Supplementary Material: Enhanced core-mantle coupling due to stratification at the top of the core

APPENDIX

We present details of the solution to the problem stated in Sect. 2 and use the assumption that $k_y = 0$. Substituting the expression for the perturbations specified in Eq. (8) into the linearized governing equations in Eq. (6-8) yields:

$$\rho_0(\mathrm{i}k_x\bar{V}\tilde{\boldsymbol{v}} + 2\boldsymbol{\Omega}\times\tilde{\boldsymbol{v}}) = -\mathrm{i}\boldsymbol{k}\tilde{p} + \tilde{\rho}'\boldsymbol{g} + \mathrm{i}k_z\bar{B}\tilde{\boldsymbol{b}}/\mu\,, \tag{S1a}$$

$$\mathrm{i}k_z\bar{B}\tilde{\boldsymbol{v}} - \mathrm{i}k_x\bar{V}\tilde{\boldsymbol{b}} - \eta k^2\tilde{\boldsymbol{b}} = \boldsymbol{0}\,, \tag{S1b}$$

$$\mathrm{i}k_x\bar{V}\tilde{\rho}' = \frac{\rho_0 N^2}{g}\tilde{v}_z\,, \tag{S1c}$$

where k denotes the magnitude of the wavenumber vector. These equations define an algebraic system for the amplitudes of the perturbations $\tilde{\boldsymbol{v}}$, $\tilde{\rho}'$, etc, which is supplemented by solenoidal conditions requiring $\boldsymbol{k}\cdot\tilde{\boldsymbol{v}} = 0$ and $\boldsymbol{k}\cdot\tilde{\boldsymbol{b}} = 0$. The unknowns in the problem include the amplitudes of the perturbations and the vertical wavenumber k_z. From the induction equation, the velocity perturbation may be expressed in terms of the magnetic one:

$$\tilde{\boldsymbol{v}} = \left(\frac{\mathrm{i}k_x\bar{V} + \eta k^2}{\mathrm{i}k_z\bar{B}}\right)\tilde{\boldsymbol{b}} = \left(\frac{\mathrm{i}k_x\bar{V} + \eta k^2}{\mathrm{i}k_z}\right)\hat{\boldsymbol{b}}\,, \tag{S2}$$

where a dimensionless magnetic perturbation was introduced in the second step, i.e. $\hat{\boldsymbol{b}} = \tilde{\boldsymbol{b}}/\bar{B}$. Using the Eq. (S1c) and $\boldsymbol{g} = -g\boldsymbol{e}_z$ the solution for the pressure perturbation is obtained from the vertical component of the momentum equation ($\boldsymbol{e}_z$-component):

$$\frac{\tilde{p}}{\rho_0} = -\frac{k_x\bar{V}}{k_z}\tilde{v}_z + V_\mathrm{A}^2\hat{b}_z + \frac{N^2}{k_x k_z\bar{V}}\tilde{v}_z\,, \tag{S3}$$

where the Alfvén velocity $V_\mathrm{A} = \bar{B}/\sqrt{\rho_0\mu}$ was introduced. Notice that the pressure perturbation does not depend on the sign of $\bar{B}$. A substitution of the latter two expressions for the velocity and pressure perturbations in the $\boldsymbol{e}_x$- and $\boldsymbol{e}_y$-component of the momentum equation and applying the solenoidal conditions

$$\tilde{v}_z = -\frac{k_x}{k_z}\tilde{v}_x\,, \qquad \tilde{b}_z = -\frac{k_x}{k_z}\tilde{b}_x \tag{S4}$$

gives a 2×2 eigenvalue problem for k_z:

$$k_x\bar{V}(-k_x\bar{V} + \mathrm{i}\eta k_z^2)\mathbf{A}\cdot\begin{bmatrix}\hat{b}_x\\ \hat{b}_y\end{bmatrix} + 2\Omega(\mathrm{i}k_x\bar{V} + \eta k_z^2)\mathbf{B}\cdot\begin{bmatrix}\hat{b}_x\\ \hat{b}_y\end{bmatrix} + k_z^2V_\mathrm{A}^2\mathbf{C}\cdot\begin{bmatrix}\hat{b}_x\\ \hat{b}_y\end{bmatrix} = \boldsymbol{0} \tag{S5a}$$

with

$$\mathbf{A} = \begin{bmatrix} 1 + \frac{k_x^2}{k_z^2}(1 - \frac{N^2}{k_x^2 \bar{V}^2}) & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 + \frac{k_x^2}{k_z^2} & 0 \\ 0 & 1 \end{bmatrix}, \tag{S5b}$$

where magnetic diffusion was neglected in the horizontal direction w.r.t. the vertical one ($\eta k^2 \approx \eta k_z^2$). These equations define the eigenvalue problem for k_z, where the eigenvectors define the amplitudes of the magnetic perturbations. Non-trivial solutions require the determinant of this matrix system to vanish, which defines a cubic equation for k_z^2. Retaining the roots of k_z^2 with $\mathrm{Im}(k_z) < 0$ gives three solutions that decay away from the boundary. We compute the roots of the cubic equation numerically using the nominal values specified in Table 1 and the corresponding eigenvectors are also determined numerically.

Hence, three solutions for the magnetic perturbation are found. However, the solution is only defined up to three constants, that means the perturbation is expressed as a linear combination of the three solutions:

$$\boldsymbol{b}(\boldsymbol{x}) = \alpha \boldsymbol{b}^{(1)}(\boldsymbol{x}) + \beta \boldsymbol{b}^{(2)}(\boldsymbol{x}) + \gamma \boldsymbol{b}^{(3)}(\boldsymbol{x}), \tag{S6}$$

where all three solution have a different spatial dependence w.r.t. the z-coordinate due to the different wavenumbers $k_z^{(i)}$. According Eq. (S2) each of the three solutions for the magnetic perturbation has a corresponding solution for the velocity perturbation.

In order to determine the yet unknown factors α, β and γ, the boundary conditions specified in Eqs. (9) and (10) are used. Neglecting terms in Eq. (9) that are second order or smaller in the perturbation gives:

$$v_z(x, y, 0) = \mathrm{i} k_x \tilde{h} \bar{V} \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}), \tag{S7}$$

where the position vector $\boldsymbol{x}$ has been restricted to the reference surface. When the mantle is an electrical insulator, we can represent the magnetic perturbation, $\boldsymbol{b}_\mathrm{M}$, as a potential field

$$\boldsymbol{b}_\mathrm{M} = -\nabla \psi_\mathrm{M}(\boldsymbol{x}), \tag{S8a}$$

where the magnetic potential satisfies $\nabla^2 \psi_\mathrm{M} = 0$. Solutions that vanishes far from the boundary ($z \to \infty$) have the form

$$\psi_\mathrm{M} = \tilde{\psi}_\mathrm{M} \exp(-k_\mathrm{T} z) \exp(\mathrm{i}\boldsymbol{k}_\mathrm{T} \cdot \boldsymbol{x}), \tag{S8b}$$

where $\tilde{\psi}_\mathrm{M}$ is an undetermined amplitude. When the magnetic continuity condition in Eq. (10) is evaluated at the reference surface ($z = 0$), the spatial dependency drops out and the following three equations result:

$$\alpha \tilde{\boldsymbol{b}}^{(1)} + \beta \tilde{\boldsymbol{b}}^{(2)} + \gamma \tilde{\boldsymbol{b}}^{(3)} = -\tilde{\psi}_\mathrm{M}(\mathrm{i}\boldsymbol{k}_\mathrm{T} - k_\mathrm{T}\boldsymbol{e}_z). \tag{S9}$$

Thus, with Eqs. (S7) and (S9) there are four equations for the unknowns α, β, γ and $\tilde{\psi}_\mathrm{M}$, which are solved numerically too. A backward substitution then yields the solutions of the perturbations of the other fields.