Domain Optimization for the Navier-Stokes Equations by an Embedding Domain Method

Thomas Slawig
Technische Universität Berlin,
Fachbereich Mathematik MA 6-2,
Strasse des 17. Juni 136,
D-10623 Berlin, Germany
email: `slawig@math.tu-berlin.de`

Abstract

Domain optimization problems for the two-dimensional stationary flow of incompressible linear-viscous fluids, i.e. the Navier-Stokes equations, are studied. An embedding domain technique which provides an equivalent formulation of the problem on a fixed domain is introduced. Existence of a solution to the domain optimization problem and Fréchet differentiability with respect to the variation of the domain of tracking type cost functionals for the velocity field are proved. A simply computable formula for the derivative of the cost functional is presented. Numerical examples show the advantages of the embedding domain method and the reliability of the derivative formula.

1 Introduction

In this work we consider domain optimization problems for the stationary Navier-Stokes equations. The aim of domain optimization is to find the shape of a domain which is optimal in the sense that a given cost functional is minimized subject to the constraint that some (partial) differential equation is satisfied. Typical features of such problems are the highly non-linear dependence and the lack of sensitivity of the cost functionals with respect to the variation of the domain.

Here we consider these problems in the field of fluid mechanics. Therefore typical constraints are the Navier-Stokes equations. We restrict our study to incompressible fluids and stationary problems in two space dimensions.

There are different types of cost functionals which are of interest in fluid mechanics, and in many types of geometrical settings domain optimization might be applied. Two examples are channel flow with a sudden expansion in machines where one tries to avoid back-flow and optimal design of airfoils for drag reduction.

The methods presented in this work allow significant reduction of computational effort necessary to perform an iterative optimization process for state equations whose solution itself is rather time-consuming, as it is the case for the Navier-Stokes equations.

The embedding domain method we used reduces the effort of discretization and assembling of the discrete systems for the changing domains during the optimization process. Moreover it provides us with a formula for the derivative of the cost functional with respect to the domain which is efficient and numerically stable.

This paper summarizes results of the author's PhD thesis [7]. It is based on a work by Kunisch and Peichl [6] who applied the same technique to the Laplace equation with mixed boundary conditions.

The outline of this paper is the following: In the second section we formulate the domain optimization problem with its geometrical setting and summarize the used basic results on the theory of the Navier-Stokes equations. In the next section we introduce the embedding domain method and derive an equivalent formulation of the Navier-Stokes equations on a fixed domain. In the fourth section we prove continuous dependence of the solution of the state equations with respect to the variation of the domain and the existence of a solution to the domain optimization problem. The central part is the fifth section which is concerned with Fréchet differentiability and the explicit formula for the derivative of the cost functional. In the last section we describe the applied numerical methods and present some results.

2 The Domain Optimization Problem

Before we formulate the domain optimization problem we describe the geometrical setting:

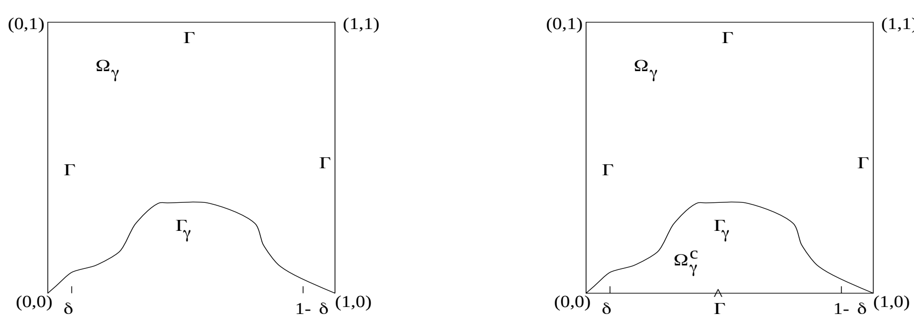

Figure 1:

We consider domains $\Omega_\gamma := \Omega(\gamma) \subset \mathbb{R}^2$ where the control parameter γ is a function defined on $I = (0, 1)$ whose graph is one part (denoted by Γ_γ) of the boundary of Ω_γ, compare the picture on the left in Fig. 1.

The remaining part $\Gamma := \partial\Omega_\gamma \setminus \overline{\Gamma}_\gamma$ is fixed and consists of the three segments $[(0,0),(0,1)]$, $[(0,1),(1,1)]$, $[(1,1),(1,0)]$. The variable part Γ_γ therefore connects the two end points of Γ, namely $(0,0)$ and $(1,0)$.

To apply the embedding domain method and to prove the explicit formula of the derivative of the cost functional we need a combination of smooth and convex polygonal boundary. Clearly Γ is a convex polygon, for Γ_γ we assume C^2 regularity.

To preserve the convexity of Ω_γ near the two transition points $(0,0),(1,0)$ we assume that γ is linear in neighbourhoods of these points. Furthermore Γ_γ shall always be in the unit square $(0,1)\times(0,1)$. Working in Sobolev spaces we assure the regularity by choosing $\gamma \in H^3(I)$, and to get existence of a solution of (2) we need boundedness in this space. Summarizing we define the set of admissible functions γ defining the variable boundary parts Γ_γ and thus the admissible domains Ω_γ by

$$\begin{aligned}\mathcal{S} \quad := \quad & \Big\{\gamma \in H^3(I) : \|\gamma\|_{H^3(I)} \le c_{\mathcal{S}}, \gamma(0)=\gamma(1)=0, c_0 \le \gamma(x) \le c_1, x \in (\delta, 1-\delta) \\ & \qquad \gamma'|_{(0,\delta)} = c^0, \gamma'|_{(1-\delta,1)} = c^1\Big\}. \end{aligned} \tag{1}$$

Here $c_0, c_1 \in (0,1), \delta \in (0, \frac{1}{2}), c_{\mathcal{S}}, c^0 \in \mathbb{R}^+, c^1 \in \mathbb{R}^-$ are fixed.

Given an observation region Ω_C which is a subset of Ω_γ for all $\gamma \in \mathcal{S}$ and a velocity field $\mathbf{u}_d \in L^2(\Omega_C)^2$ we study the following domain optimization problem:

$$\min_{\gamma \in \mathcal{S}} \mathcal{J}(\gamma) \quad := \quad \frac{1}{2}\|\mathbf{u}_\gamma - \mathbf{u}_d\|^2_{L^2(\Omega_C)^2} \tag{2}$$

where $\mathbf{u}_\gamma$ is the velocity component of a variational solution $(\mathbf{u}_\gamma, p_\gamma) \in H^1(\Omega_\gamma)^2 \times L^2_0(\Omega_\gamma)$ of the Navier-Stokes equations

$$\begin{array}{rcll} -\nu\triangle\mathbf{u}_\gamma + \mathbf{u}_\gamma\cdot\nabla\mathbf{u}_\gamma + \nabla p_\gamma & = & \mathbf{f}_\gamma & \text{in} \quad \Omega_\gamma \\ \nabla\cdot\mathbf{u}_\gamma & = & 0 & \text{in} \quad \Omega_\gamma \\ \mathbf{u}_\gamma & = & \Phi & \text{on} \quad \Gamma \\ \mathbf{u}_\gamma & = & \mathbf{0} & \text{on} \quad \Gamma_\gamma. \end{array} \tag{3}$$

In (2) an additional regularization term may be included. The space for the pressure p_γ is defined as $L^2_0(\Omega_\gamma) := \{q \in L^2(\Omega_\gamma) : \int_{\Omega_\gamma} q\,dx = 0\}$. For the inhomogeneity we assume $\mathbf{f}_\gamma \in L^\infty(\Omega_\gamma)^2$. The function Φ describing the boundary values of the velocity on Γ shall have a divergence-free extension onto Ω_γ which is in $H^2(\Omega_\gamma)^2$. We define

$$\begin{aligned} H(\Gamma) \quad := \quad & \big\{\Phi \in L^2(\Gamma)^2 : \text{ there exists } \mathbf{u}^0_\gamma \in H^2(\Omega_\gamma)^2 : \nabla\cdot\mathbf{u}^0_\gamma = 0 \text{ in } \Omega_\gamma, \\ & \quad \mathbf{u}^0_\gamma|_{\Gamma_\gamma} = \mathbf{0}, \mathbf{u}^0_\gamma|_\Gamma = \Phi\big\}. \end{aligned}$$

We summarize some needed theoretical results for the Navier-Stokes equations:

Theorem 2.1 *Let $\gamma \in \mathcal{S}, \mathbf{f}_\gamma \in L^2(\Omega_\gamma)^2$, and $\Phi \in H(\Gamma)$.*

(a) Then there exists a variational solution $(\mathbf{u}_\gamma, p_\gamma) \in H^2(\Omega_\gamma)^2 \times [H^1(\Omega_\gamma) \cap L_0^2(\Omega_\gamma)]$ *to (3) which for some* $C > 0$ *independent of* $\gamma, \mathbf{f}_\gamma$, *and* Φ *satisfies*

$$\|\mathbf{u}_\gamma\|_{H^2(\Omega_\gamma)^2} + \|p_\gamma\|_{H^1(\Omega_\gamma)} \quad \leq \quad C\left(\|\mathbf{f}_\gamma\|_{L^2(\Omega_\gamma)^2} + \|\Phi\|_{L^\infty(\Gamma)^2}\right).$$

(b) If $\nu > \nu_0 = \nu_0(\gamma, \mathbf{f}_\gamma, \Phi)$ *the solution is unique.*

Proof. Regularity and uniqueness results for completely smooth C^2 or convex polygonal boundary are standard, see e.g. [3], [5]. From [5] it can be deduced that the regularity remains valid also in our case where both boundary types are mixed. The uniform regularity for the polygonal part is stated in the same reference, whereas for the smooth part it is shown e.g. in [2, Section IV.5.]. □

3 The Embedding Domain Method

Concerning the effort of discretization and assembling of the system matrices in the numerical solution of the state equations it is much more efficient to solve them on a simple-shaped and fixed domain rather than on the changing Ω_γ during an iterative optimization process. Thus we choose a so-called "fictitious domain" $\hat{\Omega}$ satisfying $\Omega_\gamma \subset \hat{\Omega}$ for all $\gamma \in \mathcal{S}$. In our case we take $\hat{\Omega}$ as the unit square.

Remark 3.1 *A smooth transition between* Γ *and* Γ_γ *would result in a highly irregular complementary domain* $\Omega_\gamma^c = \hat{\Omega} \setminus \overline{\Omega}_\gamma$ *with not even Lipschitz boundary. Thus there would be no existence result for the state equations on* Ω_γ^c.

We now derive an equivalent formulation of the Navier-Stokes equations on $\hat{\Omega}$. For this purpose we introduce the trace operator τ_γ onto the boundary Γ_γ and extend the inhomogeneity $\mathbf{f}_\gamma$ by zero to $\hat{\Omega}$. The former boundary condition $\tau_\gamma \mathbf{u}_\gamma = \mathbf{u}_\gamma|_{\Gamma_\gamma} = \mathbf{0}$ is treated as a constraint using a Lagrange multiplier g_γ.

The *fictitious domain formulation* of the Navier-Stokes equations then is to find $(\hat{\mathbf{u}}_\gamma, \hat{p}_\gamma, g_\gamma) \in H^1(\hat{\Omega})^2 \times L_0^2(\hat{\Omega}) \times H_\gamma^*$ such that

$$\begin{aligned} -\nu \triangle \hat{\mathbf{u}}_\gamma + \hat{\mathbf{u}}_\gamma \cdot \nabla \hat{\mathbf{u}}_\gamma + \nabla \hat{p}_\gamma - \tau_\gamma^* g_\gamma &= \tilde{\mathbf{f}}_\gamma \quad \text{in} \quad H^{-1}(\hat{\Omega})^2 \\ \nabla \cdot \hat{\mathbf{u}}_\gamma &= 0 \quad \text{in} \quad L_0^2(\hat{\Omega}) \\ \tau_\gamma \hat{\mathbf{u}}_\gamma &= \mathbf{0} \quad \text{in} \quad H_\gamma. \end{aligned} \tag{4}$$

where τ_γ^* denotes the adjoint of the trace operator, and $H_\gamma := H_{00}^{1/2}(\Gamma_\gamma)^2$ is an abbreviation for the space

$$H_{00}^{1/2}(\Gamma_\gamma)^2 = \left\{\mathbf{h} \in H^{1/2}(\Gamma_\gamma)^2 : \text{there exists } \tilde{\mathbf{h}} \in H^{1/2}(\partial\Omega_\gamma)^2 : \tilde{\mathbf{h}}|_{\Gamma_\gamma} = \mathbf{h}, \tilde{\mathbf{h}}|_\Gamma = \mathbf{0}\right\}.$$

We can now prove the equivalence of problems (3) and (4):

Theorem 3.2 *Let* $\gamma \in \mathcal{S}, \mathbf{f}_\gamma \in L^2(\Omega_\gamma)^2$ *and* $\Phi \in H(\Gamma)$. *Then* $(\hat{\mathbf{u}}_\gamma, \hat{p}_\gamma, g_\gamma) \in H^1(\hat{\Omega})^2 \times L_0^2(\hat{\Omega}) \times H_\gamma^*$ *is a solution of (4) if and only if*

- $(\mathbf{u}_\gamma, p_\gamma) := (\hat{\mathbf{u}}_\gamma, \hat{p}_\gamma)|_{\Omega_\gamma} \in H^2(\Omega_\gamma)^2 \times [H_1(\Omega_\gamma) \cap L_0^2(\Omega_\gamma)]$ *solves (3),*
- $(\hat{\mathbf{u}}_\gamma, \hat{p}_\gamma)|_{\Omega_\gamma^c} = (\mathbf{0}, 0)$,
- $g_\gamma = \left(\nu \dfrac{\partial \mathbf{u}_\gamma}{\partial \mathbf{n}_\gamma} - p_\gamma \mathbf{n}_\gamma \right) \Big|_{\Gamma_\gamma}$ *in* $H^{1/2}(\Gamma_\gamma)^2$ *where* $\mathbf{n}_\gamma$ *denotes the outer (with respect to* Ω_γ*) normal vector on* Γ_γ*.*

Proof. The result is proved by testing the weak formulation of (3) with appropriate functions that vanish on Ω_γ^c, applying a uniqueness result for the homogeneous Navier-Stokes equations and Green's formula. The regularity of g_γ follows from the regularity of $\mathbf{u}_\gamma, p_\gamma$, compare Theorem 2.1. For more details see [7, Th. 3.5]. $\square$

Remark 3.3 *Here we see the first advantage of the embedding domain method: If* γ *changes only* Γ_γ *but not the whole domain has to be re-discretized. To get the discrete form of (4) only the discretized trace operator (which in principal is a one dimensional mass matrix) has to be re-assembled, and the discrete right-hand side has to be set to zero on* Ω_γ^c*. The rest of the system remains unchanged.*

4 Continuous Dependence of the Solution on the Shape of the Domain

To study convergence with respect to γ of the Lagrange multipliers $g_\gamma \in H_\gamma^*$ we introduce on H_γ the mapping

$$\mathcal{I}_\gamma \mathbf{h}(x) \quad := \quad \mathbf{h}(x, \gamma(x)) \quad \mathbf{h} \in H_\gamma, x \in I,$$

which can be shown (see [7, Th. 2.4]) to be an isomorphism between H_γ and

$$H_I \quad := \quad \left\{ \mathbf{g} \in H^{1/2}(I)^2 : \int_I \frac{\|\mathbf{g}(t)\|_2^2}{t(1-t)}\, dt < \infty \right\}.$$

We define the adjoint of $\mathcal{I}_\gamma^{-1}$ by

$$\begin{aligned} \left(\mathcal{I}_\gamma^{-1}\right)^* \quad &: \quad H_\gamma^* \to H_I^* \\ \langle \left(\mathcal{I}_\gamma^{-1}\right)^* g, \mathbf{g} \rangle_{H_I^*, H_I} \quad &:= \quad \langle g, \mathcal{I}_\gamma^{-1} \mathbf{g} \rangle_{H_\gamma^*, H_\gamma} \qquad g \in H_\gamma^*,\ \mathbf{g} \in H_I \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes dual pairings. Now we can formulate the following result:

Theorem 4.1 *Let* $\gamma, \gamma_n \in \mathcal{S}$ *with*

$$\gamma_n \to \gamma \ \text{ in } \ W^{1,\infty}(I), \qquad \tilde{\mathbf{f}}_{\gamma_n} \to \tilde{\mathbf{f}}_\gamma \ \text{ in } \ L^2(\hat{\Omega})^2,$$

and let the condition $\nu > \nu_0$ for the uniqueness of the solution to the Navier-Stokes equations be fulfilled. Then the solutions of problem (4) satisfy

$$\begin{array}{rcll} \hat{\mathbf{u}}_{\gamma_n} & \to & \hat{\mathbf{u}}_\gamma & \textit{in } H^1(\hat{\Omega})^2, \\ \hat{p}_{\gamma_n} & \to & \hat{p}_\gamma & \textit{in } L^2_0(\hat{\Omega}), \\ \left(\mathcal{I}^{-1}_{\gamma_n}\right)^* g_{\gamma_n} & \stackrel{*}{\rightharpoonup} & \left(\mathcal{I}^{-1}_{\gamma}\right)^* g_\gamma & \textit{in } H^*_I. \end{array}$$

For $\nu > \nu_1 = \nu_1(\mathbf{f}, \Phi)$ the mapping $\gamma \mapsto \hat{\mathbf{u}}_\gamma$ is Lipschitz continuous, i.e. there exists L independent of $\gamma, \bar{\gamma}$ such that

$$\|\hat{\mathbf{u}}_{\bar{\gamma}} - \hat{\mathbf{u}}_\gamma\|_{H^1(\hat{\Omega})^2} \leq L\|\bar{\gamma} - \gamma\|_{L^\infty(I)} \qquad \textit{for all } \bar{\gamma}, \gamma \in \mathcal{S}.$$

Proof. First step is to show uniform boundedness of the family of solutions for $\gamma \in \mathbb{S}$ which implies weak convergence for a subsequence and weak-$*$ convergence of the Lagrange multipliers. To show strong convergence of velocity and pressure we exploit the weak form of (4) with appropriate test functions. The different terms can be estimated exploiting their regularity. See [7, Th.3.7]. $\square$

As a consequence of this Theorem and the boundedness of $\mathcal{S}$ in $H^3(I)$ which is compactly embedded in $C^2(\bar{I})$ we now obtain:

Corollary 4.2 *The domain optimization problem has at least one solution $\gamma \in \mathcal{S}$.*

5 Fréchet Differentiability and Derivative Formula

To show differentiability we use the solution of the adjoint system of the domain optimization problem (2). We introduce a Lagrangian with two multipliers $\lambda_\gamma, \mu_\gamma$ for the constraints of the momentum and continuity equation, respectively. Then we compute the necessary optimality conditions for a saddle point of this Lagrangian which form the adjoint equations. Roughly speaking they are linearized Navier-Stokes equations: Find $(\lambda_\gamma, \mu_\gamma) \in H^1_0(\Omega_\gamma)^2 \times L^2_0(\Omega_\gamma)$ such that

$$\begin{array}{rcll} -\nu\triangle\lambda_\gamma + \nabla\mathbf{u}_\gamma \cdot \lambda_\gamma - \mathbf{u}_\gamma \cdot \nabla\lambda_\gamma + \nabla\mu_\gamma & = & -D_u\mathcal{J}(\gamma) & \text{in} \quad \Omega_\gamma \\ \nabla \cdot \lambda_\gamma & = & 0 & \text{in} \quad \Omega_\gamma \end{array} \tag{5}$$

where $\mathbf{u}_\gamma$ is the velocity component of a solution to (3). Again we derive an equivalent fictitious domain formulation by introducing an additional Lagrange multiplier χ_γ corresponding to the constraint $\tau_\gamma\lambda_\gamma = \mathbf{0}$. Then we obtain the problem:
Find $(\hat{\lambda}_\gamma, \hat{\mu}_\gamma, \chi_\gamma) \in H^1_0(\hat{\Omega})^2 \times L^2_0(\hat{\Omega}) \times H^*_\gamma$ such that

$$\begin{array}{rcll} -\nu\triangle\hat{\lambda}_\gamma + \nabla\hat{\mathbf{u}}_\gamma \cdot \hat{\lambda}_\gamma - \hat{\mathbf{u}}_\gamma \cdot \nabla\hat{\lambda}_\gamma + \nabla\hat{\mu}_\gamma - \tau^*_\gamma\chi_\gamma & = & -D_u\mathcal{J}(\gamma) & \text{in} \quad \hat{\Omega} \\ \nabla \cdot \hat{\lambda}_\gamma & = & 0 & \text{in} \quad \hat{\Omega} \\ \tau_\gamma\hat{\lambda}_\gamma & = & \mathbf{0}. & \end{array} \tag{6}$$

For the solution of the adjoint problem we can show:

Theorem 5.1 *Let $\gamma \in \mathcal{S}$ and let $(\mathbf{u}_\gamma, p_\gamma)$ be a solution to (3). Then we have:*

(a) Problem (5) has a solution $(\lambda_\gamma, \mu_\gamma) \in [H^2(\Omega_\gamma)^2 \cap H_0^1(\Omega_\gamma)^2] \times [H^1(\Omega_\gamma) \cap L_0^2(\Omega_\gamma)]$. The regularity is uniform in γ.

(b) The solution is unique for ν sufficiently large.

(c) $(\hat{\lambda}_\gamma, \hat{\mu}_\gamma, \chi_\gamma) \in H^1(\hat{\Omega})^2 \times L_0^2(\hat{\Omega}) \times H_\gamma^*$ is a solution of (6) if and only if

- $(\lambda_\gamma, \mu_\gamma) := (\hat{\lambda}_\gamma, \hat{\mu}_\gamma)|_{\Omega_\gamma}$ *is a solution of (5),*
- $(\hat{\lambda}_\gamma, \hat{\mu}_\gamma)|_{\Omega_\gamma^c} = (\mathbf{0}, 0)$,
- $\chi_\gamma = \left(\nu \dfrac{\partial \lambda_\gamma}{\partial \mathbf{n}_\gamma} - \mu_\gamma \mathbf{n}_\gamma\right)\Big|_{\Gamma_\gamma}$ *in $H^{1/2}(\Gamma_\gamma)^2$.*

Proof. Part (a) is shown in [4], (b) follows from a uniqueness result for saddle point problems, and (c) is shown as Theorem 3.2. See [7, Th. 3.9]. □

To show differentiability we characterize the set of admissible directions as

$$\mathcal{S}' := \left\{\bar{\gamma} \in H^3(I) : \text{there exists } \{t_n\}_{n\in\mathbb{N}} : t_n \downarrow 0, \gamma + t_n\bar{\gamma} \in \mathcal{S}, n \in \mathbb{N}\right\}.$$

Now we can state the main result of this paper, the Fréchet differentiability of the cost functional with respect to γ. This can be shown if the parameter ν is large enough to ensure Lipschitz continuity of the velocity vectors $\hat{\mathbf{u}}_\gamma$ with respect to γ, compare Theorem 4.1.

Theorem 5.2 *Let $\mathbf{f} \in L^\infty(\hat{\Omega})^2, \mathbf{f}_\gamma := \mathbf{f}|_{\Omega_\gamma}$ for $\gamma \in \mathcal{S}$, and $\nu > \nu_1$ as is Theorem 4.1. Then $\mathcal{J}$ is Fréchet differentiable with respect to γ, and the derivative in γ satisfies*

$$D_\gamma \mathcal{J}(\gamma)\bar{\gamma} = \frac{1}{\nu}\int_I \left[g_\gamma\big(x,\gamma(x)\big)\cdot\chi_\gamma\big(x,\gamma(x)\big) - p_\gamma\big(x,\gamma(x)\big)\mu_\gamma\big(x,\gamma(x)\big)\right]\bar{\gamma}(x)\,dx \quad (7)$$

for all $\bar{\gamma} \in \mathcal{S}'$.

Proof. Using the variational forms of the Navier-Stokes and the adjoint equations with appropriate test functions it can be shown that the directional derivative satisfies (7). Clearly the right-hand side of (7) is a bounded linear operator on $\mathcal{S}'$ and therefore the Gateaux differential. Finally it can be shown that the mapping $\gamma \mapsto D_\gamma\mathcal{J}(\gamma)$ is continuous which proves that it is the Fréchet derivative. For the details see [7, Th. 3.10]. □

We want to emphasize a second advantage of the embedding domain method:

Remark 5.3 *The Lagrange multipliers g_γ, χ_γ introduced by the embedding domain method allow to compute the derivative of $\mathcal{J}$ as the one-dimensional integral in (7) without computing normal derivatives of the velocities. In the discrete case (e.g. for finite element basis functions) the integral can be computed exactly by a simple quadrature rule.*

The assumption on ν can be generalized in the following way:

Remark 5.4 *The result of the last Theorem remains valid if the assumption $\nu > \nu_1$ is replaced by Hölder continuity of the mapping $\gamma \mapsto \hat{\mathbf{u}}_\gamma$ in the L^4 norm with an exponent $p > \frac{1}{2}$, i.e. there exist $L > 0, p > \frac{1}{2}$ independent of $\bar{\gamma}, \gamma \in \mathcal{S}$ such that*

$$\|\hat{\mathbf{u}}_{\bar{\gamma}} - \hat{\mathbf{u}}_\gamma\|_{L^4(\hat{\Omega})^2} \quad \leq \quad L\|\bar{\gamma} - \gamma\|^p_{L^\infty(I)} \qquad \text{for all } \bar{\gamma}, \gamma \in \mathcal{S}.$$

6 Numerical Methods and Results

The numerical examples presented below were computed using the formula (7) for the derivative. The state and adjoint equations were discretized using stabilized finite elements (see e.g. [1]). The discretized Navier-Stokes equations were solved using a semi-implicit scheme presented in [3]. The linear systems were solved using sparse direct solvers. As optimization routine we used a SQP method.

As example we studied a flow separation problem in a driven cavity at a Reynolds number of $Re = \frac{1}{\nu} = 500$: The computational domain is the unit square where a part of the right lateral boundary is variable. On the top boundary we have a constant horizontal velocity which is positive in y direction, on the other boundaries we have zero velocity. If the right lateral boundary is a straight line at $x = 1$ the resulting flow shows one big vortex turning clockwise.

Aim of the optimization was to split this vortex into two which are separated by a horizontal line at $y = 0.5$. To achieve this we chose this line as observation region Ω_C and minimized the cost functional

$$\mathcal{J}(\gamma) \quad := \quad \int_{\Omega_C} \|\mathbf{u}_\gamma\|_2^2 \, dx$$

The right wall between $y = 0.125$ and 0.75 was allowed to vary in the range $x \in [0.75, 1)$ with six control parameters. As start curve we used a straight line near the right lateral wall. We used no regularization.

The optimization using the gradient computed with the formula (7) reduced the cost functional to less than one percent in a few iterations. The achieved solution was "optimal" in the sense that a comparison with the solution when Γ_γ was a straight line at $x = 0.75$ showed the superiority of the optimized solution. For detailed results see Table 1 and Fig. 2 and 3.

Acknowledgements

The author would like to thank Prof. K.-H. Hoffmann, Prof. G. Leugering and Prof. F. Tröltzsch for the invitation to the Chemnitz conference, and Prof. K. Kunisch for the guidance during the author's work on his PhD thesis.

It.	Ev.	$\mathcal{J}$	$\gamma(0.125)$	$\gamma(0.25)$	$\gamma(0.375)$	$\gamma(0.5)$	$\gamma(0.625)$	$\gamma(0.75)$
	1	1.961e-04	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
1	2	1.4893e-06	0.9999	0.9999	0.9999	0.9999	0.9999	0.7500
2	3	1.211e-06	0.9999	0.9999	0.9999	0.7822	0.7500	0.7500
3	5	1.134e-06	0.9999	0.8750	0.8750	0.7661	0.7500	0.7500
4	7	1.1296e-06	0.9812	0.8758	0.8742	0.7622	0.7512	0.7500
...								
7	52	1.1296e-06	0.9812	0.8758	0.8742	0.7622	0.7512	0.7500
		1.2472e-06	0.7500	0.7500	0.7500	0.7500	0.7500	0.7500

Table 1: Convergence behaviour. In the last line the result with Γ_γ being a straight line at 0.75 is given. The cost functional value obtained by the optimization is still ten percent better. Most of the function evaluations (Ev.) were needed in the line search in the last iterations when no significant cost functional reduction was achieved.

References

[1] L.P. Franca, S.L. Frey: Stabilized Finite Element Methods: II. The Incompressible Navier-Stokes Equations, *Comp. Meth. in Appl. Mech. Eng.* **99** 1992, pp. 209-233.

[2] G.P. Galdi: *An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol.1* Springer New York 1994.

[3] V. Girault, P.-A. Raviart: *Finite Element Methods for Navier-Stokes Equations*, Springer Series in Comp. Math., Berlin New York 1986.

[4] M.D. Gunzburger, L. Hou, T.P. Svobodny: Analysis and Finite Element Approximation of Optimal Control Problems for the Stationary Navier-Stokes Equations with Distributed and Neumann Control, *Math. of Comp.* **57**, No. 195, 123-151 (1991).

[5] R.B. Kellogg, J.E. Osborn: A Regularity Result for the Stokes Problem in a Convex Polygon, *Journal on Functional Analysis*, **21**, pp. 397-431, 1976.

[6] K. Kunisch, G. Peichl: Shape Optimization for Mixed Boundary Value Problems Based on an Embedding Domain Method, to appear in *Dynamics of Continuous, Discrete and Impulsive Systems*.

[7] T. Slawig: *Domain Optimization for the stationary Stokes and Navier-Stokes Equations by an Embedding Domain Technique*, Shaker Aachen 1998.

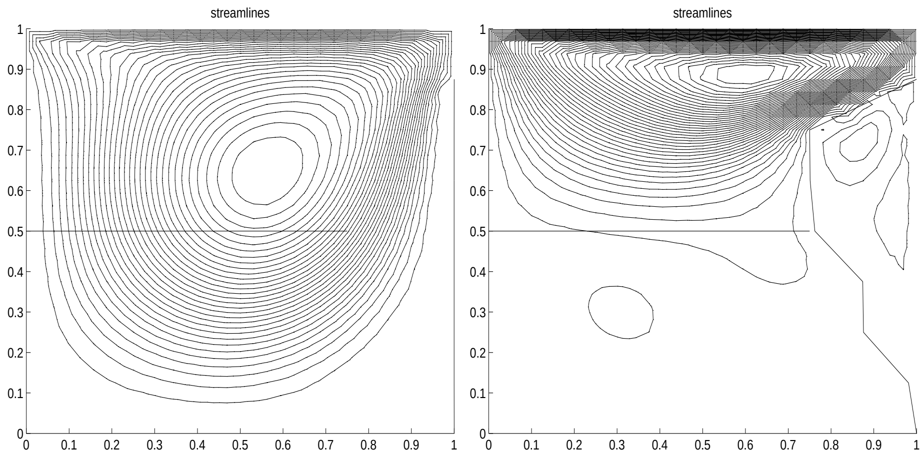

Figure 2: Streamlines for start (left) and solution curve (right). The horizontal line marks the observation region Ω_C.

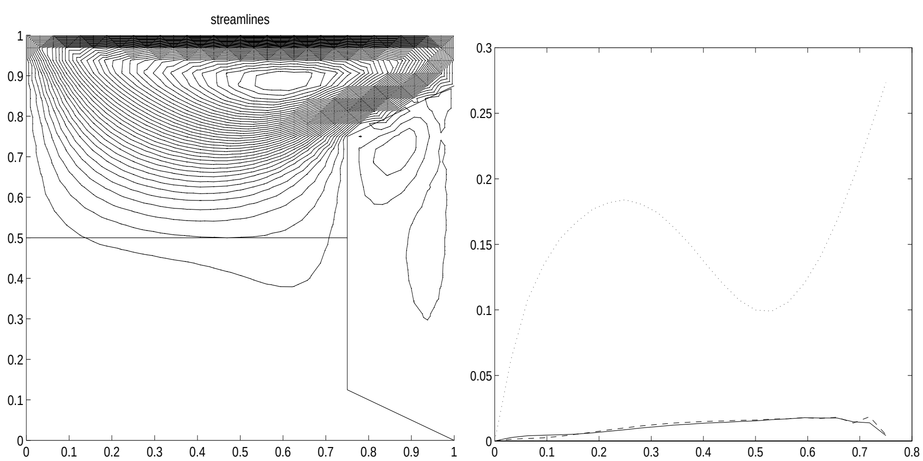

Figure 3: Left: Streamlines for straight line at $x = 0.75$, compare Table 1. Right: Euclidean norm of velocity vector on Ω_C. Dotted: start curve, solid: solution, dashed: straight line at 0.75.