$, let $\varrho_{\mathrm{c}}(r)$ be such that $\mathcal{B}(\varrho_{\mathrm{c}}(r),r)$ is critical. Then, by scale invariance (4.9), we have $\varrho_{\mathrm{c}}(r) = r^{-d}\varrho_{\mathrm{c}}(1)$. We fix $\varrho > \varrho_{\mathrm{c}}(1)$, then there exists $\varrho' < \varrho$ such that $\mathcal{B}(\varrho',1)$ is still supercritical.

For $r > d_0$, let us write $r_{\mathrm{B}}(r) = \frac{\varrho}{\varrho'}r$ and $\lambda(r) = \varrho' r^{-d}$. Then by Assumption ($\ell$) (i), (ii), and the fact that ℓ has unbounded support, $\ell(r_{\mathrm{B}}(r)) < \ell(r)$ holds for all $r > d_0$. Further, let $N_0(r)$,$\tau(r)$, and $P(r)$ be such that $r_{\mathrm{B}}(r) = \ell^{-1}(\tau(r)N_0(r)/P(r))$; such parameters exists since ℓ has unbounded support and satisfies (i) and (ii) in Assumption (ℓ). We map the Cox–Boolean model $\mathcal{C}(\lambda(r),r) = X^{\lambda(r)} \oplus B_{r/2}(o)$ to a discrete edge percolation model in the spirit of [DFMMT06, Section 3.1], control the interferences and conclude that if r is large enough, then the SINR graph $g_{(\gamma,N_0(r),\tau(r),P(r))}(X^{\lambda(r)})$ with SNR connection radius $r_{\mathrm{B}}(r)$ percolates for some $\gamma > 0$ (with probability 1, see Section 4.2.3.1).

For $n \geq 1$ and $r > d_0$, let us write $z_e = (x_e,y_e)$ for the centre of the edge e in the nearest neighbour graph of $\mathbb{Z}^2$. Let us denote the set of such edges by $E(\mathbb{Z}^2)$. Note that each z_e is an element of $\mathbb{X} = \{(x/2,y)\colon x,y \in \mathbb{Z}\} \cup \{(x,y/2)\colon x,y \in \mathbb{Z}\}$. Let us write $R_e(n,r) = [nrx_e - \frac{3}{4}nr, nrx_e + \frac{3}{4}nr] \times [nry_e - \frac{1}{4}nr, nry_e + \frac{1}{4}nr]$ if $e \in E(\mathbb{Z}^2)$ is a horizontal edge and $R_e(n,r) = [nrx_e - \frac{1}{4}nr, nrx_e + \frac{1}{4}nr] \times [nry_e - \frac{3}{4}nr, nry_e + \frac{3}{4}nr]$ if e is a vertical edge. Note that S_e is a rectangle with its edges parallel to e having length $\frac{3}{2}nr$ and its edges perpendicular to e having length $\frac{1}{2}nr$. In particular, $Q_{nr/2}(nrz_e) \subset R_e(n,r) \subset Q_{3nr/2}(nrz_e)$, and $R_e(n,r) \setminus Q_{nr/2}(nrz_e)$ is the disjoint union of two $\frac{nr}{2} \times \frac{nr}{2}$ squares, let us denote their closures by $S_e^1(n,r)$ respectively $S_e^2(n,r)$ (in an arbitrary but fixed order for each e). For any edge e in $E(\mathbb{Z}^2)$, we say that e is (n,r)-*good* if

1. $R(Q_{\frac{3}{2}nr}(nrz_e)) < \frac{3}{2}nr$, and
2. $\mathcal{C}(\lambda(r),r)$ crosses $R_e(n,r)$ in the hard direction and both $S_e^1(n,r)$ and $S_e^2(n,r)$ in the other direction.

An edge e is (n,r)-*bad* if it is not (n,r)-good. The process of (n,r)-good edges is 4-dependent as can be seen from the definition of stabilization. We write $J_{n,r}(z_e)$ for the event in (2) and $L_n(z_e)$ for the event that $\mathcal{B}(\varrho',1)$ crosses $R_e(n,1)$ in the hard direction and both $S_e^1(n,1)$ and $S_e^2(n,1)$ in the other direction.[21] Note that by scale invariance of the Poisson–Boolean model (4.7), $L_n(z_e)$ has probability equal to the one of the event that $\mathcal{B}(\lambda(r),r) = \mathcal{B}(\frac{\varrho'}{r^2},r)$ crosses $R_{e'}(n,r)$ in the hard direction and both $S_{e'}^1(n,r)$ and $S_{e'}^2(n,r)$ in the other direction for an arbitrary $e' \in E(\mathbb{Z}^2)$ and $r > 0$.

Now, let $\varepsilon > 0$. First, we fix n sufficiently large such that the probability of the event in (1) is at least $1 - \varepsilon/4$ uniformly for all $r \geq 1$ and any edge e in $nr\mathbb{Z}^2$, and that the probability of $L_n(z_e)^{\mathrm{c}}$ is also at most $\varepsilon/4$. The last condition can be satisfied thanks to the Russo–Seymour–Welsh type result, Theorem 4.1.9. Next, as observed in [HJC17, Section 7.1], the restriction of $r^{-1}\mathcal{C}(\lambda(r),r)$ to a bounded sampling window converges weakly to the corresponding restriction of $\mathcal{B}(\varrho',1)$. Now, for fixed e, the event $L_n(z_e)$ has discontinuities of measure 0 with respect to the Poisson–Boolean model. This implies that for all $r > d_0$ sufficiently large, $|\mathbb{P}(L_n(z_e)^{\mathrm{c}}) - \mathbb{P}(J_{n,r}(z_e)^{\mathrm{c}})|$ can be bounded from above by $\varepsilon/4$ uniformly in e. Now, for any e, using a union bound and the triangle inequality, we have

$$\mathbb{P}(e \text{ is } (n,r)\text{-bad}) \leq \mathbb{P}\Big(R(Q_{\frac{3}{2}rn}(z_e)) \geq \frac{3}{2}nr\Big) + \mathbb{P}(L_n(z_e)^{\mathrm{c}}) + |\mathbb{P}(L_n(z_e)^{\mathrm{c}}) - \mathbb{P}(J_{n,r}(z_e)^{\mathrm{c}})| \leq \frac{3\varepsilon}{4} < \varepsilon.$$

Applying [LSS97, Theorem 0.0], for all sufficiently large n and large enough r chosen accordingly, the process of (n,r)-good edges is stochastically dominated from below by a supercritical independent edge percolation process. Thus, the (n,r)-good sites percolate for all sufficiently large n,r.

Next, the interferences can be controlled analogously to Proposition 4.2.5. Instead of $\{I_{6n}(nz)\colon z \in \mathbb{Z}^d\}$ in Step 2 defined in Section 4.2.1.1, now one should work with the rescaled interferences $\{I_{3rn/2}(nrz_e)\colon e \in E(\mathbb{Z}^2)\}$ associated to the edges. For $n,r \geq 1$, $M > 0$, and $e \in E(\mathbb{Z}^2)$, let us write $B_{n,r,M}(e)$ for the indicator that $I_{3rn/2}(nrz_e) \leq M$. Under the assumptions (2b) or (2c), it can be proven analogously to Proposition 4.2.5 that for any pairwise distinct $e_1,\dots,e_N$,

$$\mathbb{P}(B_{n,r,M}(e_1) = 0,\dots,B_{n,r,M}(e_N) = 0) \leq q_B^N$$

[21] The precise definitions of these events are analogous to the one in Theorem 4.1.9, therefore we leave them to the reader.

for some $q_B \in [0,1)$, where for fixed, large enough n,r and $\lambda = \lambda(r)$, q_B can be made arbitrarily close to 0 by choosing M sufficiently large. Using a Peierls argument, we see that for all sufficiently large n, r (depending on n), and M (depending on n,r), the process of (n,r)-good edges e with $B_{n,r,M}(e) = 1$ percolates. Just as in [DFMMT06, Sections 3.2, 3.3], this implies percolation of $g_{(\gamma,N_0(r),\tau(r),P(r))}(X^{\lambda(r)})$ for $\gamma \in (0,\gamma^*(r))$, where

$$\gamma^*(r) = \frac{N_0(r)}{P(r)M}\Big(\frac{\ell(r)}{\ell(r_{\mathrm{B}}(r))} - 1\Big) > 0$$

(cf. (4.21), here we used again that $r_{\mathrm{B}} > r > d_0$). This holds whenever $r_{\mathrm{B}}(r) = \ell^{-1}(\tau(r)N_0(r)/P(r))$. Thus, since $\lambda(r) \downarrow 0$ as $r \to \infty$, Proposition 4.1.10 follows for small enough $\lambda > 0$. But increasing λ increases the probability of (n,r)-goodness of any edge, and it is easy to see that also the analogue of Proposition 4.2.5 works for larger $\lambda > 0$ (at the price of reducing $\gamma > 0$ without vanishing). We conclude Proposition 4.1.10 for $d = 2$.

For $d \geq 3$, one can proceed with an analogous definition of all the parameters from the first two paragraphs of the proof for $d = 2$ (adapted to the value of d), using a different discrete model. Here, we shall define a site $z \in \mathbb{Z}^d$ to be (n,r)-good if it satisfies the definition of n-goodness in Section 4.2.1.1, Step 1, but with n replaced by nr and λ by $\lambda(r)$ (in particular, with X^λ replaced by $X^{\lambda(r)}$ and $g_r(X^\lambda)$ replaced by $g_r(X^{\lambda(r)})$) everywhere. For $z \in \mathbb{Z}^d$, we write $J_{n,r}(z)$ for the event that z satisfies (2) and (3) in the definition of (n,r)-goodness. Then, for any n,r under consideration, the process of (n,r)-good sites is 7-dependent according to the definition of stabilization. Further, we let $Y^{\varrho'}$ be a Poisson point process with intensity $\varrho' = \lambda(r)r^d$, and we write $L_n(z)$ for the event that in the definition of $(n,1)$-goodness, z satisfies (2) with X^λ replaced by $Y^{\varrho'}$, and (3) with $g_r(X^\lambda)$ replaced by $g_1(Y^{\varrho'})$ everywhere. The probability of $L_n(z)$ is independent of the choice of z and tends to one as $n \to \infty$ thanks to the arguments of [HJC17, Section 5.2], since the constant intensity measure of the Poisson point process $Y^{\varrho'}$ is obviously asymptotically essentially connected. Using the scale invariance (4.7) of Poisson–Gilbert graphs, we conclude that for $z \in \mathbb{Z}^d$,

$$\mathbb{P}(z \text{ is } (n,r)\text{-bad}) \leq \mathbb{P}\Big(R(Q_{6nr}(nrz)) \geq \frac{nr}{2}\Big) + \mathbb{P}(L_n(z)^{\mathrm{c}}) + |\mathbb{P}(L_n(z)^{\mathrm{c}}) - \mathbb{P}(J_{n,r}(z)^{\mathrm{c}})|,$$

which can be made arbitrarily large by first choosing n large and then r large according to n, thanks to the weak convergence of $r^{-1}g_r(X^\lambda)$ to $g_1(Y^{\varrho'})$ as $r \to \infty$, $\lambda(r) \to 0$, $r^d\lambda(r) = \varrho'$. Thus, the proof for $d \geq 3$ can be completed analogously to the case $d = 2$, and the scale invariance (4.8) of Poisson–Gilbert graphs also implies that $\lambda_{N_0(r),\tau(r),P(r)}$ tends to zero in this coupled limit. As already indicated in Section 4.1.3.2, the proof for $d \geq 3$ is also applicable for $d = 2$. □

4.2.3. Discussion

We now discuss certain aspects of the results of Section 4.1.3, using arguments of the proofs of Section 4.2. During our discussions, we assume that $\mathbb{E}[\Lambda(Q_1)] = 1$, unless mentioned otherwise.

4.2.3.1. The probability of having an infinite cluster

For any Cox–SINR graph with $\gamma \geq 0$, the existence of an infinite cluster is a shift-invariant event. Therefore, if the stationary intensity measure Λ is also ergodic, then the probability of this event is either zero or one, and the number of infinite clusters is almost surely constant (possibly infinite), cf. [MR96, Theorem 2.1]. In particular, this holds for stabilizing Cox processes, since it is easy to derive that stabilization implies mixing and therefore also ergodicity. Without ergodicity, one can find examples where this property fails, see Section 4.2.3.3.

We note further that [DBT05] conjectured that for $d = 2$ and $\Lambda \equiv$ Leb, SINR graphs have at most one infinite cluster. However, this question is still open, also in the Cox case in general.

4.2.3.2. Phase transitions for absolutely continuous intensities

According to Section 4.1.1, if the intensity measure Λ is absolutely continuous, then $\Lambda(\mathrm{d}x) = l_x\mathrm{d}x$ for a stationary non-negative random field $l = (l_x)_{x\in\mathbb{R}^d}$. Now we argue that in this case one can verify the existence of a subcritical respectively supercritical phase based on suitable boundedness assumption on l, without requiring stabilization. For $\lambda > 0$, we write X^λ for a Cox process with intensity $\lambda\Lambda$ and Y^λ for a Poisson point process with intensity λ.

First, similarly to [S13, Section 5, proof of Theorem 3.3], we argue that if $l_{\max} = \sup\{l_x : x \in Q_1\}$ is almost surely bounded by some constant $K > 0$, then for any $N_0,\tau,P > 0$, we have $\lambda_{N_0,\tau,P} > 0$. Indeed, in this case a suitable coupling shows that for any $\gamma \geq 0$, the SINR graph of X^λ is stochastically dominated by the SNR graph of $Y^{K\lambda}$, which is subcritical for $\lambda > 0$ small enough by Theorem 4.1.5(1).

Second, if there exist $K \geq \varepsilon > 0$ such that $l_{\max} \leq K$ and $l_{\min} := \inf\{l_x : x \in Q_1\} \geq \varepsilon$, both almost surely, then for any $N_0 \geq 0$, $\tau,P > 0$, we have $\lambda_{N_0,\tau,P} < \infty$. Indeed, by Section 4.1.3.3, it suffices to show this for $N_0 > 0$. We choose a coupling according to which $Y^{\varepsilon\lambda} \subseteq X^\lambda \subseteq Y^{K\lambda}$ holds (almost surely as random subsets of $\mathbb{R}^d$). Then, the SNR graph of X^λ contains the one of $Y^{\varepsilon\lambda}$. Further, for any X_j of X^λ, the interference of a transmission from any $X_i \in X^\lambda$, $i \neq j$, to X_j is at most as large as the interference at X_j for the same transmission coming from $Y^{K\lambda}$. Now, applying Lemma 4.2.4 to $Y^{\varepsilon\lambda}$ and Proposition 4.2.5 to $Y^{K\lambda}$ and arguing as in Step 3 in Section 4.2.1.1, we conclude the claim.

In particular, the claim of Section 4.1.5.1 that the SINR graph of the modulated Poisson point process with a general random closed set Ξ has a subcritical phase for any $\lambda_1,\lambda_2 \geq 0$ and a supercritical phase if $\lambda_1,\lambda_2 > 0$ follows.

4.2.3.3. Non-stabilizing examples: zero critical intensity, mixture of both phases

Example 4.2.10 (No subcritical phase, nontrivial probability of percolation)**.** [BY13, Section 4] presented the first example of a Cox point process with the pathological property $\lambda_{\mathrm{c}}(r) = 0$. We now argue that a rich class of intensity measures has this property, and many such intensities can be constructed in a straightforward way, starting from a well-behaved intensity measure. Further, we show that for some of these examples, also $\lambda_{N_0,\tau,P} = 0$ holds for all $N_0 \geq 0$ and $\tau,P > 0$.

Indeed, for fixed $r > 0$, for any intensity measure Λ' for which the corresponding critical intensity $\lambda_{\mathrm{c}}'(r)$ is in $(0,\infty)$, the intensity measure Λ defined as $\Lambda = Z\Lambda'$, where Z is a nonnegative and unbounded random variable with $\mathbb{E}Z = 1$ that is independent of Λ', satisfies $\lambda_{\mathrm{c}}(r) = 0$. Indeed, for any $\lambda > 0$, let X^λ be a Cox process with intensity $\lambda\Lambda$. For $\lambda_0 > \lambda_{\mathrm{c}}'(r)$, the event $\{Z > \lambda_0/\lambda\}$ has positive probability, and in this event, $\lambda\Lambda > \lambda_0\Lambda'$, i.e., $\lambda\Lambda - \lambda_0\Lambda'$ is a nonnegative measure. Then, using a suitable coupling, $g_r(X^\lambda)$ includes $g_r(Y^{\lambda_0\Lambda'})$, where $Y^{\lambda_0\Lambda'}$ is a Cox point process with intensity $\lambda_0\Lambda'$, and $g_r(Y^{\lambda_0\Lambda'})$ percolates with positive probability. It follows that $\lambda > \lambda_{\mathrm{c}}(r)$. Hence, $\lambda_{\mathrm{c}}(r) = 0$. It can be shown similarly that for $\lambda > 0$ small enough, $\mathbb{P}(g_r(X^\lambda)$ percolates$) \in (0,1)$; in other words, both subcriticality and supercriticality of the Gilbert graph have positive probability. This holds even for all $\lambda > 0$ if for all $\varepsilon > 0$, $\mathbb{P}(Z > \varepsilon) < 1$. With this respect, the example of [BY13] shows a more striking degeneracy than our ones because there $\mathbb{P}(g_r(X^\lambda)$ percolates$) = 1$ holds for any $r,\lambda > 0$.

In case Λ' is asymptotically essentially connected and together with ℓ it satisfies at least one of the conditions (2a), (2b), and (2c), one can derive that for Λ, even $\lambda_{N_0,\tau,P} = 0$ holds for $N_0 \geq 0$ and $\tau,P > 0$. Indeed, by Section 4.1.3.3, it suffices to consider the case $N_0 > 0$. Then, for $\lambda_0 > 0$ sufficiently large, for the SNR graph of a Cox process with intensity $\lambda_0\Lambda'$, Lemma 4.2.4 holds. For $\lambda > 0$, we can choose $K > 0$ such that the event $\{Z \in (\lambda_0/\lambda, K\lambda_0/\lambda]\}$ has positive probability. In this event, the SNR graph of X^λ includes the one of the Cox process with intensity $\lambda_0\Lambda'$ in a suitable coupling of the two processes. Also, for $X_i,X_j \in X^\lambda$, the interference of the transmission from X_i to X_j in $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ is bounded by the interference of the same transmission in $g_{(\gamma,N_0,\tau,P)}(Y^{K\lambda_0})$, where $Y^{K\lambda_0}$ is a Cox process with intensity $K\lambda_0\Lambda'$ including X^λ. The latter interference can be controlled uniformly in X_i,X_j according to and in the sense of Proposition 4.2.5. Arguing as in Step 3 in Section 4.2.1.1, we obtain that $\gamma^*(\lambda) > 0$. It follows that $\lambda_{N_0,\tau,P} = 0$. It is also clear from the above observations about the SNR graph that for $\lambda > 0$ small, $\mathbb{P}(g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ percolates$) \in (0,1)$. One can argue similarly for r_{B} large in the case of a merely stabilizing Λ', in case (2b) or (2c) holds and ℓ has unbounded support, relying on the results of Section 4.1.3.2. $\diamond$

Nevertheless, also the class of non-stabilizing intensities Λ such that $\lambda_{\mathrm{c}}(r) > 0$ holds for all $r > 0$ is rich. According to Section 4.2.3.2, it contains the set of all absolutely continuous, uniformly bounded non-stabilizing intensities. The next example complements this with some singular and some unbounded absolutely continuous intensities that also satisfy this property.

Example 4.2.11 (Existence of a subcritical phase for non-stabilizing intensities)**.** Let us consider the construction of Example 4.2.10, with the only difference that now Z is almost surely bounded by some $K > 0$. Then, the arguments of Example 4.2.10 imply that if $\lambda_{\mathrm{c}}'(r) > 0$, then $\lambda_{\mathrm{c}}(r) > \lambda_{\mathrm{c}}'(r)/K > 0$. $\diamond$

4.2.3.4. Non-constant signal powers

[DBT05, DFMMT06, FM07, BY13] define the SINR graph under the assumption that the transmitted power of any transmitter equals the same constant $P > 0$. Clearly, making the power depend on the user's device would be a further step towards a more realistic model of a wireless network. The transmitted power could also depend on the spatial position of the user.

Some of the results of the aforementioned works are easy to extend to the case of random powers satisfying some boundedness assumptions. We demonstrate this in the case where the signal powers are i.i.d. (and thus independent of the user's spatial position). We note that some of our conclusions were already included in [KY07] for the two-dimensional Poisson case, but we will relax the assumption of [KY07] that powers are bounded, bounded away from zero, and both the minimal and the maximal power value have a positive probability.

More precisely, we choose a nonnegative random variable P with distribution $\zeta(\cdot) = \mathbb{P}(P \in \cdot)$ and we assume that the process of transmitter–power pairs is given by the independently marked point process $\mathbb{X}^\lambda = (X_i,P_i)_{i\in I}$. Here $X^\lambda = (X_i)_{i\in I}$ is a Cox process in $\mathbb{R}^d$ with intensity $\lambda\Lambda$, as before, and given X^λ, $(P_i)_{i\in I}$ are i.i.d. with distribution ζ. Then we put

$$\mathrm{SINR}((X_i,P_i),(X_j,P_j),\mathbb{X}^\lambda) := \frac{P_i\ell(|X_i - X_j|)}{N_0 + \gamma\sum_{k\notin\{i,j\}} P_k\ell(|X_k - X_j|)},$$

and we define the SINR graph on X^λ to have an edge between X_i and X_j, $i \neq j$, if and only if $\mathrm{SINR}((X_i,P_i),(X_j,P_j),\mathbb{X}^\lambda) > \tau$ and $\mathrm{SINR}((X_j,P_j),(X_i,P_i),\mathbb{X}^\lambda) > \tau$. We assume that $N_0 > 0$, Λ satisfies the same conditions as before, and ℓ satisfies (i) with $d_0 = 0$ and (iii) from Assumption (ℓ), further that $1 \geq \ell(0)$. For $\gamma = 0$, we also write SNR instead of SINR.

For $\gamma = 0$, this graph is a Gilbert graph with i.i.d. random radii $R_i = \max\{r > 0\colon \ell(r) \geq \tau N_0/P_i\}$, where we use the convention that $R_i = 0$ if $\tau N_0/P_i > \ell(0)$. Therefore, in order to analyse Cox–SINR graphs with random powers, one first needs to derive results about Cox–Gilbert graphs with random radii and the corresponding Boolean models, which are not included in [HJC17]. Although e.g. [MR96, Section 7] includes results about Boolean models with random radii driven by general stationary point processes, determining questions of percolation and coverage in the Cox case requires additional work. We therefore sticked to constant radii in the present chapter. Nevertheless, a comparison with Gilbert graphs with constant radii yields some immediate results about S(I)NR graphs with random powers.

First of all, it is clear that if $\frac{\tau N_0}{\operatorname{esssup}\zeta} \geq \ell(0)$, then the SINR graph has no edges for any $\lambda > 0$ and $\gamma \geq 0$. Let us assume that this is not the case but even $\frac{\tau N_0}{\operatorname{essinf}\zeta} < \ell(0)$. Then, if P is bounded from above and Λ is stabilizing, then the first statement of Theorem 4.1.3 implies that there exists $\lambda > 0$ such that almost surely, the SNR graph does not percolate. Obviously, the same holds for all SINR graphs with $\gamma > 0$. Note that for unbounded P the Gilbert graph may not have a subcritical phase. Indeed, it was shown in [MR96, Proposition 7.3] that if $\mathbb{E}[R_i^d] = \infty$ for $i \in I$, then, almost surely, the Boolean model covers $\mathbb{R}^d$, and thus the Gilbert graph is connected (even for $d = 1$!). This is not only true in the stabilizing Cox case but in general for stationary point processes.

Further, if P is bounded away from 0 and Λ is asymptotically essentially connected, then the second statement of Theorem 4.1.3 implies that there exists $\lambda > 0$ for which the SNR graph percolates (almost surely, cf. Section 4.2.3.1). In fact, the same holds as long as $\mathbb{P}(P > 0) > 0$. Indeed, then there exists $\varepsilon > 0$ such that $\mathbb{P}(P > \varepsilon) > 0$, and $X_\varepsilon^\lambda := \{X_i\colon P_i > \varepsilon\}$ is an independent thinning of X^λ. Therefore, by the Colouring Theorem [K93, Section 5.1], X_ε^λ is a Cox point process with asymptotically essentially connected intensity $\lambda\mathbb{P}(P > \varepsilon)\Lambda$. The Gilbert graph of X^λ contains the one of X_ε^λ, which has an infinite cluster for $\lambda > 0$ large enough. Controlling interferences using the same arguments as in Sections 4.2.3.2 and 4.2.3.3, we conclude that if also $\zeta([0,b]) = 1$ holds for some $b \in (0,\infty)$, then for large enough $\lambda > 0$, there exists $\gamma > 0$ for which the SINR graph percolates, in case Λ is asymptotically essentially connected and at least one of the conditions (2a), (2b), (2c) holds. Let us also note that for any choice of ζ, $\lambda \mapsto \gamma^*(\lambda)$ is bounded from above by $1/\tau$, thanks to the same degree bound argument as Proposition 4.1.12 for constant P. This was observed in [DBT05, Theorem 1] in the Poisson case; see Section A.3 in the Appendix for further details.

The following two interesting questions remain open for Λ asymptotically essentially connected and P unbounded. Under what conditions on ζ, Λ, and ℓ can one find $\lambda > 0$ such that the corresponding SNR graph does not percolate? Under what conditions does the SINR graph percolate for some $\gamma > 0$ for large λ?

4.2.3.5. SINR graphs with external interferers

In [BY10, Section IV] and [BY13, Section 3.4], an extended notion of SINR graphs was introduced. For two stationary point processes Φ and Φ_I, for $N_0,\tau,P > 0$ and $\gamma \geq 0$ interpreted as before, the SINR graph $g_{(\gamma,N_0,\tau,P)}(\Phi,\Phi_I)$ has vertex set Φ and an edge between two different points X_i,X_j of Φ whenever

$$\mathrm{SINR}(X_i,X_j,\Phi_I) = \frac{P\ell(|X_i - X_j|)}{N_0 + \gamma P \sum_{Z \in \Phi_I \setminus \{X_i,X_j\}} \ell(|Z - X_j|)} > \tau$$

and also the analogously defined quantity $\mathrm{SINR}(X_j,X_i,\Phi_I)$ exceeds τ. The points of Φ are called *backbone nodes* and the ones of Φ_I are the *interferers*. The motivation for studying such SINR graphs is the case $\Phi \subseteq \Phi_I$, where $\Phi_I \setminus \Phi$ is thought of as a set of *external interferers*. Then, one aims to find sufficient conditions on Φ and Φ_I under which percolation of $g_{(\gamma',N_0,\tau,P)}(\Phi,\Phi)$ for some $\gamma' > 0$ implies percolation of $g_{(\gamma'',N_0,\tau,P)}(\Phi,\Phi_I)$ for some $\gamma'' > 0$.

However, [BY10, BY13] also studied the case $\Phi \not\subseteq \Phi_I$. These papers investigated certain notions of *sub-Poisson point processes*. In particular, we say [BY13, Definition 1.2] that a point process Φ in $\mathbb{R}^d$ is *ν-weakly sub-Poisson* if $\mathbb{P}(\Phi(B) = 0) \leq \exp(-\mathbb{E}(\Phi(B)))$ for all bounded Borel sets $B \subseteq \mathbb{R}^d$, and we say that Φ is *α-weakly sub-Poisson* if

$$\mathbb{E}\Big(\prod_{i=1}^k \Phi(B_i)\Big) \leq \prod_{i=1}^k \mathbb{E}(\Phi(B_i))$$

for all $k \in \mathbb{N}$ and all pairwise disjoint bounded Borel sets $B_1,\dots,B_k$, where we used the notation $\Phi(\cdot) = \#(\Phi \cap \cdot)$. We note that a Cox point process can only be α- or ν-weakly sub-Poisson if it is a Poisson point process [BY09, Section 3.4]. The following were shown in [BY13, Theorems 3.12, 3.13] for $d = 2$ and fixed $N_0,\tau,P > 0$, with all percolation statements being meant with positive probability.

(I) If $\Phi = X^\lambda$ is a homogeneous Poisson point process with λ larger than the SNR-critical intensity $\varrho_{\mathrm{c}}(r_{\mathrm{B}})$ (cf. Section 4.2.2.2) and Φ_I is stationary α-weakly sub-Poisson with nonzero mean measure, then $g_{(\gamma,N_0,\tau,P)}(X^\lambda,\Phi_I)$ percolates for $\gamma > 0$ small enough.

(II) If Φ is stationary ν-weakly sub-Poisson with nonzero mean measure, Φ_I is α-weakly sub-Poisson and stationary, and ℓ has unbounded support, then for sufficiently large P, there is $\gamma > 0$ such that $g_{(\gamma,N_0,\tau,P)}(\Phi,\Phi_I)$ percolates.

Thus, for $\gamma > 0$, a Poisson–SINR graph with SNR-supercritical intensity preserves its infinite cluster after adding an arbitrarily large intensity of external interferers, at the price of reducing γ (without vanishing). In fact, the interferences of α-weakly sub-Poisson point processes can be controlled the same way for $d \geq 3$ as for $d = 2$ [BY13, Proposition 3.3], and thus (II) immediately extends to $d \geq 3$. Further, Theorem 4.1.5(2) implies that (I) holds for all sufficiently large λ (but not necessarily for all $\lambda > \varrho_c(r_B)$) if $d \geq 3$.

Now, the interferences generated by the sub-Poissonian external interferers were controlled in [BY13] very similarly to the proof of Proposition 4.2.5 of the present chapter. Let us also recall that the proof of this proposition requires only stabilization, but no asymptotic essential connectedness, of the intensity measure of the point process of interferers. This allows for augmenting the set of external interferers in the results of [BY13] with a stabilizing Cox process satisfying the assumptions of Theorem 4.1.5(2). On the other hand, the arguments of Sections 4.2.1.1 and 4.2.2.2 show that sub-Poissonian or Cox external interferers also do not hinder the formation of an infinite cluster if the backbone nodes form a stabilizing Cox point process with strong connectivity properties. Thus, we obtain the following generalizations of the results of [BY10, BY13] for fixed $N_0,\tau,P > 0$.

- If $\lambda > 0$, Λ is asymptotically essentially connected, Φ is a Cox process with intensity $\lambda\Lambda$, and Φ_I is the union of an α-weakly sub-Poisson point process Φ_I^0 and a Cox process Φ_I^1 with stabilizing intensity Λ', where (2a), (2b) or (2c) holds for Λ' and ℓ, then if λ is large enough (in particular, it is required that $\lambda > \lambda_c(r)$), then there exists $\gamma > 0$ such that $g_{(\gamma,N_0,\tau,P)}(\Phi,\Phi_I)$ percolates.
  Here, in order to carry out the proof analogously to Section 4.2.1.1, one needs to assume that $((\Lambda,R),(\Lambda',R'))$ are jointly stationary, where $R' = (R'_x)_{x\in\mathbb{R}^d}$ are the stabilization radii of Λ'.
- If $\lambda > 0$, ℓ has unbounded support, Λ is stabilizing, $\Phi = X^\lambda$ is a Cox process with intensity $\lambda\Lambda$, further, Φ_I is the union of an α-weakly sub-Poisson point process Φ_I^0 and a Cox process Φ_I^1 with stabilizing intensity Λ' such that Λ' and ℓ satisfy (2b) or (2c), then $g_{(\gamma,N_0,\tau,P)}(\Phi,\Phi_I)$ percolates for some $\gamma > 0$ if P is sufficiently large.
  As in the previous case, $((\Lambda,R),(\Lambda',R'))$ needs to be assumed jointly stationary.
- If supp ℓ is unbounded, Φ is ν-weakly sub-Poisson with nonzero mean measure and Φ_I is the union of an α-weakly sub-Poisson point process Φ_I^0 and a Cox process Φ_I^1 with stabilizing intensity $\lambda\Lambda'$ such that Λ' and ℓ satisfy (2b) or (2c), then $g_{(\gamma,N_0,\tau,P)}(\Phi,\Phi_I)$ percolates for some $\gamma > 0$ if P is large enough.

In these examples, it is not required that $\mathbb{E}[\Lambda'(Q_1)] = 1$, and the results also hold if the mean measure of Φ_I^0 or Φ_I^1 is zero (trivially if both are zero).

4.2.3.6. Information theoretically secure SINR graphs

[VI14] studied the following variant of the SINR graph model, called *(information theoretically) secure SINR graph* or *SSG* for short, in the two-dimensional Poisson case. Let two simple point processes Φ and Φ_E be given. We call the elements of Φ the *legitimate nodes* and the ones of Φ_E the *eavesdropper nodes*. For $P,N_0 > 0$, $\gamma \geq 0$, a path-loss function ℓ satisfying Assumption (ℓ) with $d_0 = 0$, and for $X_i,X_j \in \Phi$, $X_i \neq X_j$, we define

$$\mathrm{SINR}(X_i,\, X_j,\, \Phi) = \frac{P\ell(|X_i - X_j|)}{N_0 + \gamma P \sum_{X_k\in\Phi\setminus\{X_i,X_j\}} \ell(|X_k - X_j|)}, \tag{4.39}$$

similarly to (4.1), and for $X_i \in \Phi$ and $E \in \Phi_E$, we put

$$\mathrm{SINR}(X_i,\, E,\, \Phi) = \frac{P\ell(|X_i - E|)}{N_0 + P \sum_{X_k\in\Phi\setminus\{X_i\}} \ell(|X_k - E|).} \tag{4.40}$$

The idea of the SSG is that a secure transmission from X_i to X_j is possible if and only if the strength of the signal of X_i received by X_j is stronger than the strength of the same signal received by any of the eavesdropper nodes. Thus, the (undirected) SSG, denoted as $\mathrm{SSG}_{(\gamma,N_0,P)}(\Phi,\Phi_E)$, is defined as a graph with vertex set Φ, where $X_i,X_j \in \Phi$ are connected by an edge if and only if $\mathrm{SINR}(X_i,X_j,\Phi)$ is larger than $\mathrm{SINR}(X_i,E,\Phi)$ for all $E \in \Phi_E$ and also $\mathrm{SINR}(X_j,X_i,\Phi) > \mathrm{SINR}(X_j,E,\Phi)$ for all $E \in \Phi_E$.[22] [VI14] considered only the case $P = N_0 = 1$ because the general cases $P,N_0 > 0$ are analogous[23]. We stick to this convention and write $\mathrm{SSG}_{(\gamma)}(\Phi,\Phi_E) = \mathrm{SSG}_{(\gamma,1,1)}(\Phi,\Phi_E)$.

We present the results of [VI14] for the Poisson case, where Φ and Φ_E be two independent Poisson point processes in $\mathbb{R}^2$ with positive intensities λ, λ_E, respectively, and then we discuss their possible generalizations in the two-dimensional Cox case[24].

[22] Note that there is no factor of γ in the denominator of (4.40), which is interpreted in [VI14, Section 1.1] as follows. The code used by the legitimate nodes is not known to the eavesdroppers, who therefore cannot reduce interferences. Accordingly, it is reasonable but mathematically not necessary to assume that $\gamma \in (0,1]$. Again, the case $\gamma = 0$ is easier to handle but less realistic than the one $\gamma > 0$, as it corresponds to the assumption of perfect interference cancellation for the communication between legitimate nodes.

[23] The case $N_0 = 0$ would require a more careful treatment because SSG's with $N_0 > 0$ have no direct relation to a Boolean model, and thus the arguments of Section 4.1.3.3 do not apply to an SSG with $N_0 = 0$.

[24] The proofs of these assertions in the Poisson case involve a percolation argument on a dual lattice (cf. Section 4.3.3 for such an argument), which has no analogue for $d \geq 3$ dimensions, therefore we will stick to the case $d = 2$ in this discussion.

(I) Let $\lambda_E > 0$ be fixed. Then, there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, there exists $\gamma'(\lambda,\lambda_E) > 0$ such that for all $\gamma \in (0,\gamma'(\lambda,\lambda_E))$, $\mathrm{SSG}_{(\gamma)}(\Phi,\Phi_E)$ percolates with positive probability.

(II) Let $\lambda_E > 0$ be fixed. Then, there exists $\lambda_1 > 0$ such that for any $\lambda \in (0,\lambda_1)$ and for any $\gamma \geq 0$, almost surely, $\mathrm{SSG}_{(\gamma)}(\Phi,\Phi_E)$ does not percolate.

In order to conclude about the Cox case, we now provide some details of the proofs of these assertions in [VI14]. Both proofs involve a Peierls argument on a certain edge percolation model on $\mathbb{R}^2$, where, given $\lambda_E > 0$, percolation of open edges implies percolation in $\mathrm{SSG}_{(\gamma)}(\Phi,\Phi_E)$ for $\gamma > 0$ sufficiently small and $\lambda > 0$ sufficiently large in the case (I), and percolation of open edges implies lack of percolation in $\mathrm{SSG}_{(\gamma)}(\Phi,\Phi_E)$ for all $\lambda > 0$ sufficiently small and for any $\gamma \geq 0$ in the case (II). We will abstain from spelling out details of interference control arguments, which are very similar to the proof of Proposition 4.2.5, and also from explaining how a finite number of Peierls arguments can be combined, which is analogous to the proof of Proposition 4.2.3.

As for (I), we define a square lattice on $\mathbb{R}^2$ with edge length $s > 0$, where the two neighbouring squares of an edge a are denoted by $S_1(a),S_2(a)$ in some arbitrary (but fixed) order. We further choose a larger edge length $t > s$ satisfying $\ell(t) \leq \ell(\sqrt{5}s)/2$ and we let $Y(a)$ be the smallest square containing $\bigcup_{i=1}^4 B_t(a_i)$, with a_i, $i \in [4]$, being the four edges of the rectangle $S_1(a) \cup S_2(a)$. Now, for $\gamma > 0$, we call the edge a open if the following three conditions are all satisfied, and we call a closed otherwise.

(A) There exists at least one legitimate node in both squares $S_1(a)$, $S_2(a)$.

(B) There are no eavesdropper nodes in $Y(a)$.

(C) $\forall X_i,X_j \in \Phi \cap (S_1(a) \cup S_2(a))$, $X_i \neq X_j$, $I_j^i = \sum_{k \notin \{i,j\}} \ell(|X_k - X_j|) \leq 1/\gamma$.

We denote the events in (A), (B), and (C) by $A(a),B(a),C(a)$ respectively. For a suitable $b > 0$, the events $A(a)$ and $B(a)$ are b-dependent over different edges a, and their probabilities do not depend on a. Now, we can choose s,t sufficiently small in order to make $\mathbb{P}(B(a)^{\mathrm{c}})$ sufficiently small. Then we can choose λ large in order to make $\mathbb{P}(A(a)^{\mathrm{c}})$ small. Finally, using an interference control similar to Proposition 4.2.5, for $\varepsilon > 0$, we can choose $\gamma > 0$ small enough such that for all $N \in \mathbb{N}$, for any pairwise distinct edges $a_1,\ldots,a_N$ of the discrete model, we have $\mathbb{P}(C(a_1)^{\mathrm{c}} \cap \ldots \cap C(a_N)^{\mathrm{c}}) \leq \varepsilon^N$. These together allow for a Peierls argument, which implies percolation in the discrete model, and thanks to further deterministic arguments, percolation in $\mathrm{SSG}_{(\gamma)}(\Phi,\Phi_E)$ follows.

As for the Cox case, we shall assume that there are two independent, stationary intensities Λ,Λ_E with $\mathbb{E}[\Lambda(Q_1)] = \mathbb{E}[\Lambda_E(Q_1)] = 1$ and two parameters $\lambda,\lambda_E > 0$ such that Φ is a Cox process with intensity $\lambda\Lambda$ and Φ_E is another one with intensity $\lambda_E\Lambda_E$. In order to generalize the proof of [VI14], for small s,t, we need the properties (A) and (B) to be satisfied with high probability, in such a way that for some $b' > 0$, these properties are b'-dependent among different sites with high probability. For this, Λ,Λ_E need to be b-dependent, stabilization is not sufficient in general. Now, while if Λ_E is b-dependent, then the probability of $B(a)^{\mathrm{c}}$ can still be made arbitrarily close to zero by choosing s,t small, the same is not necessarily true for $A(a)^{\mathrm{c}}$ under a b-dependence assumption on Λ, even if one can choose λ arbitrarily large. Indeed, note that percolation of open edges implies percolation in the Gilbert graph of Φ with a certain fixed radius, which does not hold for b-dependent nonzero intensities in general. However, if we additionally assume that there exists $\varepsilon_0 > 0$ such that almost surely, $\Lambda \geq \varepsilon_0\mathrm{Leb}$ in the sense that $\Lambda - \varepsilon_0\mathrm{Leb}$ is a positive measure, then it is easy to see that $\mathbb{P}(A(a)^{\mathrm{c}})$ tends to zero as $\lambda \to \infty$.[25] Now, under the assumption that Λ satisfies the condition (2b) from Theorem 4.1.5, interference control is possible similarly to Proposition 4.2.5. It might be possible to derive a similar result under the assumption (2c) or the one (2a), however, in these cases, the proof of Proposition 4.2.5 does not guarantee that one can control the interference in discrete percolation models with small edge lengths.

For example, if Φ and Φ_E are both modulated Poisson point processes with their random closed sets being boundedly determined (e.g., Poisson–Boolean models; cf. Section 4.1.5.1) and Λ puts a positive weight both to the corresponding random closed set and its complement, then (I) holds.

The proof of (II) in [VI14] is similar to the one of (I), but with the roles of legitimate nodes and eavesdropper nodes partially interchanged. More precisely, we define a square lattice with side length $M > 0$. For an edge e of this grid, we denote the two neighbouring squares of e by $S_1(e),S_2(e)$, in an arbitrary but fixed order. Further,

[25] We expect that a suitably modified version of this assertion holds also if Λ is only asymptotically essentially connected (and b-dependent).

we choose $m \in (0,M)$ and we denote by $T_1(e)$,$T_2(e)$ the squares of side length m concentric to $S_1(e)$,$S_2(e)$, respectively. We fix $c > 0$ and we say that the edge e is open if the following three conditions are satisfied, and otherwise we say that e is closed.

(a) There exists at least one eavesdropper node in both squares $T_1(e)$, $T_2(e)$.

(b) There are no legitimate nodes in $S_1(e) \cap S_2(e)$.

(c) $\forall E \in \Phi_E \cap (T_1(e) \cup T_2(e))$, $I_E = \sum_k \ell(|X_k - E|) \leq c$.

Now, [VI14, Section 3] has verified the following statement, which is entirely deterministic and does not require that Φ_E or Φ be a Poisson point process. Given m and c, one can choose M sufficiently large such that an edge of $\mathrm{SSG}_{(0)}(\Phi,\Phi_E)$ cannot cross an open edge of this discrete percolation process. Hence, since $\mathrm{SSG}_{(\gamma)}(\Phi,\Phi_E) \preceq \mathrm{SSG}_{(0)}(\Phi,\Phi_E)$ for all $\gamma > 0$, it follows that percolation in the discrete model implies lack of percolation in $\mathrm{SSG}_{(\gamma)}(\Phi,\Phi_E)$ for all $\gamma \geq 0$. It remains to show that one can choose the parameters in such a way that the set of open edges percolates. Again, properties (a) and (b) are b-dependent for some $b > 0$, while (c) is a usual interference control property similar to the one in Proposition 4.2.5.

Now, let us fix c in (c) and write $A(e)$,$B(e)$,$C(e)$ for the events in (a), (b), and (c) corresponding to the edge e of the discrete model, respectively. Clearly, the probability of these events does not depend on the choice of e. Now, first, $\mathbb{P}(A(e)^{\mathrm{c}})$ can be made arbitrarily small by choosing m large enough. Second, for any $\varepsilon > 0$, we can choose c large enough such that given a finite upper bound λ_0 on λ, for any $N \in \mathbb{N}$ and pairwise distinct edges $e_1,\dots,e_N$ of the discrete model, we have $\mathbb{P}(C(e_1)^{\mathrm{c}} \cap \ldots \cap C(e_N)^{\mathrm{c}}) \leq \varepsilon^N$. Next, given m,c, we can choose M sufficiently large such that no edge of $\mathrm{SSG}_{(0)}(\Phi,\Phi_E)$ can cross an open edge of the discrete model. Finally, given M, we can make $\mathbb{P}(B(e)^{\mathrm{c}})$ arbitrarily small by choosing $\lambda \in (0,\lambda_0)$ sufficiently small. These together allow for a Peierls argument, implying percolation in the discrete model.[26]

Now, in the Cox case, we see the following. It is highly imaginable that edges satisfying the property (a) percolate for large m for many examples of a stabilizing intensity measure Λ_E. However, this property does not directly follow from the definition of stabilization, neither from the one of asymptotic essential connectedness. At least, we see that it holds for stabilizing intensities Λ_E such that almost surely, $\Lambda_E \geq \varepsilon_0\mathrm{Leb}$. Now, let us fix $\lambda_0 > 0$. If Λ is stabilizing and together with ℓ it satisfies (2c) or (2b), then a Peierls argument can be carried out for the process of edges satisfying (c) for all $\lambda \in (0,\lambda_0)$ simultaneously, in case m, c are sufficiently large. These assertions follow from the proof of Proposition 4.2.5 under the assumptions (2b) respectively (2c). Now, given m and c, one can make M so large that edges of $\mathrm{SSG}_{(0)}(\Phi,\Phi_E)$ cannot cross an open edge of the discrete process. Finally, for Λ stabilizing and M fixed, the probability of $B(e)^{\mathrm{c}}$ can be made arbitrarily small (independently of the choice of e) by choosing $\lambda \in (0,\lambda_0)$ sufficiently small. It may be the case that (II) also holds under the assumption (2a), however, this does not follow directly from the proof of Proposition 4.2.5 in that case, and therefore additional arguments are necessary.

As for our main examples, we see that if Φ_E is a modulated Poisson point process with a boundedly determined random closed set and positive intensity both on the random closed set and on its complement, then (II) holds under the assumption that Λ is stabilizing and (2b) or (2c) holds; see Section 4.1.5.1 for the enumeration of concrete examples. It would be highly interesting to get rid of the b-dependence assumptions and the conditions of local boundedness away from 0 for (I) and (II), and to consider also higher dimensions. One could also investigate whether the condition that Φ and Φ_E are independent can be relaxed. However, these would most likely require new proof techniques instead of just slight modifications of the ones in [VI14]. Further, we find it remarkable that while (II) is a non-percolation statement, its proof still involves an interference control on the point process of legitimate nodes, whereas in the case of usual SINR graphs, interference control is only used for verifying that percolation occurs. Thus, there arises the question whether (II) can be verified under weaker assumptions on the path-loss function and the intensity measure of the legitimate nodes.

4.2.3.7. Extending Theorem 4.1.6 to higher dimensions?

A key ingredient of the proof of Theorem 4.1.6 in Section 4.2.1.2 was the convexity argument of Lemma 4.2.8. This works only for $d = 2$, as convexity implies only that the $d-1$-dimensional Hausdorff measure (intrinsic

[26] In fact, [VI14, Section 3] indicates that the Peierls argument can be carried out for fixed $c > 0$, but we think that this is not true in general. To be more precise, we do not see why the quotient r_3 in the proof of [VI14, Theorem 3.1] should tend to zero for c fixed in the limit $\lambda \downarrow 0$.

volume) of a convex set $T \subseteq \mathbb{R}^d$ is not smaller than the one of any convex set contained in T. For the perimeters (one-dimensional Hausdorff measures), the analogous statement fails for $d \geq 3$. E.g., for $d = 3$, consider an n-gon included in a facet of the cube Q_1 and a translate of this n-gon included in the opposite facet, and connect the corresponding vertices of the n-gons in order to obtain a convex polyhedron. The perimeter of this polyhedron is at least n, thus the perimeter of convex sets embedded in Q_1 has no upper bound. Thus, if Theorem 4.1.6 holds for $d \geq 3$, its proof must be significantly different from the one presented in this thesis for $d = 2$.

For a Poisson–Voronoi tessellation S for $d \geq 2$, it is easy to see that for the singular intensity measure $\Lambda'(\mathrm{d}x) = \nu_{d-1}(S \cap \mathrm{d}x)$, for all $\alpha > 0$ we have that $\mathbb{E}[\exp(\alpha\Lambda(Q_1))] < \infty$, where ν_{d-1} denotes $d-1$-dimensional Hausdorff measure, using an analogue of Lemmas 4.2.7 and 4.2.8, and the exponential stabilization of Λ'. However, an application of the intensity measure Λ' is unclear to us, even for $d = 3$.

4.3. Proof of results about the critical interference cancellation factor

In this section we prove the results of Section 4.1.4. In Section 4.3.1 we verify Lemma 4.1.13, the only result of Section 4.1.4.1 which still requires a proof. Section 4.3.2 contains the proofs of the results of Section 4.1.4.2, in particular we prove Proposition 4.1.15 in Section 4.3.2.1 and Proposition 4.1.16 in Section 4.3.2.2. In Section 4.3.3 we sketch the proof of Corollary 4.1.17, the result of Section 4.1.4.3.

4.3.1. Proof of Lemma 4.1.13.

Let k be such that $g^{\rightarrow}_{(\gamma,N_0,\tau,P)}(\Phi)$ has in-degrees bounded by k, almost surely. Let us assume that Φ is defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and that $\omega \in \Omega$ is such that for $\Phi(\omega) = (x_i)_{i \in I_0}$, there exists $i,j \in I_0$ such that $\mathrm{SINR}(x_i,x_j,\Phi(\omega)) > \tau$ and there exist k points $x_{l_1}, \ldots, x_{l_k}$ in $\Phi(\omega) \setminus \{x_i, x_j\}$ such that $|x_{l_m} - x_i| \leq |x_j - x_i|$ for all $m \in [k]$. Then, since $\ell(|x_{l_m} - x_i|) \geq \ell(|x_j - x_i|)$ for all $m \in [k]$, $\mathrm{SINR}(x_{l_m},x_i,\Phi(\omega)) > \tau$ follows. Indeed, this is easily seen using the fact [BB09, Remark 6.1.3] that for any $n,m \in I_0$, $n \neq m$, we have

$$\mathrm{SINR}(x_n,x_m,\Phi(\omega)) = \frac{\ell(|x_n - x_m|)}{N_0 + \gamma\sum_{p \in I_0 \setminus \{n,m\}} \ell(|x_p - x_m|)} > \tau \text{ holds if and only if}$$

$$\mathrm{STINR}(x_n,x_m,\Phi(\omega)) := \frac{\ell(|x_n - x_m|)}{N_0 + \gamma\sum_{p \in I_0 \setminus \{m\}} \ell(|x_p - x_m|)} > \frac{\tau}{1+\tau\gamma}. \quad (4.41)$$

Here, STINR is a standard abbreviation for *signal-to-total received power plus noise ratio*, see e.g. [KB14], and it is clear that $\mathrm{STINR}(x_{l_m},x_i,\Phi(\omega)) \geq \mathrm{STINR}(x_j,x_i,\Phi(\omega))$ if $\ell(|x_{l_m} - x_i|) \geq \ell(|x_j - x_i|)$.

Consequently, x_i has in-degree at least $k+1$ in $g^{\rightarrow}_{(\gamma,N_0,\tau,P)}(\Phi(\omega))$. But by assumption, the set of all such ω's is contained in a $\mathbb{P}$-nullset. This implies the lemma. □

4.3.2. Proof of the results of Section 4.1.4.2

4.3.2.1. Proof of Proposition 4.1.15

We first consider the case $N_0 > 0$. It suffices to show that for fixed $N_0,\tau,P,\gamma > 0$, there is $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, $\mathbb{P}(g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ percolates$) = 0$. By Proposition 4.1.12, the statement is clear if $\gamma \geq \frac{1}{\tau}$. Else, let $N \geq 2$ be such that $\gamma \geq \frac{1}{(N-1)\tau}$. By (4.12), all in-degrees in $g^{\rightarrow}_{(\gamma,N_0,\tau,P)}(X^\lambda)$ are at most $N-1$. Let now $(Q^j)_{j=1}^\infty$ be a subdivision of $\mathbb{R}^d$ into congruent copies of $Q_{d'_0/\sqrt{d}}$, where $d'_0 = \ell^{-1}(\ell(0)/2)$ exists by Assumption (ℓ). Then, for any $j \in \mathbb{N}$, we have $\ell(|x-y|) \in [\ell(0)/2, \ell(0)]$ for all $x,y \in Q^j$.

We claim that if $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ percolates, then each Q^j containing at least one point $X_i \in X^\lambda$ from an unbounded cluster of $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ contains at most $2N+2$ points of X^λ. Indeed, otherwise, since X_i is not isolated in $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$, there exists $k \neq i$ such that $X_k \to X_i$ is an edge in $g^{\rightarrow}_{(\gamma,N_0,\tau,P)}(X^\lambda)$. Now, if at least

$2N$ points of $X^\lambda \setminus \{X_i, X_k\}$ are within distance at most d_0' from X_i, then

$$\mathrm{SINR}(X_k, X_i, X^\lambda) = \frac{\ell(|X_k - X_i|)}{N_0 + \gamma \sum_{j \neq k,i} \ell(|X_j - X_i|)} \leq \frac{\ell(0)}{2N\gamma \frac{\ell(0)}{2}} \leq \tau,$$

where in the last step we have used that $\gamma N \geq \frac{N}{(N-1)\tau} > \frac{1}{\tau}$. This implies the claim.

Since $N_0 > 0$, any edge in $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ has length at most r_{B}. Thus, if $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ percolates, then so does the process of open sites in the following site percolation model defined on the set of centres $C(Q^i)$ of the boxes Q^i, $i \in \mathbb{N}$. The site $C(Q^i)$, $i \in \mathbb{N}$, is *open* if there exists $j \in \mathbb{N}$ such that $\#(Q^j \cap X^\lambda) \leq 2N + 2$ and $\min_{x \in Q^i, y \in Q^j} |x - y| \leq r_{\mathrm{B}}$.

We now show that the process of open sites does not percolate for λ large, almost surely. This process is clearly b'-dependent for sufficiently large b' because X^λ is b-dependent, and openness of a site depends on points of X^λ in a bounded neighbourhood of the site. Thus, it suffices to show that $\mathbb{P}(C(Q^i)$ is open) tends to zero as $\lambda \to \infty$ uniformly in i. Indeed, applying dependent percolation theory [LSS97, Theorem 0.0], for large λ, the process of open sites is stochastically dominated by a subcritical independent Bernoulli site percolation process. By stationarity of X^λ, for all i we have the union bound

$$\begin{aligned}\mathbb{P}(C(Q^i) \text{ is open}) &\leq \mathbb{P}\Big(\exists j\colon \min_{x \in Q^j, y \in Q^i} |x - y| \leq r_{\mathrm{B}},\, \#(X^\lambda \cap Q^j) \leq 3N\Big)\\ &\leq C r_{\mathrm{B}}^d \mathbb{P}(\#(X^\lambda \cap Q_{d_0'/\sqrt{d}}) \leq 3N) = C r_{\mathrm{B}}^d \mathbb{E}\Big(\mathrm{e}^{-\lambda\Lambda(Q_{d_0'/\sqrt{d}})} \sum_{k=0}^{3N} \frac{\lambda\Lambda(Q_{d_0'/\sqrt{d}})^k}{k!}\Big),\end{aligned}$$

for a suitably large constant $C > 0$. Clearly, the right-hand side tends to 0 as $\lambda \to \infty$.

The case that ℓ has bounded support can be handled analogously, replacing r_{B} with $\sup \operatorname{supp} \ell$, which is a bound on the length of any edge in $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ in this case. □

4.3.2.2. Proof of Proposition 4.1.16

We fix $d = 2$ and $N_0, \tau, P > 0$, and, as in Section 4.2.1.1, for $\gamma, \lambda > 0$ we write $g_{(\gamma)}(X^\lambda) = g_{(\gamma,N_0,\tau,P)}(X^\lambda)$. Further, we fix $M > \ell(0)$, $\delta > 0$, and $c_0 > 0$ such that $\ell(r) > \tau N_0/P$ for all $r \in [0, \delta]$ and $\mathbb{P}(\Lambda(Q_{\delta/2}) > c_0) = 1$.

The proof is based on [DBT05, Section III-D] in the Poisson case. Let us summarize that proof in a way that is adaptable to the Cox case. The authors of [DBT05] constructed a square lattice with edge length $\delta/2$ with o being situated in the centre of a square. They showed that for any square of this lattice, if the number of Poisson points in the square is more than $N' = \frac{(1+2\tau\gamma)PM}{\tau^2\gamma N_0} > 0$, then all Poisson points in this square are isolated in $g_{(\gamma)}(X^\lambda)$. This also holds if X^λ is replaced by any simple point process. Let us call a square *open* if it has at most $2N'$ Poisson points and *closed* otherwise.

Next, by the independence property of the Poisson point process, any two squares are open or closed independently of each other, and thus the open sets form an independent Bernoulli site percolation process. Now, by elementary properties of the Poisson distribution [DBT05, Lemma 1], this process is subcritical for all λ sufficiently large, in which case the origin is almost surely surrounded by a circuit of closed squares. Then, the proof was concluded by verifying the following statement. If the origin is surrounded by a circuit of closed squares, then for any $X_i \in X^\lambda \cap Q_{\delta/2}$, we have $X_i \not\leftrightsquigarrow \infty$. Indeed, the statement is clear if $Q_{\delta/2}$ is itself a closed square. Else, as it was shown in [DBT05, Theorem III-D], if $X_i, X_j \in X^\lambda$ are situated on two different sides of a circuit of closed squares, then $\mathrm{SINR}(X_i, X_j, X^\lambda) \leq \tau$. This statement is entirely deterministic and remains true after replacing X^λ with any other simple stationary point process. It follows that $\mathbb{E}[\#\{X_i \in X^\lambda \cap Q_{\delta/2}\colon X_i \leftrightsquigarrow \infty \text{ in } g_{(\gamma)}(X^\lambda)\}] = 0$, and thus $\mathbb{P}(g_{(\gamma)}(X^\lambda)$ percolates$) = 0$ by stationarity.

In the b-dependent Cox case, the process of closed sites is b'-dependent for all sufficiently large b'. Our goal is to show that

$$\lim_{\lambda \to \infty} \mathbb{P}(\text{a given square is closed}) = 1. \tag{4.42}$$

Having this, by [LSS97, Theorem 0.0], the process of closed sites is stochastically dominated from below by a supercritical independent Bernoulli percolation process for large enough λ, and thus almost surely there exists an circuit of closed squares surrounding o. This allows us to conclude the proposition analogously to [DBT05, Section III-D].

Now we verify (4.42). For $\mu > 0$, we write $Y(\mu)$ for a Poisson random variable with mean μ. Let $\varepsilon \in (0,1)$. In order to simplify the notation we write $X^\lambda(\cdot) = \#(X^\lambda \cap \cdot)$. By Chebyshev's inequality, we have

$$\begin{aligned}
&\mathbb{P}\Big(\big|X^\lambda(Q_{\delta/2}(x)) - \lambda\Lambda(Q_{\delta/2}(x))]\big| > \varepsilon\lambda\Lambda(Q_{\delta/2}(x))]\Big)\\
=&\mathbb{E}\Big(\mathbb{P}\Big(\big|Y(\lambda\Lambda(Q_{\delta/2}(x))) - \mathbb{E}[Y(\lambda\Lambda(Q_{\delta/2}(x)))]\big| > \varepsilon\mathbb{E}[Y(\lambda\Lambda(Q_{\delta/2}(x)))]\Big|\Lambda\Big)\Big)\\
\leq&\mathbb{E}\Big(\mathbb{E}\Big(\frac{\mathrm{Var}(Y(\lambda\Lambda(Q_{\delta/2})))}{\varepsilon^2\mathbb{E}[Y(\lambda\Lambda(Q_{\delta/2}(x)))]^2}\Big|\Lambda\Big)\Big) = \mathbb{E}\Big(\frac{\lambda\Lambda(Q_{\delta/2}(x))}{\varepsilon^2\lambda^2\Lambda(Q_{\delta/2}(x))^2}\Big) = \frac{1}{\varepsilon^2\lambda}\mathbb{E}\Big(\frac{1}{\Lambda(Q_{\delta/2}(x))}\Big).
\end{aligned} \tag{4.43}$$

Under the assumption that $\Lambda(Q_{\delta/2}) > c_0$ almost surely, the right-hand side is finite for all $\lambda > 0$ and tends to 0 as $\lambda \to \infty$. Now, similarly to [DBT05, Section III-D], if λ satisfies

$$\lambda \geq \frac{2N'}{(1-\varepsilon)c_0}, \tag{4.44}$$

where, almost surely, the right-hand side is more than $\frac{N'}{(1-\varepsilon)\Lambda(Q_{\delta/2}(x))}$ for all $x \in \mathbb{R}^d$, then

$$\begin{aligned}
&\mathbb{P}\big(X^\lambda(Q_{\delta/2}(x)) \leq N'\big) \leq \mathbb{P}\big(X^\lambda(Q_{\delta/2}(x)) \leq (1-\varepsilon)c_0\lambda\big)\\
&\quad\leq \mathbb{P}\big(X^\lambda(Q_{\delta/2}) \leq (1-\varepsilon)\lambda\Lambda(Q_{\delta/2}(x))\big) \leq \mathbb{P}\Big(\big|X^\lambda(Q_{\delta/2}(x)) - \lambda\Lambda(Q_{\delta/2}(x))\big| > \varepsilon\lambda\Lambda(Q_{\delta/2}(x)\Big),
\end{aligned} \tag{4.45}$$

and thus by (4.43), (4.42) holds. By (4.44), $\gamma > \gamma^*(\lambda)$ holds once

$$\frac{2(1+2\tau\gamma)PM}{\tau^2\gamma N_0} \leq (1-\varepsilon)\lambda c_0,$$

or equivalently,

$$\gamma \geq \frac{2PM}{(1-\varepsilon)\tau^2 N_0\lambda c_0 - 4\tau PM}.$$

This holds if

$$\gamma \geq \frac{PM}{(1-2\varepsilon)\tau^2 N_0 \lambda c_0}, \tag{4.46}$$

for λ sufficiently large, namely for $\lambda \geq \frac{4PM}{\varepsilon\tau N_0 c_0}$. Clearly, the lower bound on the right-hand side of (4.46) is in $\mathcal{O}(1/\lambda)$. Now, for $\lambda \geq \frac{4PM}{\varepsilon\tau N_0 c_0}$ so large that the process of closed sites is stochastically dominated from below by a supercritical independent Bernoulli percolation process, the origin is almost surely surrounded by a circuit of closed squares. We conclude the proposition. □

4.3.3. Sketch of proof of Corollary 4.1.17

Since the assertion of the corollary is a lower bound on $\gamma^*(\lambda)$ for large λ, it suffices to verify it for $N_0 > 0$ (cf. Section 4.1.3.3). We fix $d = 2$, $\tau,N_0,P > 0$, $M > \ell(0)$, and $\delta > 0$ such that $\ell(\delta) > \tau N_0/P$. Further, we assume that $d_1 \in [\sup\operatorname{supp}\ell,\infty)$ such that $\Lambda(Q_{d_1})$ is bounded away from 0 (such a d_1 exists by the assumption of Corollary 4.1.17); we will make stronger assumptions on d_1 later during the proof.

In the following, we summarize the proof of the assertion $\gamma^*(\lambda) = \Omega(1/\lambda)$ from [DBT05, Section III-C] in the Poisson case and afterwards we indicate how it can be extended to the setting of Corollary 4.1.17. For $\lambda > 0$, one maps the SINR graph $g_{(N_0,\gamma,\tau,P)}(X^\lambda)$ to a square lattice $\mathcal{L}$ with edge length d_1. The dual lattice of $\mathcal{L}$, i.e., $\mathcal{L}$ shifted by the vector $(d_1/2,d_1/2)$, is denoted by $\mathcal{L}'$. Note that there is a one-to-one correspondence between the edges of $\mathcal{L}$ and the ones of $\mathcal{L}'$ by mapping an edge e of $\mathcal{L}$ to the unique edge of $\mathcal{L}'$ which crosses e. In $\mathcal{L}$, one divides each square into K^2 subsquares of size $\frac{d_1}{K} \times \frac{d_1}{K}$, where $K \in \mathbb{N}$ is defined as $K = \lceil\frac{\sqrt{5}d_1}{\delta}\rceil$. Further, for $\lambda,\gamma > 0$, one puts

$$N = \inf_{x\colon |x| \leq \sqrt{5}d_1/K} \Big\lfloor \frac{1}{\gamma M}\Big(\frac{\ell(|x|)}{\tau} - \frac{N_0}{P}\Big)\Big\rfloor = \Big\lfloor \frac{1}{\gamma M}\Big(\frac{\ell(\sqrt{5}d_1/K)}{\tau} - \frac{N_0}{P}\Big)\Big\rfloor.$$

One says that a square of $\mathcal{L}$ is *populated* if each of its subsquares contain at least one point of X^λ. Further, an edge a of $\mathcal{L}$ is *open* if both squares adjacent to a are populated and the total number of points of X^λ in all squares of $\mathcal{L}$ having at least one vertex in common with the neighbouring squares of a is at most $N+1$. An edge a' of $\mathcal{L}'$ is *open* if and only if the corresponding edge of $\mathcal{L}$ is open. The proof proceeds by the following lemma [DBT05, Lemma 2].

Lemma 4.3.1 ([DBT05])**.** *Let p denote the probability that an arbitrary edge in $\mathcal{L}'$ is closed, and let us write $q=1-p$. Then for any $q'>0$, there exists $\lambda' \in (0,\infty)$ such that for all $\lambda > \lambda'$ there exists $\gamma'(\lambda)>0$ such that for all $\gamma \in (0,\gamma'(\lambda)]$, $q<q'$. Further, $\lambda \mapsto \gamma'(\lambda)$ can be chosen such a way that $\gamma'(\lambda) = \Omega(1/\lambda)$ as $\lambda \to \infty$.*

The process of open edges in $\mathcal{L}'$ is 3-dependent thanks to the independence property of the Poisson point process. Using dependent percolation theory [LSS97], one concludes that for all sufficiently large $\lambda>0$, there exists $\gamma'(\lambda)>0$ such that for all $\gamma \in (0,\gamma'(\lambda))$, the process of open edges percolates with probability 1, and such that $\gamma'(\lambda) = \Omega(1/\lambda)$. This implies percolation in the SINR graph thanks to [DBT05, Lemmas 4, 5]. These lemmas are similar to Step 3 of the proof of Theorem 4.1.5(2), they use that ℓ has bounded support and $d_1 \geq \sup \operatorname{supp} \ell$, but they are easily seen to hold for any simple point process rather than only for the Poisson one.

Now, if X^λ is a Cox point process with intensity $\lambda\Lambda$ where Λ is b-dependent, then the process of open edges in $\mathcal{L}'$ is still b'-dependent for all sufficiently large b'. Thus, in order to conclude Corollary 4.1.17, it suffices to verify Lemma 4.3.1 under the assumptions of the corollary, for d_1 sufficiently large.

Proof of Lemma 4.3.1 under the assumptions of Corollary 4.1.17. For $p=1-q$, we estimate

$$\begin{aligned} p &= \mathbb{P}(\text{a given edge of } \mathcal{L}' \text{ is open}) \\ &\geq \mathbb{P}\big(2K^2 \text{ subsquares of area } d_1^2/K^2 \text{ have at least 1 point of } X^\lambda \text{ each, and} \\ &\qquad \text{an area of } 12d_1^2 \text{ including these subsquares has at most } N \text{ points}\big) \\ &\geq \mathbb{P}\big(12K^2 \text{ subsquares of area } d_1^2/K^2 \text{ have between 1 and } \lfloor N/(12K^2)\rfloor \text{ points of } X^\lambda \text{ each}\big) \\ &= 1-\mathbb{P}\big(\text{at least 1 of } 12K^2 \text{ subsquares of area } d_1^2/K^2 \text{ has 0 or more than } \lfloor N/(12K^2)\rfloor \text{ points}\big) \\ &\geq 1-12K^2\mathbb{P}\big(\text{a given subsquare of area } d_1^2/K^2 \text{ has 0 or more than } \lfloor N/(12K^2)\rfloor \text{ points}\big). \end{aligned} \tag{4.47}$$

Let us fix $\varepsilon>0$ and define

$$\gamma'(\lambda) = \frac{1}{12M\lambda d_1^2(1+\varepsilon)}\Big(\frac{P\ell(\sqrt{5}d_1/K)}{\tau} - N_0\Big). \tag{4.48}$$

Then we have $N = \lfloor 12\lambda d_1^2(1+\varepsilon)\rfloor$. Using this, (4.47), and the stationarity of X^λ, it suffices to show that for all sufficiently large d_1,

$$\mathbb{P}\Big(1 \leq \#\big(X^\lambda \cap Q_{d_1/K}\big) \leq \lambda(1+\varepsilon)d_1^2/K^2\Big) \tag{4.49}$$

tends to one as $\lambda \to \infty$. Indeed, then, using that $\gamma'(\lambda)$ defined in (4.48) is $\Omega(1/\lambda)$, further that the set of edges of $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ is stochastically monotone decreasing in γ, we can conclude the lemma. But for d_1 so large that $\Lambda(Q_{d_1/K})$ is also bounded away from zero, the convergence of (4.49) to zero can be verified using an estimate analogous to (4.43). □

5. Concluding discussions

In the present, last chapter of the body of this thesis, we relate the subject of our Gibbsian model for message routeing and the one of SINR percolation for Cox point processes to each other and to other models about interference limited or multihop networks. This takes the form of an extended discussion, where we do not aim to work out new mathematical methods but rather to make connections between already existing models and proof techniques, and to discuss open questions. Nevertheless, we will provide the main arguments of proofs of several assertions that do not trivially follow from earlier results but can be verified in a relatively straightforward way. The subjects that we cover in this chapter are the following.

First, we have explained the notion of SINR in Section 1.1.2, and in Section 2.3.1.1 we argued how it is related to the definition of SIR that we use in our Gibbsian model. In Section 5.1 we give a more thorough discussion about the interference related quantities, including historical remarks about the notion of SINR and the mathematical research on interference limited networks, and a discussion about the role of the path-loss function in different limiting settings. Second, it turns out that both of our main subjects are related to the results of [HJKP18] about SIR-related frustration events in highly dense multihop networks. Section 5.2 is devoted to the investigation of these relations. Finally, in Section 5.3 we discuss the most prominent open questions and conjectures related to the subject of this thesis.

5.1. Interference limited networks: historical remarks, effects of the choice of the path-loss function

Now, in Section 5.1.1 we provide some historical remarks on the notion of SI(N)R and the mathematical research about interference limited networks. In Section 5.1.2 we explain the role of the properties of the path-loss function, the one of a factor of order $1/\lambda$ in front of the interferences in different settings, and the relation between the high-density limit and the thermodynamic limit for different choices of the path-loss function.

5.1.1. Historical remarks about interference limited networks

It is common to consider the seminal paper [GK00] by Gupta and Kumar on the information-theoretic capacity of wireless networks as the origin of the mathematical research on interference limited networks. For example, [DBT05], the first paper on SINR percolation, names [GK00] as its source for the notion of SINR and the use of an SINR threshold. [GK00] had a pioneering rule in at least two respects. First, it initiated the research about the capacity of various kinds of wireless networks (see e.g. the references in [FDTT07]), which is a highly interesting subject for the engineering community. Second, it drew the attention of mathematicians, in particular probabilists, to the subject of interference limited (often multihop) networks. Providing a full overview of the mathematical research on the numerous aspects of this subject is not easy; the monographs [FM07, BB09, BB09b, H10] cover many of the investigations until 2010. Let us complement these with listing only those questions that have been studied in some of the papers or textbooks that we cite in this thesis: information-theoretic capacity [GK00, FDTT07]; coverage [BB01, BB09]; connectivity, in particular SINR percolation [DBT05, DFMMT06, FM07, KY07, BY10, V12, BY13, VI14, T18]; optimal paths in a space-time SINR graph [BBM11]; Markov chain Monte Carlo for power control and user association [BC12]; relations of the SINR process to the two-parameter Poisson–Dirichlet process [KB14]; large deviations of the interference [GT08, TL14] and of the quality of service experienced by users in the thermodynamic limit [HJKP16] or in the high-density limit [HJKP18, T16]; and our Gibbsian model for message routeing [KT18, KT19], which was investigated from the viewpoint of Markov chain Monte Carlo in [M18]. We recall that we commented on the works related more closely to the Gibbsian model in Section 2.7 and on the preliminaries of our investigations on SINR percolation in Section 1.1.2.

It is nevertheless certainly true that the notion of SI(N)R had already widely used in the engineering literature before [GK00] was published, as also the references in [GK00] show; note also that although this paper considered the notion (4.1) of SINR only with $\gamma = 1$, models with $\gamma < 1$ had also been studied earlier, see e.g. the references in [BB01, Section 2.4]. Section 5.1.2 below provides further remarks about [GK00, FDTT07].

5.1.2. Effects of the choice of the path-loss function

Now, starting from the observations of [GK00, FDTT07] about information-theoretic capacity, we explain the effect of boundedness of the path-loss function at zero on our models about message routeing and percolation, providing a comparison between the high-density limit with a bounded and the one with an unbounded path-loss function. Further, we investigate the effects of other properties of the path-loss function (integrable tails, strictly monotone decrease, sign of derivative at zero, convexity of its reciprocal) in the settings that we have considered in this thesis.

Power law decay with explosion at zero. [GK00] used the path-loss function $\ell(r) = r^{-\alpha}$, $\alpha > d = 2$ corresponding to Hertzian propagation, which has perfect scaling properties, making estimations of SI(N)R values particularly straightforward. The perfect scaling of this function ℓ was also used in [KB14] for establishing a connection between the SINR in wireless networks and the two-parameter Poisson–Dirichlet process. However, as already mentioned in Section 2.3.1.1, this path-loss function is not realistic for small distances.

Extended networks and dense networks. The paper [FDTT07] improved the results of [GK00] on the information-theoretic capacity of wireless networks, establishing a connection between the high-density limit (the case of *dense networks*) with an unbounded path-loss function that has an integrable tail at infinity and the thermodynamic limit (the case of *extended networks*) with the same path-loss function cut at the value 1. In the case of extended respectively dense networks, n users X_i, $i \in [n]$, are placed in a square of side length $\sqrt{n}$ respectively $1/\sqrt{n}$ in $\mathbb{R}^2$ uniformly and independently at random. In both models, [FDTT07] answered the following question. Given that each user chooses a destination among the users uniformly at random, what is the maximal speed $T(n)$ (measured in bits per second) at which, with high probability as $n \to \infty$, each node can simultaneously transmit towards its intended destination, using other users as relays? Here, a direct transmission from a user X_i to another user X_j is possible at rate $\log(1 + \mathrm{SINR}(X_i, X_j, \{X_k\}_{k \in [n]}))$, where the SINR is defined with some fixed $N_0 \geq 0$. (Modulo a constant factor, this rate is the *bandwidth* used for the transmission from X_i to X_j, cf. Section 2.3.1.2.) In other words there is no strict SINR threshold in this model, but low SINR values lead to low transmission rates. Note that these models formally do not involve a Poisson point process, but they model the expected behaviour of such a point process in the corresponding limit.

[FDTT07, Theorem 1] stated that for extended networks, one has $T(n) = \Theta(1/\sqrt{n})$ in case of a path-loss function of the form $\ell(r) = \min\{\mathrm{e}^{-\gamma r}/r^{\alpha}, 1\}$, where $\gamma > 0$ and $\alpha \geq 0$, or $\gamma = 0$ and $\alpha > 2$. Further, for the same choices of α and γ, [FDTT07, Theorem 2] claimed that the bound $T(n) = \Theta(1/\sqrt{n})$ also applies for dense networks in the case of $\ell(r) = \mathrm{e}^{-\gamma r} r^{-\alpha}$, i.e., for the same type of path-loss functions as in the result for extended networks but without cutting them at value one. Let us note that [GK00] verified that $T'(n) = \Omega(1/\sqrt{n \log n})$ and $T'(n) = \mathcal{O}(1/\sqrt{n})$, in case of dense networks, for the path-loss function $\ell(r) = r^{-\alpha}$, $\alpha > 2$, where $T'(n)$ is the amount of bits per second that can be transmitted on an SINR level exceeding a given threshold $\tau > 0$. In fact, in order to separate edge effects from other phenomena, [GK00] considered the case of users being situated uniformly at random on a sphere of unit area instead of in a unit disk or box. The notions of $T(n)$ and $T'(n)$ are not substantially different, and the main novelty in [FDTT07] consisted in closing the gap of order $\sqrt{\log n}$ between the upper and the lower bound in [GK00].

What follows from the results of these two papers is that with respect to information-theoretic capacity, the high-density limit with an unbounded path-loss function behaves like the rescaling of a thermodynamic limit. Informally speaking, there is a clever organization of the routeing of the messages of the n users such that the average length of a hop is of order $1/\sqrt{n}$, which is the order of nearest neighbour distances in $(X_i)_{i \in [n]}$, and the hops typically have SINR values of nonvanishing order, with high probability as $n \to \infty$. Approximately, the transmission of one message per transmitter to the corresponding receiver takes in total of order $n^{3/2}$ hops with SINR of order 1 each in the system, i.e., the sum of the inverses of all these SINR values is of order $n^{3/2}$.

Effects of the boundedness of the path-loss function in the high-density limit. The dense networks setting of [FDTT07] is rather different from the high-density limit of our Gibbsian model introduced in Section 2.1.4 with a bounded path-loss function, where we needed to rescale the interferences by a factor of constant times $1/\lambda \asymp 1/N(\lambda)$ in order to maintain SIR levels of nonvanishing order. In our setting, by continuity, the path-loss function is close to constant for small distances instead of tending to infinity. Thus without the factor $1/\lambda$ in the denominator of (2.12), the SIR of any hop in the system is $\mathcal{O}(1/\lambda)$ with high probability, even if the transmitter and the receiver of the hop are nearest neighbours in X^λ. Hence, for $k \in \mathbb{N}$, the sum of inverse SIR values of the k-hop trajectory that has the least interference penalization among all k-hop trajectories would be $\Omega(k\lambda)$ without this factor. Hence, if one wants to make the sum of the inverse SIR values of all hops of all trajectories of order $\mathcal{O}(\lambda)$, one needs to multiply the interferences by $1/\lambda$. We provided an interpretation of this rescaling in Section 2.3.1.1. Thanks to this normalization, the interference term becomes of the same order as the congestion term and (thanks to the *a priori* normalization, cf. Proposition 2.9.1) the entropy term. After this rescaling, short hops are less strongly preferred by interference penalization than before, and thus a finite maximum number of hops $k_{\max}$ turns out to be sufficient in order to keep the interference penalties small.

Summarizing, in a high-density limit, in order to maintain nonvanishing SIR levels for messages transmitted in a multihop fashion through nonvanishing distances, one either needs an unbounded path-loss function and a diverging number of hops of vanishing lengths or a bounded path-loss function and a rescaling of the interferences by $1/\lambda$ (and in the latter case, a bounded number of hops suffices). One alternative way of rescaling would be to increase the time horizon of the transmission of messages to an order of λ time slots, cf. Section 2.4.4.

We mentioned in Chapter 1 that the high-density limit considered in [HJKP18] or in our Gibbsian model is also called a mean-field or hydrodynamic limit. Indeed, in such a setting, if we consider the interference at a given receiver, the contribution of each transmitter to it is $1/\lambda$ times some random number that is bounded and bounded away from zero, i.e., of the same order for all transmitters. However, in dense networks in terms of [FDTT07], this is not true. Instead, essentially the entire interference comes from a neighbourhood of the receiver with vanishing diameter. Further, even in mean-field type high-density limits, the assumptions that one needs to make on the path-loss function depend on the questions that one investigates. For example, in Section 5.2.4 we will explain the reason of the differences between the assumptions on ℓ in the setting of [HJKP18] (which will be recalled in Section 5.2.1) and the ones in the setting of our Chapters 2 and 3.

SINR percolation with an unbounded path-loss function. Let X^λ be a Poisson point process with intensity $\lambda\mathrm{Leb}$ in $\mathbb{R}^d$, and let us assume that the path-loss function ℓ explodes at zero but it is monotone decreasing and continuous as a function $\ell\colon (0,\infty) \to [0,\infty)$, and it has integrable tails (i.e., $\int_K^\infty \ell(r) r^{d-1} \mathrm{d}r < \infty$ for some $K > 0$), e.g., $\ell(r) = r^{-\alpha}$, $\alpha > d$. Here, the interference at a given point of $\mathbb{R}^d$ is almost surely finite but it has infinite expectation [D71].[27] The results of [GK00, FDTT07] indicate that in this setting, for any $N_0,\tau,P > 0$ fixed, the SINR graph $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ should percolate for λ sufficiently large and γ sufficiently small according to λ. Indeed, roughly speaking, these two papers showed that under a suitable organization of the extended network, trajectories with hops of nearest neighbour order lengths are able to maintain nonvanishing SINR values. Further, as we have seen in Section 4.1.4.1, any SINR graph with $\gamma > 0$ is contained in the bidirectional k-nearest neighbour graph $g_{\mathbf{B}}(k,X^\lambda)$ for suitable $k \in \mathbb{N}$.

As for $d = 2$, [DBT05, Section III-D] presented simulation results suggesting that for $\ell(r) = r^{-\alpha}$, $\alpha > 2$, $\gamma^*(\lambda) > 0$ holds for all sufficiently large $\lambda > 0$, further, $\lambda \mapsto \gamma^*(\lambda)$ is monotone increasing (but still bounded by $1/\tau$). The argument for the latter property is that multiplying λ by a factor $a > 0$ is equivalent to dividing N_0 by $a^{\alpha/2}$, which certainly holds in the general Cox case as well. This is a significant difference from the case of a path-loss function satisfying Assumption (ℓ) where $\gamma^*(\lambda) = \mathcal{O}(1/\lambda)$ as $\lambda \to \infty$ in the Poisson case, according to [DBT05, Theorem 4], which also holds for some b-dependent Cox processes (cf. Section 4.1.4.2).

[27] It is easy to conclude that the same holds in the stationary Cox case considered in Chapter 4, using a splitting of the interference at a given receiver to the contribution of the Cox points in a small neighbourhood of the receiver plus the one of the points outside this neighbourhood, and letting the diameter of the neighbourhood vanish.

The main technical difficulty in verifying the existence of an infinite cluster is that an interference control in the spirit of Proposition 4.2.5 is not possible for unbounded path-loss functions, and thus new proof techniques are necessary, see Section 5.3.2.1 for further details and related open questions (also in the general Cox case).

In contrast, if $\int_K^\infty r^{d-1}\ell(r)\mathrm{d}r = \infty$ for all $K > 0$, then interferences are not almost surely finite in the Poisson case. Thus, for example for $\ell(r) = r^{-\alpha}$, $\alpha \in (0,d]$, the SINR graph has no edges if $\gamma > 0$. We expect that the same holds for all relevant examples in the Cox case.

The critical interference cancellation factor in SINR graphs with high user density. In the setting of SINR percolation for Cox processes, the point process X^λ of users is defined on $\mathbb{R}^d$ and has intensity $\lambda\Lambda$. It is clear from Campbell's theorem [K93, Section 3.2] that the expected total received power (cf. Section 2.3.1.1) increases linearly in λ. Thus, the upper and lower bounds (B) and (C) (see the beginning of Section 4.1.4) on $\gamma^*(\lambda)$ of order $1/\lambda$ as $\lambda \to \infty$ do not come as a surprise. We have verified these bounds for two-dimensional b-dependent Cox processes with intensities that are locally bounded away from 0 in a suitable way. Without this boundedness away from 0, the supercritical phase of the SNR graph may be absent, and thus the lower bound may fail, which is true for the shot-noise field and the Poisson point process modulated by a Poisson–Boolean model (or another boundedly determined random closed set) with $\min\{\lambda_1,\lambda_2\} = 0$ in some cases (see Sections 4.1.1 and 4.1.5.1). We will discuss related open problems in a more general context in Section 5.3.2.2.

Requirements on the path-loss function in the limiting regimes (1)**,** (2)**, and** (3) **of Chapter 3.** The integrability condition $\int_{\mathbb{R}^d} \ell(|x|)\mathrm{d}x < \infty$ (or equivalently, $\int_0^\infty r^{d-1}\ell(r)\mathrm{d}r < \infty$), also appeared in Chapter 3, in regime (1) (large communication areas with large transmitter–receiver distances). It guaranteed that in our large-space limit (cf. Section 3.2.2), integrals of the form $\int_{B_r(o)} \ell(|z - x_0|)\mathrm{d}z$ can be well approximated by $b = \int_{\mathbb{R}^d} \ell(|z|)\mathrm{d}z$ for all $x_0 \in W = \overline{B_r(o)}$ sufficiently far away from the boundary of W, and for x_0 close to ∂W they are not smaller than $b_0 - o(1)$, where $b_0 = \sup_{r' \in (0,\infty)} \int_{B_{r'}(o)} \ell(|y - r'e_1|)\,\mathrm{d}y \in (0,b)$ is another positive constant. Thus, for $x,y \in B_r(o)$, in our limit, $g(x,y)$ (cf. (3.3)) behaves similarly to constant times $1/\ell(|x - y|)$. Now, the convexity of $1/\ell(\cdot)$ and applications of Jensen's inequality indicate that the optimal trajectory from $x_0 \in \overline{B_r(o)}$ with $|x_0| = r_0$ to o takes $k^*(r_0)$ equal hops (cf. Corollary 3.2.2) along the straight line between x_0 and o, and macroscopic deviations from this trajectory become exponentially unlikely in our limit (cf. Proposition 3.2.3). The technical but elementary proofs in Section 3.2.2 allow us to conclude that this indication is correct.

We will discuss the role of convexity of $1/\ell(|\cdot|)$ in Section 5.3.1.3. Without the integrability of $\ell(|\cdot|)$, integrals of the form $\int_{B_r(o)} \ell(|z - x_0|)\mathrm{d}z$ for $x_0 \in W$ tend to infinity in our limit. Hence, the arguments of the proof of Theorem 3.2.1 break down in this case, and we decided not to consider this case in the present thesis.

In regime (2) (strong penalization of interference), we are working with a fixed communication area and a fixed maximal number of hops. Here, our assumption on ℓ was being strictly monotone decreasing. In this case simple geometric arguments (in the proof of Proposition 3.3.1(A)) imply that any optimal trajectory of any user with respect to interference penalization has each of its relays on the straight line segment between transmitter and receiver, further, each hop strictly decreases the distance from the base station. Further, an upgraded version of the proof of this assertion together with a standard application of the Laplace method yields Proposition 3.3.1(B), i.e., the exponentially fast decay of the probability of substantial deviations from the straight line in the limit $\gamma \to \infty$. We expect that there exist counterexamples for both parts of this proposition in case ℓ is not strictly monotone decreasing; we will discuss this in Section 5.3.1.3.

As for the global effects in regime (3) (high local density of users), the equivalent condition (3.49) for the exponential decay of the number of incoming hops per relay is easy to understand: the exponential decay holds if and only if no user has a two-hop path towards o that is better with respect to effective interference penalization than the direct hop to o. It is less clear what this condition tells about ℓ. At least, the sufficient condition (v) is rather intuitive: it implies that (3.49) holds if $(x,y) \mapsto \ell(|x - y|)$ is close to constant on $W \times W$. Note that the sufficient condition (v) has no reference to the choice of the highly densely populated subset Δ apart from the requirement that $\mathrm{Leb}(\Delta) > 0$, and also the exponential decay extends to the entire communication area W. As for the local effects, we chose $k_{\max} = 2$ and showed that if Δ has a small diameter and is situated sufficiently far away from o, then $\ell'(0) < 0$ implies that in the limit $a \to \infty$, for any transmitter, it is less favourable to choose a relay in a small neighbourhood of Δ than one in the complement of a larger neighbourhood of Δ. Our proof does not work if $\ell'(0) = 0$, but it also does not clearly indicate that the assertion is false in that case.

5.2. Connections to frustration events in the high-density limit

This section is devoted to the analysis of relations of our Gibbsian model and the ones of SINR percolation to frustration events in the high-density limit. We start with recalling the most important results of [HJKP18] and some additional observations of the author's master's thesis [T16] in Section 5.2.1. Following this, in Section 5.2.2 we argue that for some choices of the parameters, one can show that the probability of the frustration event considered in [HJKP18] tends to one, using only the degree bounds in SINR graphs (cf. Section 4.1.4), without any reference to the large deviation principles established in [HJKP18]. Afterwards, in Section 5.2.3 we use the results of [HJKP18] in order to provide sufficient conditions for the well-definedness of the version of our Gibbsian model with a "hard" interference penalization. This version was sketched in Section 2.5.1, it is the one where trajectory configurations with very low SIR levels are forbidden. Further, in Section 5.2.4 we comment on the discretization procedure introduced in Section 2.8.1, which originates from [HJKP18], but we used an altered version of it for different purposes and under more general modelling assumptions.

5.2.1. Frustration events in highly dense interference limited networks: main results of [HJKP18] and some observations of [T16]

The paper [HJKP18] describes the exponential behaviour of the probabilities of certain *frustration events* in the high-density limit. It investigates a network where the locations of the users are given very similarly to Section 2.1.1, i.e., according to a Poisson point process with diverging intensity parameter in a compact subset W of $\mathbb{R}^d$, with the additional feature of *mobility*. That is, users move along uniformly Lipschitz continuous paths in W, and a bounded time horizon is considered. The paper investigates the quality of service in terms of SIR-related quantities for communication between the base station o (which is the origin of $\mathbb{R}^d$) and the users, both in the uplink and in the downlink direction, using a direct hop or a two-hop path relaying through another user. Informally speaking, the main objects of study of the paper are frustration events of the form "more than $b\lambda$ users are strictly under quality of service level c for an amount of time longer than a with respect to a given means of communication" in the high-density limit $\lambda \to \infty$, for $a,b,c \geq 0$. Let us mention that it is clear from the proofs of [HJKP18] that the results should extend to the case of at most $k_{\max} \in \mathbb{N}$ hops in an obvious way. The formulation of the assertions becomes more involved for $k_{\max} > 2$, but we expect that this is the only reason why also the proofs get more nasty, and there is no additional technical difficulty to overcome.

Unlike in the quenched setting of Chapters 2 and 3, in [HJKP18] the only source of randomness is the one of the Poisson point process of users. The large deviations behaviour of this process determines the one of the frustration probabilities, and the main difficulty lies in the discontinuity of the dependence between the empirical measure of users and the one of frustrated users, because of which the contraction principle [DZ98, Theorem 4.2.1] cannot be applied directly. The author's master's thesis [T16] extended the main results of [HJKP18] to the setting where users have random *fadings*, interpreted as loudnesses or random signal powers (cf. Section 4.2.3.4) of the users. In order to be able to focus better on the effects of random fadings, [T16] did not include mobility of the users, but it is clear from the proofs of [HJKP18, T16] that analogues of the main assertions of both works hold in case both mobility and random fadings are included in the model.

We now recall the main results of [HJKP18] and an important observation of [T16] in a way that is most suitable for the purposes of the present chapter. That is, we omit mobility and random fadings from the model, and we focus on [HJKP18, Corollaries 1.2,1.3], the main results of that paper that have an interpretation in terms of SIR-related quantities, and we omit the presentation of the main technical result [HJKP18, Theorem 1.1] on which the proof of these two corollaries is based[28]. The main advantage of this decision is that the corollaries require much less preliminary notation than the theorem, in particular they do not need the formal

[28] In fact, the observations of Section 5.2.2 about the effect of degree bounds of SINR graphs on the model of [HJKP18] are easily seen to have analogues in the setting with mobility, but we will not spell these out, in order to keep the notation simple and similar to the one of Chapter 2.

definition of discretization. We will stick to the notation of [HJKP18] without mobility (instead of the one of [T16]). The following summary of the results of [HJKP18] is a significantly shortened version of the overview in [T16, Section 2.3], extended by [T16, Corollary 3.21].

We now turn to the mathematical setting of [HJKP18]. The users are given by a Poisson point process X^λ on $W = [-r,r]^d \subseteq \mathbb{R}^d$ for some integers $d,r \geq 1$ with an absolutely continuous intensity measure $\lambda\mu(\cdot)$, where $\lambda > 0$. Analogously to (2.1), we introduce the empirical measure

$$L_\lambda = \frac{1}{\lambda} \sum_{X_i \in X^\lambda} \delta_{X_i} \in \mathcal{M}(W).$$

We choose a path-loss function $\ell : [0,\infty) \to (0,\infty)$, assumed to be Lipschitz continuous with parameter $J_2 > 0$. Note that it is not required that ℓ be monotone decreasing. The SIR of a message transmitted by $x \in W$ and measured at $y \in W$ is defined as

$$\mathrm{SIR}(x,\, y,\, L_\lambda) = \frac{\ell(|x-y|)}{\frac{1}{\lambda}\sum_{X_i \in X^\lambda} \ell(|X_i - y|)}, \tag{5.1}$$

similarly to (2.12). Again, the denominator of (5.1) is called the *interference* at y, where the sum is scaled by $1/\lambda$, it contains also the signal strength coming from the corresponding transmitter (and even the one coming from the receiver if the receiver is contained in X^λ), and the model does not include noise, cf. Section 2.3.1.1.

[HJKP18] conducts a large deviations analysis of certain frustration events on the level of empirical measures. In particular, it shows that the most likely way for a rare event to occur is given by a certain finite Borel measure $\nu \in \mathcal{M}(W)$ that describes the most likely asymptotic configuration of users given the rare event. Aiming to formulate this, one extends the definition of SIR to arbitrary nonzero measure $\nu \in \mathcal{M}(W)$ and, consistently with (5.1), one defines

$$\mathrm{SIR}(x,\, y,\, \nu) = \frac{\ell(|x-y|)}{\int_W \ell(z-y|)\nu(\mathrm{d}z)}$$

for any $x,y \in W$. In order to keep the model flexible, it is assumed that the *quality of service* (QoS) of the direct link between x and y is given by $D(x,\, y,\, L_\lambda) = g(\mathrm{SIR}(x,\, y,\, L_\lambda))$, where $g\colon [0,\infty) \to [0,\infty)$ is a Lipschitz continuous function that is strictly monotone increasing on $[0,\varrho_+)$ and constant equal to c_+ on $[\varrho_+,\infty)$ for some $\varrho_+, c_+ > 0$. As SIR, also D is defined for general $\nu \in \mathcal{M}(W)$ via $D(x,\, y,\, \nu) = g(\mathrm{SIR}(x,\, y,\, \nu))$ and $D(x,\, y,\, \nu) = c_+$ if $\nu(W) = 0$. For example, possible choices of g include $g(r) = \min\{r, K\}$ or $g(r) = \min\{\log(1+r), K\}$ for some fixed $K > 0$.

If a user x sends a message to a user y routeing via a relay z, then the quality of the relayed transmission depends both on the SIR from x to z and the SIR from z to y. It is assumed that message transmissions are successful if the SIR values of both links, transformed by g, are above a certain threshold. In other words the assumption is that when relaying from x to y via z, the QoS can be expressed as

$$\Gamma(x,\, z,\, y,\, L_\lambda) = \min\left\{D(x,\, z,\, L_\lambda), D(z,\, y,\, L_\lambda)\right\},$$

and we define $\Gamma(x,\, z,\, y,\, \nu)$ analogously for a general $\nu \in \mathcal{M}(W)$.

In the uplink scenario, the user $X_i \in X^\lambda$ sends a message via a relay $X_j \in X^\lambda$ to the unique base station situated at the origin o of $\mathbb{R}^d$. Under an optimum relay decision, the QoS for the relayed uplink communication is defined as

$$R(X_i,\, o,\, L_\lambda) = \max\{D(X_i,\, o,\, L_\lambda),\, \max_{X_j \in X^\lambda} \Gamma(X_i,\, X_j,\, o,\, L_\lambda)\}. \tag{5.2}$$

In other words, in (5.2), the user has the possibility to connect to the base station directly, but if there exists any user such that relaying via this user offers a better connection, then the QoS is equal to the one of the best two-hop path. Similarly, in the downlink scenario, where messages are sent out from o to a user $X_i \in X^\lambda$ and relaying is again possible, the QoS for the relayed downlink communication can be expressed as

$$R(o,\, X_i,\, L_\lambda) = \max\{D(o,\, X_i,\, L_\lambda),\, \max_{X_j \in X^\lambda} \Gamma(o,\, X_j,\, X_i,\, L_\lambda)\}.$$

Extending the definition of R to arbitrary finite Borel measures $\nu \in \mathcal{M}(W)$, we write

$$R(x,\, y,\, \nu) = \max\{D(x,\, y,\, \nu),\, \nu\text{-esssup}_{z\in W}\Gamma(x,\, z,\, y,\, \nu)\}$$

for any given $x,\, y \in W$. Here ν-esssup means essential supremum with respect to ν.

The object of interest of the paper is the point process of users $X_i \in X^\lambda$ who are *frustrated* due to too low SIR. In order to describe the number of frustrated users, we define the following rescaled measure for the uplink

$$L_\lambda^{\mathrm{up}}[\tau_c] = \frac{1}{\lambda} \sum_{X_j \in X^\lambda} \delta_{X_j} \tau_c(R(X_j,\, o,\, L_\lambda)),$$

where for $c \geq 0$, $\tau_c : [0,\infty) \mapsto [0,\infty)$ is the bounded and measurable function defined as

$$\gamma \mapsto \mathbb{1}\{\gamma < c\}. \tag{5.3}$$

In particular, $L_\lambda^{\mathrm{up}}[\tau_c] \in \mathcal{M}(W)$. More generally, if $\nu \in \mathcal{M}(W)$, then $\nu^{\mathrm{up}}[\tau_c] \in \mathcal{M}(W)$ is defined via

$$\frac{\mathrm{d}\nu^{\mathrm{up}}[\tau_c]}{\mathrm{d}\nu}(x) = \tau_c(R(x,\, o,\, \nu)).$$

It is also important to consider those users who have a bad QoS for direct uplink communication separately, because if a large number of users has to communicate via a small number of relays, then communication on full bandwidth cannot be guaranteed due to congestion effects. For a general $\nu \in \mathcal{M}(W)$, the empirical measure $\nu^{\mathrm{up\text{-}dir}}[\tau_c]$ of users who have bad QoS with respect to direct communication with the base station is defined via

$$\frac{\mathrm{d}\nu^{\mathrm{up\text{-}dir}}[\tau_c]}{\mathrm{d}\nu}(x) = \tau_c(D(x,\, o,\, \nu)).$$

For the downlink, one defines $\nu^{\mathrm{do}}[\tau_c]$ via $\frac{\mathrm{d}\nu^{\mathrm{do}}[\tau_c]}{\mathrm{d}\nu}(x) = \tau_c(R(o,\, x,\, \nu))$, and analogously for $\nu^{\mathrm{do\text{-}dir}}[\tau_c]$. We write $\boldsymbol{\tau}_{\mathbf{c}} = (\tau_{c_1}, \dots ,\tau_{c_4})$,

$$L_\lambda[\boldsymbol{\tau}_{\mathbf{c}}] = (L_\lambda^{\mathrm{up}}[\tau_{c_1}],\, L_\lambda^{\mathrm{up\text{-}dir}}[\tau_{c_2}],\, L_\lambda^{\mathrm{do}}[\tau_{c_3}],\, L_\lambda^{\mathrm{do\text{-}dir}}[\tau_{c_4}]),$$

and similarly for $\nu[\boldsymbol{\tau}_{\mathbf{c}}]$. Then, for $\mathbf{b} = (b_i)_{i\in[4]}$, the event $\{L_\lambda[\boldsymbol{\tau}_{\mathbf{c}}](W) > \mathbf{b}\}$ means that more than λb_i users experience a QoS of less than c_i for all $i \in [4]$. Here, for vectors $\mathbf{a} = (a_1, \dots ,a_4)$, $\mathbf{b} = (b_1, \dots ,b_4) \in \mathbb{R}^4$, we write $\mathbf{a} < \mathbf{b}$ if $a_i < b_i$ for all $i \in [4]$. Further, we write $\mathbf{a} \leq \mathbf{b}$ if $a_i \leq b_i$ for all $i \in [4]$.[29] Let us define $[0,\mathbf{c}_+) = [0,c_+)^4$.

Let us recall the notion of relative entropy from (2.20). The main results of [HJKP18] are the following.

Corollary 5.2.1 ([HJKP18]). *Let $\mathbf{b} \in \mathbb{R}^4$ and $\mathbf{c} \in [0,\mathbf{c}_+)$. Then we have*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \mathbb{P}(L_\lambda[\boldsymbol{\tau}_{\mathbf{c}}](W) > \mathbf{b}) = - \inf_{\nu \in W\,:\; \nu[\boldsymbol{\tau}_{\mathbf{c}}](W) > \mathbf{b}} \mathcal{H}_W(\nu \mid \mu).$$

Here, $\mathcal{H}_W(\cdot \mid \mu)$ is the large deviations rate function of L_λ on scale λ on $\mathcal{M}(W)$ equipped with the weak topology, cf. [HJP18, Proposition 3.6]. Finally, probabilities of frustration events that are unlikely with respect to the intensity measure μ decay at an exponential speed.

Corollary 5.2.2 ([HJKP18]). *Let $\mathbf{b} \in \mathbb{R}^4$ and $\mathbf{c} \in [0,\mathbf{c}_+)$. If*

$$((1+\varepsilon)\mu)[\boldsymbol{\tau}_{\mathbf{c}}](W) \leq \mathbf{b} \tag{5.4}$$

holds for some $\varepsilon > 0$, then

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log \mathbb{P}(L_\lambda[\boldsymbol{\tau}_{\mathbf{c}}](W) > \mathbf{b}) < 0. \tag{5.5}$$

[29] Note that the notion of "$\leq$" is not consistent with the one of "$<$", in the sense that the condition that $\mathbf{a} \leq \mathbf{b}$ does not imply that $\mathbf{a} < \mathbf{b}$ or $\mathbf{a} = \mathbf{b}$. Indeed, take $\mathbf{a} = (0,0,0,0)$ and $\mathbf{b} = (0,1,0,0)$, then $\mathbf{a} \leq \mathbf{b}$ holds, but both $\mathbf{a} < \mathbf{b}$ and $\mathbf{a} = \mathbf{b}$ are false. However, this is the notation that was used in [HJKP18], and hence we stick to it in the present thesis.

We note that one-coordinate analogues of Corollary 5.2.2 also hold, i.e., for $i \in [4]$, if

$$((1+\varepsilon)\mu^{m_i})[\tau_{c_i}](W) \le b_i \tag{5.6}$$

holds for some $\varepsilon > 0$, then

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log \mathbb{P}(L_\lambda^{m_i}[\tau_{c_i}](W) > b_i) < 0. \tag{5.7}$$

Here, we denoted the means of communication by $m_1 =$ up, $m_2 =$ up-dir, $m_3 =$ do, and $m_4 =$ do-dir. Note that according to the definition of "$>$", already (5.7) implies (5.5).

It was observed in [T16, Section 3.11] that the one-coordinate version (5.6) of the condition (5.4) of Corollary 5.2.2 in at least one coordinate is necessary for (5.5) in the nontrivial case $\mathbf{b} \ge \mathbf{0}$, where we wrote $\mathbf{0} = (0,0,0,0)$ for the origin of $\mathbb{R}^4$. In the special case $b_i = 0$ for $i \in [4]$, [T16, Corollary 3.21] also provided an equivalent characterization of this one-coordinate condition in terms of *minimal QoS levels.* In order to present this, we define the *minimal QoS level for the uplink* as

$$S_1 = S_{\mathrm{up}} := \inf\{\widetilde{c} > 0\colon \mu^{\mathrm{up}}[\tau_{\widetilde{c}}](W) > 0\}. \tag{5.8}$$

We similarly define the minimal QoS levels for the other means of communication, denoted as $S_i = S_{m_i}$, $i = 2,3,4$, similarly to $S_1 = S_{m_1} = S_{\mathrm{up}}$, via replacing up $= m_1$ with the corresponding m_i in $\mu^{\mathrm{up}}[\tau_{\widetilde{c}}](W)$ in (5.8). Then, it is easy to see that we have

$$\begin{aligned}
S_1 = S_{\mathrm{up}} &= \inf\{\widetilde{c} > 0\colon \mu\text{-essinf}_{x\in W}\,\mu\text{-esssup}_{y\in W}\Gamma(x,y,o,\mu) < \widetilde{c}\}.\\
S_2 = S_{\text{up-dir}} &= \inf\Big\{\widetilde{c} > 0\colon \mu\text{-essinf}_{y\in W}\, g\Big(\tfrac{\ell(|y|)}{\int_W \ell(|x|)\mu(\mathrm{d}x)}\Big) < \widetilde{c}\Big\},\\
S_3 = S_{\mathrm{do}} &= \inf\{\widetilde{c} > 0\colon \mu\text{-essinf}_{x\in W}\,\mu\text{-esssup}_{y\in W}\Gamma(o,y,x,\mu) < \widetilde{c}\},\\
S_4 = S_{\text{do-dir}} &= \inf\Big\{\widetilde{c} > 0\colon \mu\text{-essinf}_{y\in W}\, g\Big(\tfrac{\ell(|y|)}{\int_W \ell(|x-y|)\mu(\mathrm{d}x)}\Big) < \widetilde{c}\Big\}.
\end{aligned}$$

We are now able to formulate [T16, Corollary 3.21].[30]

Corollary 5.2.3 ([T16]). *Let* $\mathbf{b} = (b_i)_{i\in[4]} \ge \mathbf{0}$ *and* $\mathbf{c} = (c_i)_{i\in[4]} \in [0,\mathbf{c}_+)$.

(i) The exponential decay (5.5) *holds if and only if* (5.6) *holds for at least one* $i \in [4]$.

(ii) If $i \in [4]$ *and* $b_i = 0$, *then* (5.6) *holds in the ith coordinate if and only if* $c_i < S_i$.

5.2.2. From degree bounds of SINR graphs to frustration probabilities

In fact, for some choices of the parameters $b \ge 0$ and $c \in [0,c_+)$, the degree bounds in SINR graphs (cf. Section 4.1.4.1) imply that the probability of the frustration event $\{L_\lambda^{\text{up-dir}}[\tau_c](W) > b\}$ does not decay exponentially fast, and similarly for direct downlink communication, further, relayed communication is only possible at the price of a substantial increase of the numbers of incoming hops at the relays. For direct communication, our new condition involves the parameters b, c, and $\mu(W)$ only and thus is easier to check than the general condition (5.6) (with "up-dir" respectively "do-dir" instead of "up"), which depends on much finer properties of μ and can often only be computed numerically.

By (4.12), for $\tau,\gamma > 0$, for any choice of a simple point process Φ, a noise power $N_0 \ge 0$, and a signal power $P > 0$, any vertex $X_i \in \Phi$ in the directed SINR graph $g^{\rightarrow}_{(\gamma,N_0\tau,P)}(\Phi)$ has in-degree less than $1 + 1/(\tau\gamma)$. Equivalently, the number of points $X_j \in \Phi$, $j \neq i$, such that

$$\frac{P\ell(|X_j - X_i|)}{N_0 + \gamma\sum_{k\neq i} P\ell(|X_k - X_i|)} > \frac{\tau}{1+\tau\gamma}$$

is less than $1 + 1/(\tau\gamma)$, according to (4.41). Note that these even hold if P is replaced by user-dependent power values $(P_j)_{j\in I}$, and thus our next observations in this section about the model of [HJKP18] can easily be

[30] We believe that the original formulation of this corollary in [T16] is correct, nevertheless it is somewhat tautological. Here, we present a more straightforward form of this corollary, which also relies on further observations of [T16, Section 3.11].

generalized to the setting of [T16] with random fadings (in fact, also with mobility). See Section A.3 for further details of the degree bounds in the general setting.

Further, let us choose $g(x) = \min\{x, K\}$ for $K > 0$ in the model definition of [HJKP18]. Now we claim that Corollaries 5.2.1 and 5.2.2 also hold for $\widetilde{g}(x) = x$ instead of g, which corresponds to the case when one-hop QoS values are true SIR values, using the conventions that $\mathrm{SIR}(x, y, \nu) = \infty$ for $x,y \in W$ if ν is the zero measure, and $g(\infty) = c_+$. This assertion follows from the simple fact that $g(\cdot)\mathbb{1}\{\cdot < c_+\} = \widetilde{g}(\cdot)\mathbb{1}\{\cdot < c_+\}$. Indeed, for $x,y \in W$ and $c \in [0,c+)$ (where $[0,c_+) = [0,\varrho_+)$ by the choice of g), $g(\mathrm{SIR}(x,y,L_\lambda)) < c$ implies $g(\mathrm{SIR}(x,y,L_\lambda)) = \mathrm{SIR}(x,y,L_\lambda)$ and hence $\widetilde{g}(\mathrm{SIR}(x,y,L_\lambda)) < c$, while $g(\mathrm{SIR}(x,y,L_\lambda)) \in [c,c_+)$ and $g(\mathrm{SIR}(x,y,L_\lambda)) = c_+$ both imply $\widetilde{g}(\mathrm{SIR}(x,y,L_\lambda)) \geq c$. In the following, we will consider the case of $\widetilde{g}$ (instead of a truncated g).

In this case, for $c \in [0,c_+)$ and $X_i \in X^\lambda$, in the setting of [HJKP18], $\mathrm{SIR}(X_i, o, L_\lambda)$ equals $\mathrm{STINR}(X_i, o, \Phi)$ in the sense of (4.41) and (4.1), where Φ is the union of $\{o\}$ and the simple point process X^λ supported in W, further, $N_0 = 0$, $P = 1$, $\gamma = 1/\lambda$, and, since the sum in the denominator of (5.1) also includes the transmitter, $\tau/(1 + \tau\gamma) = c$ (cf. (4.41)), i.e., $\tau = \frac{c}{1-c/\lambda}$, which tends to c as $\lambda \to \infty$. Thus, all in-degrees in the directed SINR graph $g^{\rightarrow}_{(\frac{1}{\lambda},0,\frac{c}{1-c/\lambda},1)}(\Phi)$ are less than $1 + 1/(\tau\gamma) = 1 + \frac{1-c/\lambda}{c/\lambda} = \frac{\lambda}{c}$.

In particular, less than λ/c users X_i are connected to o for direct uplink communication, i.e., satisfy $\mathrm{SIR}(X_i,o,L_\lambda) > c$. In fact, a closer inspection of the proof of the general degree bounds (cf. Section A.3) implies that the number of users X_i such that $\mathrm{SIR}(X_i,o,L_\lambda) \geq c$ is at most λ/c. Let us write $N(\lambda) := \lambda L_\lambda(W) = \#X^\lambda$, similarly to Chapter 2. Let us assume for a moment that $(X^\lambda)_{\lambda>0}$ is constructed according to Section 2.1.1. Now, let $s \in [1/c,\mu(W))$ (which is defined as $\varnothing$ if $\mu(W) \leq 1/c$). By the Poisson Law of Large Numbers [K93, Section 4.2], we have that $\lim_{\lambda\to\infty} \mathbb{P}(L_\lambda(W) > s) = 1$. Hence, $\lim_{\lambda\to\infty} \mathbb{P}(L_\lambda^{\text{up-dir}}[\tau_c](W) > s - 1/c) = 1$. In particular, $\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \mathbb{P}(L_\lambda^{\text{up-dir}}[\tau_c](W) > s - 1/c) = 0$. Thus, for $b \in [0,\mu(W) - 1/c)$, we have that $\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \mathbb{P}(L_\lambda^{\text{up-dir}}[\tau_c](W) > b) = 0$, i.e., the probability of the frustration event $\{L_\lambda^{\text{up-dir}}[\tau_c](W) > b\}$ does not decay exponentially fast (actually, it tends to one). More generally, the same arguments show that this holds as the *weak* Poisson Law of Large Numbers holds, i.e., $N(\lambda)/\lambda$ tends to $\mu(W)$ in probability as $\lambda \to \infty$.

In general SINR graphs, it is usually harder to obtain *out-degree bounds* than in-degree ones. Indeed, while in-degree bounds follow from estimating the total received power at the corresponding receiver from below by the sum of the received signal powers coming from the neighbours of the receiver in the SINR graph, in order to compute out-degrees one has to encounter interferences at many potential receivers, and these interferences are not *a priori* bounded away from 0. The special property of the high-density settings of [HJKP18, KT18, KT19] is that ℓ is assumed to be bounded and bounded away from 0, so that $\ell_{\max} = \max_{x,y\in W} \ell(|x-y|)$ and $\ell_{\min} = \min_{x,y\in W} \ell(|x-y|)$ are both positive real numbers. Hence, for any $x,y \in X^\lambda$, we have

$$\frac{\mathrm{SIR}(x,y,L_\lambda)}{\mathrm{SIR}(y,x,L_\lambda)} \in \Big[\frac{\ell_{\min}^2}{\ell_{\max}^2}, \frac{\ell_{\max}^2}{\ell_{\min}^2}\Big].$$

This way, the arguments for the direct uplink definitely also apply for the direct downlink after multiplying the SIR threshold c used for the direct uplink by a factor of at most $\frac{\ell_{\min}^2}{\ell_{\max}^2}$ and choosing b accordingly. We expect that more refined arguments can provide larger minimal factors in special cases.

The degree bounds also affect relayed communication. For the relayed uplink, for $\lambda > 0$, since at most λ/c users in X^λ can successfully transmit to o, this number is an upper bound for the total number of users who serve as a relay for at least one transmission. Hence, e.g., if one wants to guarantee that *all* users can transmit a message to o along a trajectory with SIR levels of both hops larger than c, then the average number of incoming hops of a relay must be at least $N(\lambda)/(\lambda/c)$, which almost surely tends to $c\mu(W)$. Thus, if $\mu(W)$ is large, relays must receive a large number of incoming hops on average. As we already mentioned, in real telecommunication networks, relays can typically only receive a bounded number of incoming hops. Thus, successful transmissions in such a highly congested network may not be possible to be maintained, and the increased amount of congestion leads to delays in the transmissions. Similar congestion issues also occur for relayed downlink communication, and they become even stronger in both communication directions if one increases the maximal number of hops.

5.2.3. From frustration probabilities to a strict version of the Gibbsian model

In this section we discuss the well-definedness of the version of our Gibbsian model with "hard" interference penalization, which was sketched in Section 2.5.1. To be more precise, using the notation of Chapter 2 and the convention that $\infty \times 0 = 0$, we consider the "hard" interference penalty term

$$\mathfrak{S}_{c_1} \colon \mathcal{S}_{k_{\max}}(X^\lambda) \ni s = (s^i)_{i \in I^\lambda} \mapsto \sum_{i \in I^\lambda} \infty \times \mathbb{1}\big\{\exists l \in [s^i_{-1}] \colon \mathrm{SIR}(s^i_{l-1},\, s^i_l,\, X^\lambda) \le c_1\big\}, \tag{5.9}$$

for some SIR threshold $c_1 > 0$, or the combination of $\mathfrak{S}_{c_1}$ and the "soft" penalty term $\mathfrak{S}(\cdot)$ from (2.13), i.e.,

$$\mathfrak{S}_{c_1,\mathrm{comb}}(\cdot) = \mathfrak{S}_{c_1}(\cdot) + \mathfrak{S}(\cdot). \tag{5.10}$$

In fact, for any $c_1 > 0$, $\lambda > 0$, and $k_{\max} \in \mathbb{N}$, it is not difficult to see that the probability that the sum in (5.9) is infinite for all $s \in \mathcal{S}_{k_{\max}}(X^\lambda)$ is positive; indeed, since $\ell(|\cdot - \cdot|)$ is bounded and bounded away from 0 on $W \times W$, this follows directly from the fact that $\# X^\lambda$ is an unbounded random variable. Thus, in order to define a Gibbs distribution with exponential penalty terms $\mathfrak{M}$ (2.15) for the congestion and $\mathfrak{S}_{c_1}$ respectively $\mathfrak{S}_{c_1,\mathrm{comb}}$ for the interference, we need to extend $\mathcal{S}_{k_{\max}}(X^\lambda)$ by a cemetery state Υ. That is, we choose an arbitrary set $\Upsilon \notin \mathcal{S}_{k_{\max}}(X^\lambda)$,[31] we define $\mathcal{S}_{k_{\max},*}(X^\lambda) = \mathcal{S}_{k_{\max}}(X^\lambda) \cup \{\Upsilon\}$ and we extend $\mathfrak{S}_{c_1}$ to $\mathcal{S}_{k_{\max},*}(X^\lambda)$ via setting $\mathfrak{S}_{c_1}(\Upsilon) = \infty \times \mathbb{1}\{\exists s \in \mathcal{S}_{k_{\max}}(X^\lambda) \colon \mathfrak{S}_{c_1}(s) < \infty\}$. Further, we extend $\mathfrak{S}_{c_1,\mathrm{comb}}$ to $\mathcal{S}_{k_{\max},*}(X^\lambda)$ via putting $\mathfrak{S}(\Upsilon) = 0$ and defining $\mathfrak{S}_{c_1,\mathrm{comb}}$ according to (5.10). We also put $\mathfrak{M}(\Upsilon) = 0$.

Then, for $\lambda > 0$, $\gamma > 0$, and $\beta \ge 0$, we can define the Gibbs distributions $\mathrm{P}^{\gamma,\beta,c_1}_{\lambda,X^\lambda}$ and $\mathrm{P}^{\gamma,\beta,c_1,\mathrm{comb}}_{\lambda,X^\lambda}$ on $\mathcal{S}_{k_{\max},*}(X^\lambda)$ analogously to $\mathrm{P}^{\gamma,\beta}_{\lambda,X^\lambda}$ on $\mathcal{S}_{k_{\max}}(X^\lambda)$ (cf. (2.10)) but with $\mathfrak{S}$ replaced by $\mathfrak{S}_{c_1}$ respectively $\mathfrak{S}_{c_1,\mathrm{comb}}$, also in the definition of the partition functions $Z^{\gamma,\beta,c_1}_\lambda(X^\lambda)$ respectively $Z^{\gamma,\beta,c_1,\mathrm{comb}}_\lambda(X^\lambda)$, which are the normalizing constants that make $\mathrm{P}^{\gamma,\beta,c_1}_{\lambda,X^\lambda}$ respectively $\mathrm{P}^{\gamma,\beta,c_1,\mathrm{comb}}_{\lambda,X^\lambda}$ a probability measure. Note that $\mathrm{P}^{\gamma,\beta,c_1}_{\lambda,X^\lambda}$ does not depend on the choice of γ as long as $\gamma > 0$. Further, if for all $s \in \mathcal{S}_{k_{\max}}(X^\lambda)$ we have $\mathfrak{S}_{c_1}(s) = \infty$, then $\mathrm{P}^{\gamma,\beta,c_1}_{\lambda,X^\lambda} = \mathrm{P}^{\gamma,\beta,c_1,\mathrm{comb}}_{\lambda,X^\lambda} = \delta_\Upsilon$, which is the case when for some user, no message trajectory is such that each hop has SIR larger than c_1.

Let us assume that $(X^\lambda)_{\lambda>0}$ is constructed according to Section 2.1.1, defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$. As in Chapters 2 and 3, we are interested in the behaviour of the system in the high-density limit in a quenched sense. Therefore, the first important question is whether for certain values of $c_1 > 0$, it is true that for almost all $\omega \in \Omega$, there exists $\lambda_0 = \lambda_0(\omega) > 0$ such that for all $\lambda > \lambda_0$, $\mathrm{P}^{\gamma,\beta,c_1}_{\lambda,X^\lambda(\omega)}(\Upsilon) = \mathrm{P}^{\gamma,\beta,c_1,\mathrm{comb}}_{\lambda,X^\lambda(\omega)}(\Upsilon) = 0$. This is necessary in order to derive the almost sure behaviour of the respective partition function in terms of measures on powers of W, i.e., in terms of objects that describe the routeing behaviour of the system. Now, using results of [HJKP18, T16], we show that under certain choices on the SIR threshold c_1, this assertion holds, at least along an increasing, divergent sequence of intensities $(\lambda_n)_{n \in \mathbb{N}_0}$ (independent of ω). Further, if c_1 is very close to zero, then taking a subsequence is not necessary. We make the connection between the two models formal only for $k_{\max} = 2$, since this is the case that was considered in [HJKP18, T16]. Nevertheless, we are certain that the case $k_{\max} > 2$ can be handled analogously because, as already mentioned, we expect that analogues of the results of [HJKP18, T16] also hold for $k_{\max} > 2$. Clearly, the assumptions of Chapter 2 on ℓ (i.e., continuity, nonnegativity and monotone decrease) and the ones of Section 5.2.1 (i.e., Lipschitz continuity and nonnegativity) do not exclude each other and both hold for the main examples, e.g., $\ell(s) = \min\{1, s^{-\alpha}\}$ or $\ell(s) = (1+s)^{-\alpha}$, $\alpha > 0$. Therefore, in this section we assume that all of these conditions are satisfied. Further, we let W be a compact subset of $\mathbb{R}^d$ with $\mathrm{Leb}(W) > 0$ and the absolutely continuous intensity measure $\mu \in \mathcal{M}(W)$ satisfy $\mu(W) > 0$; the results of [HJKP18] also hold in this case, see Section 5.2.4.

In the context of the one-coordinate version of Corollary 5.2.2 for the uplink communication, in case $b_1 = 0$ and $c_1 \in [0,c_+)$, the assertion (5.7) means that the probability that there exists *at least one* user who experiences QoS less than c_1 with respect to the uplink decays exponentially fast. This holds if and only if (5.6) is satisfied, which is also equivalent to the condition that $c_1 < S_1$ thanks to Corollary 5.2.3, where we recall the minimal QoS level S_1 from (5.8). Let us recall from Section 5.2.2 that Corollaries 5.2.1 and 5.2.2 (and their one-coordinate versions) also hold with $g(\mathrm{SIR}(\cdot))$ replaced by $\mathrm{SIR}(\cdot)$. We now consider this setting, where QoS levels are just

[31] More precisely, Υ shall satisfy this condition independently of λ and X^λ. This holds e.g. if Υ is a negative real number.

SIR levels. For $c_1 > 0$, $x_0 \in W$, and $\nu \in \mathcal{M}(W)$, we define the set

$$\mathfrak{A}_{c_1}(x_0,\nu) = \{x_1 \in W \colon \Gamma(x_0,\, x_1,\, o,\, \nu) > c_1\}.$$

Then, for $c_1 < S_1$, the assumption (5.6) that $\mu^{\mathrm{up}}[\tau_{c_1'}](W) = 0$ holds for some $c_1' > c_1$ implies that for any $x_0 \in W$, the set $\mathfrak{A}_{c_1}(x_0,\mu)$ is non-empty, and thus, thanks to the continuity of ℓ, it has actually positive Lebesgue measure on W. Clearly, (5.7) for the uplink with $b_1 = 0$ and $c_1 \in [0,c_+)$ is sufficient for the assertion

$$\lim_{\lambda\to\infty} \mathbb{P}(L_\lambda^{\mathrm{up}}[\tau_{c_1}](W) = 0) = 1 \tag{5.11}$$

to hold. This implies that there exists a divergent increasing sequence $(\lambda_n)_{n\in\mathbb{N}_0}$ of positive real numbers such that for $\mathbb{P}$-almost all $\omega \in \Omega$, there exists $n_0(\omega) \in \mathbb{N}$ such that for all $\mathbb{N} \ni n \geq n_0(\omega)$, we have

$$L_{\lambda_n}^{\mathrm{up}}(\omega)[\tau_{c_1}](W) = 0.$$

Indeed, this claim follows analogously to the proof of the statement that convergence in probability implies almost sure convergence along a subsequence. Indeed, let $\lambda_0 = 1$, and for $n \in \mathbb{N}$, choose $\lambda_n > \lambda_{n-1} + 1$ such that the event $A_n := \{L_{\lambda_n}^{\mathrm{up}}[\tau_{c_1}](W) > 0\}$ satisfies $\mathbb{P}(A_n) < 2^{-n}$. Then, since $\sum_{n\in\mathbb{N}_0} \mathbb{P}(A_n) < \infty$, the first Borel–Cantelli lemma implies that $\mathbb{P}(\bigcup_{n\in\mathbb{N}_0}\bigcap_{k\geq n} A_k^{\mathrm{c}}) = 1$. For $\omega \in \bigcup_{n\in\mathbb{N}_0}\bigcap_{k\geq n} A_k^{\mathrm{c}}$, there exists $n_0(\omega) \in \mathbb{N}$ such that for all $n \geq n_0(\omega)$, $L_{\lambda_n}^{\mathrm{up}}(\omega)[\tau_{c_1}](W) = 0$, which implies the claim.

For $\omega \in \bigcup_{n\in\mathbb{N}_0}\bigcap_{k\geq n} A_k^{\mathrm{c}}$, for $n > n_0(\omega)$, we have that for any $X_i \in X^{\lambda_n}(\omega)$ there is a message trajectory with at most two hops both of which have SIR (with respect to $L_{\lambda_n}(\omega)$) strictly larger than c_1. This means that for $k_{\max} = 2$, $\gamma > 0$, and $\beta \geq 0$, $\mathrm{P}^{\gamma,\beta,c_1}_{\lambda_n,X^{\lambda_n}(\omega)}(\Upsilon) = \mathrm{P}^{\gamma,\beta,c_1,\mathrm{comb}}_{\lambda_n,X^{\lambda_n}(\omega)}(\Upsilon) = 0$, in other words, the "hard" or "combined" Gibbs distributions can be represented as probability measures on $\mathcal{S}_{k_{\max}}(X^\lambda)$ (without reference to Υ).

We expect that taking a subsequence of intensities is not necessary, however, this is not guaranteed by the Borel–Cantelli lemma, and we have no arguments for eventual monotonicity of $\lambda \mapsto L_\lambda(\omega)^{\mathrm{up}}[\tau_{c_1}](W)$. On the other hand, it is clearly unnecessary to take a subsequence if c_1 is very close to zero. Indeed, we have seen that the rescaled number of users $N(\lambda)/\lambda$ tends to $\mu(W)$, almost surely. Thus, using the inequality $\mathrm{SIR}(x,\, y,\, L_\lambda) \geq \frac{\ell_{\min}\lambda}{\ell_{\max}N(\lambda)}$ for any $\lambda > 0$ and $x,y \in W$, we see that if $c_1 < \frac{\ell_{\min}}{\ell_{\max}\mu(W)}$, then for almost all $\omega \in \Omega$, there exists $\lambda_0(\omega) > 0$ such that $L_\lambda^{\mathrm{up}}[\tau_{c_1}](W) = 0$ for all $\lambda > \lambda_0(\omega)$, and thus the hard and the combined version of the Gibbsian model are well-defined for $\lambda > \lambda_0(\omega)$.

Thus, we expect that for $\gamma > 0$ and $\beta \geq 0$, the large-λ logarithmic rates of the partition functions $Z_\lambda^{\gamma,\beta,c_1}(X^\lambda)$ and $Z_\lambda^{\gamma,\beta,c_1,\mathrm{comb}}(X^\lambda)$ are given as the negative infimum of the entropy term $\mathrm{I}(\Psi)$ plus β times the congestion term $\mathrm{M}(\Psi)$ plus, in the case of $Z_\lambda^{\gamma,\beta,c_1,\mathrm{comb}}(X^\lambda)$, γ times the interference term $\mathrm{S}(\Psi)$. Here, the infimum is taken on the set of admissible trajectory settings $\Psi = ((\nu_k)_k,(\mu_m)_m)$ for which the one-hop measure ν_1 is supported only on the set $\{x_0 \in W \colon \mathrm{SIR}(x_0,o,\mu) > c_1\}$ and the two-hop measure ν_2 is supported only on the set $\mathfrak{A}_{c_1,2} = \{(x_0,x_1) \colon x_0 \in W, x_1 \in \mathfrak{A}_{c_1}(x_0,\mu)\}$ of positive Lebesgue measure. As we mentioned in Section 2.5.1, in order to carry out the proofs further topological questions need to be addressed, which is due to the discontinuity of the function $x \mapsto \infty \times \mathbb{1}_{[0,c_1]}(x)$. This goes beyond the scope of the present thesis, as well as the rigorous analysis of the minimizer(s) of the arising variational formulas.

In case $k_{\max} \geq 3$, the main change is that the set of asymptotically admissible trajectories is (in the case when we do not truncate SIR values by g)

$$\mathfrak{A}_{c_1,k_{\max}} = \{(x_0,x_1,\dots,x_{k-1}) \colon x_0,\dots,x_{k-1} \in W,\, k \in [k_{\max}],\, x_k = o,\; \min_{l\in[k]} \mathrm{SIR}(x_{l-1},\, x_l,\, \mu) > c_1\}$$

instead of $\mathfrak{A}_{c_1,2}$. For a given $c_1 > 0$, the infimum of $k_{\max} \in \mathbb{N}$ such that for any $x_0 \in W$ there exists $k \in [k_{\max}]$ and $(x_1,\dots,x_{k-1}) \in W^{k-1}$ such that $(x_0,\dots,x_{k-1}) \in \mathfrak{A}_{c_1,k_{\max}}$ is also an interesting quantity to analyse. This, in case it is finite, is interpreted as the smallest maximum number of hops $k_{\max}$ that is needed in order to guarantee that all users x_0 have an at most $k_{\max}$-hop trajectory from x_0 to o with all hops having SIR levels exceeding c_1. If it is infinite, this shows that for large λ, the threshold c_1 is very high compared to the achievable SIR levels in the system, even if hops with arbitrarily small lengths are allowed. Since $\mathrm{SIR}(x,\, y,\, L_\lambda) \leq \frac{\ell_{\max}\lambda}{\ell_{\min}N(\lambda)}$ for all $\lambda > 0$ and $x,y \in W$, this is certainly the case if $c_1 > \frac{\ell_{\max}}{\ell_{\min}\mu(W)}$.

5.2.4. Discretization in the high-density setting and related modelling questions

In this section we intend to fulfill the formidable task of discussing proof methods in non-technical terms. As we mentioned in Section 2.8.1, the discretization approach of that section originates from [HJKP18]; we used this technique in order to derive the main results of Chapter 2. However, we observed that the results of [HJKP18] hold under more general assumptions, and for the purposes of Chapter 2 these assumptions can further be relaxed. In the same time, some significant changes in the discretization procedure of [HJKP18] are also necessary in order to adapt it to the setting of Chapter 2. Let us now explain these observations in more detail.

Both in Chapter 2 and in [HJKP18] the essence of δ-discretization is the following. Using the notation of Section 2.8.1, we consider the communication area $W=[-r,r]^d$, $d\in\mathbb{N}$, and for values δ in

$$\mathbb{B}=\{b^{-n}\colon n\in\mathbb{N}_0\} \tag{5.12}$$

for $b=3$, we map the model under consideration to the collection W_δ of congruent copies of $[-r\delta,r\delta]^d$ contained in W. For the discretized model, we carry out the high-density limit $\lambda\to\infty$ followed by the limit $\mathbb{B}\ni\delta\downarrow 0$ of a vanishing fineness parameter of the discretization. Then we derive, using (semi)continuity properties of certain functions, an analogous assertion for the original model in the limit $\lambda\to\infty$ (with no reference to the discretization).

In Chapter 2, W was assumed to be a general compact subset of $\mathbb{R}^d$ with positive Lebesgue measure and containing o, and μ an absolutely continuous nonzero measure on W. These assumptions are actually also sufficient for the results of [HJKP18] to hold[32]. Indeed, under these assumptions on W, one can assume without loss of generality that $W=[-r,r]^d$ for some $r>0$ via extending μ trivially[33]. In [HJKP18], the choice $b=3$ in (5.12) is such that for any $\delta\in\mathbb{B}$, W is covered by the union of congruent δ-subcubes intersecting only at their boundaries, and o is the centre of one of these subcubes. For the same purpose, any odd integer $b>1$ would be suitable. Our proofs in Chapter 2 do not require that o be the centre of some δ-subcube, and thus b even does not need to be odd. (The assumption that $b\in\mathbb{N}$ is made in order to ensure that for all $\delta\in\mathbb{B}$, the δ-subcubes covering W are all contained in W and have trivial intersections.) The important feature of the exponential form of (5.12) is the consistency of δ-discretizations for different values $\delta\in\mathbb{B}$. In Chapter 2, in the context of standard settings, this property is reflected by parts (3) and (7) in Definition 2.8.1 and part (A) in Remark 2.8.2, while for the model of [HJKP18], it is spelt out in [T16, Lemma 2.28].

However, there is a significant difference between the model of [HJKP18] and the one of Chapter 2, which we partially already indicated in Section 2.7.1. Namely, in [HJKP18] the only source of randomness in the system is the one of the Poisson point process of users $(X^\lambda)_{\lambda>0}$, and the main results of the paper, Corollary 5.2.1 and 5.2.2, are (highly nontrivial) consequences of the large deviations behaviour of the empirical measure L_λ of users in the high-density limit $\lambda\to\infty$. In Chapter 2 we work in a quenched setting, in which the Gibbs distribution $\mathrm{P}^{\gamma,\beta}_{\lambda,X^\lambda}$ is random and defined conditional on X^λ, and our main results hold for $(\mathrm{P}^{\gamma,\beta}_{\lambda,X^\lambda(\omega)})_{\lambda>0}$ for almost all $\omega\in\Omega$. In particular, on the set of $\omega\in\Omega$ for which they hold, $L_\lambda(\omega)$ converges weakly to μ. Thus, no randomness of $\lambda\mapsto L_\lambda$ can be detected in our results about the high-density limit.

Thus, the discretization is used for two different purposes in the two settings. As already mentioned in Section 2.8.1, in the context of our Gibbsian model, the use of discretization and in particular standard settings or transmission settings is essentially a martingale convergence argument, where the superlinear increase of the congestion function η provides the uniform integrability. This approach is based on the fact that the Borel σ-field of W is generated by the union of the coarsened σ-fields $\mathcal{F}_\delta$ over $\delta\in\mathbb{B}$. Our arguments rely strongly on certain properties of the weak topology (e.g., metrizability, see the use of the Lipschitz bounded metric starting from Section 2.9.2). In contrast, informally speaking, [HJKP18] uses the discretization in order to construct an LDP for $(L_\lambda)_{\lambda>0}$ from elementary sets in the spirit of [DZ98, Section 4.1.2]. This statement is informal because the full LDP for $(L_\lambda)_{\lambda>0}$ on $\mathcal{M}(W)$ is not stated in [HJKP18], and thus it is also not clear from the context what the underlying topology should be for such a construction. This full LDP is stated and proved in [HJP18, Proposition 3.6], with respect to the τ-topology, which is finer than the weak topology (and thus the LDP also

[32] Note that the results of [HJKP18] do not require that μ be a nonzero measure, however, the case of μ being the zero measure is uninteresting for applications in telecommunications.

[33] In [HJKP18], it was even assumed that $r\in\mathbb{N}$, which has no advantage in the proofs compared to the general case $r>0$.

holds with respect to the weak topology thanks to the contraction principle). The rate function is the relative entropy with respect to the intensity measure μ. We sketched some indications of this LDP to the annealed version of the Gibbsian model in Section 2.4.5.

These two different purposes lead to two different variants of the discretization procedure. The approach of [HJKP18] was, for $\delta \in \mathbb{B}$, to map each Poisson point to the centre of the δ-discretization subcube in which it is contained. Then, the authors computed the QoS values at the images of the transmitters (and at o, which is a fixed point of the discretization) based on the SIR values with respect to this discretized point process. The difference between the true amounts of frustrated users and the discretized ones can be controlled in the limit $\delta \downarrow 0$, thanks to the fact that ℓ is Lipschitz continuous, bounded, and bounded away from 0. Further, for $\delta \in \mathbb{B}$ fixed, the δ-discretized version of the empirical measure of users satisfies an LDP on the set of finite measures on the finite space of subcube centres, with the rate function being the relative entropy with respect to the discretized version of μ on the corresponding finite space. This LDP is elementary to derive using Cramér's theorem [DZ98, Section 2.2], see [T16, Section 3.8]. It turns out that this LDP is sufficient in order to derive Corollaries 5.2.1 and 5.2.2, and also the more general technical result [HJKP18, Theorem 1.1].[34] It is an interesting open question whether using the full LDP for $(L_\lambda)_{\lambda>0}$, one can prove the main results of [HJKP18] under weaker assumptions on ℓ.

The approach of our Chapter 2 is different: here, for $k \in [k_{\max}]$, $\nu \in \mathcal{M}(W^k)$, and $\delta \in \mathbb{B}$, the discretized measure ν^δ is defined as a measure on W^k (rather than on some finite set), with the property that it equals ν restricted to the δ-coarsended σ-field $\mathcal{F}_\delta^{\otimes k}$ and has constant Lebesgue density on each element of W_δ^k. This way, one can speak about convergence of the measures L_λ^δ, $R_{\lambda,k}^\delta(S)$ ($k \in [k_{\max}]$) and $P_{\lambda,m}^\delta(S)$ ($m \in \mathbb{N}_0$) in the limit $\lambda \to \infty$ followed by the one $\delta \downarrow 0$, in the weak topology of $\mathcal{M}(W^k)$ respectively $\mathcal{M}(W)$, see Definition 2.8.1. Further, unlike in [HJKP18], we did not need to estimate the number of users experiencing QoS values less than a given number but only the entire interference penalty term. For this, Lipschitz continuity of ℓ is not required, continuity suffices (cf. Section 2.9.2). The modelling assumption that ℓ is monotone decreasing is also not needed in the proofs of Chapter 2, but it is essential in the ones of Chapter 3. Since this assumption is reasonable for describing telecommunication networks, we already made it at the beginning of Chapter 2. The boundedness of ℓ ensures that interference penalties of single hops in the system are uniformly bounded away from zero, which we find senseful from an application point of view. Finally, we kept the assumption of [HJKP18] that $\ell(|\cdot - \cdot|)$ is nonvanishing on $W \times W$, in order to avoid having trajectories with infinite interference penalty, which would have a similar effect as the strict interference penalty term introduced in Section 5.2.3.

5.3. Conjectures and open questions

We finish our concluding chapter by highlighting the most important conjectures and open questions related to the subject of the thesis. In order to emphasize their relevance we present them within one section here, even though some of them have already been mentioned in earlier sections; on the other hand, we leave out some open problems originating from earlier parts of the thesis that we find less relevant or less strongly related to our main focus. We point out the mathematical relevance of the questions, their interpretation and motivation in the context of telecommunication, the expected answer, and the main difficulties; each one whenever applicable. In Section 5.3.1 we list the questions related to our Gibbsian model and its routeing properties (using the notation of Chapters 2 and 3), and in Section 5.3.2 we enumerate the ones connected to SINR percolation for Cox processes (using the notation of Chapter 4).

5.3.1. Message routeing: a Gibbsian approach

5.3.1.1. Behaviour of the Gibbsian model in the high-density limit

Can Theorems 2.2.2 and 2.2.4, and their variants for $\beta = 0$ be derived without spatial discretization? These assertions about the behaviour of the Gibbsian model in the high-density limit do not involve

[34] In order to state the latter theorem it is necessary to have already defined elements of the discretization procedure, cf. [HJKP18, Section 1.2]. However, the same does not hold for the two corollaries, as can be seen from our summary in Section 5.2.1.

spatial discretization, unlike their proofs that we provided in the present thesis. It is therefore interesting to ask whether a proof avoiding discretization is possible. Certainly, the same question is also interesting to ask about the main results of [HJKP18] about frustration probabilities, Corollaries 5.2.1 and 5.2.2 (such a proof should not involve the key technical result [HJKP18, Theorem 1.1], which tells about discretized objects).

Does the empirical measure family $\Psi_\lambda(S) = ((R_{\lambda,k}(S))_{k\in[k_{\max}]}, (P_{\lambda,m})_{m\in\mathbb{N}_0})$ converge to the unique minimizer of the variational formula (2.35) for $\gamma > \beta = 0$? Equivalently, is the distribution of the family of empirical measures $P_{\lambda,m}(S)$ of users receiving m incoming hops ($m \in \mathbb{N}_0$) exponentially tight if the congestion term is absent? Since the variational formula (2.24) has a unique minimizer in this case as well, it is highly imaginable that the answer is positive. However, in order to verify this, new arguments have to be found, instead of Varadhan's lemma, which was used for proving the convergence for β,$\gamma > 0$. (Recall that in the *a priori* case $\gamma = \beta = 0$, the convergence of $\Psi_\lambda(S)$ to the minimizer is clear from the fact that $L_\lambda \Longrightarrow \mu$ and from the form of the minimizer (2.105).)

5.3.1.2. Extensions and variants of the model

Avoiding loops. It is a reasonable idea to modify the Gibbsian model of Section 2.1.4 via forbidding trajectories having loops (i.e., using the transmitter as a relay or visiting a relay at least twice), since erasing a loop always strictly decreases the interference penalty and never increases the congestion penalty of the trajectory family. It is plausible to think that the effect of this modification is very local and keeps the limiting behaviour of the Gibbsian system unchanged on the exponential scale. Such an assertion would imply that also in the original Gibbsian model, the effect of trajectories with loops is negligible in the high-density limit, which would provide a stronger justification for our model. Now, in the loop-free case for β,$\gamma > 0$, the difficulty is to express the combinatorics of the system in terms of standard settings or other discretized objects (cf. Section 2.8.2 for the case where loops are allowed). We have not found a closed form solution for this, although it may be possible to make certain approximations in order to obtain the same asymptotic formulas as in Section 2.9.1. (We expect that the combinatorics in the loop-free case is easier if $\gamma > \beta = 0$.)

An effective use of the relaying system. Instead of focusing only on loops, a better alternative would be to invent a routeing scheme according to which relays are effectively used, say, any hop of any admissible message trajectory decreases the distance from o and stays within a cone of a small opening angle starting at o and containing the transmitter. Indeed, from the perspective of interference penalization, loops are not the worst features that a message trajectory can exhibit; e.g., a self-loop may be less unfavourable with respect to interference penalization than a hop that increases the distance from o (although there are also counterexamples, see Example 3.5.1). Such a new model would describe a different, more regulated message routeing protocol than the one of Chapter 2 where any trajectory with at most $k_{\max}$ hops from a given user to o was allowed. Further, in the new model, the distance from o must play an important role, and therefore it is more natural to consider such a setting in a thermodynamic limit than in our high-density limit, although our large-distance limit from Section 3.2 could also be investigated in this case.

The essential question regarding time dependence in the Gibbsian model. Does our Gibbsian model introduced in Section 2.1.4 have a time dependent version that is significantly more realistic than the original one but still mathematically tractable? Here, as we discussed in Section 2.4.4, being more realistic would mean that the interferences do not only depend on the spatial positions of the transmitters but also on the trajectories of their messages. We also pointed out in the latter section that even in very simple versions of the model, the form of the minimizer(s) of the corresponding variational formula may not be amenable for analytic investigations. Versions of the model where the number of time slots is $\Theta(\lambda)$, so that the factor $1/\lambda$ in the interferences can be dropped, would also be interesting for applications. One could also consider versions where the path-loss function is unbounded and has a power law (or exponential) decay, which would lead to a different kind of high-density limit that is similar to a thermodynamic limit, cf. Section 5.1.2.

5.3.1.3. Routeing properties in the Gibbsian model

Is the minimizer (2.25) – (2.26) **amenable for analytical investigations?** In this thesis we have verified uniqueness of this minimizer corresponding to the case $\beta,\gamma > 0$, which was not yet claimed in [KT18]. However, this uniqueness does not seem to resolve the implicitness of the function C and the measure M in (2.100) – (2.102). The question is whether one can find certain properties or alternative descriptions of these quantities that make an analysis of routeing properties possible in limiting regimes, similarly to the investigation of the minimizer (2.35) in the case $\gamma > \beta = 0$ in Chapter 3. For this question, it certainly suffices to consider the special case of Section 2.1.4 corresponding to penalization of interference and congestion.

The role of convexity of $1/\ell(|\cdot|)$ **in regime** (1)**?** In the proofs of Theorem 3.2.1 (3.8) and Proposition 3.2.3 about the routeing behaviour of the system in the limiting regime (1) of large communication areas coupled with large transmitter–receiver distances, we repeatedly used Jensen's inequality based on the convexity of $1/\ell(|\cdot|)$. This convexity holds for our main examples $\ell(r) = \min\{1, r^{-\alpha}\}$ and $\ell(r) = (K + r)^{-\alpha}$, $\alpha > d$, $K > 0$. Nevertheless, it would be interesting to see whether an analogous assertion holds for path-loss functions with non-convex reciprocals. We expect that at least if $r \mapsto \ell(r)$ is convex on $[K',\infty)$ for $K' > 0$ large enough, then the most likely trajectory should still take equal hops along the straight line between transmitter and receiver. Indeed, in such a setting the scaling of the *a priori* normalization indicates that the typical length of a hop still tends to infinity in our limit (see Section 3.2.3.1), and hence it should suffice to guarantee convexity of ℓ for large distances.

A Markovian approach? The explicit form of the trajectory distribution T_{x_0} in (3.5) seems to suggest a Markovian approach in regime (1), combined with a large deviations argument for an exponential functional of the Markov chain. One might think that a large deviation principle for the empirical measure L_k of the k hop vectors $x_l - x_{l-1}$ could be the core of a proof, possibly after some spatial rescaling and under conditioning on having a fixed integral of the identity with respect to L_k. We found no way to make this route work and we conjecture that such an approach is indeed not suitable for verifying Theorem 3.2.1. The main obstacles for the Markovian approach are the following. The state space and the transition kernel of the chain depend on W and on x_0 in a particularly irregular way: they induce two different scales in the interaction of the chain and therefore also change the scale of the probabilities in a non-standard way. Another problem, which is not only technical, is that the integration area for each step is unbounded in our large-distance limit, and the steps are integrated with respect to the Lebesgue measure. Thus, in our limit, instead of working with a diverging number of steps of one Markov chain, one has to handle Markov chains with different step distributions for each $r_0 = |x_0|$, and these distributions do not converge in our limit (thanks to the diverging typical hop length).[35]

Replacing $[[x_0,o]]$ **with** $[x_0,o]$ **in** (3.38)**?** In regime (2) (limit of strong penalization of interference), (A) in Proposition 3.3.1 claims that if ℓ is strictly monotone increasing and μ is rotationally invariant, then all trajectories from $x_0 \in W$ to o that are optimal with respect to interference penalization follow the straight line *segment* $[x_0,o]$ between x_0 and o, and each hop strictly decreases the distance from o. Under the same conditions, in part (B) of the same proposition, we obtained that macroscopic deviations from the straight line $[[x_0,o]]$ have exponentially small probabilities. It seems plausible that the same assertion holds with $[[x_0,o]]$ replaced by $[x_0,o]$ in the definition of the set of k-trajectories $D_k^\varepsilon(x_0)$ deviating from the straight line by at least ε for at least one hop (3.38), for $k \in [k_{\max}]$ and $\varepsilon > 0$. However, such a stronger assertion would require a new proof. The reason is that while the original set $D_k^\varepsilon(x_0)$ is closed under the operation $x \mapsto x'$ defined in part (i) of the proof of (B) in Proposition 3.3.1 (see Section 3.3.2), the same fails with $[[x_0,o]]$ replaced by $[x_0,o]$ in (3.38).

Examples for the optimal trajectory not being straight for ℓ **not strictly monotone increasing?** It is an open question whether there exists an example where the assertion of Proposition 3.3.1(A) fails due to the missing strict monotonicity of the path-loss function ℓ. The Laplace method suggests that for such an

[35] This question originates from the first version of [KT19] (*arXiv:1801.04985v1*).

example, part (B) of the same proposition should also not be true. The most interesting would be to find such an example with μ being rotationally invariant, so that the deviation of the optimal trajectory from the straight line is caused entirely by ℓ. We have seen evidence that for such μ, realistic path-loss functions that are not strictly monotone decreasing may lead to anomalous routeing behaviour. Indeed, in Example 3.5.1 we have seen that for $\ell(r) = \min\{1, r^{-\alpha}\}$ and $k_{\max} = 2$, there are cases when the optimal two-hop trajectory of a transmitter x_0 uses a relay that is situated further away from the receiver o than x_0 itself; however, this effect is only present for x_0 situated very close to o, with its one-hop trajectory being optimal.

Local effects of regime (3) for $\ell(r) = \min\{1, r^{-\alpha}\}$? It is not clear from our approach in Section 3.4.2 whether the local effects of regime (3) of a high local density of users are present or absent in case $\ell'(0) = 0$, i.e., whether in this case it is unfavourable to choose a relay very close to the highly densely populated subset Δ, as opposed to choosing a relay significantly further away from Δ. From an application point of view, it would especially be useful to answer this question in the case of the commonly used path-loss function $\ell(r) = \min\{1, r^{-\alpha}\}$.

5.3.2. Percolation: signal-to-interference ratio percolation for Cox point processes

5.3.2.1. Phase transitions

When does it hold that $\lambda_{(N_0,\tau,P)} = \lambda_{\mathrm{c}}(r_{\mathrm{B}})$? We know from [DFMMT06, Theorem 1] that in the two-dimensional Poisson case it does. The key arguments of the proof are the following. First, the Russo–Seymour–Welsh type result, Theorem 4.1.9, holds for any supercritical Poisson–Boolean model in two dimensions. Second, by scale invariance, if $\lambda, r' > 0$ are such that the Poisson–Boolean model $X^\lambda \oplus B_{r'/2}(o)$ is supercritical, then there exists $r \in (0,r')$ such that $X^\lambda \oplus B_{r/2}(o)$ is still supercritical. For higher-dimensional Poisson–Boolean models, the scale invariance still holds, but an analogue of the Russo–Seymour–Welsh type result is not known. Further, Theorem 4.1.9 has not been verified for more general Cox point processes in $\mathbb{R}^2$. In the present thesis, in the general asymptotically essentially connected Cox case for $d \geq 2$, we replaced the discrete model used in [DFMMT06] with the process of n-good sites originating from [HJC17, Section 5.2]. According to Section 4.2.1.1, if $\lambda, r' > 0$ are such that there exists $r \in (0,r')$ such that the n-good sites defined with respect to $g_r(X^\lambda)$ percolate for all sufficiently large $n \geq 1$, then there is percolation in the SINR graph $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ with connection radius $r_{\mathrm{B}} = r'$ for large $\lambda > 0$ and accordingly chosen small $\gamma > 0$. Unless the converse of this implication holds, the approach of n-good sites cannot be used for verifying $\lambda_{(N_0,\tau,P)} = \lambda_{\mathrm{c}}(r_{\mathrm{B}})$, which may also not be true in general in the Cox case. On the other hand, in concrete cases of the intensity measure Λ, making the estimates of [HJC17, Section 5.2] more precise may be helpful for obtaining upper bounds on $\lambda_{(N_0,\tau,P)}$.

According to personal communication with C. Hirsch and B. Jahnel, the final version of [HJC17] will contain an extended notion of asymptotic essential connectedness compared to the original Definition 4.1.2, which will still imply the existence of supercritical case in the corresponding Cox–Gilbert graph. With respect to this new notion the modulated Poisson point process is asymptotically essentially connected in case $\lambda_1 > \lambda_2 = 0$ and Ξ is a *supercritical* Poisson–Boolean model in $\mathbb{R}^d$, $d \geq 2$. According to the condition (2b) in Theorem 4.1.5(2), this will probably also imply percolation in the SINR graph for large λ, small γ, and ℓ satisfying Assumption (ℓ); in order to verify this, one needs to prove the theorem under the new notion of asymptotic essential connectedness. We find it remarkable that the asymptotic essential connectedness can be verified for *any* supercritical intensity of the Poisson–Boolean model for $d \geq 2$, similarly to the Russo–Seymour–Welsh type result, Theorem 4.1.9, for $d = 2$. This assertion leads us to the conjecture that in the Poisson case, $\lambda_{N_0,\tau,P} = \lambda_{\mathrm{c}}(r_B)$ holds also for $d \geq 3$, and the same might also hold for the Poisson point process modulated by a Poisson–Boolean model in the asymptotically essentially connected case.

Is the assumption on the path-loss function in Theorem 4.1.5(2c) sharp? It would be interesting to know whether the stronger decay condition on ℓ in Theorem 4.1.5(2c) is sharp under the assumption that $\Lambda(Q_1)$ has some exponential moments. Clearly, if $\Lambda(Q_1)$ is bounded, then it suffices if $\ell(|\cdot|)$ is integrable, cf. (2b) of the same theorem. However, although our estimates in the proof of Proposition 4.2.5 under the assumption (2c) are somewhat rough, we expect that if they are not sufficient to provide any finite upper bound on the relevant exponential scale, then the interferences cannot be controlled, and thus percolation in SINR graphs with $\gamma > 0$ does not occur. Hence, we think that if $\alpha^* = \sup\{\alpha > 0\colon \mathbb{E}[\exp(\alpha\Lambda(Q_1))] < \infty\}$ is finite and Λ is not b-dependent (but asymptotically essentially connected), then there may be cases where the tail condition on ℓ in (2c) is optimal (whereas we do not expect that ℓ must be bounded, cf. Section 5.1.2). In contrast, in the b-dependent case with $\alpha^* < \infty$, we expect that integrability of ℓ is sufficient, see the next bullet point.

Let us provide some technical details. In case $N = n$, the expectation on the right-hand side of (4.31) is infinite for s sufficiently large if $\alpha^* < \infty$ but not if $\alpha^* = \infty$. Now, note that the only reason why the upper bound $K(n)$ in (4.24) is only bounded but does not vanish as $n \to \infty$ is that we did not use that $z_1,\dots,z_N$ are sufficiently far away from x. (Not using this was useful in order to handle the case (2b), where we later extended the integration area to the whole of $\mathbb{R}^d$.) Taking this property into account, one can replace $K(n)$ in (4.24) with a quantity $K'(n)$ that tends to zero at a speed depending on ℓ as $n \to \infty$, and one can replace the constraint $s \le 1/K(n)$ with the one $s = s(n) \le 1/K'(n)$. Now, for fixed $C,M > 0$, we observe that in (4.31), for $\alpha^* = \infty$ and $N = n$, $s(n) \le 1/K'(n)$ can be let slowly tend to infinity in such a way that the increase of $\mathbb{E}(\exp(Cs(n)N\Lambda(Q_1)/n))$ is still well compensated by the decay of $\exp(-s(n)NM)$. This would give a better decay condition on ℓ than the one in (2c), depending also on the speed of increase of the exponential moments of $\Lambda(Q_1)$. In particular, the case (2b) of bounded $\Lambda(Q_1)$ is precisely the one where this increase is linear in the exponential scale, and here $\int_y^\infty \ell(r)r^{d-1}\mathrm{d}r$ only needs to vanish as $y \to \infty$, instead of decaying at least linearly.

One could also ask the question whether in case $\alpha^* = 0$ one can find an unboundedly supported path-loss function that makes SINR percolation with $\gamma > 0$ possible. In this case, the use of Hölder's inequality is much more restricted than under the assumptions (2b) and (2c), and hence our proofs do not indicate whether and how one could verify a positive answer. Having the results of [JT19] about the exponential moments of planar tessellations, we see that for the most relevant two-dimensional street system models and for the main absolutely continuous examples also in higher dimensions, $\alpha^* = \infty$ holds. We are aware of uninteresting examples with $\alpha^* = 0$ or $\alpha^* \in (0,\infty)$, e.g., $\Lambda(\mathrm{d}x) = Z\mathrm{d}x$ where Z is a nonnegative random variable with $\sup\{\alpha\colon \mathbb{E}[\exp(\alpha Z)] < \infty\}$ being equal to 0 respectively being an element of $(0,\infty)$. This example is not stabilizing. In fact, it is a special case of Example 4.2.10, where its percolation behaviour was discussed.[36]

Percolation in the SINR graph for any integrable ℓ if Λ is b-dependent and has all exponential moments? We conjecture that if Λ is b-dependent with $\alpha^* > 0$, then the integrability condition in Assumption (ℓ) (iii) is sufficient for ℓ for Theorem 4.1.5 (2) respectively Proposition 4.1.10 to hold. This would be applicable e.g. for the shot-noise field, where in general one needs to take large r_{B} due to the lack of asymptotic essential connectedness. To see this, we use some observations from [JT19, Section 3.1] that were not yet present in [T18]. Namely, if Λ is b-dependent with $\alpha^* > 0$, then $\mathbb{R}^d$ is covered by congruent unit boxes (up to the boundaries) which can be subdivided into a finite number M_0 of families such that for any finite number $L \in \mathbb{N}$ of pairwise distinct elements $Q^1,\dots,Q^L$ of an arbitrary fixed family, $\Lambda_{Q^1},\dots,\Lambda_{Q^L}$ are mutually independent, which implies

$$\mathbb{E}\Big[\exp\Big(\alpha\sum_{i=1}^L \Lambda(Q^i)\Big)\Big] = \mathbb{E}\Big[\exp(\alpha\Lambda(Q_1))\Big]^L < \infty.$$

In particular, an application of Hölder's inequality for the M_0 families yields that for any $\alpha < \alpha^*$,

$$\limsup_{R\to\infty} \frac{1}{R^d} \log \mathbb{E}\big[\exp(\alpha\Lambda(Q_R)\big] < 2M_0 \log(\mathbb{E}[\exp(\alpha\Lambda(Q_1))]) < \infty. \tag{5.13}$$

This indicates that (4.27) should hold in this case (thanks to a different proof based on (5.13)), which could be used to complete the proof of Theorem 4.1.5(2) in this case.

Exponential moments for other tessellations, e.g., for the Poisson–Delaunay tessellation. A relation of two-dimensional Poisson–Voronoi, Poisson–Delaunay, and Poisson line tessellations to statistical properties of real-world street systems has been investigated, e.g., in [GFSS06]. Hence, it is highly interesting to analyse percolation properties of the SINR graph for these tessellations; see Section A.2 in the Appendix for their definitions. We will comment on the case of the Poisson line tessellation in the next bullet point.

[36] It seems to be the case that for all relevant examples of Λ, stabilization is either absent or exponentially fast or b-dependence holds. Exponentially fast stabilization might also be used to improve the decay condition on ℓ compared to the one in (2c).

B. Jahnel and the author recently verified the existence of all exponential moments of the total edge length in Q_1 for the Poisson–Delaunay tessellation in $\mathbb{R}^2$. Here, the proof [JT19, Section 2.3] turns out to be substantially more involved than in the Voronoi case, which seems to be the consequence of the fact that the Delaunay tessellation stabilizes slower than the Voronoi one [JT19, Section 3.3]. Further, a nontrivial but relatively straightforward extension [JT19, Section 2.2] of the proof for the Poisson–Voronoi tessellation yields the same assertion for the Johnson–Mehl tessellation in $\mathbb{R}^2$.[37] However, for some famous planar tessellations the question of existence of exponential moments is still open, e.g., for the Laguerre tessellation [JT19, Section 3.4].

One could also study the existence of exponential moments of the total street length for tessellation processes in three or higher dimensions, which seems mathematically more challenging but not closely related to applications in telecommunications.

Continuum percolation and SINR percolation for the Poisson line tessellation in $\mathbb{R}^2$. The Poisson line tessellation in $\mathbb{R}^2$ (see Section A.2 in the Appendix for its definition) is as important for modelling statistical properties of real-world street systems as the Poisson–Voronoi tessellation (see [GFSS06]). However, its intensity measure Λ has infinite range dependencies, in particular it is not stabilizing. It is widely believed[38] that its Gilbert graph satisfies $\lambda_{\mathrm{c}}(r) \in (0,\infty)$ for all $r > 0$, however, both the question of existence of subcritical phase and the one of existence of supercritical phase are open. Once one has verified the existence of a supercritical phase, it may become easier to conclude that there is percolation in the SINR graph with some positive γ for large λ, under additional decay assumptions on the path-loss function (without requiring compactness of its support), since a simple computation [JT19, Section 2.4] shows that $\Lambda(Q_1)$ has all exponential moments. These questions are also of interest for rectangular non-stabilizing tessellation processes that behave similarly to the Poisson line tessellation, such as Manhattan grids, the definition of which can also be found in Section A.2 in the Appendix.

Percolation in the SINR graph in case the Gilbert graph has no subcritical phase? In case the Gilbert graph has anomalously good percolation properties such as lack of a subcritical phase for all connection radii, e.g., in the example of [BY13, Section 4], it would be interesting to determine what consequences this has for percolation properties of the SINR graph. More precisely, one shall investigate the set of $\lambda > 0$ such that $\gamma^*(\lambda) > 0$. Further, for λ such that $\gamma^*(\lambda) > 0$, one could compare the value of this critical interference cancellation factor to the one corresponding to stabilizing examples, e.g., to the Poisson point process. Although the strong connectivity of the Gilbert graph for arbitrarily small positive λ and r_{B} gives rise to more potential connections in the SINR graph as well, having many edges in the Gilbert graph also indicates that interferences may be large compared to stabilizing examples with the same average spatial density of points, which may have a negative effect on the connectivity of SINR graphs for $\gamma > 0$.

The case of unbounded non-constant signal powers. As we already mentioned in the end of Section 4.2.3.4, interesting open questions in this setting are the existence of a subcritical phase in the Gilbert graph (with random radii) and the occurrence of percolation in the SINR graph for some $\lambda,\gamma > 0$, depending on the parameters Λ, ℓ, and $\zeta = \mathbb{P} \circ P^{-1}$, in case ζ is unbounded. In case the Gilbert graph has no subcritical phase, it is again interesting to ask for what values of λ it is true that $\gamma^*(\lambda) > 0$, and whether the strong connectivity of the Gilbert graph is advantegeous or malicious for percolation in the SINR graph. This question is also of importance for $d = 1$, since Gilbert graphs with unbounded radii may also percolate in one dimension.

Percolation in the SINR graph in case of an unbounded path-loss function with integrable tails? Simulation results and heuristic arguments of [DBT05] suggest that percolation in the SINR graph is possible in case ℓ is unbounded at zero and has integrable tails, cf. Section 5.1.2. Since in this case interferences have an infinite expectation, statements about interference control like Proposition 4.2.5 may not hold. In other words we expect that interferences received by points of the underlying Cox process X^λ that are not isolated in the

[37] The Johnson–Mehl tessellation is not known to be useful for modelling street systems, hence we do not include its definition in this thesis.
[38] E. Cali, C. Hirsch, B. Jahnel, and R. Patterson, personal communication (2017–2018).

SINR graph can get arbitrarily high, but for $\tau > 0$ and for $\gamma > 0$ fixed and sufficiently small, transmissions with SINR values exceeding τ are still possible along very small distances that vanish in the limit $\lambda \to \infty$. Hence, percolation properties of the SINR graph should rather depend on the ones of the smallest bidirectional k-nearest neighbour graph containing it than on the ones of the SNR graph. Indeed, the SNR graph is determined by one distance $r_{\mathrm{B}} = \ell^{-1}(\tau N_0/P)$, independently of the behaviour of ℓ near zero. Further, for fixed $\tau,P,N_0 > 0$, for ℓ unbounded, $\lambda \mapsto \gamma^*(\lambda)$ is expected to be monotone increasing (but bounded), unlike if ℓ is bounded.

Is the infinite cluster of an SINR graph unique? A positive answer was conjectured in [DBT05] for the two-dimensional Poisson case, however, the question is still open, and equally interesting in the general Cox case. Due to the infinite-range dependencies of SINR graphs, it is not clear how to generalize the arguments of the standard proof method [MR96] showing uniqueness of the infinite cluster in Gilbert graphs (with possibly random radii) to the SINR case. It has also not yet been proved that Cox–Gilbert graphs have a unique infinite cluster, but we expect that in this case some arguments of [MR96] can be applied, at least for intensity measures satisfying suitable stabilization and connectivity properties. Nevertheless, at least for Poisson point processes, for $\lambda,\gamma,N_0,\tau,P > 0$, the SINR graph $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ is a subgraph of the Gilbert graph $g_{(0,N_0,\tau,P)}(X^\lambda)$ and, thanks to the arguments of Section 4.1.4.1, also contained in the *undirected k-nearest neighbour graph* $g_{\mathbf{U}}(k,X^\lambda)$ for sufficiently large $k \in \mathbb{N}$. The latter graph [HM96, BB08] is defined on the vertex set X^λ via connecting two points $X_i,X_j \in X^\lambda$ by an edge whenever X_i is one of the k nearest points to X_j in $X^\lambda \setminus \{X_j\}$ or the other way around. For Poisson point processes, for any $k \in \mathbb{N}$, $g_{\mathbf{U}}(k,X^\lambda)$ has at most one infinite cluster [HM96], and so do Poisson–Gilbert graphs in all dimensions [MR96]. Hence, it would be surprising if the same assertion failed for the SINR graph.

Connectivity of finite SINR graphs. We have seen that SINR graphs are contained in certain Gilbert graphs and also in certain (undirected or bidirectional) k-nearest neighbour graphs, and also that among certain conditions they are well-behaved with respect to percolation. On the other hand, questions of connectivity of *finite* graphs have been studied in the Gilbert and undirected k-nearest neighbour cases (see [FM07, Chapter 3] for an overview), but we are not aware of analogous results for finite SINR graphs (apart from the ones of [V12], which are based on a non-standard notion of connectivity), while such assertions would be highly interesting both mathematically and for applications in telecommunications.

5.3.2.2. Estimates on the critical interference cancellation factor

The case of in-degrees being bounded by two. As we discussed in Section 4.1.4.1, in case $N_0,P,\tau > 0$ and $\gamma \in [\frac{1}{2\tau}, \frac{1}{\tau})$, the directed SINR graph $g^{\to}_{(N_0,\gamma,\tau,P)}(X^\lambda)$ has in-degrees bounded by 2, and thus the degrees in the undirected SINR graph $g_{(N_0,\gamma,\tau,P)}(X^\lambda)$ are also at most 2, so that connected components of the latter graph are either isolated points or paths that are infinite in zero, one or two directions. This reminds of a one-dimensional percolation process, which typically does not contain an infinite cluster. Now, the question is under what conditions on the parameters one can actually verify absence of percolation.

Proving the high-confidence result by Balister and Bollobás. To the author's best knowledge, the high-confidence result of [BB08] that the bidirectional k-nearest neighbour graph of a Poisson point process in $\mathbb{R}^2$ percolates if and only if $k \geq 5$ has still not been proved. This assertion would imply that in the two-dimensional Poisson case, for $\tau,N_0,P > 0$, $\lambda \mapsto \gamma^*(\lambda)$ is uniformly bounded by $1/(4\tau)$, and $\mathbb{P}(g_{(1/(4\tau),N_0,\tau,P)}(X^\lambda) \text{ percolates}) = 0$. This would improve the trivial bound $\gamma^*(\lambda) \leq 1/\tau$ by a factor of 4.[39] We note that percolation for the bidirectional k-nearest neighbour graph was proved in [BB08] for $k \geq 15$.

Does $\gamma^*(\lambda)$ always tend to zero as $\lambda \to \infty$? In case ℓ satisfies Assumption (ℓ), Λ is b-dependent, and $N_0 > 0$ or ℓ has compact support, it was easy to verify that $\gamma^*(\lambda) \to 0$ as $\lambda \to \infty$ (cf. Section 4.1.4.2). It would be interesting to know whether such a statement always holds for SINR graphs with unbounded edge length (i.e., in case $N_0 = 0$ and supp ℓ is not compact) and whether the b-dependence assumption can be replaced by,

[39] Numerical results of [DBT05] indicate that for a certain choice of $\tau,N_0,P > 0$, $\gamma^*(\lambda) \geq 1/(50\tau)$ occurs for some $\lambda > 0$.

e.g., stabilization. Here, Assumption (ℓ) is to be emphasized; indeed, if ℓ becomes unbounded near 0, then it is expected [DBT05] that $\lambda \mapsto \gamma^*(\lambda)$ is monotone increasing (cf. Sections 5.1.2 and 5.3.2.1).

The assertions $\gamma^*(\lambda) = \mathcal{O}(1/\lambda)$ and $\gamma^*(\lambda) = \Omega(1/\lambda)$ under weaker assumptions. We have verified these bounds (cf. Proposition 4.1.16 and Corollary 4.1.17) for two-dimensional, b-dependent Cox processes, under local positivity assumptions on Λ, and for the lower bound we also assumed that ℓ has compact support, generalizing the proofs of [DBT05] for the Poisson case. Now, a natural question is whether the b-dependence assumption can be dropped. However, we are not aware of any relevant example of Λ that is not b-dependent but locally bounded away from zero on some compact set, hence we suggest that one should rather start with investigating whether the local positivity assumption can be dropped. As for the upper bound, without this assumption, the SNR graph may not have a supercritical phase (even if Λ is b-dependent), which trivially implies that $\gamma^*(\lambda) = \mathcal{O}(1/\lambda)$ since $\lambda \mapsto \gamma^*(\lambda) \equiv 0$. The question is whether there is an intermediate case between local positivity and complete subcriticality for stabilizing Cox processes for $d = 2$. On the other hand, the lower bound certainly requires some local positivity or connectivity assumption on Λ. Indeed, if $\Lambda(Q_n)$ can be zero with positive probability for all $n > 0$, then the corresponding SNR graph may not have a supercritical phase in case of a small connection radius. We expect that in order to investigate higher-dimensional or non-b-dependent cases, one needs to employ proof techniques that are substantially different from the ones in [DBT05].

A. Appendix

A.1. Representations and strict convexity of the entropy term

We defined the entropy term $\Psi \mapsto \mathrm{I}(\Psi)$ via the formula (2.23), which we interpreted in Section 2.3.2. It is easy to see that (2.23) is equivalent to the representation in (2.89) that we used for analytical investigations. Now, we show that (2.23) is also equivalent to the expression (2.63) that arises from the combinatorics in Section 2.9.1. Afterwards, we verify strict convexity of the sum of relative entropies $\sum_{m=0}^{\infty} \mathcal{H}_W(\mu_m \mid \mu c_m)$ on its level sets.

Recall that for $k \in \mathbb{N}$ and $\xi,\eta \in \mathcal{M}(W^k)$, we have $\mathcal{H}_{W^k}(\xi|\eta) = H_{W^k}(\xi|\eta) - \xi(W^k) + \eta(W^k)$. Further, for an admissible trajectory setting $\Psi = ((\nu_k)_{k\in[k_{\max}]},(\mu_m)_{m\in\mathbb{N}_0})$, recall the measure $M = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l \nu_k = \sum_{m=0}^{\infty} m\mu_m$. Starting from the definition of $\mathrm{I}(\cdot)$, in (2.23), we compute

$$
\begin{aligned}
\mathrm{I}(\Psi) &= \sum_{k=1}^{k_{\max}} \mathcal{H}_{W^k}\big(\nu_k \mid \mu \otimes M^{\otimes(k-1)}\big) + \sum_{m=0}^{\infty} \mathcal{H}_W(\mu_m \mid \mu c_m) + \mu(W)\Big(1 - \sum_{k=1}^{k_{\max}} M(W)^{k-1}\Big) - \frac{1}{\mathrm{e}} \\
&= \sum_{k=1}^{k_{\max}} H_{W^k}\big(\nu_k \mid \mu \otimes M^{\otimes(k-1)}\big) - \sum_{k=1}^{k_{\max}} \nu_k(W^k) + \mu(W) \sum_{k=1}^{k_{\max}} M(W)^{k-1} + \sum_{m=0}^{\infty} \mathcal{H}_W(\mu_m \mid \mu c_m) \\
&\quad + \mu(W)\Big(1 - \sum_{k=1}^{k_{\max}} M(W)^{k-1}\Big) - \frac{1}{\mathrm{e}} \\
&= \sum_{k=1}^{k_{\max}} H_{W^k}\big(\nu_k \mid \mu \otimes M^{\otimes(k-1)}\big) + \sum_{m=0}^{\infty} H_W(\mu_m \mid \mu c_m) - \frac{1}{\mathrm{e}},
\end{aligned}
$$

where we used (2.42) and the fact that $\sum_{k=1}^{k_{\max}} \nu_k(W^k) = \mu(W)$ by (i) in (2.19). By the definition of the measure M, it suffices to show that

$$
\sum_{k=1}^{k_{\max}} H_{W^k}\big(\nu_k \mid \mu \otimes M^{\otimes(k-1)}\big) = \sum_{k=1}^{k_{\max}} H_{W^k}(\nu_k|\mu^{\otimes k}) - H_W(M|\mu). \tag{A.1}
$$

Clearly, if any of the sides of (A.1) is infinite, then so is the other side. Else, we verify (A.1) as follows

$$
\begin{aligned}
&\sum_{k=1}^{k_{\max}} H_{W^k}\big(\nu_k \mid \mu \otimes M^{\otimes(k-1)}\big) \\
&\quad = \sum_{k=1}^{k_{\max}} \int_{W^k} \mathrm{d}\nu_k(x_0,\dots,x_{k-1})\Big[\log \frac{\mathrm{d}\nu_k}{\mathrm{d}\mu^{\otimes k}}(x_0,\dots,x_{k-1}) - \log \frac{\mathrm{d}(\mu \otimes M^{\otimes(k-1)})}{\mathrm{d}\mu^{\otimes k}}(x_0,\dots,x_{k-1})\Big] \\
&\quad = \sum_{k=1}^{k_{\max}} \int_{W^k} \mathrm{d}\nu_k(x_0,\dots,x_{k-1})\Big[\log \frac{\mathrm{d}\nu_k}{\mathrm{d}\mu^{\otimes k}}(x_0,\dots,x_{k-1}) - \log \Big(\prod_{l=1}^{k-1} \frac{\mathrm{d}M}{\mathrm{d}\mu}(x_l)\Big)\Big] \\
&\quad = \sum_{k=1}^{k_{\max}} \int_{W^k} \mathrm{d}\nu_k(x_0,\dots,x_{k-1}) \log \frac{\mathrm{d}\nu_k}{\mathrm{d}\mu^{\otimes k}}(x_0,\dots,x_{k-1}) - \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \int_W \pi_l \nu_k(\mathrm{d}x_l) \log \frac{\mathrm{d}M}{\mathrm{d}\mu}(x_l) \\
&\quad = \sum_{k=1}^{k_{\max}} H_{W^k}(\nu_k|\mu^{\otimes k}) - H_W(M|\mu).
\end{aligned}
$$

As for the strict convexity of $\sum_{m=0}^{\infty} \mathcal{H}_W(\mu_m \mid \mu c_m)$ on its level sets, we use the identity $\sum_{m=0}^{\infty} c_m = 1$ and the one $\sum_{m=0}^{\infty} \mu_m = \mu$ for any $(\mu_m)_{m\in\mathbb{N}_0}$ that is contained in an admissible trajectory setting Ψ with $\mathrm{I}(\Psi) < \infty$, and we obtain

$$\begin{aligned}\sum_{m=0}^{\infty}\mathcal{H}_W(\mu_m\mid\mu c_m)&=\sum_{m=0}^{\infty}\int_W\Big(\mathrm{d}\mu_m\log\frac{\mathrm{d}\mu_m}{c_m\mathrm{d}\mu}-\mathrm{d}\mu_m+c_m\mathrm{d}\mu\Big)=\sum_{m=0}^{\infty}\int_W\mathrm{d}\mu_m\log\frac{\mathrm{d}\mu_m}{c_m\mathrm{d}\mu}\\&=\mu(W)\sum_{m=0}^{\infty}\frac{\mu_m(W)}{\mu(W)}\int_W\Big(\frac{\mathrm{d}\mu_m}{\mu_m(W)}\log\frac{\frac{\mathrm{d}\mu_m}{\mu_m(W)}}{\frac{\mathrm{d}\mu}{\mu(W)}}+\mathrm{d}\mu_m\log\frac{\mu_m(W)}{c_m\mu(W)}\Big)\\&=\mu(W)\sum_{m=0}^{\infty}\frac{\mu_m(W)}{\mu(W)}\mathcal{H}_W\Big(\frac{\mu_m}{\mu_m(W)}\Big|\frac{\mu}{\mu(W)}\Big)+\mu(W)\mathcal{H}_{\mathbb{N}_0}\Big(\Big(\frac{\mu_m(W)}{\mu(W)}\Big)_{m\in\mathbb{N}_0}\Big|(c_m)_{m\in\mathbb{N}_0}\Big).\end{aligned}\tag{A.2}$$

Now, by [DZ98, Proof of Lemma 6.2.12], for any Polish space X and $\nu\in\mathcal{M}(X)$, $\mathcal{H}_X(\cdot\mid\nu)$ is strictly convex on its level sets. Since W and $\mathbb{N}_0$ are Polish spaces (the latter one with any metric that generates the discrete topology), for any $m\in\mathbb{N}_0$, the sum on the right-hand side of (A.2) is a convex combination of functions $W\to[0,\infty]$ that are strictly convex in μ_m on their level sets, hence it is strictly convex in $(\mu_m)_{m\in\mathbb{N}_0}$ on its level sets. Further, since $\mu_m\mapsto\mu_m(W)$ is linear, the remaining term is also convex in $(\mu_m)_{m\in\mathbb{N}_0}$. This implies strict convexity of $\sum_{m=0}^{\infty}\mathcal{H}_W(\mu_m\mid\mu c_m)$ on its level sets. Since each term in I in (2.23) is bounded from below as well as S and M, it follows that $\Psi\mapsto\mathrm{I}(\Psi)+\gamma\mathrm{S}(\Psi)+\beta\mathrm{M}(\Psi)$ is strictly convex on its level sets for any $\beta,\gamma\geq 0$.

A.2. Tessellations based on Poisson point processes

Let us recall the Poisson point process based tessellation processes that occurred in the present thesis, i.e., the Poisson–Voronoi, the Poisson–Delaunay, and the Poisson line tessellation. We use [BB09, Chapter 4], [CGHJNP18], [HJC17, Section 3.1], [L12, Section 2], and [MS08]. Further, the definition of Manhattan grids that we present originates from personal communication with B. Jahnel and R. Patterson (2017); see [JT19, Section 1] for a more general definition with similar properties.

A *tessellation* of $\mathbb{R}^d$, $d\in\mathbb{N}$, is a countable collection of pairwise disjoint, non-empty open sets, such that the union of the closures of the open sets equals $\mathbb{R}^d$, and any compact subset of $\mathbb{R}^d$ intersects with only finitely many of the open sets. The following stochastic processes based on Poisson point processes are almost surely tessellations of the Euclidean space of the corresponding dimension.

The Poisson–Voronoi tessellation in $\mathbb{R}^d$. The Poisson–Voronoi tessellation in $\mathbb{R}^d$ is constructed in the following way. Let $X_{\mathbf{S}}=(X_i)_{i\in I}$ be a homogeneous Poisson point process with intensity $\lambda_{\mathbf{S}}>0$. The *Voronoi cell* of $X_i\in X_{\mathbf{S}}$ is defined as

$$\mathfrak{C}(X_i)=\mathfrak{C}(X_i,X_{\mathbf{S}})=\{x\in\mathbb{R}^d\colon |x-X_i|=\min_{j\in I}|x-X_j|\}.$$

X_i is called the *cell centroid* of $\mathfrak{C}(X_i)$. Then, almost surely, we have that for all $X_i\in X_{\mathbf{S}}$, $\mathfrak{C}(X_i)$ is a convex polytope, in particular it is bounded. We call the collection of cells $\{\mathfrak{C}(X_i)\}_{i\in I}$ a *Poisson–Voronoi tessellation.* If we let S be the union of edges (i.e., one-dimensional faces) of $\{\mathfrak{C}(X_i)\}_{i\in I}$, then S is a stationary segment process in $\mathbb{R}^d$. Thus, the intensity measure $\Lambda(\cdot)=\nu_1(S\cap\cdot)$ is stationary, exponentially stabilizing (see Section 4.2.1.2, or [HJC17, Section 3.1] for a more precise formulation), and asymptotically essentially connected. We call S the edge set of a Poisson–Voronoi tessellation and Λ a random intensity measure given by (the edge set of) a Poisson–Voronoi tessellation.

The Poisson–Delaunay tessellation in $\mathbb{R}^d$ and its relation to the Poisson–Voronoi tessellation. Let V be the set of vertices (0-dimensional faces) of $\{\mathfrak{C}(X_i)\}_{i\in I}$. Then, almost surely, each vertex $v\in V$ is incident to precisely $d+1$ points $Y_{v,1},\dots,Y_{v,d+1}$ of $X_{\mathbf{S}}$; these are precisely the points X_i of $X_{\mathbf{S}}$ such that $|v-X_i|=\min_{j\in I}|v-X_j|$. These $d+1$ points are the vertices of the *Delaunay cell* $\mathfrak{C}'(v)=\mathfrak{C}'(v,X_{\mathbf{S}})$, which is a d-dimensional simplex. The collection of Delaunay cells $\{\mathfrak{C}'(v)\}_{v\in V}$ is called a *Poisson–Delaunay tessellation.* Almost surely, for all $k\in[d]\cup\{0\}$, any k-dimensional face of the Poisson–Delaunay tessellation is a k-dimensional simplex. The union S' of the edges of $\{\mathfrak{C}'(v)\}_{v\in V}$ is another stationary segment process in $\mathbb{R}^d$. We call S' the edge set of a Poisson–Delaunay tessellation and $\Lambda'(\cdot) = \nu_1(S'\cap\cdot)$ a random intensity measure given by (the edge set of) a Poisson–Delaunay tessellation. Then, as [HJC17, Section 3.1] indicated, Λ' is exponentially stabilizing and asymptotically essentially connected.

The Voronoi tessellation $\{\mathfrak{C}(X_i)\}_{i\in I}$ and the Delaunay tessellation $\{\mathfrak{C}'(v)\}_{v\in V}$ based on the same Poisson point process $X_\mathbf{S}$ are *dual*, that is, given $X_\mathbf{S}$, there is a one-to-one correspondence between the k-dimensional faces of one of the tessellations and the $(d-k)$-dimensional faces of the other one for all $k \in [d] \cup \{0\}$. In particular, the vertices of the Delaunay tessellation correspond to the cells of the Voronoi one and vice versa, and for $d = 2$, there is a bijection between the sets of edges of the two tessellations.[40] The Voronoi tessellation is *normal*: almost surely, for $k \in [d] \cup \{0\}$, the number of points of $X_\mathbf{S}$ whose cells share a $(d-k)$-dimensional face of the Voronoi tessellation equals $k + 1$, independently of the choice of the face. In particular, each vertex $v \in V$ has the same degree (namely, $d + 1$). On the other hand, the Delaunay tessellation is not normal and has unbounded degrees.

This way, we have defined a Poisson–Delaunay tessellation in terms of its dual Voronoi tessellation. We note that another construction with no reference to the Voronoi tessellation is possible. Indeed, according to [BB09, Section 4.4], writing $X_\mathbf{S} = (X_i)_{i\in I}$, for X_i, $i \in I$, let us define the set $\mathfrak{D}(X_i) = \mathfrak{D}(X_i,X_\mathbf{S})$ of *Delaunay neighbours* of X_i in $X_\mathbf{S}$ as $\{X_{j_1(i)}, \dots ,X_{j_d(i)}\}$, where $\{j_1(i), \dots ,j_d(i)\}$ is the unique subset of $I \setminus \{i\}$ with d elements such that the open ball circumscribed on the points $X_i,X_{j_1(i)}, \dots ,X_{j_d(i)}$ does not contain any element of $X_\mathbf{S}$. Then, almost surely, $\mathfrak{D}(X_i)$ is well-defined for all $i \in I$. Thus, one can define an edge set S' via connecting each point of $X_\mathbf{S}$ by an edge to each of its Delaunay neighbours, and this edge set defines the tessellation equal to the Poisson–Delaunay tessellation that we defined earlier.

One can analogously define Voronoi and Delaunay tessellations for different point processes, under suitable assumptions of simpleness and nonequidistantness.

The Poisson line tessellation in $\mathbb{R}^2$ and related rectangular processes. In $\mathbb{R}^2$, any straight line e is uniquely determined by the pair (x_e,θ_e) of the signed perpendicular distance $x_e \in \mathbb{R}$ from o and the angle $\theta_e \in [0,\pi)$ between the orientation vector of e and the x-axis. For $(x,\theta) \in \mathbb{R} \times [0,\pi)$, let us write $e(x,\theta)$ for the unique line corresponding to these parameters. Let $\Phi = \{(X_i,\theta_i)\}_{i\in I}$ be a homogeneous Poisson point process on $\mathbb{R} \times [0,\pi)$ and let $S'' = \{e(X_i,\theta_i)\}_{i\in I}$. Then S'' is a stationary segment process on $\mathbb{R}^2$ consisting of infinite lines. The collection of the cells given by the crossings of the lines in S'' is called the *Poisson line tessellation* in $\mathbb{R}^2$. Defining $\Lambda''(\cdot) = \nu_1(S'' \cap \cdot)$, Λ'' is a stationary random intensity measure with infinite-range dependencies; in particular, it is not stabilizing.

Another class of tessellation processes on $\mathbb{R}^2$ defined by infinite lines and lacking stabilization is the one of *Manhattan grids*, constructed as follows. Let X^1, X^2 be two stationary independent renewal processes on $\mathbb{R}$ with arrival times $(t^1_i)_{i\in\mathbb{Z}}$ and $(t^2_i)_{i\in\mathbb{Z}}$, respectively. The collection of the cells given by the crossings of the vertical lines $x = t^1_i$ and the horizontal lines $y = t^2_i$ yields a tessellation process in $\mathbb{R}^2$, in which almost surely each cell is a rectangle. This process is stationary but not isotropic; it can be made isotropic (i.e., rotationally invariant) via rotating it by a uniform random angle in $[0,\pi)$ that is independent of the tessellation. The mathematically easiest tractable case is when both X^1 and X^2 are homogeneous Poisson processes with the same parameter, which can be seen as a simplified, rectangular version of the Poisson line tessellation.

A.3. Proof of the degree bound in general SINR graphs

In this section we follow [DBT05, Section II-A]. The degree bound result [DBT05, Theorem 1] was proven for SINR graphs with random transmitted powers (cf. Section 4.2.3.4) in the Poisson case. We now formulate and verify the same result for general simple point processes, in which case the proof does not change.

Let us fix $\tau,\gamma > 0$, $N_0 \geq 0$, and a marked point process $\mathbf{\Phi} = (X_i,P_i)_{i\in I}$ with values in $\mathbb{R}^d \times (0,\infty)$, $d \in \mathbb{N}$, such that $\Phi = (X_i)_{i\in I}$ is a simple point process in $\mathbb{R}^d$. Then we define the directed SINR graph $g^{\rightarrow}_{(\gamma,N_0,\tau)}(\mathbf{\Phi})$ to

[40] The intensity parameter $\lambda_\mathbf{S} > 0$ that corresponds to the constraint $\mathbb{E}[\Lambda'(Q_1)] = 1$ may be different from the one corresponding to $\mathbb{E}[\Lambda(Q_1)] = 1$. However, in the two-dimensional case, the two values of $\lambda_\mathbf{S}$ are equal thanks to the duality between the Voronoi and the Delaunay tessellation, cf. [MS08, Section 4.1].

be the directed graph with vertex set Φ where for any $i,j \in I, i \neq j$, there is a directed edge from X_i to X_j if and only if

$$\mathrm{SINR}((X_i,P_i),\,(X_j,P_j),\,\mathbf{\Phi}) = \frac{P_i\ell(|X_i - X_j|)}{N_0 + \gamma\sum_{k\notin\{i,j\}} P_k\ell(|X_k - X_j|)}$$

is larger than τ,[41] where the path-loss function ℓ is assumed to be nonnegative, more precisely, it is a function $\ell\colon [0,\infty) \to [0,\infty)$. As usual, the undirected SINR graph is defined as the undirected graph with vertex set Φ where for any $i,j \in I, i \neq j$, (X_i,X_j) is an edge if and only if X_i and X_j are connected by a directed edge in $g^{\rightarrow}_{(\gamma,N_0,\tau)}(\mathbf{\Phi})$ in both directions. Then, it is clear that in case P_i are constant equal to $P > 0$, $g^{\rightarrow}_{(\gamma,N_0,\tau)}(\mathbf{\Phi})$ equals $g^{\rightarrow}_{(\gamma,N_0,\tau,P)}(\Phi)$, and similarly for the undirected graphs. We have the following.

Theorem A.3.1 ([DBT05])**.** *Under the above choices of the parameters, all in-degrees in $g^{\rightarrow}_{(\gamma,N_0,\tau)}(\mathbf{\Phi})$ are less than $1 + 1/(\tau\gamma)$.*

Proof. Let us choose $i \in I$ and let us assume that X_i has in-degree $N \in \mathbb{N}_0$ in the graph $g^{\rightarrow}_{(\gamma,N_0,\tau,P)}(\mathbf{\Phi})$. We show that $N < 1+1/(\tau\gamma)$. If $N \leq 1$, then the claim holds. Else, let $X_{j_1},\dots,X_{j_N}$ be such that there is a directed edge from X_{j_k} to X_i for all $k \in [N]$. Let $j_0 \in \{j_k\colon k \in [N]\}$ be such that $P_{j_0}\ell(|X_{j_0} - X_i|) = \min_{k\in[N]} P_{j_k}\ell(|X_{j_k} - X_i|)$; this is a positive quantity because otherwise $\mathrm{SINR}((X_{j_0},P_{j_0}),(X_i,P_i),\mathbf{\Phi})$ would be equal to zero and thus smaller than τ. Then, since there is an edge from X_i to X_{j_0}, we can estimate

$$\tau < \frac{P_{j_0}\ell(|X_{j_0} - X_i|)}{N_0 + \gamma\sum_{k\notin\{i,j_0\}} P_k\ell(|X_k - X_i|)} \leq \frac{P_{j_0}\ell(|X_{j_0} - X_i|)}{\gamma(N-1)P_{j_0}\ell(|X_{j_0} - X_i|)} \leq \frac{1}{(N-1)\gamma}.$$

That is, $N < 1 + 1/(\tau\gamma)$, as required. □

[41] As in Chapter 4, we will tacitly exclude the degenerate case $N_0 = \gamma = 0$ and use the convention that for $N_0 = 0$, $\mathrm{SINR}((X_i,P_i),\,(X_j,P_j),\,\mathbf{\Phi}) > \tau$ holds if and only if $P_i\ell(|X_j - X_i|) > \gamma\sum_{k\notin\{i,j\}} P_k\ell(|X_k - X_j|)$.

A.4. Bibliographic details of the parts of this thesis

This thesis has been prepared using the following versions of the papers that it is based on:

- [KT18]: W. König and A. Tóbiás, A Gibbsian model for message routeing in highly dense multihop networks, *arXiv:1704.03499v3* (2018). Third version, submitted to *arXiv* on 13 August 2018.
- [KT19]: W. König and A. Tóbiás, Routeing properties in a Gibbsian model for highly dense multihop networks, *arXiv:1801.04985v3* (2019). Third version, submitted to *arXiv* on 15 February 2019.
- [T18]: A. Tóbiás, Signal to interference ratio percolation for Cox point processes. *arXiv:1808.09857* (2018). First version, submitted to *arXiv* on 29 August 2018.

The paper [KT18] was accepted for publication in the *ALEA Latin American Journal of Probability and Statistics* in January 2019 (after the submission of the first version of the thesis). As of today, the status of the papers [KT19, T18] is submitted but not yet accepted; see the Curriculum Vitae for further details.

Unlike the first version of the thesis, this final version also includes references to the author's joint work [JT19] with B. Jahnel, which was finished on 7 February 2019, between the submission of the first version of the thesis (on 11 December 2018) and the thesis defense. This paper has also been submitted but not yet accepted.

A.5. Statement about the author's own contribution to the co-authored parts of this thesis

The material of Chapters 2 and 3 about the Gibbsian model for message routeing in highly dense multihop networks and its routeing properties is the result of collaboration between the author's PhD supervisor W. König and the author. The text of the original papers [KT18, KT19] on which these chapters are based was written jointly by both collaborators, and the additional parts of these chapters were written by the author, based on further discussions of W. König. The author thanks C. Hirsch, B. Jahnel, M. Klimm, S. Morgenstern, R. Patterson, and M. Renger for interesting discussions and comments about the content of these chapters. The author also thanks an anonymous reviewer for pointing out some inaccuracies in an earlier version of [KT18] regarding the discretization procedure.

The content of Chapter 4 is based on the single-author paper [T18] written by the author. The author thanks W. König and B. Jahnel for checking the manuscript and B. Jahnel also for his ideas and hints used for extending Theorem 4.1.6 from small $\alpha > 0$ to all $\alpha > 0$. The author also thanks C. Hirsch and A. Hinsen (né Wapenhans) for interesting discussions and comments related to SINR percolation for Cox point processes. He thanks R. Löffler for pointing out several inaccuracies in a previous version of the manuscript.

These remarks also apply to the introductions of the respective chapters in Chapter 1. The author thanks C. Hirsch, B. Jahnel, and W. König for insightful comments on the subject of Chapter 5, which was not included in the three aforementioned papers.

Summarizing, the only co-author of the present thesis is W. König. Nevertheless, the thesis also includes references to the author's joint paper [JT19] with Benedikt Jahnel. For further details of the relation between this thesis and the papers [KT18, KT19, T18], see Section 1.3.

A.6. Index of notations

Preface to the index of notations and the index. This thesis comes with a substantial amount of notation that has to be introduced before stating and proving the results. During proofs, we have always intended to recall the necessary notation coming from earlier sections. As a second source of help for the reader, here we summarize the global notations (i.e., the ones that do not only appear in one particular proof or discussion) grouped according to the place of their first occurrence. We often do not repeat precise mathematical definitions but rather highlight the essential properties and interpretations of the entities. We followed similar principles during creating the index; we decided to include some expressions in the index at multiple places, so that the reader can find them more conveniently.

General notations and abbreviations

$\mathcal{M}(Y)$	set of finite Borel measures on the topological space Y
$B_\varrho(x)$	open ball of radius $\varrho > 0$ around $x \in X$ in the metric space (X,d_0)
d	dimension of the model being considered
o	origin of $\mathbb{R}^d$
$\#$	cardinality of a countable set (we write ∞ for countably infinite)
$\mathbb{N}$	$= \{1,2,\dots\}$
$\mathbb{N}_0$	$= \{0,1,2,\dots\}$
$[n]$	$= \{1,2,\dots,n\}$ for $n \in \mathbb{N}$; we write $[0] = \varnothing$
$a \vee b$	$= \max\{a,b\}$ $(a,b \in \mathbb{R})$
$a \wedge b$	$= \min\{a,b\}$ $(a,b \in \mathbb{R})$
A^o	interior of the set A
$\overline{A}$	closure of the set A
A^{c}	complement of the set A
∂A	boundary of the set A
$\lvert x \rvert$	Euclidean norm of $x \in \mathbb{R}^d$
$\lVert x \rVert_p$	ℓ^p-norm of $x \in \mathbb{R}^d$, $p \in [1,\infty]$
$\mathrm{dist}_p(\varphi,\psi)$	$= \inf\{\lVert x-y \rVert_p \colon x \in \varphi, y \in \psi\}$, ℓ^p-distance between $\varphi,\psi \subseteq \mathbb{R}^d$, $p \in [1,\infty]$
dist	$= \mathrm{dist}_2$
$f(x) = \mathcal{O}(g(x))$	(as $x \to \infty$, for $f,g \colon [0,\infty) \to [0,\infty)$): if $\exists K > 0$, $\exists c > 0$ such that $\forall x > K$, $f(x) \leq cg(x)$
$f(x) = \Omega(g(x))$	defined to be true if and only if $g(x) = \mathcal{O}(f(x))$
$f(x) = \Theta(g(x))$	defined to be true if and only if $f(x) = \mathcal{O}(g(x))$ and $g(x) = \mathcal{O}(f(x))$
$f(x) = o(g(x))$	$f(x)/g(x)$ tends to 0 in the corresponding limit
$\sim$	the quotient of the two sides tends to one
$\approx$	the quotient of the two sides stays bounded and bounded away from 0
A^B	set of functions with domain B mapping to A (for arbitrary sets A,B)
$\langle \nu, f \rangle$	integral of the function f against the measure ν
$\Longrightarrow$	weak convergence of finite measures
supp f	support of the function f
esssup$X$	essential supremum of the real-valued random variable X
essinf$X$	essential infimum of the real-valued random variable X
LDP	large deviation principle
Leb	Lebesgue measure (in the respective dimension)
δ_x	Dirac delta measure at $x \in \mathbb{R}^d$ $(\delta_x(A) = \mathbb{1}\{x \in A\})$
i.i.d.	independent and identically distributed
a.s.	almost surely
SIR	signal-to-interference ratio
SINR	signal-to-interference plus noise ratio
SNR	signal-to-noise ratio
STIR	signal-to-total received power ratio
STINR	signal-to-total received power plus noise ratio

Definition of the Gibbsian model (Chapter 2, used also in Chapter 3)

W	communication area: a compact subset of $\mathbb{R}^d$ with $o \in W$ and $\mathrm{Leb}(W) > 0$
μ	$\in \mathcal{M}(W)$ intensity measure, absolutely continuous with $\mu(W) > 0$
$X^\lambda = (X_i)_{i \in I^\lambda}$	Poisson point process with intensity $\lambda\mu$, $\lambda > 0$
$N(\lambda)$	$= \# I^\lambda$, number of points of X^λ
$(\Omega,\mathcal{F},\mathbb{P})$	probability space on which $(X^\lambda)_{\lambda>0}$ is defined
L_λ	rescaled empirical measure of X^λ

Message trajectory configurations; individual trajectories and their coordinates

S^i, s^i	a random respectively deterministic trajectory of a message transmitted by the user X_i
$K_i = S^i_{-1}$, s^i_{-1}	number of hops of S^i respectively s^i
S^i_0, s^i_0	$= X_i$, transmitter of the message having trajectory S^i respectively s^i
$S^i_1,\dots,S^i_{K_i-1}$, $s^i_1,\dots,s^i_{s^i_{-1}-1}$	$\in X^\lambda$, the $K_i - 1 = S^i_{-1} - 1$ resp. $s^i_{-1} - 1$ relays of the trajectory S^i respectively s^i
$S^i_{K_i}$, $s^i_{s^i_{-1}}$	$= o$, receiver of the message having trajectory S^i respectively s^i
$k_{\max}$	$\in \mathbb{N}$, maximal number of hops of a message trajectory
S, s	a trajectory configuration of the form $S = (S^i)_{i\in I^\lambda}$ respectively $s = (s^i)_{i\in I^\lambda}$
$\mathcal{S}^i_{k_{\max}}(X^\lambda)$	the set of all admissible trajectories s^i of the i-th user with at most $k_{\max}$ hops
$\mathcal{S}_{k_{\max}}(X^\lambda)$	the set of all admissible configurations of trajectories s with at most $k_{\max}$ hops each
W^k	$= W^{\{0,1,\dots,k-1\}}$

Model definition, continued

$\pi_l\nu$	l-th marginal of the measure $\nu \in \mathcal{M}(W^k)$, $l \in \{0,1,\dots,k-1\}$
$R_{\lambda,k}(s)$	rescaled emp. measure of the k-hop trajectories in the trajectory conf. s, $k \in [k_{\max}]$
f_k	generalized interference penalty functional $f_k\colon \mathcal{M}(W) \times W^k \to \mathbb{R}$
$\mathfrak{S}$	interference penalty term (generalized or the true one, depending on the context)
$m_i(s)$	number of incoming hops at the user (relay) X_i in the trajectory configuration s
η	generalized congestion penalty functional $\eta\colon \mathbb{N}_0 \to \mathbb{R}$, superlinearly increasing
$\mathfrak{M}$	congestion penalty term (generalized or true, depending on the context)
γ	> 0 prefactor for interference penalization
β	≥ 0 prefactor for congestion penalization
$\mathrm{P}^{\gamma,\beta}_{\lambda,X^\lambda}$	(quenched) Gibbs distribution on $\mathcal{S}_{k_{\max}}(X^\lambda)$
$Z^{\gamma,\beta}_\lambda(X^\lambda)$	(quenched) partition function; the normalizing constant of $\mathrm{P}^{\gamma,\beta}_{\lambda,X^\lambda}$
ℓ	path-loss function; continuous, monotone decreasing, strictly positive
$\mathrm{SIR}(X_i, x, L_\lambda)$	SIR of the transmission from $X_i \in X^\lambda$ to $x \in W$ with respect to the measure L_λ
$P_{\lambda,m}(s)$	rescaled emp. measure of users X_i with $m_i(s) = m$ for a trajectory conf. s, $m \in \mathbb{N}_0$
$\Psi_\lambda(s)$	$= ((R_{\lambda,k}(s))_{k\in[k_{\max}]},(P_{\lambda,m}(s))_{m\in\mathbb{N}_0})$

The high-density limit

ν_k	$\in \mathcal{M}(W^k)$ limiting measure of k-hop message trajectories
μ_m	$\in \mathcal{M}(W)$ limiting measure of users receiving m incoming hops
Ψ	admissible trajectory setting $\Psi = ((\nu_k)_{k=1}^{k_{\max}}, (\mu_m)_{m=0}^\infty)$
M	$= \sum_{m=0}^\infty m\mu_m = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l\nu_k$, limiting measure of all incoming hops for the admissible trajectory setting Ψ
$\mathcal{H}_V(\nu \mid \widetilde{\nu})$	relative entropy of $\nu \in \mathcal{M}(V)$ with respect to $\widetilde{\nu} \in \mathcal{M}(V)$ for a space V
$H_V(\nu \mid \widetilde{\nu})$	like $\mathcal{H}_V(\nu \mid \widetilde{\nu})$, but without subtracting the difference of total masses
condition (i)	$\sum_{k=1}^{k_{\max}} \pi_0 R_{\lambda,k}(s) = L_\lambda$ (discrete case), $\sum_{k=1}^{k_{\max}} \pi_0\nu_k = \mu$ (continuous case)
condition (ii)	$\sum_{m=0}^\infty P_{\lambda,m}(s) = L_\lambda$ (discrete case), $\sum_{m=0}^\infty \mu_m = \mu$ (continuous case)
condition (iii)	$\sum_{m=0}^\infty mP_{\lambda,m}(s) = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l R_{\lambda,k}(s)$ (discrete case), $\sum_{m=0}^\infty m\mu_m = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l\nu_k$ (continuous case)
$\mathrm{I}(\Psi)$	entropy term of Ψ
c_m	weight of the Poisson distribution with parameter $1/(\mathrm{e}\mu(W)$ at m
$\widetilde{f}_k(\cdot)$	$= f_k(\mu,\cdot)$
$\mathrm{S}(\Psi)$	limiting interference penalty term (generalized or true) of Ψ
$\mathrm{M}(\Psi)$	limiting congestion penalty term (generalized or true) of Ψ
A,B,C	Lagrange multipliers of the minimizer (functions that make the minimizer of the variational formula satisfy (i), (ii), and (iii))

$\mathcal{A}$	$= (\prod_{k=1}^{k_{\max}} \mathcal{M}(W^k)) \times \mathcal{M}(W)^{\mathbb{N}_0}$, the product space in which admissible trajectory settings are contained
$\Sigma_\lambda(s)$	$= (R_{\lambda,k}(s))_{k \in [k_{\max}]}$
Σ	asymptotic routeing strategy $\Sigma = (\nu_k)_{k=1}^{k_{\max}})$
$\mathcal{A}_0$	$= \prod_{k=1}^{k_{\max}} \mathcal{M}(W^k)$, the space in which asymptotic routeing strategies are contained
$\mathrm{S}(\Sigma)$	limiting interference penalty term (generalized or true) of Σ
$\mathrm{J}(\Sigma)$	entropy term of Σ

Proofs of the results of Chapter 2

Elements of the discretization procedure

r	> 0, a number such that $W = [-r,r]^d$ in the setting of Section 2.8.1
$\mathbb{B}$	$= \{3^{-n} \mid n \in \mathbb{N}\}$ set of discretization parameters
δ	$\in \mathbb{B}$ discretization parameter
W_δ	set of δ-subcubes
$W_1^\delta, \dots, W_{\delta^{-d}}^\delta$	elements of W_δ in an arbitrary but fixed order
$W_{\mathbb{B}}$	set of $x \in W$ such that for all $\delta \in \mathbb{B}$ there is a uniqe $j \in [\delta^{-d}]$ such that $x \in W_j^\delta$
$\mathcal{F}_\delta$	$= \sigma(W_\delta)$, the σ-field generated by W_δ
ν^δ	$\nu^\delta(\cdot) = \nu(\cdot \mid \mathcal{F}_\delta)$ conditional version of the measure ν given $\mathcal{F}_\delta$

Standard settings and their coordinates

$\underline{\Psi}$	standard setting, $\underline{\Psi} = ((\nu_k)_{k=1}^{k_{\max}}, ((\nu_k^\delta)_{k=1}^{k_{\max}})_{\delta \in \mathbb{B}}, ((\nu_k^{\delta,\lambda})_{k=1}^{k_{\max}})_{\delta \in \mathbb{B}, \lambda > 0},$ $(\mu_m)_{m=0}^\infty, ((\mu_m^\delta)_{m=0}^\infty)_{\delta \in \mathbb{B}}, ((\mu_m^{\delta,\lambda})_{m=0}^\infty)_{\delta \in \mathbb{B}, \lambda > 0}, (\mu^{\delta,\lambda})_{\delta \in \mathbb{B}, \lambda > 0})$
$\mu^{\delta,\lambda}$	$= L_\lambda^\delta$, rescaled empirical measure of δ-discretized user locations for intensity λ
$\nu_k^{\delta,\lambda}$	rescaled empirical measure of δ-discretized k-hop trajectories for intensity λ
ν_k^δ	weak limit of $\nu_k^{\delta,\lambda}$ as $\lambda \to \infty$
$\mu_m^{\delta,\lambda}$	rescaled empirical measure of δ-discretized locations of users receiving m incoming hops for intensity λ
μ_m^δ	weak limit of $\mu_m^{\delta,\lambda}$ as $\lambda \to \infty$
$P_\lambda(s)$	$= L_\lambda$, rescaled empirical measure of all trajectories for $s \in \mathcal{S}_{k_{\max}}(X^\lambda)$
$J^{\delta,\lambda}(\underline{\Psi})$	set of $s \in \mathcal{S}_{k_{\max}}(X^\lambda)$ corresponding to $\underline{\Psi}$ at the δ,λ-coordinates

Elements of proofs from Section 2.9

Ω_1	the event of probability 1 in which we prove our quenched results
$\mathrm{Lip}_1(W^k)$	the set of Lipschitz continuous functions from W^k to $\mathbb{R}$ with Lipschitz parameter at most 1 and uniform bound 1
d_k	Lipschitz bounded metric on W^k, $k \in \mathbb{N}$
d_0	product metric on $\mathcal{A}$ based on the d_k's
$\mathcal{M}_{\leq \alpha}(V)$	the set of elements ν of $\mathcal{M}(V)$ such that $\nu(V) \leq \alpha$, for a space V and $\alpha > 0$
$\mathfrak{Y}(s)$	contribution of the trajectory conf. s to the partition function $Z_\lambda^{\gamma,\beta}(X^\lambda)$
$G(\delta,\lambda)$	set of coll. of measures $\Psi^{\delta,\lambda} = ((\nu_k^{\delta,\lambda})_{k=1}^{k_{\max}}, (\mu_m^{\delta,\lambda})_{m=0}^\infty)$ with values in $\frac{1}{\lambda}\mathbb{N}_0$ s. t. $\sum_{k=1}^{k_{\max}} \pi_0 \nu_k^{\delta,\lambda} = L_\lambda^\delta$, $\sum_{m=0}^\infty \mu_m^{\delta,\lambda} = L_\lambda^\delta$, and $\sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l \nu_k^{\delta,\lambda} = \sum_{m=0}^\infty m \mu_m^{\delta,\lambda}$
$\mathcal{S}_{k_{\max},C}(X^\lambda)$	set of elements s of $\mathcal{S}_{k_{\max}}(X^\lambda)$ such that $\mathfrak{M}(s) \leq \lambda C$, for $C > 0$

$J^{\delta,\lambda}(\Psi^{\delta,\lambda})$	$= J^{\delta,\lambda}(\underline{\Psi})$ for any $\underline{\Psi}$ containing $\Psi^{\delta,\lambda} \in G(\delta,\lambda)$ at the respective coordinates
$\nu \leq \nu'$	for $\nu,\nu' \in \mathcal{M}(W^k)$, $k \in [k_{\max}]$: $\nu' - \nu \in \mathcal{M}(W^k)$
Φ	$= ((\tau_k)_{k=1}^{k_{\max}},(\sigma_m)_{m=0}^{\infty})$, perturbator for deriving the Euler–Lagrange equations
$\mathcal{M}_{\pm}(V)$	set of finite signed measures on the space V
$\mathcal{A}_{\pm}$	like $\mathcal{A}$, but contained with $\mathcal{M}_{\pm}$ instead of $\mathcal{M}$ on each coordinate
$\underline{\Sigma}$	transmission setting; collection of all ν_k, ν_k^{δ}, and $\nu_k^{\delta,\lambda}$ type coordinates in a standard setting
$K^{\delta,\lambda}(\underline{\Sigma})$	set of $s \in \mathcal{S}_{k_{\max}}(X^{\lambda})$ corresponding to $\underline{\Sigma}$ at the δ,λ-coordinates

Game-theoretic discussion of Section 2.6

n	$= N(\lambda)$, number of users
$\mathcal{S}$	$= \mathcal{S}_{k_{\max}}(X^{\lambda})$, set of admissible trajectory configurations
$\mathcal{S}^i$	$= \mathcal{S}^i_{k_{\max}}(X^{\lambda})$, $i \in [n]$, set of admissible trajectories of the i-th user
$C_i(s)$	individual cost of the i-th trajectory s^i for a trajectory configuration $s \in \mathcal{S}$
$C(s)$	total cost of the trajectory s
s^{-i}	$= (s^j)_{j\in[n],j\neq i}$
$s = (s^i, s^{-i})$	see the definition of s^{-i} above

Routeing properties: results and proofs in Chapter 3

$g(x,y)$	$= \frac{\int_W \ell(\|z-y\|)\mu(\mathrm{d}y)}{\ell(\|x-y\|)}$, $x,y \in W$: pair functional for the limiting interference penalty term
$T_{x_0}(k,x_1,\dots,x_{k-1})$	distribution of the typical trajectory from x_0 to o, where $\pi_0 T_{x_0}$ corresponds to the variable k and $\pi_l T_{x_0}$ to x_l, for $l \in [k-1]$

Regime 1: large communication areas with large transmitter–receiver distances

b	$= \int_{\mathbb{R}^d} \ell(\|x\|)\mathrm{d}x$
r	radius of the communication area $W = \overline{B_r(o)}$
r_0	$= \|x_0\|$, distance of the transmitter x_0 to the base station o
$a_k(x_0)$	summand in the sum $1/A(x_0)$ corresponding to k-hop trajectories for $k \in [k_{\max}]$, $x_0 \in W$, and defined analogously for $k \in \mathbb{N} \setminus [k_{\max}]$
ω_d	volume of the unit ball in $\mathbb{R}^d$
$k(r_0)$	number of hops from x_0 to o
t^*	unique minimizer of $t \mapsto dt + b\gamma t^{1-\alpha}$
$k^*(r_0)$	$\sim t^* r_0 / \log^{1/\alpha} r_0$, most likely number of hops from x_0 to o in regime 1
$\lim_{r,r_0}$	"our limit": $r_0 \to \infty$ with $r > r_0 \asymp r$, where $x_0 \in W = \overline{B_r(o)}$ satisfies $\|x_0\| = r_0$
$D_{\varepsilon,\delta}(k,x_0)$	set of all $(x_1,\dots,x_{k-1}) \in W^{k-1}$ such that at least δk hops (x_{l-1},x_l) deviate from their optimal length by at least $\varepsilon \log^{1/\alpha} r_0$, for $x_0 \in W$, cf. (3.13)
$s(r)$	a distance such that $r - s(r) \to \infty$ and $0 < r - s(r) = o(r)$ in our limit; s is an auxiliary function such that $\partial B_{s(r)}(o)$ is asymptotically far away from $\mathbb{R}^2 \setminus B_r(o)$
$K(r_0,r)$	index of the last hop where the trajectory from x_0 to o is at distance at least $s(r)$ away from o, and $K(r_0,r) = 0$ if such an index does not exist
b_0	$= \lim_{r\to\infty} \int_{B_r(o)} \ell(\|y - re_1\|)\,\mathrm{d}y$; limit of the smallest interference experienced by a user in W

Regime 2: strong penalization of interference

$[[x,y]]$	straight line through $x,y \in \mathbb{R}^d$
$[x,y]$	closed straight line segment between $x,y \in \mathbb{R}^d$
r	radius of the communication area $W = \overline{B_r(o)}$
$D_k^{\varepsilon}(x_0)$	set of $(x_1,\dots,x_{k-1}) \in W^{k-1}$ such that $\mathrm{dist}(x_l,[[x_0,o]]) > \varepsilon$ for at least one $l \in [k-1]$, for $k \in [k_{\max}]$, $x_0 \in W$

Regime 3: high local density of users

a	local density parameter
Δ	subset of W with $\mathrm{Leb}(W) > 0$ and a highly dense local population
μ^a	$= \mu + a\mathrm{Leb}\|_\Delta$, modified intensity measure, we consider the limit of high local density $a \to \infty$
ν_k^a, M^a, g^a	analogues of ν_k,M,g, respectively, with μ replaced by μ^a everywhere in their definitions
$g_\Delta(x,y)$	$= \frac{\int_\Delta \mathrm{d}z \ell(\|y-z\|)}{\ell(\|x-y\|)}$, effective interference term in the limit $a \to \infty$
$\ell_{\max}$	maximal value of $\ell(\| \cdot - \cdot \|)$ on $W \times W$
$\ell_{\min}$	minimal value of $\ell(\| \cdot - \cdot \|)$ on $W \times W$
$\Xi_{x_0}(A)$	$= \min_{x_1 \in A}[g_\Delta(x_0,x_1) + g_\Delta(x_1,o)]$, minimum interference penalty for a two-hop trajectory from x_0 to o via a relay in the set A, $A \subseteq W$ such that $A = \overline{A^o}$

Numerical examples

r_0^*	critical radius separating domination of one-hop and two-hop communication

SINR percolation for Cox processes (Chapter 4)

Continuum percolation for Cox processes

$\mathbb{M}$	space of Borel measures on $\mathbb{R}^d$, equipped with the evaluation σ-field
$Q_r(x)$	$= x + [-r/2,r+2]^d$, closed hypercube of side length $r \geq 0$ around $x \in \mathbb{R}^d$
Λ	intensity measure, a stationary random element of $\mathbb{M}$ with $\mathbb{E}[\Lambda(Q_1)] = 1$
X^λ	Cox process with intensity measure $\lambda\Lambda$, for $\lambda > 0$
Λ_B	restriction of Λ to the measurable set $B \subseteq \mathbb{R}^d$.
$g_r(X^\lambda)$	Gilbert graph of X^λ with connection radius r
$\lambda_{\mathrm{c}}(r)$	critical intensity for $g_r(X^\lambda)$
$R = (R_x)_{x \in \mathbb{R}^d}$	stabilization radii of the stabilizing intensity measure Λ
$R(Q_n(x))$	$= \sup_{y \in Q_n(x) \cap \mathbb{Q}^d} R_y$, for $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$
$\mathrm{supp}(\mu)$	support of the (possibly singular) measure μ

Notations of the main examples of Λ

$l = \{l_x\}_{x \in \mathbb{R}^d}$	stationary random field with $\mathbb{E}[l_o] = 1$ such that $\Lambda(\mathrm{d}x) = l_x \mathrm{d}x$, in case Λ is absolutely continuous
Ξ	random closed set, from the definition of a modulated Poisson point process (e.g., a Poisson–Boolean model)
λ_1,λ_2	≥ 0: intensities of Λ given by a modulated Poisson point process on Ξ respectively Ξ^{c}
k	$: \mathbb{R}^d \to [0,\infty)$ non-negative integrable kernel with compact support, from the definition of a shot-noise field
$X_{\mathbf{S}}$	homogeneous Poisson point process defining a shot-noise field, Poisson–Voronoi or Poisson–Delaunay tessellation, or Poisson p. p. modulated by a Poisson–Boolean model
$\lambda_{\mathbf{S}}$	intensity of $X_{\mathbf{S}}$
ν_1	one-dimensional Hausdorff measure
S	"street system", a stationary point process S with values in the space of line segments, e.g., a Poisson–Voronoi, Poisson–Delaunay or Poisson line tessellation
$\mathfrak{C}(X_i)$	Voronoi cell of $X_i \in X_{\mathbf{S}}$

SINR percolation: model definition and statement of results

ℓ	path-loss function; continuous, monotone decreasing, nonnegative
N_0	≥ 0 environmental noise
P	> 0 transmitted power (if $N_0 = 0$, we use the convention $P = 1$)
γ	≥ 0 interference cancellation factor (the case $\gamma = N_0 = 0$ is excluded)
τ	> 0 SINR threshold
$\mathrm{SINR}(X_i, X_j, X^\lambda)$,	SINR from $X_i \in X^\lambda$ to $X_j \in X^\lambda$ with respect to the Cox process X^λ
$\mathrm{SINR}(X_i, X_j, \Phi)$	defined analogously for a general simple point process Φ and $X_i,X_j \in \Phi$
$g_{(\gamma,N_0,\tau,P)}(X^\lambda)$	(undirected) SINR graph of $X^\lambda, \lambda > 0$, for the choice of parameters γ,N_0,τ,P,
$g_{(\gamma)}(X^\lambda)$	abbreviation for $g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ in case N_0,τ,P are clear from context
(X_i,X_j)	edge between X_i and X_j in a SINR graph (exists if both SINR's are larger than τ)
$g^{\rightarrow}_{(\gamma,N_0,\tau,P)}(X^\lambda)$	directed SINR graph of X^λ with parameters γ,N_0,τ,P
$X_i \to X_j$	directed edge from X_i to X_j in a directed SINR graph
$\mathrm{SNR}(X_i,X_j)$	SNR between two points X_i and X_j of a point process (independent of the other points of the process and equal to $\mathrm{SNR}(X_j,X_i)$)
$r_{\mathbb{B}}$	$= \ell^{-1}(\tau N_0/P)$, critical distance for the SNR being larger than τ
$G \preceq G'$	the two graphs G,G' have the same vertex set, and the edge set of G is contained in the one of G'
$\gamma^*(\lambda)$	supremum of all $\gamma > 0$ such that $\mathbb{P}(g_{(\gamma,N_0,\tau,P)}(X^\lambda)$ percolates$) > 0$, for $\lambda > 0$
$\lambda_{(N_0,\tau,P)}$	infimum of $\lambda > 0$ such that $\gamma^*(\lambda') > 0$ holds for all $\lambda' \geq \lambda$
Assumption (ℓ)	(i) ℓ is constant on $[0,d_0]$ for some $d_0 \geq 0$, and on $[d_0,\infty)$ it is strictly monotone decreasing as long as it does not vanish, (ii) $1 \geq \ell(0) > \tau N_0/P$, (iii) $\int_{\mathbb{R}^d} \ell(\vert x\vert)\mathrm{d}x < \infty$ (equivalently, $\int_0^\infty r^{d-1}\ell(r) < \infty$)
d_0	see Assumption (ℓ)
condition (2a)	ℓ has compact support
condition (2b)	$\Lambda(Q_1)$ is [almost surely] bounded
condition (2c)	$\Lambda(Q_1)$ has some exponential moments, and $\int_r^\infty r^{d-1}\ell(r)\mathrm{d}r = \mathcal{O}(1/r)$ as $r \to \infty$
$g_{\mathbf{B}}(k,\Phi)$	bidirectional k-nearest neighbour graph of the simple nonequidistant point process Φ

Proof of phase transitions

ℓ_a	the path-loss function ℓ shifted by $a\sqrt{d}/2$, constant equal to $\ell(0)$ on $[0,a\sqrt{d}/2]$
$I(z)$	total received power at $z \in \mathbb{R}^d$
$I_a(z)$	total received power at $z \in \mathbb{R}^d$ with ℓ replaced by ℓ_a
$A_n(z)$	$= \mathbb{1}\{z$ is n-good$\}$, $z \in \mathbb{Z}^d,n \in \mathbb{N}$
$B_{n,M}(z)$	$= \mathbb{1}\{I_{6n}(nz) \leq M\}$, $z \in \mathbb{Z}^d,n \in \mathbb{N},M > 0$
$C_{n,M}(z)$	$= A_n(z)B_{n,M}(z)$
$I_{6n}^{\mathrm{in}}(x)$	$= \sum_{X_i \in X^\lambda \cap Q_{12n\sqrt{d}}} \ell_{6n}(\vert X_i - x\vert)$, $x \in \mathbb{R}^d$, $n \geq 1$
$I_{6n}^{\mathrm{out}}(x)$	$= \sum_{X_i \in X^\lambda \setminus Q_{12n\sqrt{d}}} \ell_{6n}(\vert X_i - x\vert)$
R_o	$= \inf\{r > 0\colon Q_r \cap X_{\mathbf{S}} \neq \varnothing\}$
$X^{\lambda,*}$	Palm version of the point process X^λ
$\mathbb{M}_{\mathrm{co}}$	set of σ-finite counting measures on $\mathbb{R}^d$
$x \leftrightsquigarrow \infty$ in G	a vertex $v \in V$ is connected in an infinite connected component of the graph $G = (V,E)$
$\theta(\lambda,r)$	percolation probability of o in $g_r(X^{\lambda,*})$
$\overline{\theta}(\varrho)$	percolation prob. of the Gilbert graph $g_r(X^{\lambda,*})$ of a Poisson point process with $\lambda r^d = \varrho$

Section 5.2: notations related to [HJKP18] that were not included in Chapters 2, 3

J_2	Lipschitz continuity parameter of the path-loss function ℓ
g	$g\colon [0,\infty) \to [0,\infty)$: a Lipschitz continuous function, monotone increasing on $[0,\varrho_+)$ and constant equal to c_+ on $[\varrho_+,\infty)$ for some $\varrho_+, c_+ > 0$
QoS	quality of service
c_+	see g
ϱ_+	see g
$D(x, y, \nu)$	$= g(\mathrm{SIR}(x, y, \nu))$, QoS for a direct transmission from $x \in W$ to $y \in W$ with respect to the nonzero measure $\nu \in \mathcal{M}(W)$
$\Gamma(x, y, z, \nu)$	QoS of the two-hop path from x to z via y with respect to ν
$R(x, z, \nu)$	QoS of the optimal at most 2-hop path from x to z with respect to ν
$\tau_c(\cdot)$	$= \mathbb{1}\{\cdot < c\}$, $c \geq 0$
$\nu^{\mathrm{up}}[\tau_c]$	empirical measure of users under having QoS level less than c with respect to ν and (possibly relayed) uplink communication
$\nu^{\text{up-dir}}[\tau_c]$	the same for direct uplink communication
$\nu^{\mathrm{do}}[\tau_c]$	the same for (possibly relayed) downlink communication
$\nu^{\text{do-dir}}[\tau_c]$	the same for direct downlink communication
$\boldsymbol{\tau}_{\mathbf{c}}$	$= (\tau_{c_1}, \ldots, \tau_{c_4})$ for $\mathbf{c} = (c_1, \ldots, c_4) \in [0,\infty)^4$
$\mathbf{a} \leq \mathbf{b}$	$\Leftrightarrow a_i \leq b_i$, $\forall i \in [4]$, for $\mathbf{a} = (a_1, \ldots, a_4) \in [0,\infty)^4$, $\mathbf{b} = (b_1, \ldots, b_4) \in [0,\infty)^4$
$\mathbf{a} < \mathbf{b}$	$\Leftrightarrow a_i < b_i$, $\forall i \in [4]$
$[0,\mathbf{c}_+)$	$= [0,c_+)^4$
$\mathbf{0}$	$= (0,0,0,0) \in \mathbb{R}^4$
$S_1 = S_{\mathrm{up}}$	minimal QoS level for the uplink
$S_2 = S_{\mathrm{up-dir}}$	minimal QoS level for the direct uplink
$S_3 = S_{\mathrm{do}}$	minimal QoS level for the downlink
$S_4 = S_{\mathrm{do-dir}}$	minimal QoS level for the direct downlink
m_1,m_2,m_3,m_4	means of communication: $m_1 =$ up, $m_2 =$ up-dir, $m_3 =$ do and $m_4 =$ do-dir

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References

[BB01] F. Baccelli and B. Blaszczyszyn, On a Coverage Process Ranging from the Boolean Model to the Poisson Voronoi Tessellation With Applications to Wireless Communications, *Advances in Applied Probability* **33**, 293–323 (2001).

[BB08] P. Balister and B. Bollobás, Percolation in the k-nearest neighbor graph. Preprint, available at `http://www.memphis.edu/msci/people/pbalistr/kperc.pdf` (2008).

[BB09] F. Baccelli and B. Blaszczyszyn, *Stochastic Geometry and Wireless Networks: Volume I: Theory*, Now Publishers Inc. (2009).

[BB09b] F. Baccelli and B. Blaszczyszyn, *Stochastic Geometry and Wireless Networks: Volume II: Applications*, Now Publishers Inc. (2009).

[BBM11] F. Baccelli, B. Blaszczyszyn, and O. Mirsadeghi, Optimal paths on the space-time SINR random graph, *Adv. Appl. Probab.* **43**, 131–150 (2011).

[BC12] F. Baccelli and C. S. Chen, Self-Optimization in Mobile Cellular Networks: Power Control and User Association, *IEEE International Conference on Communications*, Cape Town (2010).

[BY09] B. Blaszczyszyn and D. Yogeshwaran, Directionally convex ordering of random measures, shot noise fields, and some applications to wireless communications, *Adv. App. Prob.*, **41**, 623—646 (2009).

[BY10] B. Blaszczyszyn and D. Yogeshwaran, Connectivity in Sub-Poisson Networks, in *Proc. of 48th Annual Allerton Conference*, see also: *arXiv:1009.5696* (2010).

[BY13] B. Blaszczyszyn and D. Yogeshwaran, Clustering and percolation of point processes, *Electron. J. Probab.* **18**, 1–20 (2013).

[CBK17] A. Chattopadhyay, B. Blaszczyszyn, and H. P. Keeler, Gibbsian On-Line Distributed Content Caching Strategy for Cellular Networks, *IEEE Transactions on Wireless Communications* **17**, 969–981, see also: *arXiv:1610.02318* (2017).

[CCS16] R. Colini-Baldeschi, Roberto Cominetti, and Marco Scarsini, On the Price of Anarchy of Highly Congested Nonatomic Network Games, in: *Algorithmic game theory*, 117–128, Lecture Notes in Comput. Sci., Springer, Berlin (2016).

[CLSK13] M. Chen, M. Ch. Liew, Z. Shao, and C. Kai, Markov Approximation for Combinatorial Network Optimization, *IEEE Trans. Inform. Theory*, **59:10** (2013).

[CGHJNP18] E. Cali, N. N. Gafur, C. Hirsch, B. Jahnel, T. En-Najjary, and R. Patterson, Percolation for D2D Networks on Street Systems, *IEEE 16th International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks (WiOpt)*, 1-6, see also: *arXiv:1801.10588* (2018).

[D71] D. J. Daley, The Definition of a Multi-Dimensional Generalization of Shot Noise, *J. Appl. Prob.* **8**, 128–135 (1971).

[DBT05] O. Dousse, F. Baccelli, and P. Thiran, Impact of Interferences on Connectivity in Ad Hoc Networks, *IEEE/ACM Trans. Networking* **1**, 425–436 (2005).

[DFMMT06] O. Dousse, M. Franceschetti, N. Macris, R. Meester, and P. Thiran, Percolation in the signal to interference ratio graph, *J. Appl. Prob.* **43**, 552–562 (2006).

[DZ98] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, 2nd edition, Springer, Berlin (1998).

[FDTT07] M. Franceschetti, O. Dousse, D. Tse, and P. Thiran, Closing the gap in the capacity of wireless networks via percolation theory, IEEE Trans. Inform. Theory **53:3**, 1009–18 (2007).

[FM07] M. Franceschetti and R. Meester, *Random Networks for Communication – From Statistical Physics to Information Systems*, Cambridge University Press (2007).

[G61] E. N. Gilbert, Random plane networks, *J. SIAM* **9**, 533–543 (1961).

[G99] G. Grimmett, *Percolation*, second edition, Springer, Berlin (1999).

[G11] H-O. Georgii, *Gibbs Measures and Phase Transitions*, De Gruyter, 2nd edition (2011).

[GFSS06] C. Gloaguen, F. Fleischer, H. Schmidt, and V. Schmidt, Fitting of Stochastic Telecommunication Network Models via Distance Measures and Monte–Carlo Tests, *Telecommunication Systems*, **31:4**, 353–377 (2006).

[GK00] P. Gupta and P. R. Kumar, The Capacity of Wireless Networks, *IEEE Trans. Inform. Theory* **46:2**, (2000).

[GKP16] S. Ghosh, M. Khrisnapur, and Y. Peres, Continuum percolation for Gaussian zeroes and Ginibre eigenvalues, *Ann. Probab.*, **44:5**, (2016).

[GT08] A. J. Ganesh and G. L. Torrisi, Large Deviations of the Interference in a Wireless Communication Model, *IEEE Trans. Inform. Theory* **54:8**, (2008).

[GZ93] H-O. Georgii and H. Zessin, Large deviations and the maximum entropy principle for marked point random fields, *Probab. Theory Relat. Fields* **96:2**, 177–204 (1993).

[H02] O. Häggström, *Finite Markov Chains and Algorithmic Applications,* Cambridge University Press (2002).

[H10] M. Haenggi, *Advanced Topics in Random Wireless Networks*, lecture notes, available at `https://www3.nd.edu/~mhaenggi/ee87021/syllabus.pdf` (2010).

[HJ17] C. Hirsch and B. Jahnel, Large deviations for the capacity in dynamic spatial relay networks, accepted for publication in *Markov Processes and Related Fields*, see also: *arXiv:1712.03763* (2017).

[HJC17] C. Hirsch, B. Jahnel, and E. Cali, Continuum percolation for Cox point processes, accepted for publication in *Stochastic Processes and their Applications*, see also: *arXiv:1710.11407* (2017).

[HJKP16] C. Hirsch, B. Jahnel, P. Keeler, and R. Patterson, Large-deviation principles for connectable receivers in wireless networks, *Advances in Applied Probability*, **48**, 1061–1094, see also: *arXiv:1506.00576* (2016).

[HJKP18] C. Hirsch, B. Jahnel, P. Keeler, and R. Patterson, Large deviations in relay-augmented wireless networks, *Queueing Systems* **88**, 349–387, see also: *arXiv:1510.04146* (2018).

[HJP18] C. Hirsch, B. Jahnel, and R. Patterson, Space-time large deviations in capacity-constrained relay networks, *ALEA Latin American Journal of Probability and Statistics* **15**, 587–615, see also: *arXiv:1609.06856* (2018).

[HM96] O. Häggström and R. Meester, Nearest Neighbor and Hard Sphere Models in Continuum Percolation, *Random Structures and Algorithms* **9**, 295–315, (1996).

[J16] S. Jansen, Continuum percolation for Gibbsian point processes with attractive interactions, *Electron. J. Probab., 21: No. 47*, 22, (2016).

[JT19] B. Jahnel and A. Tóbiás, Exponential moments for planar tessellations, *arXiv:1902.02148* (2019).

[K93] J. F. C. Kingman, *Poisson Processes*, Oxford University Press, New York (1993).

[KB14] H. P. Keeler and B. Blaszczyszyn, SINR in wireless networks and the two-parameter Poisson-Dirichlet process, *IEEE Wireless Communications Letters, IEEE*, **3:5**, 525–528 (2014).

[KT18] W. König and A. Tóbiás, A Gibbsian model for message routeing in highly dense multihop networks, accepted for publication in *ALEA Latin Americal Journal of Probability and Statistics*, see also: *arXiv:1704.03499v3* (2018).

[KT19] W. König and A. Tóbiás, Routeing properties in a Gibbsian model for highly dense multihop networks, *arXiv:1801.04985v3* (2019).

[KY07] Z. KONG and E. M. YEH, Directed Percolation in Wireless Networks with Interference and Noise, *arXiv:0712.2469* (2007).

[L12] M. N. M. VAN LIESHOUT, An introduction to planar random tessellation models, *Spatial Statistics*, **1:1**, 40–49.

[LP17] G. LAST and M. PENROSE, *Lectures on the Poisson Process*, Cambridge University Press (2017).

[LSS97] T. M. LIGGETT, R. H. SCHONMANN, and A. M. STACEY, Domination by product measures, *Ann. Probab.*, **25:1**, 71—95 (1997).

[M18] S. MORGENSTERN, *Markov Chain Monte Carlo for Message Routing*, master's thesis, TU Berlin (2018).

[MR96] R. MEESTER and R. ROY, *Continuum percolation*, Cambridge University Press (1996).

[MS08] J. MØLLER and D. STOYAN, Stochastic geometry and random tessellations, *Tessellations in the Sciences: Virtues, Techniques and Applications of Geometric Tilings*, Springer-Verlag (2008).

[NRTV07] N. NISAN, T. ROUGHGARDEN, É. TARDOS, and V. VAZIRANI (editors), *Algorithmic Game Theory*, Cambridge University Press (2007).

[S13] K. STUCKI, Continuum percolation for Gibbs point processes, *Electron. Commun. Probab.*, *18:67*, 10, (2013).

[SPW07] H. SONG, M. PENG, and W. WANG, Node Selection in Relay-based Cellular Networks, *IEEE 2007 International Symposium on Microwave, Antenna, Propagation, and EMC Technologies For Wireless Communications* (2007).

[T16] A. TÓBIÁS, *Highly dense mobile communication networks with random fadings*, master's thesis, TU Berlin, *arXiv:1606.06473* (2016).

[T18] A. TÓBIÁS, Signal to interference ratio percolation for Cox point processes, *arXiv:1808.09857* (2018).

[TL14] G. L. TORRISI and E. LEONARDI, Large deviations of the interference in the Ginibre network model, *Stochastic Systems*, **4:1**:173–205 (2014).

[V12] R. VAZE, Percolation and Connectivity on the Signal to Interference Ratio Graph, *Proc. IEEE INFOCOM 2012*:513–521 (2012).

[VI14] R. VAZE and S. IYER, Percolation on the information theoretically secure signal to interference ratio graph, *J. Appl. Probab.*, **51**:910–920 (2014).

[YCG11] C.-L. YAO, G. CHEN, and T.-D. GUO, Large deviations for the graph distance in supercritical continuum percolation, *J. Appl. Prob.*, **48:1**:154—172 (2011).