Möbius Invariant Flows of Tori in S^4

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Zusammenfassung

Diese Arbeit beschreibt, wie Bäcklund- und Darbouxtransformationen sowie Solitonenflüsse für allgemeine konforme Immersionen von Riemannschen Flächen in die 4-Sphäre definiert werden können. Dabei wird detailliert auf den Zusammenhang zu den klassischen Transformationstheorien für Isotherm- und Willmoreflächen eingegangen, die als Spezialfälle in der neuen Theorie enthalten sind.

Das Hauptinteresse dieser Arbeit gilt den globalen Eigenschaften der Transformationen im Fall konformer Immersionen von Riemannschen Flächen des Geschlechts 1. Es wird gezeigt, dass im Fall konformer Immersionen eines Torus mit Normalbündel vom Grad 0 sowohl Bäcklund- als auch Darbouxtransformationen das Willmorefunktional und das sogenannte Spektrum erhalten. Für diese Immersionen wird eine Spektralkurve definiert, welche die Menge der Darbouxtransformationen der Immersion parametrisiert und eine natürliche Interpretation als holomorphe Kurve in $\mathbb{CP}^3$ erlaubt.

Die Solitonenflüsse werden als spezielle Deformationen quaternionisch projektiver Strukturen eingeführt. Es werden Evolutionsgleichungen für die Invarianten hergeleitet, die nach dem Fundamentalsatz der Flächentheorie in der 4–dimensionalen Möbiusgeometrie eine konforme Immersion bis auf Möbiustransformation eindeutig beschreiben. Als Beispiel wird gezeigt, dass der Davey–Stewartson–Fluss für Flächen, im Fall von Zylindern, Rotationsflächen und Kegeln über Kurven in 3–dimensionalen Raumformen, den bekannten Rauch–Ring–Fluss für Raumkurven ergibt und dass analog der Novikov–Veselov–Fluss dem mKdV–Fluss für Raumkurven entspricht.

Abschließend werden die beiden folgenden Sätze bewiesen: der erste Satz besagt, dass ein Torus in der 4–Sphäre, der in einer 3–Sphäre enthalten ist, genau dann unter dem Davey–Stewartson–Fluss stationär ist, wenn er isotherm und constrained Willmore ist. Der zweite Satz zeigt, dass man den Davey–Stewartson–Fluss unter bestimmten Annahmen als Grenzwert von Darbouxtransformationen erhält.

Introduction

For the last 15 years, integrable systems methods have been successfully applied to the study of special classes of immersions in differential geometry of surfaces. The most prominent example is surely that of constant mean curvature tori (cf. [**21**] and [**1**]). Other important examples are isothermic surfaces (see e.g. [**3**] and [**15**] and the references therein) and Willmore surfaces (cf. [**13**] and [**2**]). All these surfaces come with an associated family depending on a spectral parameter. Therefore, the whole machinery of the loop group approach to integrable systems can be applied, and Bäcklund and Darboux transformations as well as a hierarchy of flows can be defined.

What This Is All About. The subject of the present text is the generalization to all immersions of some of these integrable systems concepts, like Bäcklund and Darboux transformations and flows, that were previously defined only for the special surface classes mentioned above. For this generalization it turns out to be essential to consider immersion into 4–dimensional instead of 3–dimensional space. Furthermore, it becomes clear that the right setting for the study of the flows and transformations is that of Möbius geometry rather than Euclidean geometry.

The quaternionic approach to surface theory developed in [**20**], [**2**] and [**8**] provides the principal model of 4–dimensional Möbius geometry used throughout this work for defining and studying the transformations and flows of general immersions. In the development of the general theory, the quaternionic formulation of the transformation theory of isothermic and Willmore immersions, which was pioneered in [**14**] and [**2**], proved to be very useful as a starting point and testing ground.

The focus of this text is on the global theory of flows and transformations in the case of immersions of tori with degree 0 normal bundle. This case is the only one, where the definitions of all transformations and flows, at least in its present form, apply globally. Moreover, in this case it is possible to define a spectral curve which parameterizes the Darboux transformations of a given immersion and which itself allows a natural interpretation as a holomorphic curve in $\mathbb{CP}^3$. One expects that this interpretation as a holomorphic curve in $\mathbb{CP}^3$ is the starting point for the development of an algebraic geometric theory describing all finite type tori and their transformations in terms of twistor theory and linear deformations in the Jacobian of the spectral curve.

Where This All Comes From. A first step in the direction of studying general surfaces by integrable systems methods was taken by Konopelchenko and Taimanov (see [**17**] and [**26**]), who showed that integrable evolution equations related to the modified Korteweg–de–Vries (mKdV) equation and non–linear Schrödinger (NLS) equation, the so called Novikov–Veselov (NV) and Davey–Stewartson (DS) equations, can be used to define flows on the space of immersions of tori into Euclidean 4–space. It turned out that, despite their definition in terms of Euclidean data, the flows are invariant under Möbius transformations. While this fact is quite difficult to prove in the original setting, a new, a priori Möbius invariant definition of the flows was achieved in [**4**], based on the classical lightcone model of conformal space. This definition is significantly more geometric than the old one with its origin in mathematical physics. The Möbius geometric approach to the flows, in the lightcone version as well as in the equivalent quaternionic version, is the subject of Chapters IV and V.

Another important step towards a transformation theory of general immersions was Ulrich Pinkall's course on experimental geometry of discrete curves in $\mathbb{CP}^1$ [**22**]. In the preparation of this lecture course with computer experiments, he gained the insight that a theory of Darboux and Bäcklund transformations as well as the definition of an associated family should be possible for all immersed tori with degree 0 normal bundle in $\mathbb{HP}^1$. This inspired a period of intense work of several people, namely Ulrich Pinkall, Franz Pedit, Fran Burstall, Katrin Leschke and the author, during the summer of 2002. Many of the results described in Chapter II and Chapter III originate from this collaboration.

Purpose Of This Work. The purpose of this work is to document the present state of the theory as a basis for further studies, in giving a rather complete snapshot of the results that have been obtained by now. This includes the definitions for most of the basic objects of the theory, as well as a collection of results which are thought to be important for future developments. These definitions might not all be in their ultimate form, but they serve the purpose of generalizing the classical definitions. Furthermore, many of the results which have been obtained with these definitions indicate that they point in the right direction.

Because the whole theory is still in a state of change and progress, it is not always easy to know about the future importance of its objects, results and observations and about the exact relation among them. Therefore, rather than restricting to a few highlights, it seemed to be adequate to include a wider variety of material.

The guiding principle in the choice of this material has been to concentrate on the results admitting a proper proof by the present techniques and to reduce speculation to a minimum. As a consequence, the motivation for some of the results, as well as some general considerations motivating the whole approach and tying things together, had to be omitted, because they would have been to speculative.

One exception is the discussion of the spectral curve, which clearly had to be included, even if its treatment sometimes has a speculative character (as in Sections 8 and 16.1), because it is a cornerstone of the whole theory. This is justified by the fact that, in the rest of the work, the spectral curve is only used as a motivation for some of the results. Furthermore, there are strong arguments, mainly from mathematical physics, indicating that its foundation can actually be put on a firmer ground.

Summary Of This Work. Chapter I contains the necessary background from [**20**], [**2**] and [**8**]. It is not intended to be read as a whole, but should serve as a quick reference.

In Chapter II, Bäcklund and Darboux transformations as well as an associated family are defined for all conformal immersions. It is shown that, for tori of degree 0 normal bundle, these transformations preserve the Willmore functional and the so called spectrum. For Darboux transformations, a Bianchi type permutability theorem is proven. Moreover, it is shown that, under certain assumptions, Bäcklund transformations can be obtained as a limit of Darboux transformations.

A new definition of the spectral curve of an immersed torus is given. Initially, this is done to develop a more geometric understanding of the spectrum, which is usually defined in terms of Floquet theory. It turns out that this spectral curve, although initially defined to serve a different purpose, can be most easily understood as a set parametrizing the closed Darboux transformations of the torus. Furthermore, these Darboux transformations can be used to realize the spectral curve as a holomorphic curve in $\mathbb{CP}^3$.

Chapter III explains, how the classical transformation theories of isothermic and constrained Willmore immersions are obtained as special cases of the new theory. In these special cases, the classical associated family, given by a family of flat quaternionic connections depending on one real parameter, plays a fundamental role for all classical transformation. It is proven that, for tori, the new Darboux transformations of these special surfaces, generically, can be obtained from a complexification of this family of flat connections. Furthermore, it is sketched how, for tori of degree 0 normal bundle, the holonomy of this complexified family of connections can be used to define a Riemann surface, which is naturally mapped into the spectral curve, and the bundle of eigenlines of the holonomy, which corresponds to the realization of the spectral curve as a holomorphic curve in $\mathbb{CP}^3$.

In Chapter IV, a quaternionic version of the fundamental theorem for surfaces in the conformal 4–sphere is given. Its formulation is based on the language of quaternionic projective structures. Furthermore, in Section 13, a new link to the classical lightcone model is established and the invariants arising in the quaternionic version of the fundamental theorem are put in relation to those arising in the lightcone version.

Chapter V treats flows of tori as special deformations of projective structures. The deformation theory is developed in both the quaternionic and lightcone setting. It is shown that both approaches are equivalent. The deformation formulae of [**4**] are recovered and completed (the formulae given there are incomplete, because the deformation of the normal bundle is missing). As an application, it is proven that a torus immersed into the 3–sphere is stationary under the Davey–Stewartson flow if and only if it is constrained Willmore and isothermic, which holds if and only if it is a constant mean curvature surface in some space form.

Finally, a link is made to the theory of discrete flows: it is shown that, under certain assumptions, the Davey–Stewartson flow can be obtained as a limit of Darboux transformations.

A Word About The Bibliography. I decided to included mainly references to books and papers that I am familiar with. As a result, though their ideas have been highly influential in this part of geometry, there are no references to classical differential geometers like Bianchi, Blaschke, Darboux and Thomsen. Instead, the interested reader is referred to the extensive bibliography of [**15**], which is my canonical reference, not only for the classical theory, but also for question of modern Möbius differential geometry.

CHAPTER I

Fundamentals

This chapter gives an overview of those parts of the theory developed in [**20**], [**2**] and [**8**] that are essential for the results presented in this work. The aim of the chapter is mainly to fix some notation and to serve as a quick reference.

The three sections of this chapter are Section 1, which provides the basic definitions of quaternionic holomorphic geometry, Section 2, which describes the quaternionic approach to Möbius geometry of immersions into the conformal 4–sphere $\mathbb{HP}^1$ and to Euclidean geometry of immersions into $\mathbb{R}^4 = \mathbb{H}$, and Section 3, which introduces the quaternionic holomorphic line bundles related to immersions of surfaces in Möbius and Euclidean geometry.

1. Quaternionic Holomorphic Bundles

This section collects the basic definitions and some fundamental facts concerning quaternionic holomorphic structures and connections on quaternionic vector bundles over Riemann surfaces.

1.1. Quaternionic Vector Bundles. The skew field of *quaternions*, denoted by $\mathbb{H}$, is the 4–dimensional real vector space with the basis 1, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ and the unique multiplication (i.e. real bilinear product) such that 1 is the neutral element, $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$.

The conjugate of a quaternion $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is $\bar{q} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$. Quaternionic conjugation satisfies $\overline{pq} = \bar{q}\bar{p}$. The real and imaginary part of a quaternion q is defined by

$$\operatorname{Re} q = \frac{1}{2}(q + \bar{q}) \qquad \text{and} \qquad \operatorname{Im} q = \frac{1}{2}(q - \bar{q}).$$

The subset of real quaternions $\operatorname{Re}\mathbb{H}$ consists of all real multiples of 1 and is therefore identified with $\mathbb{R}$. It is the center of $\mathbb{H}$. The subset of imaginary quaternions $\operatorname{Im}\mathbb{H}$ is the 3–dimensional real vector space with the basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ which gives rise to the identification $\operatorname{Im}\mathbb{H} = \mathbb{R}^3$.

Quaternionic Vector Spaces. A *quaternionic vector space* is a real vector space endowed with a multiplication by quaternions from the right. Let V be a quaternionic vector space. Its dual space V^* (i.e. the space of quaternionic linear forms) can be made into a quaternionic vector space by defining $\alpha\lambda := \bar{\lambda}\alpha$. However, due to the non–commutativity of $\mathbb{H}$, tensor products of quaternionic vector spaces, in particular spaces of homomorphisms between quaternionic vector spaces, have no natural structure of a quaternionic vector space.

A *complex quaternionic vector space* is a quaternionic vector space V together with a quaternionic linear endomorphism $J \in \operatorname{End}(V)$ that satisfies $J^2 = -\operatorname{Id}$. The following lemma shows that such endomorphisms correspond to complex subspaces of the complex vector space $(V, \mathbf{i})$, which is obtained when the quaternionic multiplication on V is restricted to the multiplication by the complex numbers $\mathbb{C} = \mathrm{Span}_{\mathbb{R}}\{1, \mathbf{i}\} \subset \mathbb{H}$.

LEMMA 1. *Let V be a n–dimensional quaternionic vector space. There is a 1-1–correspondence between endomorphisms $J \in \mathrm{End}(V)$ satisfying $J^2 = -\mathrm{Id}$ and $\mathbf{i}$–complex subspaces $\hat{V}$ of complex dimension n in $(V, \mathbf{i})$ not containing a quaternionic line. The correspondence assigns to an endomorphism J the subspace $\hat{V} = \{v \in V \mid Jv = v\mathbf{i}\}$.*

PROOF. Let $J \in \mathrm{End}(V)$ be an endomorphism with $J^2 = -\mathrm{Id}$ and define $\hat{V} = \{v \in V \mid Jv = v\mathbf{i}\}$. The $\mathbf{i}$–complex subspace $\hat{V}\mathbf{j}$ is the $-\mathbf{i}$–eigenspace of J and right multiplication by $\mathbf{j}$ defines an $\mathbb{R}$–linear isomorphism between $\hat{V}$ and $\hat{V}\mathbf{j}$. The projection $v^{\pm} = \frac{1}{2}(v \mp Jv\mathbf{i})$ decomposes every vector $v \in V$ into $\pm\mathbf{i}$–eigenvectors of J. Therefore $V = \hat{V} \oplus \hat{V}\mathbf{j}$ is a direct sum decomposition and both summands have complex dimension n. Conversely, every $\mathbf{i}$–complex subspaces $\hat{V}$ of dimension n that contains no quaternionic line determines a unique J. □

Every complex vector space $\hat{V}$ gives rise to a complex quaternionic vector space V which is (formally) defined as $V = \hat{V} \oplus \hat{V}\mathbf{j}$, i.e. V is the direct sum $\hat{V} \oplus \hat{V}$ with $(v, w)\mathbf{i} = (v\mathbf{i}, -w\mathbf{i})$, $(v, w)\mathbf{j} = (-w, v)$ and $J(v, w) = (v\mathbf{i}, w\mathbf{i})$. We call V the *quaternionification* of $\hat{V}$.

A complex quaternionic vector space is a bi–vector space: it has a quaternionic multiplication from the right and a complex multiplication from the left. Both multiplications are compatible.

When a complex quaternionic vector space is tensored by a complex vector space (from the left), one again obtains a complex quaternionic vector space. The space of quaternionic linear homomorphisms $\mathrm{Hom}(V, W)$ between two complex quaternionic vector spaces V and W has a natural decomposition

$$\mathrm{Hom}(V, W) = \mathrm{Hom}_+(V, W) \oplus \mathrm{Hom}_-(V, W)$$

where $\mathrm{Hom}_{\pm}(V, W) = \{A \in \mathrm{Hom}(V, W) \mid J^W A = \pm A J^V\}$. We take post–composition with J^W as the natural complex structure on $\mathrm{Hom}(V, W)$ and $\mathrm{Hom}_{\pm}(V, W)$.

Quaternionic Vector Bundles. It is straightforward to generalize the concepts of the preceding section to vector bundles over Riemann surfaces. We only give the most important definition.

DEFINITION. A *quaternionic vector bundle* over a Riemann surface M is a real vector bundle with a (fiberwise) multiplication by quaternions from the right. A *complex quaternionic vector bundle* V is a quaternionic vector bundle together with a section $J \in \Gamma(\mathrm{End}(V))$ such that $J^2 = -\mathrm{Id}$. If M is compact, the *degree* of a complex quaternionic vector bundle V on M is defined to be the degree of the underlying complex vector bundle $\hat{V}$.

The degree of a complex quaternionic vector bundle V on a compact Riemann surface can be computed by

$$\deg(V) = \frac{1}{2\pi}\int_M \langle JR^\nabla\rangle \tag{1}$$

where ∇ is a connection on V that satisfies $\nabla J = 0$ and where $\langle B\rangle = \frac{1}{4}\operatorname{tr}_\mathbb{R} B$ (for B a quaternionic endomorphism and $\operatorname{tr}_\mathbb{R} B$ the real trace of B seen as a real endomorphism).

1.2. Quaternionic Holomorphic Vector Bundles. Let M be a Riemann surface and let V be real vector bundle with a complex structure J (we have in mind a complex quaternionic bundle or the endomorphism bundle of a complex quaternionic bundle). Then there is a decomposition

$$T^*M \otimes_\mathbb{R} V = KV \oplus \bar{K}V$$

where KV and $\bar{K}V$ denote the tensor products of the canonical and anti–canonical bundle with V, that is $KV = \{\omega \in T^*M \otimes V \mid *\omega = J\omega\}$ and $\bar{K}V = \{\omega \in T^*M \otimes V \mid *\omega = -J\omega\}$. The decomposition of a V–valued 1–form $\omega \in \Omega^1(V)$ is denoted by

$$\omega = \omega' + \omega''$$

where $\omega' = \frac{1}{2}(\omega - J * \omega) \in \Gamma(KV)$ and $\omega'' = \frac{1}{2}(\omega + J * \omega) \in \Gamma(\bar{K}V)$.

Definition. A *holomorphic structure* on a complex quaternionic vector bundle (V, J) is a quaternionic linear operator

$$D\colon \Gamma(V) \to \Gamma(\bar{K}V)$$

satisfying the Leibniz rule

$$D(\psi\lambda) = (D\psi)\lambda + (\psi\lambda)''$$

for $\psi \in \Gamma(V)$ and $\lambda\colon M \to \mathbb{H}$. A complex quaternionic vector bundle with a holomorphic structure is called a *holomorphic quaternionic vector bundle*. A section $\psi \in \Gamma(V)$ is called *holomorphic* if $D\psi = 0$. The space of holomorphic sections is denoted by $H^0(V)$.

The operator $\bar\partial\colon \Gamma(V) \to \Gamma(\bar{K}V)$ defined by $\bar\partial = \frac{1}{2}(D - JDJ)$ is again a holomorphic structure. By definition, $\bar\partial$ is the J–commuting part of D. The J–anti–commuting part of D is the tensor field $Q = \frac{1}{2}(D + JDJ) \in \Gamma(\bar{K}\operatorname{End}_-(V))$ called the *Hopf field* of D.

The operator $\bar\partial$ can be seen as a complex holomorphic structure on the complex vector bundle $\hat{V}$ and the quaternionic holomorphic structure

$$D = \bar\partial + Q$$

on V is a zero order perturbation of the operator $\bar\partial$. Hence, D is elliptic and on a compact Riemann surface, the space $H^0(V)$ of holomorphic sections is finite dimensional.

Definition. Let M be a compact Riemann surface and (V, J, D) a holomorphic quaternionic vector bundle. Its *Willmore energy* (or Willmore functional) is defined by

$$W(V) = 2\int_M \langle Q \wedge *Q\rangle$$

where Q is the Hopf field of $D = \bar{\partial} + Q$ and $\langle B\rangle = \frac{1}{4}\operatorname{tr}_{\mathbb{R}} B$ (for B a quaternionic endomorphism and $\operatorname{tr}_{\mathbb{R}} B$ the real trace of B seen as a real endomorphism).

Paired Holomorphic Bundles. For every quaternionic holomorphic vector bundle V, the bundle KV^* has a canonical holomorphic structure. Both bundles are called paired according to the following definition.

Definition. A *pairing* between the complex quaternionic vector bundles V_1 and V_2 is a real linear map $(\,,)\colon V_1 \otimes_{\mathbb{R}} V_2 \to T^*M \otimes_{\mathbb{R}} \mathbb{H}$ which is pointwise non–degenerate and satisfies

$$(\psi\lambda, \varphi\mu) = \bar{\lambda}(\psi, \varphi)\mu$$

and

$$*(\psi, \varphi) = (J\psi, \varphi) = (\psi, J\varphi)$$

for $\psi \in \Gamma(V_1)$ and $\varphi \in \Gamma(V_2)$ and λ, $\mu \in \mathbb{H}$. Two holomorphic structures D_1 and D_2 on V_1 and V_2 are called *compatible* with respect to the pairing $(\,,)$ if

$$d(\psi, \varphi) = (D_1\psi \wedge \varphi) + (\psi \wedge D_2\varphi)$$

for all $\psi \in \Gamma(V_1)$ and $\varphi \in \Gamma(V_2)$. (In this formula we assume that the slot of $(\,,)$ is always served last in the wedge products.)

Note that if $(\,,)$ is a pairing between V_1 and V_2, then $\overline{(\,,)}$ is a pairing between V_2 and V_1. If one of two paired complex quaternionic line bundles V_1 and V_2 is equipped with a holomorphic structure, there is a unique compatible holomorphic structure on the other bundle.

Every complex quaternionic bundle V is paired with KV^* via

$$(\omega, \psi) := \langle \omega, \psi\rangle$$

for $\omega \in \Gamma(KV^*)$ and $\psi \in \Gamma(V)$ (where $\langle\,,\rangle$ denotes the evaluation between V^* and V). It is easy to verify that the bundle KV^* is, up to isomorphism, the unique bundle paired with V. The notion of pairing is merely introduced to emphasize the symmetry between V and KV^*. For a quaternionic holomorphic vector bundle V with holomorphic structure D, the unique compatible holomorphic structure D on KV^* is defined by

$$d\langle \omega, \psi\rangle = \langle D\omega, \psi\rangle - \langle \omega \wedge D\psi\rangle \tag{2}$$

for $\omega \in \Gamma(KV^*)$ and $\psi \in \Gamma(V)$.

In contrast to the complex theory, where all tensor products of holomorphic bundles inherit natural holomorphic structures, in the quaternionic setting, the paired bundle KV^* of a holomorphic bundle V is the only bundle that inherits a holomorphic structure from the one on V. It can be seen (see [**8**, Section 2.4]) that paired holomorphic bundles have the same Willmore functional. By the Riemann–Roch Theorem (cf. [**8**, Theorem 2.2]), on a compact Riemann surface of genus g, the dimensions of the spaces of holomorphic sections of the paired bundles V and KV^* are related by

$$\dim_{\mathbb{H}} H^0(V) - \dim_{\mathbb{H}} H^0(KV^*) = \deg(V) - (g-1)\,\mathrm{rank}(V).$$

1.3. Quaternionic Holomorphic Structures and Connections. Let ∇ be a quaternionic connection on a complex quaternionic vector bundle V. The decomposition

$$\nabla = \nabla' + \nabla''$$

into K– and $\bar{K}$–part defines a holomorphic structure

$$D = \nabla''$$

on V. Its decomposition into J–commuting and J–anti–commuting part is $D = \bar{\partial} + Q$. Similarly, the anti–holomorphic structure ∇' has the decomposition $\nabla' = \partial + A$ into J–commuting part $\partial = \frac{1}{2}(\nabla' - J\nabla' J)$ and J–anti–commuting part $A = \frac{1}{2}(\nabla' + J\nabla' J) \in \Gamma(K\,\mathrm{End}_-(V))$. Thus, every connection ∇ on a complex quaternionic vector bundle is decomposed into

$$\nabla = \partial + \bar{\partial} + A + Q. \tag{3}$$

The connection $\hat{\nabla} = \partial + \bar{\partial}$ is the underlying complex connection, i.e. the J–commuting part $\hat{\nabla} = \frac{1}{2}(\nabla - J \circ \nabla \circ J)$ of ∇. A consequence of (3) together with $\hat{\nabla} J = 0$ is

$$\nabla J = 2(*Q - *A). \tag{4}$$

Another consequence of (3) is the decomposition

$$R^{\nabla}_{+} = R^{\hat{\nabla}} + A \wedge A + Q \wedge Q \tag{5}$$

$$R^{\nabla}_{-} = d^{\hat{\nabla}}(A + Q) \tag{6}$$

of the curvature tensor of ∇ into J–commuting and J–anti–commuting parts.

It is worthwhile mentioning that every holomorphic structure $D = \bar{\partial} + Q$ on a complex holomorphic vector bundle V can be written as $D = \nabla''$ for some quaternionic connection on V: such connection can be defined by (3) after choosing an arbitrary complex anti–holomorphic structure ∂ and an arbitrary $A \in \Gamma(K\,\mathrm{End}_-(V))$ (e.g. $A \equiv 0$).

Let (V, J) be a complex quaternionic vector bundle. Then (V^*, J^*) is also a complex quaternionic vector bundle. A quaternionic connection on V gives rise to a unique connection on V^* which we denote by ∇, too. The connection ∇ on V has the decomposition $\nabla = \partial + \bar{\partial} + A + Q$ and the induced holomorphic structure on V is $D = \nabla'' = \bar{\partial} + Q$. The holomorphic structure on V^* induced by ∇ is $D = \bar{\partial} - A^*$ where $\bar{\partial}$ is the complex holomorphic structure on V^* induced by $\bar{\partial}$ on V.

In contrast to the complex case, a holomorphic structure $D = \bar{\partial} + Q$ on (V, J) does not define a canonical holomorphic structure on V^*. Some additional structure, e.g. a connection ∇ on V with $\nabla'' = D$, is needed. (The operator $\bar{\partial} - Q^*$ on V^* induced by D on V is not a holomorphic structure, but a so called mixed structure, cf. [**8**, Lemma 2.1] for a detailed discussion. In Section 3.1, the concept of mixed structure is used in the definition of the holomorphic structure on L^{-1}.)

Connections and Paired Holomorphic Structures. While a holomorphic structure D on a complex quaternionic vector bundle V does only induce a holomorphic structure on KV^*, a quaternionic connection on V induces holomorphic structures $D = \nabla''$ on V and V^* (see above). Furthermore, using the identification $\bar{K}K \cong \Lambda^2 T^*M^{\mathbb{C}}$ with $d\bar{z}\,dz \mapsto d\bar{z} \wedge dz = -dz \wedge d\bar{z}$, ∇ induces holomorphic structures $D = d^\nabla$ on KV and KV^*: the operator $d^\nabla \colon \Gamma(KV^*) \to \Gamma(\bar{K}KV^*)$ satisfies the Leibniz rule

$$d^\nabla(\omega\lambda) = (d^\nabla\omega)\lambda - \omega \wedge d\lambda = (d^\nabla\omega)\lambda + \frac{1}{2}(\omega d\lambda + J\omega * d\lambda),$$

which shows that it is indeed a holomorphic structure on KV^*. (The same argument applies to KV.) The following lemma shows that the holomorphic structures on V and KV^* (and, equivalently, those on V^* and KV) are compatible with respect to the pairings.

Lemma 2. *Let V be a complex quaternionic vector bundle, ∇ a quaternionic connection. Then the holomorphic structure $D = \nabla''$ on V and d^∇ on KV^* are compatible.*

Proof. The lemma follows from

$$d\langle\omega, \psi\rangle = \langle d^\nabla\omega, \psi\rangle - \langle\omega \wedge \nabla\psi\rangle = \langle d^\nabla\omega, \psi\rangle - \langle\omega \wedge D\psi\rangle$$

for $\omega \in \Gamma(KV^*)$ and $\psi \in \Gamma(V)$. $\square$

If ∇ is flat, the differential $\nabla\psi$ of a holomorphic section $\psi \in H^0(V)$ is again a holomorphic section $\nabla\psi \in H^0(KV)$. Furthermore, in the case of flat ∇, the Willmore functionals of the involved bundles are related by the following Lemma.

Lemma 3. *If ∇ is a flat quaternionic connection on a complex quaternionic vector bundle V, then the Willmore functionals of the holomorphic bundles V and V^* (with $D = \nabla''$) and KV^* with $D = d^\nabla$ are related by*

$$W(KV^*) = W(V) = W(V^*) + 4\pi \deg(V^*).$$

Proof. The first part follows from Lemma 2, since paired holomorphic bundles always have the same Willmore functional. The usual decomposition with respect to J of the connection ∇ on V^* is $\nabla = \hat{\nabla} + A + Q$. Flatness of ∇ implies $R^{\hat{\nabla}}J = A \wedge *A - Q \wedge *Q$ (see (5)). The degree of V^* (by equation (1)) satisfies

$$2\pi \deg(V^*) = \int_M \langle R^{\hat{\nabla}}J\rangle = \int_M \langle A \wedge *A\rangle - \langle Q \wedge *Q\rangle,$$

and the Willmore functional of V^* is defined by $W(V^*) = 2\int_M \langle Q \wedge *Q\rangle$. The holomorphic structure on V being $\bar{\partial} - A^*$, its Willmore functional satisfies $W(V) = 2\int_M \langle A \wedge *A\rangle$. This proves the second part of the formula. $\square$

Explicit Formulae. Let L be a complex quaternionic line bundle on a Riemann surface M. If $\psi \in \Gamma(L)$ is a nowhere vanishing section, then

$$J\psi = \psi N$$

with $N\colon M \to S^2$. (Note that, since every quaternionic line bundle on a surface admits a nowhere vanishing section, every complex structure on a quaternionic line bundle can be defined by this formula for some $N\colon M \to S^2$.)

If L is a holomorphic line bundle and ψ is a holomorphic section, i.e. $D\psi = 0$, the Hopf field Q of D is given by

$$Q\psi = \tfrac{1}{2}(D\psi + JD(J\psi)) = \tfrac{1}{2}\psi N dN'' \tag{7}$$

where $dN'' = \frac{1}{2}(dN + N * dN)$. This formula is useful for the calculation of the Willmore functional of L, which is

$$W(L) = 2\int_M \langle Q \wedge *Q\rangle = \tfrac{1}{2}\int_M dN'' \wedge *dN'' = \int_M |dN''|^2 \tag{8}$$

(with the identification of 2–forms with quadratic forms via $q(X) = \omega(X, JX)$ applied to $|dN''|^2$).

The following lemma relates the degree of L and the mapping degree of N.

Lemma 4. *Let L be a complex quaternionic line bundle on a compact Riemann surface M and let $\psi \in \Gamma(L)$ be nowhere vanishing section with $J\psi = \psi N$. Then* $\deg L = \deg N$ *(with* $\deg N$ *denoting the mapping degree of $N\colon M \to S^2$).*

Proof. We define a quaternionic connection on L by $\nabla\psi = 0$. Since ∇ is flat, by (5), the curvature of the underlying complex connection $\hat\nabla$ (which is given by $\nabla = \hat\nabla + A + Q$) satisfies $R^{\hat\nabla}J = A \wedge *A - Q \wedge *Q$. By (1), the degree of L is

$$\begin{aligned}\deg(L) &= \frac{1}{2\pi}\int_M \langle R^{\hat\nabla}J\rangle = \frac{1}{2\pi}\int_M \langle A \wedge *A\rangle - \langle Q \wedge *Q\rangle = \\ &= \frac{1}{8\pi}\int_M dN' \wedge *dN' - dN'' \wedge *dN''\end{aligned}$$

(where we used equation (7) and the analogous formula $A\psi = \frac{1}{2}\psi N dN'$ for A, where $dN' = \frac{1}{2}(dN - N * dN)$). The mapping degree of N is

$$\deg(N) = \frac{1}{4\pi}\int_M N^*\omega_{S^2}$$

where $N^*\omega_{S^2}$ denotes the pull back of the volume form ω_{S^2} on S^2, given by

$$N^*\omega_{S^2} = \langle NdN, dN\rangle = -\frac{1}{2}NdN \wedge dN = \frac{1}{2}(dN' \wedge *dN' - dN'' \wedge *dN'').$$

$\square$

If $\psi \in \Gamma(L)$ is a nowhere vanishing holomorphic section of a holomorphic line bundle L, every other section $\tilde\psi \in \Gamma(L)$ is of the form $\tilde\psi = \psi\lambda$ with $\lambda\colon M \to \mathbb{H}$. The section $\tilde\psi$ is holomorphic if and only if λ satisfies the equation

$$*d\lambda = Nd\lambda, \tag{9}$$

where $N\colon M \to S^2$ is defined by $J\psi = \psi N$. Denote by ∇ the flat quaternionic connection on L defined by $\nabla\psi = 0$. (Note that, since ψ is holomorphic, ∇ satisfies $D = \nabla''$.) In Lemma 2 we have seen that ∇ induces the holomorphic structure d^∇ on KL. Every section $\omega \in \Gamma(KL)$ can be written as $\omega = \psi\hat{\omega}$ with $\hat{\omega} \in \Omega^1(\mathbb{H})$ a 1–form satisfying $*\hat{\omega} = N\hat{\omega}$. The section ω is d^∇–holomorphic if and only if $d\hat{\omega} = 0$. In particular, if $\tilde{\psi} = \psi\lambda \in H^0(L)$, then $\nabla\tilde{\psi} = \psi d\lambda \in H^0(KL)$.

2. Möbius Geometry of Surfaces in $\mathbb{HP}^1$

The quaternions $\mathbb{H}$ and the quaternionic projective line $\mathbb{HP}^1$ play a similar role in 4–dimensional Euclidean and Möbius geometry as the complex numbers $\mathbb{C}$ and the complex projective line $\mathbb{CP}^1$ play in the 2–dimensional case. While the advantage of using complex numbers in 2–dimensional geometry is widely known and accepted, this is certainly not the case with the quaternions and their use in 4–dimensional geometry.

The purpose of this section is to give a quick overview of the quaternionic approach to 4–dimensional Euclidean and Möbius geometry, with a focus on the differential geometry of immersed surfaces in both geometries. The first and second subsections treat the Euclidean case, the third and forth subsections treat the Möbius case.

2.1. The Quaternions and 4–dimensional Euclidean Space. The standard scalar product $\langle v, w\rangle = \mathrm{Re}(\bar{v}w)$ makes the quaternions $\mathbb{H} = \mathbb{R}^4$ into an Euclidean vector space. As in the case of the real and complex numbers, the length $|v| = \sqrt{\langle v, v\rangle}$ with respect to this scalar product is multiplicative, i.e. it satisfies $|vw| = |v||w|$. This formula shows that, for λ, $\mu \in \mathbb{H}^*$, the map $x \in \mathbb{H} \mapsto \mu x\lambda$ is a conformal linear isomorphism. Furthermore, since $\mathbb{H}^*$ is connected, this map preserves orientation. The following lemma implies that all orientation preserving conformal linear isomorphism of $\mathbb{H}$ are of that form.

Lemma 5. *Every orientation preserving linear isometry of $\mathbb{R}^4 = \mathbb{H}$ is of the form*

$$x \in \mathbb{H} \mapsto \mu x\lambda \in \mathbb{H}$$

with λ, $\mu \in S^3$ uniquely determined up to common multiplication by -1. Such isometry leaves $\mathrm{Im}\,\mathbb{H} = \mathbb{R}^3$ invariant if and only if $\mu = \bar{\lambda}$.

Proof. For the uniqueness part of the statement it is sufficient to note that, if $\mu x\lambda = x$ holds for all x, then $\mu = \lambda = \pm 1$ (which follows from the fact that $\mathbb{R}$ is the center of $\mathbb{H}$).

The isometries $x \in \mathbb{H} \mapsto \mu x\lambda$ with λ, $\mu \in S^3$ form a subgroup of all linear isometries, which acts transitively on the unit vectors in $\mathbb{H}$. Therefore, for the existence part of the statement, it is sufficient to verify that every isometry of $\mathbb{H}$ which leaves 1 fixed (i.e. leaves $\mathrm{Im}\,\mathbb{H}$ invariant) is of the form $x \mapsto \bar{\lambda}x\lambda$. This easily follows from the fact that every orientation preserving linear isometry of $\mathbb{R}^3 = \mathrm{Im}\,\mathbb{H}$ is a rotation around an axis: let $N \in S^2 \subset \mathrm{Im}\,\mathbb{H}$ be a vector of length 1 describing the axis and let α be the angle of rotation. Then $x \mapsto \bar{\lambda}x\lambda$ with $\lambda = \cos(\frac{\alpha}{2}) - \sin(\frac{\alpha}{2})N$ is the rotation around N by α. $\square$

Remark. An immediate consequence of this lemma is that S^3 and $S^3 \times S^3$ are the universal covering groups of SO(3) and SO(4).

Lemma 6. *For every oriented real subspace $U \subset \mathbb{H}$ of dimension 2 there are unique vectors N and R satisfying $N^2 = R^2 = -1$ with the property that*

$$U = \{x \in \mathbb{H} \mid Nx = -xR\} \tag{10}$$

and that left multiplication by N rotates vectors in U by $\pi/2$ in positive direction. Then $U^\perp = \{x \in \mathbb{H} \mid Nx = xR\}$ is the orthogonal complement of U and left multiplication by N rotates vectors in $U^\perp$ by $\pi/2$ in positive direction.

Conversely, every pair of vectors N and R satisfying $N^2 = R^2 = -1$ defines, via (10)*, an oriented 2–plane.*

Proof. Assume, $U = \{x \in \mathbb{H} \mid Nx = -xR\}$ is 2–dimensional and N, R satisfy $N^2 = R^2 = -\mathrm{Id}$. Then, the 2–plane $\tilde{U} = \lambda U \mu$ with λ, $\mu \in S^3$ is of the form

$$\tilde{U} = \lambda U \mu = \{x \in \mathbb{H} \mid \tilde{N}x = -x\tilde{R}\}, \tag{11}$$

where $\tilde{N} = \lambda N \lambda^{-1}$ and $\tilde{R} = \mu^{-1} R \mu$. Because the group of orientation preserving linear isometries acts transitively on the 2–dimensional subspaces of $\mathbb{H}$, Lemma 5 and (11) imply that it is sufficient to prove the first part of the statement for one oriented subspace U, e.g. for $U = \mathbb{C}\mathbf{j} = \mathrm{Span}_{\mathbb{R}}\{\mathbf{j}, \mathbf{k}\}$ with $\mathbf{j}$, $\mathbf{k}$ a positive basis. In this example, $N = R = \mathbf{i}$ clearly is the unique choice of N, R with the required properties. Furthermore, it is obvious that $U^\perp = \mathbb{C} = \mathrm{Span}_{\mathbb{R}}\{1, \mathbf{i}\}$ is of the form $U^\perp = \{x \in \mathbb{H} \mid Nx = xR\}$ and that N acts on $U^\perp$ as a rotation by $\pi/2$.

The conversely part as well follows from (11) and the example $U = \mathbb{C}\mathbf{j}$, because every N, R with $N^2 = R^2 = -\mathrm{Id}$, by Lemma 5 and transitivity of SO(3) on S^2, can be written as $N = \lambda \mathbf{i} \lambda^{-1}$ and $R = \mu^{-1}\mathbf{i}\mu$ for λ, $\mu \in S^3$. □

Remark. Using the description

$$S^2 = \{N \in \mathrm{Im}\,\mathbb{H} \mid |N| = 1\} = \{N \in \mathbb{H} \mid N^2 = -1\} \subset \mathrm{Im}\,\mathbb{H} \tag{12}$$

of 2–spheres in $\mathrm{Im}\,\mathbb{H}$, the preceding lemma implies that the Grassmannian of oriented 2–planes in $\mathbb{R}^4$ is $S^2 \times S^2$.

The Grassmannian of oriented 2–planes in $\mathbb{R}^3$ is S^2, because a 2–plane $U = \{x \in \mathbb{H} \mid Nx = -xR\}$ is contained in $\mathrm{Im}\,\mathbb{H} = \mathbb{R}^3$ if and only if $N = R$.

2.2. Conformal Immersions into $\mathbb{H}$.

Definition. Let M be a Riemann surface. A map $f\colon M \to \mathbb{H}$ is called a *conformal immersion* if it is an immersion and if the induced metric $\langle df, df\rangle$ is compatible with the conformal structure on M.

Lemma 7. *Let M be a Riemann surface. A map $f\colon M \to \mathbb{H}$ is a conformal immersion if and only if there are N, $R\colon M \to S^2 \subset \mathrm{Im}\,\mathbb{H}$ such that*

$$*df = N df = -df R$$

and furthermore df is nowhere zero. N and R are then unique.

The proof shows that the lemma remains true if $*df = Ndf = -dfR$ is replaced with $*df = Ndf$ or $*df = -dfR$.

Proof. Assume the differential df is nowhere zero and satisfies the differential equation $*df = Ndf$ for $N\colon M \to S^2 \subset \operatorname{Im}\mathbb{H}$. By $*df = Ndf$, the differential df_p maps two vectors v, $J^M v \in T_pM$ to $df_p v$ and $Ndf_p v$. Both pairs of vectors are orthogonal and have the same length. Since df_p is not zero, it has full rank 2 and is a conformal linear map (using the fact that a linear map is conformal if and only if it preserves orthogonality). Therefore, f is a conformal immersion.

Conversely, assume f is a conformal immersion. Then Lemma 6 implies existence and uniqueness of N and R with $Ndf = -dfR$ such that left multiplication by N is a positive $\pi/2$ rotation in $T_fM = df(TM)$. Now $*df = Ndf$ follows from conformality, because the rotations by $\frac{\pi}{2}$ in TM and T_fM are compatible under df. $\square$

The maps N and R thus defined are called the *left and right normal vectors* of f. By Lemma 6, the tangent and normal bundle along f are (pointwise)

$$T_fM = \{x \in \mathbb{H} \mid Nx = -xR\} \tag{13}$$

$$\perp_f M = \{x \in \mathbb{H} \mid Nx = xR\}. \tag{14}$$

Their complex structure is given by left multiplication by N. Note that in general, both N and R are *not* sections of $\perp_f M$.

Immersions into a translate of $\operatorname{Im}\mathbb{H} = \mathbb{R}^3$ are characterized by $N = R$. In this case, the normal bundle is $\perp_f M = \operatorname{Span}_{\mathbb{R}}\{1, N\}$. In particular, N is true normal vector, namely the classical Gauss map.

We call a non–constant smooth map $f\colon M \to \mathbb{H}$ from a Riemann surface M into $\mathbb{H}$ a *branched conformal immersion* if it has a smooth($= C^\infty$) left or right normal vector, i.e. if there is $N\colon M \to S^2$ with $*df = Ndf$ or if there is $R\colon M \to S^2$ with $*df = -dfR$. The branch points, i.e. the points where the differential df has rank 0, are isolated (following from the fact that holomorphic section can only vanish at isolated points and that df can be interpreted as a holomorphic section, see the explicit formulae in Section 1.3). Away from those points, the map is a conformal immersion.

For smooth N, the equation $*df = Ndf$ is elliptic. It generalizes the classical Cauchy–Riemann equation: a map $f\colon M \to \mathbb{C} = \operatorname{Span}_{\mathbb{R}}\{1, \mathbf{i}\}$ is holomorphic if and only if $*df = \mathbf{i}df$. Therefore, the classical theory of complex functions can be seen as a degenerate case of surface theory, the study of branched conformal immersions into the plane. Otherwise stated, the theory of conformal immersion into $\mathbb{H}$ can be seen as a quaternionic valued generalization of classical function theory. This observation was the starting point of quaternionic surface theory, cf. [**20**].

Some Useful Formulae for Immersions into $\mathbb{H}$. Let $f\colon M \to \mathbb{H}$ be a conformal immersion with $*df = Ndf = -dfR$. The second fundamental form $\mathrm{II}(X,Y) = X(Y(f))^\perp$ of f is then (see [**2**, Proposition 7] for a proof)

$$\mathrm{II}(X,Y) = \frac{1}{2}(NdN(X)df(Y) + df(Y)dR(X)R). \tag{15}$$

The mean curvature vector is $\mathcal{H} = \frac{1}{2}\operatorname{tr}(\mathrm{I\!I})$ and with[1] $H = \bar{\mathcal{H}}N = R\bar{\mathcal{H}}$ we have

$$dN' = \frac{1}{2}(dN - N * dN) = -df H \qquad \text{and} \qquad dR' = \frac{1}{2}(dR - R * dR) = -H df$$

(see [**2**, Proposition 8]). The Willmore functional of the immersion f is

$$W(f) = \int_M |\mathcal{H}|^2 |df|^2 = \int_M |dN'|^2 = \int_M |dR'|^2, \tag{16}$$

where the usual identification of 2–forms with quadratic forms (via $q(X) = \omega(X, JX)$) is applied. Even if defined in terms of Euclidean quantities, for compact surfaces, the Willmore functional is invariant under Möbius transformations (see the remark below for the sketch of a direct proof, cf. Section 3.4 for a detailed discussion).

The Levi–Civita connection on $T_f M$ and the normal connection on $\perp_f M$ (which are both obtained by orthogonal projection from the connection d on the trivial $\mathbb{H}$–bundle) are compatible with the respective complex structures (which are both given by left multiplication by N). Using the classical Gauss– and Ricci–equations and the above formula for $\mathrm{I\!I}$, one can prove (see [**2**, Proposition 9]) that the curvature tensors[2] of these connections are determined by

$$K|df|^2 = \operatorname{tr}(R^{LC} J^T) = \frac{1}{2}(N^*\omega_{S^2} + R^*\omega_{S^2}) \tag{17}$$

$$K^\perp |df|^2 = \operatorname{tr}(R^\perp J^\perp) = \frac{1}{2}(N^*\omega_{S^2} - R^*\omega_{S^2}) \tag{18}$$

where $N^*\omega_{S^2}$ and $R^*\omega_{S^2}$ denote pull backs of the volume form on S^2, e.g.

$$N^*\omega_{S^2} = \langle N dN, dN\rangle = -\frac{1}{2} N dN \wedge dN = \\ = \frac{1}{2}(dN' \wedge *dN' - dN'' \wedge *dN'') = |dN'|^2 - |dN''|^2.$$

On a compact Riemann surface, the degree of $T_f M$ seen as a complex line bundle is

$$\deg(T_f M) = \frac{1}{2\pi}\int_M K|df|^2 = \deg(N) + \deg(R),$$

$\deg(N)$ and $\deg(R)$ denoting the mapping degrees of N, $R\colon M \to S^2$, e.g.

$$\deg(N) = \frac{1}{4\pi}\int_M N^*\omega_{S^2}.$$

By the Gauss–Bonnet theorem, the degree of $\deg(T_f M)$ is equal to the Euler characteristic $\chi(M)$ of the Riemann surface, i.e.

$$\deg(T_f M) = \frac{1}{2\pi}\int_M K|df|^2 = \chi(M) = 2 - 2g,$$

[1] Note that H has the opposite sign as in [**2**]!

[2] In contrast to [**2**], we use the definition $K^\perp|df|^2 = \langle R^\perp N\xi, \xi\rangle$, where $\xi \in \Gamma(\perp_f M)$ is a section of unit length.

where g denotes the genus of M. The degree of the normal bundle $\perp_f M$ seen as a complex line bundle is

$$\deg(\perp_f M) = \frac{1}{2\pi}\int_M K^{\perp}|df|^2 = \deg(N) - \deg(R). \tag{19}$$

Remark. As already mentioned above, for immersions of compact surfaces, the Willmore functional $W(f)$ is Möbius invariant. This is an immediate consequence of Lemma 19 below. A direct proof is sketched here.

The invariance of the Willmore functional under conformal transformations of $\mathbb{R}^4 = \mathbb{H}$ is obvious. Hence, for the invariance under Möbius transformations, it suffices to prove that $W(f)$ is preserved under the inversion $x \mapsto x^{-1}$. This follows from the observation that the inversion preserves $|dR''|$ and $|dN''|$ (which is easily verified) and that the Willmore functional satisfies

$$W(f) = \int_M |dR''|^2 + 4\pi\deg(R) = \int_M |dN''|^2 + 4\pi\deg(N)$$

(which follows from the formulae of this subsection), because the integrands of these integrals are invariant under Möbius transformations and the degrees are topological constants (depending on the genus and the degree of the normal bundle only).

2.3. The Quaternionic Projective Line as the Conformal 4–Sphere. We use the quaternionic projective line $\mathbb{HP}^1$ as the standard model of the conformal 4–sphere. Since $\mathbb{HP}^1$ is equipped with a canonical choice of points ∞ and 0 (the lines defined by the standard basis e_1 and e_2 of $\mathbb{H}^2$), it is favorable to work with $\mathbb{P}V$ where V is a 2–dimensional quaternionic vector space. This space is projectively equivalent to $\mathbb{HP}^1$, but the equivalence depends on the choice of a basis of V and is therefore not canonical.

Every basis v_1, v_2 of V defines a bijective map $\mathbb{P}V\backslash\{[v_1]\} \to \mathbb{H}$ with $[v_1\lambda_1 + v_2\lambda_2] \mapsto \lambda_1\lambda_2^{-1}$. This map is called the *Euclidean chart* defined by v_1, v_2, the point $[v_1]$ is the point at infinity of this Euclidean space. The atlas obtained by taking all Euclidean charts does not only define a differentiable structure, but also a conformal structure on $\mathbb{P}V$, because its transition functions are conformal transformations. The projective line $\mathbb{P}V$ with this conformal structure is conformally equivalent to the 4–sphere S^4 with its standard conformal structure (given by the metric induced from the canonical embedding $S^4 \subset \mathbb{R}^5$). A conformal diffeomorphism between both spaces can be obtain by stereographic projection. Therefore, $\mathbb{P}V$ or—after choosing a basis—$\mathbb{HP}^1$ is a model for the conformal 4–sphere.

The group $\mathrm{PGL}(V)$, which—after choosing a basis—can be identified with $\mathrm{PGL}(2,\mathbb{H})$, acts on $\mathbb{P}V$ as the group of orientation preserving conformal diffeomorphisms and is called the group of (orientation preserving) *Möbius transformations*. The geometry of $\mathbb{P}V$ together with the group of (orientation preserving) Möbius transformations is called the (oriented) *Möbius geometry* of the 4–sphere. (In the following, all Möbius transformations are, if not otherwise stated, presumed to be orientation preserving.)

The quaternionic projective line $\mathbb{HP}^1$ is equipped with the Euclidean chart induced by the standard basis e_1, e_2 of $\mathbb{H}^2$. In coordinates, this chart becomes

$$\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} \in \mathbb{HP}^1\backslash\{\infty\} \mapsto \lambda_1\lambda_2^{-1} \in \mathbb{H} \qquad \text{where} \qquad \infty = \begin{bmatrix}1\\ 0\end{bmatrix}. \tag{20}$$

Using this chart, we identify $\mathbb{H}$ with $\mathbb{HP}^1\backslash\{\infty\}$ (which is sometimes expressed by writing $\mathbb{H} \subset \mathbb{HP}^1$). The inverse of the chart is the conformal embedding

$$x \in \mathbb{H} \mapsto \begin{bmatrix}x\\ 1\end{bmatrix} \in \mathbb{HP}^1. \tag{21}$$

Remark. In contrast to the metrical 4–sphere $S^4 \subset \mathbb{R}^5$, the quaternionic projective line $\mathbb{HP}^1$ has no distinguished (globally defined) metric compatible with its conformal structure. It is therefore easier in the quaternionic model to avoid the use of metrical constructions.

The quaternionic projective line is by no means the only possible model of the conformal 4–sphere. Another model, the classical lightcone model, is introduced in Appendix A (see also [**15**] for an extensive discussion of this and the various other models of the conformal sphere.)

The Tangent Space of $\mathbb{HP}^1$. In projective differential geometry, it is convenient to use the identification

$$\mathrm{Hom}(L, V/L) = T_L\mathbb{P}V \tag{22}$$

(where V is a vector space). The idea behind this identification is simple: let $p\colon V\backslash\{0\} \to \mathbb{P}V$ be the canonical projection. Then $dp_x v = dp_{x\lambda} v\lambda$ holds for all $x \in V\backslash\{0\}$, $v \in V$ and $\lambda \in \mathbb{H}^*$ because $p(x) = p(x\lambda)$ holds for all $x \in V\backslash\{0\}$ and $\lambda \in \mathbb{H}^*$. The differential $dp_x\colon V \to T_L\mathbb{P}V$ (where $L = [x]$) is surjective and $\ker dp_x = L$. Hence, the map

$$\xi \in \mathrm{Hom}(L, V/L) \mapsto dp_x(\xi(x)) \in T_L\mathbb{P}V \tag{23}$$

is independent of the choice $x \in L$. It is injective and, because both spaces have the same dimension, it is bijective.

A Convention. Throughout this text we use this notational convention:

If V is a quaternionic rank 2 vector space, the corresponding trivial vector bundle $M \times V$ over M is denoted by the same letter V.
We do not distinguish between a map $f\colon M \to \mathbb{P}V$ from a manifold M into the projective line $\mathbb{P}V$ and the corresponding line subbundle $L \subset V$ of the trivial vector bundle V.

The Differential of Maps into Projective Space. The identification $\mathrm{Hom}(L, V/L) = T_L\mathbb{P}V$ yields the following characterization of the differential of maps into projective space.

Lemma 8. *Under the identification* (22)*, the differential of a map $L \subset V$ into the projective space $\mathbb{P}V$ is the 1–form $\delta \in \Omega^1(\mathrm{Hom}(L, V/L)$ given by*

$$\delta = \pi d_{|L}$$

(where $\pi\colon V \to V/L$ is the canonical projection). The map $L \subset V$ is an immersion if its differential $\delta \in \Omega^1(\mathrm{Hom}(L, V/L))$, seen as a bundle map

$$\delta\colon TM \to \mathrm{Hom}(L, V/L),$$

is injective at every point.

PROOF. Obviously, δ is tensorial. To prove that it corresponds to the ordinary differential of maps between manifolds, take a nowhere vanishing section $\psi \in \Gamma(L)$. Since

$$\begin{tikzcd} V\backslash\{0\} \arrow[r, "p"] & \mathbb{P}V \\ M \arrow[u, "\psi"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes, we have $df_p = dp_{\psi(p)} d\psi_p$ for all $p \in M$. Therefore, under the above isomorphism, the differential df becomes $\xi \in \Omega^1(\mathrm{Hom}(L, V/L))$ with $\xi_p(\psi(p)) = \pi_p d\psi_p$. $\square$

For example, the differential of the conformal embedding (21), under the identification $\mathrm{Hom}(L, V/L) = T_L\mathbb{P}V$, is given by the formula

$$v \in T_x\mathbb{H} = \mathbb{H} \mapsto \left(\begin{pmatrix} x \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} v \\ 0 \end{pmatrix} \mod \begin{bmatrix} x \\ 1 \end{bmatrix} \right), \tag{24}$$

which is needed in the proof of the following lemma.

LEMMA 9. *Let $L \subset V$ be a line in a quaternionic rank 2 vector space V. There is a 1-1–correspondence between 2–dimensional oriented real subspaces $U \subset \mathrm{Hom}(L, V/L)$ and between pairs of complex structures J, $\tilde{J}$ on L and V/L. The complex structures corresponding to U are the ones characterized by the property that $\tilde{J}B = BJ$ for all $B \in U$ and that post composition with $\tilde{J}$ is a rotation by $\frac{\pi}{2}$.*

The subspace $U^\perp = \{B \in \mathrm{Hom}(L, V/L) \mid \tilde{J}B = -BJ\}$ is then the orthogonal complement to U. A rotation by $\frac{\pi}{2}$ in $U^\perp$ is also given by post composition with $\tilde{J}$.

PROOF. Choosing an Euclidean chart, the statement immediately follows from Lemma 6 and the fact that (24) is the differential of a conformal map: under the correspondence between pairs of quaternions N, R satisfying $N^2 = R^2 = -1$ and between pairs of complex structures J and $\tilde{J}$ on L and V/L via

$$J \begin{pmatrix} x \\ 1 \end{pmatrix} = - \begin{pmatrix} x \\ 1 \end{pmatrix} R$$

and

$$\tilde{J} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mod L = \begin{pmatrix} 1 \\ 0 \end{pmatrix} N \mod L \qquad \text{where} \qquad L = \begin{bmatrix} x \\ 1 \end{bmatrix},$$

vectors $v \in T_x\mathbb{H} = \mathbb{H}$ with $Nv = \mp vR$ correspond, via (24), to homomorphisms $B \in \mathrm{Hom}(L, V/L)$ satisfying $\tilde{J}B = \pm BJ$. $\square$

2–Sphere in $\mathbb{HP}^1$. In Möbius geometry of surfaces, 2–spheres play a fundamental role analogous to that of 2–planes in Euclidean surface theory.

Definition. A subset $X \subset \mathbb{P}V$ is called a *2–sphere* if, with respect to one (and therefore every) Euclidean chart, it is a 2–sphere or, in the case that $\infty \in X$, a 2–plane.

Equivalently, 2–spheres can be defined as the maximal 2–dimensional totally umbilic submanifolds of $\mathbb{P}V$. The following characterization of 2–sphere is essential for our treatment of surfaces in $\mathbb{P}V$. It can be seen as the Möbius geometric analogue to Lemma 6.

Lemma 10. *Let V be a quaternionic vector space of rank 2. There is a 1-1–correspondence between*

a) pairs $\pm S \in \operatorname{End}(V)$ of endomorphisms with $S^2 = -\operatorname{Id}$,

b) pairs $\hat{V}$ and $\hat{V}\mathbf{j}$ of 2–dimensional $\mathbf{i}$–complex subspaces of $(V, \mathbf{i})$ that are not $\mathbf{j}$–invariant and

c) 2–spheres in $\mathbb{P}V$.

The correspondence assigns to $\pm S$ the subspaces $\hat{V} = \{v \in V \mid Sv = v\mathbf{i}\}$ and $\hat{V}\mathbf{j}$. It assigns to $\hat{V}$ the 2–sphere $\mathbb{P}_{\mathbb{H}}\hat{V} = \{[v] \in \mathbb{P}V \mid v \in \hat{V}\}$.

Proof. The correspondence between a) and b) is Lemma 1, therefore we only have to prove the statement for b) and c). The correspondence assigning to a subspace $\hat{V}$ the set $\mathbb{P}_{\mathbb{H}}\hat{V}$ is equivariant with respect to the natural actions of $\mathrm{GL}(2, \mathbb{H})$ and $\mathrm{PGL}(2, \mathbb{H})$. Since these actions, as well as the action of $\mathrm{PGL}(2, \mathbb{H})$ on the 2–spheres in $\mathbb{P}V$, are transitive, it is sufficient to prove that there is one pair of subspace $\hat{V}$ and $\hat{V}\mathbf{j}$ uniquely corresponding to a 2–sphere.

Let e_1, e_2 be a basis of V and take $\hat{V} = \operatorname{Span}_{\mathbb{C}}\{e_1, e_2\}$ in $(V, \mathbf{i})$. Obviously, in the Euclidean chart defined by e_1, e_2, the set $\mathbb{P}_{\mathbb{H}}\hat{V}$ becomes $\mathbb{C} \subset \mathbb{H}$, which shows that $\mathbb{P}_{\mathbb{H}}\hat{V}$ is indeed a 2–sphere. To prove the theorem, we have to check that, if $\tilde{V}$ is a complex 2–dimensional subspace of $(V, \mathbf{i})$ with $\mathbb{P}_{\mathbb{H}}\hat{V} = \mathbb{P}_{\mathbb{H}}\tilde{V}$, then $\tilde{V} = \hat{V}$ or $\tilde{V} = \hat{V}\mathbf{j}$. Because $\mathbb{P}_{\mathbb{H}}\hat{V} = \mathbb{P}_{\mathbb{H}}\tilde{V}$, we can take a basis $e_1 a$ and $e_2 b$ of $\tilde{V}$ with a, $b \in \mathbb{H}$ such that $a\lambda b^{-1} \in \mathbb{C}$ for all $\lambda \in \mathbb{C}$. The aim is to prove that

$$(*) \qquad a, b \in \mathbb{C} \qquad \textit{or} \qquad a, b \in \mathbf{j}\mathbb{C},$$

because this implies $\tilde{V} = \hat{V}$ or $\tilde{V} = \hat{V}\mathbf{j}$. To prove $(*)$, note that $a\lambda b^{-1} \in \mathbb{C}$ holds for all $\lambda \in \mathbb{C}$ if and only if the linear isometry $x \mapsto axb^{-1}$ of $\mathbb{H}$ leaves $\mathbb{C}$ invariant. This implies $ab^{-1} \in \mathbb{C}$ or $a = cb$ for $c \in \mathbb{C}$, and using this we see that the map $x \mapsto bxb^{-1}$ as well has to leave $\mathbb{C}$ invariant, which clearly is only possible if $b \in \mathbb{C}$ or $b \in \mathbb{C}\mathbf{j}$. □

Because $\mathbb{P}_{\mathbb{H}}\hat{V}$ is the fixed point set of the Möbius transformation induced by S (which is therefore an inversion at the 2–sphere $\mathbb{P}_{\mathbb{H}}\hat{V}$), we obtain the following geometric interpretation of the correspondence given in the preceding lemma:

Corollary 11. *Every 2–sphere is the fixed point set $\{[v] \in \mathbb{P}V \mid S[v] = [v]\}$ of a unique involutive Möbius transformation. This Möbius transformation is given by a pair of endomorphisms $\pm S \in \operatorname{End}(V)$ with $S^2 = -\operatorname{Id}$.*

For another, quite different, direct proof of this corollary, see [**2**, Section 3.4]. The following lemma describes the tangent spaces of a 2–sphere.

LEMMA 12. *Let V be a quaternionic rank 2 vector space and let $S \in \operatorname{End}(V)$ satisfy $S^2 = -\operatorname{Id}$. Denote by X the 2–sphere in $\mathbb{P}V$ defined by S. Its tangent space T_LX at a point $L \in \mathbb{P}V$ is (under the identification* (22)*)*

$$\operatorname{Hom}_+(L, V/L),$$

where $+$ indicates the $+$–part with respect to the complex structures induced by S on the bundles L and V/L. The normal space is $\operatorname{Hom}_-(L, V/L)$.

Post composition with S on $\operatorname{Hom}_\pm(L, V/L)$ induces a rotation by $\frac{\pi}{2}$ compatible with the conformal structure, in particular it induces an orientation.

REMARK. As indicated in the last statement of the lemma, we consider endomorphisms $S \in \operatorname{End}(V)$ with $S^2 = -\operatorname{Id}$ as *oriented 2–spheres* in $\mathbb{P}V$.

PROOF. Take ψ a section of the tautological line bundle L over X. Then $S\psi = \psi\lambda$ for a quaternionic valued function λ. This implies $Sd\psi = d\psi\lambda + \psi d\lambda$ and, taking the projection to V/L, we obtain $S\delta\psi = \delta\psi\lambda = \delta S\psi$, or simply

$$S\delta = \delta S,$$

where δ denotes the differential as in Lemma 8. This proves that $\operatorname{Hom}_+(L, V/L)$ is the tangent space. The other statements follow from Lemma 9. □

2.4. Conformal Immersions into $\mathbb{HP}^1$. With the identification introduced above, maps from a Riemann surface M into $\mathbb{HP}^1$ are identified with line subbundles $L \subset \mathbb{H}^2$ of the trivial $\mathbb{H}^2$ bundle on M. Such a map $L \subset \mathbb{H}^2$ is called a *holomorphic curve*, if there is $J \in \Gamma(\operatorname{End} L)$, such that the differential $\delta = \pi d_{|L}$ satisfies

$$*\delta = \delta J.$$

If the map takes values in $\mathbb{H} \subset \mathbb{HP}^1$, the bundle L can be written as

$$L = \psi\mathbb{H} \qquad \text{with} \qquad \psi = \begin{pmatrix} f \\ 1 \end{pmatrix}$$

for $f \colon M \to \mathbb{H}$. It is easily verified (using (24)) that L is a holomorphic curve with $J\psi = -\psi R$ if and only if $*df = -df R$. Clearly, f is immersed if and only if δ is nowhere vanishing, hence an immersed holomorphic curves into $\mathbb{HP}^1$ is the same thing as a conformal immersion into $\mathbb{HP}^1$.

It is advantageous to use the following slightly more general definition of holomorphic curves.

DEFINITION. Let (V, ∇) be a flat quaternionic rank 2 bundle. A line subbundle $L \subset V$ is called a *holomorphic curve*, if there is $J \in \Gamma(\operatorname{End} L)$ such that the differential $\delta = \pi\nabla_{|L} \in \Gamma(T^*M \otimes \operatorname{Hom}(L, V/L))$ of L satisfies

$$*\delta = \delta J.$$

L is called immersed if δ is nowhere vanishing.

Obviously, if the holomorphic curve $L \subset V$ is not a parallel subbundle of (V, ∇), the endomorphism J is uniquely defined and satisfies $J^2 = -\mathrm{Id}$.

For a rank 2 vector space V (e.g. $V = \mathbb{H}^2$), the corresponding trivial vector bundle V carries the trivial connection $\nabla = d$. As we have seen above, conformal immersions into $\mathbb{P}V$ correspond to holomorphic curves $L \subset V$ in (V, ∇).

The more general approach of considering holomorphic curves $L \subset V$ in flat vector bundles (V, ∇) instead of vector spaces allows the treatment of immersions from the universal covering $\tilde{M}$ into $\mathbb{HP}^1$ with Möbius monodromy lifting to $GL(2, \mathbb{H})$. We give here only a brief explanation of this fact (for a more detailed discussion, see Lemma 80): Since V can be trivialized on the universal covering $\tilde{M}$, an immersed holomorphic curve gives rise to an immersion $\tilde{L} \subset \mathbb{H}^2$ from $\tilde{M}$ into $\mathbb{HP}^1$ that is equivariant with respect to the monodromy representation of the group of deck transformations. Conversely, every immersion $\tilde{L} \subset \mathbb{H}^2$ from $\tilde{M}$ into $\mathbb{HP}^1$ that is equivariant with respect to a $\mathrm{GL}(2, \mathbb{H})$–representation of the group of deck transformations defines a flat bundle (V, ∇) on M (which is unique up to gauge transformation) and a holomorphic curve $L \subset V$.

For (V, ∇) a flat quaternionic rank 2 vector bundle, we use the following notations synonymously:

- $L \subset V$ is an immersed holomorphic curve in (V, ∇),
- L is a conformal immersion into (V, ∇) and
- (V, L, ∇) is a quaternionic projective structure (as defined in Section 12) with flat connection ∇.

The adjective conformal in conformal immersion might be dropped if either conformality is clear from the context, or if the conformal structure on the surface M is not important.

The Mean Curvature Sphere. The mean curvature sphere congruence of a conformal immersion is the Möbius geometric analogue to the Gauss map in Euclidean geometry of surfaces. In contrast to the Gauss map, i.e. the (oriented) tangent plane congruence of the immersion, the mean curvature sphere congruence depends on second derivatives of the immersion. It describes the unique 2–sphere congruence tangent to the immersion with the property that every 2–sphere has the same mean curvature as the immersion has at the corresponding point. While this formulation involves the metrical quantity mean curvature, it is nevertheless Möbius invariant, i.e. does not depend on the Euclidean chart chosen.

Definition. The *mean curvature sphere (congruence)* of an immersed holomorphic curve $L \subset V$ in a flat quaternionic rank 2 vector bundle (V, ∇) is the (unique) section $S \in \Gamma(\mathrm{End}(V))$ with $S^2 = -\mathrm{Id}$ that has the properties

i) $SL = L$,
ii) $*\delta = \delta S = S\delta$ and
iii) $Q_{|L} = 0$ (or, equivalently, $\operatorname{im} A \subset L$).

The *Hopf fields* of the holomorphic curve $L \subset V$ are the forms $A \in \Gamma(K\,\mathrm{End}_-(V))$ and $Q \in \Gamma(\bar{K}\,\mathrm{End}_-(V))$ defined by the usual decomposition $\nabla = \hat{\nabla} + A + Q$ with respect to S.

Theorem 2 of [**2**] states that every immersed holomorphic curve has a unique mean curvature sphere S. The first condition $SL = L$ stands for pointwise intersection of the immersion with the corresponding 2–sphere. The second condition $*\delta = \delta S = S\delta$ means that, pointwise, each 2–sphere touches the immersion with the same orientation (cf. Lemma 9 and Lemma 12). The last condition $Q_{|L} = 0$ or, equivalently, $\operatorname{im} A \subset L$ (see [**2**, Lemma 5]), expresses the fact that, with respect to one (and therefore every) Euclidean chart, each of the 2–spheres has the same mean curvature as the immersion has at the corresponding point (see the remark at the end of the section).

The bundle $L^\perp \subset V^*$ is a holomorphic curve, called the *dual curve* of $L \subset V$. Its mean curvature sphere is S^*. The Hopf fields of $L^\perp$ are $A^\perp = -Q^*$ and $Q^\perp = -A^*$. Every general property of the Hopf fields A is reflected in a corresponding property of the Hopf fields Q. This property can be obtained by passing to the dual curve $L^\perp$, i.e. by dualization using $A^\perp = -Q^*$, see e.g. the proof of Lemma 14, which is considerably simplified by the use of this principle. (Geometrically, $L^\perp$ is essentially the same immersion than L. With respect to Euclidean charts, the transformation $L \rightsquigarrow L^\perp$ corresponds to $f \rightsquigarrow \bar{f}$.)

The next lemma is a simple but important observation, which is used at various occasions in the following chapters.

LEMMA 13. *Let $L \subset V$ be a non–constant holomorphic curve in a flat bundle (V, ∇) and let $\eta \in \Omega^1(V)$ (or $\eta \in \Omega^1(\operatorname{End}(V))$) be a 1–form taking values in L. Then $d^\nabla \eta$ takes values in L if and only if $\eta \in \Gamma(KV)$ (or $\eta \in \Gamma(K \operatorname{End}(V))$).*

PROOF. The lemma is a direct consequence of

$$\pi d^\nabla \eta = \delta \wedge \eta.$$

(In the case $\eta \in \Omega^1(\operatorname{End}(V))$, from the way the differential of endomorphism valued forms is taken, one might expect another term popping up behind η. It does not, because η takes values in L and π vanishes on L.) $\square$

LEMMA 14. *The Hopf fields $A \in \Gamma(K \operatorname{End}_-(V))$ and $Q \in \Gamma(\bar{K} \operatorname{End}_-(V))$ of a conformal immersion $L \subset V$ into (V, ∇) satisfy*

$$d^\nabla * A \in \Omega^2(\mathcal{R}) \qquad \text{and} \qquad d^\nabla * Q \in \Omega^2(\mathcal{R})$$

where $\mathcal{R} = \{B \in \operatorname{End}(V) \mid L \subset \ker B, \operatorname{im} B \subset L\}$.

PROOF. Lemma 13 proves $\operatorname{im} d^\nabla * A \subset L$ and, by applying the same argument to the dual curve $L^\perp$, we obtain $(d^\nabla * Q)L = 0$. The statement follows, since ∇ is flat and therefore (4) implies $d^\nabla * Q = d^\nabla * A$. $\square$

REMARK. Since $d^\nabla * A = S d^{\hat\nabla} A$ and $d^\nabla * Q = -S d^{\hat\nabla} Q$, we also have $d^{\hat\nabla} A$, $d^{\hat\nabla} Q \in \Omega^2(\mathcal{R})$.

The *degree* of a holomorphic curve (L, J) is defined to be the degree of the complex quaternionic line bundle L^{-1}. The *Willmore functional* of the holomorphic curve (L, J) is defined by

$$W(L, J) = 2 \int_M \langle A \wedge *A \rangle. \tag{25}$$

We will see below (cf. Section 3.1) that this is the Willmore functional of the natural (Möbius invariant) holomorphic structure on L^{-1}. (Warning: the Willmore functional of a holomorphic curve (L, J), which is always denoted by $W(L, J)$, should not be confused with the Willmore functional W(L) of the holomorphic structure on L defined after fixing a point at infinity, cf. Sections 3.3 and 3.4.)

Hence, both the degree and the Willmore functional of the holomorphic curve are defined by the corresponding quantities of the quaternionic holomorphic line bundle L^{-1}. In the case of the degree, L^{-1} is preferred to L in order to be consistent with the notation in complex algebraic geometry (see [**9**]). In the case of the Willmore functional there is no other choice, since L does *not* carry a Möbius invariant quaternionic holomorphic structure.

The Tangent and the Normal Bundle. Let $L \subset V$ be a conformal immersion into a flat quaternionic rank 2 bundle (V, ∇). Using Lemma 9, the condition $*\delta = S\delta = \delta S$ in the definition of the mean curvature sphere implies that the tangent bundle of L, expressed in the homomorphism model (22) of the tangent space of projective space, is

$$\mathrm{Hom}_+(L, V/L)$$

and the normal bundle of L is

$$\mathrm{Hom}_-(L, V/L)$$

(where $\pm$–parts are taken with respect to the complex structures induced by the mean curvature sphere S). With respect to an Euclidean chart, the map (24) clearly defines an isomorphism between the tangent and normal bundles $T_f M$ and $\perp_f M$ of the Euclidean realization f and the bundles $\mathrm{Hom}_\pm(L, V/L)$.

Since $\delta \in \Gamma(K\,\mathrm{Hom}_+(L, V/L))$ is a nowhere vanishing section, on a compact Riemann surface, the degree of V, which is equal to the degree of the normal bundle $\mathrm{Hom}_-(L, V/L)$ (with complex structure given by post composition with that of V/L), satisfies

$$\deg(\mathrm{Hom}_-(L, V/L)) = \deg V = 2\deg L - \deg K. \tag{26}$$

In particular, the degree of the normal bundle is 0 if and only if $\deg L = g - 1$, i.e. if and only if L has the degree of a spin bundle (cf. Section 3.3).

Mean Curvature Sphere and Hopf Fields in Euclidean Chart. Let $L \subset V$ be an immersed holomorphic curve in $\mathbb{P}V$ (with V a quaternionic rank 2 vector space V). The choice of a basis e_1, e_2 of V defines an isomorphism of V with $\mathbb{H}^2$, such that $L \subset \mathbb{H}^2$ becomes a curve in $\mathbb{HP}^1$. Assuming that L does not go through $\infty = [e_1]$, the curve can be written as

$$L = \begin{pmatrix} f \\ 1 \end{pmatrix} \mathbb{H}$$

where $f \colon M \to \mathbb{H}$ is the corresponding immersion into $\mathbb{H}$. We usually call f the *Euclidean realization* of L with respect to the Euclidean chart given by e_1, e_2. The mean curvature sphere of L is then (as is proven in Section 7.2 of [**2**]) given by

$$S = \mathrm{Ad}\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} N & 0 \\ H & -R \end{pmatrix} \tag{27}$$

where N and R denote the left and right normal vectors of the Euclidean realization f (i.e. $*df = Ndf = -dfR$) and where H is given by $dN' = -dfH$ or $dR' = -Hdf$ (see Section 2.2). The Hopf fields are

$$2 * A = \mathrm{Ad}\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ w & dR'' \end{pmatrix} \tag{28}$$

and

$$2 * Q = \mathrm{Ad}\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} dN'' & 0 \\ -w + dH & 0 \end{pmatrix}, \tag{29}$$

where w can be computed by

$$w = \tfrac{1}{2}(-dH - R * dH + H * dN'') = -dH''^R + \tfrac{1}{2}H * dN''.$$

Remark. For a fixed point p, the endomorphism S_p defines an oriented 2–sphere. It is easily checked that the endomorphism S_p itself is the mean curvature sphere congruence of this 2–sphere considered as a conformal immersion: condition i) in the definition of the mean curvature sphere is clearly satisfied, condition ii) holds by Lemma 12 and condition iii) holds, because S_p is constant and therefore $A^{S_p} = Q^{S_p} = 0$. Because, for every Euclidean chart, formula (27) as well applies to the 2–sphere defined by S_p, we obtain that, at the point p, the immersion and the 2–sphere have the same mean curvature vector $\mathcal{H} = \bar{H}R = N\bar{H}$.

Therefore, at every point and for every Euclidean chart, the immersion has the same mean curvature vector as the corresponding sphere, which justifies the name mean curvature sphere.

3. Holomorphic Bundles Related to Immersions into $\mathbb{HP}^1$

Every immersed holomorphic curve L in a flat quaternionic rank 2 bundle (V, ∇) comes with several quaternionic holomorphic line bundles, the most important ones being L^{-1} and V/L. These holomorphic bundles are indispensable in the treatment of associated families, Bäcklund transformations and Darboux transformations (see Chapter II) for immersions of Riemann surfaces into the conformal 4–sphere.

In the case that V has trivial holonomy, i.e. in the case of $L \subset V$ an immersed holomorphic curve in $\mathbb{P}V$ with V a fixed rank 2 vector space, the holomorphic structures on the bundles L^{-1} and V/L are Möbius invariant. The curves $L \subset V$ and $L^\perp \subset V^*$ can then be obtained via Kodaira correspondence from 2–dimensional spaces of holomorphic sections of L^{-1} and V/L.

If an Euclidean chart is chosen such that L does not go through ∞, the Euclidean realization $f\colon M \to \mathbb{H}$ of L with respect to this chart can be obtained by the so called Weierstrass representation. This means that there are compatible holomorphic structures on the paired bundles $L^\perp \cong KL^{-1}$ and L, such that

$$df = (\varphi, \psi)$$

for $\varphi \in H^0(KL^{-1})$ and $\psi \in H^0(L)$. The holomorphic structures on KL^{-1} and L, and the factorization of the differential df, are of course not Möbius invariant, but depend on the choice of a point at infinity.

The first and second subsection treat the Möbius invariant bundles and the third and fourth subsection treat the bundles of the Weierstrass representation and their relation to the Möbius invariant bundles.

3.1. Holomorphic Bundle Related to a Holomorphic Curve. Let L be a holomorphic curve in (V, ∇), i.e. there is $J \in \Gamma(\mathrm{End}(L))$ such that $*\delta = \delta J$ or, equivalently,

$$\pi(\nabla + *\nabla J)_{|L} = 0.$$

Hence, L carries a quaternionic linear operator $\tilde{D} \colon \Gamma(L) \to \Omega^1(L)$ defined by

$$\tilde{D} = \frac{1}{2}(\nabla + *\nabla J)_{|L}.$$

This operator is a so called mixed structure, i.e. it satisfies $*\tilde{D} = -\tilde{D}J$ and $\tilde{D}(\psi f) = (\tilde{D}\psi)f + \frac{1}{2}(\psi df + J\psi * df)$ for $\psi \in \Gamma(L)$ and f a quaternionic function. This mixed structure induces a holomorphic structure D on the complex quaternionic line bundle L^{-1} which is defined by

$$\langle D\alpha, \psi\rangle + \langle \alpha, \tilde{D}\psi\rangle = \frac{1}{2}(d\langle \alpha, \psi\rangle + *d\langle \alpha, J\psi\rangle) \tag{30}$$

for every $\alpha \in \Gamma(L^{-1})$ and $\psi \in \Gamma(L)$. It is easily verified that the Willmore functional $W(L^{-1})$ of this holomorphic structure on L^{-1} coincides with the Willmore functional $W(L, J)$ of the holomorphic curve (L, J) (as defined in Section 2.4).

Kodaira Correspondence. In the case that (V, ∇) has trivial monodromy, i.e. in the case of $L \subset V$ a holomorphic curve in $\mathbb{P}V$ with V a fixed vector space, (the isomorphism class of) the quaternionic holomorphic line bundle L^{-1} is invariant under Möbius transformations. By definition of D on L^{-1}, each element $\alpha \in V^*$ restricts to a holomorphic section $\alpha_{|L} \in H^0(L^{-1})$. If the holomorphic curve is non–constant, the dual space V^* is a 2–dimensional linear system $H = V^* \subset H^0(L^{-1})$ without base points.

Conversely, via *Kodaira correspondence*(see [**8**, Section 2.6] for a more detailed discussion), every 2–dimensional linear system $H \subset H^0(L^{-1})$ without base points gives rise to a holomorphic curve $L \subset V = H^*$ by pointwise defining L to be the line perpendicular to the subspace in H of sections vanishing at this point. More explicitly, the holomorphic curve L can be obtained by choosing a basis α, $\beta \in H$ such that β is nowhere vanishing (this exists, since for quaternionic holomorphic bundles, the base point divisor is realized by generic sections). Then $\alpha = \beta\bar{f}$ for a quaternionic function f and, with respect to the dual basis e_1, e_2 of $V = H^*$, the holomorphic curve L is given by

$$L = \begin{pmatrix} f \\ 1 \end{pmatrix} \mathbb{H}.$$

Hence, a 2–dimensional linear system without base points uniquely determines a holomorphic curve and vice versa. The holomorphic curve L is immersed if and only if the corresponding 2–dimensional linear system has no Weierstrass points.

3.2. Holomorphic Bundles Related to an Immersed Holomorphic Curve. A holomorphic curve L in (V,∇) gives rise to a holomorphic structure on L^{-1}, which is Möbius invariant in the case of trivial holonomy. If L is immersed, the dual curve $L^\perp \subset V^*$ as well is a holomorphic curve, and by the same argument, the bundle $V/L = (L^\perp)^{-1}$ carries a holomorphic structure, which, in the case of trivial holonomy, is Möbius invariant, too.

Let L be an immersed holomorphic curve in (V,∇) and denote by S its mean curvature sphere. The usual decomposition of ∇ with respect to S is $\nabla = \bar\partial + \partial + Q + A$. The operator $\nabla'' = \bar\partial + Q$ (with $\nabla'' = \frac{1}{2}(\nabla + S * \nabla)$) makes V into a quaternionic holomorphic vector bundle. The line bundle L is invariant under ∇'', which follows from the simple computation

$$\pi\nabla''_{|L} = \pi\frac{1}{2}(\nabla + S*\nabla)_{|L} = \frac{1}{2}(\delta + S*\delta) = 0.$$

Because of $Q_{|L} = 0$, the induced holomorphic structure on L is $\nabla''_{|L} = \bar\partial_{|L}$.

By invariance of L under ∇'', the quotient $V/L = (L^\perp)^{-1}$ inherits a holomorphic structure D defined by

$$D\pi = \pi\nabla''. \tag{31}$$

The holomorphic structure D on $V/L = (L^\perp)^{-1}$ is the same than the one induced by the mixed structure $\tilde D$ on $L^\perp$. This follows from (30) applied to the holomorphic curve $L^\perp$, since

$$\begin{aligned}\langle \tilde D\alpha, \pi\psi\rangle + \langle \alpha, \pi\nabla''\psi\rangle &= \frac{1}{2}(\langle \nabla\alpha + *\nabla(S\alpha), \psi\rangle + \langle \alpha, (\nabla + S*\nabla)\psi\rangle \\ &= \frac{1}{2}(d\langle\alpha,\pi\psi\rangle + *d\langle S\alpha, \pi\psi\rangle\end{aligned}$$

for $\alpha \in \Gamma(L^\perp)$ and $\psi \in \Gamma(V)$.

The relation between the Willmore functionals of the holomorphic structures on L^{-1} and V/L is given by the following Lemma.

Lemma 15. *Let $L \subset V$ be a conformal immersion of a compact Riemann surface into (V,∇). The relation between the Willmore functionals of the holomorphic line bundles L^{-1} and V/L is*

$$W(L^{-1}) - W(V/L) = 4\pi\deg(V) = 4\pi(\deg(V/L) - \deg(L^{-1})).$$

Proof. This follows from the flatness of the connection $\nabla = \hat\nabla + A + Q$ on V, which implies $R^{\hat\nabla}S = A\wedge *A - Q\wedge *Q$ (see (5)). Using (1), the relation between the Willmore functionals follows by integration, because $A\wedge *A$ yields the Willmore functional of L^{-1} and $Q\wedge *Q$ that of V/L. □

Thus, both Willmore functionals coincide if and only if the degree of the normal bundle of the immersion is 0, e.g. for immersions into the 3–sphere.

The 1–Jet Bundle of V/L. Using the sequence

$$V \xrightarrow{\pi} V/L \to 0$$

and the jet–derivative

$$d = \pi\nabla,$$

V becomes a model for the 1–jet bundle of the quaternionic holomorphic line bundle V/L. Axiom 3 of Chapter 3 in [**8**] is satisfied, since $\ker\pi = L$ is immersed, and Axiom 4 holds by equation (31) and $d'' = (\pi\nabla)'' = \pi(\nabla'') = D\pi$. The connection ∇ on the bundle V is a flat *adapted connection* on the 1–jet bundle of V/L, meaning that ∇ is a flat connection with the property that $d = \pi\nabla$ is the jet–derivative (cf. [**8**, Lemma 3.4]).

The Bundle KL. The bundle KL is paired with L^{-1} and therefore has a holomorphic structure. This holomorphic structure can as well be defined by

$$D\omega = d^{\nabla}\omega \tag{32}$$

for $\omega \in \Gamma(KL)$, where we use again the identification $\bar{K}K \cong \Lambda^2 T^*M^{\mathbb{C}}$ via $d\bar{z}\,dz \mapsto d\bar{z}\wedge dz$. This is indeed the holomorphic structure induced by the pairing, because it satisfies equation (2), i.e.

$$d\langle\pi\alpha,\omega\rangle = d\langle\alpha,\omega\rangle = \langle\nabla\alpha\wedge\omega\rangle + \langle\alpha, d^{\nabla}\omega\rangle = \langle\underbrace{(\pi\nabla\alpha)''}_{D\pi\alpha}\wedge\omega\rangle + \langle\pi\alpha, d^{\nabla}\omega\rangle$$

for $\alpha \in \Gamma(V^*)$ and $\omega \in \Gamma(KL)$. (Here, $\pi\colon V^* \to V^*/L^{\perp}$ is the canonical projection and $(\pi\nabla''\alpha) = D\pi\alpha$ is (31) applied to $L^{-1} = V^*/L^{\perp}$.)

By Lemma 13, every closed 1–form $\omega \in \Omega^1(L)$ is a section of KL and therefore holomorphic. Furthermore, by the same lemma, the derivative $d^{\nabla}\omega$ of every $\omega \in \Gamma(KL)$ takes values in L as well, i.e. $d^{\nabla}\omega \in \Omega^2(L)$.

3.3. The Holomorphic Bundles Related to Immersed Holomorphic Curves after Fixing a Point at Infinity. Let V be a vector space and let $L \subset V$ be an immersed holomorphic curve. Denote by $H = V^* \subset H^0(L^{-1})$ the 2–dimensional linear system related to L via Kodaira correspondence. If $[\beta] \in \mathbb{P}H$ is a point such that L does not go through $\infty = [\beta]^{\perp}$, then, as a holomorphic section, $\beta \in H^0(L^{-1})$ is nowhere vanishing. Setting $\nabla\beta = 0$, β defines a flat connection ∇ on L^{-1} which satisfies $\nabla'' = D$. By Lemma 2, the connection ∇ induces compatible holomorphic structures on KL^{-1} and L. These holomorphic structures on KL^{-1} and L are not Möbius invariant, but depend on the choice of $[\beta]$. Geometrically, $[\beta]$ or, equivalently, $\infty = [\beta]^{\perp} \in \mathbb{P}V$, determines a geometry of similarities, i.e. the subgeometry belonging to the subgroup of Möbius transformations fixing the point ∞.

Choosing a basis α, β of $H = V^*$, the dual basis e_1, e_2 of V defines an Euclidean chart. In this basis we have

$$L = \psi\mathbb{H} \qquad \text{with} \qquad \psi = \begin{pmatrix} f \\ 1 \end{pmatrix} \in \Gamma(L),$$

where $f\colon M \to \mathbb{H}$ is the Euclidean realization of L with respect to this chart. Hence, seen as sections in $H^0(L^{-1})$, α and β satisfy $\alpha = \beta\bar{f}$. The sections

$\varphi = \nabla\alpha = \beta d\bar{f} \in \Gamma(KL^{-1})$ and $\psi \in \Gamma(L)$, which can be characterized by $\beta(\psi) = 1$, are holomorphic. They satisfy

$$df = (\varphi, \psi)$$

where (,) denotes the pairing between KL^{-1} and L.

LEMMA 16. *For every immersion $f\colon M \to \mathbb{H}$, there are paired holomorphic bundles KL^{-1} and L and holomorphic sections $\varphi \in H^0(KL^{-1})$ and $\psi \in H^0(L)$, such that*

$$df = (\varphi, \psi).$$

These holomorphic bundles and sections are uniquely determined by f up to isomorphism.

DEFINITION. The representation $df = (\varphi, \psi)$ of an immersion $f\colon M \to \mathbb{H}$ is called its *Weierstrass representation.*

The existence part of the lemma is clear from the above discussion, the uniqueness part is a simple exercise (cf. [**20**, Theorem 4.3] for a direct proof of Lemma 16). This notion of Weierstrass representation for conformal immersions of surfaces into Euclidean 4–space $\mathbb{R}^4 = \mathbb{H}$ is due to Pedit and Pinkall. It is a coordinate free generalization to surfaces in 4–space of the Weierstrass representation for conformal immersions into $\mathbb{R}^3$ which was first defined (locally) by Eisenhart [**6**] and Konopelchenko [**17**] and (globally) by Taimanov [**26**]. Their representation itself is a generalization of the classical Weierstrass representation for minimal surfaces in $\mathbb{R}^3$.

The Bundles $L^\perp$ and L as Paired Bundles. While the holomorphic structures defined on KL^{-1} and L depend on the choice of a point at infinity, the underlying complex quaternionic line bundles as well as the $\bar\partial$–operators induced by the holomorphic structures needed for the Weierstrass representation are Möbius invariant, which is described in the following.

The dual curve $L^\perp \subset V^*$ of an immersed holomorphic curve $L \subset V$, seen as a complex quaternionic line bundle, is paired with L. The pairing is defined by the isomorphism $\delta^\perp\colon L^\perp \to KL^{-1} \cong KV^*/L^\perp$ given by

$$\delta^\perp = \pi d_{|L^\perp}, \tag{33}$$

which is the differential of the curve $L^\perp$ in the sense of Section 2.4. Hence, for $\varphi \in \Gamma(L^\perp)$ and $\psi \in \Gamma(L)$, the pairing is given by $(\varphi, \psi) := \langle d\varphi, \psi\rangle$. (The isomorphism $\delta\colon L \to KV/L$ given by the differential of L itself induces, up to an inevitable minus sign, the same paring between $L^\perp$ and L.)

The Möbius invariant holomorphic structure on L^{-1} is defined using the mixed structure $\bar\partial_L + A$ on L induced by d (cf. Section 3.1). The holomorphic structure on L defined using a fixed point ∞ has the form $\bar\partial_L + Q$. In particular, for every choice of ∞ such that L does not go through ∞, the underlying $\bar\partial$–operator is the same Möbius invariant operator $\bar\partial_L$. A simple calculation shows that, for all compatible holomorphic structures on paired quaternionic line bundles, the underlying $\bar\partial$–operators as well are compatible. This proves the following lemma.

LEMMA 17. *The underlying $\bar\partial$–operators of the holomorphic structures induced on the paired bundles $L^\perp \cong KL^{-1}$ and L by fixing a point $[e_1] \in \mathbb{P}V$ (with the property that L does not go through $[e_1]$) are Möbius invariant, i.e. do not depend on the choice of $[e_1] \in \mathbb{P}V$.*

The holomorphic structure

$$d'' = \tfrac{1}{2}(d + S * d)$$

acts as a $\bar\partial$–operator on the invariant subbundle $L \subset V$ (see Section 3.2). This $\bar\partial$–operator is obviously the operator occurring in the preceding lemma, i.e. the operator $\bar\partial_L$. Similarly, the compatible $\bar\partial$–operator on the paired bundle $L^\perp$ can be obtained by restricting the operator $d'' = \frac{1}{2}(d + S * d)$ on V^* to $L^\perp$ (a fact that can be easily checked directly, but as well follows from Lemma 18 by applying the same arguments to the dual curve $L^\perp$).

The Weierstrass Representation as a Notion of the Geometry of Similarities. The aim of this paragraph is to clarify, why the Weierstrass representation is a notion of the geometry of similarities.

Let $f\colon M \to \mathbb{H}$ be an immersion with Weierstrass representation $df = (\varphi, \psi)$ for $\psi \in H^0(L)$ and $\varphi \in H^0(KL^{-1})$. Pairing the sections $\tilde\psi = \psi\lambda \in H^0(L)$ and $\tilde\varphi = \varphi\bar\mu \in H^0(KL^{-1})$ yields $\mu df \lambda = (\tilde\varphi, \tilde\psi)$. Hence, every immersion into $\mathbb{H}$ obtained from f by a similarity has a Weierstrass representation with the same holomorphic bundles and holomorphic sections scaled by a constant.

In order to understand how this follows from the projective model, we continue the discussion preceding Lemma 16. The holomorphic structures on L and KL^{-1} defined there depend only on the choice of a point at infinity $\infty = [\beta]^\perp$, which determines the geometry of similarities. Denote by f the Euclidean realization of L with respect to a basis α, β of V^*. The sections φ and ψ with $df = (\varphi, \psi)$ are obtained from the basis α, β by $\nabla\alpha = \varphi$ and $\beta(\psi) = 1$. The sections $\tilde\varphi = \varphi\bar\mu \in H^0(KL^{-1})$ and $\tilde\psi = \psi\lambda \in H^0(L)$ satisfy $\tilde\varphi = \nabla\tilde\alpha \in \Gamma(KL^{-1})$ and $\tilde\beta(\tilde\psi) = 1$ for $\tilde\alpha = \alpha\bar\mu + \beta\tilde\mu$ and $\tilde\beta = \beta\bar\lambda^{-1}$, which is another basis of $H = V^*$ compatible with the geometry of similarities given by $[\beta]$. Its dual basis is $\tilde e_1 = e_1\mu^{-1}$ and $\tilde e_2 = e_2\lambda - e_1\mu^{-1}\bar{\tilde\mu}\lambda$ where e_1, e_2 is the dual basis to α, β. The Euclidean realization $\tilde f$ of L with respect to the Euclidean chart defined by $\tilde e_1$, $\tilde e_2$ is $\tilde f = \mu f\lambda + \bar{\tilde\mu}\lambda$, because $L = (\tilde e_1\tilde f + \tilde e_2)\mathbb{H}$.

Spin Bundles and Immersion into $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$. A conformal immersion $f\colon M \to \mathbb{H}$ takes values in $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ (up to translation) if and only the paired holomorphic bundles KL^{-1} and L needed for the Weierstrass representation are holomorphically isomorphic, i.e. $KL^{-1} \cong L$, and the holomorphic sections needed for the Weierstrass representation coincide under this isomorphism. This is a simple consequence of Lemma 16, because f takes values in a translate of $\operatorname{Im}\mathbb{H}$ if and only if $d\bar f = -df$ if and only if f has its Weierstrass representation in the bundles KL^{-1} and L and also (by passing to the conjugate pairing) in the bundles L and KL^{-1}.

DEFINITION. A bundle L that is paired with itself is called a *spin bundle.*

Because a paring $(\,,)\colon L \otimes L \to T^*M \otimes \mathbb{H}$ satisfies $\overline{(\varphi, \psi)} = -(\psi, \varphi)$, every holomorphic section $\psi \in H^0(L)$ of a spin bundle gives rise to a closed 1–form

(ψ, ψ) with values in $\operatorname{Im}\mathbb{H}$. The underlying complex holomorphic line bundle $(\hat{L}, \bar{\partial})$ of a spin bundle is a complex holomorphic spin bundle, i.e. a square root of the canonical bundle K. In particular, the degree of a spin bundle is $\deg L = g - 1$.

3.4. Summary and Coordinate Formulae. Let V be a quaternionic rank 2 vector space and let $L \subset V$ be a conformal immersion into $\mathbb{P}V$. The bundles L^{-1} and V/L carry Möbius invariant holomorphic structures. The holomorphic curves $L \subset V$ and $L^\perp \subset V^*$ are complex quaternionic line bundles, but only carry a holomorphic structure after a point at infinity $[e_1] \in \mathbb{P}V$ is fixed. The goal of this section is to clarify the role of these bundles, to explain the geometric meaning of their Willmore functionals and to list some important holomorphic sections.

The relations between these bundles can be visualized by the diagram

$$\begin{tikzcd}
L^\perp \arrow[dr, dotted, <->] & V/L \arrow[l, squiggly] \arrow[d, squiggly] \\
L^{-1} \arrow[u, squiggly] \arrow[r, squiggly] & L
\end{tikzcd}$$

where the bundles on the same vertical level are dual to each other (e.g. $V/L = (L^\perp)^{-1}$) and the diagonal arrow stands for the pairing between $L^\perp$ and L. The wavy arrows shall indicate that every quaternionic connection on L^{-1} or V/L, by Lemma 2, induces paired holomorphic structures on $L^\perp$ and L.

Fixing a point $e_1 \in V$ such that L does not go through $[e_1]$, and $\beta \in V^*$ with $\beta(e_1) = 0$, defines holomorphic structures on $L^\perp$ and L. To simplify notation, the corresponding nowhere vanishing holomorphic sections of V/L and L^{-1} are denoted by the same letters. As in Section 3.3, we define a flat quaternionic connection on L^{-1} by $\nabla\beta = 0$. As indicated by the wavy arrows in the above diagram, this defines compatible holomorphic structures on $L^\perp \cong KL^{-1}$ and L. In the same way, a connection on V/L is defined by $\nabla e_1 = 0$. Again as indicated by the wavy arrows (this time in the upper triangle), $L^\perp$ and L inherit another pair of compatible holomorphic structures. By definition, both pairs of holomorphic structures only depend on $[e_1]$ (and not on the representative $e_1 \in [e_1]$). It is not surprising that both pairs of holomorphic structures coincide:

Lemma 18. *Let $L \subset V$ be an immersed holomorphic curve in $\mathbb{P}V$ (with V a quaternionic rank 2 vector space). Let $[e_1] \subset \mathbb{P}V$ be such that L does not go through $[e_1]$ and let $\beta \in V^*$ satisfy $\beta(e_1) = 0$. Then, the compatible holomorphic structures on the paired bundles L and $L^\perp \cong KL^{-1}$ induced (as in Lemma 2) by the connection ∇ on L^{-1} satisfying $\nabla\beta = 0$ coincide with the compatible holomorphic structures induced by the connection ∇ on V/L satisfying $\nabla e_1 = 0$.*

This lemma is proven at the end of the following paragraph. It is needed in the proof of the following lemma.

Lemma 19. *Let $L \subset V$ be an immersed holomorphic curve in $\mathbb{P}V$ (with V a quaternionic rank 2 vector space). Let L and $L^\perp \cong KL^{-1}$ be equipped with the holomorphic structures induced by the choice of a point $[e_1] \subset \mathbb{P}V$ such that L*

does not go through $[e_1]$. The Willmore functionals of the paired holomorphic line bundles L and KL^{-1} are related by

$$W(KL^{-1}) = W(L) = W(L^{-1}) + 4\pi\deg(L^{-1}) = W(V/L) + 4\pi\deg(V/L)$$

to the Willmore functionals of the Möbius invariant holomorphic line bundles L^{-1} and V/L.

Proof. The first equality holds, because paired holomorphic bundles have the same Willmore functional. Application of Lemma 3 to both triangles in the above diagram (and the fact that, by Lemma 18, both pairs of holomorphic structures coincide) implies the second and third equality. □

Coordinate Formulae. Let e_1, e_2 be a basis of V with the property that L does not go through the point $[e_1]$ and denote by α, β its dual basis. In the basis e_1, e_2

$$L = \psi\mathbb{H} \qquad \text{with} \qquad \psi = \begin{pmatrix} f \\ 1 \end{pmatrix} \in \Gamma(L),$$

where $f\colon M \to \mathbb{H}$ is the Euclidean realization of L with respect to the Euclidean chart belonging to e_1, e_2. Denote by N, $R\colon M \to S^2$ the left and right normal vectors of f given by $*df = Ndf = -dfR$.

To simplify notation, we denote the holomorphic sections of L^{-1} and V/L obtained from elements of the vector spaces V^* and V by the same letters. The sections β and $\alpha = \beta\bar{f}$ span the 2–dimensional linear system in $H^0(L^{-1})$, which yields L by Kodaira correspondence. The nowhere vanishing section β satisfies $J\beta = \beta R$. By (8) and the formulae of Section 2.2, this implies that the Willmore functional of L^{-1} (which equals the Willmore functional of the holomorphic curve as defined in Section 2.4) is given by

$$W(L^{-1}) = \int_M |dR''|^2 = \int_M (|\mathcal{H}|^2 - K + K^\perp)|df|^2. \tag{34}$$

As explained in Section 3.3, defining a connection on L^{-1} by $\nabla\beta = 0$, one obtains holomorphic structures on the paired bundles L and KL^{-1} needed for the Weierstrass representation of f (as indicated by the wavy arrows in the lower triangle of the above diagram). We have seen in Section 3.3 that the basis α, β induces holomorphic sections $\varphi = \nabla\alpha = \beta d\bar{f} \in H^0(KL^{-1})$ and $\psi \in H^0(L)$, which is the unique section of L with $\beta(\psi) = 1$, such that $df = (\varphi, \psi)$. These sections both are nowhere vanishing and satisfy $J\psi = -\psi R$ and $J\varphi = -\varphi N$. Again, from equation (8) and the formulae of Section 2.2, we obtain that the Willmore functionals of KL^{-1} and L are given by

$$W(KL^{-1}) = \int_M |dN'|^2 = \int_M |\mathcal{H}|^2|df|^2 \text{ and} \tag{35}$$

$$W(L) = \int_M |dR'|^2 = \int_M |\mathcal{H}|^2|df|^2, \tag{36}$$

i.e. they coincide with the (classical) definition (see Section 2.2) of the Willmore functional for $f\colon M \to \mathbb{H}$ in terms of Euclidean quantities .

Clearly, for the dual curve $L^\perp$, the holomorphic bundle $V/L = (L^\perp)^{-1}$ plays the same role that L^{-1} plays for L. As holomorphic sections of V/L, e_1 and e_2

satisfy $e_2 = -e_1 f$. The Kodaira embedding of the linear system spanned by e_1 and e_2 is the curve $L^\perp \subset V^*$. With respect to the basis α and β of V^*, this curve is given by

$$L^\perp = \tilde{\varphi}\mathbb{H} \qquad \text{with} \qquad \tilde{\varphi} = \begin{pmatrix} 1 \\ -\bar{f} \end{pmatrix} \in \Gamma(L^\perp).$$

The nowhere vanishing holomorphic section $e_1 \in H^0(V/L)$ satisfies $Je_1 = e_1 N$. By (8) and the formulae of Section 2.2, this implies that the Willmore functional of V/L (which equals the Willmore functional of the dual curve $L^\perp$) is given by

$$W(V/L) = \int_M |dN''|^2 = \int_M (|\mathcal{H}|^2 - K - K^\perp)|df|^2. \tag{37}$$

The same construction of Section 3.3 can be applied to the bundle V/L (which amounts to interchanging the role of L and $L^\perp$). The connection on V/L give by $\nabla e_1 = 0$ defines a holomorphic structures on $L^\perp$ and $L \cong KV/L$ (as indicated by the wavy arrows in the upper triangle of the above diagram). The section $\tilde{\varphi} \in \Gamma(L^\perp)$ satisfies $\langle \tilde{\varphi}, e_1 \rangle = 1$. It is therefore holomorphic as well as the section $\tilde{\psi} = \nabla e_2 = -e_1 df \in \Gamma(KV/L)$. Under the canonical pairing between $L^\perp$ and KV/L we obtain $(\tilde{\varphi}, \tilde{\psi}) = -df$. Since the holomorphic bundles needed for the Weierstrass representation of f are unique up to isomorphism (cf. Lemma 16), the paired holomorphic bundles KL^{-1} and L (with their holomorphic structure induced by the connection ∇ with $\nabla\beta = 0$ on L^{-1}) are indeed isomorphic to $L^\perp$ and KV/L. This proves Lemma 18. Explicitly, the isomorphisms are given by $\delta^\perp \colon L^\perp \to KL^{-1}$ which maps $\tilde{\varphi}$ to $-\varphi$ and by $\delta \colon L \to KV/L$ which maps ψ to $-\tilde{\psi}$. (Unfortunately, these isomorphisms are not compatible with the pairings, but change the paring of the sections by a minus sign.)

Immersed Tori with Degree 0 Normal Bundle. Let M be a compact Riemann surface of genus $g = 1$, i.e. a torus, and let $L \subset V$ be an immersed holomorphic curve in $\mathbb{P}V$ (with V a quaternionic rank 2 vector space). Since the canonical bundle K has degree 0, all bundles on the same side of the above diagram have the same degree.

We have seen above that an immersion $L \subset V$ has normal bundle degree 0 if and only if the bundle L itself has spin bundle degree, which—for tori—is 0. In this case, all bundles in the diagram have degree 0 and by Lemma 19 all involved holomorphic bundles, i.e. the two Möbius invariant bundles L^{-1} and V/L, and, after choosing a point at infinity, the paired bundles $L^\perp$ and L, have the same Willmore functional.

The Plücker Formula. The holomorphic sections of the bundles in the above diagram which are listed in the preceding discussion, are the ones that belong to every holomorphic curve, after choosing a basis e_1, e_2 such that L does not go through $[e_1]$. The existence of these sections does not depend on the topology of M, nor on the degree of L. It merely follows from the fact that $L \subset V$ is an immersed holomorphic curve (and V is a trivial flat vector bundle) and gives a lower bound for the dimensions of the spaces of holomorphic sections of the involved bundles.

Of course, it is an interesting question, how many more holomorphic sections these bundles can have and what additional sections mean geometrically. The Plücker formula (see [**8**]) shows that the Willmore functional gives an upper bound for the number of sections of a line bundle. More precisely,

$$W(L) \geq 4\pi\big((n+1)(n(1-g)-d)+\operatorname{ord}(H)\big),$$

where $W(L)$ is the Willmore functional of a quaternionic holomorphic line bundle L of degree d on a compact Riemann surface M of genus g and where $H \subset H^0(L)$ is a linear system of dimension $n+1$.

In particular, for a degree 0 bundle L over a torus, the Plücker formula reduces to $W(L) \geq 4\pi \operatorname{ord}(H)$. We finish the section with a list of some simple consequences of the Plücker formula for holomorphic bundles of degree 0 over a torus:

- A holomorphic bundle of Willmore functional $W < 4\pi$ can at most have a 1–dimensional space of sections. The sections have no zeros.
- A holomorphic bundle of Willmore functional $W < 8\pi$ can at most have a 2–dimensional space of sections. This linear system has no Weierstrass points (i.e. $\operatorname{ord}(H) = 0$), since otherwise, there would be a 1–dimensional linear system in H of order greater or equal to 2.

  Hence, for every 2–dimensional linear system in a degree 0 bundle over the torus with $W < 8\pi$, the holomorphic curve defined by Kodaira correspondence is immersed.
- Every conformal immersion of a torus into the quaternionic projective line $\mathbb{P}V$ with normal bundle degree 0 and with $W < 8\pi$ is injective. (If it was not injective, there would be a section of L^{-1} with at least 2 zeros, contradicting the Plücker formula.)

CHAPTER II

Transformations of Immersed Tori

This chapter discusses Bäcklund and Darboux transformations of tori immersed into 4–space. In addition, other important concepts of integrable system theory, like the spectrum and the associated family, are treated.

Section 4 is devoted to the spectrum, Section 5 treats the associated family, Section 6 the theory of Bäcklund transformations, Section 7 introduces Darboux transformations, and Section 8 describes a new approach to the spectral curve of conformally immersed tori.

It should be noted that the constructions of this chapter make essential use of the fact that we are considering immersion into 4–dimensional space. As a consequence of this, all transformations have a forward version related to the holomorphic curve L, and a backward version related to the dual holomorphic curve $L^\perp$. To simplify the presentation, we mainly discuss the forward versions, omitting the adjective forward where confusion seems impossible.

All the transformations generalize the classical transformations of isothermic immersions and of constrained Willmore immersions, cf. Chapter III. One of the main difficulties in the general theory is the lack of a spectral parameter in the associated family, and the resulting impossibility to apply standard loop group techniques. Another difficulty is that most constructions are less explicit than their classical counterparts: explicit formulae are often replaced with non–explicit solutions to differential equations, mostly related to some quaternionic holomorphic bundles.

4. The Spectrum of a Torus Immersed into $\mathbb{HP}^1$

In Chapter V, a definition of the Davey–Stewartson and Novikov–Veselov flows for tori of normal bundle degree 0 is given. The underlying soliton equations are the simplest non–trivial equations of an infinite hierarchy of commuting Hamiltonian flows. They therefore have infinitely many first integrals, the first of which is the Willmore functional of the immersion. The idea for introducing the spectrum is that it should serve as some kind of 'master first integral' for the flows which contains the information of all other first integrals.

This idea originates from the fact that, for finite–dimensional equations with a matrix Lax representation, the eigenvalues of the matrix are invariant under the evolution. Many finite dimensional integrable systems possess a matrix Lax representation depending on a so called spectral parameter, such that the eigenvalues, depending on the parameter as well, define a Riemann surface which is invariant under the flow and is called the spectral curve. This Riemann surface is an essential tool, not only in the study of finite dimensional matrix systems, but also in the finite gap theory of 1+1 dimensional soliton equations like the Korteweg–de–Vries and non–linear Schrödinger equation, where the matrix Lax representation is replaced with a Lax representation consisting of ordinary differential operators which depend on a complex parameter.

The idea to introduce the spectrum as an invariant of the soliton flows on the space of immersions of tori arose in the study of Grinevich, Schmidt and Taimanov ([**10**], [**27**], [**28**]). The major difficulty in the definition of the spectrum is that the underlying evolution equations do not admit a Lax representation with a spectral parameter. Instead, the spectrum is defined in terms of Floquet theory for partial differential operators.

In this section, the spectrum of a conformally immersed torus with normal bundle degree 0 is defined using a quaternionic version of this Floquet theoretic approach. The definition is a slight, Möbius invariant modification of the original one, but is proven to yield essentially the same notion of spectrum.

4.1. The Spectrum of a Holomorphic Line Bundle over the Torus. In this subsection, the spectrum of a quaternionic holomorphic line bundle of degree 0 over the torus is defined. It is briefly sketched along the lines of [**28**], why this spectrum can be made into a Riemann surface. Finally, a theorem is proven, which is needed (in the next subsection) to show that the Möbius invariant definition of the spectrum of a conformal immersion coincides with the old Euclidean definition.

Let L be a quaternionic holomorphic line bundle over a Riemann surface M of genus 1, which, by uniformization, can be assumed to be of the form $M = \mathbb{C}/\Gamma$ for a lattice Γ. Denote by $\tilde{L}$ the pull back of L to the universal covering $\tilde{M} = \mathbb{C}$ of M.

We call $\psi \in H^0(\tilde{L})$ a *holomorphic section with monodromy* if there is $\lambda \in \operatorname{Hom}(\Gamma, \mathbb{H}^*)$ such that $\psi_{z+\gamma} = \psi_z \lambda_\gamma$ for all $z \in \mathbb{C}$ and $\gamma \in \Gamma$. The homomorphism λ is then called the (Floquet) *multiplier* of ψ. Because Γ is commutative, λ takes values in a commutative subalgebra of $\mathbb{H}^*$, which has to be contained in $\operatorname{Span}_{\mathbb{R}}\{1, N\}$ for $N \in S^2 \subset \operatorname{Im}\mathbb{H}$. Therefore, there is $\mu \in \mathbb{H}^*$ such that the section $\psi\mu \in H^0(\tilde{L})$ has the multiplier $\mu^{-1}\lambda\mu \in \operatorname{Hom}(\Gamma, \mathbb{C}^*)$ with values in $\mathbb{C} = \operatorname{Span}_{\mathbb{R}}\{1, \mathbf{i}\}$.

Following Grinevich, Schmidt and Taimanov ([**10**], [**27**] and [**28**]), we call the set of all complex multipliers that belong to some non–trivial holomorphic section with monodromy the spectrum of L.

Definition. The *spectrum* of a holomorphic line bundle L of degree 0 is

$$\operatorname{Spec}(L) = \left\{ \lambda \in \operatorname{Hom}(\Gamma, \mathbb{C}^*) \;\middle|\; \begin{array}{c} \text{there exists } \psi \in H^0(\tilde{L}) \text{ such that } \psi_{z+\gamma} = \psi_z \lambda_\gamma \\ \text{for all } z \in \mathbb{C} \text{ and } \gamma \in \Gamma \text{ and } \psi \not\equiv 0 \end{array} \right\}.$$

The spectrum has an anti–holomorphic involution, because if $\lambda \in \operatorname{Spec}(L)$, then $\bar{\lambda} \in \operatorname{Spec}(L)$, which follows from the fact that, if there is a non–trivial section $\psi \in H^0(\tilde{L})$ satisfying $\psi_{z+\gamma} = \psi_z \lambda_\gamma$ for all $z \in \mathbb{C}$ and $\gamma \in \Gamma$, then $\psi\mathbf{j}$ satisfies $\psi_{z+\gamma}\mathbf{j} = \psi_z \mathbf{j} \bar{\lambda}_\gamma$ for all $z \in \mathbb{C}$ and $\gamma \in \Gamma$.

After choosing generators γ_1, γ_2 of the lattice Γ, one can identify

$$\operatorname{Hom}(\Gamma, \mathbb{C}^*) \cong \mathbb{C}^* \times \mathbb{C}^* \qquad \text{via} \qquad \lambda \mapsto (\lambda(\gamma_1), \lambda(\gamma_2))$$

and therefore consider $\mathrm{Spec}(L)$ as a subset of $\mathbb{C}^* \times \mathbb{C}^*$. Every $\lambda \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ can be written as $\lambda(\gamma) = \exp(2\pi\mathbf{i}(p_1 \operatorname{Re}(\gamma) + p_2 \operatorname{Im}(\gamma))$, where p_1, $p_2 \in \mathbb{C}$ are the so called quasi–momenta. Taimanov proves[1] by use of the polynomial Fredholm alternative that the set of quasi–momenta is a complex analytic subvariety of $\mathbb{C}^2$ which locally is given as the zero set of one holomorphic function (see [**27**, Theorem 3] or [**28**, Theorem 4]). For the cases we are interested in, this set is non–empty, because there exist holomorphic sections for the trivial multiplier. By a perturbation argument (see [**27**, Theorem 3]), Taimanov shows that the codimension is positive, which implies that it is 1, because the set is locally given by one analytic function. He then defines the spectral curve to be the Riemann surface obtained by normalizing the set of quasi–momenta and taking the quotient of this normalization with respect to the action of the dual lattice Γ^*. (Note that this definition is not the definition of the spectral curve we are using, cf. Section 8.)

Remark. Let ∇ be a flat connection on L with $\nabla'' = D$ and let $\psi \in \Gamma(\tilde{L})$ be a parallel section. Then, there is a multiplier $\lambda \in \mathrm{Hom}(\Gamma, \mathbb{H}^*)$ such that

$$\psi_{z+\gamma} = \psi_z \lambda_\gamma$$

for all $z \in \mathbb{C}$ and $\gamma \in \Gamma$ and, in particular, ψ is a nowhere vanishing holomorphic section with monodromy.

We have seen above that, for some $\mu \in \mathbb{H}^*$, the section $\psi\mu$ has a complex multiplier. This complex multiplier is uniquely determined by ∇ up to conjugation. Obviously, all multipliers in $\mathrm{Spec}(L)$ belonging to nowhere vanishing holomorphic sections with monodromy can be obtained by this construction.

The following Theorem 1 immediately implies that the spectra of the Möbius invariant bundles of a degree 0 torus immersed into $\mathbb{HP}^1$ essentially coincide with those of the bundles needed for a Weierstrass representation (see Section 4.2 for details).

Theorem 1. *Let L be a quaternionic holomorphic line bundle of degree 0 on a torus $\mathbb{C}/\Gamma$. Then:*

a) $\mathrm{Spec}(KL^{-1}) = \mathrm{Spec}(L)^{-1} = \{\lambda^{-1} \mid \lambda \in \mathrm{Spec}(L)\}$ *and*

b) if there is a flat connection ∇ with $\nabla'' = D$, then $\mathrm{Spec}(KL) = \mathrm{Spec}(L)$ *for KL equipped with the holomorphic structure d^∇.*

Proof. To prove part a), define

$$H_\lambda = \{\psi \in \Gamma(\tilde{L}) \mid \psi_{z+\gamma} = \psi_z \lambda_\gamma \text{ for all } z \in \mathbb{C} \text{ and } \gamma \in \Gamma\}$$

[1]Taimanov's proof (e.g. that of Theorem 4 in [**28**]), which in his paper is only carried out for surfaces in $\mathbb{R}^3$, i.e. in the case of spin bundles, works also—with slight modifications—in the more general case of arbitrary quaternionic holomorphic line bundles of degree 0: choosing a section with monodromy that is parallel with respect to a flat hermitian connection on $\hat{L}$ compatible with the $\bar{\partial}$–operator, one can easily make the link to his coordinate dependent approach and write the quaternionic holomorphic structure as a Dirac operator. As in Taimonov's article, analyticity of the set of quasi–momenta is proven by applying standard elliptic theory and the polynomial Fredholm alternative. A reference for the polynomial Fredholm alternative, which is probably more accessible than the one given in Taimanov's article, can be found in [**23**, Theorem VI.14].

and

$$\tilde{H}_\lambda = \{\omega \in \Gamma(\bar{K}\tilde{L}) \mid \omega_{z+\gamma} = \omega_z \lambda_\gamma \text{ for all } z \in \mathbb{C} \text{ and } \gamma \in \Gamma\}.$$

These sets are the spaces of sections of the complex vector bundles obtained by tensoring L and $\bar{K}L$ by the complex line bundle with multiplier λ. The holomorphic structure D induces a differential operator $D\colon H_\lambda \to \tilde{H}_\lambda$. To construct the adjoint operator, we define

$$H'_\lambda = \{\alpha \in \Omega^2(\tilde{L}^{-1}) \mid \alpha_{z+\gamma} = \alpha_z \bar{\lambda}_\gamma^{-1} \text{ for all } z \in \mathbb{C} \text{ and } \gamma \in \Gamma\}$$

and

$$\tilde{H}'_\lambda = \{\beta \in \Gamma(K\tilde{L}^{-1}) \mid \beta_{z+\gamma} = \beta_z \bar{\lambda}_\gamma^{-1} \text{ for all } z \in \mathbb{C} \text{ and } \gamma \in \Gamma\}.$$

The (real) pairings between H'_λ and H_λ and between $\tilde{H}'_\lambda$ and $\tilde{H}_\lambda$ defined by

$$\ll \alpha, \psi \gg = \int_M \mathrm{Re}(\langle \omega, \psi \rangle)$$

for $\alpha \in H'_\lambda$ and $\psi \in H_\lambda$ and

$$\ll \beta, \omega \gg = \int_M \mathrm{Re}(\langle \beta \wedge \omega \rangle)$$

for $\beta \in \tilde{H}'_\lambda$ and $\omega \in \tilde{H}_\lambda$ are non–degenerate. (The integrals are well defined on M since periods cancel.) The formula $d\langle \beta, \psi \rangle = \langle D_{KL}\beta, \psi \rangle - \langle \beta \wedge D_L \psi \rangle$ (see equation (2)) implies

$$\ll D_{KL}\beta, \psi \gg = \ll \beta, D\psi \gg$$

and the adjoint operator of D_L is $D_L^* = D_{KL}$. Since $\operatorname{coker} D \cong \ker D^*$ via the pairing, the index of D is

$$\operatorname{index}(D) = \dim \ker(D) - \dim \ker(D^*).$$

This index is 0 for all multipliers $\lambda \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$. For the trivial multiplier $\lambda \equiv 1$, this follows from the Riemann–Roch theorem (see Section 1.2). For other multipliers it follows by homotopy invariance of the index.

This proves part a): $\ker(D)$ is the set of holomorphic section of $\tilde{L}$ with monodromy λ, $\ker(D^*)$ is the set of holomorphic section of $K\tilde{L}^{-1}$ with monodromy $\bar{\lambda}^{-1}$. Since both spaces have the same dimension, $\lambda \in \mathrm{Spec}(L)$ is equivalent to $\bar{\lambda}^{-1} \in \mathrm{Spec}(KL^{-1})$, which proves the first part of the theorem because the spectrum is invariant under complex conjugation.

To prove part b), note that, by flatness of ∇, the derivative $\nabla\psi$ of a holomorphic section $\psi \in \Gamma(\tilde{L})$ is a holomorphic section in $\Gamma(K\tilde{L})$ with respect to the holomorphic structure d^∇. Obviously, $\nabla\psi$ only vanishes identically if ψ is ∇–parallel, which proves

$$(*) \qquad \mathrm{Spec}(L)\backslash\{\lambda, \bar{\lambda}\} \subset \mathrm{Spec}(KL)$$

where λ, $\bar{\lambda}$ denote the multipliers belonging to ∇–parallel sections.

The same argument applies to L^{-1} and KL^{-1}, which become holomorphic bundles by taking the holomorphic structures ∇'' and d^∇ induced from the dual connection of ∇ on L. Hence

$$(**) \qquad \mathrm{Spec}(L^{-1})\backslash\{\lambda^{-1}, \bar{\lambda}^{-1}\} \subset \mathrm{Spec}(KL^{-1}).$$

By Lemma 2, the holomorphic structures on the paired bundles L and KL^{-1} and on the paired bundles KL and L^{-1} are compatible. Using part a), $\{\lambda, \bar{\lambda}\} \subset \mathrm{Spec}(L)$ and $\{\lambda^{-1}, \bar{\lambda}^{-1}\} \subset \mathrm{Spec}(L^{-1})$ imply $\{\lambda^{-1}, \bar{\lambda}^{-1}\} \subset \mathrm{Spec}(KL^{-1})$ and $\{\lambda, \bar{\lambda}\} \subset \mathrm{Spec}(KL)$. Together with the inclusions $(*)$ and $(**)$, this yields $\mathrm{Spec}(L) \subset \mathrm{Spec}(KL)$ and $\mathrm{Spec}(L^{-1}) \subset \mathrm{Spec}(KL^{-1})$. Since, again by part a), the second inclusion is equivalent to $\mathrm{Spec}(KL) \subset \mathrm{Spec}(L)$, this proves part b) of the theorem. $\square$

4.2. The Spectrum of Immersions of Tori with Degree 0. This subsection defines the spectrum of a conformal immersion of a torus with normal bundle degree 0 into conformal 4–space. Using Theorem 1, this Möbius invariant definition is shown to essentially coincide with the old one in terms of the Weierstrass representation.

The spectrum of immersed tori was first defined [**10**], [**27**] in an Euclidean framework, for immersion of tori into $\mathbb{R}^3$, using the holomorphic structure on the spin bundle L (in an equivalent, coordinate dependent formulation involving a Dirac operator with a real potential). In [**10**], Grinevich and Schmidt prove Möbius invariance of the spectrum. The Möbius invariant approach presented here is due to Ulrich Pinkall.

Definition. Let $L \subset V$ (with V a rank 2 quaternionic vector space) be an immersion of a torus with normal bundle degree 0 into $\mathbb{P}V$. The *spectrum* of $L \subset V$ is defined to be the spectrum of the quaternionic holomorphic line bundle L^{-1}.

In the case that the bundles KL^{-1} and L are equipped with the holomorphic structures defined by the flat connection ∇ on L^{-1} corresponding to the choice of a point at ∞ (cf. Section 3.3), Theorem 1 implies that

$$\mathrm{Spec}(L^{-1}) = \mathrm{Spec}(KL^{-1}) \qquad \text{and} \qquad \mathrm{Spec}(L) = \mathrm{Spec}(KL^{-1})^{-1}.$$

By Lemma 18, the same argument can be applied to the bundle $V/L \cong (L^{\perp})^{-1}$, which yields

$$\mathrm{Spec}(V/L) = \mathrm{Spec}(L).$$

(Hence, all bundles on the same side in the diagram of Section 3.4 have the same spectrum and those on different sides have spectra related by $\lambda \mapsto \lambda^{-1}$.)

Remark. For surfaces in $S^3 \subset \mathbb{HP}^1$, when choosing a point at infinity on S^3, the holomorphic bundles KL^{-1} and L are spin bundles, i.e. $KL^{-1} \cong L$. In particular, $\mathrm{Spec}(KL^{-1}) = \mathrm{Spec}(L)$, which implies $\mathrm{Spec}(L^{-1}) = \mathrm{Spec}(L^{-1})^{-1}$. Hence, in addition to the anti–holomorphic involution $\lambda \mapsto \bar{\lambda}$, the holomorphic involution $\lambda \mapsto \lambda^{-1}$ preserves the spectrum.

5. Associated Family

This section defines the associated family for general immersions. It is quite short, because the associated family plays a far less prominent role than, for example, in the classical transformation theories of isothermic immersions and constrained Willmore immersions (cf. Chapter III). The main reason for this is that, in contrast to the classical cases, the associated family for general tori can not be parametrized by an explicit family of connections depending on a spectral parameter.

5.1. Definition. The idea of our approach to the associated family of conformal immersions into 4–space is the following: instead of changing L in a fixed space (V, ∇), the vector bundle V and the line subbundle L are left fixed and the connection ∇, i.e. the background geometry of the surrounding space, is changed (see also Section 14.1 for a discussion of the philosophy behind this approach). The associated family is obtained by taking all flat adapted connections on V seen as the 1–jet bundle of the holomorphic bundle V/L.

Recall that, for $L \subset V$ an immersed holomorphic curve in (V, ∇), the sequence $V \to V/L \to 0$ with the jet–derivative $d = \pi\nabla$ is a model for the 1–jet bundle of the quaternionic holomorphic line bundle V/L (cf. Section 3.2), and that ∇ itself is a flat adapted connection on the 1–jet bundle. Every other flat adapted connection on the 1–jet bundle is of the form $\nabla + \omega$ with $\omega \in \Omega^1(\mathrm{End}(V))$ satisfying $\pi\omega = 0$ and $d^\nabla\omega + \omega \wedge \omega = 0$.

Definition. Let $L \subset V$ be a conformal immersion into the flat quaternionic rank 2 bundle (V, ∇). The *associated family* of (V, L, ∇) is the family of immersions $(V, L, \nabla + \omega)$ with $\omega \in \Omega^1(\mathrm{End}(V))$ satisfying

$$\pi\omega = 0 \qquad \text{and} \qquad d^\nabla\omega + \omega \wedge \omega = 0.$$

By Lemma 13, every $\omega \in \Omega^1(\mathrm{End}(V))$ with $\pi\omega = 0$ and $d^\nabla\omega + \omega \wedge \omega = 0$ satisfies $*\omega = S\omega$ (where S is the mean curvature sphere of $L \subset V$), i.e.

$$\omega \in \Gamma(K\,\mathrm{Hom}(V, L)).$$

Remark. The classical associated families of isothermic surfaces and constrained Willmore surfaces (cf. Chapter III) form subsets of this general associated family which are given by algebraic formulae depending on a real parameter.

The following lemma shows that the mean curvature sphere S is the same for all immersions in the associated family.

Lemma 20. *Let $L \subset V$ be a conformal immersion into (V, ∇). Then, all immersions $(V, L, \nabla + \omega)$ in the associated family of (V, L, ∇) have the same mean curvature sphere.*

Proof. Let S be the mean curvature sphere of the conformal immersion $L \subset V$ into (V, ∇). Because $\pi\omega = 0$, all immersions in the associated family have the same differential $\delta = \pi\nabla_{|L} = \pi(\nabla + \omega)_{|L}$, and in particular the condition $*\delta = S\delta = \delta S$ is satisfied for all immersions. $(\nabla + \omega)S = \nabla S + [\omega, S]$ and $\omega \in \Gamma(K\,\mathrm{Hom}(V, L))$ imply that Q is the same for all members of the associated family. In particular, the condition $Q_{|L} = 0$ is satisfied for all immersions, which shows that S simultaneously is the mean curvature sphere of all immersions in the associated family. □

Since S is the same for all immersed curves, they all have the same degree. Because, by definition, all holomorphic curves in the associated family have the same holomorphic bundle V/L, Lemma 15 implies that they all have the same Willmore functional, which is defined to be the Willmore functional of the bundles L^{-1}.

Associated Family and Linear Systems with Monodromy. We first give a quick introduction to the concept of linear systems with monodromy, because they do not seem to be explicitly treated anywhere else. Let L be a quaternionic holomorphic line bundle on a Riemann surface M and denote by $\tilde{L}$ the pullback to the universal covering $\tilde{M}$. A quaternionic linear subspace $H \subset H^0(\tilde{L})$ is called a *linear system with monodromy* if $\gamma^* H \subset H$ holds for all deck transformations $\gamma \in \Gamma$. A finite dimensional subspace $H \subset H^0(\tilde{L})$ is a linear system with monodromy if and only if for one (every) basis ψ_1,..., ψ_n of H, there is a multiplier homomorphism $\Lambda \colon \Gamma \to \mathrm{GL}(n, \mathbb{H})$ such that

$$\gamma^* \psi_i = \sum_j \psi_j \Lambda_{ji}(\gamma)$$

for all $\gamma \in \Gamma$. The Plücker formula [**8**] applies as well in the case of finite dimensional linear systems with monodromy (with essentially the same proof). Via Kodaira correspondence, a linear system with monodromy having no base points corresponds to a holomorphic curves with monodromy in quaternionic projective spaces.

The following lemma gives an alternative characterization of the immersions in the associated family.

LEMMA 21. *Let $L \subset V$ be a conformal immersion into (V, ∇). Then, there is a 1-1–correspondence between immersions $(V, L, \nabla + \omega)$ in the associated family and 2–dimensional linear systems in $H^0(\tilde{V}/\tilde{L})$ with monodromy and without Weierstrass points.*

PROOF. Given $\nabla + \omega$, projecting the parallel section from V to V/L yields a linear system with monodromy.

Conversely, prolongation of a basis of the linear system yields a frame of $\tilde{V}$ with multiplier. The flat adapted connection $\nabla + \omega$ corresponding to the linear system is the one that makes this frame parallel. ☐

This shows that all immersions in the associated family are holomorphic curves with monodromy in $\mathbb{HP}^1$ obtained via Kodaira correspondence and dualization from 2–dimensional linear systems with monodromy of the bundle V/L. The following corollary indicates that one can only expect a few immersions in the associated family to have trivial monodromy.

COROLLARY 22. *Let $L \subset V$ be a conformal immersion of a torus of normal bundle degree 0 into (V, ∇). If L has Willmore functional $W(L^{-1}) < 8\pi$, the associated family contains at most one immersion without monodromy.*

PROOF. By Lemma 15, $W(V/L) = W(L^{-1}) < 8\pi$, and the Plücker formula implies that V/L cannot have a 3–dimensional linear system without monodromy (see Section 3.4). This proves the statement, because if there were two immersion without monodromy in the associated family, the bundle V/L would have a 3–dimensional linear system without monodromy. ☐

"Backward" Associated Family. We would like to finish the section with a brief discussion of the backward version of the associated family. Let $L \subset V$ be a conformal immersion into (V, ∇). The connection $\nabla + \eta$ on V belongs to the backward associated family if and only if the dual connection $\nabla - \eta^*$ on V^* is a flat adapted connection on the 1–jet bundle of $L^{-1} = V^*/L^\perp$, which is equivalent to the condition

$$\eta_{|L} = 0 \qquad \text{and} \qquad d^\nabla \eta + \eta \wedge \eta = 0$$

on the 1–form $\eta \in \Omega^1(\mathrm{End}(V))$. If the 1–form η satisfies this conditions, then $*\eta = \eta S$ (because $*\eta^* = S^*\eta^*$). Obviously, all immersions $(V, L, \nabla + \eta)$ of the backward associated family share the same holomorphic structure on L^{-1}.

6. Bäcklund Transformations

A Bäcklund transformation of a conformal immersion is obtained by using Möbius invariant holomorphic bundles related to the initial immersion as the bundles of the Weierstrass representation of a new immersion (or, equivalently, by using a bundle of the Weierstrass representation of the initial immersion as a Möbius invariant bundle for the new immersion).

As with some of the other transformations, it is useful to introduce the notion of forward and backward Bäcklund transformation: a backward Bäcklund transformation of L is the conjugate of a (forward) Bäcklund transformation of the dual curve of L. This terminology is justified by the fact that, from every immersed forward Bäcklund transformation of a conformal immersion, the original immersion can be recovered by applying a backward Bäcklund transformation.

Inspired by the Bäcklund transformation theory of Willmore immersions (see [**2**]), the concept of 2–step Bäcklund transformations is introduced for general immersions. It is proven that, under certain conditions, a 2–step Bäcklund transformation can be obtained by applying two 1–step Bäcklund transformations.

The last part of the section discusses the Bäcklund transformation theory for tori of normal bundle degree 0. It is shown that, in this case, the Bäcklund transformation preserves the degree, the Willmore functional and the spectrum of immersions.

6.1. Definition and General Theory. Let $L \subset V$ be a conformal immersion into $\mathbb{P}V$ where V is a quaternionic rank 2 vector space. Denote by $V^* \subset H^0(L^{-1})$ the 2–dimensional linear system of the Möbius invariant bundle L^{-1} which Kodaira corresponds to L.

Definition. A *Bäcklund transformation* of L is a (branched) immersion $g\colon \tilde{M} \to \mathbb{H}$ (with translational monodromy) satisfying

$$dg = (\beta, \omega)$$

where $\beta \in V^* \subset H^0(L^{-1})$ is nowhere vanishing and $\omega \in H^0(KL)$. A Bäcklund transformation is called *closed* if it has no monodromy.

The notion of Bäcklund transformation is not a Möbius geometric one: by choosing the point at infinity $[\beta]^{\perp}$, the Möbius symmetry is broken. Furthermore, the Bäcklund transformation has no natural interpretation as a holomorphic curve in the same quaternionic projective line.

Since L does not go through $\infty = [\beta]^{\perp}$, there is a unique section $\psi \in \Gamma(L)$ with $\beta(\psi) = 1$. After choosing a basis e_1, e_2 of V with $\beta(e_1) = 0$ and $\beta(e_2) = 1$, the section ψ takes the form

$$\psi = \begin{pmatrix} f \\ 1 \end{pmatrix},$$

where $f\colon M \to \mathbb{H}$ is the Euclidean realization of L with respect to the chart given by e_1, e_2. Clearly, every $\omega \in \Omega^1(L)$ can be written as $\omega = \psi\hat{\omega}$ with $\hat{\omega} \in \Omega^1\mathbb{H}$. Since $\omega \in \Omega^1(L)$ is an element of $H^0(KL)$ if and only if $d\omega = 0$ (see Section 3.2), we have

$$H^0(KL) = \{\psi\hat{\omega} \mid \hat{\omega} \in \Omega^1(\mathbb{H}), d\hat{\omega} = 0 \text{ and } *\hat{\omega} = -R\hat{\omega}\},$$

where R is the right normal vector of f. In particular, g is a Bäcklund transformation with $dg = (\beta, \omega)$ if and only if $*dg = -Rdg$ or, equivalently,

$$df \wedge dg = 0. \tag{38}$$

Remark. The fact that the Bäcklund transformation is defined in terms of the Weierstrass representation and that β in the definition of the Bäcklund transformation singles out a geometry of similarities in $\mathbb{P}V$ shows that the notion of Bäcklund transformations is strongly related to geometry of similarities.

Furthermore, if f, $g\colon M \to \mathbb{H}$ satisfy (38), then applying a similarity to both f and $\bar{g}$ yields a new pair of immersions $\tilde{f}$ and $\tilde{\bar{g}}$ satisfying $d\tilde{f} \wedge d\tilde{g} = 0$.

Closed Bäcklund Transformations. The following lemma is a simple but important consequence of the preceding discussion. It provides an alternative description of closed Bäcklund transformations which is essential for understanding the subtle interplay between Möbius geometry and geometry of similarities that is characteristic for Bäcklund transformation theory (see the remark below for details).

Lemma 23. *Let $L \subset V$ be a conformal immersion into $\mathbb{P}V$ (with V a quaternionic rank 2 vector space). Let $\beta \in V^* \subset H^0(L^{-1})$ be nowhere vanishing and suppose L is equipped with the holomorphic structure such that $\psi \in \Gamma(L)$ with $\beta(\psi) = 1$ is holomorphic. Then, there is a 1-1–correspondence between closed Bäcklund transformations g of the form $dg = (\beta, \omega)$ with $\omega \in H^0(KL)$ and 2–dimensional linear systems* $\operatorname{Span}\{\psi, \psi g\} \subset H^0(L)$.

Proof. Let $J\beta = \beta R$. Then, a section $\psi g \in \Gamma(L)$ is holomorphic if and only if $*dg = -Rdg$. The lemma holds, because, as we have seen in (38), this is exactly the condition for g to be a closed Bäcklund transformation of the form $dg = (\beta, \omega)$ with $\omega \in H^0(KL)$. The 1–form ω is then given by $\omega = \psi dg$. □

The promised alternative description of Bäcklund transformations will be given in the following remark. First, it is necessary to briefly analyze the relationship between the holomorphic bundles involved.

The holomorphic structure on L occurring in Lemma 23 is obviously induced from the connection ∇ defined by $\nabla\psi = 0$. By Lemma 2, this connection ∇ on L, together with the dual connection on L^{-1}, induce compatible holomorphic structures on the paired bundles L^{-1} and KL and on the paired bundles KL^{-1} and L. Since β is a holomorphic section of L^{-1} and parallel with respect to ∇, the induced holomorphic structures on the paired bundles L^{-1} and KL are the usual Möbius invariant ones (cf. Section 3.2). The induced holomorphic structures on KL^{-1} and L are the ones needed for the Weierstrass representation of L with respect to the geometry of similarities defined by β (cf. Section 3.3).

The four holomorphic structures are visualized by the diagram

$$
\begin{tikzcd}
KL^{-1} \arrow[dr, leftrightarrow, dotted] & \\
L^{-1} \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow, dotted] & L \\
& KL
\end{tikzcd}
$$

where the diagonal arrows stand for paired holomorphic bundles and the horizontal arrow stands for the dual bundles carrying the connections ∇.

From the point of view of the initial immersion L, the bundles on the lower diagonal arrow carry Möbius invariant holomorphic structures, while those on the upper arrow carry the holomorphic structures induced by the point $[\beta]^\perp$ at infinity. (For L, the upper triangle of this diagram corresponds to the lower triangle of the diagram in Section 3.4.)

Remark. We are now able to give a reinterpretation of Lemma 23 which is important for understanding the underlying mechanism of Bäcklund transformations.

By definition, g is a closed Bäcklund transformation of L if and only if the Weierstrass representation of g is obtained by pairing two sections in the Möbius invariant bundles L^{-1} and KL belonging to L.

By Lemma 23 this is equivalent to g admitting a (Möbius invariant[2]) representation $\tilde{\psi} = \psi g$ as the quotient of two holomorphic sections in $H^0(L)$, where the holomorphic structure on L is that belonging to the Weierstrass representation of L with respect to the geometry of similarities induced by the choice of a point at infinity, and where ψ is one of the sections occurring in the Weierstrass representation of L.

In short, either one uses Möbius invariant bundles belonging to L to obtain a Weierstrass representation of the Bäcklund transformation, or one uses the holomorphic bundle of a Weierstrass representation of L to obtain a Möbius geometric representation of the Bäcklund transformation.

From the point of view of the Bäcklund transformation of L, which we suppose now to be immersed, the bundles on the upper diagonal arrow carry

[2] A quotient representation is Möbius invariant, because it corresponds to taking the dualization of a curve obtained by Kodaira correspondence.

Möbius invariant holomorphic structures, while those on the lower arrow carry the holomorphic structures of a Weierstrass representation. (For the Bäcklund transformation, the lower triangle of this diagram corresponds to the upper triangle of the diagram in Section 3.4.)

Adopting the usual terminology to call the conjugate of a Bäcklund transformation of the dual curve a *backward Bäcklund transformation*, we obtain:

COROLLARY 24. *Let*

$$L = \begin{bmatrix} f \\ 1 \end{bmatrix} \subset \mathbb{H}^2$$

be a conformal immersion. If g is a closed immersed (forward) Bäcklund transformation of L, then f is a backward Bäcklund transformation of

$$L' = \begin{bmatrix} g \\ 1 \end{bmatrix} \subset \mathbb{H}^2.$$

PROOF. Because L', by Lemma 23, is obtained via Kodaira correspondence and dualization from a 2–dimensional linear system in L (with its holomorphic structure induced by β), we have $L \cong (L'^{\perp})^{-1}$. Since f can be obtained via Weierstrass representation from KL^{-1} and L, it is a backward Bäcklund transformation of L'.

An alternative proof (using the second characterization of the preceding remark) goes as follows: g has its Weierstrass representations in the bundles L^{-1} and KL. Since L is obtained via Kodaira correspondence from L^{-1}, it is a backward Bäcklund transformation of L'. □

2–Step Bäcklund Transformations. Before we define 2–step Bäcklund transformations, we derive an alternative characterization of 2–dimensional linear systems of the bundle KL.

LEMMA 25. *Let $L \subset V$ be a conformal immersion into $\mathbb{P}V$ with V a rank 2 vector space. Then, there is a 1-1–correspondence between 2–dimensional linear systems of KL and equivalence classes of 1–forms $\omega \in \Omega^1(\mathrm{End}(V))$ with non–constant kernel satisfying*

$$\pi\omega = 0 \qquad \textit{and} \qquad d^{\nabla}\omega = 0,$$

where $\pi\colon V \to V/L$ is the canonical projection and where two 1–forms ω and $\tilde{\omega}$ are called equivalent if $\tilde{\omega} = \omega A$ for a quaternionic automorphism A of V. The 1-form ω corresponding to the linear system maps a basis of V to a basis of the linear system in $H^0(KL)$.

PROOF. The lemma is a direct consequence of the fact that a form ω with the given properties maps constant vectors of V to elements of $H^0(KL)$, which, as explained in Section 3.2, are the closed forms in $\Omega^1(L)$. □

We will see later on (in Chapter III) that, for isothermic and for constrained Willmore immersions , there is a canonical choice of ω with the above properties. Furthermore, for degree 0 tori of Willmore functional $W < 8\pi$, there is only one such ω up to right multiplication by an automorphism of V.

Definition. Let $L \subset V$ be a conformal immersion into $\mathbb{P}V$ with V a rank 2 vector space. A holomorphic curve is a *2–step Bäcklund transformation* of L if it is obtained by Kodaira correspondence and dualization from a 2–dimensional linear system in $H^0(KL)$ without base points.

The following corollary to Lemma 25 gives an alternative characterization of 2–step Bäcklund transformations in the spirit of the 2–step Bäcklund transformation theory of constrained Willmore immersions in [**2**] (see also the introduction to Section 10).

Corollary 26. *The 2–step Bäcklund transformations of L are exactly the non–constant curves arising as the kernel of a 1–form $\omega \in \Omega^1(\mathrm{End}(V))$ with $\pi\omega = 0$ and $d^\nabla \omega = 0$.*

It should be mentioned that a non–constant kernel of a closed 1–form ω with values in L is automatically a holomorphic curve, because it is obtained from a linear system by Kodaira correspondence and dualization.

The name 2–step Bäcklund transformation is justified by the following lemma, which shows that applying two 1–step Bäcklund transformations yields a 2–step Bäcklund transformation, if both 1–step Bäcklund transformations are taken with respect to the same geometry of similarities on the target space of the first 1–step Bäcklund transformation.

Lemma 27. *Let $L \subset V$ be a conformal immersion into $\mathbb{P}V$ (with V a quaternionic rank 2 vector space) and let $\omega \in \Omega^1(\mathrm{End}(V))$ be a 1–form with $\pi\omega = 0$ and $d^\nabla\omega = 0$, having a 1–dimensional, non–constant kernel. Assume g is a closed, immersed Bäcklund transformation of L satisfying $dg = (\beta, \omega v)$ with $\beta \in V^*$ and $v \in V$.*

Then, the 2–step Bäcklund transformation $\ker\omega$ *of L is a 1–step Bäcklund transformation of g.*

Proof. This lemma is a direct consequence of the remark following Lemma 23, because g has its Weierstrass representation in the bundles L^{-1} and KL and $\ker\omega$ is obtained by the quotient representation from a linear system in $H^0(KL)$ which contains one of the sections needed for the Weierstrass representation. □

In contrast to the 1–step Bäcklund transformation, which interchanges the role of Möbius invariant bundles and Euclidean bundles, the notion of 2–step Bäcklund transformation is a purely Möbius geometric one. There is no Weierstrass representation involved and therefore no period problem.

6.2. Bäcklund Transformations of Degree 0 Tori. We turn now to the case of immersions of tori with degree 0 normal bundle. This is the only case where all of the holomorphic bundles related to the immersion have the same degree and Willmore functional (cf. Lemma 19).

Lemma 28. *Let $L \subset V$ be a conformal immersion of a torus with normal bundle degree 0 into $\mathbb{P}V$, where V is a quaternionic rank 2 vector space. Then, every closed, immersed Bäcklund transformation has the same Willmore functional and the same spectrum as L.*

Proof. Because the Bäcklund transformation uses the Möbius invariant holomorphic bundle L^{-1} of the initial immersion L as one of the bundles in the Weierstrass representation of the new immersion g, the statement about the Willmore functional is an immediate consequence of Lemma 19. By definition of the spectrum of an immersion, the statement about the spectrum immediately follows from Theorem 1 (see also Section 4.2). □

Remark. By a similar proof, it can be shown that the analogous lemma holds for immersed 2–step Bäcklund transformations.

Existence of Closed Bäcklund Transformations. This paragraph is devoted to the question of existence of closed Bäcklund transformations in the case of immersions of tori with normal bundle degree 0.

In the case of degree 0 tori, the Riemann–Roch theorem yields that the spaces of holomorphic sections of L^{-1} and KL have the same dimension. This dimension has to be greater or equal 2, because L^{-1} has the 2–dimensional linear system $V^* \subset H^0(L^{-1})$. We can therefore chose a 2–dimensional linear system in $H^0(KL)$. Because, in the case $W < 8\pi$, the Plücker formula implies that every 2–dimensional linear system has no Weierstrass points, we obtain:

Lemma 29. *A conformally immersed torus in $\mathbb{P}V$ with normal bundle degree 0 and Willmore functional $W < 8\pi$ has a unique (and immersed) 2–step Bäcklund transformation.*

For tori of higher Willmore functional, there is a 2–step Bäcklund transformation for every 2–dimensional linear system without base points in $H^0(KL)$ (but it is not clear, whether such linear systems exist).

The case of 1–step Bäcklund transformations is more complicated. Let $\beta \in V^* \subset H^0(L^{-1})$ be nowhere vanishing. Then there is a closed Bäcklund transformation g satisfying

$$dg = (\beta, \omega)$$

with $\omega \in H^0(KL)$ if and only if the bundle L, equipped with the holomorphic structure induced by the connection ∇ satisfying $\nabla\psi = 0$ where $\beta(\psi) = 1$, has a 2–dimensional linear system containing ψ (see Lemma 23). Because the connection ∇ on L^{-1} and L is flat, it maps holomorphic sections of L^{-1} and L to d^∇–holomorphic sections of KL^{-1} and KL. Setting $h_m := \dim_{\mathbb{H}} H^0(L^{-1}) = \dim_{\mathbb{H}} H^0(KL)$ and $h_e := \dim_{\mathbb{H}} H^0(KL^{-1}) = \dim_{\mathbb{H}} H^0(L)$ (where equality hold by the Riemann–Roch theorem), we obtain

$$|h_m - h_e| \leq 1.$$

This implies:

Lemma 30. *A conformally immersed torus in $\mathbb{P}V$ with normal bundle degree 0 and* $\dim_{\mathbb{H}} H^0(L^{-1}) \geq 3$ *has a closed 1–step Bäcklund transformation.*

In this case, there are actually many closed 1–step Bäcklund transformations, because the above procedure works for every choice of $\beta \in V^* \subset H^0(L^{-1})$ that is nowhere vanishing. The proof of the preceding lemma does not apply in the case of $W < 8\pi$, because then $\dim_{\mathbb{H}} H^0(L^{-1}) = 2$ (by the Plücker formula).

The next lemma is a different kind of existence result. It guarantees the existence of maps $g\colon M \to \mathbb{H}$ (defined on the torus) satisfying

$$dg = (\beta, \omega)$$

with $\beta \in V^* \subset H^0(L^{-1})$ and $\omega \in H^0(KL)$. Note that, in contrast to the definition of Bäcklund transformations, β is allowed to be a holomorphic section with zeros. We call such g a *generalized Bäcklund transformation* of L.

Lemma 31. *Let $L \subset V$ be a conformal immersion of a torus into $\mathbb{P}V$ (with V a rank 2 vector space). If $\omega \in \Omega^1(\mathrm{End}(V))$ is a closed form with values in L and non–trivial kernel, then there is a (possibly generalized) closed Bäcklund transformation $g\colon M \to \mathbb{H}$ of L.*

Proof. Let a, $b \in H_1(M)$ be generators of the first homology of the torus. Define $A = \int_a \omega$ and $B = \int_b \omega \in \mathrm{End}(V)$. Then, the Bäcklund transformation g defined by $dg = \langle \beta, \omega v \rangle$ with $\beta \in V^*$, $v \in V$ has vanishing periods if and only if β, v are non–trivial solutions to

$$\langle \beta, Av \rangle = \langle \beta, Bv \rangle = 0.$$

If A is invertible, the matrix $A^{-1}B$ has an eigenvector $v \in V$. Hence Av and Bv are contained in the same quaternionic line, and there is $\beta \in V^*$ such that $\langle \beta, Av \rangle = \langle \beta, Bv \rangle = 0$.

If A is not invertible, either B is invertible and the same argument can be applied after interchanging the roles of A and B, or both A and B have non–trivial kernel, in which case one can also find solutions β and v to $\langle \beta, Av \rangle = \langle \beta, Bv \rangle = 0$ and therefore a Bäcklund transformation with vanishing periods. $\square$

Generically, one can expect that at least one of the period matrices is invertible, and that $A^{-1}B$ or $B^{-1}A$ has exactly two eigenlines. In this case, there are exactly two closed Bäcklund transformations (up to similarity).

We conjecture that, for degree 0 tori with $W < 8\pi$, every generalized Bäcklund transformation is immersed and therefore a Bäcklund transformation in the proper sense. In the case that g is a branched immersion, i.e. that g admits a smooth left or right normal vector, the fact that g is immersed follows from the Plücker formula, because g can be seen as a holomorphic section of a quaternionic holomorphic line bundle (defined by the smooth normal vector) with $W < 8\pi$. In the remaining case, when g is not a branched immersion, it should be possible to prove the conjecture by essentially the same argument, using some kind of Plücker formula for bundles with non–smooth potentials.

7. Darboux Transformations

This section is devoted to a new definition of Darboux transformations for general immersion. Among the transformations of general conformal immersions, Darboux transformations seem to be the most fundamental ones: they are closely related to the new definition of the spectral curve given in Section 8, the Davey–Stewartson flow can be obtained as a limit of Darboux transformations (see Section 16), and certain Bäcklund transformations as well can be obtained as a limit of Darboux transformations (see Section 7.3).

7.1. Definition of the Darboux Transformation. Let (V, ∇) be a flat quaternionic rank 2 bundle on M and let $L \subset V$ be a conformal immersion. We discuss here several equivalent characterizations of Darboux transforms of L.

Let $L^\#$ be a subbundle of V that does not intersect the immersion L, i.e. $V = L \oplus L^\#$. With respect to this splitting, ∇ can be decomposed to

$$\nabla = \begin{pmatrix} \nabla^L & \delta^\# \\ \delta & \nabla^\# \end{pmatrix}$$

and flatness of ∇ becomes

$$0 = R^\nabla = \begin{pmatrix} R^{\nabla^L} + \delta^\# \wedge \delta & d^{\nabla^L, \nabla^\#} \delta^\# \\ d^{\nabla^\#, \nabla^L} \delta & R^{\nabla^\#} + \delta \wedge \delta^\# \end{pmatrix}. \tag{39}$$

Definition. Let $L \subset V$ be a conformal immersion into (V, ∇). A subbundle $L^\# \subset V$ of V is called a *Darboux transformation* of L if it satisfies $V = L \oplus L^\#$ and $\delta \wedge \delta^\# = 0$.

As a direct consequence of (39) we obtain:

Lemma 32. *Let $L \subset V$ be a conformal immersion into (V, ∇). A subbundle $L^\# \subset V$ with $V = L \oplus L^\#$ is a Darboux transformation of L if and only if $\nabla^\#$ is a flat connection. Thus, $L^\#$ is a Darboux transformation of L if and only if (locally) there is $\psi \in \Gamma(L^\#)$ with $\nabla\psi \in \Omega^1(L)$.*

Proof. The equation $R^{\nabla^\#} + \delta \wedge \delta^\# = 0$ shows that $L^\#$ is a Darboux transformation (i.e. $\delta \wedge \delta^\# = 0$) if and only if $\nabla^\#$ is flat. This proves the first statement.

For $\psi \in \Gamma(L^\#)$, the decomposition of ∇ implies

$$\nabla\psi = \delta^\#\psi + \nabla^\#\psi.$$

Therefore, ψ is $\nabla^\#$–parallel if and only if $\nabla\psi \in \Omega^1 L$. Since flatness of $\nabla^\#$ is equivalent to the (local) existence of a parallel section, the second statement follows. ☐

Otherwise stated, every section $\psi \in \Gamma(V)$ with $\nabla\psi \in \Omega^1(L)$ and nowhere vanishing $\pi\psi \in \Gamma(V/L)$ (where $\pi \colon V \to V/L$ is the canonical projection) defines a Darboux transformation. This directly leads to a characterization of the Darboux transformation in terms of jet theory: we have seen (in Section 3.2) that, using the sequence $V \to V/L \to 0$ with jet derivative $d = \pi\nabla$, the bundle V becomes a realization of the 1–jet bundle for the quaternionic holomorphic line bundle V/L. A section $\psi \in \Gamma(V)$ satisfies $\nabla\psi \in \Omega^1(L)$ if and only if $d\psi = 0$, which is the condition for ψ being the prolongation of the holomorphic section $\varphi = \pi\psi \in H^0(V/L)$.

Corollary 33. *A subbundle $L^\# \subset V$ with $V = L \oplus L^\#$ is a Darboux transformation of L if and only if (locally) it is obtained by prolongation of a nowhere vanishing holomorphic section of V/L.*

Because a flat line bundle $L^\#$ admits a parallel section $\psi \in \Gamma(\tilde{L}^\#)$ with monodromy (where $\tilde{L}^\#$ denotes the pullback of the bundle $L^\#$ to the universal covering $\tilde{M}$ of M), every Darboux transformation $L^\#$ of L is (globally!) obtained by prolongation of a nowhere vanishing holomorphic section with monodromy of $\tilde{V}/\tilde{L}$ (namely the prolongation of $\pi\psi \in H^0(\tilde{V}/\tilde{L})$ where $\psi \in \Gamma(\tilde{L}^\#)$ is the parallel section).

This suggests calling a bundle $L^\#$ that is defined away from isolated points a *generalized Darboux transformation* of the immersion L, if there is the prolongation $\psi \in \Gamma(\tilde{V})$ of a (non–trivial) holomorphic section $\varphi = \pi\psi \in \Gamma(\tilde{V}/\tilde{L})$ with monodromy such that

$$L^\# = \psi\mathbb{H}$$

holds away from zeros of ψ. Clearly, every $\varphi \in H^0(\tilde{V}/\tilde{L})$ with monodromy yields a generalized Darboux transformation $L^\#$, which is defined away from the (isolated) points where φ vanishes to second order. At the point of vanishing to first order of ψ, the bundles L and $L^\#$ coincide (i.e. the splitting $L \oplus L^\#$ becomes singular) and $L^\#$ is branched.

Remark. For degree 0 tori, the Plücker formula with monodromy shows that holomorphic sections with monodromy whose prolongation vanishes at some points can only exist if the Willmore functional is greater or equal 8π.

For tori, the holomorphic section in $H^0(\tilde{V}/\tilde{L})$ with monodromy can be chosen with a complex multiplier (see Section 4.1). Hence, every generalized Darboux transformation of a torus $L \subset V$ gives rise to a pair of multipliers λ, $\bar{\lambda} \in \mathrm{Spec}(V/L)$. Conversely, every multiplier $\lambda \in \mathrm{Spec}(V/L)$ is obtained from a generalized Darboux transformation. (We suppose that a generic multiplier gives rise to a unique generalized Darboux transformation $L^\#$. Moreover, since generic holomorphic sections with monodromy are supposed to have no zeros, for almost all multipliers the splitting $L \oplus L^\#$ should be non–singular and $L^\#$ a true Darboux transformation.)

Remark. The following two observations reveal relations between Darboux transformations and the associated family and Bäcklund transformations.

Let $L \subset V$ be a conformal immersion into (V, ∇). Every parallel subbundle with respect to a connection $\nabla + \omega$ in the associated family is a generalized Darboux transformation, because a $\nabla + \omega$–parallel section $\psi \in \Gamma(\tilde{V})$ satisfies

$$\nabla\psi = -\omega\psi \in \Omega^1(\tilde{L}).$$

(Since parallel sections have no zeros, these generalized Darboux transformations are everywhere defined.) For $\omega = 0$, we obtain the constant Darboux transformations. For generic ω, the holonomy of $\nabla + \omega$ has two eigenlines and we obtain two generalized Darboux transformations (see Appendix B). Conversely, every pair of generalized Darboux transformations whose underlying holomorphic sections form a 2–dimensional linear system with monodromy gives rise to an element of the associated family.

Let $L \subset V$ be a conformal immersions into $\mathbb{P}V$ with V a rank 2 vector space. If $L^\#$ is a generalized Darboux transformation of L, then there is $\psi \in \Gamma(\tilde{L}^\#)$ with $\nabla\psi \in \Gamma(K\tilde{L})$. Because $\nabla\psi$ is closed, it is a holomorphic section with monodromy of the bundle $K\tilde{L}$. In particular, for every nowhere vanishing $\beta \in V^* \subset H^0(L^{-1})$, the function $g = \beta(\psi)$ is a Bäcklund transformation of $\tilde{M}$

(but not of M, because $\nabla\psi$ is a holomorphic section with monodromy). For another (less obscure) relation between Darboux transformations and Bäcklund transformations, see Section 7.3.

Geometrical Characterization. If both L and $L^\#$ are immersions, the condition for $L^\#$ being a Darboux transformation of L allows the following geometrical interpretation, which uses the notion of left– and right–touching: two immersions *left–touch (right–touch)* at a common point if and only if, with respect to one (and therefore every) Euclidean chart, their left (right) normal vectors coincide (which, by Lemma 9, is equivalent to the fact that both tangent spaces, in the homomorphism model, have the same $\tilde{J}$ (or J)).

LEMMA 34. *Let L and $L^\#$ be immersions into (V,∇) such that $V = L \oplus L^\#$. The following are equivalent:*

i) $L^\#$ is a Darboux transformation of L.
ii) L and $L^\#$ induce the same conformal structure and the 2–sphere congruence touching L and intersecting $L^\#$ (at corresponding points) left–touches $L^\#$.
iii) L and $L^\#$ induce the same conformal structure and the 2–sphere congruence touching $L^\#$ and intersecting L right–touches L.

PROOF. With respect to the splitting $V = L \oplus L^\#$, the 2–sphere congruence S touching L and intersecting $L^\#$ is

$$S = \begin{pmatrix} J & 0 \\ 0 & \tilde{J} \end{pmatrix}$$

with $J \in \Gamma(\mathrm{End}(L))$ and $\tilde{J} \in \Gamma(\mathrm{End}(L^\#))$ satisfying $*\delta = \delta J = \tilde{J}\delta$. Now $L^\#$ induces the same conformal structure and S left–touches $L^\#$ if and only if $*\delta^\# = J\delta^\#$, which is equivalent to $\delta \wedge \delta^\# = 0$.

Similarly, the 2–sphere congruence $S^\#$ touching $L^\#$ and intersecting L is

$$S^\# = \begin{pmatrix} J & 0 \\ 0 & J^\# \end{pmatrix}$$

with $\tilde{*}\delta^\# = \delta^\# J^\# = J\delta^\#$. Now both conformal structures coincide and $S^\#$ right–touches L if and only if $\tilde{*} = *$ and $*\delta = \delta J$. This is equivalent to $\delta \wedge \delta^\# = 0$. $\square$

This geometrical characterization shows that the Darboux transformation discussed here generalizes the classical Darboux transformation for isothermic immersions (see Section 9.4): two isothermic surfaces are classical Darboux transformations of each other if and only if they have the same conformal structure and there is a 2–sphere congruence touching both immersions.

In contrast to the classical Darboux transformation, the generalization discussed here is directed: if $L^\#$ is a Darboux transformation of L, in general, L is not a Darboux transformation of $L^\#$.

REMARK. We will see in Lemma 66 that, if both L and $L^\#$ are Darboux transformations of each other or, equivalently, if L and $L^\#$ induce the same conformal structure and admit a 2–sphere congruence that touches both immersions, then both surfaces are isothermic and the Darboux transformation is a classical one.

For L and $L^{\#}$ conformal immersions into $\mathbb{P}V$ that are both contained in the same 3–dimensional sphere, $L^{\#}$ can only be a Darboux transformation of L, if both immersions are isothermic and L as well is a Darboux transformation of $L^{\#}$. This follows from the fact that, in a 3–sphere, left–touching or right–touching already implies touching.

A pair of immersed curves $V = L \oplus L^{\#}$ in (V, ∇) gives rise to the pair $V^* = L^{\perp} \oplus (L^{\#})^{\perp}$ of immersed curves in (V^*, ∇). The following corollary is a direct consequence of Lemma 34 and the fact that the role of left– and right–touching is interchanged when passing from V to V^*.

Corollary 35. *Let $V = L \oplus L^{\#}$ be two curves immersed into (V, ∇). Then, $L^{\#}$ is a Darboux transformation of L if and only if $L^{\perp}$ is a Darboux transformation of $(L^{\#})^{\perp}$.*

Backward Darboux Transformations. As with the other transformations, it is useful to introduce the following notion of backward Darboux transformations. We call $L^{\flat}$ a *backward Darboux transformation* of L if and only if $(L^{\flat})^{\perp}$ is a forward Darboux transformation of $L^{\perp}$. The preceding corollary states that, in the case that both L and $L^{\flat}$ are immersed, $L^{\flat}$ is a backward Darboux transformation of L if and only if L is a forward Darboux transformation of $L^{\flat}$.

Lemma 36. *Let L be an immersed holomorphic curve in (V, ∇). The bundle $L^{\flat}$ is a backward Darboux transformation of L if and only if (locally) there is a nowhere vanishing holomorphic section $\alpha \in H^0(L^{-1})$ such that $\psi \in \Gamma(L)$ with $\alpha(\psi) = 1$ satisfies $\nabla\psi \in \Omega^1(L^{\flat})$.*

Proof. By Corollary 33, $(L^{\flat})^{\perp}$ is a forward Darboux transformation of $L^{\perp}$ if and only if locally there is $\alpha \in H^0(L^{-1})$ whose prolongation $\tilde{\alpha} \in \Gamma(V^*)$ satisfies $(L^{\flat})^{\perp} = \tilde{\alpha}\mathbb{H}$. Recall that the prolongation $\tilde{\alpha} \in \Gamma(V^*)$ is the unique section with $\pi\tilde{\alpha} = \alpha$ and $\nabla\tilde{\alpha} \in \Omega^1(L^{\perp})$.

Let $\tilde{\alpha} \in \Gamma(V^*)$ be a section not intersecting $L^{\perp}$. Then, there is a unique section $\psi \in \Gamma(L)$ with $\langle \tilde{\alpha}, \psi \rangle = 1$, and by

$$\langle \nabla\tilde{\alpha}, \psi \rangle + \langle \tilde{\alpha}, \nabla\psi \rangle = 0$$

the section $\tilde{\alpha}$ is the prolongation of the nowhere vanishing holomorphic section $\alpha = \pi\tilde{\alpha} \in H^0(L^{-1})$, i.e. $\nabla\tilde{\alpha} \in \Omega^1(L^{\perp})$, if and only if ψ satisfies $\nabla\psi \in \Omega(L^{\flat})$ with $L^{\flat} = [\tilde{\alpha}]^{\perp}$. □

7.2. Splitting Formulae. Let (V, ∇) be a flat quaternionic rank 2 bundle, L an immersion and $L^{\#}$ an immersed Darboux transformation of L. Above we have already used the decomposition

$$\nabla = \begin{pmatrix} \nabla^L & \delta^{\#} \\ \delta & \nabla^{\#} \end{pmatrix}$$

of the connection ∇ with respect to the splitting $V = L \oplus L^{\#}$. In the same way, all endomorphisms of V can be decomposed with respect to $V = L \oplus L^{\#}$. The decomposition of the Hopf fields provides a useful tool for studying the relationship between properties of L and of the Darboux transformation $L^{\#}$. An immediate application is the proof of Lemma 39 which states that, in the case of degree 0 tori, the Darboux transformation preserves the Willmore functional. The main application is Theorem 6 on Darboux transformations of constrained Willmore immersions.

Mean Curvature Sphere and Hopf fields of L. To begin with, we derive the formulae for the mean curvature sphere and the Hopf fields of the immersion L using an arbitrary splitting $V = L \oplus L^{\#}$. (The bundle $L^{\#}$ is neither assumed to be immersed nor to be a Darboux transformation of L.)

Every 2–sphere congruence S that pointwise intersects L is of the form

$$S = \begin{pmatrix} J & H \\ 0 & \tilde{J} \end{pmatrix}$$

where J, $\tilde{J}$ and H are sections of the corresponding homomorphism bundles satisfying $J^2 = -\operatorname{Id}_L$, $\tilde{J}^2 = -\operatorname{Id}_{L^{\#}}$ and $JH + H\tilde{J} = 0$. We assume now that S touches L with the same orientation, i.e. $*\delta = \delta J = \tilde{J}\delta$. Then ∇S becomes

$$\nabla S = \begin{pmatrix} \nabla^L J - H\delta & \nabla H + \delta^{\#}\tilde{J} - J\delta^{\#} \\ 0 & \nabla^{\#}\tilde{J} + \delta H \end{pmatrix}$$

and the forms A and Q, defined by the usual decomposition $\nabla S = 2*Q - 2*A$ into $K-$ and $\bar{K}$–part with respect to the complex structure S, are then given by

$$2*A = \begin{pmatrix} 2*A^L & -\nabla H' - \overbrace{(\delta^{\#}\tilde{J} - J\delta^{\#})'}^{(*)} + \frac{1}{2}H*(\nabla^{\#}\tilde{J} + \delta H) \\ 0 & \underbrace{2*A^{\tilde{J}} - \delta H}_{=0 \text{ iff } S \text{ is mcs.}} \end{pmatrix}. \tag{40}$$

and

$$2*Q = \begin{pmatrix} \overbrace{2*Q^L - H\delta}^{=0 \text{ iff } S \text{ is mcs.}} & \nabla H'' + \overbrace{(\delta^{\#}\tilde{J} - J\delta^{\#})''}^{(**)} + \frac{1}{2}H*(\nabla^{\#}\tilde{J} + \delta H) \\ 0 & 2*Q^{\tilde{J}} \end{pmatrix} \tag{41}$$

(where $\nabla^L J = 2*Q^L - 2*A^L$ and $\nabla^{\#}\tilde{J} = 2*Q^{\tilde{J}} - 2*A^{\tilde{J}}$ as well are the decompositions into $K-$ and $\bar{K}$–parts).

Remark. If $L^{\#}$ is a Darboux transformation, the term $(**)$ vanishes, and therefore, the $()'$ in $(*)$ can be dropped.

Because a 2–sphere congruence S that touches L with the right orientation is the mean curvature sphere congruence if and only if $Q_{|L} = 0$ (or, equivalently, $\operatorname{im} A \subset L$), we have proven:

Lemma 37. *The sphere congruence S is the mean curvature sphere of L if and only if $2*Q^L = H\delta$ or, equivalently, $2*A^{\tilde{J}} = \delta H$.*

If these conditions are satisfied, the Hopf fields A and Q of the immersion L take the form

$$2*A = \begin{pmatrix} v & w \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad 2*Q = \begin{pmatrix} 0 & \tilde{w} \\ 0 & \tilde{v} \end{pmatrix}$$

with homomorphism valued 1–forms v, w, $\tilde{v}$ and $\tilde{w}$ determined by the formulae above. They satisfy $\delta \wedge v = \delta \wedge w = 0$ and $\tilde{v} \wedge \delta = \tilde{w} \wedge \delta = 0$. The differentials of $2 * A$ and $2 * Q$ are

$$d^{\nabla}(2 * A) = \begin{pmatrix} d^{\nabla^L} v + w \wedge \delta & d^{\nabla} w + v \wedge \delta^{\#} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & d^{\nabla} w + v \wedge \delta^{\#} \\ 0 & 0 \end{pmatrix} \tag{42}$$

and

$$d^{\nabla}(2 * Q) = \begin{pmatrix} 0 & d^{\nabla} \tilde{w} + \delta^{\#} \wedge \tilde{v} \\ 0 & d^{\nabla^{\#}} \tilde{v} + \delta \wedge \tilde{w} \end{pmatrix} = \begin{pmatrix} 0 & d^{\nabla} \tilde{w} + \delta^{\#} \wedge \tilde{v} \\ 0 & 0 \end{pmatrix}. \tag{43}$$

(where $d^{\nabla^L} v + w \wedge \delta = 0$ and $d^{\nabla^{\#}} \tilde{v} + \delta \wedge \tilde{w} = 0$ by Lemma 14).

Mean Curvature Sphere and Hopf fields of $L^{\#}$. The second bundle $L^{\#}$ in the splitting $V = L \oplus L^{\#}$ is now assumed to be immersed and to be a Darboux transformation of the immersion L, i.e. $\delta \wedge \delta^{\#} = 0$.

The mean curvature sphere $S^{\#}$ of $L^{\#}$ takes the form

$$S^{\#} = \begin{pmatrix} J & 0 \\ H^{\#} & J^{\#} \end{pmatrix}$$

with $*\delta^{\#} = \delta^{\#} J^{\#} = J \delta^{\#}$ and $H^{\#} J + J^{\#} H^{\#} = 0$. (Note that the bundle $L^{\#}$ has the two complex structures $\tilde{J}$ and $J^{\#}$!)

Since the formulae for the Hopf fields of L were derived without using the Darboux condition, they can as well be applied to calculate the Hopf fields $A^{\#}$ and $Q^{\#}$ of $L^{\#}$. These are

$$2 * A^{\#} = \begin{pmatrix} 0 & 0 \\ -(\nabla H^{\#})'^{J^{\#}} + \frac{1}{2} H^{\#} * (\nabla J + \delta^{\#} H^{\#}) & 2 * A^{J^{\#}} \end{pmatrix}$$

and

$$2 * Q^{\#} = \begin{pmatrix} 2 * Q^L & 0 \\ (\nabla H^{\#})''^{J^{\#}} + (\tilde{J} - J^{\#})\delta + \frac{1}{2} H^{\#} * (\nabla J + \delta^{\#} H^{\#}) & 0 \end{pmatrix}$$

where $\nabla^{\#} J^{\#} = 2 * Q^{J^{\#}} - 2 * A^{J^{\#}}$. Note that we used here $(\delta J - J^{\#}\delta)'^{J^{\#}} = 0$, which holds because $*(\tilde{J} - J^{\#})\delta = (-\operatorname{Id} - J^{\#}\tilde{J})\delta = -J^{\#}(\tilde{J} - J^{\#})\delta$.

Lemma 38. *The mean curvature sphere condition is $2 * A^L = \delta^{\#} H^{\#}$ or, equivalently, $2 * Q^{J^{\#}} = H^{\#} \delta^{\#}$.*

If S is the mean curvature sphere, the Hopf forms of $L^{\#}$ can be written as

$$2 * A^{\#} = \begin{pmatrix} 0 & 0 \\ w^{\#} & v^{\#} \end{pmatrix} \qquad \text{and} \qquad 2 * Q^{\#} = \begin{pmatrix} \tilde{v}^{\#} & 0 \\ \tilde{w}^{\#} & 0 \end{pmatrix}$$

with forms $v^{\#}$, $w^{\#}$, $\tilde{v}^{\#}$ and $\tilde{w}^{\#}$ determined by the formulae above. They satisfy $\delta^{\#} \wedge v^{\#} = \delta^{\#} \wedge w^{\#} = 0$ and $\tilde{v}^{\#} \wedge \delta^{\#} = \tilde{w}^{\#} \wedge \delta^{\#} = 0$. The differentials of $2 * A^{\#}$ and $2 * Q^{\#}$ are (where the second equality holds by Lemma 14)

(44)
$$d^{\nabla}(2 * A^{\#}) = \begin{pmatrix} 0 & 0 \\ d^{\nabla} w^{\#} + v^{\#} \wedge \delta & d^{\nabla} v^{\#} + w^{\#} \wedge \delta^{\#} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ d^{\nabla} w^{\#} + v^{\#} \wedge \delta & 0 \end{pmatrix}$$

and

(45)
$$d^\nabla(2*Q^\#) = \begin{pmatrix} d^\nabla\tilde{v}^\# + \delta^\# \wedge \tilde{w}^\# & 0 \\ d^\nabla\tilde{w}^\# + \delta \wedge \tilde{v}^\# & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ d^\nabla\tilde{w}^\# + \delta \wedge \tilde{v}^\# & 0 \end{pmatrix}.$$

REMARK. The A's and Q's of the complex structures J, $\tilde{J}$ and $J^\#$ can be used to compute the Willmore functionals of several holomorphic line bundles induced by the immersed holomorphic curves L and $L^\#$: the Willmore functional of L^{-1} can be computed from A^L, that of V/L from $Q^{\tilde{J}}$ and that of KL from $Q^{J^\#}$ (this last fact is shown below, in the proof of Lemma 39). The Willmore functional of $(L^\#)^{-1}$ can be calculated using $A^{J^\#}$ and that of $V/L^\#$ can be calculated using Q^L.

Application: the Darboux Transformation Preserves the Willmore Functional. Let M be a torus and let $V = L \oplus L^\#$ be two immersion into (V, ∇) such that $L^\#$ is a Darboux transformation of L. Then L and $L^\#$ have the same degree, since $J\delta^\# = \delta^\# J^\#$ and $\delta^\#_X$ for X a nowhere vanishing vector field is a (topological) isomorphism of complex quaternionic line bundles.

LEMMA 39. *Let $V = L \oplus L^\#$ be two immersion of a torus into (V, ∇) such that $L^\#$ is a Darboux transformation of L. If the immersion L has normal bundle degree 0, the immersion $L^\#$ has the same Willmore functional and normal bundle degree 0.*

PROOF. If L has normal bundle degree 0, the holomorphic curve L has degree 0 (see Section 2.4). By the discussion above, the immersion $L^\#$ as well has degree 0, and therefore normal bundle degree 0.

The map $\delta^\# \colon L^\# \to KL$ clearly is an isomorphism of complex quaternionic line bundles. It becomes a holomorphic isomorphism when $L^\#$ is equipped with the holomorphic structure $D = \frac{1}{2}(\nabla^\# + J^\# * \nabla^\#)$. To prove this, it suffices to note that a local $\nabla^\#$–parallel section $\psi \in \Gamma(L^\#)$ (which exists by Lemma 32) is mapped to $\delta^\#\psi = \nabla\psi$, which is a d^∇–closed local section of KL and therefore holomorphic (by equation (32)). Since the bundle KL is paired with L^{-1} it has the same Willmore functional.

Flatness of $\nabla^\#$ and $\nabla^\# J^\# = 2 * Q^{J^\#} - 2 * A^{J^\#}$ implies
$$R^{\hat{\nabla}^\#} + A^{J^\#} \wedge A^{J^\#} + Q^{J^\#} \wedge Q^{J^\#} = 0$$
and therefore
$$0 = \deg(L^\#, J^\#) = \frac{1}{2\pi}\int_M \langle R^{\hat{\nabla}^\#} J^\# \rangle = \frac{1}{2\pi}\int_M \langle A^{J^\#} \wedge *A^{J^\#} \rangle - \langle Q^{J^\#} \wedge *Q^{J^\#} \rangle.$$
Since
$$W((L^\#)^{-1}) = 2\int_M \langle A^{J^\#} \wedge *A^{J^\#} \rangle$$
(where we are using $A^\#_{|L^\#} = A^{J^\#}$) and
$$W(L^\#, D) = 2\int_M \langle Q^{J^\#} \wedge *Q^{J^\#} \rangle,$$

the Willmore functionals $W((L^\#)^{-1})$ and $W(L^{-1}) = W(KL) = W(L^\#, D)$ coincide. $\square$

Application: the Willmore Equation. By Equation (42), the immersion $L \subset V$ is Willmore if and only if $d^\nabla w + v \wedge \delta^\# = 0$. Since $v = 2 * A^L$, this is equivalent to

$$d^\nabla W = 0 \qquad \text{for} \qquad W = w - *\delta^\#. \tag{46}$$

Similarly, by Equation (45), the immersion $L^\#$ is Willmore if and only if $d^{\nabla^\#}\tilde{w}^\# + \delta \wedge \tilde{v}^\# = 0$. Since $\tilde{v}^\# = 2 * Q^L$, this condition is equivalent to

$$d^\nabla W^\# = 0 \qquad \text{for} \qquad W^\# = \tilde{w}^\# - *\delta. \tag{47}$$

The following lemma is needed for the proof of Theorem 6.

LEMMA 40. *For a splitting $V = L \oplus L^\#$ with L and $L^\#$ immersions such that $L^\#$ is a Darboux Transformation of L we have*

$$W\delta = \delta^\# W^\#.$$

PROOF. The formulae of W and $W^\#$ are

$$W = -\nabla H' - \delta^\# \tilde{J} + \frac{1}{2} H * (\underbrace{\nabla^\# \tilde{J} + \delta H}_{2*Q^{\tilde{J}}})$$

and

$$W^\# = (\nabla H^\#)''^{J^\#} - J^\#\delta + \frac{1}{2} H^\# * (\underbrace{\nabla J + \delta^\# H^\#}_{2*Q^L = H\delta}).$$

One auxiliary equation is obtained by calculating $d^{\nabla^L}\nabla^L J$ once using the formula $R^{\nabla^L} + \delta^\# \wedge \delta = 0$, i.e.

$$d^{\nabla^L}\nabla^L J = -[\delta^\# \wedge \delta, J] = \delta^\#(J^\# - \tilde{J}) \wedge \delta,$$

and once using the decomposition $\nabla^L J = 2 * Q^L - 2 * A^L$, which yields

$$\begin{aligned} d^{\nabla^L}\nabla^L J = d^{\nabla^L}(2 * Q^L - 2 * A^L) + \\ = d^{\nabla^L}(H\delta - \delta^\# H^\#) = \nabla H \wedge \delta + \delta^\# \wedge \nabla H^\#. \end{aligned}$$

With the standard identification of 2–forms with quadratic forms one obtains

$$2\delta^\#(\mathrm{Id} + J^\# \tilde{J})\delta = (\nabla H \tilde{J} - *\nabla H)\delta + \delta^\#(*\nabla H^\# - J^\# \nabla H^\#). \tag{$*$}$$

Another auxiliary equation is obtained by taking the covariant derivative of $JH + H\tilde{J} = 0$. This implies

$$J\nabla H + (2 * Q^L - 2 * A^L)H + \nabla H \tilde{J} + H(2 * Q^{\tilde{J}} - 2 * A^{\tilde{J}}) = 0$$

and using $2 * Q^L = H\delta$, $2 * A^L = \delta^\# H^\#$ and $2 * A^{\tilde{J}} = \delta H$ we obtain

$$J\nabla H - \delta^\# H^\# H + \nabla H \tilde{J} + H 2 * Q^{\tilde{J}} = 0. \tag{$**$}$$

The second auxiliary equation allows to rewrite W as

$$-2JW = -2 * W = *\nabla H + J\nabla H + 2 * \delta^\# \tilde{J} + H2 * Q^{\tilde{J}}$$
$$\overset{(**)}{=} *\nabla H + 2 * \delta^\# \tilde{J} + \delta^\# H^\# H - \nabla H \tilde{J}.$$

The first auxiliary equation implies

$$\begin{aligned}-2JW\delta &= -(\nabla H\tilde{J} - *\nabla H)\delta + 2 * \delta^\# \tilde{J}\delta + \delta^\# H^\# H\delta \\ &\overset{(*)}{=} -2\delta^\#(\mathrm{Id} + J^\# \tilde{J})\delta + \delta^\#(*\nabla H^\# - J^\# \nabla H^\#) + 2 * \delta^\# \tilde{J}\delta + \delta^\# H^\# H\delta \\ &= -2\delta^\#\delta + \delta^\#(*\nabla H^\# - J^\# \nabla H^\#) - \delta^\# J^\# H^\# H * \delta\end{aligned}$$

Since the formula for $W^\#$ implies

$$-2J^\# W^\# = -J^\# \nabla H^\# + *\nabla H^\# - 2\delta - J^\# H^\# H * \delta,$$

we obtain $-2J\delta^\# W^\# = -2\delta^\# J^\# W^\# = -2JW\delta$ which proves the statement. $\square$

REMARK. This proof is modeled after the proof in [**2**, Section 9.1].

7.3. Darboux Transformations in Euclidean Chart. Let $f\colon M \to \mathbb{H}$ and $g\colon M \to \mathbb{H}$ be immersions into $\mathbb{H} \subset \mathbb{HP}^1$. The aim of this section is to derive a condition for g being a Darboux transformation of f. Let

$$\psi = \begin{pmatrix} f \\ 1 \end{pmatrix} \in \Gamma(V) \qquad \text{and} \qquad \psi^\# = \begin{pmatrix} g \\ 1 \end{pmatrix} \in \Gamma(V).$$

be the corresponding sections of $V = \mathbb{H}^2$ and denote by $L = \psi\mathbb{H}$ and $L^\# = \psi^\#\mathbb{H}$ the induced subbundles. We assume that both immersion do not intersect, i.e. $V = L \oplus L^\#$. The ingredients of the usual decomposition

$$\nabla = \begin{pmatrix} \nabla^L & \delta^\# \\ \delta & \nabla^\# \end{pmatrix}$$

of ∇ with respect to the splitting $V = L \oplus L^\#$ are determined by

$$\nabla \begin{pmatrix} f \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} f \\ 1 \end{pmatrix} (f-g)^{-1} df}_{\nabla^L \psi} + \underbrace{\begin{pmatrix} g \\ 1 \end{pmatrix} (g-f)^{-1} df}_{\delta\psi}$$

and

$$\nabla \begin{pmatrix} g \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} f \\ 1 \end{pmatrix} (f-g)^{-1} dg}_{\delta^\# \psi^\#} + \underbrace{\begin{pmatrix} g \\ 1 \end{pmatrix} (g-f)^{-1} dg}_{\nabla^\# \psi^\#}.$$

The endomorphisms J and $\tilde{J}$ defined by $*\delta = \delta J = \tilde{J}\delta$ are

$$J\psi = -\psi R \qquad \text{and} \qquad \tilde{J}\psi^\# = \psi^\#(g-f)^{-1} N (g-f)$$

where $*df = Ndf = -dfR$. Similarly, $J^\#$ and $\tilde{J}^\#$ satisfying $*\delta^\# = \delta J^\# = \tilde{J}^\#\delta$ are

$$J^\#\psi^\# = -\psi^\# R_g \qquad \text{and} \qquad \tilde{J}^\#\psi = \psi(f-g)^{-1} N_g (f-g)$$

where $*dg = N_g dg = -dgR_g$.

Lemma 41. *Let $f\colon M \to \mathbb{H}$ and $g = f + v\colon M \to \mathbb{H}$ be nowhere intersecting immersions into $\mathbb{H} \subset \mathbb{HP}^1$. The following are equivalent:*

i) g is a Darboux transformation of f.
ii) $N_g = -(g-f)R(g-f)^{-1}$
iii) The form dfv^{-1} satisfies the Maurer–Cartan equation

$$d(dfv^{-1}) + dfv^{-1} \wedge dfv^{-1} = 0.$$

Proof. By definition, $L^\#$ is a Darboux transformation of L if and only if $\delta \wedge \delta^\# = 0$. This is equivalent to $J = \tilde{J}^\#$, which, in the light of the formulae for J and $\tilde{J}^\#$ above, becomes ii).

On the other hand, by Lemma 32, $L^\#$ is a Darboux transformation of L if and only if $\nabla^\#$ is flat. Since $\alpha = v^{-1}dg$ is a connection form of $\nabla^\#$, flatness is equivalent to $d\alpha + \alpha \wedge \alpha = 0$. A short calculation using $g = f + v$ shows that this again is equivalent to iii). □

This last condition shows that a Darboux transformation of f is obtained from a solution dfv^{-1} to the Maurer–Cartan equation by the formula $g = f + v$. Otherwise stated, every Darboux transformation of f corresponds to a flat connection. This point of view is made precise in Theorem 2.

Application: Bäcklund Transformations as a Limit of Darboux Transformations. Let $L \subset V$ be a conformal immersion into $\mathbb{P}V$ and let $L^{\#_t} \subset V$ be a family of Darboux transformations of L with the property that $L^{\#_0}$ is a fixed point.

Take e_1, e_2 a basis of V such that $L^{\#_0} = e_2\mathbb{H}$ and such that $\infty = [e_1]$ does not lie on $L^{\#_t}$ for small t. Let f be the Euclidean realization of L with respect to e_1, e_2 and let $g_t = f + v_t$ that of $L^{\#_t}$. In the proof of Lemma 41 we have seen that $\alpha = v^{-1}dg$ satisfies $d\alpha + \alpha \wedge \alpha = 0$. Since $\alpha_0 = 0$ this implies

$$d\dot{\alpha} = 0,$$

where $\dot{}$ denotes the derivative with respect to t at $t = 0$, and from $g_0 = 0$ we obtain

$$d(f^{-1}) \wedge d\dot{g} = 0.$$

Hence, by (38), $\dot{g}$ is a Bäcklund transformation of f^{-1}.

At least for isothermic surfaces, locally, a family $L^{\#_t}$ with the properties mentioned above always exists (see Lemma 68, where we prove that every dual surface can be obtained as a limit of Darboux transformations).

Remark. For general tori we expect that, in the normalization of the spectral curve (as defined in Section 8), there are points belonging to constant Darboux transformations. At those points there would exist a family $L^{\#_t}$ of Darboux transformations, which, in the limit, gives rise to closed Bäcklund transformations.

Application: Darboux Transformations of Cylinders. As an application of Lemma 41 we study Darboux transformations of cylinders. Let γ, $\tilde{\gamma}\colon I \subset \mathbb{R} \to \operatorname{Im}\mathbb{H}$ be two space curves and define $f(s,t) = s + \gamma(t)$ and $g(s,t) = s + \alpha + \tilde{\gamma}(t)$ with $\alpha \in \mathbb{R}$ the cylinders over the curves.

We assume that γ, $\tilde{\gamma}$ are parametrized with respect to arc length such that f and g are conformal immersions of $\mathbb{R} \times I$ into $\mathbb{H}$. Then $*df = Tdf = dfT$ where $T = \gamma'$ and $*dg = \tilde{T}dg = dg\tilde{T}$ with $\tilde{T} = \tilde{\gamma}'$. (This shows that a cylinder is an isothermic immersion which is dual to itself.)

Setting $\tilde{\gamma} = \gamma + v$, we have $g = f + v + \alpha$ and by Lemma 41 g is a Darboux transformation of f if and only if v satisfies the Ricatti equation

$$v' = (v+\alpha)T(v+\alpha)^{-1} - T. \tag{48}$$

Using the decomposition $v = v_+ + v_-$ into T–commuting and anti–commuting parts, this equation becomes

$$\begin{aligned}
v' &= \frac{1}{\alpha^2 + |v|^2}(v+\alpha)T(-v+\alpha) - T \\
&= \frac{1}{\alpha^2 + |v|^2}(v_+ + v_- + \alpha)(-v_+ + v_- + \alpha)T - T \\
&= \frac{1}{\alpha^2 + |v|^2}(-v_+^2 + v_-^2 + \alpha^2 + 2\alpha v_- + 2v_+v_-)T - T \\
&= \frac{2v_-^2 + 2\alpha v_- - 2v_-v_+}{\alpha^2 + |v|^2}T = \frac{2(v_-^2 + v_+^2) + 2\alpha v_- - 2vv_+}{\alpha^2 + |v|^2}T.
\end{aligned}$$

Since $v_-T = v \times T$ and $v_+T = -\langle v, T\rangle$, we obtain that (48) is equivalent to

$$v' = \frac{-2|v|^2}{\alpha^2 + |v|^2}T + \frac{2\alpha}{\alpha^2 + |v|^2}v \times T + \frac{2\langle v, T\rangle}{\alpha^2 + |v|^2}v. \tag{49}$$

This is equation (2.16) of [**16**] (where r and l there correspond to α and $|v|$ here). Therefore, g is a Darboux transformation of f if and only if the curve $\tilde{\gamma}$ is obtained from γ by Darboux transformation[3] of space curves as described in [**16**].

Let γ be a space curve parametrized with respect to arc length. For every $\alpha \in \mathbb{R}$ and every initial value $v(0) \in \operatorname{Im}\mathbb{H}$, there is a unique solution $v(t)$ to (49). It is easily verified that v has constant length and that $\tilde{\gamma} = \gamma + v$ as well is parametrized with respect to arc length. This suggest that the Darboux transformation for space curves can be interpreted in terms of a generalized tractrix construction (see the end of Section 2.2 in [**16**]).

If f and g are two cylinders of the form above such that g is Darboux transformation of f, then f is a Darboux transformation of g if and only if $\alpha = 0$. This follows from (48), because f and g are Darboux transformations of each other if and only if

$$\tilde{T} = (v+\alpha)T(v+\alpha)^{-1} = (v+\alpha)^{-1}T(v+\alpha),$$

and therefore the condition on a Darboux transformation to be forward and backward is $(v+\alpha)^2T = T(v+\alpha)^2$. This last equation is equivalent to $\alpha v_- = 0$, which again is equivalent to $\alpha = 0$. (The case $v_- = 0$, or equivalently $v \| T$, is ignored, because then (48) implies $v' = 0$ and therefore, as well by $v \| T$, γ has to be a straight line.)

[3] The transformation in [**16**] is actually called a Bäcklund transformation, but we prefer to call it a Darboux transformation to be consistent with the our terminology.

7.4. The Main Theorems. In this subsection, two important theorems about Darboux transformations are proven. The first theorem is essential for the new approach to the spectral curve of conformally immersed tori (in Section 8). The second theorem is a (Bianchi–type) permutability theorem.

A Correspondence Between Splittings and Adapted Connections. Let (V, ∇) be a flat quaternionic rank 2 bundle and let $L \subset V$ be an immersion. As usual, we denote by $\mathcal{R}$ the vector bundle

$$\mathcal{R} = \{W \in \operatorname{End}(V) \mid \operatorname{im} W \subset L \text{ and } L \subset \ker W\}.$$

The sections of $\mathcal{R}$ act on the space of splittings $V = L \oplus L^{\#}$ by

$$\check{L} = (\operatorname{Id} + W)L^{\#} \tag{50}$$

for $W \in \Gamma(\mathcal{R})$. This makes the set $\{$splittings $V = L \oplus L^{\#}\}$ into an affine space with vector space $\Gamma(\mathcal{R})$.

There is another affine space naturally arising in this context, namely the space of adapted connections on the bundle V/L: a connection ∇ on V/L is called adapted if $\nabla'' = D$, where D is the natural holomorphic structure on V/L and where ∇'' denotes the $\bar{K}$–part of ∇ with respect to the natural complex structure on V/L.

The next theorem shows that there is a natural identification of both affine spaces.

Theorem 2. *Let $L \subset V$ be an conformal immersion into a flat quaternionic rank 2 bundle (V, ∇). Every splitting $V = L \oplus L^{\#}$, using the isomorphism $\pi \colon L^{\#} \to V/L$, induces a connection $\nabla^{\#}$ on $V/L \cong L^{\#}$. This connection is adapted.*

The correspondence assigning to a splitting the induced adapted connection is an affine bijection between the space of splittings and the space of adapted connections on V/L. The splitting $V = L \oplus \check{L}$ given by $\check{L} = (\operatorname{Id} + W)L^{\#}$ for $W \in \Gamma(\mathcal{R})$ yields the connection

$$\check{\nabla} = \nabla^{\#} + \delta W.$$

In particular, Darboux transformations of L correspond to flat adapted connections on V/L.

Proof. The connection $\nabla^{\#}$ is indeed adapted: the holomorphic structure on V/L is defined by $D\pi = (\pi\nabla)'' = \pi(\nabla'')$ where ∇'' is the $\bar{K}$–part of ∇ on V with respect to the mean curvature sphere S of L. Since S induces the natural complex structure on V/L, we have $\pi(\nabla'') = (\nabla^{\#})''\pi$ where $(\nabla^{\#})''$ is calculated with respect to the complex structure on V/L. Therefore, $D\pi = (\nabla^{\#})''\pi$, and $D = (\nabla^{\#})''$ follows by surjectivity of π.

For $\psi \in \Gamma(L^{\#})$ and $W \in \Gamma(\mathcal{R})$ we have

$$\pi\nabla(\operatorname{Id} + W)\psi = \pi\nabla\psi + \delta W\psi.$$

Using that $\pi \colon L^{\#} \to V/L$ and $\pi \colon \check{L} \to V/L$ are isomorphisms, this becomes

$$\check{\nabla}\pi(\operatorname{Id} + W)\psi = \nabla^{\#}\pi\psi + \delta W\psi$$

and with $\pi(\mathrm{Id}+W) = \pi$ (again using the isomorphism $\pi\colon L^{\#} \to V/L$) one obtains

$$\check{\nabla} = \nabla^{\#} + \delta W.$$

This equation shows at the same time that the correspondence is bijective and affine. The statement about Darboux transformations follows directly from Lemma 32. $\square$

A Permutability Theorem. The principal reason for introducing the notion of generalized Darboux transformation are the following theorem and lemma.

THEOREM 3. *Let $L \subset V$ be a conformal immersion into $\mathbb{P}V$ (with V a rank 2 vector space). Let $L^{\#}$ and $\check{L}$ be two different immersed Darboux transformation of L. Then, there is a subbundle $\check{L}^{\#}$ which is simultaneously a generalized Darboux transformation of $\check{L}$ and of $L^{\#}$.*

As usual, we visualize this permutability theorem by the diagram

(51)
$$\begin{tikzcd} \check{L} \arrow[r, dotted] & \check{L}^{\#} \\ L \arrow[u] \arrow[r] & L^{\#} \arrow[u, dotted] \end{tikzcd}$$

where the continuous arrows indicate the initial Darboux transformations and the dotted arrows indicate the generalized Darboux transformations, whose existence is guaranteed by the permutability theorem.

PROOF. Let $\psi \in \Gamma(\tilde{L}^{\#})$ and $\check{\psi} \in \Gamma(\tilde{\check{L}})$ be the prolongations of holomorphic sections of $\tilde{V}/\tilde{L}$ with monodromy, which define the Darboux transformations. Because $\nabla\psi$, $\nabla\check{\psi} \in \Gamma(K\tilde{L})$, there is $\chi\colon \tilde{M} \to \mathbb{H}^*$ such that $\nabla\check{\psi} = (\nabla\psi)\chi$. Then

$$\nabla(\check{\psi} - \psi\chi) = \nabla\check{\psi} - (\nabla\psi)\chi - \psi d\chi = -\psi d\chi \in \Omega^1(L^{\#})$$

and $\check{\psi} - \psi\chi \in \Gamma(\check{L}^{\#})$ is the prolongation of a holomorphic section of $\tilde{V}/\tilde{L}^{\#}$. It therefore gives rise to a generalized Darboux transformation $\check{L}^{\#} = (\check{\psi} - \psi\chi)\mathbb{H}$ of $L^{\#}$.

By interchanging the role of $L^{\#}$ and $\check{L}$, the same argument shows that $\check{L}^{\#}$ thus defined is a Darboux transformation of $\check{L}$ as well. $\square$

REMARK. The definition of $\check{L}^{\#}$ can be formulated more invariantly as follows: define the nowhere vanishing section $B \in \Gamma(\mathrm{Hom}(\check{L}, L^{\#}))$ by

$$\check{\delta} = \delta^{\#} B.$$

Then the generalized Darboux transformation of $\check{L}$ and $L^{\#}$ from the preceding theorem is given by $\check{L}^{\#} := \mathrm{im}(\mathrm{Id}_{\check{L}} - B)$.

The proof of the following lemma uses essentially the same idea as the last proof.

LEMMA 42. *Let $L \subset V$ be a conformal immersion of a torus of degree 0 and let $L^{\#}$ be an immersed Darboux transformation of L. Then*

$$\mathrm{Spec}(L^{\#}) = \mathrm{Spec}(L).$$

PROOF. The Darboux transformation $L^{\#}$ is obtained by prolongation of a holomorphic section in V/L with multiplier $\lambda \in \operatorname{Spec}(L)$. Let $\check{L}$ be a generalized Darboux transformation with multiplier $\mu \in \operatorname{Spec}(L)$ (we do not assume $\check{L}$ to be immersed!). As in the proof of the preceding Theorem, for $\psi \in \Gamma(\tilde{L}^{\#})$ and $\check{\psi} \in \Gamma(\tilde{\check{L}})$ with $\nabla\psi, \nabla\check{\psi} \in \Omega^1(\tilde{L})$, there is χ such that $\nabla(\check{\psi}-\psi\chi) \in \Omega^1(L^{\#})$. The section $\check{\psi}-\psi\chi$ is not identically zero (because $\lambda \neq \mu$) and has the multiplier μ, too. Because every multiplier $\mu \in \operatorname{Spec}(V/L)\backslash\{\lambda, \bar{\lambda}\}$ gives rise to a generalized Darboux transformation $\check{L}^{\#}$ of $L^{\#}$, we have $\mu \in \operatorname{Spec}(V/L^{\#})$. Hence, $\lambda, \bar{\lambda}$ are the only points from $\operatorname{Spec}(V/L)$ which might be missing in $\operatorname{Spec}(V/L^{\#})$, so by continuity $\operatorname{Spec}(V/L) \subset \operatorname{Spec}(V/L^{\#})$. The other inclusion follows by passing to the dual bundles (using the fact that, as a consequence of Theorem 1, the spectra of the duals curve are related by $\lambda \mapsto \lambda^{-1}$ to those of the original curve, see Section 4.2). $\square$

8. Spectral Curve of a Torus Immersed into $\mathbb{HP}^1$

The reader should be warned that this section is of speculative character!

A new definition of the spectral curve of a conformally immersed torus with degree 0 normal bundle is proposed. This definition is definitely not in its ultimate form. For example, the spectral curve as presently defined necessarily has horrible singularities (at least one of them is a whole $\mathbb{HP}^n$ with $n \geq 1$ contained in the spectral "curve"). But one should expect that, by a suitable normalization procedure, the spectral curve can be made into a proper Riemann surface.

Initially, the motivation for this new definition was to develop a more geometric understanding of the (Floquet theoretic) spectrum of Section 4. (The original idea leading to this new definition is explained in Section 8.2.) But then, a fundamental new interpretation of the spectral curve has emerged from the new notion of Darboux transformation of Section 7: Theorem 2 immediately implies that the spectral curve of an immersion parametrizes its Darboux transformations, which, in particular, yields a realization of the spectral curve as a holomorphic curve in $\mathbb{CP}^3$ (see Section 8.4).

The first subsection defines an infinite dimensional manifold $\mathcal{M}$ that contains all spectral curves. In the second subsection, the spectral curve is defined as an analytic subset of $\mathcal{M}$. The third subsection contains a naive treatment of the different kinds of bad (singular) points in the spectral curve. At the end a definition of good points is given. (We expect that, starting from those good points, one can define the desingularization of the spectral curve mentioned above.) In the forth and last subsection, it is shown that the spectral curve defined for a conformal immersion of a torus with normal bundle degree 0 into $\mathbb{HP}^1$ admits a natural realization as a holomorphic curve in $\mathbb{CP}^3$.

8.1. The Manifold $\mathcal{M}$. For M a compact surface and L a quaternionic line bundle over M we define

$$\mathcal{M} = \left\{ (\nabla, S) \,\middle|\, \begin{array}{c} \nabla \text{ a flat quaternionic connection on } L, \\ S \in \Gamma(\operatorname{End}(L)) \text{ a section satisfying } S^2 = -\operatorname{Id} \text{ and } \nabla S = 0 \end{array} \right\}.$$

The aim of this section is to 'prove' that $\mathcal{M}$ is an infinite dimensional (Frechet) manifold. It is a subset of the space $\mathcal{A}\times\mathcal{S}$ where $\mathcal{A}$ denotes the affine space of quaternionic connections on L and $\mathcal{S}=\{S\in\Gamma(\mathrm{End}(L)) \mid S^2=-\mathrm{Id}\}$. The space $\mathcal{S}$ is a manifold. A local chart at $S\in\mathcal{S}$ is given by

$$B\in\Gamma(\mathrm{End}_-(L))\mapsto(\mathrm{Id}+B)S(\mathrm{Id}+B)^{-1}\in\mathcal{S}. \tag{52}$$

By $\mathcal{S}_0$ we denote the subset of $S\in\mathcal{S}$ with the property that the complex quaternionic line bundle (L,S) has degree 0. (Since every quaternionic line bundle is topologically trivial, $\mathcal{S}$ is diffeomorphic to the set of maps $N\colon M\to S^2$. The subset $\mathcal{S}_0$ is the connected component diffeomorphic to the set of maps of mapping degree 0.)

For $(\nabla,S)\in\mathcal{M}$, since ∇ is a flat S–complex connection on the complex quaternionic line bundle, (L,S) has degree 0 and $S\in\mathcal{S}_0$. The projection

$$(\nabla,S)\in\mathcal{M}\mapsto S\in\mathcal{S}_0$$

is clearly surjective and the action of $G=\{\alpha+\beta\mathbf{i}\in\Omega^1\mathbb{C}\mid d\alpha=d\beta=0\}$ on $\mathcal{M}$ via

$$(\nabla,S,\alpha+\beta\mathbf{i})\in\mathcal{M}\times G\mapsto(\nabla+\alpha+\beta S,S)\in\mathcal{M}$$

preserves the fibers and is free and simply transitive. This makes $\mathcal{M}$ into a G principal fiber bundle.

To prove that $\mathcal{M}$ is a manifold, we define the local chart at $(\nabla,S)\in\mathcal{M}$ by

$$\begin{aligned}&(B,\alpha+\beta\mathbf{i})\in\Gamma(\mathrm{End}_-(L))\times G\mapsto\\ &\qquad\big((\mathrm{Id}+B)\circ(\nabla+\alpha+\beta S)\circ(\mathrm{Id}+B)^{-1},(\mathrm{Id}+B)S(\mathrm{Id}+B)^{-1}\big)\in\mathcal{M}.\end{aligned} \tag{53}$$

This map is injective and differentiable and, because (52) is a chart, it has a differentiable inverse, which shows that $\mathcal{M}$ is indeed a submanifold of $\mathcal{A}\times\mathcal{S}_0$ (a fact that could as well be proven using the implicit function theorem). Moreover, this map is a trivialization for the G principal fiber bundle $\mathcal{M}\to\mathcal{S}_0$.

It is easy to see that the locally defined flat connection induced by the charts of the form (53) patch together to a globally defined flat connection on $\mathcal{M}$, its horizontal distribution at (∇,S) being the tangent space to the embedding

$$B\in\Gamma(\mathrm{End}_-(L))\mapsto\big((\mathrm{Id}+B)\circ\nabla\circ(\mathrm{Id}+B)^{-1},(\mathrm{Id}+B)S(\mathrm{Id}+B)^{-1}\big)\in\mathcal{M}.$$

The tangent space of $\mathcal{M}$ at the point $(\nabla,S)\in\mathcal{M}$ is

$$T_{(\nabla,S)}\mathcal{M}=\left\{(\omega,Y)\in\Omega^1(\mathrm{End}(L))\times\Gamma(\mathrm{End}(L))\,\middle|\,\begin{matrix}d^\nabla\omega=0,\ SY+YS=0\\ \omega_-=\frac12\nabla YS\end{matrix}\right\}.$$

The vertical space of the projection $(\nabla,S)\in\mathcal{M}\mapsto S\in\mathcal{S}_0$ at $(\nabla,S)\in\mathcal{M}$ is

$$\mathcal{V}_{(\nabla,S)}=\{(\omega,0)\in\Omega^1(\mathrm{End}(L))\times\Gamma(\mathrm{End}(L))\mid d^\nabla\omega=0\text{ and }[\omega,S]=0\}$$

and the horizontal space of the flat connection on $\mathcal{M}$ at $(\nabla,S)\in\mathcal{M}$ is

$$\mathcal{H}_{(\nabla,S)}=\{(\tfrac12\nabla YS,Y)\in\Omega^1(\mathrm{End}(L))\times\Gamma(\mathrm{End}(L))\mid SY+YS=0\}.$$

LEMMA 43. *Every tangent vector $(\omega,Y)\in T_{(\nabla,S)}\mathcal{M}$ can be written as the tangent vector to a curve in $\mathcal{M}$ of the form $t\mapsto(G_t\circ\nabla\circ G_t^{-1},G_tSG_t^{-1})$ where $G_t\in\Gamma(\mathrm{End}(\tilde L))$ with $G_0=\mathrm{Id}$ is a family of gauge transformations defined on the universal covering $\tilde M$ of M.*

Proof. Clearly, the horizontal part of a tangent vector $(\omega, Y) \in T_{(\nabla,S)}$ can be written as a tangent vector to a curve $t \mapsto (\check{G}_t \circ \nabla \circ \check{G}_t^{-1}, \check{G}_t S \check{G}_t^{-1})$ with $\check{G}_t \in \Gamma(\mathrm{End}(L))$ a gauge transformation defined on M. Since all flat complex connections on the complex quaternionic line bundle (L, S) are gauge equivalent over the universal covering $\tilde{M}$, the vertical part of the vector can be written as the tangent vector to a curve $t \mapsto (\hat{G}_t \circ \nabla \circ \hat{G}_t^{-1}, S)$ with $\hat{G}_t \in \Gamma(\mathrm{End}_+(\tilde{L}))$ a family of gauge transformations defined on $\tilde{M}$. This proves the statement, since $G_t = \hat{G}_t \check{G}_t$ yields a curve in $\mathcal{M}$ with tangent vector $(\omega, Y) \in T_{(\nabla,S)}$. $\square$

The complex structure

$$J_{(\nabla,S)}(\omega, Y) = (\omega S, YS) \tag{54}$$

on $\mathcal{M}$ is integrable, since the differential at $(B, \alpha + \beta\mathbf{i})$ of the chart (53) at the point $(\nabla, S) \in \mathcal{M}$

$$\begin{aligned}
&(\dot\alpha + \dot\beta\mathbf{i}, \dot B) \in G \times \Gamma(\mathrm{End}_-(L)) \mapsto \\
&\big(-\nabla \dot B(\mathrm{Id} + B)^{-1} + \nabla B(\mathrm{Id} + B)^{-1}\dot B(\mathrm{Id} + B)^{-1} + \dot B(\alpha + \beta S)(\mathrm{Id} + B)^{-1} + \\
&+ (\mathrm{Id} + B)(\dot\alpha + \dot\beta S)(\mathrm{Id} + B)^{-1} - (\mathrm{Id} + B)(\alpha + \beta S)(\mathrm{Id} + B)^{-1}\dot B(\mathrm{Id} + B)^{-1}, \\
&\qquad \dot B S(\mathrm{Id} + B)^{-1} - (\mathrm{Id} + B)S(\mathrm{Id} + B)^{-1}\dot B(\mathrm{Id} + B)^{-1}\big) \in T\mathcal{M}
\end{aligned} \tag{55}$$

is obviously complex linear, meaning it is compatible with the right action of S on $(\dot\alpha + \dot\beta\mathbf{i}, \dot B)$ and the right action of $(\mathrm{Id} + B)S(\mathrm{Id} + B)^{-1}$ on the tangent vector to $\mathcal{M}$.

The Genus 1 Case. We assume now that the surface M is a torus. In the next section, the spectral curve of a holomorphic line bundle on a torus (equipped with the structure of a Riemann surface) is defined as a subset of $\mathcal{M}$. In this definition, ∇ appears to be the essential object while S seems to play an auxiliary role. It is therefore interesting to gain a better understanding of the projection

$$(\nabla, S) \in \mathcal{M} \mapsto \nabla \in \mathcal{A}_0$$

(with $\mathcal{A}_0$ denoting the subset of flat connections in $\mathcal{A}$). Compared to the projection $\mathcal{M} \to \mathcal{S}_0$, the projection $\mathcal{M} \to \mathcal{A}_0$ is much more difficult to understand, because the type of fibers depends on ∇.

Lemma 44. *The fiber of the projection $(\nabla, S) \in \mathcal{M} \mapsto \nabla \in \mathcal{A}_0$ is $\{\pm S\}$ if the holonomy of ∇ is not real. It is diffeomorphic to $\mathbb{CP}^1$ if the holonomy is real.*

Proof. For a flat quaternionic connection ∇ on L, there are two possible cases: either, the holonomy $H(\gamma)$ of ∇ (seen as a parallel section $H(\gamma)$ of $\mathrm{End}(L)$ as described in Appendix B) is real for all deck transformations $\gamma \in \Gamma$, or the parallel automorphisms $H(\gamma)$ with $\gamma \in \Gamma$ form a non–real commutative subgroup of all automorphisms of L. In the first case, every $S_p \in \mathrm{End}(L_p)$ with $S_p^2 = -\mathrm{Id}$ extends to a parallel section $S \in \Gamma(\mathrm{End}(L))$ with $S^2 = -\mathrm{Id}$. In the second case, every $H(\gamma)$ is of the form $H(\gamma) = a\,\mathrm{Id} + bS$, where $\pm S \in \Gamma(\mathrm{End}(L))$ are the only parallel sections of $\mathrm{End}(L)$ with $S^2 = -\mathrm{Id}$. This proves the lemma. $\square$

The Differential of the Holonomy Map. For a curve $t \mapsto (\nabla^t, S^t) \in \mathcal{M}$ of the form

$$t \mapsto (\nabla^t, S^t) = (G_t \circ \nabla \circ G_t^{-1}, G_t S G_t^{-1})$$

with $G_0 = \mathrm{Id}$ and G_t defined on the universal covering $\tilde{M}$, the holonomy of ∇^t is

$$H^t(\gamma) = \gamma^* G_t \, H(\gamma) \, G_t^{-1} \tag{56}$$

for $\gamma \in \Gamma$ a deck transformation (and H the holonomy of ∇).

Lemma 45. *The differential of the holonomy map*

$$H \colon \mathcal{M} \to \mathrm{Hom}(\Gamma, \Gamma(\mathrm{GL}(L)))$$

satisfies

$$dH_{(\nabla,S)} J^{T\mathcal{M}}(\omega, Y) = (dH_{(\nabla,S)}(\omega, Y))S \tag{a}$$

and

$$(dH(\gamma)_{(\nabla,S)}(\omega, Y))_+ = -H \int_\gamma \omega_+ \tag{b}$$

for all $(\nabla, S) \in \mathcal{M}$ *and* $(\omega, Y) \in T_{(\nabla,S)}\mathcal{M}$ *and* $\gamma \in \Gamma$.

Proof. We denote by $()^{\cdot}$ the derivatives in the direction

$$(\omega, Y) = (-\nabla \dot{G}, [\dot{G}, S])$$

of the tangent vector to the above curve (note that by Lemma 43 all tangent vectors to $\mathcal{M}$ are of that form). By $()'$ we denote the derivatives in the direction

$$J(\omega, Y) = (\omega S, YS) = (-\nabla G', [G', S])$$

where G' has to be of the form $G' = \dot{G}S + a + bS$. Using (56), the derivative of $H(\gamma)$ in the direction of (ω, Y) is

$$\dot{H}(\gamma) = \gamma^* \dot{G} \, H(\gamma) - H(\gamma)\dot{G},$$

which proves (b). Similarly, we have

$$H(\gamma)' = \gamma^* G' \, H(\gamma) - H(\gamma)G',$$

and together, both equations imply $H(\gamma)' = \dot{H}(\gamma)S$, which proves (a). □

Corollary 46. *The function* $h = a + b\mathbf{i}$ *defined by* $H = a + bS$ *is a holomorphic function from* $\mathcal{M}$ *into* $\mathrm{Hom}(\Gamma, \mathbb{C}^*)$. *Furthermore, away from the points in* $\mathcal{M}$ *with real holonomy, we have* $dS \circ J = \dot{S}S$.

Proof. We use the same notation as in the proof of the preceding lemma. Part (a) of the preceding lemma, i.e. $H' = \dot{H}S$, yields

$$a' + b'S + bS' = (\dot{a} + \dot{b}S + b\dot{S})S.$$

The $+$–part of this equation yields holomorphicity of h and at the points with $b \neq 0$, the $-$–part proves $S' = \dot{S}S$, i.e. $dS \circ J = \dot{S}S$. □

Remark. S can therefore be seen as a meromorphic function on $\mathcal{M}$ defined away from the points with real holonomy.

8.2. The Spectral Curve Σ of a Holomorphic Line Bundle. The above definition of $\mathcal{M}$ involves neither a complex structure on M nor on L. If such complex structures are fixed, i.e. if M is a Riemann surface (with $*$ acting on cotangent vectors) and L a complex quaternionic line bundle (which complex structure J), then there is a canonical map

$$(\nabla, S) \in \mathcal{M} \mapsto \tfrac{1}{2}(\nabla + J * \nabla)$$

to the space of quaternionic holomorphic structures on L. The inverse image of a holomorphic structure D under this map is the spectral curve of the holomorphic line bundle (L, J, D). While this definition makes sense for all Riemann surfaces, the name "curve" is probably only justified in the case of degree 0 bundles over a torus.

Definition. Let (L, J, D) be a quaternionic holomorphic line bundle of degree 0 over a torus M. The *Spectral Curve* of L is

$$\Sigma = \{(\nabla, S) \in \mathcal{M} \mid \nabla'' = \tfrac{1}{2}(\nabla + J * \nabla) = D\}.$$

This definition was proposed by Ulrich Pinkall. The relation to the spectrum of Section 4 is easily explained. Ignoring the multipliers in $\operatorname{Spec}(L)$ that belong to holomorphic sections of $\tilde{L}$ with zeros, elements of $\operatorname{Spec}(L)$ are obtained from flat connections ∇ on L satisfying $\nabla'' = D$ (see the remark in Section 4.1). If the holonomy of a connection ∇ is not real, it gives rise to two different multipliers (each of which corresponds to one of the two parallel S's belonging to ∇ as in Lemma 44), otherwise there is of course only one. An advantage of considering pairs (∇, S) of connections with parallel endomorphisms (instead of ∇'s alone) is that, in the case of non–real holonomy, S singles out one of the multipliers and one obtains a true map from Σ to $\operatorname{Spec}(L)$ which, of course, is the map h described above. (There are other "advantages", too. For example, S is essential for the definition of the complex structure on the manifold $\mathcal{M}$.)

Remark. It is clear that, in passing from $\operatorname{Spec}(L)$ to Σ, on the one hand, all multipliers that do not admit nowhere vanishing holomorphic sections are lost. On the other hand, real points in $\operatorname{Spec}(L)$ are "blown up" to whole $\mathbb{CP}^1$'s and multipliers admitting a higher dimensional space of sections are "blown up" to $\mathbb{CP}^n$'s (or even $\mathbb{HP}^n \times \mathbb{CP}^1$'s if the holonomy is real). Nevertheless, away from such "bad" points it is expected that both $\operatorname{Spec}(L)$ (or rather the quotient of its logarithm) and Σ have the same normalization.

The *formal tangent space* of Σ at (∇, S) is

$$T_{(\nabla,S)}\Sigma = \left\{ (\omega, Y) \in \Omega^1(\operatorname{End}(L)) \times \Gamma(\operatorname{End}(L)) \,\middle|\, \begin{matrix} d^\nabla \omega = 0,\ SY + YS = 0 \\ \omega'' = 0,\ \omega_- = \tfrac{1}{2}\nabla Y S \end{matrix} \right\}.$$

We conjecture that the following can be shown:

1. $\Sigma \subset \mathcal{M}$ is an analytic set. (It can be seen that it is the zero set of a holomorphic section of a complex line bundle over $\mathcal{M}$).
2. Since $d^\nabla \omega = 0$ on $\Gamma(K \operatorname{End}(L))$ is an elliptic equation, the dimension of the formal tangent space is finite. Its (complex) dimension is always greater or equal 1, at generic points it is 1.

(3) Taimanov defines a spectral curve by taking the desingularization of the set of quasi–momenta (after passing to the quotient of the action of the dual lattice), cf. Section 4.1. The link between our spectral curve and Taimanov's spectral curve is given by taking the logarithm of the map $h\colon \Sigma \to \mathrm{Spec}(L)$. We expect this map to be more or less bijective and that both complex sets have the same normalization.

8.3. The Good and the Bad (Points). Let Σ be the spectral curve of a holomorphic line bundle L of degree 0 over a Riemann surface of genus $g = 1$. At most points, we expect Σ to be a Riemann surface and the map $h\colon \Sigma \to \mathrm{Spec}(L)$ from the spectral curve to the spectrum to be bijective and non–singular. This subsection gives a naive description of the points, where this does not hold.

Regular Points. To start with, we analyze further the points where Σ has the right dimension, i.e. $\dim_{\mathbb{C}} T_{(\nabla,S)}\Sigma = 1$.

Definition. A point $(\nabla, S) \in \Sigma$ is called *regular* if the holonomy of ∇ is not real and if $\dim_{\mathbb{C}} T_{(\nabla,S)}\Sigma = 1$.

The points $(\nabla, S) \in \Sigma$ with real holonomy are excluded, because at such points Σ contains a complex projective line full of S'es which are parallel with respect to ∇. This is clarified by the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & T_{(\nabla,S)}\Sigma & \longrightarrow & B & \longrightarrow & 0 \\ & & Y & \mapsto & (0,Y) & & & & \\ & & & & (\omega, Y) & \mapsto & \omega & & \end{array} \tag{57}$$

where $A = \{Y \in \Gamma(\mathrm{End}_-(L)) \mid \nabla Y = 0\}$ (which is 0 if and only if the holonomy is not real) and $B = \{\omega \in \Gamma(K\,\mathrm{End}(L)) \mid d^\nabla\omega = 0,\ \omega_- \text{ exact}\}$.

At the points of Σ with $\dim_{\mathbb{C}} T_{(\nabla,S)}\Sigma = 1$, either the holonomy is not real, i.e. $\{Y \in \Gamma(\mathrm{End}_-(L)) \mid \nabla Y = 0\} = 0$. Then, by the exact sequence, $T_{(\nabla,S)}\Sigma \cong \{\omega \in \Gamma(K\,\mathrm{End}(L)) \mid d^\nabla\omega = 0,\ \omega_- \text{ exact}\}$. Or the holonomy is real and the sequence implies $\{\omega \in \Gamma(K\,\mathrm{End}(L)) \mid d^\nabla\omega = 0,\ \omega_- \text{ exact}\} = 0$ and $T_{(\nabla,S)} \cong \{Y \in \Gamma(\mathrm{End}_-(L))\}$. In the first case, which is the case of regular points, there are only $\pm S$ belonging to ∇. Along a variation of the point in Σ, the change of S is prescribed by that of ∇. In the second case, a variation of the point in Σ changes S but leaves ∇ fixed.

Lemma 47. *Let E be a complex line bundle on the torus $\mathbb{C}/\Gamma$ and let ∇ be a flat connection with holonomy $\mu \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$. If a form $\omega \in \Omega^1(\tilde{E})$ with $\gamma^*\omega = \omega\lambda_\gamma$ for all $\gamma \in \Gamma$ and $\mu \neq \lambda \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ satisfies $d^\nabla\omega = 0$, then it can be integrated preserving monodromy, i.e. there is $\psi \in \Gamma(\tilde{E})$ with $\gamma^*\psi = \psi\lambda_\gamma$ for all $\gamma \in \Gamma$ such that $\nabla\psi = \omega$.*

Proof. Since ∇ is flat, there exists $\tilde{\psi} \in \Gamma(\tilde{E})$ with $\nabla\tilde{\psi} = \omega$. Denote by $\varphi \in \Gamma(\tilde{E})$ a non–trivial parallel section. The idea of the proof is to show that for the right choice of $c \in \mathbb{C}$, the section $\psi = \tilde{\psi} + \varphi c$ has the desired multiplier.

For every $\gamma \in \Gamma$ the section $\gamma^*\tilde{\psi} - \tilde{\psi}\lambda_\gamma$ of $\tilde{E}$ is a parallel and there is β_γ such that

$$\gamma^*\tilde{\psi} = \tilde{\psi}\lambda_\gamma + \varphi\beta_\gamma. \tag{$*$}$$

Applying $(*)$ twice and using $\gamma^*\varphi = \varphi\mu_\gamma$ yields

$$(\gamma+\gamma')^*\tilde\psi = \gamma^*(\tilde\psi\lambda_{\gamma'} + \varphi\beta_{\gamma'}) = \tilde\psi\lambda_\gamma\lambda_{\gamma'} + \varphi\beta_\gamma\lambda_{\gamma'} + \varphi\mu_\gamma\beta_{\gamma'}$$

for γ, $\gamma' \in \Gamma$. On the other hand, applying $(*)$ directly to $\gamma+\gamma'$ yields

$$(\gamma+\gamma')^*\tilde\psi = \tilde\psi\lambda_{\gamma+\gamma'} + \varphi\beta_{\gamma+\gamma'}.$$

Both equations together with $\lambda_{\gamma+\gamma'} = \lambda_\gamma\lambda_{\gamma'}$ imply

$$\beta_{\gamma+\gamma'} = \beta_\gamma\lambda_{\gamma'} + \mu_\gamma\beta_{\gamma'}$$

for all γ, $\gamma' \in \Gamma$. By commutativity of Γ we also have $\beta_{\gamma+\gamma'} = \beta_{\gamma'}\lambda_\gamma + \mu_{\gamma'}\beta_\gamma$ which implies

$$(**) \qquad \beta_\gamma(\lambda_{\gamma'} - \mu_{\gamma'}) = \beta_{\gamma'}(\lambda_\gamma - \mu_\gamma).$$

The condition for $\psi = \tilde\psi + \varphi c$ to have the multiplier λ is

$$(\tilde\psi + \varphi c)\lambda_\gamma = \tilde\psi\lambda_\gamma + \varphi\beta_\gamma + \varphi\mu_\gamma c$$

or $c(\lambda_\gamma - \mu_\gamma) = \beta_\gamma$. Now, setting $c := \frac{\beta_\gamma}{\lambda_\gamma - \mu_\gamma}$ for γ with $\lambda_\gamma - \mu_\gamma \neq 0$, the section ψ has the right monodromy: for γ' with $\lambda_{\gamma'} - \mu_{\gamma'} \neq 0$ the above condition is satisfied by $(**)$. And for γ' with $\lambda_{\gamma'} - \mu_{\gamma'} = 0$ the above condition is satisfies because $(**)$ implies $\beta_{\gamma'} = 0$. $\square$

Corollary 48. *The holonomy at $(\nabla, S) \in \Sigma$ is non–real if and only if*

$$T_{(\nabla,S)}\Sigma \cong \{\omega \in \Gamma(K\,\mathrm{End}(L)) \mid d^\nabla\omega = 0\}.$$

Proof. Assume the holonomy is non–real. Then, the preceding lemma applied to the form $\omega_- \in \Omega^1(\mathrm{End}_-(L))$ yields

$$\{\omega \in \Gamma(K\,\mathrm{End}(L)) \mid d^\nabla\omega = 0\} \cong \{\omega \in \Gamma(K\,\mathrm{End}(L)) \mid d^\nabla\omega = 0,\ \omega_- \text{ exact}\},$$

and by the exact sequence

$$(*) \qquad T_{(\nabla,S)}\Sigma \cong \{\omega \in \Gamma(K\,\mathrm{End}(L)) \mid d^\nabla\omega = 0,\ \omega_- \text{ exact}\}.$$

Conversely, if $T_{(\nabla,S)}\Sigma \cong \{\omega \in \Gamma(K\,\mathrm{End}(L)) \mid d^\nabla\omega = 0\}$, in particular $(*)$ has to hold and the exact sequence implies that the holonomy is non–real. $\square$

Lemma 49. *For a quaternionic connection ∇ on a complex quaternionic line bundle L of degree 0*

$$\dim_{\mathbb{R}}\{\omega \in \Gamma(K\,\mathrm{End}(L)) \mid d^\nabla\omega = 0\} = \dim_{\mathbb{R}}\{B \in \Gamma(\mathrm{End}(L)) \mid *\nabla B = BJ\}.$$

Proof. This proof uses essentially the same Riemann–Roch type argument than that of Theorem 1, Part a). The operator

$$d^\nabla \colon \Gamma(K\,\mathrm{End}(L)) \to \Omega^2(\mathrm{End}(L))$$

is elliptic. With respect to the non–degenerate pairings

$$\ll \beta, \omega \gg = \int_M \langle \beta \wedge \omega \rangle$$

for $\beta \in \Gamma(\mathrm{End}(L)\bar K$ and $\omega \in \Gamma(K\,\mathrm{End}(L))$ and

$$\ll B, \alpha \gg = \int_M \langle B, \alpha \rangle$$

for $B \in \Gamma(\mathrm{End}(L))$ and $\alpha \in \Omega^2(\mathrm{End}(L))$, the negative of its adjoint operator is

$$\tilde{D}\colon \Gamma(\mathrm{End}(L)) \to \Gamma(\mathrm{End}(L)\bar{K})$$

with $\tilde{D}B = \frac{1}{2}(\nabla B + \nabla JB)$ and the index of d^∇ is

$$\mathrm{index}(d^\nabla) = \dim_{\mathbb{R}} \ker(d^\nabla) - \dim_{\mathbb{R}} \ker(\tilde{D}).$$

By the (complex) Riemann–Roch theorem, the index is zero for complex connections (i.e. connections with $\nabla J = 0$), and by homotopy invariance it is zero for all quaternionic connections. $\square$

THEOREM 4. *A point $(\nabla, S) \in \Sigma$ is regular if and only if*

$$\dim_{\mathbb{R}}\{\omega \in \Gamma(K\,\mathrm{End}(L)) \mid d^\nabla \omega = 0\} = 2$$

or, equivalently,

$$\dim_{\mathbb{R}}\{B \in \Gamma(\mathrm{End}(L)) \mid *\nabla B = \nabla BJ\} = 2.$$

PROOF. It is clear (by definition of regular and Corollary 48) that if the point is regular, the (real) dimension of $\{\omega \in \Gamma(K\,\mathrm{End}(L)) \mid d^\nabla\omega = 0\}$ is 2. The second equivalence is Lemma 49.

Conversely, assume the (real) dimension of one (hence both) vector spaces is 2. The holonomy is then non–real, since the set of parallel endomorphisms is a subset of $\{B \in \Gamma(\mathrm{End}(L)) \mid *\nabla B = BJ\}$. By Corollary 48 the point is regular. $\square$

REMARK. A point $(\nabla, S) \in \Sigma$ is regular if the space of solutions to the equation

$$*\nabla B = \nabla BJ \tag{58}$$

for $B \in \Gamma(\mathrm{End}(L))$ is $\mathrm{Span}_{\mathbb{R}}\{\mathrm{Id}, S\}$.

Good Points. Good points are those regular points, where the holonomy map h is injective. The following lemma shows that the notions injective and infinitesimally injective coincide.

LEMMA 50. *Let L be a holomorphic line bundle on the torus $M = \mathbb{C}/\Gamma$. Let $(\nabla, S) \in \Sigma$ be a point with non–real holonomy in the spectral curve of L. Then, the differential $dh_{(\nabla,S)}$ is not injective if and only if there is $(\tilde{\nabla}, \tilde{S}) \in \Sigma$ with $h(\nabla, S) = h(\tilde{\nabla}, \tilde{S})$ and $(\nabla, S) \neq (\tilde{\nabla}, \tilde{S})$.*

PROOF. Let $(\omega, Y) \in \ker dh_{(\nabla,S)}$ be non–zero. Part (b) of Lemma 45 implies

$$0 = dh(\gamma)_{(\nabla,S)}(\omega, Y) = -h\int_\gamma \omega_+$$

for all $\gamma \in \Gamma$. Hence ω_+ is closed and, ω_- being always closed (because of the condition $\omega_- = \frac{1}{2}\nabla YS$) we can find a section $Z \in \Gamma(\mathrm{End}(L))$ with $\nabla Z = \omega$. This section Z satisfies $*\nabla Z = J\nabla Z$.

Define $G = Z\mu + \mathrm{Id}$ with $\mu = a + bS \in \mathbb{C}$. For small μ, the endomorphism G is invertible and

$$\tilde{\nabla} = G \circ \nabla \circ G^{-1} = \nabla - (\nabla G)G^{-1}$$

is a flat connection with $\tilde{\nabla}'' = D$. It is gauge equivalent to ∇, and therefore $h(\nabla, S) = h(\tilde{\nabla}, \tilde{S})$ for $\tilde{S} = GSG^{-1}$. Since G is not parallel (because $\omega = 0$ is not possible for points with non–real holonomy), the point $(\tilde{\nabla}, \tilde{S}) \in \Sigma$ is different from (∇, S). This proves one direction of the statement.

Conversely, assume that $(\nabla, S) \neq (\tilde{\nabla}, \tilde{S})$ satisfy $h(\nabla, S) = h(\tilde{\nabla}, \tilde{S})$. Since both connections are flat, there is a section $\psi \in \Gamma(\mathrm{End}(\tilde{L}))$ with monodromy satisfying $\nabla\psi = 0$ and $S\psi = \psi\mathbf{i}$, and a section $\tilde{\psi} \in \Gamma(\mathrm{End}(\tilde{L}))$ with monodromy such that $\tilde{\nabla}\tilde{\psi} = 0$ and $\tilde{S}\tilde{\psi} = \tilde{\psi}\mathbf{i}$. Because the multipliers off both sections coincide, the section $B \in \Gamma(\mathrm{End}(L))$ defined by $B\psi = \tilde{\psi}$ has no monodromy. Since both connections are compatible with the holomorphic structure D and B satisfies $\tilde{\nabla} = B \circ \nabla B^{-1} = \nabla - (\nabla B)B^{-1}$, we have

$$*\nabla B = J\nabla B$$

(where ∇B does not vanish identically, because the holonomy of ∇ is not real). Setting $(\omega, Y) = (\nabla B, -2B_-S)$, we obtain a tangent vector in $T_{(\nabla,S)}\Sigma$ which is contained in $\ker h_{(\nabla,S)}$. □

Remark. The second part of the proof can be clarified by the following heuristic argument: if there are two different connections with the same monodromy, the (complex!) space of holomorphic sections of L with a corresponding multiplier has dimension greater or equal to 2. To every nowhere vanishing sections corresponds a point in the spectral curve. In particular, there is a variation of (∇, S) in Σ of points in the spectral curve. By construction, its derivative (ω, Y) lies in the kernel of $dh_{(\nabla,S)}$.

Definition. A point $(\nabla, S) \in \Sigma$ is called *good* if and only if every solution $B \in \Gamma(\mathrm{End}(L))$ to $*\nabla B = J\nabla B$ is parallel and every solution to $*\nabla B = \nabla BJ$ is parallel. A point is called *bad* if it is not good.

Otherwise stated, a point $(\nabla, S) \in \Sigma$ of the spectral curve is called good if and only if

$$\begin{aligned}\dim_{\mathbb{R}}&\{B \in \Gamma(\mathrm{End}(L)) \mid *\nabla B = J\nabla B\}\\ &= \dim_{\mathbb{R}}\{B \in \Gamma(\mathrm{End}(L)) \mid *\nabla B = \nabla BJ\} = 2.\end{aligned}$$

By Theorem 4, a good point is in particular a regular point (and therefore has non–real monodromy).

8.4. The Eigenspace Curve. It is explained, how the spectral curve of a conformal immersion $L \subset \mathbb{H}^2$, which is defined to be that of the holomorphic line bundle $\mathbb{H}^2/L$, can be realized as a holomorphic curve in $\mathbb{CP}^3$.

Definition. The *spectral curve* of a conformal immersion $L \subset V$ of a torus with normal bundle degree 0 into (V, ∇) is defined to be the spectral curve Σ of the holomorphic line bundle V/L.

Let $L \subset \mathbb{H}^2$ be a conformal immersion with normal bundle degree 0 of the torus M into $\mathbb{HP}^1$. By Theorem 2, every point of the spectral curve Σ corresponds to a Darboux transformation of the initial immersion. This defines a map

$$\tilde{\mathcal{E}} \colon M \times \Sigma \to \mathbb{HP}^1$$

with the property that, for every $(\nabla, S) \in \Sigma$, the Darboux transformation corresponding to (∇, S) is $p \in M \mapsto \tilde{\mathcal{E}}(p, (\nabla, S))$.

The aim of this section is to prove that for every fixed point $p \in M$ of the torus, the map $\tilde{\mathcal{E}}_p := \tilde{\mathcal{E}}(p, \_)\colon \Sigma \to \mathbb{HP}^1$ is conformal and has a holomorphic twistor lift $\mathcal{E}_p\colon \Sigma \to \mathbb{CP}^3$. We call the holomorphic curve $\mathcal{E}_p$ the *eigenspace curve* of the immersion at $p \in M$.

Remark. The name eigenspace curve is justified by the fact that, in the case of isothermic and constrained Willmore tori, this eigenspace curve is related to the eigenspaces of the holonomy of the respective complex family of flat connections, see e.g. the last paragraph of Section 9.4.

Theorem 5. *The eigenspace curve $\mathcal{E}_p\colon \Sigma \to \mathbb{CP}^3$ of the conformal immersion $L \subset \mathbb{H}^2$ at every point $p \in M$ of the torus M is holomorphic.*

Proof. Let $(\nabla, S) \in \Sigma$ be a point in Σ and let (∇^t, S^t), $(\nabla^s, S^s) \in \Sigma$ be two curves in Σ with $(\nabla^0, S^0) = (\nabla, S) \in \Sigma$ such that

$$(\nabla, S)' = (\nabla, S)^{\cdot} S \in T_{(\nabla,S)}\Sigma$$

(where $'$ and $^{\cdot}$ denote the derivatives with respect to s and t at 0).

Denote by $L^\#$ the Darboux transformation corresponding to (∇, S). By Theorem 2, there is a family of endomorphisms $W^t \in \Gamma(\mathrm{Hom}(L^\#, L)) = \Gamma(\mathcal{R})$ such that the Darboux transformations $\tilde{\mathcal{E}}(\_, (\nabla^t, S^t))$ along the curve (∇^t, S^t) are given by

$$L^{\#_t} = (\mathrm{Id} + W^t) L^\#.$$

Similarly, there is $W^s \in \Gamma(\mathrm{Hom}(L^\#, L))$ describing the Darboux transformations $L^{\#_s} = (\mathrm{Id} + W^s) L^\#$ along (∇^s, S^s).

By $\mathrm{Hom}(L^\#, V/L^\#) = \mathrm{Hom}(L^\#, L)$, the derivative of the curve $\tilde{\mathcal{E}}_p$ in the direction of (∇^t, S^t) is

$$\delta^{\tilde{\mathcal{E}}_p} = \dot{W}$$

and, by Theorem 2, the derivative in the direction of (∇^s, S^s) is

$$*\delta^{\tilde{\mathcal{E}}_p} = W' = \dot{W} S,$$

which proves, that $\tilde{\mathcal{E}}_p$ is conformal, i.e. $*\delta^{\tilde{\mathcal{E}}_p} = \delta^{\tilde{\mathcal{E}}_p} S$.

Under the isomorphisms in the commuting diagram

$$\begin{tikzcd} L^\# \arrow[rr, "\mathrm{Id} + W^t"] \arrow[dr, "\pi"'] & & L^{\#_t} \arrow[dl, "\pi"] \\ & V/L, & \end{tikzcd}$$

the family of complex structures S^t on V/L defines a family of complex structures on $L^\#$, which we denote by $S^{\#_t}$, and a complex structure $\tilde{S}^{\#_t}$ on each $L^{\#_t}$. In particular, we have

$$\tilde{S}^{\#_t}(\mathrm{Id} + W^t) = (\mathrm{Id} + W^t) S^{\#_t}.$$

Denote by $Z^t \in \Gamma(\mathrm{End}_-(L^\#))$ the family of endomorphisms such that

$$S^{\#_t}(\mathrm{Id} + Z^t) = (\mathrm{Id} + Z^t) S.$$

Then, $\dot{S}^{\#} + S\dot{Z} = \dot{Z}S$ implies

$$(*) \qquad \dot{Z} = \tfrac{1}{2}S\dot{S}^{\#}.$$

For every $p \in M$, the $\tilde{S}$–complex structures are the ones belonging to the conformal immersion $\tilde{\mathcal{E}}_p$. The twistor lift is therefore given by the $\mathbf{i}$–eigenspaces of $\tilde{S}$ on $L^{\#_t}$.

Let $\psi \in \Gamma(L^{\#})$ be a section with $S\psi = \psi\mathbf{i}$. Then, for every $p \in M$,

$$\psi^t = (\mathrm{Id} + W^t)(\mathrm{Id} + Z^t)\psi(p)$$

is a section of the complex line subbundle bundle of $\mathbb{C}^4 = (\mathbb{H}^2, \mathbf{i})$ that defines the twistor lift $\mathcal{E}_p$ of $\tilde{\mathcal{E}}_p$ defined along the curve (∇^t, S^t). The analogous construction yields a section

$$\psi^s = (\mathrm{Id} + W^s)(\mathrm{Id} + Z^s)\psi(p)$$

of that line bundle over (∇^s, S^s).

This implies

$$\dot{\psi} = (\dot{W} + \dot{Z})\psi$$

and, using $W' = *\delta^{\tilde{\mathcal{E}}_p} = \dot{W}S$ and $Z' = \dot{Z}S$ (which follows from equation $(*)$ and $S^{\#'} = \dot{S}^{\#}S$)

$$\psi' = (W' + Z')\psi = (\dot{W} + \dot{Z})S\psi = \dot{\psi}\mathbf{i}.$$

This proves the statement. $\square$

CHAPTER III

Special Surface Classes

This chapter treats isothermic immersions and constrained Willmore immersions, which make up two important special classes of surfaces occurring in a Möbius geometric context. Both classes form integrable systems in the sense that they admit an associated family depending on a real parameter. The principal task of this chapter is to describe, how their classical transformation theories arise as special cases of the new transformations of Chapter II.

Two classes of immersions which are both isothermic and constrained Willmore, namely that of surfaces obtained from elastic curves in 3–dimensional space forms and that of constant mean curvature surfaces in 3–dimensional space forms, are treated in Sections 3.2 and 3.3 of Appendix A and in Sections 15.2 and 15.3 on flows.

9. Isothermic Surfaces

Isothermic immersions and their transformations have already been investigated at the end of the 19^{th} century. At the end of the 20^{th} century, in the framework of integrable systems theory, they anew have become an object of intense study. See [**3**] or [**15**] for a survey of isothermic surfaces.

In this section, a quaternionic characterization of isothermic surfaces, developed by Burstall, Ferus, Hertrich–Jeromin, Leschke, Pedit and Pinkall, is given. It is explained in detail, how isothermic surfaces fit into the new transformation theory and how they can be characterized in terms of transformations.

9.1. Definition and General Properties. Let (V, ∇) be a flat quaternionic rank 2 vector bundle and let $L \subset V$ be a conformal immersion. We use the following definition of isothermic surfaces.

Definition. A conformal immersion $L \subset V$ into (V, ∇) is called *isothermic*, if there is a non–trivial 1–form $\omega \in \Omega^1(\mathcal{R})$ satisfying $d^\nabla \omega = 0$, where, as usual, $\mathcal{R} = \{W \in \operatorname{End}(V) \mid \operatorname{im} W \subset L \text{ and } L \subset \ker W\}$.

The relation to the classical definition of isothermic surface is discussed in Section 9.3.

Lemma 51. *Let $L \subset V$ be an immersion into (V, ∇). Then:*

i) For a 1–form $\omega \in \Omega^1(\mathcal{R})$ with $d^\nabla \omega = 0$, every connection $\nabla + \rho\omega$ with $\rho \in \mathbb{R}$ is flat.

ii) For a 1–form $\omega \in \Omega^1(\operatorname{End} V)$, the conditions

$$\omega \in \Omega^1 \mathcal{R} \quad \textit{and} \quad d^\nabla \omega = 0$$

are equivalent to $\nabla + \omega$ and $\nabla - \omega^*$ being flat adapted connections on the 1–jet bundles $V \to V/L$ and $V^* \to V^*/L^\perp$ of the quaternionic holomorphic line bundles V/L and $L^{-1} = V^*/L^\perp$.

iii) Every 1–form $\omega \in \Omega^1\mathcal{R}$ with $d^\nabla \omega = 0$ satisfies

$$\omega \in \Gamma(K\mathcal{R}_+).$$

PROOF. i) follows from

$$R^{\nabla+\rho\omega} = R^\nabla + \rho d^\nabla \omega + \rho^2 \omega \wedge \omega = 0.$$

ii) With the characterization in Section 5 of the flat adapted connections on the respective 1–jet bundles, it suffices to check that

$$\omega \in \Omega^1\mathcal{R} \quad \text{and} \quad d^\nabla \omega = 0$$

is equivalent to

$$d^\nabla \omega + \omega \wedge \omega = 0 \quad \text{and} \quad \pi\omega = 0 \quad \text{and} \quad \omega L = 0,$$

which is of course evident. iii) is a direct consequence of ii), because, by Section 5, $*\omega = S\omega$ holds for $\nabla + \omega$ a flat adapted connection on the 1–jet bundle $V \to V/L \to 0$ of V/L and $*\omega = \omega S$ holds for $\nabla - \omega^*$ a flat adapted on the 1–jet bundle $V^* \to V^*/L^\perp \to 0$ of $L^{-1} = V^*/L^\perp$. □

The Holomorphic Quadratic Differential $\omega\delta$. The aim of this paragraph is to prove that if $L \subset V$ is an isothermic immersion and $\omega \in \Omega^1(\mathcal{R})$ satisfies $d^\nabla \omega = 0$, the form $\omega\delta$ is a holomorphic quadratic differential.

LEMMA 52. *Let $L \subset V$ be a conformal immersion into a flat bundle (V, ∇) and let $\eta \in \Gamma(K\mathcal{R}_+)$. Then $\eta\delta \in \Gamma(K^2 \operatorname{End}(L)) = \Gamma(K^2)$ is holomorphic if and only if*

$$d^{\hat\nabla} \eta = 0.$$

PROOF. The idea is to choose a holomorphic vector field X, i.e. a vector field X with $[X, JX] = 0$, and to prove that $d^{\hat\nabla}\eta = 0$ if and only if $\eta_X \delta_X \in \Gamma(\operatorname{End}(L))$ is holomorphic.

Let $V = L \oplus \check L$ be a splitting and let

$$\hat\nabla = \begin{pmatrix} \nabla^L & \check\delta \\ \delta & \check\nabla \end{pmatrix} \qquad \text{and} \qquad \eta = \begin{pmatrix} 0 & \check\eta \\ 0 & 0 \end{pmatrix}$$

be the connection $\hat\nabla$ and the form η written with respect to the splitting. Then

$$d^{\hat\nabla}\eta = \begin{pmatrix} 0 & d^{\nabla^L, \check\nabla}\check\eta \\ 0 & 0 \end{pmatrix}.$$

For X holomorphic, we have

$$\begin{aligned} d^{\nabla^L,\check\nabla}_{JX,X}\check\eta &= \nabla^L_{JX} \circ \eta_X - \eta_X \circ \check\nabla_{JX} - \nabla^L_X \circ \eta_{JX} + \eta_{JX} \circ \check\nabla_X \\ &= 2 * (\nabla^L)''_X \circ \eta_X - \eta_X \circ 2 * \hat\nabla''_X. \end{aligned} \tag{59}$$

Equation (5) implies $R^{\hat\nabla} L \subset L$ or, equivalently, $d^{\check\nabla,\nabla^L}\delta = 0$, and the analogous computation as for δ yields

$$0 = d^{\check\nabla,\nabla^L}_{JX,X}\delta_X = 2 * \hat\nabla''_X \circ \delta_X - \delta_X \circ 2 * (\nabla^L)''_X. \tag{60}$$

Equations (59) and (60) together prove that $\eta_X\delta_X$ commutes with $(\nabla^L)''$ if and only if $d^{\nabla^L,\check{\nabla}}\check{\eta} = 0$ or, equivalently, $d^{\hat{\nabla}}\eta = 0$. Because $\eta_X\delta_X$ is holomorphic if and only if it commutes with $(\nabla^L)''$, this proves the statement. $\square$

COROLLARY 53. *Let $L \subset V$ be an isothermic immersion with $\omega \in \Omega^1(\mathcal{R})$ satisfying $d^\nabla\omega = 0$. Then, $\omega\delta$ is a holomorphic quadratic differential.*

PROOF. It suffices to check that Lemma 52 applies to a form $\omega \in \Omega^1(\mathcal{R})$ with $d^\nabla\omega = 0$: by part iii) of Lemma 51, such form is in $\Gamma(K\mathcal{R}_+)$ and therefore satisfies

$$0 = d^\nabla\omega = \underbrace{d^{\hat{\nabla}}\omega}_{+} + \underbrace{A\wedge\omega + \omega\wedge Q}_{-}.$$

$\square$

On the torus, a closed 1–form $\omega \in \Omega^1(\mathcal{R})$ has no zeros, because the quadratic differential $\omega\delta$ is a holomorphic section of the bundle K^2 which, for tori, is holomorphically trivial.

Uniqueness of ω. Let $L \subset V$ be an isothermic immersion. Except for the round sphere, the form $\omega \in \Omega^1(\mathcal{R})$ with $d^\nabla\omega = 0$ is unique up to scaling by a real factor.

LEMMA 54. *If L is not totally umbilic and if there is $\omega \in \Omega^1\mathcal{R}$ with $d^\nabla\omega = 0$, then every other $\tilde{\omega}$ with these properties is of the form $\tilde{\omega} = \rho\omega$ with $\rho \in \mathbb{R}$.*

PROOF. Assume ω and $\tilde{\omega}$ have the given property. Corollary 53 shows that ω vanishes at isolated points. Because L is not totally umbilic, A or Q is non–zero at some point. By passing to $L^\perp$ if necessary, we can assume that A does not vanish identically.

Part iii) of Lemma 51 implies that, away from possible zeros of ω, the form $\tilde{\omega}$ can be written as

$$\tilde{\omega} = (a + bS)\omega$$

for smooth real functions a and b. The derivative of this equation is

$$0 = d^\nabla\tilde{\omega} = (da + dbS)\wedge\omega - b2 * A\wedge\omega.$$

Decomposition into $+$ and $-$ part shows $(da + dbS)\wedge\omega = 0$ and $b2 * A\wedge\omega = 0$. Because A is by assumption non–zero at some point[1], the second equation implies that b has to vanish on an open set. The first equation is equivalent to $a + ib$ holomorphic, which proves the lemma, because a holomorphic function whose imaginary part vanishes on some open set has to be constant. $\square$

9.2. Euclidean Characterization of Isothermic Surfaces in Terms of Dual Surfaces. Suppose (V,∇) has no monodromy, i.e. $V \cong \mathbb{H}^2$. With respect to the Euclidean chart defined by the basis e_1, e_2, the immersion takes the form

$$L = \begin{pmatrix} f \\ 1 \end{pmatrix}\mathbb{H},$$

[1]By Lemma 86, if A is non–zero at some point, $A_{|L}$ as well is non–zero at some point.

with f its Euclidean realization, and every 1–form $\omega \in \Omega^1\mathcal{R}$ can be written as

$$\omega = \operatorname{Ad}\begin{pmatrix}1 & f\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ \hat\omega & 0\end{pmatrix} \tag{61}$$

with $\hat\omega \in \Omega^1\mathbb{H}$ a quaternionic 1–form. Since

$$\begin{aligned} d^\nabla\omega &= \operatorname{Ad}\begin{pmatrix}1 & f\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ d\hat\omega & 0\end{pmatrix} + \operatorname{Ad}\begin{pmatrix}1 & f\\ 0 & 1\end{pmatrix}\left[\begin{pmatrix}0 & df\\ 0 & 0\end{pmatrix}\wedge\begin{pmatrix}0 & 0\\ \hat\omega & 0\end{pmatrix}\right]\\ &= \operatorname{Ad}\begin{pmatrix}1 & f\\ 0 & 1\end{pmatrix}\begin{pmatrix}df\wedge\hat\omega & 0\\ d\hat\omega & \hat\omega\wedge df\end{pmatrix}, \end{aligned} \tag{62}$$

the condition $d^\nabla\omega = 0$ is equivalent to $d\hat\omega = 0$ and $df\wedge\hat\omega = \hat\omega\wedge df = 0$. A map $g\colon \tilde M \to \mathbb{H}$ with $dg = \hat\omega$ is then a dual surface according to the following definition.

Definition. Let $f\colon M \to \mathbb{H}$ be an immersion. A *dual surface* of f is a (branched) immersion $g\colon \tilde M \to \mathbb{H}$ (with translational monodromy) satisfying

$$df\wedge dg = dg\wedge df = 0.$$

A more geometric characterization of dual surfaces is the following: g is a dual surface if and only if, at every point, $\bar g$ has the same tangent space as f with the opposite orientation, i.e. $*d\bar g = -Nd\bar g = d\bar g R$, whereas $*df = Ndf = -dfR$. In the classical literature, such $\bar g$ is also called a *Christoffel transformation* of the immersion f, cf. [**15**].

The following lemma is a direct consequence of (61) and (62).

Lemma 55. *Let $L \subset \mathbb{H}^2$ be a conformal immersion. With respect to a Euclidean chart, every closed form $\omega \in \Omega^1(\mathcal{R})$, via (61), corresponds to a dual surface g.*

In particular, the immersion L is isothermic if and only if its representative $f\colon M \to \mathbb{H}$ with respect to one (and therefore every) Euclidean chart admits a dual surface g.

Together with Lemma 54, this implies:

Corollary 56. *If $f\colon M \to \mathbb{H}$ is an isothermic immersion that is not totally umbilic, its dual surface is unique up to translation and scaling by a real factor.*

While the notion of isothermic immersion is Möbius invariant, the notion of dual surface is not. The differential of a dual surface g with respect to the Euclidean chart given by a basis v, w of V is

$$dg = \beta\omega v,$$

where $\beta(v) = 0$ and $\beta(w) = 1$. Hence, ω is an object that incorporates the right transformation behavior of the differentials of dual surfaces under Möbius transformations.

Remark. In the Euclidean setting, the holomorphic quadratic differential $\omega\delta$ of Corollary 53 can be expressed follows: let

$$L = \psi\mathbb{H} \qquad \text{with} \qquad \psi = \begin{pmatrix}f\\ 1\end{pmatrix}$$

be an isothermic immersion with closed form

$$\omega = \mathrm{Ad}\begin{pmatrix}1 & f\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ dg & 0\end{pmatrix}.$$

Then, the holomorphic quadratic differential of Corollary 53 is given by

$$\omega\delta\,\psi = \psi\, dgdf. \tag{63}$$

9.3. Classical Definition of Isothermic Surfaces.

DEFINITION. An immersion $f\colon M \to \mathbb{H}$ is *classically isothermic* if, at every non–umbilic point, it locally admits conformal curvature line coordinates.

In codimension 2, the notion of curvature lines is only well defined if the normal bundle is flat. Therefore, in the definition of classically isothermic it is implicitly assumed that the immersion has a flat normal bundle. The notion of classically isothermic immersions is invariant under Möbius transformations, because curvature lines are a Möbius invariant notion.

The following lemma provides the link between our definition and the classical definition of isothermic immersions.

LEMMA 57. *Let $f\colon M \to \mathbb{H}$ be a conformal immersion. For conformal curvature line coordinates $z = x + iy$, the form*

$$\hat{\omega} = f_x^{-1}dx - f_y^{-1}dy \tag{64}$$

is closed and integrates to an immersed dual surface.

Locally, this defines a 1-1–correspondence between conformal curvature line coordinates[2] *and between immersed dual surfaces. In particular, an immersion $f\colon M \to \mathbb{H}$ is classically isothermic if and only if, away from umbilics, it (locally) admits an immersed dual surface.*

SKETCH OF A PROOF. We do only give the idea of a proof, the details of which can be found in [**15**].

The details of the proof that, for $z = x + iy$ conformal curvature line coordinates, the form (64) is closed and satisfies $df \wedge \hat{\omega} = \hat{\omega} \wedge df = 0$, can be found in [**15**, 5.2.1]. The proof that every immersed dual surface has a differential of the form (64) for conformal curvature line coordinates $z = x + iy$ is given in [**15**, 5.2.6 and 5.2.7].

It remains to be checked that the conformal curvature line coordinates are uniquely determined by the dual surface. To see this, let g be a dual surface whose differential is of the form (64) for conformal curvature line coordinates $z = x + iy$. The quadratic differential $\omega\delta$ (see Corollary 53) belonging to g via (61) is of the form $q\,dz^2$. By (63),

$$\omega(\tfrac{\partial}{\partial x})\delta(\tfrac{\partial}{\partial x})\,\psi = \psi\,\tfrac{\partial g}{\partial x}\tfrac{\partial f}{\partial x} = \psi\,\tfrac{\partial f}{\partial x}^{-1}\tfrac{\partial f}{\partial x} = \psi,$$

and $q = 1$. Hence, the quadratic differential is dz^2, and because only the coordinates $\pm z + a$ induce the same quadratic form, the conformal curvature line coordinates are uniquely determined by the dual surface (up to sign and translation). $\square$

[2]Here we identify conformal coordinates z and $\tilde{z}$ if they differ by a translation and / or sign only, i.e. $\tilde{z} = \pm z + a$ for $a \in \mathbb{C}$.

Together with Lemma 55, we obtain the following alternative characterization of classically isothermic immersions:

Corollary 58. *A conformal immersion $L \subset V$ into (V, ∇) is classically isothermic if and only if, at every non–umbilic point, there locally is a nowhere vanishing closed 1–form $\omega \in \Omega^1(\mathcal{R})$.*

Because conformal curvature line coordinates of isothermic surfaces (that are not totally umbilic) are unique up to scaling and rotation by $\pi/2$, Lemma 57 provides an alternative proof of Corollary 56 (and, equivalently, of Lemma 54).

In the proof of Lemma 57 we have also seen that:

Corollary 59. *Let $L \subset V$ be isothermic with non–vanishing $\omega \in \Omega^1(\mathcal{R})$. Then, the Null directions of the imaginary part of the holomorphic quadratic differential $\omega\delta$ are curvature line directions.*

Isothermic vs. Classically Isothermic. The relation between isothermic and classically isothermic is as follows: if the immersion is isothermic, then, away from the zeros of ω, Lemma 57 guarantees the local existence of conformal curvature line coordinates (even at umbilic points). Furthermore, the conformal curvature line coordinates have to patch together nicely in the sense that they are the Null directions of the imaginary part of the holomorphic quadratic differential $\omega\delta$ of Corollary 53.

On the torus, ω has no zeros (see Corollary 53). Hence, every isothermic immersion of a torus admits a global conformal vector field which is tangential to one of the curvature line directions. This yields:

Lemma 60. *A classically isothermic immersion of a torus is isothermic if and only if the conformal curvature line coordinates extend through the branch points and are tangential to a global conformal vector field.*

Remark. For a conformal immersion of a torus to be isothermic, it is not sufficient that there are local conformal curvature line coordinates at every point (including umbilics). The following example of an (in our sense) non–isothermic torus, which admits conformal curvature line coordinates at every point, was communicated to the author by Udo Hertrich–Jeromin: take three non–intersecting round spheres centered at the corners of a triangle and glue them, by using three pieces of round cylinders, such that the resulting torus locally is a smooth rotational surface.

For Riemann surfaces other than tori, the definition of isothermic given here is probably not the right globalization of classically isothermic. For example, in the case of conformal immersions of the sphere one should allow the quadratic differential $\omega\delta$ to be meromorphic ω. At present time, there seems to be no satisfactory global definition of isothermic immersions in the case of general Riemann surfaces.

9.4. Transformations of Isothermic Surfaces. In this section, the classical transformation theory of isothermic surfaces is described as a special case of the transformation theory for general surfaces. It turns out that isothermic surfaces can be characterized in terms of every transformation.

It is worthwhile noting that, in the development of the transformations for general surfaces, the classical transformation theory of isothermic surfaces—especially in its 'quaternionified' version—served as a prototype for large parts of the theory.

Associated Family. The following characterization of isothermic surfaces is a direct consequence of part ii) of Lemma 51.

LEMMA 61. *A holomorphic curve $L \subset V$ in (V, ∇) is isothermic if and only if there is a non–trivial 1–form $\omega \in \Omega^1(\operatorname{End} V)$ such that $(V, L, \nabla + \omega)$ is contained in the forward and backward associated family of $L \subset V$.*

Let L be isothermic with $\omega \in \Omega^1(\mathcal{R})$ closed. The family of immersions

$$(V, L, \nabla + \rho\omega)$$

with $\rho \in \mathbb{R}$ is called the *classical associated family of isothermic surfaces.* Its elements are again isothermic with the same ω. By part i) and ii) of Lemma 51, it forms a subset of the associated family as defined in Section 5. If L is not totally umbilic, Lemma 54 implies that the classical associated family actually is the intersection of the forward and backward associated family. The members of the classical associated family of isothermic surfaces are sometimes called Bianchi–, Calapso– or T–transformations of L, cf. [**15**].

2–Step Bäcklund Transformation. The characterization of isothermic tori without monodromy in terms of 2–step Bäcklund transformations is a direct consequence of our definition of isothermic surfaces and of the characterization of 2–step Bäcklund transformations in terms of 1–forms.

LEMMA 62. *Let $L \subset V$ be a conformal immersion of a torus into $\mathbb{P}V$ with V a rank 2 vector space. Then, L is isothermic if and only if it is a 2–step Bäcklund transformation of itself.*

PROOF. Assume L is isothermic with $\omega \in \Omega^1(\mathcal{R})$ closed. Because ω has no zeros (by Corollary 53), L is the kernel of the closed form ω with values in L. Hence, it is a 2–step Bäcklund transformation of itself.

Conversely, assume that L is a 2–step Bäcklund transformation of itself. Then, there is a closed form $\omega \in \Omega^1(\operatorname{End}(V))$ with values in L having L as its kernel. This shows that L is isothermic. □

This immediately implies:

COROLLARY 63. *Let $L \subset V$ be a conformal immersion of a torus into $\mathbb{P}V$ with V a rank 2 vector space. Then, L is isothermic if and only if the quaternionic holomorphic bundles V/L and KL are isomorphic.*

PROOF. L is a 2–step Bäcklund transformation of itself if and only if it can be obtained via Kodaira correspondence and dualization of a 2 dimensional linear system in KL without Weierstrass points. This proves the statement, because L can always be obtained via Kodaira correspondence and dualization of a 2 dimensional linear system in V/L. □

This corollary can be generalized to the case of conformal immersions of arbitrary Riemann surfaces.

LEMMA 64. *An immersed holomorphic curve L in (V, ∇) is isothermic if and only if there is a holomorphic bundle map $B\colon V/L \to KL$.*

PROOF. Assume L is isothermic. Then ω gives rise to a quaternionic linear map $B\colon V/L \to KL$ in the obvious way: $B\pi\psi := \omega\psi$ for $\psi \in \Gamma(V)$ (where $\pi\colon V \to V/L$ is the canonical projection). This map commutes with the respective J's. Using $d^\nabla\omega = 0$ and the definition of the holomorphic structure on KL we obtain that B is holomorphic, because

$$D(B\pi\psi) = d^\nabla(\underbrace{B\pi\psi}_{\omega\psi}) = \underbrace{d^\nabla\omega}_{0}\psi - \omega\wedge\nabla\psi = -B\wedge\pi\nabla\psi = -B\wedge D\pi\psi = B(D\pi\psi).$$

Conversely, given B, the form $\omega \in \Gamma(\mathcal{R})$ is defined by $\omega := B\pi$. The same calculation as above shows that $D(B\pi\psi) = B(D\pi\psi)$ for all $\psi \in \Gamma(V)$ implies $(d^\nabla\omega)\psi = 0$ for all $\psi \in \Gamma(V)$, i.e. ω is closed. □

1–Step Bäcklund Transformation, Dual Surfaces. In the case of tori, Corollary 63 implies:

LEMMA 65. *Let $L \subset V$ be a conformal immersion of a torus into $\mathbb{P}V$ with V a rank 2 vector space. Assume KL admits a nowhere vanishing section. Then L is isothermic if and only if one (and therefore every) immersed forward Bäcklund transformation with monodromy is at the same time a backward Bäcklund transformation.*

PROOF. Forward Bäcklund transformations have their Weierstrass representation in the bundles L^{-1} and KL and backward Bäcklund transformations have theirs in $KL^\perp$ and V/L. Hence, Lemma 16 implies that one (every) forward Bäcklund transformation is at the same time backward if and only if V/L is isomorphic to KL. □

The fact that on general Riemann surfaces a closed 1–form $\omega \in \Omega^1(\mathcal{R})$ can (sometimes must) have zeros seems to be an obstruction to giving a rigorous generalization of this lemma to all Riemann surfaces. Using Lemma 64, one could at least formulate a local version with the additional assumption that the isothermic ω has no zeros.

It should be noticed that a dual surface is a special kind of forward and backward Bäcklund transformation. Let $L \subset V$ be an isothermic immersion into $\mathbb{P}V$ with V a rank 2 vector space. Let v, w be a basis of V and denote by α, β its dual basis. Then g is a dual surface with respect to the Euclidean chart given by v, w if and only if, with respect to the pairing between L^{-1} and KL, we have

$$dg = (\beta, \omega)$$

for $\omega \in H^0(KL)$ and, with respect to the pairing between $KL^\perp$ and V/L, we have

$$dg = (\tilde{\omega}, v)$$

with $\tilde{\omega} \in H^0(KL^\perp)$. This follows from the fact that, if f is the Euclidean realization of L with respect to the chart given by v, w, the first equation is equivalent to $df \wedge dg = 0$ and the second is equivalent to $dg \wedge df = 0$, which can be derived similarly as (38).

The Classical Darboux Transformation of Isothermic Surfaces. The aim of this paragraph is to discuss the relation between the classical Darboux transformation of isothermic surfaces and the new Darboux transformation of Section 7. As with the other transformations, isothermic surfaces can be characterized in terms of Darboux transformations as the case of immersions admitting a Darboux transformation which is at the same time forward and backward.

DEFINITION. Let L be an isothermic immersion into (V, ∇) with $\omega \in \Gamma(\mathcal{R})$ a non–trivial closed form. A bundle $L^\# \subset V$ is called a *classical isothermic Darboux transformation* of L if it is parallel with respect to $\nabla + \rho\omega$ for $\rho \in \mathbb{R}$.

Obviously, classical isothermic Darboux transformations are generalized Darboux transformations in the new sense: if $\psi \in \Gamma(\tilde{L}^\#)$ is a $\nabla + \rho\omega$–parallel section, then $\nabla\psi = -\rho\omega\psi \in \Omega^1(\tilde{L})$.

REMARK. On a torus, because all holonomies commute, there is at least one parallel subbundle of $\nabla + \rho\omega$ for every ρ. This subbundle defines a global generalized Darboux transformation. If $\rho \neq 0$ and the subbundle is a proper Darboux transformation (i.e. never intersects L), then it is immersed, because on the torus ω has no zeros (cf. Corollary 53).

To simplify the statement of the next lemma, we need the definition of a Darboux pair.

DEFINITION. Let (V, ∇) be a flat quaternionic rank two bundle and let L, $L^\# \subset V$ be two immersed holomorphic curves with $V = L \oplus L^\#$. Then L and $L^\#$ are called a *Darboux pair* if $L^\#$ is at the same time a forward and backward Darboux transformations of L.

Otherwise said, L and $L^\#$ form a Darboux pair if and only if $L^\#$ is a Darboux transformation of L and vice versa. By Lemma 34, L and $L^\#$ form a Darboux pair if they are conformally equivalent and if there is a 2–sphere congruence touching both immersion. (This last characterization of Darboux pairs is the classical definition.)

LEMMA 66. *Let (V, ∇) be a flat quaternionic rank two bundle.*

a) If L, $L^\# \subset V$ form a Darboux pair, then both immersions are isothermic. Furthermore, they are classical isothermic Darboux transformations of each other. The closed forms $\omega \in \Omega^1(\mathcal{R})$ and $\omega^\# \in \Omega^1(\mathcal{R}^\#)$ are nowhere vanishing.

b) If $L \subset V$ is an isothermic immersion and $V = L \oplus L^\#$ a splitting with $L^\#$ an immersed classical isothermic Darboux transformation of L, then L and $L^\#$ form a Darboux pair. In particular, $L^\#$ is isothermic.

PROOF. a) With respect to the splitting $V = L \oplus L^\#$ we have

$$\nabla = \begin{pmatrix} \nabla^L & \delta^\# \\ \delta & \nabla^\# \end{pmatrix}.$$

The condition that L and $L^{\#}$ are Darboux transformations of each other is $\delta \wedge \delta^{\#} = \delta^{\#} \wedge \delta = 0$. The form

$$\omega = \begin{pmatrix} 0 & \delta^{\#} \\ 0 & 0 \end{pmatrix}$$

is in $\omega \in \Omega^1(\mathcal{R})$ and, by equation (39), its derivative is

$$d^{\nabla}\omega = \begin{pmatrix} \delta^{\#} \wedge \delta & d^{\nabla^L, \nabla^{\#}} \delta^{\#} \\ 0 & \delta \wedge \delta^{\#} \end{pmatrix} = 0.$$

This shows that L is isothermic. By definition of ω, it is clear that the bundle $L^{\#}$ is parallel with respect to the connection $\nabla - \omega$. This shows that $L^{\#}$ is a classical isothermic Darboux transformation of L. The analogous construction shows that $L^{\#}$ is isothermic. ω (and analogously $\omega^{\#}$) is nowhere vanishing, because $L^{\#}$ is immersed.

b) Since $L^{\#}$ is a Darboux transformation (in the new sense) of L, we only have to prove that L is a Darboux transformation of $L^{\#}$. Let $\omega \in \Omega^1(\mathcal{R})$ be a 1–from satisfying $d^{\nabla}\omega = 0$. With respect to the splitting $V = L \oplus L^{\#}$ we have

$$\nabla = \begin{pmatrix} \nabla^L & \delta^{\#} \\ \delta & \nabla^{\#} \end{pmatrix} \qquad \text{and} \qquad \omega = \begin{pmatrix} 0 & \hat{\omega} \\ 0 & 0 \end{pmatrix},$$

and the fact that $L^{\#}$ is $\nabla + \rho\omega$–parallel implies $\delta^{\#} + \rho\hat{\omega} = 0$. Because $\hat{\omega}$ satisfies $\delta \wedge \hat{\omega} = \hat{\omega} \wedge \delta = 0$ we obtain that $\delta^{\#} \wedge \delta = 0$ and L is a Darboux transformation of $L^{\#}$. Hence, L and $L^{\#}$ form a Darboux pair. Part a) shows that $L^{\#}$ is isothermic. □

Remark. It might be of interest that for tori (using Corollary 53) the proof of part a) does as well show the following: if $L \subset V$ is an immersed holomorphic curve in (V, ∇) and if there is a splitting $V = L \oplus L^{\#}$ with $\delta \wedge \delta^{\#} = \delta^{\#} \wedge \delta = 0$, then either $L^{\#}$ is ∇–parallel or it is immersed and L and $L^{\#}$ form a Darboux pair.

The preceding lemma immediately yields a characterization of classically isothermic surfaces.

Corollary 67. *Let L be a conformal immersion into (V, ∇). Then L is classically isothermic if and only if at every non–umbilic point there locally is a splitting $V = L \oplus L^{\#}$ with $L^{\#}$ an immersion that is at the same time a forward and backward Darboux transformation of L (i.e. L and $L^{\#}$ form a Darboux pair).*

Proof. Recall that L is classically isothermic if and only if away from umbilics there locally is a nowhere vanishing closed 1–form $\omega \in \Omega^1(\mathcal{R})$ (see Corollary 58).

One implication in the statement follows from part a) of Lemma 66.

To show that the other implication follows from part b), one only has to fix $\rho \in \mathbb{R}^*$ and to choose a local $\nabla + \rho\omega$–parallel subbundle $L^{\#}$ of V that does not intersect L. Because ω in nowhere vanishing, $L^{\#}$ is immersed. □

Remark. It would be interesting to derive a global characterization of isothermic tori similar to the preceding corollary. The problem is to prove the existence of $L^\#$ for a given ω. In general, by taking a parallel subbundle with respect to $\nabla + \rho\omega$, which always exist, because on the torus all holonomies commute, one only obtains a generalized Darboux transformation. (For degree 0 tori, one expects that most points of the spectrum correspond to good points of the spectral curve. This would immediately imply that, for almost all ρ, one obtains a proper Darboux transformation.)

The next Lemma, which can also be found in [**15**], shows that every dual surface is a limit of Darboux transformations (for an analogous result about Bäcklund transformations of general immersions, cf. Section 7.3).

LEMMA 68. *Every isothermic dual surface can be obtained as a limit of classical isothermic Darboux transformations.*

PROOF. Let $L \subset V$ be an isothermic immersion (with V a rank 2 vector space). Let f be the Euclidean realization of $L \subset V$ with respect to the Euclidean chart defined by the basis e_1, e_2 of V, i.e., with respect to this basis, we have

$$L = \begin{pmatrix} f \\ 1 \end{pmatrix} \mathbb{H}.$$

We prove that every dual surface of f^{-1} can be obtained as a limit of Darboux transformations (we choose f^{-1} instead of f in order to be consistent with the discussion in Section 7.3). For an arbitrary dual surface μ of f^{-1}, a closed form $\omega \in \Omega^1(\mathcal{R})$ can be defined by

$$\omega = \begin{pmatrix} 1 & 0 \\ f^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & d\mu \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -f^{-1} & 1 \end{pmatrix}.$$

Let F_ρ be a solution to the equation

$$dF_\rho = -\rho\omega F_\rho \tag{65}$$

normalized by $F_\rho(p_0) = \mathrm{Id}$ for a fixed point $p_0 \in M$. Then, for every $v \in V$, $L^\#_\rho = F_\rho v\mathbb{H}$ is a family of classical Darboux transformations with $L^\#_0 = [v]$ constant. The differential of (65) with respect to ρ at $\rho = 0$ is

$$d\dot{F}v = -\omega v. \tag{66}$$

We consider now $L^\#_\rho = F_\rho e_2 \mathbb{H}$ and write

$$L^\#_p = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \mathbb{H} = \begin{pmatrix} g \\ 1 \end{pmatrix} \mathbb{H} \qquad \text{where} \qquad \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = F_\rho e_2.$$

Then $dg = dg_1\, g_2^{-1} - g_1 g_2^{-1}\, dg_2\, g_2^{-1}$ and, because for $\rho = 0$ we have $g_1 = 0$ and $g_2 = 1$, the differential of dg with respect to ρ at $\rho = 0$ is

$$d\dot{g} = d\dot{g}_1.$$

By equation (66), we have $d\dot{g}_1 = -d\mu$ and the limit $\dot{g}$ of Darboux transformations is the dual surface $-\mu$ of f^{-1}. $\square$

The New Darboux Transformation Applied to Isothermic Immersion. In the preceding section we have seen that an isothermic immersion L into (V, ∇) admits an associated family of flat connections $\nabla + \rho\omega$ with a real parameter $\rho \in \mathbb{R}$. The classical Darboux transformations arise as parallel subbundles of these connections.

Replacing the real parameter by a complex parameter $\rho = a + bT$, with T the complex linear endomorphism of $(V, \mathbf{i})$ defined by $T\psi = \psi\mathbf{i}$, gives rise to a complex 1–parameter family

$$\nabla^\rho = \nabla + \rho\omega \tag{67}$$

of flat complex connections (the connections for $\rho \in \mathbb{C}\backslash\mathbb{R}$ are not quaternionic linear, because T is not quaternionic linear). Every (local) ∇^ρ–parallel section $\psi \in \Gamma(V)$ not intersecting L gives rise to a Darboux transformation, because

$$\nabla\psi = -\rho\omega\psi = -\omega\psi(a + b\mathbf{i}) \in \Omega^1(L). \tag{68}$$

The following Lemma shows that, except for bad points of the spectral curve, all Darboux transformations of isothermic surfaces are obtained from parallel subbundles of ∇^ρ for some complex parameter ρ.

Lemma 69. *Let L be an isothermic immersion of a torus into (V, ∇) with $\omega \in \Gamma(\mathcal{R})$ a non–trivial closed form. If $L^\#$ is a Darboux transformation of L corresponding to a good point of the spectral curve, the $L^\#$ is the quaternionification of a parallel subbundle of $\nabla + \rho\omega$ for some complex parameter $\rho \in \mathbb{C}$.*

Proof. With respect to the splitting $V = L \oplus L^\#$, the form ω can be written as

$$\omega = \begin{pmatrix} 0 & \hat{\omega} \\ 0 & 0 \end{pmatrix}.$$

Closedness of ω is equivalent to $\delta \wedge \hat{\omega} = \hat{\omega} \wedge \delta = 0$ and $d^{\nabla^L, \nabla^\#}\hat{\omega} = 0$. Since ω is nowhere vanishing, $\delta^\#$ can be written as

$$\delta^\# = \hat{\omega}B$$

with $B \in \Gamma(\mathrm{End}(L^\#))$. Now, $d^{\nabla^L, \nabla^\#}\hat{\omega} = d^{\nabla^L, \nabla^\#}\delta^\# = 0$ implies

$$*\nabla^\# B = \tilde{J}\nabla^\# B.$$

Since $L^\#$ corresponds to a good point $(\nabla^\#, S) \in \Sigma$ of the spectral curve, the endomorphism B has to be parallel. Therefore, $B = a + bS$ for a, $b \in \mathbb{R}$. Now take a parallel section (with monodromy) $\psi \in \Gamma(\tilde{L}^\#)$ such that $S\psi = \psi\mathbf{i}$. Then

$$\nabla\psi = \delta^\#\psi = \omega B\psi = \omega\psi(a + b\mathbf{i}) = \underbrace{(a + bT)}_{=:-\rho}\omega\psi.$$

This proves the claim, since ψ defines a $\nabla + \rho\omega$–parallel subbundle. $\square$

Remark. The Darboux transformations of the preceding lemma are immersed, because on a torus, ω has no zeros (see Corollary 53).

The following lemma shows that all Darboux transformations of an isothermic immersion L obtained from $\nabla + \rho\omega$–parallel subbundles for complex ρ are again isothermic. In particular, all Darboux transformations of a torus corresponding to good points of the spectral curve are again isothermic.

LEMMA 70. *Let L be an isothermic immersion into (V, ∇) with $\omega \in \Gamma(\mathcal{R})$ a non–trivial closed form. Let $V = L \oplus L^{\#}$ be a splitting with $L^{\#}$ a Darboux transformation of L obtained from a $\nabla + \rho\omega$–parallel subbundle for $\rho \in \mathbb{C}$. Then $L^{\#}$ is isothermic.*

PROOF. We have to find a closed form

$$\omega^{\#} = \begin{pmatrix} 0 & 0 \\ \hat{\omega}^{\#} & 0 \end{pmatrix}.$$

Closedness of $\omega^{\#}$ is equivalent to $d^{\nabla^{\#}, \nabla^{L}} \hat{\omega}^{\#} = 0$ and $\hat{\omega}^{\#} \wedge \delta^{\#} = \delta^{\#} \wedge \hat{\omega}^{\#} = 0$. Writing

$$\hat{\omega}^{\#} = B\delta$$

for some endomorphism $B \in \Gamma(\mathrm{End}(L^{\#}))$, the closedness condition for $\omega^{\#}$ becomes equivalent to $*\nabla^{\#} B = (\nabla^{\#} B)\tilde{J}$ and $B\tilde{J} = J^{\#} B$. We claim that such B can be found.

To prove the claim, take $\psi \in \Gamma(\tilde{L}^{\#})$ a parallel section with monodromy of ∇^{ρ}. For $\rho = a + bT$ we obtain

$$\nabla\psi = -(a + bT)\omega\psi = \delta^{\#}\psi$$

and defining a quaternionic linear endomorphism $S \in \Gamma(\mathrm{End}(L^{\#}))$ by $S\psi = \psi\mathbf{i}$ we obtain

$$-\omega(a + bS) = \delta^{\#}.$$

By applying $*$ we get $-\omega\tilde{J}(a + bS) = \delta^{\#} J^{\#} = -\omega(a + bS)J^{\#}$. This implies $\tilde{J}(a + bS) = (a + bS)J^{\#}$ and setting $B = (a + bS)^{-1}$ proves the claim, since S is parallel. □

Spectral Curve of Isothermic Surfaces. In the case of isothermic immersions, with their 1–parameter family of flat connections, one can apply many techniques from integrable systems theory. It is for example fairly standard in this setting (see e.g. [**7**], [**8**] or [**24**]) to define a spectral curve by the following procedure.

Let $L \subset \mathbb{H}^2$ be an isothermic immersion of a torus. For every point p of the torus, the holonomy of ∇^{ρ} depends holomorphically on ρ. Generically, the holonomy has four distinct eigenlines, and one expects that, if L is a torus of normal bundle degree 0, there is a unique Riemann surface Σ' which is a four–fold branched covering of the complex plane and admits a holomorphic map

$$\mathcal{E}' \colon \Sigma' \to \mathbb{CP}^3$$

parameterizing the eigenlines of the holonomies of ∇^{ρ}. Because over the torus all holonomies commute, this construction does not depend on the choice of the point p on the torus.

The points of Σ' correspond to ∇^{ρ}–parallel complex subbundles of $(\mathbb{H}^2, \mathbf{i})$ whose quaternionifications are generalized Darboux transformations of L (as can be seen using (68)). Its parallel sections have a unique complex multiplier, which defines a map from Σ' to $\mathrm{Spec}(\mathbb{H}^2/L)$.

For the elements of an open subset $\mathring{\Sigma}'$ of Σ', the quaternionic line subbundle thus obtained does not intersect L, and therefore is an immersed Darboux transformation $L^\#$ (in the proper sense) of L. Because the bundle $L^\#$ is equipped with a distinguished complex structure S inherited from the complex line bundle, this defines a map from $\mathring{\Sigma}'$ to the spectral curve Σ of $\mathbb{H}^2/L$ as defined in Section 8.2. Under this map, $\mathcal{E}'$ corresponds to the eigenspace curve $\mathcal{E}$ described in Section 8.4.

Remark. One should expect that Σ' and Σ do essentially coincide and that the above map extends to an analytic isomorphism defined away from some singular parts of both sets.

10. Constrained Willmore Surfaces

A conformal immersion L into $\mathbb{HP}^1$ is called Willmore if it is a critical point of the Willmore functional under all variations of L. It is called constrained Willmore if it is a critical point of the Willmore functional under those variations of L that preserve the conformal structure of the underlying Riemann surface. For a survey of Willmore surfaces see e.g. [**29**]. Constrained Willmore surfaces are scarcely mentioned in the literature ([**4**] is one of a few references).

The aim of this section is to show that the associated family, spectral curve and Darboux transformation theory of constrained Willmore immersions is a special case of the general approach of Chapter II. (It is obvious from the definitions that the same holds for the Bäcklund transformation theory, which is described[3] in detail in [**2**] and therefore omitted here, except for a short remark at the end of Section 10.1.)

10.1. Definition. The Euler–Lagrange equation in the case of Willmore immersions is derived in [**2**, Section 6]. Due to analytic difficulties, it is much more subtle to give a rigorous derivation of the Euler–Lagrange equation for constrained Willmore surfaces. Instead, we use the supposed Euler–Lagrange equation as the definition of constrained Willmore immersions. It can then be easily verified (by techniques analogous to those of Section 6 in [**2**]) that a constrained Willmore immersion according to our definition is actually a critical point of the Willmore functional under all conformal variations. At least for tori it is possible to prove the converse, i.e. every critical point of the Willmore functional under conformal variations satisfies our equation (a reference to a proof of this fact is given in the last paragraph of Section 3.1 in Appendix A).

Definition. A conformal immersion $L \subset V$ into (V, ∇) is called *constrained Willmore* if and only if there is $\eta \in \Omega^1(\mathcal{R}_+)$ such that

$$d^\nabla(2 * A + \eta) = 0. \tag{69}$$

It is called *Willmore* if $d^\nabla * A = 0$.

[3]Constrained Willmore surfaces are not explicitly treated in [**2**], but the essential definitions of the Bäcklund transformation theory of constrained Willmore immersions can be obtained from those in the Willmore case by simply replacing the Hopf fields A and Q with the forms $\tilde{A}$ and $\tilde{Q}$ of Section 10.2.

REMARK. i) Equation (69) is equivalent to

$$d^\nabla(2 * Q + \eta) = 0, \tag{70}$$

because $d^\nabla * Q = d^\nabla * A$ (following from (4) by flatness of ∇).

ii) If a conformal immersion $L \subset V$ is isothermic and constrained Willmore with $\omega \in \Omega^1(\mathcal{R})$ closed, all forms $\eta + \rho\omega \in \Omega^1(\mathcal{R}_+)$ with $\rho \in \mathbb{R}$ do as well satisfy equation (69). For non–isothermic constrained Willmore surfaces, the form η is unique.

iii) Lemma 13 and Lemma 14 together imply

$$\eta \in \Gamma(K\mathcal{R}_+).$$

iv) The quadratic differential $\eta\delta \in \Gamma(K^2)$ is holomorphic. This follows from Lemma 52, because

$$0 = d^\nabla(2 * A + \eta) = \underbrace{d^{\hat\nabla}\eta}_{+} + \underbrace{2Sd^{\hat\nabla}A + A \wedge \eta + \eta \wedge Q}_{-}$$

and therefore $d^{\hat\nabla}\eta = 0$.

For tori, part iv) of the remark implies that $\eta\delta$ is constant[4]. In the case of the sphere, it implies that η has to vanish identically, i.e. the notions of Willmore and constrained Willmore immersions coincide (which is not surprising, because there is only one conformal structure on the sphere).

REMARK. For constrained Willmore immersions, by Lemma 25, the closed 1–form

$$\omega = 2 * A + \eta,$$

yields a 2–dimensional linear system in KL and therefore can be used to single out special 1–step and 2–step Bäcklund transformations. The Bäcklund transformations of [**2**] are defined using this form ω, for example, the 2–step Bäcklund transformation $\ker\omega$ is the one defined there. (Similarly, the closed form $2 * Q + \eta$ can be used to compute backward Bäcklund transformations.)

10.2. Associated Family and Spectral Curve. As in the case of isothermic surfaces, there is an associated family of constrained Willmore surfaces which is given by an explicit algebraic expression depending on a real parameter. This family is part of the general associated family, i.e. it consists of flat adapted connection on the 1–jet bundle V of V/L. All immersion in this family are again constrained Willmore. Furthermore, the real parameter can be complexified, which yields an alternative description of the spectral curve of constrained Willmore tori.

Let $L \subset V$ be a constrained Willmore immersion into (V, ∇). The *constrained Willmore associated family* of (V, L, ∇) is the family (V, L, ∇^λ) with

[4] In the lightcone picture, this fact substantially simplifies the equations defining constrained Willmore surfaces. It is mainly this simplification which allows to give a simple proof of the fact that, for tori, the defining equation for constrained Willmore surfaces is indeed the Euler–Lagrange equation of the constrained variational problem, see the last paragraph of Section 3.1 in Appendix A for a reference.

$$\nabla^\lambda = \nabla + (\lambda - 1)\tilde{A} \tag{71}$$

for $\lambda = a + bS$ with a, $b \in \mathbb{R}$ and $a^2 + b^2 = 1$. The form $\tilde{A}$ is defined by $\tilde{A} = A - \frac{1}{2} * \eta$ where $\eta \in \Omega^1(\mathcal{R})$ is a 1–form with $d^\nabla(2 * A + \eta) = 0$. The forms $\tilde{A}$ and $\tilde{Q} = Q - \frac{1}{2} * \eta$ satisfy $*\tilde{A} = S\tilde{A}$ and $*\tilde{Q} = \tilde{Q}S$. Furthermore, $\nabla S = 2 * \tilde{Q} - 2 * \tilde{A}$.

The family ∇^λ with $\lambda \in S^1$ is contained in the general associated family, i.e. it is a flat adapted connection on the 1–jet bundle V of V/L. Since $\pi\tilde{A} = 0$, it suffices to prove that

$$d^\nabla(\lambda - 1)\tilde{A} + (\lambda - 1)\tilde{A} \wedge (\lambda - 1)\tilde{A} = 0.$$

From $d^\nabla * \tilde{A} = 0$ and $\tilde{A} = -S * \tilde{A}$ we obtain $d^\nabla \tilde{A} = 2 * \tilde{A} \wedge * \tilde{A} = 2\tilde{A} \wedge \tilde{A}$. Hence

$$\begin{aligned} d^\nabla(\lambda - 1)\tilde{A} &+ (\lambda - 1)\tilde{A} \wedge (\lambda - 1)\tilde{A} = \\ &= 2(\lambda - 1)\tilde{A} \wedge \tilde{A} - 2b * \tilde{A} \wedge \tilde{A} + (\lambda - 1)\tilde{A} \wedge (\lambda - 1)\tilde{A} \\ &= 2(a - 1)\tilde{A} \wedge \tilde{A} + (\lambda - 1)(A - \tfrac{1}{2} * \eta) \wedge (\lambda - 1)(A - \tfrac{1}{2} * \eta) \end{aligned}$$

(which, using the fact that the first η vanishes by type, becomes)

$$\begin{aligned} &= 2(a - 1)\tilde{A} \wedge \tilde{A} + (\lambda - 1)(\bar{\lambda} - 1)(A - \tfrac{1}{2} * \eta) \wedge (A - \tfrac{1}{2} * \eta) \\ &= 2(a - 1)\tilde{A} \wedge \tilde{A} + (|\lambda|^2 - 2a + 1)\tilde{A} \wedge \tilde{A} \\ &= (|\lambda|^2 - 1)\tilde{A} \wedge \tilde{A} = 0 \end{aligned}$$

and ∇^λ indeed is a family of flat connections for all $\lambda \in S^1$. A similar calculation shows that, for every $\lambda \in S^1$,

$$d^{\nabla^\lambda}(2 * A^\lambda + \lambda\eta) = 0$$

(where $A^\lambda = \lambda A$ denotes the Hopf field of ∇^λ), i.e. all immersions in the constrained Willmore associated family are constrained Willmore.

Remark. Obviously, the family of flat connection

$$\nabla^\lambda = \nabla + \tilde{Q}(\lambda - 1) \tag{72}$$

is contained in the backward associated family.

As in the case of Willmore connections (cf. equation (109) of [**8**]), the parameter λ can be complexified. This yields the family of $\mathbf{i}$–complex connections

$$\nabla^\mu = \nabla + \tfrac{\mu-1}{2}(1 - TS)\tilde{A} + \tfrac{\mu^{-1}-1}{2}(1 + TS)\tilde{A} \tag{73}$$

on $(V, \mathbf{i})$ with $\mu = a + bT \in \mathbb{C}^*$ (where as usual, T denotes the complex endomorphisms given by $T\psi = \psi\mathbf{i}$). For $\mu = a + Tb \in S^1$, we have $\nabla^\mu = \nabla^\lambda$ with $\lambda = a + bS$. All of the complex connections ∇^μ are flat, because their curvature depends holomorphically on μ and vanishes for $\mu \in S^1$.

Every (local) ∇^μ–parallel section $\psi \in \Gamma(V)$ yields a Darboux transformation of L, because

$$\nabla\psi = -(\tfrac{\mu-1}{2}(1 - TS)A + \tfrac{\mu^{-1}-1}{2}(1 + TS)A)\psi \in \Omega^1(L).$$

For tori of degree 0, as explained in the last paragraph of Section 9, one expects that the holonomy of the family of flat connections ∇^μ can be used to give an alternative description of the spectral curve and the eigenline curve of a constrained Willmore immersion. There is a certain hope that one can prove (e.g. by similar methods as used in [**8**] for the case of constant mean curvature surfaces) that constrained Willmore surfaces of degree 0 are finite type, i.e. have a compact normalization of the spectral curve (cf. Section 16.1).

10.3. Darboux Transformations. The aim of this section is to prove that, for immersions of tori, essentially all Darboux transformations of constrained Willmore surfaces are constrained Willmore, and that all these Darboux transformation corresponding to good points of the spectral curve can be obtained from parallel subbundles with respect to a connection ∇^μ for $\mu \in \mathbb{C}^*$.

Unfortunately, the following theorem is not exactly the result we would like to have, because it states that, if the Darboux transformation $L^\#$ corresponding to a regular point is constrained Willmore, then L as well is constrained Willmore. We were rather expecting a result stating that if L is constrained Willmore, then $L^\#$ is as well constrained Willmore.

THEOREM 6. *Let L, $L^\#$ be conformal immersions with normal bundle degree 0 of a torus into (V,∇) and assume $L^\#$ is a Darboux transformation of L. If $L^\#$ is (constrained) Willmore and corresponds to a regular point of the spectral curve of V/L, then L as well is (constrained) Willmore.*

PROOF. The crucial point of the proof is that, by Lemma 40, the forms W and $W^\#$ of (46) and (47) can be written as $W = \delta^\# B$ and $W^\# = B\delta$ for the same endomorphism B of $L^\#$.

The immersion L is constrained Willmore if and only if there is a 1–form $\eta \in \Omega^1(\mathcal{R}_+)$ such that $d^\nabla(2 * A + \eta) = 0$. As usual we write

$$\nabla = \begin{pmatrix} \nabla^L & \delta^\# \\ \delta & \nabla^\# \end{pmatrix}$$

with respect to the splitting $V = L \oplus L^\#$. Then, $\eta \in \Omega^1(\mathcal{R}_+)$ and $d^\nabla \eta$ take the form

$$\eta = \begin{pmatrix} 0 & \hat\eta \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad d^\nabla \eta = \begin{pmatrix} \hat\eta \wedge \delta & d^{\nabla^L,\nabla^\#}\hat\eta \\ 0 & \delta \wedge \hat\eta \end{pmatrix}.$$

With (42) and (46) we obtain that L is constrained Willmore if only if $\hat\eta = \delta^\# C$ for an endomorphism C of $L^\#$ satisfying $J^\# C = C\tilde J$ and $d^{\nabla^L,\nabla^\#}(\delta^\#(B+C)) = 0$ or, equivalently,

$$*\nabla^\#(B + C) = J^\#\nabla^\#(B + C). \tag{74}$$

The immersion $L^\#$ is constrained Willmore if and only if there is a 1–form $\eta^\# \in \Omega^1(\mathcal{R}^\#{}_+)$ such that $d^\nabla(2 * Q^\# + \eta^\#) = 0$. Then, $\eta^\# \in \Omega^1(\mathcal{R}^\#{}_+)$ and $d^\nabla\eta$ take the form

$$\eta = \begin{pmatrix} 0 & 0 \\ \hat\eta^\# & 0 \end{pmatrix} \qquad \text{and} \qquad d^\nabla \eta = \begin{pmatrix} \delta \wedge \hat\eta^\# & 0 \\ d^{\nabla^\#,\nabla^L}\hat\eta^\# & \hat\eta^\# \wedge \delta^\# \end{pmatrix}.$$

With (45) and (47) we obtain that $L^\#$ is constrained Willmore if only if $\hat\eta = D\delta$ for an endomorphism D of $L^\#$ satisfying $J^\# D = D\tilde J$ and $d^{\nabla^\#,\nabla^L}((B+D)\delta) = 0$ or, equivalently,

$$*\nabla^\#(B+D) = \nabla^\#(B+D)\tilde J. \tag{75}$$

Now, if $L^\#$ corresponds to a regular point of the spectral curve of V/L, by Theorem 4, every solution $B+D$ to (75) has to be parallel and therefore does as well solve (74). This proves the statement. ☐

The following Lemma states that, if we are in the situation of the preceding theorem, then $L^\#$ can be obtained from a parallel subbundle ∇^μ for $\mu \in \mathbb{C}^*$.

Lemma 71. *Let L, $L^\#$ be conformal immersions with normal bundle degree 0 of a torus into (V,∇) and assume $L^\#$ is a Darboux transformation of L that is constrained Willmore and corresponds to a regular point of the spectral curve of V/L. Then, $L^\#$ is obtained by quaternionification of a parallel line subbundle of ∇^μ for $\mu\in\mathbb{C}^*$.*

Proof. Let $\psi \in \Gamma(L^\#)$ be a $\nabla^\#$–parallel section, i.e.

$$\nabla\psi = \delta^\#\psi.$$

We can assume that ψ has a complex multiplier.

The first step is to calculate the form $\tilde A$. It is given by

$$2*\tilde A\psi = (W + *\delta^\# + \hat\eta)\psi.$$

Here, W is the 1–form from (46) which, by Lemma 40, can be written as $W = \delta^\# B$, and $\hat\eta$ is the 1–form from the proof of Theorem 6 which can be written as $\hat\eta = \delta^\# C$ for C satisfying $\nabla^\#(B+C) = 0$.

With $\tilde B = B + C$, we obtain

$$\tilde A\psi = \tfrac12\delta^\#(\mathrm{Id} - J^\#\tilde B)\psi.$$

Now

$$\begin{aligned}
\nabla^\mu\psi &= \delta^\#\psi + \left(\tfrac{\mathrm{Id}-TS}{2}(\mu-1) + \tfrac{\mathrm{Id}+TS}{2}(\mu^{-1}-1)\right)\tfrac12\delta^\#(\mathrm{Id}-J^\#\tilde B)\psi\\
&= \delta^\#\left(\tfrac{\mathrm{Id}-TJ^\#}{2}\left(\psi + \tfrac{\mu-1}{2}(\mathrm{Id}-J^\#\tilde B)\psi\right) + \tfrac{\mathrm{Id}+TJ^\#}{2}\left(\psi + \tfrac{\mu^{-1}-1}{2}(\mathrm{Id}-J^\#\tilde B)\psi\right)\right)\\
&= \delta^\#\left(\tfrac{\mathrm{Id}-TJ^\#}{2}\left(\tfrac{\mu+1}{2}\psi - \tfrac{\mu-1}{2}J^\#\tilde B\psi\right) + \tfrac{\mathrm{Id}+TJ^\#}{2}\left(\tfrac{\mu^{-1}+1}{2}\psi - \tfrac{\mu^{-1}-1}{2}J^\#\tilde B\psi\right)\right)
\end{aligned}$$

and, because $\frac{\mathrm{Id}-TJ^\#}{2}$ and $\frac{\mathrm{Id}+TJ^\#}{2}$ project to different spaces, we obtain that $\nabla^\mu\psi = 0$ is equivalent to the pair of equations

$$\begin{aligned}
\tfrac{\mathrm{Id}-TJ^\#}{2}\left(\tfrac{\mu+1}{2}\psi - \tfrac{\mu-1}{2}J^\#\tilde B\psi\right) &= 0\\
\tfrac{\mathrm{Id}+TJ^\#}{2}\left(\tfrac{\mu^{-1}+1}{2}\psi - \tfrac{\mu^{-1}-1}{2}J^\#\tilde B\psi\right) &= 0.
\end{aligned}$$

These equations are equivalent to

$$\begin{aligned}
\tfrac{\mu+1}{2}\psi - \tfrac{\mu-1}{2}J^\#\tilde B\psi - \tfrac{\mu+1}{2}TJ^\#\psi - \tfrac{\mu-1}{2}T\tilde B\psi &= 0\\
\tfrac{\mu+1}{2}\psi + \tfrac{\mu-1}{2}J^\#\tilde B\psi + \tfrac{\mu+1}{2}TJ^\#\psi - \tfrac{\mu-1}{2}T\tilde B\psi &= 0
\end{aligned}$$

and, by taking the sum and difference of both equations, we obtain

$$\begin{aligned} \tfrac{\mu+1}{2}\psi &= \tfrac{\mu-1}{2}T\tilde{B}\psi \\ \tfrac{\mu-1}{2}\tilde{B}\psi &= -\tfrac{\mu+1}{2}T\psi. \end{aligned}$$

Now, obviously both equations are equivalent to

$$T\tilde{B}\psi = \tfrac{\mu+1}{\mu-1}\psi.$$

Because $\tilde{B}$ is parallel, the holonomy is not real, and ψ has a complex multiplier, the endomorphism $T\tilde{B}$ is given by $T\tilde{B}\psi = \tilde{\mu}\psi$ for $\tilde{\mu} \in \mathbb{C}$. The lemma is proven if there is $\mu \in \mathbb{C}^*$ with

$$\tilde{\mu} = \tfrac{\mu+1}{\mu-1}.$$

Such $\mu \in \mathbb{C}^*$ is given by

$$\mu = \tfrac{\tilde{\mu}+1}{\tilde{\mu}-1},$$

except for $\tilde{\mu} = \pm 1$. (What does this $\tilde{\mu} = \pm 1$ mean???) $\square$

CHAPTER IV

Projective Structures and the Fundamental Theorem

The purpose of this chapter is to derive a quaternionic version of the fundamental theorem of surface theory in the conformal 4–sphere. The fundamental theorem clarifies, which data is needed to determine an immersion into the conformal 4–sphere uniquely up to Möbius transformation (see also the historical remark below). The invariants thus obtained play an important role in the next chapter on flows.

The chapter starts, in Section 11, with a discussion of complex projective structures. Section 12, with its discussion of the fundamental theorem in the quaternionic setting, represents the core of the chapter. In Section 13, a new link between the quaternionic model and the lightcone model (see Appendix A) of the conformal 4–sphere is established and the invariants arising in the respective versions of the fundamental theorem are compared. Both versions of the fundamental theorem—as formulated here—use the concept of projective structures on Riemann surfaces.

A projective structure on a Riemann surface M consists of a vector bundle of rank n on M, a line subbundle of the vector bundle and a connection on the vector bundle satisfying an additional condition. Projectively, the vector bundle describes a family of projective spaces along M, the line subbundle marks a point in every projective space, and the connection defines a map from the tangent space of M to the tangent space of the projective space at the point described by the line bundle. The additional condition in the definition of projective structures is that this map has to be injective at every point and therefore identifies every tangent space of M with a subspace of the tangent space of the projective space at the corresponding point.

In the following sections, the underlying vector bundles of complex and quaternionic projective structures are assumed to be of rank 2, because this is the case needed for applications in 4–dimensional Möbius geometry. The real projective structures arising in Appendix A (in the form of Minkowski space bundles) are allowed to have arbitrary rank, the case corresponding to 4–dimensional Möbius geometry is that of rank 6 Minkowski space bundles.

The quaternionic projective structures introduced in Section 12 and the projective structures used in the lightcone version of the fundamental theorem (in Appendix A) are flat and therefore correspond to conformal immersions with monodromy. The discussion of complex projective structures in Section 11 does also include the non–flat case, which is essential for the application of complex projective structures in Section 12.

A Short Remark on the History of the Fundamental Theorem of Surface Theory in a Möbius Geometric Setting. In view of F. Klein's "Erlanger Programm", which proposes to consider geometry as the study of invariants of transformation groups, a very natural object of investigation in submanifold theory are the differential invariants of submanifolds with respect to the transformation group belonging to the geometry. The quest is to find a set of invariants that uniquely characterizes a submanifold up to elements of the transformation group. In Euclidean geometry, the first and second fundamental form provide such invariants: the classical Bonnet theorem, which we refer to as the fundamental theorem of submanifold theory in Euclidean space, states that a submanifold is uniquely determined up to Euclidean motion by its fundamental forms, which have to satisfy a compatibility condition called the Gauss–Codazzi–Ricci equation. Moreover, the fundamental theorem states that, conversely, every solution to the Gauss–Codazzi–Ricci equation (locally) defines a submanifold.

It is not surprising that—from the end of the 19^{th} century on—there have been diverse approaches to the problem of finding a complete set of Möbius invariants for surface theory and general submanifold theory (i.e. a Möbius geometric fundamental theorem). Early versions of the fundamental theorem were given by P. Calapso (based on work by A. Tresse from 1892) and R. Rothe. A version using the lightcone model has been derived by G. Thomson. Later on, the fundamental new methods introduced by E. Cartan have been applied to the problem by Cartan himself and, in the suite, by various other persons, for example by K. Yano and Y. Muto. (For a more detailed discussion of these early—as well as more recent—approaches to the problem and for references to the original literature, see [**5**] and [**15**].)

There are some interesting points about the development of the fundamental theorem in the Möbius setting: firstly, there has been surprisingly little interaction between the different actors in the game; many approaches seem to be quite unknown to other differential geometers. In particular, even after 100 years of efforts by many persons, a "final" (generally accepted) version of the fundamental theorem has not been established by the end of the 20^{th} century. Secondly, many approaches do not work in the general case, but only away from umbilic points (which is due to difficulties arising from the fact that, unlike Euclidean geometry, which is a first order geometry, conformal geometry is a second order geometry). Thirdly, many authors, like Thomson or Yano and Muto, take as a complete set of invariants a whole bunch of tensors (or forms). Their choice of invariants, which apparently is not optimal, seems to lack a simple geometric interpretation and furthermore has to satisfy complicated compatibility equations.

The quaternionic version of the fundamental theorem presented here is a generalization to surfaces in the 4–sphere of a (quaternionic) version of the fundamental theorem for surfaces in the 3–sphere derived by Pedit and Pinkall in 2000. The lightcone version from Appendix A is due to Burstall, Pedit and Pinkall (cf. [**4**]). Both versions, which are essentially equivalent for surfaces in the 4–sphere (as we are going to prove below), have the advantage that the invariants admit a direct geometrical interpretation and that the integrability equations are very similar to the usual compatibility equations arising in Euclidean geometry (see in particular the last remark of Section 2 in Appendix A). Furthermore, there is no problem at umbilic points. A disadvantage of the given fundamental theorems is that the invariants depend on a nowhere vanishing conformal vector field[1]. This is of course no problem for local considerations (or for global considerations, if the underlying Riemann surface is a torus), but in general this dependence on a conformal vector field is surely not completely satisfying.

A more conceptual and invariant approach to the fundamental theorem for general submanifolds is currently being worked out by Burstall and Calderbank (cf. [**5**]). Their approach works for all submanifolds (no conformal vector field is needed and there is no problem at umbilic points) of arbitrary dimension and codimension and their invariants admit a direct geometrical interpretation. The more conceptual approach results in a gain of geometric clarity (see for example their treatment of transformations of isothermic surfaces), but is technically less elementary than the "low tech" approach of [**4**]. For surfaces equipped with a nowhere vanishing conformal vector field, the invariants of [**5**] correspond directly to the ones defined in [**4**].

11. Complex Projective Structures

The purpose of introducing the concept of complex projective structures on Riemann surfaces is twofold. Firstly, the notion of complex projective structure is needed in the formulation of the quaternionic version of the fundamental theorem in Section 12. Secondly, the understanding of complex projective structures should simplify that of quaternionic projective structures.

The concept of complex projective structures is also of interest in itself. In particular flat complex projective structures frequently appear (under various names) in the literature on uniformization theory, Teichmüller theory and gauge theory over Riemann surfaces.

11.1. Definition and Classification.

Let M be a Riemann surface.

DEFINITION. A *complex projective structure* on M is a triple $\mathcal{P}_{\mathbb{C}} = (V, L, \nabla)$ where V is a complex rank 2 bundle, L is a line subbundle of V and ∇ is a connection on V such that the derivative $\delta = \pi\nabla_{|L} \in \Gamma(T^*M \otimes \operatorname{Hom}(L, V/L))$ (where $\pi\colon V \to V/L$ is the canonical projection) is pointwise a bijective map

$$\delta\colon TM \to \operatorname{Hom}(L, V/L)$$

that is compatible with the Riemann surface structure on M in the sense that

$$*\delta = \mathbf{i}\delta,$$

i.e. δ is a nowhere vanishing section of $K \operatorname{Hom}(L, V/L)$.

[1]The first quaternionic versions (Theorem 8) does of course not depend an the choice of a conformal vector field, but the second quaternionic version (Theorem 9), which is the one used later on, and the lightcone version (Theorem 15) do.

A complex projective structure $\mathcal{P}_{\mathbb{C}} = (V, L, \nabla)$ can be regarded as a Cartan connection modeled on $\mathbb{CP}^1$: the bundle V is—projectively—interpreted as a bundle of $\mathbb{CP}^1$'s with marked points given by the line bundle L. The connection ∇ defines the map $\delta\colon TM \to \operatorname{Hom}(L, V/L)$ which identifies the tangent spaces of the Riemann surface with those of the $\mathbb{CP}^1$'s, i.e. with $\operatorname{Hom}(L, V/L) \cong T_L\mathbb{P}(V)$, respecting the involved complex structures on both bundles.

Two complex projective structures $\mathcal{P}_{\mathbb{C}} = (V, L, \nabla)$ and $\tilde{\mathcal{P}_{\mathbb{C}}} = (\tilde{V}, \tilde{L}, \tilde{\nabla})$ are called *gauge equivalent*, denoted by $\mathcal{P}_{\mathbb{C}} \cong \tilde{\mathcal{P}_{\mathbb{C}}}$, if there is a bundle isomorphism $G\colon V \to \tilde{V}$ with $GL = \tilde{L}$ and $\nabla = G^{-1} \circ \tilde{\nabla} \circ G$. Two complex projective structures are called *equivalent*, denoted by $\mathcal{P}_{\mathbb{C}} \sim \tilde{\mathcal{P}_{\mathbb{C}}}$, if there is a complex line bundle E with connection ∇^E such that

$$(V \otimes E, L \otimes E, \nabla \otimes \operatorname{Id}_E + \operatorname{Id}_V \otimes \nabla^E) \cong (\tilde{V}, \tilde{L}, \tilde{\nabla}).$$

If L and $\tilde{L}$ are (topologically) isomorphic bundles (which is always the case for non–compact M and which is equivalent to $\deg L = \deg \tilde{L}$ for compact M), then, clearly, $(V, L, \nabla) \sim (\tilde{V}, \tilde{L}, \tilde{\nabla})$ holds if and only if there is a complex 1–form $\eta \in \Omega^1\mathbb{C}$ such that $(V, L, \nabla + \eta \operatorname{Id}) \cong (\tilde{V}, \tilde{L}, \tilde{\nabla})$.

Definition. A complex projective structure $\mathcal{P}_{\mathbb{C}} = (V, L, \nabla)$ is called
- i) *torsion free* if $RL \subset L$,
- ii) *normal* if $R \in \Omega^2(\mathbb{C}\operatorname{Id} \oplus \mathcal{R})$ and
- iii) *flat* if ∇ is projectively flat, i.e. $R \in \Omega^2(\mathbb{C}\operatorname{Id})$

with R the curvature of ∇ and $\mathcal{R} = \{A \in \operatorname{End}(V) \mid \operatorname{im} A \subset L \text{ and } L \subset \ker A\}$.

All three properties are well defined on equivalence classes of complex projective structures.

Moduli Spaces of Complex Projective Structures. Let (V, L, ∇) be a torsion free complex projective structure. Every $\omega \in \Omega^1\mathcal{R}$ defines a new torsion free complex projective structure by $(V, L, \nabla + \omega)$. It is indeed a complex projective structure, because $\pi(\nabla + \omega)_{|L} = \pi\nabla_{|L} = \delta$ is not changed. To prove that $(V, L, \nabla + \omega)$ is again torsion free, we chose a splitting $V = L \oplus \check{L}$. With respect to this splitting, ∇ and ω take the form

$$\nabla = \begin{pmatrix} \nabla^L & \check{\delta} \\ \delta & \check{\nabla} \end{pmatrix} \qquad \text{and} \qquad \omega = \begin{pmatrix} 0 & \check{\omega} \\ 0 & 0 \end{pmatrix}$$

and therefore, by $R^{\nabla+\omega} = R^{\nabla} + d^{\nabla}\omega + \omega \wedge \omega$, the curvature of $\nabla + \omega$ is

$$R^{\nabla+\omega} = R^{\nabla} + \begin{pmatrix} \check{\omega} \wedge \delta & d^{\nabla^L, \check{\nabla}}\check{\omega} \\ 0 & \delta \wedge \check{\omega} \end{pmatrix}, \tag{76}$$

which in particular shows that $(V, L, \nabla + \omega)$ really is torsion free. (This fact does also follow directly from $\pi R^{\nabla+\omega}{}_{|L} = \delta \wedge \omega_{|L} + \pi\omega \wedge \delta = 0$. We nevertheless give the splitting argument, because equation (76) is needed in the sequel.) The following theorem shows that this construction yields one representative in every equivalence class of complex projective structures.

Theorem 7. *Let $\mathcal{P}_{\mathbb{C}} = (V, L, \nabla)$ and $\tilde{\mathcal{P}_{\mathbb{C}}} = (\tilde{V}, \tilde{L}, \tilde{\nabla})$ be two torsion free complex projective structures. Then, there is a unique 1–form $\omega \in \Omega^1\mathcal{R}$, such that $(V, L, \nabla + \omega) \sim (\tilde{V}, \tilde{L}, \tilde{\nabla})$.*

PROOF. After tensoring $(\tilde{V}, \tilde{L}, \tilde{\nabla})$ by a complex line bundle E, we can assume that there is an isomorphism $g\colon L \to \tilde{L}$. In order to extend g to an isomorphism from V to $\tilde{V}$, we fix splittings $V = L \oplus L'$ and $\tilde{V} = \tilde{L} \oplus \tilde{L}'$. Since $\delta\colon L \to KV/L \cong KL'$ and $\tilde{\delta}\colon \tilde{L} \to K\tilde{V}/\tilde{L} \cong K\tilde{L}'$ are injective, the isomorphism g induces a unique isomorphism $g'\colon L' \to \tilde{L}'$ making the diagram

$$\begin{tikzcd} L \arrow[r, "g"] \arrow[d, "\delta"'] & \tilde{L} \arrow[d, "\tilde{\delta}"] \\ KL' \arrow[r, "g'"'] & K\tilde{L}' \end{tikzcd}$$

commutative. The isomorphisms g and g' together induce an isomorphism from $V = L \oplus L'$ to $\tilde{V} = \tilde{L} \oplus \tilde{L}'$. Using this isomorphism, we replace $(\tilde{V}, \tilde{L}, \tilde{\nabla})$ with a gauge equivalent complex projective structure $(V, L, \nabla + \omega)$, where the 1–form $\omega \in \Omega^1(\operatorname{End} V)$, by construction, satisfies $\omega L \subset L$.

Since both complex projective structures are torsion free, the equation $R^{\nabla+\omega} = R^\nabla + d^\nabla\omega + \omega \wedge \omega$ implies $\pi d^\nabla \omega_{|L} = 0$. With respect to the splitting, ω takes the form

$$\omega = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix},$$

and we obtain $0 = \pi d^\nabla \omega_{|L} = \delta \wedge \alpha + \gamma \wedge \delta$, which yields equality of the $\bar{K}$–parts of the forms α and $\gamma \in \Omega^1\mathbb{C}$.

If we apply a gauge transformation G of $V = L \oplus L'$ which has the form

$$G = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

the connection $\tilde{\nabla} = \nabla + \omega$ changes to

$$G^{-1} \circ \tilde{\nabla} \circ G = \tilde{\nabla} + \begin{pmatrix} -b\delta & * \\ 0 & \delta b \end{pmatrix}.$$

Therefore, for the right choice of b, both diagonal entries α and $\gamma \in \Omega^1\mathbb{C}$ of ω have the same K–parts and the initial complex projective structure $(\tilde{V}, \tilde{L}, \tilde{\nabla})$ is equivalent to $(V, L, \nabla + \omega)$ with a form $\omega \in \Omega^1(\mathbb{C}\operatorname{Id} \oplus \mathcal{R})$. Because adding a complex 1–form yields an equivalent complex projective structure, we can assume $\omega \in \Omega^1\mathcal{R}$. This completes the existence part of the proof.

To prove uniqueness of ω, we have to show that $(V, L, \nabla + \omega) \sim (V, L, \nabla)$ with $\omega \in \Omega^1\mathcal{R}$ implies $\omega = 0$. Firstly, $(V, L, \nabla + \omega) \sim (V, L, \nabla)$ is equivalent to $(V, L, \nabla + \omega + \eta \operatorname{Id}) \cong (V, L, \nabla)$ with $\eta \in \Omega^1\mathbb{C}$, i.e. for some gauge transformation G we have $\nabla G = G(\omega + \eta \operatorname{Id})$. With respect to a splitting $V = L \oplus \check{L}$ this reads

$$\begin{pmatrix} da - b\delta & \nabla b + \check{\delta}(c-a) \\ \delta(a-c) & \delta b + dc \end{pmatrix} = \begin{pmatrix} a\eta & a\check{\omega} + b\eta \\ 0 & c\eta \end{pmatrix}$$

where $G = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $\omega = \begin{pmatrix} 0 & \check{\omega} \\ 0 & 0 \end{pmatrix}$ and $\nabla = \begin{pmatrix} \nabla^L & \check{\delta} \\ \delta & \check{\nabla} \end{pmatrix}$. From the lower left corner we obtain $a = c$. The diagonal then implies $b = 0$ and from the upper right corner we get $\check{\omega} = 0$. $\square$

Theorem 7 shows that, if $\mathcal{P}_{\mathbb{C}} = (V, L, \nabla)$ is a torsion free complex projective structure, then $\nabla \mapsto \nabla + \omega$ with $\omega \in \Omega^1\mathcal{R}$ induces a simply transitive action of $\Omega^1\mathcal{R}$ on the equivalence classes of torsion free complex projective structures.

COROLLARY 72. *Let M be a Riemann surface with a torsion free complex projective structure $\mathcal{P}_{\mathbb{C}} = (V, L, \nabla)$. The set of equivalence classes of torsion free complex projective structures on M is an affine space with underlying vector space $\Omega^1\mathcal{R}$.*

Equation (76) shows that, for every torsion free complex projective structure (V, L, ∇), there is a unique $\eta \in \Gamma(\bar{K}\mathcal{R})$ such that $(V, L, \nabla + \eta)$ is normal. The projective structure $(V, L, \nabla + \eta)$ is then called the *normalization* of (V, L, ∇).

We will see in Corollary 77 that every Riemann surface M admits a flat (hence normal) complex projective structure. Given one normal complex projective structure (V, L, ∇), Theorem 7 together with equation (76) imply that every other normal complex projective structure on M is equivalent to $(V, L, \nabla + \omega)$ for a unique $\omega \in \Gamma(K\mathcal{R})$. Using the isomorphism $\Gamma(K\mathcal{R}) \to \Gamma(K^2)$ with $\omega \mapsto \omega\delta$ we obtain:

COROLLARY 73. *The set of equivalence classes of normal complex projective structures on a Riemann surface M is an affine space with vector space $\Gamma(K^2)$.*

Suppose (V, L, ∇) is flat, then again by Theorem 7 and equation (76) we see that every other flat complex projective structure is equivalent to $(V, L, \nabla + \omega)$ for a unique $\omega \in \Gamma(K\mathcal{R})$ with $d^{\nabla, \hat{\nabla}}\omega = 0$. As in the proof of Lemma 52 (which works for torsion free complex projective structures) one can show that this last condition is equivalent to $\omega\delta \in \Gamma(K^2)$ being holomorphic, i.e. $\omega\delta \in H^0(K^2)$.

COROLLARY 74. *The set of equivalence classes of flat complex projective structures on a Riemann surface M is an affine space with vector space $H^0(K^2)$.*

11.2. Flat Complex Projective Structures.

LEMMA 75. *Let $\mathcal{P}_{\mathbb{C}} = (V, L, \nabla)$ be a flat complex projective structure on a Riemann surface M. Then there is an equivalent complex projective structure $\tilde{\mathcal{P}_{\mathbb{C}}} = (\tilde{V}, \tilde{L}, \tilde{\nabla})$ with flat connection $\tilde{\nabla}$ admitting a non–trivial parallel determinant form, i.e. section of $\Lambda^2\tilde{V}^*$.*

PROOF. After tensoring by a line bundle (E, ∇^E), we can assume that Λ^2V^* admits a nowhere vanishing section which we call $det \in \Gamma(\Lambda^2V^*)$. This section $det \in \Gamma(\Lambda^2V^*)$ becomes parallel for the equivalent complex projective structure $\tilde{\mathcal{P}_{\mathbb{C}}} = (V, L, \nabla + \eta \operatorname{Id})$ with $\eta = \frac{1}{2}\alpha$, where $\alpha \in \Omega^1\mathbb{C}$ is the connection form given by $\nabla det = det \cdot \alpha$.

Because $\tilde{\mathcal{P}_{\mathbb{C}}}$ is a flat complex projective structure, the connection $\nabla + \eta \operatorname{Id}$ is projectively flat, i.e. there is $\omega \in \Omega^2(\mathbb{C})$ such that $R^{\nabla + \eta \operatorname{Id}} = \omega \operatorname{Id}$. The connection on Λ^2V^* induced by $\nabla + \eta \operatorname{Id}$ being flat, it is clear that ω has to vanish identically. □

REMARK. On a compact Riemann surface, if the connection ∇ of a flat complex projective structure $\mathcal{P}_{\mathbb{C}} = (V, L, \nabla)$ is flat, the bundle L has the degree of a spin bundle, i.e. $\deg L = \frac{1}{2} \deg K$. This follows, because $\delta\colon L \to KV/L$ is an isomorphism and therefore $\deg V = \deg L + \deg V/L = 2 \deg L - \deg K$.

From now on, we restrict our attention to those representatives of an equivalence class of flat complex projective structures, which have a flat connection. Two such complex projective structures $\mathcal{P}_{\mathbb{C}} = (V, L, \nabla)$ and $\tilde{\mathcal{P}}_{\mathbb{C}} = (\tilde{V}, \tilde{L}, \tilde{\nabla})$ with flat connections are equivalent if one is obtained from the other by tensoring by a flat complex line bundle and applying a gauge transformation.

Furthermore, in view of the preceding corollary, one could restrict to those representatives of flat complex projective structures, which admit a parallel determinant form. Two such structures are equivalent if one is obtained from the other by applying a gauge transformation and by tensoring by a $\mathbb{Z}_2$–bundle, which is a flat bundle squaring to the trivial bundle, i.e. a complexified flat real line bundle.

Flat Complex Projective Structures and Development Maps. Let $\mathcal{P}_{\mathbb{C}} = (V, L, \nabla)$ be a complex projective structure with ∇ a flat connection. Fixing a trivialization of $\tilde{V} = \pi^* V$ (where $\pi\colon \tilde{M} \to M$ is the universal covering of M) yields an isomorphisms of $\tilde{V}$ with the trivial $\mathbb{C}^2$ bundle on $\tilde{M}$. As explained in Appendix B, the bundle (V, ∇) can be obtained from this $\mathbb{C}^2$ bundle by identifying fibers using the monodromy representation $\rho\colon \Gamma \to \mathrm{GL}(2, \mathbb{C})$ of the group of deck transformations Γ. The subbundle $\tilde{L} = \pi^* L$ is equivariant with respect to ρ, meaning $\tilde{L}_{\gamma p} = \rho(\gamma)\tilde{L}_p$ for all $p \in \tilde{M}$ and $\gamma \in \Gamma$. Projectively, this defines a holomorphic immersion $\Delta\colon \tilde{M} \to \mathbb{CP}^1$ with Möbius monodromy, i.e. $\Delta(\gamma p) = \varrho(\gamma)\Delta(p)$ for all $p \in \tilde{M}$ and $\gamma \in \Gamma$ with $\varrho = [\rho]\colon \Gamma \to \mathrm{PGL}(2, \mathbb{C})$.

Definition. A *development map* of a Riemann surface M is a holomorphic immersion $\Delta\colon \tilde{M} \to \mathbb{CP}^1$ with Möbius monodromy, i.e. there is a representation $\varrho\colon \Gamma \to \mathrm{PGL}(2, \mathbb{C})$ such that $\Delta(\gamma p) = \varrho(\gamma)\delta(p)$ for all $p \in \tilde{M}$ and $\gamma \in \Gamma$.

A development map Δ defines an atlas on M whose coordinate changes are Möbius transformations. Such atlas is, what Gunning [**11**] calls a projective structure, and what Kulkarni and Pinkall [**19**] call a Möbius structure. It can be verified by analytic continuation that equivalence classes of atlases whose coordinate changes are Möbius transformations uniquely correspond to a Möbius equivalence classes of development maps.

The development map Δ defined above by trivializing $\tilde{V}$ depends, up to Möbius transformation, only on the equivalence class of $\mathcal{P}_{\mathbb{C}}$: if on passes to another trivialization, Δ changes by a Möbius transformation and ϱ gets conjugated. If one passes to a complex projective structure $\tilde{\mathcal{P}}_{\mathbb{C}} = (\tilde{V}, \tilde{L}, \tilde{\nabla})$ obtained from $\mathcal{P}_{\mathbb{C}}$ by tensoring by a flat complex line bundle, ρ gets multiplied by a character $\chi\colon \Gamma \to \mathbb{C}^*$, but Δ and $\varrho = [\rho]$ do not change.

Remark. It should be noted that the monodromy of a connection ∇ admitting a parallel determinant form takes values in $\mathrm{SL}(2, \mathbb{C})$, and that tensoring by a $\mathbb{Z}_2$–bundle amounts to multiplying the monodromy representation by a character $\chi\colon \Gamma \to \{\pm 1\}$.

By now we have seen that an equivalence class of flat complex projective structures gives rise to a unique development map (up to projective transformation). The following lemma shows that the converse is also true.

LEMMA 76. *Let M be a Riemann surface. There is a 1-1–correspondence between equivalence classes of flat complex projective structures and between Möbius equivalence classes of development maps.*

PROOF. It suffices to shows that every development map $\Delta\colon \tilde{M} \to \mathbb{CP}^1$ is obtained from a flat complex projective structure $\mathcal{P}_{\mathbb{C}}$. The map Δ defines a subbundle $\tilde{L} \subset \mathbb{C}^2$ of the trivial $\mathbb{C}^2$ bundle on $\tilde{M}$. In [**12**] (page 235), Gunning proves that the monodromy representation $\varrho\colon \Gamma \to \mathrm{PGL}(2, \mathbb{C})$ can be lifted to a representation $\rho\colon \Gamma \to \mathrm{GL}(2, \mathbb{C})$, i.e. $\varrho = [\rho]$. Taking the quotient of the trivial $\mathbb{C}^2$ bundle on $\tilde{M}$ with respect to ρ defines a flat bundle V on M with a subbundle L (cf. Lemma 111). By construction, $\mathcal{P}_{\mathbb{C}} = (V, L, \nabla)$ is a flat complex projective structure on M. □

Lemma 76 together with the uniformization theorem yields:

COROLLARY 77. *Every Riemann surface admits a flat complex projective structure.*

11.3. The Splitting Induced by a Coordinate.

LEMMA 78. *Let $\mathcal{P}_{\mathbb{C}} = (V, L, \nabla)$ be a torsion free complex projective structure on a Riemann surface and let X be a nowhere vanishing holomorphic vector field on M. There is a unique splitting $V = L \oplus L'$ such that $\delta_X\colon L \to L'$ is a gauge transformation (where L and L' are equipped with the connections induced from ∇ by the splitting).*

PROOF. With respect to a splitting $V = L \oplus \check{L}$, the connection ∇ takes the form
$$\nabla = \begin{pmatrix} \nabla^L & \check{\delta} \\ \delta & \check{\nabla} \end{pmatrix}.$$
Since $\mathcal{P}_{\mathbb{C}}$ is torsion free, $d^{\check{\nabla},\nabla^L}\delta = 0$. By holomorphicity of X (i.e. $[X, JX] = 0$), this implies
$$0 = \nabla_X \delta_{JX} - \nabla_{JX}\delta_X = \check{\nabla}_X \circ \delta_{JX} - \delta_{JX} \circ \nabla^L_X - \check{\nabla}_{JX} \circ \delta_X + \delta_X \circ \nabla^L_{JX} =$$
$$= 2\mathbf{i}\check{\nabla}''_X \circ \delta_X - 2\mathbf{i}\delta_X \circ (\nabla^L)''_X.$$
Otherwise stated, for every splitting, δ_X is holomorphic with respect to the $\bar{K}$–parts of both connections. Changing the splitting corresponds to applying a gauge transformation of the form
$$G = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$
Such gauge transformation changes ∇ to
$$\tilde{\nabla} = \nabla + G^{-1}\nabla G = \nabla + \begin{pmatrix} -b\delta & * \\ 0 & \delta b \end{pmatrix},$$
which shows that there is a unique $b \in \Gamma(\mathrm{Hom}((\check{L}, L))$ with the property that the connections on L and $\check{L}$ induced by the splitting coincide under the isomorphism $\delta_X\colon L \to \check{L}$. □

Therefore, with respect to this splitting, and under identifying of L with $\check{L}$ via δ_X, every torsion free complex projective structure takes the form

$$\nabla = \begin{pmatrix} \nabla^L & \mathfrak{c} \\ dz & \nabla^L \end{pmatrix},$$

where dz is the 1–form dual to X and $\mathfrak{c}$ is a complex 1–form. A short computation (or equation (76)) shows that the curvature of ∇ is

$$R^\nabla = \begin{pmatrix} R^{\nabla^L} + \mathfrak{c} \wedge dz & d\mathfrak{c} \\ 0 & R^{\nabla^L} - \mathfrak{c} \wedge dz \end{pmatrix}.$$

The normalization of the complex projective structure (V, L, ∇) is $(V, L, \tilde{\nabla})$ with

$$\tilde{\nabla} = \begin{pmatrix} \nabla^L & \mathfrak{c}' \\ dz & \nabla^L \end{pmatrix},$$

where $\mathfrak{c}'$ denotes the K–part of $\mathfrak{c}$.

12. Quaternionic Projective Structures

This section is devoted to a quaternionic version of the fundamental theorem of surfaces theory in the conformal 4–sphere. The theorem is formulated using the notion of quaternionic projective structures. A quaternionic projective structure (V, L, ∇) is essentially the same thing as a holomorphic curve $L \subset V$ in a quaternionic rank 2 bundle (V, ∇) (as introduced in Section 2.4), except that here the connection ∇ is only required to be projectively flat.

The advantage of this new terminology is that it emphasizes the projective nature of the involved objects as well as the analogy to the case of flat complex projective structures.

12.1. Definition and Immersions with Möbius Monodromy.

Definition. Let M be a Riemann surface. A *quaternionic projective structure* on M is a triple $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$ where V is a quaternionic rank 2 bundle, L a line subbundle of V and ∇ a connection on V such that

i) there is a 2–form $\omega \in \Omega^2\mathbb{R}$ with $R^\nabla = \omega \operatorname{Id}$, i.e. ∇ is projectively flat, and
ii) the derivative $\delta = \pi\nabla_{|L} \in \Gamma(T^*M \otimes \operatorname{Hom}(L, V/L))$ of L (where, as usual, $\pi\colon V \to V/L$ is the canonical projection), pointwise, is an injective map

$$\delta\colon TM \to \operatorname{Hom}(L, V/L).$$

The definition is the quaternionic analogue to that of a flat complex projective structure. We restrict our study of quaternionic projective structures to the flat case, because this is the one corresponding to immersions with monodromy into $\mathbb{HP}^1$ (cf. Lemma 80).

Two quaternionic projective structures $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$ and $\tilde{\mathcal{P}}_{\mathbb{H}} = (\tilde{V}, \tilde{L}, \tilde{\nabla})$ are called *equivalent*, denoted by $\mathcal{P}_{\mathbb{H}} \approx \tilde{\mathcal{P}}_{\mathbb{H}}$, if there is a real line bundle E with connection ∇^E such that

$$(V \otimes E, L \otimes E, \nabla \otimes \operatorname{Id}_E + \operatorname{Id}_V \otimes \nabla^E) \cong (\tilde{V}, \tilde{L}, \tilde{\nabla}),$$

where $\cong$, as in the complex case, denotes gauge equivalence of both quaternionic projective structures.

Representatives with Flat Connection of an Equivalence Class of Quaternionic Projective Structures. The following lemma shows that, as in the case of flat complex projective structures (see Lemma 75), every equivalence class of quaternionic projective structures contains representatives with a flat connection ∇ admitting a "parallel determinant form" in the sense to be specified now.

As usual (cf. [**15**]), the determinant of a quaternionic endomorphism A of a quaternionic rank 2 vector space V is defined to be the determinant of A seen as a complex endomorphism of the complex rank 4 vector space $(V, \mathbf{i})$. Otherwise stated, the determinant of A is defined by its action on $\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$. Since eigenvalues occur in pairs of complex conjugate numbers, the determinant is always greater or equal to zero.

Let (V, ∇) be a quaternionic rank 2 vector bundle with connection. The bundle $\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$ has the real structure $\tau\colon \Lambda^4_{\mathbb{C}}(V, \mathbf{i}) \to \Lambda^4_{\mathbb{C}}(V, \mathbf{i})$ which is given by $\tau(v_1 \wedge ... \wedge v_4) = v_1\mathbf{j} \wedge ... \wedge v_4\mathbf{j}$. The connection on $\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$ induced by ∇ commutes with the real structure τ, and therefore restricts to a connection on the real line bundle $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$. The real line bundle $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$ is orientable. Each fiber $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V_p, i)$ has a distinguished *positive* direction defined by $v_1 \wedge v_2 \wedge v_1\mathbf{j} \wedge v_2\mathbf{j}$, where v_1, v_2 forms a basis of V_p.

Remark. A parallel section of the line bundle $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$ is the right object corresponding to the parallel determinant form in the complex case: if ∇ is flat and admits a parallel section of $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$, its monodromy is in $\mathrm{SL}(2, \mathbb{H})$, which is defined to be the group of quaternionic endomorphisms of $\mathbb{H}^2$ with determinant 1.

Lemma 79. *Let (V, L, ∇) be a quaternionic projective structure. Then, there is an equivalent quaternionic projective structure $(V, L, \nabla + \eta \operatorname{Id})$ with $\eta \in \Omega^1\mathbb{R}$, such that $R^{\nabla + \eta \operatorname{Id}} = 0$ and $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$ admits a $(\nabla + \eta \operatorname{Id})$–parallel section.*

Proof. Because $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V, i)$ is orientable, there is a nowhere vanishing section $\zeta \in \Gamma(\operatorname{Re}\Lambda^4_{\mathbb{C}}(V, i))$. The curvature of ∇ on $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V, i)$ is $R = d\alpha \operatorname{Id}$, where $\alpha \in \Omega^1\mathbb{R}$ is the connection form given by $\nabla\zeta = \zeta\alpha$. Since the curvature of ∇ on V is $R^\nabla = \omega \operatorname{Id}$, we have $4\omega = d\alpha$, and $\eta = -\frac{1}{4}\alpha$ yields an equivalent quaternionic projective structure $(V, L, \nabla + \eta \operatorname{Id})$ with the property that $\nabla + \eta \operatorname{Id}$ is flat and that $\zeta \in \Gamma(\operatorname{Re}\Lambda^4_{\mathbb{C}}(V, i))$ is parallel with respect to $\nabla + \eta \operatorname{Id}$. □

Two quaternionic projective structures $\mathcal{P}_{\mathbb{H}}$ and $\tilde{\mathcal{P}}_{\mathbb{H}}$ with flat connections are equivalent, if one is obtained from the other by applying a gauge transformation and tensoring by a flat real line bundle. Similarly, two quaternionic projective structures $\mathcal{P}_{\mathbb{H}}$ and $\tilde{\mathcal{P}}_{\mathbb{H}}$ with flat connections and parallel sections of $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$ are equivalent, if one is obtained from the other by applying a gauge transformation and tensoring by a a $\mathbb{Z}_2$–bundle, i.e. a flat real line bundle squaring to the trivial bundle.

Conformal Immersion with Möbius Monodromy. The aim of this paragraph is to prove that every equivalence class of quaternionic projective structures on a Riemann surface M gives rise to a conformal immersion with Möbius monodromy of the universal covering $\tilde{M}$ into $\mathbb{HP}^1$. The following lemma can be seen as a quaternionic version of Lemma 76 (i.e. the fact that, up to equivalence, flat complex projective structures correspond to development maps). The proofs of both lemmas are analogous.

Let $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$ be a quaternionic projective structure with flat ∇. Trivializing the bundle $\tilde{V} = \pi^* V$ on the universal covering $\tilde{M}$ of M yields an isomorphisms of $\tilde{V}$ with the trivial $\mathbb{H}^2$ bundle on $\tilde{M}$. The bundle $\tilde{L} = \pi^* L$ is a subbundle of the trivial $\mathbb{H}^2$ bundle on $\tilde{M}$ which is equivariant with respect to the monodromy representation $\rho\colon \Gamma \to \mathrm{GL}(2, \mathbb{H})$, i.e. $\tilde{L}_{\gamma p} = \rho(\gamma)\tilde{L}_p$ for all $p \in \tilde{M}$ and all deck transformations $\gamma \in \Gamma$.

Projectively, according to the following definition, the bundle $\tilde{L}$ is a conformal immersion $f\colon \tilde{M} \to \mathbb{HP}^1$ with Möbius monodromy.

Definition. A *conformal immersion with Möbius monodromy* is a conformal immersion $f\colon \tilde{M} \to \mathbb{HP}^1$ of the universal covering $\tilde{M}$ with

$$f(\gamma p) = \varrho(\gamma) f(p)$$

for all $p \in \tilde{M}$ and $\gamma \in \Gamma$, where $\varrho\colon \Gamma \to \mathrm{PGL}(2, \mathbb{H})$ is a representation of the group of deck transformations.

The immersion f with Möbius monodromy is uniquely determined (up to Möbius transformation) by the equivalence class of $\mathcal{P}_{\mathbb{H}}$: a gauge transformation and a different choice of initial value of the trivialization changes $\tilde{L}$ by an element of $\mathrm{GL}(2, \mathbb{H})$ while the monodromy representations gets conjugated. If (V, L, ∇) gets tensored by a flat real line bundle, ρ gets multiplied by a character $\chi\colon \Gamma \to \mathbb{R}^*$ and $\tilde{L}$ does not change at all.

Conversely, a conformal immersion $f\colon \tilde{M} \to \mathbb{HP}^1$ with Möbius monodromy $\varrho\colon \Gamma \to \mathrm{PGL}(2, \mathbb{H})$ that admits a lift $\rho\colon \Gamma \to \mathrm{GL}(2, \mathbb{H})$ with $\varrho = [\rho]$ gives rise (cf. Lemma 111) to a unique equivalence class of quaternionic projective structures $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$. A Möbius equivalent immersion $\tilde{f}$ gives rise to the same equivalence class of quaternionic projective structures, and we have proven the following lemma.

Lemma 80. *There is a 1-1–correspondence between equivalence classes of quaternionic projective structures on a Riemann surface M and Möbius equivalence classes of conformal immersions $f\colon \tilde{M} \to \mathbb{HP}^1$ with Möbius monodromy $\varrho\colon \Gamma \to \mathrm{PGL}(2, \mathbb{H})$ admitting a lift to* $\mathrm{GL}(2, \mathbb{H})$.

Remark. Ulrich Pinkall gave an example of a torus with Möbius monodromy not admitting a lift to $\mathrm{GL}(2, \mathbb{H})$.

12.2. Decomposition of Quaternionic Projective Structure. To derive the fundamental theorem of surface theory in quaternionic projective space we use the mean curvature sphere and decompose a quaternionic projective structure into an underlying complex projective structure and the Hopf fields.

The following lemma is a direct consequence of the fact that every immersed holomorphic curve $L \subset V$ in a flat rank 2 bundle (V, ∇) admits a unique mean curvature sphere (cf. Section 2.4).

Lemma 81. *A quaternionic projective structure $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$ admits a unique $S \in \Gamma(\mathrm{End}\, V)$ with $S^2 = -\,\mathrm{Id}$ satisfying $*\delta = S\delta = \delta S$ and $Q_{|L} = 0$.*

PROOF. By Lemma 79, there is $\eta \in \Omega^1\mathbb{R}$ such that $\nabla + \eta\,\mathrm{Id}$ is flat. For $\tilde{\mathcal{P}_{\mathbb{H}}} = (V, L, \nabla + \eta\,\mathrm{Id})$, the lemma follows from Section 2.4, because the mean curvature sphere S of the holomorphic curve $L \subset V$ in the flat rank 2 bundle $(V, \nabla + \eta\,\mathrm{Id})$ has the desired properties. The same S as well has the right properties with respect to $\mathcal{P}_{\mathbb{H}}$, because δ and Q coincide for both connections. □

We call S the *mean curvature sphere* of the quaternionic projective structure $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$. It gives rise to the usual decomposition $\nabla = \hat{\nabla} + A + Q$ into the S–complex connection $\hat{\nabla}$ and the S–anti linear forms A and Q.

A quaternionic projective structure $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$ is uniquely determined by the following data:

a) the *induced complex projective structure* $\mathcal{P}_{\mathbb{C}} = (\hat{V}, \hat{L}, \hat{\nabla})$ defined by

$$\hat{V} = \{\psi \in V \mid S\psi = \psi\mathbf{i}\}$$
$$\hat{L} = L \cap \hat{V}$$
$$\hat{\nabla} = \frac{1}{2}(\nabla - S\nabla S)$$

and

b) the *Hopf fields* $A \in \Gamma(K\,\mathrm{End}_-\,V)$ and $Q \in \Gamma(\bar{K}\,\mathrm{End}_-\,V)$.

If $\mathcal{P}_{\mathbb{H}}$ gets tensored by a real line bundle, the equivalent quaternionic projective structure $\tilde{\mathcal{P}_{\mathbb{H}}}$ thus obtained induces the complex projective structure $\tilde{\mathcal{P}_{\mathbb{C}}}$ obtained from $\mathcal{P}_{\mathbb{C}}$ by tensoring by the same real line bundle. The differentials $\tilde{A}$ and $\tilde{Q}$ of $\tilde{\mathcal{P}_{\mathbb{H}}}$ are the ones of $\mathcal{P}_{\mathbb{H}}$ tensored by identity.

Let $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$ be a quaternionic projective structure. By (5) and (6), the projective flatness of ∇, i.e. $R^{\nabla} = \omega\,\mathrm{Id}$ with $\omega \in \Omega^2\mathbb{R}$, is equivalent to

$$R^{\hat{\nabla}} + A \wedge A + Q \wedge Q = \omega\,\mathrm{Id} \tag{77}$$

$$d^{\hat{\nabla}}A + d^{\hat{\nabla}}Q = 0. \tag{78}$$

This is a compatibility equation that the data of every quaternionic projective structure $\mathcal{P}_{\mathbb{H}}$ has to satisfy.

In the following, we analyze the meaning of the induced complex projective structure and the Hopf fields, and show, which minimal set of data is needed to uniquely characterize a quaternionic projective structure. This minimal set of data, together with the compatibility conditions that has to be satisfied by the data, makes up the fundamental theorem of Möbius geometry of surfaces in the 4–sphere.

The Induced Complex Projective Structure. Equation (77) immediately implies that the complex projective structure $\mathcal{P}_{\mathbb{C}} = (\hat{V}, \hat{L}, \hat{\nabla})$ induced by the quaternionic projective structure $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$ is torsion free.

In the following, we call two complex projective structures $\approx$–*equivalent*, if one is obtained from the other by tensoring by a real line bundle and applying a gauge transformation. Obviously, $\approx$–equivalence implies $\sim$–equivalence. The $\approx$–equivalence class of $\mathcal{P}_{\mathbb{C}}$ is exactly the set of complex projective structures that can be obtained from the quaternionic projective structures equivalent to $\mathcal{P}_{\mathbb{H}}$.

We will analyze now, which additional information is encoded in this $\approx$–equivalence class. To do so, we need an equivalence relation on complex line bundles. Two complex line bundles (E,∇) and $(\tilde{E},\tilde{\nabla})$ are called *$\approx$–equivalent* if they satisfy $(E,\nabla)\cong(\tilde{E},\tilde{\nabla}+\alpha)$ where $\cong$ as usual denotes gauge equivalence and $\alpha\in\Omega^1\mathbb{R}$ is a real 1–form. The following lemma shows that this equivalence relation is strongly related to the one on complex projective structures carrying the same name.

LEMMA 82. *The correspondence assigning to a complex projective structure $\mathcal{P}_\mathbb{C}=(\hat{V},\hat{L},\hat{\nabla})$ its $\sim$–equivalence class and the $\approx$–equivalence class of the complex line bundle $(E,\nabla^E)=(\Lambda^2\hat{V},\Lambda^2\hat{\nabla})$ induces a 1-1–correspondence between*

$$\{\text{ complex projective structures } \mathcal{P}_\mathbb{C}\}_{/\approx}$$

and

$$\{\text{ complex projective structures } \mathcal{P}_\mathbb{C}\}_{/\sim}\times$$
$$\times\{\text{ complex line bundle } (E,\nabla) \text{ of even degree}\}_{/\approx}.$$

PROOF. Obviously, the correspondence does not depend on the representative of the $\approx$–equivalence class of $(\hat{V},\hat{L},\hat{\nabla})$ and is therefore well defined.

Assume we are given the $\sim$–equivalence class of $\mathcal{P}_\mathbb{C}=(\hat{V},\hat{L},\hat{\nabla})$ and the $\approx$–equivalence class of (E,∇^E). Since E has even degree, after tensoring $\mathcal{P}_\mathbb{C}$ by a complex line bundle, we can assume that E and $\Lambda^2\hat{V}$ have the same degree. Then, by adding a complex 1–form to the connection ∇, we can arrange gauge equivalence $E\cong\Lambda^2\hat{V}$. Obviously, the equivalent complex projective structure $\tilde{\mathcal{P}_\mathbb{C}}=(\hat{V},\hat{L},\hat{\nabla})$ thus obtained 'corresponds' to (E,∇). It is not unique, but can be tensored by a $\mathbb{Z}_2$–bundle. Replacing the bundle (E,∇^E) with the equivalent bundle $(E,\nabla^E+\alpha)$ for a real 1–form α, we replace $\hat{\nabla}$ with $\hat{\nabla}+\frac{1}{2}\alpha$ to obtain a $\approx$–equivalent complex projective structure. $\square$

REMARK. The picture described in the discussion following Lemma 78 provides a good way to understand what is going on in the preceding proof: the diagonal connection of the complex projective structure (with respect to the splitting induced by a nowhere vanishing holomorphic vector field) is a square root of (E,∇). As such, it is uniquely determined by (E,∇) up to tensoring by a $\mathbb{Z}_2$–bundle. By Theorem 7, after fixing a root, the upper right corner of $\hat{\nabla}$ still leaves enough freedom to obtain one complex projective structure from every possible $\sim$–equivalence class of complex projective structures. Briefly said, the role of (E,∇) is to single out special representatives in the $\sim$–equivalence classes of complex projective structures.

Hence, the additional information contained in the $\approx$–equivalence class of $(\hat{V},\hat{L},\hat{\nabla})$ (in contrast to its $\sim$–equivalence class) is the $\approx$–equivalence class of the complex line bundle (E,∇). We will see in Lemma 91 that this bundle is isomorphic to the normal bundle of the immersion corresponding to the underlying quaternionic projective structure.

A nowhere vanishing section of the bundle $\operatorname{Re}\Lambda^4_\mathbb{C}(V,\mathbf{i})$ defines an isomorphism $\operatorname{Re}\Lambda^4_\mathbb{C}(V,\mathbf{i})\cong\mathbb{R}$. The following lemma shows that sections of $\operatorname{Re}\Lambda^4_\mathbb{C}(V,\mathbf{i})$ can be identified with hermitian form on $E=\Lambda^2\hat{V}$.

Lemma 83. *The map assigning to a positive section of* $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V,\mathbf{i})$*, via the induced identification* $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V,\mathbf{i}) \cong \mathbb{R}$*, the positive hermitian form on* $\Lambda^2\hat{V}$ *given by*

$$\langle \psi\wedge\varphi, \psi\wedge\varphi\rangle_{\Lambda^2\hat{V}} := \psi\mathbf{j}\wedge\varphi\mathbf{j}\wedge\psi\wedge\varphi$$

where ψ*,* $\varphi \in \Gamma(\hat{V})$*, is bijective. The section of* $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V,\mathbf{i})$ *is parallel if and only if the hermitian form is parallel.*

Proof. As a non–trivial map between real line bundles, the map clearly is bijective. Assume the section of $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V,\mathbf{i})$ is parallel. Then

$$d\langle b_1, b_2\rangle_{\Lambda^2\hat{V}} = d(\overline{b_1}\wedge b_2) = \overline{\nabla b_1}\wedge b_2 + \overline{b_1}\wedge\nabla b_2$$

for b_1, $b_2 \in \Gamma(\Lambda^2\hat{V})$. Since $A+Q \in \Omega^1(\operatorname{End}_-(V))$, we obtain

$$d\langle b_1, b_2\rangle_{\Lambda^2\hat{V}} = \overline{\hat{\nabla} b_1}\wedge b_2 + \overline{b_1}\wedge\hat{\nabla} b_2 = \langle\hat{\nabla} b_1, b_2\rangle_{\Lambda^2\hat{V}} + \langle b_1, \hat{\nabla} b_2\rangle_{\Lambda^2\hat{V}},$$

i.e. the induced metric is parallel. To prove the converse, assume $\langle\,,\rangle_{\Lambda^2\hat{V}}$ is parallel and the section of $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V,\mathbf{i})$ is not. Then, there is an equivalent connection $\nabla+\alpha$ with $\alpha\in\Omega^1\mathbb{R}$, such that the section of $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V,\mathbf{i})$ is parallel and the metric is parallel with respect to $\hat{\nabla}$ and $\hat{\nabla}+\alpha$, which implies $\alpha = 0$. □

Corollary 84. *Let* (V, L, ∇) *be a quaternionic projective structure. Then* $E = \Lambda^2\hat{V}$ *admits a non–trivial parallel hermitian form if and only if* ∇ *admits a parallel section of* $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V,\mathbf{i})$ *(and in particular is flat).*

By the preceding corollary and Lemma 79, parallel hermitian forms on the complex line bundle $(E,\nabla) = (\Lambda^2\hat{V}, \Lambda^2\hat{\nabla})$ exist for special representatives of the $\approx$–equivalence class of a quaternionic projective structure. These representatives are unique up to tensoring by a $\mathbb{Z}_2$–bundle and applying a gauge transformation.

Similarly, the following lemma shows that there are special representatives of the $\approx$–equivalence class of (E,∇) admitting a parallel hermitian form, and that these special representatives are all gauge equivalent.

Lemma 85. *The* $\approx$*–equivalence class of a complex line bundle* (E,∇) *has a representative admitting a parallel hermitian form. This representative is uniquely determined up to gauge transformation.*

Proof. Let $\langle\,,\rangle$ be a positive definite hermitian form on E. There is a unique $\alpha\in\Omega^1(\mathbb{R})$, given by $\alpha = \frac{1}{2}\beta$ with $\beta\in\Omega^1(\mathbb{R})$ defined by $\nabla\langle\,,\rangle = \langle\,,\rangle\beta$, such that $\nabla+\alpha$ leaves $\langle\,,\rangle$ parallel. The bundle of hermitian forms on E being an oriented real line bundle, every other positive definite hermitian form can be written as $\langle\,,\tilde{\rangle} = e^{2u}\langle\,,\rangle$ for a real function u. If $\nabla+\tilde{\alpha}$ leaves $\langle\,,\tilde{\rangle}$ parallel, then $\tilde{\alpha} = \alpha + du$ since $\psi\in E\mapsto e^{-u}\psi\in E$ is a gauge transformation and isometry between $(E, \nabla+\alpha, \langle\,,\rangle)$ and $(E, \nabla+\tilde{\alpha}, \langle\,,\tilde{\rangle})$. □

Remark. It can be easily verified that, on a compact Riemann surface, the $\approx$–equivalence class of (E,∇) is uniquely determined by $\operatorname{Im} R^\nabla \in \Omega^1(\mathbb{R}\mathbf{i})$ together with the holomorphic equivalence class of the holomorphic structure $\bar\partial = (\nabla+\alpha)''$ induced by the hermitian connection $\nabla+\alpha$.

The following diagram should clarify the relation between special representatives of the $\approx$–equivalence classes of a quaternionic projective structure $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$, of the induced complex projective structure $\mathcal{P}_{\mathbb{C}} = (\hat{V}, \hat{L}, \hat{\nabla})$ and of the complex line bundle $(E, \nabla) = (\Lambda^2\hat{V}, \Lambda^2\hat{\nabla})$.

quaternionic projective structure	complex projective structure	complex line bundle
(V, L, ∇) $\approx$ means: tensor by real line bundle and gauge transformation	$(\hat{V}, \hat{L}, \hat{\nabla})$ $\approx$ means: tensor by real line bundle and gauge transformation	(E, ∇) $\approx$ means: add real 1–form and gauge transformation
(V, L, ∇) with ∇ flat $\approx$ means: tensor by flat real line bundle and gauge transformation	$(\hat{V}, \hat{L}, \hat{\nabla})$ with $\rightsquigarrow$ $\approx$ means: tensor by flat real line bundle and gauge transformation	(E, ∇) s.th. ∇ on hermitian forms is flat $\approx$ means: add a closed real 1–form and gauge transformation
(V, L, ∇) with parallel section of $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$ $\approx$ means: tensor by $\mathbb{Z}_2$–bundle and gauge transformation	$(\hat{V}, \hat{L}, \hat{\nabla})$ with $\rightsquigarrow$ $\approx$ means: tensor by $\mathbb{Z}_2$–bundle and gauge transformation	(E, ∇) admitting a parallel hermitian forms $\approx$ means: gauge transformation

The Hopf Fields. The aim of this paragraph is to show that the relevant information contained in the Hopf fields A and Q of a quaternionic projective structure $\mathcal{P}_{\mathbb{H}}$ is the trace free second fundamental form of the corresponding immersion.

LEMMA 86. *Let $(\hat{V}, \hat{L}, \hat{\nabla})$ be a complex projective structure. Denote by V and L the bundles obtained by quaternionification of $\hat{V}$ and $\hat{L}$. For every 1–form $\mathfrak{a} \in \Gamma(K\operatorname{End}_-(L))$, there is a unique 1–form $A \in \Gamma(K\operatorname{End}(V))$ with*

$$d^{\hat{\nabla}}A \in \Omega^2(\mathcal{R}) \quad \textit{such that } \operatorname{im} A \subset L \textit{ and } A_{|L} = \mathfrak{a}.$$

Similarly, for every 1–form $\mathfrak{q} \in \Gamma(\bar{K}\operatorname{End}_-(V/L))$, there is a unique 1–form $Q \in \Gamma(K\operatorname{End}(V))$ with

$$d^{\hat{\nabla}}Q \in \Omega^2(\mathcal{R}) \quad \textit{such that } Q_{|L} = 0 \textit{ and } \pi Q_{|V/L} = \mathfrak{q}.$$

Both A and Q are in $\Omega^1(\operatorname{End}_-(V))$.

PROOF. We prove the statement for A. The statement for Q follows by passing to the dual curve. Denote by $\tilde{A} \in \Gamma(K\operatorname{End}_-(V))$ an arbitrary form satisfying $\operatorname{im}\tilde{A} \subset L$ and $\tilde{A}_{|L} = \mathfrak{a}$. We have to shows that there is a unique $B \in \Gamma(K\mathcal{R})$ such that $A = \tilde{A} + B$ satisfies $d^{\hat{\nabla}}A \in \Omega^2(\mathcal{R})$. By Lemma 13, the 2–form $d^{\hat{\nabla}}A$ takes values in L. Its restriction to L is

$$(d^{\hat{\nabla}}A)_{|L} = (d^{\hat{\nabla}}\tilde{A})_{|L} + B \wedge \delta.$$

This shows that there is a unique B such that $(d^{\hat{\nabla}}A)_{|L} = 0$. $\square$

By definition of the mean curvature sphere, the Hopf fields $A \in \Gamma(K \operatorname{End}_- V)$ and $Q \in \Gamma(\bar{K} \operatorname{End}_- V)$ of a quaternionic projective structure (V, L, ∇), satisfy $\operatorname{im} A \subset L$ and $Q_{|L} = 0$. Furthermore, by Lemma 14, the differentials of A and Q are in $\Omega^1(\mathcal{R})$. Hence, by the preceding lemma, A and Q are uniquely determined by $A_{|L} \in \Gamma(K \operatorname{End}_- L)$ and $\pi Q_{|V/L} \in \Gamma(\bar{K} \operatorname{End}_- V/L)$, meaning that, with the help of the induced complex projective structure, A and Q can be computed from $A_{|L}$ and $\pi Q_{|V/L}$.

The first possible interpretation of the role played by $A_{|L}$ and $\pi Q_{|V/L}$ is to see it as the analogue of the trace free second fundamental form of an immersion: if ∇ is flat, locally (V, L, ∇) can be represented with respect to an Euclidean chart: one can chose a parallel frame for ∇, thus defining an identification $V \cong \mathbb{H}^2$, such that

$$L = \begin{pmatrix} f \\ 1 \end{pmatrix} \mathbb{H}.$$

The normal bundle $\perp_f M$ of the immersion f into $\mathbb{H}$ is identified with the normal bundle $\operatorname{Hom}_-(L, V/L)$ of $L \subset V$ via equation (24), i.e.

$$\perp_f M \to \operatorname{Hom}_-(L, V/L) \qquad v \in \perp_f M \mapsto (\begin{pmatrix} f \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} v \\ 0 \end{pmatrix} \bmod L).$$

Equation 15 implies that the trace free second fundamental form of f satisfies $\mathring{\mathrm{II}}(X, Y) = \frac{1}{2}(N dN''(X) df(Y) + df(Y) dR''(X) R)$ (where N and R denote the left and right normal vectors of f). Using the formulae for A and Q in terms of the Euclidean data of f (cf. Section 2.4), we obtain that (under identification of the normal bundles)

$$\mathring{\mathrm{II}}(X, Y) = \pi Q_X \delta_Y - \delta_Y A_X. \tag{79}$$

Another possible interpretation of the role played by $A_{|L}$ and $\pi Q_{|V/L}$ is in terms of holomorphic geometry: the complex projective structure $\mathcal{P}_{\mathbb{C}} = (\hat{V}, \hat{L}, \hat{\nabla})$ induces $\bar{\partial}$–operators on $\hat{L}$ and $\hat{V}/\hat{L}$, which, together with $A_{|L}$ and $\pi Q_{|V/L}$, yield the usual quaternionic projective structures on V/L and L^{-1}. It should be noted that different representatives of one equivalence class of quaternionic projective structures yield non–isomorphic quaternionic projective structures, because the $\bar{\partial}$–operators for two representatives are not isomorphic. If one considers only those representatives of quaternionic projective structures that admit a parallel section of $\operatorname{Re} \Lambda^2_{\mathbb{C}}(V, \mathbf{i})$, the complex holomorphic line bundles are uniquely determined up to tensoring by a $\mathbb{Z}_2$–bundle.

The Fundamental Theorem. The following theorem is the fundamental theorem of Möbius geometry of surfaces in quaternionic projective space. It states, which minimal data is needed to uniquely determine a quaternionic projective structure (this is the uniqueness part of the theorem). Furthermore, it gives a compatibility condition satisfied by this data, and it states that one can start with such data satisfying the compatibility condition to obtain a quaternionic projective structure (which is the existence part of the theorem).

THEOREM 8 (Fundamental Theorem). *Let M be a Riemann surface. There is a 1-1–correspondence between*

(1) equivalence classes of quaternionic projective structures $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$ on M and

(2) pairs of data given by

a) a $\approx$–equivalence class of torsion free complex projective structures $\mathcal{P}_{\mathbb{C}} = (\hat{V}, \hat{L}, \hat{\nabla})$ and

b) 1–forms $A_{|L} \in \Gamma(K \operatorname{End}_- L)$ and $\pi Q_{|V/L} \in \Gamma(\bar{K} \operatorname{End}_- V/L)$ (where V and L are the complex quaternionic bundles obtained by quaternionification of $\hat{V}$ and $\hat{L}$)

such that the compatibility equations

$$R^{\hat{\nabla}} + A \wedge A + Q \wedge Q = \omega \operatorname{Id}$$
$$d^{\hat{\nabla}} A + d^{\hat{\nabla}} Q = 0$$

are satisfies for a real 2–form $\omega \in \Omega^2\mathbb{R}$. (Here A and $Q \in \Omega^1(\operatorname{End}(V))$ are the forms which, by Lemma 86, are uniquely determined from $A_{|L}$ and $\pi Q_{|V/L}$.)

The data assigned to a quaternionic projective structures (V, L, ∇) under this correspondence are a) the induced complex projective structure and b) the restrictions of the Hopf differentials A and Q to the respective bundles.

PROOF. The correspondence is well defined, because it does not depend on the choice of representative in the equivalence class of quaternionic projective structures. We have already seen above that the a quaternionic projective structure is uniquely determined by the given data. The compatibility conditions are equations (77) and (78).

What is left to do is to explain, how the data satisfying the compatibility condition gives rise to a quaternionic projective structure. From $(\hat{V}, \hat{L}, \hat{\nabla})$, one can recover V and L by quaternionification $V = \hat{V} \oplus \hat{V}\mathbf{j}$ and $L = \hat{L} \oplus \hat{L}\mathbf{j}$ and Lemma 86 shows that $A_{|L}$ and $\pi Q_{|V/L}$ uniquely determine $A \in \Gamma(K \operatorname{End}_- V)$ and $Q \in \Gamma(\bar{K} \operatorname{End}_- V)$. This yields a connection $\nabla = \hat{\nabla} + A + Q$ on V, which, by the compatibility equations, is projectively flat. □

12.3. Refined Decomposition of Quaternionic Projective Structures using a Splitting. For applications it is necessary to have a more refined decomposition of the data describing a quaternionic projective structure. We will derive this using a splitting induced by a nowhere vanishing holomorphic vector field on M (cf. Lemma 78). The advantage of using this splitting is that it considerably simplifies the formulae. A disadvantage is that the only compact Riemann surfaces globally admitting such nowhere vanishing holomorphic vector fields are those of genus 1, i.e. tori.

Let M be a torus (or an open Riemann surface) with a nowhere vanishing holomorphic vector field X and dual 1–form dz. Let (V, L, ∇) be a quaternionic projective structure on M with flat ∇. By Lemma 78 there is a unique splitting $\hat{V} = \hat{L} \oplus \hat{L}'$ such that $\delta_X \colon \hat{L} \to \hat{L}'$ is a gauge equivalence with respect to the connections on $\hat{L}$ and $\hat{L}'$ induced by the splitting. Under the identification of $\hat{L}$ and $\hat{L}'$ via δ_X, we have

$$\hat{\nabla} = \begin{pmatrix} \nabla^L & \mathfrak{c} \\ dz & \nabla^L \end{pmatrix} \qquad A = \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ 0 & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & \mathfrak{p} \\ 0 & \mathfrak{q} \end{pmatrix}$$

for a complex 1–form $\mathfrak{c}$ and $\mathfrak{a} = A_{|L}$, $\mathfrak{b} \in \Gamma(K \operatorname{End}_- L)$ and $\mathfrak{q} = \pi Q_{|V/L}$, $\mathfrak{p} \in \Gamma(\bar{K} \operatorname{End}_- L)$ (where we use the identification between L and V/L induced by δ_X). Therewith, the compatibility equations (77) and (78)

$$\begin{aligned} &R^{\hat{\nabla}} + A \wedge A + Q \wedge Q = 0 \quad \text{ and} \\ &d^{\hat{\nabla}} A + d^{\hat{\nabla}} Q = 0 \end{aligned}$$

become

$$\begin{pmatrix} R^{\nabla^L} + \mathfrak{c}'' \wedge dz & d\mathfrak{c} \\ 0 & R^{\nabla^L} - \mathfrak{c}'' \wedge dz \end{pmatrix} + \begin{pmatrix} \mathfrak{a} \wedge \mathfrak{a} & \mathfrak{a} \wedge \mathfrak{b} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathfrak{p} \wedge \mathfrak{q} \\ 0 & \mathfrak{q} \wedge \mathfrak{q} \end{pmatrix} = 0,$$

$$\begin{pmatrix} d^{\nabla^L}\mathfrak{a} + \mathfrak{b} \wedge dz & d^{\nabla^L}\mathfrak{b} + \mathfrak{a} \wedge \mathfrak{c}' \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & d^{\nabla^L}\mathfrak{p} + \mathfrak{c}' \wedge \mathfrak{q} \\ 0 & d^{\nabla^L}\mathfrak{q} + dz \wedge \mathfrak{p} \end{pmatrix} = 0.$$

These equations are equivalent to

$$d\mathfrak{c} + \mathfrak{a} \wedge \mathfrak{b} + \mathfrak{p} \wedge \mathfrak{q} = 0, \tag{80}$$

$$R^{\nabla^L} + \frac{1}{2}(\mathfrak{a} \wedge \mathfrak{a} + \mathfrak{q} \wedge \mathfrak{q}) = 0, \tag{81}$$

$$d^{\nabla^L}\mathfrak{b} + d^{\nabla^L}\mathfrak{p} + \mathfrak{a} \wedge \mathfrak{c}' + \mathfrak{c}' \wedge \mathfrak{q} = 0, \tag{82}$$

where the forms $\mathfrak{b}$ and $\mathfrak{p}$ as well as $\mathfrak{c}''$ are determined by ∇^L, $\mathfrak{a}$, $\mathfrak{q}$ and $\mathfrak{c}'$ via

$$\mathfrak{c}'' \wedge dz = \frac{1}{2}(\mathfrak{q} \wedge \mathfrak{q} - \mathfrak{a} \wedge \mathfrak{a}), \tag{83}$$

$$d^{\nabla^L}\mathfrak{a} + \mathfrak{b} \wedge dz = 0, \tag{84}$$

$$d^{\nabla^L}\mathfrak{q} + dz \wedge \mathfrak{p} = 0. \tag{85}$$

Equations (80), (81) and (82) are called *Gauss equation*, *Ricci equation* and *Codazzi equation* (in that order). The system of all three equations together will be referred to as the *Gauss–Codazzi–Ricci equation*.

REMARK. Note that equations (84) and (85) reprove Lemma 86: given a complex projective structure and $\mathfrak{a}$ (resp. $\mathfrak{q}$), there is a unique $\mathfrak{b}$ (resp. $\mathfrak{p}$) such that the corresponding A (resp. Q) satisfies $(d^{\hat{\nabla}} A)_{|L} = 0$ (resp. $\pi d^{\hat{\nabla}} Q = 0$).

We are now able to state a version of the fundamental theorem in terms of the data (L, ∇^L), $\mathfrak{c}'$, $\mathfrak{a}$ and $\mathfrak{q}$.

THEOREM 9 (Fundamental Theorem - Splitted Version). *Let M be a Riemann surface with a nowhere vanishing holomorphic vector field X. There is a 1–1–correspondence between quaternionic projective structures $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$ on M with flat ∇ and between the data $(\hat{L}, \nabla^L)$, $\mathfrak{c}'$, $\mathfrak{a}$ and $\mathfrak{q}$ satisfying the Gauss–Codazzi–Ricci equation* (80)*,* (81) *and* (82) *(where $\mathfrak{c}''$, $\mathfrak{b}$ and $\mathfrak{p}$ are determined by* (83)*,* (84) *and* (85)*).*

Representation of Data with respect to a Frame. In this paragraph it is shown how the data of a quaternionic projective structure can be written with respect to a frame. The formulae thus obtained are the ones used in the next chapter: though less invariant, they seem to be better suited for explicit calculations with the data.

Let M be a Riemann surface equipped with a nowhere vanishing holomorphic vector field X and dual 1–form dz. Let $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$ be a quaternionic projective structure on M with flat ∇. Using the splitting with respect to X (cf. Lemma 78), $\mathcal{P}_{\mathbb{H}}$ induces a complex connection ∇^L on the line bundle $\hat{L}$.

We assume that $(\hat{L}, \nabla^L)$ is equipped with a positive definite parallel hermitian form, which, by Corollary 84 and the isomorphism

$$\hat{L}^2 \cong \Lambda^2 \hat{V} \qquad \psi^2 \mapsto \psi \wedge \delta_X \psi,$$

corresponds to a positive parallel section of $\operatorname{Re} \Lambda^4_{\mathbb{C}}(V, \mathbf{i})$. Furthermore, we assume that $\hat{L}$ admits a nowhere vanishing section $\psi \in \Gamma(\hat{L})$, which we chose to be of constant length. Such section always exists on open Riemann surfaces. For closed Riemann surfaces, it exists if and only if $\hat{L}$ has degree 0, which, for tori, is equivalent to normal bundle degree 0.

Using the splitting of $\hat{V}$, the section $\psi \in \Gamma(\hat{L})$ defines a second section $\varphi = \delta_X \psi \in \Gamma(\hat{V})$. With respect to the frame ψ, φ for $\hat{V}$, the data of Theorem 9 can be written as

$$\nabla^L \psi = \psi(g\, dz - \bar{g}\, d\bar{z}), \qquad \mathfrak{c} = -\frac{1}{2} c\, dz + e\, d\bar{z},$$

and

$$\begin{aligned} \mathfrak{a}\psi &= \psi a\, dz\mathbf{j} \qquad & \mathfrak{b}\psi &= \psi b\, dz\mathbf{j}, \qquad \text{and} \\ \mathfrak{q}\psi &= \psi \bar{q}\, d\bar{z}\mathbf{j} \qquad & \mathfrak{p}\psi &= \psi \bar{p}\, d\bar{z}\mathbf{j} \end{aligned}$$

with complex functions g, c, e, a, b, q and p. The connection ∇ is given by

$$\nabla(\psi, \varphi) = (\psi, \varphi) \begin{pmatrix} g\, dz - \bar{g}\, d\bar{z} + a\, dzj & -\frac{1}{2} c\, dz + e\, d\bar{z} + b\, dzj + \bar{p}\, d\bar{z}j \\ dz & g\, dz - \bar{g}\, d\bar{z} + \bar{q}\, d\bar{z}j \end{pmatrix}. \tag{86}$$

Summary. We finish this section with a brief summary of the necessary data defining a quaternionic projective structure $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$ with a parallel section of $\operatorname{Re} \Lambda^4_{\mathbb{C}}(V, \mathbf{i})$ on a Riemann surface M. Suppose M is equipped with a nowhere vanishing holomorphic vector field X so that we can use the refined data described in this section.

i) First data is a hermitian complex line bundle $(\hat{L}, \nabla^L)$. This bundle is a square root of the bundle (E, ∇^E) (which is supposed to become $\Lambda^2 \hat{V}$ of the induced complex projective structure). The bundle E will be shown (cf. Lemma 92) to be gauge equivalent to the *Möbius normal bundle* of the immersion corresponding to the quaternionic projective structure.

ii) Then there is $\mathfrak{c}'$ which defines a normalized complex projective structure $(\hat{V}, \hat{L}, \hat{\nabla})$ with $\hat{V} = \hat{L} \oplus \hat{L}$ (where we consider $\hat{L}$ as a subbundle $\hat{L} \subset \hat{V}$ by taking the first summand) and

$$\hat{\nabla}^{\tilde{}} = \begin{pmatrix} \nabla^L & \mathfrak{c}' \\ dz & \nabla^L \end{pmatrix}.$$

The data $\mathfrak{c}'$ is called the *Schwarzian derivative* of the quaternionic projective structure with respect to the vector field X.

iii) Last there are the differentials $\mathfrak{a}$ and $\mathfrak{q}$. They give $A_{|L}$ and $\pi Q_{|V/L}$ and by equations (83), (84) and (85) they determine $\mathfrak{c}''$ and $\mathfrak{b}$ and $\mathfrak{p}$ and therefore $\hat{\nabla}$ and A and Q. We have seen that the information given by $\mathfrak{a}$ and $\mathfrak{q}$ is that of the *trace free second fundamental form*. Another possible interpretation is that $\mathfrak{a}$ and $\mathfrak{q}$ define the Möbius invariant holomorphic structures on the bundle L^{-1} and V/L. (Note that $\hat{\nabla}^{\tilde{}}$ already defines the underlying $\bar{\partial}$ operators on both bundles.)

13. Link to the Lightcone Model

This section provides a link between the quaternionic and the classical lightcone model of surface theory in conformal 4–space. The link presented here seems to be new. A different approach, which uses quaternionic hermitian forms to establish such link between both models, is explained in [**15**].

13.1. Some Linear Algebra. The main idea behind the link between both models of the conformal 4–sphere is a simple observation from linear algebra. For a 2–dimensional quaternionic vector space V, every element of the Grassmannian of complex 2–planes in $(V, \mathbf{i})$ corresponds either to a point or an oriented 2–sphere in $\mathbb{P}V$, depending on whether the 2–plane is a quaternionic line or not. When the Grassmannian is, as usual, represented as the Null quadric with respect to a non–degenerate bilinear form $\langle\,,\rangle$ on $\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$, the quaternionic lines admit a natural interpretation as the real part of this quadric with respect to a real structure τ. This establishes the link to the lightcone model, because the restriction of $\langle\,,\rangle$ to the real part of $\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ is a Minkowski metric.

The Real Structure and the Bilinear Form. Let V be a quaternionic rank 2 vector space. The vector space $\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ has the real structure τ given by $\tau(v_1 \wedge v_2) = v_1\mathbf{j} \wedge v_2\mathbf{j}$. It is compatible with the real structure on $\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$ given by $\tau(v_1 \wedge v_2 \wedge v_3 \wedge v_4) = v_1\mathbf{j} \wedge v_2\mathbf{j} \wedge v_3\mathbf{j} \wedge v_4\mathbf{j}$ (see also the discussion preceding Lemma 79) in the sense that $\tau(b_1 \wedge b_2) = \tau b_1 \wedge \tau b_2$ for all b_1, $b_2 \in \Lambda^2_{\mathbb{C}}(V, \mathbf{i})$. In the following, both real structures will be denoted by $\bar{b} := \tau b$, too. We denote by $\mathcal{V} = \operatorname{Re} \Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ the 6–dimensional real vector space consisting of real elements of $\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ with respect to τ. Clearly, the complexification $\mathcal{V}^{\mathbb{C}}$ of $\mathcal{V}$ is again $\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$.

In the following, we fix a positive element of $\operatorname{Re} \Lambda^4_{\mathbb{C}}(V, \mathbf{i})$ (recall that the positive direction in $\operatorname{Re} \Lambda^4_{\mathbb{C}}(V, \mathbf{i})$ is defined by $v_1 \wedge v_2 \wedge v_1\mathbf{j} \wedge v_2\mathbf{j}$ for an arbitrary basis v_1, v_2 of V). This defines an isomorphism $\operatorname{Re} \Lambda^4_{\mathbb{C}}(V, \mathbf{i}) \cong \mathbb{R}$ as well as an isomorphism $\Lambda^4_{\mathbb{C}}(V, \mathbf{i}) \cong \mathbb{C}$.

LEMMA 87. *Using this isomorphism* $\operatorname{Re} \Lambda^4_{\mathbb{C}}(V, \mathbf{i}) \cong \mathbb{R}$*, the* $\wedge$*–product becomes a complex symmetric bilinear form on* $\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ *which we denoted by* $\langle\,,\rangle$*. Its restriction to* $\mathcal{V} = \operatorname{Re} \Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ *is a Minkowski metric.*

PROOF. The bilinear form $\langle\,,\rangle$ is obviously non–degenerate on $\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$. The restriction to $\mathcal{V}$ is a real symmetric bilinear form. To prove that it is a Minkowski metric, we take a basis v_1, v_2 of V. Then $b_1 = v_1 \wedge v_1 j$ and $b_2 = v_2 \wedge v_2\mathbf{j}$ are two real lightlike vectors in $\mathcal{V}$ with $\langle b_1, b_2\rangle = v_1 \wedge v_1\mathbf{j} \wedge v_2 \wedge v_2\mathbf{j} < 0$ and therefore span a 2–dimensional Minkowski space. The vectors $v_1 \wedge v_2$ and $v_1 \wedge v_2\mathbf{j}$ both are Null vectors in $\mathcal{V}^{\mathbb{C}}$, orthogonal to each other and orthogonal to the real 2–dimensional Minkowski space $\operatorname{Span}\{b_1, b_2\}$. Since each of the two vectors has positive scalar product with its conjugate, their real and imaginary parts have to be spacelike vectors. They form an orthogonal basis of spacelike vectors, and therefore span a 4–dimensional Euclidean vector space, which is the orthogonal complement of the 2–dimensional Minkowski space $\operatorname{Span}\{b_1, b_2\}$. This shows that the metric on $\mathcal{V}$ is Minkowski. ☐

An element $b \in \Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ satisfies $\langle b, b\rangle = b \wedge b = 0$ if and only if it is decomposable, i.e. $b = v_1 \wedge v_2$ for v_1, $v_2 \in V$. This defines a 1-1–correspondence between Null lines in $\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ and 2–dimensional complex subspaces in $(V, \mathbf{i})$, realizing the Grassmannian of complex 2–planes in $(V, \mathbf{i})$ as the projective Null quadric of the complex bilinear form $\langle\,,\rangle$ in $\mathbb{P}\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$. This quadric is usually called the *Plücker quadric*.

Identification of Points. The real part of the Plücker quadric is the projectivization $\mathbb{P}\mathscr{L}$ of the lightcone $\mathscr{L} \subset \mathcal{V}$ in the 6–dimensional Minkowski space $\mathcal{V} = \operatorname{Re}\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$. The elements of this real part correspond to the complex 2–planes in $(V, \mathbf{i})$ that are invariant under multiplication by $\mathbf{j}$ or, equivalently, to the quaternionic lines in V, which yields the bijection

$$[v] \in \mathbb{P}V \mapsto [v \wedge v\mathbf{j}] \in \mathbb{P}\mathscr{L} \tag{87}$$

from $\mathbb{P}V$ to the projectivized lightcone $\mathbb{P}\mathscr{L}$, and therefore provides a very natural link between the quaternionic model (cf. Section 2.3) and the classical lightcone model (cf. Appendix A) of the conformal 4–sphere.

Identification of 2–Spheres. The non–real elements of the null quadric in $\mathbb{P}\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ correspond to the complex 2–planes in $(V, \mathbf{i})$ that are not invariant under multiplication by $\mathbf{j}$. By Lemma 10 and Lemma 12, the elements of this set uniquely describe the oriented 2–sphere in $\mathbb{P}V$.

Denote by $\hat{V}$ a 2–dimensional complex plane in $(V, \mathbf{i})$ defining a 2–sphere, and let ψ, φ be a basis of $\hat{V}$. The line $[\psi\wedge\varphi] \in \mathbb{P}\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ uniquely determines $\hat{V}$. Furthermore, the real and imaginary part of the complex null vector $\psi\wedge\varphi$ form a (real) orthonormal basis of a 2–dimensional real vector space, which we call $\mathcal{W}^{\perp}$. A complex structure J on $\mathcal{W}^{\perp}$ compatible with the metric is obtained by setting $J\psi\wedge\varphi = \psi\wedge\varphi\mathbf{i}$. Then, for $n = \frac{1}{2}\operatorname{Re}(\psi\wedge\varphi)$, we have $\psi\wedge\varphi = \frac{1}{2}(n - Jn\mathbf{i})$. Equivalently, instead of prescribing J, one can define an orientation on $\mathcal{W}^{\perp}$ by declaring that $\operatorname{Im}(\psi \wedge \varphi)$, $\operatorname{Re}(\psi \wedge \varphi)$ forms a positive basis.

Obviously, the vector space $\mathcal{W}^{\perp}$ and J (i.e. the orientation) only depend on the complex line in $\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ given by $\psi \wedge \varphi$. Different elements in this line correspond to different positive orthogonal bases of $\mathcal{W}^{\perp}$.

LEMMA 88. *A line $[\psi\wedge\varphi] \in \mathbb{P}\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ defines at the same time an oriented 2–sphere in $\mathbb{P}V$ and, via polarity, an oriented 2–sphere $\mathbb{P}(\mathcal{W}\cap\mathscr{L})$ in $\mathbb{P}\mathscr{L}$. Under the bijection* (87) *from $\mathbb{P}V$ to $\mathbb{P}\mathscr{L}$, both 2–spheres are identified.*

Proof. To prove this, it suffices to note that for a basis ψ, φ of $\hat{V}$, the vectors $\psi \wedge \psi\mathbf{j}$ and $\varphi \wedge \varphi\mathbf{j}$ are mapped into $\mathcal{W}$, which is the real vector space orthogonal to $\psi \wedge \varphi$. Since every vector in $\hat{V}\backslash\{0\}$ can be complemented to a basis and since the line given by $[\psi \wedge \varphi]$ is independent of the basis of $\hat{V}$, every point of the 2–sphere in $\mathbb{P}V$ goes to a point of $\mathbb{P}(\mathcal{W} \cap \mathscr{L})$. □

Let $\psi \wedge \varphi \in \Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ define an oriented 2–sphere. The same 2–sphere with the inverse orientation is then given by $\psi\mathbf{j} \wedge \varphi\mathbf{j}$. In the quaternionic case, this changes $\hat{V} = \mathrm{Span}_{\mathbb{C}}\{\psi, \varphi\}$ to $\hat{V}\mathbf{j}$. In the lightcone case, this changes J to $-J$, which can be seen by conjugating $J\psi \wedge \varphi = \psi \wedge \varphi\mathbf{i}$.

Tangent and Normal Spaces of 2–Spheres. The link between the quaternionic and the lightcone model is given by the bijection (87). Its differential describes the corresponding bijection

$$B \in \mathrm{Hom}(L, V/L) \mapsto \\ (\psi \wedge \psi\mathbf{j} \mapsto B\psi \wedge \psi\mathbf{j} + \psi \wedge B\psi\mathbf{j} \mod \mathcal{L}) \in \mathrm{Hom}(\mathcal{L}, \mathcal{L}^{\perp}/\mathcal{L}) \tag{88}$$

between the tangent spaces at $L = \psi\mathbb{H} \in \mathbb{P}V$ and $\mathcal{L} = \psi \wedge \psi\mathbf{j}\mathbb{R} \in \mathbb{P}\mathscr{L}$.

Lemma 89. *Let S be an oriented 2–sphere in $\mathbb{P}V$. The induced isomorphism between the normal space $\mathrm{Hom}_-(L, V/L)$ at $L = \psi\mathbb{H} \in \mathbb{P}V$ and $\mathrm{Hom}(\mathcal{L}, \mathcal{W}^{\perp})$ at $\mathcal{L} = \psi \wedge \psi\mathbf{j}\mathbb{R} \in \mathbb{P}\mathscr{L}$ is compatible with the complex structures.*

Proof. By Lemma 10 and Lemma 12, an oriented 2–sphere through $L = \psi\mathbb{H} \in \mathbb{P}V$ is given by $\hat{V} \subset (V, \mathbf{i})$. For every non–trivial $B \in \mathrm{Hom}_-(L, V/L)$, one can chose a basis ψ, φ of $\hat{V}$ such that

$$B\psi = \varphi\mathbf{j} \mod L.$$

The corresponding normal vector in the lightcone model is then

$$(\psi \wedge \psi\mathbf{j} \mapsto \underbrace{\varphi\mathbf{j} \wedge \psi\mathbf{j} - \psi \wedge \varphi}_{-2\,\mathrm{Re}(\psi\wedge\varphi)}) \in \mathrm{Hom}(\mathcal{L}, \mathcal{W}^{\perp}).$$

The complex structures of both versions of the normal bundle are compatible, because under the isomorphism, SB goes to

$$\psi \wedge \psi\mathbf{j} \mapsto 2\,\mathrm{Im}(\psi \wedge \varphi) = -2J\,\mathrm{Re}(\psi \wedge \varphi).$$

□

Remark. The analogous result for tangent vectors is the following: $B\psi = \varphi$ goes to

$$(\psi \wedge \psi\mathbf{j} \mapsto \underbrace{\varphi \wedge \psi\mathbf{j} + \psi \wedge \varphi\mathbf{j}}_{2\,\mathrm{Re}(\varphi\wedge\psi\mathbf{j})} \mod \mathcal{L}) \in \mathrm{Hom}(\mathcal{L}, \mathcal{L}^{\perp}/\mathcal{L})$$

and SB goes to

$$(\psi \wedge \psi\mathbf{j} \mapsto \underbrace{\varphi\mathbf{i} \wedge \psi\mathbf{j} + \psi \wedge \varphi\mathbf{i}\mathbf{j}}_{-2\,\mathrm{Im}(\varphi\wedge\psi\mathbf{j})} \mod \mathcal{L}) \in \mathrm{Hom}(\mathcal{L}, \mathcal{L}^{\perp}/\mathcal{L})$$

.

13.2. From Quaternionic Projective Structures to the Lightcone Model. This subsection describes the link between quaternionic projective structures and conformal immersions into Minkowski space bundles induced by the algebraic version of the preceding subsection. Furthermore, the relation between the invariants arising in the fundamental theorems is described.

Link between the Projective Structures. Let (V, L, ∇) be a quaternionic projective structure. We suppose that $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$ has a fixed positive parallel section, in particular ∇ is flat. We have seen that the section of $\operatorname{Re}\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$ defines a non–degenerate bilinear product $\langle\,,\rangle$ on the vector bundle $\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$, which restricts to a Lorentzian metric on the 6–dimensional real vector bundle $\mathcal{V} = \operatorname{Re}\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ (see Lemma 87). The connection induced by ∇ on $\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ commutes with τ, and therefore defines a connection on $\mathcal{V}$ which is denoted by ∇, too. The metric $\langle\,,\rangle$ is parallel with respect to this connection. If (V, ∇) has trivial monodromy, then $(\mathcal{V}, \nabla)$ is a trivial Minkowski space bundle.

Remark. The converse is not true. It can happen that the $\mathrm{SL}(2, \mathbb{H})$ monodromy is non–trivial, but acts trivially on $\operatorname{Re}\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$. This can be repaired by tensoring by a $\mathbb{Z}_2$–bundle.

The fiberwise bijection (87) from $\mathbb{P}V$ to $\mathbb{P}\mathscr{L}$ maps the quaternionic line subbundle $L \subset V$ to a real line subbundle $\mathcal{L} \subset \mathscr{L}$ of the lightcone in $\mathcal{V}$. The map $\mathcal{L} \subset \mathscr{L}$ is a conformal immersion into the Minkowski space bundle $(\mathcal{V}, \nabla)$.

As described in Section 13.1, the mean curvature sphere congruence $\hat{V}$ of (V, L, ∇) defines a 2–sphere congruence $\mathcal{W}$ in $(\mathcal{V}, \nabla)$, which is, as one should expect, the mean curvature sphere congruence of $\mathcal{L} \subset \mathscr{L}$.

Link between the Invariants. Let X be a nowhere vanishing holomorphic vector field on the underlying Riemann surface M. In both models, such vector field is needed to defined the data (in the quaternionic model, for the splitting of $\hat{V}$, cf. Section 12.3, and in the lightcone case for the normalized section, cf. Appendix A). For a nowhere vanishing section $\psi \in \Gamma(\hat{L})$, there is frame ψ, φ, where $\varphi = \delta_X\psi \in \Gamma(\hat{V})$ is calculated using the splitting with respect to X (cf. Lemma 78). Globally, such section exists if and only if the bundle $\hat{L}$ is topologically trivial, which, for tori, is equivalent to normal bundle degree 0. To simplify the formulae, we assume that ψ has constant length with respect to the metric on $\hat{L}$ induced by that on $\Lambda^2\hat{V}$ (cf. Corollary 84) by the isomorphism

$$\hat{L}^2 \cong \Lambda^2\hat{V} \quad \psi^2 \mapsto \psi \wedge \delta_X\psi. \tag{89}$$

With respect to this frame ψ, φ, the connection ∇ is given by equation (86), which is

$$\nabla(\psi, \varphi) = (\psi, \varphi)\left(\begin{pmatrix} g\,dz - \bar{g}\,d\bar{z} & -\frac{1}{2}c\,dz + e\,d\bar{z} \\ dz & g\,dz - \bar{g}\,d\bar{z} \end{pmatrix} + \begin{pmatrix} a\,dzj & b\,dzj + \bar{p}\,d\bar{z}j \\ 0 & \bar{q}\,d\bar{z}j \end{pmatrix}\right),$$

where dz denotes the 1–form dual to X, and z is a coordinate on the universal covering of M. In the following, all z– and $\bar{z}$–derivatives of sections of the bundle $\mathcal{V}^{\mathbb{C}} = \Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ are taken with respect to ∇, e.g.

$$\Psi_z = \tfrac{1}{2}(\nabla_X\Psi - \nabla_{JX}\Psi\mathbf{i}).$$

In contrast to z, the z– and $\bar{z}$–derivatives are well defined on M itself.

Let $\Psi = \psi \wedge \psi\mathbf{j} \in \Gamma(\mathcal{L})$ be the induced section of $\mathcal{L} \subset \mathscr{L}$. A simple calculation using the frame equation shows $\Psi_z = \varphi \wedge \psi\mathbf{j}$. This implies conformality, i.e. $\langle \Psi_z, \Psi_z \rangle = 0$. Furthermore, $\langle \Psi_z, \Psi_{\bar{z}} \rangle = \psi \wedge \varphi \wedge \psi\mathbf{j} \wedge \varphi\mathbf{j}$ or, equivalently,

$$|\nabla\Psi|^2 = 2\,\psi \wedge \varphi \wedge \psi\mathbf{j} \wedge \varphi\mathbf{j}\,|dz|^2. \tag{90}$$

Taking $\psi \in \Gamma(\hat{L})$ with $\langle \psi, \psi \rangle_{\hat{L}} = 1/\sqrt{2}$, where the metric on $\hat{L}$ is induced by the isomorphism (89), the section Ψ is normalized. Such section $\psi \in \Gamma(\hat{L})$ is only determined up to multiplication by a S^1–valued function, which is not surprising, because it also defines a section $\xi = \psi \wedge \varphi \in \Gamma((\mathcal{W}^\perp)^\mathbb{C})$ of the complexified Möbius normal bundle. This section can be written as $\xi = \frac{1}{2}(n - Jn\mathbf{i})$ where $n \in \Gamma(\mathcal{W}^\perp)$ is a real section. Because $\bar{\xi} \wedge \xi = \psi\mathbf{j} \wedge \varphi\mathbf{j} \wedge \psi \wedge \varphi$, the section n has length 1. Prescribing such section $n \in \Gamma(\mathcal{W}^\perp)$ determines ψ up to sign.

Remark. If $\psi \in \Gamma(\hat{L})$ is a section such that $\Psi = \psi \wedge \psi\mathbf{j} \in \Gamma(\mathcal{L})$ is normalized, the normalized frame for $\mathcal{W}^\mathbb{C}$ (see Section 1.1 of Appendix A) is $\Psi = \psi\wedge\psi\mathbf{j}$, $\Psi_z = \varphi \wedge \psi\mathbf{j}$, $\Psi_{\bar{z}} = \psi \wedge \varphi\mathbf{j}$ and $\hat{\Psi} = 2\varphi \wedge \varphi\mathbf{j}$. Furthermore, a frame for $\mathcal{W}^{\perp\mathbb{C}}$ is given by $\xi = \psi \wedge \varphi$ and $\bar{\xi} = \psi\mathbf{j} \wedge \varphi\mathbf{j}$.

To analyze the relation between the data a, q and c from Section 12.3 and between κ and c from Appendix A, we have to calculate

$$\begin{aligned}\Psi_{zz} &= (\varphi \wedge \psi\mathbf{j})_z = \varphi_z \wedge \psi\mathbf{j} + \varphi \wedge \psi_{\bar{z}}\mathbf{j}\\ &= (\varphi g - \frac{1}{2}\psi c + \varphi\mathbf{j}q) \wedge \psi\mathbf{j} + \varphi \wedge (-\psi\bar{g} + \psi a\mathbf{j})\mathbf{j}\\ &= -\frac{1}{2}\underbrace{\psi \wedge \psi\mathbf{j}}_{\Psi}\,c + \varphi\mathbf{j} \wedge \psi\mathbf{j}\,q - \varphi \wedge \psi\,a.\end{aligned}$$

For $\kappa = \xi a - \bar{\xi} q$ (with $\xi = \psi \wedge \varphi$) we obtain

$$\Psi_{zz} + \tfrac{c}{2}\Psi = \kappa.$$

Since this is equation (148), which defines the invariants from Appendix A, we have proven the following lemma.

Lemma 90. *Let X be holomorphic vector field, and $\psi \in \Gamma(\hat{L})$ a section such that $\Psi = \psi \wedge \psi\mathbf{j} \in \Gamma(\mathcal{L})$ is normalized. Then, the invariants in the lightcone setting are c and $\kappa = \xi a - \bar{\xi} q$ where $\xi = \psi\wedge\varphi$ and c, a and q are the quaternionic invariants with respect to the frame ψ, φ of $\hat{V}$.*

To complete the link between both models, we still have to analyze the relation between the normal bundles in both models. In Section 13.1 an isomorphism is defined between the projective normal bundles $\mathrm{Hom}_-(L, V/L)$ and $\mathrm{Hom}(\mathcal{L}, \mathcal{W}^\perp)$. Without further choices, these bundles do not come with a fiber metric and connection. But, in the presence of a fixed holomorphic vector field X, we have the normalized splitting of $\hat{V}$ and the normalized lift in $\Gamma(\mathcal{L})$.

Firstly, using the metric on $\hat{L}$, we get an isomorphism

$$\psi^2 = \psi \wedge \varphi \in \hat{L}^2 = \Lambda^2\hat{V} \mapsto (\tilde{\psi}\mathbf{j} \in \hat{L}\mathbf{j} \mapsto \varphi\langle\tilde{\psi}, \psi\rangle_{\hat{L}}) \in \mathrm{Hom}_-(L, V/L).$$

Secondly, the normalized section $\Psi \in \Gamma(\mathcal{L})$ induces an isomorphism between the projective normal bundle $\mathrm{Hom}(\mathcal{L}, \mathcal{W}^\perp)$ in the lightcone model and between the Möbius normal bundle $\mathcal{W}^\perp$.

LEMMA 91. *Let X be a holomorphic vector field. Then there is the commutative diagram of isomorphisms*

$$\begin{tikzcd} \mathrm{Hom}_-(L, V/L) \arrow[r] & \mathrm{Hom}(\mathcal{L}, \mathcal{W}^\perp) \arrow[d] \\ \Lambda^2 \hat{V} \arrow[u] \arrow[r, "\sqrt{2}\,\mathrm{Re}"] & \mathcal{W}^\perp \end{tikzcd}$$

where the upper horizontal arrow is the canonical isomorphism from Section 13.1 and the vertical arrows are the above isomorphisms induces by the normalization with respect to X. Furthermore, the lower horizontal arrow is an isometry and compatible with the connections.

PROOF. To proof the commutativity of the diagram, we take an element $\psi \wedge \varphi \in \Gamma(\Lambda^2 \hat{V})$ of length $\frac{1}{\sqrt{2}}$ (such that $\Psi = \psi \wedge \psi \mathbf{j} \in \Gamma(\mathcal{L})$ is the normalized section) and check whether its image under the composition

$$\Lambda^2 \hat{V} \overset{1.)}{\to} \mathrm{Hom}_-(L, V/L) \overset{2.)}{\to} \mathrm{Hom}(\mathcal{L}, \mathcal{W}^\perp) \overset{3.)}{\to} \mathcal{W}^\perp$$

is $\sqrt{2}\,\mathrm{Re}(\psi \wedge \varphi)$. The image under 1.) is the homomorphism

$$\tilde{\psi}\mathbf{j} \in \hat{L}\mathbf{j} \mapsto \varphi \langle \tilde{\psi}, \psi \rangle_{\hat{L}}.$$

Its image under 2.) is the homomorphism

$$\tilde{\psi} \wedge \tilde{\psi}\mathbf{j} \mapsto -\varphi \langle \tilde{\psi}, \psi \rangle_{\hat{L}} \mathbf{j} \wedge \tilde{\psi}\mathbf{j} + \tilde{\psi} \wedge \varphi \langle \tilde{\psi}, \psi \rangle_{\hat{L}}$$

and its image under 3.) is

$$\frac{1}{\sqrt{2}}(-\varphi\mathbf{j} \wedge \psi\mathbf{j} + \psi \wedge \varphi) = \sqrt{2}\,\mathrm{Re}(\psi \wedge \varphi).$$

This proves commutativity.

We have seen above that $\psi \wedge \varphi = \frac{1}{2}(n - Jn\mathbf{i})$ for $n \in \Gamma(\mathcal{W}^\perp)$ a section of length 1. Therefore, $\sqrt{2}\,\mathrm{Re}(\psi \wedge \varphi) = \frac{1}{\sqrt{2}} n$ has length $\frac{1}{\sqrt{2}}$ and the lower horizontal arrow is an isometry.

It remains to prove that the lower horizontal arrow is a gauge equivalence. From Equation (86) we obtain the formula

$$\hat{\nabla}(\psi \wedge \varphi) = \psi \wedge \varphi(2g\,dz - 2\bar{g}\,d\bar{z})$$

for the connection $\hat{\nabla}$ on $\Lambda^2 \hat{V}$, as well as

$$\begin{aligned} \nabla\xi &= \nabla\psi \wedge \varphi + \psi \wedge \nabla\varphi \\ &= (\psi(g\,dz - \bar{g}\,d\bar{z}) + \psi\mathbf{j}\,\bar{a}\,dz) \wedge \varphi + \\ &\qquad + \psi \wedge (\varphi(g\,dz - \bar{g}\,d\bar{z}) + \psi\mathbf{j}(\bar{b}\,d\bar{z} + p\,dz) + \varphi\mathbf{j}\,q\,dz). \end{aligned}$$

This shows that the normal connection D on $\mathcal{W}^\perp$ is given by

$$D\xi = \xi(2g\,dz - 2\bar{g}\,d\bar{z}).$$

Therefore, because Re is a gauge equivalence from $(((\mathcal{W}^\perp)^{\mathbb{C}})^{(1,0)}, D)$ to $(\mathcal{W}, D)$, the lower horizontal arrow as well is a gauge equivalence. $\square$

Link to the Metrical Normal Bundles. Fixing a basis e_1, e_2 of the quaternionic rank 2 vector space V (i.e. an isomorphism $V \cong \mathbb{H}^2$) determines a Euclidean chart of $\mathbb{P}V$, whose inverse is the embedding

$$x \in \mathbb{H} \to [e_1 x + e_2] \in \mathbb{P}V.$$

A Minkowski metric on $\mathcal{V} = \operatorname{Re}\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ is defined, via Lemma 87, by

$$1 \cong 2e_1\mathbf{j} \wedge e_2\mathbf{j} \wedge e_1 \wedge e_2 \in \operatorname{Re}\Lambda^4_{\mathbb{C}}(V, \mathbf{i}).$$

Under the identification (87) of $\mathbb{P}V$ with the projective lightcone $\mathbb{P}\mathscr{L}$ of $\mathcal{V}$, the point $[e_1 x + e_2] \in \mathbb{P}V$ goes to

$$(91)\quad \Phi = 2\operatorname{Re}(e_1 \wedge e_2\mathbf{j}(x_0 + x_1\mathbf{i})) + 2\operatorname{Re}(e_1\mathbf{j} \wedge e_2\mathbf{j}(x_2 + x_3\mathbf{i})) + \\ + e_1 \wedge e_1\mathbf{j}|x|^2 + e_2 \wedge e_2\mathbf{j}$$

where $x = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$. This yields an isometry from $\mathbb{H}$ to the Euclidean space form $\mathcal{S}_\infty$ with $\infty = 2e_1 \wedge e_1\mathbf{j}$ (cf. Section 2 of Appendix A).

Remark. In formula (91), the isometry from $\mathbb{H}$ to $\mathcal{S}_\infty$ is defined in terms of a basis of complex Null vectors. For the convenience of the reader, we give here the same map with respect to the real orthonormal basis $v_0, ..., v_5$ of $\mathcal{V}$ (with v_0 a lightlike vector) defined by

$$\begin{aligned} e_1 \wedge e_1\mathbf{j} &= \frac{1}{2}(v_0 - v_5) & e_2 \wedge e_2\mathbf{j} &= \frac{1}{2}(v_0 + v_5) \\ e_1 \wedge e_2\mathbf{j} &= \frac{1}{2}(v_1 - v_2\mathbf{i}) & e_2 \wedge e_1\mathbf{j} &= \frac{1}{2}(v_1 + v_2\mathbf{i}) \\ e_1\mathbf{j} \wedge e_2\mathbf{j} &= \frac{1}{2}(v_3 - v_4\mathbf{i}) & e_1 \wedge e_2 &= \frac{1}{2}(v_3 + v_4\mathbf{i}). \end{aligned}$$

With respect to this basis, Φ takes the form

$$\Phi = \left(\frac{1 + |x|^2}{2}, x_0, x_1, x_2, x_3, \frac{1 - |x|^2}{2}\right),$$

which is exactly the embedding of Euclidean space into the lightcone used in [**15**, Section 1.4.3].

With respect to e_1, e_2, an immersion $L \subset V$ into $\mathbb{P}V$ takes the form

$$L = \psi\mathbb{H} \quad \text{with} \quad \psi = \begin{pmatrix} f \\ 1 \end{pmatrix},$$

where $f\colon M \to \mathbb{H}$ is the Euclidean realization of L with respect to the Euclidean chart defined by e_1, e_2. Under the identification (87), the section ψ goes to

$$(92)\quad \Phi = 2\operatorname{Re}(e_1 \wedge e_2\mathbf{j}(f_0 + f_1\mathbf{i})) + 2\operatorname{Re}(e_1\mathbf{j} \wedge e_2\mathbf{j}(f_2 + f_3\mathbf{i})) + \\ + e_1 \wedge e_1\mathbf{j}|f|^2 + e_2 \wedge e_2\mathbf{j}$$

where $f = f_0 + \mathbf{i}f_1 + \mathbf{j}f_2 + \mathbf{k}f_3$. The section $\Phi \in \Gamma(\mathcal{L})$ is the metrical lift of the conformal immersion $\mathcal{L} \subset \mathscr{L}$ with respect to the Euclidean space form $\mathcal{S}_\infty$ with $\infty = 2e_1 \wedge e_1\mathbf{j}$.

Because (92) is the restriction of the isometry (91) to $f\colon M \to \mathbb{H}$, the conformal immersions $f\colon M \to \mathbb{H}$ and $\Phi\colon M \to \mathcal{S}_\infty$ have gauge equivalent metrical normal bundles.

Together with Lemma 91 and Lemma 103, this proves:

LEMMA 92. *Let (V, L, ∇) be a quaternionic projective structure with $V = \mathbb{H}^2$ and $\nabla = d$. Then, the complex line bundle $\Lambda^2\hat{V}$ with the connection $\hat{\nabla}$ is gauge equivalent to the metrical normal bundle of L with respect to an arbitrary Euclidean chart.*

CHAPTER V

Flows

This last chapter is about flows of tori in the conformal 4–space. In contrast to the integrable transformations of Chapter II, the flows discussed here are flows in the proper sense, i.e. they are given by evolution equations depending on a real parameter. The focus is on the evolution related to the Davey–Stewartson hierarchy, an important hierarchy of soliton equations.

The use of the Davey–Stewartson hierarchy in the context of surface theory was introduced by Konopelchenko [**18**], who defined the flows in an Euclidean setting, as a deformation of the Dirac potentials arising in the Weierstrass representation. A more geometric definition of the flows has been given by Burstall, Pedit and Pinkall [**4**]. Their approach defines the flows in a Möbius geometric setting, but at the same time allows a simple treatment of the resulting flows in the metrical subgeometries. In this Möbius geometric setting, it becomes an obvious fact that the flows are Möbius invariant, which is quite hard to prove in the original Euclidean setting.

Section 14 contains the necessary foundations for the treatment of flows. A set of formulae for conformal deformations is derived in both the quaternionic and the lightcone model. The concept of projective structures as a unifying theme allows to apply the same ideas in both cases. The deformation formulae in the lightcone case are easier to handle, but the link (of Section 13) between both models allows a simple translation between both sets of deformations formulae. For the purist supporter of the quaternionic approach to surface theory, the fact that the lightcone formulae are simpler to use is probably somewhat dissatisfactory. Until some better formulae are found, he might still use the lightcone formulae as a purely formal tool to simplify calculations with the quaternionic data, ignoring their origin from the lightcone model.

In Section 15, the Davey–Stewartson and Novikov–Veselov flows are defined. It is shown that in the case of surfaces obtained from curves in 3–dimensional space forms, the evolution of the invariants reduces to the well known nonlinear Schrödinger equation (NLS) and the modified Korteweg–de–Vries equation (mKdV). Furthermore, in the case of constant mean curvature (CMC) surfaces in 3–dimensional space forms, one immediately recovers the evolution of CMC surfaces described by Pinkall and Sterling [**21**]. The main result of this section is Theorem 13, which, together with a theorem by Richter, shows that CMC tori in 3–dimensional space forms can be characterized by the fact that they are stationary under the Davey–Stewartson flow.

Finally, Section 16 brings together the Davey-Stewartson flow with the discrete flows, i.e. the Darboux transformations discussed in Section 7. It is proven that the continuous flows, under certain assumptions, can be obtained as a limit of Darboux transformations.

A Short Remark on Integrable Flows. With the word flow appearing already in the title of this thesis, it seems to be appropriate to give a short explanation of the notion of flow, which we use exclusively in the context of integrable evolution equations.

Integrable evolution equations are infinite dimensional analogues of finite dimensional completely integrable systems in the sense of Arnold–Liouville: an integrable evolution equation is a Hamiltonian equation admitting an infinite family of first integrals (symmetries) that commute pairwise and allow to find explicit solutions to the equation (because the equation gets linearized on the level sets of the integrals). Clearly, every Hamiltonian vector field that belongs to one of the first integrals gives rise to another integrable evolution equation with the same family of integrals. The family of all evolution equations arising this way is usually referred to as a hierarchy of integrable evolution equations, the flows of such family of Hamiltonian vector fields are the ones we have in mind when we talk about flows.

In general, the equations of a hierarchy are related to isospectral deformations of solutions to some linear equation. They are usually defined as a commutativity condition for ordinary differential operators (in the case of differential operator valued Lax equations) or matrix polynomials (in the case of loop algebra valued Lax equations).

A quite different approach for defining a hierarchy is a geometrical recursion scheme as used in [**21**] and [**4**] for treating the Sinh–Gordon and Korteweg–de–Vries hierarchies. In this geometrical approach, one considers both "extrinsic" deformations of an object (curve, surface) in space (given by the normal deformation) and "intrinsic" deformations (given by the deformation of the differential invariants). The hierarchy of flows is defined recursively by cleverly interchanging the role of extrinsic and intrinsic deformations: one starts with a simple extrinsic deformation, calculates the corresponding intrinsic deformation of the invariants and uses this to define a new extrinsic deformation and so on.

This remark on flows is mainly included as a motivation. In the following, a discussion of the Hamiltonian aspect of the theory is completely missing. Our discussion of the flows is in the spirit of the geometrical approach described in the last paragraph. Because the right recursion scheme is not clear by now, it is not yet possible to give a geometrical description of the whole Davey–Stewartson hierarchy (only the first 4 flows are discussed below). Nevertheless, it is quite clear that a whole hierarchy can be defined (for example, because it can be done by the method of Konopelchenko and Taimanov, cf. [**17**], [**18**] and [**26**], using the ordinary Davey–Stewartson hierarchy as an evolution of Dirac potentials).

14. Deformations of Projective Structures

The aim of this section is to derive a set of formulae for conformal deformations of immersions. This is done in both the quaternionic and the lightcone model. In both cases, a conformal immersion is represented as a projective structure on a Riemann surface, which consists of a vector bundle with a flat connection ∇ describing the 'background geometry' and a line subbundle describing the immersion: in the quaternionic case, the immersion is given by a quaternionic projective structure (V, L, ∇) with a flat connection (see Section 12), in the lightcone case it is given by a Minkowski space bundle $(\mathcal{V}, \nabla)$ and a subbundle $\mathcal{L} \subset \mathscr{L}$ of the lightcone bundle (see Appendix A).

The approach for studying conformal deformations is essentially the same in both models: the vector bundle and its line subbundle are kept fixed, while the flat connection is deformed by a family of gauge transformations. Practically, this is achieved by deforming the invariants with respect to a normalized frame of the projective structure.

14.1. Deformations of Quaternionic Projective Structures – The Invariant Part. An important concept in our treatment of flows is the following philosophy due to Ulrich Pinkall: in the study of deformations of immersions, one can either leave the background geometry (V, ∇) fixed and vary the immersion L, which we call the *Schrödinger picture*, or one can leave V and L fixed and vary the background geometry ∇, which we call the *Heisenberg picture*[1].

Why the Heisenberg Picture can be Applied. The following theorem shows that all conformal immersions of a Riemann surface with the same degree can be obtained by fixing V, L and S and changing ∇. (This theorem can be seen as a quaternionic analogue to Theorem 7.)

THEOREM 10. *Let $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$ and $\tilde{\mathcal{P}_{\mathbb{H}}} = (\tilde{V}, \tilde{L}, \tilde{\nabla})$ be two quaternionic projective structures of the same degree on a Riemann surface M. Then, the quaternionic projective structure $\tilde{\mathcal{P}_{\mathbb{H}}}$ is gauge equivalent to $(V, L, \nabla + \omega)$ for a form $\omega \in \Omega^1(\mathrm{End}(V))$. The gauge transformation can be chosen such that (V, L, ∇) and $(V, L, \nabla + \omega)$ have the same mean curvature sphere.*

PROOF. Denote by $\hat{V}$ and $\hat{L}$ the complex subbundles corresponding to the mean curvature sphere S of $L \subset V$. Similarly, denote by $\hat{\tilde{V}}$ and $\hat{\tilde{L}}$ the complex subbundles corresponding to the mean curvature sphere $\tilde{S}$ of $\tilde{L} \subset \tilde{V}$. Since both immersions have the same degree, there is an isomorphisms $g \colon \hat{L} \to \hat{\tilde{L}}$. Choosing splittings $\hat{V} = \hat{L} \oplus L'$ and $\hat{\tilde{V}} = \hat{\tilde{L}} \oplus \tilde{L}'$, the fact that both immersions are conformal implies that there is an isomorphism $g' \colon L' \to \tilde{L}'$ making the diagram

$$\begin{tikzcd} \hat{L} \arrow[r, "g"] \arrow[d, "\delta"'] & \hat{\tilde{L}} \arrow[d, "\tilde{\delta}"] \\ KL' \arrow[r, "g'"'] & K\tilde{L}' \end{tikzcd}$$

commutative. The gauge transformation $G \colon V \to \tilde{V}$ defined by g and g' satisfies $\tilde{L} = GL$ and $\tilde{S} = GSG^{-1}$. Therefore, the quaternionic projective structure

[1] The Heisenberg picture has already been applied in our treatment of the associated family of Section 5, where not only V and L, but also the mean curvature sphere S were left fixed.

$(V, L, \nabla + \omega)$, with $\omega \in \Omega^1(\mathrm{End}(V))$ defined by $\nabla + \omega = G^{-1} \circ \nabla \circ G$, is gauge equivalent to $(\tilde{V}, \tilde{L}, \tilde{\nabla})$ and has the same mean curvature sphere. $\square$

REMARK. It should be noted that the gauge transformation G as well as the form ω constructed in the preceding lemma are not unique: there is still the freedom of one $\mathbb{C}$–valued function in the choice of the splittings and the freedom of one $\mathbb{C}^*$–valued function in the choice the g. By using splittings induced by a conformal vector field X (cf. Lemma 78), the first choice can be fixed. Furthermore, in the case that $\mathcal{P}_{\mathbb{H}}$ and $\tilde{\mathcal{P}_{\mathbb{H}}}$ are equipped with parallel sections of $\mathrm{Re}\,\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$ and $\mathrm{Re}\,\Lambda^4_{\mathbb{C}}(\tilde{V}, \mathbf{i})$ respectively, one can chose g respecting the induced metrics on $\hat{L}$ and $\tilde{L}\hat{}$ (cf. Lemma 83). Then G is uniquely determined up to multiplication by a S^1–valued function.

LEMMA 93. *The two quaternionic projective structures* $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$ *and* $\tilde{\mathcal{P}_{\mathbb{H}}} = (V, L, \nabla + \omega)$ *(where* $\omega \in \Omega^1(\mathrm{End}(V))$*) have the same mean curvature sphere if and only if* ω *satisfies* $\omega'' L \subset L$, $\omega_- L \subset L$ *and* $\omega''_- L = 0$.

PROOF. Denote by S the mean curvature sphere of $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$ characterized by $*\delta = S\delta = \delta S$ and $Q_{|L} = 0$. The differential of L with respect to $\tilde{\nabla} = \nabla + \omega$ is $\tilde{\delta} = \delta + \pi\omega_{|L}$. Therefore, $*\tilde{\delta} = S\tilde{\delta}$ is equivalent to $\pi\omega''{}_{|L} = 0$, which again is equivalent to $\omega'' L \subset L$. In the same way, $S\tilde{\delta} = \tilde{\delta} S$ is equivalent to $\omega_- L \subset L$. Finally, the Hopf field of $\tilde{Q}$ of $\tilde{\nabla}$ being $\tilde{Q} = Q + \omega''_-$, the condition $\tilde{Q}_{|L} = 0$ is equivalent to $\omega''_- L = 0$. $\square$

Deformations and Infinitesimal Deformations. Let M be a Riemann surface. In the Schrödinger picture, a conformal deformation is given by a family $L_t \subset V$ with $t \in I \subset \mathbb{R}$ of holomorphic curves in (V, ∇). The link to the Heisenberg picture is given by Theorem 10, which shows that there is a family of gauge transformations $G_t \colon V \to V$ satisfying $L_t = G_t L$, such that every G_t is compatible with the mean curvature spheres. Otherwise stated, the family of conformal immersions is given by the family of quaternionic projective structures

$$(V, L, \nabla^t = \nabla + G_t^{-1} \nabla G_t), \tag{93}$$

which do all have the same mean curvature sphere.

Infinitesimally, this deformation is given by $\dot{\nabla} = \nabla Y$, where the endomorphism $Y = \dot{G} \in \Gamma(\mathrm{End}(V))$ is the infinitesimal gauge transformation belonging to the family G_t of gauge transformations (the dot, as usual, denoting the t–derivative at $t = 0$). By Lemma 93, the infinitesimal condition for Y to preserve the mean curvature sphere S is $(\nabla Y)'' L \subset L$, $(\nabla Y)_- L \subset L$ and $(\nabla Y)''_- L = 0$.

For every point $p \in M$, the projective derivative at $t = 0$ of the curve $L_t(p) = G_t(p) L(p)$ is given by $\pi Y(p)_{|L(p)} \in \mathrm{Hom}(L(p), V(p)/L(p))$. Therefore, we call $\pi Y_{|L} \in \Gamma(\mathrm{Hom}(L, V/L))$ the *deformation vector field* of the infinitesimal gauge transformation $Y \in \Gamma(\mathrm{End}(V))$. The vector fields

$$\pi(Y_-)_{|L} \in \Gamma(\mathrm{Hom}_-(L, V/L)) \quad \text{and} \quad \pi(Y_+)_{|L} \in \Gamma(\mathrm{Hom}_+(L, V/L))$$

are called the *normal– and tangential deformation* vector fields.

Definition. Let $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$ be a quaternionic projective structure. We call $Y \in \Gamma(\mathrm{End}(V))$ an *admissible infinitesimal gauge transformation* of $\mathcal{P}_{\mathbb{H}}$ with deformation vector field $\pi Y_{|L} \in \Gamma(\mathrm{Hom}(L, V/L))$ if

$$(\nabla Y)''_+ L \subset L, \quad (\nabla Y)_- L \subset L \quad \text{and} \quad (\nabla Y)''_- L = 0$$

are satisfied. The infinitesimal deformation of $\mathcal{P}_{\mathbb{H}}$ is then given by $\dot{\nabla} = \nabla Y$.

In the preceding discussion we have seen that, for the infinitesimal study of conformal deformation, it is sufficient to consider admissible infinitesimal gauge transformations. In the next section we explain, which deformation vector fields belong to admissible infinitesimal gauge transformations, and how far a deformation vector field determines an admissible infinitesimal gauge transformation.

14.2. Deformations of Quaternionic Projective Structures – Explicit Deformation Formulae with respect to a Frame. In this section, we derive the condition on a vector field $\tilde{Y} \in \Gamma(\mathrm{Hom}(L, V/L))$ to be the deformation vector field $\tilde{Y} = \pi Y_{|L}$ of an admissible infinitesimal gauge transformation $Y \in \Gamma(\mathrm{End}(V))$, and we give explicit formulae how $\pi Y_{|L}$ determines Y.

One difficulty in the treatment of deformations is that the admissible infinitesimal gauge transformation $Y \in \Gamma(\mathrm{End}(V))$ is not uniquely determined by the deformation vector field $\tilde{Y} = \pi Y_{|L}$. The situation here is quite analogous to that of Theorem 10: Y is not determined by $\pi Y_{|L}$ just like the gauge transformation G in the theorem in not determined by the conformal deformation (see the remark following Theorem 10). But like the choice of the gauge transformation G can be significantly reduced by using the splitting induced by a nowhere vanishing holomorphic vector field X and the metric induced by parallel sections of the bundles $\mathrm{Re}\,\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$, the choice of admissible infinitesimal gauge transformations $Y \in \Gamma(\mathrm{End}(V))$ belonging to $\tilde{Y} = \pi Y_{|L}$ can be reduced by imposing the analogous conditions on Y. The remaining freedom geometrically corresponds to (infinitesimal) gauge transformations of the normal bundle.

We therefore assume that, in the Schrödinger picture, the deformations are given by line subbundles $L_t \subset V$ of a vector bundle (V, ∇) admitting a parallel section of $\mathrm{Re}\,\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$. For the explicit calculation of (infinitesimal) deformations of quaternionic projective structures, we work with the frames described at the end of Section 12.3.

The Frame Equations. Let $\mathcal{P}_{\mathbb{H}} = (V, L, \nabla)$ be a quaternionic projective structure of normal bundle degree 0 on a torus M with the property that $\mathrm{Re}\,\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$ is equipped with a positive parallel section. Let X be a nowhere vanishing holomorphic vector field on M.

Let $\psi \in \Gamma(\hat{L})$ be a section of constant length (where the metric on $\hat{L}$ is induced by the section of $\mathrm{Re}\,\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$ as explained in the remark following Theorem 9) and denote by $\varphi = \delta_X \psi \in \Gamma(\hat{V})$ the corresponding section calculated with respect to the splitting induced by X. We call such frame ψ, φ respecting the splitting and metric an *adapted frame* for a quaternionic projective structure (V, L, ∇) (with respect to X and the metric). With respect to the frame, the connection ∇ takes the form (see equation (86)),

$$\nabla(\psi,\varphi)=(\psi,\varphi)\left(\begin{pmatrix} g\,dz-\bar g\,d\bar z & -\frac12 c\,dz+e\,d\bar z\\ dz & g\,dz-\bar g\,d\bar z\end{pmatrix}+\begin{pmatrix} a\,dzj & b\,dzj+\bar p\,d\bar z j\\ 0 & \bar q\,d\bar z j\end{pmatrix}\right)$$

where, as usual, z denotes the coordinate on the universal covering corresponding to X.

LEMMA 94. *The data describing a quaternionic projective structure with respect to an adapted frame has to satisfy the integrability equations (or Gauss–, Codazzi– and Ricci–equations)*

$$\begin{gathered}-\tfrac12 c_{\bar z}-e_z+a\overline{\mathcal{D}_{\bar z}a}+q\mathcal{D}_z\bar q=0\\ \mathcal{D}_{\bar z}\mathcal{D}_{\bar z}a+\tfrac12\bar c a=-\mathcal{D}_z\mathcal{D}_z\bar q-\tfrac12 c\bar q\\ |q|^2-|a|^2=2(g_{\bar z}+\bar g_z),\end{gathered}$$

where we use the operator $\mathcal{D}f=df+f(2g\,dz-2\bar g\,d\bar z)$. Furthermore, e, b and p are determined by

$$\begin{aligned}e&=-\frac12(|a|^2+|q|^2)\\ b&=\mathcal{D}_{\bar z}a\\ \bar p&=-\mathcal{D}_z\bar q.\end{aligned}$$

It is easy to verify that, changing the adapted frame ψ, φ to the new adapted frame $\tilde\psi=\psi e^{\theta\mathbf{i}}$ and $\tilde\varphi=\varphi e^{\theta\mathbf{i}}$ with θ a real function, the data a, q, c and g change to

$$\tilde a=ae^{-2\theta\mathbf{i}},\qquad \tilde q=qe^{2\theta\mathbf{i}},\qquad \tilde c=c\quad\text{and}\quad \tilde g=g+\mathbf{i}\theta_z. \tag{94}$$

Deformations. The aim of this paragraph is to analyze the infinitesimal condition on an endomorphism $Y\in\Gamma(\mathrm{End}(V))$ to be the admissible infinitesimal gauge transformation $Y=\dot G$ of a family of gauge transformation G_t respecting the splitting and metric.

Let $L_t\subset V$ be a conformal deformation with $L_0=L$ (in the Schrödinger picture) of the quaternionic projective structure (V,L,∇). By Theorem 10 and the following remark, there is a family G_t of gauge transformations of V with $G_0=\mathrm{Id}$, such that, for all t, the sections $\psi^t=G_t\psi$ and $\varphi^t=G_t\varphi$ form an adapted frame. In particular, G_t respects the splitting and metric.

With respect to the frame ψ, φ, every $Y\in\Gamma(\mathrm{End}(V))$ takes the form

$$Y=\begin{pmatrix} x_{11}+y_{11}\mathbf{j} & x_{12}+y_{12}\mathbf{j}\\ \omega+\sigma\mathbf{j} & x_{22}+y_{22}\mathbf{j}\end{pmatrix} \tag{95}$$

for complex functions x_{ij}, y_{ij}, ω and σ and, using the above formula for ∇ with respect to the frame, one can easily verify that

$$\nabla Y=\begin{pmatrix} ul & ur\\ ll & lr\end{pmatrix}$$

with lower left corner

(96)
$$ll=d\omega+\mathcal{D}\sigma\mathbf{j}+dz(x_{11}+y_{11}\mathbf{j})+\mathbf{j}q\,dz(\omega+\sigma\mathbf{j})-(x_{22}+y_{22}\mathbf{j})dz-(\omega+\sigma\mathbf{j})a\,dz\mathbf{j},$$

upper left corner

(97)
$$ul = dx_{11} + \mathcal{D}y_{11}\mathbf{j} + a\,dz\mathbf{j}(x_{11} + y_{11}\mathbf{j}) + (-\tfrac{1}{2}c\,dz + e\,d\bar{z} + b\,dz\mathbf{j} + \mathbf{j}p\,dz)(\omega + \sigma\mathbf{j}) - \\ - (x_{11} + y_{11}\mathbf{j})a\,dz\mathbf{j} - (x_{12} + y_{12}\mathbf{j})dz,$$

and lower right corner

(98)
$$lr = dx_{22} + \mathcal{D}y_{22}\mathbf{j} + dz(x_{12} + y_{12}\mathbf{j}) + jq\,dz(x_{22} + y_{22}\mathbf{j}) - \\ - (\omega + \sigma\mathbf{j})(-\tfrac{1}{2}c\,dz + e\,d\bar{z} + b\,dz\mathbf{j} + \mathbf{j}p\,dz) - (x_{22} + y_{22}\mathbf{j})\mathbf{j}q\,dz.$$

Furthermore, the $K\,\mathrm{End}_+$–part of the upper right corner is

(99)
$$(ur)'_+ = (x_{12z} + \tfrac{1}{2}c(x_{11} - x_{22}) - a\bar{y}_{21} - b\bar{y}_{22} + qy_{12} + py_{11})dz$$

The first condition on Y to be an adapted infinitesimal gauge transformation is $(\nabla Y)''_+ L \subset L$, which can be seen as conformality of the deformation. By equating the $\bar{K}\,\mathrm{End}_+$–part of (96) to 0, this condition becomes

(100)
$$\omega_{\bar{z}} = \bar{q}\bar{\sigma} - \sigma\bar{a}.$$

By equating the End_-–part of (96) to 0, the second condition $(\nabla Y)_- L \subset L$ which states that S infinitesimally stays a touching sphere becomes

(101)
$$y_{11} = a\omega - \mathcal{D}_z\sigma \quad \text{and} \quad y_{22} = \bar{q}\bar{\omega} + \mathcal{D}_{\bar{z}}\sigma.$$

By equating the $\bar{K}\,\mathrm{End}_-$–part of (97) (and the $K\,\mathrm{End}_-$–part of (98)) to 0, the third condition $(\nabla Y)''_- L = 0$ (or, equivalently, $\mathrm{im}(\nabla Y)'_- \subset L$) which states that S actually stays the mean curvature sphere becomes

(102)
$$y_{12} = \mathcal{D}_{\bar{z}}y_{11} + \bar{p}\bar{\omega} + e\sigma$$

(103)
$$= -\mathcal{D}_z y_{22} + b\omega + e\sigma.$$

The condition that the splitting is preserved, by setting the $K\,\mathrm{End}_+$–part of (96) to 0 and by equating the $K\,\mathrm{End}_+$–parts of (97) and (98), becomes

(104)
$$w_z + x_{11} - x_{22} = 0$$

(105)
$$x_{12} = \tfrac{1}{2}(\underbrace{(x_{11} - x_{22})_z}_{-\omega_{zz}} - c\omega - a\bar{y}_{11} - qy_{22} - b\bar{\sigma} - p\sigma).$$

(Note that the $\bar{K}\,\mathrm{End}_+$–parts of (97) and (98) coincide automatically, a fact which is already used in the proofs of Theorem 7 or Lemma 78.)

Finally, the condition that the metric is preserved (or rather that all sections ψ^t have constant length, which is equivalent to the End_+–part of (97) being an imaginary 1–form) by a straightforward computation becomes

$$(\chi + \bar{\chi})_z = 0$$

where $\chi = x_{11} + x_{22}$. This is equivalent to χ having constant real part. Since in the end we want all sections to have constant length $\langle\psi_t, \psi_t\rangle_{\hat{L}_t} = 1/\sqrt{2}$, we will only allow purely imaginary χ. By (104), the function χ determines x_{11} and x_{22} via

(106)
$$x_{11} = \tfrac{1}{2}(\chi - \omega_z) \quad \text{and} \quad x_{22} = \tfrac{1}{2}(\chi + \omega_z).$$

Equations (101), (102), (103), (105) and (106) show that an adapted infinitesimal gauge transformation Y preserving the splitting and metric is uniquely determined, when the normal deformation σ, the tangential deformation ω and χ are prescribed. We will see below that the purely imaginary function χ describes the infinitesimal rotation of the normal frame.

The infinitesimal deformations $\dot{a}$, $\dot{q}$, $\dot{c}$ and $\dot{g}$ of the invariants can be computed[2] from the $K\,\mathrm{End}_-$–part of (97), the $\bar{K}\,\mathrm{End}_-$–part of (98), the $K\,\mathrm{End}_+$–part of (99) and, for $\dot{g}$, half of the sums of the $K\,\mathrm{End}_+$–parts of (97) and (98). We obtain the following theorem.

THEOREM 11. *Let M be a torus equipped with a nowhere vanishing holomorphic vector field X. Let (V, L, ∇) be a quaternionic projective structure of normal bundle degree 0 on M equipped with a positive parallel section of $\mathrm{Re}\,\Lambda^4_{\mathbb{C}}(V, \mathbf{i})$. Furthermore, let a, q, c and g denote the data with respect to an adapted frame.*

Then, complex functions σ and ω solving (100), i.e.

$$\omega_{\bar{z}} = \bar{q}\bar{\sigma} - \sigma\bar{a},$$

together with the purely imaginary function χ uniquely determine an adapted infinitesimal gauge transformation $Y \in \Gamma(\mathrm{End}(V))$ which preserves the splitting induced by X and the metric. The corresponding infinitesimal deformation $\dot{\nabla} = \nabla Y$ is described by

$$\begin{aligned}
\dot{a} &= \omega\mathcal{D}_z a + \bar{\omega}\mathcal{D}_{\bar{z}} a - \mathcal{D}_z\mathcal{D}_z\sigma - \tfrac{1}{2}c\sigma + \frac{a}{2}(3\omega_z - \bar{\omega}_{\bar{z}}) - a\Sigma \\
\dot{q} &= \omega\overline{\mathcal{D}_{\bar{z}}\bar{q}} + \bar{\omega}\overline{\mathcal{D}_z\bar{q}} + \overline{\mathcal{D}_{\bar{z}}\mathcal{D}_{\bar{z}}\sigma} + \frac{1}{2}c\bar{\sigma} + \frac{q}{2}(3\omega_z - \bar{\omega}_{\bar{z}}) + q\Sigma \\
\dot{c} &= \omega c_z + \bar{\omega}c_{\bar{z}} + \omega_{zzz} + 2c\omega_z - 8aq\omega_{\bar{z}} + \\
&\qquad + 3(q\mathcal{D}_{\bar{z}}\mathcal{D}_z\sigma - (\overline{\mathcal{D}_z\bar{q}})(\mathcal{D}_z\sigma)) + 3((\mathcal{D}_{\bar{z}}a)(\overline{\mathcal{D}_{\bar{z}}\sigma}) - a\overline{\mathcal{D}_z\mathcal{D}_{\bar{z}}\sigma}) + \\
&\qquad\quad + (\bar{\sigma}\mathcal{D}_{\bar{z}}\mathcal{D}_z a - (\overline{\mathcal{D}_z\sigma})(\mathcal{D}_z a)) + ((\mathcal{D}_{\bar{z}}\sigma)(\overline{\mathcal{D}_{\bar{z}}\bar{q}}) - \sigma\overline{\mathcal{D}_z\mathcal{D}_{\bar{z}}\bar{q}}) \\
\dot{g} &= \frac{1}{2}(\chi_z + q(\mathcal{D}_{\bar{z}}\sigma) - (\overline{\mathcal{D}_z\bar{q}})\sigma + a(\overline{\mathcal{D}_z\sigma}) - (\mathcal{D}_{\bar{z}}a)\bar{\sigma} + (|q|^2 - |a|^2)\bar{\omega})
\end{aligned}$$

where, as above, we set $\mathcal{D}f = df + (2g\,dz - 2\bar{g}\,d\bar{z})f$, and where z is the coordinate on the universal covering corresponding the X.

REMARK. Equation (100) shows that, given the normal deformation σ, the tangential deformation ω has to solve a $\bar{\partial}$–equation. Locally, such equation can always be solved, its solution being unique up to adding a holomorphic function. Globally, on the torus, a solution ω to the $\bar{\partial}$–equation can be found if and only if $\int_M \bar{q}\bar{\sigma} - \sigma\bar{a} = 0$ (a fact that can be easily verified using Fourier series). Such solution is then unique up to adding a constant.

Unfortunately, the deformation formulae in this section are quite complicated. For practical purposes, they are surely outperformed by the lightcone formulae of Section 14.3. The quaternionic formulae are nevertheless included for two reasons. Firstly, because they might be helpful in the search for more

[2]The necessary computations are straightforward, but, in particular in the case of $\dot{c}$, quite painful.

invariant quaternionic deformation formulae. The various attempts for finding such more invariant approach which is still suited for computations did, until now, not lead to significant progress. But this will hopefully change in the future.

Secondly, using the new link to the lightcone model of Section 13, one immediately verifies (see Section 14.4) that both sets of deformations formulae are equivalent. This fact suggests that the lightcone formulae can be used as a purely formal tool that simplifies manipulations with the quaternionic data[3].

14.3. Deformations in the Lightcone Model. Let M be a Riemann surface with a nowhere vanishing holomorphic vector field X. In the Schrödinger picture, a deformation of conformal immersions is given by a family $\mathcal{L}_t \subset \mathscr{L}$ (with $t \in I \subset \mathbb{R}$) of conformal immersions into the lightcone of a Minkowski space bundle $(\mathcal{V}, \nabla)$. In the Heisenberg picture, $\mathcal{V}$ and $\mathcal{L}$ are left fixed while the deformation is realized by considering a family of connections ∇^t. As in the quaternionic case, the Heisenberg picture is more suited for our purpose. The Schrödinger picture is mainly used to simplify the derivation of the right formulae in the Heisenberg picture.

A deformation preserving the conformal structure and normalization is most easily described in the Schrödinger picture as follows. Let Ψ denote the lift of a conformal immersion $\mathcal{L} \subset \mathscr{L}$ into $(\mathcal{V}, \nabla)$ normalized with respect to the coordinate z on the universal covering which corresponds to X. An arbitrary infinitesimal deformation is of the form

$$\dot{\Psi} = a\Psi + b\Psi_z + \bar{b}\Psi_{\bar{z}} + \varrho$$

with real function a, complex function b and $\varrho \in \Gamma(W^\perp)$. The deformation $\dot{\Psi}$ hat no $\hat{\Psi}$–component, since we deform tangential to the lightcone. The infinitesimal condition $\langle \dot{\Psi}_z, \Psi_z \rangle = 0$ for preserving the conformal structure is equivalent to

$$\bar{b}_z = 2\langle \varrho, \kappa \rangle, \tag{107}$$

and the condition $\mathrm{Re}\langle \dot{\Psi}_z, \Psi_{\bar{z}} \rangle = 0$ for preserving the normalization is equivalent to

$$a = -\operatorname{Re} b_z.$$

Equation (107) corresponds to equations (100) in the quaternionic case. It shows, how the *normal deformation* ϱ determines the *tangential deformation* b via solving a $\bar{\partial}$–problem (see also the Remark following Theorem 11).

THEOREM 12. *Let M be a Riemann surface with a nowhere vanishing holomorphic vector field X. Let $\mathcal{L} \subset \mathcal{V}$ be a conformal immersion of M into a Minkowski space bundle $(\mathcal{V}, \nabla)$ with normal bundle $\mathcal{W}^\perp$ of degree 0. An infinitesimal conformal deformation of $\mathcal{L}$ is uniquely determined by a normal variation $\varrho \in \Gamma(\mathcal{W}^\perp)$ and a tangential variation b satisfying* (107)*, i.e.*

$$\bar{b}_z = 2\langle \varrho, \kappa \rangle,$$

[3]Actually, in the attempt to find such formal tool, the author wrote down the formula $\kappa = \xi a - \bar{\xi} q$, as a purely formal object with the right transformation behavior, about one year before it became justified by the new link between the quaternionic and the lightcone invariants of Section 13.

together with a normal bundle rotation given by $(\dot{\xi})^{\perp} = J\xi\tilde{\chi}$*, where* $\xi \in \Gamma(\mathcal{W}^{\perp\mathbb{C}})$ *is a section with* $\langle \xi, \xi \rangle = 0$ *and* $\langle \xi, \bar{\xi} \rangle = \frac{1}{2}$ *and where* $\tilde{\chi}$ *is a real function.*

The deformation of the data κ*,* c *and* D *is then described by*

$$\begin{aligned} \dot{\kappa} &= D_z D_z \varrho + \tfrac{c}{2}\varrho + (\tfrac{3}{2}b_z - \tfrac{1}{2}\bar{b}_{\bar{z}})\kappa + bD_z\kappa + \bar{b}D_{\bar{z}}\kappa - J\kappa\tilde{\chi}, \\ \dot{c} &= b_{zzz} + 2cb_z + bc_z + \bar{b}c_{\bar{z}} + 16\langle \varrho, \bar{\kappa}\rangle\langle \kappa, \kappa\rangle + \\ &\qquad 6(\langle D_{\bar{z}} D_z \varrho, \kappa\rangle - \langle D_z\varrho, D_{\bar{z}}\kappa\rangle) + 2(\langle D_{\bar{z}}\varrho, D_z\kappa\rangle - \langle \varrho, D_{\bar{z}} D_z \kappa\rangle), \\ \dot{D}_z\xi &= (\tilde{\chi}_z + 2\langle J\varrho, D_{\bar{z}}\kappa\rangle + 2\langle J\kappa, \bar{b}\bar{\kappa} + D_{\bar{z}}\varrho\rangle)J\xi. \end{aligned}$$

The quantities ϱ, b and $\tilde{\chi}$ completely describe the infinitesimal deformation $\dot{\Psi}$ of the conformal immersion as well as the frame Ψ, Ψ_z, $\Psi_{\bar{z}}$, $\hat{\Psi}$, ξ and $\bar{\xi}$ in the Schrödinger picture. The resulting infinitesimal deformation $\dot{\kappa}$, $\dot{c}$ and $\dot{D}$ of the invariants describes the deformation in the Heisenberg picture. While the equation for $\dot{\Psi}$ in the Schrödinger picture extrinsically describes the conformal deformation, in the Heisenberg picture the extrinsic information is lost, because the deformation is only determined up to Möbius transformations.

Proof. The deformation of the frame Ψ, Ψ_z, $\Psi_{\bar{z}}$ and $\hat{\Psi}$ as well as that of the invariants κ and c is computed in Section 4.2 of [**4**]. We therefore concentrate on the deformation of the normal connection D, which is missing in [**4**]. We also have to derive a slightly modified version of the deformation equation for κ which takes into account the normal bundle rotation.

As in the quaternionic case, the passage from the Schrödinger picture to the Heisenberg picture can be easily understood by the following consideration: the deformation $\mathcal{L}_t \subset \mathcal{V}$ (with $(\mathcal{V}, \nabla)$ fixed) in the Schrödinger picture can be represented as $\mathcal{L}_t = \mathcal{G}_t\mathcal{L}$ where $\mathcal{G}_t$ denotes a gauge transformation of $\mathcal{V}$ preserving the metric and normalized frame of the mean curvature sphere. In the Heisenberg picture, the deformation is realized by viewing the fixed bundle $\mathcal{L}$ as conformal immersion into the family of Minkowski space bundles $(\mathcal{V}, \nabla^t)$ with $\nabla^t = \nabla + \mathcal{G}_t^{-1}\nabla\mathcal{G}_t$.

If we denote by κ_t the conformal Hopf differential of $\mathcal{L}$ with respect to ∇^t, then

$$\mathcal{G}_t\kappa^t = \Psi^t_{zz} + \tfrac{1}{2}c^t\Psi^t.$$

Hence,

$$(\dot{\mathcal{G}}\kappa)^{\perp} + \dot{\kappa} = \dot{\Psi}^{\perp}_{zz} + \tfrac{1}{2}c\varrho.$$

Using $(\dot{\mathcal{G}}\xi)^{\perp} = (\dot{\xi})^{\perp} = J\xi\tilde{\chi}$ and $\kappa = \xi a - \bar{\xi}q$ we obtain $(\dot{\mathcal{G}}\kappa)^{\perp} = J\kappa\tilde{\chi}$. Together with equation (52) of [**4**], we obtain the equation for the deformation of κ.

In order to compute the deformation $\dot{D}$ of the normal connection D, we need to know the infinitesimal deformation $\dot{\xi}$ of $\xi_t = \mathcal{G}_t\xi$. Its $\mathcal{W}^{\perp}$–part is

$$(\dot{\xi})^{\perp} = J\xi\tilde{\chi} \tag{108}$$

and the $\mathcal{W}$–part (cf. [**4**, (51)]) is

$$(\dot{\xi})^{\top} = \langle \xi, \tau\rangle\Psi + \langle \xi, \varrho\rangle\hat{\Psi} - 2\langle \xi, b\kappa + D_z\varrho\rangle\Psi_{\bar{z}} - 2\langle \xi, \bar{b}\bar{\kappa} + D_{\bar{z}}\varrho\rangle\Psi_z \tag{109}$$

(where τ, which we do not need in the sequel, denotes the normal deformation of $\hat{\Psi}_t = \mathcal{G}_t\hat{\Psi}$).

Denote by D^t the normal connection with respect to ∇^t. Then, by the frame equation (151),

$$\mathcal{G}_t D^t_z \xi = \xi^t_z - 2\Psi^t \langle D_{\bar z}\kappa^t, \xi^t\rangle + 2\Psi^t_{\bar z}\langle \kappa^t, \xi^t\rangle.$$

The normal part of the t–derivative at $t = 0$ is

$$(\dot{\mathcal{G}} D_z \xi)^\perp + \dot D_z \xi = (\dot\xi_z)^\perp - 2\varrho\langle D_{\bar z}\kappa, \xi\rangle + 2(\bar b\bar\kappa + D_{\bar z}\varrho)\langle\kappa, \xi\rangle,$$

where $(\dot\Psi_{\bar z})^\perp = \bar b\bar\kappa + D_{\bar z}\varrho$ holds by equation (49) of [**4**].

Equations (108) and (109) can be used to compute

$$(\dot\xi_z)^\perp = JD_z\xi\tilde\chi + J\xi\tilde\chi_z + 2\langle\xi,\varrho\rangle D_{\bar z}\kappa - 2\langle\xi, \bar b\bar\kappa + D_{\bar z}\varrho\rangle\kappa.$$

From $D_z\xi = \xi 2g$ and $(\dot{\mathcal{G}}\xi)^\perp = (\dot\xi)^\perp = J\xi\tilde\chi$, we obtain $(\dot{\mathcal{G}} D_z\xi)^\perp = D_z J\xi\tilde\chi$, which, together with the last two equations, implies

$$\dot D_z\xi = J\xi\tilde\chi_z + 2(\langle\xi,\varrho\rangle D_{\bar z}\kappa - \langle\xi, D_{\bar z}\kappa\rangle\varrho) + 2(\langle\xi,\kappa\rangle(\bar b\bar\kappa + D_{\bar z}\varrho) - \langle\xi, \bar b\bar\kappa + D_{\bar z}\varrho\rangle\kappa).$$

The above formula for $\dot D_z\xi$ follows by the same argument as used to obtain equation (154') from (154). $\square$

Remark. Instead of complex normal vectors, one could as well work with real normal vectors $n \in \Gamma(\mathcal{W}^\perp)$ of length 1. Then $\xi = \frac{1}{2}(n - Jn\mathbf{i})$ is a complex normal vector as used above and $\dot\xi^\perp = J\xi\tilde\chi$ is equivalent to $\dot n^\perp = Jn\tilde\chi$. The deformation of D expressed with respect to n is given by the same formula

$$\dot D_z n = (\tilde\chi_z + 2\langle J\varrho, D_{\bar z}\kappa\rangle + 2\langle J\kappa, \bar b\bar\kappa + D_{\bar z}\varrho\rangle)Jn.$$

Deformations in Metrical Subgeometries. Now let $\Phi\colon M \to S_{v_0}$ be a conformal immersion into a space form, let z be a conformal chart and denote by H, q and u the invariants of Φ with respect to z (cf. (156), (157) and (158)). A general infinitesimal deformation of Φ is of the form

$$\dot\Phi = b\Phi_z + \bar b\Phi_{\bar z} + \rho$$

where b is a complex function and $\rho \in \Gamma(\mathcal{T}^\perp)$ a section of the metrical normal bundle. Note that we deform tangentially to the conic section S_{v_0} and therefore $\dot\Phi$ has no Φ– and v_0–component.

The resulting deformation of Φ_z is

$$\dot\Phi_z = -\tfrac{G}{2}e^{2u}\bar b\Phi + (2u_z b + b_z - \langle H,\rho\rangle)\Phi_z + (\bar b_z - 2e^{-2u}\langle q,\rho\rangle)\Phi_{\bar z} + \tfrac12 e^{2u}\bar b v_0 + bq + \tfrac12 e^{2u}\bar b H + \nabla^\perp_z\rho.$$

The deformation is conformal if $\langle\dot\Phi_z, \Phi_z\rangle = 0$, i.e. if the normal deformation determines the tangent deformation by

$$\bar b_z = 2e^{-2u}\langle q,\rho\rangle.$$

Thus, for a conformal deformation we have

$$\dot\Phi_z = -\tfrac{G}{2}e^{2u}\bar b\Phi + (2u_z b + b_z - \langle H,\rho\rangle)\Phi_z + \tfrac12 e^{2u}\bar b v_0 + bq + \tfrac12 e^{2u}\bar b H + \nabla^\perp_z\rho. \tag{110}$$

The resulting deformation of the conformally normalized lift $\Psi = e^{-u}\Phi$ (with respect to z) is

$$\begin{aligned}\dot\Psi &= e^{-u}\dot\Phi - e^{-u}\dot u\Phi \\ &= e^{-u}(b\Phi_z + \bar b\Phi_{\bar z} + \rho) - \dot u\Psi \\ &= \underbrace{(u_z b + u_{\bar z}\bar b - \dot u - \langle\rho, H\rangle)}_{a}\Psi + b\Psi_z + \bar b\Psi_{\bar z} + \underbrace{e^{-u}(\rho + \langle\rho, H\rangle\Phi)}_{\varrho}\end{aligned}$$

where we used $e^{-u}\Phi_z = \Psi_z + u_z\Psi$. Therefore, the conformal normal variation is $\varrho = e^{-u}\hat\rho$ (where $\hat{}$ denotes the isomorphism of Lemma 103) and the tangential variation b is the same as in the metrical setting. Since $\hat{}$ is an isomorphism of the normal bundles we have

$$\bar b_z = 2e^{-2u}\langle q, \rho\rangle_{\mathcal{T}^\perp} = 2\langle e^{-u}q, e^{-u}\rho\rangle_{\mathcal{T}^\perp} = 2\langle\kappa, \varrho\rangle_{\mathcal{W}^\perp}$$

where we used $\kappa = e^{-u}\hat q$ (cf. Lemma 106). Thus, the $\bar\partial$–problem in the metrical setting is equivalent to the one in the conformal setting. We have proven the following Lemma.

LEMMA 95. *Let $\Phi\colon M \to S_{v_0}$ be a conformal immersion into a space form, z a conformal chart and*

$$\dot\Phi = b\Phi_z + \bar b\Phi_{\bar z} + \rho$$

an infinitesimal conformal deformation. Then, the corresponding deformation of the conformally normalized lift $\Psi = e^{-u}\Phi$ is

$$\dot\Psi = a\Psi + b\Psi_z + \bar b\Psi_{\bar z} + \varrho$$

with $a = -\operatorname{Re}(b_z)$ and $\varrho = e^{-u}\hat\rho$ ($\hat{}$ denoting the isomorphism of Lemma 103). The tangential deformation b is determined (up to holomorphic function) by the normal deformation via

$$\bar b_z = 2e^{-2u}\langle q, \rho\rangle_{\mathcal{T}^\perp} = 2\langle\kappa, \varrho\rangle_{\mathcal{W}^\perp}.$$

Since $a = -\operatorname{Re}(b_z)$, we have the following deformation of u

$$\dot u = u_z b + u_{\bar z}\bar b + \operatorname{Re}(b_z) - \langle\rho, H\rangle.$$

In order to compute the deformation of H and q, we use (110), the frame equations $\Phi_{zz} = q + 2u_z\Phi_z$ and $\Phi_{z\bar z} = \frac{1}{2}e^{2u}(H + v_0 - G\Phi)$ and the metrical Codazzi equation (cf. Lemma 104).

First, we calculate the deformation of q. For this, we calculate

$$\begin{aligned}(\dot\Phi_z)_z^\perp &= (2u_z b + b_z - \langle H, \rho\rangle)q + \nabla_z^\perp(bq + \tfrac{1}{2}e^{2u}\bar bH + \nabla_z^\perp\rho) \\ &= (2u_z b + b_z - \langle H, \rho\rangle)q + \\ &\qquad + b\nabla_z^\perp q + b_z q + \underbrace{\tfrac{1}{2}e^{2u}\bar b\nabla_z^\perp H}_{\bar b\nabla_{\bar z}^\perp q} + e^{2u}u_z\bar bH + \tfrac{1}{2}e^{2u}\bar b_z H + \nabla_z^\perp\nabla_z^\perp\rho\end{aligned}$$

$$(\Phi_{zz})^{\cdot\perp} = \dot q^\perp + 2u_z\dot\Phi_z^{\perp} = \dot q^\perp + 2u_z(bq + \tfrac{1}{2}e^{2u}\bar bH + \nabla_z^\perp\rho)$$

and $(\dot\Phi_z)_z = (\Phi_{zz})^\cdot$ implies

$$\dot q^\perp = b\nabla_z^\perp q + \bar b\nabla_{\bar z}^\perp q + 2b_z q + \langle q, \rho\rangle H - \langle H, \rho\rangle q + \nabla_z^\perp\nabla_z^\perp\rho - 2u_z\nabla_z^\perp\rho.$$

In order to calculate the deformation of H we need

$$
\begin{aligned}
(\dot\Phi_z)^\perp_{\bar z} &= \tfrac12 e^{2u}(2u_z b + b_z - \langle H,\rho\rangle)H + \nabla^\perp_{\bar z}(bq + \tfrac12 e^{2u}\bar b H + \nabla^\perp_z\rho)\\
&= \tfrac12 e^{2u}(2u_z b + b_z - \langle H,\rho\rangle)H +\\
&\qquad \underbrace{b\nabla^\perp_{\bar z} q}_{\frac12 e^{2u} b\nabla^\perp_z H} + b_{\bar z} q + \tfrac12 e^{2u}\bar b\nabla^\perp_{\bar z} H + e^{2u}u_{\bar z}\bar b H + \tfrac12 e^{2u}\bar b_{\bar z} H + \nabla^\perp_{\bar z}\nabla^\perp_z\rho\\
(\Phi_{z\bar z})^{\cdot\perp} &= \tfrac12 e^{2u}(\dot H^\perp - G\rho) + e^{2u}\dot u H\\
&= \tfrac12 e^{2u}(\dot H^\perp - G\rho) + e^{2u}(u_z b + u_{\bar z}\bar b + \mathrm{Re}(b_z) - \langle\rho, H\rangle)H.
\end{aligned}
$$

Now $(\dot\Phi_z)_{\bar z} = (\Phi_{z\bar z})^{\cdot}$ implies

$$
\dot H^\perp = b\nabla^\perp_z H + \bar b\nabla^\perp_{\bar z} H + \langle H,\rho\rangle H + 4e^{-4u}\langle\bar q,\rho\rangle q + G\rho + 2e^{-2u}\nabla^\perp_{\bar z}\nabla^\perp_z\rho
$$

and we have proven

Lemma 96. *Let $\Phi\colon M \to S_{v_0}$ be a conformal immersion into a space form, z a conformal chart and $\dot\Phi = b\Phi_z + \bar b\Phi_{\bar z} + \rho$ an infinitesimal conformal variation. The deformation of the invariants is then give by*

$$
\begin{aligned}
\dot u &= u_z b + u_{\bar z}\bar b + \mathrm{Re}(b_z) - \langle\rho, H\rangle,\\
\dot q^\perp &= b\nabla^\perp_z q + \bar b\nabla^\perp_{\bar z} q + 2b_z q + \langle q,\rho\rangle H - \langle H,\rho\rangle q + \nabla^\perp_z\nabla^\perp_z\rho - 2u_z\nabla^\perp_z\rho,\\
\dot H^\perp &= b\nabla^\perp_z H + \bar b\nabla^\perp_{\bar z} H + \langle H,\rho\rangle H + 4e^{-4u}\langle\bar q,\rho\rangle q + G\rho + 2e^{-2u}\nabla^\perp_{\bar z}\nabla^\perp_z\rho.
\end{aligned}
$$

In this paragraph, to simplify the presentation, we did not apply the Heisenberg picture and did not consider normal bundle rotation. As a consequence, the evolution of the normal connection is not computed. The missing formulae can easily be derived (as in the proof of Theorem 12) from the formulae of the preceding lemma. In the case of normal bundle rotation $\tilde\chi = 0$, one gets no extra term in these formulae (i.e. only $\dot q^\perp$ and $\dot H^\perp$ have to be replaced with $\dot q$ and $\dot H$ to get the right formulae in the Heisenberg model). The evolution of the metrical normal connection can then be obtained from the evolution of the conformal normal connection, because both normal bundles are gauge equivalent.

We are mainly interested in the case of deformations in 3–dimensional space forms, because the deformation formulae are only applied to study flows of constant mean curvature surfaces (cf. Section 15.3). In this case, no modification of the formulae is necessary, because the normal bundle is trivial.

14.4. Link between the Quaternionic and Lightcone Deformation Formulae. The aim of this section is to describe the link between the deformations of quaternionic projective structures as in Theorem 11 and the deformations of conformal immersions in the lightcone model as in Theorem 12.

Let (V, L, ∇) be a quaternionic projective structure with parallel section of $\mathrm{Re}\,\Lambda^4_{\mathbb C}(V,\mathbf i)$ on a Riemann surface M equipped with a holomorphic vector field X. Let ψ, φ be an adapted frame, i.e. $\psi \in \Gamma(\hat L)$ and $\varphi = \delta_X\psi$ (with respect to the splitting induced by X) and $\langle\psi,\psi\rangle_{\hat L} = 1/\sqrt2$. Then $\Psi = \psi\wedge\psi\mathbf j$ is the normalized section of the corresponding immersion into the Minkowski space bundle $\mathcal V = \mathrm{Re}^2_{\mathbb C}(V,\mathbf i)$.

In Section 13 we have seen, that ψ and φ define the frame $\Psi = \psi \wedge \psi\mathbf{j}$, $\Psi_z = \varphi \wedge \psi\mathbf{j}$, $\Psi_{\bar z} = \psi \wedge \varphi\mathbf{j}$ and $\hat\Psi = 2\varphi \wedge \varphi\mathbf{j}$ of the (complexified) mean curvature sphere $\mathcal{W}^{\mathbb{C}}$ and a frame $\xi = \psi \wedge \varphi$ and $\bar\xi = \psi\mathbf{j} \wedge \varphi\mathbf{j}$ of the (complexified) Möbius normal bundle $\mathcal{W}^{\perp\mathbb{C}}$.

Furthermore, we have seen that the relation between the invariants in the quaternionic setting and those in the lightcone setting is described by $\kappa = \xi a - \bar\xi q$ and $D\xi = \xi(2g\,dz - 2\bar g\,d\bar z)$ (and the fact that c is the same in both cases).

Assume $Y \in \Gamma(\mathrm{End}(V))$ is an adapted infinitesimal gauge transformation which arises as $Y = \dot G$ where G_t is a family of gauge transformations with the property that $\psi_t = G_t\psi$ and $\varphi_t = G_t\varphi$ for all t is an adapted frame of a quaternionic projective structure $(V, L_t = \psi_t\mathbb{H}, \nabla)$. Then, the action of G_t on $\mathcal{V} = \mathrm{Re}\,\Lambda^2_{\mathbb{C}}(V, \mathbf{i})$ defines a family of gauge transformations $\mathcal{G}_t$ of $\mathcal{V}$ compatible with the metric and this family describes the conformal deformation in the lightcone model.

The deformation of Ψ is $\Psi_t = \mathcal{G}_t\Psi = G_t\psi \wedge G_t\psi\mathbf{j}$, hence (using (95))

(111)

$$\begin{aligned}\dot\Psi &= Y\psi \wedge \psi\mathbf{j} + \psi \wedge Y\psi\mathbf{j}\\ &= (\psi(x_{11} + y_{11}\mathbf{j}) + \varphi(\omega + \sigma\mathbf{j})) \wedge \psi\mathbf{j} + \psi \wedge (\psi(x_{11} + y_{11}\mathbf{j}) + \varphi(\omega + \sigma\mathbf{j}))\mathbf{j}\\ &= \underbrace{\psi \wedge \psi\mathbf{j}}_{\Psi}(x_{11} + \bar x_{11}) + \underbrace{\varphi \wedge \psi\mathbf{j}}_{\Psi_z}\omega + \underbrace{\psi \wedge \varphi\mathbf{j}}_{\Psi_{\bar z}}\bar\omega + \underbrace{\psi \wedge \varphi(-\sigma) + \varphi\mathbf{j} \wedge \psi\mathbf{j}\bar\sigma}_{\varrho}.\end{aligned}$$

Thus $b = \omega$ and $\varrho = -(\xi\sigma + \bar\xi\bar\sigma)$. The deformation of ξ is $\xi_t = \mathcal{G}_t\xi = G_t\psi \wedge G_t\varphi$, hence (again using formula (95) and $\chi = x_{11} + x_{22}$)

(112)

$$\begin{aligned}\dot\xi &= Y\psi \wedge \varphi + \psi \wedge Y\varphi\\ &= (\psi(x_{11} + y_{11}\mathbf{j}) + \varphi(\omega + \sigma\mathbf{j})) \wedge \varphi + \psi \wedge (\psi(x_{12} + y_{12}\mathbf{j}) + \varphi(x_{22} + y_{22}\mathbf{j}))\\ &= \underbrace{\psi \wedge \varphi}_{\xi}\underbrace{(x_{11} + x_{22})}_{\chi} + \underbrace{\psi \wedge \psi\mathbf{j}}_{\Psi}\bar y_{12} - \underbrace{\varphi \wedge \varphi\mathbf{j}}_{\frac{1}{2}\hat\Psi}\bar\sigma + \underbrace{\psi \wedge \varphi\mathbf{j}}_{\Psi_{\bar z}}\bar y_{22} - \underbrace{\varphi \wedge \psi\mathbf{j}}_{\Psi_z}\bar y_{11}.\end{aligned}$$

This shows that $\chi = \tilde\chi\mathbf{i}$ and we have proven:

LEMMA 97. *If an infinitesimal conformal deformation in the quaternionic setting (as in Theorem 11) is given by σ, ω and χ, the corresponding deformation in the lightcone setting (as in Theorem 12) is given by $\varrho = -(\xi\sigma + \bar\xi\bar\sigma)$, $b = \omega$ and $\tilde\chi = -\chi\mathbf{i}$.*

REMARK. It is easy to check (though not necessary, because it is clear from the construction) that the $\bar\partial$–problem relating the normal– and tangential deformations, as well as the deformation formulae of the invariants from Theorems 11 and 12 correspond under the translation formulae for the data, i.e. $\kappa = \xi a - \bar\xi q$, $c = c$ and $D\xi = \xi(2g\,dz - 2\bar g\,d\bar z)$, and the translation formulae of the preceding lemma.

15. The Davey–Stewartson Hierarchy

The Davey–Stewartson (DS) hierarchy is a 2+1–dimensional generalization of the hierarchy belonging to the non–linear Schrödinger equation (NLS). Both are important examples of soliton hierarchies, i.e. infinite families of commuting Hamiltonian flows. The members of these families are non–linear equations occurring as the compatibility condition of a pair of linear equations. In the case of the DS– and NLS–hierarchy, one of the linear operators is a Dirac operator, and the n^{th} flow of the hierarchies is a n^{th}–order deformation of the potentials of this Dirac operator.

It is well known that one can, geometrically, define evolution equations for curves in 3–dimensional space forms in such a way that the complex curvature of the curves evolves under the equations of the NLS–hierarchy. An interesting feature is that, while the definition of the even order flows depends on the complex structure of the normal bundle, which does only exist in the codimension two case, the odd order flows can be defined for curves in space forms of arbitrary dimension. The corresponding odd order soliton equations are the equations of the modified Korteweg–de–Vries (mKdV) hierarchy.

Konopelchenko discovered (see [**17**], [**18**]) that the equations of the DS hierarchy can be seen as deformations of surfaces in 4–dimensional space in a similar way as the equations of the NLS hierarchy can be interpreted as deformations of curves in 3–dimensional space forms. His approach uses the Weierstrass representation of surfaces: the equations of the DS hierarchy are used to deform the potentials of the Dirac operators, which occur in the Weierstrass representation. While Konopelchenko's discussion is purely local, Taimanov points out in [**26**] that the evolution equations can in fact be globally defined for immersions of tori. The Novikov–Veselov (NV) hierarchy, which is a 2+1–dimensional generalization of the mKdV hierarchy, consists of the odd–order flows of the DS hierarchy. As in the case of curves, the odd order flows can be defined for all codimensions.

This section starts with the definition of the flows of the DS hierarchy for immersions of tori. As an example, it is explained that for special surfaces obtained from curves (i.e. cylinders, rotational surfaces and cones), one recovers the evolution equations for space curves related to the NLS hierarchy. Furthermore, it is proven that the Novikov–Veselov flow preserves constant mean curvature surfaces and, in the Euclidean case, coincides with one of the flows used in [**21**]. The last part of the section deals with a result on the tori in 3–space which are stationary under the DS flow.

15.1. The Flows. We only give a brief description of the flows, for more details, cf. [**4**]. We use the lightcone setting, because this simplifies the formulae significantly. The corresponding formulae for the quaternionic case can be easily derived with the help of the correspondence discussed in Section 14.4.

Let $\mathcal{L} \subset \mathcal{V}$ be a conformal immersion into a Minkowski space bundle $(\mathcal{V}, \nabla)$ over a torus M with a fixed holomorphic vector field X (and z the corresponding coordinate on the universal covering). We use Theorem 12 to describe conformal deformation of $\mathcal{L}$.

The role of the vector field X (or the corresponding coordinate z) in the treatment of flows is twofold: on the one hand, it is needed to define the invariants (of Chapter IV or Appendix A) and to calculate derivatives occurring in the evolution equations. On the other hand, X is needed to actually define the flows: all flows (except the 0^{th} flow) defined below depend on the choice of a nowhere vanishing holomorphic vector field (or, equivalently, the choice of a coordinate on the universal covering). In principle it is possible that the vector field used to define the invariants is not the same than the one used to define the flows, but for simplicity, in the following definitions, we assume that both vector fields coincide (see also the proof of Lemma 98, where we use flow vector fields that are not coordinate vector fields). Nevertheless, in order to emphasize the dependence of the flow on a vector field, we sometimes call the vector field used in the definition of the flows the *flow vector field*.

The Normal Bundle Rotation or 0^{th} Flow. The *normal bundle rotation flow* or *0^{th} flow* is defined by $\varrho = 0$, $b = 0$ and $\tilde{\chi}$ a real function. This means, there is no normal and tangential deformation (i.e. the immersion itself is not deformed), but only a rotation of the normal bundle frame. Theorem 12 implies that the resulting deformation of the invariants is

$$\begin{aligned} \dot{\kappa} &= -\tilde{\chi} J\kappa, \\ \dot{c} &= 0, \\ \dot{D} &= d\tilde{\chi} J. \end{aligned} \tag{113}$$

Remark. We have seen in Theorem 12 that the infinitesimal rotation of the normal bundle given by $\tilde{\chi}$ is independent of the normal– and tangential deformation ϱ and b. In the definition of the higher flows we will therefore always demand that $\tilde{\chi}$ is zero. Geometrically, this normalization means that, in the Schrödinger picture, the normal frame evolves by parallel transport in the normal bundle along the parameter t of the evolution.

The Reparametrization or 1^{st} Flow. The next flow is the *reparametrization flow* or *1^{st} flow*. It is obtained by setting $\varrho = 0$, $b \in \mathbb{C}$ and $\tilde{\chi} = 0$, meaning there is no normal deformation, but only a reparametrization of the conformal immersion. The invariants evolve by

$$\begin{aligned} \dot{\kappa} &= bD_z\kappa + \bar{b}D_{\bar{z}}\kappa, \\ \dot{c} &= bc_z + \bar{b}c_{\bar{z}}, \\ \dot{D}_z &= 2\bar{b}\langle J\kappa, \bar{\kappa}\rangle J. \end{aligned} \tag{114}$$

The Davey–Stewartson or 2^{nd} Flow. The *Davey–Stewartson flow* or *2^{nd} flow* is obtained by taking the normal deformation

$$\varrho = 2\,\mathrm{Re}(J\kappa). \tag{115}$$

By (107), the tangential deformation has to satisfy

$$\bar{b}_z = 2\langle J\bar{\kappa}, \kappa\rangle.$$

On a torus, this equation has a solution b if and only if $\int_M \langle J\bar{\kappa}, \kappa\rangle = 0$ (see the remark following Theorem 11), which by equation (154) is equivalent to the normal bundle degree of the immersion being 0. The solution b is then uniquely defined up to adding a constant. Otherwise stated, for tori of normal bundle degree 0, the 2^{nd} flow is well defined up to reparametrization flow.

The formulae for the evolution of the invariants under the DS flow are not given here (but can be found in the proof of Lemma 98). It is worthwhile noting that, in the general case, there is no explicit solution b to the $\bar\partial$–problem and therefore, the evolution of the invariants as well is non–explicit. An explicit solution to the $\bar\partial$–problem, namely $b = 0$, and explicit deformation formulae can be given in the case of tori with flat normal bundle (e.g. isothermic tori).

Remark. By Lemma 95 and Lemma 106, the normal deformation of the flow with respect to a space form geometry is

$$\rho = 2\,\mathrm{Re}(Jq) = 2\,\mathrm{Re}(J\,\mathrm{I\!I}(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z})), \tag{116}$$

i.e. $\dot\Phi^\perp = \frac{1}{2}J(\mathrm{I\!I}_{XX} - \mathrm{I\!I}_{YY}) = J\,\mathrm{I\!I}^0(X, X)$ where $X = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar z}$ and $Y = JX$. Therefore, in the metrical setting, the normal deformation is given by the trace free second fundamental form evaluated on the flow vector field and rotated by 90 degree in the normal bundle. This can be seen as the analog to the smoke ring flow on curves parametrized with respect to arc length in 3–dimensional space forms, see also Subsection 15.2.

The Novikov–Veselov or 3$^{\text{rd}}$ Flow. The *Novikov–Veselov flow* or *3rd flow* is obtained by taking the normal deformation

$$\varrho = 2\,\mathrm{Re}(D_z\kappa). \tag{117}$$

By (107), the tangential deformation has to satisfy

$$\bar b_z = 2\langle D_z\kappa + D_{\bar z}\bar\kappa, \kappa\rangle = \langle \kappa, \kappa\rangle_z + 2\langle D_{\bar z}\bar\kappa, \kappa\rangle.$$

In contrast to that of the Davey–Stewartson flow, this $\bar\partial$–problem can always be solved by the remark following Theorem 11, because $\int_M \langle\kappa, \kappa\rangle_z = 0$ and $\int_M \langle D_{\bar z}\bar\kappa, \kappa\rangle = 0$. Both integrals vanish, because the integrands are exact forms (which is clear in the case of the first integrand and which follows by using the Gauss equation (152) in the case of the second integrand).

For isothermic surfaces, if z is a conformal curvature line coordinate,

$$b = \langle\kappa, \kappa\rangle + \tfrac{c}{4} \tag{118}$$

solves the $\bar\partial$–problem (as can be checked using the Gauss equation (152)). For general immersions, the $\bar\partial$–problem can not be solved explicitly and the resulting evolution of the invariants as well is not explicit.

An important difference to the Davey–Stewartson flow is that the Novikov–Veselov flow can be defined for surfaces in arbitrary codimension. In particular, it leaves surfaces in S^3 invariant.

Remark. By Lemma 95 and Lemma 106, the normal deformation of the flow with respect to a space form geometry is

$$\rho = 2\,\mathrm{Re}(\nabla^\perp_z(qe^{-u})e^u). \tag{119}$$

General Properties of the Flows. For tori, the 2$^{\text{nd}}$ and 3$^{\text{rd}}$ flow are only well defined up to reparametrization flow, because one can always add a constant to the solution b to the $\bar\partial$–problem. Furthermore, as mentioned above, all flows (except the 0$^{\text{th}}$ flow) depend on the choice of a nowhere vanishing holomorphic vector field.

Although we are not able to define higher flows, we expect that this can be done (as in the Euclidean setting, using the original DS hierarchy). These higher flows shall be well defined up to lower flows only, which is always the case with soliton equations.

We finish the general discussion of flows with a list of important properties which are, in the case of the flows defined above, either evident or proven in [**4**], and which are expected to hold as well for all higher flows:

- the conformal type of the torus, as well as the monodromy of its immersion, are preserved (this is clear, because we are only considering conformal deformations, i.e. b's solving the $\bar{\partial}$–problem, and since we deform by infinitesimal gauge transformations),
- the Willmore functional of the immersion is preserved (to be honest, for the NV flow this is only proven in the case of surfaces with flat normal bundle, e.g. isothermic surfaces and surfaces in S^3),
- the odd–order flows leave the set of surfaces in S^3 invariant,
- all flows leave the sets of constrained Willmore immersions and of isothermic immersions invariant. (For isothermic surfaces, the $\bar{\partial}$–problem can be solved explicitly.)

Remark. In principle it makes sense to define the flows locally (as it is done by Konopelchenko), the 'only' problem being that locally there is a huge amount of conformal coordinates (flow vector fields) as well as solution to the $\bar{\partial}$–equation.

On the torus, the conformal vector fields (or coordinates on the universal covering) are unique up to rotation and translation and the solutions to the $\bar{\partial}$–problem are unique up to constant.

One might wish as well to define flows for other compact Riemann surfaces than tori. It is quite clear that, using the present approach, this cannot work without some modification, because there simply are no nowhere vanishing holomorphic vector fields, which are needed for the definition of the invariants (in both the quaternionic and the lightcone setting) and for the definition of the flows.

We expect that for the sphere, with its 3–dimensional space of holomorphic vector fields (all of which do necessarily have 2 zeros), one can actually find some modification of our approach which allows to define some kind of flows. These should yield a new characterization of soliton spheres.

For compact surfaces of genus $g \geq 2$, which do not admit any global holomorphic vector fields at all, it is difficult to imagine, how a global version of the flows could possibly look like. It is clear that, in this case, a less coordinate dependent approach, like in Section 14.1 or in [**5**], would be required for the definition of global flows.

15.2. Example: Flows of Curves. As an example, it is shown that the classical flows of curves in space forms can be recovered as special cases of the flows of the Davey–Stewartson hierarchy.

Let $\gamma(x)$ be a curve parametrized with respect to arc length in a 3–dimensional space form. In Lemma 109, it is shown that γ gives rise to a conformal immersion by the cylinder, cone or rotational surface construction and that the conformal invariants of this immersion, with respect to $z = x + iy$ for x an arc length parameter of γ, are given by

$$\kappa = \tfrac{1}{4}(\kappa_1 n_1 + \kappa_2 n_2)$$

and

$$c = \tfrac{1}{4}(\kappa_1^2 + \kappa_2^2) + \tfrac{G}{2},$$

where G is the curvature of the space form, κ_1, κ_2 the curvature of γ with respect to an orthonormal parallel frame N_1, N_2 of the normal bundle and where n_1, n_2 is a parallel frame of the Möbius normal bundle obtained from N_1, N_2.

Davey–Stewartson and Smoke Ring Flow. By (116), the normal deformation of the Davey–Stewartson flow, in the Euclidean setting of the space form $\mathcal{S}_{v_0}$ used in the proof of Lemma 109, is $\rho = 2\operatorname{Re}(Jq)$. With the tangential deformation $b = 0$ (which solves the $\bar\partial$–problem, because our immersion has a flat normal bundle), and using the formulae for q in the proof of Lemma 109, it follows that the resulting deformation of the curve γ obtained by setting $y = 0$ is

$$\dot\gamma = \tfrac{1}{2}\gamma' \times \gamma''. \tag{120}$$

This is the deformation of the *smoke ring flow* on curves, which is obtained by 90 degree rotation of the curvature vector. The formula for the deformation of the conformal Hopf differential of Theorem 12 implies

$$\dot\kappa = \tfrac{J}{2}(\kappa'' + (24|\kappa|^2 + G)\kappa)$$

(where $()'$ denotes the $\frac{\partial}{\partial x}$–derivative). In particular, the complex curvature $\boldsymbol{\psi} = \kappa_1 + i\kappa_2$ of the curve in a space form of curvature G evolves by the non–linear Schrödinger equation

$$\dot{\boldsymbol{\psi}} = \tfrac{1}{4}(\boldsymbol{\psi}''' + (\tfrac{1}{2}|\boldsymbol{\psi}|^2 + G)\boldsymbol{\psi}'). \tag{121}$$

From the classical theory of the smoke ring flow, it is clear that the evolution is by curves in the same space form, which, by Corollary 110, is equivalent to $c = 4|\kappa|^2 + \frac{G}{2} = \frac{1}{4}|\boldsymbol{\psi}|^2 + \frac{G}{2}$ being invariant under the flow.

By (160), a curve is elastic if and only if it is stationary under the smoke ring flow in the sense that the smoke ring flow acts by isometries of the space form and rotation of the normal bundle only. It is free elastic if and only if it acts by isometries of the space form only (without rotation of the normal bundle).

Remark. In view of the fact that the Davey–Stewartson flow and the smoke ring flow are defined by a very similar recipe, it is not surprising that the former reduces to the later in the case of surfaces obtained from curves. The surface flow can be seen as a generalization of the curve flow in the same way as the Davey–Stewartson equation is a $2+1$–generalization of the non–linear Schrödinger equation.

Novikov–Veselov and mKdV Flow. By (119), the normal deformation of the Novikov–Veselov flow, in the Euclidean setting of the space form $\mathcal{S}_{v_0}$ used in the proof of Lemma 109, is $\rho = 2\,\mathrm{Re}(\nabla_z^{\perp}(qe^{-u})e^u)$. Because the immersion is isothermic, (118) implies that the tangential deformation b has to satisfy

$$b = |\kappa|^2 + \tfrac{c}{4} + \mu = \tfrac{1}{8}|\boldsymbol{\psi}|^2 + \tfrac{G}{8} + \mu$$

for some complex constant μ. It turns out convenient to choose $\mu = -\frac{G}{8}$. Using the formulae for q in the proof of Lemma 109, we obtain that the resulting deformation of the curve γ, i.e. for $y = 0$, is

$$\begin{aligned}\dot\gamma = \dot\Phi(.,0) &= \rho + b\Phi_x(.,0)\\ &= \tfrac{1}{4}(\nabla_T^{\perp}\nabla_T T + \tfrac{1}{2}|\boldsymbol{\psi}|^2 T)\\ &= \tfrac{1}{4}(\nabla_T\nabla_T T - \langle\nabla_T\nabla_T T, T\rangle T + \tfrac{1}{2}|\boldsymbol{\psi}|^2 T)\\ &= \tfrac{1}{4}(\nabla_T\nabla_T T + \tfrac{3}{2}\langle\nabla_T T, \nabla_T T\rangle)\end{aligned}$$

(where ∇ denotes the Levi–Civita–Connection of $\mathcal{S}_{v_0}$). The deformation of the curve is therefore given by

$$\dot\gamma = \tfrac{1}{4}(T'' + \tfrac{3}{2}\langle T', T'\rangle T), \tag{122}$$

which is the usual formula for the *modified Korteweg–de–Vries (mKdV) flow* on curves in a space form of curvature G. The formula for the deformation of the conformal Hopf differential of Theorem 12 implies

$$\dot\kappa = \tfrac{1}{4}(\kappa''' + (24|\kappa|^2 + G)\kappa')$$

(where $()'$ denotes the $\frac{\partial}{\partial x}$–derivative). In particular, the complex curvature $\boldsymbol{\psi} = \kappa_1 + i\kappa_2$ of the curve in a space form of curvature G evolves by the mKdV equation

$$\dot{\boldsymbol{\psi}} = \tfrac{1}{4}(\boldsymbol{\psi}''' + (\tfrac{3}{2}|\boldsymbol{\psi}|^2 + G)\boldsymbol{\psi}'). \tag{123}$$

It is clear from the classical theory of the mKdV flow that the evolution is by curves in the same space form, which, by Corollary 110, is equivalent to $c = 4|\kappa|^2 + \frac{G}{2} = \frac{1}{4}|\boldsymbol{\psi}|^2 + \frac{G}{2}$ being invariant under the flow.

15.3. Example: Flows of CMC Surfaces. Another interesting example for the study of flows are the constant mean curvature (CMC) surfaces. The aim of this subsection is to prove that, for a flow vector field intersecting the curvature line directions at constant angle, the Novikov-Veselov flow induces an evolution of CMC surfaces in the same space form. Under this evolution, the mean curvature and the metrical Hopf differential are constant. It is shown that, in the case of CMC surface in Euclidean space, one of the flows used in [**21**] is obtained.

In Section 3.3 of Appendix A we show that, for a CMC surface in a 3–dimensional space form $\mathcal{S}_{v_0}$ of curvature G, there is a conformal coordinate z such that the Hopf differential q with respect to z is constant. Such coordinate is a rotated conformal curvature line coordinate.

Using $\kappa = qe^{-u}$ (cf. equation (167)), the normal deformation of the Novikov–Veselov flow in the space form $\mathcal{S}_{v_0}$, which is given by (119), becomes

$$\rho = 2\,\mathrm{Re}((qe^{-u})_z e^u) = -2\,\mathrm{Re}(qu_z). \tag{124}$$

For the tangential deformation we have to solve the $\bar\partial$–equation $\bar b_z = 2e^{-2u}q\rho$ (cf. Lemma 95). It is easy to check (using the Gauss equation (165)) that

$$b = \bar q^2 e^{-2u} + \tfrac{1}{2}(u_{zz} - u_z^2 + Hq + \mu)$$

with $\mu \in \mathbb{C}$ is a solution to the $\bar\partial$–problem. By Lemma 96, the resulting evolution of the metric u is

$$\begin{aligned}\dot u &= u_z b + u_{\bar z}\bar b + \operatorname{Re}(b_z) - \rho H\\ &= \operatorname{Re}(2u_z b + b_z - 2qHu_z)\\ &= \operatorname{Re}(2u_z\bar q^2 e^{-2u} + u_z(u_{zz} - u_z^2 + Hq + \mu) -\\ &\qquad - 2u_z\bar q^2 e^{-2u} + \tfrac{1}{2}u_{zzz} - u_z u_{zz} - 2qHu_z)\\ &= \operatorname{Re}(\tfrac{1}{2}u_{zzz} - u_z^3),\end{aligned}$$

where we have put $\mu = Hq$, the evolution of the mean curvature H is

$$\begin{aligned}\dot H &= bH_z + \bar b H_{\bar z} + H^2\rho + 4e^{-4u}|q|^2\rho + G\rho + 2e^{-2u}\rho_{z\bar z}\\ &= 2e^{-2u}((\tfrac{1}{2}e^{2u}(H^2 + G) + 2e^{-2u}|q|^2)\rho + \rho_{z\bar z})\\ &= -4e^{-2u}\operatorname{Re}((\tfrac{1}{2}e^{2u}(H^2+G) + 2e^{-2u}|q|^2)qu_z + qu_{zz\bar z})\\ &= -4e^{-2u}\operatorname{Re}(q(\tfrac{1}{4}e^{2u}(H^2+G) - e^{-2u}|q|^2 + u_{z\bar z})_z) = 0,\end{aligned}$$

where the last equality holds by the Gauss equation (165), and the evolution of the Hopf differential q is

$$\begin{aligned}\dot q &= bq_z + \bar b q_{\bar z} + 2b_z q + \rho_{zz} - 2u_z\rho_z\\ &= 2(-2u_z\bar q^2 e^{-2u} + \tfrac{1}{2}u_{zzz} - u_z u_{zz})q - (qu_{zzz} + \bar q u_{\bar z zz}) + 2u_z(qu_{zz} + \bar q u_{\bar z z})\\ &= 2(-2u_z\bar q^2 e^{-2u})q - \bar q u_{\bar z zz} + 2u_z\bar q u_{\bar z z}\\ &\overset{(*)}{=} 2(-2u_z\bar q^2 e^{-2u})q + \bar q u_z(2|q|^2 e^{-2u} + \tfrac{1}{2}e^{2u}(H^2+G)) +\\ &\qquad + 2u_z\bar q(|q|^2 e^{-2u} - \tfrac{1}{4}e^{2u}(H^2+G)) = 0,\end{aligned}$$

where $(*)$ holds by the Gauss equation (165). This show that the evolution under the Novikov–Veselov flow is by CMC surfaces in the same space form and with constant mean curvature.

Remark. The normal deformation (124) and the evolution of u coincide with those in [**21**], see equations 3(1) and 3(9) there.

15.4. Stationary Tori. We call a torus *stationary* under the n^{th} flow, if, for all flow vector fields X, the deformation vector field of the n^{th} flow is a linear combination of the deformation vector fields of the lower order flows.

For example, totally umbilic surfaces are the only stationary surfaces for the 0^{th} flow (with constant $\tilde\chi$). Similarly, a surface is stationary under the 1^{st} flow if and only if it is homogeneous, i.e. the orbit of a 2–parameter group of Möbius transformations, for example a cyclide of Dupin.

The aim of this section is to study those tori in S^3 which are stationary under the Davey–Stewartson flow, i.e. those tori which do not evolve under the Davey–Stewartson flow up to reparametrization, Möbius transformation and rotation of the normal bundle. The rest of this section is devoted to the proof and discussion of the following theorem.

THEOREM 13. *An conformal immersion with monodromy of a torus into the conformal 4–sphere which takes values in a 3–sphere is stationary under the Davey–Stewartson flow if and only if it is isothermic*[4] *and constrained Willmore.*

By a theorem of J. Richter (see [**4**, Section 3.4]), an immersion of a torus into the 3–sphere is constrained Willmore and isothermic if and only if it is a constant mean curvature (CMC) surface with respect to a space form geometry.

The following lemma provides the general equation characterizing the tori stationary under the DS flow.

LEMMA 98. *A conformal immersion with Möbius monodromy of a torus into the conformal 4–space with normal bundle degree 0 is stationary under the Davey–Stewartson flow if and only if there are complex constants α and β and a complex function μ such that the invariants of the torus, with respect to a conformal coordinate z on the universal covering of the torus, satisfy*

$$J(D_zD_z\kappa + \frac{c}{2}\kappa) + \frac{3}{2}b_z\kappa + bD_z\kappa = \alpha D_z\kappa + \bar\beta D_{\bar z}\kappa - \mu J\kappa, \tag{125}$$

$$J(D_zD_z\bar\kappa + \frac{c}{2}\bar\kappa) - \frac{1}{2}\bar b_{\bar z}\kappa + \bar b D_{\bar z}\kappa = \beta D_z\kappa + \bar\alpha D_{\bar z}\kappa - \bar\mu J\kappa, \tag{126}$$

$$\begin{aligned} &b_{zzz} + 2cb_z + bc_z + 16\langle J\kappa, \bar\kappa\rangle\langle\kappa,\kappa\rangle + \\ &\qquad +8(\langle D_{\bar z}D_zJ\kappa,\kappa\rangle - \langle D_zJ\kappa, D_{\bar z}\kappa\rangle) = \alpha c_z + \bar\beta c_{\bar z}, \end{aligned} \tag{127}$$

$$\begin{aligned} &\bar b c_{\bar z} + 6(\langle D_{\bar z}D_zJ\bar\kappa,\kappa\rangle - \langle D_zJ\bar\kappa, D_{\bar z}\kappa\rangle) + \\ &\qquad +2(\langle D_{\bar z}J\bar\kappa, D_z\kappa\rangle - \langle J\bar\kappa, D_{\bar z}D_z\kappa\rangle) = \beta c_z + \bar\alpha c_{\bar z}, \end{aligned} \tag{128}$$

$$0 = 2\bar\beta\langle\bar\kappa, J\kappa\rangle + (\mu)_z, \tag{129}$$

$$2\langle\bar\kappa, J\kappa\rangle\bar b + 2\langle D_{\bar z}\bar\kappa,\ \kappa\rangle - 2\langle D_{\bar z}\kappa,\bar\kappa\rangle = 2\bar\alpha\langle\bar\kappa, J\kappa\rangle + (\bar\mu)_z, \tag{130}$$

where b is a solution to the $\bar\partial$–equation $\bar b_z = 2\langle J\bar\kappa,\kappa\rangle$.

PROOF. We fix a coordinate z on the universal covering of the torus. Because stationary means that, for all such coordinates, the second flow is linearly dependent of the lower flows, we need to consider all possible coordinates on the universal cover. It is sufficient to consider all coordinates $\tilde z = ze^{-\mathbf{i}\theta}$ for $\theta \in \mathbb{R}$, because scaling the coordinate by a real factor does only change the speed of the flows (and adding a constant does not change anything, because the flows do only depend on the corresponding holomorphic vector field).

From (149) and (150), we obtain that the invariants with respect to the new coordinate are $\tilde\kappa = \kappa e^{2i\theta}$ and $\tilde c = ce^{2i\theta}$. Furthermore $\tilde b = be^{i\theta}$ solves the $\bar\partial$–problem for the tangential deformation with respect to $\tilde z$ if b solves that with respect to z. Using this and

$$\frac{\partial}{\partial\tilde z} = \frac{\partial}{\partial z}e^{\mathbf{i}\theta},$$

[4]There is a little gap at the end of the proof: in the case that a stationary torus is Willmore, we can only prove that it has to be classically isothermic. Because these tori actually have to satisfy a much stronger condition than classically isothermic, we believe that they also have to be isothermic in the global sense.

Theorem 12 implies that the DS flow with respect to $\tilde{z}$ (with this choice of $\tilde{b}$ and with $\tilde{\chi} = 0$) is given by

$$\begin{aligned}
\dot{\kappa} &= e^{2i\theta}(J(D_z D_z \kappa + \frac{c}{2}\kappa) + \frac{3}{2} b_z \kappa + b D_z \kappa) + \\
&\qquad e^{-2i\theta}(J(D_z D_z \bar{\kappa} + \frac{c}{2}\bar{\kappa}) - \frac{1}{2}\bar{b}_{\bar{z}}\kappa + \bar{b} D_{\bar{z}}\kappa) \\
\dot{c} &= e^{2i\theta}(b_{zzz} + 2cb_z + bc_z + 16\langle J\kappa, \bar{\kappa}\rangle\langle \kappa, \kappa\rangle + \\
&\qquad\qquad + 8(\langle D_{\bar{z}} D_z J\kappa, \kappa\rangle - \langle D_z J\kappa, D_{\bar{z}}\kappa\rangle)) + \\
&\quad e^{-2i\theta}(\bar{b}c_{\bar{z}} + 6(\langle D_{\bar{z}} D_z J\bar{\kappa}, \kappa\rangle - \langle D_z J\bar{\kappa}, D_{\bar{z}}\kappa\rangle) + \\
&\qquad\qquad + 2(\langle D_{\bar{z}} J\bar{\kappa}, D_z\kappa\rangle - \langle J\bar{\kappa}, D_{\bar{z}} D_z \kappa\rangle)), \\
\dot{D}_z &= e^{-2i\theta}(2\langle \bar{\kappa}, J\kappa\rangle \bar{b} + 2\langle D_{\bar{z}}\bar{\kappa},\ \kappa\rangle - 2\langle D_{\bar{z}}\kappa, \bar{\kappa}\rangle).
\end{aligned}$$

The immersion is stationary, if there is a complex function $\alpha(\theta)$ depending on θ and a real function $f(\theta, p)$ depending on θ and the point $p \in M$ of the torus, such that, for all θ, the above deformation equals the linear combination

$$\begin{aligned}
\dot{\kappa} &= \alpha(\theta)e^{\mathbf{i}\theta} D_z \kappa + \bar{\alpha}(\theta)e^{-\mathbf{i}\theta} D_{\bar{z}}\kappa - fJ\kappa, \\
\dot{c} &= \alpha(\theta)e^{\mathbf{i}\theta} c_z + \bar{\alpha}(\theta)e^{-\mathbf{i}\theta} c_{\bar{z}} \\
\dot{D}_z &= 2\bar{\alpha}(\theta)e^{-\mathbf{i}\theta}\langle J\kappa, \bar{\kappa}\rangle J + f(\theta)_z J.
\end{aligned}$$

of reparametrization– and normal bundle rotation flow.

The function $f(\theta, p)$ can be written as

$$f(\theta) = \sum_{k\in\mathbb{Z}} f_k e^{ik\theta},$$

where f_k are functions on the torus satisfying $f_{-k} = \bar{f}_k$ for all $k \in \mathbb{Z}$, and $\alpha(\theta)$ can be written as

$$\alpha(\theta)e^{i\theta} = \sum_{k\in\mathbb{Z}} \alpha_k e^{ik\theta}$$

where the α_k are complex constants.

A Fourier decomposition with respect to θ immediately yields the condition for the immersion to be stationary under the DS flow, which proves the lemma. The constants α, β in the statement of the lemma are the Fourier coefficients α_2 and α_{-2}, and the function μ is the coefficient f_2. □

We are now able to prove the Theorem above.

Proof (of Theorem 13). Immersions taking values in a 3–sphere have a flat normal bundle, $\langle J\bar{\kappa}, \kappa\rangle = 0$. In particular, $b = 0$ is a solution to the $\bar{\partial}$–equation $\bar{b}_z = 2\langle J\bar{\kappa}, \kappa\rangle$. This immediately simplifies all of the equations in Lemma 98. For example, since we are on a torus, (129) is equivalent to μ being constant.

Because the immersion takes values in a 3–sphere, there (locally) is a parallel section $n \in \Gamma(\mathcal{W}^{\perp})$ of the normal bundle with length 1. Then

$$\langle \kappa, n\rangle = 0. \tag{131}$$

Setting $\xi = \frac{1}{2}(n - Jn\mathbf{i})$, we have $\kappa = \xi a - \bar{\xi} q$ and (131) is equivalent to $a = q$, i.e. $\kappa = \xi a - \bar{\xi} a = -Jn\mathbf{i}\, a$. This has several important consequences: firstly, the equations (125) and (126) can be decomposed into Jn– and n–part, which yields

$$D_z D_z \kappa + \frac{c}{2}\kappa = -\mu\kappa, \tag{125'}$$

$$D_z D_z \bar{\kappa} + \frac{c}{2}\bar{\kappa} = -\bar{\mu}\kappa, \tag{126'}$$

and

$$\alpha D_z \kappa + \bar{\beta} D_{\bar{z}} \kappa = 0, \tag{125''}$$

$$\beta D_z \kappa + \bar{\alpha} D_{\bar{z}} \kappa = 0. \tag{126''}$$

Secondly, the remaining scalar product terms in equations (127) and (128) vanish and the equations become

$$\alpha c_z + \bar{\beta} c_{\bar{z}} = 0, \tag{127'}$$

$$\beta c_z + \bar{\alpha} c_{\bar{z}} = 0. \tag{128'}$$

The equations (125”), (126”), (127’) and (128’) are solves by $\alpha = \beta = 0$ (or, in the case of equivariant surfaces by some other α and β).

In the case $\mu \neq 0$, equation (126’), together with the Codazzi equation, implies $\mathrm{Im}(\bar{\mu}\kappa) = 0$, which shows that, after rotation of the coordinate, κ can be assumed to be real and the immersion is isothermic. Furthermore,

$$D_z D_z \kappa + \frac{c}{2}\kappa = -\mu\kappa$$

shows that the immersion is constrained Willmore (see equation (159)). (Note that the remaining equations (125’) and (130) are automatically satisfied if κ and μ are real and (126’) is satisfied).

In the case that $\mu = 0$, the immersion is Willmore. It can be easily verified that equation (130), which for immersions into the 3–sphere becomes

$$\langle D_{\bar{z}} \bar{\kappa},\ \kappa \rangle - \langle D_{\bar{z}} \kappa, \bar{\kappa} \rangle = 0, \tag{130'}$$

implies that, away from umbilic points, a rotation of z yields conformal curvature line coordinates. In particular, the immersion is classically isothermic (see Section 9.3). Furthermore, equation 125’ is then automatically satisfied, because on every connected component of the set of non–umbilic points, κ can be written as a real section of the normal bundle times a constant complex number.

It is immediately checked (by reversing the argumentation) that an isothermic constrained Willmore torus in a 3–dimensional space form is stationary. □

16. DS Flow as a Limit of Darboux Transformations

In this section it is proven that, at least in the case of special immersions, the reparametrization and Davey–Stewartson flows arise as a limit of Darboux transformations. The special immersions we have in mind here are the so called finite type tori, which we introduce in the following (speculative) section.

16.1. A Short Note on Finite Type Tori. This section, which is purely speculative, is merely included to place the result of the next theorem in the right context.

Definition. A conformal immersion of a torus into the conformal 4–sphere $\mathbb{HP}^1$ with normal bundle of degree 0 is called *finite type*, if there is a compact Riemann surface $\tilde{\Sigma}$ and an analytic diffeomorphism $f\colon \tilde{\Sigma}\backslash\{x_1, ..., x_n\} \to \mathring{\Sigma}$ defined away from finite many point, where $\mathring{\Sigma} = \Sigma\backslash\{\text{bad points}\}$.

Remark. We conjecture that the following facts can be proven:

(1) For every point $p \in M$, the eigenspace curve $\mathcal{E}_p\colon \Sigma \subset \mathcal{M} \to \mathbb{CP}^3$ and the map $S_p\colon \Sigma \subset \mathcal{M} \to \mathbb{CP}^1$ extend holomorphically to $\tilde{\Sigma}$.
(2) The map $p \in M \mapsto \mathcal{E}_p\mathcal{E}_0^{-1} \in Jac(\Sigma)$ is linear. In particular, all finite type tori can be obtained from algebraic curves (with special real symmetries) in $\mathbb{CP}^3$ and linear deformations tangent to the Abel map (at infinity). Furthermore, the higher flows arise as linear deformations in the direction of the Frenet flag of the Abel map (at infinity).
(3) Constrained Willmore tori are finite type. In the case of CMC–tori in Euclidean space, this has been proven by Pinkall and Sterling, cf. [**21**]. The general case of constrained Willmore tori in the 3–sphere is dealt with in [**25**].

The first point, namely the fact that the eigenspace curve can be holomorphically extended to infinity, is the motivation for the assumption of the next theorem: it shows that, if the eigenspace curve can be extended through infinity, the deformation vector fields of the reparametrization and the Davey–Stewartson flow can be described in terms of Darboux transformations.

16.2. The Theorem.

Theorem 14. *Let $L \subset V$ be a conformal immersion of a torus (of degree 0) into $\mathbb{P}V$, where V is a quaternionic rank 2 vector space. We assume that there is a smooth family of immersions $L_t \subset V$, such that $L_0 = L$ and L_t for $t \neq 0$ is a Darboux transformations of L, and such that there is a point $p \in M$ with the property that the t–derivative of $L_t(p)$ for $t = 0$ is non–zero. Then,*

a) the first derivative of L_t at $t = 0$ is a nowhere vanishing tangential (conformal) vector field, and
b) the second derivative at $t = 0$ of the common Darboux transformation of L_t and L_{-t} (that exists by Theorem 3) has as its normal component that of the Davey–Stewartson flow.

The rest of the section it devoted to the proof of this theorem.

Step 1: a Setting for the Proof. To start with, we chose a splitting $V = L \oplus \check{L}$, where $\check{L}$ is a fixed point. With respect to this splitting, the trivial connection ∇ on V takes the form

$$\nabla = \begin{pmatrix} \nabla^L & 0 \\ \delta & \check{\nabla} \end{pmatrix}.$$

We assume that none of the immersions L_t goes through $\check{L}$. Then

$$L_t = (\mathrm{Id} + W_t)L$$

with $W_t \in \Gamma(\mathrm{Hom}(L, \check{L}))$. For $\psi \in \Gamma(L)$, we have

$$\nabla(\mathrm{Id} + W_t)\psi = \underbrace{\nabla^L \psi}_{\in L} + \underbrace{(\delta\psi + \check{\nabla}(W_t\psi))}_{\in \check{L}} \tag{132}$$

$$= \underbrace{(\mathrm{Id} + W_t)\nabla^L \psi}_{\in L_t} + \underbrace{(\delta\psi + (\nabla W_t)\psi)}_{=\delta^t(\mathrm{Id} + W_t)\psi \in \check{L}}, \tag{133}$$

where δ^t denotes the derivative of the immersion L_t. By Lemma 32 and (132), the condition for L_t to be a Darboux transformation of L is that there is $\psi_t \in \Gamma(\tilde{L})$ (where, as usual, $\tilde{L}$ denotes the pull–back of L to the universal covering of M) with

$$\delta\psi_t + (\check{\nabla} W_t \psi_t) = 0. \tag{134}$$

Define $\beta_t \in \Omega^1(\mathrm{End}(L))$ by $\nabla^L \psi_t = -\beta_t \psi_t$. Since

$$\nabla(\mathrm{Id} + W_t)\psi_t = \nabla^L \psi_t = -\beta_t \psi_t \in \Omega^1(L)$$

is a closed form, Lemma 13 implies $\beta_t \in \Gamma(K \,\mathrm{End}(L))$.

Remark. The connection $\nabla^L + \beta_t$ is gauge equivalent to the flat connection $\nabla^{\#_t}$ on L_t, the gauge equivalence being $L \overset{\mathrm{Id} + W_t}{\longrightarrow} L^t$.

With this definition, equation (134) becomes

$$\delta + \nabla W_t = W_t \beta_t. \tag{134'}$$

Step 2: the Deformation Vector Field $\dot{W}$ is Tangential and Conformal. The deformation vector field of L_t is

$$\dot{W} \in \Gamma(\mathrm{Hom}(L, \check{L} = V/L))$$

(with $\dot{}$ denoting the derivative with respect to t at $t = 0$). We have to prove that $\dot{W}$ is in Hom_+ on the non–empty open set $M' = \{p \in M \mid \dot{W}(p) \neq 0\}$. On M', because $W_0 = 0$, the family of endomorphisms W_t takes the form

$$W_t = t\tilde{W}_t$$

where $\tilde{W}_t$ is nowhere vanishing. Then, (134') implies

$$t\beta_t = \tilde{W}_t^{-1}(\delta + t\nabla\tilde{W}_t),$$

proving that $\beta_t = \frac{1}{t}\tilde{\beta}_t$ where the family of 1–forms $\tilde{\beta}_t \in \Gamma(K \,\mathrm{End}(L))$ smoothly extends through $t = 0$. Evaluation of (134') at $t = 0$ yields

$$\delta = \tilde{W}_0 \tilde{\beta}_0$$

and, since both δ and β_t are of type K, $\dot{W} = \tilde{W}_0$ has to be in Hom_+, which proves that the deformation vector field is tangential. In particular, there is a vector field X on M satisfying $\dot{W} = \delta_X$.

To prove that $\dot{W}$ (or rather X) is conformal, we use the fact that all immersion L_t are conformal, i.e.

$$*\delta^t = \delta^t \tilde{J}^t$$

for a family of complex structures $\tilde{J}^t$ on the bundles L_t. This is equivalent to

$$*\delta^t(\mathrm{Id} + W_t) = \delta^t(\mathrm{Id} + W_t)J^t \tag{135}$$

(where $J^t = (\mathrm{Id} + W_t)^{-1}\tilde{J}^t(\mathrm{Id} + W_t)$ are the corresponding complex structures on L under the gauge transformation $(\mathrm{Id} + W_t)$). Equation (133) implies that $\delta^t(\mathrm{Id} + W_t) = \delta + \nabla W_t$, and the infinitesimal version of equation (135) is

$$\tfrac{1}{2}(\nabla\dot{W} + *\nabla\dot{W}J) = -\tfrac{1}{2} * \delta\dot{J}. \tag{136}$$

In particular, taking the +–part of this equation yields

$$(\hat{\nabla}\dot{W})'' = \frac{1}{2}(\hat{\nabla}(\delta_X) + J * \hat{\nabla}(\delta_X)) = 0 \tag{137}$$

(where $\hat{\nabla}\dot{W}$ denotes the covariant derivative of $\dot{W}$ with respect to the +–parts of the connections $\check{\nabla}$ and ∇^L). To see that (137) is equivalent to conformality of the vector field X given by $\dot{W} = \delta_X$, we use $d^{\hat{\nabla}}\delta = 0$ (which follows by taking the +–part of $d^{\nabla}\delta = 0$). This equation implies

$$\begin{aligned} 0 &= (d^{\hat{\nabla}}\delta)_{(JX,X)} = \hat{\nabla}_{JX}(\delta_X) - \hat{\nabla}_X(\delta_{JX}) - \delta_{[JX,X]} \\ &= (*\hat{\nabla})_X(\delta_X) - J\hat{\nabla}_X(\delta_X) - \delta_{[JX,X]} = 2 * (\hat{\nabla}(\delta_X))''_X - \delta_{[JX,X]}. \end{aligned} \tag{138}$$

Because $\dot{W} = \delta_X$, (137) and (138) together imply $[JX, X] = 0$, which is equivalent to X holomorphic. Since we are on the torus, this implies that X and $\dot{W} = \delta_X$ have no zeroes.

Step 3: the Second Derivative of W. In order to compute the normal component of the second derivative of W, we put

$$W_t = tW_1 + t^2 W_2 + ...$$

and

$$\beta_t = \tfrac{1}{t}\beta_0 + \beta_1 + t\beta_2 ...$$

(Since $\dot{W} = \delta_X$ has no zeros, this formula for β is justified by the discussion at the beginning of Step 2.) Then, the t^0–term of equation (134') is (as already used above)

$$\delta = W_1\beta_0,$$

which, by $W_1 = \delta_X$, implies $\beta_0(X) = 1$.

The t^1–term of (134') is

$$\nabla W_1 = W_1\beta_1 + W_2\beta_0.$$

Because $W_1 \in \Gamma(\mathrm{Hom}_+(L, \check{L}))$ and $\beta_1 \in \Gamma(K\,\mathrm{End}(L))$, this equation implies

$$(\nabla W_1)'' = \frac{1}{2}(\nabla W_1 + *J\nabla W_1) = (W_2)_-\beta_0,$$

and, by plugging in X, we obtain

$$(W_2)_- = (\nabla W_1)''_X = (\nabla\delta_X)''_X.$$

Hence, by conformality of X, which we have seen (in equation (138)) to be equivalent to $(\hat{\nabla}(\delta_X))'' = 0$, the –-part of W_2 satisfies

$$(W_2)_- = (\hat{\nabla}(\delta_X) + (\check{A} + \check{Q})\delta_X - \delta_X(A^L + Q^L))''_X = \check{Q}_X\delta_X - \delta_X Q^L_X. \tag{139}$$

Step 4: the 2–fold Darboux Transformation. By Theorem 3, the common Darboux transformation of L_t and L_{-t} can be calculated as follows: for sections ψ_t and $\psi_{-t} \in \Gamma(L)$ satisfying $\nabla(\mathrm{Id} + W_t)\psi_t$, $\nabla(\mathrm{Id} + W_{-t})\psi_{-t} \in \Gamma(\tilde{L})$, there is a function χ_t such that

$$\nabla(\mathrm{Id} + W_t)\psi_t = \nabla(\mathrm{Id} + W_{-t})\psi_{-t}\chi_t. \tag{140}$$

The common Darboux transformation is then given by

$$L^{\#_{t,-t}} = ((\mathrm{Id} + W_{-t})\psi_{-t}\chi_t - (\mathrm{Id} + W_t)\psi_t)\mathbb{H}$$

or, equivalently, by

$$L^{\#_{t,-t}} = ((\mathrm{Id} + W_{-t})B_t - (\mathrm{Id} + W_t))L,$$

when the endomorphism $B_t \in \Gamma(\mathrm{End}(L))$ is defined by $B_t\psi_t = \psi_{-t}\chi_t$. Otherwise stated, with respect to the splitting $V = L \oplus \check{L}$ we have

$$L^{\#_{t,-t}} = (\mathrm{Id} + R_t)L$$

with $R_t \in \Gamma(\mathrm{End}(L))$ defined by

$$R_t = (W_{-t}B_t - W_t)(B_t - \mathrm{Id})^{-1}. \tag{141}$$

Step 5: the t–Derivative of B_t. The aim is now to calculate the second derivative of R_t with respect to t at $t = 0$. To do this, we have to understand the t–dependence of B_t. Equation (133) yields

$$\nabla(\mathrm{Id} + W_t)\psi = \underbrace{(\mathrm{Id} + W_t)(\nabla^L\psi + W_t^{-1}(\delta + \nabla W_t)\psi)}_{\in L_t} + \underbrace{-W_t^{-1}(\delta + \nabla W_t)\psi)}_{=\delta^{\#_t}(\mathrm{Id} + W_t)\psi \in L},$$

where $\delta^{\#_t}$ denotes the derivative of L^t represented with respect to the splitting $V = L \oplus L^t$. In particular, using equation (140), we obtain

$$-W_t^{-1}(\delta + \nabla W_t) = -W_{-t}^{-1}(\delta + \nabla W_{-t})B_t. \tag{142}$$

Replacing W_t by $t\tilde{W}_t$, this implies

$$-\tilde{W}_t^{-1}(\delta + t\nabla\tilde{W}_t) = \tilde{W}_{-t}^{-1}(\delta - t\nabla\tilde{W}_{-t})B_t, \tag{143}$$

which shows that B_t smoothly extends through $t = 0$ with $B_0 = -\mathrm{Id}$, i.e. B_t takes the form $B_t = (t\tilde{B}_t - \mathrm{Id})$ with $\tilde{B}_t \in \Gamma(\mathrm{End}(L))$ smooth in t. Plugging this into the last equation yields

$$-\tilde{W}_t^{-1}(\delta + t\nabla\tilde{W}_t) = \tilde{W}_{-t}^{-1}(\delta - t\nabla\tilde{W}_{-t})(t\tilde{B}_t - \mathrm{Id}).$$

The t–derivative of this equation at $t = 0$ is

$$\tilde{W}_0^{-1}\dot{\tilde{W}}\tilde{W}_0^{-1}\delta - \tilde{W}_0^{-1}\nabla\tilde{W}_0 = \tilde{W}_0^{-1}\dot{\tilde{W}}\tilde{W}_0^{-1}\delta(-\mathrm{Id}) - \tilde{W}_0^{-1}\nabla\tilde{W}_0(-\mathrm{Id}) + \tilde{W}_0^{-1}\delta\tilde{B}_0,$$

which implies, using $\tilde{W}_0 = \dot{W} = \delta_X$,

$$\tilde{B}_0 = 2\tilde{W}_0^{-1}(\dot{\tilde{W}} - \nabla_X\tilde{W}_0). \tag{144}$$

Step 6: the t–Derivative of R_t. Equation (141), with $W_t = t\tilde{W}_t$ and $B_t = (t\tilde{B}_t - \mathrm{Id})$, becomes

$$R_t = -t(\tilde{W}_{-t}(t\tilde{B}_t - \mathrm{Id}) + \tilde{W}_t)(t\tilde{B}_t - 2\,\mathrm{Id})^{-1}. \tag{145}$$

Therefore, the second t–derivative $\ddot{R}$ of R_t at $t = 0$ is

$$\ddot{R} = 2\dot{\tilde{W}} + \tilde{W}_0\tilde{B}_0. \tag{146}$$

Using equation (144) and equation (139) (and $\dot{\tilde{W}} = W_2$), we obtain

$$\begin{aligned}(\ddot{R})_- &= 4(\dot{\tilde{W}})_- - 2(\nabla_X\tilde{W}_0)_- \\ &= 4(\check{Q}_X\delta_X - \delta_X Q_X^L) - 2((\check{A}_X + \check{Q}_X)\delta_X - \delta_X(A_X^L + Q_X^L)) \\ &= 2(\check{Q}_X\delta_X - \check{A}_X\delta_X - \delta_X Q_X^L + \delta_X A_X^L).\end{aligned}$$

Equations (40) and (41) imply that

$$\check{A}_X\delta_X + \delta_X Q_X^L = -\tfrac{1}{2}(*\delta_X H\delta_X + \delta_X H * \delta_X) = 0.$$

Hence

$$(\ddot{R})_- = 2(\check{Q}_X\delta_X + \delta_X A_X^L). \tag{147}$$

Step 7: Link to the Davey–Stewartson Flow. To make the link to the Davey–Stewartson flow, we need to describe the normal deformation of the Davey–Stewartson flow in the quaternionic setting. The normal deformation of the Davey–Stewartson flow is obtained by taking the 90° rotation of the trace free second fundamental form evaluated on the flow vector field, i.e. (using formula (79) for the trace free second fundamental form)

$$\pi Y_{|L} = J(\check{Q}_Z\delta_Z - \delta_Z A_Z^L)$$

where Z is the flow vector field. For $Z = X + JX$ we obtain

$$\pi Y_{|L} = J(\check{Q}_Z\delta_Z - \delta_Z A_Z^L) = (\ddot{R})_-$$

which proves the statement.

APPENDIX A

The Lightcone Model

This appendix is devoted to another model of the conformal 4–sphere, the classical lightcone model, which dates back to Darboux. This model is technically more elementary than the quaternionic model. Its use substantially simplifies the treatment of flow in Chapter V. Furthermore, spherical and hyperbolic geometry can be easily integrated in this model, which is interesting for the treatment of constant mean curvature surfaces. In Section 13, a very natural link between the quaternionic model and the lightcone model is established.

1. Surfaces in the Lightcone Model of the Conformal n–Sphere

The lightcone model for the conformal n–sphere is $\mathbb{P}\mathscr{L}$, the projectivized lightcone $\mathscr{L} \subset \mathcal{V}$ in a $n+2$–dimensional Minkowski space $\mathcal{V}$, which we always assume to be oriented and time oriented.

Every map $f\colon M \to \mathbb{P}\mathscr{L}$ from a manifold M into the projectivized lightcone corresponds uniquely to a line subbundle $\mathcal{L} \subset \mathscr{L}$. As in the quaternionic setting, we do not distinguish between maps into $\mathbb{P}\mathscr{L}$ and subbundles $\mathcal{L} \subset \mathscr{L}$, but mostly prefer the line subbundle notation.

Lemma 99. *A map $\mathcal{L} \subset \mathscr{L}$ of a manifold M into the projectivized lightcone $\mathbb{P}\mathscr{L}$ is an immersion if and only if for one (every) nowhere vanishing section $\Phi \in \Gamma(\mathcal{L})$, the form $g = \langle d\Phi, d\Phi\rangle$ is a Riemannian metric on M. The section $\tilde{\Phi} = \Phi\lambda$ obtained by scaling Φ by a real function λ induces the conformally equivalent metric $\tilde{g} = \langle d\tilde{\Phi}, d\tilde{\Phi}\rangle = \lambda^2 g$.*

Proof. We only have to prove that g is Riemannian if $\mathcal{L} \subset \mathscr{L}$ is immersed. Assume, g is not a Riemannian metric. Then, there must be a vector $X \in T_pM$ such that $d\Phi_{|p}(X)$ is not spacelike. Since $d\Phi_{|p}(X)$ is orthogonal to the lightlike vector $\Phi(p)$, it cannot be timelike. Hence, it must me lightlike, too. But two orthogonal lightlike vectors in a Minkowski space are parallel and $\mathcal{L} \subset \mathscr{L}$ cannot be immersed. Therefore, if $\mathcal{L} \subset \mathscr{L}$ is immersed, the induced metric g is Riemannian. □

This lemma shows in particular that the projectivized lightcone $\mathbb{P}\mathscr{L}$ itself has a conformal structure. To prove that it is conformally equivalent to S^n, we chose an orthonormal basis. This defines an isometry $\mathcal{V} \cong \mathbb{R}^{n+1,1}$, where $\mathbb{R}^{n+1,1}$ denotes $\mathbb{R}^{n+2}$ equipped with the metric $\langle v, w\rangle = -v_0w_0 + v_1w_1 + ... + v_{n+1}w_{n+1}$. This isometry gives rise to the isometric embedding

$$x \in S^n \subset \mathbb{R}^{n+1} \to (1, x) \in \mathbb{R}^{n+1,1} \cong \mathcal{V}$$

into the lightcone. It projects to the conformal diffeomorphism $x \mapsto [1, x]$ from S^n to $\mathbb{P}\mathscr{L}$. Therefore, $\mathbb{P}\mathscr{L}$ is indeed a model for the conformal n–sphere.

The identity component $O_o(V) \cong O_o(n+1,1)$ of the group of isometries of $\mathcal{V} \cong \mathbb{R}^{n+1,1}$ acts on $\mathbb{P}(\mathscr{L})$ as the group of (orientation preserving) Möbius transformations.

One nice feature of the lightcone model is the following consequence of Lemma 99.

Corollary 100. *Let $\mathcal{L} \subset \mathscr{L}$ be an immersion of M into $\mathbb{P}\mathscr{L}$. There is a 1-1–correspondence between the nowhere vanishing future pointing sections and between the metrics on M that represent the induced conformal structure.*

We restrict now our attention to immersions of a Riemann surface M into the projective lightcone $\mathbb{P}\mathscr{L}$ of a Minkowski vector space $\mathcal{V}$.

Definition. A map $\mathcal{L} \subset \mathscr{L}$ of a Riemann surface M into $\mathbb{P}\mathscr{L}$ is called a *conformal immersion* if it is an immersion and if the induced conformal structure is compatible with that already given on the Riemann surface M.

The *mean curvature sphere* congruence of a conformal immersion into $\mathbb{P}\mathscr{L}$ is defined by

$$\mathcal{W} = \mathrm{Span}\{\Phi, \mathrm{im}\, d\Phi, \Phi_{z\bar{z}}\},$$

where $\Phi \in \Gamma(\mathcal{L})$ is an arbitrary nowhere vanishing section and z is a coordinate on M. It is easy to verify, that $\mathcal{W}$ does not depend on the choice of Φ and z. The bundle $\mathcal{W}$ is a 4–dimensional Minkowski space bundle (which immediately follows from $\langle \Phi, \Phi_{z\bar{z}}\rangle = -\langle \Phi_{\bar{z}}, \Phi_z\rangle < 0$). Therefore, $\mathbb{P}(\mathcal{W} \cap \mathscr{L})$ is a 2–sphere congruence in $\mathbb{P}\mathscr{L}$. Pointwise, the 2–spheres are tangent to the immersion. The name mean curvature sphere congruence is justified by the fact that, in every space form geometry, the immersion and the 2–spheres of the congruence have the same mean curvature (see Corollary 105).

The bundle $\mathcal{W}^\perp$ is called the *Möbius normal bundle*. It carries a positive definite metric. The connection D on $\mathcal{W}^\perp$ defined by orthogonal projection of the derivative in $\mathcal{V}$ is called the *normal connection*. This connection is compatible with the metric.

1.1. A Möbius Invariant Normalization. Suppose, the Riemann surface M is equipped with a nowhere vanishing holomorphic vector field X. Then, every conformal immersion $\mathcal{L} \subset \mathscr{L}$ of M into $\mathbb{P}\mathscr{L}$ has a unique future pointing section $\Psi \in \Gamma(\mathcal{L})$ such that X has length 1 with respect to the induced metric, i.e. $\langle d\Psi(X), d\Psi(X)\rangle = 1$.

Definition. The section $\Psi \in \Gamma(\mathcal{L})$ with $\langle d\Psi(X), d\Psi(X)\rangle = 1$ is called the *normalized lift* of $\mathcal{L} \subset \mathscr{L}$ with respect to the holomorphic vector field X.

Remark. Define dz to be the 1–form dual to X. Then, the condition for Ψ being the normalized lift becomes $|d\Psi| = |dz|$.

To simplify notation, we often prefer to use the corresponding coordinate z on the universal covering $\tilde{M}$ instead of X. It should be noted that the 1–form dz as well as the derivatives $\frac{\partial}{\partial z} = \frac{1}{2}(X - iJ^M X)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(X + iJ^M X)$ with respect to z are well defined on the torus itself.

The normalization is Möbius invariant. It was first used by Burstall, Pedit and Pinkall in [**4**] to defined a new set of invariants for immersions into the lightcone model $\mathbb{P}\mathscr{L}$ of the conformal n–sphere. These invariants are essential in their treatment of Möbius invariant flows on surfaces in $\mathbb{P}\mathscr{L}$.

There is a unique section $\hat{\Psi} \in \Gamma(\mathcal{W})$ of the mean curvature sphere congruence of $\mathcal{L} \subset \mathscr{L}$ satisfying $\langle \hat{\Psi}, \hat{\Psi} \rangle = 0$, $\langle \Psi, \hat{\Psi} \rangle = -1$ and $\langle d\Psi, \hat{\Psi} \rangle = 0$. This yields the normalized frame $(\Psi, \Psi_z, \Psi_{\bar{z}}, \hat{\Psi})$ for $\mathcal{W}^{\mathbb{C}}$. With respect to this frame, the scalar product of $\mathcal{W}$ is given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Since Ψ_{zz} is orthogonal to Ψ, Ψ_z and $\Psi_{\bar{z}}$, there is a complex function c and a section $\kappa \in \Gamma((\mathcal{W}^{\perp})^{\mathbb{C}})$ such that

$$\Psi_{zz} + \tfrac{c}{2}\Psi = \kappa. \tag{148}$$

This inhomogeneous Hill's equation is the fundamental equation of the approach to Möbius geometry of surfaces developed in [**4**]. The quantities κ and c are Möbius invariants (depending on z) of the immersion called the *conformal Hopf differential* and the *Schwarzian derivative* of the immersion. The fundamental theorem below states that these invariants, together with the normal connection D on $\mathcal{W}^{\perp}$, determine the immersion up to Möbius transformation.

A short computation (see Equations (22) to (24) in [**4**]) shows that the dependence of the invariants on the choice of coordinate is as follows: if $\tilde{\kappa}$ and $\tilde{c}$ are the invariants with respect to another coordinate $\tilde{z}$, then

$$\tilde{\kappa}\frac{d\tilde{z}^2}{|d\tilde{z}|} = \kappa\frac{dz^2}{|dz|} \tag{149}$$

and

$$\tilde{c}d\tilde{z}^2 = (c - S_z(\tilde{z}))dz^2 \tag{150}$$

where $S_z(\tilde{z})$ denotes the classical Schwarzian derivative of $\tilde{z}$ with respect to z. Recall that the classical Schwarzian derivative

$$S_z(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

of a holomorphic function f uniquely determines f up to Möbius transformation of $\mathbb{CP}^1$ (see e.g. [**4**], where the $\mathbb{CP}^1$ theory is presented as a "baby model" of higher dimensional conformal surface theory).

1.2. The Fundamental Theorem. To simplify the formulation of the fundamental theorem of Möbius geometry of immersion into $\mathbb{P}\mathscr{L}$, we adopt a slightly more general point of view. From now on, instead of taking a fixed Minkowski vector space $\mathcal{W}$ as the target space, we prefer to take a flat vector bundle with some additional structure.

DEFINITION. Let M be a Riemann surface. A *Minkowski space bundle* $(\mathcal{V}, \nabla)$ on M is an oriented vector bundle $\mathcal{W}$ with a flat connection ∇ and a Lorentzian fiber metric $\langle\, , \rangle$ that is compatible with ∇ and has a fixed time orientation. By a *conformal immersion* of M into (the lightcone of) a Minkowski space bundle $(\mathcal{V}, \nabla)$ we mean a line subbundle $\mathcal{L} \subset \mathscr{L}$ of the lightcone bundle $\mathscr{L} \subset \mathcal{V}$ such that every nowhere vanishing section $\Phi \in \Gamma(\mathcal{L})$ induces a compatible metric $g = \langle \nabla\Phi, \nabla\Phi \rangle$ on M.

This notion is mainly introduced to simplify the statement of the fundamental theorem. Furthermore, it emphasizes the similarities to the treatment of quaternionic projective structures.

Of course, in the new terminology, a conformal immersion $\mathcal{L} \subset \mathscr{L}_{\mathbb{R}^{n+1,1}}$ of M into the projectivization $\mathbb{P}\mathscr{L}_{\mathbb{R}^{n+1,1}}$ of the lightcone $\mathscr{L}_{\mathbb{R}^{n+1,1}} \subset \mathbb{R}^{n+1,1}$ becomes a conformal immersion into the (lightcone of the) trivial Minkowski space bundle $\mathcal{V} = \mathbb{R}^{n+1,1}$ with the connection d.

The following analogue to Lemma 76 and 80 can be proven by essentially the same method (which is based on Lemma 111).

Lemma 101. *There is a 1–1-correspondence between equivalence classes of conformal immersions $\mathcal{L} \subset \mathscr{L}$ into Minkowski space bundles $(\mathcal{V}, \nabla)$ and Möbius equivalence classes of conformal immersions with Möbius monodromy of the universal covering $\tilde{M}$ into $\mathbb{P}\mathscr{L}_{\mathbb{R}^{n+1,1}}$.*

The notion of conformal immersion with Möbius monodromy used here is the obvious one (see the definition preceding Lemma 80 for the quaternionic analogue of this notion). Furthermore, two conformal immersions into Minkowski space bundles are called *equivalent*, if there is a gauge equivalence respecting the metrics and the line subbundles.

An advantage compared to the quaternionic case is that really all conformal immersions of $\tilde{M}$ into $\mathbb{P}\mathscr{L}_{\mathbb{R}^{n+1,1}}$ with Möbius monodromy can be represented this way. A lifting problem as in Lemma 80 does not occur, because the connected component of $O_o(n+1,1)$ really is the Möbius group, while $\mathrm{SL}(2,\mathbb{H})$ is its universal covering.

Let $\mathcal{L} \subset \mathscr{L}$ be a conformal immersion of a Riemann surface into a Minkowski space bundle $(\mathcal{V}, \nabla)$. Let X be a nowhere vanishing holomorphic vector field on M and denote by z the corresponding conformal coordinate on the universal covering $\tilde{M}$. Let $\psi \in \Gamma(\mathcal{L})$ be the normalized lift of $\mathcal{L} \subset \mathscr{L}$ with respect to X. The construction of the mean curvature sphere congruence $\mathcal{W}$ and of the invariants κ and c carries over verbatim to the more general setting, when we use the convention that all derivatives are taken using the connection ∇. Let $(\Psi, \Psi_z, \Psi_{\bar{z}}, \hat{\Psi})$ be the frame of $\mathcal{W}^{\mathbb{C}}$ introduced above and let $n \in \Gamma(\mathcal{W}^{\perp})$ be a section of the Möbius normal bundle. A straightforward calculation (which is carried out in [**4**, Section 2.3]) shows that the frame equations are

$$\begin{aligned} \Psi_{zz} &= -\tfrac{c}{2}\Psi + \kappa \\ \Psi_{z\bar{z}} &= -|\kappa|^2\Psi + \tfrac{1}{2}\hat{\Psi} \\ \hat{\Psi}_z &= -2|\kappa|^2\Psi_z - c\Psi_{\bar{z}} + 2D_{\bar{z}}\kappa \\ n_z &= 2\langle D_{\bar{z}}\kappa, n\rangle\Psi - 2\langle \kappa, n\rangle\Psi_{\bar{z}} + D_z n, \end{aligned} \tag{151}$$

and that the integrability equations (which express the flatness of ∇, i.e. the commutativity of the second derivatives, with respect to the frame) are

$$\tfrac{1}{2}c_{\bar{z}} = 2|\kappa|_z^2 + \langle D_z\bar{\kappa}, \kappa\rangle - \langle \bar{\kappa}, D_z\kappa\rangle \tag{152}$$

$$\mathrm{Im}(D_{\bar{z}}D_{\bar{z}}\kappa + \tfrac{\bar{c}}{2}\kappa) = 0 \tag{153}$$

$$R^D_{z\bar{z}}n = 2\langle\bar{\kappa}, n\rangle\kappa - 2\langle\kappa, n\rangle\bar{\kappa}. \tag{154}$$

The three integrability equations are called the *conformal Gauss, Codazzi* and *Ricci equation.*

REMARK. In dimension 4, the normal bundle is 2–dimensional and has a complex structure J compatible with the metric and orientation. It is easy to verify that for all vectors n, w_1 and $w_2 \in W^{\perp\mathbb{C}}$, we have

$$\langle w_1, n\rangle w_2 - \langle w_2, n\rangle w_1 = \langle Jw_1, w_2\rangle Jn.$$

Therefore, in the case of immersions into the conformal 4–space, the Ricci equation (154) takes the simpler form

$$R^D_{z\bar{z}} = 2\langle J\bar{\kappa}, \kappa\rangle J. \tag{154'}$$

Conversely, one can start with the data $(\mathcal{W}^\perp, D)$, c and $\kappa \in \Gamma((\mathcal{W}^\perp)^\mathbb{C})$ satisfying these integrability equations, and therewith, using the frame equations, define a flat connection, which is proven by the same calculation as needed for the prove of the integrability equations. We obtain:

THEOREM 15 (Fundamental Theorem, Lightcone Version). *Let M be a Riemann surface equipped with a nowhere vanishing conformal vector field X. There is a 1–1–correspondence between equivalence classes of conformal immersions $\mathcal{L} \subset \mathscr{L}$ of M into a Minkowski space bundle $(\mathcal{V}, \nabla)$ and equivalence classes of data $(\mathcal{W}^\perp, D)$, c and $\kappa \in \Gamma((\mathcal{W}^\perp)^\mathbb{C})$ (where by equivalence of the data we mean gauge equivalence of $(\mathcal{W}^\perp, D)$ compatible with κ) satisfying the conformal Gauss, Codazzi and Ricci equations.*

2. Metrical Subgeometries of Möbius Geometry

This section describes, how the space form geometries can be realized as conic sections in the lightcone model, thus providing a very efficient way to consider spherical, Euclidean and hyperbolic geometry as subgeometries of Möbius geometry. In particular, this yields simple formulae expressing the Möbius invariants of a surface in terms of the metrical invariants. It seems to be considerably more difficult to derive corresponding formulae in the quaternionic setting.

2.1. Space forms as Metrical Subgeometries. Let $\mathcal{V}$ be a Minkowski space of dimension $n+2$. For $v_0 \in \mathcal{V}\backslash\{0\}$, define the conic section

$$\mathcal{S}_{v_0} = \{x \in \mathscr{L} \mid \langle x, v_0\rangle = -1\}.$$

$\mathcal{S}_{v_0}$ with the induced metric is a space form of curvature $G = -\langle v_0, v_0\rangle$. In the case $G \neq 0$, the space form $\mathcal{S}_{v_0}$ is the sphere or hyperboloid

$$\mathcal{S}_{v_0} = \{x \in \mathcal{V} \mid \langle x, v_0\rangle = -1 \text{ and } \langle x - p, x - p\rangle = 1/G\}$$

where we take $p = -\frac{v_0}{\langle v_0, v_0\rangle}$ as the origin in the affine space $\{x \mid \langle x, v_0\rangle = -1\}$.

An m–sphere in $\mathbb{P}\mathscr{L}$ is given by $\mathbb{P}(\mathcal{U}\cap\mathscr{L})$, where $\mathcal{U}$ is a $m+2$–dimensional subspace $\mathcal{U}\subset\mathcal{V}$ of index one. Equivalently, it can be given via polarity by the $n-m$–dimensional Euclidean subspace $\mathcal{U}^\perp$. The part of the m–sphere lying in $\mathcal{S}_{v_0}$ is the submanifold $\mathcal{S}_{v_0}\cap\mathcal{U} = \{x\in\mathscr{L}\cap\mathcal{U} \mid \langle x, v_0\rangle = -1\}$.

Lemma 102. *The mean curvature vector of the m–sphere $\mathcal{S}_{v_0}\cap\mathcal{U}$ in $\mathcal{S}_{v_0}$ at the point $x_0\in\mathcal{S}_{v_0}\cap\mathcal{U}$ is*

$$H_{x_0} = -v_0^\perp - \langle v_0^\perp, v_0^\perp\rangle x_0,$$

where $y = y^\top + y^\perp$ denotes the decomposition of a vector with respect to $\mathcal{U}\oplus\mathcal{U}^\perp$.

Proof. The tangent space of $\mathcal{S}_{v_0}$ at x_0 is $T_{x_0}\mathcal{S}_{v_0} = \mathrm{Span}\{x_0, v_0\}^\perp$. The Levi–Civita connection of $\mathcal{S}_{v_0}$ is

$$\nabla^{LC}_X Y = d_X Y + \langle d_X Y, x_0\rangle\tilde{v}_0 + \langle d_X Y, \tilde{v}_0\rangle x_0 \tag{155}$$

where $X, Y\in\Gamma(T\mathcal{S}_{v_0})$ and $\tilde{v}_0 = v_0 + \frac{\langle v_0, v_0\rangle}{2}x_0$. The tangent and normal bundles of the manifold $M = \mathcal{S}_{v_0}\cap\mathcal{U}$ are

$$T_{x_0}M = T_{x_0}\mathcal{S}_{v_0}\cap\mathcal{U}$$

$$N_{x_0}M = T_{x_0}\mathcal{S}_{v_0}\cap\mathcal{U}^\perp.$$

We take an orthonormal basis $V_1, ..., V_m$ of $T_{x_0}M$ and extend it to tangential vector fields on M by

$$\tilde{V}_i(x) = V_i + \langle V_i, x\rangle w(x) + \langle V_i, w(x)\rangle x$$

where $w(x) = v_0^\top + \frac{\langle v_0^\top, v_0^\top\rangle}{2}x$. Then

$$d_{\tilde{V}_i}\tilde{V}_{i|x_0} = v_0^\top + \langle v_0^\top, v_0^\top\rangle x_0$$

and using (155) we have

$$\nabla^{LC}_{\tilde{V}_i}\tilde{V}_{i|x_0} = -v_0^\perp - \langle v_0^\perp, v_0^\perp\rangle x_0$$

and $H_{x_0} = \frac{1}{m}\sum_i(\nabla^{LC}_{\tilde{V}_i}\tilde{V}_{i|x_0})^\perp$ satisfies the given formula. ☐

2.2. Metrical Invariants of a Conformal Immersion. Let $\mathcal{S}_{v_0}$ be a space form. Every conformal immersion $\mathcal{L}\subset\mathscr{L}$ into $\mathbb{P}(\mathscr{L})$ admits a unique *metrical lift* $\Phi\in\Gamma(\mathcal{L})$ with $\langle\Phi, v_0\rangle = -1$ defined away from the points where $\mathcal{L}$ intersects the infinity boundary $\mathbb{P}(v_0^\perp\cap\mathscr{L})$ of the space form. The section Φ can be seen as a map $\Phi\colon M\to\mathcal{S}_{v_0}$ into the space form.

The *tangent plane congruence* is $\mathcal{T} = \mathrm{Span}\{\Phi, d\Phi, v_0\}$. The mean curvature of $\mathcal{T}$ in $\mathcal{S}_{v_0}$ is indeed 0, which follows from Lemma 102 since $v_0\in\mathcal{T}$. The orthogonal complement $\mathcal{T}^\perp$ of $\mathcal{T}$ is called the *metrical normal bundle*.

The metrical invariants of a surface in a space form are the induced *metric* or (first fundamental form)

$$g = \langle d\Phi, d\Phi\rangle,$$

the *second fundamental form*

$$\mathrm{I\!I}(X, Y) = (\nabla^{LC}_X Y)^\perp$$

(where X, Y denote vector fields on M) and the *metrical normal connection*

$$\nabla^{\perp}_X n = (\nabla^{LC}_X n)^{\perp}$$

(where X is a vector field on M and $n \in \Gamma(\mathcal{T}^{\perp})$ a section of the metrical normal bundle). The second fundamental form has the decomposition

$$\mathrm{I\!I} = Hg + \mathrm{I\!I}^0 \tag{156}$$

into its trace free part $\mathrm{I\!I}^0$ and the part given by the *mean curvature vector* H.

As bundles with metrical connections, the metrical normal bundle $\mathcal{T}^{\perp}$ and the Möbius normal bundle $\mathcal{W}^{\perp}$ are gauge equivalent, as stated by the following lemma, whose proof is a straight forward computation using equation (155).

LEMMA 103. *Let $\Phi\colon M \to \mathcal{S}_{v_0}$ be the metrical lift of a conformal immersion into a space form $\mathcal{S}_{v_0}$. Then the map*

$$n \in \mathcal{T}^{\perp} \mapsto \langle H, n\rangle \Phi + n \in \mathcal{W}^{\perp}$$

is an isometry between the metrical normal bundle $\mathcal{T}^{\perp}$ and the Möbius normal bundle $\mathcal{W}^{\perp}$. It is compatible with the connections $\nabla^{\perp}$ and D on the bundles $\mathcal{T}^{\perp}$ and $\mathcal{W}^{\perp}$.

With respect to a conformal chart z, the invariants are usually represented by the real function u defined by

$$g = e^{2u}|dz|^2, \tag{157}$$

the real section H of the normal bundle $\mathcal{T}^{\perp}$ (which is independent of z), the complex section of $(\mathcal{T}^{\perp})^{\mathbb{C}}$

$$q = \mathrm{I\!I}(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}) \tag{158}$$

of the normal bundle and the normal connection $\nabla^{\perp}$.

The form $(\mathrm{I\!I}^0)^{(2,0)} = q\, dz^2$, i.e. the $(2,0)$–part of the trace free second fundamental form $\mathrm{I\!I}^0$, is usually called the *metrical Hopf differential* of the immersion. With this definition, equation 156 becomes

$$\mathrm{I\!I} = Hg + q\, dz^2 + \bar{q}\, d\bar{z}^2. \tag{156'}$$

Choosing a coordinate z does also fix a frame $(\Phi, \Phi_z, \Phi_{\bar{z}}, v_0)$ of the bundle $\mathcal{T}^{\mathbb{C}}$. Its frame and integrability equations are given in the following lemma.

LEMMA 104. *Let $\Phi\colon M \to \mathcal{S}_{v_0}$ be the metrical lift of a conformal immersion into a space form $\mathcal{S}_{v_0}$ of curvature $G = -\langle v_0, v_0\rangle$ and denote by z a conformal chart. Then*

$$\Phi_{z\bar{z}} = \tfrac{1}{2} e^{2u}(H + v_0 - G\Phi)$$
$$\Phi_{zz} = \mathrm{I\!I}(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}) + 2u_z \Phi_z$$

where u defines the metric $g = e^{2u}|dz|^2$ induced by the immersion, H denotes its mean curvature vector, and $\mathrm{I\!I}$ its second fundamental. For a section $n \in \Gamma(\mathcal{T}^{\perp})$ of the metrical normal bundle we have

$$n_z = \nabla^{\perp}_z n - \langle H, n\rangle \Phi_z - 2e^{-2u}\langle \mathrm{I\!I}(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}), n\rangle \Phi_{\bar{z}}.$$

The metrical Gauss, Codazzi and Ricci equations are

$$u_{z\bar z} - |q|^2 e^{-2u} + \tfrac14 e^{2u}(|H|^2 + G) = 0$$
$$\nabla^\perp_{\bar z} q = \tfrac12 e^{2u}\nabla^\perp_z H$$
$$R^\perp_{z\bar z} n = 2e^{-2u}(\langle \bar q, n\rangle q - \langle q, n\rangle \bar q)$$

where $\mathbb{I} = Hg + q\,dz^2 + \bar q\,d\bar z^2$.

Proof. The formula (155) for the Levi–Civita connection of $\mathcal{S}_{v_0}$ shows

$$\tfrac12 e^{2u} H = \Phi_{z\bar z} + \langle \Phi_{z\bar z}, \Phi\rangle \tilde v_0 + \langle \Phi_{z\bar z}, \tilde v_0\rangle \Phi.$$

(Note that one does not have to project on the normal bundle of the immersion in $T\mathcal{S}_{v_0}$, because, by conformality, $\Phi_{z\bar z}$ is orthogonal to Φ_z and $\Phi_{\bar z}$.) Using

$$\langle \Phi_{z\bar z}, \Phi\rangle = -\langle \Phi_z, \Phi_{\bar z}\rangle = -\tfrac12 e^{2u}$$

and

$$\langle \Phi_{z\bar z}, \tilde v_0\rangle = \langle \Phi_{z\bar z}, v_0 + \tfrac{\langle v_0, v_0\rangle}{2}\Phi\rangle = \tfrac{\langle v_0, v_0\rangle}{2}\langle \Phi_{z\bar z}, \Phi\rangle = -\tfrac{\langle v_0, v_0\rangle}{2}\tfrac12 e^{2u},$$

we obtain the first statement.

Since Φ_{zz} is orthogonal to Φ and v_0 we have $\nabla^{LC}_{\frac{\partial}{\partial z}}\Phi_z = \Phi_{zz}$ and

$$\mathbb{I}(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}) = \Phi_{zz} - 2e^{-2u}(\langle \Phi_{zz}, \Phi_z\rangle \Phi_{\bar z} + \langle \Phi_{zz}, \Phi_{\bar z}\rangle \Phi_z).$$

Now

$$\langle \Phi_{zz}, \Phi_z\rangle = 0$$

and

$$\langle \Phi_{zz}, \Phi_{\bar z}\rangle = \langle \Phi_z, \Phi_{\bar z}\rangle_z = e^{2u} u_z$$

yields the second statement. The third formula follows by developing the tangential part of n_z with respect to the frame $(\Phi, \Phi_z, \Phi_{\bar z}, v_0)$.

The integrability equations are equivalent to $\Phi_{zz\bar z} = \Phi_{z\bar z z}$ and $n_{z\bar z} = n_{\bar z z}$. More precisely, the Gauss equation is obtained by comparing the coefficients of Φ_z in the first equation, the Codazzi equation is the normal part of the first equation. The Ricci equation is the normal part of $n_{z\bar z} = n_{\bar z z}$. □

Because the integrability equations correspond to flatness of the connection on $\mathcal{V}$, it is clear that a set of data satisfying the integrability equations for $G \in \mathbb{R}$ defines a flat connection and therefore an immersion into a space form of curvature G. This yields the following metrical version of the fundamental theorem.

Theorem 16 (Fundamental Theorem, Metrical Version). *Locally, with respect to a chart z, there is a 1–1-correspondence between equivalence classes of conformal immersions into a space form and between equivalence classes of sets of data $(\mathcal{T}^\perp, \nabla^\perp)$, $H \in \Gamma(\mathcal{T}^\perp)$, $q \in \Gamma((\mathcal{T}^\perp)^{\mathbb{C}})$ and u satisfying the Gauss, Codazzi and Ricci equations, where two immersions are considered to be equivalent, if one is obtained from the other by an isometry of the space form, and where two sets of data are equivalent, if one is obtained from the other by applying a gauge transformation compatible with H and q.*

COROLLARY 105. *The mean curvature sphere* $\mathcal{W} = \mathrm{Span}\{\Phi, \Phi_z, \Phi_{\bar z}, \Phi_{z\bar z}\}$ *has the same mean curvature vector as* $\Phi\colon M \to \mathcal{S}_{v_0}$.

PROOF. A short computation shows that $v_0^\perp = v_0 - a\Phi - b\Phi_{z\bar z}$ with $a = |H|^2 + \langle v_0, v_0\rangle - 2$ and $b = 2e^{-2u}$. Using Lemma 102, one can easily see that the mean curvature of $\mathcal{W}$ coincides with that of Φ. □

While the relation between the metrical and Möbius normal bundles is clear by Lemma 103, the relation between the other invariants is given by the following lemma.

LEMMA 106. *Let* $\Phi\colon M \to \mathcal{S}_{v_0}$ *be a conformal immersion into a space form and* z *a conformal chart. Denote by* u, H *and* $\mathrm{I\!I}$ *the metrical data and by* κ *and* c *the conformal data of* Φ *with respect to* z. *Then*

$$\tfrac{1}{2}c = u_{zz} - u_z^2 + \langle H, \mathrm{I\!I}(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z})\rangle$$

and

$$\kappa \tfrac{dz^2}{|dz|} = \widehat{\tfrac{\mathrm{I\!I}^{2,0}}{|d\Phi|}}$$

(where $\widehat{\ }$ *denotes the isomorphism of Lemma 103).*

With q as defined in (158), the preceding formulae take the simpler form

$$\tfrac{1}{2}c = u_{zz} - u_z^2 + \langle H, q\rangle \qquad \text{and} \qquad \kappa = \hat{q}e^{-u}.$$

PROOF. Using $\Psi = e^{-u}\Phi$ and Lemma 104 we obtain

$$\begin{aligned}\Psi_{zz} &= e^{-u}\Phi_{zz} - 2e^{-u}u_z\Phi_z - e^{-u}(u_{zz} - u_z^2)\Phi\\ &= e^{-u}\,\mathrm{I\!I}(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}) - e^{-u}(u_{zz} - u_z^2)\Phi\\ &= e^{-u}(\mathrm{I\!I}(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}) + \langle \mathrm{I\!I}(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}), H\rangle\Phi) - (\langle \mathrm{I\!I}(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}), H\rangle + u_{zz} - u_z^2)\Psi,\end{aligned}$$

where the first term is the image of $e^{-u}\,\mathrm{I\!I}(\frac{\partial}{\partial z}, \frac{\partial}{\partial z})$ under the isomorphism of Lemma 103 between metrical and conformal normal bundle. □

REMARK. A simple calculation shows that the conformal Gauss equation is obtained by taking a $\bar z$–derivative of the metrical Gauss equation, while the conformal Codazzi and Ricci equations are equivalent to their metrical counterparts.

3. Special Surface Classes in the Lightcone Model

This section gives the characterization of special surface classes in the lightcone setting. The results are needed in Section 15 on flows. The first subsection treats isothermic and constrained Willmore surfaces, the second subsection immersions obtained from curves and the third subsection constant mean curvature surfaces.

3.1. Isothermic and Constrained Willmore Surfaces. A brief characterization of isothermic and constrained Willmore surfaces in the lightcone setting is given in this subsection. For a more detailed discussion see [**4**, Section 3.4], for a treatment in the quaternionic setting, see Chapter III.

Isothermic Surfaces. Let $\mathcal{L} \subset \mathcal{V}$ be a conformal immersion of a Riemann surface M into a Minkowski space bundle $(\mathcal{V}, \nabla)$ and let X be a nowhere vanishing holomorphic vector field with corresponding conformal coordinate z on the universal covering M.

LEMMA 107. *The coordinate z is a conformal curvature line coordinate if and only if the conformal Hopf differential κ with respect to z is a real section $\kappa \in \Gamma(\mathcal{W}^{\perp})$ of the Möbius normal bundle.*

PROOF. By Lemma 106 we have to prove that, with respect to an arbitrary space form geometry, z is a conformal curvature line coordinate if and only if the coefficient q of the metrical Hopf differential is real. This follows directly from $\mathbb{I} = Hg + q\,dz^2 + \bar{q}\,d\bar{z}^2$, because for $z = x + \mathbf{i}y$ we obtain

$$\mathbb{I}(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}) = q\mathbf{i} + \overline{q\mathbf{i}} = (q - \bar{q})\mathbf{i}.$$

Thus, the condition for x, y being curvature line coordinates, i.e. $\mathbb{I}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = 0$, is indeed that q is real. $\square$

We obtain the following characterization of isothermic immersions of tori, where by isothermic we mean isothermic in the global sense as defined in Section 9 (see in particular Lemma 60).

LEMMA 108. *An immersion of a torus is isothermic if and only if there is a global conformal vector field such that κ, with respect to this vector field, is real.*

One can immediately seen from the Gauss–Codazzi–Ricci equations that, for every solution κ, c and D with real κ, one obtains a whole 1–parameter family of solution κ, $c + r$ and D with $r \in \mathbb{R}$. This is the associated family of isothermic surfaces.

Constrained Willmore Surfaces. It is proven in [**4**] (see equations (34) and (54) there) that a conformal immersion of a torus is constrained Willmore if and only if

$$D_{\bar{z}} D_{\bar{z}} \kappa + \tfrac{1}{2}\bar{c}\kappa = \mathrm{Re}(\lambda\kappa), \tag{159}$$

where $\lambda \in \mathbb{C}$ and where z is a conformal coordinate on the universal cover. (For other Riemann surfaces, λ has to be replaced with a holomorphic function.)

3.2. Surfaces from Curves. Every curve parametrized with respect to arc length in a 3–dimensional space form gives rise to a conformal immersion into $\mathbb{R}^4$ and therefore into the lightcone. The immersion is obtained by applying a cylinder, cone or rotational surface construction (as explained below in (162), (163) and (164)) to a curve in Euclidean space, the (metrical) sphere or hyperbolic space. The main goal of this subsection is to prove the following lemma relating the metrical invariants of the curve to the conformal invariants of the immersion.

LEMMA 109. *Let γ be a regular curve in a space form of curvature G. Let κ_1, κ_2 be its curvature with respect to a parallel orthonormal frame N_1, N_2 of the normal bundle. Then, with respect to the coordinate $z = x + iy$ where x is an* *arc length parameter of γ, the conformal immersion obtained via the cylinder, cone or rotational surface construction*[1] *has the conformal Hopf differential*

$$\kappa = \tfrac{1}{4}(\kappa_1 n_1 + \kappa_2 n_2)$$

(where n_1, n_2 is a parallel frame of the Möbius normal bundle canonically obtained from N_1, N_2) and the Schwarzian derivative

$$c = \tfrac{1}{4}(\kappa_1^2 + \kappa_2^2) + \tfrac{G}{2}.$$

Before we prove the lemma, we draw some simple consequences. Using Theorem 15 (and the fundamental theorem for curves in space forms), we immediately obtain the following characterization of immersions obtained from curves:

Corollary 110. *A conformal immersions can be obtained from a curve by the above constructions if and only if it is isothermic and admits a conformal curvature line coordinate $z = x + iy$ such that the conformal Hopf differential κ with respect to z is parallel in y–direction and the Schwarzian derivative satisfies $c = 4|\kappa|^2 + \frac{G}{2}$ for $G \in \mathbb{R}$.*

Therefore, the isothermic surface associated family applied to a conformal immersion that is obtained from a curve in a space form of curvature G consists of a family of conformal immersions that are again obtained from curves. For spectral parameter r (cf. Section 3.1) the corresponding curve is obtained by integrating the same curvatures κ_1, κ_2 with respect to a space form of curvature $G + \frac{r}{2}$.

A conformal immersion obtained from a curve is constrained Willmore if it satisfies (159) for a constant[2] λ, which is equivalent to

$$\boldsymbol{\psi}'' + (\tfrac{1}{2}|\boldsymbol{\psi}|^2 + G + \mu)\boldsymbol{\psi} = 0 \qquad (160)$$

for $\mu \in \mathbb{R}$, where $()'$ denotes the derivative with respect to the arc length parameter x of the curve and where $\boldsymbol{\psi} = \kappa_1 + \kappa_2\mathbf{i}$ denotes the complex curvature of the curve with respect to its parallel frame.

Definition. A curve in a space form of curvature G whose complex curvature satisfies (160) for $\mu \in \mathbb{R}$ is called *elastic*. It is called *free elastic* if $\mu = 0$.

Because for conformal immersions obtained from curves, equation (159) is equivalent to (160), the conformal immersion is constrained Willmore if and only if the curve is elastic. Furthermore, the conformal immersion if Willmore if and only if the curve is free elastic (because both cases correspond to the case of vanishing of the respective constant).

Remark. This correspondence is not surprising, because free elastic and elastic curves admit a variational characterization analogous to that of Willmore and constrained Willmore immersions: free elastic curves are the critical points of the functional $\int |\boldsymbol{\psi}|^2$ under all variations preserving arc length, and elastic

[1] In order to keep the statement of the lemma short, these constructions are explained only in the proof, see equations (162),(163) and (164).

[2] For conformal immersions obtained from curves, the holomorphic function λ in (159) is automatically constant, because its real part does only depend on x.

curves are the critical point of the same variational problem under the constraint that the end points are fixed.

The Proof. The rest of this subsection is devoted to the proof of Lemma 109. Unfortunately, we do not know any better than proving the lemma by a case by case study, i.e. all three types of space forms are treated separately. The idea is the same in all three cases: starting from a curve, we define a surface in $\mathbb{R}^4$ and calculate the mean curvature vector H and as well as u and q representing the metric and Hopf differential with respect to a coordinate z (cf. equations (156), (157) and (158)).

The conformal invariants are obtained by the correspondence between the metrical and conformal invariants of Section 2 of this appendix, after embedding $\mathbb{R}^4$ as a conic section $\mathcal{S}_{v_0}$ into the lightcone model. For this, one can use for example the isometric embedding

$$F\colon \mathbb{R}^4 \to \mathbb{R}^{6,1} \quad x \in \mathbb{R}^4 \mapsto (1 + \tfrac{x\cdot x}{4}, x, 1 - \tfrac{x\cdot x}{4}), \tag{161}$$

which maps $\mathbb{R}^4$ into the space form $\mathcal{S}_{v_0}$ with $v_0 = \frac{1}{2}(1, 0, ..., 0, -1)$. Its differential is given by $dF_{|x}v = (\frac{x\cdot v}{2}, v, \frac{x\cdot v}{2})$.

To simplify the proof, we only consider curves in the space forms with curvature $G \in \{0, \pm 1\}$.

Cylinders over Curves in Euclidean Space. Denote by $\gamma\colon I \to \mathbb{R}^3$ a curve parametrized with respect to arc length. The *cylinder* over γ is the immersion

$$f(x, y) = (\gamma(x), y). \tag{162}$$

Since f is an isometric immersion, the induced metric with respect to $z = x+iy$ is given by $u = 0$. The mean curvature vector of f is

$$H = \tfrac{1}{2}(\tfrac{\partial^2 f}{\partial x^2} + \tfrac{\partial^2 f}{\partial y^2})^\perp = \tfrac{1}{2}(\kappa_1 N_1 + \kappa_2 N_2),$$

where N_1, N_2 is a parallel orthonormal frame of the normal bundle with curvature κ_1, κ_2, i.e. $\gamma'' = \kappa_1 N_1 + \kappa_2 N_2$. The Hopf differential is given by

$$q = \mathrm{I\!I}(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}) = \tfrac{1}{4}(\tfrac{\partial^2 f}{\partial x^2} - \tfrac{\partial^2 f}{\partial y^2})^\perp - \tfrac{i}{2}\tfrac{\partial^2 f}{\partial x \partial y}^\perp = \tfrac{1}{4}(\kappa_1 N_1 + \kappa_2 N_2).$$

The normal connection $\nabla^\perp$ of the cylinder is determined by the fact that N_1, N_2 is a parallel orthonormal frame.

Denoting by the same letters the metrical data of the image of f under the embedding $F\colon \mathbb{R}^4 \to \mathbb{R}^{6,1}$ and by $n_i = \widehat{dF(N_i)}$ the parallel frame of the Möbius normal bundle $\mathcal{W}^\perp$ (where $\hat{}$ is the isomorphism of Lemma 103), Lemma 106 yields

$$c = \tfrac{1}{4}(\kappa_1^2 + \kappa_2^2)$$
$$\kappa = \tfrac{1}{4}(\kappa_1 n_1 + \kappa_2 n_2).$$

Cones over Curves in the 3–Sphere. Let $\gamma\colon I \to S^3 \subset \mathbb{R}^4$ be a curve parametrized with respect to arc length. The *cone* over γ is the immersion

$$f(x, y) = e^y \gamma(x). \tag{163}$$

(In contrast to cylinders and rotational surfaces, the cone construction yields surfaces that are not metrically equivariant, since inner translation in y–direction corresponds to homotheties of $\mathbb{R}^4$!)

If N_1, N_2 is a parallel orthonormal frame of the normal bundle, (γ, T, N_1, N_2) is a SO(4)–frame satisfying the frame equation

$$(\gamma, T, N_1, N_2)' = (\gamma, T, N_1, N_2) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\kappa_1 & -\kappa_2 \\ 0 & \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 & 0 \end{pmatrix},$$

where κ_1, κ_2 is the curvature of the curve with respect to the parallel orthonormal frame N_1, N_2 of the normal bundle.

The metric induced by f is $g = e^{2y}|dz|^2$, thus $u(x, y) = y$, its mean curvature vector is

$$H = \tfrac{1}{2e^{2y}}(\tfrac{\partial^2 f}{\partial x^2} + \tfrac{\partial^2 f}{\partial y^2})^\perp = \tfrac{1}{2e^y}(\kappa_1 N_1 + \kappa_2 N_2)$$

and the Hopf differential is given by

$$q = \mathrm{I\!I}(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}) = \tfrac{1}{4}(\tfrac{\partial^2 f}{\partial x^2} - \tfrac{\partial^2 f}{\partial y^2})^\perp - \tfrac{i}{2}\tfrac{\partial^2 f}{\partial x \partial y}^\perp = \tfrac{1}{4}e^y(\kappa_1 N_1 + \kappa_2 N_2).$$

As in the case of cylinders, after embedding $\mathbb{R}^4$ into the lightcone, Lemma 106 implies

$$\kappa = \tfrac{1}{4}(\kappa_1 n_1 + \kappa_2 n_2)$$
$$c = \tfrac{1}{4}(\kappa_1^2 + \kappa_2^2) + \tfrac{1}{2}.$$

Rotational Surfaces from Curves in Hyperbolic Space. We use the half space $\{(x_1, x_2, r) \in \mathbb{R}^3 \mid r > 0\}$ with the hyperbolic metric $g = \frac{1}{r^2}\langle\, , \rangle_{\mathbb{R}^3}$ as a model for hyperbolic space. The Levi–Civita connection of g is

$$\nabla_X Y = d_X Y - \tfrac{1}{r}(X_3 Y + Y_3 X) + \tfrac{1}{r}\langle X, Y\rangle_{\mathbb{R}^3}\tfrac{\partial}{\partial r}$$

for X and Y vector fields on the half space and X_3 and Y_3 their r–components.

Denote by γ a curve in the half space parametrized with respect to arc length. Let $T = \gamma'$ and take N_1, N_2 a ∇–parallel orthonormal frame of the normal bundle. The frame equation is then

$$\nabla_{\gamma'}(T, N_1, N_2) = (T, N_1, N_2) \begin{pmatrix} 0 & -\kappa_1 & -\kappa_2 \\ \kappa_1 & 0 & 0 \\ \kappa_2 & 0 & 0 \end{pmatrix},$$

and the formula for ∇ implies

$$T' = \kappa_1 N_1 + \kappa_2 N_2 + 2\tfrac{r'}{r}T - r\tfrac{\partial}{\partial r}$$
$$N_i' = \tfrac{r'}{r}N_i + (\tfrac{1}{r}(N_i)_3 - \kappa_i)T$$

where $(N_i)_3$ denotes the r–component of N_i. Setting $\tilde{N}_i = \frac{1}{r}N_i$ we have $\tilde{N}_i' = \frac{1}{r}(N_i' - \frac{r'}{r}N_i)$. Thus $\tilde{N}_i' = \frac{1}{r}((\tilde{N}_i)_3 - \kappa_i)T$ and $\tilde{N}_1$, $\tilde{N}_2$ is a d–parallel frame of the normal bundle.

From $\gamma(x) = (\gamma_1(x), \gamma_2(x), r(x))$ we obtain the *rotational surface*

$$f(x, y) = (\gamma_1(x), \gamma_2(x), r(x)\cos(y), r(x)\sin(y)). \tag{164}$$

We assume that f is a conformal immersion, which is equivalent to γ being parametrized with respect to hyperbolic arc length.

The tangent vectors of f are

$$f_x = (\gamma_1', \gamma_2', r'\cos(y), r'\sin(y)) \quad \text{and} \quad f_y = (0, 0, -r\sin(y), r\cos(y))$$

and the second derivatives

$$f_{xx} = (\gamma_1'', \gamma_2'', r''\cos(y), r''\sin(y)), \qquad f_{yy} = (0, 0, -r\cos(y), -r\sin(y))$$

and

$$f_{xy} = (0, 0, -r'\sin(y), r'\cos(y)).$$

The induced metric of f is $g = r^2|dz|^2$, thus $u(x,y) = \ln(r(x))$. Its mean curvature vector is

$$\begin{aligned} H &= \tfrac{1}{2r^2}(\tfrac{\partial^2 f}{\partial x^2} + \tfrac{\partial^2 f}{\partial y^2})^\perp \\ &= \tfrac{1}{2r^2}((\gamma_1'', \gamma_2'', r''\cos(y), r''\sin(y)) + (0,0,-r\cos(y), -r\sin(y)))^\perp \\ &= \tfrac{1}{2r^2}R(y)(T' - r\tfrac{\partial}{\partial r})^\perp = \tfrac{1}{2r}R(y)(\kappa_1\tilde{N}_1 + \kappa_2\tilde{N}_2 + 2\tfrac{r'}{r^2}T - 2\tfrac{\partial}{\partial r})^\perp, \end{aligned}$$

where we embed $\mathbb{R}^3$ as $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$ and use

$$R(y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(y) & -\sin(y) \\ 0 & 0 & \sin(y) & \cos(y) \end{pmatrix}.$$

(Note that the $^\perp$ can be dropped, because the given expression is obviously perpendicular to f_y and it is perpendicular to $f_x = T$ by $\langle T, T\rangle_{\mathbb{R}^3} = r^2$ and $\langle T, \frac{\partial}{\partial r}\rangle_{\mathbb{R}^3} = r'$.)

The Hopf differential of f is given by

$$\begin{aligned} q = \mathbb{I}(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}) &= \tfrac{1}{4}(\tfrac{\partial^2 f}{\partial x^2} - \tfrac{\partial^2 f}{\partial y^2})^\perp - \tfrac{i}{2}\tfrac{\partial^2 f}{\partial x \partial y}^\perp \\ &= \tfrac{1}{4}((\gamma_1'', \gamma_2'', r''\cos(y), r''\sin(y)) - (0,0,-r\cos(y), -r\sin(y)))^\perp \\ &= \tfrac{1}{4}R(y)(T' + r\tfrac{\partial}{\partial r})^\perp = \tfrac{r}{4}R(y)(\kappa_1\tilde{N}_1 + \kappa_2\tilde{N}_2 + 2\tfrac{r'}{r^2}T)^\perp \\ &= \tfrac{r}{4}R(y)(\kappa_1\tilde{N}_1 + \kappa_2\tilde{N}_2). \end{aligned}$$

Denoting by n_i the image of the parallel normal field $R(y)\tilde{N}_i(x)$ of the immersion f under the embedding of $\mathbb{R}^4$ into the lightcone and the isomorphism of the normal bundles from Lemma 103, Lemma 106 implies

$$(*) \qquad \kappa = \tfrac{1}{4}(\kappa_1 n_1 + \kappa_2 n_2).$$

For the calculation of c via $\frac{1}{2}c = u_{zz} - u_z^2 + \langle H, \mathbb{I}(\frac{\partial}{\partial z}, \frac{\partial}{\partial z})\rangle$ (cf. Lemma 106), we need

$$\langle H, q\rangle_{\mathbb{R}^4} = \tfrac{1}{8}(\kappa_1^2 + \kappa_2^2) - \tfrac{1}{4}\langle \tfrac{\partial}{\partial r}, \kappa_1\tilde{N}_1 + \kappa_2\tilde{N}_2\rangle_{\mathbb{R}^3},$$

which, using

$$r'' = \langle T', \tfrac{\partial}{\partial r}\rangle_{\mathbb{R}^3} = r(\kappa_1(\tilde{N}_1)_3 + \kappa_2(\tilde{N}_2)_3) + 2\tfrac{r'}{r}r' - r$$

calculates to

$$\langle H, q\rangle_{\mathbb{R}^4} = \tfrac{1}{8}(\kappa_1^2 + \kappa_2^2) - \tfrac{1}{4}(\tfrac{r''}{r} + 1 - 2(\tfrac{r'}{r})^2).$$

Since $u = \ln r$ we have $u_z = \frac{1}{2}\frac{r'}{r}$ and $u_{zz} = \frac{1}{4}\frac{r''r - r'^2}{r^2}$, thus

$$u_{zz} - (u_z)^2 = \tfrac{1}{4}\tfrac{r''r - 2r'^2}{r^2}$$

and

$$\tfrac{c}{2} = \langle H, q\rangle_{\mathbb{R}^4} + u_{zz} - (u_z)^2 = \tfrac{1}{8}(\kappa_1^2 + \kappa_2^2) - \tfrac{1}{4}.$$

Hence $c = \frac{1}{4}(\kappa_1^2 + \kappa_2^2) - \frac{1}{2}$.

Together with $(*)$ we obtain the formulae

$$\kappa = \tfrac{1}{4}(\kappa_1 n_1 + \kappa_2 n_2)$$
$$c = \tfrac{1}{4}(\kappa_1^2 + \kappa_2^2) - \tfrac{1}{2}$$

expressing the conformal data of a rotational surface obtained from a curve in hyperbolic 3–space in terms of the invariants of the curve.

3.3. Constant Mean Curvature Surfaces. Constant mean curvature (CMC) surfaces in 3–dimensional space forms are a very important class of isothermic, constrained Willmore surfaces. In this subsection, the formulae of Section 2 of this Appendix are applied to give a uniform approach for the study of CMC surfaces in space forms.

Let $\mathcal{S}_{v_0}$ be a 3–dimensional space form of curvature G and let z be a conformal coordinate. Because the normal bundle of immersions into 3–dimensional space is trivial, the mean curvature H and the Hopf differential q with respect to z can be considered as function. Lemma 104 yields the following metrical Gauss and Codazzi equations

$$u_{z\bar{z}} - |q|^2 e^{-2u} + \tfrac{1}{4}e^{2u}(H^2 + G) = 0, \tag{165}$$

$$q_{\bar{z}} = \tfrac{1}{2}e^{2u}H_z. \tag{166}$$

By Lemma 106, the conformal invariants expressed in u, H and q become

$$\kappa = qe^{-u}, \tag{167}$$

$$c = 2(u_{zz} - u_z^2 + Hq). \tag{168}$$

An immersion is CMC if the mean curvature H is constant. By (166) this is equivalent to q being holomorphic. In particular, away from umbilics, one can choose a coordinate such that q is constant, e.g. $q = 1$. This shows that CMC immersions are classically isothermic. (For CMC tori, there are no umbilics and q constant can be achieved for a global coordinate, which implies that CMC tori are isothermic in the global sense.)

Assume now $q = 1$. In the Möbius geometric setting, the isothermic surface associated family is given by the family of immersions with the invariants κ and $c + r$ for $r \in \mathbb{R}$ (see Section 3.1 of Appendix A). In the metrical setting, one obtains the family of immersions with the invariants u, $H + \frac{r}{2}$ and $q = 1$ in the space forms of curvature G_r given by $(H + \frac{r}{2})^2 + G_r = H^2 + G$ (to prove this fact, it suffices to check that, for all r, the metrical data solves the Gauss equation and gives the right conformal data). Hence, for CMC surfaces, the isothermic surface associated family becomes *Lawson correspondence.*

To see that CMC surfaces are constrained Willmore, it is (by (159)) sufficient to check that, for $q = 1$, the conformal data satisfies the equation

$$\kappa_{\bar{z}\bar{z}} + \tfrac{\bar{c}}{2}\kappa = H\kappa.$$

This equation is derived in [**4**, Section 3.4] for the proof of the Theorem by Richter stating that a torus in the conformal 3–sphere is isothermic and constrained Willmore if and only if it is CMC in some space form.

Remark. For CMC surfaces with conformal curvature line coordinates, the Gauss equation (165) is the only remaining compatibility equation. With the right coordinate and after scaling the metric of the space form (or, in the lightcone model, v_0 and the immersions) we can always assume $q = \frac{1}{2}$ and $H^2 + G \in \{0, \pm 1\}$. This simplifies the Gauss equation to

$$\begin{aligned}
&2u_{z\bar{z}} + \sinh(2u) = 0 && \text{if } H^2 + G = 1 \\
&u_{z\bar{z}} = \tfrac{1}{4}e^{-2u} && \text{if } H^2 + G = 0 \\
&2u_{z\bar{z}} = \cosh(2u) && \text{if } H^2 + G = -1,
\end{aligned}$$

which are the Sinh–Gordon, Liouville and Cosh–Gordon equations.

APPENDIX B

Holonomy of Flat Connections on the Torus

Let (V, ∇) be a flat vector bundle on a surface M. For a point $p \in M$ and a loop γ based at p, the *holonomy* of ∇ is defined to be the endomorphism

$$H_p(\gamma)\colon V_p \to V_p$$

obtained by parallel transporting vectors in V_p along γ. The flatness of ∇ is equivalent to the fact that H_p does only depend on the homotopy class of γ. This gives rise to the holonomy representation

$$H_p\colon \pi_1(M,p) \to \mathrm{GL}(V_p)$$

of the fundamental group. For another point $q \in M$, the holonomy representation $H_q\colon \pi_1(M,q) \to \mathrm{GL}(V_q)$ is obtained from that at p by conjugation by the parallel transport $\mathcal{P}_\eta$ along a path η joining p and q, i.e. $H_q = \mathcal{P}_\eta H_p \mathcal{P}_\eta^{-1}$. It should be noted that

- parallel section of V correspond to common fixed points of $H_p(\gamma)$ for all γ,
- parallel subbundles of V correspond to subspaces in V_p that are invariant under $H_p(\gamma)$ for all γ and
- parallel sections of the endomorphism bundle of V correspond to endomorphisms of V_p that commute with $H_p(\gamma)$ for all γ.

The Case of Abelian Fundamental Group. In the case of abelian fundamental group, e.g. for M a torus, the image of $\pi_1(M,p)$ under the holonomy representation H_p is a commutative subgroup of $\mathrm{GL}(V_p)$. One simple but important consequence of this fact is that every eigenspace of $H_p(\gamma)$ for one $\gamma \in \pi_1(M,p)$ gives rise to a parallel subbundle (because by commutativity of $\pi_1(M,p)$, the eigenspaces of $H_p(\gamma)$ are invariant under $H_p(\gamma')$ for all $\gamma' \in \pi_1(M,p)$).

Another consequence of an Abelian fundamental group is that, by extension via parallel transport, every $H_p(\gamma)$ itself yields a parallel section of the endomorphism bundle. Obviously, for $\gamma' = \eta\gamma\eta^{-1} \in \pi_1(M,q)$ conjugate to $\gamma \in \pi_1(M,p)$ by a path η joining p and q, the element $H_q(\gamma') = \mathcal{P}_\eta H_p \mathcal{P}_\eta^{-1}$, gives rise to the same parallel section as $H_p(\gamma)$. At this point it is advantageous to use the group of deck transformations Γ of the universal covering $\tilde{M}$, which, in the case of Abelian fundamental group, is canonically isomorphic to $\pi_1(M,p)$ for every $p \in M$. Because the elements in $\pi_1(M,p)$ and $\pi_1(M,q)$ determined by the same $\gamma \in \Gamma$ are conjugate with respect to a path joining p and q, the parallel section of the endomorphism bundle determined by $H_p(\gamma)$ does not depend on the choice of base point p but only on γ. Therefore, the holonomy representation can be considered as a group homomorphism

$$H\colon \Gamma \to \Gamma^{\parallel}(\mathrm{GL}(V))$$

into the set of parallel automorphisms $\Gamma^{\parallel}(\mathrm{GL}(V)) \subset \Gamma(\mathrm{End}(V))$ of V.

Monodromy Uniquely Determines a Flat Vector Bundle. On the universal covering $\pi\colon \tilde{M} \to M$ of M, every flat vector bundle V on M can be trivialized, i.e. there is a global parallel frame field $\underline{s}$ for $\tilde{V} = \pi^*V$ on $\tilde{M}$ which defines an isomorphism between $\tilde{V}$ and the trivial $\mathbb{K}^n$ bundle over $\tilde{M}$ (where $\mathbb{K}$ denotes the base field of the vector bundle V and n its rank).

The frame $\underline{s}$ gives rise to a multiplier homomorphism $\lambda\colon \Gamma \to \mathrm{GL}(n, \mathbb{K})$ defined by

$$\gamma^*\underline{s} = \underline{s}\lambda(\gamma)$$

for all elements $\gamma \in \Gamma$ of the group of deck transformations. This homomorphism λ is also called the *monodromy representation* (of ∇ with respect to the trivialization $\underline{s}$). By the following lemma, λ determines V up to gauge equivalence.

LEMMA 111. *There is a 1-1–correspondence between gauge equivalence classes of flat $\mathbb{K}$–vector bundles of rank n over M and between conjugacy classes of representations $\lambda\colon \Gamma \to \mathrm{GL}(n, \mathbb{K})$ of the group of deck transformations Γ.*

PROOF. We have already seen that the choice of a parallel frame field $\underline{s}$ gives rise to a multiplier λ, which gets conjugated when the the frame field is changed. Clearly, gauge equivalent bundles have conjugate multipliers.

Conversely, a multiplier defines a vector bundle V on M by taking the quotient of the trivial $\mathbb{K}^n$ bundle $\tilde{M} \times \mathbb{K}^n$ on the universal covering $\tilde{M}$ with respect to the action $\Gamma \times \tilde{M} \times \mathbb{K}^n \to \tilde{M} \times \mathbb{K}^n$ given by $(\gamma, (\tilde{p}, v)) \mapsto (\gamma\tilde{p}, \lambda(\gamma)v)$. It is clear that conjugate multipliers yield equivalent bundles. Furthermore, it is clear by construction that every flat vector bundle can be obtained that way. $\square$

REMARK. Let $p \in M$ and $\tilde{p} \in \pi^{-1}\{p\}$. The choice of $\tilde{p}$ defines an isomorphism $\Gamma \cong \pi_1(M, p)$, which we denote by $\gamma \mapsto \gamma_{\tilde{p}}$. Using this isomorphism, we have the following relation between the holonomy of ∇ and the monodromy of the parallel frame $\underline{s}$:

$$H_p(\gamma_{\tilde{p}})\underline{s}(\tilde{p}) = \gamma^*\underline{s}(\tilde{p}) = \underline{s}(\tilde{p})\lambda(\gamma)$$

for all $\gamma \in \Gamma$. In the case of Abelian fundamental group, we have

$$H(\gamma)\underline{s} = \underline{s}\lambda(\gamma)$$

for all $\gamma \in \Gamma$, where we use $H\colon \Gamma \to \Gamma^{\parallel}(\mathrm{GL}(V))$ as described in the preceding paragraph.

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