Quadratic growth rates of geodesics on F structures with a link to polygonal billiards

vorgelegt von
Diplom-Mathematiker
Martin Schmoll
aus Günsterode

Vom Fachbereich 3 - Mathematik
der Technischen Universität Berlin
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften
- Dr.-rer.-nat.-

genehmigte Dissertation

1. Berichter: Prof. Dr. Ruedi Seiler
2. Berichter: Priv. Doz. Dr. Tyll Krüger
3. Berichter: Priv. Doz. Dr. Jörg Schmeling
Zus. Berichter: Prof. PhD. Serge Troubetzkoy

Tag der wissenschaftlichen Aussprache: 23.2.2000

Berlin 2000

Acknowledgments

During the preparation of this work, I have visited and worked at many places, thus there are a lot of persons and institutions to which I owe thanks. I would like to do that chronologically. First of all I have to thank Dr. habil. Tyll Krüger, Prof. Serge Troubetzkoy and Prof. Philipe Blanchard for making it possible for me to join the DFG project “Ergodic theory and analysis”. During the preparation of this thesis it was always a pleasure to work together with these three persons. Even at the beginning, during the long time when it seemed nothing would work out well, they always gave me the feeling that it is worthwhile to continue. I thank the department of mathematics at the university of Birmingham in Alabama for the hospitality during my visit in the fall of 1997.[1] The first half of the year 1998 I could work at the beautiful Max Planck Institute for mathematics in the sciences in Leipzig thanks to Prof. Stephan Müller, Prof. Jürgen Jost and Prof. Eberhardt Zeidler. The substantial part of the work was created at the CNRS in Luminy (Marseille). Starting in march 1999 I had the opportunity to work at the CPT in the group of Prof. Ricardo Lima. Discussions with Pascal Hubert led me to the right way to solve some problems, Alex Eskin explained me his ideas and how to use Ratners theorem. It is a pleasure for me to thank both of them at this place. Serge Troubetzkoy has made the effort to read the English text for correction and has pointed me to some unprecise mathematical statements. And last but not least I have to thank my mother to keep all the years something for me, which is worth to be called a “home”.

[1] This journey was made possible by the NATO grant CRG 941044

Introduction

The study of the billiard motion in a n-gon is an old subject of dynamical systems. In this work we study questions related to the asymptotic growth rate of the prime geodesic spectrum in a polygonal billiard. In the case of a marked Torus (arising as phase surface of the billiard in a rectangle) we give explicit results. We begin by sketching the objects which are necessary to state the results. The phase space of a polygonal billiard is the product space of the n-gon with S^1 (modulo some identifications on the boundaries). This phase space splits in isometric invariant surfaces and the restriction of the flow to these is linear. The flow on the invariant surfaces can be viewed as the geodesic (directional) flow of the flat metric induced from the n-gon. There are two points one has to observe: the first is that the closed geodesics arise in families of geodesics of equal length. Geometrically these are cylinders and their boundary consists of trajectories each of which contains at least one singular point. Singular points can be viewed as copies of the vertices of the n-gon, in a geometric description singular points are conical singularities of the flat metric, i.e. points around which the total angle is $2\pi n$, where $n > 1$. Trajectories connecting these singular points are called saddle connections. Their importance is seen in the fact that each maximal cylinder of periodic trajectories is bounded by two or more saddle connections, which have at most the length of the periodic trajectories in the cylinder. The second point causes really difficult problems.

The applicability of useful methods to study the dynamical behavior of a polygonal billiard divides them into two classes: either the flow invariant surfaces are all compact or they are all non compact. Compactness or non compactness of these surfaces is seen to be equivalent to the rationality of all the (inner) angles of the n-gon (modulo π) or the non rationality of one of them. One of the (few) general results which holds for all n-gons is contained in the work "The growth rate for the number of singular and periodic orbits for a polygonal billiard" of Katok [Kat]. He proves that the the number of saddle connections shorter than a given length T growth sub-exponentially in T. By the above remark the growth rate for closed geodesics is less or equal to the growth rate of saddle connections, but in general it is not even clear that a single closed geodesic exists at all.

For rational n-gons the flow on the invariant surfaces could be understood as flow of the horizontal foliation on a compact oriented surface S together with some special complex atlas $\mathcal{A}$: all the coordinate changes are of the form $z \longmapsto z + c$ with $c \in \mathbb{C}$. These are just translations, therefore they are named translation structures (or surfaces). By means of this atlas we have have a complex structure and we can pull back dz and $dz \otimes d\overline{z}$ to get a holomorphic differential and a flat Riemann metric with conic singularities on S. The metric induces a volume which can be also seen as pullback of $\frac{i}{2} dz \wedge d\overline{z}$ with respect to the Atlas $\mathcal{A}$. The horizontal flow is defined to be the flow of the vectorfield X_h given in $\mathcal{A}$ coordinates by $X_h(x) = (1, 0)$ it is the geodesic flow with respect to the induced metric in the positive real direction on $(S, \mathcal{A})$. Also there is a

$SL_2(\mathbb{R})$ action on that structures given by the post composition of the chart maps by an element $A \in SL_2(\mathbb{R})$ viewed as linear map on $\mathbb{R}^2$.

Most of the statements which can be proved for translation structures are true for structures defined by maximal atlases $\mathcal{A}$ where coordinate changes of the form $z \longmapsto \pm z + c$ are allowed. These are called F structures. The differences to translation structures are that the vectorfield X_h is defined only locally (the integral leaves might not be orientable) and the $SL_2(\mathbb{R})$ action has to be replaced by a $PSL_2(\mathbb{R})$ action. Clearly a translation structure defines an F structure, thus we use this terminology from now on. Under weak assumptions F structures of genus g have well studied finite dimensional moduli spaces which are certain subspaces of the cotangent space to the moduli space $\mathcal{M}_g$ of genus g surfaces. The unit cotangent bundle $\mathcal{QD}(g) := U^*\mathcal{M}_g$ can be compactified in a way that the boundary consists of F structures of lower genus. The surfaces in the boundary are not necessarily connected. They can have nodes and punctures which arise from pinching closed curves. By using the $SL_2(\mathbb{R})$ action one is able to move a given F structure to a boundary point without destroying the existence of closed geodesics. It can be shown that one can always find limit points with respect to this $SL_2(\mathbb{R})$ action which contain some special F structures: the torus, the sphere with four punctures or some "degenerated" F structure. It is well known that the first two have a lot of periodic trajectories. If one approximates a "degenerated" F structure one already has a nonsingular closed geodesic, on the approximating surfaces. Precisely on the line of this idea H. Masur [Msr86] was able to prove

Theorem 0.1 (H.Masur) *For every F structure u there is a dense subset of directions in $\theta \in S^1$, such that the horizontal flow on $\exp(i\theta)u$ contains one or more cylinders of periodic trajectories.*

and furthermore in [BGKT] it is proved

Theorem 0.2 (M. Boshernitzan, G. Galperin, T. Krüger, S. Troubetzkoy) *The periodic orbits of the billiard in a rational n-gon are dense in the phase space.*

The arguments of their proof can be carried over verbatim to the general case of F structures. Thus in the tangent space to every F surface there is a dense set of points, such that a geodesic hitting that point is closed.

We can associate a pair of vectors $\pm v_{\mathcal{C}} = l(\mathcal{C})\exp(i\theta) \in \mathbb{C} \cong \mathbb{R}^2$ to every closed cylinder $\mathcal{C}$ (saddle connections as well) of the horizontal flow in $\exp(i\theta)u$ which has the length $l(\mathcal{C})$ with respect to the natural distance on u. In this way for every F structure u there are sets of vectors (with multiplicities) $\mathcal{V}_{PO/SC}(u)$ associated to closed orbits or saddle connections. For $u \in \mathcal{QD}(g)$ let

$$N_{PO}(u,T) := |\{v \in \mathcal{V}_{PO}(u) : |v| < T\}|$$

be the growth function of $\mathcal{V}_{PO}(u)$ and $N_{SC}(u,T)$ the analogously defined function for saddle connections on u. Again Masur [Msr88, Msr90] proved the following Tchebychev type theorem:

Theorem 0.3 (H. Masur) *For every $u \in \mathcal{QD}(g)$ exist positive constants $c^{min}_{SC/PO}(u)$, $c^{max}_{SC/PO}(u)$ such that for $T > T_u$*

$$c^{min}_{SC/PO}(u)T^2 < N_{PO/SC}(u,T) < c^{max}_{SC/PO}(u)T^2.$$

In particular the billiard in a rational n-gon has a dense set of directions with periodic trajectories, moreover their number grows quadratically with the length. The natural question if or when this Tchebychev theorem can be replaced by an exact quadratic counting theorem is in general open and one of the central motivations of this work. Under strong assumptions on the geometry of an F structure u Veech [Vch89, Vch92] was able to prove that $\lim_{T\to\infty} \frac{N_{PO/SC}(u,T)}{T^2}$ exists. Namely this is the case if there exists "a lot" of affine maps of u. An affine map ϕ on u is a map which is locally in the charts of the natural atlas $\mathcal{A}_u$ affine. The set of affine maps of u is a group under composition of maps. We denote that group by $Aff(u)$. The maps $\phi \in Aff(u)$ are differentiable with constant derivative defined modulo ± 1. It can be proved that "taking derivatives" defines a homomorphism $Aff(u) \xrightarrow{d} PSL_2(\mathbb{R})$ and the image of d is called the Veech group $V(u)$ of u. We say u has "a lot" of affine maps if the Veech group $V(u)$ is a lattice in $PSL_2(\mathbb{R})$. F structures with this property are also called Veech surfaces. Veech has shown [Vch89]:

Theorem 0.4 (W. Veech) *Let u be an F structure and $V(u)$ a lattice, then*

$$\lim_{T\to\infty} \frac{N_{PO/SC}(u,T)}{T^2} = c_{PO/SC}(u).$$

The constant $c_{PO}(u)$ is computable by knowledge of the length and width of the maximal cylinders of closed geodesics in finitely many directions. The number of directions equals the number of cusps in $V(u)$. Veech was able to prove that the F structures associated to the billiard in a regular n-gon G^n_{reg} have a lattice Veech group $V(G^n_{reg})$. He computes the asymptotic constants $c_{PO}(G^n_{reg})$ in [Vch92]. After this breakthrough other papers about the subject occur, some new examples of Veech surfaces are found by Vorobetz [Vrb96b]. That it is not easy to find Veech billiards might be seen in the result Kenyon and Smillie [KeSm98], they have proven recently

Theorem 0.5 (R. Kenyon, J. Smillie) *Let T be an acute non isoscele rational triangle with angles α, β and γ are of the form $\pi p_1/q$, $\pi p_2/q$ and $\pi p_3/q$ with $q \leq 10000$. Then T is a lattice polygon if and only if (α, β, γ) is one of the cases:*

$$\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}\right) \text{ or } \left(\frac{\pi}{5}, \frac{\pi}{3}, \frac{7\pi}{15}\right) \text{ or } \left(\frac{2\pi}{9}, \frac{\pi}{3}, \frac{4\pi}{9}\right).$$

Because there are approximately 100.000.000.000 triangles which fulfill $q \leq 10000$ they conjecture that the restriction on q is not necessary.

Coverings of Veech surfaces are again Veech surfaces [GutJdg97, Vrb96b], which has the consequence that every moduli space of F structures contains a dense set of Veech surfaces with Veech groups commensurable to $PSL_2(\mathbb{Z})$. On the other hand by results of Veech [Vch86] and Masur [Msr86] about the ergodicity of the geodesic flow in the moduli spaces the generic point cannot have a big Veech group. Thus to find bigger sets where the Tchebychev theorem can be improved to a limit theorem one has to use another idea. Veech [Vch98] suggests one way by using a general measure theoretical Ansatz, which we describe roughly now. (In the following we restrict our statement to periodic orbits and omit the index of $\mathcal{V}(u)$, but the statements are true for saddle connections as well). Veech observed [Vch98] that counting quadratic growth rates of periodic orbits on u is the same as to evaluate the integral (limits)

$$\lim_{t\to\infty} \int_0^{2\pi} \sum_{v\in\mathcal{V}(u)} f(a_t \exp(i\theta)u)d\theta \tag{1}$$

with

$$a_t = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \in SL_2(\mathbb{R}), \qquad f \in C_0^\infty(\mathbb{R}_+^2) \text{ and } \mathbb{R}_+^2 := \mathbb{R}\times\mathbb{R}_+.$$

The crucial point is that it is not clear that the limit exists pointwise, but as a function of u it might be regular enough to be evaluated as an expectational value with respect to a measure on some spaces of F structures. To make this precise, the integrand above defines a transformation

$$\hat{f}(u) := \sum_{v\in\mathcal{V}(u)} f(v)$$

of functions on $\mathbb{R}^2$ to functions on the space $\mathcal{M}$ of F structures. Under the hypothesis that

(A) there is a $SL_2(\mathbb{R})$ action on $\mathcal{M}$ which is equivariant with respect to $\mathcal{V}$: i.e. $g\mathcal{V}(u) = \mathcal{V}(gu)$, $\forall g \in SL_2(\mathbb{R})$ and

(B) there exists an $SL_2(\mathbb{R})$ invariant ergodic probability measure μ on $\mathcal{M}$.

(C$_\mu$) $\hat{f} \in L^{1+\epsilon}(\mathcal{M},\ \mu)$.

Veech proves (in more generality) the following

Theorem 0.6 (Siegel Veech formula) *Suppose $(\mathcal{M},\ \mu)$ is a space of F structures with the above properties. Then there exists a constant $c_{\mathcal{V},\mu}$ such that*

$$\int_{\mathcal{M}} \hat{f}(u)d\mu(u) = c_{\mathcal{V},\mu} \int_{\mathbb{R}^2} f(x,y)dxdy \quad \textit{for any } f \in C_0^\infty(\mathbb{R}_+^2). \tag{2}$$

In fact, to prove this one only needs $\hat{f} \in L^1(\mathcal{M},\ \mu)$. To built a bridge from the theorem to the above limit, one uses property (B), to apply an ergodic theorem in the spirit of

$$\lim_{t\to\infty} \int_0^{2\pi} \hat{f}(a_t \exp(i\theta)u)d\theta = \int_{\mathcal{M}} \hat{f}(u)d\mu(u) \quad \mu \text{ a.e. } .$$

Setting $f := \chi_R$ to be the characteristic function of some rectangle $R \subset \mathbb{R}^2$ and demanding further:
(**D**) For every $u \in \mathcal{M}$ there is a constant $c_{\mathcal{V}}^{max}(u) < \infty$ such that $N_{\mathcal{V}}(v,T) < c_{\mathcal{V}}^{max}(u)T^2$ for all v in an open neighborhood $U(u)$ of u.
Eskin and Masur ([EskMsr98] Proposition 3.2) were able to prove

Theorem 0.7 *Suppose $(\mathcal{M},\mu)$, $\mathcal{V}(\cdot)$ have the properties (A), (B), (C_μ) and (D), then*

$$\lim_{T\to\infty} \frac{N_{\mathcal{V}}(u,T)}{T^2} = \pi c_{\mathcal{V},\mu} \quad \mu \text{ a.e. on } \mathcal{M}$$

where $c_{\mathcal{V},\mu}$ is as in the Theorem 0.6.

The theorem applies to spaces $\mathcal{Q}^1(g,P) \subset \mathcal{QD}^1(g,n)$ of F structures with fixed singularity pattern P. These possess an ergodic (with respect to a_t) $SL_2(\mathbb{R})$ invariant probability measure μ_0 (cf. [Vch86, Vch90])(topologically the $\mathcal{Q}^1(g,P)$ give a stratification of $\mathcal{QD}^1(g,n)$ if (m,P) varies). It is naturally to ask the questions:
1. Can one compute the constants $c_{\mathcal{V},\mu}$?
2. Do there exist other $SL_2(\mathbb{R})$ invariant, ergodic probability measures on $\mathcal{Q}^1(g,P)$?
3. Is $\limsup_{t\to\infty} \frac{N_{\mathcal{V}}(u,T)}{T^2} \le \pi c_{\mathcal{V},\mu}$ for all $u \in \mathcal{Q}^1(g,P)$?
If the third statement is true and the computations of the $c_{\mathcal{V},\mu}$ could be done, then this would imply some control of the growth rates of saddle connections on non rational polygonal billiards. How good the control is depends on the behavior of the constants $c_{\mathcal{V},\mu}$ as functions on the genus g of u. Question two and three are answered in the fourth chapter in a positive way for the translation tori with n marked points. The second question also has a well known positive answer, the $SL_2(\mathbb{R})$ orbits $(SL_2(\mathbb{R}).u \cong SL_2(\mathbb{R})/V(u),\mu)$ of Veech surfaces u together with image of the Haar measure μ in their moduli spaces give examples. In these cases the constants $c_{\mathcal{V},\mu}$ reflecting the growth rates pointwise, which in turn forces the next question:
4. Are there moduli or parameter spaces of F structures where the limits (1) exist pointwise?

A. Eskin [Esk98] has observed that $SL_2(\mathbb{R})$ invariant parameter spaces of branched coverings of translation tori are examples for homogeneous spaces of F structures. He uses Ratners classification theorem for ergodic measures on homogeneous spaces to obtain pointwise limits of the growth rates in this spaces. The examples in chapter four, the n marked two tori are taken out of the set of homogeneous spaces which are spaces of torus coverings. But in chapter four the calculations are done without using Ratners theorem, just by using more or less elementary formalisms. Nevertheless explicit results on the growth constants as function of the parameter space are obtained. If we take the two marked torus $\mathbb{T}^2_x := \mathbb{R}^2/\mathbb{Z}^2$ with marked points 0 and x, then the parameter space (without taking care on the $SL_2(\mathbb{R})$ action) is simply the torus $\mathbb{T}^2$ itself: x can be any point on the torus, except the point 0. The later does not affect the results, so $x = 0$ is viewed as an allowed constellation and then $\mathbb{T}^2$ is the parameter space. In the case of saddle connections (SC) on two marked tori, for the function $x \longmapsto c_{SC}(x)$ it turns out that:

$$\begin{aligned} \left[\frac{p_1}{n}, \frac{p_2}{n}\right] &\longmapsto \frac{6}{\pi}\left(1 + \prod_{p|n \ prime}\left(1 - \frac{1}{p^2}\right)^{-1} \sum_{\gcd(i,n)=1}\left(\frac{1}{i^2} - \frac{1}{n^2}\right)\right) \quad \text{on } \mathbb{Q}^2 \\ [x,y] &\longmapsto \frac{6}{\pi} + \pi \quad \text{for } (x,y) \notin \mathbb{Q}^2 \end{aligned} \tag{3}$$

(where we assume $\gcd(p_1, p_2, n) = 1$). This function is continuous at non rational points, thus the above formula approaches $\frac{6}{\pi} + \pi$ as $\left(\frac{p_1}{n}, \frac{p_2}{n}\right)$ converges to a non rational number. This continuity can be proven by a direct estimate, it is not restricted to 2 markings. It also holds for general n markings with other limiting constants and sets of course. For any n the limiting constants as functions of the marking are maximal exactely at the points of continuity. In the case of rational two markings the Veech groups are calculated, too. A simple argument shows that every Veech surface coming from a rational marking is isomorphic to one of the family $F = \left\{\mathbb{T}^2_{[\frac{1}{n},0]} : \ n \in \mathbb{N}\right\}$. Therefore all the Veech groups of rationally marked tori are isomorphic to one of the Veech groups $V_{[\frac{1}{n},0]}$ associated to the family of surfaces F. The asymptotic formula of Veech and an asymptotic counting of lattice points in the plane is used to compute the index $\left[V_{[\frac{1}{n},0]} : SL_2(\mathbb{Z})\right]$. Alternatively we compute the same index by using the isotropy subgroup $\mathit{Aff}_x(u) \subset \mathit{Aff}(u)$ of a given point in $x \in u$. With the help of the description of the Veech groups for rational 2 markings one is able to rediscover the index of some classical congruence subgroups of $SL_2(\mathbb{R})$.

In the case of general n markings one of the main objects was to understand if there is something in between purely rational markings and non rational markings which has not a maximal growth rate. This question also could be answered with "yes" (see Theorem 4.16 and the preparatory Lemma 4.2). Unfortunately the elementary methods discovered can not (simply) taken over to prove something on families of branched coverings of tori. But the use of Ratners theorem allows one to compute the Siegel Veech constants for branched coverings of tori (see the forthcoming paper [EMS]).

Description of the chapters

Chapter 1

We introduce F structures, translation surfaces and their basic properties. In the second part we describe the moduli and Teichmüller spaces of F structures, as well as certain strata $\mathcal{QD}(g,P)$ of them given by fixing the order of the singular points. We present an outline of the construction of a manifold structure for each connected component $\mathcal{Q}(g,P)$ of $\mathcal{QD}(g,P)$ following Veechs paper [Vch90]. Our presentation is precise up to some calculations which can be found in Veechs paper, the aim is making the construction more transparent. We use the manifold structure to establish the local stability of saddle connections and cylinders of closed geodesics in $\mathcal{Q}(g,P)$.

Chapter 2

We define affine maps, the affine group $Aff(u)$ and the Veech group $V(u)$ with respect to a F structure u. We prove all (known) general properties of $Aff(u)$ and $V(u)$, as well as show geometrical consequences, if $V(u)$ contains a parabolic element. In the second part we study the relation of $V(u)$ and $V(v)$ given a covering map $u \xrightarrow{\pi} v$. We generalize a result of Gutkin and Judge stating that if a translation structure u has a Veech group commensurable to $SL_2(\mathbb{Z})$ then there exists a translation map $u \xrightarrow{\phi} \mathbb{T}^2$ to the torus eventually branched over one point. We show: if u is a half translation structure and $V(u)$ is commensurable to $PSL_2(\mathbb{Z})$, then there exists a flat map $u \xrightarrow{\pi} \mathbb{P}^2$ to the pillow $\mathbb{P}^2$. Metrically the pillow is the surface of genus zero obtained by gluing together two copies of a rectangle along their boundaries. It has four metric singularities around which the total angle is π. From the existence of a v_{min} it follows, that the Veech group of a covering of v_{min} is commensurable to $V(v_{min})$. Finally we explain the counting formula of Veech for F structures u with a lattice Veech group $V(u)$.

Chapter 3

We explain the method of Eskin and Masur [EskMsr98] founded on a new idea of Veech [Vch98] to prove that almost every half translation surface in the moduli spaces $\mathcal{Q}(g,n,P)$ has asymptotic quadratic growth rates for non singular closed geodesics, saddle connections and other objects which fulfill some necessary properties. “Almost everywhere” has to be understood with respect to some measure μ which is ergodic with respect to the $SL_2(\mathbb{R})$ operation on $\mathcal{Q}(g,P)$, in particular it is true for the “Liouville measure” μ_0 introduced and studied by Masur [Msr82] and Veech [Vch90]. We present the idea of Eskin how to use these methods for homogeneous spaces of F structures, to obtain not only almost everywhere but pointwise results.

Chapter 4

The behavior of the quadratic growth rates of saddle connections and closed geodesics on (homogeneous) spaces parametrizing F structures of n marked tori is described. These spaces are themselves tori (eventually of higher dimension) and they parameterize simply different constellations of the marked points. By the method of counting lattice points in the plane limit quadratic growth rates for every marking are established. We compute the Veech groups of the rationally two marked tori and use the results from counting lattice points to calculate their indices in $SL_2(\mathbb{Z})$. As a byproduct the index of the classical congruence groups $\Gamma_1(n)$ in $SL_2(\mathbb{Z})$ can be computed. We give formulas for the asymptotic growth constants on two marked tori. Furthermore the sets where the growth constants are maximal are described and it is proved that these have measure one (with respect to the Lebesgue measure).

Abstract

This work is centered around the problem to find growth constants for the growth rates of certain kinds of geodesics, on translation surfaces or more generally F structures. This kind of problem is motivated by the study of polygonal billiards. We develop the necessary formal language by using quadratic differentials and explain in which way a polygonal billiard gives rise to an F structure. Since in all known attempts to attack the counting problems Teichmüller- and Modulispaces of F structures will occur we explain their natural manifold structure. Every F structure has a symmetry group, called the affine group and associated to it the Veech group. The properties of the affine and the Veech group are collected. The importance of the affine group is seen in the fact that one is able to parameterize all geodesics on an F structure and finally compute their quadratic asymptotic if the affine group of an F structure is big. Eskin and Masur, inspired by an idea of W. Veech, recently found a way to prove the existence of asymptotic growth constants for bigger classes of F structures. We present an outline of this theory together with a description how to use Ratners theorem if one has a homogeneous space parameterizing F structures. Finally we compute quadratic growth constants of saddle connections and closed geodesics on two marked tori as functions of the relative marking. It turns out that the index of the associated Veech groups in SL(2,Z) is the most important ingredient to evaluate the growth constants, it is computed in two different ways. Some properties of these functions are shown in the more general context of spaces of n-marked tori.

Zusammenfassung

Die Absicht der vorliegenden Arbeit ist es darzustellen, wie quadratische Wachstumskonstanten für (gewisse Arten von) Geodäten auf F Strukturen berechnet werden können. F Strukturen sind grob gesprochen Flächen zusammen mit einem quadratischen Differential. Sie stellen einen allgemeinen formalen Rahmen zum Studium der polygonalen Billiards dar, der genaue Zusammenhang wird in der Arbeit beschrieben. Die Eigenschaften von F Strukturen im Hinblick auf das Studium ihrer Geodäten, ebenso wie die Mannigfaltigkeiten-Struktur ihrer Modulräume, werden besprochen. Der $SL_2(\mathbb{R})$-Orbit jeder F Struktur besitzt eine Symmetriegruppe, die sogenannte affine Gruppe. Aus der affinen Gruppe lässt sich eine weitere Gruppe, die Veech-Gruppe, ableiten. W. Veech konnte für F Strukturen, deren Veech-Gruppe ein Gitter ist die quadratischen, asymptotischen Konstanten für Wachstumsraten von Geodäten, berechnen. Diese Theorie wird näher beleuchtet und die bekannten Resultate zur Struktur der affinen- sowie der Veech Gruppe werden bewiesen. und für Anwendungen auf F Strukturen werden einige Verfeinerungen angegeben. Es folgt eine Darstellung der von Eskin, Masur und Veech entwickelten Idee, die fast sichere Existenz von quadratischen Wachstumskonstanten in Modulräumen von F Strukturen unter Benutzung von Siegel-Veech Maßen zu zeigen. Sind die Parameterräume der F Strukturen $SL_2(\mathbb{R})$-invariante homogene Räume, dann liefert Ratners Klassifikation von ergodischen Maßen sogar punktweise Resultate. Diese Ergebnisse können auf gewisse polygonale Billiards angewendet werden. Der abschließende Teil der Arbeit besteht darin, die quadratischen Wachstumsraten für markierte Tori zu berechnen, beziehungsweise deren Existenz und Verhalten in Abhängigkeit von der Markierung zu verstehen. Enthalten in diesem Abschnitt sind auch Beschreibungen der affinen- und der Veech-Gruppe von zweifach rational markierten Tori. Der Index dieser Veech-Gruppen in $SL_2(\mathbb{Z})$ spielt eine wesentliche Rolle bei der Berchnung der asymptotischen Konstanten, er wird auf zwei verschiedene Weisen berechnet. Außerdem werden die Wachstumskonstanten von Sattel-Verbindungen und periodischen Bahnen als Funktionen der relativen Markierung auf dem Parameterraum zweifach markierter Tori angegeben.

Contents

1 F structures on surfaces **13**

1.1 Definition and basic properties of F structures 13

1.2 Geodesics on F structures 17

1.3 The translation structure of a polygonal billiard 18

1.4 A manifold structure on moduli spaces of F surfaces 21

1.5 Deformation invariance of periodic families in $\mathcal{Q}(g, P)$ 28

2 Quadratic growth rates on Veech surfaces **30**

2.1 The affine group *Aff(u)* 30

2.2 Affine coverings and the Veech group 36

2.3 Counting on Veech surfaces 40

3 Computing asymptotic growth rates with Siegel measures **45**

4 Marked tori **54**

4.1 Notation and preparation 54

4.2 Quadratic growth rates of lattices and point distributions in $\mathbb{R}^2$ 57

4.3 The Veech group of $\mathbb{T}^2_{(\frac{p_1}{q_1}, \frac{p_2}{q_2})}$ and its index in $SL_2(\mathbb{Z})$ 62

4.4 Quadratic growth rates and constants on marked tori 68

4.4.1 The 2 marked torus $\mathbb{T}^2_x$ 68

4.4.2 The general case 71

4.4.3 Branched coverings of marked tori 74

Chapter 1

F structures on surfaces

1.1 Definition and basic properties of F structures

We start to develop the basic geometric objects in the study of rational polygonal billiards. For this we always assume S_g is a compact oriented surface of genus g and $O_n \subset S_g$ is a set of n points. $S_{g,n}$ is S_n where the set O_n is removed. If we do not say it explicitly S_g is assumed to have no boundary. We define

Definition 1.1 (F structure) *A F structure u is a surface $S_{g,n}$ together with a maximal complex atlas $\mathcal{A}$ that has only coordinate changes of the form $z \longmapsto \pm z + c$ with $c \in \mathbb{C}$. If $\mathcal{A}$ can be reduced to contain only coordinate changes $z \longmapsto z + c$ then the structure is called a translation structure, if not we will say it is a half translation structure. u is called positive, if it induces the given orientation on $S_{g,n}$.*

Remark: The name F structure was introduced by W. Veech. Maybe the "F" in the name referres to "flat" since, as we will see in a moment, an F structure induces a natural flat metric. We denote the set of positive F structures on $S_{g,n}$ by $\Omega^+(S_{g,n})$.

an F structure induces some canonical geometric objects on $S_{g,n}$: by means of charts from $\mathcal{A}$ we can pull back all tensors on $\mathbb{C}$ which are invariant under the complex linear map $-id$. These are

- a Riemannian flat metric g_u as the pullback of $dz \otimes d\overline{z}$ which in turn induces
- a volume form $dvol_u$. This can also be viewed as the pullback of $\frac{i}{2} dz \wedge d\overline{z}$.
- A complex structure $\mathcal{J}_u$ on $S_{g,n}$.
- One can also pull back the quadratic differential dz^2 to obtain a globally non vanishing and holomorphic (w.r.t. $\mathcal{J}_u$) quadratic differential q_u on $S_{g,n}$. Moreover in the case of a translation structure dz is pulled back to induce a non vanishing holomorphic differential α on $S_{g,n}$.

- q_u or α_u induce for every $\theta \in S^1$ foliations $\mathcal{F}_\theta(u)$ defined by the solutions of the equation $Re(\exp(i\theta)q)v_z = 0$ or $Im(\exp(i\theta)\alpha)v_z = 0$ respectively with $v_z \in T_zS_{g,n}$. The special cases $\theta = \frac{\pi}{2}$ and $\theta = 0$ induce the horizontal $\mathcal{F}_h$ and vertical $\mathcal{F}_v$ foliation. In the case of the one form α the foliation can be oriented since the above equations are fulfilled by the canonical vector field associated to $\exp(i\theta)\alpha$ by means of the metric g_u.

On $\Omega^{(+)}(S_{g,n})$ we have a natural $GL_2(\mathbb{R})$ operation. For $u = (S_{g,n}, \mathcal{A}) \in \Omega^{(+)}(S_{g,n})$ it is given by post composition $A \circ \phi$ of chart maps ϕ from $\mathcal{A}$ by an element $A \in GL_2(\mathbb{R})$. Formally we write Au for the resulting F structure. Since

$$SL_2(\mathbb{R}) = \{A \in GL_2(\mathbb{R}) : \det(A) = 1\}$$

$SL_2(\mathbb{R})$ operates on $\Omega(S_{g,n})$ and preserves the natural volume of the F structures. Both operations obviously map translation structures to translation structures. Since by the definition of F structures $Au = -Au$ for $A \in GL_2(\mathbb{R})$ the operations descend to operations of $PGL_2(\mathbb{R})$ and $PSL_2(\mathbb{R})$ respectively. The operation of $GL_2(\mathbb{R})$ on the canonical quadratic differential for example is given by

$$\begin{aligned} q_{Au} &= (A \circ \phi)^* dz^2 = \phi^* \left((a^2 + c^2)\, dx^2 + i\,(b^2 + d^2)\, dy^2\right) = \\ &= (a^2 + c^2)\, \phi^* dx^2 + i\,(b^2 + d^2)\, \phi^* dy^2 \end{aligned}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$$

and ϕ is a chart map of the natural atlas.

An F structure u is reducible to a translation structure iff there is a θ, so that the foliation $\mathcal{F}_\theta(u)$ is orientable. That is it is the integral foliation of a vectorfield. We saw above that this is true if u is a translation structure. On the other hand $\mathcal{F}_\theta(u)$ is orientable then also the foliation $\mathcal{F}_{\theta+\frac{\pi}{2}}(u)$ is by the complex structure $\mathcal{J}_u$. By integration this defines a translation atlas for $exp(i\theta)u$ and by the $SL_2(\mathbb{R})$ operation for u as well.

Definition 1.2 *A F map $\phi : u \longrightarrow v$ between two F structures $u = (S_{g,n}, \mathcal{A})$ and $v = (S_{g',n'}, \mathcal{A}')$ is a continuous map $\phi : S_{g,n} \longrightarrow S_{g',n'}$ so that the maximal atlas on $S_{g,n}$ generated by $\{\psi \circ \phi : \psi \in \mathcal{A}'$ chart map $\}$ is equal to $\mathcal{A}$.*

Thus locally in natural coordinates $\phi(z) = \pm z + c$ with a constant $c \in \mathbb{C}$ and by definition all the natural geometric objects on u defined above are pull backs of the ones on v. For example

$$\phi^* q_v = q_u \quad \text{and} \quad \phi^* \mathcal{J}_v = \mathcal{J}_u.$$

It follows ϕ is a holomorphic covering map.

For practical reasons it is important to answer the question under what circumstances an F structure on $S_{g,n}$ can be continued to the whole surface S_g.

Definition 1.3 *A (positive) F structure $u = (S_{g,n}, \mathcal{A}) \in \Omega^{(+)}(S_{g,n})$ is called admissible, if the completion of $S_{g,n}$ with respect to the natural flat metric g_u is S_g.*

Proposition 1.1 *[Vch86] If u is admissible, then*

1 The complex structure $\mathcal{J}_u$ extends to S_n

2 The quadratic differential q_u extends to a meromorpic quadratic differential on S_n with at worst simple poles.

A direct consequence of this is that the Riemann metric g_u can be continued to a singular metric on S_g. It has conic singularities in the points where the quadratic differential q_u is zero or has a simple pole. Around these singular points the total angle is $k\pi$ where $k \geq 1$, thus locally in polar coordinates:

$$g_u = dr^2 + \left(\frac{k}{2}\right)^2 r^2 d\varphi^2$$

If q_u has a pole, g_u has a pole, too (of first order if u is admissible). This in turn causes the finiteness of the (volume) integral:

$$||q_u|| := \int\limits_{S_n} dvol_u < \infty$$

A quadratic differential q (on S_g) is called admissible if the integral above is finite. On the set of admissible quadratic differentials (on S_g) $||\cdot||$ defines a norm.

If an admissible F structure contains a translation structure, then the continuation of the associated one form α can have only zeroes in O_n but no poles. Moreover around each conic singularity the total angle is $2k\pi$ with $k \in \mathbb{N}$. On the other hand a meromorphic quadratic differential q with finite norm $||q||$ on S_g and zeroes and poles in $Z \subset O_n$ (Z is allowed to be a proper subset) defines an admissible F structure on $S_{g,n}$ by the following construction of an atlas: locally in a neighborhood U_{z_0} of $z_0 \in S_{g,n}$ take charts by choosing a branch of the square root $\sqrt{q}$ of q and define:

$$\zeta_{z_0}(z) = \int\limits_{\gamma_{z_0}^z} \sqrt{q}$$

where $\gamma_{z_0}^z$ is a path from z_0 to z in U_{z_0}. $\zeta_{z_0}(z)$ does not depend on the path because $\sqrt{q}$ is holomorphic. Of course the neighborhoods U_{z_0} must be starshaped, to avoid difficulties with the poles. It is clear by the definition of the ζ coordinates that the coordinate changes are of the form $\zeta \longmapsto \pm\zeta + c$ with a constant $c \in \mathbb{C}$. The maximal atlas $\mathcal{A}_q$ on $S_{g,n}$ associated to this coordinates defines an F structure which is admissible by definition.

Let $\Omega^+(g,n)$ be the set of all admissible, positive F structures on S_g with singular points only in O_n. The topological group

$$H^+(g,n) := \{\phi \in homeo(S_g) : \phi \text{ orientation preserving and } \phi(O_n) = O_n\}$$

operates on $\Omega^+(g,n)$ by the rule

$$\phi_* u = u \circ \phi^{-1} \quad u \in \Omega^+(g,n)$$

as well as the subgroup $H_0^+(g,n) \subset H^+(g,n)$ of homeomorphisms homotopic to the identity. $\Omega^+(g,n)$ can be topologized by the distance (cf Veech [Vch86])

$$d(u_1,u_2) = \sup_{x\in S_{g,n}} \sup_{\substack{(U_j,f_j)\in u_j \\ x\in U_1 \bigcap U_2}} \limsup_{y\to x} L\left(\left(\frac{f_1(y)-f_1(x)}{f_2(y)-f_2(x)}\right)^2\right) \tag{1.1}$$

($L(z) = |ln|z| + i \arg z|$ with $-\pi \le \arg z < \pi$.) The distance is sensitive to changes of the volume of the F structure and also recognizes the direction. The following example might illustrate that. Take a cylinder $\mathcal{C} := (0,w) \times [0,h]/(x,0) \sim (x,h)$ of width w and hight h with the natural translation structure $u_{\mathcal{C}}$ induced by dz. There are natural linear operations on $\mathcal{C}$, the simplest is stretching $\mathcal{C}$ by

$$v_t := \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}$$

or to say it in another way, take the new F structure $v_t u_{\mathcal{C}}$ on $\mathcal{C}$. Since the transformation v_t is linear, to calculate the distance $d(v_t u_{\mathcal{C}}, u_{\mathcal{C}})$, it is enough to evaluate the argument of the logarithm L in one point of $\mathcal{C}$. Thus the result is $d(v_t u_{\mathcal{C}}, u_{\mathcal{C}}) = 2t$. For the rotation

$$r_\theta := \exp(i\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

we compute $d(r_\theta u_{\mathcal{C}}, u_{\mathcal{C}}) = 2\theta$. And finally for

$$a_t := \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}$$

the distance $d(a_t u_{\mathcal{C}}, u_{\mathcal{C}})$ is $2t$. Now the considerations don't use the fact that u is a cylinder, they are true for any F structure u. Hence $d(v_t u, u) = 2t$, $d(r_\theta u, u) = 2\theta$ and $d(a_t u, u) = 2t$ for all F structures u.

Veech [Vch86] has proved that the $H^+_{(0)}(g,n)$ invariant pseudometrics $D_{(0)}(u_1,u_2) := \inf_{\phi\in H^+_{(0)}(g,n)} d(u_1, u_2\circ\phi^{-1})$ on $\Omega^+(g,n)$ induce complete metrics on the "Teichmüllerspace"

$$QD(g,n) := \Omega^+(g,n)/H_0^+(g,n) \tag{1.2}$$

and the modulispace

$$\mathcal{QD}(g,n) := \Omega^+(g,n)/H^+(g,n) \tag{1.3}$$

Remark 1.1 *Since for any $\phi \in H^+(g,n)$ the natural atlas of $u \circ \phi$ is just the pullback of one of the u, all the geometric objects defined above are canonically identified under the operation of $H^+(g,n)$. In this sense we can speak for example of a metric $g_{[u]}$ for the class $[u] \in \mathcal{QD}(g,n)$.*
Since the $PGL_2(\mathbb{R})$ and $PSL_2(\mathbb{R})$ action on $\Omega^+(g,n)$ are from the left and the $H^+(g,n)$ action is from the right they commute and both actions descend to $QD(g,n)$ and $\mathcal{QD}(g,n)$ respectively.

Given an F structure $u = (S_{g,n}, \omega) \in \Omega^+(g,n)$ (ω denotes the holomorphic one form on $S_{g,n}$ which describes the flat structure) and a surface M together with a differentiable map $M \xrightarrow{f} S_g$.If f has only finitely many critical points, the pullback $f^*\omega$ defines a holomorphic one form on M and therefore of course an F structure. By the operation of the associated groups of positive homeomorphisms on $(M, f^*\omega)$ and $(S_{g,n}, \omega)$ the map f^* on F structures descends to a map of classes. Varying ω f^* becomes a map of Teichmüller- and/or Moduli-spaces

$$\mathcal{QD}(g,n) \xrightarrow{f^*} \mathcal{QD}(M).$$

The definition of the metric D implies $D_M(f^*[\omega_1], f^*[\omega_2]) \leq D_{S_{g,n}}([\omega_1],[\omega_2])$, hence the map f^* is a continuous map between Modulispaces. This is of course true if one assumes M to be a two dimensional submanifold of $S_{g,n}$ and f is the natural inclusion. The induced map can be viewed as a restriction map of the F structure to the given submanifold M.

If the object to that one restricts is not a submanifold of $S_{g,n}$, but a subspace depending on the F structure then the restriction map might not be defined on the whole space $\mathcal{QD}(g,n)$. For an example see the proof of Proposition 1.11.

1.2 Geodesics on F structures

There are two special kinds of geodesics on F surfaces $u = (S_g, q)$ with respect to the canonical flat metric g_u. The first one are called *saddle connections*, these are geodesics with respect to g_u which connect two singular points $p_1, p_2 \in O_n$ and contain no other singular point. The set of saddle connections on u is denoted with $SC(u)$. The second are *closed geodesics* c which do not contain singular points. Because the metric g_u is locally Euclidean there is a neighborhood of c which is isometric to the cylinder $\mathcal{C} = c \times (0,\epsilon)$ (with the Euclidean metric) and therefore the closed geodesics occur in parallel families. Each cylinder of closed geodesics is contained in a maximal one, which must have singular points on its boundary, otherwise it could be extended. Thus the boundaries of each maximal cylinder of closed geodesics consists of saddle connections. We denote the set of maximal cylinders of closed geodesics on u with $PO(u)$. If we do not want to precise about what we are speaking: singular geodesics (= saddle connections) or (cylinders of) closed geodesics, we will say just "geodesic".

Since the F structure recognizes the direction (modulo ± 1) as well as the length of a $s \in SC(u)$ or of a $\mathcal{C} \in PO(u)$ we can associate a vector $\pm v(s) \in \mathbb{R}^2$ $(\pm v(\mathcal{C}) \in \mathbb{R}^2)$ to each. The length of each $v(s)$ $v(\mathcal{C})$ is the g_u length of s (of any periodic orbit in $\mathcal{C}$) and its direction is the direction of s $(\mathcal{C})$. The set of such vectors given an F structure u is denoted by $\mathcal{V}_{(\cdot)}(u)$ where $\cdot = sc$ for saddle connections and $\cdot = po$ for cylinders of closed geodesics. Since geodesics are defined in terms of the metric g_u and $g_u = g_{\exp(i\theta)u}, \quad \forall\, \theta \in [0, 2\pi]$ we have

$$\mathcal{V}_{(\cdot)}(u) = \mathcal{V}_{(\cdot)}(\exp(i\theta)u), \quad \forall\, \theta \in [0, 2\pi].$$

Furthermore, by the above definition of the $GL_2(\mathbb{R})$ operation on F structures, it is clear that

$$g\mathcal{V}_{(\cdot)}(u) = \mathcal{V}_{(\cdot)}(gu) \quad \forall\, g \in GL_2(\mathbb{R}) \tag{1.4}$$

with respect to the linear operation of $GL_2(\mathbb{R})$ on $\mathbb{R}^2$. From this it directly follows that the $GL_2(\mathbb{R})$ action on F structures induces a linear action on the geodesics: if s is a geodesic on u, then As is a geodesic of Au $(A \in GL_2(\mathbb{R}))$ and $\pm v(As) = \pm Av(s)$. We have:

Proposition 1.2 *The length $l(s)$ of a geodesic s, the width $w(\mathcal{C})$ and the volume $vol(\mathcal{C})$ of a cylinder of closed geodesics $\mathcal{C}$ on an F structure u are continuous functions with respect to the operation of $GL_2(\mathbb{R})$ on u.*

If we restrict the operation to $SL_2(\mathbb{R})$, the volume of cylinders of geodesics is invariant. Since (see remark 1.1) $\mathcal{V}_{(\cdot)}(u) = \mathcal{V}_{(\cdot)}(u \circ \phi)$. Thus $\mathcal{V}_{(\cdot)}(u)$ depends only on the class $[u]$ of u in the Teichmüller or moduli space.

There are subsets of $PO(u)$ which are also of interest. The set of *regular geodesics* is the set of all maximal cylinders of closed geodesics which are bounded by a single saddle connection on each boundary component. All the maximal cylinders of closed geodesics which have more than one saddle connection on a boundary component are called *irregular*.

1.3 The translation structure of a polygonal billiard

Let $G_n \subset \mathbb{C}$ a n-gon with an oriented boundary and denote the vector of inner angles by $\alpha = (\alpha_1, ..., \alpha_n)$. The phase space of the billiard in G_n is the topological quotient $G_n \times S^1/\sim$, where the equivalence relation $\sim$ is defined as:

$$(x, \theta) \sim (x', \theta')$$

if $x = x' \in v$ with $v \in \partial G_n$ and $\theta' = r_v\theta$ where r_v is the reflection at the edge e. Since this equivalence relation is only well defined when x is not a vertex point, we assume that the vertices are removed from G_n. Thus the connected components of this phase space are all of the form $G_n \times S_{G_n}\theta$ where S_{G_n} is the reflection group generated by the edges of G_n. If we take the translation structure $\exp(i\theta)dz$ on each copy $G_n \times \theta$ of the n-gon G_n in the phase space, by the rule of the identification they glue together to give a global differential on each invariant surface. If θ is not parallel to the fixpoint line of some reflection in S_{G_n}, the invariant surfaces are isometric to $G_n \times S_{G_n}/\sim$ where $(x,r) \sim (x',r')$ if $x = x' \in v$ with $v \in \partial G_n$ and $r^{-1} \circ r' = r_v$. In the other case the number $|S_{G_n}\theta|$ is exactly half the order $\mathrm{ord}(S_{G_n})$ of S_{G_n}. The resulting invariant surfaces are doubly covered by the former ones because $S_{G_n}\theta \cong S_{G_n}/(r_\theta)$ (r_θ is the reflection on the line defined by θ). On the billiard table this is the same as moving the particle parallel to one of the edges of G_n. On the surface $G_n \times S_{G_n}/\sim$ this means that a particle starting on one of the copies of G_n and parallel to one of its edges will meet only "half" of the surface.

Remark 1.2 *The global symmetry of order two generated by the two fold covering*

$$G_n \times S_{G_n}/\sim \xrightarrow{\pi} G_n \times (S_{G_n}/(r_\theta))/\sim$$

can, together with a Poincaré recurrence theorem be used to prove that trajectories perpendicular to an edge of G_n are almost sure periodic. For compact invariant surfaces (the "rational angle" case) this is done in [VGSt92]. A more general recurrence theorem is used by Serge Troubetzkoy to obtain some results in the "non rational angle" case [Tr96, Tr97].

The billiard dynamics on each invariant surface is given by the integral trajectories of the horizontal vector field X_h associated to the differential above i.e. in $p = (x,\theta) \in G_n \times S^1$ we have $X_h(p) = \exp(i\theta)$. By means of the above identification this is the constant vectorfield $X(p) = \exp(i\theta)$ on the surface $G_n \times S_{G_n}/\sim$. Thus we see the billiard dynamics in a given direction θ is just the geodesic flow with respect to the induced flat metric in direction $\exp(i\theta)$ on the surface $G_n \times S_{G_n}/\sim$. Since each transverse interval to the flow has invariant length we have an invariant Lebesgue measure for the directional flow and a flow invariant product measure on the phase space. To see which invariant surfaces can be compactified we prove

Proposition 1.3 *The reflection group S_{G_n} generated by reflections in the edges of G_n has finite order exactly if $\alpha \in \mathbb{Q}^n\pi$, where α is the vector of inner angles (defined above). If $\frac{\alpha}{2\pi} = (\frac{p_1}{q_1}, ..., \frac{p_n}{q_n}) \in \mathbb{Q}^n$ with $\gcd(p_i, q_i) = 1$ then:*

$$\mathrm{ord}(S_{G_n}) = \mathrm{lcm}(q_1, ..., q_n)$$

Proof: Assume there exists a non rational angle between two edges of G_n, then the reflection group generated by these edges is infinite cyclic, thus in the case of a non rational $\frac{\alpha}{2\pi}$ $\mathrm{ord}(S_{G_n})$ is infinite. To treat the rational case, we observe that two consecutive reflections $s_i \circ s_j$ at the edges e_i and e_j cause a rotation r_θ of twice the (inner) angle between e_i and e_j: $\theta = 2\angle(e_i, e_j)$. If all the angles between the edges are rational, say $\angle(e_i, e_j) = \pi \frac{p_{ij}}{q_{ij}}$ ($\gcd(p_{ij}, q_{ij}) = 1$), then the rotation group R generated by the edges contains the rotation r_θ with $\theta_0 = \frac{2\pi}{\operatorname{lcm} q_{ij}}$ since this number is an integer linear combination of the rotation angles $\pi \frac{p_{ij}}{q_{ij}}$. Thus the rotation group R contains the group R_{θ_0} generated by θ_0 which in turn contains every generator of R so they are identical. Since R has by construction index two in the reflection group S_{G_n} the Proposition is proved.

□

Since the order of S_{G_n} is equal to the number of copies of the n-gon G_n in the surface $G_n \times S_{G_n}/\sim$ we can compactify this translation structure for rational n-gons. To do this we have to put in correctly copies of the vertex points of G_n in $G_n \times S_{G_n}/\sim$. By construction a vertex $v_i \in G_n$ with inner angle $2\pi \frac{p_i}{q_i}$ results a conic singularity on $G_n \times S_{G_n}/\sim$ with cone angle $2\pi p_i$. Because there are S_{G_n} copies of G_n in $G_n \times S_{G_n}/\sim$ there exist exactly $\frac{|S_{G_n}|}{q_i}$ conic singularities associated to the vertex v_i on the compactified translation structure, which we call u_{G_n}. Thus if G_n is rational, u_{G_n} is an admissible translation surface and we can calculate its genus:

Proposition 1.4 *Let G_n be a rational n-gon with an inner angle vector given by $\frac{\alpha}{2\pi} = \left(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\right) \in \mathbb{Q}^n$ where $\gcd(p_i, q_i) = 1$. Then the Euler characteristic of the associated translation surface u_{G_n} is:*

$$\chi(u_{G_n}) = 2(1-g) = |S_{G_n}| \sum_{i=1}^{n} \frac{1-p_i}{q_i} \tag{1.5}$$

here S_{G_n} is the reflection group generated by the edges of G_n and g is the genus of u_{G_n}.

Proof: By construction the surface u_{G_n} has a combinatorial cell subdivision by copies of G_n. We have:

$$|\{faces\}| = |S_{G_n}| \qquad |\{edges\}| = \frac{n}{2}|S_{G_n}| \qquad |\{vertices\}| = |S_{G_n}| \sum_{i=1}^{n} \frac{1}{q_i}$$

Thus by Eulers formula

$$\chi(u_{G_n}) = |\{faces\}| - |\{edges\}| + |\{vertices\}| = |S_{G_n}| \left(1 - \frac{n}{2} + \sum_{i=1}^{n} \frac{1}{q_i}\right)$$

and the identity $\sum_{i=1}^{n} \frac{p_i}{q_i} = \frac{1}{2}(n-2)$ for the inner angles in G_n we finally get

$$\chi(u_{G_n}) = |S_{G_n}| \left(\sum_{i=1}^{n} \frac{1}{q_i} - \sum_{i=1}^{n} \frac{p_i}{q_i}\right) = |S_{G_n}| \sum_{i=1}^{n} \frac{1-p_i}{q_i}$$

$\square$

Example: Let G_n be the triangle given by $\left(\frac{1}{2n}, \frac{1}{2n}, \frac{n-2}{2n}\right) 2\pi$ where $n \in \mathbb{N}$, then

$$\chi(u_{G_n}) = \begin{cases} 4-n & \text{if n is even} \\ 3-n & \text{if n is uneven} \end{cases}$$

and for the genus $g(u_{G_n})$ of the translation surface we have

$$g(u_{G_n}) = \left[\frac{n-1}{2}\right] \tag{1.6}$$

Back to the general case: the subgroup $R_{G_n} \subset S_{G_n}$ of all rotations operates (as a subgroup of S_{G_n}) on u_{G_n} and the quotient under this operation is a singular Euclidean surface which can be viewed as two copies of G_n identified along their edges. This is the smallest compact and singular Euclidean surface one can associate to every polygonal billiard, but it has the disadvantage that it is not a translation structure, because the total angles around the singular points are never integer multiples of 2π. Therefore the $SL_2(\mathbb{R})$ operation defined above is not well defined on these surfaces, so that most of the arguments to prove statements about the geodesic flow do not work. Of course one can take the simplest translation structure covering the surface in view, (which is u_{G_n} for rational n-gons) but only for the price of non compactness in the case of non rational n-gons.

1.4 A manifold structure on moduli spaces of F surfaces

In this section we describe a manifold structure on moduli spaces of F surfaces. This is done by Veech [Vch86, Vch90] and also by Masur [Msr82] before. We follow the line of Veechs paper [Vch90], but omit the proofs which are too technical. From the viewpoint of algebraic geometry (see Kontsevich and Zorich [KoZo]) the manifold structures can be described using techniques like deformation theory and the Gauss Manin connection. But these techniques are not in the baggage of the most people studying polygonal billiards, thus it is more natural to use Veechs explicit construction. The motivation to present this material is the use of the moduli spaces in the theory of Eskin and Masur as well as to use it for the deformation results in the next section.

Let Σ_g be a compact Riemann surface of genus g. Again if we remove the set O_n of n points out of Σ_g the resulting surface is denoted by $\Sigma_{g,n}$. For the rest of the chapter we make the assumption $3g-3+n > 0$ corresponding Teichmüller space $\mathcal{T}(g,n)$ is homeomorphic to a ball of complex dimension $3g-3+n$. It follows more or less directly from the original construction of O. Teichmüller that the Teichmüller space of all pairs (Σ_g, q), where q is an integrable, meromorphic quadratic differential on Σ_g with poles and zeroes only in O_n is the cotangential space to $\mathcal{T}(g,n)$. Thus $T^*\mathcal{T}(g,n)$ is exactly the space of positive admissible F structures on $\Sigma_{g,n}$. The natural refinement is to prescribe the number and orders of singular points of the associated metric, or what is the same: prescribe the number of poles and zeroes of a given order for the natural quadratic differential on the flat surface. With $O_n := (p_1, ..., p_n)$ we define $\Omega^+(g, q_1, ..., q_m) \subset \Omega^+(g,n)$ to be the set of all $u = (\Sigma_g, \mathcal{A})$ where g_u has cone singularities of order $q_i\pi$ in p_i. We make no restrictions and allow $q_i = 2$ to be just a "marked point". Since by the Gauss Bonett theorem $\sum_{i=1}^n (q_i - 2) = 4g - 4$, for every partition P of $4g-4$ in n integers bigger or equal than -1, we can define spaces $(\Omega^+(g,P), d(\cdot,\cdot))$ with the distance (1.1) in this way (for some decompositions P these spaces might be empty, see the following discussion). It is well known (cf [MS93]) that the associated Teichmüller spaces $QD(g,P)$ define a stratification of the spaces $\Omega^+(g,n)$ as the decomposition P varies. There are two possible ways to define the quotient moduli space: one can take classes with respect to the group $H^+(g,n)$ or one can take the subgroup $H^+(g,P) \subset H^+(g,n)$ of homeomorphisms which fix the sets of singularities of the same kind in O_n. This is a matter of taste since $H^+(g,P)$ is of finite index in $H^+(g,n)$ and so one quotient space is a finite covering of the other. The spaces $QD(g,P)$ are not necessarily connected (see [KoZo]), so finally we define $Q(g,P)$ to be a connected component of $QD(g,P)$. These spaces are studied by Veech [Vch90] and Masur [Msr82, MS93]. We restate the main constructions and results, following the line of Veechs paper. Every half translation structure, given by the data $(\Sigma_g, q) \in \Omega^+(g,P)$, has a two sheeted covering (see Masur [Msr82]):

$$\Sigma_{g'} \xrightarrow{pr} \Sigma_{g,n}$$

which is branched over all zeroes of g_u of uneven degree. The pullback pr^*q of q is holomorphic (the poles are removed) and results an orientable foliation, thus pr^*q is the square of a holomorphic one form α on $\Sigma_{g'}$. At first using this observation we restrict our treatment to spaces of translation structures $M(g,P)$ defined as the Teichmüller space of pairs $(\Sigma_{g,n}, \alpha)$ where the holomorphic one form α has a zero of order q_i in each point $p_i \in O_n \subset \Sigma_g$ and $P = (2q_1, ..., 2q_n)$ is a decomposition of $4g-4$ by even natural numbers including 0. Because by Hopf's theorem $2g-2 = \sum_{i=1}^n q_i$ if α is a holomorphic one form with zeroes of order $q_i + 1$ on Σ_g. If we put the question the other way round: given some singularity datum P and prescribed orientability $\epsilon = \pm 1$ ($\epsilon = -1$ for non orientable foliations), is the space $QD(g,P)$ ($g = 1 + \sum_{i=1}^n q_i$) non empty? Masur and Smillie [MS93] have proven, that the only exceptions are $P = (\emptyset; -)$, $P = (1, -1; -)$, $P = (3, 1; -)$ and $P = (4; -)$ cannot be realized by quadratic differentials. The exception $P = (\emptyset; -)$ says that on the torus there is no nonorientable foliation defined by a quadratic differential. Since the orders of the zeroes are not changed under the $SL_2(\mathbb{R})$ operation on (half)translation structures all the above spaces are invariant under the $SL_2(\mathbb{R})$ operation. To study the spaces $M(g,P)$ let (S_g, α) be some representative and $O_n = \{p_1, ..., p_n\} \in S_g$ be the set of zeroes of α with orders $o_\alpha(p_i) = q_i + 1$. Then, as is proven in Veech [Vch90], it is possible to find for almost

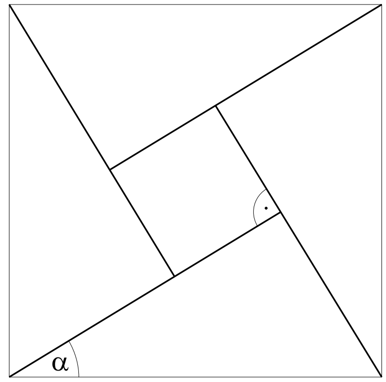

Figure 1.1: A weaving on the torus (parallel boundaries have to be identified).

every $\exp(i\theta) \in S^1$ a cell subdivision of S_g with the following properties:

1. The faces are rectangles with respect to the Euclidean metric structure together with the induced orientation.

2. The edges are parts of horizontal or vertical trajectories with respect to α_θ where $\alpha_\theta := \exp(i\theta)\alpha$. Each edge is contained in a maximal edge. One boundary point of each maximal edge is in the set O_n the other is on a perpendicular edge, but not on its boundary.

3. The vertices are the boundary points of the edges and there exist only finitely many.

4. Every point $p_i \in O_n$ is the boundary of $2(ord(p_i)+1)$ vertical and $2(ord(p_i)+1)$ horizontal edges.

A one skeleton of S_g inducing a cell subdivision of the translation structure (S_g, α_θ) with the above properties is called a *weaving* for (S_g, α_θ).

To describe $M(g, P)$ locally the main observation is that fixing the length of the edges in a consistent way, all the opposite sides of the "rectangles" have to be of the same length. This characterizes (the class of) (S_g, α_θ) in $M(g, P)$. The consistence condition can be formulated in terms of a coboundary operator on the dual

$$\mathcal{C}_0^* \xrightarrow{\partial^*} \mathcal{C}_1^* \xrightarrow{\partial^*} \mathcal{C}_2^* \tag{1.7}$$

of the chain complex of real vectorspaces

$$\mathcal{C}_2 \xrightarrow{\partial} \mathcal{C}_1 \xrightarrow{\partial} \mathcal{C}_0 \tag{1.8}$$

generated by the the cell subdivision of S_g above. A measurement of length on each edge in the weaving defines a homomorphism on the $\mathbb{R}$ vectorspace $\mathcal{C}_1$ generated by the edges and therefore an element $\phi \in \mathcal{C}_1^*$. From now on the vectorspaces $\mathcal{C}_i^*$ and $\mathcal{C}_i$

are identified by means of $e^l(e_m) = \delta_{l,m}$ for generators (= i cells) $e_m \in \mathcal{C}_i$. If we define $\phi_{h,v} \in \mathcal{C}_1$ to be ϕ on horizontal (vertical) edges and zero on vertical (horizontal) ones, the consistency condition becomes:

$$\partial^* \phi_{h,v} = 0.$$

Horizontal and vertical one cycles are defined as $\mathcal{Z}^1_{h/v} := \ker \partial^* \cap \mathcal{C}_{h,v}$ where $\mathcal{C}_{h,v}$ is the linear span of the horizontal (vertical) edges in $\mathcal{C}_1$. There is an elementary calculation of the dimensions by using the above cell subdivision.

Proposition 1.5

$$\begin{aligned} \dim \mathcal{C}_h = \dim \mathcal{C}_v &= 2g - 2 + n + |\{faces\}| \\ \dim \mathcal{Z}^1_h = \dim \mathcal{Z}^1_v &= 2g - 1 + n \end{aligned} \tag{1.9}$$

Proof: See Veech [Vch90] Lemma 4.5 and Lemma 4.10

□

Now the relative cohomolgy groups $H^1(S_g, O_n; \mathbb{R})$ can be defined in terms of (1.7)

$$H^1(S_g, O_n; \mathbb{R}) = \frac{\ker \partial^* \cap \mathcal{C}_1}{\partial^* \mathcal{C}_0(O_n)^\perp}$$

$\mathcal{C}_0(O_n)$ is the subspace of cocycles generated by O_n. We have natural projection maps

$$\mathcal{Z}^1_{h,v} \xrightarrow{pr} H^1(S_g, O_n; \mathbb{R})$$

and since $H^1(S_g, O_n; \mathbb{R})$ is a topological invariant we can compute its dimension using the long exact cohomology sequence associated to the sequence of space pairs

$$O_n \xrightarrow{i} (S_g, O_n) \xrightarrow{\pi} S_g$$

Here i is the inclusion on the second factor and π is the projection on the first factor of the space pair (S_g, O_n).

$$0 \longrightarrow \underbrace{H^0(S_g, O_n; \mathbb{R})}_{\text{dim=0}} \xrightarrow{\pi^*} \underbrace{H^0(S_g; \mathbb{R})}_{\text{dim=1}} \xrightarrow{i^*} \underbrace{H^0(O_n; \mathbb{R})}_{\text{dim}=|O_n|=n} \xrightarrow{\partial^*}$$

$$\xrightarrow{\partial^*} \underbrace{H^1(S_g, O_n; \mathbb{R})}_{\text{dim=?}} \xrightarrow{\pi^*} \underbrace{H^1(S_g; \mathbb{R})}_{\text{dim}=2g} \xrightarrow{i^*} H^1(O_n; \mathbb{R}) = 0$$

$$\Rightarrow \quad \dim_{\mathbb{R}} H^1(S_g, O_n; \mathbb{R}) = 2g - 1 + n. \tag{1.10}$$

Now $\alpha \in \mathcal{Z}^1_{h,v}$ and $pr(\alpha) = 0$ means $\alpha = \partial^* s$, with $s(x) = 0,\ \forall x \in O_n$. A calculation shows that every cycle of (co)vertices which maps under ∂^* to a horizontal (vertical) cocycle is zero on boundaries of vertical (horizontal) edges. The definition of a weaving implies $s = 0$ and therefore $\alpha = 0$. Thus the homomorphism pr is injective and by the equal dimensions of the spaces pr is an *isomorphism*.

Let us fix $(S_g, \alpha_0) \in \Omega^+(g,n)$ with a weaving Λ. Since the horizontal and vertical cycles $\mathcal{Z}^1_{h,v}$ are identified with respect to pr there is a canonical isomorphism $\mathcal{J} : \mathcal{Z}^1_h \longrightarrow \mathcal{Z}^1_v$. This defines a complex multiplication on $\mathcal{Z}^1 := \mathcal{Z}^1_h \oplus \mathcal{Z}^1_v$ and because of

$$\mathcal{Z}^1 \cong H^1(S_g, O_n; \mathbb{R}) \oplus iH^1(S_g, O_n; \mathbb{R}) \cong H^1(S_g, O_n; \mathbb{C}) \tag{1.11}$$

pr extends to a canonical $\mathbb{C}$ linear isomorphism again denoted by pr. The one form α_0 defines an element $[\alpha_0]_h + [\alpha_0]_v \in H^1(S_g, O_n; \mathbb{C})$ by integration along the edges of Λ (these lengths are all positive by the definition of Λ).

Now we define a new translation structure by picking an element $[\beta] \in H^1(S_g, O_n; \mathbb{C})$ and associate the new length $\beta(e_{h,v})$ to horizontal (vertical) edges respectively. This makes sense only, if we restrict to cohomology classes which assign a positive length to every edge. The set of all such classes is a (positive) cone in $H^1(S_g, O_n; \mathbb{C})$, it depends on the weaving Λ of course and is called $C^+(\Lambda)$. The so defined map $\Phi_\Lambda : C^+(\Lambda) \longrightarrow \Omega^+(g,n)$ has the properties:

1. $[\Phi_\Lambda(\beta)] = \beta$ for all $\beta \in C^+(\Lambda)$ especially $[\Phi_\Lambda(\alpha_0)] = \alpha_0$.
2. Λ is a weaving for $\Phi_\Lambda(\beta)$ if $\beta \in C^+(\Lambda)$

Moreover since $H^+_0(g,n)$ operates trivially on $H^1(S_g, O_n; \mathbb{R})$ Φ_Λ descends to a map $\Phi_\Lambda : C^+(\Lambda) \longrightarrow M(g,n)$ with:

1. Φ_Λ is a continuous map with respect to the natural topology on $C^+(\Lambda)$ and the topology induced by the Teichmüller metric 1.1
2. Φ_Λ is injective and open.

The main problem is to show that the map Φ_Λ is continuous with respect to the Teichmüller metric (1.1). Veech solved this by constructing the map $\Phi_\Lambda : C^+(\Lambda) \longrightarrow \Omega^+(g,n)$ in a way that guarantees the continuity and then establishes all the other properties. Roughly speaking given a weaving Λ one constructs the map Φ_Λ starting with the Euclidean rectangles defined by the cell subdivision associated to Λ and the flat structure u. Given $\alpha \in C^+(\Lambda)$ for each rectangle R α defines new lengths on the edges and so a new rectangle R_α. The difference between R and R_α is measured by constructing a natural piecewise linear map $R \xrightarrow{\phi} R_\alpha$ and define $\delta(R, R_\alpha) := \max \left|\ln \frac{d\phi}{ds}\right|$, where $\frac{d\phi}{ds} > 0$ is the locally constant arclength derivative. δ is equivalent to the Teichmüller distance d 1.1 on rectangles R. Now the deformed rectangles R_α define a new flat atlas on S_g and so a new F structure with the same singularity data as u. By taking the maximum over all rectangles of Λ^c δ defines a distance between u and u_α again denoted with δ. This δ is equivalent to the Teichmüller distance d on

$\Omega^+(g,n)$. From this construction the continuity of the map Φ_Λ and the other properties mentioned above follow. The injectivity of Φ_Λ is proved in Lemma 5.11. of [Vch90] and the openness is Proposition 6.1. in the same paper.

The construction of the local parameterizations modeled on $H^1(S_g, O_n; \mathbb{C})$ yields charts for almost all translation surfaces, the only exceptions are the translation structures $u = [(S_g, \alpha)] \in M(g,n)$ on which there is no weaving. But since $SL_2(\mathbb{R})$ operates on $M(g,n)$ by $u \mapsto Au$ with $A \in SL_2(\mathbb{R})$ we can rotate u to a structure $\exp(i\theta)u$ that has a weaving. Thus our set of charts covers all points of $M(g,n)$. Since the maps pr associating a cohomology class to a cycle are linear it follows that changes of charts resulting from changing the weaving are complex linear maps.

The Lebesgue measure μ, on the vectorspace $H^1(S_g, O_n; \mathbb{C}) \cong \mathbb{C}^{2g-1+n}$ normed by assigning covolume one to the lattice $H^1(S_g, O_n; \mathbb{Z})$ defines a positive smooth measure μ_0 on $M(g,P)$.

Noting that the natural volume $\frac{i}{2}\int_{S_g} \omega \wedge \bar{\omega}$ of a translation structure $u = (S_g, \omega)$ is the intersection form $[Im(\omega), Re(\omega)]$ evaluated on the cycles $(Im(\omega), Re(\omega)) \in \mathcal{Z}^1_h \oplus \mathcal{Z}^1_v$ in $H^1(S_g, O_n; \mathbb{C})$ yields the function

$$\begin{array}{ccc} M(g,P) & \stackrel{A}{\longrightarrow} & \mathbb{R}_+ \\ (S_g,\omega) & \longmapsto & \frac{i}{2}\int_{S_g} \omega \wedge \bar{\omega} \end{array} \tag{1.12}$$

On the other hand it can be proven (Veech [Vch90] Proposition 4.19) that $[Im(\omega), Re(\omega)]$ is the sum of the volumes of the faces of the cell decomposition defined by the weaving Λ with respect to the natural metric on u. Thus A is a real analytic function.

Since the operation of $H^+(g,n)$ on $H^1(S_g, O_n; \mathbb{R})$ is linear, maps generators to integer multiples of generators and has $H_0^+(g,n)$ as isotropy group, it descends to an operation of the mapping class group $\Gamma(g,n) := H^+(g,n)/H_0^+(g,n)$ represented by $GL_{(2g-1+n)}(\mathbb{Z})$ acting on $\mathbb{R}^{2g-1+n} \cong H^1(S_g, O_n; \mathbb{R})$. Analogously the mapping class group $\Gamma(g,n)$ operates on $\mathbb{R}^{2(2g-1+n)} \cong \mathbb{R}^2 \otimes H^1(S_g, O_n; \mathbb{R}) \cong H^1(S_g, O_n; \mathbb{C})$ represented by elements of $GL_{2(2g-1+n)}(\mathbb{Z})$. With respect to the manifold coordinates of Veech described above the operation of $SL_2(\mathbb{R})$ is represented by the linear action of $SL_2(\mathbb{R})$ on the first factor of $\mathbb{R}^2 \otimes H^1(S_g, O_n; \mathbb{R}) \cong H^1(S_g, O_n; \mathbb{C})$. By this fact the measure μ_0 is $SL_2(\mathbb{R})$ invariant by definition.

Theorem 1.6 (W. Veech) *(1) $M(g,P)$ is a complex affine manifold of dimension $2g-1+n$.*

(2) The function $A : M(g,P) \longmapsto \mathbb{R}_+$ is real analytic, positive and without critical points, therefore

(3) the level sets of A are real analytic submanifolds of $M(g,P)$, they are $SL_2(\mathbb{R})$ and $\Gamma(g,n)$ invariant.

(4) μ_0 is a smooth measure on $M(g,P)$, it is $SL_2(\mathbb{R})$ and $\Gamma(g,n)$ invariant.

The description of the manifold structure of the Teichmüller spaces $M(g,P)$ results a manifold structure of the moduli spaces $\mathcal{M}(g,P) := M(g,P)/\Gamma(g,n)$. To describe the spaces of quadratic differentials which are not the square of a holomorphic one form, let $(\Sigma_{g,n}, q) \in \Omega^+(g,n)$ be a half translation structure and

$$\Sigma_{g'} \xrightarrow{\pi} \Sigma_{g,n}$$

the orientation covering. Then there exists a holomorphic one form α on $\Sigma_{g'}$ such that $\pi^* q = \alpha^2$ and $i^*\alpha = -\alpha$ for the sheet exchange involution i on $\Sigma_{g'}$ characterized by $\pi \circ i = \pi$. The general idea is to study subspaces $Scs(i) := \{u \in \Omega^+(g,P) : i^*\omega_u = -\omega_u\}$ for some involution i of S_g. The images of the subspaces $Scs(i)$ in the Teichmüller $M(g,P)$ or moduli space $\mathcal{M}(g,P)$ are analytic submanifolds of codimesion one. Applied to the sheet exchange map i of the orientation covering above results the generalization of the above theorem

Theorem 1.7 (W. Veech) *The connected components $\mathcal{Q}(g,P)$ of the moduli spaces $\mathcal{QD}(g,P)$ of positive admissible F structures on $S_{g,n}$ with metric singularities of order q_i in $p_i \in O_n$ for $i = 1, ..., n$ are complex affine manifolds with respect to the above defined atlas. Their dimension is*

$$\dim_{\mathbb{C}} \mathcal{Q}(g,P) = 2g - 1 + l$$

if they parameterize spaces of differentials and

$$\dim_{\mathbb{C}} \mathcal{Q}(g,P) = 2g - 2 + l$$

if not. The function $A : \mathcal{Q}(g,P) \longrightarrow \mathbb{R}_+$ is real analytic and the level sets

$$\mathcal{Q}^1(g,P) := A^{-1}(1)$$

are real analytic $SL_2(\mathbb{R})$ and $\Gamma(g,n)$ invariant submanifolds. The measure μ_1 on $\mathcal{Q}^1(g,P)$ defined by

$$\mu_1 \times dA = \mu_0$$

is preserved under the $SL_2(\mathbb{R})$ operation.

There are two important theorems about the measure μ_1 and the $SL_2(\mathbb{R})$ operation on $\mathcal{Q}^1(g,P)$ proved by Veech in [Vch90], [Vch86] and partly by Masur in [Msr82]

Theorem 1.8 (W. Veech; H. Masur) *The μ_1 volume of $\mathcal{Q}^1(g,P)$ is finite.*

Theorem 1.9 (W. Veech; H. Masur) *The one parameter group*

$$\left\{ \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} : t \in \mathbb{R} \right\} \subset SL_2(\mathbb{R})$$

acts ergodically on $(\mathcal{Q}^1(g,P),\ \mu_1)$.

Corollary 1.10 *The natural tensors q_m, g_m, $dvol_m$ for $m = [u] \in \mathcal{Q}^1(g,P)$ are real analytic in m with respect to the above constructed manifold structure for $\mathcal{Q}^1(g,P)$.*

Proof: This follows for q_m from the definitions of the charts. The metric g_m and the volume $dvol_m$ is expressible in local coefficients of q_m.

□

1.5 Deformation invariance of periodic families in $\mathcal{Q}(g,P)$

First we want precise what are the objects we speak about. Take an F structure u representing a class $[u] \in \mathcal{Q}(g,P)$. Then a cylinder of closed geodesics $\mathcal{C}$ on u is defined as a cylinder of closed geodesics with respect to the metric g_u on u. We do not assume that $\mathcal{C}$ is defined by the horizontal foliation on u. Since the metric D on $\mathcal{Q}(g,P)$ is sensitive to changes of the direction this would destroy the proposed stability of $\mathcal{C}$. The same remark holds for saddle connections on u.

Proposition 1.11 *For each maximal cylinder of periodic orbits $\mathcal{C}$ and each saddle connection $\mathcal{S}$ on $m = [u] \in \mathcal{Q}^{(1)}(g,P)$ there exists open neighborhoods $U_{\mathcal{C}}(m)$ and $U_{\mathcal{S}}(m)$ of u in $\mathcal{Q}^{(1)}(g,P)$ to which these objects deform real analytically. The length $l(\mathcal{C})$, $l(\mathcal{S})$ width $w(\mathcal{C})$ and area$(\mathcal{C})$ are real analytic functions on $U_{\mathcal{C}}$, $U_{\mathcal{S}}$ respectively.*

Proof: Take a point $[u] \in \mathcal{Q}(g,P)$ and fix a maximal cylinder $\mathcal{C}$ of closed geodesics on u. Take a chart from the atlas described in the last section covering a neighborhood of $[u] \in \mathcal{Q}(g,P)$. Remember that this is the same as giving a certain cell subdivision of u, where the faces are rectangles. A neighborhood of $[u]$ in $\mathcal{Q}(g,P)$ is defined by changing the length and the heights of these rectangles in a consistent manner. By this the length, the width and therefore the volume of $\mathcal{C}$ are changing real analytically with the class $[u]$ in this coordinates. This implies of course the stability of cylinders of closed geodesics in $\mathcal{Q}(g,P)$ and $\mathcal{Q}^{(1)}(g,P)$ as well. This proves the claim for families of periodic orbits. To treat the case of saddle connections, we fix a saddle connection s on u and take the following rectangle: on both (singular) endpoints of s we take intervals perpendicular to s. Connecting points of both intervals by lines parallel to s gives a rectangle of the same length as s and a certain width, which can be chosen maximal. No one uses the same argument as above by replacing the cylinder with the rectangle.

□

Remark: The width of a maximal cylinder of closed geodesics $\mathcal{C}$ is defined by the distance of two singular points on its boundary measured perpendicular to the direction of the closed orbits. This distance becomes zero by a finite movement of the singularities induced by changing the F structure in $\mathcal{Q}(g,P)$. Thus the existence of a deformation of $\mathcal{C}$ as a cylinder of closed orbits is restricted to a neighborhood of u in $\mathcal{Q}(g,P)$. The same holds for the rectangle of saddle connections, in exact if the width of the rectangle collapses then the saddle connection is divided in more than one.

Corollary 1.12 *Let $T > 0$ and $m \in \mathcal{Q}^1(g,P)$, then there exist open neighborhoods $U_{SC}(m,T)$, $U_{PO}(m,T)$ of m such that*

$$N_{SC}(m,T) \leq N_{SC}(w,T) \quad \forall w \in U_{SC}(m,T) \tag{1.13}$$

$$N_{PO}(m,T) \leq N_{PO}(w,T) \quad \forall w \in U_{PO}(m,T) \tag{1.14}$$

Proof: By theorem 1.11 we have the stability of each single saddle connection s or cylinder of periodic orbits $\mathcal{C}$ in some open sets in $\mathcal{Q}^1(g,P)$ and their length changes continuously in these neighborhoods. For each $s \in SC(m)$ ($\mathcal{C} \in PO(m)$) with $l(s) < T$ ($l(\mathcal{C}) < T$) respectively, we take neighborhoods $U(s)$ ($U(\mathcal{C})$) , so that $l(s_w) < T \quad \forall w \in U(s)$ ($l(\mathcal{C}_w) < T \quad \forall w \in U(\mathcal{C})$). Since there are only finitely many saddle connections or families of periodic orbits of length smaller than T on m the sets

$$U_{SC}(m,T) := \bigcap_{\{s \in SC(m):\ l(s)<T\}} U(s)$$

$$U_{PO}(m,T) := \bigcap_{\{\mathcal{C} \in PO(m):\ l(\mathcal{C})<T\}} U(\mathcal{C})$$

are open in $\mathcal{Q}^1(g,P)$ and the corollary is proved.

□

Cylinders of periodic orbits will vanish if the two boundary components consisting of saddle connections are deformed into each other, to give a new set of saddle connections. The number of saddle connections is not changed by this deformation, but their length might be smaller. We assume that singular points are not moved into another, otherwise we would leave the space $\mathcal{Q}^1(g,P)$. By this process we observe that it is in principle possible that the number of saddle connections under a given length T on a point $m \in \mathcal{Q}^1(g,P)$ increases drastically in a neighborhood of this point. Saddle connections can vanish only by means of the following process: a geodesic triangle degenerates to a geodesic line consisting of two saddle connections.

Chapter 2

Quadratic growth rates on Veech surfaces

Ten years ago W. Veech [Vch89] found a way to gave explicit growth constants on surfaces which possess a lattice of affine maps. To explain his idea we begin with the definition of the affine group of a translation structure and general properties of it.

2.1 The affine group *Aff(u)*

Notation: Let L be either the group $PGL_2(\mathbb{R})$ if we speak about F structures or the group $GL_2(\mathbb{R})$ if we speak about translation structures. In the same sense G denotes the group $SL_2(\mathbb{R})$ or $PSL_2(\mathbb{R})$.

Definition 2.1 *Let $u \in \Omega(g,\ n)$ and $v \in \Omega(g',\ n')$ then an orientation preserving homeomorphism $\phi : S_g \longrightarrow S_{g'}$ is called **affine**, if*

- $\phi^{-1}(O_{n'}) \subset O_n$
- *ϕ is affine in natural coordinates of u and v, that is for any two charts $S_{g,n} \supset U \xrightarrow{f} \mathbb{R}^2$ and $S_{g',n'} \supset V \xrightarrow{h} \mathbb{R}^2$ exist affine linear maps $\mathbf{A}_{(f,h)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with:*

$$\begin{array}{ccc} S_{g,n} \supset U & \xrightarrow{\phi} & \phi(V) \cap U \subset S_{g',n'} \\ f\big\downarrow & & \big\downarrow h \\ \mathbb{R}^2 & \xrightarrow[\mathbf{A}_{(f,h)}]{} & \mathbb{R}^2 \end{array} \tag{2.1}$$

We denote the set of affine maps between u and v with $Aff(u,v)$, if $u = v$ we write $Aff(u)$.

Remarks: Since $\phi \in Aff(u,v)$ is locally affine linear, ϕ is differentiable. The derivative is (locally) constant and determined up to the factor ± 1, thus it is an element of $PGL_2(\mathbb{R})$. Therefore the differential d defines a map

$$d : Aff(u,v) \longrightarrow PGL_2(\mathbb{R})$$

If we take a representative $A \in GL_2(\mathbb{R})$ of $[d\phi]$ then the set theoretical identity id of $S_{g,n}$ induces an affine map $id : Au \longrightarrow u$ with differential A^{-1}. Thus $\phi : Au \longrightarrow v$ is an affine map with derivative $\pm id$ and therefore a F map as defined in 1.2:

Proposition 2.1 *Every affine map $\phi \in Aff(u,v)$ defines F maps $\phi : Au \longrightarrow v$ and $\phi : u \longrightarrow Av$ with $A \in [d\phi]$.*

Proposition 2.2 *If*

$$\begin{array}{ccc} S_{g_1} & \xrightarrow{f} & S_{g_2} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle h} \\ S_{g_3} & \xrightarrow[g]{} & S_{g_4} \end{array} \tag{2.2}$$

is a commutative rectangle of maps and assume three of them are affine maps or F maps with respect to given F structures u_i on S_{g_i} $(i = 1, ..., 4)$, then the last one is an affine map, F map respectively.

Proof: In the case of affine maps one can use the last proposition to change the F structures by elements of $SL_2(\mathbb{R})$ so that all the affine maps become F maps. Since F maps are complex analytic maps they are covering maps, eventually branched over the singular points. Covering maps are one to one away from their branch points, thus the fourth map has to be a F map, too. The statement for affine maps follows by changing the F structures to the ones we started with.

□

Each map $\phi \in Aff(u,v)$ induces surjective maps on the sets of saddle connections $SC(u) \xrightarrow{\phi_*} SC(v)$ and on the set of closed geodesics $PO(u) \xrightarrow{\phi_*} PO(v)$. If $u = v$ it can be said more because

Proposition 2.3 *The set of affine maps on u is a group under composition of maps.*

The prove is obvious and the map d becomes a group homomorphism onto

$$V(u) = \{A \in L : A = \pm d\phi,\ \phi \in Aff(u)\} \tag{2.3}$$

$V(u)$ is called the $Veech\ group$ of u. The kernel $N(u) := \ker(d)$ is a group, too. Thus there is an exact sequence:

$$1 \longrightarrow N(u) \longrightarrow Aff(u) \xrightarrow{d} V(u) \longrightarrow 1$$

$Aff(u)$ has a canonical subgroup $Aff_0(u)$ defined as :

$$Aff_0(u) := \{\phi \in Aff(u) : \phi|_{O_n} = id\}. \tag{2.4}$$

All the properties which we will state and prove will hold for both groups, but $Aff(u)$ makes the connection to the moduli spaces $\mathcal{QD}(g,n)$ defined in the first chapter more canonical. The Veech group operates as a subgroup of L on F structures

$$v \longrightarrow A.v \quad A \in L \quad v \in \Omega(S_{g,n})$$

as well as the affine group:

$$v \longrightarrow \phi_* v \equiv v \circ \phi^{-1} \quad \phi \in Aff(u) \quad v \in \Omega(S_{g,n}).$$

On u itself one has by definition

$$d\phi.u = \phi_* u = u \circ \phi^{-1} \quad \phi \in Aff(u) \tag{2.5}$$

The general properties of $Aff(u)$ and $V(u)$ are:

Proposition 2.4 *1. Let $u \in \Omega^+(g,n)$ an F structure $\phi \in Aff(u)$, then $d\phi \in G$. Thus $V(u) \subset G$.*

2. $V(u)$ is a discrete subgroup of G.

3. $V(gu) = gV(u)g^{-1} \quad \forall\ u \in \Omega(g,n)$ and $\forall\ g \in L$

4. The trace $tr(A)$ of each $A \in V(u)$ is an algebraic number.

5. If $\phi \in Aff(u)$ and $d\phi \neq 1$ then ϕ is not homotopic to the identity relative O_n.

6. $Aff(u)$ is a discrete subgroup of $H^+(g,n)$.

7. If u has singular points, $N(u)$ is a finite group of F maps, holomorphic with respect to $\mathcal{J}_u$.

Proof:

1. We can assume that the natural volume of u is 1. Then

$$1 = \int_{\Sigma_{g,n}} \omega_u = \int_{\phi^{-1}(\Sigma_{g,n})} \phi_*^{-1}\omega_u = \int_{\Sigma_{g,n}} det(d\phi)\omega_u =$$

$$= det(d\phi) \int_{\Sigma_{g,n}} \omega_u = det(d\phi)$$

2. The length function $l_{gu}(s)$ of each saddle connection $s \in SC(gu)$ is a continuous function in $g \in G$. If $g \in V(u)$ then g induces a self map of $SC(u)$ But the length spectrum of $SC(u)$ can not have accumulation points, that would contradict the finiteness of the number of saddle connections under a given length.

3. This follows from

$$gag^{-1}g.u = ga.u = g.u \ \forall a \in V(u), \ u \in \Omega(g,n) \text{ and } \ g \in L$$

4. Let $[u] \in Q(g,P)$ then $Aff(u)$ is a subgroup of the mapping class group $\Gamma(g,n)$ operating on $Q(g,P)$. In coordinates modeled on $H^1(S_g, O_n; \mathbb{R})$, $\Gamma(g,P)$ is up to conjugation represented by $GL_{2(2g-1+n)}(\mathbb{Z})$. As stated in the chapter on the spaces of quadratic differentials, if $A \in V(u)$: A operates linearly on the first factor of $\mathbb{R}^2 \otimes H^1(S_g, O_n; \mathbb{R})$. Thus it has to be in $GL_{2(2g-1+n)}(\mathbb{R})$. But since every element of the Veech group is represented by an element of the mapping class group (Equation 2.5) it is conjugated to an element of $GL_{2(2g-1+n)}(\mathbb{Z})$. Thus there is an equation $\sum_{i=0}^{n} p_i A^i = 0$ with $p_i \in \mathbb{Z}$ proving the claim.

5. If ϕ is homotopic to the identity modulo O_n, then it is already the identity restricted to O_n. Let $A = d\phi \in V(u)$ if we assume $A \neq id$ then A induces a non trivial map on the set of saddle connections $SC(u)$. Let $s \in SC(u)$ with $\phi(s) \neq s$, these are two geodesics with respect to the Euclidean metric on u connecting the same points. Thus they cannot be homotopic. This proves the claim.

6. If not there is a sequence of maps $\phi_i \in Aff(u)$ converging to some $\phi \in H_0^+(g,n)$. Thus $\phi_i \circ \phi_{i-1}$ converges to the identity and must be homotopic to the identity for i large enough.

7. An element $\phi \in N(u)$ is a map locally represented by $z \longrightarrow \pm z + c$ where $c \in \mathbb{C}$. Thus the map is holomorphic, even more it is an isometry with respect to the natural metric g_u. We know ϕ, if we know what it does with one fixed saddle connection s. By the assumption on the existence of singular points there exist saddle connections on u. Since ϕ is an isometry and there are only finitely many saddle connections of the same length on u (by Masur's quadratic bound on the growth rate) there can be only finitely many different maps in $N(u)$. The assumption on the existence on singular points is important, the translation torus without singular points has a continuous family of F maps, since it is a Lie group.

□

Remarks: The determinant $\det(d\phi)$ of an affine map $\phi \in Aff(u)$ is a well defined map, even if the Veech group is a projective group, because $GL(2,\mathbb{R}) \xrightarrow{\det} \mathbb{R}$ is invariant under multiplication with $-id \in GL(2,\mathbb{R})$.

If

$$A = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R})$$

then A is called *parabolic* if $|tr(A)| = |a + d| = 2$, *hyperbolic* if $|tr(A)| > 2$ and *elliptic* if $|tr(A)| < 2$. Thus rotations and reflections are elliptic.

For the order of the group $N(u)$ one obtains the canonical estimate

$$|N(u)| \leq \min_{\{s_1 \text{singularity}\}} \{\nu(s_1)\, \mathrm{ord}(s_1)\}.$$

The s_i are the end points of the chosen saddle connection s $\mathrm{ord}(s_1)$ is the order of the singular point s_1 (with respect to the metric g_u) and $\nu(s_1)$ is the number of singular points of order $\mathrm{ord}(s_1)$. Consequently, if there is a singularity s_1 on u with $\nu(s_1) = 1$ and $\mathrm{ord}(s_1) = 2$, then $|N(u)| \leq 2$.

Proposition 2.5 ([Vch89]) *The Veech group $V(u)$ of an F structure u is never co-compact in G.*

Proof: We have to show that there is a sequence in $G/V(u)$ without limit points.
To find it we take a direction $\theta \in [0, 2\pi]$ on u so that $\mathcal{F}_\theta(u)$ contains a saddle connection s.

$$a_t \;=\; \pm \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \quad \in G \tag{2.6}$$

operates on $r_\theta.u$ and contracts the saddle connection e^{-t}. If we assume $G/V(u)$ is compact then there exists a subsequence of

$$a_t r_\theta.u \in G.u, \quad t \in \mathbb{R}_+$$

which converges modulo $V(u)r_\theta.u \cong r_\theta.u$, but this in turn implies that there is a sequence

$$b_i a_{t_i} r_\theta.u \longrightarrow u_\infty \in G.u$$

Thus the saddle connection s converges to some saddle connection s_∞ on u_∞ and by the continuity of the length function $l_{u_\infty}(\cdot)$: $l_{u_\infty}(s) > 0$ a contradiction.

□

Lemma 2.6 *Let $u_{G_n} = G_n \times S_{G_n}/\sim$ denote the F structure of a billiard in a n-gon as constructed in chapter 1. The reflection group S_{G_n} operates as a group of affine maps on u_{G_n}. The derivatives in $V(u_{G_n})$ are elliptic elements of G.*
In particular for the F structure associated to a non rational n-gon the orders of the affine and the Veech group are infinite.

Proof: By definition S_{G_n} operates from the left on $u_{G_n} = G_n \times S_{G_n}/\sim$ inducing affine maps ϕ_g with derivative g:

$$u_{G_n} \xrightarrow{\phi_g} u_{G_n} \cong gG_n \times S_{G_n}/\sim$$

Since the elements in S_{G_n} are all reflections or rotations, they are elliptic.

□

Lemma 2.7 (W. Veech [Vch89]) *Let $u \in \Omega(g,n)$, $\phi \in Aff(u)$ and $A \in \{\pm d\phi\}$. Is $v \in \mathbb{R}^2 \backslash \{(0,0)\}$ an eigenvector to the eigenvalue 1 of A, then if $d\phi \neq \pm id$, every trajectory of $\mathcal{F}_v(u)$ is closed or a saddle connection. In this case $S_{g,n}$ is decomposed in $n < \infty$ maximal cylinders $\mathcal{C}_1$,..., $\mathcal{C}_n$ of periodic trajectories, bounded by saddle connections. Furthermore there is a $k > 0$, such that $\phi^k|_{\mathcal{C}_j}$ is a linear Dehn twist with $\phi^k|_{\partial\mathcal{C}_j} = id$.*

Proof: Let O_n be the set of singular points of u and $d\phi \neq \pm id$. There are only finitely many trajectories of $\mathcal{F}_v(u)$ starting or ending in each singular point of O_n, they are called separatrices. Since the leaves of $\mathcal{F}_v(u)$ are mapped to leaves under ϕ there has to exist a $k > 0$ with $\phi^k(l_v) = l_v$, for every separatrix $l_v \in \mathcal{F}_v(u)$. Since ϕ^k is continuous, the boundary point of l_v is a fixpoint and we have $\phi^k|_{l_v} = id$. Either l_v is dense in some open set of u, in which case ϕ^k is $\pm$ identity on that open set by continuity, or it is a saddle connection. In the first case $d\phi^k = \pm id$ contradicting our assumption. Thus every separatrix is a saddle connection, moreover every leaf has to be compact: if not there is one that approaches a saddle connection and therefore again $\phi^k = \pm id$ contradicting the assumption. The only compact leaves are saddle connections or periodic orbits. Since there exists only finitely many singular points, there are only finitely many maximal cylinders of periodic trajectories $\mathcal{C}_1$,..., $\mathcal{C}_n$. Because ϕ^k is the identity on their boundary components it is a linear Dehn twist restricted to each cylinder. If we isometrically identify $\mathcal{C}_i$ with $(0,w) \times [0,h]/(x,0) \sim (x,h)$ then ϕ^k is represented as

$$\phi^k \quad = \quad \begin{pmatrix} 1 & 0 \\ m\frac{h}{w} & 1 \end{pmatrix} \pmod{h} \tag{2.7}$$

$m \in \mathbb{Z}$ and $\pmod{h}$ has to be understood with respect to the off diagonal element.

□

2.2 Affine coverings and the Veech group

At beginning the results on the behavior of the Veech group under covering maps are presented. The material is contained in the papers of Gutkin and Judge [GutJdg97] as well as in Vorobetz [Vrb96b]. First a definition

Definition 2.2 *Two F covers $u \xrightarrow{\pi} v$, $u \xrightarrow{\psi} w$ are called lower equivalent, if there is an F isomorphism $v \xrightarrow{\phi} w$ with $\phi \circ \pi = \psi$. Two F coverings $u \xrightarrow{\pi} v$, $w \xrightarrow{\psi} v$ of v are called upper equivalent, if there is an F isomorphism $u \xrightarrow{\phi} w$ with $\phi \circ \psi = \pi$.*

Note that this is an equivalence relation. Given a F covering $u \xrightarrow{\pi} v$ of two F structures u, v every $A \in GL_2(\mathbb{R})$ defines a new F covering, given by

$$Au \xrightarrow{\pi} Av.$$

As well as every $\psi \in \mathit{Aff}(u)$ then ψ defines a new F covering by

$$u \xrightarrow{\psi} d\psi^{-1}u \xrightarrow{\pi} d\psi^{-1}v.$$

This defines an operation of $\mathit{Aff}(u)$ on coverings (of fixed singularity and topological type) and since equivalence classes of covers are mapped to equivalence classes under this operation $\mathit{Aff}(u)$ the operation descends. Now if $u \xrightarrow{\pi} v$ is lower equivalent to $u \xrightarrow{\pi\circ\psi} d\psi^{-1}v$, then by definition there exists an F isomorphism $v \xrightarrow{\phi} d\psi^{-1}v$. But this is an affine map $\phi \in \mathit{Aff}(v)$ with derivative $d\phi = d\psi$. Thus the isotropy subgroup $\mathit{Aff}_\pi(u) \subset \mathit{Aff}(u)$ of the operation of $\mathit{Aff}(u)$ on lower equivalence classes consist exactly of the elements which descend to $\mathit{Aff}(v)$, especially $V(u) \cap V(v) = d\mathit{Aff}_\pi(u) \cap V(v)$. The analogous result holds for the operation of $\mathit{Aff}(v)$ on F coverings. To use these observations one has to prove the finiteness of the set of isomorphy classes of covers. We recall the result obtained by Vorobetz [Vrb96a] and Gutkin and Judge [GutJdg97]:

Lemma 2.8 *Let $u = (S_{g,n}, q)$ be an admissible F structure with singularities (maybe of order zero) in $O_n = S^c_{g,n} \subset S_g$.*

1. *Up to isomorphy there exist only finitely many translation coverings of u of fixed degree d branched only over O_n, their number is smaller then $d!^{3(2g-2+n)}$*
2. *The same is true for translation structures of fixed degree that are covered by u.*

Proof: Fix a geodesic triangulation of u that is:

- the faces are triangles with respect to the Euclidean metric structure on u,
- the vertices are the vertices of the triangles and equal the points in O_n,
- the edges are saddle connections.

For the existence of such a triangulation see [GutJdg97]. Further let us assume that we have a translation covering $\pi : v \longrightarrow u$ of u with degree d, than the preimage of the geodesic triangulation of u defines a geodesic triangulation on v. The preimage of each edge or face consists of d isometric copies of itself. Because by definition the edges of a triangle in the cover have to be mapped to edges, for each edge there are $d!$ possibilities how preimages of the triangles border on it can be identified. The kind of identification can change only at branch points which are contained in the points O_n. Thus $d!^{|edges|}$ gives an estimate for the number of covers of u of degree d branched over O_n. But because of the construction of the triangulation $|\{vertices\}| = |O_n| = n$, $2|\{edges\}| = 3|\{faces\}|$ the number of edges is related to the genus g of S_g by Eulers formula

$$2(1-g) = n + (\frac{2}{3} - 1)|\{edges\}| \quad \Rightarrow \quad |\{edges\}| = 3(2g - 2 + n)$$

For the second statement assume we have two translation coverings $\pi : u \longrightarrow v$ and $\psi : u \longrightarrow w$ of degree d covered by u. Then the set of points on w where $\pi \circ \psi^{-1}$ defines a (bijective) map is open and closed and if it is not empty it contains all non singular points because w is connected. Thus the only question is how many possibilities there are for one non singular point of u to be mapped under a covering map of degree d. For this let $\epsilon > 0$ be so small that the ϵ balls around the singular points on u do not intersect. Further fix a direction θ and take the intervals of length ϵ starting at a singular point in direction θ. These are $q+1$ intervals for each metric singularity of order q. By the argument above, the number of different images of the (nonsingular) endpoints of the intervals can achieve, limits the number of possible translation structures covered by u. Since by construction the number of points is the number of singularities counted with multiplicity: $2g - 2 + n$, we have 2^{2g+n-3} possible covers.

□

From this it follows that

Corollary 2.9 *Given a translation covering $u \xrightarrow{\pi} v$, or an affine map $u \xrightarrow{\phi} w$ with $d\phi = A$ than*

1. *$V(u) \cap V(v)$ has finite index in $V(u)$ and $V(v)$, thus $V(u)$ and $V(v)$ are commensurate.*

2. *Since $V(Au) = AV(u)A^{-1}$ and there is a* F *cover*

$$Au \xrightarrow{id} u \xrightarrow{\phi} w$$

$V(w) \cap AV(u)A^{-1}$ has finite index in $V(w)$. Thus $V(u)$ and $V(v)$ are commensurable.

3. *If $V(v)$ contains a unipotent subgroup Γ, then $V(u) \cap \Gamma$ is of finite index in Γ and*

4. *if $V(u)$ contains a unipotent subgroup Γ, then $V(v) \cap \Gamma$ is of finite index in Γ.*

Remark: A subgroup of $SL_2(\mathbb{R})$ is called *unipotent*, if it contains only elements which have all eigenvalues equal to one.

Example: Take a rectangle and identify parallel boundarys to become a translation torus, named $\mathbb{T}^2$. The Veech group of $\mathbb{T}^2$ is $SL_2(\mathbb{Z})$. Now represent the element $-id \in SL_2(\mathbb{Z})$ as derivative of the following affine map on $\mathbb{T}^2$: rotate the rectangle about 180 degrees and translate the result over the given one. This map has exactly four fixed points on $\mathbb{T}^2$ represented on the rectangle by the vertices, the middle points of the edges and the center. Taking the quotient half translation structure with respect to this involution we get a surface of genus 0, with four conic singularities each having an angle π. This can be viewed as a sphere with four horns and we will call it the pillow $\mathbb{P}^2$. The Veech group of the pillow is by construction $V(\mathbb{P}^2) = SL_2(\mathbb{Z})/(\pm id) \cong PSL_2(\mathbb{Z})$.

For coverings of the pillow $\mathbb{P}^2$ and the translation torus $\mathbb{T}^2$ it can be said more:

Theorem 2.10 ([GutJdg97] Theorem 5.4) *If u is a translation structure, then the following are equivalent:*

1. *The groups $V(u)$ and $SL(2,\mathbb{Z})$ are commensurable.*
2. *There exists a translation covering $u \xrightarrow{\pi} \mathbb{T}^2$, branched only over one point.*
3. *There exists an Euclidean parallelogram that tiles u by translations.*

Theorem 2.11 *Let u be a half translation structure, then the following statements are equivalent:*

1. *The groups $V(u)$ and $PSL(2,\mathbb{Z})$ are commensurable.*
2. *There exists a translation covering $u \xrightarrow{\pi} \mathbb{P}^2$.*
3. *There exists a tiling of u by an Euclidean parallelogram, where the edges can be identified by translations and rotations about 180 degrees.*

Proof: For the prove we assume the equivalences in the case of a translation structure u are true, the proof is contained in the paper [GutJdg97] of Gutkin and Judge. Let u be a half translation surface whose structure is not reducible to a translation structure. Then there exists a connected two fold covering map $v \xrightarrow{\psi} u$ (see Masur [Msr82]) where v is a translation surface (with the induced translation structure) and which has at least one branch point p. We assume all preimages of singular points in u are singular in v even if their degree is 0. Further there is a canonical involution i on v with the branch points of ψ as fixed point set. Since i is an isometry of order two its derivative is equal to $-id$. Now assume we have a tiling of u by parallelograms, which is not a translation tiling but contains identifications by rotations about 180 degrees. Then the u cannot be a translation structure, because the foliations parallel to the edges of the rectangles are not orientable. Thus the orientation covering as above is connected and the involution i preserves the parallelograms of the induced tiling, in the sense that it maps parallelograms to parallelograms. Also the tiling is now a translation tiling, otherwise the covering v is not the translation covering. By Gutkin and Judge we have a translation map $v \xrightarrow{\phi} \mathbb{T}^2$ to the translation torus $\mathbb{T}^2$ which is defined as the quotient of the parallelogram by identifying parallel edges. Since i preserves rectangles, i descends to an affine involution of the torus with derivative $-id$ and one fixed point given by the branch point of ϕ (represented by the vertex points of the parallelogram). This is exactly the affine map in the construction of the pillow above. Thus the map ϕ descends to an half translation map $u \xrightarrow{\pi} \mathbb{P}^2$. By corollary 2.9 (1) from the existence of the map π it follows that $V(u)$ is commensurable to $PSL_2(\mathbb{Z})$ and the tiling of u by parallelograms is obtained by the preimages of $\mathbb{P}^2$. Starting with the assumption u has a Veech group commensurable to $PSL_2(\mathbb{Z})$, we will again use an orientation cover v. The Veech group of $V(v)$ is a subgroup commensurable to $SL_2(\mathbb{Z})$ since the involution i has derivative $-id$. Again using the result of Gutkin and Judge, as well as the above construction proves the theorem.

□

One can refine the first theorem by using an observation of Vorobetz [Vrb96b]

Proposition 2.12 (Ya. B. Vorobetz) *Given a translation covering $u \xrightarrow{\pi} v$ of degree d. If the lower affine equivalence class of degree d coverings defined by π is the only one, then $V(u) \subset V(v)$. This is the case if there exists a direction on v with exactly one saddle connection. In particular for each translation covering $u \xrightarrow{\pi} \mathbb{T}^2$ of the torus branched over one marked point: $V(u) \subset SL_2(\mathbb{Z})$.*

Proof: If $A \in V(u)$ then there is a affine map ϕ such that $u \xrightarrow{\phi} Au \xrightarrow{\pi} Av$ is a F map. $\pi \circ \phi$ is by hypotheses in the same class as $u \xrightarrow{\pi} v$, thus there exists a F map $v \xrightarrow{\psi} Av$, it follows $V(u) \subset V(v)$. If there exists a direction which contains exactly one saddle connection on v, then the same is true for all covers $u \xrightarrow{\pi} w$ of degree d. By the proof of Lemma 2.8 w is isomorphic to v.

□

Lemma 2.13 *Each u such that $[u] \in \mathcal{Q}(g, P)$ has projective rational coordinates is a covering of the torus or the pillow.*

Proof: All the horizontal and vertical cycles have (modulo a common factor) rational length in this case, thus on u there is a common subdivision by rectangles of the same length and hight.

□

2.3 Counting on Veech surfaces

In a remarkable work W. Veech [Vch89] found that F structures u where $V(u)$ is a lattice have quadratic growth constants, which can be calculated by information about the closed cylinders of geodesics in finitely many directions and on the isotropy subgroup of $V(u)$ fixing these directions. Moreover Veech was able to prove that the invariant surfaces of the billiard in a regular n-gon always have a lattice Veech group. He computes the associated asymptotic constants later in [Vch92] using his formula.

The idea is the following: the image of $SL_2(\mathbb{R})u$ of an F structure $u \in \Omega(g,n)$ in the modulispace $\mathcal{QD}(g,n)$ is isomorphic to $SL_2(\mathbb{R})/V(u)$, because by the definition of the Veech group Au is identified with u in $\mathcal{QD}(g,n)$ if $A \in V(u)$. If $V(u)$ is a lattice, then by Proposition 2.5 it is not cocompact, but has always finite covolume (with respect to the Poincaré metric). By the general theory $SL_2(\mathbb{R})/V(u)$ has finitely many "cusps" $[\Lambda_i]$, $i = 1, ..., k$. Group theoretically cusps are conjugation classes (in V(u)) of maximal unipotent (parabolic) subgroups of $V(u)$. As subgroups of $SL_2(\mathbb{R})$ the cusps are all conjugated to the group

$$\left\{ \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} : j \in \mathbb{Z} \right\}$$

The two main observations if the Veech group has finite covolume are:

Theorem 2.14 *[Vch89] (Veech alternative) Let u be an F structure with a lattice Veech group $V(u)$. Then for a given $\theta \in S^1$ the leaves of the foliation $\mathcal{F}_\theta(u)$ are either all compact or the foliation is uniquely ergodic and every leave is dense on u. Moreover if the leaves in direction θ are compact then there exists a maximal unipotent subgroup $\Lambda \subset V(u)$ which, represented as linear maps on $\mathbb{R}^2$, have eigenvalues in direction θ.*

Thus this Λ fixes the foliation $\mathcal{F}_\theta(u)$. Because up to conjugation in $V(u)$ there are only finitely many such isotropy groups Λ we have finitely many directions $\theta_1, ..., \theta_k$ with closed foliations. Let $v_1, ..., v_k \in \mathbb{R}^2$ be a set of vectors representing the above directions. Then each v_i is an eigen direction for the group Λ_i and for every other closed direction represented by the vector v there exists an $A \in V(u)$ with $A(v) = v_i$ iff the isotropy group of v is conjugated to Λ_i. By Lemma 2.7 if $1 \neq A \in V(u)$ fixes a direction θ, then the foliation $\mathcal{F}_\theta(u)$ splits in n cylinders $\mathcal{C}_1, ..., \mathcal{C}_n$ of closed geodesics with the hights $h(\mathcal{C}_i)$ and widths $w(\mathcal{C}_i)$. If v is the eigenvector of A and $v^\perp$ a vector orthogonal to v, then

$$Av_i^\perp = v_i^\perp + m_i \frac{h(\mathcal{C}_i)}{w(\mathcal{C}_i)} v_i.$$

With respect to this coordinates A operates on $\mathcal{C}_i$ as:

$$A = \begin{pmatrix} 1 & m_i \frac{h(\mathcal{C}_i)}{w(\mathcal{C}_i)} \\ 0 & 1 \end{pmatrix} \pmod{h(\mathcal{C}_i)}$$

If we assume A is the generator of the parabolic subgroup Λ_i in $V(u)$ with v as an eigenvalue, then m_i is the index of Λ_i in the group of Dehn twists of $\mathcal{C}_i$ generated by $\left((x,y) \longmapsto \left(x, y + \frac{h(\mathcal{C}_i)}{w(\mathcal{C}_i)}\right)\right)$. If we define m to be the least common integer multiple of $\left(\frac{h(\mathcal{C}_1)}{w(\mathcal{C}_1)}, ..., \frac{h(\mathcal{C}_n)}{w(\mathcal{C}_n)}\right)$ then

$$A_0 = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \quad (\text{mod } \operatorname{lcm}\{h(\mathcal{C}_1), ..., h(\mathcal{C}_n)\})$$

defines a Dehn twist on each cylinder $\mathcal{C}_i$, and this is the smallest m possible. Furthermore the linear maps on the cylinders glue together to give a global affine map $\phi \in \mathit{Aff}(u)$ with $d\phi = A_0$, thus $A_0 = A$. The discussion also results a necessary condition for a flat surface to have a lattice Veech group: for every periodic direction with cylinder decomposition $\mathcal{C}_1, ..., \mathcal{C}_n$

$$\frac{h(\mathcal{C}_i)}{w(\mathcal{C}_i)} \frac{w(\mathcal{C}_j)}{h(\mathcal{C}_j)} \in \mathbb{Q}.$$

Vorobetz ([Vrb96b] Theorem 4.5) found examples of polygonal billiard surfaces for which this criterion is not fulfilled. Now we can parametrice all periodic cylinders in terms of the affine group $\mathit{Aff}(u)$ and following Veech write down the ζ function counting the length of cylinders of prime geodesics for each cusp $[\Lambda_i]$ separately:

$$\zeta_i(u, s) = \sum_{\phi \in \mathit{Aff}(u)/d^{-1}\Lambda_i} \sum_{j=1}^{n_i} \frac{1}{h(\phi \mathcal{C}_i)^s} \tag{2.8}$$

Using this and an Ikehara Tauberian theorem Veech was able to prove:

Theorem 2.15 (W. Veech [Vch89]) *Let $u \in \Omega(g, n)$ and $V(u)$ a lattice and $N_{PO}([\Lambda_i], u, T)$ the number of cylinders of closed geodesics of length smaller than T with isotropy group conjugated to Λ_i, then*

$$\lim_{T \to \infty} \frac{N_{PO}([\Lambda_i], u, T)}{T^2} = Res_{s=1} \zeta_i(u, 2s) = \frac{1}{Vol(\mathbb{H}/V(u))} \sum_{j=1}^{n_i} \frac{m_{ij}}{area(\mathcal{C}_{ij})} \tag{2.9}$$

and

$$c_{PO}(u) = \lim_{T \to \infty} \frac{N_{PO}(u, T)}{T^2} = \frac{1}{Vol(\mathbb{H}/V(u))} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{m_{ij}}{area(\mathcal{C}_{ij})} \tag{2.10}$$

where k is the number of cusps $[\Lambda_i]$ in $V(u)$ and n_i is the number of maximal cylinders of periodic orbits in an eigendirection of a maximal parabolic subgroup conjugated to Λ_i.

Remark 2.1 *Because $gV(u)g^{-1} = V(gu)$ for all $g \in SL_2(\mathbb{R})$, the cusps of $V(u)$ and $V(gu)$ are conjugated. Thus the group indices $m_{ij}(u) = m_{ij}(gu)$ are invariant under* *the operation of* $SL_2(\mathbb{R})$. *Because of* $\mathcal{C}_{ij}(gu) = g\mathcal{C}_{ij}(u)$ *the same holds for the areas of the maximal periodic cylinders. This implies that the asymptotic formula above is the same for every* F *structure in the Teichmüller disc* $SL_2(\mathbb{R})/V(u)$.

If we take the $GL_2^+(\mathbb{R})$ *orbit of* u *then the same is true up to the fact that the areas of the closed cylinders have to be multiplied by* $\det(g)$ *for* gu $g \in GL_2^+(\mathbb{R})$. *Thus*

$$c_{PO}(gu) = \det(g)^{-1} c_{PO}(u)$$

.

Since, if u *is a Veech surface, all the saddle connections of* u *bound cylinders of closed geodesics a similar asymptotic formula could be written down for saddle connections.*

By now there are ways to compute the asymptotic growth rates $N_{PO}(u,T)$ without using the Tauberian Theorem. Gutkin and Judge [GutJdg97] have used the mixing property of the geodesic flow on negatively curved surfaces.

At the end of this chapter some results on isotropy subgroups of the affine group are discussed. They will be useful in Chapter 4.

Definition 2.3 *Let* $u = (S_g, \mathcal{A})$ *be an* F *structure and* $x \in S_g$. *Then*

$$\mathit{Aff}_x(u) := \{g \in \mathit{Aff}(u) : gx = x\}$$

and

$$V_x(u) := d \circ \mathit{Aff}_x(u) \subset V(u).$$

By general group theory these are subgroups of $\mathit{Aff}(u)$, $V(u)$ *respectively.*

The following is a direct consequence from the definitions and elementary group theory.

Proposition 2.16 *Let* u *be an* F *structure and* u_x *the* F *structure* u *where the point* x *is marked, that is* x *is supposed to be a singular point (of degree 0) of the natural flat metric. Then*

$$[\mathit{Aff}(u) : \mathit{Aff}_x(u)] = \mathrm{ord}(\mathit{Aff}(u)/\mathit{Aff}_x(u)) = |O_{\mathit{Aff}(u)}(x)| \tag{2.11}$$

and

$$[V(u) : V_x(u)] = \mathrm{ord}(V(u)/V_x(u)) \tag{2.12}$$

where $O_{\mathit{Aff}(u)}(x) := \{y \in S_g : y \in \mathit{Aff}(u).x\}$ *is the orbit of* x *under* $\mathit{Aff}(u)$ *and* G/H *denotes the set of right cosets* xH *of the subgroup* $H \subset G$.

Proof: See for example Chapter I §5 in S. Langs book on algebra [Lang].

□

Now, with the assumptions as above, we can define a new F structure u_x. Take a nonsingular $x \in S_g$ and view it as a singularity of order 0 in the Euclidean metric on u.

Lemma 2.17 *Let u be an F structure and u_x as above. If x is the only singularity of order 0 then*

$$Aff(u_x) = Aff_x(u) \quad and \quad V(u_x) = V_x(u) \tag{2.13}$$

Proof: Since x is the only singularity of order 0 it has to be a fixpoint of the affine maps on u_x. The resulting affine maps are the affine maps of u which fix x.

□

Remark 2.2 *The restriction to one point of order zero is important. If one drops that condition the affine group of u_x might be bigger than the one of u.*

Take for example the torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ with singularities of order zero in the points $[0, 1/2], [1/2, 0]$ and $[0, 0]$ (the notation $[x]$ indicates the class of $x \in \mathbb{R}^2$ on the torus $\mathbb{T}^2$). We denote that marked torus by $\mathbb{T}^2_{[1/2]}$, it defines a covering of degree four of $\mathbb{T}^2$, by $\mathbb{T}^2_{[1/2]} \ni [x] \longmapsto [4x] \in \mathbb{T}^2$. Gutkin and Judge's result (Theorem 2.10) implies $V(\mathbb{T}^2_{[1/2]}) \subset SL_2(\mathbb{Z})$. On the other hand

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; A^{\perp} := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \notin V(\mathbb{T}^2_{[1/2]}) \tag{2.14}$$

because both corresponding Dehn twists would map $[0, 1/2]$ and $[1/2, 0]$ to $[1/2, 1/2]$. Thus $V(\mathbb{T}^2_{[1/2]}) \neq SL_2(\mathbb{R})$. Since

$$A_{2m} := \begin{pmatrix} 1 & 2m \\ 0 & 1 \end{pmatrix}, \; A^{\perp}_{2m} := \begin{pmatrix} 1 & 0 \\ 1 & 2m \end{pmatrix} \text{ and } R := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in V(\mathbb{T}^2_{[1/2]}) \quad \forall m \in \mathbb{Z} \tag{2.15}$$

$V(\mathbb{T}^2_{[1/2]})$ is generated by A_2 and R. If we also mark the point $[1/2, 1/2]$ on $\mathbb{T}^2_{[1/2]}$ and denote the resulting translation torus by $\mathbb{T}^2_{[1/2,1/2]}$, then A^m, $R \in V(\mathbb{T}^2_{[1/2,1/2]})$ $\forall m \in \mathbb{Z}$, thus $V(\mathbb{T}^2_{[1/2,1/2]}) \cong SL_2(\mathbb{Z})$.

Proposition 2.18 *If u, v are F structures and $\phi \in Aff(u, v)$, then*

$$Aff(u) \cong Aff(v) \quad and \quad V(u) \cong V(v) \tag{2.16}$$

In fact the groups are conjugated. Thus if u is a Veech surface: $c_{PO/SC}(u) = c_{PO/SC}(v)$.

Proof: If there exists an affine map $\phi \in \mathit{Aff}(u,v)$ and if $\psi \in \mathit{Aff}(u)$, then $\phi \circ \psi \circ \phi^{-1} \in \mathit{Aff}(v)$. For the Veech groups we have for $A \in V(u)$, $d\phi \circ A \circ d\phi^{-1} \in V(v)$. That these maps induce isomorphisms can be computed as in the proof of Proposition 2.4 part 3. Another possibility to prove the proposition is realizing that u and v are points on the same Teichmüller disc, since $v = d\phi.u$.

□

One can apply the proposition to marked F structures

Corollary 2.19 *Let u be an F structure $x \in S_g$ and u_x as above. Then for every $y \in O_{\mathit{Aff}(u)}(x)$*

$$\mathit{Aff}(u_x) \cong \mathit{Aff}(u_y) \quad \textit{and} \quad V(u_x) \cong V(u_y) \tag{2.17}$$

the isomorphisms are by conjugations. If u_x is a Veech surface:

$$c_{PO/SC}(u_x) = c_{PO/SC}(u_y) \quad \forall y \in O_{\mathit{Aff}(u)}(x)$$

Proof: By hypothesis there exists an affine map $\phi_{xy} \in \mathit{Aff}(u)$ with $\phi(x) = y$. This defines an affine isomorphism $\phi_{xy} \in \mathit{Aff}(u_x, u_y)$.

□

Chapter 3

Computing asymptotic growth rates with Siegel measures

This chapter is an introduction to the ideas of Veech [Vch98], Eskin and Masur [EskMsr98] to compute asymptotic constants with the help of Siegel measures. Since the results obtained by that viewpoint are essential to motivate the calculations in the next chapter we give a short outline of some of the main statements, ideas or sketches of the proofs.

Let $\mathcal{M}$ be a space with a $SL_2(\mathbb{R})$ action and an $SL_2(\mathbb{R})$ invariant Borel probability measure μ. If μ is ergodic we call it (following Veech [Vch98] Definition 6.3) a "Siegel measure". To obtain the most general results the functions $\mathcal{V}(u)$ earlier used for the vectors associated to saddle connections or cylinders of periodic orbits of a F structure u are generalized to set valued functions $\mathcal{M} \xrightarrow{\mathcal{V}} \mathbb{R}^2 - \{(0,0)\}$ with the properties

(A) For all $g \in SL(2,\mathbb{R})$ and $u \in \mathcal{M}$: $\quad \mathcal{V}(gu) = g\mathcal{V}(u)$.

(B) For every $u \in \mathcal{M}$ there is a constant $c_{\mathcal{V}}^{max}(u) < \infty$ such that $N_{\mathcal{V}}(v,T) < c_{\mathcal{V}}^{max}(u)T^2$ for all v in an open neighborhood $U(u)$ of u.

($\mathbf{C}_\mu$) There exists $T > 0$ and $\epsilon > 0$, such that the function $\mathcal{M} \xrightarrow{N_{\mathcal{V}}(\cdot,T)} \mathbb{R}^2 - \{(0,0)\}$ is in $L^{1+\epsilon}(\mathcal{M},\mu)$.

The following Lemma (Lemma 3.4. in [EskMsr98]) is basic for the calculation of the growth rates. It is in a way a simplification of a more general argument contained in Veechs paper [Vch98]. As before denote

$$a_t := \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \text{ and } r_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Lemma 3.1 (A. Eskin, C. McMullen) *For $f \in C_0^\infty(\mathbb{R}_+^2)$ and every $\epsilon > 0$ there exists $T_0 > 0$ such that*

$$\left| e^{2t} \int_o^{2\pi} f(a_t r_\theta v) d\theta - \frac{1}{2\pi ||v|| e^{-t}} \int_{-\infty}^{\infty} f(x, ||v|| e^{-t}) dx \right| < \epsilon \tag{3.1}$$

for every t with $e^t > T_0$ and every $v \in \mathbb{R}^2$ with $||v|| > T_0$.

The proof of this lemma is not that hard and it has a nice heuristics, which is recalled now:

Suppose $f = \chi_R$ where $R := [-b, b] \times [c, d]$ with $b > 0$ and $0 < c < d$. Then

$$\int_{-\infty}^{\infty} f(x, ||v|| e^{-t}) dx = \begin{cases} \frac{b}{\pi ||v|| e^{-t}} & \text{if } ce^t \le ||v|| \le de^t \\ 0 & \text{otherwise} \end{cases}.$$

Since $a_t^{-1} R = [-e^{-t} b, e^{-t} b] \times [e^t c, e^t d]$ the integral $\int_o^{2\pi} f(a_t r_\theta v) d\theta$ is non zero only if $r_\theta v \in a_t^{-1} R$. In this case it is $\frac{1}{||v|| \theta}$ times the length of the intersection of the circle of radius $||v||$ with the rectangle $a_t^{-1} R$. The intersection of a circle with $a_t^{-1} R$ is empty if its radius $||v||$ is outside $[e^t c, e^t d]$ and essentially the horizontal segment $[-e^{-t} b, e^{-t} b] \times \{||v||\}$. Thus

$$\int_o^{2\pi} f(a_t r_\theta v) d\theta \approx \frac{b e^t}{\pi ||v||} = \frac{e^{-t}}{2\pi ||v||} \int_{-\infty}^{\infty} f(x, ||v|| e^{-t}) dx.$$

Since the estimate becomes arbitrary good as t increases the lemma is immediate for the characteristic function $f = \chi_R$.

From this lemma one can derive directly the following:

Proposition 3.2 (A. Eskin, H. Masur) *Suppose $\mathcal{V}$ satisfies condition (A). Then there exists an absolute constant $c < \infty$ such that for any $T > 0$ and any $u \in \mathcal{M}$,*

$$N_{\mathcal{V}}(u, 2T) - N_{\mathcal{V}}(u, T) \le cT^2 \int_0^{\infty} N_{\mathcal{V}}(a_t r_\theta u, 4) d\theta$$

where $t = \log(T)$.

To obtain more one needs the "Siegel Veech" formula. To state it we have to transform non negative functions on $\mathbb{R}_+^2$ to functions on $\mathcal{M}$.

Definition 3.1 *For a non negative function $f \in C_0^\infty(\mathbb{R}_+^2)$ let the $Veechtransform$ be defined as*

$$\hat{f}(u) := \sum_{v \in \mathcal{V}(u)} f(v).$$

If one assumes $\mathcal{V}$ has property (B), then the sum is finite and $\hat{f}$ is a bounded function on compact subsets of $\mathcal{M}$. A more general version of the following theorem is proved in Veech [Vch98]

Theorem 3.3 (Siegel Veech formula) *Given a space* $(\mathcal{M},\ \mu)$ *with an* $SL_2(\mathbb{R})$ *action as above and* $\mathcal{V}$ *with properties (A), (B) and* (C_μ)*. Then there exists a constant* $c_{\mathcal{V},\mu}$ *such that for any* $f \in C_0^\infty(\mathbb{R}_+^2)$,

$$\int_{\mathcal{M}} \hat{f}(u)d\mu(u) = c_{\mathcal{V},\mu} \int_{\mathbb{R}^2} f(x,y)dxdy. \tag{3.2}$$

The constant $c_{\mathcal{V},\mu}$ *is always non negative; it is zero if and only if* $\hat{f} = 0$ μ *almost everywhere.*

Proof: We recall the presentation contained in the paper of Eskin and Masur [EskMsr98]. As mentioned above, it is itself a reduced version of Veechs proof, but the arguments are the same. The difference is the more general notion of a Siegel measure. By the assumptions (B), (C_μ) and because μ is a probability measure: $\hat{f} \in L^1(\mathcal{M}, \mu)$.

Define the linear functional Φ by

$$\Phi(f) = \int_{\mathcal{M}} \hat{f}(u)d\mu(u).$$

Since the measure μ is invariant with respect to the action of $SL_2(\mathbb{R})$ on $\mathcal{M}$, Φ is itself invariant under the $SL_2(\mathbb{R})$ action on continuous functions on $\mathbb{R}^2$. Φ is positive as well, that is $f \geq 0$ implies $\Phi(f) \geq 0$. From the general theory of such functionals it follows that for all $f \in C_0^\infty(\mathbb{R}^2)$,

$$\Phi(f) = a_\mu f(0) + b_\mu \int_{\mathbb{R}^2} f(x,y)dxdy \tag{3.3}$$

with constants $a_\mu, b_\mu \geq 0$ (positivity), not dependent of f. To prove that $a_\mu = 0$ choose a non negative function $\phi \in C_0^\infty(\mathbb{R}^2)$ so that $\phi(0,0) = 1$ and $\phi \leq 1$ and define the sequence of functions $f_j(x,y) := \phi(jx, jy)f(x,y)$. Then for all $j \in \mathbb{N}$ $f_j(0,0) = f(0,0)$ and $\int_{\mathbb{R}^2} f_j \to 0$ by the theorem of dominated convergence. For fixed $u \in \mathcal{M}$ $\lim_{j\to\infty} \hat{f}_j(u) = 0$, since $\hat{f}(u)$ is bounded and $(0,0) \neq \mathcal{V}(u)$. By the definition of linear functional Φ and the fact that $\hat{f} \in L^1(\mathcal{M}, \mu)$, again by dominated convergence we get $\Phi(f_j) \to 0$. Putting $\hat{f}_j$ into the equation (3.3) and taking the limit $j \to \infty$ shows: $a_\mu = 0$. Finally let $c_{\mathcal{V},\mu} := b$.

$\square$

Theorem 3.4 (A. Eskin, H. Masur) *Suppose $(\mathcal{M}, \mu)$, $\mathcal{V}(\cdot)$ have the properties (A), (B) and (C_μ), then*

$$\lim_{T\to\infty} \frac{N_{\mathcal{V}}(u,T)}{T^2} = \pi c_{\mathcal{V},\mu} \quad \mu \text{ a.e.}$$

$c_{\mathcal{V},\mu}$ as in Theorem 3.3.

Proof: We present the proof from [EskMsr98] (Proposition 3.2.). To begin let

$$J_f(y) := \frac{1}{2y} \int_{-\infty}^{\infty} f(x,y)dx.$$

As a map $J_{(\cdot)} : C_0^\infty(\mathbb{R}_+^2) \longrightarrow C_0^\infty(\mathbb{R}_+)$ J is surjective. Let χ be the characteristic function of the interval $[1,2]$, $\epsilon > 0$ be arbitrary and the constant c as in condition (B). Choose functions $h_\pm \in C_0^\infty(\mathbb{R}_+)$ so that $h_- \leq \chi \leq h_+$ and $\int_0^\infty y(h_+(y) - h_-(y))dy \leq \epsilon/c$. We can find $f_\pm \in C_0^\infty(\mathbb{R}_+^2)$ so that $J_{f_\pm} = h_\pm$. Finally we choose a non negative function $\phi \in C_0^\infty(\mathbb{R}_+^2)$ with $\int_{\mathbb{R}} \phi = 1$ and support so close to zero that $J_{\phi * f_-} \leq \chi \leq J_{\phi * f_+}$, where $(\phi * f)(v) := \int_{-\infty}^{\infty} \phi(t) f(a_{-t}v)dt$. Then for $v \in \mathbb{R}^2$ and $\tau > 0$,

$$J_{\phi * f_-}(||v||e^{-\tau}) \leq \chi(||v||e^{-\tau}) \leq J_{\phi * f_+}(||v||e^{-\tau})$$

Then with the help of Lemma 3.1, for sufficiently large τ

$$e^{2\tau} \int_o^{2\pi} \phi * f_-(a_\tau r_\theta v)d\theta - \epsilon/c \leq \chi(||v||e^{-\tau}) \leq e^{2\tau} \int_o^{2\pi} \phi * f_+(a_\tau r_\theta v)d\theta + \epsilon/c. \tag{3.4}$$

Summing over $v \in \mathcal{V}(u)$ and noting that that by condition (B), the number of nonzero terms with $\chi(||v||e^{-\tau}) \neq 0$, we get

$$e^{2\tau} \int_o^{2\pi} \phi * \hat{f}_-(a_\tau r_\theta u)d\theta - \epsilon e^{2\tau} \leq N_{\mathcal{V}}(u, 2e^\tau) - N_{\mathcal{V}}(u, e^\tau) \leq$$

$$\leq e^{2\tau} \int_o^{2\pi} \phi * \hat{f}_+(a_\tau r_\theta u)d\theta + \epsilon e^{2\tau}. \tag{3.5}$$

Since we did not define μ generic let us remark that it is needed to use Nevos Theorem and that a μ generic set contains a set of μ measure one. Now if we choose the F structure u to be μ generic, then the Theorem of Nevo (see Theorem 1.5. in [EskMsr98]) implies

$$\lim_{\tau\to\infty} \int_o^{2\pi} \phi * \hat{f}_\pm(a_\tau r_\theta u)d\theta = \int_{\mathcal{M}} \hat{f}_\pm(u)d\mu(u) = c_{\mathcal{V},\mu} \int_{\mathbb{R}^2} f_\pm(x,y)dxdy. \tag{3.6}$$

The second equality above is simply the Siegel Veech formula. By construction we can evaluate the right hand side of (3.6)

$$\int_{\mathbb{R}^2} f_\pm(x,y)dxdy = 2\pi \int_0^\infty y J_{f_\pm}(y)dy = 2\pi(3/2+E). \tag{3.7}$$

If we assume without loss of generality that the local constant c in (B) is larger than 1, then $E < \epsilon$. Dividing (3.5) by $e^{2\tau}$, taking the limit for $\tau \to \infty$ and using (3.6) and (3.7) gives

$$\begin{aligned} 3\pi c_{\mathcal{V},\mu} - (2\pi c_{\mathcal{V},\mu}+1)\epsilon &\leq \liminf_{\tau\to\infty} \frac{N_{\mathcal{V}}(u,2e^\tau) - N_{\mathcal{V}}(u,e^\tau)}{e^{2\tau}} \\ &\leq \limsup_{\tau\to\infty} \frac{N_{\mathcal{V}}(u,2e^\tau) - N_{\mathcal{V}}(u,e^\tau)}{e^{2\tau}} \\ &\leq 3\pi c_{\mathcal{V},\mu} + (2\pi c_{\mathcal{V},\mu}+1)\epsilon. \end{aligned} \tag{3.8}$$

Since $\epsilon > 0$ is arbitrary this implies

$$\lim_{T\to\infty} \frac{1}{T^2}\left(N_{\mathcal{V}}(u,2T) - N_{\mathcal{V}}(u,T)\right) = 3\pi c_{\mathcal{V},\mu}$$

where $T := e^\tau$. To finish the proof, since by property (B) and the above equation

$$\frac{1}{T^2}\left(N_{\mathcal{V}}\left(u,\frac{T}{2^i}\right) - N_{\mathcal{V}}\left(u,\frac{T}{2^{i+1}}\right)\right) < \frac{3c}{4^i}$$

we can take the sum

$$\sum_{i=0}^{n} \frac{1}{T^2}\left(N_{\mathcal{V}}\left(u,\frac{T}{2^i}\right) - N_{\mathcal{V}}\left(u,\frac{T}{2^{i+1}}\right)\right) = \frac{N_{\mathcal{V}}(u,T)}{T^2} - \frac{N_{\mathcal{V}}\left(u,\frac{T}{2^{n+1}}\right)}{\left(\frac{T}{2^{n+1}}\right)^2}4^{-n-1} \tag{3.9}$$

and the limits in n and T, we obtain using the dominated convergence theorem:

$$\begin{aligned} \lim_{T\to\infty} \frac{N_{\mathcal{V}}(u,T)}{T^2} &= \lim_{T\to\infty} \sum_{i=0}^{\infty} \frac{1}{T^2}\left(N_{\mathcal{V}}\left(u,\frac{T}{2^i}\right) - N_{\mathcal{V}}\left(u,\frac{T}{2^{i+1}}\right)\right) = \\ &= \frac{3}{4}\pi c_{\mathcal{V},\mu} \sum_{i=0}^{\infty} \frac{1}{4^i} = \pi c_{\mathcal{V},\mu}. \end{aligned} \tag{3.10}$$

□

Now one can use this general framework for spaces of (normed) quadratic differentials $(\mathcal{Q}^1(g,P),\mu)$.

Corollary 3.5 *Let $u \in \mathcal{Q}^1(g,P)$ and μ be a Siegel measure on $\mathcal{Q}^1(g,P)$, then*

$$\lim_{T\to\infty} \frac{N_{\mathcal{V}}(u,T)}{T^2} = \pi c_{\mathcal{V},\mu} \quad \mu \text{ a.e.}$$

for the sets $\mathcal{V}$ of cylinders of closed geodesics, saddle connections or regular and irregular cylinders of closed geodesics.

Property (B) is of the map $\mathcal{V}$ is obtained by Masur's Theorem on the quadratic growth rate stated in the introduction, slightly refined in the sense that the bounding constant can be chosen uniformly on compact sets in the moduli spaces. Property (C_μ) is checked in the paper [EskMsr98] of Eskin and Masur. If μ is the "Liouville measure" μ_1 introduced in the section on the manifold structure of spaces of F structures, then a.e. with respect to μ_1 says that the set of points in $\mathcal{Q}(g, P)$, which have growth limits is "bigger" than a dense set. Thus Eskin and Masurs result contains more points than just the torus or pillow coverings (see 2.13) which are "only" dense in $\mathcal{Q}(g, P)$. Now let μ be the image of the Haar measure on $PSL_2(\mathbb{R})$ under the quotient

$$PSL_2(\mathbb{R}) \xrightarrow{\pi} PSL_2(\mathbb{R})/V(u)$$

where $V(u)$ is the Veech group of a Veech surface $u \in \mathcal{Q}(g, P)$. Taking the $PSL_2(\mathbb{R})$ invariant embedding (the Teichmüllerdisc) $PSL_2(\mathbb{R}).u$ of $PSL_2(\mathbb{R})/V(u)$ in $\mathcal{Q}(g, P)$ together with the "Siegel measure" $\frac{\mu}{vol_\mu(PSL_2(\mathbb{R})/V(u))}$ guaranty by the corollary limits "only" almost everywhere, whereas the "old" theory of Veech (see Theorem 2.15 and the remark after) proves pointwise limits. Thus in this case the new theory seems to be weaker than the old one. But in fact the more subtle approach of Veech (see [Vch98] Theorem 15.10 and Theorem 16.1) reproves that the limits exist pointwise in these cases. Moreover A. Eskin observed that if the spaces $\mathcal{M}$ as above are homogeneous spaces, than Ratners ergodic measure classification (see for example [EskMoSh, Ra91]) on homogeneous spaces can be used to obtain pointwise results. The consequences of Ratners Theorem for homogeneous spaces of F structures are mostly not written down yet, partly the forthcoming paper [EMS] will make use of the theory. The main point is that the right hand side of

$$\lim_{i\to\infty} \int_0^\pi f(a_{t_i} r_\theta u) d\theta = \int_{\mathcal{P}} f d\mu_{\mathcal{P}}.$$

is an integral over the ergodic measure μ as t_i tends to infinity. Then one can apply the Siegel Veech formula to this integral. In terms of weak convergence of measures the above equation means $\lim_{t\to\infty} a_t^*\mu = \mu_\infty$. If one assumes this sort of convergence on a homogeneous space $\mathcal{M}$ with a $SL_2(\mathbb{R})$ action there is the following important observation:

Proposition 3.6 *Assume μ is a measure on a homogeneous space $\mathcal{P} := L/\Gamma$ with an $SL_2(\mathbb{R})$ action and $\lim_{t\to\infty} a_t^*\mu = \mu_\infty$, where the limit is taken in the sense of weak convergence. Then there is a one parameter unipotent subgroup $U \in SL_2(\mathbb{R})$ under which μ_∞ is invariant.*

Proof: Suppose there is a sequence $\mu_i := a_{t_i}^*\mu$ converging to μ_∞, then there exists a sequence $\{\theta_i\}_{i\in\mathbb{N}}$ with $\lim_{i\to\infty}\theta_i = 0$ such that

$$\begin{aligned}\lim_{i\to\infty} a_{t_i}^{-1} r_{\theta_i} a_{t_i} &= \lim_{i\to\infty}\begin{pmatrix} e^{t_i} & 0 \\ 0 & e^{-t_i}\end{pmatrix}\begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i\end{pmatrix}\begin{pmatrix} e^{-t_i} & 0 \\ 0 & e^{t_i}\end{pmatrix} = \\ &= \lim_{i\to\infty}\begin{pmatrix} \cos\theta_i & e^{2t_i}\sin\theta_i \\ -e^{-2t_i}\sin\theta_i & \cos\theta_i\end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1\end{pmatrix} =: u \end{aligned} \tag{3.11}$$

This is immediate since given $\{t_i\}$ with $t_i \to \infty$ we can choose θ_i so, that $e^{2t_i}\sin\theta_i \to a$ and $\theta_i \to 0$. The other entries in the matrix clearly converge to the entries of u. By definition for all $f \in C_0^0(\mathcal{P})$

$$\mu(f) = \int_0^\pi f(r_\theta u)d\theta = \int_0^\pi f(r_{\theta_i} r_\theta u)d\theta$$

thus

$$r_{\theta_i}^*\mu = \mu \quad \Rightarrow \quad a_{t_i}^* r_{\theta_i}^* \mu = a_{t_i}^*\mu \quad \Rightarrow \quad lim_{i\to\infty}\left(a_{t_i}^* r_{\theta_i}^* a_{t_i}^{-1,*}\right) a_{t_i}^*\mu = a_{t_i}^*\mu$$

$$\Rightarrow \quad u^*\mu_\infty = \mu_\infty \tag{3.12}$$

u defines a one parameter unipotent subgroup

$$U = U(u) := \left\{\begin{pmatrix} 1 & ta \\ 0 & 1\end{pmatrix} : t \in \mathbb{R}\right\} \subset SL_2(\mathbb{R})$$

and since by assumption $SL_2(\mathbb{R})$ is a subgroup of $\mathcal{P}$, U is also a subgroup of $\mathcal{P}$. It is easy to see that μ_∞ is invariant under U as well.

$\square$

Now one defines

$$\mathcal{M}_U(\mathcal{P}) := \{\nu \text{ probability measure on } \mathcal{P} : u^*\nu = \nu \quad \forall u \in U\}.$$

To proceed one hopes to find a closed subgroup F of L containing U which behaves good with respect to the lattice Γ. Furthermore it should be small enough to ensure that U acts ergodically on it, with respect to a natural ergodic measure. This is indeed possible and Ratners theorem says that all ergodic measures are obtained in this way. To continue one needs the following definitions (this discussion is essentially contained in [EskMoSh]):

Definition 3.2 *Let $\mathcal{H}$ be the set of all subgroups F of L such that:*

- *$F \cap \Gamma$ is a lattice in F and*

- *the subgroup generated by all unipotent one parameter subgroups of L contained in F acts ergodically on $\pi(F) \cong F/(F \cap \Gamma)$ (with respect to the F invariant probability measure).*

One knows (see [Ra91] Theorem 1.1)

Proposition 3.7 (M. Ratner) *$\mathcal{H}$ is a countable set.*

Let U be a unipotent one parameter subgroup of L and $F \in \mathcal{H}$. Now one has to kick out the set of points g in the orbit of Fg containing Ug which are already in an orbit of a group of smaller dimension. Therefore one defines:

$$\begin{aligned} N(U,F) &:= \{g \in L : U \subset g^{-1}Fg\} \\ S(U,F) &:= \bigcup \{N(U,F') : F' \in \mathcal{H},\ F' \subset F,\ \dim F' < \dim F\} \end{aligned}$$

One has

Lemma 3.8 *Let $g \in L$ and $F \in \mathcal{H}$. Then $g \in N(U,F) \backslash S(U,F)$ if and only if the group $g^{-1}Fg$ is the smallest closed subgroup of L which contains U and whose orbit through $\pi(g)$ is closed in $\mathcal{P} = L/\Gamma$. Moreover in this case the action of U on $\pi(F)g$ is ergodic with respect to a finite $g^{-1}Fg$ invariant measure.*

A consequence of this is

$$\pi(N(U,F)\backslash S(U,F)) = \pi(N(U,F))\backslash \pi(S(U,F)) \quad \forall F \in \mathcal{H}.$$

Finally by Ratners theorem (cf [Ra91]) there is a close connection between ergodic measures in $\mathcal{M}_U(\mathcal{P})$ and orbits of subgroups F out of $\mathcal{H}$:

Theorem 3.9 (M. Ratner) *For every ergodic $\mu \in \mathcal{M}_U(\mathcal{P})$ there exists an $F \in \mathcal{H}$ and a $g \in L$ such that μ is $g^{-1}Fg$ invariant and $\mu(\pi(F)g) = 1$.*

In particular, spaces of coverings of marked tori branched over the marked points, where the marked points are varying, give examples of such homogeneous spaces of F structures. They contain billiards with a wall as described below. Such billiards are studied by A. Eskin, H. Masur and the author [EMS] in a forthcoming paper. The parameter space of n marked tori for example is given by $SL_2(\mathbb{R}) \ltimes \mathbb{R}^{2n}/SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^{2n}$. We remark that the values of the Siegel Veech constants are not known, it is one goal of the above mentioned paper to compute them for spaces of torus coverings. The next chapter contains computations of the Siegel Veech constants for spaces of marked tori. The methods which are used are elementary, just by counting quadratic growth rates of certain lattice points in $\mathbb{R}^2$. We are able to prove some properties of the Siegel Veech constants in this case (see also the introduction). Most important the maximality of the constants for the Haar measure on the whole parameter spaces is proved. It seems

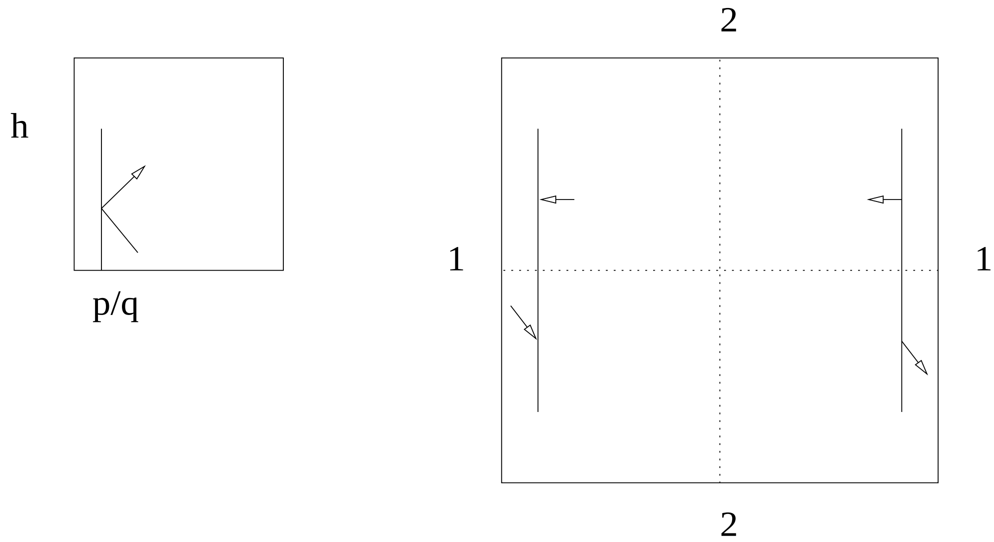

Figure 3.1: The phase space P (right) of a billiard with a wall at $x = p/q$ (left) is a torus covering.

not impossible that the results in the next chapter are true for covering spaces of two marked tori, or equivalently to the spaces generated by a billiards with a (rational) wall, as shown in Figure 3.1. The right picture shows the phase space P of the billiard in the square with a wall on its left hand side. In the phase space the wall becomes two slits, which have to be identified as the arrows indicate. The phase space is a torus covering if and only if the wall sits on a rational x coordinate, say $x = p/q$ with $\gcd(p, q) = 1$. It has degree q, if we assume the length of the boundaries of the billiard square are one. To see this one can take the map

$$\mathbb{R}^2/2\mathbb{Z}^2 \ni [x, y] \longmapsto [qx, y] \in \mathbb{R}^2/2\mathbb{Z}^2$$

which defines the covering. The images of the endpoints of the slits are the two points over which the covering is branched. The two conical singularities at the endpoints of the slits have, by the rule of the identification total cone angle 4π. Varying the two branchpoints causes a two parameter family of torus coverings. Changing the hight of the wall in the billiard square gives a one parameter subfamily. The above described theory establishes quadratic growth limits for every point in the parameter spaces in both cases.

At the end of this section we note a further consequence of Theorem 3.4 (or Corollary 3.5)

Corollary 3.10 (A. Eskin, H. Masur) *Let $\mathcal{V}$ denote the vectors associated to irregular closed cylinders of geodesics on $u \in \mathcal{Q}(g, P)$ and let μ_1 the "Liouville measure" on $\mathcal{Q}(g, P)$. Then $c_{\mathcal{V},\mu} = 0$, thus the quadratic growth rate of irregular cylinders of closed geodesics is zero almost everywhere.*

This is in general not the case for torus coverings or other Veech surfaces.

Chapter 4

Marked tori

4.1 Notation and preparation

For $x_1, ..., x_n \in \mathbb{R}$ let $\mathbb{T}^2_{([x_0],[x_1],...,[x_{n-1}])}$ be the translation torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ marked at the points $[x_0], [x_1], ..., [x_{n-1}] \in \mathbb{T}^2$. Without any restriction of the generality we can assume $[x_0] = 0 + \mathbb{Z}^2$ and we write $\mathbb{T}^2_{[x_1,...,x_{n-1}]}$ instead of $\mathbb{T}^2_{([x_0],[x_1],...,[x_{n-1}])}$. Our calculations are restricted to the torus which is isometric to $\mathbb{R}^2/\mathbb{Z}^2$ with the induced Euclidean metric. With respect to this choice the periodic directions (of the geodesic flow) are exactly the rational ones. This is by no means a restriction to the case of the general translation torus, if one replaces rational direction by periodic direction. Whenever we speak about a vector v_s associated to a saddle connection s (or a cylinder of periodic orbits) this is the vector in $\mathbb{R}^2$ that has the same direction and the same length as s with respect to the natural covering map $\mathbb{R}^2 \to \mathbb{T}^2$. By the results of Veech [Vch89] or Gutkin and Judge [GutJdg97] $\mathbb{T}^2_{[x_1,...,x_{n-1}]}$ is a torus covering exactly if for all $i \in \{1, ..., n-1\}$ the marked points x_i have rational coordinates. So $\mathbb{T}^2_{[x_1,...,x_{n-1}]}$ is a Veech surface if and only if x_i is rational for all $i \in \{1, ..., n-1\}$. The result of Vorobetz [Vrb96b] (Proposition 5.6) states in this case $V\left(\mathbb{T}^2_{[x_1,...,x_{n-1}]}\right) \subset SL_2(\mathbb{Z})$, where $V\left(\mathbb{T}^2_{[x_1,...,x_{n-1}]}\right)$ denotes the Veech group of $\mathbb{T}^2_{[x_1,...,x_{n-1}]}$.

Important convention: if we write a rational vector $(\frac{p}{n}, \frac{q}{n})$, we automatically assume that $\gcd(p, q, n) = 1$.

Definition 4.1

1. *Two points* $[x], [y] \in \mathbb{T}^2$ *are said to be* **relatively rational**, *if* $x - y \in \mathbb{Q}^2$.
2. $SC_{(x,y)}$ *denotes the set of saddle connections between the points* $[x] \in \mathbb{T}^2$ *and* $[y] \in \mathbb{T}^2$.

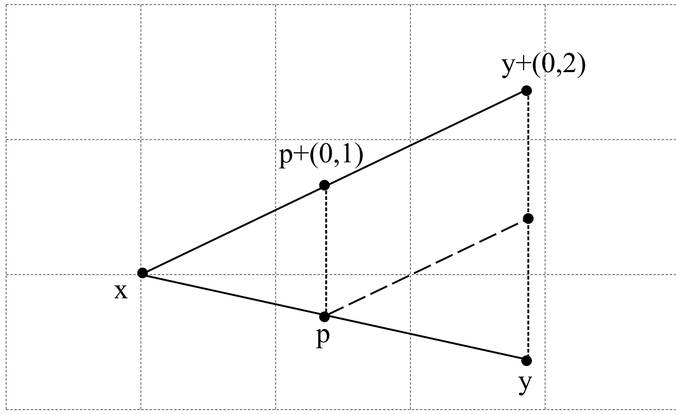

Figure 4.1: The figure indicates for $r = 2$: If $|py| = r|xy|$ then r has to be rational.

3. *$l(s)$ is the length of a $s \in SC_{(x,y)}$ and finally if $[p] \in s \in SC_{(x,y)}$ then $l^x_s(p)$ $(l^y_s(p))$ denotes the distance between $[x]$ ($[y]$) and $[p]$,* **measured along** *s.*

Relatively rational is obviously an equivalence relation. Every marking splits up in classes of relatively rational points.

Proposition 4.1 *Let $[x], [y] \in \mathbb{T}^2$ and $s_1, s_2 \in SC_{(x,y)}$. Then for all $p \in s_1 \cap s_2$:*

$$l^x_{s_i}(p) = \frac{q_p}{n} l(s_i) \quad \text{where } q_p \in \mathbb{Z} \text{ with } 1 \leq q_p \leq n = |\{p : p \in s_1 \cap s_2\}|$$

holds. The ratio q_p/n does not depend on the choice of the saddle connection s_1 or s_2, we call it the "d relation" (as a shorthand for division relation) of p [w.r.t. $SC_{(x,y)}$]. The distance $d_{s_i}(p_1, p_2)$ between two consecutive intersection points p_1, p_2 is

$$d_{s_i}(p_1, p_2) = \frac{1}{n}(l^x_{s_i}(p_2) - l^x_{s_i}(p_1)) \text{ for } i = 1 \text{ or } i = 2$$

measured always along s_1 or s_2.

Proof: We take representatives in $\mathbb{R}^2$, $[x]$ is supposed to be represented by $(0, 0)$. The saddle connections s_i are represented by the lines $\tilde{s}_i$, starting at the origin. Since we have to identify points modulo $\mathbb{Z}^2$, the statement follows already from the Strahlensatz (see Figure 1) of elementary geometry.

□

Definition 4.2 *A* **lattice** *G is a subset of $\mathbb{R}^2$ of the form*

$$G = (p_1 + q_1\mathbb{Z}, p_2 + q_2\mathbb{Z}) \quad p_i, q_i \in \mathbb{R}^+$$

A **point distribution** *is a finite union of lattices. The set of* **visible points** *G^V of of a point distribution G consists of all points $p \in G$ such there exists no $0 < |\lambda| < 1$ and another point $q \in G$ with $q = \lambda p$.*

Loosely speaking: There is no other lattice point of G on the line between the origin and the points $\pm p$. For the correct counting of periodic orbits and saddle connections we need further:

Definition 4.3 *If a point distribution G is contained in another point distribution H then the visible points H^{V_G} of H with respect to G are those $p \in H^V$ for which there is a $\lambda \neq 0$ such that $\lambda p \in G$. In this case G will be called H* **complete** *if $H^{V_G} = G^V$.*
G is called **closed**, *if $mG \subset G$ for any integer $m \in \mathbb{Z}\backslash\{0\}$ such there exist two points $x, y \in G$ with $x = my$.*

Let $[x], [y]$ be as in the above Proposition, again we put $0 = x \in \mathbb{R}^2$. Then the set of saddle connections $SC_{(x,y)}$ is described by the "lattice of vectors" $G_{(x,y)} = y + \mathbb{Z}^2$. Where under a lattice of vectors (instead of points) we understand the set of all lines from the origin to the points of a given lattice. When we speak about the length of elements (or "lines") v in some point distribution, we always use the Euclidean norm $\|v\|$.

Lemma 4.2 *Let $[x], [y], [z] \in \mathbb{T}^2$ be different points*

$[z] \in s \in SC_{(x,y)}$ is an intersection point of two saddle connections, if and only if $l_s^x(z) \in \frac{p}{q} l(s)$ $1 \leq p \leq q$ $\quad p, q \in \mathbb{N}$

If $[z]$ is the intersection point of two saddle connections in $SC_{(x,y)}$ then the following relation for the for the lattices of vectors holds

$$\frac{q}{p} G_{(x,z)} \cap G_{(x,y)} = qG_{(x,y)}$$

where p/q is the d relation of z and $qG_{(x,y)} := y + q\mathbb{Z}^2$.

Proof: Because of Proposition (1) it is enough to show: if

$$[z] \in s \in SC_{(x,y)} \text{ and } l_s^x(z) \in \frac{p}{q} l(s) \quad 1 \leq p \leq q \quad p, q \in \mathbb{N}$$

then there exists some other saddle connection $s \in SC_{(x,y)}$, passing through $[z]$. This follows in fact from the second statement, which we prove now. By assuming $x = 0$ we have $z = \frac{p}{q} y$ and therefore the condition that $z + (k, l)$ with $(k, l) \in \mathbb{Z}^2$ lies on a saddle connection represented by $y + (m, n)$ with $(m, n) \in \mathbb{Z}^2$ is:

$$z + (k, l) = \frac{p}{q} y + (k, l) = \lambda(y + (m, n)) \quad \text{with } 0 < \lambda < 1.$$

Solutions of this equation which are in $G_{(x,y)} = y + \mathbb{Z}^2$ are exactly given by $\lambda = \frac{p}{q}$ together with pairs of (m, n) which are multiples of q, so we have:

$$G_{(x,z)} \cap \frac{q}{p} G_{(x,y)} = \frac{p}{q} y + p\mathbb{Z}^2 = pG_{(x,z)}.$$

The statement follows by multiplying with $\frac{q}{p}$.

$\square$

The Lemma is viewed as the generalization of the following observation on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ (with the origin as a marked point): a point p is rational if and only if it is in the intersection of saddle connections. If this is the case the saddle connections through p gave rise to a lattice (by taking representatives in $\mathbb{R}^2$). The moral of the Lemma is: when we count saddle connections for general markings all markings which are lying on some saddle connection and dividing the length of it rationally will cause a lattice of intersection points. This is important as we will see later, this phenomena decreases the asymptotic constant of that marking. Because of this observation we also have to take care about the definition of "non rational" markings in $\mathbb{T}^{2^n}$, at least if we want to count saddle connections. Since we want the non rational markings to be the set of markings where the growth rates are maximal. So for saddle connections we have no other choice than to define :

$$\mathbb{T}^{2^n}_{nrat(sc)} = \left\{ \begin{array}{c} (x_1, ...x_n) \in \mathbb{T}^{2^n} : x_i \in s \in SC(x_j, x_k) \text{ for at most one } s, \text{ for} \\ \text{all triples } (i,j,k) \in \{1, ..., n\} \text{ with pairwise distinct entries.} \end{array} \right\} \tag{4.1}$$

We define further for saddle connections the set of "rational n markings" to be

$$\mathbb{T}^{2^n}_{rat(sc)} = (\mathbb{T}^{2^n}_{nrat(sc)})^C$$

This is of course the set of all n markings, where there exists at least one marked point which lies on more than one saddle connection. Though the set of non rational n markings is a set of full Lebesque measure on $\mathbb{T}^{2^n}$ (see Theorem 4.16)(because to end up with a non rational marking any further marked point on $\mathbb{T}^2_{(x_0,...,x_{n-1})}$ can be taken out of the set of all points which are on at most one saddle connection). For periodic orbits we define rational and non rational in the usual way:

$$\mathbb{T}^{2^n}_{rat(po)} = \{[x] : x \in \mathbb{Q}^{2^n}\}.$$

4.2 Quadratic growth rates of lattices and point distributions in $\mathbb{R}^2$

To compute the quadratic growth rates we need some preparatory Lemmas. The first is quite well known and uses an argument going back to Gauss. If $G \subset \mathbb{R}^2$ (in general a lattice) we define for $T > 0$

$$N(G^{(V)}, T) := |G^{(V)} \cap B(0,T)|.$$

Now we try to evaluate the right hand side of the last equation for the simplest kind of lattices:

Lemma 4.3 *Let* $G = \{(x, y) \in \mathbb{R}^2 : (x, y) = (p_1 + q_1\mathbb{Z}, p_2 + q_2\mathbb{Z})\}$. *Then*

$$N(G, T) = |G \cap B(T, 0)| = \frac{\pi}{q_1 q_2} T^2 + o(T^2).$$

Proof: To begin we treat the case $p_1 = p_2 = 0$. By drawing horizontal and vertical lines through the lattice points, we got a decomposition of the plane in rectangles with sidelengthes (q_1, q_2). Every lattice point of G can be viewed as the upper right vertex of one rectangle. So the number of points of G in the concentric circle $B(0, T)$ of radius T is determined by the number of rectangles in that circle up to a certain correction term coming from the fact that a rectangle might be only partly inside the ball. But we have after increasing the radius of the ball by the length of the diagonal of the rectangles $\sqrt{q_1^2 + q_2^2}$ the estimate:

$$N(G, T) \leq \frac{Vol\left(B\left(0, T + \sqrt{q_1^2 + q_2^2}\right)\right)}{q_1 q_2}$$

And by decreasing the radius with the same amount:

$$\frac{Vol\left(B\left(0, T - \sqrt{q_1^2 + q_2^2}\right)\right)}{q_1 q_2}) \leq N(G, T)$$

Putting together these two estimates we see:

$$\left| N(G, T) - \frac{\pi}{q_1 q_2} T^2 \right| \leq \frac{\pi}{q_1 q_2} \left(2T \sqrt{q_1^2 + q_2^2} + q_1^2 + q_2^2 \right)$$

In the general case we have the estimate:

$$\begin{aligned} \left| (G - (p_1, p_2)) \cap B\left(0, T - \sqrt{p_1^2 + p_2^2}\right) \right| \leq |G \cap B(0, T)| \leq \\ \leq \left| (G - (p_1, p_2)) \cap B\left(0, T + \sqrt{p_1^2 + p_2^2}\right) \right| \end{aligned} \tag{4.2}$$

the statement follows easily.

□

Before we state the next Proposition let us make a general remark on our strategy of counting. We are interested in counting for example prime closed geodesics, so we associate a certain point distribution $G \subset \mathbb{Z}^2$ to all geodesics in question. The point distribution G does not reflect a priory the prime geodesics, but it will be chosen in a way that its quadratic asymptotic as in the last Lemma can be calculated easily. The points in G which are associated to prime geodesics will be the visible points of G with respect to $\mathbb{Z}^2$. The price we have to pay is that we cannot relate the distribution of visible points and all points of (the completed and closed) G just by the factor $\frac{6}{\pi^2}$. We have to choose our multiplicities with respect to G:

Proposition 4.4 *Let $G \subset \mathbb{Z}^2$ be a $\mathbb{Z}^2$ complete and closed lattice. Let $K \subset \mathbb{Z}\backslash\{0\}$ be the set of numbers l such that $lG^V \subset G$. Then the following relation holds*

$$N(G^V,T) = N(G,T)\left(\sum_{l\in K}\frac{1}{l^2}\right)^{-1} + o(T^2) \tag{4.3}$$

between the numbers of points of G and the number of visible points G^V of G. For the lattice $G = \mathbb{Z}^2$ this implies:

$$N(\mathbb{Z}^2,T) = N(\mathbb{Z}^{2^V},T)\frac{3}{\pi^2} + o(T^2) \tag{4.4}$$

Proof: First let us check that there is no $x \in G$ that is not an integer multiple of some $y \in G^V$. Assume there is, then there exists a $z \in G^V$ and a rational number $\frac{p}{q}$ with $x = \frac{p}{q}z$. So q divides z, a contradiction to the $\mathbb{Z}^2$ completeness of G. Since G is closed and every element in G is some K-multiple of an element in G^V we have:

$$N(G,T) = \sum_{l\in K} N\left(G^V, \frac{T}{l}\right) + o(T^2) = \left(\sum_{l\in K}\frac{1}{l^2}\right) N(G^V,T) + o(T^2). \tag{4.5}$$

The lattice $G = \mathbb{Z}^2$ is obviously $\mathbb{Z}^2$ complete and closed. We have $K = \mathbb{Z}\backslash\{0\}$, so

$$\begin{aligned} N(\mathbb{Z}^2,T) &= \left(\sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty}\frac{1}{l^2}\right) N(\mathbb{Z}^{2^V},T) + o(T^2) = \\ &= 2\left(\sum_{l=1}^{\infty}\frac{1}{l^2}\right) N(\mathbb{Z}^{2^V},T) + o(T^2) = \frac{\pi^2}{3}N(\mathbb{Z}^{2^V},T) + o(T^2). \end{aligned}$$

□

Counting only a certain percentage p of the length of each line in a point distribution G is equivalent to counting the length of vectors in the point distribution $pG := \{pv : v \in G\}$. Thus we have the relation:

$$N(pG,T) = N\left(G, \frac{T}{p}\right) = \frac{1}{p^2}N(G,T) + o(T^2). \tag{4.6}$$

Remark: As is seen above the relating factor between cylinders of geodesics and prime geodesics will be $\frac{3}{\pi^2}$ instead of $\frac{6}{\pi^2}$ because we don't distinguish between the two directions a prime geodesic might have. Our decision to do that is based on the point of view : count only (the length of) the geometric object.

Definition 4.4 *We define the set of lattices*

$$G_{(q_1,p_1,q_2,p_2)} := (q_1\mathbb{Z} + p_1, q_2\mathbb{Z} + p_2) \subset \mathbb{Z}^2 \text{ with } p_1, q_1, p_2, q_2 \in \mathbb{Z}$$

and $\gcd(p_i, q_i) = 1$. *Further for* $n \in \mathbb{N}$ *and a divisor* m *of* n *define* I_m^n *to be the set of numbers which are relatively prime to* $\frac{n}{m}$

$$I_m^n = \left\{ j \in \{1, 2, ..., \frac{n}{m} - 1\} : \gcd(j, \frac{n}{m}) = 1 \right\}$$

(if $m = 1$ *we write shortly* I^n *instead of* I_m^n*) and the point distributions*

$$G_{I_m^n} := \bigcup_{k \in I_m^n} G_{(n,mk,n,0)} \subset \mathbb{Z}^2.$$

With this conventions

Lemma 4.5 *The point distributions* G_{I^n} *are for all* $n \in \mathbb{N}$ *complete with respect to* $\mathbb{Z}^2$ *and closed. We have:*

$$N(G_{I^n}, T) = N(G^V_{I^n}, T) 2 \left[\sum_{k \in I^n} \sum_{l=1}^{\infty} \frac{1}{(nl + k)^2} \right]. \tag{4.7}$$

Especially for all prime numbers p *we have*

$$N(G_{I^p}, T) = N(G^V_{I^p}, T) \left[\frac{\pi^2}{3} \left(1 - \frac{1}{p^2} \right) \right]. \tag{4.8}$$

Proof: First we try to find all integers which map G_{I^n} under element wise multiplication to G_{I^n} itself. For $l \in \mathbb{Z}$ and $p \in G_{I^n}$ we have $lp \in G_{I_m^n}$, iff

$$l \equiv r \pmod{n} \quad \text{with } r \in I^n.$$

Therefore l has to be in $n\mathbb{Z} + r$ with $r \in I^n$. Obviously multiplication with a number from this set maps points of G_{I^n} again to G_{I^n}. So if the point distributions are complete with respect to $\mathbb{Z}^2$ they are also closed. To see the last statement we take $l \in \mathbb{Z} \backslash \{0\}$ and some $(p_1, p_2) \in \mathbb{Z}^2$ with $l(p_1, p_2) \in G_{I^n}$. This is equivalent to $l(p_1, p_2) = (kn + r, mn)$ with $r \in I^n$ and $k, m \in \mathbb{Z}$. The equation in the first coordinate says that l and p_1 are relatively prime to n and from this it follows that p_2 has to be a multiple of n. So (p_1, p_2) is already contained in G_{I^n} and the completeness follows. Using equation (4.5) from Proposition 4.4 gives the first part of the Proposition. For the second part we simply observe that because p is prime $I^p = \{1, 2, ..., p - 1\}$:

$$\sum_{k \in I^p} \sum_{l=1}^{\infty} \frac{1}{(pl + k)^2} = \sum_{l=1}^{\infty} \frac{1}{l^2} - \sum_{l=1}^{\infty} \frac{1}{(pl)^2}.$$

$\square$

Remark: Obviously the lattices I_m^n are for $m \neq 1$ never $\mathbb{Z}^2$ complete.

Before we compute the relating factor of equation (4.7) in general, an observation is helpful: Let $\chi_0 : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ be the principal character modulo n, that is $\chi_0(l) = 1$ if $\gcd(l, n) = 1$ and is equal to 0 if not, then obviously:

$$\sum_{k \in I^n} \sum_{l=1}^{\infty} \frac{1}{(nl+k)^2} = \sum_{l=1}^{\infty} \frac{\chi_0(l)}{l^2} = L(\chi_0, 2)$$

where $L(\chi_0, 2)$ is the L series with respect to χ_0 evaluated at 2. The following is well known (see for example exercise (3.d) page 166 in [Ten95] or Serre [S])

Proposition 4.6 *If s is a complex number with real part bigger than 1, then:*

$$L(\chi_0, s) = \sum_{l=1}^{\infty} \frac{\chi_0(l)}{l^s} = \zeta(s) \prod_{p|n \ prime} \left(1 - \frac{1}{p^s}\right). \tag{4.9}$$

Proof: The condition on the real part on s guarantees absolute convergence of the series so that the following manipulations are justified. We start with the right hand side:

$$\begin{aligned}\zeta(s) \prod_{p|n \ prime} \left(1 - \frac{1}{p^s}\right) &= \sum_{l=1}^{\infty} \frac{1}{l^s} \left(\sum_{m|n} \frac{\mu(m)}{m^s} \right) = \\ &= \sum_{k=1}^{\infty} \frac{1}{k^s} \left(\sum_{m|\gcd(k,n)} \frac{\mu(m)}{m^s} \right) = \sum_{k=1}^{\infty} \frac{\chi_0(k)}{k^s}.\end{aligned} \tag{4.10}$$

We have used the well known Möbius function μ and the fact that

$$\sum_{d|n} \mu(d) = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } n > 1 \end{cases}$$

to reorder the sum in a useful way.

$\square$

Evaluating the above L series for $s = 2$ we find

$$N(G_{I^n}, T) = N(G_{I^n}^V, T) \frac{\pi^2}{3} \left[\prod_{p|n \ prime} \left(1 - \frac{1}{p^2}\right) \right]. \tag{4.11}$$

The left hand side of this equation is easy to compute with the help of Lemma 4.3 and the definition of G_{I^n}

$$N(G_{I^n}, T) = \frac{\pi}{n^2} |I^n| T^2 + o(T^2). \tag{4.12}$$

We use Eulers φ function to identify

$$|I^n| = \varphi(n) = n \prod_{p|n \ prime} \left(1 - \frac{1}{p}\right) \tag{4.13}$$

and write finally

$$\begin{aligned} \lim_{T\to\infty} \frac{N(G^V_{I^n}, T)}{T^2} &= \frac{3}{\pi}\frac{\varphi(n)}{n^2} \prod_{p|n \ prime} \left(1 - \frac{1}{p^2}\right)^{-1} = \\ &= \frac{3}{n\pi} \prod_{p|n \ prime} \left(1 + \frac{1}{p}\right)^{-1}. \end{aligned} \tag{4.14}$$

For later use we show

Proposition 4.7 *G_{I^n} contains the $\mathbb{Z}^2$ completion of the lattice $G_{(n,r,n,0)}$ $=\{(n\mathbb{Z} + r, n\mathbb{Z}) : \ r \in I^n\}$.*

Proof: It is enough to show that a multiple of each point $(kn + l, mn) \in G_{I^n}$ is in $G_{(n,r,n,0)}$. That is we have to solve $rx \equiv l \pmod n$. But this is always possible since r and l are relativ prime with respect to n.

□

4.3 The Veech group of $\mathbb{T}^2_{(\frac{p_1}{q_1},\frac{p_2}{q_2})}$ and its index in $SL_2(\mathbb{Z})$

We now describe the affine group $Aff(\mathbb{T}^2_{[x]})$ for a rational point $x \in \mathbb{Q}^2$.

Proposition 4.8 *Assume $(\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2}), (\frac{1}{2}, 0) \neq x \in \mathbb{Q}^2$. Then*

$$Aff(\mathbb{T}^2_x) = \left\{ z \mapsto Az + v : A \in SL_2(\mathbb{Z}) \ \text{with} \ \left\{ \begin{array}{lll} Ax \equiv -x & \pmod{\mathbb{Z}^2} & \text{if } v = x \\ Ax \equiv x & \pmod{\mathbb{Z}^2} & \text{if } v = 0 \end{array} \right\} \right.$$

For the three exceptional points we have

$$Aff(\mathbb{T}^2_{x_2}) = \left\{ z \mapsto Az + v : A \in SL_2(\mathbb{Z}) \ \text{with} \ \left\{ \begin{array}{ll} Ax \equiv x & \pmod{\mathbb{Z}^2} \\ v = x & \end{array} \right\} \right.$$

Proof: $A \in SL_2(\mathbb{Z})$ is a consequence of Proposition 5.6 in Vorobetz [Vrb96b]. The only thing which is left to prove is the statement on the translation vector v. It is defined only modulo $\mathbb{Z}^2$, so we can assume $v \in [0,1)^2$. The subgroup $N(\mathbb{T}^2_x)$ of pure translations is trivial if $x \neq (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2}), (\frac{1}{2}, 0)$:
If the lattice $\mathbb{Z}^2_x = x + \mathbb{Z}^2$ generated by x should be translated to $\mathbb{Z}^2$ by v and by the same translation $\mathbb{Z}^2$ to $\mathbb{Z}^2_x$, then $v \equiv x \equiv -x \pmod{\mathbb{Z}^2}$ must hold. Thus either v is

trivial, or x is one of the exceptions. In the general case the same problem comes up if the map should interchange the lattices $\mathbb{Z}^2$ and $\mathbb{Z}^2_x$. The only possibility is using A to map $\mathbb{Z}^2_x$ to $\mathbb{Z}^2_{-x}$ and then to translate this into $\mathbb{Z}^2$. Thus we have to fulfill the conditions:

$$Ax \equiv -x \pmod{\mathbb{Z}^2} \text{ and } v = x \quad \text{or} \quad Ax \equiv x \pmod{\mathbb{Z}^2} \text{ and } v = 0$$

For the three exceptions there is in fact only one condition, because in these cases we have $-x \equiv x \pmod{\mathbb{Z}^2}$, so $-id$ is automatically contained in these affine groups, as well as the translation $v = x$.

□

Theorem 4.9 *The Veech group $V(\mathbb{T}^2_x)$ where $x = (\frac{p_1}{q_1}, \frac{p_2}{q_2})$*

- *for $1 \leq \gcd(q_1, q_2) < \min(q_1, q_2)$ is*

$$V(\mathbb{T}^2_x) = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{array}{llll} a \equiv 1 & \pmod{q_1} & b \equiv 0 & \pmod{q_2} \\ c \equiv 0 & \pmod{q_1} & d \equiv 1 & \pmod{q_2} \end{array} \right\}$$

- *for $q_1 | q_2$ (or $q_2 | q_1$ analogously) is*

$$V(\mathbb{T}^2_x) = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{array}{c} a + \frac{q_1 p_2}{q_2} b \equiv 1 \pmod{q_1} \\ b \equiv 0 \pmod{\frac{q_2}{q_1}} \\ c \equiv 0 \pmod{q_1} \\ d \equiv 1 \pmod{q_2} \end{array} \right\}$$

- *for $q_1 = q_2$ is*

$$V(\mathbb{T}^2_x) = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{q_1} \right\}$$

  In the case $q_1 = 2$ that is $(\frac{p_1}{q_1}, \frac{p_2}{q_2}) = (\frac{1}{2}, \frac{1}{2})$ we have to add the rotation:

$$r_{\frac{\pi}{2}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- *for $p_2 = 0$ (or equivalently $p_1 = 0$) is*

$$V(\mathbb{T}^2_x) = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \pmod{q_1} \right\}$$

Proof: We evaluate the condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{p_1}{q_1} \\ \frac{p_2}{q_2} \end{pmatrix} \equiv \pm \begin{pmatrix} \frac{p_1}{q_1} \\ \frac{p_2}{q_2} \end{pmatrix} \pmod{\mathbb{Z}^2}$$

on the coefficients $a, b, c, d \in \mathbb{Z}$. If for example $p_2 = 0$ then we have

$$\begin{pmatrix} \frac{p_1}{q_1} a \\ \frac{p_1}{q_1} c \end{pmatrix} \equiv \pm \begin{pmatrix} \frac{p_1}{q_1} \\ 0 \end{pmatrix} \pmod{\mathbb{Z}^2}.$$

It follows directly that $a \equiv \pm 1 \pmod{q_1}$, $c \equiv 0 \pmod{q_1}$ and $b \in \mathbb{Z}$ is arbitrary. Since the determinant of the matrix is always 1 we have the condition $ad \equiv 1 \pmod{q_1}$. Therefore $d \equiv \pm 1 \pmod{q_1}$, if $a \equiv \pm 1 \pmod{q_1}$. The other three cases are equally simple, so we omit them.

□

We prove now that the all 2 marked tori are affine equivalent to one from a very special family, if the marking is rational.

Proposition 4.10 *If $x = (\frac{p}{n}, \frac{q}{n})$ $(\gcd(p_1, p_2, n) = 1)$, then $\mathbb{T}^2_x$ is isomorphic to $\mathbb{T}^2_{(\frac{1}{n}, 0)}$. More precisely: there exists an affine map of $\mathbb{T}^2$ represented by an element of $SL_2(\mathbb{Z})$ which maps $\left[\frac{p}{n}, \frac{q}{n}\right]$ to $\left[\frac{1}{n}, 0\right]$ (and preserves [0]).*

Proof: Since the linear Dehn twist represented by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and the rotation of 90 degrees generates the affine group of $\mathbb{T}^2$ as well as its Veech group $V(\mathbb{T}^2) = SL_2(\mathbb{Z})$ it is enough to show that given a point $[p_1/n, p_2/n] \in \mathbb{T}^2$ with $\gcd(p_1, p_2, n) = 1$ one can map it to the point $[1/n, 0] \in \mathbb{T}^2$ by a linear map from $SL_2(\mathbb{Z})$. Let $c := \gcd(p_1, p_2)$ then we have the equation $\frac{p_2}{c} p_1 - \frac{p_1}{c} p_2 = 0$. If one is viewing the vector $(-\frac{p_2}{c}, \frac{p_1}{c})$ as the second row of a matrix $A \in SL_2(\mathbb{R})$ then the vector (a, b) representing the first row has to fulfill the condition $1 = \det(A) = a\frac{p_1}{c} + b\frac{p_2}{c}$ This equation has solutions $a, b \in \mathbb{Z}$ because $\frac{p_1}{c}$ and $\frac{p_2}{c}$ are relatively prime. Taking solutions a, b defines an $A \in SL_2(\mathbb{R})$ with $A(p_1, p_2) = (c, 0)$ where c is relatively prime to n. Thus the congruence $kc \equiv 1 \pmod{n}$ has a solution $k \in \mathbb{Z}$ and one can use the linear Dehn twists

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \begin{pmatrix} c \\ 0 \end{pmatrix} \equiv \begin{pmatrix} c \\ 1 \end{pmatrix} \pmod{n}$$

and

$$\begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \pmod{n}$$

to obtain the result up to a rotation of 90 degrees.

□

Corollary 4.11 *The Veech groups $V(\mathbb{T}^2_x) \subset SL_2(\mathbb{Z})$ for rational points $x = (\frac{p_1}{n}, \frac{p_2}{n})$ with $\gcd(p_1, p_2, n) = 1$ are all isomorphic. The isomorphism is given by conjugation with an element of $SL_2(\mathbb{Z})$.*

Proof: This fact is a direct consequence of Corollary 2.19.

□

Remark 4.1 *Since every rational number x has a unique representation as above, we can restrict our considerations to the families of markings defined by*

$$x_n := \left(\frac{1}{n}, 0\right) \quad \text{or} \quad x_{(n,n)} := \left(\frac{1}{n}, \frac{1}{n}\right) \quad \text{where} \quad 1 \neq n \in \mathbb{N}$$

and to the corresponding tori. Here we choose the family $\mathbb{T}^2_{x_n}$.

To compute the index $[SL_2(\mathbb{Z}) : V(\mathbb{T}^2_{[\frac{p}{n},\frac{q}{n}]})]$, the first observation is that the only affine map $\phi \in \mathit{Aff}(\mathbb{T}^2_{[\frac{p}{n},\frac{q}{n}]})$ ($\gcd(p, q, n) = 1$) which interchanges $[0]$ and $[\frac{p}{n}, \frac{q}{n}]$ is given by rotation on 180 degrees and a translation by the vector $(\frac{p}{n}, \frac{q}{n})$ (for $n > 2$). Since it maps $[\frac{p}{n}, \frac{q}{n}]$ to 0 it is of no interest for the following discussion. Up to this map ϕ, $\mathit{Aff}(\mathbb{T}^2_{[\frac{p}{n},\frac{q}{n}]})$ can be viewed as the isotropy subgroup $\mathit{Aff}_{[\frac{p}{n},\frac{q}{n}]}(\mathbb{T}^2) \subset \mathit{Aff}(\mathbb{T}^2)$ defined in 2.3 and studied in Proposition 2.16. Lemma 4.10 above implies together with Proposition 2.16 for $n > 2$

$$\begin{aligned} [\mathit{Aff}_{[\frac{p}{n},\frac{q}{n}]}(\mathbb{T}^2) : \mathit{Aff}(\mathbb{T}^2)] &= |\{(a, b) : 0 < a, b \leq n \text{ and } \gcd(a, b, n) = 1\}| \\ &= n^2 \prod_{p|n \ prime} \left(1 - \frac{1}{p^2}\right). \end{aligned} \tag{4.15}$$

The last equation on the cardinality of the set of pairs (a, b) is well known. $d\phi = -id$ is normal in $Sl_2(\mathbb{Z})$ and therefore also in $V(\mathbb{T}^2_{[\frac{p}{n},\frac{q}{n}]})$, thus

$$V_{[\frac{p}{n},\frac{q}{n}]}(\mathbb{T}^2) \cong V(\mathbb{T}^2_{[\frac{p}{n},\frac{q}{n}]})/(-id).$$

Since $-id$ generates a subgroup of order 2 and the subgroup $N(\mathbb{T}^2_{[\frac{p}{n},\frac{q}{n}]}) \subset \mathit{Aff}(\mathbb{T}^2_{[\frac{p}{n},\frac{q}{n}]})$ of pure translations is trivial if $n \neq 2$ we have

$$[SL_2(\mathbb{Z}) : V(\mathbb{T}^2_{[\frac{p}{n},\frac{q}{n}]})] = \frac{n^2}{2} \prod_{p|n \ prime} \left(1 - \frac{1}{p^2}\right).$$

For $n = 2$ the index in question is 3 because $-id$ is already an element in $V_{[\frac{p}{n},\frac{q}{n}]}(\mathbb{T}^2)$.

In contrast to this calculation there is a second way to compute the index of the Veech groups using the asymptotic formula of Veech and the results from the last section. To begin, the Veech groups are presented as subgroups of $SL_2(\mathbb{Z})$ but they can also be viewed as subgroups of its quotient modulo $\pm id$: $PSL(2, \mathbb{Z})$. Since both equivalence classes modulo $-id$ are always in each $V(\mathbb{T}^2_{x_n})$ all the following statements make sense. We denote by $V_p(\mathbb{T}^2_{x_n})$ the image of $V(\mathbb{T}^2_{x_n})$ in $PSL(2,\mathbb{R})$. The orbits $V_p(\mathbb{T}^2_{x_n})(\pm v_{\mathcal{C}_i}) \in \mathbb{R}^2$, $i = 1, ..., k(x_n)$ ($k(x_n)$ is the number of different orbits) of the vectors $\pm v_{\mathcal{C}_i}$ associated to a maximal cylinder $\mathcal{C}_i$ of closed geodesics which fills the torus under $V_p(\mathbb{T}^2_{x_n})$ give always the same asymptotic constant (for the formula see Proposition 6.3 in [GutJdg97]):

$$\lim_{T\to\infty} \frac{N(V_p(\mathbb{T}^2_{x_n})(\pm v_{\mathcal{C}_i}), T)}{T^2} = Vol\left(\mathbb{H}/V_p(\mathbb{T}^2_{x_n})\right)^{-1} \frac{[Aff(\mathcal{C}_i) : Aff_0(\mathcal{C}_i)]}{area(\mathcal{C}_i)}$$

Since each cylinder $\mathcal{C}_i$ fills the torus we have $area(\mathcal{C}_i) = 1$ and $[Aff(\mathcal{C}_i) : Aff_0(\mathcal{C}_i)] = 1$. Here $Aff_0(\mathcal{C}_i)$ is the group generated by linear Dehn twists around $\mathcal{C}_i$. Moreover since x_n is rational $\mathbb{T}^2_{x_n}$ is a covering of a 1 marked torus

$$\begin{aligned} Vol\left(\mathbb{H}/V_p(\mathbb{T}^2_{x_n})\right) &= [PSL_2(\mathbb{Z}) : V_p(\mathbb{T}^2_{x_n})]\, Vol\left(\mathbb{H}/PSL_2(\mathbb{Z})\right) = \\ &= \frac{\pi}{3}[SL_2(\mathbb{Z}) : V(\mathbb{T}^2_{x_n})]. \end{aligned} \tag{4.16}$$

The other side of Veech's asymptotic formula is the quadratic growth rate of the lengths of the maximal cylinders that fill the torus. Their directions are exactly the directions of lines in $\mathbb{R}^2$ which begin at the origin and cross some point of the lattice $x_n + \mathbb{Z}^2$. To get a set of integer points which represent (eventually multiples of the length of) the closed cylinders in question we simply multiply with n and get the well known lattice $G_{(n,1,n,0)} = n\mathbb{Z}^2 + (1,0) \subset \mathbb{Z}^2$. Since we are on the torus $\mathbb{R}^2/\mathbb{Z}^2$ to count the length spectrum of the cylinders correctly we have to look for the visible points of the completion of $G_{(n,1,n,0)}$ with respect to $\mathbb{Z}^2$. This completion, by Proposition 4.7, is the point distribution G_{I^n}, thus we finally have:

$$\lim_{T\to\infty} \frac{N(G^V_{I^n}, T)}{T^2} = \frac{3k(x_n)}{\pi}[SL_2(\mathbb{Z}) : V(\mathbb{T}^2_{x_n})]^{-1}. \tag{4.17}$$

What is left to calculate is $k(x_n)$:

Proposition 4.12

$$k(x_n) = \begin{cases} \frac{1}{2}\varphi(n) = \frac{n}{2}\prod_{p|n \ prime}\left(1 - \frac{1}{p}\right) & if \quad n > 2 \\ 1 & if \quad n = 2 \end{cases}. \tag{4.18}$$

Proof: We observe that the ratio of the length of the two saddle connections connecting $[0]$ and $[x_n]$ and bounding the same cylinder of periodic orbits is an invariant under the operation of the Veech group. Furthermore modulo $V(\mathbb{T}^2_{x_n})$ we can represent every periodic orbit or saddle connection by a point in $Q_n = \{(k,l) \in \mathbb{Z}^2 : \ 0 < k, l \leq n\}$. Thus the directions with only one periodic family are represented by:

$$\{(i,n) \in Q_n : \gcd(i,n) = 1\}.$$

The line from $(0,0)$ to (i,n) intersects exactly one point of the set $x_n + \mathbb{Z}^2$ characterized by

$$\left(x\frac{i}{n}, x\right) \quad \text{with } xi \equiv 1 \pmod{n}.$$

If $i \in \{1, 2, ..., n-1\}$ runs through the numbers relatively prime to n, then x does this as well. So the above regarded length ratio is in any case one of the numbers

$$\frac{\min\{x, n-x\}}{\max\{x, n-x\}} \quad \text{with } x \in I^n.$$

We see, exactly two relatively prime numbers modulo n, namely x and $n-x$ having the same invariant. If this invariant is complete, which will be shown in the next Lemma, then the number of not conjugated orbits under the operation of the Veech group is given by

$$\frac{1}{2}\varphi(n) = \frac{1}{2}\left|\{x \in \{1, 2, ..., n-1\} : \ \gcd(x, n) = 1\}\right| = \frac{n}{2}\prod_{p|n \ prime}\left(1 - \frac{1}{p}\right).$$

The exceptional case $n = 2$ is trivial because there exists only one direction of one cylinder in Q_2.

□

Lemma 4.13 *Let $\mathbb{T}^2_x$ be a rational marked torus. Two directions v_1, v_2 with only one periodic cylinder are in the same orbit under the operation of the Veech group, if they have the same ratio*

$$I(v_i) = \frac{\min\{|s_1^{v_i}|, |s_2^{v_i}|\}}{\max\{|s_1^{v_i}|, |s_2^{v_i}|\}} \quad i = 1, 2.$$

Here $s_j^{v_i}$ $j = 1, 2$ are the two saddle connections which are on the boundary of the cylinder in direction v_i.

Proof: Obviously the condition is necessary all that is left is to show that it is sufficient. That is we have to find an element of the Veech group that is a map between the two directions with the same invariant. Forgetting for a moment the marked point x we can map any periodic direction to any other by elements of $SL_2(\mathbb{Z})$. Especially the saddle connections in question are mapped on one another. Because the invariant is the same for both possible orientations of the saddle connections and $-id \in SL_2(\mathbb{Z})$ we have found a map, which also transports one marked point to the other.

□

Now we can compute the index in question again:

Corollary 4.14 *Let $x = (\frac{p_1}{n}, \frac{p_2}{n})$, $n > 2$ then*

$$[SL_2(\mathbb{Z}) : V(\mathbb{T}^2_x)] = \frac{n^2}{2}\prod_{p|n \ prime}\left(1 - \frac{1}{p^2}\right) \tag{4.19}$$

holds. In the case $n = 2$ the index is 3.

Proof: From equality (4.17) together with $k(x_n) = \frac{1}{2}\varphi(n)$ for $n > 2$ it follows

$$[SL_2(\mathbb{Z}) : V(\mathbb{T}^2_x)] = \frac{3}{2\pi}\varphi(n) \left[\lim_{T\to\infty} \frac{N(G^V_{I^n}, T)}{T^2}\right]^{-1}. \tag{4.20}$$

Then put in the right hand side of equation (4.14) to get the result. If $n = 2$ then $k(x_2) = 1$ and we end up with 3.

□

Remark 4.2 *The two ways of computing the index of the Veech groups of two marked tori might be used to obtain the asymptotic formula 4.14 counting lattice points without using the second section. One can take the first index computation and the Veech formula together with the knowledge about the number $k(x_n)$ (see Proposition 4.12) of cusps to compute $\lim_{T\to\infty} N(G^V_{I^n}, T)/T^2$ backwards.*

Nevertheless the first given way to compute the index of the Veech groups seems to be a very simple method to calculate the index of the well known groups (for example compare S.Lang [Lang])

$$\Gamma_1(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \pmod{n} \right\}$$

in $SL_2(\mathbb{Z})$. Either the $\Gamma_1(n)$ are subgroups of index two in $V(\mathbb{T}^2_{x_n})$ if $n > 2$, or in the case $n = 2$ $-id \in \Gamma_1(2)$ and $V(\mathbb{T}^2_{x_2})$ is isomorphic to $\Gamma_1(2)$. Thus:

$$[SL_2(\mathbb{Z}) : \Gamma_1(n)] = \begin{cases} n^2 \prod_{p|n \ prime} \left(1 - \frac{1}{p^2}\right) & \text{if } n > 2 \\ 3 & \text{if } n = 2 \end{cases}. \tag{4.21}$$

4.4 Quadratic growth rates and constants on marked tori

4.4.1 The 2 marked torus $\mathbb{T}^2_x$

We begin with a general observation concerning the directions of periodic families on the torus. These directions are exactly all the "rational" directions and this fact will not change how many points we will ever mark and where we will ever place them on the torus. The only thing that changes is the number of periodic families in each rational direction, but they are all of equal length. In the case of the two marked torus $\mathbb{T}^2_x$ there are always two families in any rational direction if x is non rational. In this case there are saddle connections connecting $[0]$ and $[x]$ but which bound no periodic family because they are not in a rational direction. For rational markings x this can never happen.

Collecting things for rational x: either we have a rational direction with two periodic families and the bounding saddle connections connecting the same marked point, or we have only one family but then the two bounding saddle connections will connect the marked points $[0]$ and $[x]$. Thus we see counting the length $N_{po}(\mathbb{T}^2_x, T)$ of cylinders of periodic trajectories for non rational markings x on $\mathbb{T}^2_x$ is easy:

$$\lim_{T\to\infty} \frac{N_{po}(\mathbb{T}^2_x, T)}{T^2} = 2 \lim_{T\to\infty} \frac{N_{po}(\mathbb{T}^2, T)}{T^2} = \frac{6}{\pi} \quad \forall x \notin \mathbb{T}^2_{rat}. \tag{4.22}$$

To treat the rational case we have to subtract the directions where only one family is from the doubled number of periodic orbits of the one marked torus. Let us denote the number of single cylinders on $\mathbb{T}^2_x$ with length smaller than T by $N_{po,[x]}(\mathbb{T}^2_x, T)$. Thus we write:

$$N_{po}(\mathbb{T}^2_x, T) = 2N_{po}(\mathbb{T}^2, T) - N_{po,[x]}(\mathbb{T}^2_x, T) + o(T^2).$$

With the help of the discussion at the end of the last paragraph we compute for a rational point $x = (\frac{p_1}{n}, \frac{p_2}{n})$:

$$\begin{aligned} \lim_{T\to\infty} \frac{N_{po}(\mathbb{T}^2_x, T)}{T^2} &= \lim_{T\to\infty} \frac{1}{T^2} \left(2N(\mathbb{Z}^{2^V}, T) - N(G^V_{I^n}, T)\right) = \\ &= \frac{6}{\pi} \left(1 - \frac{1}{2n} \prod_{p|n \ prime} \left(1 + \frac{1}{p}\right)^{-1}\right). \end{aligned} \tag{4.23}$$

To count saddle connections is less easy, but again using the last paragraph (and the discussion before equation (4.17) we can write down the associated point distributions:

$$\begin{aligned} &\lim_{T\to\infty} \frac{N_{sc}(\mathbb{T}^2_x, T)}{T^2} = \\ &\lim_{T\to\infty} \frac{1}{T^2} \left(2N(\mathbb{Z}^{2^V}, T) - 2N(G^V_{I^n}, T)\right) + \\ + \ &\lim_{T\to\infty} \frac{1}{T^2} \sum_{i\in I^n} \left(\frac{1}{i^2} + \frac{1}{(n-i)^2}\right) \frac{n^2}{|I^n|} N(G^V_{I^n}, T) = \cdots \end{aligned}$$

Here 4.14 and Proposition 4.12 is used. The last term needs some explanation. We have seen that $N(G^V_{I^n}, T)$ already counts the length of the simple periodic cylinders. It counts as well the length of of their boundaries that is the sum of the length of the two bounding saddle connections. By Proposition 4.12 there are $\frac{1}{2}\varphi(n)$ different families of orbits of saddle connections labeled by their length ratios. Equation (4.17) shows these orbits under the Veech group all have the same growth rate, namely $\frac{2}{\varphi(n)} N(G^V_{I^n}, T)$. The relation of the length of a saddle connection bounding a family to the length of the family is more or less the invariant I. Anyway it is one of the numbers $\frac{i}{n}$ or $\frac{n-i}{n}$ where $i \in I^n$. With the help of equation (4.6) one finds the above expression. Continuing the evaluation:

$$
\begin{aligned}
\cdots &= \frac{6}{\pi} - \frac{6}{n\pi} \prod_{p|n\ prime} \left(1+\frac{1}{p}\right)^{-1} + \frac{6}{\pi} \prod_{p|n\ prim} \left(1-\frac{1}{p^2}\right)^{-1} \left(\sum_{i\in I^n} \frac{1}{i^2}\right) = \\
&= \frac{6}{\pi}\left(1 + \prod_{p|n\ prim}\left(1+\frac{1}{p}\right)^{-1}\left(\prod_{p|n\ prim}\left(1-\frac{1}{p}\right)^{-1}\sum_{i\in I^n}\frac{1}{i^2} - \frac{1}{n}\right)\right) = \\
&= \frac{6}{\pi}\left(1 + \prod_{p|n\ prim}\left(1-\frac{1}{p^2}\right)^{-1}\sum_{i\in I^n}\left(\frac{1}{i^2}-\frac{1}{n^2}\right)\right). \qquad (4.24)
\end{aligned}
$$

The last equality follows with the help of equation (4.13). For non rational markings we have:

$$
\begin{aligned}
N_{sc}(\mathbb{T}_x^2, T) &= 2N(\mathbb{Z}^{2^V}, T) + N(x\mathbb{Z}^2, T) + o(T^2) = \\
&= \frac{6}{\pi}T^2 + \pi T^2 + o(T^2). \qquad (4.25)
\end{aligned}
$$

Summarizing we have:

Theorem 4.15 *The limits*

$$\lim_{T\to\infty} \frac{N_{po/sc}(\mathbb{T}_x^2, T)}{T^2}$$

exist for all x and we have the estimates

$$\lim_{T\to\infty} \frac{N_{sc}(\mathbb{T}_x^2, T)}{T^2} \leq \lim_{T\to\infty} \frac{N_{sc}(\mathbb{T}_y^2, T)}{T^2} = \frac{6}{\pi} + \pi$$

and

$$\lim_{T\to\infty} \frac{N_{po}(\mathbb{T}_x^2, T)}{T^2} \leq \lim_{T\to\infty} \frac{N_{po}(\mathbb{T}_y^2, T)}{T^2} = \frac{6}{\pi}$$

for y non rational. The inequalities are strict if x is rational. Moreover for each sequence of rational points $\{x_n\}_{n=0}^{\infty}$ converging to a non rational point x:

$$\lim_{n\to\infty}\left(\lim_{T\to\infty} \frac{N_{sc/po}(\mathbb{T}_{x_n}^2, T)}{T^2}\right) = \lim_{T\to\infty} \frac{N_{sc/po}(\mathbb{T}_x^2, T)}{T^2}.$$

This is the continuity of the quadratic constants at non rational points.

Proof: The existence of the limits is proved in the equations (4.22), (4.23), (4.24) and (4.25), even more the computations gave explicitly the values of the limits. In the case of saddle connections it is indeed not easy to read the continuity of the growth rate function out of the expression (4.24) directly (in fact before the author found the proof below he believes the formula was wrong). By comparing equation (4.25) (the limiting constant) to the first line of (4.24) we have to show:

$$\lim_{n\to\infty} \prod_{p|n\ prim} \left(1-\frac{1}{p^2}\right)^{-1} \left(\sum_{i\in I^n} \frac{1}{i^2}\right) = \frac{\pi^2}{6}.$$

By using the equation (4.6), which states

$$\sum_{i\in I^n} \sum_{l=0}^{\infty} \frac{1}{(nl+i)^2} = \frac{\pi^2}{6} \prod_{p|n\ prime} \left(1-\frac{1}{p^2}\right)$$

and the following estimates

$$\sum_{i\in I^n} \left(\frac{1}{i^2}+\frac{\pi^2}{6}\frac{1}{4n^2}\right) < \sum_{i\in I^n} \sum_{l=0}^{\infty} \frac{1}{(nl+i)^2} < \sum_{i\in I^n} \left(\frac{1}{i^2}+\frac{\pi^2}{6}\frac{1}{n^2}\right)$$

we have:

$$\begin{aligned} \frac{\pi^2}{6}\left(1-\frac{1}{n}\prod_{p|n\ prime}\left(1+\frac{1}{p}\right)^{-1}\right) &< \sum_{i\in I^n}\frac{1}{i^2} \prod_{p|n\ prime}\left(1-\frac{1}{p^2}\right)^{-1} < \\ &< \frac{\pi^2}{6}\left(1-\frac{1}{4n}\prod_{p|n\ prime}\left(1+\frac{1}{p}\right)^{-1}\right). \end{aligned} \tag{4.26}$$

Clearly if $n \to \infty$ the right and left hand side of the inequalities converge to $\frac{\pi^2}{6}$. In the case of periodic orbits the continuity follows immediately from equation (4.23).

□

4.4.2 The general case

We generalize the results of the last section to arbitrary many markings, with the exception that we do not try to find explicit formulas for rational markings. For the proof of the main theorem it is helpful to make the following definition:

Definition 4.5 *Let $SC_{(x,y)}$ and $SC_{(x,z)}$ be two sets of saddle connections and suppose y is located on a lattice of saddle connections out of $SC_{(x,z)}$. Then let $G_{(x,y)}^{\subset(x,z)} \subset G_{(x,y)}$ be the lattice of vectors which are parallel to vectors from the lattice $G_{(x,z)}$. Analogously let $G_{(x,z)}^{\supset(x,y)} \subset G_{(x,z)}$ be the lattice of vectors which are parallel to vectors from $G_{(x,y)}$.*

It follows from Corollary 4.2 that these are indeed lattices. The "$\subset$" sign ("$\supset$" respectively) in the notation refers to the intuitive point of view that a vector $v \in G_{(x,y)}^{\subset(x,z)}$ $\left(v \in G_{(x,z)}^{\supset(x,y)}\right)$ lying on $w \in G_{(x,z)}$ $(w \in G_{(x,y)})$ is shorter (longer) than w.

Theorem 4.16 *1. [continuity] Let $\mathbb{T}^2_{[x_1,...,x_{n-1}]}$ be a n marked translation torus. Then the limits*

$$\lim_{T\to\infty}\frac{N_{sc/po}(\mathbb{T}^2_{[x_1,...,x_{n-1}]},T)}{T^2}$$

exist.

2. [$c^{max}_{sc/po}$ on non rational markings] The set ${\mathbb{T}^2}^n_{nrat(sc/po)}$ has full Lebesgue measure on $(\mathbb{T}^2)^n$ and the inequalities

$$\lim_{T\to\infty}\frac{N_{sc}(\mathbb{T}^2_{[x_1,...,x_{n-1}]},T)}{T^2}\le\lim_{T\to\infty}\frac{N_{sc}(\mathbb{T}^2_{[y_1,...,y_{n-1}]},T)}{T^2}=\frac{n(n+1)}{2}\pi+\frac{3n}{\pi}$$

and

$$\lim_{T\to\infty}\frac{N_{po}(\mathbb{T}^2_{[x_1,...,x_{n-1}]},T)}{T^2}\le\lim_{T\to\infty}\frac{N_{po}(\mathbb{T}^2_{[y_1,...,y_{n-1}]},T)}{T^2}=\frac{3n}{\pi}$$

are true for all $(y_0,...,y_{n-1})\in{\mathbb{T}^2}^n_{nrat(sc/po)}$. In both cases (sc's and po's) the inequality is strict, if $(x_0,...,x_n)\in{\mathbb{T}^2}^{n+1}_{rat(sc/po)}$

3. [continuity at non rational points]
If $\{(x^i_0,...,x^i_n)\}^\infty_{i=0}$ $(x^i_0,...,x^i_n)\in{\mathbb{T}^2}^n_{rat(sc/po)}$ is a sequence with $\lim_{i\to\infty}(x^i_0,...,x^i_n)=(y_0,...,y_{n-1})\in{\mathbb{T}^2}^n_{nrat(sc/po)}$ then

$$\lim_{i\to\infty}\left(\lim_{T\to\infty}\frac{N_{sc/po}(\mathbb{T}^2_{[x^i_1,...,x^i_n]},T)}{T^2}\right)=\lim_{T\to\infty}\frac{N_{sc/po}(\mathbb{T}^2_{[y_1,...,y_n]},T)}{T^2}. \tag{4.27}$$

Proof: We prove the statements by induction over the number n of marked points. To start for one point there is nothing to show, moreover we have already seen the proof for $n=2$. So we assume the statements are true for all n markings $(x_0,...,x_{n-1})$ of the torus. We add another marking x_n. If the resulting marking is non rational i.e. $(x_0,...,x_n)\in{\mathbb{T}^2}^{n+1}_{nrat(sc)}$ (for saddle connections) we have

$$N_{sc}(\mathbb{T}^2_{[x_1,...,x_n]},T)=N_{sc}(\mathbb{T}^2_{[x_1,...,x_{n-1}]},T)+\sum_{i=0}^{n}N(SC_{(x_i,x_n)},T)$$

and by the definition of non rational points we have

$$\lim_{T\to\infty}\frac{N(SC_{(x_i,x_n)},T)}{T^2}=\pi$$

for all $i\in\{0,1,...,n-1\}$. In the case $i=n$ we have saddle connections that are starting and ending in x_n, but these bound periodic trajectories. So we have to take care of multiplicities when counting these

$$\lim_{T\to\infty}\frac{N(SC_{(x_n,x_n)},T)}{T^2}=\frac{3}{\pi}$$

so finally

$$\lim_{T\to\infty}\frac{N_{sc}(\mathbb{T}^2_{[x_1,\ldots,x_n]},T)}{T^2} = \lim_{T\to\infty}\frac{N_{sc}(\mathbb{T}^2_{[x_1,\ldots,x_{n-1}]},T)}{T^2}+n\pi+\frac{3}{\pi} = \frac{n(n+1)}{2}\pi+\frac{3n}{\pi}.$$

Since a saddle connection starting from x_n bounds a cylinder of periodic trajectories only if it ends in x_n too, we have to add in each rational direction a new periodic family. So the growth rate for periodic orbits is

$$\lim_{T\to\infty}\frac{N_{po}(\mathbb{T}^2_{[x_1,\ldots,x_n]},T)}{T^2}=\lim_{T\to\infty}\frac{N_{po}(\mathbb{T}^2_{[x_1,\ldots,x_{n-1}]},T)}{T^2}+\frac{3}{\pi}=\frac{3n}{\pi}.$$

If the marking $(x_1,..,x_n)$ is rational it is divided in equivalence classes of relatively rational points. The set of these classes is denoted by $\mathcal{K}_{relrat}(x_1,...,x_n)$. For each class the Veech theory holds (see [Vch89, Vch92, Vch98]) and guarantees that all the limits exist. For the case of the inequalities between the limits we observe that if the marked points are rational there exists always a direction in which there is only one cylinder of closed trajectories. The orbit of this direction under the operation of the Veech group leads to a positive asymptotic growth constant (Proposition 6.1 in [GutJdg97]). But since this causes "multiplicities" (we have to count only visible points) the limit growth rates are in both cases (po's and sc's) smaller than the ones for non rational markings. Thus we have already shown that if one class has more than one point than the growth rates are smaller than for non rational markings. Finally to compute the growth rates in the case of periodic orbits, we observe that each class $k\in\mathcal{K}_{relrat}$ defines a marked Veech torus $\mathbb{T}^2_k$, so we have:

$$N_{po}(\mathbb{T}^2_{[x_1,\ldots,x_n]},T))=\sum_{k\in\mathcal{K}_{relrat}(x_1,\ldots,x_n)}N_{po}(\mathbb{T}^2_k,T)+o(T^2). \tag{4.28}$$

Here we use the fact that between points of different classes there are never saddle connections with rational slope (in rational direction).

Thus it remains to count the saddle connections between points of different classes. To do this let $x_i\notin p_{x_n}$ be a marked point and $SC^{(x_0,\ldots,x_{n-1})}_{(x_i,x_j)}$ the set of saddle connections connecting x_i and x_j on the n marked torus $\mathbb{T}^2_{[x_1,\ldots,x_{n-1}]}$.

Two different things can happen:
First more than one saddle connection from the set $SC_{(x_i,x_n)}$ is parallel to (more precisely "is on top of") one of the saddle connections in the set $SC^{(x_0,\ldots,x_{n-1})}_{(x_i,x_j)}$ (with end point x_j). The set of all indices j of such x_j are denoted with I.

For the second case, more than one saddle connection from $SC_{(x_i,x_n)}$ is parallel to one out of $SC^{(x_0,\ldots,x_{n-1})}_{(x_i,x_j)}$. As above denote the set of all indices of such x_j with K. By Corollary 4.2 the sets of saddle connections for which this holds are already lattices (of vectors) that are described by $G^{\subset(x_i,x_n)}_{(x_i,x_j)}$ or $G^{\supset(x_i,x_j)}_{(x_i,x_n)}$ respectively. Because these are lattices they have limit growth rates and moreover we can write:

$$N(SC^{(x_0,\dots,x_n)}_{(x_i,x_n)},T) = N(G_{(x_i,x_n)},T) - \sum_{x_j\in I} N(G^{\subset(x_i,x_n)}_{(x_i,x_j)},T) \tag{4.29}$$

and for all $j \in K$

$$N(SC^{(x_0,\dots,x_n)}_{(x_i,x_j)},T) = N(SC^{(x_0,\dots,x_{n-1})}_{(x_i,x_j)},T) - N(G^{\supset(x_i,x_j)}_{(x_i,x_n)},T). \tag{4.30}$$

After these changes for all $x_i \notin p_{x_n}$ the induction is complete. From the two equalities it follows immediately: If intersections of the above kind exist (so the marking is rational in our sense) than the associated growth rates are smaller than for a non rational marking. The continuity follows from the fact that by approximating a non rational marking all the denominators of all the d relations are growing over all bounds. By Lemma 4.2 the "periods" of the intersection lattices do the same.

□

4.4.3 Branched coverings of marked tori

We are by the simplicity of our methods not able to conclude anything about (families of) branched coverings of tori, without using the existence of the growth rate limits by the results [EMS] of Eskin and Masur. Even the sets where the growth rate is the biggest is not seen as just the preimage of the non rational markings on the torus, but it has to be contained in this set. This is because the geometry and combinatorics of the covering might cause new sets where the growth rates are different. Geometrically the reason for this is that the length spectrum of the cylinders of periodic trajectories is connected in a way to the saddle connections bounding them, which depends on the covering and where the singular points are. If one makes the simplifying assumption that all inverse images of the marked points are itself marked then from Theorem 4.16 one can conclude:

Corollary 4.17 *Let* $\mathbb{V} \xrightarrow{\pi} \mathbb{T}^2_{[x_1,\dots,x_n]}$ *a covering eventually branched over* $(x_1,\dots,x_n)$ *with the induced translation structure* π^*dz *. Further denote by* $\mathbb{V}_C$ *the translation structure where each inverse image of a marked point is itself marked . Then, after rescaling the volume of* $\mathbb{V}$ *(with respect to the induced flat metric) to one, we have:*

$$\lim_{T\to\infty} \frac{N_{sc}(\mathbb{V}_C,T)}{T^2} = deg(\pi)^2 \lim_{T\to\infty} \frac{N_{sc}(\mathbb{T}^2_{[x_1,\dots,x_n]},T)}{T^2} \tag{4.31}$$

($deg(\pi)$ is the degree of π). Moreover if some of the markings are artificial (i.e. not singular points of the induced metric) then the growth rates after "unmark" any subset of these points will be smaller than the above one. So we have

$$\limsup_{T\to\infty} \frac{N_{sc}(\mathbb{V},T)}{T^2} \le deg(\pi)^2 \left(\frac{n(n+1)}{2}\pi + \frac{3n}{\pi}\right). \tag{4.32}$$

Remark: If one includes the above mentioned result of Eskin and Masur [EMS] then the $\limsup$ in equation (4.32) can be replaced by lim. Similarly one can give the corresponding estimate for maximal periodic cylinders.

Proof: The statement is a direct consequence of the fact that each saddle connection on the torus has exactly $deg(\pi)$ preimages on $\mathbb{V}_C$ and the Theorem 4.16. The second part is clear, because by removing nonsingular markings, the directions in which there are saddle connections are not changed. But in each such direction the number of saddle connections could decrease and their length increase.

$\square$

Remark: With the results of Veech [Vch90] every translation surface can be approximated by torus coverings in spaces of differentials $\mathcal{Q}(g, P)$. Since periodic cylinders and saddle connections are locally stable with respect to deformations in this spaces we have some sort of control on their numbers by the Corollary. But because of the increasing number of artificial markings while approximating a general translation surface or equivalently the increasing degree of our covering, the estimate above is to weak to predict for example quadratic bounds of the growth rates of po's or sc's in general.

List of Figures

1.1 A weaving on the torus (parallel boundaries have to be identified). . . . 23

3.1 The phase space P (right) of a billiard with a wall at $x = p/q$ (left) is a torus covering. 53

4.1 The figure indicates for $r = 2$: If $|py| = r|xy|$ then r has to be rational. 55

Bibliography

[BGKT] M. Boshernitzan, G. Galperin, T. Krüger, S. Troubetzkoy, *Periodic billiard orbits are dense in rational polygons*. Trans. Amer. Math. Soc. 350 (1998), no. 9, 3523-3535.

[EaGd97] C. Earle, F. Gardiner, *Teichmüller discs and Veechs F-structures*. Contemporary mathematics Vol 201,(1997) 165-189.

[Esk98] Alex Eskin, *Counting problems and semisimple groups*. Doc. Math. J. DMV , Extra Volume ICM II (1998) 539-552.

[EskMsr98] Alex Eskin, Howard Masur, *Pointwise asymptotic formulas on flat surfaces*. Preprint (1998) 37 pages.

[EMS] Alex Eskin, Howard Masur, Martin Schmoll *The Siegel Veech constants on spaces of torus coverings*. In preparation.

[EskMcM] Alex Eskin, Curt McMullen, *Mixing, counting and equidistribution in Lie groups*. Duke Math. J. 71 (1993) 181-209

[EskMoSh] Alex Eskin, S. Mozes and N. Shah, *Unipotent flows and counting lattice points on homogeneous spaces*. Annals of Math. 143(1996) 253-299.

[Gut] E. Gutkin, *Billiards in almost integrable polyhedral surfaces*. Ergodic Theory and Dynamical Systems 4(1984) 569-584.

[GutJdg96] E. Gutkin, C. Judge, *The geometry and arithmetic of translation surfaces with applications to polygonal billiards*. Math. Res. Lett.3 (1996) 391-403.

[GutJdg97] E.Gutkin, C. Judge, *Affine mappings of translation surfaces*. Geometry and Arithmetic, preprint 97.

[GuTr96] Serge Troubetzkoy, Eugene Gutkin, *Directional flows and strong recurrence for polygonal billiards*. Proceedings of the International Congress of Dynamical Systems, Montevideo, Uruguay.

[HuMsr82] John Hubbard, Howard Masur, *Quadratic differentials and foliations*. Acta Math. 142 (1979), no. 3-4, 221-274.

[HuSch98] Pascal Hubert, Thomas Schmidt, *Veech groups and polygonal coverings.* Preprint (1998).

[Kat] A. Katok, *The growth rate for the number of singular and periodic orbits for a polygonal billiard.* Comm. in Mathematical Physics 111, pg. 151-160 (1987).

[KeSm98] Richard Kenyon, John Smillie, *Billiards on rational-angled triangles.* Preprint (1998).

[KMS] S. Kerckhoff,H. Masur, J. Smillie, *Ergodicity of Billiard flows and quadratic differentials.* Annals of Math. 124 (1986) 293-311.

[KoZo] Kontsevich, Maxim; Zorich, Anton, Lyaponov exponents and Hodge theory preprint (1997)

[Lang] Serge Lang, *Algebra.* Second edition. Addison-Wesley Publishing Co., Reading, Mass., 1984. xv+714 pp. ISBN: 0-201-05487-6.

[Lang] Serge Lang, *Introduction to modular forms.* Grundlehren der mathematischen Wissenschaften 222, Springer Verlag 1976.

[Msr82] Howars Masur, *Interval exchange transformations and measured foliations.* Ann. of Math., 115 (1982) 169-200.

[Msr86] Howard Masur, *Closed trajectories for quadratic differentials with an application to billiards.* Duke Math. J. 53 (1986) 307-314.

[Msr88] Howard Masur, *Lower bounds for the number of saddle connections and closed trajectories of a quadratic differential.* In Holomorphic Functions and Moduli 1, 215-228, MSRI Publ. 10, Springer Verlag 1988.

[Msr90] Howard Masur, *The growth rate of trajectories of a quadratic differential.* Ergodic Theory and Dynamical Systems 10 (1990) 151-176.

[MS93] Howard Masur, John Smillie, *Quadratic differentials with prescribed singularities and pseudo Anosov diffeomorphisms.* Comment. Math. Helvetici 68 (1993) 289-307.

[Ra91] Marina Ratner, *On Raghunathan's measure conjecture.* Ann. of Math. (2) 134 (1991), no. 3, 545-607.

[S] J.P. Serre, *Cours d'arithmétique.* Le Mathématicien, Presses Universitaires de France.

[Shah] Nimish A. Shah, *Limit distributions of expanding translates of certain orbits on homogeneous spaces.* Proc. Indian Acad. Sci. Math. Sci. 106 (1996), no. 2 105-125.

[Schu] Georg Schumacher, Wolf Barth und andere, *The theory of Teichmüller spaces, a view towards moduli spaces of Kähler manifolds.* In Complex Manifolds, Springer Verlag ,Berlin - New York ,(1990) 251- 310.

[Si89] Karl-Ludwig Siegel, *Lectures on the Geometry of Numbers.* Springer-Verlag, Berlin (1989).

[Str84] Kurt Strebel, *Quadratic Differentials, Grenzgebiete der Mathematik.* Springer-Verlag, Berlin (1984).

[Ten95] G. Tenenbaum, *Introduction à la théorie analytique et probabiliste des nombres.* Cours Spécialisés, Société Mathématique de France, (1995).

[Th88] William Peter Thurston, *On the geometry and dynamics of diffeomorphisms on surfaces.* Bulletin of the American Mathematical Society, Volume 19, Nb.2 (1988), 417-432.

[Tr96] Serge Troubetzkoy, *Recurrence and periodic billiard orbits in polygons.* Preprint (1996) 22 pages.

[Tr97] Serge Troubetzkoy, *Periodic billiard orbits in rank 1 polygons.* Preprint (1997) 20 pages.

[Vch86] W. Veech, *The Teichmüller geodesic flow.* Annals of mathematics 64, 124 (1986) 441-530.

[Vch89] W. Veech, *Teichmüller curves in Moduli space, Eisenstein series, and an application to triangular billiards.* Invent. Math. 97 (1989) 553-583.

[Vch90] W. Veech, *Moduli spaces of quadratic Differentials.* Journal D'Analyse Mathématique Vol. 55 (1990) 341-379.

[Vch92] W. Veech, *The billiard in the regular polygon.* Geom. and Funct. Analysis 2 (1992) 341-379.

[Vch93] W. Veech, *Flat Surfaces.* American Journal of Mathematics 115 (1993) 589-689.

[Vch95] W. Veech, *Geometric realizations of hyperelliptic curves.* in Chaos, Dynamics and Fractals, Plenum (1995).

[Vch95] W. Veech, *Geometric realizations of hyperelliptic curves, 2.* Preprint (1995).

[Vch98] W. Veech, *Siegel measures* Ann. Math. (2) 148, No.3, 895-944 (1998).

[Vrb96a] Y. Vorobets, *Planar structures and billiards in rational polygons.* Russ. Math. Surveys 51 (1996) 177-178.

[Vrb96b] Y. Vorobets, *Planar structures and billiards in rational polygons: The Veech alternative*. Russ. Math. Surveys 51 (1996) 779-817.

[Vrb97] Y. Vorobets, *Ergodicity of billiards in polygons: The Veech alternative*. Sbornik Mathematics, 188:3 (1997) 389-434.

[VGSt92] Y. Vorobets, G.A. Galperin, A.M. Stepin, *Periodic billiard trajectories in polygons; generating mechanisms*. Russ. Math. Surveys 47.3 (1992) 5-80.

[ZK] A.N. Zemlyakov, A.B. Katok, *Topological transitivity of Billiards in polygons*. Matem. Zametki 18(2)(1975) 291-300. English translation in Math. notes 18(2) (1976) 760-764.